Recall :

- $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.

- = slope of tangent to the graph of $f(x)$.

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§3 : algebraic identities for finding derivatives.

- $(f+g)' = f' + g'$
- $(f-g)' = f' - g'$
- $(cf)' = c(f')$, $c \in \mathbb{R}$

- $(x^{n})' = n \cdot x^{n-1}$ (power rule)

- $(e^x)' = e^x$

- $(a^x)' = a^x \cdot \ln a$ (a is a positive real number)

- $\lim_{h \to 0} \dfrac{e^h - 1}{h} = 1$

- Quiz on Monday - IVT, 2.7, 2.8, 3.1, 3.2, 3.3

- **Product Rule:**

$$(f\cdot g)' = \lim_{h\to 0} \frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x)}{h}$$

add & subtract $f(x+h)\cdot g(x)$

$$= \lim_{h\to 0} \frac{f(x+h)\cdot g(x+h) - f(x+h)\cdot g(x) + f(x+h)\cdot g(x) - f(x)\cdot g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)\left[g(x+h) - g(x)\right] + \left(f(x+h) - f(x)\right)\cdot g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)\cdot\left[g(x+h) - g(x)\right]}{h} + \lim_{h\to 0} \frac{\left[f(x+h) - f(x)\right]}{h}\cdot g(x)$$

$$= \lim_{h\to 0} f(x+h)\cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\cdot \lim_{h\to 0} g(x)$$

$$(f\cdot g)' = f(x)\cdot g'(x) + f'(x)\cdot g(x)$$

- Product Rule: $(f \cdot g)' = f'g + g'f$

- Quotient Rule: $\left(\frac{f}{g}\right)' = \frac{g \cdot f' - f \cdot g'}{g^2}$

eg: Find $(e^x \cdot x)'$.

Ans
$$(e^x x)' = (e^x)' \cdot x + e^x \cdot (x')$$
$$= e^x \cdot x + e^x \cdot 1$$
$$= e^x \cdot x + e^x$$
$$= e^x (x+1)$$

[Interesting exercise: Use product rule to prove power rule]

Q. Find the point on the curve $(x \cdot 2^x)$ at which the tangent is horizontal.

A. tangent horizontal $\Rightarrow$ derivative is 0.

$f(x) = x \cdot 2^x$

$f'(x) = 0 \quad \leftarrow$ Solve for x.

$$f'(x) = (x \cdot 2^x)'$$

$$= (x)' \cdot 2^x + x \cdot (2^x)'$$

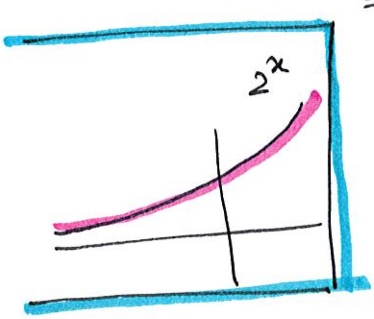
$$= 1 \cdot 2^x + x \cdot 2^x \cdot \ln 2$$

$$= 2^x (1 + x \cdot \ln 2)$$

want $f'(x) = 0$

$$\Rightarrow 2^x \cdot (1 + x \cdot \ln 2) = 0$$

$\Rightarrow \underbrace{2^x = 0}$ or $\underbrace{1 + x \ln 2 = 0}$



this has no solution

$\Rightarrow x \cdot \ln 2 = -1$

$\Rightarrow \boxed{x = -1/\ln 2}$

<u>Ans</u>: the tangent to $x \cdot 2^x$ is horizontal

at $x = -1/\ln 2$.

$f(x) = x \cdot 2^x$

$$y = f(-1/\ln 2)$$

$$= \left(-1/\ln 2\right) \cdot 2^{-1/\ln 2}$$

$$= \frac{-1}{\ln 2} \cdot \frac{1}{2^{1/\ln 2}}$$

Claim: $2^{1/\ln 2} = e$

Quotient Rule: $\left(\frac{f}{g}\right)' = \frac{g \cdot f' - f \cdot g'}{g^2}$

eg: Find $\left(\frac{e^x}{x}\right)'$

Ans:

Method 1) $\left(\frac{e^x}{x}\right)' = \frac{x \cdot (e^x)' - e^x \cdot (x)'}{x^2}$

