Today I might finally understand the Bockstein SS:

Start with
$$\begin{tikzcd} H^*(X;\mathbb{Z}) \arrow[rr] & & H^*(X;\mathbb{Z}) \arrow[dl] \\ & H^*(X;\mathbb{Z}/2) \arrow[ul] & \end{tikzcd}$$
This is nothing but the long exact sequence corresponding to $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$ written in a contorted fashion.

What is E^r? $E^1 = H^*(X;\mathbb{Z}/2)$

E^2 page:
$$\begin{tikzcd} 2H^*(X;\mathbb{Z}) \arrow[rr] & & 2H^*(X;\mathbb{Z}) \arrow[dl] \\ & E^2 \arrow[ul] & \end{tikzcd}$$
$$E^2 = \ker\left(H^*(X;\mathbb{Z}/2) \to H^{*+1}(X;\mathbb{Z}) \to H^{*+1}(X;\mathbb{Z}/2)\right) / \mathrm{Im}$$
$$[\omega] \bmod 2 \mapsto \left[\frac{d\omega}{2}\right] \mapsto \left[\frac{d\omega}{2}\right] \bmod 2$$

E^r page:
$$\begin{tikzcd} 2^r H^*(X;\mathbb{Z}) \arrow[rr] & & 2^r H^*(X;\mathbb{Z}) \arrow[dl] \\ & E^r \arrow[ul] & \end{tikzcd}$$
$$E^r = \ker / \mathrm{Im} \qquad [\omega] \bmod 2 \mapsto \left[\frac{d\omega}{2^r}\right] \mapsto \left[\frac{d\omega}{2^r}\right] \bmod 2$$

Prop: If $[\omega] \in H^*(X;\mathbb{Z})$ has no torsion and is lift of a non-zero $[\omega] \bmod 2 \in H^*(X;\mathbb{Z}/2)$ then $[\omega] \bmod 2$ represents a non-zero element in E^∞.

Proof: This is because $d\omega$ itself is 0 so $\left[\frac{d\omega}{2^r}\right] \bmod 2$ is 0 $\forall r$.

If $\left[\frac{d\tau}{2^r}\right] = [\omega]$ then $2^r\omega$ would be exact in $H^*(X;\mathbb{Z})$, but as $[\omega]$ has no torsion this forces $[\omega]=0$.

Prop: $[\omega]$ generates $\mathbb{Z}/2^r$ summand $\in H^{n+1}(X;\mathbb{Z}) \Rightarrow \exists\, [\omega'] \in H^n(X;\mathbb{Z}/2)$ and $[\omega''] \in H^{n+1}(X;\mathbb{Z}/2)$ non-zero such that $d_k[\omega'] = 0 = d_k[\omega'']$ $k<r$, $d_n\omega' = \omega''$.

Proof: Existence of ω', ω'' is thanks to the Universal Coefficient Theorem:
$$H^n(X;\mathbb{Z}/2) = H^n(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/2 \oplus \underbrace{\mathrm{Tor}(H^{n+1}(X;\mathbb{Z}); \mathbb{Z}/2)}_{\rightsquigarrow\, \omega'}$$
$$H^{n+1}(X;\mathbb{Z}/2) = \underbrace{H^{n+1}(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/2}_{\rightsquigarrow\, \omega''} \oplus \dots$$

$$d_r\omega' = \left[\frac{d\tau}{2^r}\right] \bmod 2 \qquad \tau \in C^n(X;\mathbb{Z}) \text{ is a lift of } \omega'$$

This requires us to understand how Tor generates elements in UCT.

