Ch 3

Let $Q \in qCat$.
- objects of hQ : elements $1 \xrightarrow{a} Q$
- mor of hQ : homotopy classes of 1-simp in Q

$$\begin{tikzcd} & y \arrow[d, equal] \\ x \arrow[ur, "f"] \arrow[r, "f'"'] & y \end{tikzcd} \equiv f \sim f'$$

Classical fact :
J = small category

$$h(Q^J) \neq (hQ)^J$$

"homotopy coherent diagrams" $\neq$ "homotopy commutative diagrams"

But there is a map

$$Q^J \times J \xrightarrow{ev} Q$$

apply $h : qCat \longrightarrow Cat$

$$h(Q^J \times J) \cong h(Q^J) \times hJ$$

$$\cong h(Q^J) \times J \longrightarrow hQ$$

$$\rightsquigarrow \quad h(Q^J) \longrightarrow (hQ)^J$$

For $J = \mathbb{2} = \Delta[1] = $ "$0 \to 1$"

Lemma: Q any q-Cat $\quad h(Q^{\mathbb{2}}) \longrightarrow (hQ)^{\mathbb{2}} \in Cat$

The smothering functor is
- surjective on objects
- full
- conservative (reflects invertibility of objects)

Def^n: A functor of strict 1-cats is smothering if it has 3 RLP wrt to the three morphisms : $\{ \phi \hookrightarrow \mathbb{1},\ \mathbb{1} + \mathbb{1} \hookrightarrow \mathbb{2},\ \mathbb{2} \hookrightarrow \mathbb{I} \}$

Proof: The fibers of a smothering functor are connected groupoids.

Lemma: Given a pullback of qCat with p an isofib

$$\begin{tikzcd} E \times_B A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

$h: qCat \to Cat$ does not preserve pullbacks but

$$h(E \times_B A) \longrightarrow hE \times_{hB} hA$$

is smothering.

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def^n

A an ∞-category

- $A^{\mathbb{2}} :=$ ∞-category of arrows

$$\exists \quad A^{\mathbb{2}} = A^{\Delta[1]} \underset{\substack{(p_1,\, p_0) \\ \parallel \\ (\mathrm{dom},\, \mathrm{cod})}}{\longrightarrow\!\!\!\!\rightarrow} A^{\overset{\mathbb{1}+\mathbb{1}}{\parallel}\atop{\partial\Delta[1]}} \cong A \times A$$

This comes with a canonical 2-cell in hK

$$\begin{tikzcd} A^{\mathbb{2}} \arrow[r, bend left, "p_0"] \arrow[r, bend right, "p_1"'] \arrow[r, phantom, "\Downarrow"] & A \end{tikzcd}$$

Simplicial cotensors in $\mathcal{K}$ satisfy :

$$\begin{tikzcd} \mathrm{Fun}(X, A^{\mathbb{2}}) \arrow[r, phantom, "\cong"] \arrow[d, "{(p_0,p_1)}"'] & \mathrm{Fun}(X, A)^{\mathbb{2}} \arrow[d, "{(ev_0,\, ev_1)}"] \\ \mathrm{Fun}(X, A \times A) \arrow[r, phantom, "\cong"] & \mathrm{Fun}(X,A) \times \mathrm{Fun}(X,A) \end{tikzcd}$$

when $X = A^{\mathbb{2}}$, $\mathrm{im}(\mathrm{id}_X)$ is the 2-cell.

Universal Property of $\begin{tikzcd} A^{2} \arrow[r, bend left, "p_0"] \arrow[r, bend right, "p_1"'] \arrow[r, phantom, "\Downarrow"] & A \end{tikzcd}$ in hK

Prop: $\begin{tikzcd} A^{2} \arrow[r, bend left, "p_0"] \arrow[r, bend right, "p_1"'] \arrow[r, phantom, "\Downarrow\iota"] & A \end{tikzcd}$ has a weak UP that supplies 3-operations in $h\mathcal{K}$:

1-cell induction:

$$\begin{tikzcd} X \arrow[d, bend right=60, "t"'] \arrow[d, bend left=60, "s"] \arrow[d, phantom, "\overset{\alpha}{\Leftarrow}"] \\ A \end{tikzcd} = \begin{tikzcd} X \arrow[d, dashed] \\ A^{2} \arrow[d, bend right=60, "p_1"'] \arrow[d, bend left=60, "p_0"] \arrow[d, phantom, "\Leftarrow"] \\ A \end{tikzcd}$$

2-cell induction:

Given $X \overset{a}{\underset{b}{\rightrightarrows}} A^{2}$ and τ_0, τ_1 s.t.

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ A^{2} \arrow[dr, "p_1"'] \arrow[rr, phantom, "\overset{\tau_1}{\Leftarrow}"] & & A^{2} \arrow[dl, "p_1"'] \arrow[dl, bend left=60, "p_0"] \arrow[dl, phantom, "\overset{\kappa}{\Leftarrow}", bend left=20] \\ & A & \end{tikzcd} = \begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ A^{2} \arrow[dr, "p_0"] \arrow[dr, bend right=60, "p_1"'] \arrow[dr, phantom, "\Downarrow\kappa", bend right=20] \arrow[rr, phantom, "\underset{\tau_0}{\Leftarrow}"] & & A^{2} \arrow[dl, "p_0"] \\ & A & \end{tikzcd}$$

i.e. $\exists$ a 2-cell $\begin{tikzcd} X \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] \arrow[r, phantom, "\Downarrow\tau"] & A^{2} \end{tikzcd}$ s.t. $p_1\tau = \tau_1$
$p_0\tau = \tau_0$

2-cell conservativity:

If τ_0, τ_1 are invertible then so is τ_1.

Proof uses the fact that $hFun(X, A^{2}) \longrightarrow (hFun(X, A))^{2}$ is smothering.

Prop: Whiskering with the generic 2-cell over A defines a bijection

$$\left\{ \begin{tikzcd} X \arrow[r, bend left=50, "s"] \arrow[r, bend right=50, "t"'] \arrow[r, phantom, "\Downarrow\alpha"] & A \end{tikzcd} \right\} \xleftarrow{\cong} \left\{ \begin{tikzcd} & X \arrow[dl, "t"'] \arrow[dr, "s"] \arrow[d, dashed, "a"] & \\ A \arrow[rr, phantom, "\Leftarrow"] & A^{2} \arrow[l] \arrow[r] & A \end{tikzcd} \right\}$$

fibered iso :

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] \arrow[d, bend right, "a"'] \arrow[d, bend left, "a'"] \arrow[d, phantom, "\underset{\gamma}{\cong}"] & \\ A & A^{2} \arrow[l] \arrow[r] & A \end{tikzcd}$$

$p_1 \gamma = id_t \qquad p_2 \gamma = id_s$

Prop: The weak UP of A^{2} characterizes it up to equiv in $\mathcal{K}/_{A \times A}$.