Thm Serre: $H^*(K(\mathbb{Z}/2,n);\mathbb{Z}/2)$ is a polynomial ring over $\mathbb{Z}/2$ with generators, $\{Sq^I(\iota_n)\}$ where I is an admissible index of excess $<n$, $H^n(K(\mathbb{Z}/2,n);\mathbb{Z}/2)=\mathbb{Z}/2$ in

excess $(i_1,i_2,\cdots)$
$= (i_1-2i_2)+(i_2-2i_3)+\cdots+i_n$

$n=1$: $H^*(K(\mathbb{Z}/2,1);\mathbb{Z}/2) = H^*(\mathbb{RP}^\infty,\mathbb{Z}/2) = \mathbb{Z}/2[x]$

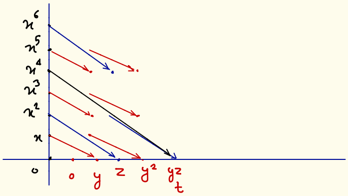
Sq^0 is the only excess 0 index

$n=1$ We have a fibration $K(\mathbb{Z}/2,1)\longrightarrow * \longrightarrow K(\mathbb{Z}/2,2)$

Serre SS:

Indices with excess at most 1 are

$0, (1), (2,1), (4,2,1), \dots$



$d_2x=y \qquad d_2x^n = nx^{n-1}y = \begin{cases} x^{n-1}y & n \text{ odd} \\ 0 & \text{else}\end{cases}$

$d_2(x^ny) = nx^{n-1}y^2$

$d_2(x^ny^2) = n\cdot x^{n-1}y^3$

$d_3(x^2)=z \qquad d_3(x^{2n}) = n\,x^{2n-2}z$

Here's how the SS runs:

1) On page 2, $x^{2n+1}\longrightarrow x^{2n}y$ and the two die
2) xy has to die on page 2 but it is not hit so $xy\longrightarrow y^2$
3) So $x^2\longrightarrow z$ on page 3.

Enter the transgressions.

Transgressions:

Homology:

$E^2_{0,n} = H_n(F;\mathbb{Z}/2)$

Subsequent differentials kill elements in $E^2_{0,n}$, consequently $E^2_{0,n}=H_n(F;\mathbb{Z}/2)\twoheadrightarrow E^\infty_{0,n} = E^2_{0,n}/\sum \mathrm{im}(d)$

By the remark in the margin, $E^\infty_{0,n}\hookrightarrow F_0\hookrightarrow H_n(E,\mathbb{Z}/2)$

Composing we have

$$H_n(F;\mathbb{Z}/2)\twoheadrightarrow E^\infty_{0,n}\hookrightarrow H_n(E,\mathbb{Z}/2)$$

Doing the exact same thing on the bottom row we get

$$H_n(E,\mathbb{Z}/2)\twoheadrightarrow E^\infty_{n,0}\hookrightarrow H_n(B;\mathbb{Z}/2)$$

These maps are nothing but i_*,π_*, $F\xrightarrow{i}E\xrightarrow{\pi}B$ (Why?)

We have wrong way maps:

$$\begin{array}{ccc} E^\infty_{0,n+1} & \xrightarrow{d_{n+1}} & E^\infty_{n,0} \\ \downarrow & & \uparrow \\ H_{n+1}(B) = E^2_{0,n+1} & & E^2_{n,0} = H_n(F) \end{array}$$

Useful: Differentials in a SS go the wrong way i.e. from quotient to sub.

eg: In cohomological SS $d^r: E^{p,q}_r \to E^{p+r,q+r-1}_r$

So we would have

$$0\to E^{n,0}_\infty\to F_0\to E^{n-1,1}_\infty\to 0$$
$$\downarrow$$
$$F_1$$
$$\downarrow$$
$$0\to E^{n-2,2}_\infty\to F_2\to E^{n-3,3}_\infty\to 0$$
$$\downarrow$$
$$F_3$$
$$\vdots$$

$$0\subseteq F_0\subseteq F_1\subseteq F_2\subseteq\cdots\subseteq F_n = H^n(E)$$

Differentials are the connecting homomorphisms!?!

This is the transgression, $\tau: E^{\infty}_{n+1,0} \to E^{\infty}_{0,n}$, and $x \in H_{n+1}(B)$ if $\tau(x)$ is defined & $d^i x = 0$ for $i \leq n$.

Alternate def^n: $\pi: (E,F) \longrightarrow (B,*)$

Let $x \in H_{n+1}(B,*)$ be in the image of π_*. Pick $x' \in H_n(E,F)$ such that $\pi_*(x') = x$.

Define: $\tau'(x) = \partial x'$.

