Derived Algebraic Geometry

↳ aim: deal with degenerate situations / non-transversal intersection

Want: "Loop space $\mathcal{L}X$" for a variety X

· need derived fibre product, "resolve" rings of functions

Let $\operatorname{char}(k)=0$

- Functions in DAG form commutative graded algebras

In affine case:

$$\cdots \longrightarrow A^{-2} \longrightarrow A^{-1} \longrightarrow A^{\leq 0} \longrightarrow 0 \longrightarrow 0 \qquad \in \mathrm{cdga}^{\leq 0}$$

Ex: Koszul complexes: $\mathbb{A}^n \xhookrightarrow{F} \mathbb{A}^m$ $\quad F=(f_1,\dots,f_m)$

$k[x_0,\dots,x_n,y_1,\dots,y_m]$ $\quad dy_i = f_i$, $|x_i|=0$, $|y_i|=-1$

Non Ex: X smooth affine variety

$$\mathcal{O}_X \xrightarrow{d_{DR}} \Omega^1_X \longrightarrow \cdots \quad \Omega^d X$$

Derived affine scheme:

$\mathrm{Aff} = (\mathrm{cdga}^{\leq 0})^{op}$

formally: $\operatorname{Spec} A \in \mathrm{Aff} \leftrightarrow A \in (\mathrm{cdga}^{\leq 0})$

· $U=\operatorname{Spec} A$, $\mathrm{QCoh}(U) = \mathrm{Mod}\, A$ $\quad$ dg-modules over A

since A is connective $\tau^{\leq k} M$ is again an A-module

eg: if $A = k[x_1,\dots,x_n]$, $|x_i|=0$

this can be thought of as $0 \longrightarrow A \longrightarrow 0$

· Spec A is to be thought of geometrically.

- Given a cdga A, $H^0(A)$ is a commutative algebra with a map $A \longrightarrow H^0(A)$

dualizing $\quad \operatorname{Spec} H^0(A) \hookrightarrow \operatorname{Spec} A$

should think of this as an infinitesimal thickening

Derivations : $A \xrightarrow{\delta} M$ k-linear satisfying (graded) Leibniz

Kahler differentials : Ω^1_A - universal derivation $A \xrightarrow{d_{DR}} \Omega^1_A$

$$\mathrm{Der}_k(A,M) \simeq \mathrm{Hom}_A(\Omega^1_A, M)$$

Caution: Ω^1_- does not respect quasi-isomorphisms, but does so for quasi-free cdga's eg. Koszul.

eg: Koszul cdga $A = k[x_1, \dots, x_n, y_1, \dots, y_m]$ $dy_i = f_i$

$$\Omega^1_A = \langle \underbrace{d_{DR}x_1, \dots, d_{DR}x_n}_{0}, \underbrace{d_{DR}y_1, \dots, d_{DR}y_m}_{-1} \rangle$$

with differential $d(d_{DR}y_i) := d_{DR}(dy_i) = d_{DR}f_i = \sum_j \frac{\partial f_i}{\partial x_j} d_{DR}x_j$

So Ω^1_A for Koszul cdga knows about dF.

If $F \neq$ submersion, there is cohomology in deg -1 of Ω^1_A.

To generalize to non-affine setting:

Functor of points has to be valued in 'spaces' or 'homotopy types' or '∞-grpds' : Spc

· because we need to invert quasi-iso, need to have Yoneda lemma.

Def: A pre-stack is an ∞-functor

$$\mathrm{Aff}^{op} = \mathrm{cdga}^{\leq 0} \longrightarrow \mathrm{Spc}$$

PrStk : ∞-category of such

Ex:

- $\mathrm{Spec}\,A\,(B) = \mathrm{Map}(A,B)$ so $\mathrm{Aff} \hookrightarrow \mathrm{PrStk}$
- G grp in PrStk, then

$$\left| \;\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} G \times G \rightrightarrows G \longrightarrow * \;\right| \xrightarrow[\text{realization}]{\text{geometric}} BG$$

PrStk has an internal Hom, right adjoint to $*$

If $K \in \mathrm{Spc}$, get a constant PreStk $A \longmapsto K$

So $\mathrm{Spc} \hookrightarrow \mathrm{PrStk}$

$\underline{\mathrm{Map}}(S^1, X) =: LX$ $\qquad S^1 = \bigcirc =$
$$\begin{tikzcd} \circ\ \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \circ \arrow[r] & S^1 \arrow[ul, phantom, "\lrcorner"] \end{tikzcd}$$

IS

$$\underline{\mathrm{Map}}(*, X) \times_{\mathrm{Map}(* \circ, X)} \mathrm{Map}(\circ, X) \simeq X \times_{X \times X} X$$

$X = \mathrm{spec}\,A$, A commutative ring

$$LX = A \overset{L}{\underset{A \otimes A}{\otimes}} A = HH_*(A)$$

Quasi-coherent sheaves on pre-stack:

<u>Defn</u> qCoh sheaves on X = compatible system on affines mapping to X

$$\begin{tikzcd} \operatorname{Spec} B \arrow[rr, bend left, "j"] \arrow[d, "f"'] & & X \\ \operatorname{Spec} A \arrow[urr, "i"'] & & \end{tikzcd} \qquad = \lim_{(\mathrm{Aff}/X)^{op}} QCoh(U) \longleftarrow \text{computed}$$

$f^* i^* \mathcal{F} = j^* \mathcal{F}$

Rmk: Comparison to QCoh on schemes
→ traditionally defined by gluing in the Zariski topology (or faithfully flat top fppf)
→ Need to check this definition satisfies Zariski descent.

Eg: X_{DR} : de Rham associated to X

$$X_{DR}(A) = X\left(H^0(A)^{red}\right)$$

- $\exists$ a map $X \longrightarrow X_{DR}$ that identifies infinitesimally nearby points

$D\text{-mod}(X) := QCoh(X_{dR})$ if X smooth $\simeq$ usual D-mods
so that $\Gamma(X_{DR}, \mathcal{O}_{X_{DR}}) \simeq H^*_{DR}(X)$