Facts about quasi-categories :

- Category of qCats + maps being simplicial sets <u>is enriched over qCats</u>.

  i.e. the maps $\{A \to B\}$ between qCats coincides with the set of vertices of the qCat $B^A =: \mathrm{Fun}(A,B)$.

- $\exists$ special maps : isofib, equiv, trivial fib, isofib.

$\equiv$ There are many strict 1-cats of ∞-cats + functors between them that admit these same functors.

Def An <u>∞-cosmos</u> $\mathcal{K}$ is a category that is enriched over qCats

- objects A, B are ∞-categories
- functor-spaces $\mathrm{Fun}(A,B)$ s.t. vertices are the maps in the underlying 1-cat $\mathcal{K}$.
- specific classes of isofib. satisfying closedness wrt ...
- completeness wrt certain limits
- equivalences have sections.

Recall $f: A \to B$ in $\mathcal{K}$ is equiv if $\forall X \in \mathcal{K}$

$$\mathrm{Fun}(X,A) \xrightarrow{\simeq} \mathrm{Fun}(X,B)$$ is a trivial fib [equiv + isofib]

or equiv. $A \xrightarrow{f} B$ is an equivalence iff

$\exists\; g : B \to A$

$$\begin{tikzcd} & A & \\ A \arrow[ur, equal] \arrow[r] \arrow[dr, bend right, "gF"'] & A^{\mathbb{I}} \arrow[u] \arrow[d] \\ & {} \end{tikzcd} \qquad \begin{tikzcd} & B & \\ B \arrow[ur, equal] \arrow[r] \arrow[dr, bend right, "fg"'] & B^{\mathbb{I}} \arrow[u] \arrow[d] \\ & {} \end{tikzcd}$$

2- categories

eg: 2-categor Cat : categories A, B, C

functors $f: A \longrightarrow B$

natural transforms
$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$$

$\equiv$ Any ∞-cosmos $\mathcal{K}$ has an associated homotopy 2-category (or strict 2-cat) analogous to Cat.

Def The homotopy 2-category $h\mathcal{K}$ of $\mathcal{K}$:

- objects are same as $\mathcal{K}$
- 1-cells —— " ——
- 2-cells $\begin{tikzcd} A \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$ are homotopy classes of 1-simplices in $\mathrm{Fun}(A,B)$

i.e. $\underset{(\text{in } h\mathcal{K})}{\mathrm{Fun}^h(A,B)} = \underset{(\text{in } \mathcal{K})}{ho\,(\mathrm{Fun}(A,B))}$

Rmk (on 2-cells) A natural transformation $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\alpha\Downarrow"] & B \end{tikzcd}$ in $h\mathcal{K}$

is represented by $\mathbb{2} \xrightarrow{\alpha} \mathrm{Fun}(A,B)$.

- A natural iso (an invertible 2-cell)

$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right] \arrow[r, phantom, "\cong\Downarrow\alpha"] & B \end{tikzcd}$ in $h\mathcal{K}$ is represented by

$$\begin{tikzcd} \mathbb{2} \arrow[r, "\alpha"] \arrow[d, hook] & \mathrm{Fun}(A,B) \\ \mathbb{I} \arrow[ur, dashed, "\alpha"'] & \end{tikzcd}$$

Rmk: Cat is an ∞-cosmos. It homotopy 2-category is itself.

Prop: $h\mathcal{K}$ has a terminal object, in a 2-categorical sense.

Proof: $\mathcal{K}$ has a terminal object 1 in a simplicially enriched sense (axiom)
i.e. $\mathrm{Fun}(X, 1) \cong 1$

Apply $q\mathrm{Cat} \xrightarrow{ho} \mathrm{Cat}$ which sends $\mathrm{Fun}(A,B) \longrightarrow \mathrm{Fun}^h(A,B)$

$$\begin{array}{ccc} \mathrm{Fun}(X,1) & \longmapsto & \mathrm{Fun}^h(X,1) \\ \wr\!\wr & & \wr\!\wr \\ 1 & \longmapsto & 1 \in \mathrm{Cat} \end{array}$$

Prop: hK has products in 2-categorical sense.

Proof: follows from the fact that ho- preserves products.

- Any 1-category has a notion of iso between objects.
- Any 2-category " " equiv "

Def: Equiv:
- obj A, B
- 1-cells $A \xrightarrow{f} B$, $B \xrightarrow{f} A$
- invertible 2-cells,

$$\begin{tikzcd} A \arrow[r, bend left, equal, ""{name=U, below}] \arrow[r, bend right, "gf"', ""{name=D, above}] \arrow[from=U, to=D, phantom, "\cong\Downarrow\alpha"] & A \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, bend left, ""{name=U, below}] \arrow[r, bend right, equal, ""{name=D, above}] \arrow[from=U, to=D, phantom, "\cong\Downarrow\beta"] & B \end{tikzcd}$$

Prop: $A \xrightarrow{f} B$ is an equiv in $\mathcal{K}$ iff its an equiv in $h\mathcal{K}$.

Proof. $\Rightarrow$ $A \xrightarrow{f} B$ equiv in $\mathcal{K}$ means $\exists\ B \xrightarrow{g} B$

with

$$\begin{tikzcd} & A \\ A \arrow[ur, equal] \arrow[r, "\sim"] \arrow[dr] & A^{I} \arrow[u] \arrow[d] \\ & A \end{tikzcd} \qquad \begin{tikzcd} & B \\ B \arrow[ur] \arrow[r, "\beta"] \arrow[dr, equal] & B^{I} \arrow[u] \arrow[d] \\ & B \end{tikzcd}$$

$\Leftarrow$ **Claim:** $\begin{tikzcd} C \arrow[r, bend left, "h"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\cong\Downarrow"] & D \end{tikzcd} \in hK$ then h equiv $\Rightarrow$ k equiv

Proof:

$$\begin{tikzcd} & C \arrow[dl, "h"'] \arrow[d] \arrow[dr, "k"] & \\ D & D^{I} \arrow[l, two heads, "\sim"] \arrow[r, two heads, "\sim"'] & D \end{tikzcd}$$

Now, if f, g are equiv in hK

$\Rightarrow gf,\ fg \cong \mathrm{id}$ in hK

Then by a 2-out-of-6 property for equiv. in $\mathcal{K}$ we're done.

Lem: 2 functors preserve equivalences.

Lem: Any simplicially enriched functor $\mathcal{K} \xrightarrow{F} \mathcal{L}$ gives a 2-functor $h\mathcal{K} \to h\mathcal{L}$.

Cor: $A \xleftarrow{\sim} A'$, $B \xrightarrow{\sim} B'$ in an ∞-cosmos then

$$\mathrm{Fun}(A,B) \xrightarrow{\sim} \mathrm{Fun}(A',B') \quad \in q\mathcal{C}at.$$