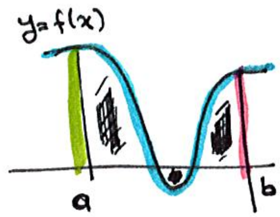
Wednesday, Nov 20, 2019

- Quiz on Monday on Sections 4.7, 5.3.

- Recall:

$$\int_a^b f(x) \cdot dx = \text{Signed area "under" the curve } y = f(x) \text{ from } a \text{ to } b$$



$$= \text{Reimann sum}$$

$$= F(b) - F(a), \text{ where } F(x) \text{ is any anti-derivative of } f(x).$$

- Integrals behave nicely with respect to addition, subtraction, scalar multiplication but <u>not</u> products, quotients, composition.

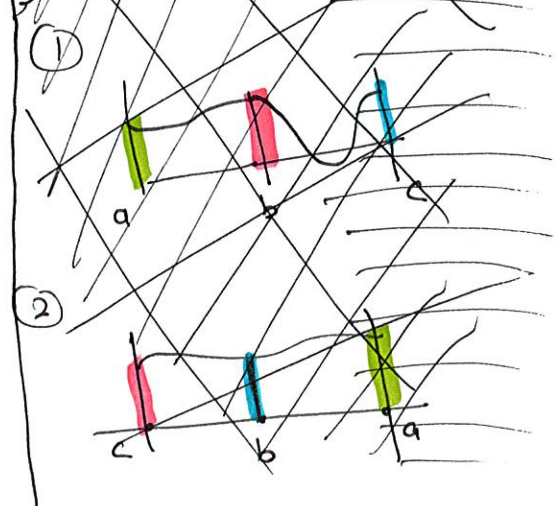
$$\int_a^b f(x) + g(x)\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$$

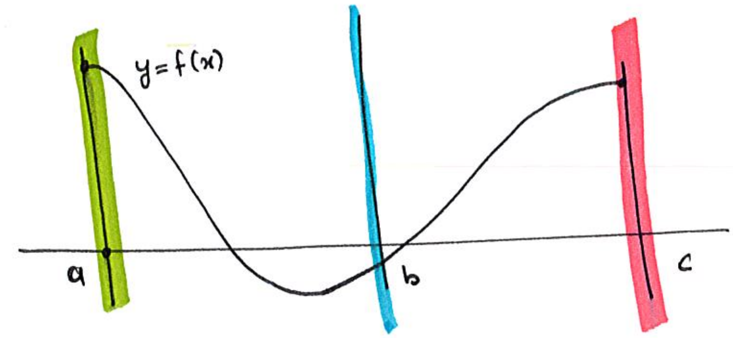
$$\int_a^b f(x) - g(x)\, dx = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx$$

$$\int_a^b c \cdot f(x)\, dx = c \int_a^b f(x)\, dx$$

1. $\int_a^b f(x)\, dx + \int_b^c f(x)\, dx = \int_a^c f(x)\, dx$

2. $\int_c^a f(x)\, dx - \int_c^b f(x)\, dx = \int_b^a f(x)\, dx$





① $$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

② $$\int_a^c f(x)\,dx - \int_a^b f(x)\,dx = \int_b^c f(x)\,dx$$

Aside : Appendix E, Q. 43

Compute $$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left(\frac{i}{n}\right)^2 .$$

