How to glue donuts

2 day class - Imagining higher dimensional objects

Day 1 Making friends with the 3-sphere

Day 2 donuts

Use the chat window without restraint

except when I ask a question

for answering questions we'll be using

CHAT BLASTS

CHAT BLASTS

1. when I ask a question, type your answer in the chat window, but

   do not press enter

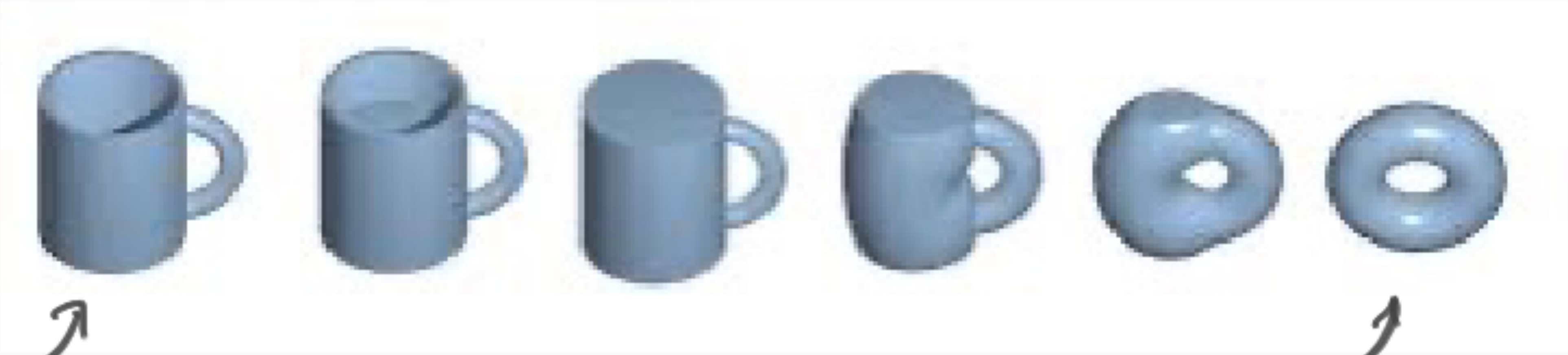
2. when I say 3-2-1 go everyone presses enter at the same time

Warning: I'm a topologist

Everything

1. is superflexible
2. can pass through objects

Topologists only care about "holes"



coffee mug = donut

Spheres

2-dim sphere $= \{(x,y,z) : x^2+y^2+z^2=1\}$

It is only the surface

Describe

1-sphere

0-sphere

1-sphere

$\{(x,y) : x^2+y^2=1\}$

circle $= S^1$

0-sphere

$\{(x) : x^2=+1\}$

$\parallel$

$x=1$ and $x=-1$

$S^0 = *$ $\quad *$

you are here

0-deg of freedom

Goal: Understand $S^3 = \{(x,y,z,t) : x^2+y^2+z^2+t^2=1\}$

3 - points = 0 - dim manifold

n - points = 0 - dim manifold

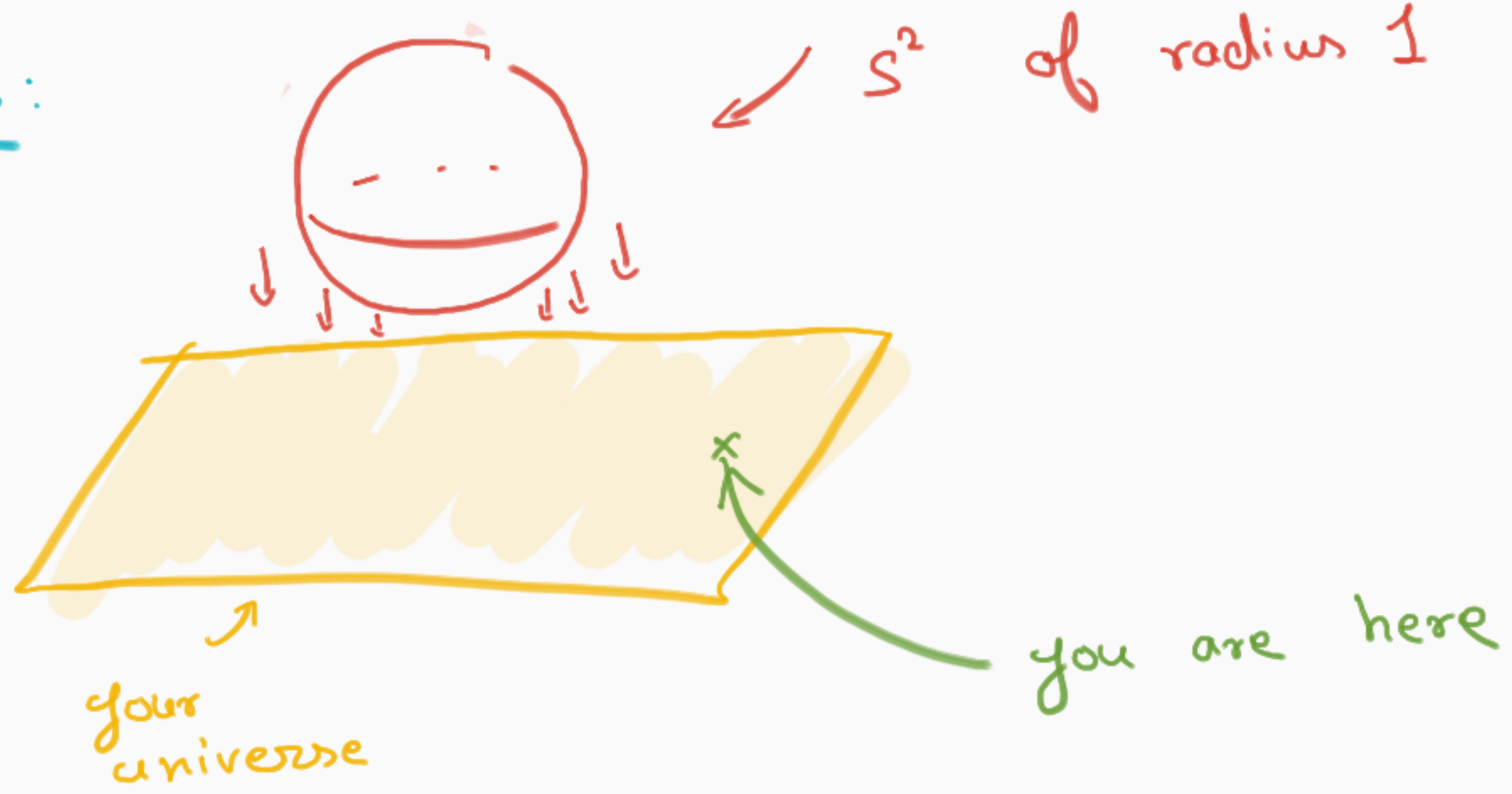
2 - points = 0 - dim sphere

Circle = S^1 = 1-sphere = 1-dim manifold

S^2 = 2 - " - = 2 - "

