Call an element $Sq^i \in \mathcal{Sq}$ decomposible if it belongs to the subalgebra generated by $Sq^0, Sq^1, \dots, Sq^{i-1}$. Call it indecomposible otherwise.

Prop: Sq^i is indecomposible iff $i = 2^k$.

Proof. Suppose $i = 2^k$

Look at $x^i \in H^i(\mathbb{RP}^\infty; \mathbb{Z}/2)$. $Sq\, x^i = (Sq\, x)^i = (x + x^2)^i = x^i + x^{2i}$

$\Rightarrow Sq^j x^i \begin{cases} = 0 & \text{if } 0 < j < i \\ = x^{2i} & \text{if } j = i \end{cases}$ and hence Sq^i is indecomposible

Suppose $i \neq 2^k$. Let $i = 2^k + j$ for some such that $2^k > j$

Then

$$Sq^{2^k} Sq^j = \underbrace{\binom{2^k - 1}{j}}_{\text{odd}} Sq^i + \sum_{c > 0} \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c$$

and hence we are done.

And so $\mathcal{Sq}$ is generated as a algebra by Sq^{2^i} (not freely).

Prop: $f : S^{2n-1} \longrightarrow S^n$ can have Hopf invariant 1 only if $n = 2^k$.

Proof: Hopf invariant of $f = 1 \Rightarrow Sq^n$ is indecomposible.

Prop: Every element Sq^I other than Sq^0 is nilpotent.

Proof: This is remarkable result. I want to see this happen.

$Sq^1 Sq^1 = 0$

$Sq^2 Sq^2 = \cancel{\binom{2-1}{2}} Sq^4 + \binom{2-2}{2-2} Sq^3 Sq^1$

$Sq^2 Sq^2 Sq^2 = Sq^2 Sq^3 Sq^1$

$= \left[\binom{3-1}{2} Sq^5 + \binom{1}{0} Sq^4 Sq^1 \right] Sq^1 = Sq^5 Sq^1$

$(Sq^2)^4 = Sq^2 Sq^5 Sq^1$

$= \left[\binom{5-1}{2} Sq^7 Sq^0 + \binom{3}{0} Sq^6 Sq^1 \right] Sq^1 = 0$

$(Sq^2 Sq^1)^{n+1} = Sq^2 (Sq^3)^n Sq^1$

$Sq^3 Sq^3 = \binom{3-1}{3} Sq^6 + \binom{1}{1} Sq^5 Sq^1 = Sq^5 Sq^1$ $\quad (Sq^3)^3 = (Sq^3 Sq^5) Sq^1 = 0$

$= \left[\binom{4}{3} Sq^8 + \binom{3}{1} Sq^7 Sq^1 \right] Sq^1$

Lemma: If Sq^I is contained in some finite subalgebra of $\mathcal{S}q$ then Sq^I is nilpotent.
This is because Sq^I has +ve degree and hence some power of it should lie outside any finite subalgebra if it were not nilpotent.

Let $A(t)$ be the subalgebra of $\mathcal{S}q$ generated by $Sq^0, Sq^1, Sq^2, \ldots, Sq^{2^t}, Sq^{2^{t+1}}$. $A(t)$ is clearly a Hopf algebra itself. Look at $A(t)^*$.
We have a natural surjection of Hopf algebras.

$$\pi: \mathcal{S}q^* \longrightarrow A(t)^*$$
$$(Sq^i)^* \longmapsto (Sq^i)^*$$

Let $J_t = \mathcal{S}q^* \{ (\xi_1)^{2^{t+1}}, (\xi_2)^{2^t}, \cdots, (\xi_t)^4, (\xi_{t+1})^2, \xi_{t+2}, \xi_{t+3}, \cdots \}$

$(\xi_1)^{2^{t+1}*} = Sq^{2^{t+1}}$

$(\xi_2)^{2^t *} = Sq^{2^{t+1}} Sq^{2^t}$

$\vdots$

$(\xi_{t+1})^{2*} = Sq^{2^{t+1}} Sq^{2^t} \cdots Sq^2$

$(\xi_{t+2})^* = Sq^{2^{t+1}} Sq^{2^t} \cdots Sq^2 Sq^1$

Because $Sq^{2^{t+k}}$ does not lie in $A(t)$ we see that $J_t \in \ker \pi$
Because $\mathcal{S}q^*$ is simply the polynomial ring $\xi_1^i \notin J_t$ for $i < 2^{t+1}$
and as $(\xi_1^i)^* = Sq^i$, we see that $\ker \pi \in J_t$

$$\Rightarrow A(t) \cong \left(\mathcal{S}q^* / J_t \right)^*$$

The RHS is clearly finite and so $A(t)$ is finite.
Because $Sq^I \in A(t)$ for some t each Sq^I is nilpotent.

