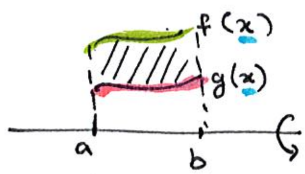
Wednesday, Dec 04, 2019

Review:

Section 6.2: Volume = $\int$ cross-sectional area

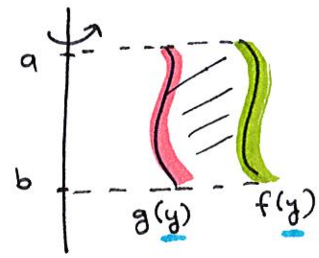
Special case: Solid of revolution

Rotate about x-axis:



$$V = \int_a^b \pi \left[f(x)^2 - g(x)^2 \right] dx$$

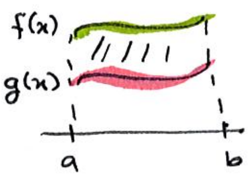
Rotate about y-axis:

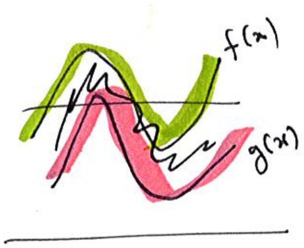


$$V = \int_a^b \pi \left[f(y)^2 - g(y)^2 \right] dy$$

Section 6.1: • if f(x) is above g(x),

Area between f(x) and g(x) = $\int_a^b f(x) - g(x)\, dx$





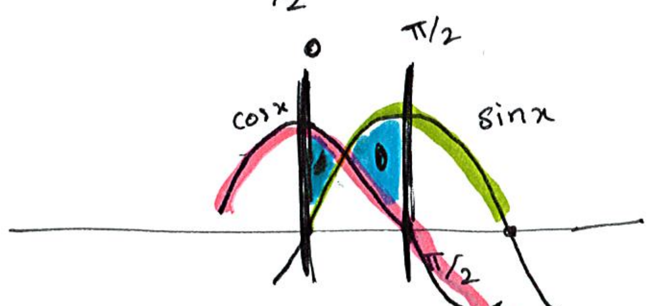
• More generally, remember that

1) integrals compute signed areas

2) $\int_a^b = \int_a^c + \int_c^b$

eg: Q. Compute area between sin x and cos x between x = 0 and $\pi/2$.

A.



Split into two regions: 0 to $\pi/4$ *, $\pi/4$ to $\pi/2$ *

$\underbrace{\qquad}_{\cos x \text{ is above } \sin x}$ $\underbrace{\qquad}_{\sin x \text{ is above } \cos x}$

*: **Point of intersection**:

$\sin x = \cos x$ in 0 to π

$\Rightarrow \quad x = \pi/4$

$$\text{Area} = \int_0^{\pi/4} \cos x - \sin x \, dx + \int_{\pi/4}^{\pi/2} \sin x - \cos x \, dx$$

$$= \Big[\sin x + \cos x\Big]_0^{\pi/4} + \Big[-\cos x - \sin x\Big]_{\pi/4}^{\pi/2}$$

$$= \begin{bmatrix} \sin \frac{\pi}{4} + \cos \pi/4 \\ - \sin 0 - \cos 0 \end{bmatrix} + \begin{bmatrix} - \cos \pi/2 - \sin \pi/2 \\ - \left(- \cos \pi/4 - \sin \pi/4\right) \end{bmatrix}$$

$$= \left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1\right] + \left[\begin{matrix} -0 - 1 \\ - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) \end{matrix}\right]$$

$$= \frac{2}{\sqrt{2}} - 1 - 1 + \frac{2}{\sqrt{2}}$$

$$= \sqrt{2} - 1 - 1 + \sqrt{2}$$

$$= 2\sqrt{2} - 2 .$$

Big Review:

Basics:

- Graphs of standard functions
- Log, exp, identities
- $\sin^2 x + \cos^2 x = 1$
- Inverse functions $\left(y = f^{-1}(x) \equiv f(y) = x\right)$
- Derivatives of standard functions.

