Review from previous class

- Various notions of limits

① $\lim_{x \to a} f(x)$ ② $\lim_{x \to a^+} f(x)$ ③ $\lim_{x \to a^-} f(x)$

④ $\lim_{x \to \infty} f(x)$ ~~lim~~ ⑤ $\lim_{x \to -\infty} f(x)$

- In general, it is not the case that $\lim_{x \to a} f(x)$ and $f(a)$ are equal

- We say that a function is continuous if at $x = a$

$$\lim_{x \to a} f(x) = f(a).$$

- All the standard functions are continuous wherever they are defined.

(standard functions: polynomials, trig, exp, log, inverse trig)

- Asymptotes:

① Vertical asymptote $\boxed{x = a}$

if $\lim_{x \to a^{\pm}} f(x) = \pm\infty$

② Horizontal asymptote $\boxed{y = L}$

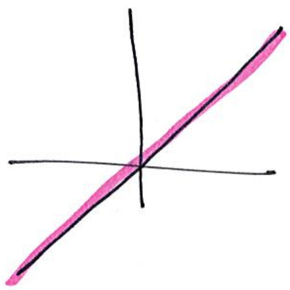
if $\lim_{x \to \pm\infty} f(x) = L$

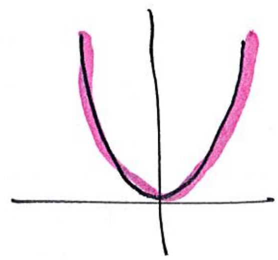
~~For standar~~

- For standard functions, you should know the <u>graphs</u> very well to find their limits.

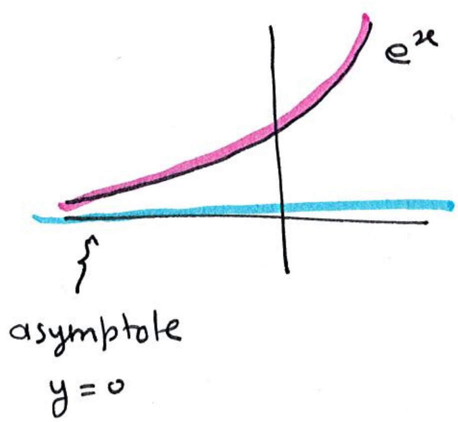
- **Polynomials** :

  - $f(x) = x$
  - $f(x) = x^2$

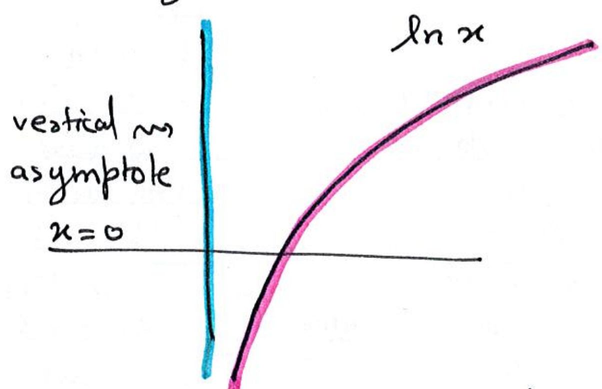




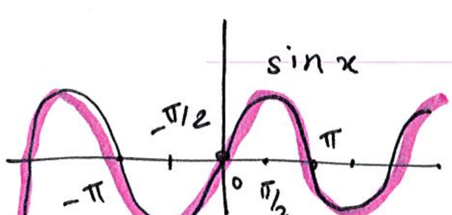
- **Exponential** :

