Today: Sections 2.7, 2.8.

- Webwork HW02 is open.

- Almost every concept is some kind of a limit.

$$\text{every concept} \to \text{derivatives},\ \text{integrals}$$

derivatives → tangents to graphs

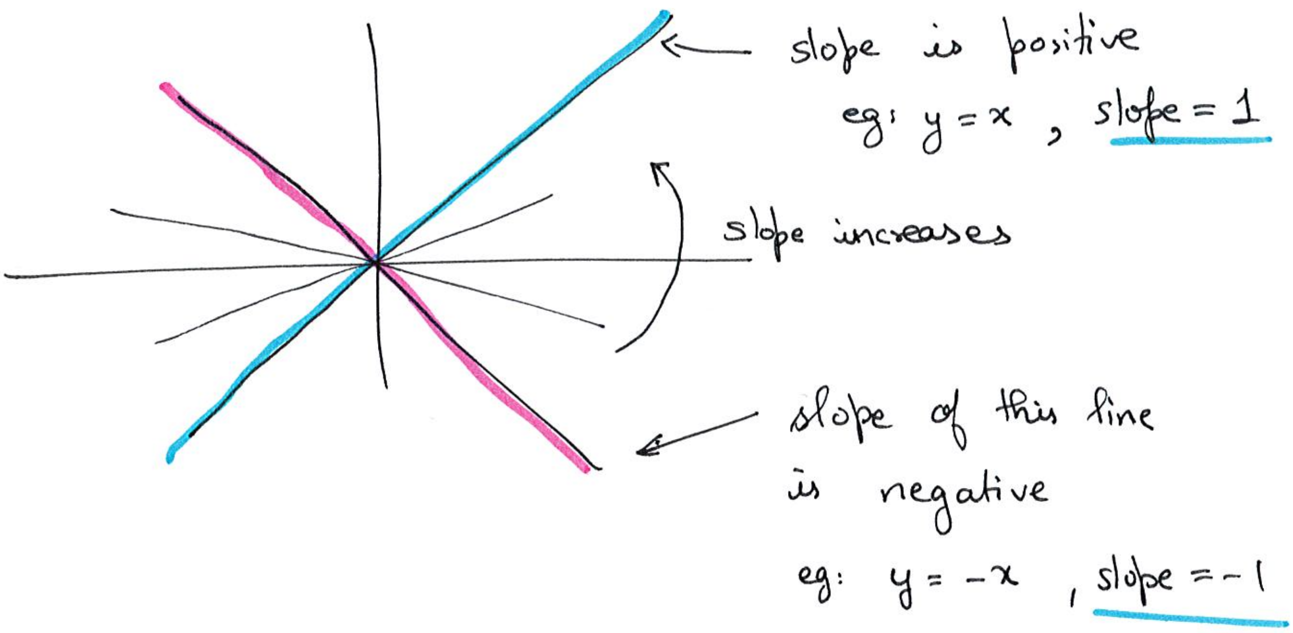
2.7, 2.8 tangents to graphs

Review: slope

