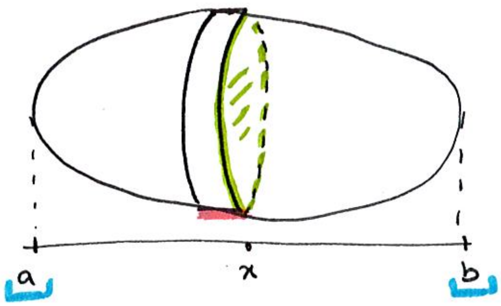
Monday, Dec 02, 2019

Recall :

Section 6.2 : Volume of "cylinder"

V = Area of base × height

Volume of solid



$$\approx \sum \underbrace{A(x)}_{\substack{\shortparallel \\ \text{Area of} \\ \text{cross-section}}} \cdot \Delta x$$

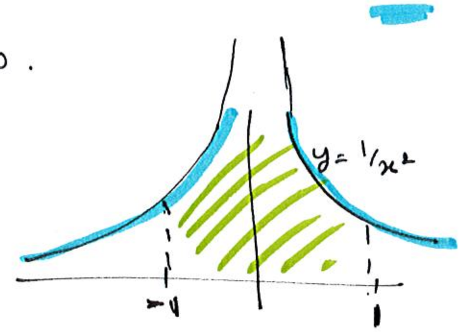
$$= \int_a^b A(x) \cdot dx$$

Aside This in Chapter 7.

- Q. Does $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\, dx$ exist ?

A: For us, it does not exist, because $\frac{1}{x^2}$ ~~is not~~ goes to ∞ at $x = 0$.

- Such an integral is called an improper integral.



- this integral is defined as

$$\int_{-1}^{1} \frac{1}{x^2}\, dx = \int_{-1}^{0} \frac{1}{x^2}\, dx + \int_{0}^{1} \frac{1}{x^2}\, dx$$

(01)

this is defined as $= \lim_{t \to 0^-} \int_{-1}^{t} \frac{1}{x^2}\, dx + \lim_{t \to 0^+} \int_{t}^{1} \frac{1}{x^2}\, dx$

<u>Note</u> : · you cannot just use the anti-derivatives directly as $\frac{1}{x^2}$ is ∞ in the range of integration.

· The fundamental theorem of calculus <u>ONLY applies</u> <u>to continuous functions.</u>

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the book uses the formula

$$\int \frac{1}{x}\, dx = \ln|x| + C.$$

· For the exam, use $\ln|x|$ instead of just $\ln x$.

<u>Back to 6.2</u> :

Volumes of solids $= \int A(x)\, dx$

· To find volumes :

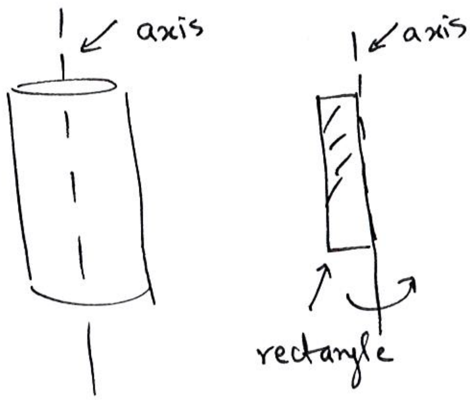
Step 1 : Describe the cross sectional area as a function of x (sometimes as a function of y)

Step 2 : Compute the definite integral using the bounds of the solid. $\int_a^b A(x)\, dx$

Standard example : Solid of Revolution

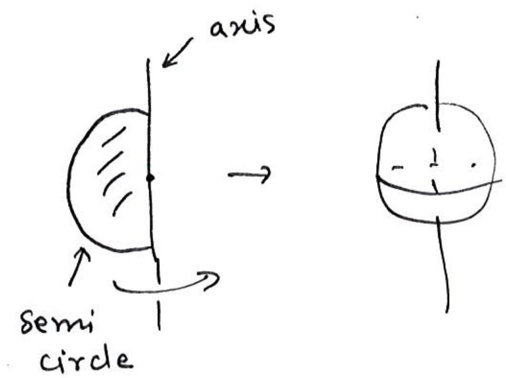
rotate the "area" about the axis to get a solid object.

