- Midterm: Friday 7:00 – 9:00pm

  see OWL for classroom assigments.

  Midterm covers everything upto and including implicit differentiation

  Sec. 3.5

- Solve previous year's midterms

- Review implicit differentiation

Q. $\tan(x-y) = \dfrac{y}{1+x^2}$ find y'.

A: Implicit differentiation.

Differentiate both sides

$$\left(\tan(x-y)\right)' = \left(\frac{y}{1+x^2}\right)'$$

chain Rule

$$= \sec^2(x-y)\cdot(x-y)'$$

$$= \sec^2(x-y)\cdot(x'-y')$$

$$= \sec^2(x-y)\cdot(1-y')$$

by quotient rule

$$= \frac{(1+x^2)\cdot y' - y\cdot(1+x^2)'}{(1+x^2)^2}$$

by power rule

$$= \frac{(1+x^2)\cdot y' - y\cdot 2x}{(1+x^2)^2}$$

$$\Rightarrow \sec^2(x-y)\cdot(1-y') = \frac{(1+x^2)\cdot y' - y\,2x}{(1+x^2)^2}$$

· Isolate y'

$$\Rightarrow \quad \sec^2(x-y) - \sec^2(x-y)\cdot y' = \frac{(1+x^2)\cdot y'}{(1+x^2)^2} - \frac{y\cdot 2x}{(1+x^2)^2}$$

$$\Rightarrow \quad -\frac{(1+x^2)}{(1+x^2)^2}\cdot y' - \sec^2(x-y)\cdot y' = -\sec^2(x-y) - \frac{y\cdot 2x}{(1+x^2)^2}$$

$$\Rightarrow \quad y'\left(\frac{-(1+x^2)}{(1+x^2)^{\cancel{2}}} - \sec^2(x-y)\right) = -\sec^2(x-y) - \frac{2xy}{(1+x^2)^2}$$

$$\Rightarrow \quad y' = \frac{\sec^2(x-y) + \frac{2xy}{(1+x^2)^2}}{\frac{1}{(1+x^2)} + \sec^2(x-y)}$$

<u>Review EVERYTHING!</u>

1. Finding derivatives
2. Continuity
3. Limits
4. Properties of functions (domain, range etc)

① Derivatives :

Derivatives of standard functions

- Power rule : $(x^n)' = n \cdot x^{n-1}$
- Exponential functions : $(a^x)' = a^x \cdot \ln a$
- Trig function : $(\sin x)' = \cos x$

  $(\cos x)' = -\sin x$

  $(\tan x)' = \sec^2 x$
- Logarithm : $(\ln x)' = \frac{1}{x}$
- Inverse trig : $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$

  $(\arctan x)' = \frac{1}{1+x^2}$

Rules of differentiations

- product rule : $(fg)' = f'g + g'f$
- quotient rule : $(f/g)' = \frac{g \cdot f' - f \cdot g'}{g^2}$
- chain rule : $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$
- implicit diff :

2. · Tangents : • slope of tangent at $x = a$ of $y = f(x)$ $= f'(a)$

• slope of normal at $x = a$ of $y = f(x)$ $= \dfrac{-1}{f'(a)}$

• Basic definition : ① $\lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h} = f'(a)$

② $\lim_{x \to a} \dfrac{f(x) - f(a)}{x - a} = f'(a)$

• if $f'(a)$ exists, we say $f(x)$ is differentiable at $x = a$.

② Continuity :

• $f(x)$ is continuous at $x = a$ if

1) $\lim_{x \to a} f(x)$ exists
- $\lim_{x \to a^-} f(x)$ exists
- $\lim_{x \to a^+} f(x)$ exists
- the two limits are equal

2) $f(a) = \lim_{x \to a} f(x)$

Standard question : continuity of piecewise functions.

• Facts : ① Standard functions are continuous wherever

Theorem they are defined

② Intermediate Value theorem

(Use for showing solutions exist).

