Homotopy categories :

Def : A an ∞-category. Its homotopy category is

$$\mathrm{ho}A := \mathrm{Fun}^h(1, A) \qquad 1 = \text{terminal object}$$

i.e.

ob (ho A) = $1 \xrightarrow{a} A$ are "elements" of A.

mor (ho A) = $f : a \longrightarrow b$

$$\begin{tikzcd} 1 \arrow[r, bend left=50, "a"{name=U}] \arrow[r, bend right=50, "b"'{name=D}] & A \arrow[from=U, to=D, Rightarrow, "f"'] \end{tikzcd}$$

By Yoneda Lemma :

An object is uniquely determined upto iso by its "generalized elements" $X \xrightarrow{a} A$

But we're interested in studying ∞-categories upto equiv. & not an iso.

- By 2-category Yoneda :

$A \xrightarrow{f} B$ is an equiv if its an equiv on all generalized elements $\forall X$. $\mathrm{Hom}(X, A) \xrightarrow[\sim]{} \mathrm{Hom}(X, B)$ is an equiv.

- We've shown that equiv between ∞-cats coincide with 2-cat equiv visible in hK.

Initial / terminal elements in A, limits, colimits

Def : A = ∞-category

An initial element is left adjoint to $A \xrightarrow{!} 1$

A terminal element is right adjoint to $A \xrightarrow{!} 1$.

$$\begin{tikzcd} 1 \arrow[r, bend left, "i"] \arrow[r, phantom, "\perp"] & A \arrow[l, bend left, "!"] \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, bend left, "!"] \arrow[r, phantom, "\perp"] & A \arrow[l, bend left, "t"] \end{tikzcd}$$

Initial : $1 \xrightarrow{i} A$, counit :

$$\begin{tikzcd} & 1 \arrow[dr, "i"] & \\ A \arrow[ur, "!"] \arrow[rr, equal] & \Downarrow & A \end{tikzcd}$$

+ 1- triangle equality.

Prop: $A \simeq A'$. Then A has an initial element iff A' does.
In which case its preserved under equiv.

Prop: Initial elements are "representably initial"

i.e. $X \xrightarrow{!} 1 \xrightarrow{i} A$ is initial in $\mathrm{Fun}(X, A)$.

i) $\forall X$ $$\begin{tikzcd} \mathbb{1} \cong \mathrm{Fun}(X,1) \arrow[r, shift left=2, "i_*"] \arrow[r, phantom, "\perp"] & \mathrm{Fun}(X,A) \arrow[l, shift left=2, "!"] \end{tikzcd} \in q\mathcal{C}at$$

ii) $\forall X$ $$\begin{tikzcd} \mathbb{1} \cong \mathrm{Fun}^h(X,1) \arrow[r, shift left=2, "i_*"] \arrow[r, phantom, "\perp"] & \mathrm{Fun}(X,A) \arrow[l, shift left=2, "!"] \end{tikzcd} \in \mathcal{C}at$$

Prop: Any two initial elements of A are iso in hoA.

Limits, Colimits:

2 kinds of diagrams: 1) diagrams indexed by a simplicial set J

($\mathcal{K}$ cartesian closed) 2) diagrams indexed by other ∞-category J.

Def: ∞-category of J-indexed diagrams valued in A: $A^J : \mathcal{K}_J^{op} \times \mathcal{K}_A \longrightarrow \mathcal{K}$

$$A^J := \begin{cases} \text{i) simplicial co-tensor} \\ \text{ii) used cartesian closedness of } \mathcal{K}. \end{cases}$$

- $J \xrightarrow{!} 1$ induces "constant functor" $A \xrightarrow{\Delta} A^J$

Def: An ∞-cat A has all colims of shape J if Δ has a left adjoint.
all limits of shape J if it has a right adjoint.