eg :



Cylinder

eg :



Sphere

Q. Find the volume of ~~regio~~ solid obtained by revolving the region between

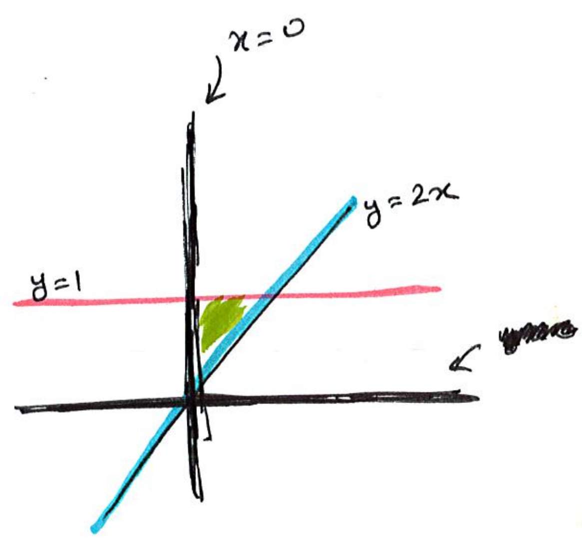
$x = 0$

$y = 1$

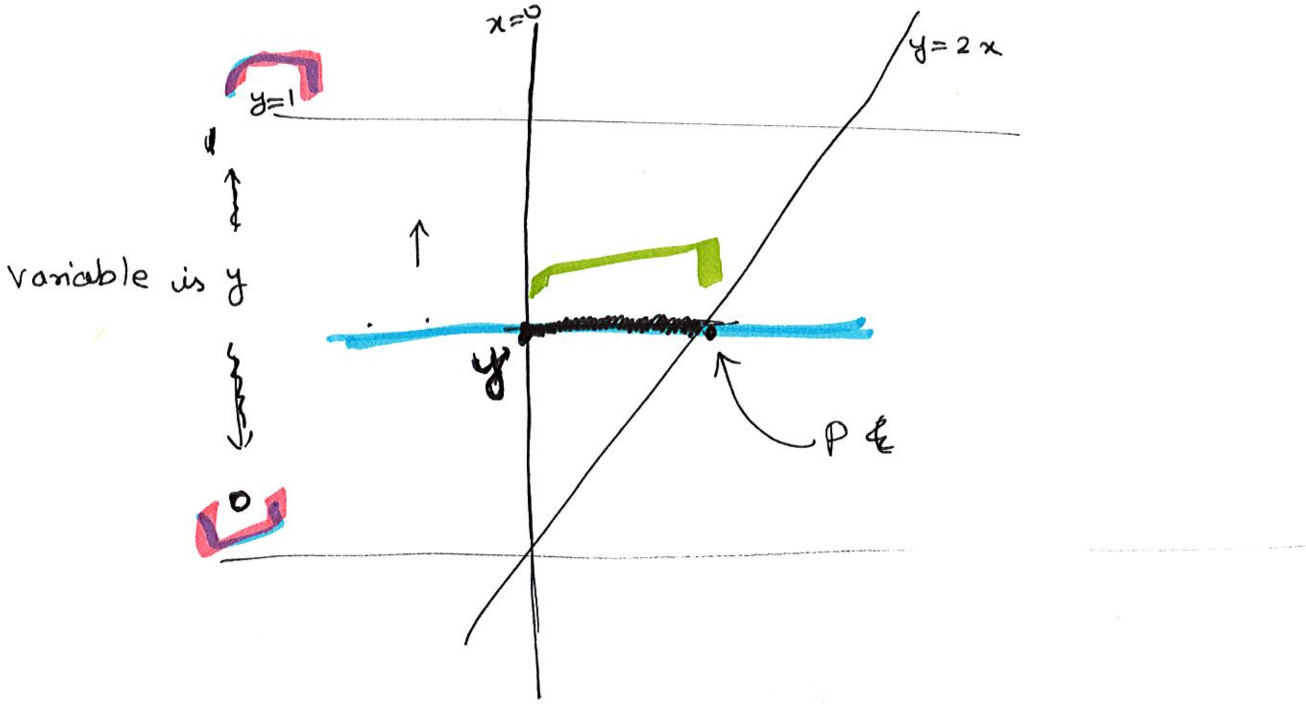
$y = 2x$

about the y-axis.

An. Step 0 : Draw the region



Step 1 : • Use the axis as your primary variables (for problems ~~&~~ involving solids of revolution)



• Take cross-section perpendicular to the axis

||

circle with radius = x-co-ordinate of P in terms of y

||

circle with radius $\frac{y}{2}$ (as P is on the line $y=2x$, so $x=\frac{y}{2}$)

$$\text{Area} = \pi r^2 = \pi\left(\frac{y}{2}\right)^2$$

$$\text{Volume} = \int_0^1 \pi \cdot \left(\frac{y}{2}\right)^2 \cdot dy$$

