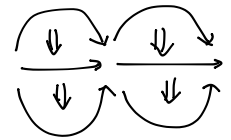
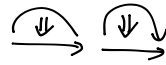
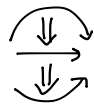


Recall: $\text{Cat} = 2\text{-category}$. categories, functors, natural transformations

\mathcal{K} - an ∞ -cosmos,
 $\mathcal{H}\mathcal{K} = 2\text{-category}$ ∞ -categories, ∞ -functors, ∞ -natural transformations

- In 2-categories:

2 cells compose vertically, horizontally, middle four interchange, pasting diagrams



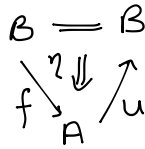
whiskering

$$X \xrightarrow{r} A \begin{array}{c} \xrightarrow{f} \\ \downarrow \alpha \\ \xrightarrow{g} \end{array} B \xrightarrow{p} C = X \begin{array}{c} \xrightarrow{r f p} \\ \downarrow r \alpha p \\ \xrightarrow{r g p} \end{array} C$$

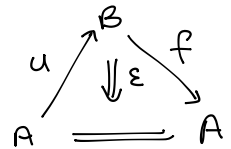
Defⁿ: Adjunction: adjunction between ∞ -category in a fixed ∞ -cosmos \mathcal{K} :

- ∞ -categories A, B
- ∞ -functors $A \xrightarrow{u} B$, $B \xrightarrow{f} A$
- ∞ -nat tran

unit



counit



Triangle identity

1)

2)

We say " $f \dashv u$ " \rightarrow f is left-adjoint to u .

Prop: adjunctions compose.

$$\begin{array}{ccc} A & \xleftarrow{f} B & \xleftarrow{f'} C \\ \text{\tiny \perp} & & \text{\tiny \perp} \\ A & \xrightarrow{u} B & \xrightarrow{u'} C \end{array} \Rightarrow \begin{array}{ccc} A & & C \\ \text{\tiny \perp} & & \\ A & \xrightarrow{uu'} & C \end{array}$$

Prop: Uniqueness of adjoints

$$\begin{aligned} 1) & f \dashv u, f' \dashv u \Rightarrow f \cong f' \\ 2) & f \cong f', f \dashv u \Rightarrow f' \dashv u \end{aligned}$$

Proof: 1)

$$\begin{array}{ccc} A & \xleftarrow{f, f'} B \\ \text{\tiny \perp} & & \\ A & \xrightarrow{u} B \end{array}$$

units : $\text{id}_B \xrightarrow{\eta} uf, \text{id}_B \xrightarrow{\eta'} uf'$

counits : $fu \xrightarrow{\varepsilon} \text{id}_A, f'u \xrightarrow{\varepsilon'} \text{id}_A$

Construct isos :

$$\begin{array}{ccc} B & \xleftarrow{f} B \\ f' \downarrow \eta' & \nearrow \varepsilon & \downarrow f \\ A & \xrightarrow{\quad} & A \end{array}$$

$$\begin{array}{ccc} \overline{f} & \xrightarrow{\eta} & \overline{f'} \\ \downarrow & \nearrow \varepsilon' & \downarrow \\ f & \xrightarrow{\quad} & f' \end{array}$$

Prop: An equiv of ∞ -categories :

$$\begin{array}{ccc} A & \xrightleftharpoons[g]{g} B & + \quad A \xrightleftharpoons[\cong]{fg} A + \quad B \xrightleftharpoons[\cong]{gf} B \end{array}$$

can be promoted to an adjoint equivalence by modifying just one of the two cells.

A be an ∞ -category. $A^2 = \infty$ -category of arrows in A

we've maps

$$\begin{array}{ccc} \bullet & \dashrightarrow & A^2 \\ \downarrow \wr & & \downarrow \text{codomain} \\ A & \dashrightarrow & A \\ & \text{domain} & \end{array}$$

both isofib

Pullback = composable arrows

$$A^2 \times_A A^2$$

\exists adjunctions

$$\begin{array}{ccc}
 & \xleftarrow{\text{pair with id domain}} & \\
 A^2 \times_A A^2 & \xrightarrow[\perp]{\perp} & A^2 \\
 & \xleftarrow{\text{pair with id domain}} &
 \end{array}$$

Proof: $\mathcal{A} = \cdot \rightarrow \cdot$ $\mathcal{B} = \cdot \rightarrow \cdot \rightarrow \cdot$

we have
adjunctions

$$\begin{array}{ccc}
 & \xleftarrow{s'} & \\
 \mathcal{A} & \xrightarrow[\perp]{\perp s'} & \mathcal{B} \\
 & \xleftarrow[s^\circ]{\perp} &
 \end{array}$$

Prove this.