S^3 = 3 - sphere = 3-dim manifold

Flatlands:



Q. What would a 3-sphere passing through our universe look like?

More about 3-spheres:

inequality

2-ball: $\{(x,y) : x^2+y^2 \le 1\}$

$\boxed{\text{ball = filled}}$

(manifold with a boundary)

3-ball $= \{(x,y,z) : x^2+y^2+z^2 \le 1\}$

1-ball $= \{x : x^2 \le 1\}$ = Segment

$[-1, 1]$

Making spheres from balls

- 1-dim

1-ball

1-sphere

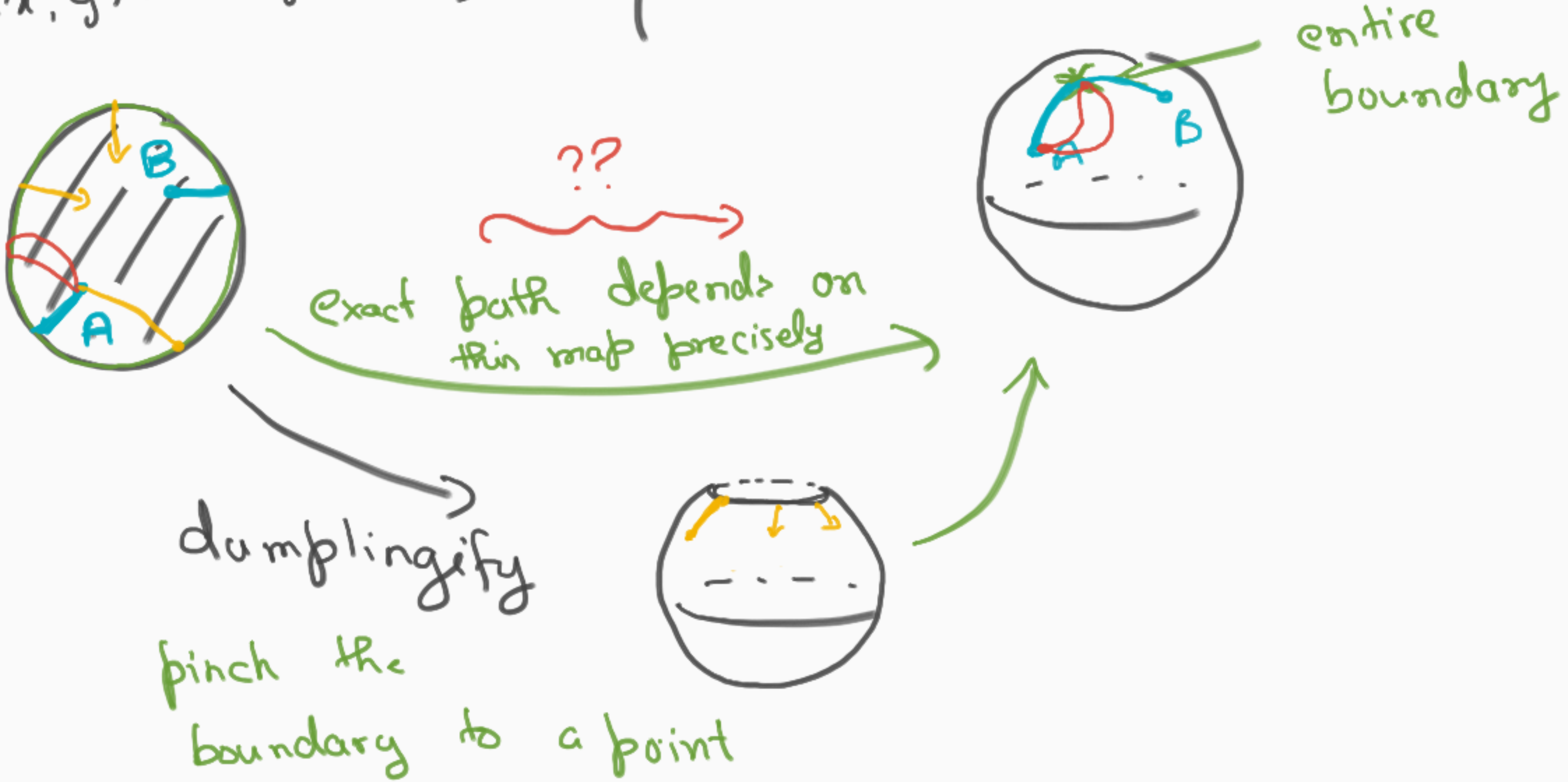
glue ends together

2-ball

$$\{(x,y): x^2+y^2 \leq 1\}$$

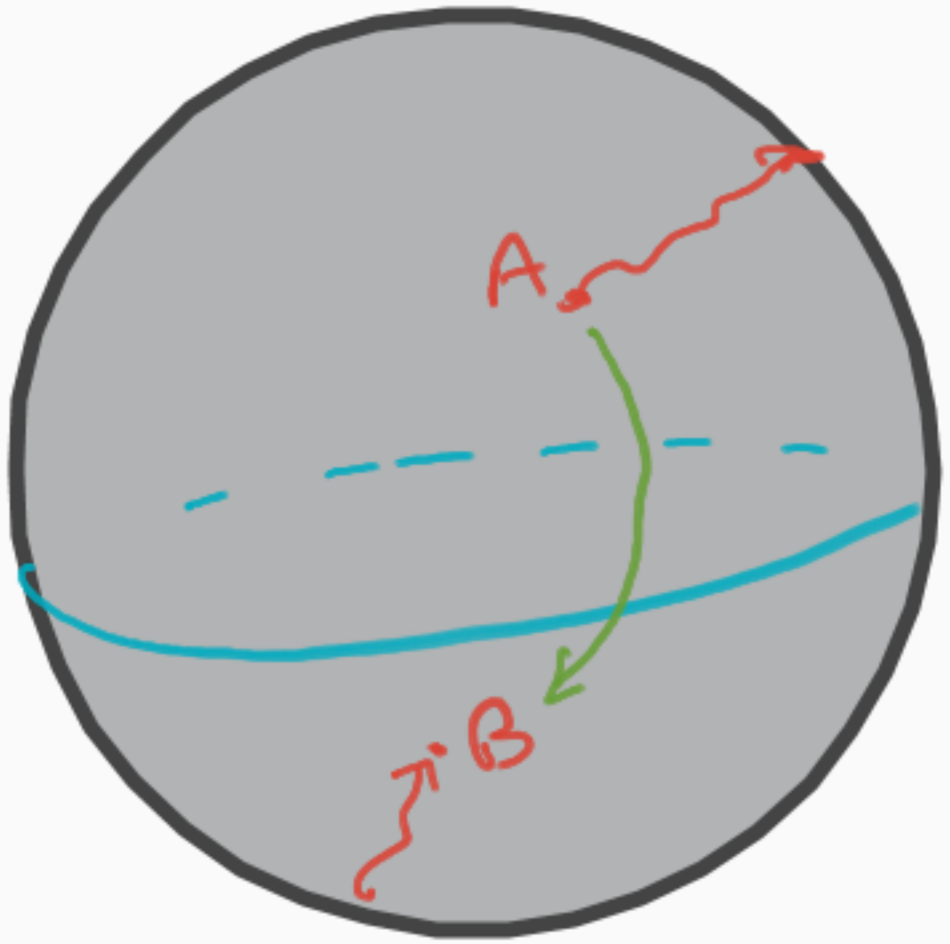
2-sphere

$$\{(x,y,z): x^2+y^2+z^2=1\}$$



3 - ball

$$\{(x,y,z) \mid x^2+y^2+z^2 \le 1\}$$



3 - sphere

- Take the 3-ball
- Squish the boundary together
  ↳ form quotient topology

everything in boundary of the ball is squished together

