Monday, Nov 25, 2019

Recall:

- $\int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt$

- $\int_a^a f(t)\,dt = 0$
- $\int_a^b f(t)\,dt = -\int_b^a f(t)\,dt$

Fundamental Theorem of Calculus

- $\int_a^b f(t)\,dt = F(b) - F(a)$, where F is any antiderivative of f

- $\left(\int_a^x f(t)\,dt \right)' = f(x)$

Chain Rule →

$$\left(\int_{g(x)}^{h(x)} f(t)\,dt \right)' = f(h(x)) \cdot h'(x) - f(g(x)) \cdot g'(x).$$

- Section 5.3

Q.41) Find $\int_0^4 2^s\,ds$. →

- Find any antiderivative of 2^s.
- $\cancel{f(x}$ $f'(s) = 2^s$

we know

$(2^s)' = 2^s \cdot \ln 2$

$\Rightarrow \dfrac{(2^s)'}{\ln 2} = 2^s$

$\Rightarrow \left(\dfrac{2^s}{\ln 2}\right)' = 2^s$

$\Rightarrow$ antiderivative of $2^s = \dfrac{2^s}{\ln 2}$

|||

$\int 2^s\,ds = \dfrac{2^s}{\ln 2} + C$

$\Rightarrow \int_0^4 2^s\,ds = \dfrac{2^s}{\ln 2}\Big|_0^4$

$= \dfrac{2^4}{\ln 2} - \dfrac{2^0}{\ln 2}$

$(c \cdot f)' = c \cdot (f')$

- $\ln 2$ is a constant
- so is $\frac{1}{\ln 2}$

$$\left(\frac{2^s}{\ln 2}\right)' = \frac{(2^s)'}{\ln 2}$$

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eg: Find $\int_0^4 x^n \, dx$. $(n \neq -1)$

- Find anti-derivative of x^n
- $F'(x) = x^n$

we know

$$(x^{n+1})' = (n+1) \cdot x^n$$

$$\Rightarrow \frac{(x^{n+1})'}{n+1} = x^n$$

$$\Rightarrow \left(\frac{x^{n+1}}{n+1}\right)' = x^n$$

- $F(x) = \frac{x^{n+1}}{n+1}$

$$\Rightarrow \int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

$$\Rightarrow \int_0^4 x^n \, dx = \frac{x^{n+1}}{n+1} \Big|_0^4 = \frac{4^{n+1}}{n+1} - \frac{0}{n+1}$$

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(02)

Derivatives		Anti-Derivatives	
$F(x)$	$F'(x)$	$f(x)$	$\int f(x)\,dx$
x^n	$n.x^{n-1}$	x^{n-1}	$\frac{x^n}{n}+C$
$\ln x$	$\frac{1}{x}$	$\frac{1}{x}$	$\ln x + C$
e^x	e^x	e^x	$e^x + C$
a^x	$a^x.\ln a$	a^x	$\frac{a^x}{\ln a}+C$
$\sin x$	$\cos x$	$\cos x$	$\sin x + C$
$\cos x$	$-\sin x$	$\sin x$	$-\cos x + C$
$\tan x$	$\sec^2 x$	$\sec^2 x$	$\tan x + C$
$\arctan x$	$\frac{1}{1+x^2}$	$\frac{1}{1+x^2}$	$\arctan x + C$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x + C$

$$\int f(x)\,dx = F(x) \xrightarrow{\text{derivative}} f(x) = F'(x)$$

$$F(x) \xleftarrow[\text{antiderivative}]{} f(x)$$

Section 5.3)

Q.13) Given $F(x) = \int_1^{e^x} \ln t \, dt$. Find $F'(x)$.

A :

$$\left(\int_1^x \ln t \, dt \right)' = \ln x$$

Chain Rule

$$\left(\int_1^{e^x} \ln t \, dt \right)' = (\ln e^x) \cdot (e^x)'$$

$$= x \cdot e^x$$

more generally,

$$\left(\int_{g(x)}^{h(x)} f(t) \, dt \right)' = \underbrace{f(h(x)) \cdot h'(x)} - \underbrace{f(g(x)) \cdot g'(x)}$$

These extra terms are by Chain Rule.

Ch 5.5 : Basic u-substitution

cf. · We are reversing chain rule

Goal is still to find anti-derivatives.

