What is a Spectrum?

Basic Notions Seminar

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Table of contents

Introduction

Overview

Q. What is a *spectrum*?

1. Worst named object in mathematics.
2. Objects representing cohomology theories.

$$ho(\mathbf{Spectra}) = \mathbf{CohomologyTheories}$$

3. Abelianized/stabilized spaces
   - **Spectra** behave like chain complexes of spaces
   - **Spectra** is enriched over abelian groups
   - **Spectra** is triangulated.
4. Ring spectra encode homotopy coherent deformations of rings.

Notation

Space	locally compact Hausdorff space with a **chosen basepoint**
Spaces	category of spaces, basepoint preserving continuous maps
CW	subcategory of CW-complexes with a chose basepoint
$[X, Y]$	(based) homotopy classes of maps $X \to Y$
$X \vee Y$	X and Y glued at their respective basepoint
$X \wedge Y$	$(X \times Y)/(X \vee Y)$
ΣX	reduced suspension
ΩX	loop space of X at the basepoint

Review: Smash product of spaces

$(\mathbf{Spaces}, \wedge)$ is a closed symmetric monoidal category. In particular,

$$\mathbf{Spaces}(X \wedge Y, Z) \cong \mathbf{Spaces}(X, \mathbf{Spaces}(Y, Z))$$

and $X \wedge Y \cong Y \wedge X$.

$$S^1 \wedge X \cong \Sigma X$$
$$\mathbf{Spaces}(\Sigma X, Y) \cong \mathbf{Spaces}(X, \Omega Y)$$
$$S^n \wedge S^m \cong S^{n+m}$$

where S^m is the m-dimensional sphere.

Cohomologies

Cohomology theory
A (reduced) *cohomology theory* is a collection of contravariant functor

$$\mathrm{E}^* : \mathbf{Spaces}^{op} \to \mathbf{GradedAbelianGroups}$$

satisfying the Eilenberg–Steenrod axioms:

- homotopy invariance:

$$X \simeq Y \implies \mathrm{E}^i(X) \cong \mathrm{E}^i(Y) \text{ for all } i \in \mathbb{Z}$$

- excision: if A is a sub CW-complex of X, then there exists a long exact sequence

$$\cdots \to \mathrm{E}^i(X/A) \to \mathrm{E}^i(X) \to \mathrm{E}^i(A) \to \mathrm{E}^{i+1}(X/A) \to \dots$$

- $\mathrm{E}^i(\star) = 0 \;$ for all $\; i \in \mathbb{Z}$
- other technical axioms

Brown Representability

Brown Representability Theorem
For every cohomology theory E^* there is a sequence of spaces,

$$\mathsf{E}_\bullet = \{\mathsf{E}_i\}_{i \in \mathbb{Z}}$$

unique up to homotopy, satisfying

$$\mathsf{E}^i(X) \cong [X, \mathsf{E}_i]$$

for all $i \in \mathbb{Z}$ and all $X \in \mathbf{CW}$.

We say that the sequence $\mathsf{E}_\bullet$ *represents* the cohomology theory E^*.

Ω-spectra

Ω-spectrum

With E^* and $\mathsf{E}_\bullet$ as above, applying excision to the inclusion $X \subset Cone(X)$ we get,

$$\begin{aligned} \mathsf{E}^n(X) &\cong \mathsf{E}^{n+1}(\Sigma X) \\ &\cong [\Sigma X, \mathsf{E}_{n+1}] \\ &\cong [X, \Omega \mathsf{E}_{n+1}] \\ \implies \quad [X, \mathsf{E}_n] &\cong [X, \Omega \mathsf{E}_{n+1}] \\ \implies \quad \mathsf{E}_n &\simeq \Omega \mathsf{E}_{n+1} \end{aligned}$$

Ω-spectrum

Ω-spectrum
An Ω-*spectrum* is a sequence of spaces

$$\mathsf{E}_\bullet = \{\mathsf{E}_i\}_{i \in \mathbb{Z}}$$

satisfying

$$\mathsf{E}_i \xrightarrow{\simeq} \Omega \mathsf{E}_{i+1}$$

for all $i \in \mathbb{Z}$.