Why 3-sphere is extra special:

$$\{x^2+y^2+z^2+t^2 = (360)^2\} = S^3$$

gluing 3-ball boundary
$x^2+y^2+z^2 \le (360)^2$

On the boundary, all directions do nothing. So we get a map from 3-sphere to set of rotations of $\mathbb{R}^3$

Given a vector

$$\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Ans:

rotate along $\vec{v}$ counter clockwise, angle $= \|\vec{v}\|$

Recall:

Spheres (hollow)

$$S^n = \{(x_1, x_2, \dots, x_{n+})\ :\ x_1^2 + x_2^2 + \dots + x_{n+1}^2 = 1\}$$

S^0 = 2 points

S^1 = circle

S^2 = 2-sphere , S^3 = ???

Balls (solid)

$$B^n = \{(x_1, x_2, \dots, x_n)\ :\ x_1^2 + x_2^2 + \dots + x_n^2 \leq 1\}$$

B^1 = segment $[-1, 1]$

B^2 = disk

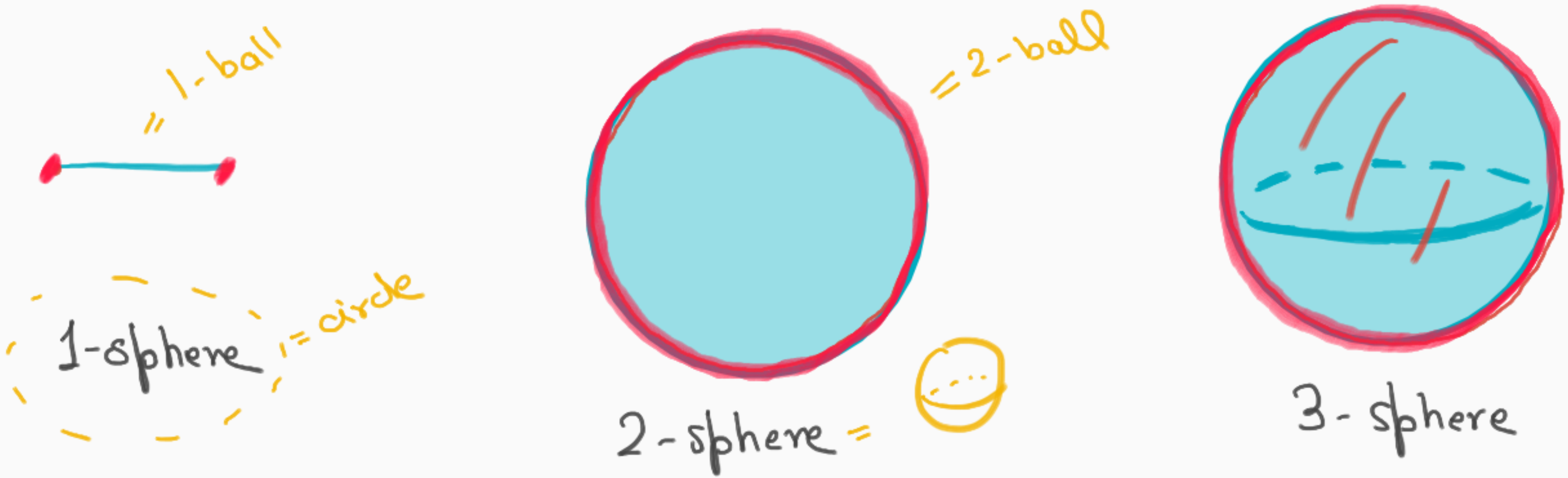
B^3 = ball

from balls to spheres (quotient topology)