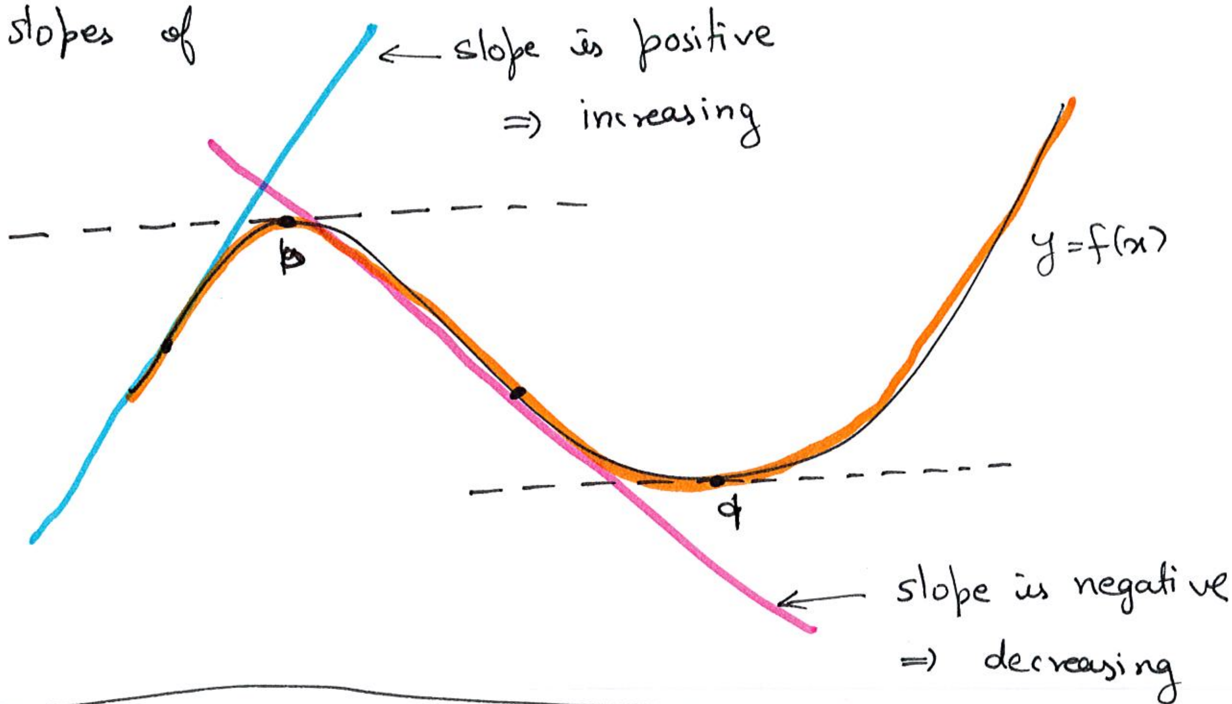
equation of the line

~~is~~ $$(y - y_0) = \frac{y_1 - y_0}{x_1 - x_0} \cdot (x - x_0)$$

$$\text{slope} = \frac{y_1 - y_0}{x_1 - x_0}.$$



slopes of



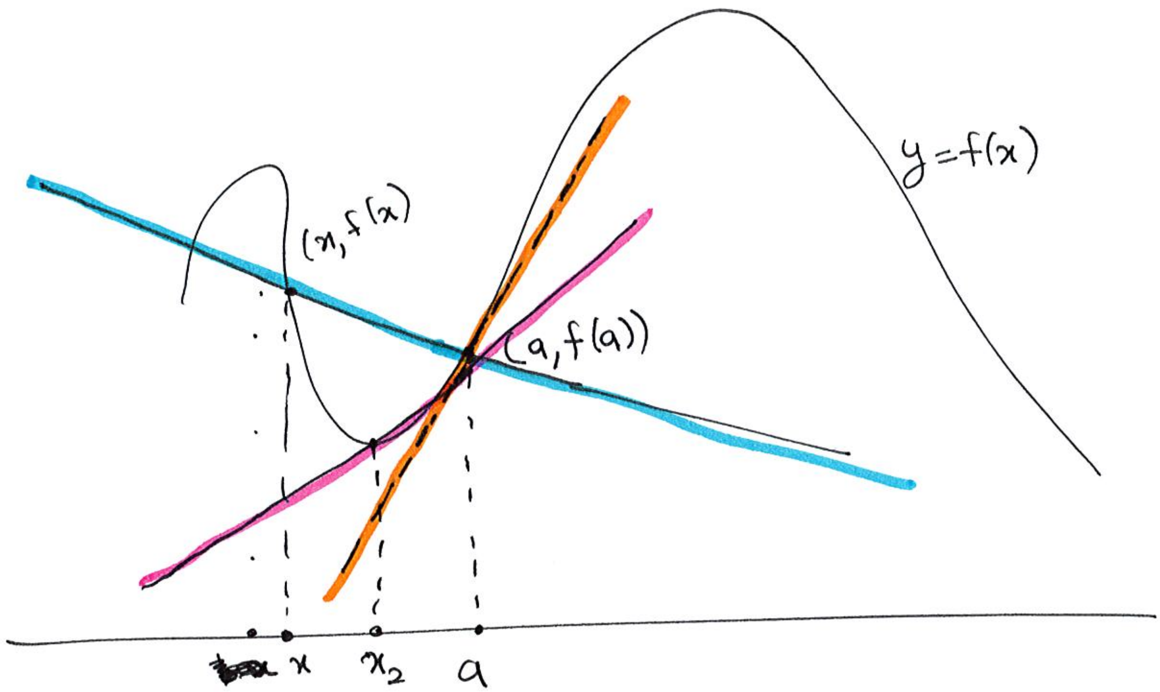
- if function is increasing $\iff$ slope of tangent is positive

