∞ - cosmos

"universe in which ∞-cats live as objects"

axiomatization of properties of qCat + isofib, equiv, trivial fib
$\twoheadrightarrow$, $\xrightarrow{\sim}$, $\overset{\sim}{\twoheadrightarrow}$

Def^n: ∞-cosmos is a simplicially enriched category $\mathcal{K}$ i.e.
- objects $A, B, \dots$ called ∞-categories
- functor spaces $\mathrm{Fun}(A,B)$ which are quasi-categories

+ a specified class of isofibrations denoted "$\twoheadrightarrow$" satisfying :

(write $f: A \to B$ for a vertex in $\mathrm{Fun}(A,B)$)

i) Completeness: $\mathcal{K}$ has a terminal object, products, pullbacks of isofibrations, limits of towers of isofibrations, cotensors with simplicial sets.

ii) Isofibrations contain - isos, maps with codomain $\mathbb{1}$
closed under - pullback, products, limits of towers,

- $f: A \twoheadrightarrow B$ isofib $\Rightarrow$ $\mathrm{Fun}(X,A) \twoheadrightarrow \mathrm{Fun}(X,B)$ isofib

if $U \hookrightarrow V$ sSet, $f \in \mathrm{Fun}(A,B) \Rightarrow A^V \twoheadrightarrow A^U \times_{B^U} B^V$

Def: A map $f: A \to B \in \mathrm{Fun}(A,B)_0$ is an equivalence
iff $f_*: \mathrm{Fun}(X,A) \to \mathrm{Fun}(X,B)$ is an equivalence.

Similarly for trivial fibration.

iii) Cofibrancy: trivial fibrations split.

Examples of ∞-cosmoi :

- $qCat$ – objects are quasi-categories
- CSS – " complete segal spaces
- Segal Segal spaces
- 1-comp 1-cosimplicial sets

} models of $(\infty,1)$-categories
=
weak categories with
- 0-cells = obj
- 1-cells
- 2-cells

$\vdots$

2-cells & higher are invertible

<u>Examples</u> which aren't $(\infty,1)$:

- Θ_n - spaces
- CSS_n = n-fold Segal spaces
- n-comp
- Cat , Kan , ...

Digression :

A simplicial category $\mathcal{K}$: $\Delta^{op} \longrightarrow Cat$

$$K_0 \rightleftarrows K_1 \rightleftarrows K_2 \rightleftarrows \cdots \quad \in Cat$$

- $ob\, K_0 = ob\, \mathcal{K}$
- face, degeneracy maps are id on objects

we think of K_n as a category with
obj = objects of $\mathcal{K}$
arrows = n-arrows $\rightsquigarrow$ n-simplex in $Fun(A,B)$

Def: Cotensors $\quad B \in \mathcal{K} \qquad U \in sSet$

B^U is defined by

$$\forall A \in \mathcal{K}, \quad Fun(A, B^U) \cong Fun(A, B)^U$$

Other limits are limits in the underlying category $\mathcal{K}_0$ with enriched universal properties.

eg: $$Fun(X, A \times B) \xrightarrow{\cong} F(X,A) \times F(X,B) \quad \text{as} \quad qCat \subseteq sSet$$

Prop: $qCat$ is an ∞-cosmos.

Proof: Leibniz exponential stuff
$\Rightarrow$ A, B $qCat$ then $Fun(A,B) := B^A$ is a $qCat$

Lemma: For $f : A \to B \in Fun(A,B)_0$ TFAE:
1) f is an equivalence
2) f is a homotopy equivalence relative to cotensoring with $\mathbb{I}$ i.e. $\exists\ g : B \to A$ s.t. and α, β

$$\begin{tikzcd}
 & A \\
A \arrow[ur, equal] \arrow[r, "\alpha"] \arrow[dr, "gf"'] & A^{\mathbb{I}} \arrow[u, "ev_0"'] \arrow[d, "ev_1"] \\
 & A
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & B \\
B \arrow[ur, equal] \arrow[r, "\beta"] \arrow[dr, "fg"'] & B^{\mathbb{I}} \arrow[u, "ev_0"'] \arrow[d, "ev_1"] \\
 & B
\end{tikzcd}$$

Proof: 1) $\Rightarrow$ 2)

Let f be an equiv $\Rightarrow$ $\exists$ an equiv $\in$ qCat

$$f_*: \mathrm{Fun}(B,A) \longrightarrow \mathrm{Fun}(B,B)$$

$\exists$ inverse $\quad \mathrm{qCat} \ni \tilde{g} : \mathrm{Fun}(B,B) \longrightarrow \mathrm{Fun}(B,A)$

let $\quad g = \tilde{g}(1_B)$

Similarly for the homotopy $\quad B \longrightarrow B^{\mathbb{I}}$.

For the other homotopy use the maps $\quad \mathrm{Fun}(A,A) \xrightarrow[\sim]{f_*} \mathrm{Fun}(A,B)$

$$1_A \quad gf \longmapsto f \cong fgf$$

Def^n: An ∞-cosmos is cartesian closed if $\forall\ A, B, C \in \mathcal{K}$ if $\exists\ C^A, C^B \in \mathcal{K}$ with natural isoms

$$\mathrm{Fun}(A \times B, C) \cong \mathrm{Fun}(A, C^B) \cong \mathrm{Fun}(B, C^A)$$

and $(-)^A$ preserve isofib.

Prop: Cat is an ∞-cosmos :

- obj = categories
- hom = nerves of B^A
- equiv = equivalences of categories
- trivial fib = surjective equiv.
- isofib :

$$\begin{tikzcd} \mathbb{1} \arrow[r, "a"] \arrow[d] & A \arrow[d] \\ \mathbb{I} \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$