· eg: $(e^x)' = e^x$

$(e^{(x^2)})' = e^{x^2} \cdot (x^2)'$ Chain Rule

$= e^{x^2} \cdot 2x$

$\Rightarrow \int e^{x^2} \cdot 2x \cdot dx = e^{x^2} + C$

Q. Starting from $\int e^{x^2} 2x \, dx$ how do we get e^{x^2} ?

A: Method of u-substitution

· $\int e^{x^2} \cdot 2x \cdot dx$ Let $u = x^2$

then $du = (x^2)' dx$

$du = 2x \, dx$

$= \int e^u \cdot du$

$= e^u + C$

$= e^{x^2} + C$

Method of u-substitution:

When to use:

- if all else fails do u-sub (for us)
- look for a function and its derivative in your expression.

$$e^{x^2} \cdot 2x$$

why? $\quad f(g(x))' = f'(g(x)) \cdot g'(x)$. $\quad$ Chain Rule

eg: Find. $\displaystyle\int \tan x \, dx$

A: $\displaystyle\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$

our goal is to absorb the "derivative" inside du.

$$u = \cos x$$

$$du = (\cos x)' \cdot dx$$

$$= -\sin x \cdot dx$$

$$\Rightarrow -du = \sin x \cdot dx$$

$$\int \tan dx = \int \frac{\sin x}{\cos x} \, dx = \int \frac{-du}{u} = -\ln u + C = -\ln(\cos x) + C$$

- Look for $g'(x).dx$ ← these steps require you to know all the standard derivatives by heart
- substitute $u = g(x)$
- reduce to a simpler problem, and use the standard anti-derivatives.

eg: $\int \frac{x^2}{1+x^3}\, dx$

Looking for $g'(x)\,dx$

Method 1:

$u = x^3$

$du = 3x^2\,dx$

$\frac{du}{3} = x^2 dx$

Plugging back in

$\int \frac{du/3}{1+u}$

$= \frac{1}{3}\int \frac{du}{1+u}$

$v = 1+u$
$dv = (1+u)'.du$

$v = 1+u$
$dv = du$

$= \frac{1}{3}\int \frac{dv}{v} = \frac{1}{3}\ln v + C$

$= \frac{1}{3}\ln(1+u) + C = \frac{1}{3}\ln(1+x^3) + C$

Method 2: $\int \frac{x^2}{1+x^3}\, dx$

$u = x^3+1$

$du = 3x^2\,dx$

$\frac{du}{3} = x^2\,dx$

Plugging back in

$\int \frac{du/3}{u}$

$= \frac{1}{3}\int \frac{du}{u}$

$= \frac{1}{3}\ln u + C$

$= \frac{1}{3}\ln(1+x^3) + C$

Q. find $\int \sec x \cdot \tan x \, dx$

$$= \int \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} \, dx$$

We're looking for $g'(x) \cdot dx$

$u = g(x)$

$du = g'(x) \cdot dx$

$$u = \cos x$$

$$du = (\cos x)' \cdot dx$$

$$= -\sin x \, dx$$

$$-du = \sin x \cdot dx$$

Plugging in for u, du

$$= \int \frac{-du}{u \cdot u}$$

$$= -\int \frac{1}{u^2} \, du$$

$$= -\int u^{-2} \, du$$

$$= - \cdot \frac{u^{-1}}{-1} + C$$

Plug back in $u = \cos x$

$$= \frac{1}{\cos x} + C$$

$$\int \sec x \tan x \, dx = \sec x + C$$

- Look for $g'(x).dx$

- Substitute $u = g(x)$

$$du = g'(x)\, dx$$

- Change the original function (in x) to a function in u.

  ↓ do something

- find the simplified integral

← Solve $\int \dots du$

- Plug back $u = g(x)$.

eg:

$$\int \frac{x\, dx}{\sqrt{1 - x^2}}$$

$u = x^2$

$du = 2x.dx$

$\frac{du}{2} = x.dx$

$$= \int \frac{du/2}{\sqrt{1-u}}$$

$$= \frac{1}{2} \int \frac{du}{\sqrt{1-u}}$$

$$\int \frac{dx}{\sqrt{1-x^2}}$$

$$= \arcsin x + c$$

$(1-u)' = -1$

- Linear functions $ax+b$ have constant derivatives

- ~~use~~ these are used a lot in u-sub

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$v = 1 - u$

$dv = (1-u)' du$

$= -1\, du$

$dv = -du \Rightarrow du = -dv$

$$= \int \frac{1}{2} \frac{du}{\sqrt{1-u}} = \int \frac{1}{2} \left(\frac{-dv}{\sqrt{v}} \right)$$

$$= \frac{-1}{2} \int \frac{dv}{\sqrt{v}}$$

$$= \frac{-1}{2} \int v^{-1/2}\, dv$$

$$= \frac{-1}{2} \cdot \frac{v^{-1/2+1}}{-1/2+1} + C$$

$$= \frac{-1}{2} \cdot \frac{(1-u)^{(-1/2+1)}}{-1/2+1} + C$$

$$= \frac{-1}{2} \cdot \frac{(1-x^2)^{1/2}}{1/2} + C$$

$$\int \frac{x\, dx}{\sqrt{1-x^2}} = -\sqrt{1-x^2} + C$$