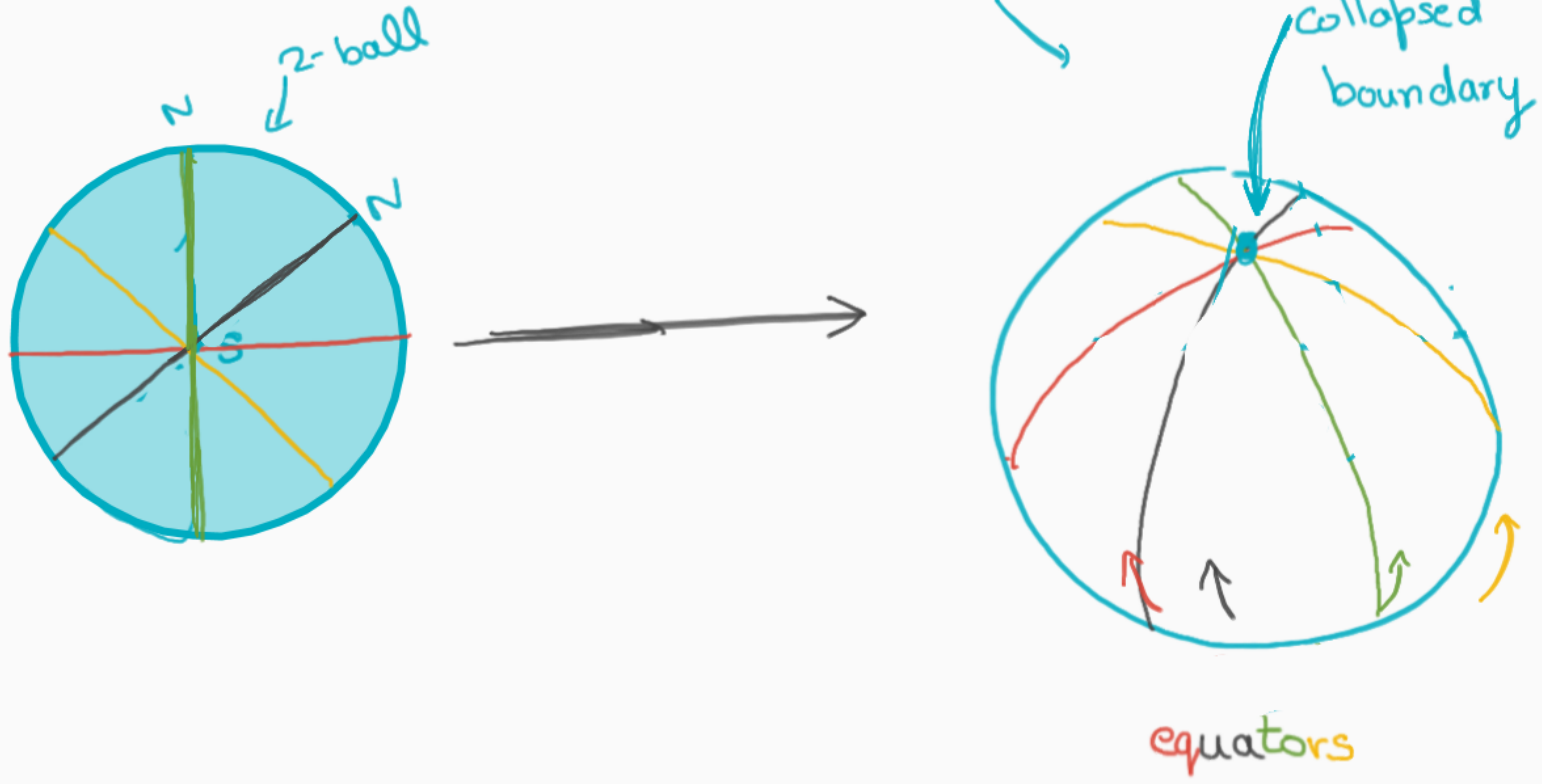
from n-ball you can create an n-sphere

by squishing the boundary together

thereby creating an n-dim dumpling



Warm-up exercise



Aside: 3-sphere encodes rotations of $\mathbb{R}^3$

$$\text{3-ball of radius } 360 \xrightarrow{Rot} \text{rotations of } \mathbb{R}$$

$\parallel$

$\{(x,y,z) : x^2+y^2+z^2 \leq 360^2\}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \vec{v} \longmapsto \text{Counter clockwise rotation by } \|\vec{v}\| \text{ degrees in the direction of } \vec{v}.$$

This map sends the entire boundary to a single point, hence defines a map from 3-sphere to rotations of $\mathbb{R}^3$

This map is not injective

Q. How many points are there in the preimage $Rot^{-1}(\mathbb{0})$?

$\mathbb{0}$ = 0-rotation i.e. no rotation

$Rot: \underline{\text{3-sphere}} \longrightarrow \text{rotations of } \mathbb{R}^3.$

- For what $\vec{v}$ does rotating by $\|v\|$ degrees give you no rotation? $\|\vec{v}\|=0$ or $\|\vec{v}\|=360$

$\|v\| = 0 \longrightarrow \vec{v} = 0$

$\|v\| = 360 \longrightarrow \vec{v}$ is on the boundary

when pushed to sphere, all such $\vec{v}$'s become the same point

Ball of $\boxed{\text{radius } 360}$

Sphere by collapsing the boundary

free variable

3-sphere: $\{ \underbrace{a + bi + cj + dk}_{} : a^2 + b^2 + c^2 + d^2 = 1 \}$

where $i^2 = j^2 = k^2 = -1$,

$ij = k$

$jk = i$

$ki = j$

extra condition

quaternions of length 1

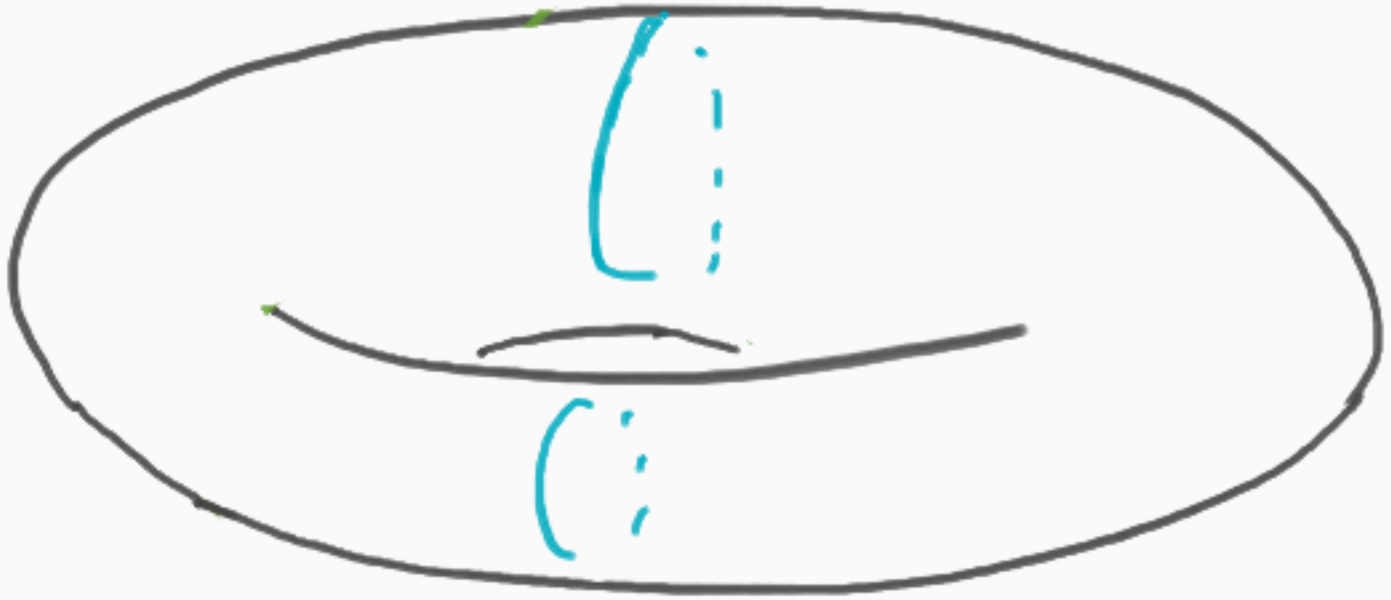
$\shortparallel$

$\mathbb{H} = S^3$

[quaternions capture rotations in $\mathbb{R}^3$]

