Notes from Dwyer And Wilkerson, Elementary Geometric Structure of Compact Lie Groups

Unless otherwise stated G - compact connected Lie group

Prop: $\phi: \mathbb{R}^n \longrightarrow T_eG$ vector space map $\Rightarrow$, if G abelian or $n=1$ $\exists\, \psi: \mathbb{R}^n \longrightarrow G$ group homo, such that $d\psi_e = \phi$.
Proved by using exp maps for a left G-invariant metric.

Prop. Every compact abelian group is isomorphic to $\mathbb{Z}^n$.
Proof: By the above proposition we have a homomorphism $T_eG \cong \mathbb{R}^n \xrightarrow{exp} G$. Now exp map is a local diffeo at 0, by being a group map it becomes a local diffeo at each pt. Again by symmetry this map would be proper and hence a covering map. The only discrete subgroups of $\mathbb{R}^n$ are $\mathbb{Z}^k$ and by compactness $k=n$.

Prop: T, T' tori $\mathrm{Hom}_{\mathbb{Z}}(T,T') \xrightarrow{\pi_1} \mathrm{Hom}_{\mathbb{Z}}(\pi_1 T, \pi_1 T')$ is an isomorphism.
Proof: T, T' are $K(\pi,1)$ spaces.

Prop. $1 \to T \longrightarrow G \longrightarrow T' \longrightarrow 1$ exact $\Rightarrow$ G is a torus.
Proof: $1 \to \mathbb{Z}/3 \to S_3 \longrightarrow \mathbb{Z}/2 \longrightarrow 1$ shows that connectedness is important else extension of an abelian group by another group can be non-abelian.

Prop. Every Lie group contains a non-trivial torus as a subgroup.

Lefschetz No. $f: X \longrightarrow X$, $\mathcal{L}(f) := \sum_i (-1)^i \, tr_{\mathbb{Q}}(H^i(f;\mathbb{Q}))$ X is always a finite CW cplx

Lefschetz fixed pt thm: If $f: X \longrightarrow X$ has no fixed points then $\mathcal{L}(f) = 0$. In particular as $\mathcal{L}(\mathrm{id}_X) = \chi(X)$ if $f: X \to X$ is homotopic to identity and has no fixed points $\chi(X) = 0$.

Cor: If G is a compact non-trivial connected Lie group then $\chi(G) = 0$.

Prop: $f: A\cup_B C \longrightarrow A\cup_B C$ $f(A)\subseteq A$, $f(C)\subseteq C$ then $\mathcal{L}(f) = \mathcal{L}(f|_A)+\mathcal{L}(f|_C)-\mathcal{L}(f|_B)$
Proof by Mayer Vietoris.

Prop: $F\to E\to B$ filtration then $\chi(E)=\chi(F)\chi(B)$
Proof by looking at the Serre SS.

Prop: M compact manifold. $G \curvearrowright M$ then $\mathcal{L}(g) = \chi(M^g)$.
Note that this is not true for arbitrary diffeo $f: X\to X$ eg: $f: S^1\to S^1$ which is homotopic to identity and whose graph looks like  then $\mathcal{L}(f)=1-1=0$, $\chi(S^1)^f = \chi(*)=1$. Somehow f coming from action of a Lie group is important.

Look at $\langle g\rangle\subseteq G$ and let H be its closure, then $H\leq G$ is a Lie Subgroup (?) and $M^g=M^H$.

Claim: M^H is a closed submanifold of M.

The closed part would follow from the fact that fixed points are $\Delta^{-1}\phi(M\times H)$, the tough part is showing submanifold.

$$\begin{array}{ccccc} M\times H & \xrightarrow{\phi} & M\times M & \xleftarrow{\Delta} & M \\ (m,h) & \longmapsto & (m,hm) & & \\ & & (m,m) & \longleftarrow & m \end{array}$$

Let $\langle,\rangle$ be an arbitrary Riemannian metric on M then define: $(,)_x : T_xM\times T_xM\longrightarrow\mathbb{R}$ as

$$(a,b)_x = \int_h \langle h_*a, h_*b\rangle\, dh$$

Then clearly $(,)$ is a H-invariant inner product on M.
We can give local charts for M^H at x as $\left[\exp_{(,)}\left(\bigcap_{g\in G}\ker g_{*x}\right)\right.$
which makes M^H a submanifold.

Note that by the proof of the above claim we can assume g acts via isometry.
So look at a tubular neighborhood of A in X, say T. Wlog assume T is a unit bundle, so that $f(T)\subseteq T$.
Let $U = X\setminus A$, $V=T$, $U\cap V = T\setminus A$, then $\mathcal{L}(f|_X) = \mathcal{L}(f|_{X\setminus A})+\mathcal{L}(f|_T)-\mathcal{L}(f|_{T\setminus A})$
as all the fixed points are in A, $\mathcal{L}(f|_{X\setminus A}) = \mathcal{L}(f|_{T\setminus A})=0$,
$\Rightarrow \mathcal{L}(f)=\mathcal{L}(f|_T)=\mathcal{L}(f|_A)=\chi(A)$ (This can be seen by breaking T into local patches)

Note that the only property of f used in the above proof was that it was an isometry.
Conversely a th^m by Myres and Steenrod states that the space of isometries is a Lie group.