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3) Limits : You must know

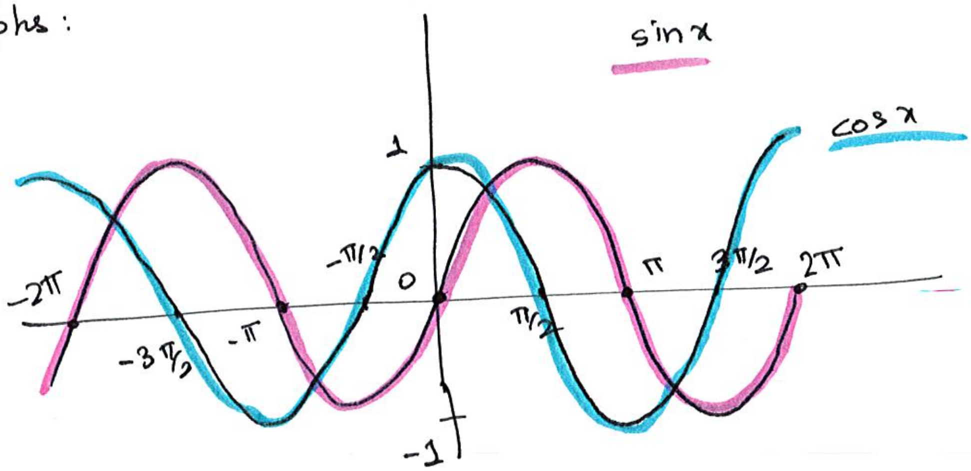
1) graphs of standard functions

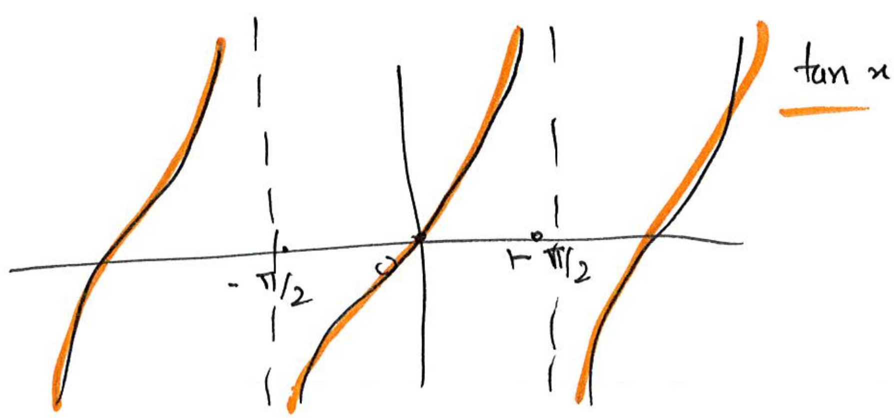
