Siegel mass Formula - Jacob Lurie

Quadratic forms - when are two related to each other by a change of coords?
↳ Depends on the ring of coefficients.

If ring $= \mathbb{R}$ ⇝ signature classifies this
ring $= \mathbb{Z}, \mathbb{Z}/n$ ⇝ ?

Let q, q' be +ve definite quadratic forms $/\mathbb{Z}$
We say q & q' are <u>in the same genus</u> if they're equivalent mod N for $N > 0$.

$O_q(R) = \{A \in GL_n(R) : q = q \cdot A\}$

$O_q(\mathbb{R}) \supseteq O_q(\mathbb{Z})$ ⇝ preserves certain lattice
compact finite

Def: $\text{Mass}(q) = \sum_{q' \in \text{genus}(q)} 1/|O_{q'}(\mathbb{Z})|$ ← natural way to count objects in the presence of symmetry.

- q is said to be unimodular if it is non-degenerate mod p all p.
  ↳ Facts: ① # variables ≥ 8
  ② all in the same genus

Th^m: Mass formula for unimodular forms

$$\sum_{\substack{q \text{ unimodular} \\ \text{in } n\text{-variables}}} \frac{1}{|O_q(\mathbb{Z})|} = \frac{\zeta(2)\,\zeta(4)\cdots\zeta(n-2)\,\zeta(\frac{n}{2})}{\text{Vol}(S^1)\,\text{Vol}(S^2)\cdots\text{Vol}(S^n)}$$

eg: $n=8$. $\text{RHS} = \frac{1}{2^{14}3^5 5^2 7}$ ← Denom = |Weyl group of E_8|

Mass formula ⇒ ∃ a unique unimodular form in 8 variables
the one coming from E_8-lattice

$n=32$ $\text{RHS} \simeq 4 \times 10^7$ ⇒ ∃ millions of unimodular quadratic forms.

§ Let q, q' be in same genus

$\Rightarrow$ for each $N>0$, $\exists\ A_n \in Gl_n(\mathbb{Z}/N)$ such that $q = q' \circ A_n$

Walog there are compatible $A = \{A_N\}_{N>0} \in GL_n(\hat{\mathbb{Z}})$ $\leftarrow$ profinite completion of $\mathbb{Z}$ $= \varprojlim \mathbb{Z}/n\mathbb{Z} = \prod_p \mathbb{Z}_p$

$\Rightarrow q$ and q' are equivalent over $\mathbb{Q}_p$ $\forall p$

$\xRightarrow{\text{Hasse Minkowski}}$ q and q' are equivalent over $\mathbb{Q}$

$\Rightarrow q = q' \circ B$ for some $B \in Gl_n(\mathbb{Q})$

$\Rightarrow q = q' \circ A = q \circ B^{-1} \circ A \Rightarrow B^{-1} \circ A \in O_q(\mathbb{A}^{fin})$ $\rightsquigarrow$ ring of finite adeles

But $B^{-1} \circ A$ is only well defined upto $O_q \cdots$ $\qquad \mathbb{A}^{fin} = \hat{\mathbb{Z}} \otimes \mathbb{Q} \subseteq \prod_p \mathbb{Q}_p$

$\Rightarrow B^{-1} \circ A \in O_q(\mathbb{Q}) \backslash O_q(\mathbb{A}^{fin}) / O_q(\hat{\mathbb{Z}})$

$\Rightarrow$ $\exists$ a bijection between the number of quadratic forms in a genus $=$ # double cosets

§ $\mathbb{A}$ = ring of all adeles $\leftarrow$ locally compact topological ring
$:= \mathbb{R} \times \mathbb{A}^{fin}$

The double cosets is replaced by $\underbrace{O_q(\mathbb{Q})}_{\text{discrete}} \backslash \underbrace{O_q(\mathbb{A})}_{\substack{\text{locally compact group,} \\ \text{has a Haar measure } \mu}} / O_q(\hat{\mathbb{Z}} \times \mathbb{R})$ $\rightarrow$ open subgroup, compact

We're counting # orbits of $O_q(\hat{\mathbb{Z}} \times \mathbb{R}) \circlearrowright O_q(\mathbb{Q}) \backslash O_q(\mathbb{A})$

Unravelling : $\quad$ Mass $(q) = \dfrac{\mu(O_q(\mathbb{Q}) \backslash O_q(\mathbb{A}))}{\mu(O_q(\hat{\mathbb{Z}} \times \mathbb{R}))} =: X$

$SO_q(\mathbb{R}) \subseteq O_q(\mathbb{R})$ determinant 1.

