Comma Categories:

Def: To any ∞-functors $C \xrightarrow{g} A \xleftarrow{f} B$ form,

$$\begin{tikzcd}
\mathrm{Hom}_A(f,g) \arrow[r, "\phi"] \arrow[d, "{(p_1, p_1)}"'] & A^{2} \arrow[d] \\
C \times B \arrow[r, "g \times f"'] & A \times A
\end{tikzcd}$$

- elements of $\mathrm{Hom}_A(f,g)$ are triples $(c \in C,\ b \in B,\ \alpha : fb \longrightarrow gc \in A)$.

∞-comma category: $\quad \mathrm{Hom}_A(f,g) \xrightarrow{(p_1, p_0)} C \times B$

This has a canonical 2 cell called the comma cone

$$\begin{tikzcd}
 & \mathrm{Hom}_A(f,g) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\
C \arrow[dr, "g"'] & \arrow[l, phantom, "\overset{\phi}{\Leftarrow}"] & B \arrow[dl, "f"] \\
 & A &
\end{tikzcd}$$

eg: A^{2} is a comma category $\mathrm{Hom}_A(id_A, id_A)$.

As before we get 3-operations in hK :

1-cell induction :

$$\begin{tikzcd}
 & X \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\
C \arrow[dr, "g"'] & \Leftarrow & B \arrow[dl, "f"] \\
 & A &
\end{tikzcd}
=
\begin{tikzcd}
 & X \arrow[d, dashed] & \\
 & \mathrm{Hom}_A(f,g) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\
C \arrow[dr, "g"'] & \Leftarrow & B \arrow[dl, "f"] \\
 & A &
\end{tikzcd}$$

2-cell induction, 2 cell conservativity.

- There are weak UPs which uniquely determine $\mathrm{Hom}_A(f,g) \twoheadrightarrow C \times B$.

Prop: The comma cone induces a (fibered) bijection:

$$\left\{ \begin{tikzcd} & X \arrow[dl, "c"'] \arrow[dr, "b"] & \\ C \arrow[rr, phantom, "\overset{x}{\Leftarrow}"] \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\ & A & \end{tikzcd} \right\} \longleftrightarrow \left\{ \begin{tikzcd} & X \arrow[dl] \arrow[dr] \arrow[d, dashed] & \\ C & \mathrm{Hom}_A(f,g) \arrow[l, "p_1"] \arrow[r, "p_0"'] & B \end{tikzcd} \right\}$$

$\shortparallel$

fibered iso classes of maps of spans

$$\begin{tikzcd} X \arrow[rr, "x"] \arrow[dr, Rightarrow] & & \mathrm{Hom}_A(f,g) \arrow[dl, Rightarrow] \\ & C \times B & \end{tikzcd}$$

Def^n: $1 \overset{f}{\underset{g}{\rightrightarrows}} A$

then $\mathrm{Hom}_A(f,g)$ is called an *internal mapping space*

i.e.

$$\begin{tikzcd} \mathrm{Hom}_A(f,g) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A^2 \arrow[d] \\ 1 \arrow[r, "f \times g"'] & A \times A \end{tikzcd}$$

- $\mathrm{Hom}_A(x,y)$ is a <u>discrete</u> object in $\mathcal{K}$.

    every 2-cell with codomain $\mathrm{Hom}_A(f,g)$ is invertible.

    i.e. this is a ∞-groupoid / Kan complex.

