# Representability of commas :

**Th$^m$:** Given functor $C \xrightarrow{g} A \xleftarrow{f} B$ and $C \xrightarrow{r} A$ $\exists$ a bijection :

$$\left\{ \begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "r"] \arrow[r, "g"'] \arrow[r, phantom, shift left=3, "\Downarrow \rho"] & A \end{tikzcd} \quad \text{absolute right lifting} \right\} \xleftrightarrow{\ \cong\ } \left\{ \mathrm{Hom}_B(B, r) \simeq_{C \times B} \mathrm{Hom}_A(f, g) \right\}$$

**Th$^m$:** Given $C \xrightarrow{g} A \xleftarrow{f} B$ $\exists$ an absolute right lifting through $f$ iff

$$\begin{tikzcd} C \arrow[r, dashed, bend right, "i"'] \arrow[r, phantom, "\perp"] & \mathrm{Hom}_A(f,g) \arrow[l, bend right] \end{tikzcd}$$

admits a <u>right adjoint right inverse</u> in which case

$r := p_0 \cdot i$ is the representing functor.

**Def$^n$:** A <u>right adjoint right inverse</u> (rari) for $f : B \longrightarrow A$ is a right adjoint $i$ s.t. the counit $fi \overset{\varepsilon}{\Longrightarrow} \mathrm{id}_A$ is the identity 2-cell ($fi = \mathrm{id}_A$)

By triangle identities : $f\eta = \mathrm{id}_f$ , $\eta i = \mathrm{id}_i$ .

**Def:** $A \in \mathcal{K}$. A slice $\infty$-cosmos over $A$

- obs = isofib $B \overset{p}{\twoheadrightarrow} A$ , $C \overset{q}{\twoheadrightarrow} A$

- mor =

$$\begin{tikzcd} \mathrm{Fun}_A(B \overset{p}{\twoheadrightarrow} A, C \overset{q}{\twoheadrightarrow} A) \arrow[r] \arrow[d] & \mathrm{Fun}(B, C) \arrow[d, two heads, "q"] \\ \mathbb{1} \arrow[r, "p"'] & \mathrm{Fun}(B, A) \end{tikzcd}$$

- $X \longrightarrow A$ defines a map $\mathcal{K}/A \longrightarrow \mathcal{K}/X$

- A RARI to $f : B \to A$ is exactly a terminal element of $f$ over $A$.

## §4. Adjunctions:

**Prop:**
- $B \underset{u}{\overset{f}{\rightleftarrows}} A$ are adjoint $f \dashv u$ iff $\mathrm{Hom}_A(f,A) \simeq_{A\times B} \mathrm{Hom}_B(B,u)$.
- Given any $X \xrightarrow{a} A$, $Y \xrightarrow{b} B$, $\mathrm{Hom}_A(fb,a) \simeq_{X\times Y} \mathrm{Hom}_B(b,ua)$.
- $f \dashv g \Rightarrow 1 \xrightarrow{\varepsilon a} \mathrm{Hom}_A(f,a)$ is terminal, $1 \xrightarrow{\eta b} \mathrm{Hom}_B(b,u)$ is initial.

**Thm: Adjoint functor theorem:**
$B \xrightarrow{f} A$ admits a right adjoint iff $\mathrm{Hom}_A(f,A) \xrightarrow{p_1} A$ has a terminal element over $A$. Dually $f$ has a left adjoint iff $\mathrm{Hom}_A(A,f) \xrightarrow{p_0} A$ has an initial element over $A$.

## Limits & Colimits:

**Def:** Given $1 \xrightarrow{d} A^J$, $\infty$-category of

cones over $d$:

$$\begin{tikzcd}
& \mathrm{Hom}_{A^J}(\Delta,d) \arrow[dl, "p_1"'] \arrow[dr, "p_0"] & \\
1 \arrow[dr, "d"'] \arrow[rr, phantom, "\overset{\phi}{\Leftarrow}"] & & A \arrow[dl, "\Delta"] \\
& A^J &
\end{tikzcd}$$

cones under $d$:

$$\begin{tikzcd}
& \mathrm{Hom}_{A^J}(d,\Delta) \arrow[dl] \arrow[dr] & \\
A \arrow[dr, "\Delta"'] \arrow[rr, phantom, "\overset{\phi}{\Leftarrow}"] & & 1 \arrow[dl, "d"] \\
& A^J &
\end{tikzcd}$$

**Prop:**
- $d$ has a limit (colimit) iff $\mathrm{Hom}_{A^J}(\Delta,d)$ $(\mathrm{Hom}_{A^J}(d,\Delta))$ is right representable (left).
- $d$ has a limit iff $\mathrm{Hom}_{A^J}(\Delta,d)$ has a terminal element.