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eg: $A(1)$ = algebra generated by Sq^0, Sq^1, Sq^2

$A(1)^* \cong \mathbb{Z}/2 [\xi_1, \xi_2] \big/ \langle \xi_1^4, \xi_2^2 \rangle \cong \underbrace{\mathbb{Z}/2[\xi_1]/\xi_1^4}_{\text{4 elements}} \times \underbrace{\mathbb{Z}/2 [\xi_2]/\xi_2^2}_{\text{2 elements}}$ as an algebra

$A(1)$ is 8 dimensional

In general $\dim A(n) = 2^1 \cdot 2^2 \cdots 2^{n+1} = 2^{(n+1)(n+2)/2}$

Basis for $A(1)$: $\boxed{1}, \boxed{Sq^1}, \boxed{Sq^2}$, $\boxed{Sq^2 Sq^1}$, $Sq^1 Sq^2 = \boxed{Sq^3}$, $Sq^2 Sq^2 = \binom{0}{0} Sq^{4-1} Sq^1$

$(Sq^2)^3 = Sq^2 Sq^3 Sq^1 = \left[\binom{2}{2} Sq^5 + \binom{1}{0} Sq^4 Sq^1 \right] Sq^1 = \boxed{Sq^5 Sq^1} = \boxed{Sq^3 Sq^1}$

$Sq^2 Sq^3 = \boxed{Sq^5 + Sq^4 Sq^1}$ $\rightsquigarrow$ $\xi_1^2 \xi_2$ $\quad \xi_1^3 \xi_2$ $\quad \xi_1 \xi_2$ Duals

A useful lemma:

For $\alpha \in H^n(X; \underset{G}{\mathbb{Z}/2^k})$ look at the fibration $Y \to X \xrightarrow{\alpha} K(G,n)$. Extend this to $K(G,n-1) \to Y \to X$. Then in the Serre SS (assume $\mathbb{Z}/2$ coefficients). Then $i_{n-1} \in H^{n-1}(K(G,n-1))$ transgresses to α.

Proof: Let's try the oldest trick in topology.

Let X^{n-1} be the $n-1$ skeleton of X. Quotient it out. $X' = X^n / X^{n-1}$

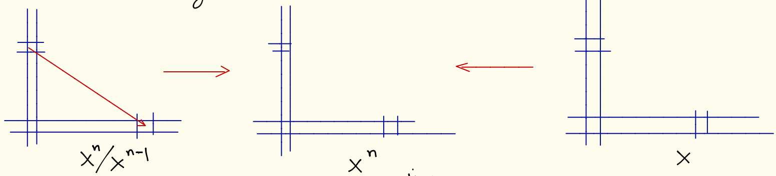
$$\begin{tikzcd}
K(G,n-1) \arrow[r] \arrow[dr] \arrow[ddr] & Y \arrow[r] & X \arrow[r, "\alpha"] & K(G,n) \\
& Y' \arrow[u] \arrow[r] \arrow[d] & X^n \arrow[u] \arrow[ur, "\alpha"] \arrow[d] & \\
& Y'' \arrow[r] & X^n/X^{n-1} \arrow[uur, "\alpha"'] &
\end{tikzcd}$$

$$H^n(X^n) \leftarrow H^n(X) \leftarrow 0$$

$$H^n(X^n/X^{n-1}) \longrightarrow H^n(X^n) \longrightarrow \underset{0}{\underset{\shortparallel}{H^n(X^{n-1})}}$$

By naturality of the constructions everything in this diagram commutes. Further the SS's would also commute.

Now X^n/X^{n-1} is a wedge of spheres and the statement is easy to prove for this situation. Assuming this,



$$\begin{tikzcd}
i_{n-1} \arrow[r] \arrow[d] & i_{n-1} \arrow[d, dashed, "\text{should exist}"] & i_{n-1} \arrow[l] \arrow[d, dashed] \\
\alpha \arrow[r, "\text{epic}"'] & \alpha & \alpha \arrow[l, "\text{monic}"]
\end{tikzcd}$$

We have also implicitly used the fact that $K(G,n-1)$ is $n-1$ connected.

Now for the case of Spheres: $K(G,n-1) \longrightarrow Y \longrightarrow S^n \longrightarrow K(G,n)$

$$\to \underset{0}{\underset{\shortparallel}{\pi_{n+1} K(G,n)}} \longrightarrow \pi_n(Y) \longrightarrow \pi_n(S^n) \xrightarrow{\cong} \pi_n(K(G,n)) \quad \Rightarrow Y \text{ is } n\text{-connected.}$$

This means that i_{n-1} has to support a non-trivial differential, but the only option is transgression!