function is decreasing $\iff$ slope ~~is~~ of tangent is negative

at local maxima/minima eg. p, q $\implies$ slope is 0.

A subtle point, we do this in next chapter

Tangents as limits:



- To find the slope of the tangent to the graph at $(a, f(a))$

- Draw a "secant" = line passing through $(x, f(x))$ and $(a, f(a))$

- as we move x toward a, the secant moves toward the tangent at a.

- slope of the tangent at $(a, f(a))$ $= \lim_{x \to a}$ slope of the secant between $(x, f(x))$ & $(a, f(a))$

- Slope of line through $(a, f(a))$ and $(x, f(x))$ $= \dfrac{f(x) - f(a)}{x - a}$

$\Rightarrow$ slope of the tangent at $(a, f(a))$ $= \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$

__Definition__ : the derivative of $f(x)$ at a is

. denoted 1) $f'(a)$

2) $\left.\dfrac{d}{dx} f(x)\right|_{x=a}$

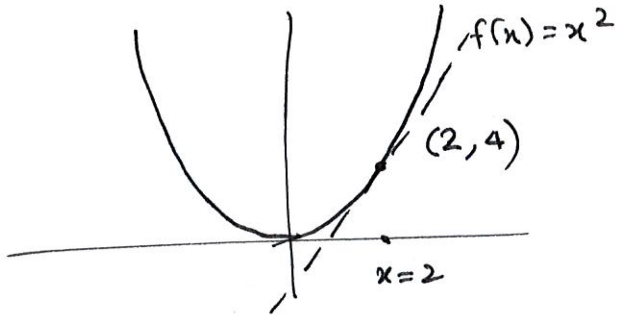
__Cor__ : ~~if the fun~~

function is increasing $\Longleftrightarrow$ derivative > 0

function is decreasing $\Longleftrightarrow$ derivative < 0

minima / max $\Rightarrow$ derivative $= 0$

eg: Find derivative of $f(x) = x^2$ at $x = 2$



Solution (slope of tangent line)
• derivative

$$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$$

$$= \lim_{x \to 2} \frac{x^2 - 2^2}{x - 2}$$

$$= \lim_{x \to 2} \frac{(x-2)(x+2)}{(x-2)}$$

$$= \lim_{x \to 2} (x+2)$$

$$= 2 + 2 = 4$$

• Equation of the tangent line at $(2, 4)$: $(y - 4) = 4 \cdot (x - 2)$

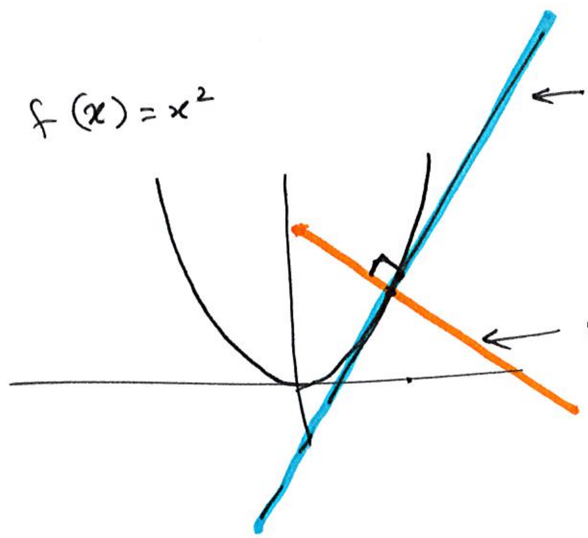
· ~~Slope of a line~~

equation of a line through (a, b) with slope m $\equiv$ $(y-b) = m \cdot (x-a)$

· the normal line at the point $(a, f(a))$ is the line perpendicular to the tangent.

