Relationship between Hochschild and Cyclic homology using "periodicity" operator

Analogous to the Gysin sequence for S^1 spaces.

$S^1 \to E \to B$ fibration $\Rightarrow$ $\cdots \to H_q(E) \to H_q(B) \to H_{q-2}(B) \to H_{q-1}(E) \to \cdots$

Th^m (Connes): For any associative algebra A, $\exists$ natural LES,

$$\cdots \to HH_n(A) \xrightarrow{I} HC_n(A) \xrightarrow{S} HC_{n-2}(A) \xrightarrow{B} HH_{n-1}(A) \xrightarrow{I} \cdots$$

Proof: Consider the first two columns of $CC_{*,*}(A)$ — call them $CC(A)^{\{2\}}$

Think of this as a subcomplex of $CC_*(A)$

$$0 \to CC(A)^{\{2\}} \xrightarrow{I} CC(A) \xrightarrow{S} CC(A)[2,0] \to 0$$

$$CC(A)[2,0]_{p,q} = CC_{p-2,q}(A)$$

Rmk: 1) If A is unital, b' is acyclic. So $CC(A)^{\{2\}}$ is quasi-iso to first column

2) For non-unital algebra I, homology of $CC(A)^{\{2\}}$ is a way to describe the Hoschild homology of I,

→ when A is augemented $(A \underset{\eta}{\overset{g}{\rightleftarrows}} k)$ so that

$A = I \times k$ so that $A/kA = \bar{A} = I$ then

$$HH_n(I) := \operatorname{coker}\left(HH_n(k) \xrightarrow{\eta_*} HH_n(A)\right)$$

Prop: For any k-algebra I (not necessarily unital) there is an exact sequence

$$\cdots \to HH_n^{naive}(I) \to HH_n(I) \to HH_n^{bar}(I) \to HH_{n-1}^{naive}(I) \to \cdots$$

$HH_n^{naive}(I) = H_*(C_n(I), b)$, $HH_n^{bar}(I) = H_n(C_*(I), 1\otimes b')$

$$HH_n(I) := H_n\left((I_+ \otimes I^{\otimes *}, b)_{red}\right)$$

Proof: For $n=0$, $I_+ \otimes I^{\otimes 0} = I$

For $n>0$, $I_+ \otimes I^{\otimes n} \cong I^{\otimes n+1} \oplus I^{\otimes n}$

$k \supseteq \mathbb{Q}$ The map $I : HH_*(A) \longrightarrow HC_*(A)$ is induced by the natural projection

$$HC_*(A) \cong H^\lambda_*(A)$$

$$p : A^{\otimes(n+1)} \longrightarrow A^{\otimes(n+1)}/(1-t) =: C^\lambda_n(A)$$

The map $B : C^\lambda_n(A) \longrightarrow A^{\otimes(n+1)} = (1-t)\circ S \circ N$

$$(a_0 \cdots a_n) \longmapsto \sum_{i=0}^{n} (-1)^{ni} (1, a_i, \dots, a_n, a_0, \dots, a_{i-1}) + \sum_{i=1}^{n+1} (-1)^{n(i-1)} (a_{i-1}, 1, a_i, \dots, a_n, a_0, \dots)$$

Set $\beta : C_n(A) \longrightarrow C_{n-1}(A) = \sum_{i=0}^{n} (-1)^i \, i \, d_i$

$$b^{[2]} : C_n(A) \longrightarrow C_{n-2}(A) = \sum_{0 \le i < j \le n} (-1)^{i+j} d_i d_j$$

Lemma: a) $b^{[2]} = b\beta + \beta b$

b) $[b, b^{[2]}] = 0$

Proof: a) Using $d_i d_j = d_j d_{i+1}$ for $i \geq j$ we get

$$b\beta = \sum_{0 \le i < j \le n} (-1)^{i+j} (j-i) d_i d_j$$

$$\beta b = \sum_{0 \le i < j \le n} (-1)^{(i+j)} (i-j+1) \, d_i d_j$$

Th^m: $\mathbb{Q} \subseteq k$, $1_A \in A$. Let $x \in C_n(A) := A^{\otimes(n+1)}$ such that $p(x) = \bar{x} \in C^\lambda_n(A) (= A^{\otimes n+1}/(1-t))$ is a cycle in $(C^\lambda_*(A), b)$, then

$$\underset{H^\lambda_n(A)}{\underset{\cap}{S([\bar{x}])}} := \left[-\frac{1}{n(n-1)} \, b^{[2]}(x) \right] \in H^\lambda_{n-2}(A)$$

Proof: Since $\bar{x}$ is a cycle, then $\exists \alpha = (x, y, z, \dots) \in C_n(A) \oplus C_{n-1}(A) \oplus \cdots$ a cycle in $\mathrm{Tot}\, CC(A)$

By def, $S(\alpha) = (z, \dots) \in C_{n-2}(A) \oplus \cdots$ and the image in $H^\lambda_{n-2}(A)$ is $[\bar{z}]$

$$\begin{array}{ccccc} C_n(A) \ni x & & & & \\ \big\downarrow -b & & & & \\ -bx = (1-t)y & \xleftarrow{1-t} & y \in C_{n-1}(A) & & \\ & & \big\downarrow b' & & \\ & & b'(y) = N(z) & \longleftarrow & z \in C_{n-2}(A) \end{array}$$

The homotopy (h,h') $h := -1/n \sum_{i=0}^{n} it^i$, $h' := \frac{1}{n-1}\,\mathrm{id}$ gives

$y = -hb(x)$, $z := h'(b'(y)) = -h'b'hb(x)$

...

