Some Topological SS

1. Barr or Rothenberg - Steenrod SS

Given a principal G-fibration $G \to E \xrightarrow{\pi} B$, let p be prime.
Then $\exists$ first quadrant SS with

$$E_2^{r,s} = \mathrm{Ext}^{r,s}_{H_*(G;\mathbb{F}_p)}(H_*(E);\mathbb{F}_p) \Rightarrow H^{r+s}(B;\mathbb{F}_p)$$

$H_*(E)$ is a right $H_*(G)$ module as $G \curvearrowright E$

Sketch of construction

1. Cartan Eilenberg resolution on $C_*(E)$ as $C_*(G)$ modules - free resolution, in homology this yields a free resolution of $H_*(E)$ as a $H_*(G)$ modules, call this P.
2. $\cancel{C_*(E) \otimes_{C_*(G)} \mathbb{F}_p = C_*(B) \Longrightarrow}$ $P_\cdot \otimes_{C(G)_*} \mathbb{F}_p \simeq C_*(B)$ as total complexes

   Work it out
3. $P_\cdot \otimes_{C(G)_*} \mathbb{F}_p$ is a bicomplex whose total homology computes $H_*(B;\mathbb{F}_p)$
4. Filter vertically and then horizontally
$$(E^1, d^1) = H_*(P_\cdot) \otimes_{H_*(G)} \mathbb{F}_p$$
5. Consider $\mathrm{Hom}_{C_*(G)}(P_\cdot(E);\mathbb{F}_p)$
$$= \mathrm{Hom}_{C_*(G)}(P_\cdot(E) \otimes_{C_*(G)} \mathbb{F}_p ; \mathbb{F}_p)$$ this a co-chain complex computing $H^*(B;\mathbb{F}_p)$
6. vertical, horizontal
$$E_1 = \mathrm{Hom}_{H_*(G)}(H_*(E);\mathbb{F}_p)$$
$$E_2 = \mathrm{Ext}_{H_*(G)}(H_*(E);\mathbb{F}_p)$$

2. Eilenberg Moore SS

$F \to E \to B$ fiber bundle, $\pi_1(B) \curvearrowright H_*(B;\mathbb{F}_p)$ trivially then,

$\exists$ second quadrant cohomologial SS

$$E_2 = \mathrm{Tor}_{-r,s}^{H^*(B,\mathbb{F}_p)}(H^*(E,\mathbb{F}_p)) \Rightarrow H^{r+s}(F;\mathbb{F}_p) \quad \text{(convergence is non-trivial)}$$

Sketch of proof:

$$\begin{array}{ccccc} F & \to & E & \to & B \\ \uparrow & & & & \\ \Omega E & \to & \Omega B & & \end{array} \quad \rightsquigarrow$$

Ref: McCleary
User's guide

Q. Is the E_* term for Serre SS for $\Omega B \to F \to E$ is the E_2-term for EMSS.

Q. $H^*(BU(n);\mathbb{F}_p)$, $H^*(\Omega S^{2n+1};\mathbb{F}_p)$

→ If $F \to E \to B$ fibration, $\pi_1(B)$ acting trivially on $H^*(F;\mathbb{Z}/p)$ then $\exists$ proj. resolution of $C^*(E;\mathbb{Z}/p)$ then $\exists$ proj resolution of $C^*(E,\mathbb{Z}/p)$ $P.\pi$ as a $C^*(B;\mathbb{Z}/p)$ module then $\mathrm{Tot}(P.(\pi) \otimes_{C^*(B;\mathbb{Z}/p)} \mathbb{Z}/p) \simeq C^*(F)$

We use $\mathbb{Z}/p$ because we want to use Kunneth theorem somehow.

Cartan Eilenberg Resolution:

Given a DGA: $0 \to E^0 \xrightarrow{d} E^1 \xrightarrow{d} E^2 \to \cdots$ $\qquad Z^p = \ker(E^p \xrightarrow{d} E^{p+1})$ $\qquad B^p = \mathrm{im}(E^{p-1} \to E^p)$

Then by repeatedly applying the Horseshoe Lemma we get a resolution $P^{r,s}$ of E^* such that good things happen when we take horizontal cohomologies

1) $Z^\delta(P^{*,s})$ is a resolution of Z^s
2) $B^\delta(P^{*,s})$ is a resolution of B^s
3) $H^\delta(P^{*,s})$ is a resolution of H^s
4) $P^{*,s}$ is a resolution of E^s
5) $\mathrm{Tot}(P^{*,*})$ is quasi-isomorphic to E^*.