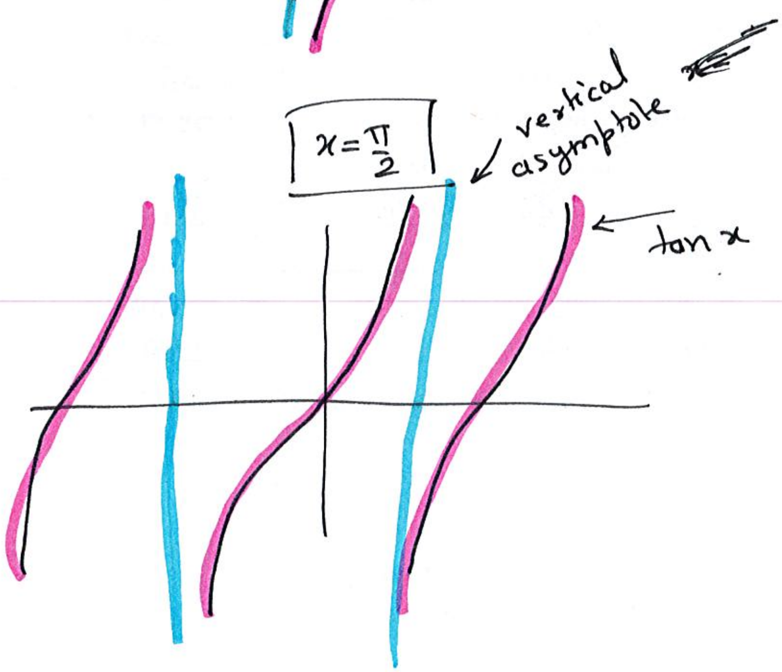


Logarithm



- Trigonometric





- In general, it is very hard, sometimes impossible to find limits.

- Instead, we'll learn some tricks.

- § 2.3 onwards

  - Algebraic methods of finding

    - $\lim_{x \to a} f(x)$ | $\lim_{x \to \infty} f(x)$

    $\downarrow$

  - Method 1: plug in $x = a$ and hope for the best

eg:

$$\lim_{x \to 0} \frac{e^x}{x^2 - 1} \rightsquigarrow \text{Plugging in } x = 0$$

$$\frac{e^0}{0-1} = \frac{1}{-1} = -1$$

$$= \frac{e^0}{0-1}$$

$$= \frac{1}{-1}$$

$$= -1$$

Method 2 :

eg: $\lim_{x \to 1} \frac{e^x}{x^2-1}$ ⇝ Plugging in $x=1$

$\frac{e^1}{1-1} = \frac{e}{0}$

we cannot divide by 0.

- if denominator is 0 but numerator is not 0.
- $\boxed{\text{Answer} = \pm\infty}$
- To get the precise answer, you'll have to plug in values of x close to 1.

Method 3 :

eg: $\lim_{x \to 1} \frac{x^2-3x+2}{x^2-1}$

Plugging in $x=1$

$\frac{1-3+2}{1-1} = \frac{0}{0}$

- If both numerator and denominator become 0, do some algebraic manipulation.
-

in this case, try factoring

$$\frac{x^2-3x+2}{x^2-1} = \frac{\cancel{(x-1)}(x-2)}{\cancel{(x-1)}(x+1)}$$

$$= \frac{x-2}{x+1}$$

$$\lim_{x\to 1} \frac{x^2-3x+2}{x^2-1} = \lim_{x\to 1} \frac{x-2}{x+1}$$

$$= \frac{1-2}{1+1}$$ Plugging in $x=1$

$$= \frac{-1}{2}$$

Check

$$\frac{1-2}{1+1} = \frac{-1}{2}$$

Method III. 2)

e.g.: $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{\sqrt{x}}$

Plug in $\frac{\sqrt{0+1}-1}{\sqrt{0}} = \frac{0}{0}$

Multiplying by the conjugate:

$$\frac{\sqrt{x+1}-1}{\sqrt{x}} = \frac{\sqrt{x+1}-1}{\sqrt{x}} \cdot \frac{(\sqrt{x+1}+1)}{(\sqrt{x+1}+1)}$$

