Deep Learning & Applied AI

Linear algebra revisited

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Linear algebra is the study of linear maps on finite dimensional vector spaces

Linear algebra is about matrices as much as astronomy is about telescopes

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- **distributive properties**: $a(u + v) = au + av$ and $(a + b)v = av + bv$ for all $a, b \in \mathbb{R}$ and all $u, v \in V$

Example: Lists of numbers

$\mathbb{R}^n$ is defined to be the set of all n-long sequences of numbers in $\mathbb{R}$:

$$\mathbb{R}^n = \{(x_1, \ldots, x_n) : x_j \in \mathbb{R} \text{ for } j = 1, 2, \ldots, n\}$$

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Addition and multiplication are defined as expected:

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$
$$\lambda(x_1, x_2, \ldots, x_n) = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n)$$

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With these definitions, $\mathbb{R}^n$ is a vector space

Example: Functions

Consider the set of all functions $f : [0, 1] \to \mathbb{R}$ with the standard definitions for sum and scalar product:

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The above forms a vector space. In fact, any set of functions $f : S \to \mathbb{R}$ with $S \neq \emptyset$ (Q: why?) and the definitions above forms a vector space.

Elements of a vector space (called vectors)
are not necessarily lists

A vector space is an abstract entity whose elements
might be lists, functions, or weird objects

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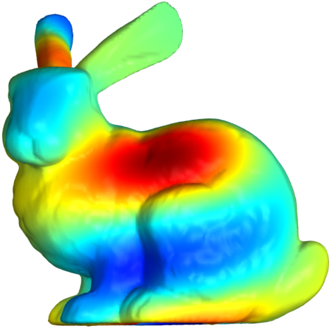

We can still use linear algebra to study functions on surfaces

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A subset $U \subset V$ is a subspace of V if it is a vector space (using the same operations defined for V)

In particular:

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- $\{(x_1, x_2, 0) : x_1, x_2 \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$
- The set of piecewise-linear functions is a subspace of all functions $f : \mathbb{R} \to \mathbb{R}$

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- $v_1, \ldots, v_n \in V$ are linearly independent if and only if each $v \in \operatorname{span}(v_1, \ldots, v_n)$ has only one representation as a linear combination of $v_1, \ldots, v_n$

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So every vector $v \in V$ can be expressed uniquely as a linear combination

$$v = \sum_{i=1}^{n} \alpha_i v_i$$

You can think of a basis as the minimal set of vectors that generates the entire space

Example: Bases

- $(1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$ is a basis of $\mathbb{R}^n$ called the standard basis; its vectors are called the indicator vectors.

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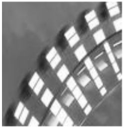
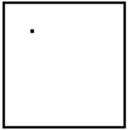
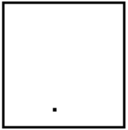
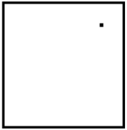
- $$f_1(x) = \begin{cases} 1 & \text{if } x = x_1 \\ 0 & \text{else} \end{cases}$$
  $$f_2(x) = \begin{cases} 1 & \text{if } x = x_2 \\ 0 & \text{else} \end{cases}$$
  $$\vdots$$

  is the standard basis for the set of functions $f : \mathbb{R} \to \mathbb{R}$; the basis vectors are also called indicator functions

Examples

An image expressed in the standard basis:

 $= \alpha_1$  $+\alpha_2$  $+\alpha_3$  $+ \cdots$

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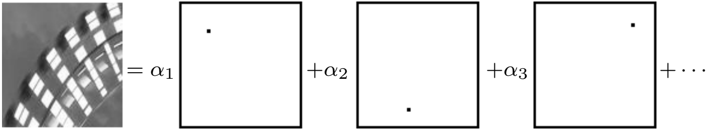
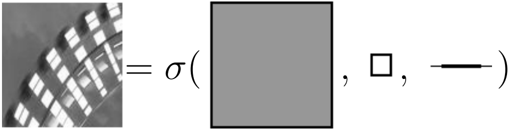


The same image, expressed in terms of a nonlinear map σ:



The image is **not** in the span of the three features.

