DERIVATIVES OF LOGARITHMIC FUNCTIONS

(Sect 3.6)

RMK:

- RECALL THAT IF f IS DIFFERENTIABLE AND HAS AN INVERSE f^{-1}, WE CAN DO THE FOLLOWING:

$$f\left(f^{-1}(x)\right) = x$$

$$\frac{d}{dx}\left(f\left(f^{-1}(x)\right)\right) = \frac{d}{dx}(x)$$

$$f'\left(f^{-1}(x)\right) \cdot \frac{d}{dx}\left(f^{-1}(x)\right) = 1$$

$$\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

- RECALL THAT IF $f(x) = b^x$, THEN $f^{-1}(x) = \log_b(x)$.

AND $f'(x) = \ln(b) \cdot b^x$.

- THIS IMPLIES THAT:

$$\frac{d}{dx}\left(\log_b(x)\right) = \frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'(f^{-1}(x))} =$$

$$= \frac{1}{\ln(b) \cdot b^{(\log_b(x))}} =$$

$$= \frac{1}{\ln(b) \cdot x}$$

THEOREM:

- $$\frac{d}{dx}\left(\log_b(x)\right) = \frac{1}{x \cdot \ln(b)}$$

- IN PARTICULAR: $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$.

EX: 1. FIND $\frac{d}{dx} \ln(\sin(x))$.

Sol: $$\frac{d}{dx} \ln(\sin(x)) = \frac{1}{\sin(x)} \cdot \frac{d}{dx}(\sin(x)) = \frac{\cos(x)}{\sin(x)} =$$

$$= \cot(x).$$

2. DIFFERENTIATE $f(x) = \log_{10}(2 + \sin(x))$

Sol: $$f'(x) = \frac{1}{(2+\sin(x)) \cdot \ln(10)} \cdot \cos(x).$$

LOGARITHMIC DIFFERENTIATION:

RMK:

- SUPPOSE WE WANT TO DIFFERENTIATE $f(x) = x^x$.
- f IS NOT AN EXPONENTIAL FUNCTION OR A POWER FUNCTION, SO IT LOOKS LIKE WE ARE STUCKED...
- HOWEVER, $\ln(f(x)) = \ln(x^x) = x \cdot \ln(x)$.

  THIS IS A PRODUCT, SO WE KNOW HOW TO DIFFERENTIATE IT.

- NOTE THE FOLLOWING:

$$\frac{d}{dx}\left(\ln(f(x))\right) = \frac{1}{f(x)} \cdot f'(x) = \frac{f'(x)}{f(x)}.$$

- SO IN OUR CASE, WE CAN DIFFERENTIATE

$$\ln(f(x)) = x \cdot \ln(x)$$

ON BOTH SIDES, TO OBTAIN

$$\frac{f'(x)}{f(x)} = \ln(x) + x \cdot \frac{1}{x} = \ln(x) + 1$$

USING THE FACT THAT $f(x) = x^x$, WE GET:

$$\frac{f'(x)}{x^x} = \ln(x) + 1$$

SO

$$f'(x) = x^x\left(\ln(x) + 1\right).$$

METHOD (LOGARITHMIC DIFFERENTIATION).

1. TAKE THE NATURAL LOGARITHM OF BOTH SIDES OF AN EQUATION
$$y = f(x).$$

2. DIFFERENTIATE BOTH SIDES.

3. SOLVE FOR y'.

EX: DIFFERENTIATE $f(x) = \dfrac{x^{3/4} \cdot \sqrt{x^2+1}}{(3x+2)^5}$.

