Transfer Map

Given a fiber bundle, smooth, $F \to E \xrightarrow{p} B$ we wish to define a stable map $\tau : \Sigma^N B_+ \longrightarrow \Sigma^N E_+$ which satisfies certain identities. F - closed, oriented manifold.

Let $i : E \hookrightarrow \mathbb{R}^N$ be an embedding. Extend this to an embedding $(p \times i) : E \hookrightarrow B \times \mathbb{R}^N$ with normal bundle ν.
Let $T_F E \subseteq TE$ be the space of vertical fibers i.e. $T_F E = \ker (p_* : TE \longrightarrow TB)$
Let W be a vertical vector field i.e. section of $T_F E$,

Claim: $T_F E \oplus \nu$ is a trivial bundle on E and hence isomorphic to $E \times \mathbb{R}^N$.
Proof: We can extend $p_* : TE \to TB$ to $\tilde{p}_* : TE \oplus \nu \longrightarrow TB$ trivially. Then we have $T_F E \oplus \nu = \ker \tilde{p}_*$.
But from the commutative diagram

$$\begin{tikzcd} TE \oplus \nu \arrow[rr, "\cong"] \arrow[dr, "\tilde{p}_*"'] & & TB \times T\mathbb{R}^N|_E \arrow[dl] \\ & TB & \end{tikzcd}$$

we see that
$T_F E \oplus \nu \cong T\mathbb{R}^N|_E \cong E \times \mathbb{R}^N$

Thom space of trivial bundle over B:

$$\mathrm{Th}\begin{pmatrix} B \times \mathbb{R}^N \\ \downarrow \\ B \end{pmatrix} = B \times D^N / B \times S^N \cong \frac{B \times S^N}{B} \cong \frac{B \times S^N \sqcup S^N}{B \sqcup S^N} \cong \frac{B_+ \times S^N}{B_+ \vee S^N} = \Sigma B_+$$

first compactify each fiber then shrink

Pontryagin - Thom map:
For a unit disk bundle $\mathbb{D} \to E \to X$ sitting inside M we have the Pontryagin Thom map

1pt compactification $\rightsquigarrow$ $M \cup \{\infty\} \xrightarrow[PT]{} \mathrm{Th}(E) = E / \partial E$ which is injection on the interior of E and trivial on everything else.

Now define the transfer map to be:

$$\tau_W : \Sigma^N B_+ \cong \mathrm{Th}(B \times \mathbb{R}^N) \xrightarrow[PT]{} \mathrm{Th}(\nu) \longrightarrow \mathrm{Th}(\nu \oplus T_F E) = \mathrm{Th}(E \times \mathbb{R}^N) \cong \Sigma E_+$$
$$v \longmapsto v + W$$

Property 1: τ_W defines a stable homotopy class of map from $B_+ \longrightarrow E_+$, in particular it is independant of the choice of embedding (because embeddings stabilize)

Property 2: τ_W is independant of W (because any two sections are homotopic to each other)

Because of property 2, we can define τ using $W=0$ and use a transversal section for computations.

Property 3: If W has no zeroes $\tau_W = 0$. This is because we can scale W up so that it is everywhere greater than 1, so that $v+W$ is always of length >1 and so goes to a point in the Thom space.

Property 4: $\hat{B} \subseteq B$ be a submanifold. Let $\hat{E} = E|_{\hat{B}}$ then the following diagram commutes:

$$\begin{tikzcd} \Sigma^N \hat{E}_+ \arrow[r, hook, "j"] & \Sigma^N E_+ \\ \Sigma^n \hat{B}_+ \arrow[u, "\hat{\tau}"] \arrow[r, hook, "i"] & \Sigma^N B_+ \arrow[u, "\tau"'] \end{tikzcd}$$

Which implies, $j_* \circ \hat{\tau}_* = \tau_* \circ i_*$ and $\hat{\tau}^* \circ j^* = i^* \circ \tau^*$

The only thing to note here is that for $E \hookrightarrow B \times \mathbb{R}^n$ we have

$$\begin{tikzcd} \hat{E} \arrow[r, hook] \arrow[dr] & E \arrow[r, hook] & B \times \mathbb{R}^N \\ & & \hat{B} \times \mathbb{R}^N \arrow[u, hook] \end{tikzcd}$$

and we have a commutative diagram

$$\begin{tikzcd} Th(B \times \mathbb{R}^n) \arrow[r] & Th(\nu) \arrow[r] & Th(\nu \oplus T_F E) = Th(E \times \mathbb{R}^n) \\ Th(\hat{B} \times \mathbb{R}^n) \arrow[u, hook] \arrow[r] & Th(\hat{\nu}) \arrow[u] \arrow[r] & Th(\nu \oplus T_F \hat{E}) - Th(\hat{E} \times \mathbb{R}^n) \arrow[u] \end{tikzcd}$$

