Review of Limits and Continuity

- A function $f(x)$ is continuous at $x=a$ if

$$\lim_{x \to a} f(x) = f(a)$$

- All standard functions are continuous, wherever they are defined.

  eg: x^n, e^x, $\ln x$, $\arcsin x$ etc.

- There is no general method for finding limits.
- The following are some tricks to finding limits
  - Plugging in values if the function is continuous
  - Multiplying by conjugate for radicals.
  - Squeeze theorem
  - Graphs of standard function.
- $\lim_{x\to\infty} \frac{\text{polynomial}}{\text{polynomial}}$ (or $\lim_{x\to -\infty}$) divide by ~~biggest~~ ~~term~~ highest degree term

- Very important:

$\frac{\text{constant}}{0}$ → limit $= \infty$

→ limit $= -\infty$

→ limit ~~d~~ does not exist

$\frac{0}{0}$ or $\frac{\infty}{\infty}$

- limit might still exist
- needs more simplification.

Problems

Section 2.3)

Q. Show $\lim\limits_{x \to 0} \frac{|x|}{x}$ does not exist.

A.

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

$$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = -1$$

(left of 0 means $x < 0$)

$$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = 1$$

(right of 0 means $x > 0$)

as $\lim_{x \to 0^-} \frac{|x|}{x} \neq \lim_{x \to 0^+} \frac{|x|}{x}$

$\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

Section 2.5

19) Show

$$f(x) = \begin{cases} x+3 & x \leq -1 \\ 2^x & x > -1 \end{cases}$$

is discontinuous at $x = -1$

$\lim_{x \to -1^-} f(x)$	$\lim_{x \to -1^+} f(x)$	$f(-1)$
$= \lim_{x \to (-1)^-} (x+3)$	$= \lim_{x \to (-1)^+} 2^x$	$= (-1)+3$
$= (-1)+3$	$= 2^{(-1)}$	$= 2$
$= 2$	$= 1/2$	

$\lim_{x \to (-1)^-} f(x) \neq \lim_{x \to (-1)^+} f(x) \Rightarrow f(x)$ is discontinuous at $x = -1$.

Section 2.6)

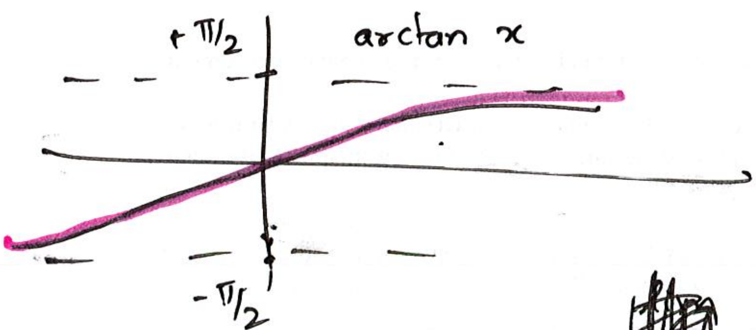
$$\lim_{x \to 2^+} \arctan\left(\frac{1}{x-2}\right)$$

$$= \arctan\left(\lim_{x \to 2^+} \frac{1}{x-2}\right)$$ as $\arctan x$ is continuous

Plugging in $x=2$, $\frac{1}{0}$

$$\lim_{x \to 2^+} \frac{1}{x-2} = \infty, -\infty, \text{d.n.e.}$$

$$\lim_{x \to 2^+} \frac{1}{x-2} = \infty$$ as $x-2>0$ when x is to right of 2.



$$\arctan(\infty) = \frac{\pi}{2}$$

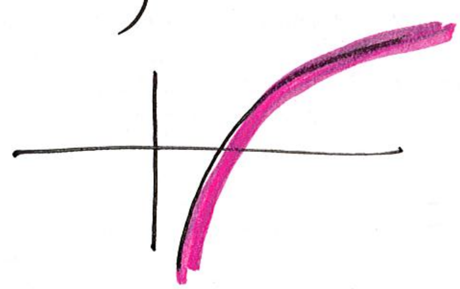
Q. $\lim_{x \to 2^-} \arctan\left(\frac{1}{x-2}\right)$

$\lim_{x \to 2^-} \frac{1}{x-2} = \underline{-\infty}$ when x to left of 2 $x-2<0$

$\lim_{x \to 2^-} \arctan\left(\frac{1}{x-2}\right) = -\frac{\pi}{2}$

Q. $\lim_{x \to 0^+} \arctan(\ln x) = ?$

Graph of $\ln x$:



$\lim_{x \to 0^+} \ln x = -\infty$

$\Rightarrow \lim_{x \to 0^+} \arctan(\ln x) = -\frac{\pi}{2}$

as $\lim_{x \to -\infty} \arctan x = -\frac{\pi}{2}$

Section 2.6

28) $$\lim_{x \to \infty} \underbrace{\sqrt{4x^2+3x}}_{\downarrow\ \infty} - \underbrace{2x}_{\downarrow\ \infty}$$

$$\infty - \infty$$

Just like: $\frac{0}{0}$, $\frac{\infty}{\infty}$, the form $\infty - \infty$ is indeterminate.

