

Homotopy categories

Def: A an ∞ -category. Its homotopy category is

$$h_0 A := \text{Fun}^h(1, A) \quad 1 = \text{terminal object}$$

i.e.

$$\begin{aligned} \text{ob}(h_0 A) &= 1 \xrightarrow{a} A \text{ are "elements" of } A \\ \text{mor}(h_0 A) &= f: a \rightarrow b \quad 1 \begin{array}{c} \xrightarrow{f} \\ \Downarrow \\ \xrightarrow{b} \end{array} A \end{aligned}$$

By Yoneda Lemma:

An object is uniquely determined up to iso by its "generalized elements" $X \xrightarrow{a} A$

But we're interested in studying ∞ -categories up to equiv. & not an iso.

• By 2-category Yoneda:

$A \xrightarrow{f} B$ is an equiv if its an equiv on all generalized elements
 $\forall X \quad \text{Hom}(X, A) \xrightarrow{\sim} \text{Hom}(X, B)$ is an equiv.

• We've shown that equiv between ∞ -cats coincide with 2-cat equiv visible in hK .

Initial / terminal elements in A , limits, colimits

Def:

$A = \infty$ -category

An initial element is left adjoint to $A \xrightarrow{!} 1$

A terminal element is right adjoint to $A \xrightarrow{!} 1$.

$$1 \begin{array}{c} \xleftarrow{i} \\ \perp \\ \xleftarrow{!} \end{array} A$$

$$1 \begin{array}{c} \xleftarrow{i} \\ \perp \\ \xleftarrow{!} \end{array} A$$

Initial: $1 \xrightarrow{i} A$, counit: $A \begin{array}{c} \xrightarrow{!} 1 \\ \Downarrow \\ \xrightarrow{i} \end{array} A$ + 1-triangle equality.

Prop: $A \simeq A'$. Then A has an initial element iff A' does.
In which case it's preserved under equiv

Prop: Initial elements are "representably initial."

i.e. $X \xrightarrow{!} 1 \xrightarrow{i} A$ is initial in $\text{Fun}(X, A)$.

$$\text{i)} \quad \forall X \quad 1 \cong \text{Fun}(X, 1) \xrightleftharpoons[\perp]{!_*} \text{Fun}(X, A) \in \mathbf{qCat}$$

$$\text{ii)} \quad \forall X \quad 1 \cong \text{Fun}^h(X, 1) \xrightleftharpoons[\perp]{!_*} \text{Fun}(X, A) \in \mathbf{Cat}$$

Prop: Any two initial elements of A are iso in $\text{ho}A$.

Limits, Colimits:

2 kinds of diagrams: i) diagrams indexed by a simplicial set J

(\mathcal{K} cartesian closed) 2) diagrams indexed by other ∞ -category J .

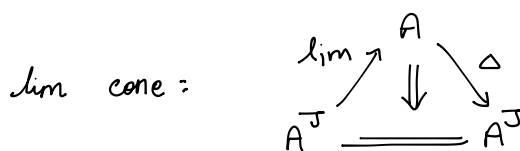
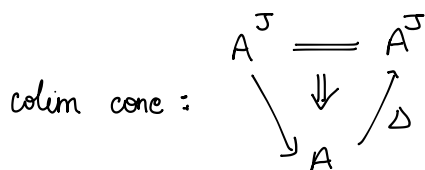
Def: ∞ -category of J -indexed diagrams valued in A : $\bar{A}: \mathcal{K}_J^{\text{op}} \times \mathcal{K}_A \longrightarrow \mathcal{K}$

$$A^J := \begin{cases} \text{i) simplicial } \infty\text{-tensor} \\ \text{ii) used cartesian closedness of } \mathcal{K}. \end{cases}$$

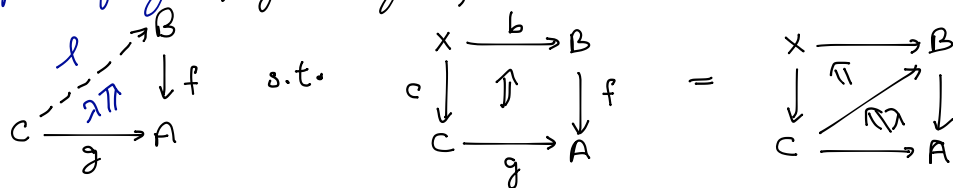
$J \xrightarrow{!} 1$ induces "constant functor" $A \xrightarrow{\Delta} A^J$

Def: An ∞ -cat A has all colimits of shape J if Δ has a left adjoint.
all limits of shape J if it has a right adjoint.

$$\begin{array}{ccc} & \xleftarrow{\text{colim}} & \\ & \perp & \\ A & \xrightarrow{\Delta} & A^J \\ & \perp & \\ & \xleftarrow{\text{lim}} & \end{array}$$



def: In 2-category $\mathcal{H}\mathcal{K}$, given a co-span $C \xrightarrow{g} A \xleftarrow{f} B$ an absolute left-lifting of g through f is

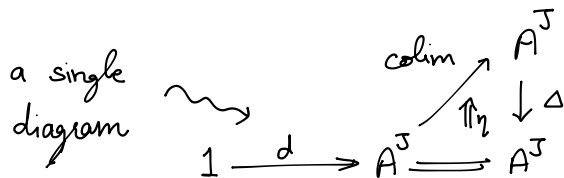


for absolute right lifting reverse the 2-cells.

Lem If $\begin{array}{ccc} & B & \\ \nearrow \eta & \downarrow f & \\ C & \xrightarrow{g} & A \end{array}$ is an absolute left lifting diagram so is $\begin{array}{ccc} & B & \\ \nearrow \eta & \downarrow f & \\ X & \longrightarrow & C \xrightarrow{\quad} A \end{array}$

Prop: $g \cdot \text{id}_B \Rightarrow u f$ is unit of an adjunction iff $\begin{array}{ccc} & A & \\ \nearrow t & \downarrow u & \\ B & \xrightarrow{=} & B \end{array}$ is an absolute left lifting diagram. similarly for counit.

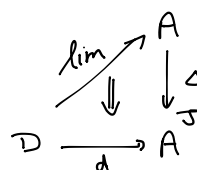
Note: colimit cone is an absolute left lifting diagram



Def: Let $D \xrightarrow{d} A^J$ be a family of J -indexed diagrams of A .

d colim is an absolute left lifting $\begin{array}{ccc} & A & \\ \nearrow \text{colim} & \downarrow \Delta & \\ D & \xrightarrow{d} & A^J \end{array}$

lim right



Notation: $\Delta = [0] \rightleftharpoons [1] \rightrightarrows [2] \rightarrow \dots$

\cap
 $\Delta_+ = \Delta \cup [-1]$ \nwarrow initial

\cap
 $\Delta_{-\infty} = [-1] \xleftarrow{s^{-1}} [0] \xleftarrow{s^{-1}} [1] \dots$

Prop: $A = \infty\text{-cat}$. If A admits augmentation + splitting has a colimit,
the colimit given by the augmentation

$$\begin{array}{ccccc}
 & & & A & \\
 & & \swarrow \text{co}[-1] & \downarrow \Delta & \\
 A^{\Delta_{-\infty}} & \longrightarrow & A^{\Delta_+^{\text{op}}} & \xrightarrow{\quad} & A^{\Delta^{\text{op}}} \\
 & & \uparrow & &
 \end{array}$$