Eilenberg Mclane Spaces

$H^n(K(G,n);G) \cong \mathrm{Hom}_{\mathbb{Z}}(G,G)$

$\quad i_n \longleftarrow\!\shortmid \ id$

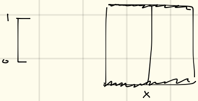
Prop: $\exists$ bijection of sets $[K(G_1,n),K(G_2,n)]_* = \mathrm{Hom}(G_1,G_2) = H^n(K(G_1,n);G_2)$

$\varphi \longmapsto [\varphi_*(\pi_n)] \longmapsto \varphi^* i_n(K(G_2,n))$

Proof by construction. Extend maps skeleton-wise for one direction.
For the other direction use obstruction theory.

given $\varphi_1,\varphi_2 : K(G_1,n) \longrightarrow K(G_2,n)$ s.t. $\varphi_{1*}\pi_n \cong \varphi_{2*}\pi_n$

want to construct a homotopy from φ_1 to φ_2.



$\xrightarrow[?]{h} K(G_2,n)$

$K(G_1,n)$

h defined on $K(G,\times n)\times\{0,1\} \cup * \times [0,1]$
Need to extend h to other cells.
Start with $(\vee S^n)\times[0,1]$ and do induction.

Prop: Any $K(G,n)$ is homotopy equivalent to a $K(G,n)$ built by starting with a wedge of S^n.

Proof: Assume $K(G,n)$ is a CW complex with $n-1$ skeleton X_{n-1}. We will show $\bar{K}(G,n)$ obtained by collapsing X_{n-1} to a pt is a homotopy equivalence.
Need to construct a map $f: \bar{K}(G,n) \longrightarrow K(G,n)$. Assume $\bar{K}(G,n) = K(G,n) \underset{X_{n-1}}{\cup} \mathrm{Cone}(X_{n-1})$
Again do this using induction and obstruction theory. Construct & Check a homotopy inverse.

Cor: $K[G,n]$ is unique upto homotopy.
Q. Can we do this using Yoneda ?

Cor: G abelian $\Rightarrow K(G,n)$ has group structure upto homotopy.

i.e. $\exists \mu: K(G,n)\times K(G,n) \longrightarrow K(G,n)$, id, inverse only upto homotopy.

Proof:

$$+: G\times G \longrightarrow G \qquad\qquad \mu: K(G\times G,n) = K(G,n)\times K(G,n) \longrightarrow K(G,n)$$

$$-: G \longrightarrow G \qquad \Rightarrow \qquad -: K(G,n) \longrightarrow K(G,n)$$

$$0: \{id\} \longrightarrow G \qquad\qquad 0: K(\{id\},n) = * \longrightarrow K(G,n)$$

Cor: $[X, K[G,n]]_*$ is a group.

Th^m: $[X, K(G,n)]_* \cong \tilde{H}^n(X,G)$ naturally. $\rightsquigarrow$ Brown Representability

Proof: Consider finite CW complexes, pointed.

$[\Sigma X, K(G,n+1)]_* = [X, \Omega K(G,n+1)]_* = [X, K(G,n)]_*$

$A \subseteq X \Rightarrow \exists$ exact seq $[X \cup CA, K(G,n)]_* \longrightarrow [X, K(G,n)]_* \longrightarrow [A, K(G,n)]_*$

So $[X, K(G,n)]_*$ is a cohomology theory.

We have a natural map $[X, K(G,n)]_* \xrightarrow{\xi} H^n(X,G)$

When $X = S^k$, ξ is an iso. $\varphi \longmapsto \varphi^* \iota_n$

So X is an isomorphism for all X.

Cohomology Operations

Def^n (Cohomology operation):

ξ of type (n_1, n_2, G_1, G_2) is a natural map: $\psi: H^{n_1}(X,G_1) \longrightarrow H^{n_2}(X,G_2)$

(Not required for ψ to be a group homomorphism)

eg: $H^n(X,\mathbb{Z}) \longrightarrow H^{kn}(X,\mathbb{Z})$ not additive, but $H^n(X, \mathbb{Z}/p) \longrightarrow H^{pn}(X, \mathbb{Z}/p)$ is additive

$\alpha \longmapsto \alpha^k$ $\qquad\qquad$ $\alpha \longmapsto \alpha^p$

Th^m: $(n_1, n_2, G_1, G_2) \xleftrightarrow{1-1} K(n_1, G_1) \longrightarrow K(n_2, G_2)$

Proof. By Yoneda's lemma.