· slope of normal $= -\dfrac{1}{\text{slope of tangent}} = \dfrac{-1}{f'(a)}$

eg: $f(x) = x^2$



← tangent : slope $= 4$

← normal : slope $= \dfrac{-1}{4}$

equation of the normal line :

$$\boxed{(y-4) = \frac{-1}{4}(x-2)}$$

tangent line :

$$\boxed{(y-4) = 4(x-2)}$$

$$\boxed{f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$

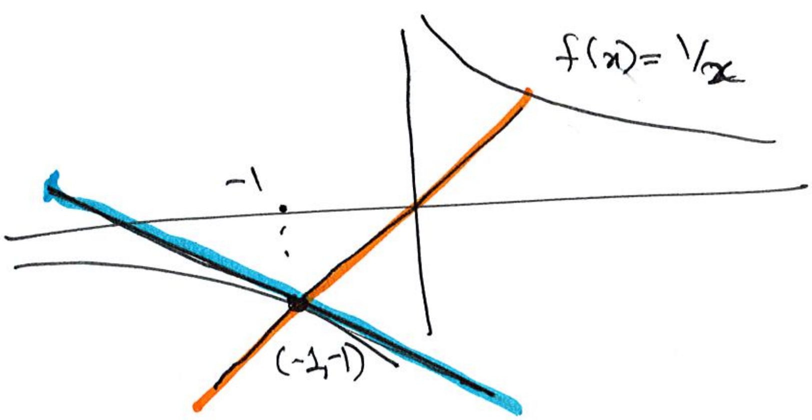
• ~~the~~ Let $x = a + h$ for some real number h

as $x \to a$, $h \to 0$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{(a+h) - a}$$

$$\boxed{f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

eg: Find slope of tangent to $f(x) = \frac{1}{x}$ at $x = -1$



$$f'(-1) = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h}$$

$$= \lim_{h \to 0} \frac{1/(-1+h) - (1/-1)}{h} \qquad \left(f(x) = \frac{1}{x}\right)$$

$$= \lim_{h \to 0} \frac{1/(-1+h) + 1}{h} = \lim_{h \to 0} \frac{\frac{1}{-1+h} + \frac{-1+h}{-1+h}}{h}$$

$$= \lim_{h \to 0} \frac{\left(\frac{1 + (-1+h)}{-1+h}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\left(\frac{h}{-1+h}\right)}{h}$$

$$= \lim_{h \to 0} \frac{h/(-1+h)}{h} = \lim_{h \to 0} \frac{h}{h \cdot (1+h)}$$

$$\frac{a/b}{c} = \frac{a}{b \cdot c}$$

eg: $\frac{1/2}{2} = 1/4 = \frac{1}{2 \cdot 2}$

$$= \lim_{h \to 0} \frac{1}{-1+h}$$

$$= -1$$

- equation of tangent line $(-1,-1)$ (slope $= -1$) : $(y - (-1)) = (-1)(x - (-1))$

$\Rightarrow$ $\boxed{y + 1 = -x - 1}$

- equation normal line :

slope of normal $= \frac{-1}{-1} = 1$

$(y - (-1)) = 1 \cdot (x - (-1))$

$\Rightarrow$ $y + 1 = x + 1$

$\Rightarrow$ $\boxed{y = x}$

Derivative ~~at a of f(x)~~ is of $f(x)$ at $x = a$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

this is now a "function"

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

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eg: Find $f'(x)$ of $f(x) = \sqrt{x}$.