2) Standard identities / theorems

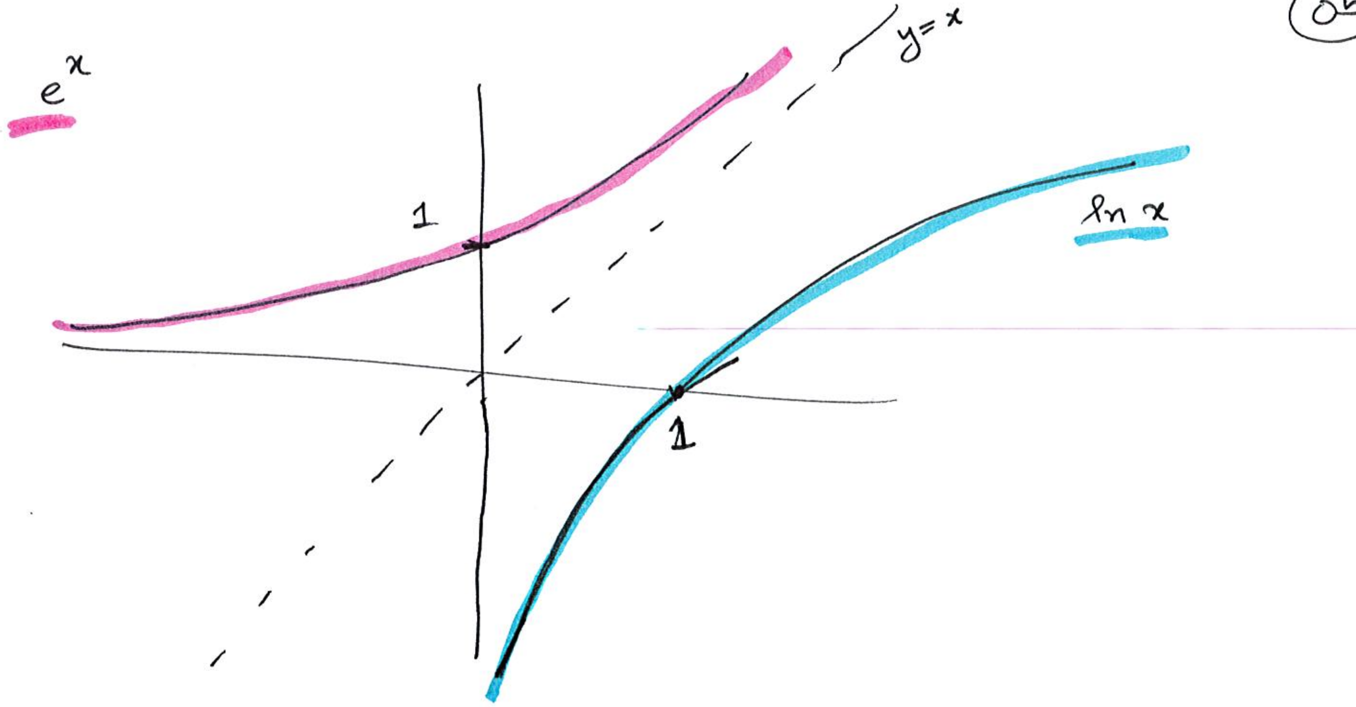
• Squeeze theorem
(typically used for trig limits)

• $\lim\limits_{h \to 0} \dfrac{\sin h}{h} = 1$

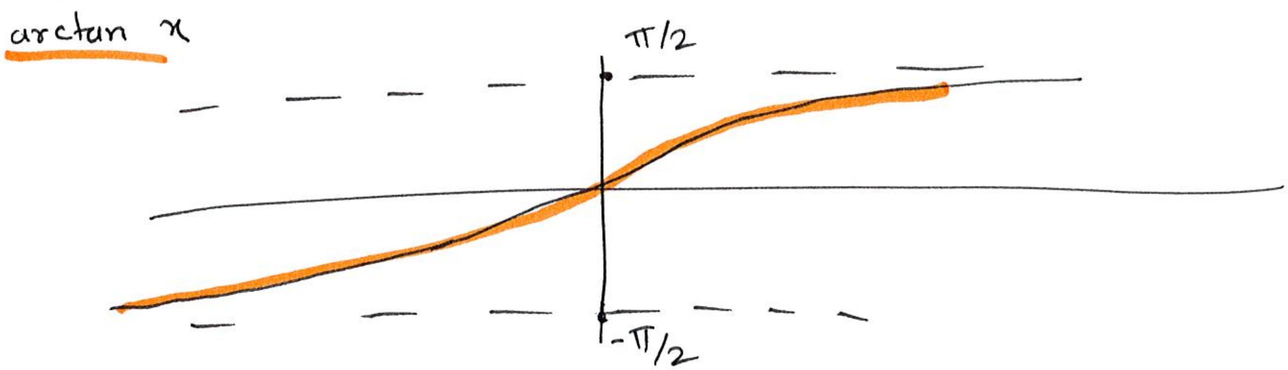
graphs :