Claim: τ' is well defined in $E^{\infty}_{0,n}$, $\operatorname{im} \pi_* = E^{\infty}_{n+1,0}$ and $\tau' = \tau$.

Again I do not see how to prove this. :(

CoHomology:

We simply reverse the arrows to get

$$\begin{tikzcd} E^{\infty}_{0,n} \arrow[r] \arrow[d, hook] & E^{\infty}_{n+1,0} \\ H^n(F) = E^2_{0,n} \arrow[r, dashed, "\tau"'] & E^2_{n+1,0} = H^{n+1}(B) \arrow[u] \end{tikzcd}$$ transgression

But now because the second def^n is natural, we get

$$\begin{tikzcd} \tilde{H}^{n-1}(F) \arrow[r, "\delta"] \arrow[rr, dashed, bend right, "\tau"'] & H^n(E,F) & \tilde{H}^n(B) \arrow[l, "\pi^*"'] \end{tikzcd}$$

Both δ, π^* commute with Sq^i and so we get:

Prop: If $x \in H^n(F)$ is transgressive then so is $Sq^i(x) \in H^{n+i}(F)$.

Back to $K(\mathbb{Z}/2, 2)$:

$$x^{2^n} = Sq^{2^{n-1}} Sq^{2^{n-2}} \cdots Sq^1 x$$

And x is transgressive, so invoking Borel's theorem we find that $H^*(K(\mathbb{Z}/2,2);\mathbb{Z}/2)$ is a free polynomial algebra generated over $Sq^{2^{n-1}} Sq^{2^{n-2}} \cdots Sq^1 y$. This method will now generalize.

$n=m$ Assume the theorem to be true. $H^*(K(\mathbb{Z}/2,m);\mathbb{Z}/2) = \mathbb{Z}_2[Sq^I(i_m)]$ excess of $I < m$

$n = m+1$

To prove the inductive step we need to prove that $Sq^I(i_m)$ forms a minimal generating set, with excess of $I \leq m$. Which amounts to showing two things:

$Sq^{I_1}(i_m) \cdots Sq^{I_k}(i_m)$ for $e(I_k) \leq m$ form a $\mathbb{Z}/2$ basis for $H^*(K(\mathbb{Z}/2,m);\mathbb{Z}/2)$

$I_1 < I_2 \cdots < I_k$

First let us enumerate the possible generators:

$e(0)$: Sq^0 $\qquad Sq^1(i_1) = [2] \qquad Sq^2Sq^1(i_1) = [4] \qquad Sq^4Sq^2Sq^1(i_1) = [8]$

$e(1)$: $Sq^1 \quad Sq^2Sq^1 \quad Sq^4Sq^2Sq^1 \quad \cdots \rightsquigarrow Sq^1(i_2) = [2,1],\ Sq^2Sq^1(i_2) = [4,1] \quad Sq^4Sq^2Sq^1(i_2) = [8,1]$

$e(2)$: $Sq^3Sq^1 \quad Sq^6Sq^3Sq^1 \quad \cdots \rightsquigarrow Sq^2(i_3) = [2,2,1],\ Sq^3Sq^1(i_3) = Sq^3([2,1,1])$

$Sq^2 \quad Sq^4Sq^2 \quad Sq^5Sq^2Sq^1$ $\qquad Sq^2Sq^1(i_3) = Sq^2[2,1,1] \qquad = [4,2,1]$

$= [4,1,1] + [2,2,2] \qquad Sq^4Sq^2(i_3) = Sq^4([2,2,1])$

$= [4,4,1]$

for $i < j_1 + j_2 + \cdots$

$Sq^i(x_1^{j_1}x_2^{j_2}\cdots)$

$= \sum \binom{j_1}{k_1}\binom{j_2}{k_2}\cdots x_1^{j_1+k_1} x_2^{j_2+k_2}\cdots$

$k_1 + k_2 + \cdots = i$

$e(I) = m-1 \qquad Sq^{i_1}Sq^{i_2}\cdots Sq^{i_k}Sq^{i_{k+1}}\cdots$

$e(I') = m \qquad Sq^{i_1+2^k}Sq^{i_2+2^{k-1}}\cdots Sq^{i_k+2}Sq^{i_{k+1}+1}\cdots$

Suppose $I = (i_1, i_2, \ldots)$ admissible with excess $m-1$.

$[Sq^I(i_m)]^2 = Sq^{|I|+m}Sq^I(i_m)$ $\qquad$ excess of $(|I|, i_1, i_2, \ldots) = 2(|I|+m) - (|I| + m + |I|)$

$= m$

$[Sq^I(i_m)]^4 = Sq^{2|I|+2m}Sq^{|I|+m}Sq^I(i_m)$

So every polynomial over $Sq^I(i_m)_{:\,|I|<m}$ can be written in terms of monomials in $Sq^I(i_m)_{:\,|I|\le m}$. Furthermore this representation is unique.

