

# Limits

Def<sup>n</sup> An  $\infty$ -cat has all limits of shape  $J$  if

$$\begin{array}{ccc} & & A \\ \exists \nearrow & \Downarrow \lambda & \downarrow \Delta \\ A^J & \xrightarrow{id} & A^J \end{array}$$

an absolute right lifting of  $id$  through  $\Delta$ .

A particular limit of a  $J$ -shaped diagram  $1 \xrightarrow{d} A^J$

$$\begin{array}{ccc} & & A \\ \lambda \nearrow & \Downarrow \lambda & \downarrow \Delta \\ 1 & \xrightarrow{\quad} & A^J \end{array}$$

an absolute right lifting.

Universal property of  $(\lambda, \lambda)$ :

$$\begin{array}{ccc} \text{Given} & \begin{array}{ccc} X & \xrightarrow{\quad} & A \\ \downarrow & \Downarrow \lambda & \downarrow \Delta \\ 1 & \xrightarrow{d} & A^J \end{array} & = & \begin{array}{ccc} X & \xrightarrow{\quad} & A \\ \downarrow \exists \nearrow & \Downarrow \lambda & \downarrow \Delta \\ 1 & \xrightarrow{\quad} & A^J \end{array} \end{array}$$

Goal: Right adjoints preserve limits.

Lemma: Let  $\begin{array}{ccc} & B & \\ \lambda \nearrow & \Downarrow \lambda & \downarrow f \\ C & \xrightarrow{g} & A \end{array}$  be absolute right lifting diagram

Given

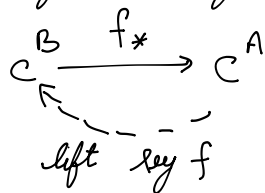
$$\begin{array}{ccc} & D & \\ \searrow K & \Downarrow K & \downarrow \\ & B & \\ \lambda \nearrow & \Downarrow \lambda & \downarrow \\ C & \xrightarrow{\quad} & A \end{array}$$

$K$  is an absolute right lifting

iff  $\lambda \circ K$  is.

(This generalizes pullbacks)

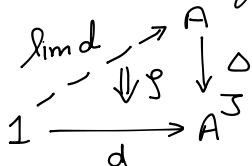
Digression: right liftings are right adjoints



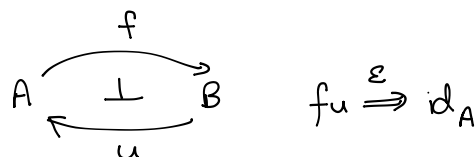
and  $\gamma$  is a counit.

Th<sup>m</sup>: Right adjoints preserve limits.

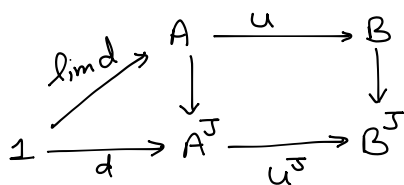
Proof: Given a limit diagram



and



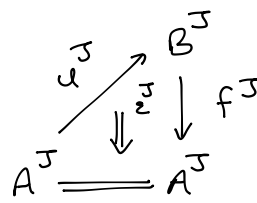
Claim we need to show



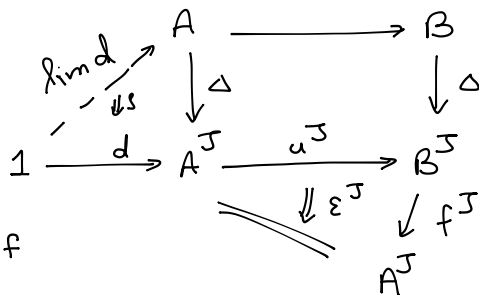
Proof:  $f \dashv u$  with counit  $\varepsilon : fu \Rightarrow \text{id}_A$

$\Rightarrow f^J \dashv u^J$  with counit  $\varepsilon^J$

i.e. we've a right lifting diagram

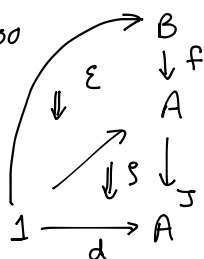


To prove this consider



and simplify this too

and both  $\varepsilon, \varepsilon^J$  are also right liftings.



## Initial Functor

Idea Consider  $1 \xrightarrow{d} A^J$ ,  $J \in \mathbf{qCat}$ ,  $J$  has an initial element  $1 \xrightarrow{i} J$   
Then  $\lim d = d(i) \in A$ .

Def  $I \xrightarrow{K} J \in \mathbf{qCat}$  is initial (or in  $\mathcal{K}$  if  $\mathcal{K}$  is cart. closed)  
is initial if  $J$ -indexed limits exist iff restricted  $I$ -indexed limits exist + limits coincide.

Prop: If  $I \xrightarrow{K} J$  has a right adjoint then  $K$  is initial.