Replacing O_q by SO_q changes X by a power of 2.

§ $\exists$ a canonical Haar measure on $SO_q(\mathbb{A})$

$$SO_q(\mathbb{A}) = SO_q(\mathbb{R}) \times \prod_p^{res} SO_q(\mathbb{Q}_p)$$

$SO_q(\mathbb{R}) \rightsquigarrow V_{\mathbb{R}}$ = space of left-invariant top forms on $SO_q(\mathbb{R})$ $\rightsquigarrow$ 1-dim vector space

$\cup|$

$V_{\mathbb{Q}}$ = . . . algebraic differential forms over $\mathbb{Q}$

$\cap|$

$SO_q(\mathbb{Q}_p) \rightsquigarrow$ similar structures over this p-adic analytic Lie group over $\mathbb{Q}_p$

Let $\overset{\#}{\omega} \in SO(\mathbb{Q})$. determines a measure $\mu_{\omega,\infty}$ on $SO_q(\mathbb{R})$

$\mu_{\omega,p}$ on $SO_q(\mathbb{Q}_p)$

Def $\mu_{tam} = \mu_{\omega,\infty} \times \prod_p \mu_{\omega,p}$

$\rightsquigarrow$ if we scale ω by a constant 5 the measures get multiplied by $\|5\|_p$ whose product cancel out.

$$\mu_{tam}(SO_q(\hat{\mathbb{Z}} \times \mathbb{R})) = \mu_{tam}(SO_q(\mathbb{R}) \times \prod_p SO_q(\mathbb{Z}_p))$$

Thm Mass formula – Tamagawa-Weil version

$$\mu_{tam}(SO_q(\mathbb{Q}) \backslash SO_q(\mathbb{A})) = 2$$

If we replace SO_q by $Spin_q$ we get 1.

Conjecture (Weil) - Proved by Langlands, Lai, Kottwitz

G - semisimple, simply connected algebraic group over $\mathbb{Q}$

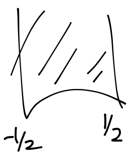
$$\mu_{tam}(G(\mathbb{Q}) \backslash G(\mathbb{A})) = 1$$

Problem $\mathcal{H} = \{x+iy : y>0\}$ area form $= \dfrac{dx\,dy}{y^2}$ $SL_2(\mathbb{Z}) \circlearrowright \mathcal{H}$

hyperbolic plane

Q. Compute area $\left(\mathcal{H}/SL_2(\mathbb{Z})\right)$ using

1) Calculus
2) Weyl's conjecture for $G = SL_2(\mathbb{Z})$

→ 

$$\text{Area} = \int_{-1/2}^{1/2}\int_{\sqrt{1-x^2}}^{\infty} \frac{dx\,dy}{y^2} = \pi/3$$

→ What is $SL_2(\mathbb{A})$? $\mathbb{A} = \mathbb{R}\times\mathbb{A}^{fin} = \mathbb{R}\times\hat{\mathbb{Z}}\otimes\mathbb{Q} = \mathbb{R}\times\left(\prod_p \mathbb{Z}_p\right)\otimes\mathbb{Q}$

↳ We must have $SL_2(\mathbb{A}) = SL_2(\mathbb{R}) \times \prod^{res} SL_2(\mathbb{Q}_p)$