Prop: $\mathbb{T}^n \curvearrowright X$, finite CW complex, action without fixed points $\Rightarrow \chi(X)=0$.
We need to invoke the fact that $\mathbb{T}^n$ has an element α such that $\langle\alpha\rangle$ is dense in $\mathbb{T}^n$.

Prop: $T \curvearrowright M$ closed manifold, then $\chi(M)=\chi(M^T)$.
Again follow from existence of α with $\langle\alpha\rangle$ dense in T.

Prop: T, T' tori $\leq G$ such that $\chi(G/T)\neq 0$, $\chi(G/T')\neq 0$. Then $\exists g\in G$ s.t. $T'=gTg^{-1}$

Proof: $T' \curvearrowright G/T$ by left multiplication $\Rightarrow \chi(G/T)^{T'} = \chi(G/T)\neq 0$

$\Rightarrow \exists\, g\in G$ s.t. $\forall h\in T'$, $hg/T = g/T$

i.e. $T' \subseteq gTg^{-1}$

By symmetry, $gTg^{-1}=T'$

Prop: $T\leq G$ is maximal iff $[C_G(T):T]$ is finite. (T is almost self-centralising)

Proof: Look at fiber bundle: $C_G(T)/T \hookrightarrow N_G(T)/T \longrightarrow N_G(T)/C_G(T)$

$$\Rightarrow \chi(N_G(T)/T) = \chi(C_G(T)/T)\cdot \chi(N_G(T)/C_G(T))$$

$$= \chi(C_G(T)/T)\cdot\left(\# N_G(T)/C_G(T)\right)$$ as these are inner auts

Now, $T\curvearrowright G/T$ by left multiplication and $(G/T)^T = N_G(T)/T$ and so we get
$\chi(G/T) = \chi(N_G(T)/T)$ so that

$$\chi(G/T) = \chi(C_G(T)/T)\cdot \# N_G(T)/C_G(T)$$

If $[C_G(T):T]$ finite then $\chi(G/T)\neq 0$

If $[C_G(T):T]$ not finite $\Rightarrow \dim(C_G(T)/T)>0 \Rightarrow \chi(C_G(T)/T)=0$.

Th^m: Every compact Lie group has a maximal torus.

Proof: Induction on $\dim G$. Let $T\leq G$ be a torus. Then by induction $C_G(T)/T$ contains a torus T'. Let $\overline{T}$ be inverse image in $C_G(T)$. So we have $1\to T\to \overline{T}\to T'\to 1$
$\Rightarrow \overline{T}$ is a torus. $[C_G(\overline{T}):\overline{T}] \leq [C_G(T):\overline{T}]$. So remains to show $[C_G(T):\overline{T}]$ is finite.
But this is because $\chi(C_G(T)/\overline{T}) = \chi((C_G(T)/T)/\overline{T})\neq 0$ as $\overline{T}$ is maximal in $C_G(T)/T$.
And so $\overline{T}$ is maximal in $C_G(T)$ and $C_G(\overline{T}) = C_{C_G(T)}(\overline{T})$ and so
$[C_G(\overline{T}):\overline{T}] = [C_{C_G}(\overline{T}):\overline{T}] < \infty$.

Prop: $T \leq T' \leq G \Rightarrow T$ cannot be maximal.

$T'/T \hookrightarrow G/T \longrightarrow G/T'$ is a fibration which gives $\chi(G/T) = \chi(G/T') \cdot \chi(T/T') = \chi(G/T') \cdot 0 = 0$.

Prop: T maximal $\Rightarrow C_G(T) = T$, for G-connected.

Proof: Let $a \in C_G(T)$, we wish to say $a \in T$. Look at $A = \langle a, T \rangle$. Then it suffices to say that $\exists g \in G$ such that $gAg^{-1} \leq T$ because we already have gTg^{-1} is itself a maximal torus. So $gAg^{-1} \subseteq T$ would imply $gTg^{-1} = T$ and $a \in g^{-1}Tg = T$.

Finding such a g is same as finding a gT such that $agT = gT$ $\forall a \in A$ i.e. suffices to show that $(G/T)^A$ is non-empty. $\chi[(G/T)^A] = \chi[((G/T)^{\langle a \rangle})^T] = \chi[(G/T)^{\langle a \rangle}] = \chi(G/T) \neq 0$ (as $\langle a \rangle$ homotopic to identity)

Prop: If $S \leq G$ is a torus, and S commutes with a then $\langle S, a \rangle \subseteq T$ for some maximal torus T. In particular $a = e$ gives every torus is contained in a maximal torus and $S = \{e\}$ gives every element is contained in some maximal torus.

