Bott Periodicity

$G \subseteq U(n)$ compact Lie subgroup.

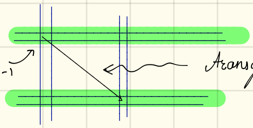
Th^m (Milnor) Existense of universal G-bundle $EG \to BG$.

Characteristic Classes:

if $U(n) \to E \to B$ is principal, look at

$$\underset{\overset{\shortparallel}{S^{2n-1}}}{U(n)/U(n-1)} \to E/U(n-1) \to B$$ this is a fiber bundle,

In Serre SS



i_{2n-1} transgression $d(i_{2n-1})$ is the Euler class e_{2n} of E

We know $H^*(BU(n);\mathbb{Z})$ is polynomial in n-variables, we are trying to identify these.

Given $\underset{\xi_1}{U(n) \to E_n \to B}$ and $\underset{\xi_2}{U(m) \to E_m \to B}$. Let $\xi_1 \times \xi_2 := \Delta^*(E_n \times E_m) \times_{U(n)\times U(m)} U(n+m)$.

Th^m. Chern classes exist & are unique. $c(\xi_1 \times \xi_2) = c(\xi_1) \cup c(\xi_2)$.

Proof. $U(n) \hookrightarrow \xi \to B$, $U(1) \hookrightarrow S^\infty \to \mathbb{C}P^\infty$

Consider the hom: $\boxtimes : U(n) \times U(1) \longrightarrow U(n)$

$$(A, z) \longmapsto zI_n \cdot A$$

Consider $U(n)$ bundle over $B \times \mathbb{C}P^\infty$ $\quad (\xi \times \gamma) \times_{U(n)\times U(1)} U(n) \longrightarrow B \times \mathbb{C}P^\infty =: \xi \boxtimes \gamma$

Let $c_n(\xi \boxtimes \gamma) \in H^{2n}(B \times \mathbb{C}P^\infty)$ By Künneth,

$$\overset{\shortparallel}{x^n \cdot H^0(B) + x^{n-2} H^2(B) + \cdots + 1 \cdot H^{2n}(B)}$$

i.e. $c_n(\xi \boxtimes \gamma) = c_0(\xi)\, x^n + \cdots + c_n(\xi)\, x^0$

Define $\rightsquigarrow$ are the Chern Classes.

1. $c_i = 0$ for $i > n$
2. Natural
3. To find $c_0(\xi)$ pull back ξ to a point
   Restriction of $\xi \boxtimes \eta$ to $pt \times \mathbb{C}P^\infty$ is $\eta^{\times n}$ and $e(\eta^{\times n}) = e(\eta)^n = x^n$
   $\Rightarrow c_0(\xi) = 1$.
4. Using $(\xi \times \eta) \boxtimes \gamma = (\xi \boxtimes \gamma) \times (\eta \boxtimes \gamma)$
   $e((\xi \times \eta) \boxtimes \gamma) = e(\xi \boxtimes \gamma) \smile e(\eta \boxtimes \gamma)$
   $\Rightarrow c(\xi \times \eta) = c(\xi)\, c(\eta)$

5. To find c_n include $B\times *$ into $B\times \mathbb{CP}^\infty$.

Uniqueness:

Consider $\theta: BU(n-1)\times BU(1) \longrightarrow BU(n)$ with fiber $U(n)/U(n-1)\times U(1) = \mathbb{CP}^{n-1}$

$$\begin{array}{ccc} BU(n-1)\times BU(1) & \longrightarrow & BU(n) \\ \| & & \| \\ Gr_{n-1}(\mathbb{C}^\infty)\times \mathbb{CP}^\infty & & Gr_n(\mathbb{C}^\infty) \end{array}$$

gives us
$$\begin{array}{ccccc} \mathbb{CP}^{n-1} & \hookrightarrow & BU(n-1)\times BU(1) & \longrightarrow & BU(n) \\ & & \| & & \\ \{L\} & \hookrightarrow & \left\{\begin{array}{c} V^n\subseteq \mathbb{C}^\infty \\ L\subseteq V \end{array}\right\} & \longrightarrow & \{V^n\} \end{array}$$

Consider Serre SS,

as every thing is in even degree there are no-differentials.

$\Rightarrow \quad H^*(BU(n)) \hookrightarrow H^*(BU(n-1))\otimes H^*(BU(1))$ by the edge homomorphism

$\Rightarrow \quad H^*(BU(n)) \hookrightarrow [H^*(BU(1))]^{\otimes n} \rightsquigarrow \Rightarrow$ Uniqueness

$U(n) \longleftarrow U(1)^{\otimes n} : \Delta$

$$\Delta^*(c(n)) = (1+x_1)(1+x_2)\cdots(1+x_n)$$
$$= (1+\sigma_1+\sigma_2+\cdots+\sigma_n) \rightsquigarrow \text{elementary Symmetric } f^n s$$

$$\mathbb{Z}[x_1,\dots x_n]^{\Sigma_n} = [H^*(BU(1))^{\otimes n}]^{\Sigma_n} = \mathbb{Z}[\sigma_1,\dots\sigma_n]$$

$\Rightarrow$ Chern classes in $H^*(BU(n))$ map under Δ^* to all symmetric $f^n s$.