- If $E_\bullet$ represents a cohomology theory, then $E_\bullet$ is an Ω-spectrum.
- Conversely, one can show that if $\mathsf{E}_\bullet$ is an Ω-spectrum then

  $$X \mapsto [\mathsf{E}_\bullet, X]$$

  is a reduced cohomology theory.

Examples: Singular cohomology

The reduced singular cohomology theory with coefficients in $\mathbb{Z}$, $\mathrm{H}\mathbb{Z}^*$, is represented by the Eilenberg–Maclane spectrum

$$\mathrm{H}\mathbb{Z}_n = \begin{cases} K(\mathbb{Z}, n) & \text{if } n \geq 0 \\ \star & \text{if } n < 0 \end{cases}$$

where $K(\mathbb{Z}, n)$ is any space satisfying

$$\pi_i K(\mathbb{Z}, n) = \begin{cases} \mathbb{Z} & \text{if } i = n \\ 0 & \text{otherwise.} \end{cases}$$

Examples: Complex K-theory

The reduced complex K-theory, KU^*, is represented by the spectrum

$$\mathrm{KU}_n = \begin{cases} \mathbb{Z} \times BU & \text{if } n \text{ is even} \\ U & \text{if } n \text{ is odd,} \end{cases}$$

where

$$U := \mathrm{colim}_n U(n)$$
$$BU := \mathrm{colim}_n BU(n)$$

and $BU(n) = \mathrm{Gr}(n, \mathbb{C}^\infty)$ is the complex n-plane Grasmannian in $\mathbb{C}^\infty$.

Bott periodicity theorem says that $\mathrm{KU}_\bullet$ is an Ω-spectrum.

Examples

- One can similarly define
  1. Eilenberg–Maclane spectra for other coefficients and
  2. KO, reduced real K-theory spectrum.
- But it is difficult to come up with other natural examples of Ω-spectra. It is not easy to come up with examples of *infinite loop spaces*.
- A sequential spectrum is a simplified version of an Ω-spectrum that is easy to define.

Sequential Spectra

Sequential spectra

Sequential spectrum
A *sequential spectrum* E is

- a sequence of spaces $\{\mathsf{E}_i\}_{i\in\mathbb{Z}}$
- along with *structure maps*

$$\Sigma\mathsf{E}_n \to \mathsf{E}_{n+1}$$

  for all $n \in \mathbb{Z}$.

Unlike with Ω-spectrum, we do not impose any conditions on the structure maps $\Sigma\mathsf{E}_n \to \mathsf{E}_{n+1}$.

Examples

- An Ω-spectrum is a sequential spectrum

$$\mathsf{E}_\bullet = \mathsf{E}_n \xrightarrow{\simeq} \Omega\mathsf{E}_{n+1} \quad \rightsquigarrow \quad \Sigma\mathsf{E}_n \to \mathsf{E}_{n+1}$$

- For any space X, define the spectrum $\Sigma^\infty X_\bullet$ as

$$(\Sigma^\infty X)_n := \begin{cases} \Sigma^n X & \text{if } n \geq 0 \\ \star & \text{if } n < 0. \end{cases}$$

  with structure maps given by the natural isomorphism $\Sigma(\Sigma^n X) \to \Sigma^{n+1} X$.