Def^n: Any functor $A \xrightarrow{f} B$ may be represented as an ∞-cat in 2 ways:

left rep:

$$\begin{tikzcd} & \mathrm{Hom}_B(f,B) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\ B \arrow[rr, phantom, "\Leftarrow", near start, yshift=12pt] & & A \arrow[ll, "f"] \end{tikzcd}$$

right rep:

$$\begin{tikzcd} & \mathrm{Hom}_B(B,f) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\ A \arrow[rr, "f"] & & B \end{tikzcd}$$

A comma $\mathrm{Hom}_A(f,g) \twoheadrightarrow C\times B$ is

left representable: if $\exists\ B \xrightarrow{\ell} C$ s.t.

$$\begin{tikzcd}[column sep=small] \mathrm{Hom}_A(f,g) \arrow[rr, "\simeq"] \arrow[dr, two heads] & & \mathrm{Hom}_C(\ell, C) \arrow[dl, two heads] \\ & C\times B & \end{tikzcd}$$

right representable: if $\exists\ C \xrightarrow{r} B$ s.t.

$$\begin{tikzcd}[column sep=small] \mathrm{Hom}_A(f,g) \arrow[rr, "\simeq"] \arrow[dr, two heads] & & \mathrm{Hom}_B(B, r) \arrow[dl, two heads] \\ & C\times B & \end{tikzcd}$$

Th^m:

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\Rightarrow\rho"] & A \end{tikzcd}$$

is an absolute right lifting

iff induced map of commas y is an equiv.

$$\begin{tikzcd}[column sep=small] & \mathrm{Hom}(B,r) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] \arrow[d, phantom, "\xLeftarrow{\phi}"] & \\ C \arrow[dr, "g"'] & B \arrow[l] \arrow[d, phantom, "\xLeftarrow{\rho}"] & B \arrow[ll] \arrow[dl, "f"] \\ & A & \end{tikzcd} \quad = \quad \begin{tikzcd}[column sep=small] & \mathrm{Hom}_B(B,r) \arrow[d, dashed, "\exists\ y"] & \\ & \mathrm{Hom}_A(f,g) \arrow[dl] \arrow[dr] \arrow[d, phantom, "\xLeftarrow{\phi}"] & \\ C \arrow[r, "g"'] & A & B \arrow[l, "f"] \end{tikzcd}$$

Th^m: $\exists$ a bijection :

$$\left\{ \begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\Downarrow\rho"] & A \end{tikzcd} \right\} \xrightarrow{\ \simeq\ } \left\{ \begin{tikzcd}[column sep=small] & \mathrm{Hom}_B(B,r) \arrow[dl] \arrow[dr] \arrow[d, "y"] & \\ C & \mathrm{Hom}_A(f,g) \arrow[l] \arrow[r] & B \end{tikzcd} \right\} \Big/ \text{fibered iso}$$

ρ absolute right lifting iff y is an $\simeq$.

Lemma $\exists$ an adjunction $\begin{tikzcd} A \arrow[r, dashed, bend right, "i"'] \arrow[r, phantom, "\perp"] & \mathrm{Hom}_B(B,f) \arrow[l, bend right, "p_1"'] \end{tikzcd}$, counit $=$ id

unit : $\mathrm{id} \xRightarrow{u} i p_1$

s.t. $p_1 u = \mathrm{id}$, $p_0 u = \phi$.

Cor: (cheap Yoneda lemma) Given $A \underset{g}{\overset{f}{\rightrightarrows}} B$ we get a bijection

$$\left\{ \begin{tikzcd} A \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] \arrow[from=U, to=D, phantom, "\Downarrow\alpha"] & B \end{tikzcd} \right\} \longleftrightarrow \left\{ \begin{tikzcd} & \mathrm{Hom}_B(B,f) \arrow[dl] \arrow[dr] \arrow[dd, dashed, "a"] & \\ A & & B \\ & \mathrm{Hom}_B(B,g) \arrow[ul] \arrow[ur] & \end{tikzcd} \right\} \Big/ \text{fibered iso}$$

$$\updownarrow$$

$$\left\{ \begin{tikzcd} & \mathrm{Hom}_B(g,B) \arrow[dl] \arrow[dr] \arrow[dd, dashed, "a"] & \\ B & & A \\ & \mathrm{Hom}_B(f,g) \arrow[ul] \arrow[ur] & \end{tikzcd} \right\} \Big/ \text{fibered iso}$$