Recall:

- derivative $f'(a)$ = slope of tangent line to the graph of $f(x)$ at $x=a$.

Rules of differentiation: f, g differentiable functions.

- $(f+g)' = f' + g'$
- $(f-g)' = f' - g'$
- $(cf)' = c \cdot f'$ c is a real number.

- $(fg)' = fg' + g' f$
- $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - g'f}{g^2}$
- $\boxed{(f \circ g)' = f'(g) \cdot g'}$ Chain Rule

- $(e^x)' = e^x$
- $(a^x)' = a^x \cdot \ln a$

- $(\sin x)' = \cos x$
- $(\cos x)' = -\sin x$
- - - -

Important limits:

① $\lim\limits_{h \to 0} \dfrac{e^h - 1}{h} = 1$ ② $\lim\limits_{h \to 0} \dfrac{\sin h}{h} = 1$ ③ $\lim\limits_{h \to 0} \dfrac{\cos h - 1}{h} = 0$

~~Section 3.3~~

~~39. Find~~ $\lim\limits_{x \to 0} \dfrac{\sin(3x)}{5x}$ ~~If we plug in, we get~~ $\dfrac{0}{0}$.

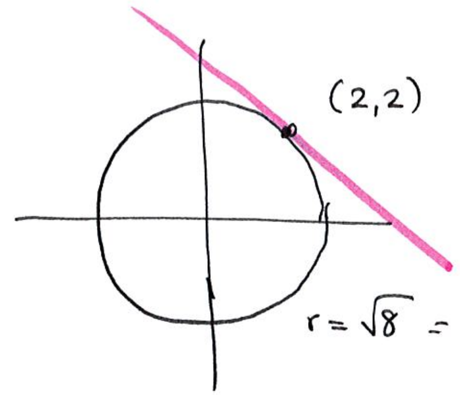
~~A: we want to use~~ $\boxed{\lim\limits_{h \to 0} \dfrac{\sin h}{h} = 1}$

3.5 Implicit differentiation :

eg: Find tangent to

$$x^2+y^2=8 .$$

at the point $(2,2)$.



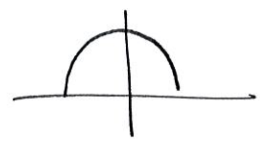
Ans:

$$x^2+y^2=8$$

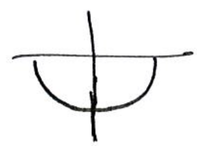
$$\Rightarrow \quad y^2 = 8-x^2$$

$$\Rightarrow \quad y = \pm\sqrt{8-x^2}$$

$\longrightarrow$ $y=\sqrt{8-x^2}$



$\longrightarrow$ $y=-\sqrt{8-x^2}$



for $(2,2)$ we look at

$$y=\sqrt{8-x^2}$$

$$y' = \left(\sqrt{8-x^2}\right)'$$

$$= \left(\left(8-x^2\right)^{1/2}\right)'$$

By Chain Rule

$$= \frac{1}{2}\cdot\left(8-x^2\right)^{-1/2}\cdot\left(8-x^2\right)'$$

outermost function $= x^{1/2}$

derivative $= \frac{1}{2}\cdot x^{-1/2}$

$$\boxed{y' = \frac{1}{2}\left(8-x^2\right)^{-1/2}\cdot(-2x)}$$

at $(2,2)$, $x=2$

$$\Rightarrow \quad y' = \frac{1}{2}\cdot\left(8-2^2\right)^{-1/2}\cdot(-2\cdot 2)$$

$$y' = \frac{1}{2} \cdot (4)^{-1/2} \cdot (-4)$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{4}} \cdot -4$$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot -4$$

$\boxed{y' = -1}$ = slope of the tangent line

$\Rightarrow$ equation of tangent : $(y-2) = (-1) \cdot (x-2)$.

---

eg: Find tangent to

$$x^2 + y^2 = 8$$

at the point $(2,2)$.

Ans : Goal : find $\frac{dy}{dx}$

by treating y as a ~~variable~~ function of x.