Conversely, every $Sq^I(i_m)_{:\,|I|=m}$ can be written in the above by just reversing the process!!

This is the most incredible combinatorial proof I've ever seen.

The thing works for $K(\mathbb{Z},n)$ too.

$K(\mathbb{Z},1) \sim S^1 \qquad H^*(K(\mathbb{Z},1);\mathbb{Z}/2) \cong \mathbb{Z}/2[i_1]/i_1^2$

$K(\mathbb{Z}/2,2) \sim \mathbb{CP}^\infty \qquad H^*(K(\mathbb{Z},2);\mathbb{Z}/2) \cong \mathbb{Z}/2[i_2]$, Generating set consists of $i_2, Sq^2 i_2, Sq^4Sq^2 i_2, \ldots$

$H^*(K(\mathbb{Z},3);\mathbb{Z}/2) \cong \mathbb{Z}/2[i_3, Sq^2 i_3, Sq^4Sq^2 i_3, \ldots]$

$H^*(K(\mathbb{Z},n);\mathbb{Z}/2) \cong \mathbb{Z}/2[Sq^I i_n]$ $\qquad$ excess$(I) < n$, I does not contain Sq^1 & admissible.

(We can look at the natural map $K(\mathbb{Z},n) \longrightarrow K(\mathbb{Z}/2,n)$. This induces a map $H^*(K(\mathbb{Z}/2,n);\mathbb{Z}/2) \longrightarrow H^*(K(\mathbb{Z},n);\mathbb{Z}/2)$ and the corresponding SS)

For $K(\mathbb{Z}/2^k;n)$ look at the fibration coming from $\mathbb{Z} \xrightarrow{2^k} \mathbb{Z} \to \mathbb{Z}/2^k$

$$K(\mathbb{Z},n) \longrightarrow K(\mathbb{Z},n) \longrightarrow K(\mathbb{Z}/2^k,n) \longrightarrow K(\mathbb{Z},n+1)$$

what map is this?

Cannot really use Borel's th^m here as $K(\mathbb{Z},n)$ is not contractible.

$$\begin{array}{c} K(\mathbb{Z},1) \\ \| \\ \Omega K(\mathbb{Z},2) \end{array} \xrightarrow{2^k} K(\mathbb{Z},1) \longrightarrow K(\mathbb{Z}/2^k,1) \longrightarrow K(\mathbb{Z},2) \xrightarrow{2^k} K(\mathbb{Z},2)$$

$H_1(K(\mathbb{Z}/2^k,1);\mathbb{Z}) = \mathbb{Z}/2^k$

$H^1(K(\mathbb{Z}/2^k,1);\mathbb{Z}/2)$

$= \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2^k, \mathbb{Z}/2)$

$= \mathbb{Z}/2$

$$\begin{tikzcd}
1 & \mathbb{Z}/2 \arrow[drr] & & \mathbb{Z}/2 \arrow[drr] & & \mathbb{Z}/2 \arrow[drr] & & \mathbb{Z}/2 \arrow[drr] & & \\
0 & \mathbb{Z}/2 & & \mathbb{Z}/2 & & \mathbb{Z}/2 & & \mathbb{Z}/2 & & \\
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 &
\end{tikzcd}$$

So the differential has to be 0. $H^i(K(\mathbb{Z}/2^k,1);\mathbb{Z}/2) \cong \mathbb{Z}/2 \quad \forall i$

Further the multiplicative structure will stay $\Rightarrow H^*(K(\mathbb{Z}/2^k,1);\mathbb{Z}/2) \cong H^*(S^1;\mathbb{Z}/2) \otimes H^*(\mathbb{CP}^\infty;\mathbb{Z}/2)$

Also as modules over the Steenrod algebra

Now using Borel' th^m on the fibration

$$K(\mathbb{Z}/2^k,n+1) \longrightarrow * \longrightarrow K(\mathbb{Z}/2^k,n)$$

and fitting this between

$$\begin{tikzcd}
K(\mathbb{Z},n+1) \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & K(\mathbb{Z},n) \arrow[d] \\
K(\mathbb{Z}/2^k,n+1) \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & K(\mathbb{Z}/2^k,n) \arrow[d] \\
K(\mathbb{Z},n+2) \arrow[r] & * \arrow[r] & K(\mathbb{Z},n+1)
\end{tikzcd}$$

we get $\quad H^*(K(\mathbb{Z}/2^k;n);\mathbb{Z}/2) \cong H^*(K(\mathbb{Z},n);\mathbb{Z}/2) \otimes H^*(K(\mathbb{Z},n+1);\mathbb{Z}/2)$