$(a-b)(a+b) = a^2-b^2$

$$= \frac{(\sqrt{x+1})^2 - 1^2}{\sqrt{x}\cdot(\sqrt{x+1}+1)}$$

$$= \frac{x+\cancel{1}-\cancel{1}}{\sqrt{x}\cdot(\sqrt{x+1}+1)}$$

$$= \frac{x^{\cancel{1}}}{\sqrt{x}\cdot(\sqrt{x+1}+1)}$$

$\leftarrow \frac{x}{\sqrt{x}} = \sqrt{x}$

$= \frac{x^1}{x^{1/2}} = x^{1-1/2} = x^{1/2} = \sqrt{x}$

$$= \frac{\sqrt{x}}{\sqrt{x+1}+1}$$

- $\frac{\sqrt{x+1}-1}{\sqrt{x}} = \frac{\sqrt{x}}{\sqrt{x+1}+1}$

$\lim_{x\to 0}$. Plugging in $x=0$

$\frac{0}{\sqrt{0+1}+1} = \frac{0}{2} = 0$

$$\lim_{x\to 0}\frac{\sqrt{x+1}-1}{\sqrt{x}} = \lim_{x\to 0}\frac{\sqrt{x}}{\sqrt{x+1}+1}$$

$$= \frac{0}{\sqrt{0+1}+1} \quad \text{Plugging in } x=0$$

$$= \frac{0}{2} = 0 \quad ■$$

- Later on : • we'll do L'Hospital's rule for type $\frac{0}{0}$

- ~~Find lim~~

- Standard functions are continuous.
what this also means is that we push limits inside the function.

eg: $\lim_{x\to 0} e^{\sin x} = e^{\lim_{x\to 0}\sin x} = e^{\sin 0}$ $\boxed{\text{Plugging in } x=0}$

$= e^0 = 1$ ■

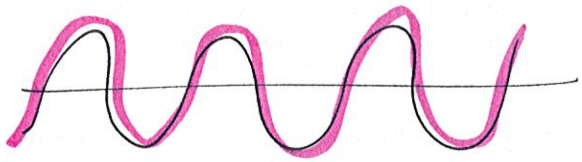
eg: $\lim_{x \to 0} \sin\left(\frac{1}{x}\right) \neq$

Plug in $x = 0$

$\sin\left(\frac{1}{0}\right)$

looks like

$\lim_{x \to \pm\infty} \sin(x)$ → this limit does not exist



- $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist.

eg. $\lim_{x \to 0} \frac{1}{x} = \pm\infty$

same is true for $\lim_{x \to 0} \sin\left(\frac{20}{x}\right)$

eg:

$$\lim_{x \to 0} \sin\left(\underbrace{\frac{\sqrt{x+1} - 1}{\sqrt{x}}}\right)$$

do the conjugate method in inside

Plugging in $\frac{0}{0}$

$\sqrt{a} - \sqrt{b}$ has conjugate $\sqrt{a} + \sqrt{b}$

$$= \lim_{x \to 0} \sin\left(\frac{\sqrt{x}}{\sqrt{x+1} + 1}\right)$$

$$= \sin\left(\lim_{x \to 0}\left(\frac{\sqrt{x}}{\sqrt{x+1} + 1}\right)\right)$$ because sin is continuous

$$= \sin\left(\frac{0}{\sqrt{0+1} + 1}\right)$$ plugging in

$$= \sin 0 = 0$$

Theorem: If f is a continuous function then

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

Method 4) · Squeeze theorem:

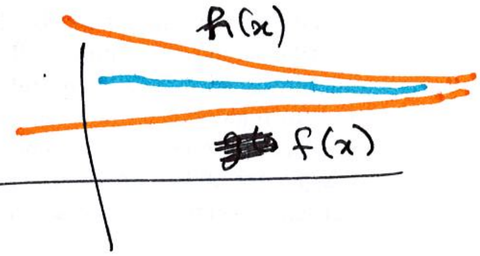
if $f(x) \leq g(x) \leq h(x)$

for some functions f, g, h such that

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L$$



Typically used for trig functions

- ~~Trig fn~~ $-1 \leq \sin x \leq 1$

  $-1 \leq \cos x \leq 1$

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eg: $\lim_{x \to 0} x \cdot \sin\left(\frac{1}{x}\right)$

