Nov 18, 2019

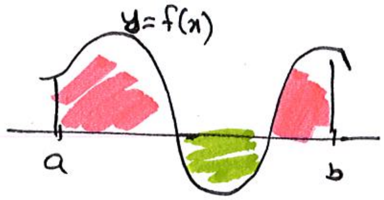
§5 Integrals

- $\int_a^b f(x)\cdot dx = $ signed area between $y=f(x)$ and the x-axis

5.1, 5.2

signed area $\Rightarrow$ above the x-axis is positive area

below the x-axis is negative area



- precise definition involves Riemann sums

these look like $\lim_{n\to\infty}\left(\sum_{i=0}^{n-1} \frac{1}{n}\cdot f(\quad)\right)$

or $\lim_{\Delta x\to 0}\left(\sum_{i=0}^{n-1} \Delta x\cdot f(\quad)\right)$

- not useful for computations.

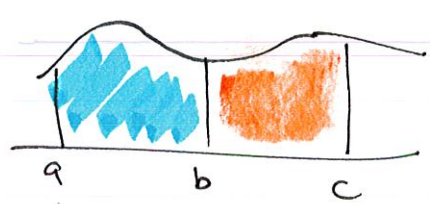
5.3, 5.4, — Computations of integrals

- Geometric properties

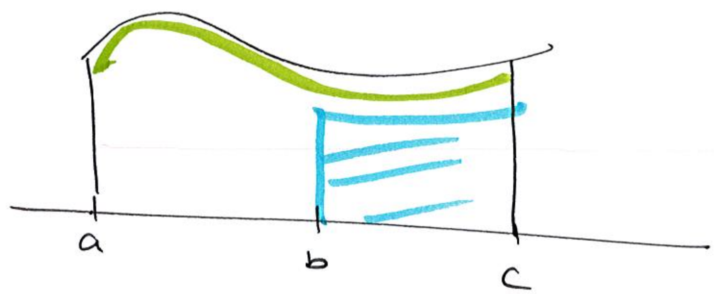
upper limit →

lower limit →

$$\boxed{\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx}$$



$$\int_a^c f(x)\,dx - \int_b^c f(x)\,dx = \int_a^b f(x)\,dx$$



Compare

$(f+g)' = f' + g'$ • $\displaystyle\int_a^b f(x) + g(x)\ dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

$(f-g)' = f' - g'$ • $\displaystyle\int_a^b f(x) - g(x)\ dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

$(cf)' = c \cdot f'$ • $\displaystyle\int_a^b c \cdot f(x)\,dx = c \cdot \int_a^b f(x)\,dx$

↓ product rule → integration by parts (X not in this course)

↓ chain rule → u-substitution

Q. How do we compute integrals?

A. Fundamental theorem of calculus.

<u>Recall</u>: Antiderivatives

- $F'(x) = f(x)$ then $F(x)$ is <u>an</u> antiderivative of $f(x)$.

$$\begin{tikzcd} F(x) \arrow[r, bend left, "\text{derivative}"] & f(x) \arrow[l, bend left, "\text{anti-derivative}"] \end{tikzcd}$$

- Anti-derivatives to be careful about

$f(x)$	$F(x)$
$x^n, n \neq -1$	$\frac{x^{n+1}}{n+1} + C$
$\frac{1}{x}$	$\ln x + C$
$\frac{1}{1+x^2}$	$\arctan x + C$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x + C$

- others you guess

$\sec^2 x$	$\tan x + C$
$\sin x$	$-\cos x + C$

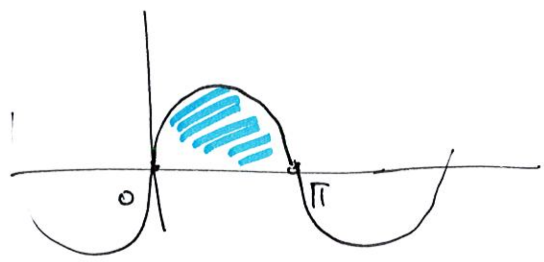
· Fundamental theorem of calculus v.1

$$\int_a^b f(x) \cdot dx = F(b) - F(a)$$ where F is any antiderivative of $f(x)$.

