

Comma Categories:

Def: To any ∞ -functors $C \xrightarrow{g} A \xleftarrow{f} B$ form,

$$\begin{array}{ccc} \text{Hom}_A(f, g) & \xrightarrow{\phi} & A^2 \\ (p_1, p_0) \downarrow & & \downarrow \\ C \times B & \xrightarrow{g \times f} & A \times A \end{array}$$

• elements of $\text{Hom}_A(f, g)$ are triples $(c \in C, b \in B, \alpha: fb \rightarrow gc \in A)$.

∞ -comma category: $\text{Hom}_A(f, g) \xrightarrow{(p_1, p_0)} C \times B$

This has a canonical 2 cell called the comma cone

$$\begin{array}{ccccc} & & \text{Hom}_A(f, g) & & \\ p_1 \swarrow & & \xleftarrow{\phi} & & \searrow p_0 \\ C & & & & B \\ & \searrow g & & \swarrow f & \\ & & A & & \end{array}$$

eg: A^2 is a comma category $\text{Hom}_A(\text{id}_A, \text{id}_A)$.

As before we get 3-operations in $\mathcal{h}K$:

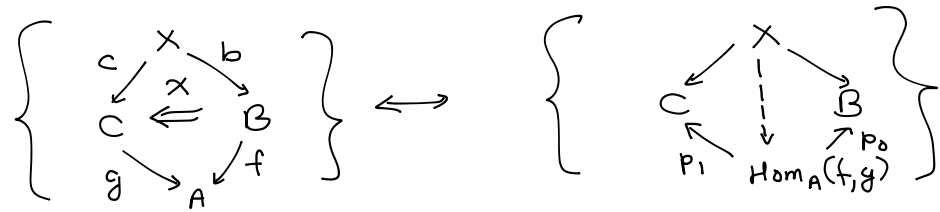
1-cell induction:

$$\begin{array}{ccc} & X & \\ p_1 \swarrow & & \searrow p_0 \\ C & \xleftarrow{\quad} & B \\ & \searrow g & \swarrow f \\ & & A \end{array} = \begin{array}{ccc} & \downarrow X & \\ & \text{Hom}_A(f, g) & \\ p_1 \swarrow & \xleftarrow{\quad} & \searrow p_0 \\ C & \xleftarrow{\quad} & B \\ & \searrow g & \swarrow f \\ & & A \end{array}$$

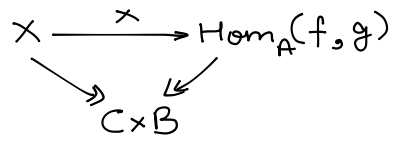
2-cell induction, 2 cell conservativity.

• These are weak UPs which uniquely determine $\text{Hom}_A(f, g) \rightarrow C \times B$.

Prop: The comma cone induces a (fibred) bijection:

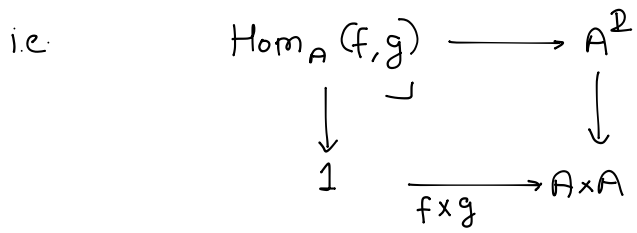


fibred iso " classes of maps of spans



Def: $1 \begin{smallmatrix} \xrightarrow{f} \\ \xrightarrow{g} \end{smallmatrix} A$

then $Hom_A(f, g)$ is called an internal mapping space.

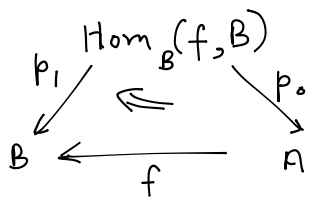


- $Hom_A(x, y)$ is a discrete object in \mathcal{K} .
every 2-cell with codomain $Hom_A(f, g)$ is invertible.

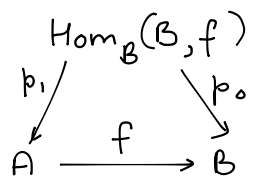
i.e. this is a ∞ -groupoid / Kan complex.

Def: Any functor $A \xrightarrow{f} B$ may be represented as an ∞ -cat in 2 ways:

left rep:



right rep:



A comma $\text{Hom}_A(f, g) \longrightarrow C \times B$ is

left representable: if $\exists B \xrightarrow{l} C$ s.t. $\text{Hom}_A(f, g) \simeq \text{Hom}_C(l, C)$

$$\searrow \quad \swarrow$$

$C \times B$

right representable: if $\exists C \xrightarrow{r} B$ s.t.

$$\text{Hom}_A(f, g) \simeq \text{Hom}_B(B, r)$$

$\searrow \quad \swarrow$
 $C \times B$

Th^m:

$\begin{array}{ccc} & & B \\ & \nearrow r & \downarrow f \\ C & \xrightarrow{g} & A \end{array}$ is an absolute right lifting
iff induced map of commas y is an equiv.

$$\begin{array}{ccc} & \text{Hom}(B, r) & \\ p_1 \swarrow & \xleftarrow{\phi} & \searrow p_2 \\ C & \xleftarrow{\quad} & B \\ g \searrow & \xleftarrow{\quad} & \swarrow f \\ & A & \end{array}$$

=

$$\begin{array}{ccc} & \text{Hom}_B(B, r) & \\ \exists \downarrow y & & \\ & \text{Hom}_A(f, g) & \\ C \swarrow & \xleftarrow{\phi} & \searrow B \\ g \searrow & \xleftarrow{\quad} & \swarrow f \\ & A & \end{array}$$

Th^m: \exists a bijection:

$$\left\{ \begin{array}{ccc} & & B \\ & \nearrow r & \downarrow f \\ C & \xrightarrow{g} & A \end{array} \right\} \xrightarrow{\simeq} \left\{ \begin{array}{ccc} & \text{Hom}_B(B, r) & \\ C \swarrow & y \downarrow & \searrow B \\ & \text{Hom}_A(f, g) & \end{array} \right\} / \text{fibered iso}$$

$\{$ absolute right lifting iff y is an \simeq .

Lemma \exists an adjunction $A \xleftarrow[p_1]{\quad} \text{Hom}_B(B, f)$, counit = id
unit: $\text{id} \xrightarrow{u} ip_1$
s.t. $p_1 u = \text{id}$, $p_2 u = \phi$.

Cot: (cheap Yoneda lemma) given $A \xrightleftharpoons[g]{f} B$ we get a bijection