Techniques.

- Derivatives using product rule, chain rule, quotient rule, implicit differentiation, logarithmic diff
- Antiderivatives (guess, check)
- u-substitution
- derivatives of integrals (Fundamental theorem of Calculus V2)
- limits —
  - L'Hopital's rule
  - Logarithmic differentiation
  - Squeeze theorem
  - $\frac{\text{polynomial}}{\text{polynomial}}$
  - Limit laws

Applications :

- Finding min / max
  - ↳ derivatives, critical points
    - ↳ local ↙ Second derivative test
    - ↳ absolute ↓ checking "y-values" at critical points and endpoints

- Optimization problems
  - ↳ • Convert word problems into min/max problem.
    - • Solve min / max problem
    - • Sometimes instead of second derivative test, you can find <u>sign of first derivative</u> around critical points.

- Areas of regions : $\int_a^b f(x) - g(x) \, dx$

- Volumes of solids : $\int$ cross-section area
  - ↳ Solids of revolution :

    $$\text{cross-sectional areas} = \pi R^2 - \pi r^2$$

- Sketching graphs of complicated functions
  - ↳ increasing / decreasing, concavity, asymptotes etc.

Extra bits :

- Domain , range of functions , odd-even
- Riemann sums , Appendix E
- Related rates
- Intermediate Value theorem
- ~~Sketching graphs of complicated functions~~

Fundamental theorem of calculus:

essentially : FT says "integration and differention are inverse operations"

V1

• $f(x) \xrightarrow{\text{differentiation}} f'(x) \xrightarrow{\text{integration}} \boxed{\int f'(x)\,dx = f(x) + c}$

usually written as

$\int g(x)\,dx = G(x) + c$, where $G(x)$ is any anti-derivative of $g(x)$

V2

• $f(x) \xrightarrow{\text{integerate}} \boxed{\int_a^x f(t)\,dt \xrightarrow{\text{differentiate}} f(x)}$

V. important : (1) lower bound is a constant, upper bound is • x.

(2) use different variable for bounds, and inside the integral

$$\left(\int_a^{g(x)} f(t)\,dt\right)' = f(g(x))\cdot g'(x) \qquad \text{(Chain Rule)}$$

$$\left(\int_{h(x)}^{g(x)} f(t)\,dt\right)' = \left(\int_a^{g(x)} f(t)\,dt\right)' + \left(\int_{h(x)}^{a} f(t)\cdot dt\right)'$$

Chain Rule + Fundamental Theorem

Pset 07 on Webwork,

Q.2 Step 1: Graph $f(x)$

Step 2: Break $\int_a^b f(x)\cdot dx$ into intervals according to the definition of the piecewise function

Step 3: integrals = signed area ~~under~~ between the graph and the x-axis.

eg: Find $\int_0^3 |x-1|\,dx$

$$|x-1| = \begin{cases} (x-1) & x \geq 1 \\ -(x-1) & x \leq 1 \end{cases}$$

$$= \int_0^1 |x-1|\,dx + \int_1^3 |x-1|\,dx$$

$$= \int_0^1 -(x-1)\,dx + \int_1^3 (x-1)\,dx$$

Q. If $f(x) = \int_2^x \ln t \, dt$

find $f^{-1}(0)$.

A: If $a = f^{-1}(0)$

then $f(a) = 0$.

$$\Rightarrow \int_2^a \ln t \, dt = 0$$

this is 0 when $\underline{a = 2}$.