$$= \pi \int_0^1 \frac{y^2}{4}\, dy$$

$$= \frac{\pi}{4} \int_0^1 y^2\, dy$$

$$= \frac{\pi}{4} \cdot \frac{y^3}{3} \Big|_0^1$$

$$= \frac{\pi}{4} \cdot \left[\frac{1^3}{3} - \frac{0^3}{3} \right]$$

$$= \frac{\pi}{12}$$

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if you are rotating about y-axis, write x as a function of y

$$\int A(y)\, dy$$

if you are rotating about x-axis, write y as a function of x

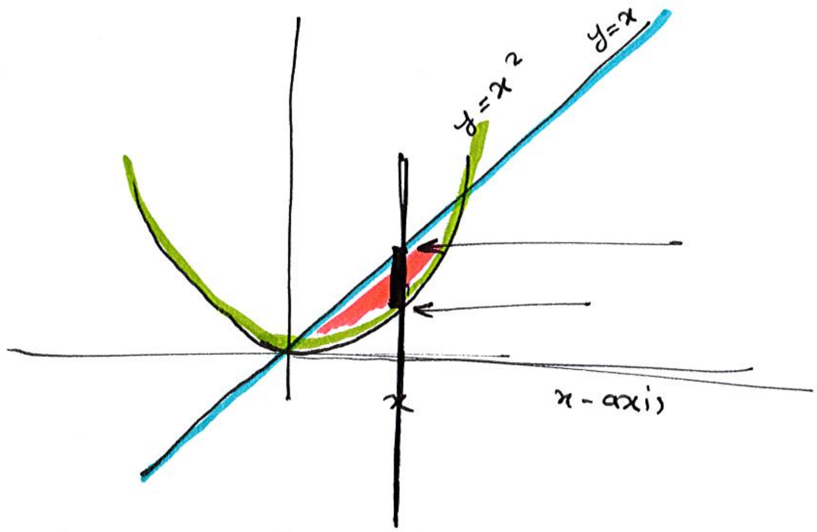
$$\int A(x)\, dx.$$

- When dealing with solids of revolution cross-section is always circular $(= \pi r^2)$

Q.1) Find volume of ~~reg~~ solid obtained by rotating the region bounded by $y = x$ and $y = x^2$ about the x-axis.

