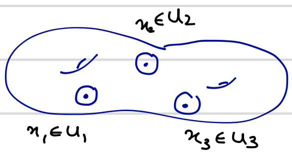
Over $\mathbb{C}$, $\mathrm{Ran}\, X = \{\text{non-empty finite } S \subseteq X\}$

$\mathcal{F}$ = cosheaf of spaces on $\mathrm{Ran}(X)$

$$\mathcal{F}(\mathrm{Ran}(X)) = \mathrm{Map}(X, BG) \approx \mathrm{Bun}_G(X)$$

$$\mathcal{F}(\mathrm{Ran}(U_1, \dots, U_n)) = \mathrm{Map}_c(U_1 \cup U_2 \cdots \cup U_n, BG) \approx \prod_{i=1}^{n} \Omega^2 BG$$

eg:

X : 

"costalk" = G-bundles on X which are trivial outside the 3 points x_1, x_2, x_3

__Over any field__ :

$$\mathrm{Ran}_G(X) = \{(P, S, \gamma)\}$$

P: G-bundle on X

S: finite subset of X

γ: trivialization of P outside X

$Y \longrightarrow \mathrm{Ran}\, X$ = non-empty finite set of maps $Y \longrightarrow X$

Non-abelian Poincare duality in Algebraic Geometry:

$\mathrm{Ran}_G(X) \longrightarrow \mathrm{Bun}_G(X)$ induces an iso on étale cohomology.

Reason: "fibres are contractible"

$$\begin{tikzcd}
\mathrm{Rat}(X,G) \arrow[r] \arrow[d] & \mathrm{Ran}_G(X) \arrow[d] \\
* \arrow[r] & \mathrm{Bun}_G(X)
\end{tikzcd}$$

Rational

Rat(G) classifies finite rational maps $X \longrightarrow G$

eg: $G = GL_1 = \mathbb{A}^1 \setminus \{0\} \subseteq \mathbb{A}^1$

$\mathrm{Rat}(X) \approx K(X)$ meromorphic functions = affine space

$\mathrm{Rat}(X, \mathbb{G}_m) \approx K(X) \setminus \{0\}$

corresponds to the constant function 0

Ex. $G = GL_n \subseteq M_{n,n}$

$$\begin{array}{ccc} Rat(X, M_{n,n}) & = & M_{n,n}(K(X)) \\ \cup\!| & & \downarrow \det \\ Rat(X, GL_n) & & K(X) \end{array}$$

Fix a divisor $D \subseteq X$,

consider $M_{n,n}(\Gamma(X, \mathcal{O}(D))$ has dim $\approx n^2 \deg(D)$ ← Riemann Roch

$\downarrow \det$

$\Gamma(X, \mathcal{O}(nD))$ has dim $\approx n \deg(D)$

§

$H^*(Ran_G(X); \underline{\mathbb{Q}}_\ell)$ ← constant sheaf

$\cong$

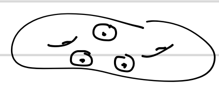
$H^*(Ran(X); R\pi_* \underline{\mathbb{Q}}_\ell)$

$\quad\quad\quad\quad\quad\quad\quad \stackrel{..}{\mathcal{A}} \rightsquigarrow$ push-forward sheaf

$\pi : Ran_G(X) \longrightarrow Ran(X)$ forget

- Fix $s \in Ran(X)$ $\quad s = \{x_1, \cdots, x_n\}$

$$Ran_G(X) \underset{Ran(X)}{\times} \{s\} = \prod_{x \in S} G(K_x) / G(\mathcal{O}_x)$$



$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad ||$

$Gr_{G,x}$ ← Affine grassmannian at the point x.

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad ||$

Projective Ind-scheme (projective algebraic variety of ∞-dim)

We think of s as a map $* \longrightarrow Ran\, X$]

$$\underset{\text{stalk}}{H^*(s^*\mathcal{A})} = H^*\left(\prod_{x \in S} Gr_{G,x}\right) \approx \bigotimes_{x \in S} H^*(Gr_{G,x})$$

- we say $\mathcal{A}$ is factorizable.

Let Y be an algebraic variety over $\mathbb{F}_q$, $\mathcal{F}$ on Y a sheaf

Th^m: Grothendieck Lefschetz trace formula

$$Tr(\varphi \mid H^*_c(X; \mathcal{F})) = \sum_{x \in Y(\mathbb{F}_q)} Tr(\varphi \mid x^* \mathcal{F})$$

Dual version:

$$Tr(\varphi^{-1} \mid H^*(Y; H^*(Y, \mathcal{F})) = \sum_{x \in Y(\mathbb{F}_q)} Tr(\varphi^{-1} \mid x^! \mathcal{F})$$

Question: Can this formula when $Y = Ran_G(X)$ and $\mathcal{F} = \mathcal{A}$?

What are the costalks : $x^! \mathcal{A}$? $\quad$ Ans: No, costalks all 0.

Why not? $\mathrm{Ran}(X)$ not a scheme, infinite dim

$$\mathrm{Ran}(X) = \bigcup_n \mathrm{Ran}(X)_{\leq n} = \{S \subseteq X, |S| \leq n\}$$

Workaround: Replace A by a "reduced" variant $\mathcal{A}_{red}$ satisfying

- $H^*(s^* \mathcal{A}_{red}) = \bigotimes_{x \in S} H^*_{red}(G_{Gr,x})$

G-semi-simple $\Rightarrow G_{Gr,x}$ connected

- $H^*(\mathrm{Ran}(X); \mathcal{A}_{red}) = H^*_{red}(\mathrm{Bun}_G(X))$

For this sheaf G-L does hold:

$$\begin{aligned}
\mathrm{Tr}(\varphi^{-1} \mid H^*(\mathrm{Ran}(X); \mathcal{A}_{red})) &= \sum_{S \subseteq X} \mathrm{Tr}(\varphi^{-1} \mid s^! \mathcal{A}_{red}) \\
\| \qquad\qquad & \qquad\qquad \| \\
1 + \mathrm{Tr}(\varphi^{-1} \mid H^*(\mathrm{Bun}_G(X))) \qquad & \quad 1 + \sum_{S \subseteq X} \prod_{x \in S} \mathrm{Tr}(\varphi^{-1} \mid x^! A) \\
&= \prod_{x \in X} \left(1 + \mathrm{Tr}(\varphi^{-1} \mid x^! \mathcal{A}_{red})\right)
\end{aligned}$$

- $\mathcal{A}_{red}$ is an example of factorization algebra on X

$\Updownarrow$ over $\mathbb{C}$

local system of E_2-algebras (non-unital)

$$H^*(x^* \mathcal{A}_{red}) = H^*_{red}(Gr_{G,x}) = H^*_{red}(\Omega^2 BG)$$

$$x^* \mathcal{A}_{red} \approx C^*_{red}(\Omega^2 BG)$$

Koszul duality for non-unital E_2-alg $\iff$ Verdier duality on $\mathrm{Ran}(X)$

$$C_*^{red}(\Omega^2 BG) \xrightarrow{\text{Koszul}} C^*_{red}(BG)$$

... going back ... $= \prod_{x \in X} \mathrm{Tr}(\varphi^{-1} \mid H^*(BG_x))$