Cartesian fibrations:

Aside: What are Spaces + Spectra as ∞-cats?

$$\text{Top Enriched Category} \xrightarrow[\text{some nerve}]{N} \text{QCat}$$

Both Top, Sp are both Top enriched.
We can then move from QCat to other $(\infty,1)$-models.

Idea: $E \overset{p}{\twoheadrightarrow} B$ isofib. $\Rightarrow$

$$\begin{tikzcd} E_b \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d, two heads] \\ 1 \arrow[r, "b"'] & B \end{tikzcd}$$

fibers E_b are ∞-cats

In a $\left\{\begin{array}{l}\text{cocartesian} \\ \text{cartesian}\end{array}\right.$ fibration arrows in B act $\left\{\begin{array}{l}\text{covariantly} \\ \text{contravariantly}\end{array}\right.$ on fibers

In cocartesian fib: can lift 2 cells along p with specified lift on domain.

i.e.

$$\begin{tikzcd} E_b \arrow[r] \arrow[d, two heads] & E \arrow[d, two heads, "p"] \\ 1 \arrow[r, bend left, "b"] \arrow[r, bend right, "b'"'] \arrow[r, phantom, "\Downarrow\beta"] & B \end{tikzcd}$$

β lifts to

$$\begin{tikzcd} E_b \arrow[rr, bend left] \arrow[rr, dashed] \arrow[dr, dashed] \arrow[dd, two heads] & & E \arrow[dd, two heads, "p"] \\ & E_{b'} \arrow[ur] \arrow[dl] & \\ 1 \arrow[rr, "b'"'] & & B \end{tikzcd}$$

Def^n: $E \overset{p}{\twoheadrightarrow} B$ iso fib. A natural transform $\begin{tikzcd} X \arrow[r, bend left, "e'"] \arrow[r, bend right, "e"'] \arrow[r, phantom, "\Downarrow\chi"] & E \end{tikzcd}$ is p-cartesian

i) induction: given $\begin{tikzcd} X \arrow[r, bend left, "e''"] \arrow[r, bend right, "e"'] \arrow[r, phantom, "\Downarrow\tau"] & E \end{tikzcd}$ and $\begin{tikzcd} X \arrow[r, bend left, "pe''"] \arrow[r, bend right, "pe'"'] \arrow[r, phantom, "\Downarrow\gamma"] & B \end{tikzcd}$

so that $p\tau = p\gamma \circ \chi$

then $\exists\ \hat{\gamma} : e'' \Rightarrow e$ s.t. $p\hat{\gamma} = \gamma$

"χ is universal lift of $p\chi$ with codomain e".

2) conservativity: if $\xi : e \Rightarrow e'$ s.t. $\chi\xi = \chi$ and $p\xi = id_{pe'}$
then ξ is invertible.

Lemma: $\chi, \chi' : e'' \Rightarrow e'$ are p-cartesian and $p\chi = p\chi'$ then
$\exists\ \xi : e'' \xrightarrow{\simeq} e'$ s.t. $p\xi = id$.

Def: An isofib $E \overset{p}{\twoheadrightarrow} B$ is *cartesian* if

1) lifting: given 2-cell $\beta : b \Rightarrow pe$

$$
\begin{tikzcd}
X \arrow[r, "e"] \arrow[dr, bend right, "b"'] \arrow[dr, phantom, "\beta \Nearrow"] & E \arrow[d, "p"] \\
& B
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, bend left, "e"] \arrow[r, bend right, dashed, "\beta^*(e)"'] \arrow[r, phantom, "\Uparrow \chi_\beta"] & E \arrow[d, "p"] \\
& B
\end{tikzcd}
\qquad
\begin{matrix} p\chi_\beta = \beta \\ p\beta^*(e) = b \end{matrix}
$$

χ_β is p-cartesian

2) p-cartesian transforms are closed under restrictions along functors

$$
\begin{tikzcd}
X \arrow[r, bend left, "e"] \arrow[r, bend right, "e'"'] \arrow[r, phantom, "\Downarrow \chi"] & E
\end{tikzcd}
\ p\text{-cart} \Rightarrow \text{so is}\
\begin{tikzcd}
Y \arrow[r, "f"] & X \arrow[r, bend left, "e'"] \arrow[r, bend right, "e"'] \arrow[r, phantom, "\Downarrow \chi"] & E
\end{tikzcd}
\ \text{is also.}
$$

Prop:
- $E \overset{p}{\twoheadrightarrow} B \overset{q}{\twoheadrightarrow} A$ $\quad q, p$ cart $\Rightarrow$ so is qp.
- $\begin{tikzcd} X \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow \chi"] & E \end{tikzcd}$ $\quad qp$ cartesian $\Leftrightarrow$ χ is p-cart, $p\chi$ is q-cart.