• Note: $SL_2(\mathbb{R})$ acts transitively on $\mathcal{H}$ via mobius transforms

$$\frac{ai+b}{ci+d} = i \Rightarrow ai+b = di-c$$

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \quad +a^2+b^2 = 1 \quad \Rightarrow \text{stabilizer of } i = SO_2(\mathbb{R})$$

$$\Rightarrow \mathcal{H} = SL_2(\mathbb{R})/SO_2(\mathbb{R})$$

We have an inclusion:

$$\begin{array}{ccccc} SL_2(\mathbb{Q}) & \hookrightarrow & SL_2(\mathbb{A}) & = & SL_2(\mathbb{R}) \times \prod_p^{res} SL_2(\mathbb{Q}_p) \\ \uparrow & & \uparrow & & \cup\!\shortmid \\ SL_2(\mathbb{Z}) & \longrightarrow & SL_2(\mathbb{R}\times\hat{\mathbb{Z}}) & = & SL_2(\mathbb{R}) \times \prod_p SL_2(\mathbb{Z}_p) \end{array}$$

$$\begin{array}{ccc} SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R}\times\hat{\mathbb{Z}}) & \longrightarrow & SL_2(\mathbb{Q})\backslash SL_2(\mathbb{A}) \\ \downarrow & & \\ SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R}) & & \\ \downarrow & & \\ SL_2(\mathbb{Z})\backslash \mathcal{H} & & \end{array}$$

Fiber $= \prod_p SL_2(\mathbb{Z}_p)$ ⇝

This map is an iso   Why?

↓ The vertical arrows pushforward the Tamagawa measure to a multiple of the Hyperbolic measure, as Hyperbolic is the only $SL_2(\mathbb{R})$ invariant measure on $\mathcal{H}$

So the question reduces to –

Given a "rational" invariant top form on Sl_2.

How does one get a measure on $SL_2(\mathbb{Z}_p)$?

- Given a p-adic manifold X of dim n, pick a "chart" $\varphi : U \subseteq X \xrightarrow{\cong} \mathbb{Q}_p^n$

  $\varphi = (g_1, \cdots, g_n)$, g_i's algebraic

  Then $\mathrm{Vol}(U) = \int |\varphi^* \omega|_p$ where ω = top form on $\mathbb{Q}_p^n$ which makes volume of $\mathbb{Z}_p^n$ equal 1.

  $\Downarrow$

  volume of $(p\mathbb{Z}_p)^n = \frac{1}{p^n}$

  - Some technical point about ω being defined over $\mathbb{Z}$.

Problem (Hensel's lemma)

Let $g_1, g_2, g_3 : SL_2(\mathbb{Q}_p) \longrightarrow \mathbb{Q}_p^3$ be 3 algebraic functions sending $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \longmapsto (0,0,0)$

defined over $\mathbb{Z}$ with g_1', g_2', g_3' linearly independent.

then (g_1, g_2, g_3) sends $SL_2(p\mathbb{Z}_p)$ isomorphically to $(p\mathbb{Z}_p)^3$.

- Assuming this as $\mathrm{Vol}(p\mathbb{Z}_p)^3 = \frac{1}{p^3}$

  so that $\mathrm{Vol}(SL_2(\mathbb{Z}_p)) = \dfrac{\underline{\text{index}} \text{ of } SL_2(p\mathbb{Z}_p) \text{ inside } SL_2(\mathbb{Z}_p)}{p^3}$

  $\mu_{\omega,p}(Sl_2(\mathbb{Z}_p)) = 1 - \frac{1}{p^2}$

$\Rightarrow$ Some technical measure theoretic arguments later

$$\mu = \prod_p \mu_{\omega,p} = \prod_p \left(1 - \frac{1}{p^2}\right) = \frac{1}{\zeta(2)} = \boxed{6/\pi^2}$$

This is the volume of the fiber of the first map

The fiber of the 2^{nd} map $= \mathrm{Vol}\left(SO_2(\mathbb{R}) / SO_2(\mathbb{R}) \cap SL_2(\mathbb{Z})\right)$

under the Tamagawa measure pushed forward

Turns out this is 2 times the hyperbolic measure Why?