- Multiply and divide by $\sqrt{4x^2+3x} + 2x$

$$\lim_{x \to \infty} \sqrt{4x^2+3x} - 2x$$

$$= \lim_{x \to \infty} \left(\sqrt{4x^2+3x} - 2x\right) \frac{\left(\sqrt{4x^2+3x} + 2x\right)}{\left(\sqrt{4x^2+3x} + 2x\right)}$$

$$= \lim_{x \to \infty} \frac{\left(\sqrt{4x^2+3x}\right)^2 - (2x)^2}{\sqrt{4x^2+3x} + 2x}$$

$$\boxed{(a+b)(a-b) = a^2 - b^2}$$

$$= \lim_{x \to \infty} \frac{\cancel{4x^2} + 3x - \cancel{4x^2}}{\sqrt{4x^2+3x} + 2x}$$

$$= \lim_{x\to\infty} \frac{3x}{\sqrt{4x^2+3x}+2x}$$

$3x \to$ deg 1

$\sqrt{4x^2} \to$ deg 1

$\sqrt{3x} \to$ deg 1/2

$2x \to$ deg 1

Divide numerator / denominator by x

$$= \lim_{x\to\infty} \frac{3\cancel{x}/\cancel{x}}{\left(\sqrt{4x^2+3x}+2x\right)/x}$$

$$= \lim_{x\to\infty} \frac{3}{\frac{\sqrt{4x^2+3x}}{x}+\frac{2\cancel{x}}{\cancel{x}}}$$

$$= \lim_{x\to\infty} \frac{3}{\frac{\sqrt{4x^2+3x}}{x}+2}$$

eg: $\boxed{\text{Aside}}$

eg: $\dfrac{x}{\sqrt{x^4+1}+\cdots}$

$\sqrt{x^4} \approx x^2$

deg 2

eg:

$$\lim_{x\to\infty} \frac{1}{\sqrt[3]{x^2+1}}$$

$\sqrt[3]{x^2} \approx x^{2/3}$

deg 2/3

$$= \lim_{x \to \infty} \frac{3}{\sqrt{\frac{4x^2 + 3x}{x^2}} + 2}$$

$$= \lim_{x \to \infty} \frac{3}{\sqrt{\frac{4x^2}{x^2} + \frac{3x}{x^2}} + 2}$$

$$= \lim_{x \to \infty} \frac{3}{\sqrt{4 + 3/x} + \cancel{3}2}$$

$$= \frac{3}{\sqrt{4+0} + 2} \quad \text{as } \lim_{x \to \infty} \frac{3}{x} = 0$$

$$= \frac{3}{2+2}$$

$$= \frac{3}{4}$$

$$\boxed{\lim_{x \to \infty} \sqrt{4x^2+3x} - 2x = \frac{3}{4}}$$

25) $$\lim_{x \to \infty} \frac{\sqrt{x + 3x^2}}{4x - 1}$$

largest degree $= 1$

Divide by x, numerator & denom.

...

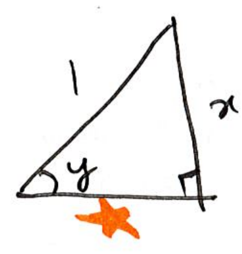
Section 1.5

70) Find $\tan(\sin^{-1}(x))$. $\left(= \tan(\arcsin x) \right)$

Ans: Let $y = \sin^{-1} x$

$\Rightarrow$ Applying sin to both sides

$$\boxed{\sin y = x}$$



as $\sin = \dfrac{\text{opposite}}{\text{hypotenuse}}$

By pythagoras,

adjacent side $= \sqrt{1^2 - x^2}$

$= \sqrt{1 - x^2}$

Back to the problem

$\tan(\sin^{-1}(x)) = \tan y$ (as $y = \sin^{-1} x$)

$$\tan y = \frac{\text{opposite side}}{\text{adjacent side}}$$

$$= \frac{x}{\sqrt{1-x^2}}$$ using the triangle

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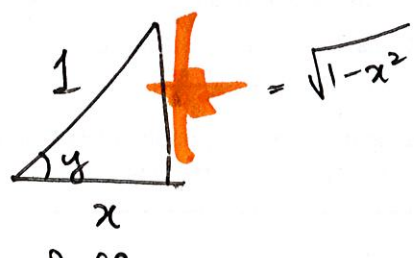
72) Find ~~tan~~ $\sin(2 \arccos x)$.