Properties of integrals:

Three variants of integrals	$\int f(x)\,dx$	$\int_a^b f(x)\,dx$	$\int_a^x f(t)\,dt$
	formal antiderivative	· number	· function
	not a function!	· signed area	

Fundamental theorem version 1

Fundamental theorem version 2

Geometric properties:

$$\int_a^a f(t)\,dt = 0$$

$$\int_a^b f(t)\,dt = -\int_b^a f(t)\,dt$$

$$\int_a^b f(t)\,dt + \int_b^c f(t)\,dt = \int_a^c f(t)\,dt$$

$$\int_a^c f(t)\,dt = \int_a^{\boxed{b}} f(t)\,dt + \int_{\boxed{b}}^c f(t)\,dt$$

Algebraic properties:

integrals behave nicely with sums, differences, scalar multiplication BUT NOT products, $\underbrace{\text{compositions}}$.

u-subsitution

- get rid of constants
- if we see $g'(x)\,dx$ then the substitution
  $$u = g(x)$$
  simplifies the integral. (sometimes)
- Note: sometimes $g'(x)$ can have terms in the denominator

eg: $\dfrac{dx}{x} = \dfrac{1}{x}\cdot dx \rightsquigarrow u = \ln x$ | $\dfrac{dx}{\sqrt{x}} \rightsquigarrow u = \sqrt{x}$

eg: $\cancel{\int \frac{\ln x}{x}\, dx}$

Challenging Problems

eg : Q. $\displaystyle\int \frac{x^3}{x^2+1}\, dx$

Attempt 1:

$u = x^3$

$du = 3x^2\, dx$

$\frac{du}{3} = x^2\, dx$

$\frac{x^3}{x^2+1} \rightsquigarrow \frac{x}{x^2+1} \cdot \underbrace{x^2\, dx}$ ✗

Attempt 2:

$\rightsquigarrow \underbrace{x^3 \cdot \frac{1}{x^2+1}\, dx} \longrightarrow$

$u = \arctan x$

$du = \frac{1}{1+x^2}\, dx$

$x^3 = \tan^3 u = (\tan u)^3$

$u = \arctan x$

$\Rightarrow \tan u = x$

$\Rightarrow \tan^3 u = x^3$ ✓

$\frac{x^3}{x^2+1} \rightsquigarrow \frac{x^2}{x^2+1} \cdot \underbrace{x \cdot dx}$

$\Downarrow$

$g'(x) \cdot dx$

Attempt 3:

$u = x^2+1$

$du = 2x\, dx$

$u = g(x)$

or

$u = x^2$

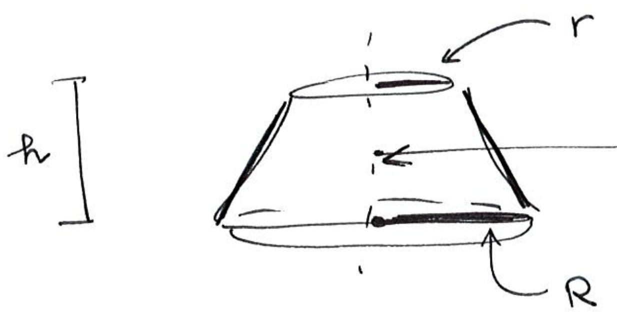
$du = 2x\, dx$ ✓

Q. Excercise

$\displaystyle\int u \cdot \sqrt{u+1}\, dx$

Q. $\displaystyle\int \frac{u}{1+u}\, du$

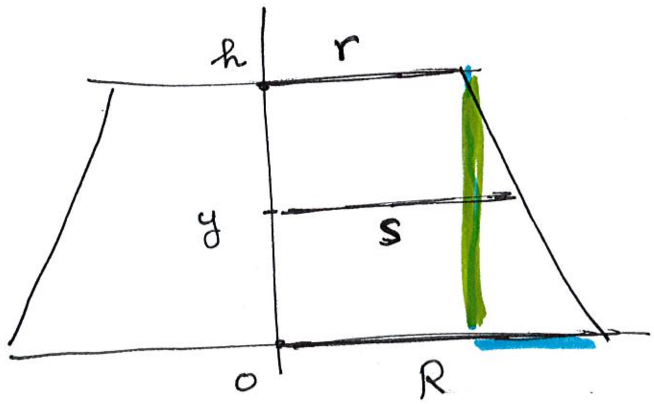
Pset 07, Problem 28 — Challenging question



Note: this is a ~~surface~~ solid of revolution

- Cross sectional area $= \pi s^2$

- Need to write s as a function of y.



