## Derivations and differential forms:

**Antisymmetrization map $\varepsilon_n$:** $S_n \ni \sigma \quad \sigma\cdot(m, a_1 \cdots a_n) := (m, a_{\sigma^{-1}(1)}, \cdots, a_{\sigma^{-1}(n)})$

extend this action by $k$-linearity on the group algebra $k[S_n]$ on $C_n(R,M)$

anti-symm element $\quad \varepsilon_n := \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, \sigma \in k[S_n]$

$$\varepsilon_n : M \otimes \Lambda^n R \longrightarrow C_n(R,M)$$

$$(m \otimes a_1 \wedge \cdots \wedge a_n) \longmapsto \varepsilon_n(m; a_1 \cdots a_n)$$

Prop:

$$\begin{tikzcd} M \otimes \Lambda^n R \arrow[r, "\varepsilon_n"] \arrow[d, "\delta"'] & C_n(R,M) \arrow[d, "b"] \\ M \otimes \Lambda^{n-1} R \arrow[r, "\varepsilon_{n-1}"] & C_{n-1}(R,M) \end{tikzcd}$$

commutes, where

$$\delta(m \otimes a_1 \wedge \cdots \wedge a_n) := \sum_{i=1}^{n} (-1)^{i+1} [m, a_i] \otimes a_1 \wedge \cdots \wedge \hat{a}_i \wedge \cdots \wedge a_n$$

Chevalley-Eilenberg inner derivation

$$+ \sum_{1 \le i < j \le n} (-1)^{i+j}\, m \otimes [a_i, a_j] \wedge a_1 \wedge \cdots \wedge \hat{a}_i \wedge \cdots \wedge \hat{a}_j \wedge \cdots \wedge a_n$$

Proof: $\quad n = 1 \quad \varepsilon_0 \cdot \delta(m \otimes a_1) = m a_1 - a_1 m \quad$ both $\varepsilon_0$ and $\varepsilon_1$ are trivial

$\qquad b \cdot \varepsilon_1 (m \otimes a_1) = m a_1 - a_1 m$

Assume true upto $n$: $\quad b \cdot \varepsilon_n = \varepsilon_{n-1} \cdot \delta$

Want to understand $\varepsilon_{n+1}$:

Let $\quad \underline{a} = (m = a_0, a_1, \cdots, a_n)$

$(\underline{a}, y) = (m, a_1, \cdots, a_n, y) \qquad a_i, y \in R, \; m \in M$

$$s(y) : C_n(R,M) \longrightarrow C_{n+1}(R,M)$$

$$(m, a_1, \cdots, a_n) \longmapsto \sum_{i=0}^{n} (-1)^i \underbrace{(m, \cdots, a_i, y, a_{i+1}, \cdots, a_n)}_{=: s_i(y)(\underline{a})}$$

Claim: $\quad \varepsilon_{n+1}(\underline{a}, y) = (-1)^n s(y)\, \varepsilon_n(\underline{a})$

Proof: $\quad s(y)\, \varepsilon_n(\underline{a}) = \sum_{i=0}^{n} (-1)^n s_i(y)\, \varepsilon_n(\underline{a}) = \sum_{i=0}^{n} (-1)^i s_i(y) \sum_{\sigma \in S_n} (-1)^{\sigma}\, \sigma(\underline{a})$

$$= \sum_{i=0,\, \sigma \in S_n}^{n} (-1)^{i+\sigma} s_i(y)\, \sigma(\underline{a})$$

$$= \sum_{i=0,\, \sigma\in S_n}^{n} (-1)^{i+\sigma} \left(a_0, a_{\sigma^{-1}(1)}, \dots, a_{\sigma^{-1}(i)}, y, a_{\sigma^{-1}(i+1)}, \dots, a_{\sigma^{-1}(n)}\right)$$

Let $\tau \in S_{n+1}$ $\quad \tau^{-1}(j) = \begin{cases} \sigma^{-1}(j) & j \le i \\ n+1 & j = i \\ \sigma^{-1}(j-1) & j > i \end{cases}$ $\quad$ how are $\tau$ and $\sigma$ related?

$$\tau^{-1} = (i{+}1 \;\; i{+}2 \dots n{+}1) \cdot \sigma^{-1}$$

$$(-1)^{\tau} = (-1)^{\tau^{-1}} = (-1)^{n-i} \cdot (-1)^{\sigma^{-1}} = (-1)^{n+i} (-1)^{\sigma} = (-1)^n \cdot (-1)^{i+\sigma}$$

$$= \sum_{\tau\in S_{n+1}} (-1)^n \cdot (-1)^{\tau} \cdot \tau(\underline{a}, y)$$

$$= (-1)^n \, \varepsilon_{n+1}(\underline{a}, y)$$

Proof incomplete, fill gaps.