Cutting things in half: torus

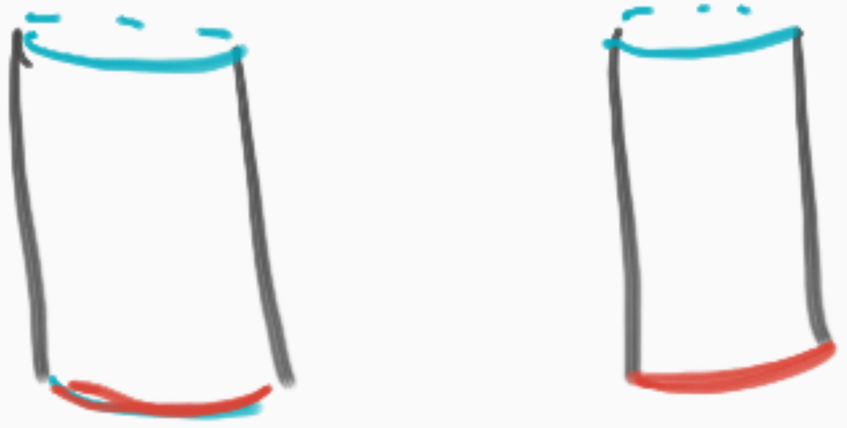
= hollow donut
= inner tube



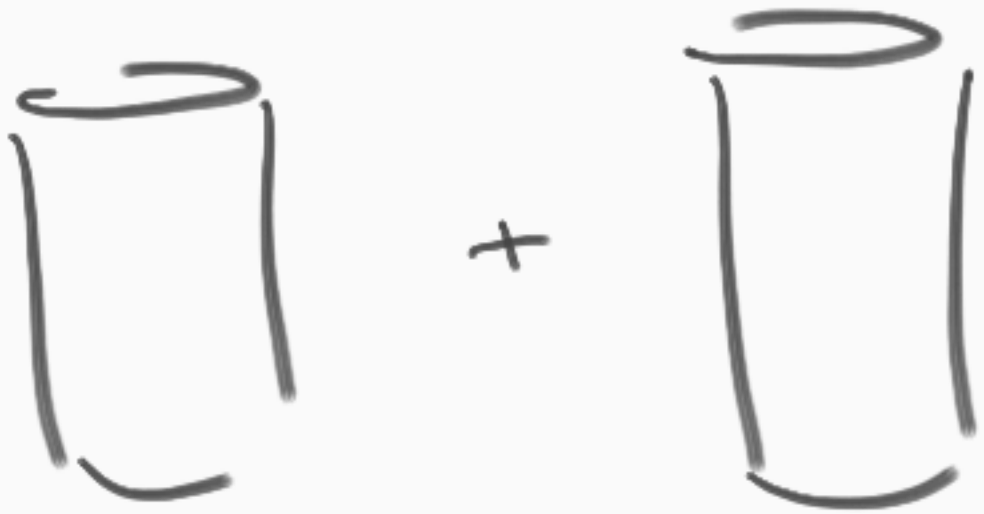
cut vertically in half

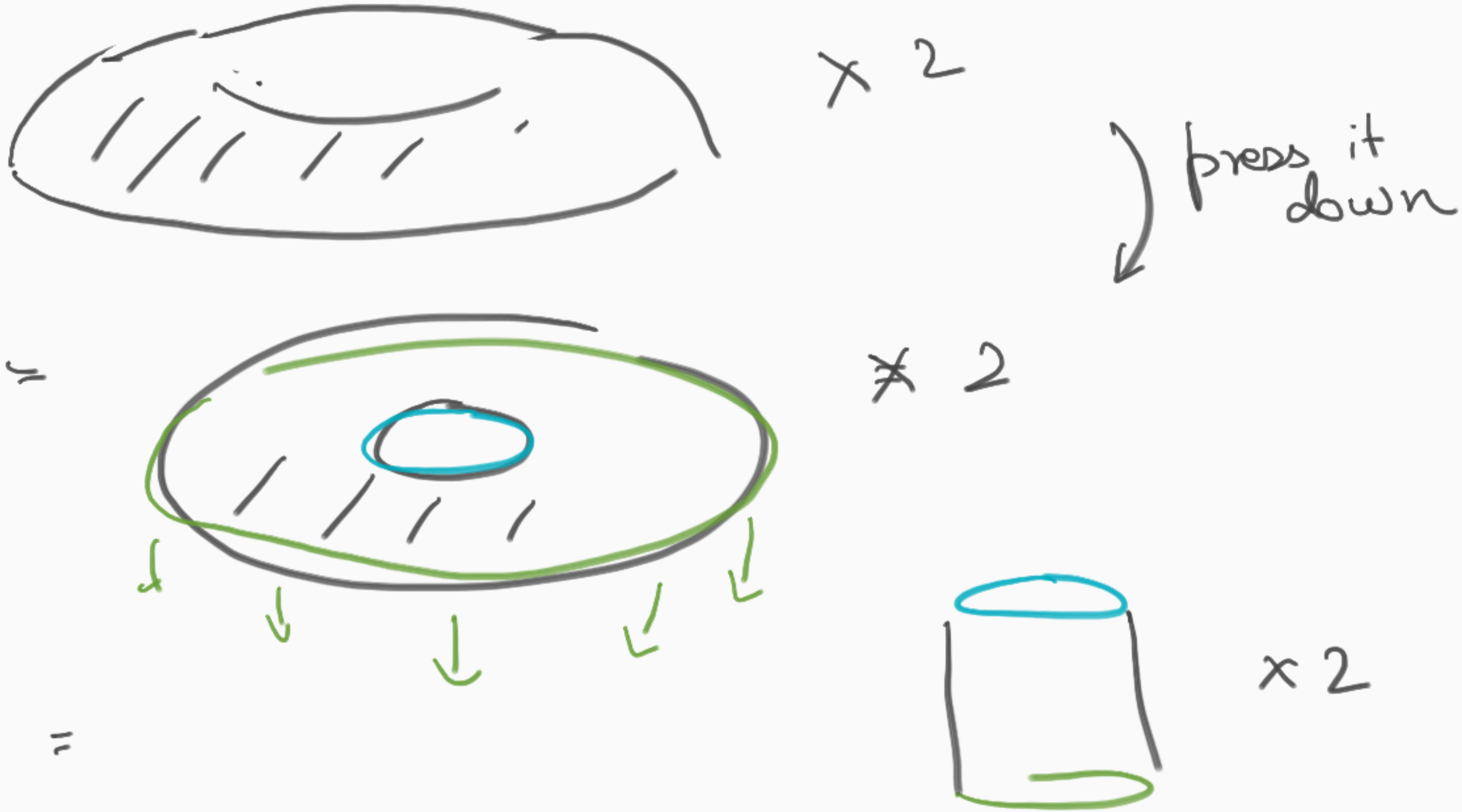
cut horizontally

Q. What are the two pieces?



2 cylinders
=
2 tubes




x 2
press it down
x 2
x 2

Cutting things in half: Klein bottle

Horizontal cut

chop vertically

2 different questions :