Q.2 about the y-axis

A.1) Step 0 : Draw the regions:



Step 1 : Primary variable : x

Cross-sectional ~~area~~ = segment rotated about a point not containing it

= Area of bigger circle − area of smaller circle

$= \pi R^2 - \pi r^2$

$= \pi x^2 - \pi (x^2)^2$

Step 2, Form the integral:

$$\int_{?}^{?} \pi x^2 - \pi x^4 \cdot dx$$

To find the bounds : find points of intersection of $y = x^2$ and $y = x$

$$\equiv \quad x^2 = x$$

$$\Rightarrow \quad x^2 - x = 0$$

$$\Rightarrow \quad x(x-1) = 0$$

$$\Rightarrow \quad x = 0 \quad \text{or} \quad x = 1$$

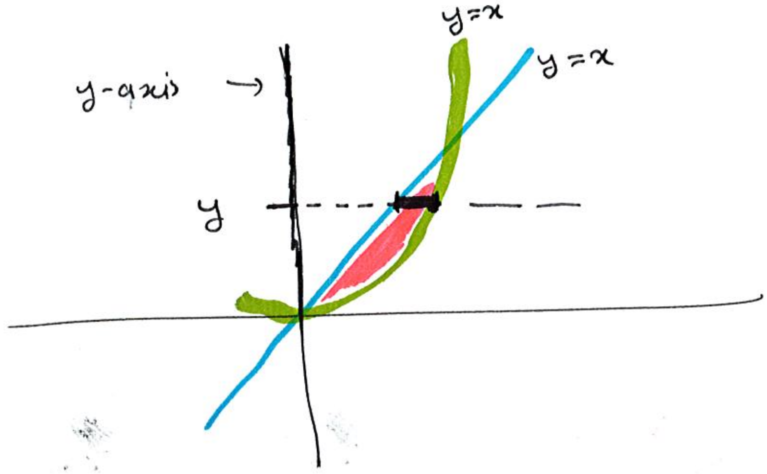
$$\Rightarrow \quad \text{Volume} = \int_0^1 \left(\pi x^2 - \pi x^4\right) dx$$

Summary: if we have a solid bounded by 2 - curves : cross-sectional area is

$$= \pi R^2 - \pi r^2$$

Q.2 Same curves, but now about y-axis

$y = x$, $y = x^2$



- Primary variable $= y$
- Cross section =

$$A(y) = \text{Area} = \pi R^2 - \pi r^2$$

$y = x^2$ $\Rightarrow \sqrt{y} = x$, $R = \sqrt{y}$

$y = x$, $x = y$, $r = y$

Note: when we rotate about the y-axis, the radius is the x-co-ordinate

$$= \pi (\sqrt{y})^2 - \pi \cdot y^2$$

Step 3:

$$V = \int_{0?}^{?} \pi (\sqrt{y})^2 - \pi y^2 \, dy$$

- Points of intersection: $x = 0$, $x = 1$

use any curve to find y : eg, $y = x$

$x = 0$, $y = 0$; $x = 1$, $y = 1$

eg $y = x^2$

$x = 0$, $y = 0$; $x = 1$, $y = 1^2 = 1$

$$\boxed{V = \int_0^1 \pi (\sqrt{y})^2 - \pi y^2 \, dy}$$

Q.1)

$$V = \int_0^1 \pi x^2 - \pi x^4 \, dx$$

Q.2)

$$V = \int_0^1 \pi y - \pi y^2 \, dy$$

(08)

Q.1)

$$V = \pi \int_0^1 x^2 - x^4 \, dx$$

$$= \pi \left(\frac{x^3}{3} - \frac{x^5}{5} \right) \Big|_0^1$$

$$= \pi \left(\frac{1}{3} - \frac{1}{5} \right)$$

$$= \frac{2\pi}{15}$$

Q.2)