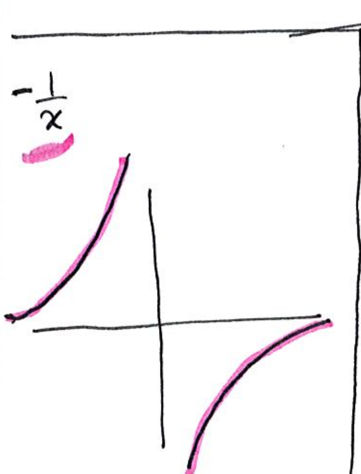




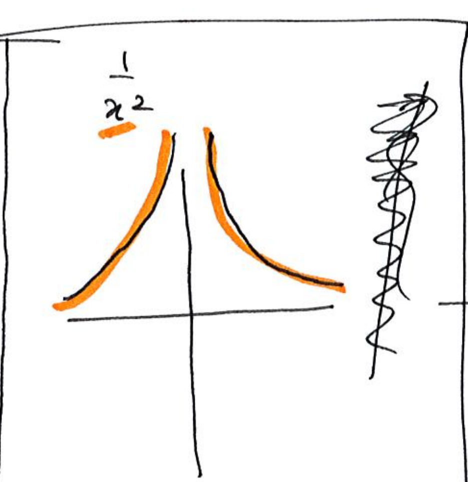


$$\lim_{x\to-\infty} e^x = 0$$

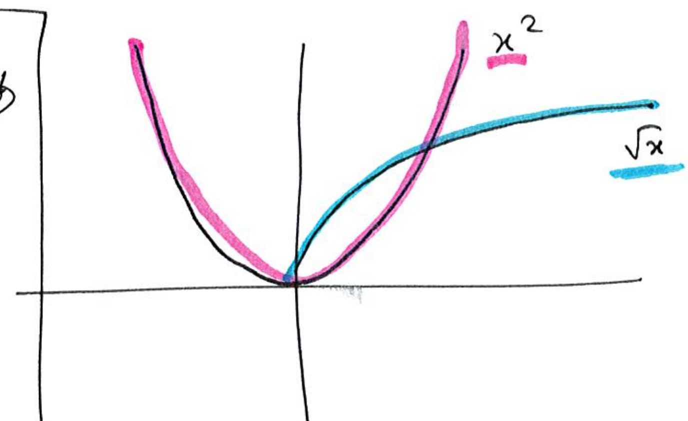




$$\lim_{x\to 0} -\frac{1}{x} = \text{d.n.e}$$



$$\lim_{x\to 0} \frac{1}{x^2} = +\infty$$



- $\lim_{x \to a} \square \qquad \lim_{x \to a^+} \square \qquad \lim_{x \to a^-} \square$

- Plug in $x=a$

Case 1: Result is a number.

we're done

Case 2: Result is $\frac{c}{0}$, c is a non-zero number

$+\infty \qquad -\infty \qquad$ d.n.e.

- Plug in values near $x=a$. and check the signs.

if either sided limit is $+\infty$ or $-\infty$ then $x=a$ is a vertical asymptote

Case 3: Result is $\underbrace{\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty}_{\text{indeterminate forms}}$, etc.

- algebraic manipulation
- multiply by conjugate
- squeeze theorem

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x ~~[illegible]~~ ~~[illegible]~~

- $\lim_{x \to \infty} f(x)$ | $\lim_{x \to -\infty} f(x)$

  - look at graphs/ behaviors of functions at ∞
  - squeeze theorem
  - $\frac{\text{polynomial}}{\text{polynomial}}$

- if $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$

  then $y = L$ is a horizontal asymptote.

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4) Properties of functions

- Domain of $f(x)$: Possible "x-values"
- Range of $f(x)$: Possible "y-values"

- Exponential

  log

  trig identities

$$\sin^2 x + \cos^2 x = 1$$
$$\tan^2 x + 1 = \sec^2 x$$

$$\sin(2x) = 2\sin x \cdot \cos x$$
$$\cos(2x) = 2\cos^2 x - 1$$
$$\cos(2x) = 1 - 2\sin^2 x$$

- Inverse functions

  $\ln x \longleftrightarrow e^x$: Find inverse of $f(x)$.