$f = e^x$

$g = x$

$\frac{g \cdot f' - f \cdot g'}{g^2}$

$= \frac{x \cdot e^x - e^x \cdot 1}{x^2}$

$= \frac{x \cdot e^x - e^x}{x^2}$

$= \frac{e^x \cdot (x-1)}{x^2}$

Method 2) $\left(\frac{e^x}{x}\right)' = \left(e^x \cdot x^{-1}\right)'$

$= (e^x)' \cdot x^{-1} + e^x \cdot (x^{-1})'$

$= e^x \cdot x^{-1} + e^x \cdot (-1) \cdot x^{-2}$

$= e^x \cdot \frac{1}{x} - \frac{e^x}{x^2}$

Trig functions:

$$(\sin x)' = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \cancel{\lim_{h\to 0} \sin x \cdot \cos h + \sin h \cdot \cos - s}$$

$$= \lim_{h\to 0} \frac{\sin x \cdot \cos h + \sin h \cdot \cos x - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x\,[\cos h - 1] + \sin h \cdot \cos x}{h}$$

$$= \lim_{h\to 0} \sin x \cdot \frac{\cos h - 1}{h} + \lim_{h\to 0} \frac{\sin h}{h} \cdot \cos x$$

$$= \sin x \cdot \underbrace{\lim_{h\to 0} \frac{\cos h - 1}{h}}_{= 0} + \cos x \cdot \underbrace{\lim_{h\to 0} \frac{\sin h}{h}}_{= 1}$$

Turns out :

$$= \sin x \cdot 0 + \cos x \cdot 1$$

$$\boxed{(\sin x)' = \cos x}$$

$$(\cos x)' = \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h \to 0} \frac{\cos x \cdot \cos h - \sin x \cdot \sin h - \cos x}{h}$$

$$= \lim_{h \to 0} \frac{\cos x(\cos h - 1)}{h} - \lim_{h \to 0} \frac{\sin x \cdot \sin h}{h}$$

$$= \cos x \lim_{h \to 0} \frac{\cos h - 1}{h} - \sin x \cdot \lim_{h \to 0} \frac{\sin h}{h}$$

$$= -\sin x$$

$$\boxed{(\cos x)' = -\sin x}$$

$(\sin x)' = \cos x$ $(\cos x)' = -\sin x$	$\lim_{h \to 0} \frac{\sin h}{h} = 1$	$\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$

Aside: why is $2^{1/\ln 2} = e$?

Ans: $2^{1/\ln 2} = \sqrt[\ln 2]{2}$

= the number which when raised to the power $(\ln 2)$ gives us 2

$= e \qquad (\text{as } e^{\ln 2} = 2)$

- **Theorems**: $\displaystyle\lim_{h \to 0} \frac{\sin h}{h} = 1$ $\quad\Big|\quad$ $\displaystyle\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$.

"Proof": Squeeze theorem &

Use some geometry to show $\cos h \leq \dfrac{\sin h}{h} \leq 1$

$$\lim_{h \to 0} \cos h = \cos 0 = 1 \qquad \Big| \qquad \lim_{h \to 0} 1 = 1$$

By Squeeze theorem

$$\lim_{h \to 0} \frac{\sin h}{h} = 1$$

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)'$$

quotient rule

$$\left(\frac{f}{g}\right)' = \frac{g \cdot f' - f \cdot g'}{g^2}$$

$f = \sin x$

$g = \cos x$

$$\cancel{= \frac{(\cos x)(\sin x)' - (\sin x)(\cos x)'}{\cos^2 x}}$$

$$= \frac{(\cos x)(\sin x)' - (\sin x) \cdot (\cos x)'}{\cos^2 x}$$

$$= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$(\tan x)' = \sec^2 x$$

Trig ~~Derivatives~~ identities

$(\sin x)' = \cos x$

$(\cos x)' = -\sin x$

$(\tan x)' = \sec^2 x$

$\sin^2 x + \cos^2 x = 1$

$\sin(2x) = 2 \cdot \cos x \cdot \sin x$

$\cos(2x) = 2\cos^2 x - 1$

$\cos(2x) = 1 - 2\sin^2 x$

- $\lim_{h \to 0} \frac{\sin h}{h} = 1$
- $\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$

