# Weil Conjecture over Function Fields

Function Fields $\longrightarrow$ Finite extension of $\mathbb{F}_p(t)$

**Conjecture** $G$ semisimple, simply connected alg group over a function field $K$. Then $\mu_{Tam}\left(G(K)\backslash G(\mathbb{A})\right) = 1$

- A function field is of the form $K(X)$, $X$ an algebraic curve over a finite field $\mathbb{F}_q$.

| Number fields | Function Fields $X/\mathbb{F}_q$ |
|---|---|
| $\mathbb{Q}$ | $\mathbb{F}_q(t)$, $K = K(X)$ |
| prime number $p$ | closed point $x \in X$ |
| $\mathbb{F}_p$ | $\kappa(x) \leftarrow$ finite extension of $\mathbb{F}_q$ |
| $\mathbb{Z}_p$ | completed local ring $\mathcal{O}_x \cong \kappa(x)[[u]]$ |
| $\mathbb{Q}_p = \mathbb{Z}_p[p^{-1}]$ | $K_x \cong \kappa(x)((u))$ |
| $\mathbb{A} = \mathbb{R} \times \prod^{res} \mathbb{Q}_p$ | $\mathbb{A} = \prod^{res}_{x \in X} K_x$ $\leftarrow$ locally compact field |
| $SO_q(\mathbb{Q})\backslash SO_q(\mathbb{A})$ | $G(K)\backslash G(\mathbb{A})$ |
| $\mu_{tam}$ | $\mu_{tam}$ |
| $q$ quadratic form $/\mathbb{Z}$ | Group scheme $G \longrightarrow X$ (affine, smooth, connected fibers) (eg: $G = X \times SL_n$) but cannot assume that every fiber is semisimple |
| $SO_q(\mathbb{R} \times \hat{\mathbb{Z}})$ | $G(\prod_{x \in X} \mathcal{O}_x)$ |
| $SO_q(\mathbb{Q})\backslash SO_q(\mathbb{A})/SO_q(\hat{\mathbb{Z}} \times \mathbb{R})$ | $G(K)\backslash G(\mathbb{A})/G(\prod_{x \in X} \mathcal{O}_x)$ |
| quadratic forms $q'$ in the genus of $q$ | Principal $G$ bundles |

**Equivalent Formulation**

$$\sum_{P \text{ principal } G\text{-bundles}} \frac{1}{|\mathrm{Aut}\, P|} \cdot q^{-d} = \prod_{x \in X} \frac{|\kappa(x)|^{\dim(G)}}{|G(\kappa(x))|}$$

Convergence issue, $\infty$-terms on the left

$\mathrm{Bun}_G(X)$ = moduli <u>stack</u> of $G$-bundles

$$[Y, \mathrm{Bun}_G(X)] \xleftrightarrow{1-1} \text{principal } G\text{-bundles on } X \times Y$$

Goal: Compute # points of $\mathrm{Bun}_G(X)$ (defined over $\mathbb{F}_q$)

Weil $\rightarrow$ $Z$ projective alg. variety over $\mathbb{F}_q$.

Q. How big is $Z(\mathbb{F}_q)$ (= set of solutions of polynomial eq$^n$s)

$$Z(\mathbb{F}_q) \subseteq Z(\overline{\mathbb{F}}_q)$$

Frobenius:

$$\begin{array}{ccc} Z & \xrightarrow{\varphi} & Z \\ \cap & & \cap \\ \mathbb{P}^n & \longrightarrow & \mathbb{P}^n \\ [x_0 : \cdots x_n] & \longmapsto & [x_0^q : \cdots : x_n^q] \end{array}$$

$Z(\mathbb{F}_q)$ is precisely the set of fixed points under the Frobenius action.

Heuristic: $|Z(\mathbb{F}_q)| = \sum (-1)^i \mathrm{Tr}(\varphi \mid H^i(Z))$ Lefschetz fixed point formula ??

$= \mathrm{Tr}(\varphi \mid H^*(Z))$

Was proven by Grothendieck $\longrightarrow$ ① RHS cohomology groups are the etale Cohomology groups / $\ell$-adic coh

② If $Z$ is not projective we replace $H^*$ by $H_c^*$, compactly supported

- $Z$ smooth, we have a Poincaré duality:
  of dim

$$H_c^i(Z) \simeq H^{2d-i}(Z)^* \quad \longleftarrow \text{Not equivariant wrt the Frobenius hence the } q^{-d}$$

$$\Rightarrow \quad q^{-d}(\# Z(\mathbb{F}_q)) = \mathrm{Tr}(\varphi^{-1} \mid H^*(Z))$$

§ Replace $Z \longrightarrow \mathrm{Bun}_G(X)$ smooth algebraic stack of dim $d$.

Lefschetz fixed point kind of formula hold for $\mathrm{Bun}_G(X)$ so that
LHS

$$q^{-d} \sum_{G\text{-bundles}} \frac{1}{|\mathrm{Aut}\, P|} = \mathrm{Tr}(\varphi^{-1} \mid H^*(\mathrm{Bun}_G X))$$

RHS

$$\prod_{x\in X} \frac{|k(x)|^{\dim(G)}}{|G(k(x))|} = \prod_{x\in X} \mathrm{Tr}\left(\varphi^{-1} \mid H^*(\mathrm{Bun}_G(\{x\}))\right)$$

" Heuristic $\mathrm{Bun}_G(X) = \prod_{x\in X}^{\mathrm{cont}} \mathrm{Bun}_G(\{x\})$ "

" $\Rightarrow$ By Kunneth formula, $H^*(\mathrm{Bun}_G(X)) \cong \bigotimes_{x\in X}^{\mathrm{cont}} H^*(\mathrm{Bun}_G(\{x\}))$ "

$$\mathrm{Tr}\left(\varphi^{-1} \mid H^*(\mathrm{Bun}_G(X))\right) = \prod_{x\in X} \mathrm{Tr}\left(\varphi^{-1} \mid \mathrm{Bun}_G(\{x\})\right)$$

Next: Make the $\otimes^{\mathrm{cont}}$ precise using factorization homology (for algebraic curves/$\mathbb{F}_q$)