Recall :

Wednesday, Nov 13, 2019

- $F'(x) = f(x) \Rightarrow F(x)$ is an antiderivative of $f(x)$.

$$F(x) \underset{\text{anti-derivative}}{\overset{\text{derivative}}{\rightleftarrows}} f(x)$$

- anti-derivatives are not unique: if $F(x)$ is an antiderivative of $f(x)$ then so is $F(x) + C$.

- you should know derivatives of standard functions VERY WELL to be able to find anti-derivatives.

- Method of finding anti-derivatives :
  - Guess
  - Check
  - Repeat.

- Be careful with your algebra, small changes can lead to drastically different derivatives.

$F'(x)$	$F(x)$
$\frac{1}{x}$	$\ln x + C$
$x^n,\ n \neq -1$	$\frac{x^{n+1}}{n+1}$
$\frac{1}{1+x^2}$	$\arctan x + C$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x + C$

Appendix E

Sigma notation

$$\sum_{i=a}^{b} f(a) = f(a) + f(a+1) + \dots + f(b)$$

"summation from $i = a$ to b"

eg: Telescoping series

$$\sum_{i=3}^{100} \left(\sqrt{(i+1)} - \sqrt{i}\right) = \left(\cancel{\sqrt{4}} - \sqrt{3}\right) \qquad i=3$$

$$+ \left(\sqrt{5} - \cancel{\sqrt{4}}\right) \qquad i=4$$

$$+ \left(\cancel{\sqrt{6}} - \sqrt{5}\right) \qquad i=5$$

$$+ \; \vdots \; +$$

$$\boxed{+\left(\sqrt{101}\right.} \left. - \cancel{\sqrt{100}}\right) \qquad i=100$$

alternating terms cancel

$$= \boxed{\sqrt{101}} - \sqrt{3}$$

• 5.1, 5.2) Defining integrals using limits.

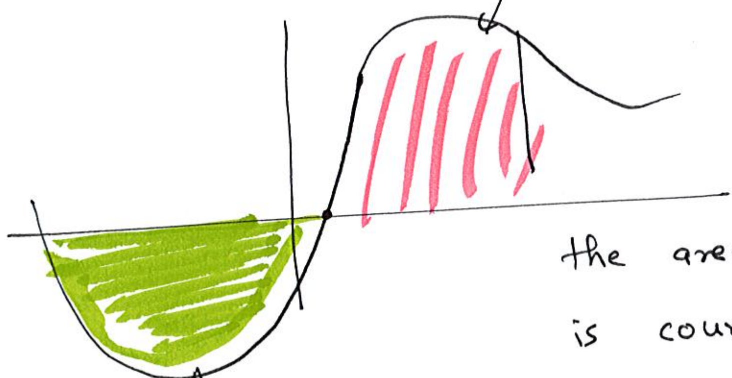
• We will define integrals as $\lim_{n\to\infty} \underbrace{\left(\sum_{\cdots}^{\cdots}\right)}_{\text{Riemann Sum}}$.

• But we cannot use this definition to compute anything.

• Instead, we compute integrals using Fundamental theorem of Calculus.

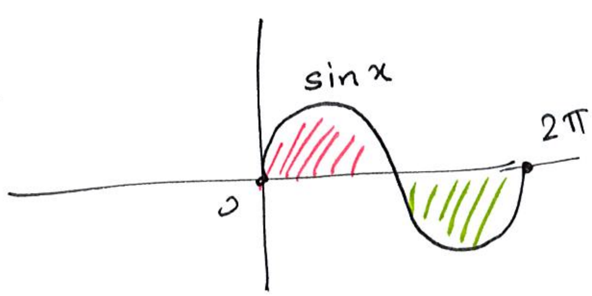
- derivative ⟶ slope of tangent of the graph $y=f(x)$
- integral ⟶ area under the graph $y=f(x)$

- Signed area :



the area under the x-axis is counted as negative area

eg :