$$V = \pi \int_0^1 y - y^2 \, dy$$

$$= \pi \left(\frac{y^2}{2} - \frac{y^3}{3} \right) \Big|_0^1$$

$$= \pi \left(\frac{1}{2} - \frac{1}{3} \right)$$

$$= \frac{\pi}{6}$$

Also see example 8

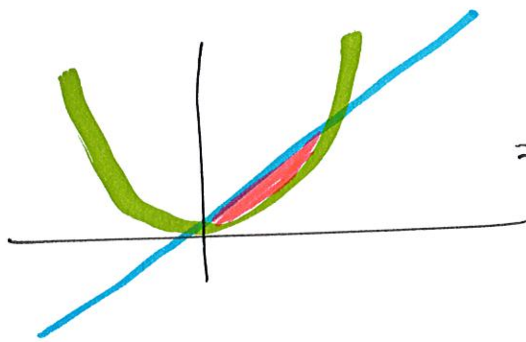
from Ch 6.2

Section 6.1 : Areas of regions

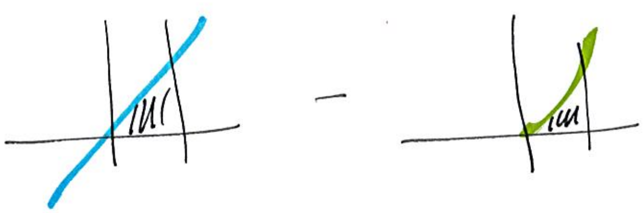
Central idea :

- integrals only complute signed areas between $y=f(x)$ and x-axis.
- for a more complicated region, we break the break the region into smaller pieces.

Q: Find area of region bounded by $y=x$ and $y=x^2$.



= Area under $y=x$ − Area under $y=x^2$



$$= \int_0^1 x\,dx - \int_0^1 x^2\,dx$$

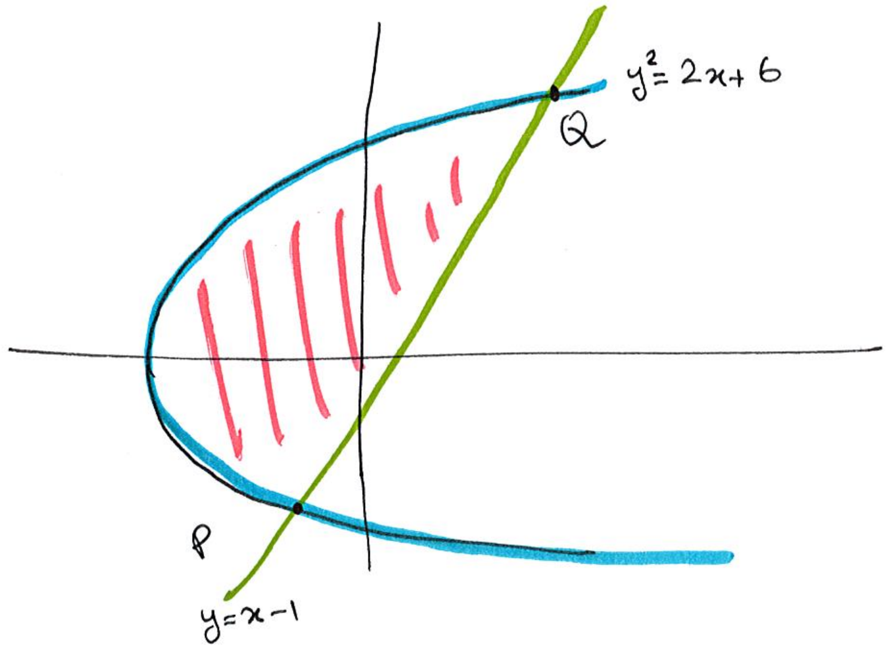
$$= \frac{x^2}{2}\Big|_0^1 - \frac{x^3}{3}\Big|_0^1$$

$$= \frac{1}{2} - \frac{1}{3}$$

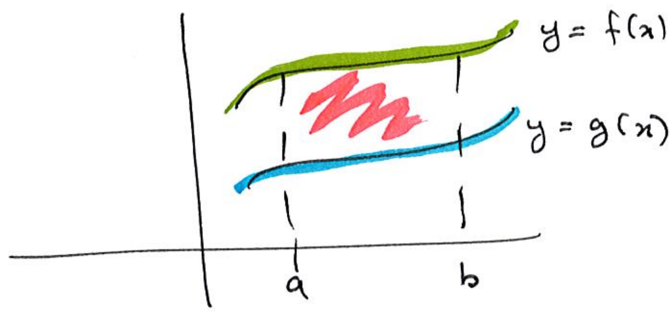
$$= \frac{1}{6}$$

Q. Find area enclosed between

$y = x - 1$ and $y^2 = 2x + 6$



- New "formula":

