Wednesday, Nov 27, 2019

Recall: u-substitution

- technique for simplifying integrals
- Reverses chain Rule: $f \circ g(x)' = f'(g(x)) \cdot g'(x)$

$$\Rightarrow \int f'(g(x)) \cdot g'(x)\, dx = f(g(x)) + C$$

- If we see $g'(x) \cdot dx$ then sometimes substituting

$$u = g(x)$$
$$du = g'(x)\, dx$$

simplifies the integral.

- Note that you have to search for the pattern $g'(x) \cdot dx$

So you should know derivatives of standard functions VERY WEL

- We can often use linear substitutions

$$\boxed{\begin{aligned} u &= ax + b \\ du &= a \cdot dx \end{aligned}}$$

to simplify integrals

- Sometimes you have to do multiple u-substitutions / u-substitution and some algebraic simplifications etc.

- $g'(x)\, dx$ can sometimes involve terms in the denominator

Q. $\int \frac{\ln x}{x}\, dx \rightsquigarrow \underbrace{\frac{1}{x} dx \rightsquigarrow \frac{1}{x} = (\ln x)'}_{\text{This suggests}}$

$$u = \ln x$$
$$du = (\ln x)' \cdot dx$$
$$du = \frac{1}{x} \cdot dx$$

$$\int \frac{\ln x}{x}\,dx = \int u\,du$$

$$= \frac{u^2}{2} + c \qquad \text{Power Rule}$$

$$= \frac{(\ln x)^2}{2} + c$$

Q. $\frac{1}{\sqrt{x}}\,dx$

$\Downarrow$

$x^{-1/2}\,dx$

By Power Rule

$(x^{1/2})' = \frac{1}{2} \cdot x^{-1/2}$

$\shortparallel$

$(\sqrt{x})'$

$\rightarrow$ $\frac{1}{\sqrt{x}}\,dx$ suggests the substitution

$u = \sqrt{x}$

$du = \frac{1}{2} \cdot \frac{1}{\sqrt{x}} \cdot du$

Q: $\frac{dx}{1+x^2}$ $\longrightarrow$ $u = \arctan x$

$du = \frac{1}{1+x^2}\,dx$

Q: $\frac{dx}{\sqrt{1-x^2}}$ $\longrightarrow$ $u = \arcsin x$

$du = \frac{1}{\sqrt{1-x^2}}\,dx$

Q: $\int \frac{x}{1+x^2}\,dx$

? is $\cancel{\frac{x}{1+x^2} = \frac{x}{1} + \frac{x}{x^2}}$

$= \int x \cdot \frac{1}{1+x^2} \cdot dx \neq \cancel{\int x \int \frac{1}{1+x^2}}$

$\downarrow$

There are two possible u-substitutions

$x = \left(\frac{x^2}{2}\right)'$

$\left[\begin{array}{l} u = x^2 \\ du = 2x\,dx \end{array}\right.$

instead we'll do

$\left[\begin{array}{l} u = x^2+1 \\ du = 2x \cdot dx \\ \frac{du}{2} = x \cdot dx \end{array}\right.$

$\int x \cdot \frac{1}{1+x^2}\,dx = \int \frac{1}{1+x^2} \cdot (x\,dx)$

$= \int \frac{1}{u} \cdot du/2$

$= \frac{1}{2} \ln u + C$

$= \frac{1}{2} \ln(x^2+1) + C$

$\frac{1}{1+x^2} = (\arctan x)'$

$u = \arctan x \Rightarrow \tan u = x$

$du = \frac{1}{1+x^2}\,dx$

$\int x \cdot \frac{1}{1+x^2}\,dx = \int \tan u\,du$

$= \int \frac{\sin u}{\cos u} \cdot du$

$v = \cos u$

$dv = -\sin u\,du$

$= \int -\frac{dv}{v}$

$= -\ln v + C$

$= -\ln(\cos u) + C$

$= -\ln(\cos(\arctan x)) + C$

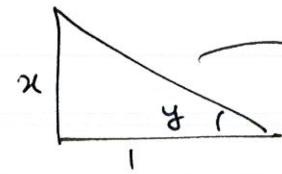
Aside

<u>Review</u>

$$\cos(\arctan x) = ?$$

Let $y = \arctan x$

$\Rightarrow \quad \tan y = x$



By Pythagoras $\sqrt{1+x^2}$

$$\Rightarrow \quad \cos y = \frac{1}{\sqrt{1+x^2}}$$

$$\Rightarrow \quad \cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}$$

Back to the question:

$$\ln(\cos(\arctan x)) = \ln\left(\frac{1}{\sqrt{1+x^2}}\right)$$

$$= \cancel{\ln 1} - \ln\left(\sqrt{1+x^2}\right)$$

$$= -\ln\left((1+x^2)^{1/2}\right)$$

$$= -\frac{1}{2}\ln(1+x^2)$$

U-subs are used to get rid of annoying constants:

eg: $\int \frac{1}{x+1}\,dx$

$u = x+1$

$du = 1 \cdot dx$

$= \int \frac{du}{u}$

$= \ln u + C$

$= \ln(1+x) + C$

eg: $\int \frac{1}{\sqrt{1-4x^2}}\,dx$

$= \int \frac{1}{\sqrt{1-(2x)^2}}\,dx$

$u = 2x$

$du = 2 \cdot dx$

$\frac{du}{2} = dx$

$= \int \frac{1}{\sqrt{1-u^2}} \cdot du/2$

$= \frac{1}{2} \int \frac{1}{\sqrt{1-u^2}}\,du$

$= \frac{1}{2} \cdot \arcsin(u) + C$

$= \frac{1}{2} \cdot \arcsin(2x) + C$

Special cases of

$$\boxed{\begin{aligned} u &= ax+b \\ du &= a \cdot dx \end{aligned}}$$

we absorb the constants in the new variable

Q. for $\int \frac{1}{\sqrt{1-4x^2}}\,dx$ why don't we use $u = 4x^2$?

$\underline{du} = 8x \cdot dx$

But there is no $\underline{x\,dx}$ in my original equation

Q. for $\int \frac{x}{\sqrt{1-4x^2}}\,dx$ $\longrightarrow$ Note: $\underline{x\,dx}$ suggests instead of $\underline{u = x^2}$ we can do

$$\begin{cases} u = 1-4x^2 \\ du = -8x \cdot dx \\ -\frac{du}{8} = x\,dx \end{cases}$$

$$= \int \frac{-du/8}{\sqrt{u}}$$

$$= -\frac{1}{8}\int \frac{du}{\sqrt{u}}$$

$$= -\frac{1}{8}\,\frac{u^{1/2}}{1/2} + C$$ By Power Rule

$$= -\frac{1}{4}\cdot u^{1/2} + C$$

$$= -\frac{1}{4}\cdot\sqrt{1-4x^2} + C$$

Q. $\cancel{\int \frac{x}{1+x^4}\,dx}$ $\int \frac{x}{1+x^4}\,dx$

Note:

① $x\,dx$ suggests $u = x^2$

② $x^4 = (x^2)^2$

$$u = x^2$$

$$\Rightarrow du = 2x\,dx$$

$$\Rightarrow \frac{du}{2} = x\,dx$$

$$= \int \frac{du/2}{1+u^2} \qquad \left(\text{as } x^4 = (x^2)^2\right)$$

$$= \frac{1}{2}\int \frac{du}{1+u^2}$$

$$= \frac{1}{2}\arctan u + C$$

$$\int \frac{x}{1+x^4}\,dx = \frac{1}{2}\arctan(x^2) + C$$

- Next: ~~in~~definite integrals $\int_a^b f(x)\,dx$

  using u-sub

  - If you do a u-sub

$$u = g(x)$$

then we change the bounds

$$\underbrace{\int_a^b}_{\text{bounds for } x} dx \longrightarrow \underbrace{\int_{g(a)}^{g(b)}}_{\text{bounds for } u} du$$

eg:

$$\int_1^2 e^{(x^2)} \cdot x\,dx = \int_{1^2}^{2^2} e^u \cdot du/2$$

$u = x^2$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$x=1$, $u = 1^2$

$x=2$, $u = 2^2$

$$= \frac{1}{2}\int_1^4 e^u\,du$$

$$= \frac{1}{2}\left(e^u\right)\Big|_1^4$$

$$= \frac{1}{2}\left(e^4 - e^1\right).$$

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eg: $\displaystyle\int_1^e \frac{\ln x}{x}\, dx$

$\displaystyle = \int_1^e \ln x \cdot \frac{1}{x} \cdot dx$

$u = \ln x$

$du = \frac{1}{x} dx$

$x = 1$, $u = \ln 1 = 0$

$x = e$, $u = \ln e = 1$

$\displaystyle = \int_0^1 u \; du$

$\displaystyle = \frac{u^2}{2} \Big|_0^1$

$\displaystyle = \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2}$

- Another way to do these is to go back to the original variable and the original bounds

eg: $\displaystyle\int_1^e \frac{\ln x}{x}\, dx \xrightarrow[\text{Forget the bounds}]{} \int \frac{\ln x}{x}\, dx \xrightarrow{u\text{-sub}} \int u\, du$