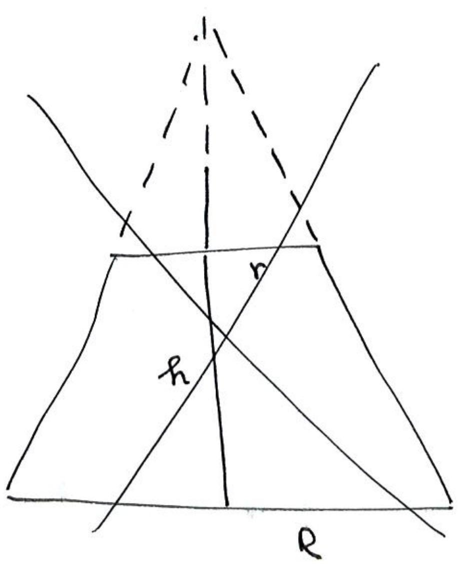
$s = ?$ when $y = 0$, $s = R$

when $y = h$, $s = r$

So at a general point y,

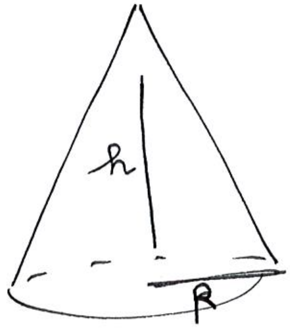
$$s(y) = R + y \cdot \left(\frac{r-R}{h}\right)$$

$$\int_0^h \pi \left(R + y \cdot \frac{(r-R)}{h}\right)^2 \cdot dy$$

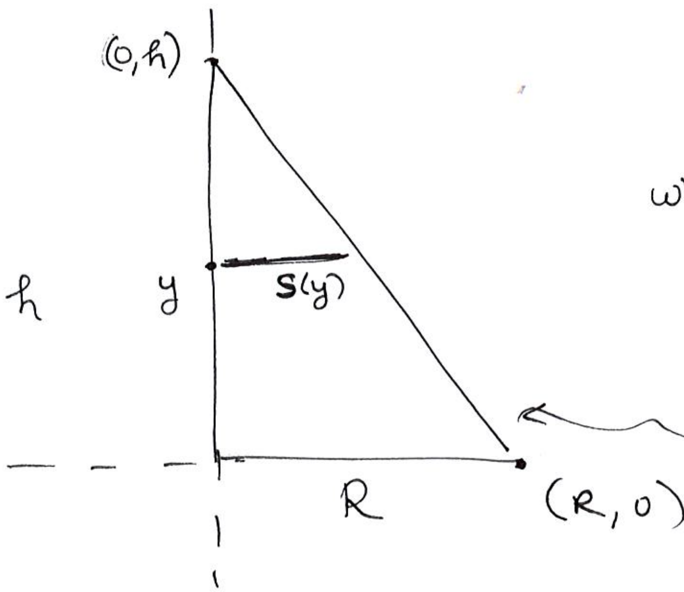


Standard question

Q.



Find volume of this



what is S(y)?

equation of line:

$$(y-h) = \frac{(h-0)}{(0-R)}(x-0)$$

$$y-h = -\frac{h}{R}\cdot x$$

$$\boxed{-\frac{R}{h}(y-h) = x}$$

$$S(y) = -\frac{R}{h}(y-h)$$

$$V = \int_0^h \pi\left(-\frac{R}{h}(y-h)\right)^2 dy$$

$$= \pi \frac{R^2}{h^2} \int_0^h (y-h)^2 \, dy$$

$$= \pi \frac{R^2}{h^2} \int_{-h}^{0} u^2 du$$

$u = y - h$

$du = dy$

$y = 0, \; u = -h$

$y = h, \; u = 0$

$$= \pi \frac{R^2}{h^2} \cdot \left. \frac{u^3}{3} \right|_{-h}^{0}$$

$$= \pi \frac{R^2}{h^2} \left(\frac{h^3}{3} \right)$$

$$= \frac{\pi R^2 h}{3}.$$