Q. Find $\lim_{h \to 0} \frac{\sin(2h)}{h}$.

A. Method 1)

$$\lim_{h \to 0} \frac{\sin(2h)}{h} = \lim_{h \to 0} \frac{\sin(2h)}{h} \cdot \frac{2}{2}$$

$$= \lim_{h \to 0} \frac{\sin(2h)}{(2h)} \cdot 2$$

Let $k = 2h$
as $h \to 0$, $k \to 0$

$$= \lim_{k \to 0} \frac{\sin(k)}{k} \cdot 2$$

$$= 1 \cdot 2$$

$$= 2$$

Method 2)

$$\lim_{h\to 0} \frac{\sin(2h)}{h} = \lim_{h\to 0} \frac{2\sin h\cdot\cos h}{h}$$

$$= 2\lim_{h\to 0} \frac{\sin h\cdot\cos h}{h}$$

$$\frac{a\cdot b}{c} = \frac{a}{c}\cdot b$$

$$= 2\lim_{h\to 0} \frac{\sin h}{h}\cdot\cos h$$

$$= 2\underbrace{\lim_{h\to 0} \frac{\sin h}{h}}\cdot\underbrace{\lim_{h\to 0} \cos h}$$

$$= 2\cdot\underbrace{1}\cdot\underbrace{1}$$

- Plug in $h=0$
- cos is continuous at 0.

- $\lim_{h\to 0} \frac{\sin h}{h} = 1$

Fact

- $\lim_{h\to 0} \frac{\sin\sqrt{h}}{\sqrt{h}} = 1$

Let $k=\sqrt{h}$ ..

- $\lim_{h\to\infty} \frac{\sin h}{h}$

$-1 \le \sin h \le 1$

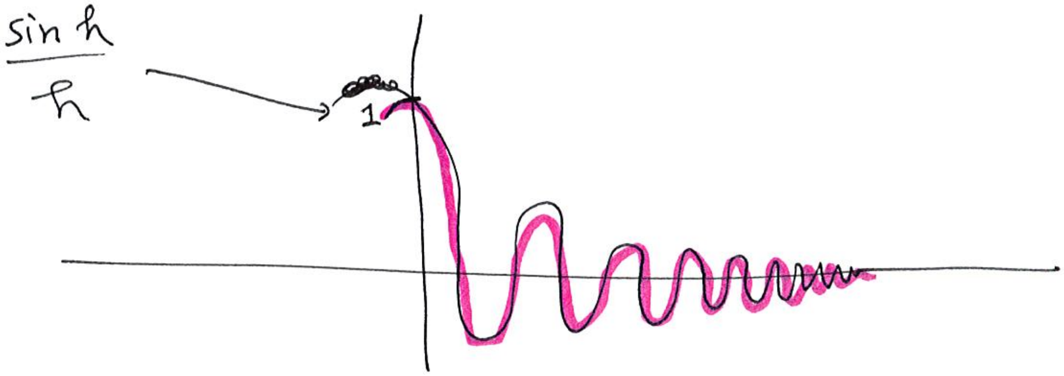
Divide both sides by h

$$-\frac{1}{h} \le \frac{\sin h}{h} \le \frac{1}{h}$$

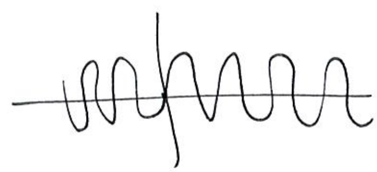
$$\lim_{h\to\infty} -\frac{1}{h} = 0 \qquad \lim_{h\to\infty} \frac{1}{h} = 0$$

$$\Rightarrow \lim_{h\to\infty} \frac{\sin h}{h} = 0$$

$$\lim_{h\to\infty} \sin h = \text{d.n.e}$$



Q. $\lim_{h \to \infty} \sin(2h) = \text{d.n.e}$



Q. $\lim_{h \to \infty} \frac{\sin(2h)}{h} = \lim_{h \to \infty} \underbrace{\frac{\sin(2h)}{(2h)}}_{\downarrow \, 0} \cdot 2$

$= 0 \cdot 2 = 0$

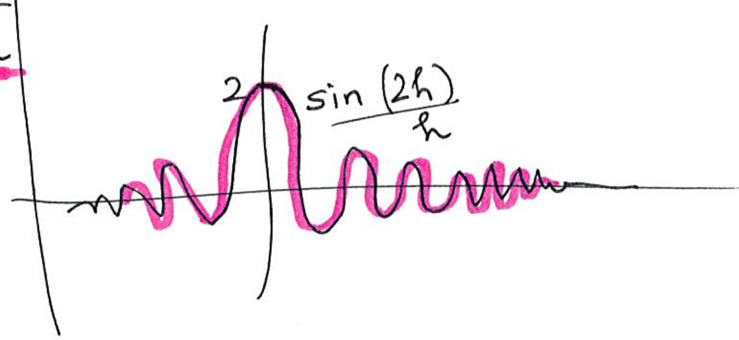
$-1 \le \sin(2h) \le 1$

divide by h

$$\frac{-1}{h} \le \frac{\sin(2h)}{h} \le \frac{1}{h}$$

$\lim_{h \to \infty} \frac{-1}{h} = 0$ $\quad$ $\lim_{h \to \infty} \frac{1}{h} = 0$ $\Rightarrow$ By Squeeze theorem $\lim_{h \to \infty} \frac{\sin(2h)}{h} = 0$