Implicit differentiation

• Differentiate both sides with respect to x.

$$x^2+y^2=8 \xrightarrow[\frac{d}{dx}]{} \frac{d}{dx}\left(x^2+y^2\right) = \frac{d}{dx} 8$$

$$\Rightarrow \frac{d(x^2)}{dx} + \frac{d(y^2)}{dx} = \frac{d}{dx} \cdot 8$$

$$\Rightarrow 2x + 2y \cdot \frac{dy}{dx} = \frac{d}{dx} \cdot 8$$

Recall:

$$\frac{d}{dx} f(x) = f'(x)$$

We are treating y as a function of x

say $y = f(x) \Rightarrow \frac{d}{dx} y^2 = \frac{d}{dx}(f(x))^2$

$$= 2 \cdot f(x) \cdot f'(x)$$

$$= 2 \cdot y \cdot y'$$

$$\Rightarrow \quad 2x + 2yy' = 0$$

$$\Rightarrow \quad 2yy' = -2x$$

$$\Rightarrow \quad y' = \frac{-2x}{2y}$$

$$\boxed{y' = -\frac{x}{y}}$$

at $(2,2)$, $y' = -\frac{2}{2} = -1$

$\boxed{y' = -1}$ = slope of tangent.

$\Rightarrow$ equation ~~slope~~ of tangent =

$$(y-2) = -1 \cdot (x-2)$$

Q. Find y' if

$$\cos(xy) = 1 + \sin y.$$

Ans: Differentiating both sides w.r.t. x,

$$\frac{d}{dx} \cos(xy) = \frac{d}{dx}(1 + \sin y)$$

(04)

Chain Rule

$$\Rightarrow -\sin(xy)\cdot(xy)' = (\sin y)'$$

$\cos' = -\sin$

$$\Rightarrow -\sin(xy)\cdot[x'y + y'x] = \cos(y)\cdot y'$$

$$\Rightarrow -\sin(xy)\left[y + \underline{y'}\cdot x\right] = \cos(y)\cdot \underline{y'}$$

$$\Rightarrow -\sin(xy)\cdot y - \sin(xy)\cdot y'\cdot x = \cos(y)\cdot y'$$

$$\Rightarrow -\sin(xy)\cdot y = \sin(xy)\cdot y'\cdot x + \cos y\cdot y'$$

$$= y'\left[\sin(xy)\cdot x + \cos y\right]$$

$$\Rightarrow \boxed{y' = \frac{-\sin(xy)\cdot y}{\sin(xy)\cdot x + \cos y}}$$

~~eg.~~ Th^m $y = \ln x$. Find y'.

~~Ans~~ Proof: $y = \ln x \Rightarrow y = \log_e x$

$$\Rightarrow e^y = x$$

Differentiate both sides ~~w.r.t. x~~

$$\Rightarrow \frac{d}{dx}(e^y) = 1$$

By chain Rule

$$\Rightarrow e^y\cdot y' = 1$$

$$\Rightarrow y' = \frac{1}{e^y} = \frac{1}{x}$$

$$\boxed{\frac{d}{dx}\ln x = \frac{1}{x}}$$

• eg: $y = \arcsin x$ Find y'.

Ans: $y = \arcsin x \Rightarrow y = \sin^{-1} x$

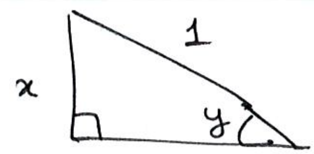
$\Rightarrow \sin y = x$

Apply $\frac{d}{dx}$ to both sides

$\Rightarrow (\sin y)' = 1$

$\Rightarrow \cos y \cdot y' = 1$

$\Rightarrow y' = \dfrac{1}{\cos y}$



By Pythagorus

$= \sqrt{1-x^2}$

$$\boxed{\Rightarrow (\arcsin x)' = \frac{1}{\sqrt{1-x^2}}}$$

---

eg: $y = \arctan x \Rightarrow$ Find y'.

Ans: $y = \arctan x \Rightarrow \tan y = x$

$= \tan^{-1} x$

Apply $\frac{d}{dx}$ to both sides

$\sec^2 y \cdot y' = 1$

$\Rightarrow y' = \dfrac{1}{\sec^2 y} = \dfrac{1}{1+\tan^2 y} = \dfrac{1}{1+x^2}$

$\Rightarrow (\arctan x)' = \dfrac{1}{1+x^2}$

Recall:

$\sin^2 x + \cos^2 x = 1$

Dvide by $\cos^2 x$

$\tan^2 x + 1 = \sec^2 x$

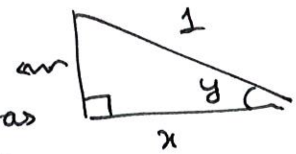
eg: $y = \arccos x$

Ans: $\cos y = x$

Apply $\frac{d}{dx}$ to both sides

$$-\sin y \cdot y' = 1$$

$$\Rightarrow \quad y' = \frac{1}{-\sin y}$$



By Pythagoras $= \sqrt{1-x^2}$

$$\boxed{(\arccos x)' = \frac{-1}{\sqrt{1-x^2}}}$$

- Implicit differentiation.
- $(\ln x)' = \frac{1}{x}$
- $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$
- $(\arctan x)' = \frac{1}{1+x^2}$