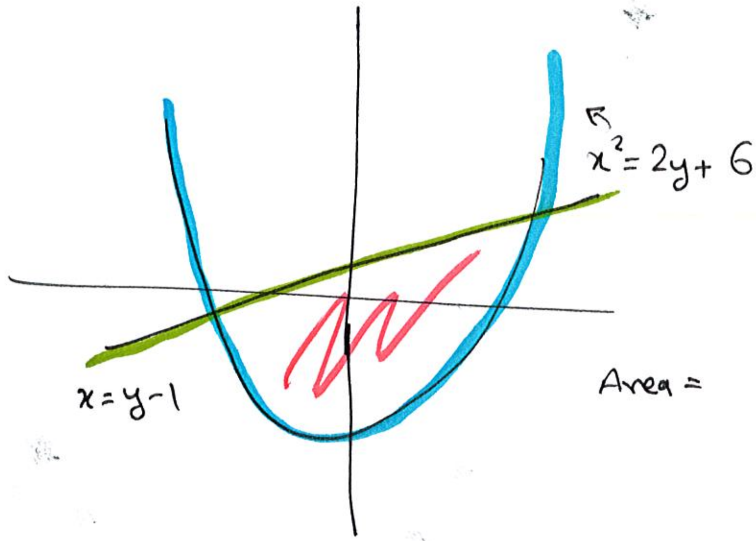


Area between $y = f(x)$ and $y = g(x)$ from a to b $= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