- this is only applicable if $f(x)$ is continuous on (a,b).

- ~~Not~~ $\int_a^b f(x) \cdot dx$ is a number.
  - the x in here is an internal variable
  - it goes away when we find the integral.

eg: Find $\int_0^{\pi} \sin x \cdot dx$



· an antiderivative of $\sin x = -\cos x$

$$\int_0^{\pi} \sin x \, dx = -\cos(\pi) - (-\cos(0)) = -\cos x \Big|_0^{\pi}$$

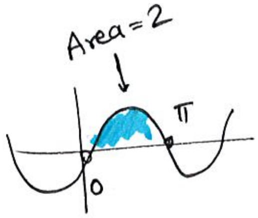
$$= -(-1) - (-1)$$

$$= 1 + 1$$

$$= 2$$

$$\int_0^{\pi} \sin x \, dx = -\cos x + c \Big|_0^{\pi}$$

$$= \left(-\cos(\pi) + \underline{c}\right) - \left(-\cos 0 + \underline{c}\right)$$



$$= -\cos \pi - (-\cos(0))$$

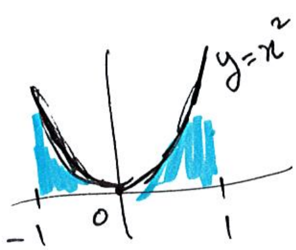
$$= 2$$

- when finding $\int_a^b f(x) \cdot dx$ ~~i.e. when we have~~

the $\underline{\underline{+c}}$ part of the antiderivative cancels.

Q. Find <u>area under the curve of</u> $y = x^2$ from -1 to 1.

A.



$$\int_{-1}^{1} x^2 \, dx = \frac{1}{3}x^3 \Big|_{-1}^{1}$$

$$= +\frac{1}{3} \cdot 1^3 - \left(+\frac{1}{3} \cdot (-1)^3\right)$$

- Area, volume
  ↓
  integrals

- slopes, rates of change, min / max
  ↓
  derivatives

$$= \frac{1}{3} + \frac{1}{3}$$

$$= \frac{2}{3}$$

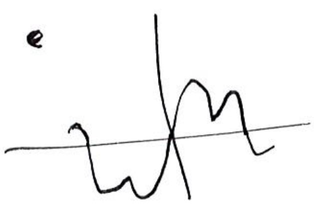
We could have done $\displaystyle\int_{-1}^{1} x^2\,dx = 2 \cdot \int_{0}^{1} x^2\,dx$