Def: $y \in R$, $\quad \mathrm{ad}(y) : C_n(R,M) \longrightarrow C_n(R,M)$

$$(a_0, a_1, \dots, a_n) \longmapsto \sum_{i=0}^{n} (a_0, \dots, a_{i-1}, [y, a_i], \dots, a_n)$$

Then one has:

$$b \cdot \varepsilon_{n+1}(\underline{a}, y) = (-1)^n \, b \cdot s(y) \cdot \varepsilon_n(\underline{a}).$$

$$= (-1)^n \left(-\mathrm{ad}(y) - s(y)\, b\right) \varepsilon_n(\underline{a}) \qquad \llcorner\, b \cdot s(y) + s(y) \cdot b = -\mathrm{ad}(y) \quad \text{proved later}$$

$$= (-1)^{n+1} \mathrm{ad}(y)\, \varepsilon_n(\underline{a}) + (-1)^{n+1} s(y)\, \varepsilon_{n-1}(\delta(\underline{a})) \qquad \llcorner \text{ by induction}$$

$$= (-1)^{n+1} \mathrm{ad}(y)\, \varepsilon_n(\underline{a}) + \varepsilon_n(\delta(\underline{a}), y) \qquad \llcorner \text{ as } \varepsilon \text{ is additive}$$

Claim: $b \cdot s(y) + s(y) \cdot b = -\mathrm{ad}(y)$

Proof: we have $d_i s_i - d_i s_{i-1} (a_0, a_1, \dots, a_n) = (a_0, a_1, \dots, -[y, a_i], a_{i+1}, \dots, a_n)$
almost a pre-simplicial homotopy.

Remark: 1) $\mathrm{ad}(y)$ induces $0$ in homology.

2) If $R$ is commutative and $M$ is symmetric, $\delta = 0$
So $b \cdot \varepsilon_n = 0 \Rightarrow \mathrm{Im}(\varepsilon_n : M \otimes \Lambda^n R \longrightarrow C_n(R,M)) \subseteq \ker(b)$
hence we get a map: $\quad M \otimes \Lambda^n R \xrightarrow{\varepsilon_n} H_n(R,M)$

Next we want to factor this through $M \otimes \Lambda^n \Omega_{R/k}$ (Kähler)

**Derivation:** $d: R \to M$, $d(ab) = a\,db + da.b$

**Universal:** universal if $\forall$ derivation $\delta: R \to N$, $\exists!$ R-linear map s.t. triangle commutes:

$$\begin{tikzcd} R \arrow[r, "d"] \arrow[dr, "\delta"'] & M \arrow[d, dashed, "\phi"] \\ & N \end{tikzcd}$$

When R- unital, commutative k-algebra

Universal derivation construction: $I := \ker(R \otimes R \xrightarrow{\eta} R)$

$M := I/I^2$ ⟿ Symmetric R-module: $I = R(x\otimes 1 - 1 \otimes x)$

$a(x\otimes 1 - 1\otimes x) - (x\otimes 1 - 1 \otimes x)a$

$= (a\otimes 1 - 1\otimes a)\cdot(x\otimes 1 - 1\otimes x)$

$d: R \longrightarrow M$

$x \longmapsto 1\otimes x - x\otimes 1$

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**Kähler differential:**

Def$^n$: $\Omega^1_{R|k}$ := R-module generated by $da$, $a\in R$ $\quad \Omega^0_{R|k} := R$

relations: $dab - a\,db - (da)\,b$

$\Omega^n_{R|k} := R(dx_1 \wedge \cdots \wedge dx_n)$

$\Omega^*_{R|k} := \bigoplus_{n\geq 0} \Omega^n_{R|k}$

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If $R$ is not unital, define $\tilde{R} := k \oplus R$, $(\lambda, x)\cdot(\mu, y) = (\lambda\mu, \lambda y + \mu x + xy)$