<u>Sol^n</u>:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \qquad \left(\text{as } f(x) = \sqrt{x}\right)$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{\left(\sqrt{x+h}\right)^2 - \left(\sqrt{x}\right)^2}{h \cdot \left(\sqrt{x+h} + \sqrt{x}\right)} \qquad (a-b)(a+b) = a^2 - b^2$$

$$= \lim_{h \to 0} \frac{x + h - x}{h \cdot \left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{h}{h \cdot (\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{(\sqrt{x+h} + \sqrt{x})}$$

$$= \frac{1}{\sqrt{x+0} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}} \qquad \boxed{\Rightarrow \quad \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}} \quad \text{Ans}$$

Q. Find $f'(x)$ for $f(x) = \dfrac{x}{2+x}$.

Ans: $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\left(\dfrac{x+h}{x+h+2} - \dfrac{x}{x+2}\right)}{h} \qquad \boxed{\text{as } f(x) = \frac{x}{2+x}}$$

$$= \lim_{h \to 0} \frac{\left(\dfrac{(x+h)(x+2) - x(x+h+2)}{(x+h+2)(x+2)}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\left(\dfrac{x^2 + h \cdot x + 2 \cdot x + 2 \cdot h - x^2 - xh - 2x}{(x+h+2) \cdot (x+2)}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\left(\dfrac{2h}{(x+h+2)(x+2)} \right)}{h}$$

$$= \lim_{h \to 0} \frac{2\cancel{h}}{\cancel{h} \cdot (x+h+2)(x+2)}$$

$$= \lim_{h \to 0} \frac{2}{(x+h+2)(x+2)}$$

$$= \frac{2}{(x+2)^2}$$

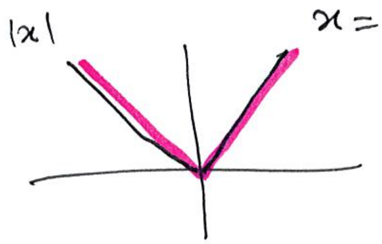
$$\boxed{\frac{d}{dx}\left(\frac{x}{x+2}\right)' = \frac{2}{(x+2)^2}}$$

- A function is called <u>differentiable</u> at $x = a$ if $f'(a)$ exists (and is not $\pm\infty$)

i.e. $\displaystyle\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ exists

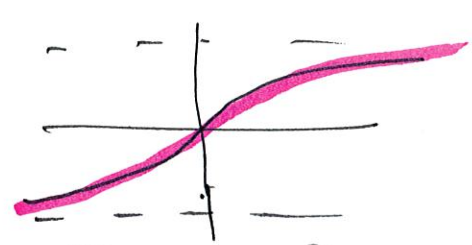
- example of non-differentiable function

$f(x) = |x|$ is not differentiable at $x = 0$



<u>Aside</u>:

arctan x



Domain $= \mathbb{R}$

Range $= \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$f(x) = \lvert x \rvert$
finding derivative at $x=0$

Proof:

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{|h| - 0}{h}$$

$$= \lim_{h\to 0} \frac{|h|}{h}$$

$$\lim_{h\to 0^-} \frac{|h|}{h} \qquad \Bigg| \qquad \lim_{h\to 0^+} \frac{|h|}{h}$$

$$= \lim_{h\to 0^-} \frac{-h}{h} \qquad \Bigg| \qquad = \lim_{h\to 0} \frac{h}{h}$$

$$= -1 \qquad \Bigg| \qquad = 1$$

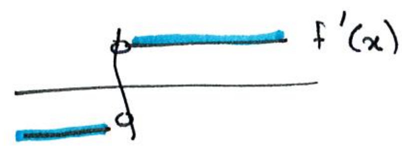
$$\lim_{h\to 0^-} \frac{|h|}{h} \neq \lim_{h\to 0^+} \frac{|h|}{h}$$

hence $\lim_{h\to 0} \frac{|h|}{h}$ d.n.e

hence $|x|$ is not differentiable at $x=0$

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$



- Differentiable functions are continuous but continuous functions are ~~not differentiable.~~ not necessarily differentiable. (eg: $|x|$)

- Now that derivative is a function ~~of~~, we can iteratively ~~fin func~~ find derivatives

$$f(x) \xrightarrow{\text{derivative}} f'(x) \xrightarrow{\text{derivative}} f''(x) \xrightarrow{\cdots} f'''(x)$$

$$f'(x) = \frac{d}{dx} f(x) \qquad f''(x) = \frac{d^2 f(x)}{dx^2} \qquad f'''(x) = \frac{d^3 f(x)}{dx^3}$$

eg: position ⟶ velocity ⟶ acceleration ⟶ jerk ⟶

- we know to find limits

- an example of a limit ~~is~~ a derivative = slope of the tangent to the graph of a function