The Bocksteins:

$$0 \to \mathbb{Z}/2 \xrightarrow{2^k} \mathbb{Z}/2^{k+1} \longrightarrow \mathbb{Z}/2^k \to 0$$

gives us

$$\longrightarrow H^n(X, \mathbb{Z}/2^{k+1}) \longrightarrow H^n(X;\mathbb{Z}/2^k) \underbrace{\longrightarrow}_{\text{This is the } k^{th} \text{ Bockstein}} H^{n-1}(X;\mathbb{Z}/2) \longrightarrow \cdots$$

β_k is a stable cohomology operation

For $\alpha \in H^n(X;\mathbb{Z}/2)$ suppose $\exists$ a lifts $\alpha', \alpha'' \in H^n(X;\mathbb{Z}/2^k)$.

$$\begin{tikzcd}
H^n(X;\mathbb{Z}/2^k) \arrow[r] \arrow[d, "2"] & H^n(X;\mathbb{Z}/2^{k-1}) \arrow[r, "\beta_{k-1}"] \arrow[d, "2"'] & H^{n+1}(X;\mathbb{Z}/2) \arrow[d, equal] \\
H^n(X;\mathbb{Z}/2^{k+1}) \arrow[r] & H^n(X;\mathbb{Z}/2^k) \arrow[r, "\beta_k"] \arrow[d] & H^{n+1}(X;\mathbb{Z}/2) \\
& H^n(X;\mathbb{Z}/2) &
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/2 \arrow[r, "2^{k-1}"] \arrow[d] & \mathbb{Z}/2^k \arrow[r] \arrow[d, "2"] & \mathbb{Z}/2^{k-1} \arrow[r] \arrow[d, "2"] & 0 \\
0 \arrow[r] & \mathbb{Z}/2 \arrow[r, "2^k"] & \mathbb{Z}/2^{k+1} \arrow[r] & \mathbb{Z}/2^k \arrow[r] & 0
\end{tikzcd}$$

The above diagram tells us $\beta_k(\alpha') - \beta_k(\alpha'') \in \beta_{k-1} H^n(X;\mathbb{Z}/2^{k-1})$

When does a lift exist? Look at

$$\begin{tikzcd}
\alpha_k \in H^n(X;\mathbb{Z}/2^k) \arrow[r] \arrow[d, mapsto] \arrow[d] & H^n(X;\mathbb{Z}/2^{k-1}) \arrow[r, "\beta_{k-1}"] & H^{n+1}(X;\mathbb{Z}/2) \\
\alpha \in H^n(X;\mathbb{Z}/2) & \alpha_{k-1} \arrow[l, bend left] &
\end{tikzcd}$$

If $\alpha \in H^n(X;\mathbb{Z}/2)$ has a lift α_k in $H^n(X;\mathbb{Z}/2^k)$ then we can push it down to $H^n(X;\mathbb{Z}/2^{k-1})$ to get a lift α_{k-1}

And by exactness we would get $\beta_{k-1}(\alpha_{k-1}) = 0$.

Converse is true for the same reason i.e. $\exists \alpha_k$ iff $\exists \alpha_{k-1}$ s.t. $\beta_{k-1}(\alpha_{k-1}) = 0$

These maps are precisely the Bockstein operations.

Odd torsion of S^3:

We have the Hopf fibration $S^1 \to S^3 \to S^1$ giving us $\pi_i(S^3) \cong \pi_i(S^2)$ for $i \geq 2$

We need to determine $H^*(K(\mathbb{Z}/p, 2))$. Now we have the fibration $K(\mathbb{Z}/p,1) \to * \to K(\mathbb{Z}/p, 2)$

$K(\mathbb{Z}/p,1)$ can be chosen to be the Lenz space. $S^\infty/\sim$ $\quad (z_1 : z_2 : z_3 : \cdots) \sim (e^{2\pi i/p} z_1, e^{2\pi i/p} z_2, e^{2\pi i/p} z_3, \dots)$

If we quotient out the entire S^1 action we get $\mathbb{CP}^\infty$. So we have a fibration $S^1 \to K(\mathbb{Z}/p,1) \to \mathbb{CP}^\infty$.

The Serre SS, given that $H^1(K(\mathbb{Z}/p,1)) \cong \mathbb{Z}/p$ gives $\tilde{H}^*(K(\mathbb{Z}/p,1);\mathbb{Z}) \cong \mathbb{Z}/p$ in even dimensions.

The product structure on $\mathbb{CP}^\infty$ caries over to give $\boxed{\tilde{H}^{2i}(K(\mathbb{Z}/p,1);\mathbb{Z}) \cong \mathbb{Z}/p\, x^i}$

In $\mathbb{Z}/p$ coefficients, $H^*(K(\mathbb{Z}/p,1);\mathbb{Z}/p) \cong \mathbb{Z}/p \quad \forall *$