Brown's lemma

"Path space of Y"
$$\begin{tikzcd} Y \arrow[r, "\Delta"] \arrow[dr, "\sim"'] & Y \times Y \\ & PY \arrow[u] \end{tikzcd}$$

project $PY \to Y \times Y \rightrightarrows Y$ the two endpoints

Every map $X \to Y$ factors as
$$\begin{tikzcd} & X \times_Y PY \arrow[r] \arrow[dl, shift left, "s"] & Y \\ X \arrow[ur, shift left, "\sim"'] \end{tikzcd}$$

Which gives us the Ken Brown's lemma
- Every map factors into section of trivial fibration followed by fibration
- It is a w.e. iff the fibration is a trivial fibration
- If $F : \mathcal{V} \to \mathcal{C}$ is a functor from a CFO to a category with w.e. then if it takes trivial fibs to w.e., it is a homotopy functor.

- Kan complexes of bounded size form a CFO

$X \to Y$ trivial fib $\Leftrightarrow$
$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, hook] & X \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Blumberg - Mandell

2-out-of-6 in a CFO $\Leftrightarrow$ 2-out-of-3 + the retract of a w.e is a w.e.

Lie groupoids

$\mathcal{V}$ category of spaces, has finite limits (small category)

contains a subcategory of covers
- every iso is a cover
- retract of a cover is a cover
- pullbacks of a cover is a cover
- $X \xrightarrow{f} Y \xrightarrow{g} Z$, $X \xrightarrow{gf} Z$ If f, gf are covers then g is a cover

eg:
- $\mathcal{V}$ = finite sets

  cover = surjective functions
- $\mathcal{V}$ = complex analytic spaces

  cover = surjective submersion
- $\mathcal{V}$ = Banach analytic spaces
- $s\mathcal{V}$ category of simplicial spaces

Def: $f: X_i \longrightarrow Y_i$ is a fibration of simplicial spaces if $\forall n > 0$, $0 \leq i \leq n$,

$$X_n \longrightarrow \mathrm{Hom}(\Lambda^n_i, X) \underset{\mathrm{Hom}(\Lambda^n_i, Y)}{\times} Y_n$$

for a trivial fibration: replace Λ^n_i by $\partial\Delta^n$.

k dimensional groupoid:

A simplicial space $X_\bullet$ is a k-groupoid if

$$X_n \longrightarrow \mathrm{Hom}(\Lambda^n_i, X) \qquad n > 0, \quad 0 \leq i \leq n$$

is a cover, and an iso if $n > k$.

eg: $n = 1$ $\quad X_1 \rightrightarrows X_0$ the two boundary maps are covers.

- k-groupoids are $(k+1)$-coskeletal. $\quad X_n \cong \mathrm{Hom}(sk_{k+1}\Delta^n, X)$

$k = 1$

X_0 = space of objects of X

X_1 = space of morphisms

$X_2 \xrightarrow{\simeq} \mathrm{Hom}(\Lambda^2_1, X)$ = space of pairs of composable morphisms

$\xrightarrow{\simeq} \mathrm{Hom}(\Lambda^2_0, X)$ = left inverse

$\xrightarrow{\sim} \mathrm{Hom}(\Lambda^2_2, X)$ = right inverse

Λ^* = differential graded algebra

$A^i = 0$ if $i \leq -k$