- Observe : this is a Riemann sum

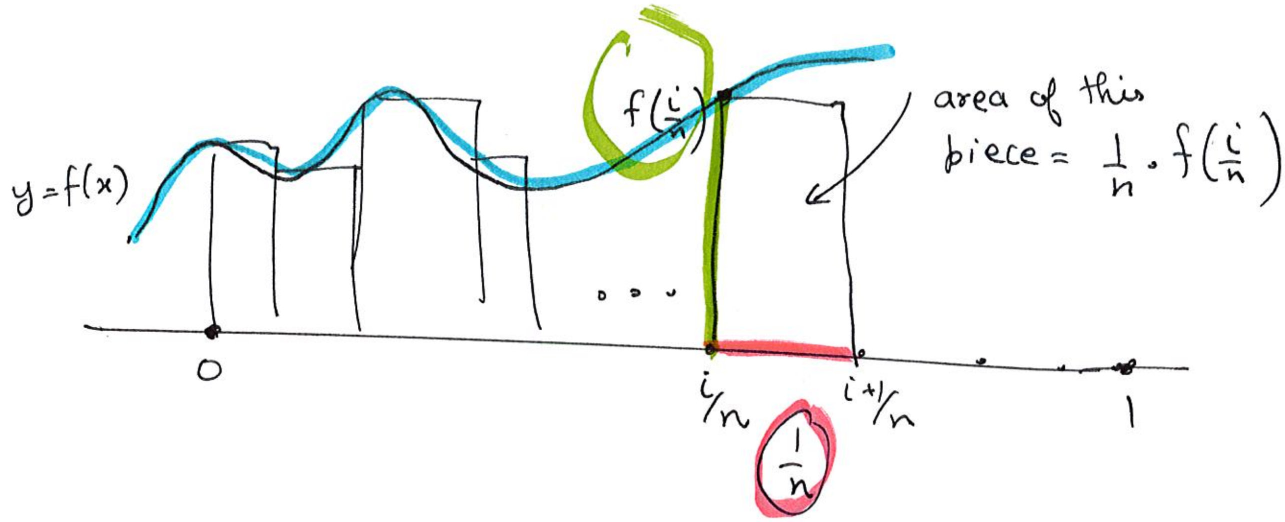
hence, it is representing some integral

Recall : $$\int_a^b f(x)\cdot dx = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\cdot f\left(\frac{i}{n}\right)$$

break [0,1] into n-terms

- $\frac{1}{n}$ = length of the interval

- $f\left(\frac{i}{n}\right)$ = height of the function at $\frac{i}{n}$



$$\text{total area of all pieces} = \sum_{i=1}^{n} \boxed{\frac{1}{n} \cdot f\left(\frac{i}{n}\right)}$$

$$\text{shrink breadth to } 0 = \underbrace{\lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{n} f\left(\frac{i}{n}\right)}$$

Compare to the question:

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{n} \cdot \left(\frac{i}{n}\right)^2$$

$\Rightarrow \quad f(x) = x^2$

$\Rightarrow$ Riemann Sum $= \displaystyle\int_0^1 x^2 \cdot dx$

Q. What are the upper and lower ~~bou~~ limits into the integral ?

$\boxed{f\left(\frac{1}{n}\right)}$ $\boxed{f\left(\frac{n}{n}\right)}$

$$\sum_{i=1}^{n} \frac{1}{n}\left(\frac{i}{n}\right)^2 = \frac{1}{n}\cdot\left(\frac{1}{n}\right)^2 + \frac{1}{n}\cdot\left(\frac{2}{n}\right)^2 + \cdots + \frac{1}{n}\cdot\left(\frac{n}{n}\right)^2$$

Starts at $\frac{1}{n}$ $\xrightarrow{\lim_{n\to\infty}}$ $\underline{0}$

ends at $\frac{n}{n}$ $\xrightarrow{\lim_{n\to\infty}}$ $\underline{1}$

- look at what inside $f(\_\_\_)$ at the lowest value of i and the largest value of i
- take $\lim_{n\to\infty}$.

Back to integrals:

- Compute $\int_0^1 (e^x - 2\sin x).\, dx$

Ans: $= \int_0^1 e^x\, dx - \int_0^1 2\sin x\, dx$

$= \int_0^1 e^x dx - 2\int_0^1 \sin x\, dx$

$= e^x \Big|_0^1 - 2 \cdot (-\cos x)\Big|_0^1$

$= \left[e^1 - e^0\right] - 2\left[-\cos(1) - (-\cos(0))\right]$

$= [e^1 - 1] - 2[-\cos 1 + 1]$.

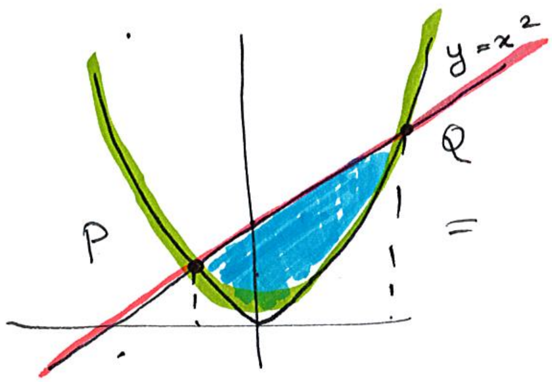
- when finding $\int_a^b f(x)\, dx$ we typically don't put "+c"