Next we want to show $H^*(BU(n)) \subseteq \left((H^*(BU(1))^{\otimes n}\right)^{\Sigma_n}$

i.e. need to show $\sigma^*\Delta^* = \Delta^*$ where $\sigma: U(1)^{\times n} \longrightarrow U(1)^{\times n}$ is a permutation.

$\exists\ \tilde{\sigma}\in U(n)$ s.t. $\Delta\circ\sigma = (\tilde{\sigma}^{-1}\cdot - \cdot\tilde{\sigma})\cdot\Delta =: c_{\tilde{\sigma}}\cdot\Delta$

But as $U(n)$ is connected $\quad c_{\tilde{\sigma}} \simeq id \quad \Rightarrow \quad \Delta\cdot\sigma \simeq \Delta$

$$\Rightarrow \quad H^*(BU(n)) = \mathbb{Z}[c_1,\cdots c_n]$$

Th^m: $\quad H^*(BO(n);\mathbb{Z}/2) = \mathbb{Z}/2[w_1,\cdots w_n]$

Bott Periodicity

$H^*(BU(n);\mathbb{Z}) = \mathbb{Z}[c_1 \cdots c_n]$

$H^*(BO(n);\mathbb{Z}/2) = \mathbb{Z}[\omega_1 \cdots \omega_n]$

$H^*(U(n);\mathbb{Z}) = \Lambda(x_1, x_3, \cdots, x_{2n-1})$ with each x_i primitive

$\rightarrow$ All classes are natural in n i.e. consider $U(n) \hookrightarrow U(n+1)$, $O(n) \hookrightarrow O(n+1)$, these inclusions maps ω_i to ω_i and c_i to c_i, x_i to x_i

Let $U = U(\infty) = \varinjlim U(n)$, $O = \varinjlim O(n)$

$BU = \varinjlim BU(n)$, $BO = \varinjlim BO(n)$

and by a Mittag-Leffler argument,

$H^*(BU;\mathbb{Z}) = \mathbb{Z}[c_1 \cdots]$ $\quad H^*(U) = \mathbb{Z}[x_1, x_3, \cdots]$ $\quad H^*(BO, \mathbb{Z}) = \Lambda[\omega_1, \omega_2, \cdots]$

Geometrically: BU: $\quad H = \bigoplus_{i=1}^{\infty} \mathbb{C}e_i \ \bigoplus_{i=1}^{\infty} \mathbb{C}f_i$

$$\begin{tikzcd}
Gr(0,0) \arrow[r] \arrow[d, hook] & Gr(1, \mathbb{C}f_1) \arrow[r, hook] \arrow[d, hook] & Gr(2, \mathbb{C}f_1 \oplus \mathbb{C}f_2) \arrow[r, hook] \arrow[d, hook] & \cdots \\
G(0, \mathbb{C}e_1) \arrow[r, hook] \arrow[d, hook] & Gr(1, \mathbb{C}f_1 + \mathbb{C}e_1) \arrow[r, hook] \arrow[d, hook] & Gr(2, \mathbb{C}f_1 + \mathbb{C}f_2 + \mathbb{C}e_1) \arrow[d, hook] & \\
Gr(0, \mathbb{C}e_1 + \mathbb{C}e_2) \arrow[r, hook] \arrow[d] & Gr(1, \mathbb{C}f_1 + \mathbb{C}e_1 + \mathbb{C}e_2) \arrow[r, hook] \arrow[d] & Gr(2, \mathbb{C}f_1 + \mathbb{C}f_2 + \mathbb{C}e_1 + \mathbb{C}e_2) \arrow[d] & \\
BU(0) \arrow[r] & BU(1) \arrow[r] & BU(2) \arrow[r, "\lim"] & BU
\end{tikzcd}$$

$\xrightarrow{\lim}$

$\rightarrow$ BO, BU are H-spaces via the $U(n) \times U(m) \xrightarrow{\mu} U(n+m)$ maps.

This product is homotopy commutative

$$\begin{tikzcd}
BU(i) \times BU(j) \arrow[r, "\mu"] \arrow[d, "\tau"] & BU(i+j) \\
BU(j) \times BU(i) \arrow[ur, "\mu"'] &
\end{tikzcd}$$

commutative upto homotopy

$H^*(BU(n);\mathbb{Z}) \underset{\Delta}{\cong} \left(H^*(BU(1),\mathbb{Z})^{\otimes n}\right)^{\Sigma_n}$

true for $\mathbb{Z}$ replaced by $\mathbb{Z}/p$

$\Rightarrow$ Dualizing

$$\begin{tikzcd}
H_*(BU(1);\mathbb{Z})^{\otimes n} \arrow[r, "\Delta"] \arrow[dr] & H_*(BU(n);\mathbb{Z}) \\
& H_*(BU(1);\mathbb{Z})^{\otimes n} / \Sigma_n \arrow[u]
\end{tikzcd}$$

By comparing ranks & freeness this map is an isomorphism.