Plugging $0 \cdot \sin\left(\frac{1}{0}\right)$

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

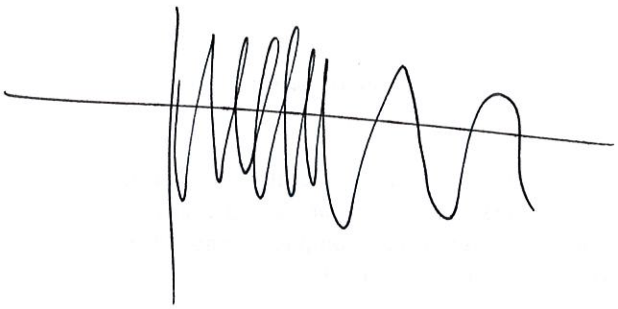
Multiplying both sides by x

$$\Rightarrow \quad -x \leq x \cdot \sin\left(\frac{1}{x}\right) \leq x$$

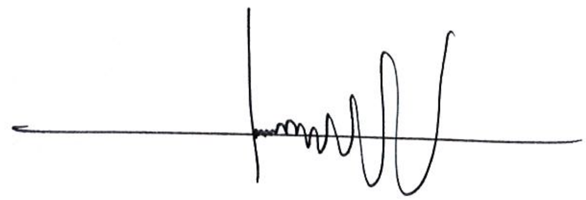
$\lim_{x \to 0} -x = 0$ $\qquad$ $\lim_{x \to 0} x = 0$

$\Rightarrow$ By Squeeze theorem, $\lim_{x \to 0} x \cdot \sin\left(\frac{1}{x}\right) = 0$

$\sin \frac{1}{x}$



$x \sin\left(\frac{1}{x}\right)$



- **Big picture:**

$\lim_{x \to a} \square$

a = real numbers

- Plug in $x = a$

1) If we get a number then that's the answer

2) If we get $\frac{\text{constant}}{0}$ (wing it)
   - ∞
   - $-\infty$
   - limit does not exist

3) If we get $\frac{0}{0}$
   - needs some manipulation

     eg. ① factor the numerator/denominator

     ② multiply by conjugate

     ③ squeeze theorem

- Trig functions are bounded

$$-1 \le \sin x \le 1$$

$$-1 \le \cos x \le 1$$

~~eg:~~ $\cancel{x^2 \sec\left(\frac{\pi}{x}\right)}$

$\cancel{x^3 \tan}$

eg: $\lim\limits_{x \to 0} \underbrace{x^2 \sin\left(\frac{\pi}{x^2}\right)}$

Plug in $0 \cdot \sin\left(\frac{\pi}{0^2}\right)$

- Need to use Squeeze theorem

$$-1 \le \sin\left(\frac{\pi}{x^2}\right) \le 1$$

Multiply by x^2

$$-x^2 \le x^2 \cdot \sin\left(\frac{\pi}{x^2}\right) \le x^2$$

$\lim\limits_{x \to 0} -x^2 = 0$ $\qquad$ $\lim\limits_{x \to 0} x^2 = 0$

By Squeeze theorem,

$$\lim_{x \to 0} x^2 \cdot \sin\left(\frac{\pi}{x^2}\right) = 0$$

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eg: $\lim\limits_{x \to 1} x^2 \cdot \sin\left(\frac{\pi}{x^2}\right) = 1 \cdot \sin\left(\frac{\pi}{1}\right) = 1 \cdot 0 = 0$