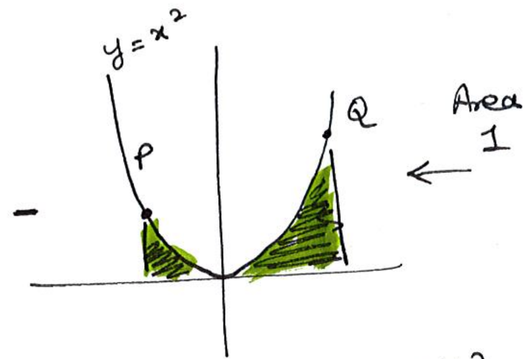
$\downarrow$

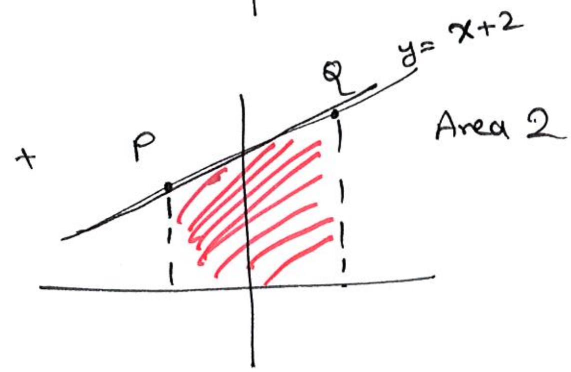
we are given both the upper and lower bounds.

Q. Compute area bounded by the curves $y = x^2$ and $y = x + 2$ | area = integrals

A:







from graph

Required area = Area 2 − Area 1

$$= \int_{?}^{?} (x+2) \cdot dx - \int_{?}^{?} x^2 \, dx$$

- Find P, Q : ~~eq~~ points of intersection of $y = x^2$ and $y = x + 2$

- equating the two $\quad x^2 = x + 2$

$$\Rightarrow x^2 - x - 2 = 0$$

$$\Rightarrow (x+1)(x-2) = 0$$

$$\Rightarrow x + 1 = 0 \quad \text{or} \quad x - 2 = 0$$

$\Rightarrow x = -1$ or $x = 2$

$\uparrow$ x-coordinate of P

$\uparrow$ x-coordinate of Q

Plugging these back in

Required area $= \displaystyle\int_{-1}^{2} (x+2)\,dx - \int_{-1}^{2} x^2\,dx$

$$= \int_{-1}^{2} x\,dx + \int_{-1}^{2} 2\,dx - \int_{-1}^{2} x^2\,dx$$

(Power rule)

$$= \frac{x^2}{2}\Big|_{-1}^{2} + 2x\Big|_{-1}^{2} - \frac{x^3}{3}\Big|_{-1}^{2}$$

$$= \left(\frac{2^2}{2} - \frac{(-1)^2}{2}\right) + \left(2\cdot 2 - 2(-1)\right) - \left(\frac{2^3}{3} - \frac{(-1)^3}{3}\right)$$

Notation :

- $\int_a^b f(x)\,dx$ is called a definite integral

area

- $\int f(x)\,dx$ is called an indefinite integral

as we are not given any bounds, is just a formal symbol.

$\int f(x)\,dx$ is the general anti-derivate.

eg : $\int \frac{1}{x}\,dx = \ln x + c$

- in indefinite integrals, we keep the "+c"
- the internal variable does not go away for indefinite integrals.
  $\parallel$
  (antiderivative)

$\int \frac{1}{x}\,dx = \ln x + c$ | $\int_1^2 \frac{1}{x}\,dx = \ln 2 - \ln 1$

5.3 Second version of fundamental theorem of calculus

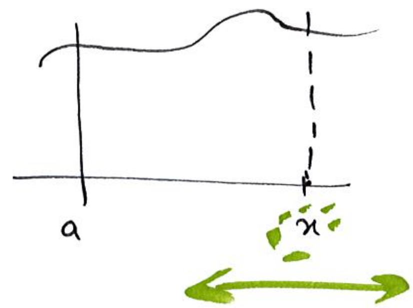
- $\displaystyle\int_a^b f(x)\,dx$ = number

- Switch the internal variable to t

$$\int_a^b f(t)\,dt$$

- Create a function

$$F(x) = \int_a^x f(t)\,dt$$



by allowing the <u>upper</u> bound to vary along the x-axis.