$$= 2 \cdot \int_{-1}^{0} x^2 \cdot dx$$

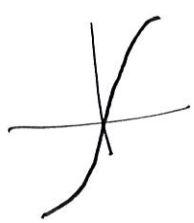
- Odd functions

$f(x) = -f(-x)$

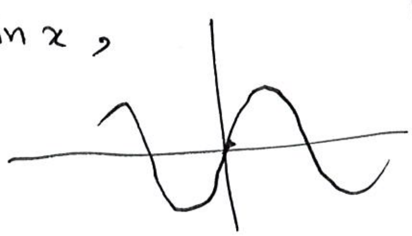
symmetric about the origin



eg: x^3,



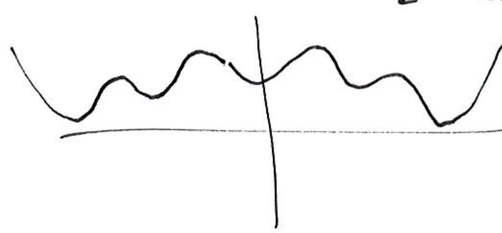
$\sin x$,



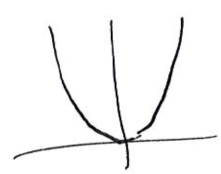
- Even functions

$f(x) = f(-x)$

Symmetric about the y-axis



eg: x^2,

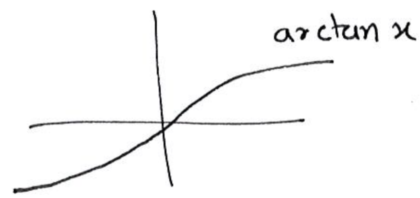


$\cos x$,

$$\int_{-a}^{a} f(x)\,dx$$ — • if $f(x)$ is odd this integral is 0

• if $f(x)$ is even this integral is $2\int_{0}^{a} f(x)\,.dx$

eg: $$\int_{-10}^{10} \arctan x \, dx = 0$$ as arctan x is an odd function



• examples of odd function : x^n , n odd
$\sin x$,
$\tan x$, $\arctan x$, $\arcsin x$.

eg: $x^3 + \arctan x$ ← • sums of odd functions is odd.

eg. $x^2 \cdot \sin x$ ← • product of an odd and an even function is odd

eg: $2 \cdot \sin x$ • Scalar multiple of odd functions is odd

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eg. $x \cdot \sin x$ is even • product of two odd functions is even

Q. How to test if a function is odd, even or neither?

A. Plug in $-x$ instead of x and see if the function changes sign

eg: • $x^2 + \sin x \xrightarrow{-x} (-x)^2 + \sin(-x)$

$= x^2 - \sin x$

neither $x^2 + \sin x$

nor $-(x^2 + \sin x)$

$\Rightarrow$ $x^2 + \sin x$ is neither odd nor even.

• $x^2 \cdot \sin x \xrightarrow{-x} (-x)^2 \cdot \sin(-x)$

$= x^2 \cdot (-\sin x)$

$= -x^2 \cdot \sin x$

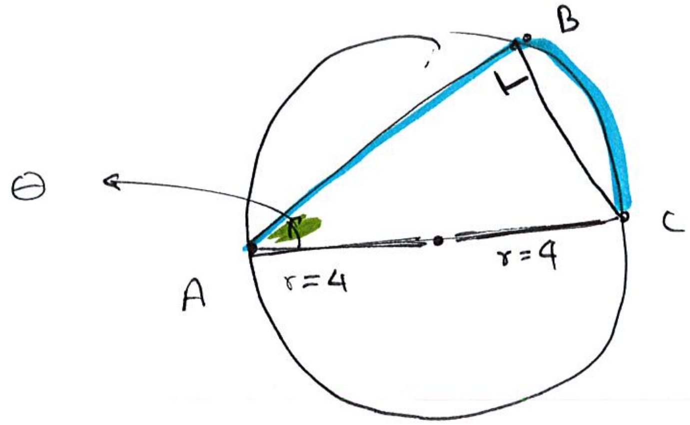
odd function

• $x^2 \cdot \cos x \xrightarrow{-x} (-x)^2 \cdot \cos(-x)$

$= x^2 \cdot \cos x$

~~odd~~ even function

Q. from Webwork Q·30 P6



- A, C are fixed
- we are going
  1) from A to B in as straight line with speed 5
- 2) from B to C along the circumference with speed 10
- radius = 4.

Goal: find the shortest / longest time taken. on such path.

Ans: Quantity to optimize :

← ~~first~~ first goal is to find a mathematical expression of this

= time to walk along AB + time to walk along BC

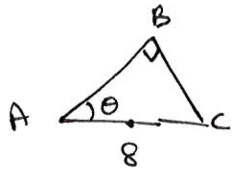
$$= \frac{\text{length of } AB}{5} + \frac{\text{length of } BC}{10}$$

$\frac{\text{distance}}{\text{speed}}$

- Let θ be the angle at A

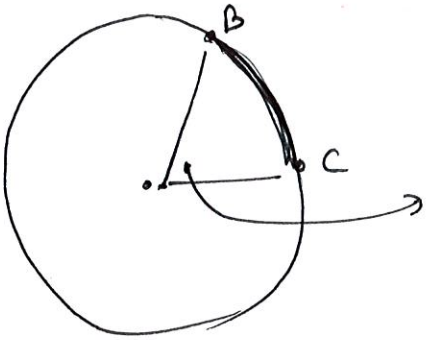
- we want expressions involving only 1 parameter / variable