$\mathrm{Tor}(H^{n+1}(X;\mathbb{Z});\mathbb{Z}/2)$ is giving us an element in $H^n(X;\mathbb{Z}/2)$. This is nothing but the connecting homomorphism in the diagram:
$$\begin{tikzcd} 0 \arrow[r] & Z^n \arrow[r] \arrow[d, "0"'] & C^n(X;\mathbb{Z}) \arrow[r] \arrow[d] & B^{n+1}(X) \arrow[r] \arrow[d, "0"] & 0 \\ 0 \arrow[r] & Z^{n+1} \arrow[r] & C^{n+1}(X;\mathbb{Z}) \arrow[r] & B^{n+2}(X) \arrow[r] & 0 \end{tikzcd}$$
$\rightsquigarrow$
$$\begin{tikzcd} 0 \arrow[r] & Z^n \otimes \mathbb{Z}/2 \arrow[r] \arrow[d, "0"'] & C^n(X;\mathbb{Z}/2) \arrow[r] \arrow[d] & B^{n+1}(X) \otimes \mathbb{Z}/2 \arrow[r] \arrow[d, "0"] & 0 \\ 0 \arrow[r] & Z^{n+1} \otimes \mathbb{Z}/2 \arrow[r] & C^{n+1}(X;\mathbb{Z}/2) \arrow[r] & B^{n+2}(X) \otimes \mathbb{Z}/2 \arrow[r] & 0 \end{tikzcd}$$
$\rightsquigarrow$
$$B^{n+1}(X) \otimes \mathbb{Z}/2 \to Z^{n+1}(X) \otimes \mathbb{Z}/2 \qquad B^n \otimes \mathbb{Z}/2 \to \underbrace{Z^n \otimes \mathbb{Z}/2}_{H^n(X;\mathbb{Z}) \otimes \mathbb{Z}/2} \to H^n(X;\mathbb{Z}/2) \to \underbrace{B^{n+1}(X) \otimes \mathbb{Z}/2 \to Z^{n+1}(X) \otimes \mathbb{Z}/2}_{\mathrm{Tor}(H^{n+1}(X);\mathbb{Z}/2)}$$

$H^{n+1}(X;\mathbb{Z}) = \mathbb{Z}/2^r\,\omega \oplus \cdots \Rightarrow \exists\, \tau \in C^n(X)$ s.t. $d\tau = 2^r\omega$ and $\tau \neq 2\tau'$ for any τ'.

By UCT: $\omega \in \ker\left(B^{n+1}(X)\otimes \mathbb{Z}/2 \longrightarrow Z^{n+1}(X)\otimes \mathbb{Z}/2\right)$

This kernel is precisely generated by $d\tau\otimes 1$

Back to our Bockstein, $d_k\omega' = \left[\frac{d\tau}{2^k}\right] \bmod 2 = \left[\frac{2^r\omega}{2^k}\right] \bmod 2 = \begin{cases} 0 & \text{if } k<r \\ \omega'' & \text{if } k=r \end{cases}$

$$C^n(X) \to C^n(X)\otimes \mathbb{Z}/2 \subseteq B^{n+1}(X)\otimes\mathbb{Z}/2$$
$$\tau \longmapsto \omega' \quad (= d\tau\otimes 1)$$

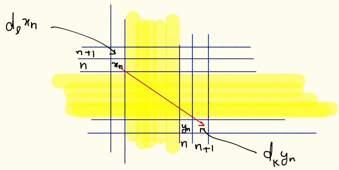
Prop: If $H^k(X;\mathbb{Z}/2)=0$ for $k<n$ and $H^n(X;\mathbb{Z}/2)\cong \mathbb{Z}/2\,\omega$ then $H^n(X;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } d_k\omega = 0\ \forall k, \\ \mathbb{Z}/2^n & \text{if } d_k\omega=0\ \forall k<n,\ d_n\omega\neq 0. \end{cases}$ + odd torsion

Proof: Direct corollary of above propositions.

This proposition will be used in spectral sequences. So we need to understand how the Bocksteins behave under transgressions.

In particular we can look at $K(\mathbb{Z}/2^n, m)$ and let i_m be generator of H^m. Then we would get $d_n i_m \neq 0$ but $d_{n-i} i_m = 0$.