Derivative computations

eg Harder question :

$$y = x^x$$

Take ln of both sides

$$\ln y = \ln(x^x)$$

$$\Rightarrow \ln y = x \ln x$$

Apply $\frac{d}{dx}$ to both sides

$$\Rightarrow (\ln y)' = x' \cdot \ln x + x \cdot (\ln x)'$$

$$= 1 \cdot \ln x + x \cdot \frac{1}{x}$$

$$= \ln x + 1$$

$$\Rightarrow \frac{1}{y} \cdot y' = \ln x + 1$$

(By Chain Rule)

$$\Rightarrow y' = y(\ln x + 1)$$

$$(x^x)' = x^x \cdot (\ln x + 1)$$

Recall :

$y = x^n$	$y = a^x$
$y' = n \cdot x^{n-1}$	$y' = a^x \cdot \ln a$
Power is constant	base is constant

(08)

Q. $y = \sqrt{\sin\sqrt{x}}$

Chain Rule

$y' = \left((\sin\sqrt{x})^{1/2} \right)'$

$= \frac{1}{2}\left((\sin\sqrt{x})^{-1/2} \right)$

$\cdot (\sin\sqrt{x})'$

Chain rule

$= \frac{1}{2} \sin(\sqrt{x})^{-1/2}$

$\cdot \cos\sqrt{x} \cdot (\sqrt{x})'$

$= \frac{1}{2} (\sin\sqrt{x})^{-1/2}$

$\cdot \cos\sqrt{x}$

$\cdot (x^{1/2})'$

$= \frac{1}{2} (\sin\sqrt{x})^{-1/2} \cos\sqrt{x} \cdot$

$\frac{1}{2} \cdot x^{-1/2}$

$= \frac{1}{4} (\sin\sqrt{x}\cdot)^{-1/2} \cdot \cos\sqrt{x}$

$\cdot x^{-1/2}$

(start from the outermost ~~function~~ expression and move inward)

Q. $y = (\sin\sqrt{x}) \cdot x^2$

Product Rule

$y' = \left((\sin\sqrt{x}) \cdot x^2 \right)'$

$= (\sin\sqrt{x})' \cdot x^2 + (\sin\sqrt{x})(x^2)'$

Chain Rule

$\cos\sqrt{x} \cdot (\sqrt{x})'$

$= \cos\sqrt{x} \cdot \frac{1}{2} \cdot x^{-1/2}$

$2x$

$= \cos\sqrt{x} \cdot \frac{1}{2} \cdot x^{-1/2} \cdot x^2 + (\sin\sqrt{x}) \cdot 2x$

$= \cos\sqrt{x} \cdot \frac{1}{2} \cdot x^{3/2} + (\sin\sqrt{x}) \cdot 2x$

Q: $y = \sqrt{\sin\sqrt{x}}$

$\Rightarrow y^2 = \sin\sqrt{x}$

Apply $\frac{d}{dx}$ to both sides

$\Rightarrow (y^2)' = (\sin\sqrt{x})'$

$\Rightarrow 2y \cdot y' = \cos\sqrt{x} \cdot (\sqrt{x})'$

$= \cos\sqrt{x} \cdot (x^{1/2})'$

$= \cos\sqrt{x} \cdot x^{-1/2} \cdot \frac{1}{2}$

$\Rightarrow y' = \frac{1}{2y} \cdot \cos\sqrt{x} \cdot x^{-1/2} \cdot \frac{1}{2}$

$= \frac{1}{4} \frac{1}{\sqrt{\sin\sqrt{x}}} \cdot \cos\sqrt{x} \cdot x^{-1/2}$

Q ~~Hard~~ $y = \tan(\arcsin x)$

Ans: $y' = \tan'(\arcsin x) \cdot (\arcsin x)'$

$= \sec^2(\arcsin x) \cdot \frac{1}{\sqrt{1-x^2}}$

Extra work

$= \frac{1}{\cos^2(\arcsin x)} \cdot \frac{1}{\sqrt{1-x^2}}$

Algebraic method

$$= \frac{1}{1-\sin^2(\arcsin x)} \cdot \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{1}{1-\left(\sin(\arcsin x)\right)^2} \cdot \frac{1}{\sqrt{1-x^2}}$$