Properties of Sequential Spectra

- If $\mathsf{E}_\bullet$ is a sequential spectrum, define

$$\mathsf{E}^n(X) := \operatorname{colim}_k [\Sigma^k X, \mathsf{E}_{n+k}]$$

  where the maps in the colimit are induced by

$$[\Sigma^k X, \mathsf{E}_{n+k}] \to [\Sigma^{k+1} X, \Sigma \mathsf{E}_{n+k}] \to [\Sigma^{k+1} X, \mathsf{E}_{n+k+1}].$$

  Then E^* is a cohomology theory.
- Example: $\Sigma^\infty X^n(S^0)$ is the n^{th} stable homotopy groups of X.
- From a sequential spectrum you can construct an Ω-spectrum by

$$R\mathsf{E}_n := \operatorname{colim}_k \Omega^k \mathsf{E}_{n+k}$$

Stable Homotopy Category

SqSpec = category of sequential spectra, morphisms are maps between sequences that commute with the structure maps

1. **SqSpec** is enriched over abelian groups
2. $(\mathbf{SqSpec}, \Sigma)$ is triangulated, where $(\Sigma \mathsf{E})_{n+1} := \mathsf{E}_n$.
3. There is a model structure on the category of sequential spectra in which the fibrant objects are Ω-spectra.
4. *Stable homotopy theory* is the study of the category $ho(\mathbf{SqSpec})$.

Symmetric Spectra

Product structures

A (reduced) *multiplicative* cohomology theory is a cohomology theory E^* valued in graded algebras

$$\begin{tikzcd} & \mathbf{GradedAlgebras} \arrow[d] \\ \mathsf{E}^* : \mathbf{Spaces}^{op} \arrow[r] \arrow[ur, "\exists"] & \mathbf{GradedAbelianGroups} \end{tikzcd}$$

More explicitly, there is a "cup product"

$$\smile : \mathsf{E}^i(X) \otimes \mathsf{E}^j(X) \to \mathsf{E}^{i+j}(X)$$

Product of Sequential spectrum

Q. Is there a corresponding multiplicative structure on the level of spectra?

Q. Is there a natural "smash product" $\wedge$ on **SqSpec** such that $(\mathbf{SqSpec}, \wedge)$ is a symmetric closed monoidal category i.e.

$$\mathbf{SqSpec}(\mathsf{E} \wedge \mathsf{F}, \mathsf{G}) \cong \mathbf{SqSpec}(\mathsf{E}, (\mathsf{F}, \mathsf{G}))?$$

A. No! The problem lies with the structure maps. There is no nice choice for

$$\Sigma(\mathsf{E}_n \wedge \mathsf{F}_n) \to \mathsf{E}_{n+1} \wedge \mathsf{F}_{n+1}$$

Product of Sequential spectrum

- However, it *is* possible to define a smash product "up to homotopy" so that

$$[\mathsf{E} \wedge \mathsf{F}, \mathsf{G}] \cong [\mathsf{E}, (\mathsf{F}, \mathsf{G})].$$

- This is enough to do computations and run spectral sequences.
- But this is not good for doing categorical constructins, which are need, in particular, in algebraic K-theory.
- In 1985, Bökstedt defined the *Topological Hochschild homology* assuming the existence of a smash product on spectra. And *THH* provides a very powerful tool for computing algebraic K-theory.

Genuine smash product

1997 EKMM [1]

2000 Symmetric Spectra [2]

2000 Orthogonal Spectra [3]

2002 Equivariant orthogonal spectra [4]

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[1] *Rings, modules, and algebras in stable homotopy theory*, A.D.Elmendorf, I.Kriz, M.A.Mandell, J.P.May

[2] *Symmetric spectra*, M.Hovey, B.Shipley, J.Smith

[3] *Model categories of diagram spectra*, M.A.Mandell, J.P.May, S.Schwede, B.Shipley

[4] *Equivariant orthogonal spectra and S-modules*, M. A.Mandell, J.P.May

Symmetric spectrum
A *symmetric spectrum* is

1. a sequence of spaces $\{\mathrm{E}_i\}_{i \in \mathbb{Z}_{\geq 0}}$
2. along with *structure maps*

$$\Sigma \mathrm{E}_n \to \mathrm{E}_{n+1}$$

   for all $n \in \mathbb{Z}_{\geq 0}$
3. an action of the n^{th} symmetric group Sym_n on E_n

such that the induced structure maps

$$S^k \wedge \mathrm{E}_n \to \mathrm{E}_{n+k}$$

are $Sym_k \times Sym_n$ equivariant, where we think of S^k as the one-point compactification of the natural k-dimensional representation of Sym_k.