  $\sin x \longleftrightarrow \arcsin x$

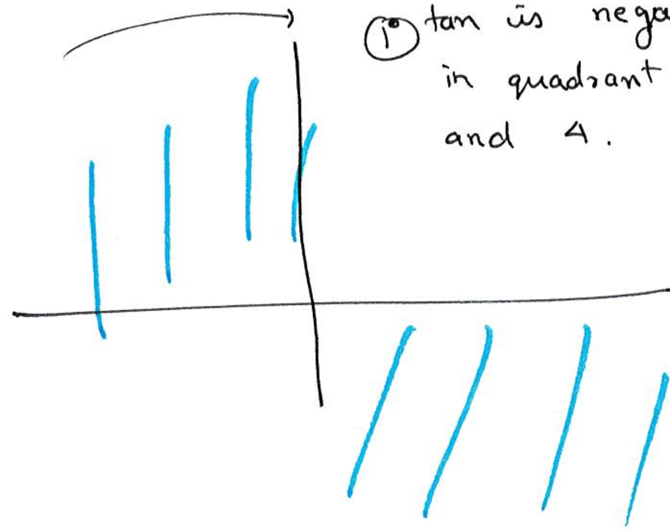
trig inverse trig

Q. Find $\cos\left(\arctan\left(-\frac{4}{3}\right)\right)$

A: Let $y = \arctan\left(-\frac{4}{3}\right)$

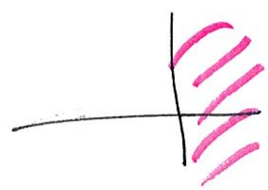
take $\tan(\cdot)$ of both sides

$\Rightarrow \boxed{\tan y = -\frac{4}{3}}$



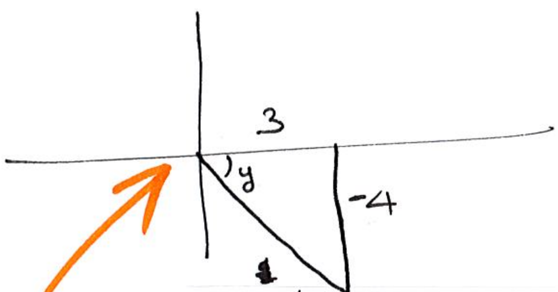
① tan is negative in quadrant 2 and 4.

② range of arctan $= \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



$=$ quadrants 1 and 4

$\Leftarrow$ y is in the 4th quadrant



$\downarrow$

Pythagoras $\Rightarrow$ hyp $= 5$

$\cos\left(\arctan\left(-\frac{4}{3}\right)\right) = \dfrac{\text{adjacent side}}{\text{hypotenuse}}$

$\downarrow$

$= \cos(y) \qquad = \frac{3}{5}$

(∵ cos is positive in the 4th quadrant)

Q. Find $\lim\limits_{t \to -\infty} \dfrac{\sqrt{9t^2+t-2}}{t-3}$ $\left[\dfrac{\text{"polynomial"}}{\text{bolynomial}}\right.$

Ans: is -3

- Find highest degree term (in denominator) $\boxed{t}$

- Divide both numerator & denominator by t

$$\lim_{t \to -\infty} \frac{\sqrt{9t^2+t-2} \,/t}{t-3 \;/t}$$

$$= \lim_{t \to -\infty} \frac{\sqrt{(9t^2+t-2)/t^2}}{(t-3)/t}$$

$$= \lim_{t \to -\infty} \frac{\sqrt{\frac{9t^2}{t^2} + \frac{t}{t^2} - \frac{2}{t^2}}}{\frac{t}{t} - \frac{3}{t}}$$

$$= \lim_{t \to -\infty} \frac{\sqrt{9 + \frac{1}{t} - 2/t^2}}{1 - 3/t}$$

$\left(\text{as } \lim\limits_{t \to -\infty} \dfrac{1}{t^n} = 0\right)$

$$= \frac{\sqrt{9+0-0}}{1-0}$$

$$= 3$$

Very subtle problem

$\sqrt{1/(-2)^2} = \sqrt{1/4}$

$= 1/2$

$\neq 1/-2$

- In practice, if you have square roots check for signs!

