dg-categories as non-commutative spaces

Motivation: Examples of dg-categories

- Fix field k, all functors are fully derived

Ex 2: X a smooth variety $\longrightarrow$ $QCoh(X)$ a dg category

- ob = complexes of $\mathcal{O}_X$-mod, quasi-coherent
- we = quasi-iso
- mor = $\mathrm{Hom}^{\cdot}(F,G)$ Hom complexes
  ↳ cohomology computes Ext groups

$\exists\, X, Y$ s.t. $X \not\simeq Y$ but $QCoh(X) \simeq QCoh(Y)$
does not preserve tensor product

so cannot recover X from $QCoh(X)$
but can remember:

- Hochschild chains $HH_*(QCoh(X)) \overset{HKR}{\simeq} \Gamma(X, \oplus \Omega^p_X[p])$
  (lose information on cohomology)
- algebraic K-theory

Ex 2: X: nice topological space, connected, based

$Loc(X) = Fun^{\infty}(X_\bullet, Vect_k) \simeq C_*(\Omega X)\text{-}Mod$

Kan-complex, dg-vector spaces, dg-modules

eg: k_X = constant local system

- $Hom(k_X, k_X) \simeq C^*(X;k)$
- $HH_*(Loc\, X) \simeq C_*(LX)$ Goodwillie–Jones
  $\exists\, S^1$-action on both sides:
  and the above iso is S^1-equivariant

Ex 3: R is a dga, Mod_R – dg-modules

$HH_*(Mod_R)$ carries an S^1-action

- No $\otimes_R$ on Mod_R, no products but HH_* plays the role of "functions on the loop space" or a "de Rham complex with funny grading"

(dg category = category enriched over chain complexes)

Big dg categories: complete, cocomplete, stable, presentable

- $\emptyset \longrightarrow *$ is an equiv (initial → final)
- $$\begin{tikzcd} \Omega_* X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & * \arrow[d] \\ * \arrow[r] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma X \end{tikzcd}$$

Stability $\Leftrightarrow$ these are inverse to each other

presentable:
- $\exists$ a set of small objects
- every object can be built out of small obj via filtered colimits.

eg: in homological algebra: $\Sigma = [1]$ and $\Omega = [-1]$

'dg-cat' - Big dg cat

DG_{cont} - ∞-category of (big) dg cats with continuous dg-functors

- symmetric monoidal with internal hom called Fun with unit $Vect_k$

- A dg-category $\mathcal{C}$ is dualizable if $\exists$ a dg-category $\mathcal{C}^\vee$ and
$Vect_k \xrightarrow{co} \mathcal{C}^\vee \otimes \mathcal{C}$, $ev: \mathcal{C} \otimes \mathcal{C}^\vee \longrightarrow Vect_k$
such that
$$\mathcal{C} \xrightarrow{Id_{\mathcal{C}} \otimes co} \mathcal{C} \otimes \mathcal{C}^\vee \otimes \mathcal{C} \xrightarrow{ev \otimes Id_{\mathcal{C}}} \mathcal{C} \simeq Id_{\mathcal{C}}$$
$$\mathcal{C}^\vee \xrightarrow[co \otimes Id_{\mathcal{C}^\vee}]{} \mathcal{C}^\vee \otimes \mathcal{C} \otimes \mathcal{C}^\vee \xrightarrow{Id_{\mathcal{C}^\vee} \otimes ev} \mathcal{C}^\vee \simeq Id_{\mathcal{C}^\vee}$$

- $\forall \mathcal{D}$, $\mathcal{C}^\vee \otimes \mathcal{D} \xrightarrow{\simeq} Fun(\mathcal{C}, \mathcal{D})$

Ex: X = nice top space, $Loc(X)$ is dualizable
$$Loc(X) \otimes Loc(Y) \xrightarrow{\simeq} Loc(X \times Y)$$
$$E, F \longmapsto E \boxtimes F := p_X^* E \times p_Y^* F$$

Th^m:
$$Loc(*) = Vect_k \xrightarrow{p^*} Loc(X) \xrightarrow[\text{induction on sys}]{\Delta_!} Loc(X \times X) \simeq Loc(X) \otimes Loc(X)$$
$$Loc(X) \times Loc(X) \simeq Loc(X \times X) \xrightarrow[\text{restriction}]{\Delta^*} Loc(X) \xrightarrow{p_!} Loc(*) = Vect$$

HH_* of dualisable dg cats

$\mathcal{C}$ dualizable : $HH_*(\mathcal{C}) := \mathrm{Tr}(\mathrm{Id}_{\mathcal{C}})$

$F \circlearrowright \mathcal{C}$

$$\mathrm{Tr}(F) : \mathrm{Vect} \xrightarrow{\mathrm{co}} \mathcal{C}^\vee \otimes \mathcal{C} \xrightarrow{\mathrm{Id}_{\mathcal{C}^\vee} \otimes \mathcal{C}} \mathcal{C}^\vee \otimes \mathcal{C} \simeq \mathcal{C} \otimes \mathcal{C}^\vee \xrightarrow{\mathrm{ev}} \mathrm{Vect}$$

$$k \longmapsto \cdots \longrightarrow (\mathrm{Tr}\, F)(k)$$

Ex:

$$\mathrm{Vect} \xrightarrow{p^*} \mathrm{Loc}(X) \xrightarrow{\Delta_!} \mathrm{Loc}(X \times X) \xrightarrow{\Delta^*} \mathrm{Loc}(X) \xrightarrow{p_!} \mathrm{Vect}$$

$$\begin{tikzcd}
LX \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X \arrow[r, "p_X"] \arrow[d, "\Delta"] & * \\
X \arrow[r, "\Delta"] \arrow[d, "p_X"] & X \times X & \\
* & &
\end{tikzcd}$$

Local systems satisfy base change : $\Delta^* \Delta_! \simeq \pi_! \pi^*$

$$HH_*(\mathrm{Loc}(X)) \simeq p_{X!}\, \pi_!\, \pi^* p_X^*(k) \simeq p_{LX\,!}(k_{LX}) \simeq C_*(LX)$$

Ex: $\mathcal{C} = \mathrm{Mod}_R$, $\mathcal{C}^\vee = \mathrm{Mod}_{R^{op}}$

$$\mathrm{ev} : \mathrm{Mod}_R \otimes \mathrm{Mod}_{R^{op}} \longrightarrow \mathrm{Vect}$$
$$M, N \longmapsto M \otimes_R N$$

$$\mathrm{co} : \mathrm{Vect} \longrightarrow \mathrm{Mod}_{R^{op}} \otimes \mathrm{Mod}_R$$
$$k \longmapsto R$$

$$HH_*(\mathrm{Mod}_R) = R \overset{L}{\underset{R^{op} \otimes R}{\otimes}} R$$