$$= \frac{1}{1-x^2} \cdot \frac{1}{\sqrt{1-x^2}}$$

$$= (1-x^2)^{-3/2}$$

~~Geometric method~~

~~cos~~(

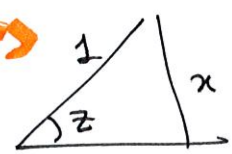
← by trig identity

$\cos^2 + \sin^2 = 1$

Q. $y = \tan(\arcsin x)$

Let $z = \arcsin x$ → $\boxed{y = \tan z}$

$\Rightarrow \sin z = x$



By pythagoras

$\sqrt{1-x^2}$

$$y = \tan z = \frac{x}{\sqrt{1-x^2}}$$

$$\boxed{y = \frac{x}{\sqrt{1-x^2}}}$$

$$y' = \left(\frac{x}{\sqrt{1-x^2}}\right)'$$

$$= \frac{\sqrt{1-x^2}\cdot x' - x\cdot\left(\sqrt{1-x^2}\right)'}{\left(\sqrt{1-x^2}\right)^2}$$

quotient rule

$$\frac{f}{g} = \frac{g\cdot f' - f\cdot g'}{g^2}$$

$$= \frac{\sqrt{1-x^2} - x\cdot\left(\sqrt{1-x^2}\right)'}{(1-x^2)}$$

$$\left(\sqrt{1-x^2}\right)' = \left((\cancel{x}1-x^2)^{1/2}\right)'$$

By chain Rule

$$= \frac{1}{2}\cdot(1-x^2)^{-1/2}\cdot(1-x^2)'$$

$$= \frac{1}{\cancel{2}}\cdot(1-x^2)^{-1/2}\cdot(-\cancel{2}x)$$

$$= \frac{\sqrt{1-x^2} - x\cdot(-\cancel{2}x)(1-x^2)^{-1/2}}{(1-x^2)}$$

$$= \frac{\sqrt{1-x^2} + x^2(1-x^2)^{-1/2}}{1-x^2}$$

~~Q:- Find~~ ~~$y = e^{x} + e^{-x}$~~

Q. ~~E~~ $y = \cos\left(e^{\sqrt{\tan 3x}}\right)$

All Chain Rules

$$y' = \cancel{\cos} -\sin\left(e^{\sqrt{\tan 3x}}\right)\cdot\left(e^{\sqrt{\tan 3x}}\right)'$$

$$= -\,\cancel{\sin\left(e^{\sqrt{\tan}}\right)}\quad e^{\sqrt{\tan(3x)}}\cdot\left(\sqrt{\tan(3x)}\right)'$$

$$(\tan(3x))^{-1/2}\cdot\frac{1}{2}\cdot(\tan(3x))'$$

$$\sec^2(3x)\cdot \underbrace{(3x)'}_{3}$$

$$= -\sin\left(e^{\sqrt{\tan 3x}}\right)\cdot e^{\sqrt{\tan 3x}}\cdot \frac{1}{2}\left(\tan(3x)\right)^{-1/2}\cdot \sec^2(3x)\cdot 3$$

Q. Find y' for $\dfrac{x^2}{x+y} = y^2+1$.

Ans: Apply $\frac{d}{dx}$ to both sides

$$\underbrace{\left(\frac{x^2}{x+y}\right)'}_{} = \underbrace{(y^2+1)'}_{}$$

$$\frac{(x+y)\cdot(x^2)' - x^2\cdot(x+y)'}{(x+y)^2}$$

$$(y^2)' + 0$$

$$= 2y\cdot y' \quad \text{Chain Rule}$$

$$= \frac{(x+y)\cdot 2x - x^2\cdot(1+y')}{(x+y)^2}$$

$$\Rightarrow \frac{(x+y)\cdot 2x - x^2(1+y')}{(x+y)^2} = 2y\cdot y'$$

$$\Rightarrow \quad (x+y)\cdot 2x - x^2(1+\underline{y'}) = 2y\cdot \underline{y'}(x+y)^2$$

$$\Rightarrow \quad (x+y)\cdot 2x - x^2 - x^2\underline{y'} = 2y\cdot\underline{y'}\cdot(x+y)^2$$

$$\Rightarrow \quad (x+y)\cdot 2x - x^2 = x^2y' + 2y\cdot y'\cdot(x+y)^2$$

$$= y'\left(x^2 + 2y(x+y)^2\right)$$

$$\Rightarrow \quad y' = \frac{(x+y)\cdot 2x - x^2}{x^2 + 2y(x+y)^2}$$