E_n algebras and vertex models

Two conjectures of Kontsevich:

A sensible definition of QFT involves a $\begin{matrix}\mathcal{F}\\ \downarrow\\ X\end{matrix}$ bundle of fields

Conjecture A [Kontsevich]

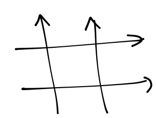
given a QFT on $\mathbb{R}^d$ which has translation & dialation symmetries, then translation invariant forms with values in $\mathcal{F}$ gets an action of E_d (E_d-alg in complex over $\mathbb{R}$)

- If A is an E_d-algebra in $Cplexes_{\mathbb{R}}$ then $A[d-1]$ has an L_∞-algebra structure. Presumably, the higher brackets are related to renormalization of combinatorial issues.

Deformation theoretic interpretation: look at the formal deformation problem of $T \in \mathcal{M}_{QFT}$ governed by an L_∞-algebra.

Discretized version:

state everything for $d=2$. Consider a square lattice in $\mathbb{R}^2$



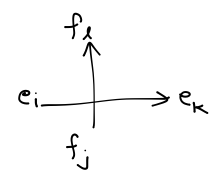
H, V are k-vector spaces.

$H = k\langle e_i \mid i \in I\rangle$

$V = k\langle f_j \mid j \in J\rangle$

$R(e_i \otimes f_j) = R_{ij}^{kl}\, e_k \otimes f_l$

Think of R_{ij}^{kl} = probability of having the following config



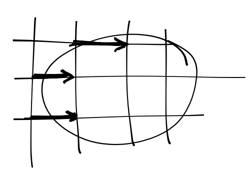
or weight associated to the config ↑

Game: Counting configs in a finite region with these weights (& prescribed boundary conditions; state sum)

Space of boundary states:

U = bounded region in $\mathbb{R}^2$
open convex

$\partial_+^h U$ = horizontal incoming edges
$= \{ e \text{ has edge} \mid e \cap V \neq \phi,\ s(e) \notin U \}$



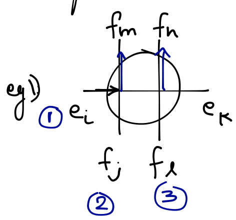
Similarly, ∂_+^v, ∂_-^h, ∂_-^v.

$$\partial_+^h u \cong \partial_-^h u\,, \qquad \partial_+^v u \cong \partial_-^v u$$

$$W(u) = \mathrm{End}\left(H^{\otimes \partial_+^h u} \otimes V^{\otimes \partial_+^v u}\right)$$

Partition function : $\quad Z_u : k \longrightarrow W(u)$

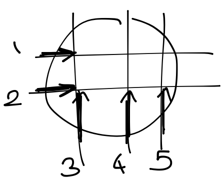
eg 1)



$$W(u) = \mathrm{End}(H \otimes V^{\otimes 2})$$

$$Z_u = \sum_s R_{ij}^{sm} R_{sl}^{kn} = \text{matrix multiplication } R^{1,2} R^{1,3}$$

eg 2)



$$Z_u = R^{2,3} R^{2,4} R^{1,3} R^{1,4} R^{2,3}$$

- $u \subseteq u' \rightsquigarrow Z_{u,u'} : W(u) \longrightarrow W(u')$
  $$A \longmapsto A \cdot \left(R^{u' \setminus u}\right)$$

$$W(\mathbb{R}^2) := \operatorname{colim}_u W(u) \qquad \mathbb{Z}^2 \circlearrowright W, \quad W(u) \longmapsto W(u + (1,0))$$

Conjecture B [Kontsevich / Upside down conjecture]
$C_{-*}(\mathbb{Z}^2, W(\mathbb{R}^2))$ is acted on by $\mathbb{Z}^2$.

- W can be turned into a prefactorization algebra.

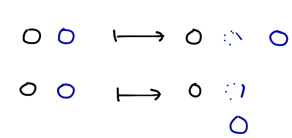
Theorem Conjecture B is true.

Proof. Category of Discretized disks

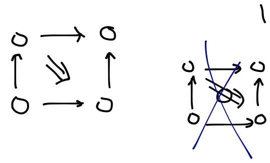
$\mathbb{D}_2$ (2,1) category

obj : finite unions of disks in $\mathbb{R}^2$

1-mor : generated by 1) inclusions

2) 

2 mor : homotopies



On this category, there is an E_2 monoidal structure :

Dunn Lurie : E_2-algebras $\cong E_1$ algebras (E_1 algebras)

In $\mathbb{D}_2$: $A \leq_h B$ if A is on the left of B
$A \leq_v B$ " " " bottom "

$A \otimes_h B := A \amalg (B + (m,0))$, where m minimal so that $A \leq_h B + (m,0)$
$A \otimes_v B := A \amalg (B + (0,m))$, " $A \leq_v B + (0,m)$

Consequence : $ho(\mathbb{D}_2)$ is braided monoidal. (this is an E_2 structure)

E_n algebras from E_n monoidal functors

$\mathcal{C}, \mathcal{D}$ E_n-monoidal $(\infty, 1)$ categories
$F : \mathcal{C} \longrightarrow \mathcal{D}$ E_n-monoidal functor
$\Rightarrow$ $\mathrm{colim}(F)$ is an E_n-algebra in $\mathcal{D}$
(as colim is a ?Kan extension)

$\mathcal{C} = \mathbb{D}_2$, $\mathcal{D} = \mathrm{Complex}_k$

$$\begin{tikzcd} \mathbb{D}_2 \arrow[r, "F"] \arrow[d] & \mathrm{Complex}_k \\ B\mathbb{Z}^2 \arrow[ur, dashed] \arrow[d] \arrow[u, phantom, "\Downarrow"] & \\ * \arrow[uur, dashed] & \end{tikzcd}$$

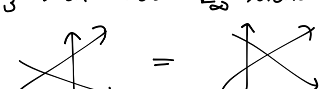
E_2 monoidal

$$\mathrm{colim}\, F = C_{-*}(\mathbb{Z}^2, \mathrm{colim}\, F(u)) \circlearrowleft_{E_2}$$

W defines a braided monoidal functor

$$\mathbb{D}_2 \longrightarrow ho(\mathbb{D}_2) \xrightarrow{W} \mathrm{Vect} \longrightarrow \mathrm{Complex}_k$$

$\Rightarrow$ $\mathrm{colim} \cong C_{-*}(\mathbb{Z}^2, W(\mathbb{R}^2))$ which proves the conjecture !!!

Remarks : • When $n \geq 3$, the E_n-algebra structure turns out to be an E_∞-algebra structures.
Q. Are there examples which produce E_3 but not E_∞ structures.
• $H = V$, $R \in GL(V^{\otimes 2})$ satisfies YBE 
what does it say for the E_2-algebra structure? $\rightarrow$