$$\begin{tikzcd} A \arrow[r, "\Delta" description] & A^J \arrow[l, bend right=50, "\mathrm{colim}"', ""{name=U}] \arrow[l, bend left=50, "\lim", ""{name=D}] \end{tikzcd}$$
$\perp$ $\perp$

colim cone :

$$\begin{tikzcd} A^J \arrow[rr, equal] \arrow[dr] \arrow[rr, phantom, shift right=4, "\Downarrow"] & & A^J \\ & A \arrow[ur, "\Delta"'] & \end{tikzcd}$$

lim cone :

$$\begin{tikzcd} & A \arrow[dr, "\Delta"] \arrow[d, phantom, "\Downarrow"] & \\ A^J \arrow[ur, "\lim"] \arrow[rr, equal] & {} & A^J \end{tikzcd}$$

def^n: In 2-category hK, given a co-span $C \xrightarrow{g} A \xleftarrow{f} B$ an absolute left-lifting of g through f is

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, dashed, "l"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\lambda\Uparrow"] & A \end{tikzcd}$$

s.t.

$$\begin{tikzcd} X \arrow[r, "b"] \arrow[d, "c"'] \arrow[dr, phantom, "\Uparrow"] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & A \end{tikzcd} = \begin{tikzcd} X \arrow[r] \arrow[d] \arrow[r, phantom, shift right=3, "\Uparrow"] & B \arrow[d] \\ C \arrow[ur] \arrow[r] \arrow[r, phantom, shift left=3, "\Uparrow\lambda"] & A \end{tikzcd}$$

for absolute right lifting reverse the 2-cells.

Lem If

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, dashed] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3, "\Uparrow\lambda"] & A \end{tikzcd}$$

is an absolute left lifting diagram so is

$$\begin{tikzcd} & & B \arrow[d] \\ X \arrow[r] & C \arrow[ur, dashed] \arrow[r] \arrow[ur, phantom, shift right=3, "\Uparrow\lambda"] & A \end{tikzcd}$$

Prop: $\eta : id_B \Rightarrow uf$ is unit of an adjunction iff

$$\begin{tikzcd} & A \arrow[d, "u"] \\ B \arrow[ur, "f"] \arrow[r, equal] \arrow[ur, phantom, shift right=3, "\Uparrow\eta"] & B \end{tikzcd}$$

is an absolute left lifting diagram.

similarly for counit.

Note: colimit cone is an absolute left lifting diagram

a single diagram $\rightsquigarrow$

$$\begin{tikzcd} & & A^J \arrow[d, "\Delta"] \\ 1 \arrow[r, "d"] & A^J \arrow[ur, "colim"] \arrow[r, equal] \arrow[ur, phantom, shift right=3, "\Uparrow\eta"] & A^J \end{tikzcd}$$

Def: Let $D \xrightarrow{d} A^J$ be a family of J indexed diagrams of A.

A colim is an absolute left lifting :

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ D \arrow[ur, "colim"] \arrow[r, "d"'] \arrow[ur, phantom, shift right=3, "\lambda\Uparrow"] & A^J \end{tikzcd}$$

lim right :

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ D \arrow[ur, "lim"] \arrow[r, "d"'] \arrow[ur, phantom, shift right=3, "\Downarrow"] & A^J \end{tikzcd}$$

Notation:

$$\begin{tikzcd} \Delta = {[0]} \arrow[r, shift left] \arrow[r, shift right] & {[1]} \arrow[l] \arrow[r, shift left] \arrow[r] \arrow[r, shift right] & {[2]} \arrow[l] \arrow[r] & \cdots \end{tikzcd}$$

$\cap$

$$\Delta_+ = \Delta \cup \overset{\swarrow \text{ initial}}{[-1]}$$

$\cap$

$$\begin{tikzcd} \Delta_{-\infty} = {[-1]} & {[0]} \arrow[l, bend right, "s^{-1}"'] & {[1]} \arrow[l, bend right, "s^{-1}"'] & \cdots \end{tikzcd}$$

Prop: $A = \infty$-cat. If A admits augmentation + splitting has a colimit, the colimit given by the augmentation

$$\begin{tikzcd} & & A \arrow[d, "\Delta"] \\ A^{\Delta_{-\infty}} \arrow[r] & A^{\Delta^{op}_{+}} \arrow[ur, "ev_{[-1]}"] \arrow[r] \arrow[ur, phantom, shift left=3, "\Uparrow"] & A^{\Delta^{op}} \end{tikzcd}$$

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