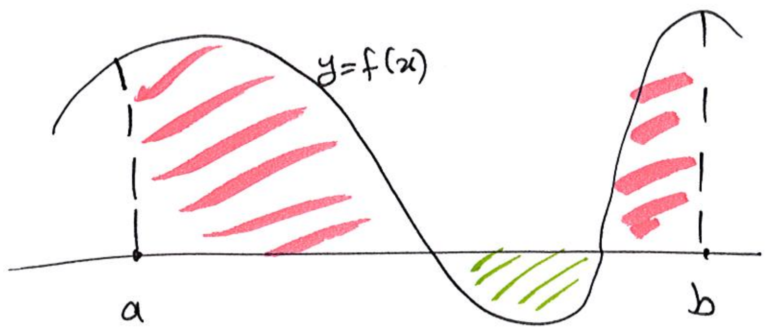
total ~~area b~~ signed area between x-axis and $\sin x$ from 0 to 2π

Pink region = positive
= - green region ⟹ Total signed area = 0.

Definition : ~~The~~

$$\int_a^b f(x)\,dx = \text{signed area between the } x\text{-axis and the curve } y=f(x) \text{ from } x=a \text{ to } x=b.$$

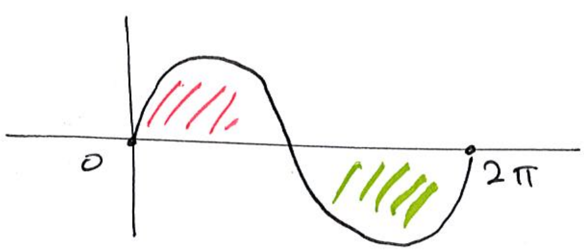
- "integral of $f(x)$ from a to b"



- definite integral.

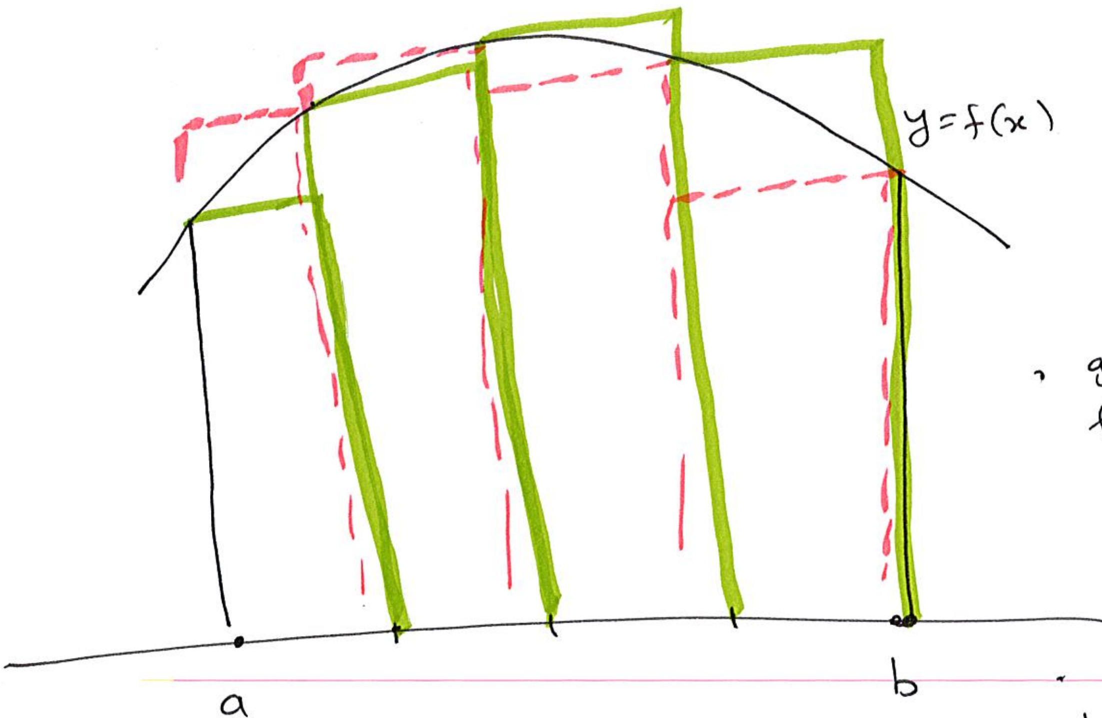
- definite integrals are <u>numbers</u>.