Example: Eilenberg–Maclane space

Dold–Thom theorem
The infinite symmetric product of S^n, $SP(S^n)$, is a $K(\mathbb{Z}, n)$ space.

Infinite symmetric product For a space X with a basepoint $\star$, the infinite symmetric product of X is

$$\left(\bigsqcup_i X^i / Sym_i\right) / \sim$$

where

$$\{x_1, \ldots, x_{i-1}, \star, x_{i+1}, \ldots x_n\} \sim \{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots x_n\}$$

Eilenberg–Maclane spectrum
The Eilenberg–Maclane symmetric spectrum is defined as

$$\mathrm{H}\mathbb{Z}_n := SP(S^n)$$

with the natural extension of the action of Sym_n on S^n to $SP(S^n)$.

Smash Product

Smash Product

Given two symmetric spectra $\mathsf{E}_\bullet$ and $\mathsf{F}_\bullet$, define $(\mathsf{E} \wedge \mathsf{F})_n$ to be the coequalizer of the two maps

$$\begin{array}{c} \bigvee_{i+1+j=n} Sym_n^+ \bigwedge_{Sym_i \times Sym_1 \times Sym_j} \mathsf{E}_i \wedge S^1 \wedge \mathsf{F}_j \\ \downarrow\downarrow \\ \bigvee_{p+q=n} Sym_n^+ \bigwedge_{Sym_p \times Sym_q} \mathsf{E}_p \wedge \mathsf{F}_q \end{array}$$

induced from

$$\mathsf{E}_i \wedge S^1 \wedge \mathsf{F}_j \to \mathsf{E}_{i+1} \wedge \mathsf{F}_j$$
$$\mathsf{E}_i \wedge S^1 \wedge \mathsf{F}_j \to \mathsf{E}_i \wedge \mathsf{F}_{j+1}$$

Smash Product

- Sym_n^+ is Sym_n thought of an a discrete space with an extra basepoint added.
- $X \wedge_G Y$ denotes the quotient by the diagonal action i.e. $(x, y) \sim (xg, g^{-1}y)$ for $g \in G$.

Smash Product

Given two symmetric spectra $E_\bullet$ and $F_\bullet$, define $(E \wedge F)_n$ to be the coequalizer of the two maps

$$\begin{array}{c} \bigvee_{i+1+j=n} Sym_n^+ \bigwedge_{Sym_i \times Sym_1 \times Sym_j} E_i \wedge S^1 \wedge F_j \\ \downarrow\downarrow \\ \bigvee_{p+q=n} Sym_n^+ \bigwedge_{Sym_p \times Sym_q} E_p \wedge F_q \end{array}$$

induced from

$$E_i \wedge S^1 \wedge F_j \to E_{i+1} \wedge F_j$$
$$E_i \wedge S^1 \wedge F_j \to E_i \wedge F_{j+1}$$

- Under this smash product, the category **symSpec** of symmetric spectra a closed symmetric monoidal category!!!
- For orthogonal spectra, we use the orthogonal $O(n)$ or unitary groups $U(n)$ instead of Sym_n.
- All these different models of spectra are Quillen equivalent. In particular, they have the same homotopy category i.e. the category $ho(\mathbf{Spectra})$ does not depend on what model of **Spectra**!!!

Final comments: Ring Spectra

- Once we have a product, we can define *ring spectra* which are monoid objects with units in the category of spectra.
- The notion of commutativity or associativity is too rigid for spectra. Instead, we can require these ring spectra to be "commutative up to higher coherences" and "associative up to higher coherences". These are called E_∞ and A_∞ ringed spectra respectively.

Thank you!