for $t < 0$

num: $\sqrt{9t^2+t-2} > 0$

denominator: $t - 3 < 0$

$\Rightarrow$ answer $= -3$

Q. $\displaystyle\lim_{t \to \infty} \frac{\sqrt{9t^2+t-2}}{3-t} = -3$

Exercise

$\displaystyle\lim_{t \to -\infty} \frac{\sqrt{9t^2+t-2}}{3-t} = 3$

· There is a square root

· check sign

$t \to \infty$ Plug in large positive t

$t = 100$ numerator : $\sqrt{9t^2+t-2} > 0$

denominator : $3-t < 0$

sign = negative

·

·

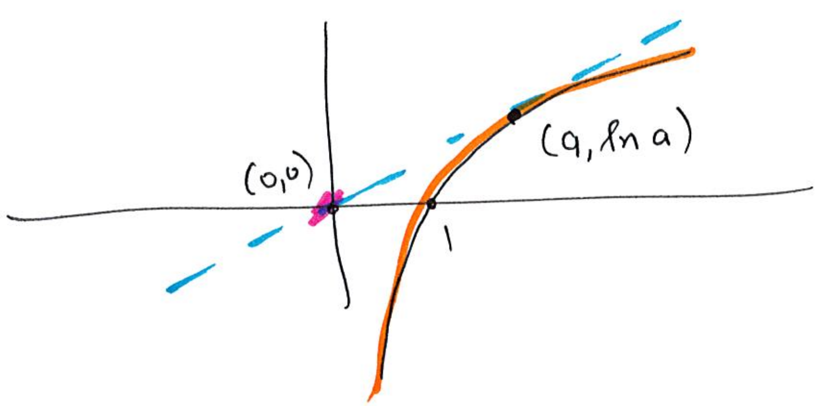
$t \to -\infty$ Plug in small negative t

$t = -100$ N : $\sqrt{9t^2+t-2} > 0$

D : $3-t > 0$

$\Rightarrow$ sign is positive

Q. Find <u>tangent</u> line to $\boxed{y = \ln x}$ that passes through (0,0).



<u>Careful</u>: (0,0) is not on the graph

Ans: • $(\ln x)' = \frac{1}{x}$

• we knows y-intercept is 0 so we expect an equation $y = mx$

• Suppose the tangency is at $(a, \ln a)$

Slope of tangent at $(a, \ln a)$ $= \frac{1}{a}$

$\Rightarrow$ equation of tangent : $(y - \ln a) = \frac{1}{a} \cdot (x - a)$

tangent passes throug $(0,0)$ $\Rightarrow$ $(0 - \ln a) = \frac{1}{a} \cdot (0 - a)$

$\Rightarrow$ $-\ln a = -\frac{a}{a}$

$\Rightarrow$ $\ln a = \frac{a}{a} = 1$

$\Rightarrow$ $\boxed{a = e}$

Point of tangency $= (e, \ln e) = (e, 1)$

tangent line : $(y - \cancel{\ln e}) = \frac{1}{e} \cdot (x - \cancel{1})$

$\Rightarrow$ $\boxed{y = \frac{x}{e}}$

DO **NOT** DO THIS.

$(\ln x)' = \frac{1}{x}$

$\rightarrow$ slope ~~eq^n~~ of tangent $= \frac{1}{x}$

eq^n $\quad y = \frac{1}{x} x + b$ $\rightarrow$ (slope or intercept cannot involve x or y)

~~has to be constant~~

$y = 1 + b$ ✗