$$\begin{array}{cccccccccccc} 0 & \to & E^0 & \to & E^1 & \to & E^2 & \to & E^3 & \to & \cdots & \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & & \\ 0 & \to & P^{0,0} & \to & P^{0,1} & \to & P^{0,2} & \to & P^{0,3} & \to & \cdots & \xrightarrow{d} \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & & \delta\uparrow \\ 0 & \to & P^{1,0} & \to & P^{1,1} & \to & 1,2 & \to & 1,3 & \to & \cdots & \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & & \\ 0 & \to & P^{2,0} & \to & P^{2,1} & \to & P^{2,2} & \to & 2,3 & \to & \cdots & \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & & \\ & & \vdots & & \vdots & & \vdots & & \vdots & & & \end{array}$$

Note that is not always a first quadrant spectral complex

Barr Spectral Sequence :

$G \supseteq E$ and hence $C_*(E)$ becomes a $C_*(G)$ module. Look at the C-E projective resolution here.

$P_{*,*} \longrightarrow C_*(E)$. Hom into $\mathbb{F}_p$ in each slot to get a map

$$\mathrm{Hom}_{C_*(G)}(P_{**}, \mathbb{F}_p) \longleftarrow \mathrm{Hom}(C_*(E); \mathbb{F}_p) \cong (C_*(E))^{C_*(G)} \sim C_*(B)$$

Claim : $\mathbb{F}_p$ is an injective module over $C_*(G)$ and hence the above map remains a quasi-iso.

First note that $\mathbb{F}_p$ is a direct summand of $C_*(G)$ and hence is projective. Then $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{F}_p, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{F}_p$ and hence $\mathbb{F}_p$ is injective.

Now that we have a double complex computing $C_*(B)$ we can run the SS starting with d:

$$E_1^{r,s} = H_d^r(\mathrm{Hom}_{C_*(G)}(P_{*_1}, \mathbb{F}_p))$$

Is it okay to change $C_*(G)$ to $H_*(G)$?

$$= \mathrm{Hom}_{H_*(G)}(H_d^r(P_{*,s}), \mathbb{F}_p)$$

as $\mathbb{F}_p$ injective

$$\begin{tikzcd}
0 \arrow[r] & (P_{1,0}, \mathbb{F}_p) \arrow[r] & (P_{1,1}, \mathbb{F}_p) \arrow[r] & (P_{1,2}; \mathbb{F}_p) \arrow[r] & {} \\
0 \arrow[r] & (P_{0,0}, \mathbb{F}_p) \arrow[r, "d"] \arrow[u, "\delta"] & (P_{0,1}, \mathbb{F}_p) \arrow[r] \arrow[u] & (P_{0,2}, \mathbb{F}_p) \arrow[r] \arrow[u] & {} \\
0 \arrow[r] & (C_0(E); \mathbb{F}_p) \arrow[r, "d"] \arrow[u, "\delta"] & (C_1(E); \mathbb{F}_p) \arrow[r] \arrow[u] & (C_2(E), \mathbb{F}_p) \arrow[r] \arrow[u] & {} \\
0 \arrow[r] & C_0(B) \arrow[r, "d"] \arrow[u, phantom, "\wr"] & C_1(B) \arrow[r, "d"] \arrow[u, phantom, "\wr"] & C_2(B) \arrow[r] \arrow[u, phantom, "\wr"] & {}
\end{tikzcd}$$

Requires flatness. See Macleary

$$E_2^{r,s} = H_\delta^s(\mathrm{Hom}_{H_*(G)}(H_d^r(P_{*,s}), \mathbb{F}_p))$$

$$= \mathrm{Ext}^{r,s}_{H_*(G)}(H_*(E), \mathbb{F}_p)$$ as $H_d^r(P_{*,s})$ was a resolution of $H_*(E)$:) □

Note that an arbitrary ring R is not injective in general over itself.

We have a corresponding spectral sequence for the homology groups also.

$$E^2_{r,s} = \mathrm{Tor}^{H_*(G)}_{r,s}(H_*(E); \mathbb{F}_p) \Longrightarrow H_{r+s}(B)$$

→ Start with $P_{**} \xrightarrow{\sim} C_*(E)$ as $C_*(G)$ modules. As $\mathbb{F}_p$ is flat we would get

$$P_{**} \otimes \mathbb{F}_p \xrightarrow{\sim} C_*(E) \otimes_{C_*(G)} \mathbb{F}_p$$

→ Run the double complex spectral sequence

$$E^1_{r,s} = H_d^r(P_{*,s} \otimes_{C_*} \mathbb{F}_p)$$

$$= H_d^r(P_{*,s}) \otimes_{H_*} \mathbb{F}_p$$ as $\mathbb{F}_p$ is flat as a $C_*(G)$ module

$$E^2_{r,s} = H_\delta^s(H_d^r(P_{*,s}) \otimes \mathbb{F}_p) = \mathrm{Tor}^{H_*(G)}_{r,s}($$