Def^n (Stable cohomology operation):

ξ type (k, G_1, G_2) is collection of cohomology operations of type $\{n, n+k, G_1, G_2\}$ such that

$$\begin{tikzcd}
H^{n+1}(\Sigma X, G_1) \arrow[r, "\psi_{n+1}"] \arrow[d, "\cong"'] & H^{n+k+1}(\Sigma X, G_2) \arrow[d, "\cong"] \\
H^n(X, G_1) \arrow[r, "\psi_n"] & H^{n+k}(X, G_2)
\end{tikzcd}$$

i.e. $\hat{\psi}_{n,n+k,G_1,G_2} = \Omega \hat{\psi}_{n+1,n+k+1,G_1,G_2}$ $\qquad$ $\hat{\psi}_{n,n+k,G_1,G_2} \in [K(n,G_1), K(n+k,G_2)]$.

Cor: Stable cohomology operations are additive.

eg: 1) Bockstein: $H^n(X, \mathbb{Z}/k) \longrightarrow H^{n+1}(X, \mathbb{Z})$

Consider $\mathbb{Z} \xrightarrow{k} \mathbb{Z} \longrightarrow \mathbb{Z}/k \rightsquigarrow H^n(X, \mathbb{Z}/k) \xrightarrow{\delta} H^{n+1}(X, \mathbb{Z})$

Check. - Naturality is straight forward.

2) P^k of type $(2pk-2k, \mathbb{Z}/p, \mathbb{Z}/p)$

on $H^{2k}(X;\mathbb{Z}/p)$ it is given by $\alpha \longrightarrow \alpha^p$

Steenrod power operation ⇝

Stability of the Bockstein:

$X \to Y \to Y \sqcup CX$ and $0 \to G \to H \to K \to 0$ gives us a complex whose rows and colums are exact:

$$\begin{tikzcd}
 & \arrow[d] & \arrow[d] & \arrow[d] & \arrow[d] & \\
\cdots & H^*(X;G) \arrow[l] \arrow[d] & H^*(Y,G) \arrow[l] \arrow[d] & H^*(Y\cup CX,G) \arrow[l] \arrow[d] & H^{*-1}(X,G) \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\cdots & X;H \arrow[l] \arrow[d] & Y,H \arrow[l] \arrow[d] & Y\sqcup CX,H \arrow[l] \arrow[d] & {}^{*-1}(X;H) \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\cdots & X;K \arrow[l] \arrow[d] & Y;K \arrow[l] \arrow[d] & Y\cup CX;K \arrow[l] \arrow[d] & {}^{*-1}(X;K) \arrow[l] \arrow[d] & \cdots \arrow[l] \\
 & {} & {} & {} & {} &
\end{tikzcd}$$

Use $Y = CX$,

$0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \longrightarrow \mathbb{Z}/p \to 0$

Stability of the Bockstein is then just commutativity of a square here.

From now on we only consider $(*, \mathbb{Z}/2, \mathbb{Z}/2)$.

Note that there is a multiplication defined on these by composition, so that we have a ring structure.

Def^n: mod-p Steenrod Algebra: $\mathcal{A}_p$ (See - Steenrod - Cohomology Operations)

graded $\mathbb{Z}/p$ algebra of stable cohomology operations of $H^*(-;\mathbb{Z}/p)$.

Th^m: $(p=2)$ $\mathcal{A}_2$ has following structure:

H is generated as an $\mathbb{Z}/2$-algebra by operations Sq^k of deg k; $k \geq 0$

a) $Sq^0 = \mathrm{Id}$; $Sq^k(x) = x^2$ if $|x| = k$

b) $Sq^i(x) = 0$ if $|x| < i$ (instability condition)

c) $Sq^k(x \cup y) = \sum_{i+j=k} Sq^i(x) \cup Sq^j(y)$. (Cartan formula)

d) $Sq^a Sq^b = \sum_{j=2}^{[a/2]} \binom{b-1-j}{a-2j} Sq^{a+b-j} Sq^j$ if $a < 2b$ (Adem Relations)

e) Sq^1 = Bockstein (Compute Bockstein SS)

$\mathcal{A}$ is a cocommutative Hopf algebra, hence $\mathcal{A}^*$ is a commutative algebra $\mathcal{A}^* = \mathbb{Z}/2[\xi_i]$ $|\xi_i| = 2^i - 1$

Th^m: A vector space basis for $\mathcal{A}$ is given by monomials $\{Sq^0, Sq^{i_1}\cdots Sq^{i_k}\}$ $\quad i_s \geq 2 i_{s+1}$, called admissible basis.

given $Sq^I := Sq^{i_1}\cdots Sq^{i_k}$, the excess $e(I) := (i_1-2i_2)+(i_2-2i_3)+\cdots+(i_{k-1}-2i_k)+i_k = i_1-(i_2+i_3\cdots+i_k)$

deg $d(I) := i_1+\cdots+i_k$.