eg:



$$\int_0^{2\pi} \sin x \, . \, dx = 0 \, .$$

- The only regions we know how to compute areas of are rectangles, circles.
- For more general shapes, we approximate using rectangles.



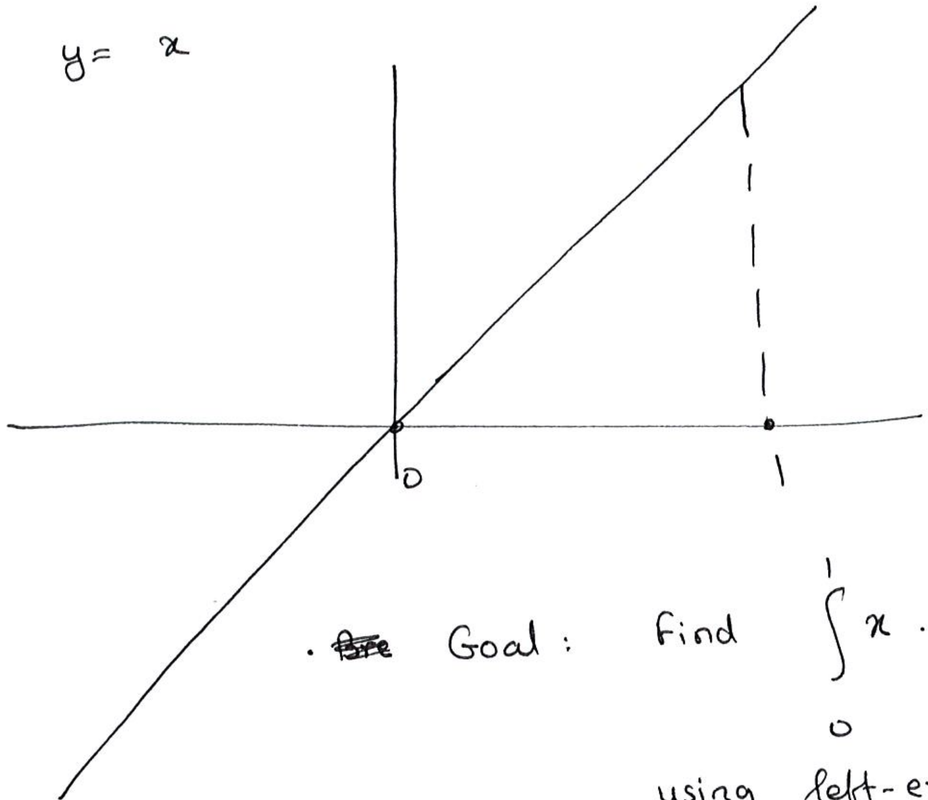
- green rectangles have height = y value of left endpoint

or

pink rectangles height = right endpoint

- make the subdivision of $[a,b]$ finer and finer. See Pg 369
- in the <u>limiting case</u> we reach the area under the curve