Property 5: For the fiber bundle $F \longrightarrow E \times X \xrightarrow{p \times 1} B \times X$ the transfer map is given by

$$(\tau \times 1): \Sigma^N (B \times X)_+ = \Sigma^N B_+ \wedge X_+ \xrightarrow{\tau \wedge 1} \Sigma^N E_+ \wedge X_+$$

$$(E \times X) \hookrightarrow B \times X \times \mathbb{R}^N \rightsquigarrow Th(B \times X \times \mathbb{R}^N) \longrightarrow Th(\nu \times X) \longrightarrow Th(\nu \times X \oplus T_F E \times X) \longrightarrow Th(E \times X \times \mathbb{R}^N)$$

Property 6: For $\alpha \in H^i(B)$, $\beta \in H^j(E)$ $\quad \tau_W^*(p^*(\alpha) \cup \beta) = \alpha \cup \tau_W^* \beta$

Look at the pullback diagram:

$$\begin{tikzcd} E \arrow[r, "1 \times p"] \arrow[d, "p"'] & E \times B \arrow[d, "p \times 1"] \\ B \arrow[r, "\Delta"] & B \times B \end{tikzcd}$$

as Δ is an inclusion we get,

$$\begin{tikzcd} \Sigma^N E_+ \arrow[r, "1 \times \Sigma^N p"] & \Sigma^N E_+ \wedge \Sigma^N B_+ \\ \Sigma^N B_+ \arrow[u, "\hat{\tau}"] \arrow[r, "\Delta"] & \Sigma^N B_+ \wedge \Sigma^N B_+ \arrow[u, "\tau"'] \end{tikzcd}$$

Looking at homologies we get the required result.

Property 7: If $B = *$ then τ_W^* is simply multiplication by $\chi(F)$.

$F \longrightarrow F \longrightarrow *$ We need an embedding $F \overset{k}{\hookrightarrow} \mathbb{R}^N$ then the map τ is composition of

$$S^N = \mathbb{R}^N \cup \{\infty\} \xrightarrow{\phi} Th(\nu) \xrightarrow{\psi} Th(\nu \oplus TF)$$

Claim: $\phi^*: H^N(S^N) \longleftarrow H^N(Th(\nu)) \cong H^{N-(N-k)}(F)$ is identity on each connected component of F.

Proof. If $F = F_1 \sqcup F_2$ then $Th(\nu(F_1 \sqcup F_2)) = Th(\nu(F_1)) \vee Th(\nu(F_2))$ and the map $S^N \to \nu(F_1 \sqcup F_2)$ factors through $S^N \to S^N \vee S^N \longrightarrow Th(\nu(F_1)) \vee Th(\nu(F_2))$. So it suffices to prove the claim for F connected.

It is easier to argue this using differential forms. The Thom class u_ν is a form supported compactly near F and $\int u_\nu = 1$ where integral is taken over each fiber. Let ω_F be the top cohomology class of F, then $H^N(\nu(F))$ is generated by $\omega_F \wedge u$. To find what $\phi^*(\omega_F \wedge u)$ should be simply integrate: $\int_{S^N} \omega_F \wedge u = \underbrace{\int_{\nu} \omega_F \wedge u}_{\text{local + patching}} = \int_F \omega \cdot \int_{\text{fiber}} u = 1$

Claim: $\psi^*: H^N(Th(\nu \oplus TF)) \longrightarrow H^N(Th(\nu))$ is given by multiplication by $\chi(F)$.

Proof. This follows from the fact that $u_{V \oplus W} = u_V \cup u_W$ and $\overset{\text{euler class of } F}{\overset{\shortparallel}{u_{TF}}} \cap [F] = \chi(F)$.

Property 8: $p_* \circ \tau_* = \text{multiplication by } \chi(F) = \tau^* \circ p^*$

Pick $\alpha \in H^i(B)$, $\beta = 1 \in H^0(E)$ then we have

$$\tau^*(p^*(\alpha) \cup 1) = \alpha \cup \tau^*(1)$$

Look at
$$\begin{tikzcd} * \arrow[r, hook] & B \\ F \arrow[u] \arrow[r] & E \arrow[u] \end{tikzcd}$$
then we have on H^0 the map between cohomologies is given by

$$\begin{tikzcd} H^0(*) & H^0(B) \arrow[l] \\ H^0(F) \arrow[u, "\tau^*"] & H^0(E) \arrow[l] \arrow[u, "\tau^*"'] \end{tikzcd}$$

This we know to be $\chi(F)$

and so this has to be $\chi(F)$.

Q. How to argue for homologies?