$$= \int_a^b f(x) - g(x) \cdot dx$$

We can find areas above $g(x)$ and below $f(x)$

Great solution: flip the x, y axes



$$\text{Area} = \int_{?}^{?} (x+1) - \left(\frac{x^2-6}{2}\right) dx$$

$x = y - 1$

$\boxed{x + 1 = y}$

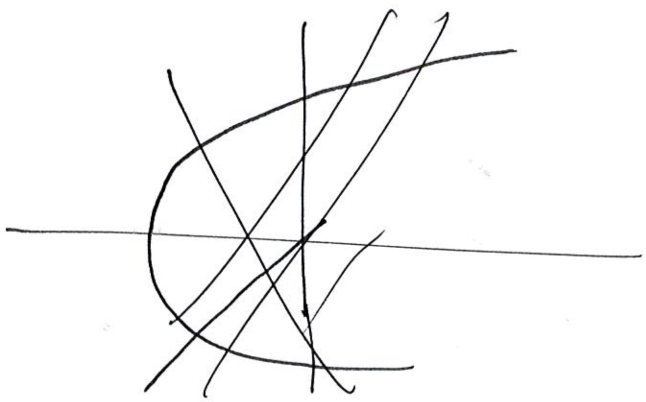
$x^2 = 2y + 6$

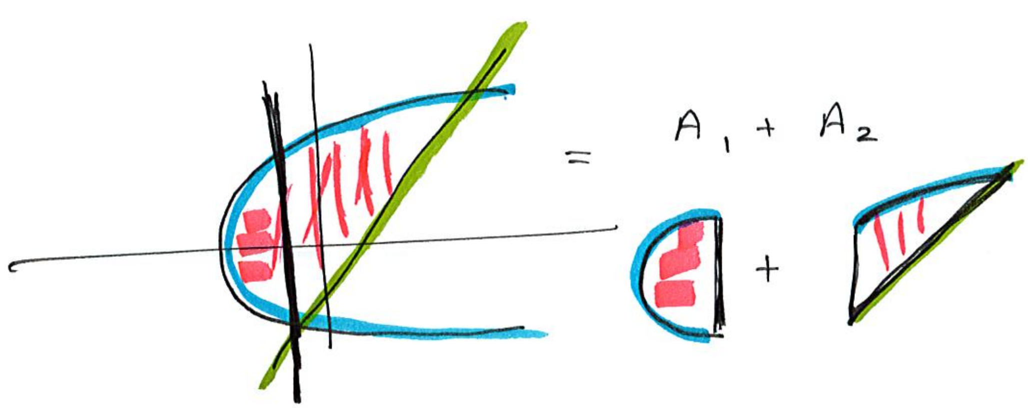
$x^2 - 6 = 2y$

$\boxed{\frac{x^2-6}{2} = y}$

the bounds are obtained by finding points of intersection.

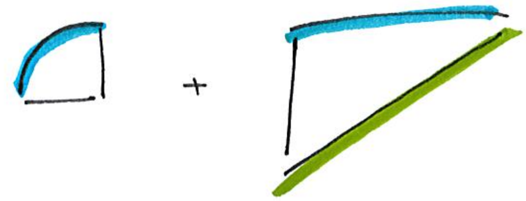
The solution in the book:





$$\cancel{x^2 = 2y+6}$$
$$\cancel{x^2 - 6 = 2y}$$
$$\cancel{\frac{x^2-6}{2} = y}$$

$$= 2A_3 + A_2$$



then find A_3, A_2 using integrals

$$\boxed{y^2 = 2x+6}$$

$$y = \sqrt{2x+6}$$

$$\boxed{y = x-1}$$

$$= \int^{?} \cancel{\frac{x^2-6}{2}} \qquad A_3 \qquad\qquad A_2$$

$$= 2 \cdot \int_{?}^{?} \sqrt{2x+6} + \int_{?}^{?} \sqrt{2x+6} - (x-1)\, dx$$

- what to take out of this problem:

① "horizontally break the region if the left or the right boundary is not vertical"

$$\int_a^b = \int_a^c + \int_c^b$$

② Use $\int f(x) - g(x)\, dx$ to find areas between curves

- this formula works even when $g(x)$ is below the x-axis

3) $\int_a^b f(x)\,dx$ is the <u>signed area</u>

So if the ~~area~~ region is below the the x-axis, you might need to fix the sign in the end.

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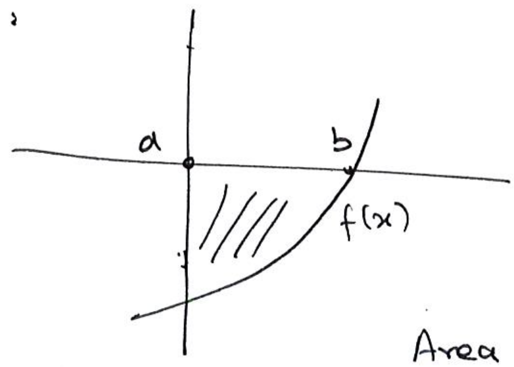
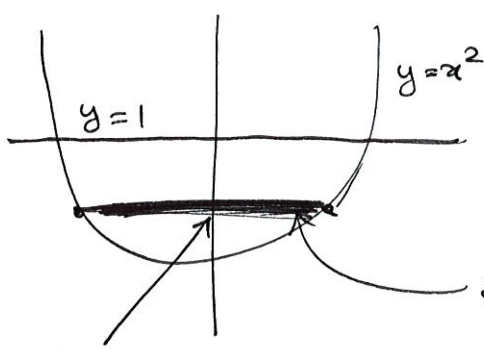
eg:



$$\text{Area} = -\int_a^b f(x)\,dx$$

Pset 7 26)



$2\sqrt{\cancel{x}y}$

Side $= S(y) = 2\sqrt{y}$

$A(y) = S^2 =$