eg : $y = x$

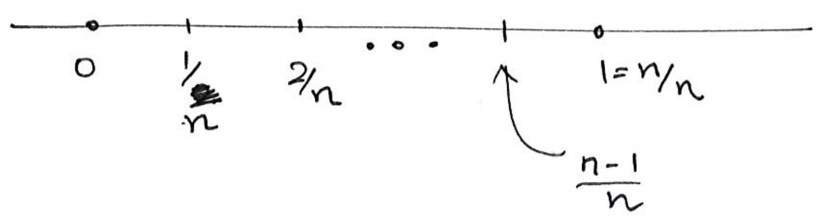


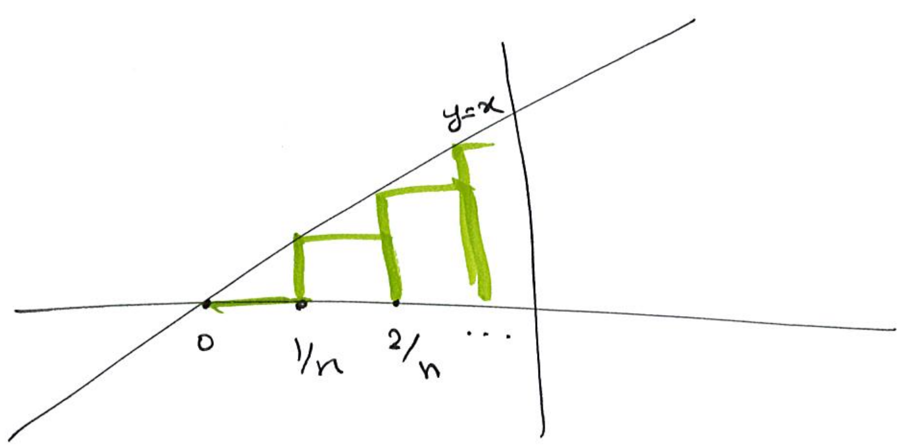
· ~~Are~~ Goal : Find $\int_0^1 x \cdot dx$

using left-end point rectangles

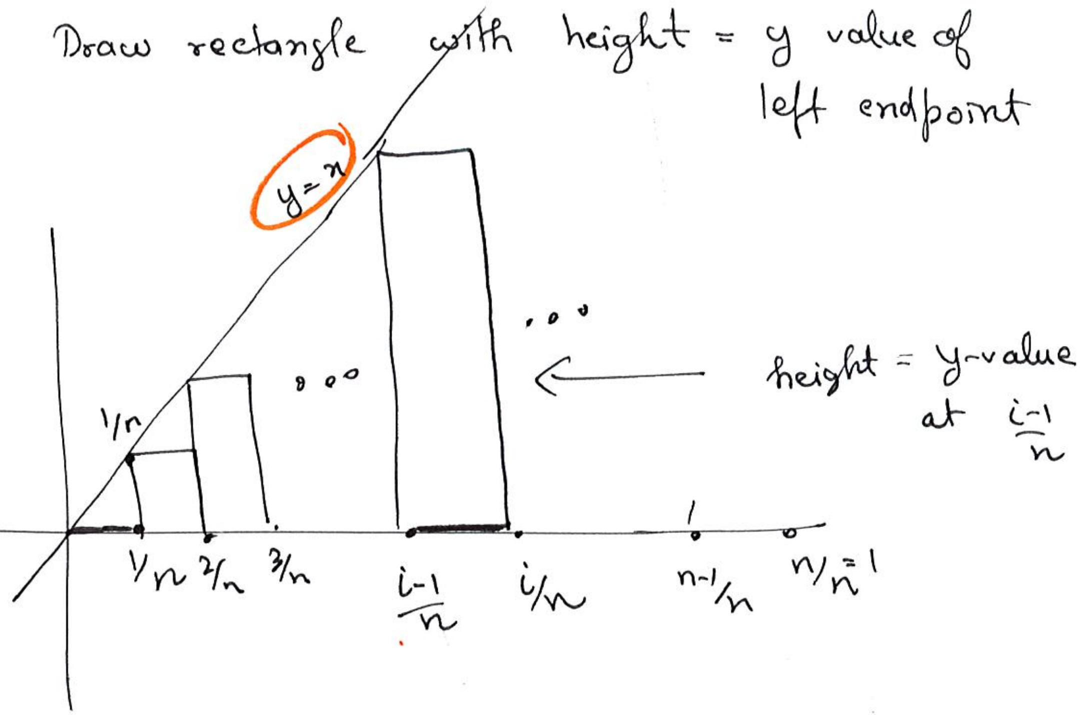
Step 1 : · Break $[0,1]$ into n pieces.