Q. What do you get when you cut the Klein bottle in half?

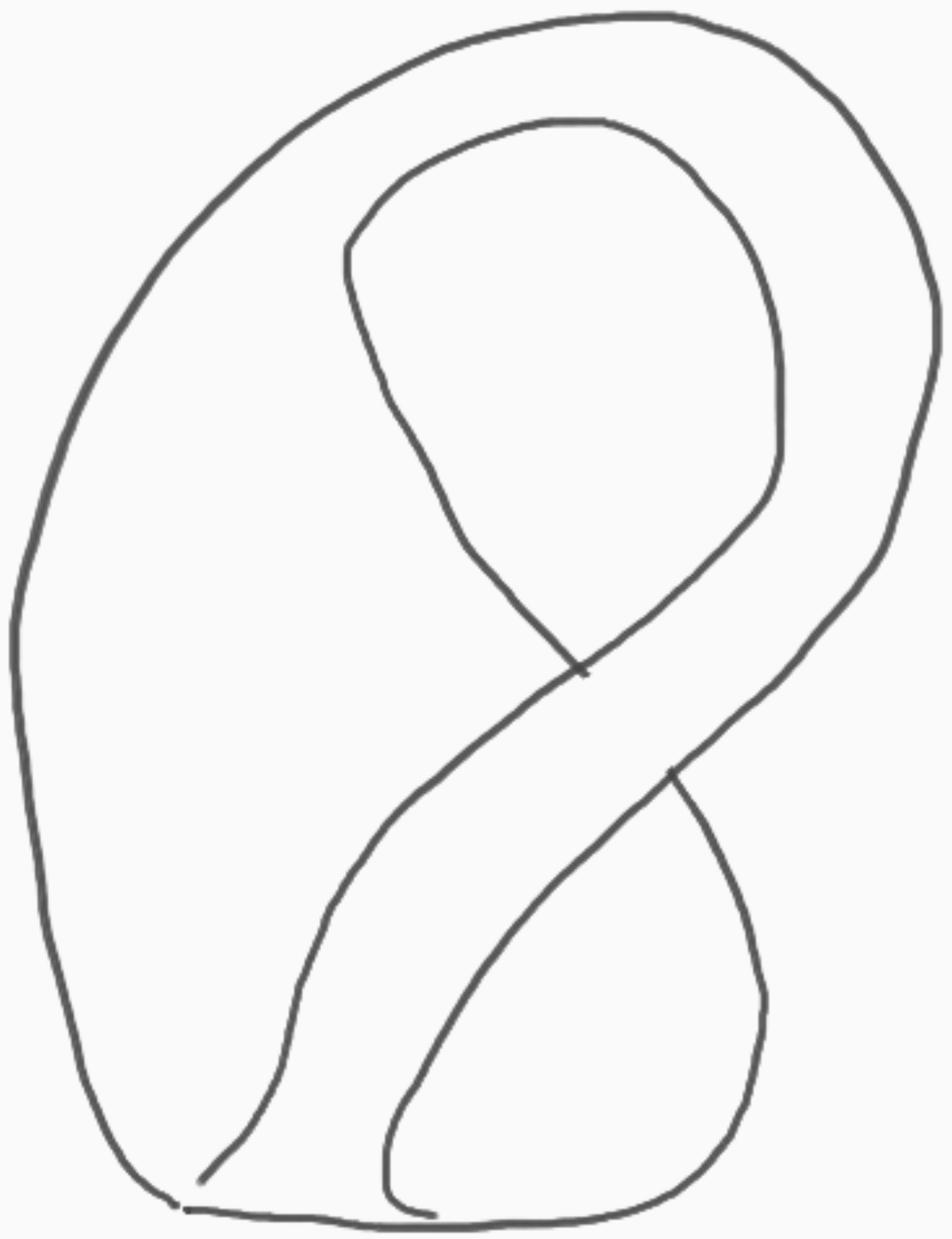
Cutting things in half: Klein bottle

2 mobius strips

chop vertically

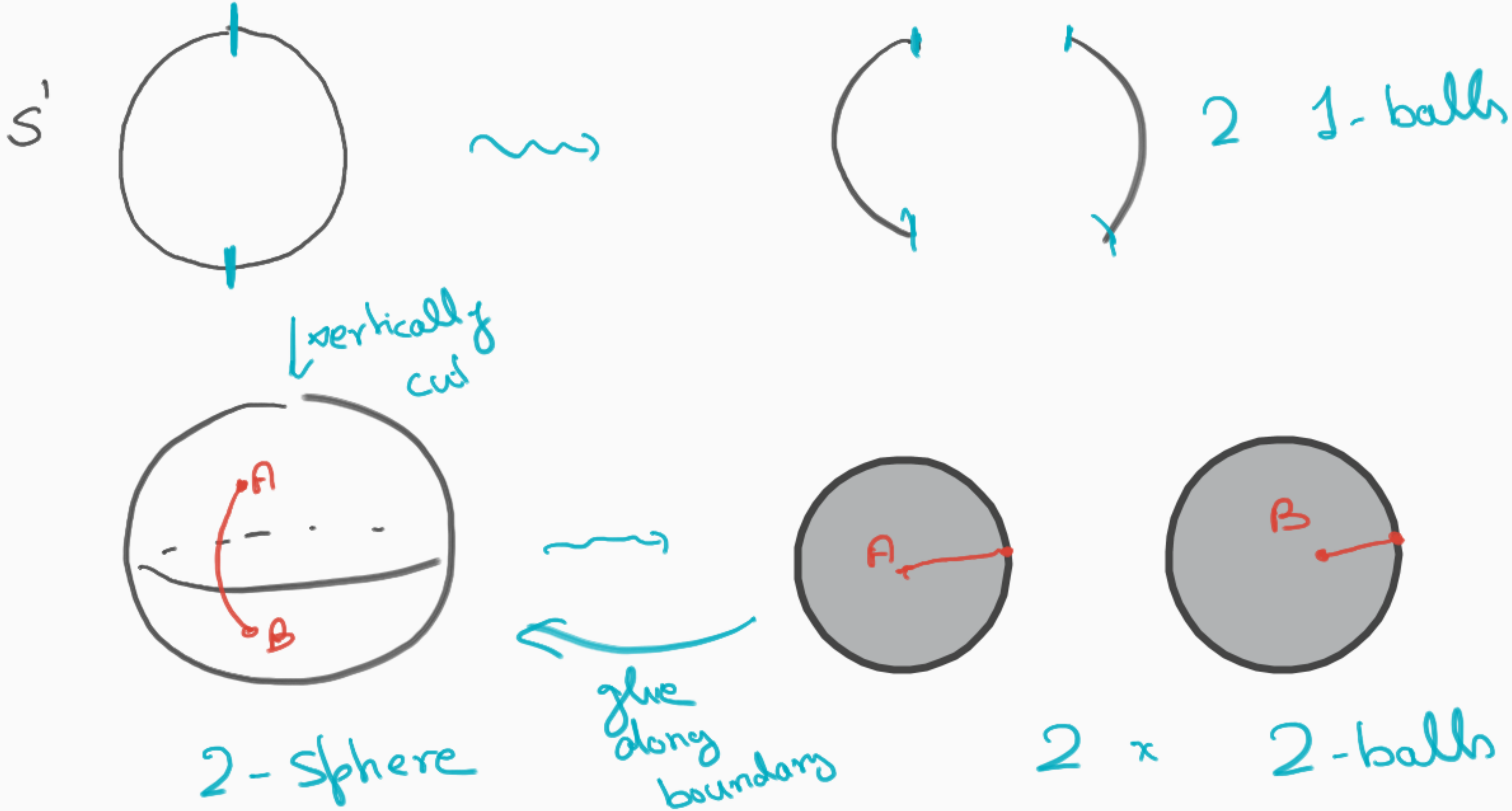
2 different questions :