Recall:

$$\begin{tikzcd}
& M\otimes\Lambda^n R \arrow[r, "s"] \arrow[dl] \arrow[d, "\varepsilon_n"] & M\otimes\Lambda^{n-1}R \arrow[d, "\varepsilon_{n-1}"] \\
M\otimes\Omega^n_{R|k} \arrow[d] & C_n(M,R) \arrow[r, "b"] \arrow[dl] & C_{n-1}(M,R) \\
H_n(R,M) & &
\end{tikzcd}$$

Under the assumption A-unital, commutative

$\varepsilon_n : \Omega^n_{A|k} \longrightarrow HH_n(A)$

Goal: To show commutativity of

$$\begin{tikzcd}
\Omega^n_{A|k} \arrow[r, "d"] \arrow[d, "\varepsilon_n"] & \Omega^{n+1}_{A|k} \arrow[d, "\varepsilon_{n+1}"] \\
HH_n(A) \arrow[r, "B"] & HH_{n+1}(A)
\end{tikzcd}$$

(Rinehart, Connes)

$H_n((\Omega^*_{A|k}, d)) := H^n_{dR}(A|_k)$ i.e. $H^n_{dR}(\operatorname{Spec} A)$

$\pi_n : C_n(A) \longrightarrow \Omega^n_{A|k}$ inducing $HH_n(A) \xrightarrow{\pi_n} \Omega^n_{A|k}$

s.t. $\pi_n \varepsilon_n = (n!)\,\mathrm{id}$

We have commutative diagram

$$\begin{tikzcd}
A\otimes\Lambda^n(A) \arrow[r, "d"] \arrow[d, "\varepsilon_n"] & A\otimes\Lambda^{n+1}(A) \arrow[d, "\varepsilon_{n+1}"] \\
C_n(A) \arrow[r, "B"] & C_{n+1}(A)
\end{tikzcd}$$

Prop: For any unital commutative k-algebra A the following diagram commutes

$$\begin{tikzcd}
HH_n(A) \arrow[r, "B"] \arrow[d, "\pi_n"] & HH_{n+1}(A) \arrow[d, "\pi_{n+1}"] \\
\Omega^n_{A|k} \arrow[r, "(n+1)d"] & \Omega^{n+1}_{A|k}
\end{tikzcd}$$

Cor: A unital, comm. $\exists$ functorial map $(I\circ\varepsilon_n) : \Omega^n_{A|k}/d\Omega^{n-1}_{A|k} \longrightarrow HC_n(A)$ which split injective when $k \supseteq \mathbb{Q}$ and making the following diagram commute

$$\begin{tikzcd}
\arrow[r] & \Omega^{n-1}_{A|k}/d \arrow[r, "d"] \arrow[d, "\tilde\varepsilon_{n-1}"] & \Omega^n_{A|k} \arrow[r, two heads] \arrow[d, "\varepsilon_n"] & \Omega^n_{A|k}/d \arrow[r, "0"] \arrow[d, "\tilde\varepsilon_n"] & \Omega^{n-2}_{A|k}/d\Omega^{n-3} \arrow[d, "\tilde\varepsilon_{n-2}"] & \\
\arrow[r, "S"] & HC_{n-1}(A) \arrow[r, "B"] & HH_n(A) \arrow[r, "I"] & HC_n(A) \arrow[r, "S"] & HC_{n-2}(A) \arrow[r, "B"] &
\end{tikzcd}$$

Relationship with Deligne Cohomology:

$$\underset{\mathbb{Z}\cdot(2\pi i)^n}{\mathbb{Z}(n)} \longrightarrow \Omega^0_{A/\mathbb{C}} \xrightarrow{d} \Omega^1 \longrightarrow \dots \longrightarrow \Omega^{n-1}$$

Deligne Complex

$$\tilde{\mathcal{D}}(A)_{..} \qquad \begin{tikzcd} \Omega^2_{A|k} & \Omega^1_{A|k} \arrow[l, "d"'] \arrow[d, "0"] & \Omega^0_{A|k} \arrow[l, "d"'] \\ \Omega^1_{A|k} \arrow[u, phantom] & \Omega^0_{A|k} \arrow[l, "d"'] & \\ \Omega^0_{A|k} & & \end{tikzcd}$$

with vertical arrows $\Omega^2_{A|k} \to \Omega^1_{A|k} \to \Omega^0_{A|k}$ in the first column.

Reduced Deligne Complex

Under the assumption $k \supseteq \mathbb{Q}$ there is a map of double complexes

$$\left(\frac{1}{n!}\pi_n\right)_n \quad B(A)_{..} \longrightarrow \tilde{\mathcal{D}}(A)_{..} = \text{bicomplex of truncated DeRham Complexes}$$

this follows from $\pi_n : C_n(A) \longrightarrow \Omega^n_{A|k}$ $\quad \pi_n \cdot b = 0$

and the fact that $\pi\ (C_*(A), b) \longrightarrow (\Omega^*_{A|k}, 0)$ is a map of complexes

So one gets, $(CC_*(A), b, B) \longrightarrow (\tilde{D}(A), 0, d)$

Prop: $k \supseteq \mathbb{Q}$, A commutative, unital, the π-map induces a canonical map

$$HC_n(A) \xrightarrow[\frac{1}{n!}\pi_n]{} \Omega^n_{A|k} / d\Omega^{n-1} \oplus H^{n-2}_{dR}(A) \oplus H^{n-4}_{dR}(A) \oplus \dots$$