$\Omega^*_{\tilde{R}|k} := \bigoplus_{n\geq 1} \Omega^n_{\tilde{R}|k}$ where $\Omega^n_{\tilde{R}|k} := \tilde{R} \otimes R^{\otimes n}$

$R \circlearrowright \Omega^*_{\tilde{R}|k}$ by $x(a_0, \dots, a_n) = xa_0 \otimes \cdots \otimes a_n$

$(a_0, \dots, a_n)x = \sum_{i=1}^{n-1} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes x + a_0 \otimes \cdots \otimes a_n x$

$d(a_0 \otimes a_1 \otimes \cdots \otimes a_n) := 1_{\tilde{R}} \otimes \underline{a_0} \otimes \cdots \otimes a_n$ $\quad \underline{a_0}$ is R-part of $a_0$

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The things in this section work well when R is commutative with unit. In general, what part works is not clear to me.

$d: \Omega^n_{R|k} \longrightarrow \Omega^{n+1}_{R|k}$, $d^2 = 0$

$x_0 \cdot dx_1 \wedge \cdots dx_n \longmapsto dx_0\, dx_1 \wedge \cdots \wedge dx_n$

$dx_1 \wedge \cdots dx_n \longmapsto 0$

$b: \Omega^n_{R|k} \longrightarrow \Omega^{n-1}_{R|k}$, $b^2 = 0$

$\omega\, dx \longmapsto (-1)^{|\omega|}[\omega, x]$

$dx \longmapsto 0 \quad \forall x \in R$

$x \longmapsto 0$

Claim : $R$ unital commutative $\Rightarrow$ $d: R \longrightarrow \Omega^1_{R/k}$ is the universal derivation
$x \longmapsto dx$

$$\Updownarrow$$

$$I/I^2 \xrightarrow{\cong} \Omega^1_{R|k}$$
$$(1\otimes x - x\otimes 1) \longmapsto dx$$

Prof: $\exists$ a canonical $R$-linear map
$$\mathrm{Hom}_R(\Omega^1_{R|k}, M) \xrightarrow{\cong} \mathrm{Der}_k(R,M) \qquad f \longmapsto f\cdot d$$

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Prop: If $R$-commutative, $HH_1(R) \cong \Omega^1_{R|k}$
If further $M$ is symmetric, $H_1(R,M) \cong M \otimes_R \Omega^1_{R/k}$

Proof: $HH_1(R) = R\otimes R \big/ ab\otimes c - a\otimes bc + ca\otimes b$ $\qquad [a\otimes b] \leftrightarrow adb$

$$\downarrow$$
$$\Omega^1_{R|k}$$

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Prop: For any commutative $k$-algebra $R$, symmetric $R$-module $M$ the anti-symmetric map $\varepsilon_n : M\otimes \Lambda^n R \longrightarrow H_n(R,M)$ factors through

$$\begin{tikzcd} M\otimes \Lambda^n R \arrow[rr, "\varepsilon_n"] \arrow[dr] & & H_n(R,M) \\ & M\otimes \Omega^n_{R|k} \arrow[ur] & \end{tikzcd}$$

Proof: Suffices to show

$$\varepsilon_n(mx, y, a_3, \cdots, a_{n+1}) + \varepsilon_n(my, x, a_3, \cdots, a_{n+1}) - \varepsilon_n(m, xy, a_3, \cdots, a_{n+1}) = b(\cdots)$$
$$(\,0 \text{ in } H_n(R,M))$$

Why?

Claim: For $k$-algebra $R$, $M \in R^e\text{-Mod}$

$$\pi_n : C_n(R, M) \twoheadrightarrow M \otimes \Omega^n_{R|k}$$ what is this $\otimes$ over? $R$? $k$?

$$a_0, a_1, \dots, a_n \longmapsto a_0\, da_1 \wedge \dots \wedge da_n$$ Has to be $R$ to make sense.

Then, $\pi \circ b = 0$

Cor: $\pi_n : H_n(R, M) \longrightarrow M \otimes \Omega^n_{R|k}$

Prop: $R$-commutative, $M$-symmetric $\Rightarrow$ $\pi_n \cdot \varepsilon_n$ is multiplication by $n!$ on $M \otimes \Omega^n_{R|k}$