· length of each piece $= \frac{1}{n}$



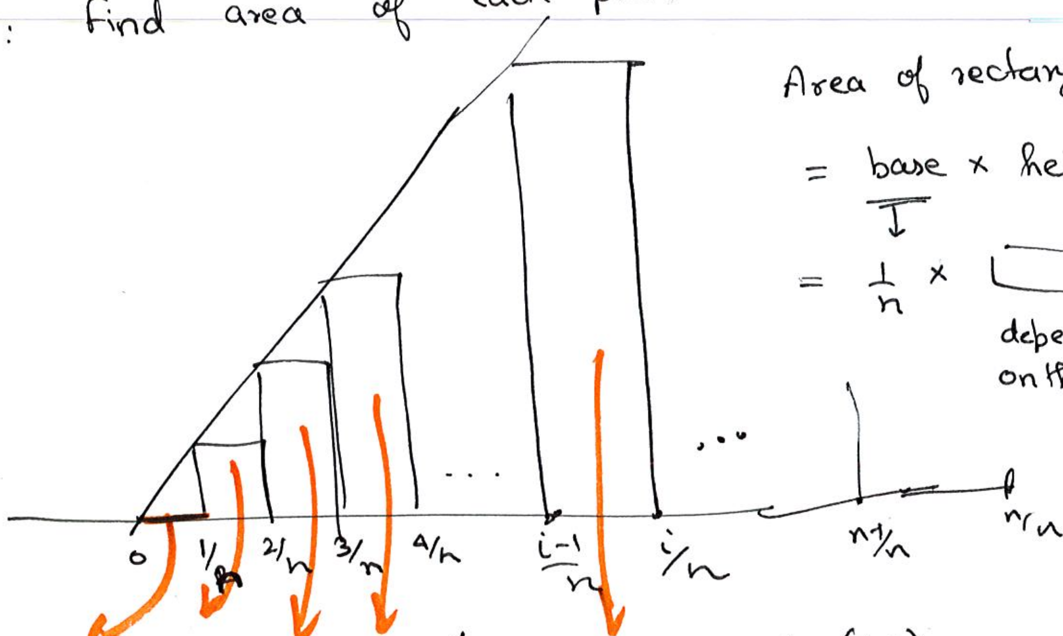


Step 2: Draw rectangle with height = y value of left endpoint



height ~~ove~~ of the rectangle over the interval $\left[\frac{i-1}{n}, \frac{i}{n}\right]$ $= \frac{i-1}{n}$

Step 3: Find area of each piece



Area of rectangle

$= \text{base} \times \text{height}$

$= \frac{1}{n} \times \boxed{}$

depends on the piece

$$\text{Area} = 0 + \frac{1}{n^2} + \frac{1}{n}\cdot\frac{2}{n} + \frac{1}{n}\cdot\frac{3}{n} + \cdots + \frac{1}{n}\cdot\frac{(i-1)}{n} + \cdots + \frac{1}{n}\cdot\frac{(n-1)}{n}$$

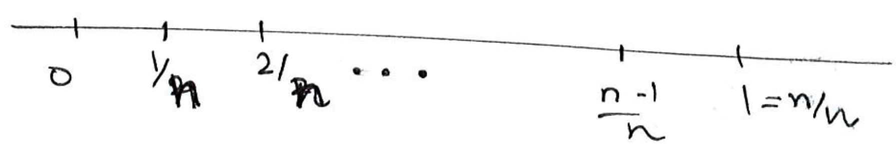
$$= \sum_{i=0}^{n-1} \frac{1}{n} \cdot \frac{i}{n}$$