Sol: (APPLY LN)

1. $$\ln(f(x)) = \ln\left(\frac{x^{3/4} \cdot \sqrt{x^2+1}}{(3x+2)^5}\right) =$$

$$= \ln\left(x^{3/4} \cdot \sqrt{x^2+1}\right) - \ln\left((3x+2)^5\right) =$$

$$= \ln\left(x^{3/4}\right) + \ln\left(\sqrt{x^2+1}\right) - \ln\left((3x+2)^5\right) =$$

$$= \frac{3}{4}\ln(x) + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

2. (DIFFERENTIATE)

$$\frac{f'(x)}{f(x)} = \frac{3}{4} \cdot \frac{1}{x} + \frac{1}{2} \cdot \frac{1}{x^2+1} \cdot 2x - 5 \cdot \frac{1}{3x+2} \cdot 3$$

$$= \frac{3}{4x} + \frac{1}{x^2+1} - \frac{15}{3x+2}$$

3. (SOLVE FOR f')

$$f'(x) = f(x) \cdot \left(\frac{3}{4x} + \frac{1}{x^2+1} - \frac{15}{3x+2} \right) =$$

$$= \frac{x^{3/4} \sqrt{x^2+1}}{(3x+2)^5} \cdot \left(\frac{3}{4x} + \frac{1}{x^2+1} - \frac{15}{3x+2} \right).$$

EX: DIFFERENTIATE $f(x) = x^{\sqrt{x}}$.

Sol:

- $\ln(f(x)) = \sqrt{x} \cdot \ln(x)$

- $\dfrac{f'(x)}{f(x)} = 1/2 \dfrac{1}{\sqrt{x}} \ln(x) + \sqrt{x} \cdot \dfrac{1}{x}$

$$= \frac{\ln(x)}{2\sqrt{x}} + \frac{1}{\sqrt{x}} =$$

$$= \frac{\ln(x)+2}{2\sqrt{x}}.$$

- $f'(x) = x^{\sqrt{x}} \cdot \left(\dfrac{\ln(x)+2}{2\sqrt{x}} \right).$

RMK

- RECALL THAT IF $f(x) = \ln(x)$

THEN $f'(x) = 1/x$.

- THIS MEANS THAT:

$$1 = \frac{1}{1} = f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} =$$

$$= \lim_{h \to 0} \frac{\ln(1+h) - \overbrace{\ln(1)}^{0}}{h} =$$

$$= \lim_{h \to 0} \frac{\ln(1+h)}{h} =$$

$$= \lim_{h \to 0} \frac{1}{h} \cdot \ln(1+h) =$$

$$= \lim_{h \to 0} \ln\left((1+h)^{1/h}\right).$$

- SO $\lim_{h \to 0} \ln\left((1+h)^{1/h}\right) = 1$.

- SINCE e^x IS CONTINUOUS:

$$\lim_{h\to 0} e^{\left(\ln\left((1+h)^{1/h}\right)\right)} = e^{\left(\lim_{h\to 0} \ln\left((1+h)^{1/h}\right)\right)} = e^1 = e$$

- AND THIS IS JUST $\lim_{h\to 0} (1+h)^{1/h}$.

SO $\lim_{h\to 0} (1+h)^{1/h} = e$.

- NOW IF $x = 1/h$, THEN IF $h \to 0^+$, $x \to \infty$

SO:

$$\lim_{x\to\infty} \left(1 + 1/x\right)^x = e$$

THEOREM:

$$\lim_{x\to\infty} \left(1 + 1/x\right)^x = e \quad , \quad \lim_{x\to 0} (1+x)^{1/x} = e$$

RELATED RATES (Sect 3.9)

IDEA: COMPUTE THE RATE OF CHANGE OF ONE QUANTITY IN TERMS OF THE RATE OF CHANGE OF ANOTHER QUANTITY.

Ex: 1. AIR IS BEING PUMPED INTO A SPHERICAL BALOON, SO THAT ITS VOLUME INCREASES AT A RATE OF $100\ cm^3/s$. HOW FAST IS THE RADIOUS OF THE BALOON INCREASING WHEN THE DIAMETER IS 50 cm?

Sol:

- LET V BE THE VOLUME OF THE BALOON. AND LET r BE ITS RADIOUS.

- BOTH OF THESE ARE FUNCTIONS OF TIME t.

- SINCE THE BALOON IS SPHERICAL, WE HAVE:

$$V = \frac{4}{3}\pi r^3 .$$

- WE KNOW THAT $\frac{dV}{dt} = 100\ cm^3/s$,

WE ARE ASKED TO FIND $\frac{dr}{dt}$

WHEN $2r = 50$ cm, (i.e. $r = 25$ cm).