Ans Let $y = \arccos x$

Applying cos to both sides

$$\boxed{\cos y = x}$$



By Pythagoras

opposite side $= \sqrt{1-x^2}$

Back to the question,

$$\sin(2 \arccos x)$$

$= \sin(2y)$ as $y = \arccos x$

$= 2 \sin y \cdot \cos y$ ~~by~~ (double angle formula)

$= 2 \cdot \sqrt{1-x^2} \cdot x$

$\cos 2y = 2\cos^2 y - 1$
$\cos 2y = 1 - 2\sin^2 y$
$\sin 2y = 2 \sin y \cdot \cos y$

$$\boxed{\sin(2 \arccos x) = 2 \cdot \sqrt{1-x^2} \cdot x}$$ Ans

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75 Find domain, range

$$g(x) = 2 \cdot \sin^{-1}(3x+1)$$

- $\sin^{-1}(x)$

Domain: $[-1, 1]$

Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

- $2 \cdot \sin^{-1}(3x+1)$

Domain:

$$-1 \le 3x+1 \le 1$$

Subtract 1 from all sides

$$\Rightarrow -1-1 \le 3x \le 1-1$$

$$\Rightarrow -2 \le 3x \le 0$$

Divide by 3

$$-\frac{2}{3} \le x \le 0$$

$$\text{Domain} = \left[-\frac{2}{3}, 0\right]$$

Range

Scale range of $\sin^{-1}$ by 2

$$\text{Range} = \left[2 \cdot \left(-\frac{\pi}{2}\right), \left(\frac{\pi}{2}\right) \cdot 2\right]$$

$$= [-\pi, \pi]$$

IVT : if $f(x)$ is continuous,

$a < b$ are numbers,

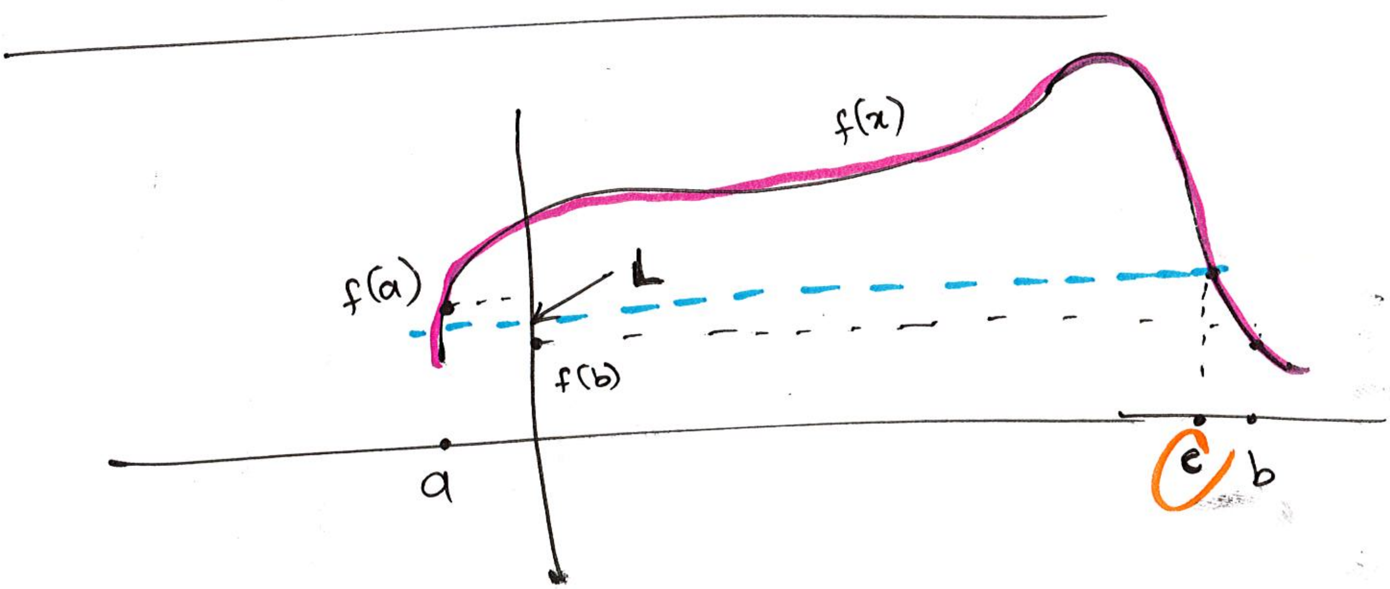
if L lies between $f(a)$ and $f(b)$

then there is a real number

$a \le c \le b$

such that

$$f(c) = L$$



IVT is the ~~...~~ mathematically precise way of saying that continuous functions have no jumps.

· Used to show existence of a solution

Q. Show that there exist a solution to

$$\cos x = x$$

between 0 and $\pi/2$.

A: A solution to

$$\cos x = x$$

is same as a solution to

$$\cos x - x = 0$$

- Let $f(x) = \cos x - x$, Let $L = 0$.

- Want to show $f(x) = 0$ for some x between $0, \pi/2$.

- Need to show 0 is between $f(0)$ and $f(\pi/2)$

- $f(0) = \cos(0) - 0 = 1 - 0 = 1$

$$f(\pi/2) = \cos\left(\frac{\pi}{2}\right) - \frac{\pi}{2} = 0 - \frac{\pi}{2} = -\pi/2$$

0 lies between 1 and $-\frac{\pi}{2}$

$\Rightarrow$ By IVT, as <u>$f(x)$ is continuous</u>

there is a c between $0, \frac{\pi}{2}$

such that $f(c) = 0$

$\Leftrightarrow$ $\cos(c) - c = 0$

$\Leftrightarrow$ $\cos(c) = c$. $\blacksquare$