Proof: Pick a maximal torus T then this is the same as showing $(G/T)^{\langle S, a \rangle}$ is not empty.

Def: $W := N_G(T)/T$ for any maximal torus T of G

Prop: W is well defined and finite.

As any two tori are conjugates of each other W is well defined. $N_G(T)/T \longleftarrow \underset{\text{discrete}}{N_G(T)/C_G(T)}$

Prop: $|W| = \chi(G/T)$

Proof $W = N_G(T)/T = (G/T)^T$. $|W| = \#(G/T)^T = \chi((G/T)^T) = \chi(G/T)$

Prop: If G is not connected with the identity component G_e then $G/G_e \cong W_G/W_{G_e} = \pi_0 G$

Prop $(Bi)^*: H^*(BG, \mathbb{Q}) \longrightarrow H^*(BT, \mathbb{Q})$ is injective.

Proof: $T \hookrightarrow G \longrightarrow \underbrace{G/T \longrightarrow BT \longrightarrow BG}_{\text{fibration}}$ so that because $\chi(G/T) \neq 0$ by invoking the transfer map we get injectivity.

Prop Conjugation action of W on $H^*(BT; \mathbb{Q})$ is faithful.

Proof: $N_G(T) \curvearrowright T$ by conjugation and as $C_G(T) = T$, $W = N_G(T)/T \curvearrowright T$ faithfully.

Prop: 1) Rank of $H^*(BG;\mathbb{Q})$ which is a polynomial algebra = rank of T = rank of $H^*(BT;\mathbb{Q})$.

2) $H^*(G/T;\mathbb{Q})$ is concentrated in even dimensions.

3) $H^*(BG,\mathbb{Q}) \cong H^*(BT;\mathbb{Q})^W$

Claim: The image of $(Bi)^*: H^*(BG;\mathbb{Q}) \longrightarrow H^*(BT;\mathbb{Q})$ lands in $H^*(BT;\mathbb{Q})^W$

Proof: Let $g \in N_G(T)$ and g denote also the action of g by conjugation. So we wish to show:

$BH \xrightarrow{g} BH \xrightarrow{Bi} BG$ is homotopic to Bi.

But every automorphism of BH is homotopic to id as BH is a classifying space of H.

Q. Does this make sense? This also implies $(Bi)^*$ lands in $H^*(BT;\mathbb{Q})^{\mathrm{Aut}[BH \hookrightarrow BG]}$

So we have $H^*(BG;\mathbb{Q}) \xrightarrow{(Bi)^*} H^*(BT;\mathbb{Q})^W \hookrightarrow H^*(BT;\mathbb{Q})$

Proof: Use Serre SS and some algebra (how?) to say that $H^*(BT;\mathbb{Q})$ is infact a finitely generated free module over $H^*(BG;\mathbb{Q})$, with say t generators.

Now invoke Eilenberg-Moore on $G/T \to BT \to BG$ to get $E_2^{p,q} = \mathrm{Tor}_{-p,q}^{H_*(BG,\mathbb{Q})}(H_*(BT;\mathbb{Q});\mathbb{Q})$

(Can we do this? $\pi_1(BG) = G \neq 0$, Do we need G to act trivially on $H_*(G/T)$?)

But as $H_*(BT)$ is free over $H_*(BG)$ only $\mathrm{Tor}_{0,*}$ survives and we get

$$H_*(G/T;\mathbb{Q}) \cong H_*(BT;\mathbb{Q}) \otimes_{H^*(BG;\mathbb{Q})} \mathbb{Q} \cong \mathbb{Q}^t$$

Now $H^*(BT)$, $H^*(BG)$ has only evenly graded terms. This proves 2)

And so $\chi(G/T) = \dim H^*(G/T;\mathbb{Q}) = t = |W|$

1) is also an algebraic argument that I don't understand. Lets assume it.

So as rings $H^*(BG;\mathbb{Q}) \cong H^*(BT;\mathbb{Q})$ but $H^*(BT,\mathbb{Q}) \cong H^*(BG;\mathbb{Q})\langle y_1 \cdots y_t \rangle$

Let $F(R)$ denote field of fractions of ring R, then we have, as module over $H^*(BG;\mathbb{Q})$.

$F(H^*(BG)) \subseteq F(H^*(BT)^W) \subseteq F(H^*(BT))$ and the two extreme ones are isomorphic, so that both the extensions are algebraic. By construction, $[F(H^*(BT)) : F(H^*(BG))] = |W|$ so by Galois th^m, $F(H^*(BG)) = F(H^*(BT)^W)$. From here some argument about integral extensions will give the answer.

Also, Bruhat decomposition gives you the cell structure.

For the sake of completeness:

Prof:
- All Borel subgroups of G are conjugate to each other.
- $N_G(B) = B \implies G/B$ = Set of all Borel Subgroups of G.
- $P \subseteq G$ any subgroup such that G/P is a smooth manifold (projective variety) implies P contains some Borel subgroup.
- $W = N_G(T)/T$ acts on the space of all Borel Subgroups containing T (How?)