<u>Step 4</u>: make the subdivisions finer, limiting case we get the integral

$$\int_0^1 x \cdot dx = \lim_{n \to \infty} \left(\sum_{i=0}^{n-1} \frac{1}{n} \cdot \frac{i}{n} \right)$$

why $\frac{n-1}{n}$:

- divide $[0,1]$ into n parts
- each part has length $1/n$



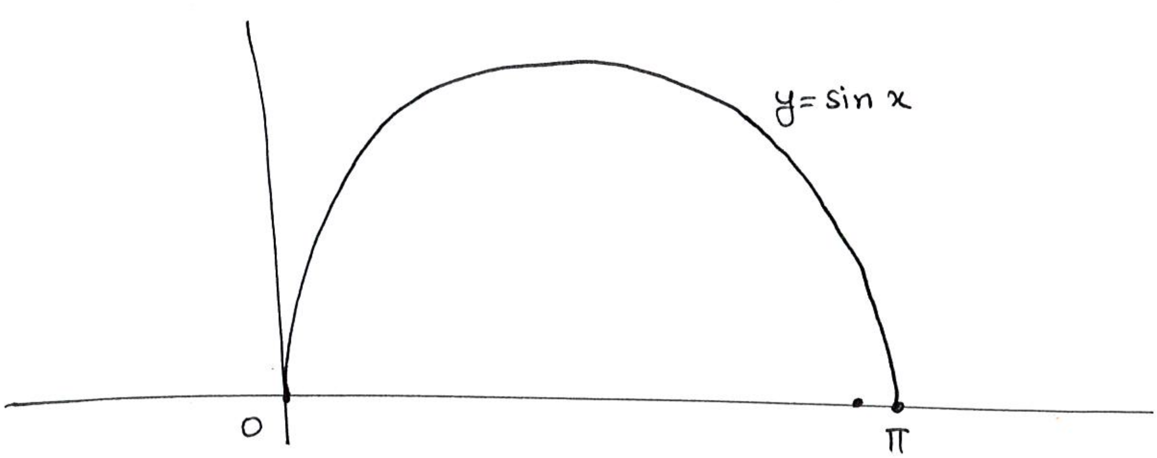
- $\lim_{n \to \infty}$ is making the width of the rectangles smaller and smaller.

$$\lim_{n \to \infty} \sum_{i=0}^{n-1} \Big(\qquad \Big)$$ is called a Riemann sum

- You don't need to find integrals using Riemann sums.

But you do need to be able to convert integrals into Riemann sums and vice versa.

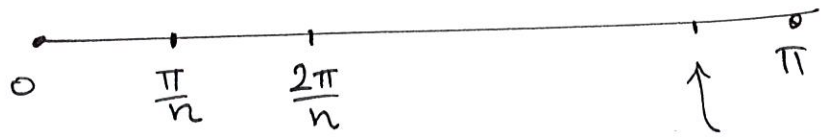
eg :



Goal: write $\int_0^{\pi} \sin x \, dx$ as a Riemann sum.