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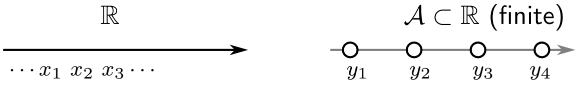

$f : \mathbb{R} \to \mathbb{R}$
infinite dimensional
(functional analysis)

$f : \mathcal{A} \to \mathbb{R}$
finite dimensional
(linear algebra)

Linear maps

A linear map from V to W is a function $T : V \to W$ with the properties:

- **additivity**: $T(u + v) = Tu + Tv$ for all $u, v \in V$
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- a map $T : \mathbb{R}^n \to \mathbb{R}^m$, defined as

$$T(x_1, \dots, x_n) = (A_{1,1}x_1 + \dots + A_{1,n}x_n, \dots, A_{m,1}x_1 + \dots + A_{m,n}x_n)$$

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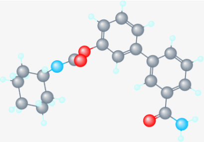
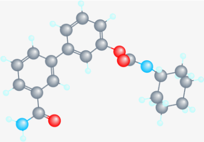
Reflection operation on an image:

$$T : \mathbb{R}^2 \to \mathbb{R}^2\,, \quad T(x, y) = (-x, y)$$



$\Rightarrow$



Linear maps as a vector space

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If $T : U \to V$ and $S : V \to W$, their product $ST : U \to W$ is defined by

$$(ST)(u) = S(Tu)$$

In other words, ST is just the usual composition $S \circ T$ of two functions

Algebraic properties of products of linear maps

- **associativity**: $(T_1T_2)T_3 = T_1(T_2T_3)$
- **identity**: $TI = IT = T$
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Keep in mind that composition of linear maps is not commutative, i.e.

$$ST \neq TS$$

in general (although there are special cases)

Example: Take $Sf = f'$ and $(Tf)(x) = x^2 f(x)$

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$$\mathbf{T} = \begin{pmatrix} T_{1,1} & \cdots & T_{1,n} \\ \vdots & & \vdots \\ T_{m,1} & \cdots & T_{m,n} \end{pmatrix}$$

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Hence each column of $\mathbf{T}$ contains the linear combination coefficients for the image via T of a basis vector from V

Matrices

Consider a linear map $T : V \to W$, a basis $v_1, \ldots, v_n \in V$ and a basis $w_1, \ldots, w_m \in W$.

The matrix of T in these bases is the $m \times n$ array of values in $\mathbb{R}$

$$\mathbf{T} = \begin{pmatrix} T_{1,1} & \cdots & T_{1,n} \\ \vdots & & \vdots \\ T_{m,1} & \cdots & T_{m,n} \end{pmatrix}$$

whose entries $T_{i,j}$ are defined by

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In other words, the matrix encodes how basis vectors are mapped, and this is enough to map all other vectors in their span, since:

$$Tv = T(\sum_j \alpha_j v_j) = \sum_j T(\alpha_j v_j) = \sum_j \alpha_j Tv_j$$

The matrix is a representation for a linear map, and it depends on the choice of bases

Matrix of a vector

Suppose $v \in V$ is an arbitrary vector, while $v_1, \ldots, v_n$ is a basis of V. The matrix of v wrt this basis is the $n \times 1$ matrix:

$$\mathbf{v} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

so that

$$v = c_1 v_1 + \cdots c_n v_n$$

Once again, we see that the matrix depends on the choice of basis for V

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  Q4: do we need the same bases for $S : U \to V$ and $T : V \to W$?

Product of "map matrix" and "vector matrix"

$$\underbrace{\begin{pmatrix} T_{1,1} & \cdots & T_{1,n} \\ \vdots & & \vdots \\ T_{m,1} & \cdots & T_{m,n} \end{pmatrix}}_{\mathbf{T}} \underbrace{\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}}_{\mathbf{c}} = \sum_{j=1}^{n} c_j \underbrace{\begin{pmatrix} T_{1,j} \\ \vdots \\ T_{m,j} \end{pmatrix}}_{\mathrm{Tv_j\ wrt\ (w_1,\ldots,w_m)}}$$

Because recall that, for bases $v_1, \ldots, v_n \in V$ and $w_1, \ldots, w_m \in W$:

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We see then that vector $c = \sum_j c_j v_j$ is mapped to $Tc = \sum_j c_j T v_j$.

In other words, matrix product is behaving as expected.

Suggested reading

Sections 1.A – 3.D of the textbook:

S. Axler, "Linear algebra done right – 3rd edition". Springer, 2015