- WE TAKE DERIVATIVE W.R.T. t IN THE FORMULA RELATING ~~RADIOUS~~ RADIUS AND VOLUME:

$$\frac{dV}{dt} = \frac{4}{3}\pi\, 3r^2 \cdot \frac{dr}{dt}$$

$$= 4\pi r^2 \frac{dr}{dt} .$$

- So $$\frac{dr}{dt} = \frac{dV}{dt} \cdot \frac{1}{4\pi r^2}$$

- $$\frac{dr}{dt} = 100\,cm^3/s \cdot \frac{1}{4\pi r^2}$$

- So $\frac{dr}{dt}$ AT $r = 25$ cm IS

$$100\,cm^3/s \cdot \frac{1}{4\pi \cdot (25\,cm)^2} =$$

$$= \frac{100}{4 \cdot \pi \cdot 25 \cdot 25}\,cm = \frac{100}{100} \cdot \frac{1}{\pi 25}\,cm$$

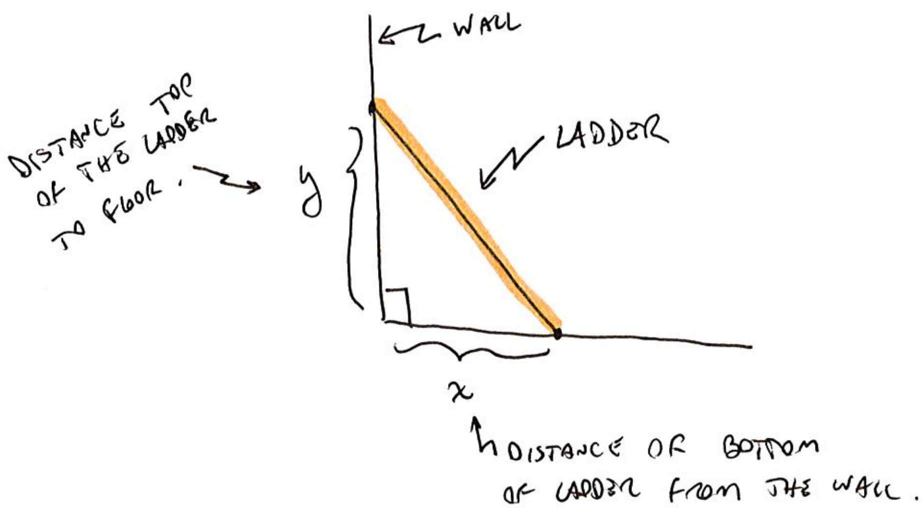
$$= \boxed{\frac{1}{\pi \cdot 25}\,cm}.$$

2. A LADDER 10 ft LONG RESTS AGAINST A VERTICAL WALL. IF THE BOTTOM OF THE LADDER SLIDES AWAY FROM THE WALL AT A RATE OF 1 ft/s, HOW FAST IS THE TOP OF THE LADDER SLIDING

DOWN THE WALL WHEN THE BOTTOM OF THE LADDER IS 6 ft FROM THE WALL?

Sol:

- LET'S START BY DRAWING A DIAGRAM TO MAKE SURE WE UNDERTAND THE SITUATION:



- x AND y DEPEND ON TIME t.

- $\dfrac{dx}{dt} = 1 \text{ ft/s}$

- $\dfrac{dy}{dt} = ?$ WHEN $x = 6 \text{ ft}$.

- $\sqrt{x^2 + y^2} = 10 \text{ ft}$.