$\displaystyle = \frac{u^2}{2} + C$

$\displaystyle = \frac{(\ln x)^2}{2} + C$

Plug the bounds back in

$\displaystyle\int_1^e \frac{\ln x}{x}\, dx$

$\displaystyle = \frac{(\ln x)^2}{2} \Big|_1^e$

$\displaystyle = \frac{(\ln e)^2}{2} - \frac{(\ln 1)^2}{2} = \frac{1}{2}$

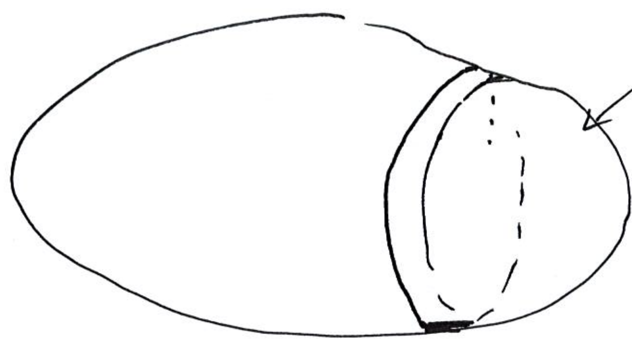
Next : 6.1 , 6.2

We'll start with 6.2

$\hookrightarrow$ find volumes of 3D objects

Idea:

Want to find Volume



$\downarrow$

Break this into cylinders

(See Pg. 440)

$\uparrow$

cylinder

Volume of this cylinder $\rightarrow$

= Area of base × height

Volume = Area of cross-section × height

$$= \underbrace{A(x)}_{\substack{\text{area of}\\ \text{cross-section}\\ \text{at some}\\ \text{point } x}} \cdot \underbrace{\Delta x}_{= \text{height}}$$

x - axis

Total volume = sum of all these cylinders (in the limiting case when the height $\to 0$)

$$= \lim_{\Delta x \to 0} \left(\sum_i A(x_i) \cdot \Delta x \right)$$

$$\text{Volume} = \int_a^b A(x)\, dx$$

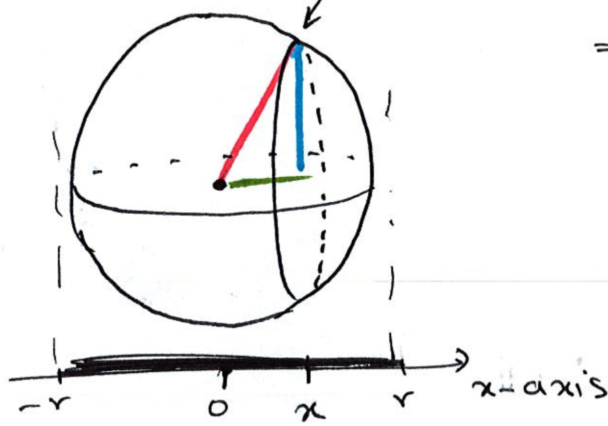
Summary : To find volume of a shape, express the cross-sectional area as a function of either x (or y) ; $A(x)$

then volume $= \int_a^b A(x)\,dx$

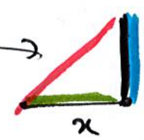
eg: Volume of a solid sphere of radius r :

cross-section at x = circle

$\Rightarrow$ area $= \pi(\text{radius})^2$ of this circle



radius of sphere $= r$



radius of the circle

By pythagoras : $\sqrt{r^2 - x^2}$

$\Rightarrow$ area of cross-section $= \pi\left(\sqrt{r^2-x^2}\right)^2$

$= \pi(r^2 - x^2)$

$$\Rightarrow V = \int_?^? \pi(r^2-x^2)\cdot dx$$

The bounds: smallest x-coordinate for the sphere $= -r$

largest x-coordinate for the sphere $= r$

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$$V = \int_{-r}^{r} \pi (r^2 - x^2)\, dx$$

$$= \pi \int_{-r}^{r} r^2 - x^2\, dx$$

$$= \pi \left[\int_{-r}^{r} r^2\, dx - \int_{-r}^{r} x^2\, dx \right]$$

$$= \pi \left[r^2 \int_{-r}^{r} 1 \cdot dx - \int_{-r}^{r} x^2\, dx \right]$$

$$= \pi \left[r^2 \cdot x \Big|_{-r}^{r} - \frac{x^3}{3} \Big|_{-r}^{r} \right]$$

$$= \pi \left[r^2 \cdot r - r^2 \cdot (-r) - \left[\frac{r^3}{3} - \frac{(-r)^3}{3} \right] \right]$$

$$= \pi \left[r^3 + r^3 - \frac{r^3}{3} - \frac{r^3}{3} \right]$$

$$= \pi r^3 \left[1 + 1 - \frac{1}{3} - \frac{1}{3} \right]$$

$$= \frac{4}{3} \pi r^3$$