Note: triangle with one side a diameter inscribed in a circle is a right angle triangle

$\Rightarrow$ 

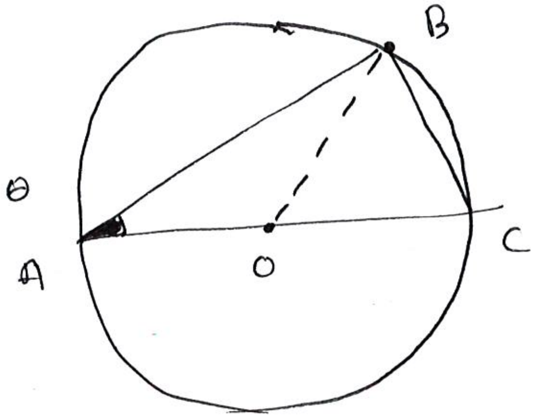
$$\boxed{\begin{aligned}\text{length of } AB &= AC \cdot \cos\theta \\ &= 8 \cdot \cos\theta\end{aligned}}$$

- **Fact #2**



arclength $BC = r \cdot$ the angle at the center

$= 4 \cdot (\qquad)$



Given $BAC = \theta$,
what is $BOC = ?$

Fact #3 $\angle BOC = 2 \cdot \angle BAC$

$= 2\theta$

$$\Rightarrow \boxed{\begin{aligned}\text{arclength } BC &= 4 \cdot (2\theta) \\ &= \text{radius} \cdot \text{angle at the center}\end{aligned}}$$

$$\text{Quantity to optimize} = \frac{\text{length of } AB}{5} + \frac{\text{arclength of } BC}{10}$$

$$f(\theta) = \boxed{\frac{8 \cdot \cos\theta}{5}} + \boxed{\frac{4 \cdot 2\theta}{10}}$$

New question : Find min/max of

$$\frac{8\cos\theta}{5} + \frac{8\theta}{10}$$

for $0 \le \theta \le \frac{\pi}{2}$

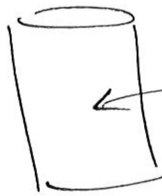
$B = C$ the line AB is vertical.

Now solve this absolute min/max problem

- Find critical points
- Compare values of the function at critical and end points.

- You should know the process for ~~finding~~ solving optimization problems.
  - ↳ Converting from word problems to math is the hardest part
  - You ~~alwa~~ (almost) always have to invent your own variables.

Q.29, Pet6



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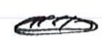
Volume $= 130 = \pi r^2 \cdot h$

$\Rightarrow h = \dfrac{130}{\pi r^2}$

Quantity to minimize = surface area

=  + 

$= 2\pi r \cdot h + \pi r^2$

$$f(r) = 2\pi r \cdot \frac{130}{\pi r^2} + \pi r^2$$

$$f(r) = \frac{260}{r} + \pi r^2$$

$$f'(r) = -\frac{260}{r^2} + 2\pi r$$

$f'(r) = 0 \Rightarrow \dfrac{260}{r^2} = 2\pi r$

$\Rightarrow \dfrac{260}{2\pi} = r^3$

$\Rightarrow r = \sqrt[3]{\dfrac{130}{\pi}}$

$$f''(r) = \frac{2 \cdot 260}{r^3} + 2\pi$$

at $r = \sqrt[3]{\frac{130}{\pi}}$, $f''(r) = \frac{2.260}{\left(\sqrt[3]{\frac{130}{\pi}}\right)^3} + 2\pi > 0$

$f''(r) > 0$

$\Rightarrow$ f is concave up at $r = \sqrt[3]{\frac{130}{\pi}}$

$\Rightarrow$ it is minima.