But the map $\Delta(BU\times\cdots\times BU) = \mu(BU(1), BU(1), \cdots, BU(1))$

Prop: $H_*(BU;\mathbb{Z})$ is the symmetric algebra on $H_*(BU(1);\mathbb{Z})$ under the map $BU(1)\hookrightarrow BU$.

$$H_*(BU;\mathbb{Z}) \cong \mathrm{Sym}(\mathbb{Z}\langle 1; b_1; b_2; b_3; \dots\rangle) \quad |b_i| = i$$
$$= \mathbb{Z}[b_1, b_2, \dots] \quad \text{monomials of length } k \text{ come from } H_*(BU(k);\mathbb{Z})$$
$$H_*(BO;\mathbb{Z}) = \mathbb{Z}/2[d_1, d_2, \dots]$$

$$H_*(U(n);\mathbb{Z}) = \Lambda(x_1 \dots x_{2n-1})$$

Generating complex for $U(n)$:

$$S^1\times \mathbb{CP}^{n-1} \longrightarrow U(n)$$
$$(\lambda, L) \longmapsto \left(\begin{array}{c|c} \lambda & 0 \\ \hline 0 & I_{L^\perp}\end{array}\right)$$

Factors through $S^1\times\mathbb{CP}^{n-1}/1\times\mathbb{CP}^{n-1} \simeq \Sigma(\mathbb{CP}^{n-1}_+)$

gives a commutative diagram:

$$\begin{array}{ccccc} \Sigma\mathbb{CP}^{n-1}_+ & \hookrightarrow & \Sigma\mathbb{CP}^n_+ & \longrightarrow & S^{2n+1} \\ \downarrow & & \downarrow & & \| \\ U(n) & \hookrightarrow & U(n+1) & \longrightarrow & S^{2n+1} \end{array} \Big\} \text{ Prove}$$

Cor: x_i are image of $H_*(\Sigma\mathbb{CP}^{n-1}_+;\mathbb{Z})$

And true also for $n=\infty$.

Note: $\exists\, S^1\times\mathbb{CP}^{n-1} \longrightarrow SU(n)$

$$(\lambda, L) \longmapsto \left(\begin{array}{c|c} \lambda & 0 \\ \hline 0 & Id_{L^\perp}\end{array}\right)\left(\begin{array}{c|c} \lambda^{-1}e_1 & \\ \hline & Id\end{array}\right)$$

Factors through

$S^1\times\mathbb{CP}^{n-1}/S^1\times 1 \cup 1\times\mathbb{CP}^{n-1}$ so $\Sigma\mathbb{CP}^{n-1}$ is a generating complex.

By Eilenberg Moore, $H^*(\Omega SU(n);\mathbb{Z}) = \Gamma(y_1, y_2, \dots, y_{n-1}) \quad |y_i| = 2i$ as Hopf algebras

$$\Rightarrow H_*(\Omega SU(n);\mathbb{Z}) = \mathbb{Z}[x_1 \dots x_{n-1}] \quad |x_i| = 2i$$

Because $\Omega SU(n) = \Omega^2 BSU(n)$ H_* is commutative, then we argue by freeness. Think ↗

Generating complex: $\Sigma\mathbb{CP}^{n-1} \xrightarrow{g} SU(n)$

gives $\mathbb{CP}^{n-1} \longrightarrow \Omega SU(n)$ ← Because of transgressions

Cor: $H_*(\Omega SU;\mathbb{Z}) \cong \mathbb{Z}[x_1, x_2, \dots]$ x_i image of $H_*(\mathbb{C}P^\infty)$.

Now we are lacking a commutative diagram

Th^m: $\Omega SU \cong BU$ as H-spaces

(Bott Periodicity) $\Omega^2(\mathbb{Z}\times BU) \cong \mathbb{Z}\times BU$

$$\begin{tikzcd} BU \arrow[rr, "\beta"] & & \Omega SU \\ & \mathbb{C}P^\infty \arrow[ul, "\tilde{f}"] \arrow[ur, "\tilde{g}"'] & \end{tikzcd}$$

β – H space map

$\hat{f}, \hat{g}$ generating complex maps

$$\Sigma BU(k) \longrightarrow SU$$

$$(\lambda; V) \longmapsto \left(\begin{array}{c|c} \lambda_V & 0 \\ \hline 0 & Id_{V^\perp} \end{array}\right)\left(\begin{array}{c|c} \lambda^{-1}_{e_1\cdots e_k} & 0 \\ \hline 0 & Id_{e_{k+1}} \end{array}\right)$$

Check commutativity, H-space structure.