Q. what do you get when you cut the Klein bottle in half?

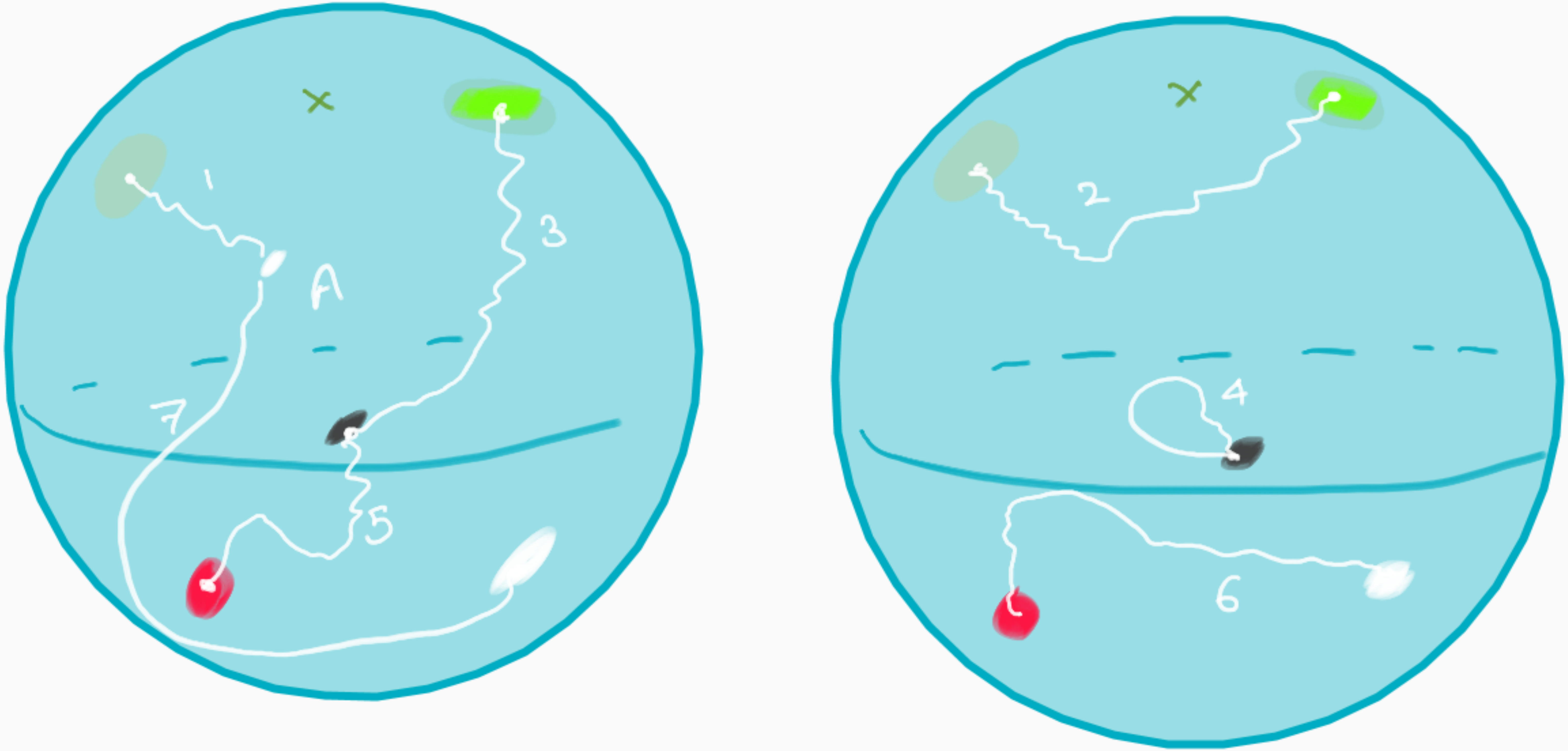


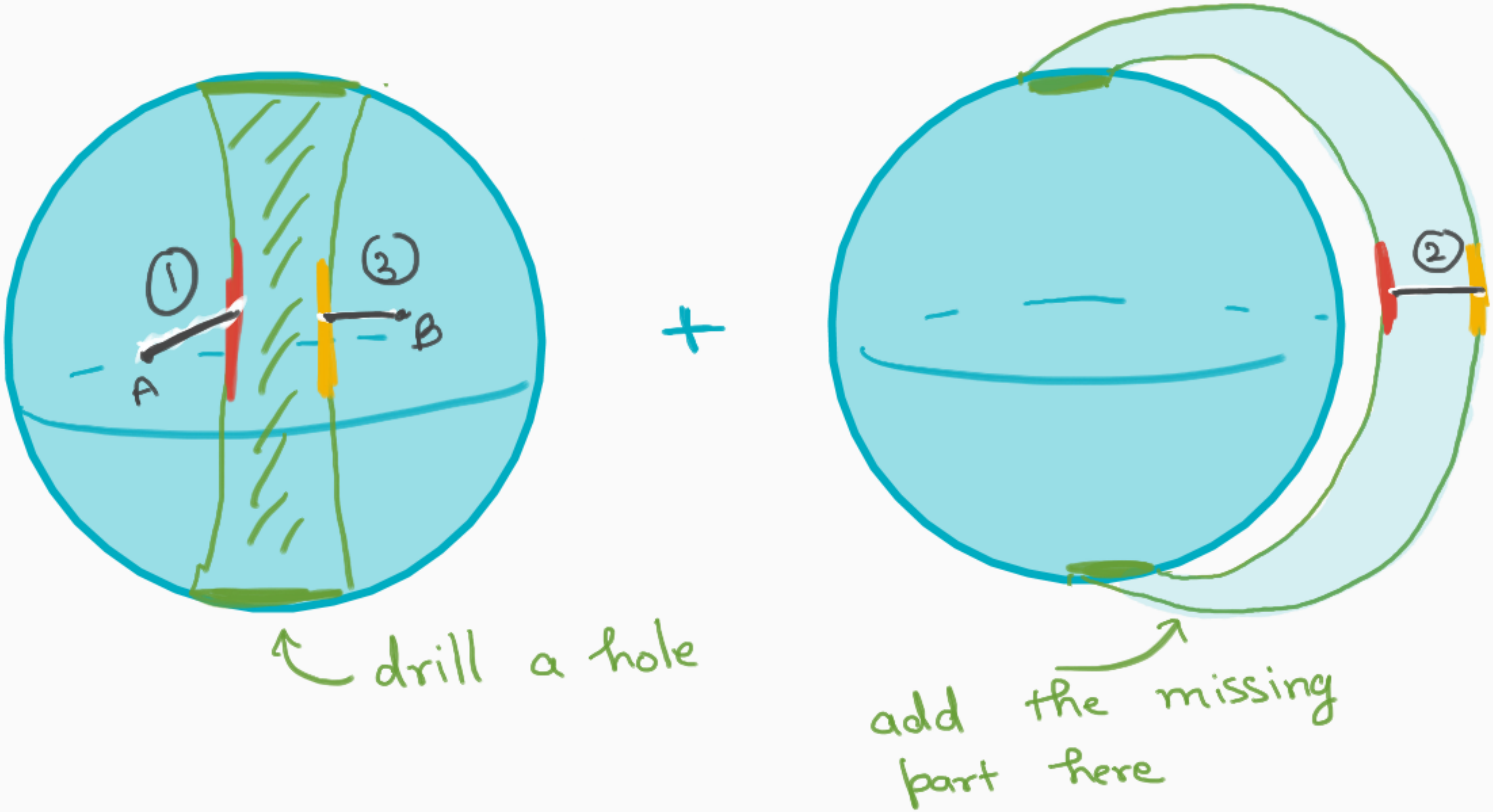
Cross section of a Klein bottle is a Mobius strip

Cutting things in half: spheres



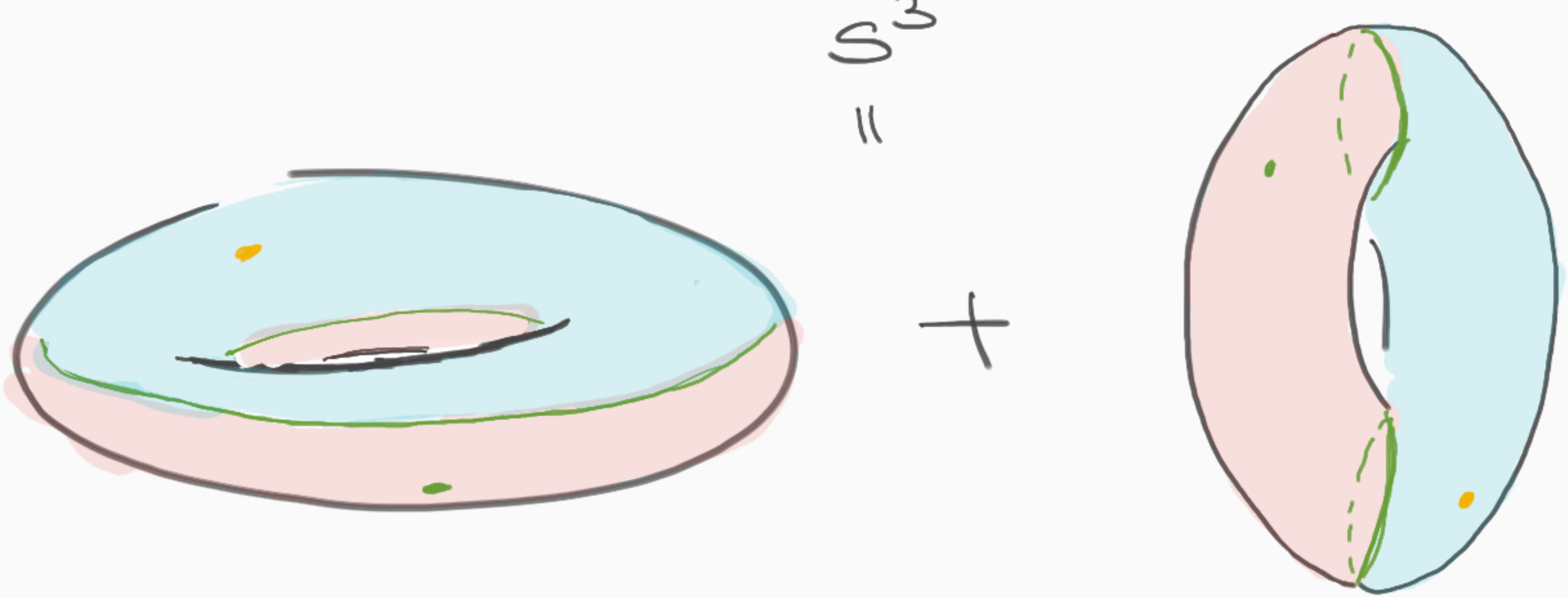
S^3 $\xrightarrow{\text{cut in half}}$ 2 3-balls





this is still S^3.

these are both donuts (or coffee cups!)



This is an example of a Heegaard splitting

Audience questions:

Q. Can we glue donuts with more than one I hole?

A. Yes. Repeat the last step.

Q. What if we glue a donut to a copy of itself along the boundary?

A. You get $S^1 \times S^2$ instead of S^3