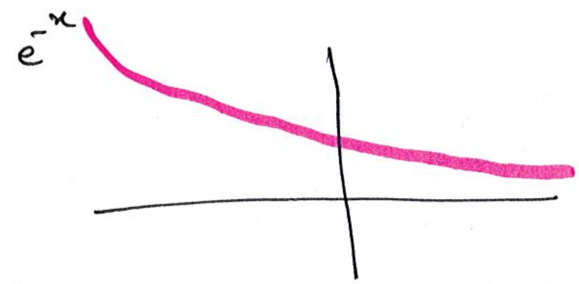
Plug in

§ Finding $\lim_{x \to \infty}$ or $\lim_{a \to -\infty}$

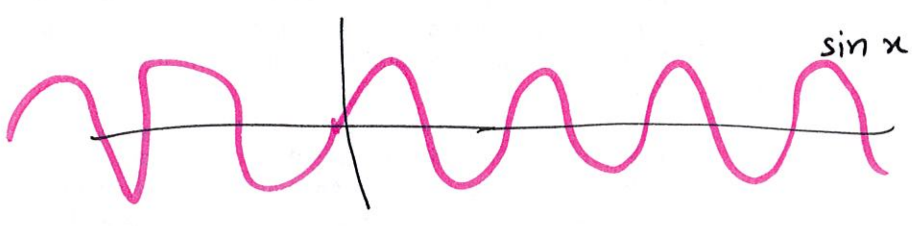
- DO NOT PLUG IN ∞ or $-\infty$

- try to find graphs

$\lim_{x \to \infty} e^{-x} = 0$ by looking the graph



- $\lim_{x \to \infty} \sin(x)$ does not exist



§ Squeeze theorem still applies here

eg: $\lim_{x \to \infty} \frac{\sin(x)}{x}$

$$f(x) \le g(x) \le h(x)$$

and $$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} h(x) = L$$

then $$\lim_{x \to \infty} g(x) = L$$

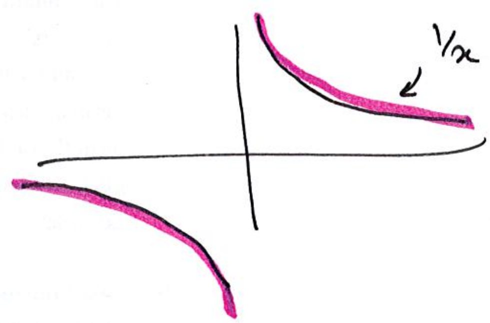
Sol^n:

$$-1 \le \sin x \le 1$$

Divide both sides by x

$$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$$

$$\lim_{x \to \infty} -\frac{1}{x} = 0 \qquad \lim_{x \to \infty} \frac{1}{x} = 0$$



$$\Rightarrow \lim_{x \to \infty} \frac{\sin x}{x} = 0$$

by Squeeze theorem

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§ $\cancel{\lim_{x \to \infty} \frac{\text{polynomial}}{\text{polynomial}}}$

eg: $\lim_{x \to \infty} e^{-x} \cdot \sin x$

Sol^n:

$$-1 \le \sin x \le 1$$

Multiply by e^{-x}

$$-e^{-x} \le \cancel{\sin x}\ e^{-x} \cdot \sin x \le e^{-x}$$

$$\lim_{x \to \infty} -e^{-x} = 0 \qquad \lim_{x \to \infty} e^{-x} = 0$$

By squeeze theorem,

$$\lim_{x \to \infty} e^{-x} \cdot \sin x = 0$$

$e^{-x} \cdot \sin x$

damped harmonic oscillator

eg:

$$\lim_{x \to \infty} \frac{x^2+1}{x^2+2}$$

$$= \lim_{x \to \infty} \frac{(x^2+1)/x^2}{(x^2+2)/x^2}$$

- x^2 is the largest degree term
- divide numerator / denominator by x^2

$$= \lim_{x \to \infty} \frac{\frac{x^2}{x^2} + 1/x^2}{x^2/x^2 + 2/x^2}$$

$$= \lim_{x \to \infty} \frac{1 + 1/x^2}{1 + 2/x^2}$$

$$= \frac{1}{1}$$

$$= 1$$

$$\lim_{x \to \infty} \frac{1}{x^n} = 0$$

$$\lim_{x \to -\infty} \frac{1}{x^n} = 0$$

- you can do this much more quickly by looking at <u>only</u> the largest degree terms.

eg.