Ans : **Step 1 :** • Break $[0, \pi]$ into n - intervals

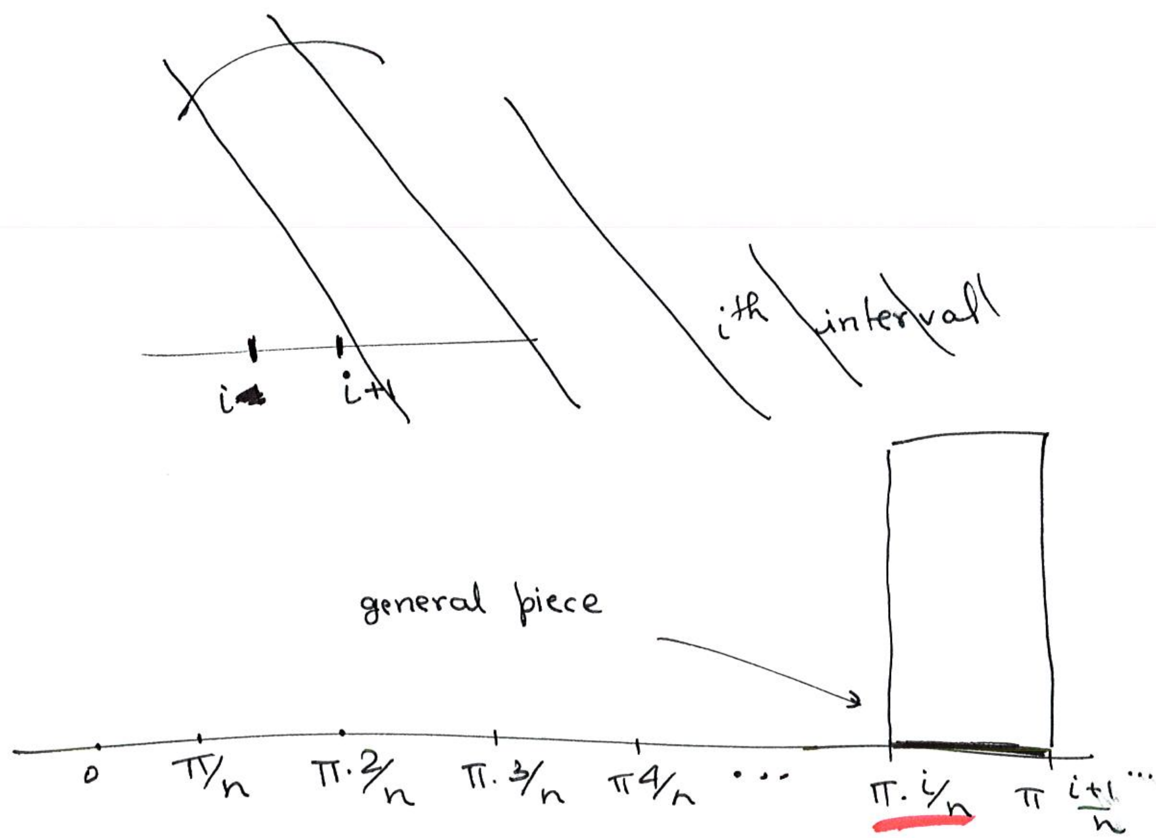
• each interval will have length π/n .



$$\pi - \frac{\pi}{n}$$
$$= \pi\left(1 - \frac{1}{n}\right)$$
$$= \pi \cdot \frac{n-1}{n}$$

Step 2 : ~~to~~ Draw rectangles over each interval with height = left endpoint's y-value

Step 3 : area of these rectangles = base × height



height of this rectangle = y-value of left endpoint

$$= \sin\left(\pi \cdot \frac{i}{n}\right).$$

base of this rectangle = $\frac{\pi}{n}$

area $= \frac{\pi}{n} \cdot \sin\left(\frac{\pi \cdot i}{n}\right)$

Step 4 :

$$\Rightarrow \int_0^{\pi} \sin x \cdot dx = \lim_{n \to \infty} \left(\sum_{i=0}^{n-1} \frac{\pi}{n} \cdot \sin\left(\frac{\pi}{n} \cdot i\right) \right)$$

- ~~More~~ We can just formalize this process.