- FROM THE LAST EQUATION:

  - $x^2+y^2 = 100 \text{ ft}^2$

    $\cancel{2x + 2y \frac{dy}{dt} =}$

  - $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

  - $\frac{dy}{dt} = -\frac{x}{y} \cdot \frac{dx}{dt}$

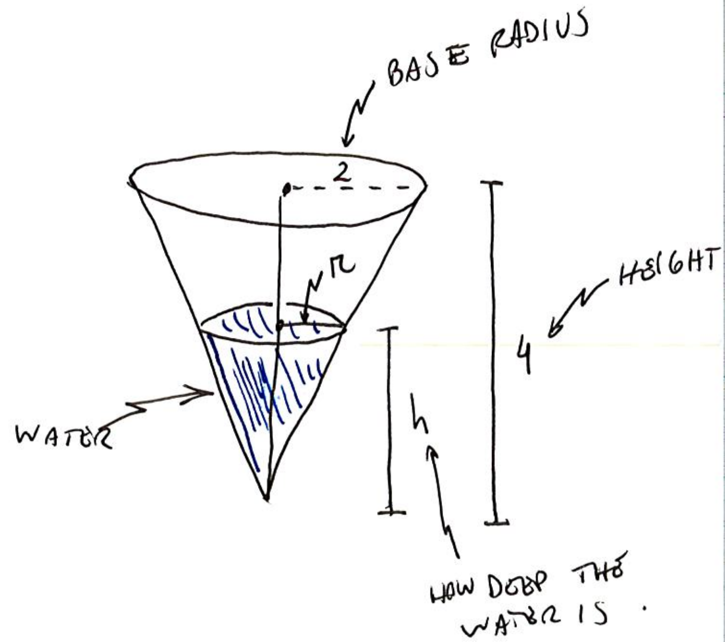
- IF $x = 6 \text{ ft}$, THEN $\sqrt{6^2+y^2} = 10 \text{ ft}$, SO $y = 8 \text{ ft}$.

- SO $\frac{dy}{dt}$ AT $x = 6 \text{ ft}$ IS $-\frac{6}{8} = \boxed{-3/4 \text{ ft/s}}$

3. A WATER TANK HAS THE SHAPE OF AN INVERTED CIRCULAR CONE WITH BASE RADIUS 2 m AND HEIGHT 4 m. IF WATER IS BEING PUMPED INTO THE TANK AT A RATE OF 2 m^3/min, FIND THE RATE AT WHICH THE WATER LEVEL IS RISING WHEN THE WATER IS 3 m DEEP.

Sol:

- DIAGRAM!



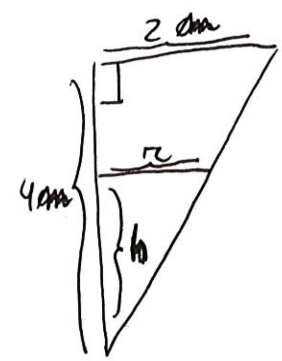
- LET V BE THE VOLUME OF WATER.
- LET r BE THE RADIUS OF THE WATER SURFACE.
- LET h BE THE WATER HEIGHT.

- $\dfrac{dV}{dt} = 2\ m^3/min$.

- $\dfrac{dh}{dt} = ?$ WHEN h IS $3m$

- $V = \frac{1}{3}\pi r^2 h$. HOW DOES r RELATE TO h?

- $r/h = 2/4$ SO $r = \dfrac{h}{2}$



SO $V = \frac{1}{3}\cdot\pi\cdot\left(\dfrac{h}{2}\right)^2\cdot h = \dfrac{\pi}{12}\cdot h^3$

- SO $\dfrac{dV}{dt} = \dfrac{\pi}{\cancel{12}_{4}}\, \cancel{3}^{1}\cdot h^2\cdot\dfrac{dh}{dt}$

SO $\dfrac{dh}{dt} = \dfrac{4}{\pi\cdot h^2}\cdot\dfrac{dV}{dt} = \dfrac{4}{\pi\cdot h^2}\cdot 2\ cm^3/min = \dfrac{8}{\pi\cdot h^2}\cdot cm^3/min$

- THUS $\dfrac{dh}{dt}$ AT $h = 3cm$ IS : $\dfrac{8}{9\pi}\ m/min$.