- $\lim_{x \to \infty} \frac{x^2+1}{x^2+2} = \lim_{x \to \infty} \frac{x^2}{x^2} = \frac{1}{1}$

Be careful

- $\lim_{x \to \infty} \frac{x^3+x+1}{x+x^2+2} = \lim_{x \to \infty} x^3 = \infty$

§ $\lim\limits_{x \to \infty} \dfrac{\text{polynomials}}{\text{polynomials}}$

eg: $\lim\limits_{x \to \infty} \dfrac{x^3 + x + 2}{x + x^2 + 1}$

- Look for the largest degree term
- Divide numerator & denominator by that term

$$\lim_{x \to \infty} \frac{x^3 + x + 2}{x + x^2 + 1} = \lim_{x \to \infty} \frac{(x^3 + x + 2)/x^3}{(x + x^2 + 1)/x^3}$$

$$= \lim_{x \to \infty} \frac{x^3/x^3 + x/x^3 + 2/x^3}{x/x^3 + x^2/x^3 + 1/x^3}$$

$$= \lim_{x \to \infty} \frac{1 + 1/x^2 + 2/x^3}{1/x^2 + 1/x + 1/x^3}$$

$$\boxed{\lim_{x \to \infty} \frac{1}{x^n} = 0}$$

$$= \frac{1 + 0 + 0}{0 + 0 + 0}$$

$$= \pm \infty$$

$$\boxed{\lim_{x \to 0} \frac{1}{x^2} = \infty}$$

Summarize

$\lim_{x \to \infty}$

$\lim_{x \to -\infty}$

- graphs
- squeeze theorem
- $\frac{\text{polynomial}}{\text{polynomial}}$ → divide by largest degree term (both numerator/denominator)

Intermediate Value theorem

- Theorem about continuous ~~theorem~~ functions

- if f is continuous,

~~f(a)~~ $a \leq b$

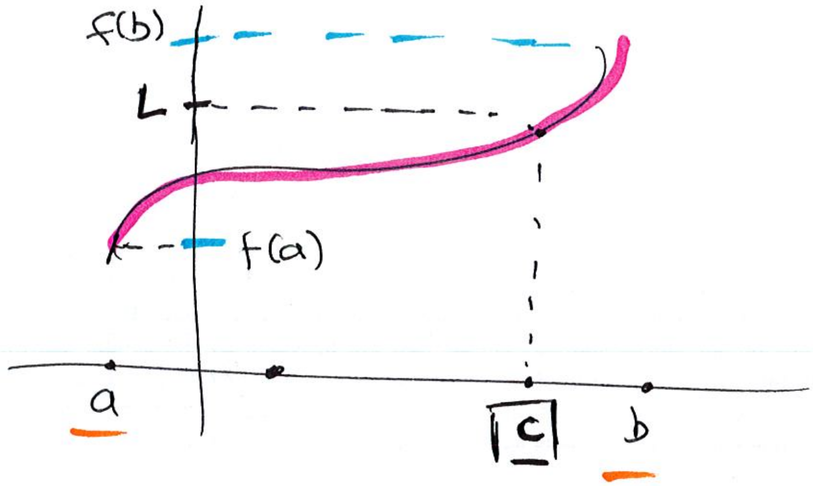
L is a number between

$f(a)$ and $f(b)$

then there is $\boxed{c}$ between a, b

such that

$f(c) = L$



eg: ~~Eg~~ Show $x^3 - x = 1$ has a root (zeroes) between 0 and 10.

Sol^n: Let $f(x) = x^3 - x - 1$

$f(0) = -1$

$f(10) = 1000 - 10 - 1$

$= 989$

0 lies between -1 and 989

$\Rightarrow$ By IVT, there is a c between 0 and 10 such that

$\cancel{f(x)=}\ f(c) = 0$

because polynomials are continuous

$\Rightarrow$ $x^3 - x - 1$ has a root between 0 and 10.