$\frac{\sin(2h)}{h}$ $\quad$ $\sin 2h \cdot \frac{1}{h}$



e

eg: ~~lim~~ for squeeze theorem

$$\lim_{x \to \infty} e^{-x} \cdot \sin x$$

$$\lim_{h \to 0} h \cdot \sin\left(\frac{1}{h}\right)$$

$$\lim_{h \to \infty} \frac{\sin h}{h}$$

For applying squeeze theorem: (typically)

- there is a $\sin x$ or $\cos x$
- there is another function that goes to 0.

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eg: $\lim_{h \to \infty} h \cdot \sin(h)$

wrong method: $-1 \leq \sin h \leq 1$

multiply by h

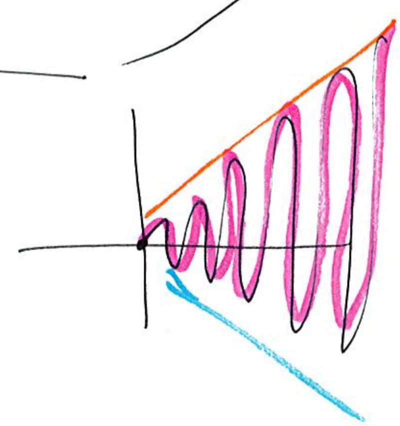
$$-h \leq h \sin h \leq h$$

$\lim_{h \to \infty} -h = -\infty$ $\lim_{h \to \infty} h = \infty$

No. conclusion

$\lim_{h \to \infty} h = \infty$ $\sin h$ oscillates

$\Rightarrow \lim_{h \to \infty} h(\sin h)$ does not exist.



Q. $\lim\limits_{h \to 0} h \cdot \sin\left(\frac{1}{h}\right)$

A:

$$-1 \leq \sin\left(\frac{1}{h}\right) \leq 1$$

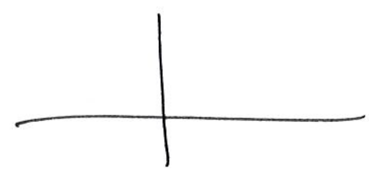
Multiply b.s. by h

$$-h \leq h \cdot \sin \frac{1}{h} \leq h$$

$\lim\limits_{h \to 0} -h = 0$ $\qquad$ $\lim\limits_{h \to 0} h = 0$

By Squeeze theorem, as both these limits agree

$$\boxed{\lim_{h \to 0} h \cdot \sin\left(\frac{1}{h}\right) = 0}$$



Q. $\lim\limits_{h \to 0} \frac{\sin h}{h} = 1$

Q. $\lim\limits_{h \to \infty} \frac{\sin h}{h} = 0$

By Squeeze theorem

Back to derivatives

Q. $(\sec x)' = \left(\frac{1}{\cos x}\right)'$

$$= \frac{\cos x \cdot (1)' - (\cos x)' \cdot 1}{(\cos x)^2}$$

$\left(\frac{f}{g}\right)' = \frac{g \cdot f' - f \cdot g'}{g^2}$

$f = 1$

$g = \cos x$

$$= \frac{0 - (-\sin x)\cdot 1}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x}$$

$$= \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}$$

$$(\sec x)' = \tan x \cdot \sec x$$

Q. Find the point at which the normal to ~~sin x~~ $\frac{e^x}{x}$ is parallel to $y = \frac{1}{2}x + 1$.

A: slope of normal $= -\frac{1}{f'(x)}$ | slope of $y = \frac{1}{2}x + 1$ is $\frac{1}{2}$.

want $-\frac{1}{f'(x)} = \frac{1}{2}$ (parallel $\Rightarrow$ same slope)

$$\Rightarrow \boxed{f'(x) = -2}$$

$$f'(x) = \left(\frac{e^x}{x}\right)'$$

$$= \frac{x\cdot(e^x)' - e^x\cdot(x)'}{x^2}$$

$$= \frac{x\cdot e^x - e^x}{x^2}$$

$$\left(\frac{f}{g}\right)' = \frac{g\cdot f' - f\cdot g'}{g^2}$$

$f = e^x$

$g = x$

$$\Rightarrow \quad \frac{x \cdot e^x - e^x}{x^2} = \cancel{} - 2$$

~~In~~ a (↑ Bad situation. oops.
I should've picked a
better problem)

solve for x.