eg:

$$\int_1^x \frac{1}{t}\cdot dt = \ln t \Big|_1^x$$

$$= \ln x - \ln 1$$

$$\int_1^x \frac{1}{t}\,dt = \ln x.$$

$\displaystyle\int_1^x f(t)\,dt$ = area under $y = f(t)$ from 1 to x

Theorem: If $F(x) = \int_a^x f(t)\,dt$

then $F'(x) = f(x)$. ■

eg: $\int_1^x \frac{1}{t}\,dt = \ln x$

$(\ln x)' = \frac{1}{x}$

. This is useful only for finding derivatives of integral

Q: Find derivative of $\int_{10}^{x} \arctan(t)\,dt$

A $F(x) = \int_{10}^{x} \arctan(t)\,dt$

$\Rightarrow F'(x) = \arctan x$ by the fundamental theorem eg: find derivative of $\int_{10}^{x} t\, dt$

By fundamenal theorem	By actual computation
$F(x) = \int_{10}^{x} t\, dt$	$F(x) = \int_{10}^{x} t\, dt$
$f'(x) = x$	$= \left.\frac{t^2}{2}\right\vert_{10}^{x}$
	$= \frac{x^2}{2} - \frac{10^2}{2}$
	$F'(x) = x - 0 = x$

• Q: Find derivative of $\int_{10}^{\sin x} \ln(t)\, dt$.

A: $F(x) = \int_{10}^{x} \ln(t)\, dt$, $F'(x) = \ln x$

$$\int_{10}^{\sin x} \ln(t)\, dt = F(\sin x)$$

$$\Rightarrow \left(\int_{10}^{\sin x} \ln(t)\, dt\right)' = \left(F(\sin x)\right)'$$

$= F'(\sin x) \cdot \sin' x$ by Chain Rule

$= \ln(\sin x) \cdot \cos x$ ∎

<u>Summary</u>: derivative of $\displaystyle\int_a^{g(x)} f(t) \cdot dt = f(g(x)) \cdot g'(x)$

<u>note</u>: lower bound is a constant.

- Instead of memorising this, remember to use chain rule.

- Back to geometry of integrals:

$$\int_a^a f(x)\,dx = 0$$ as lower & upper bounds are the same

~~we~~ when $a > b$, we define

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

when we switch bounds, we flip signs.

<u>Why</u>: So that $\displaystyle\int_a^b f(x)dx + \int_b^a f(x)dx = \int_a^a f(x)dx = 0$

Q. Find derivative of $\displaystyle\int_x^{10} \ln t \, dt$

$\underline{\underline{A}}$:

$$\left(\int_x^{10} \ln t \; dt\right)' = \left(-\int_{10}^{x} \ln t \; dt\right)'$$

$$= -\left(\int_{10}^{x} \ln t \; dt\right)'$$

$$= -\ln x$$

$\underline{\underline{Q}}$ Find derivative of $\displaystyle\int_{\sin x}^{10} \ln t \; dt$

$\underline{\underline{A}}$

$$\left(\int_{\sin x}^{10} \ln t \; dt\right)' = -\left(+\int_{10}^{\sin x} \ln t \; dt\right)'$$

$$= -\ln(\sin x) \cdot \cos x$$

<u>Summary</u> :

$$\left(\int_{h(x)}^{a} f(t) \; dt\right)' = -f(h(x)) \cdot h'(x)$$

Q. Find derivative of $\displaystyle\int_{x^2}^{\sin x} e^{t}\, dt$.

A: $$\left(\int_{x^2}^{\sin x} e^{t}\, dt\right)' = \left(\int_{x^2}^{a} e^{t}\, dt + \int_{a}^{\sin x} e^{t}\, dt\right)'$$

$$= \left(-\int_{a}^{x^2} e^{t}\, dt\right)' + \left(\int_{a}^{\sin x} e^{t}\, dt\right)'$$

• Fundamental theorem of calculus + Chain Rule

$$= -\,e^{(x^2)} \cdot (x^2)' + e^{(\sin x)} \cdot \sin' x$$

$$= -\,e^{(x^2)} \cdot 2x + e^{\sin x} \cdot \cos x$$

$$\left(\int_{g(x)}^{h(x)} f(t) \cdot dt\right)' = f(h(x)) \cdot h'(x) - f(g(x)) \cdot g'(x)$$

By Fundamental theorem + Chain Rule

Fundamental theorem of Calculus:

2 versions

Version 1:

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

where F is an antiderivative of f.

- we use this for finding integrals.

Version 2:

$$\boxed{\left(\int_a^x f(t)\,dt\right)' = f(x)}$$

. do not use same variable internally and externally

- used for finding derivatives of integrals

$$\boxed{\left(\int_{g(x)}^{h(x)} f(t)\,dt\right)' = f(h(x))\cdot h'(x) - f(g(x))\cdot g'(x)}$$

(In notes - Please add ( )' to the left hand side for the fundamental thm)

Sorry