We have a spectral sequence $\overset{x_n}{K(\mathbb{Z}/2^\ell,n)} \longrightarrow \overset{z_n}{K(\mathbb{Z}/2^{k+\ell},n)} \longrightarrow \overset{y_n}{K(\mathbb{Z}/2^k,n)}$ and we have a forced transgression $d_\ell x_n \longmapsto d_k y_n$



This hold in generality.

In $K(\mathbb{Z}/2^{k+\ell},n)$ we then have

$$z_n \longleftarrow\!\shortmid\, y_n$$
$$d_\ell x_n \longleftarrow\!\shortmid\, d_{k+\ell} z_n$$

Prop: Suppose we have a spectral sequence diagram as above for arbitrary fibration $F\to E\to B$ and now $x_n \in H^n(F;\mathbb{Z}/2)$ and $y_n\in H^n(B,\mathbb{Z}/2)$ and x_n transgresses to $d_k y_n$ then if $z_n \longleftarrow\!\shortmid\, y_n$ then $d_{k+\ell} z_n \longmapsto d_\ell x_n$.

Proof: This proof is possibly terribly flawed. But it gives an insight into what is happening. The point is x_n and $d_\ell x_n$, y_n and $d_k y_n$ are really the same elements in $\mathbb{Z}$-coefficients. They manifest as two terms because of the Universal Coefficient Theorem.

$$\begin{tikzcd}
\overset{x_n}{H^n(F;\mathbb{Z})\otimes\mathbb{Z}/2} \arrow[r] \arrow[rr, dashed, bend left] \arrow[ddrr, bend right] & H^n(F;\mathbb{Z}/2) \arrow[r] \arrow[d] & \mathrm{Tor}(H^{n+1}(F);\mathbb{Z}/2) \\
\overset{z_n}{H^n(E;\mathbb{Z})\otimes\mathbb{Z}/2} \arrow[r] & H^n(E;\mathbb{Z}/2) \arrow[r] & \mathrm{Tor}(H^{n+1}(E);\mathbb{Z}/2) \\
\overset{y_n}{H^n(B;\mathbb{Z})\otimes\mathbb{Z}/2} \arrow[r] \arrow[rr, dashed, bend left] & H^n(B;\mathbb{Z}/2) \arrow[r] \arrow[u] & \mathrm{Tor}(H^{n+1}(B);\mathbb{Z}/2)
\end{tikzcd}$$

The solid arrow is the transgression

The dotted arrows are the Bocksteins.

And because $y_n \mapsto z_n$ the Bockstein on z_n is like composing the two Bocksteins and so should be d_{k+l}.

Look at the corresponding $\mathbb{Z}$-coefficient transgressions. We have an $x_n' \in H^{n+1}(F;\mathbb{Z})$, $y_n' \in H^{n+1}(B,\mathbb{Z})$. The fact these have torsion $\Rightarrow 2^l x_n' = d\tau$, $2^k y_n' = d\sigma$ for some τ and σ. And $\tau\otimes 1$ transgresses to $y_n'\otimes 1$. Now there cannot be a transgression from x_n' to y_n' as both are in H^{n+1}. Because $\tau\otimes 1$ is transgressive x_n', y_n' should survive the entire SS and so they contribute to either a $\mathbb{Z}/2^{k+l}$ or $\mathbb{Z}/2^l + \mathbb{Z}/2^k$. Now if it was the later then going back to $\mathbb{Z}/2$ coefficients we see that x_n could not transgress. So $H^{n+1}(E;\mathbb{Z}) = \mathbb{Z}/2^{k+l} \oplus \dots$ and the mappings are canonical.

Q. What is the relationship between the Bocksteins and the Steenrod Squares?

A. $d_1 = Sq^1$. So clearly commutativity is out of the picture. The higher Bocksteins are then going to be hard to handle. So we'd use d_1 and the above lemma as much as possible.