Main Th^m "Internal char of cartesian fibrations"

$$
E \overset{p}{\twoheadrightarrow} B \quad \rightsquigarrow \quad
\begin{tikzcd}
& E \arrow[d, dashed, "i"] & \\
& \mathrm{Hom}_B(B, p) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[d, phantom, "\overset{\phi}{\Leftarrow}"] & \\
E \arrow[rr, "p"'] & & B
\end{tikzcd}
$$

Notation:

$$
\begin{tikzcd}
 & E^2 \arrow[d, dashed, "k"] \arrow[ddl, bend right, "p_1"'] \arrow[ddr, bend left, "pp_0"] & \\
 & \mathrm{Hom}_B(B,p) \arrow[dl] \arrow[dr] & \\
E \arrow[rr, "p"'] & \arrow[u, phantom, "\overset{\Phi}{\Leftarrow}"] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
E^2 \arrow[d, bend right=50, "p_1"'] \arrow[d, bend left=50, "p_0"] \arrow[d, phantom, "\overset{\Leftarrow}{k}"] \\
E \arrow[d, two heads, "p"] \\
B
\end{tikzcd}
$$

Thm: $E \overset{p}{\twoheadrightarrow} B$ iso fib. TFAE

i) p cartesian

ii)
$$
\begin{tikzcd}
E \arrow[rr, "i"] \arrow[dr, two heads, "p"'] & \arrow[d, phantom, "\perp"] & \mathrm{Hom}_p(B,p) \arrow[dl, two heads, "p_0"] \arrow[ll, dashed, bend left, "r"] \\
 & B &
\end{tikzcd}
$$
$\exists$ right adjoint over B

iii)
$$
\begin{tikzcd}
E^2 \arrow[r, bend left, "k"] \arrow[r, phantom, "\perp"] & \mathrm{Hom}_B(B,p) \arrow[l, dashed, bend left, "\bar{r}"]
\end{tikzcd}
$$
admits a RARI (with invertible counit)

Cor: $\mathcal{K} \xrightarrow{F} \mathcal{L}$ be a cosmological functor $\Rightarrow$ F preserves cartesian fibrations.

Lem: $E \overset{p}{\twoheadrightarrow} B$ is a cartesian fib iff the comma cone

$$
\begin{tikzcd}
\mathrm{Hom}_B(B,p) \arrow[r, "p_1"] \arrow[dr, bend right, "p_0"'] & E \arrow[d, two heads, "p"] \\
 & B
\end{tikzcd}
$$
has a lift $=$
$$
\begin{tikzcd}
\mathrm{Hom}_B(B,p) \arrow[r, bend left, "p_1"] \arrow[r, bend right, "r"'] \arrow[r, phantom, "\Uparrow \chi"] & E \arrow[d, two heads, "p"] \\
 & B
\end{tikzcd}
$$

such that χf is cartesian $\forall$ $X \xrightarrow{f} \mathrm{Hom}_B(B,p)$.

Lem: Consider
$$
\begin{tikzcd}
F \arrow[r, "f"] \arrow[d, two heads, "q"'] \arrow[dr, phantom, "\lrcorner" very near start] & E \arrow[d, two heads, "p"] \\
A \arrow[r, "g"'] & B
\end{tikzcd}
$$
p cartesian $\Rightarrow$ q- cartesian.

$X \overset{\psi \Downarrow}{\rightrightarrows} F$ q-cartesian $\Leftrightarrow$ $X \overset{\psi \Downarrow}{\rightrightarrows} F \xrightarrow{g} E$ is p-cartesian.

if f, g are equiv then q-cartesian $\Rightarrow$ p- cartesian.

Prop: A an ∞-category, $A^{2} \xrightarrow{p_0} A$ the domain projection functor is a cartesian fib.
Moreover,

$$\begin{tikzcd} X \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\psi"] & A \end{tikzcd} \text{ is } p_0\text{-cartesian iff } \begin{tikzcd} X \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow x"] & A^{2} \arrow[r, "p_1"] & A \end{tikzcd} \text{ is an iso.}$$

Prop: If A has all pullbacks then the codomain proj is a cartesian fib $A^{2} \xrightarrow{p_1} A$.

Def: $\mathcal{K}$ be an ∞-cosmos. $\exists$ an ∞-cosmos $\mathcal{K}^{co}$

- ob $\mathcal{K}^{co}$ = ob $\mathcal{K}$
- functors $A \longrightarrow B$ are the same
- $\mathrm{Fun}_{\mathcal{K}^{co}}(A,B) := \mathrm{Fun}_{\mathcal{K}}(A,B)^{op}$

(A qCat is a functor : $\Delta^{op} \xrightarrow{Q} \mathrm{Set}$
we can precompose with the functor $\Delta^{op} \longrightarrow \Delta^{op}$
sending $[n] \longmapsto [n]$ reversing the order.)

Then $h(\mathcal{K}^{co}) = (h\mathcal{K})^{co}$

$$\begin{tikzcd} A \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd} \rightsquigarrow \begin{tikzcd} A \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Uparrow\alpha"] & B \end{tikzcd}$$

Def: $E \overset{p}{\twoheadrightarrow} B$ is a cocartesian fib iff its p-cartesian fibration in $\mathcal{K}^{co}$.

Def: A bi-fibration is an isofib that is both cartesian & co-cartesian.

Prop: Let $E \overset{p}{\twoheadrightarrow} B$ be a bi-fibration and $\begin{tikzcd} X \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] \arrow[r, phantom, "\Downarrow\beta"] & B & E \arrow[l, two heads, "p"'] \end{tikzcd}$

Then $\exists$ an adjunction

$$\begin{tikzcd} E_a \arrow[rr, bend left, "\Sigma\beta"] \arrow[rr, phantom, "\perp"] \arrow[dr] & & E_b \arrow[ll, bend left, "\beta^*"] \arrow[dl] \\ & X & \end{tikzcd}$$

E_a, E_b fibers over a, b resp.

Special case
$\begin{tikzcd} 1 \arrow[r, bend left, "0"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Downarrow"] & 2 & E \arrow[l, two heads] \end{tikzcd}$ $\leftarrow$ Lurie's def^n of adjunction : bifibration over 2.

$\rightsquigarrow \begin{tikzcd} E_0 \arrow[r, bend left] \arrow[r, phantom, "\perp"] & E_1 \arrow[l, bend left] \end{tikzcd}$