Th^m (Serre): $H^*((\mathbb{Z}/2;n);\mathbb{Z}/2)$ is a polynomial algebra on generators $Sq^I(\iota_n)$ where I is admissible of excess $<n$.

eg. $K((\mathbb{Z}/2,1),1) = \mathbb{Z}/2[Sq^0(\iota_1)]$

$K((\mathbb{Z}/2,2),2) = \mathbb{Z}/2[\iota_2, Sq^1\iota_2, Sq^2Sq^1\iota_2, Sq^4Sq^2Sq^1\iota_2, \cdots]$

Note: not all cohomology operations are stable eg. $(Sq^2Sq^1\iota_2)^2$ but "products" of stable operations.

Lemma: $F\to E\to B$ Serre SS, $d_k: E^{0,k-1}\longrightarrow E^{k,0}$ transgression. $\alpha\in H^{k-1}(F)$, $\beta\in H^k(B)$

if $d_n[\alpha]=[\beta]$ then $d_{n+k}(\theta(\alpha))=\theta(\beta)$ for θ deg k cohomology operation, stable.

Proof: $H^{k-1}(F)\xrightarrow{\delta} H^k(E,F)$

$\uparrow \pi^*$

$\tilde{H}^k(B)$

$E_k^{0,k-1} = \{\alpha\in H^{k-1}(F) \mid \delta(\alpha) = \operatorname{Im}\pi^*\}$

$d_k\alpha = [(\pi^*)^{-1}\delta\alpha]$

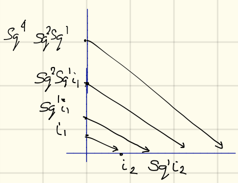
Prove this

$d_k(\alpha)=\beta \Rightarrow \pi^*\beta = \delta\alpha \Rightarrow \pi^*(\theta\beta) = \delta(\theta\alpha) \Rightarrow d_{n+k}(\theta(\alpha)) = [\theta(\beta)]$.

Proof: by induction. $n=1$ $\quad K(\mathbb{Z}/2;1) = \mathbb{RP}^\infty$

$\exists$ fibrations $K(\mathbb{Z}/2;n)\to * \to K(\mathbb{Z}/2,n+1)$ ~ Use Borel's th^m

eg: $K(\mathbb{Z}/2;1)\to * \to K(\mathbb{Z}/2,2)$



These are the fundamental seq of generators $\quad \begin{cases} Sq^1\iota_1 = \iota_1^2 \\ Sq^2Sq^1\iota_1 = \iota_1^4 \end{cases}$

So by Borel's th^m $\quad H^*(K(\mathbb{Z}/2,2);\mathbb{Z}/2) = \mathbb{Z}/2[Sq^I]$ $\quad e(I)<2$.

Assume for $K(\mathbb{Z}/2,n)$. Given an admissible $I=(i_1\cdots i_k)$, let $I_s := (i_{s+1},\cdots,i_k)$ $\quad s\geq 0$.

$\hat{I}$ = smallest s such that $e(I_s) < e(I)$ i.e. $I = (2i_2, 2i_3, \ldots, 2i_{s-1}, i_s, i_{s+1},\cdots,i_k)$

$i_s > 2i_{s+1}$

if $e(I) = m \Rightarrow i_s - (i_{s+1} + \cdots + i_k) = m$

$\Rightarrow i_s = \deg \hat{I} + m$

$\{(\hat{I}, s) \mid e(\hat{I}) < n, s \geq 0\} \longleftrightarrow \{I \mid e(I) < n+1\}$

Under this bijection

$$d_{d(I)+n+1}\left(\left(Sq^{\hat{I}}(i_n)\right)^{2^s}\right) = d_{d(I)+n+1}\left(Sq^{I}(i_n)\right) = Sq^{I}(d_{n+1} i_n) = Sq^{I}(i_{n+1})$$

$Sq^{\hat{I}}(i_n)^{2^s}$ form a fundamental system of generators.

$\Rightarrow$ By Borel's th^m; $H^*(K(\mathbb{Z}/2, n+1); \mathbb{Z}/2) = \mathbb{Z}/2\,[Sq^{I}(i_{n+1})]$ $e(I) < n+1$