$$\int_a^b f(x)\, dx = \lim_{n \to \infty} \left(\sum_{i=0}^{n-1} \frac{b}{n} \cdot f\left(\frac{b}{n} \cdot i\right) \right)$$

take finer and finer rectangles

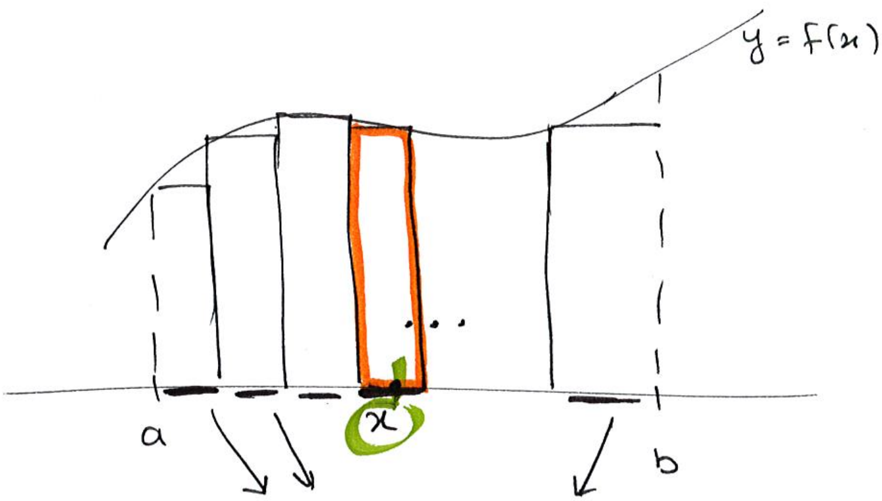
Sum all the rectangles

width of each ractangle

y-value of the ~~lft~~ left endpoint

Q. where does "dx" come from?

A.



- Call these finer interval "Δx"

"small change in x"

- height of the rectangle "at some x"

is approximately $f(x)$

- area ~~height~~ of orange rectangle $\approx f(x) \cdot \Delta x$

- adding all these rectangle ares "$\displaystyle\sum_{x \to a}^{b} f(x) \cdot \Delta x$"

- taking limit "$\Delta x \to 0$"

$$\text{"} \underbrace{\lim_{\Delta x \to 0} \sum_{x=a}^{b}} f(x) \cdot \underline{\Delta x} \text{"} = \int_{\underline{a}}^{b} f(x) \cdot \underline{dx}$$

Q. $\lim_{x \to a} \dfrac{f(x) - f(a)}{x - a} = f'(a)$

$$= \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

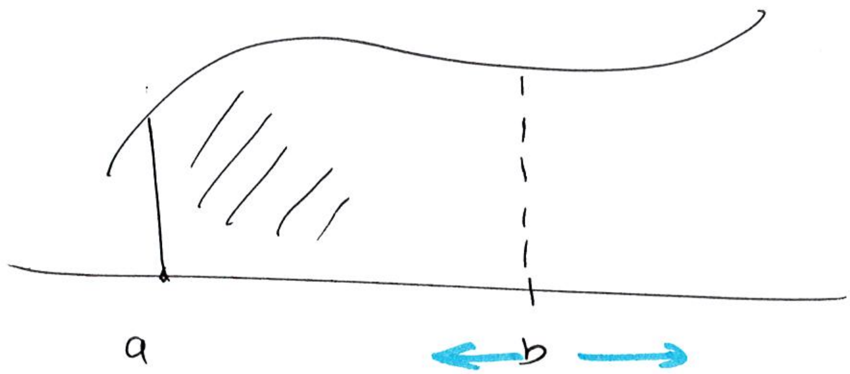
$$\Delta x = dx$$

$$\frac{d}{dx} f(x)$$

5.3) Change definite integrals into functions.

- $\int_a^b f(x)\, dx$ find area under the curve $y = f(x)$ from a to b

- $\int_a^b f(t)\, dt$ same thing x, t are ~~dummy~~ internal variables



- we allow the "upper bound" to vary

$$\int_a^x f(t) \, dt$$ ← this becomes a function of x

Theorem : The Fundamental theorem of Calculus

This function $\int_a^x f(t)\, dt$ is an anti-derivative of $f(x)$.

∎