Recall: $\mathcal{C}at$ = 2 category categories, functors, natural transformations

$\mathcal{K}$ - an ∞-cosmos,
$h\mathcal{K}$ = 2 category ∞-categories, ∞-functors, ∞-natural transformations

- In 2-categories :

2 cells compose vertically, horizontally, middle four interchange, pasting diagrams

$\alpha \circ \beta$

$$\begin{tikzcd} \bullet \arrow[r, bend left=60, ""{name=U, below}] \arrow[r, ""{name=M}] \arrow[r, bend right=60, ""{name=D, above}] & \bullet \arrow[Rightarrow, from=U, to=M] \arrow[Rightarrow, from=M, to=D] \end{tikzcd}$$

$\alpha * \beta$

$$\begin{tikzcd} \bullet \arrow[r, bend left=60, ""{name=U1, below}] \arrow[r, ""{name=D1, above}] & \bullet \arrow[r, bend left=60, ""{name=U2, below}] \arrow[r, ""{name=D2, above}] & \bullet \arrow[Rightarrow, from=U1, to=D1] \arrow[Rightarrow, from=U2, to=D2] \end{tikzcd}$$

$$\begin{tikzcd} \bullet \arrow[r, bend left=60, ""{name=U1, below}] \arrow[r, ""{name=M1}] \arrow[r, bend right=60, ""{name=D1, above}] & \bullet \arrow[r, bend left=60, ""{name=U2, below}] \arrow[r, ""{name=M2}] \arrow[r, bend right=60, ""{name=D2, above}] & \bullet \arrow[Rightarrow, from=U1, to=M1] \arrow[Rightarrow, from=M1, to=D1] \arrow[Rightarrow, from=U2, to=M2] \arrow[Rightarrow, from=M2, to=D2] \end{tikzcd}$$

whiskering

$$\begin{tikzcd} X \arrow[r, "h"] & A \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, "k"] & C \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[rr, bend left=40, "hfk"] \arrow[rr, bend right=40, "hgk"'] \arrow[rr, phantom, "h\alpha k"] & & C \end{tikzcd}$$

Def^n: Adjunction : adjunction between ∞-category in a fixed ∞-cosmos $\mathcal{K}$:
- ∞-categories A, B
- ∞-functors $A \xrightarrow{u} B$, $B \xrightarrow{f} A$
- ∞-nat tran

unit

$$\begin{tikzcd} B \arrow[rr, equal] \arrow[dr, "f"'] & \arrow[d, phantom, "\eta\Downarrow"] & B \\ & A \arrow[ur, "u"'] & \end{tikzcd}$$

counit

$$\begin{tikzcd} & B \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\varepsilon"] & \\ A \arrow[ur, "u"] \arrow[rr, equal] & \, & A \end{tikzcd}$$

triangle identity

1)

$$\begin{tikzcd} B \arrow[rr, equal] \arrow[dr, "f"'] & \arrow[d, phantom, "\eta"] & B \arrow[dr, "f"] \arrow[d, phantom, "\varepsilon"] & \\ & A \arrow[ur] \arrow[rr, equal] & \, & A \end{tikzcd} \quad = \quad \begin{tikzcd} B \arrow[r, bend left=40, "f"] \arrow[r, bend right=40, "f"'] \arrow[r, phantom, "\parallel id_f"] & A \end{tikzcd}$$

2)

$$\begin{tikzcd} & B \arrow[rr, equal] \arrow[dr] \arrow[d, phantom, "\varepsilon"] & \arrow[d, phantom, "\eta"] & B \\ A \arrow[ur] \arrow[rr, equal] & \, & A \arrow[ur] & \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, bend left=40, "u"] \arrow[r, bend right=40, "u"'] \arrow[r, phantom, "\parallel id_u"] & B \end{tikzcd}$$

We say "$f \dashv u$" $\rightarrow$ f is left-adjoint to u.

Prop: Adjunctions compose.

$$\begin{tikzcd} A \arrow[r, bend right, "u"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "f"'] \arrow[r, bend right, "u'"'] \arrow[r, phantom, "\perp"] & C \arrow[l, bend right, "f'"'] \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} A \arrow[r, bend right, "uu'"'] \arrow[r, phantom, "\perp"] & C \arrow[l, bend right, "ff'"'] \end{tikzcd}$$

Prop: Uniqueness of adjoints

1) $f \dashv u$, $f' \dashv u \Rightarrow f \cong f'$.
2) $f \cong f'$, $f \dashv u \Rightarrow f' \dashv u$

Proof: 1)

$$\begin{tikzcd} A \arrow[r, bend right, "u"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend right, "{f,f'}"'] \end{tikzcd}$$

units : $\mathrm{id}_B \overset{\eta}{\Longrightarrow} uf$, $\mathrm{id}_B \overset{\eta'}{\Longrightarrow} uf'$

counits : $fu \overset{\varepsilon}{\Longrightarrow} \mathrm{id}_A$, $f'u \overset{\varepsilon'}{\Longrightarrow} \mathrm{id}_A$

Construct isos :

$$\begin{tikzcd} B \arrow[r, equal] \arrow[d, "f'"'] & B \arrow[dr, "f"] \arrow[d, phantom, "\varepsilon"] & \\ A \arrow[ur] \arrow[rr] \arrow[u, phantom, "\eta'"] & & A \end{tikzcd} \qquad \begin{tikzcd} B \arrow[rr, equal] \arrow[dr, "f"'] & \arrow[d, phantom, "\eta"] & B \arrow[dr, "f'"] \arrow[d, phantom, "\varepsilon'"] & \\ & A \arrow[ur] \arrow[rr, equal] & & A \end{tikzcd}$$

Prop: An equiv of ∞-categories :

$$\begin{tikzcd} A \arrow[r, bend left, "g"] & B \arrow[l, bend left, "f"] \end{tikzcd} + \begin{tikzcd} A \arrow[r, bend left, "fg"] \arrow[r, bend right, equal] \arrow[r, phantom, "\cong \Downarrow \alpha"] & A \end{tikzcd} + \begin{tikzcd} B \arrow[r, bend left, equal] \arrow[r, bend right, "gf"'] \arrow[r, phantom, "\cong \Downarrow \beta"] & B \end{tikzcd}$$

can be promoted to an adjoint equivalence by modifying just one of the two cells.

A be an ∞-category. $A^2 = \infty$-category of arrows in A

we've maps

$$\begin{tikzcd} \cdot \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & A^2 \arrow[d, "\text{codomain}"] \\ A^2 \arrow[r, two heads, "\text{domain}"'] & A \end{tikzcd}$$

both isofib

Pullback = composable arrows

$$= A^2 \underset{A}{\times} A^2$$

$\exists$ adjunctions

$$\begin{tikzcd} A^{\mathbb{2}} \times_A A^{\mathbb{2}} \arrow[r, "\circ"] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & A^{\mathbb{2}} \arrow[l, bend right=50, "\text{pair with id domain}"'] \arrow[l, bend left=50, "\text{pair with id domain}"] \end{tikzcd}$$

Proof: $\mathbb{2} = \cdot \to \cdot$ $\qquad$ $\mathbb{3} = \cdot \to \cdot \to \cdot$

we have adjunctions

$$\begin{tikzcd} \mathbb{2} \arrow[r, "\delta^1"] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathbb{3} \arrow[l, bend right=50, "s^1"'] \arrow[l, bend left=50, "s^0"] \end{tikzcd}$$

Prove this.