METHOD (PROBLEM SOLVING STRATEGY):

1. READ THE PROBLEM CAREFULLY.

2. DRAW A DIAGRAM, IF POSSIBLE.

3. INTRODUCE NOTATION. ASSIGN SYMBOLS TO ALL QUANTITIES.

4. EXPRESS THE GIVEN INFORMATION AND THE REQUIRED RATE IN TERMS OF DERIVATIVES.

5. WRITE AN EQUATION THAT RELATES THE QUANTITIES OF THE PROBLEM.

6. DIFFERENTIATE AND SOLVE FOR THE UNKNOWN.

EX:

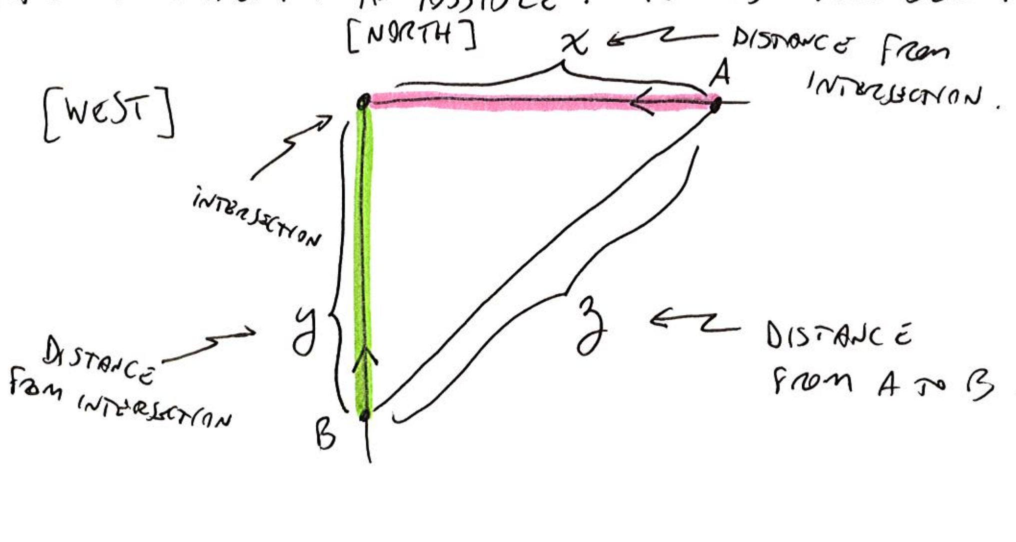
CAR A IS TRAVELING WEST AT 50 mi/h, AND CAR B IS TRAVELING NORTH AT 60 mi/h. BOTH ARE HEADED FOR THE INTERSECTION OF THE TWO ROADS.

AT WHAT RATE ARE THE CARS APPROACHING EACH OTHER WHEN CAR A IS 0.3 mi AND CAR B IS 0.4 mi FROM THE INTERSECTION?

Sol:

- (1) READ THE PROBLEM CAREFULLY! OK.

- (2) DRAW A DIAGRAM IF POSSIBLE. IT IS POSSIBLE:



• (3) INTRODUCE NOTATION. WE INTRODUCED SOME NOTATION IN THE DIAGRAM.

• (4) EXPRESS GIVEN INFO AND REQUIRED RATE IN TERMS OF DERIVATIVES:

- $\dfrac{dx}{dt} = -50 \text{ mi}/\text{h}$

- $\dfrac{dy}{dt} = -60 \text{ mi}/\text{h}$

- FIND $\dfrac{dz}{dt}$ WHEN $x = 0.3$ mi
  $y = 0.4$ mi.

• (5) RELATE QUANTITIES USING AN EQUATION. IN THIS CASE, WE CAN USE:

$$x^2 + y^2 = z^2 .$$

• (6) DIFFERENTIATE AND SOLVE:

- $$2x \cdot \frac{dx}{dt} + 2y\,\frac{dy}{dt} = 2z\,\frac{dz}{dt}$$

- So $$\frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\,\frac{dy}{dt}\right).$$

11

- SUBSTITUTING $x = 0.3$ mi, $y = 0.4$ mi

  in $x^2 + y^2 = z^2$ GIVES $z = 0.5$ mi.

- SO: $\dfrac{dz}{dt} = \dfrac{1}{0.5}\left(0.3(-50) + 0.4(-60)\right) = -78 \text{ mi/h}$

WHEN $x = 0.3$ mi

$y = 0.4$ mi.