

$\infty$ -categories, model independent theory

- Emily Riehl

### Lecture 01:

Goal: develop  $\infty$ -categories from first principles.

- "synthetic" not "analytic"
- apply simultaneously to many models
- not at all hand-wavy. axiomatic.
- compatible with other approaches
- more categorically sophisticated, less technical
- work in a 2-category (strict) :  $\infty$ -cats, functors, natural transforms

Q. What is an  $\infty$ -category?

- generalization of 1-category
- weak  $\infty$ -dimensional categories
- ...

• Special case : one example / model for  $\infty$ -categories

• Quasi-categories - Joyal "analytic"

Sets : category theory  $::$  Quasi-categories :  $\infty$ -categories

Def A quasi-category is a simplicial set  $A$  s.t.

$$\begin{array}{l} k\text{-horn} \hookleftarrow \Lambda^k[n] \longrightarrow A \\ \quad \quad \quad \downarrow \quad \quad \nearrow \\ n\text{-simplex} \hookleftarrow \Delta[n] \end{array}$$

$\forall n \geq 2$   
 $\forall 0 < k < n$   
 "inner horns can be filled"

Similar to a Kan-complex :

$$\text{s.t.} \quad \begin{array}{l} \Lambda^k[n] \longrightarrow A \\ \quad \quad \quad \downarrow \quad \quad \nearrow \\ \Delta[n] \end{array}$$

$\forall n \geq 1$   
 $\forall 0 \leq k \leq n$

Examples: 1) Kan-complexes are  $q$ -category  
 The " $\infty$ -groupoids".

2)  $\mathcal{C}$  be a 1-category.  
 (abuse of notation)  $\hookrightarrow \mathcal{C}$  be its nerve also,  $\mathcal{C}[n] = n$ -composable arrows

Prop: Nerves of categories are quasi-categories.

The horns are filled in because of associativity of composition.

\* Nerves of categories are 2-coskeletal.

• Nerve is an embedding

$$1\text{-lat} \xrightarrow{\quad \perp \quad} \mathcal{Q} \text{ lat} \subseteq \mathbf{sSet}$$

def<sup>n</sup>:  $A$  = a quasi-category  
 homotopy category of  $A$  :  $ob = A_0$   
 $mor =$  homotopy classes of parallel 1-simplices

$$f \sim g \text{ iff } \begin{array}{ccc} & f \nearrow & Y \\ X & & \parallel \\ & g \searrow & Y \end{array} \quad \text{or} \quad \begin{array}{ccc} X & & f \searrow \\ \parallel & & Y \\ X & & g \nearrow \end{array}$$

comp :  $h = g \circ f$  iff  $\exists$  a 2 simplex

$$\begin{array}{ccc} f \nearrow & & g \searrow \\ & \parallel & \\ h \longrightarrow & & \end{array}$$

\* If  $A = \text{nerve of } \mathcal{C}$  then  $ho A = \mathcal{C}$ .

Prop:

$ho : sSet \rightarrow Cat$  is left adjoint to the nerve.

i.e.

a functor  $ho X \rightarrow \mathcal{C}$  extends uniquely to a simplicial map  $X \rightarrow \mathcal{C}$ .

Features of quasi-categories :

•  $sSet$  is cartesian closed

$$\{X \times Y \rightarrow Z\} \cong \{X \rightarrow Z^Y\}$$

$$\bullet X = \Delta[n]$$

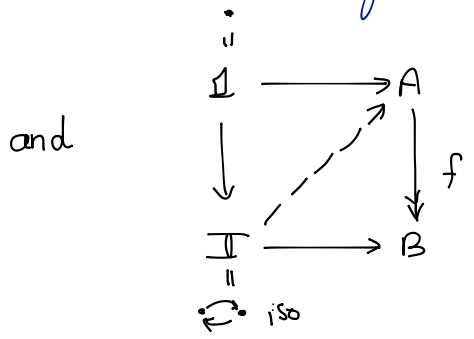
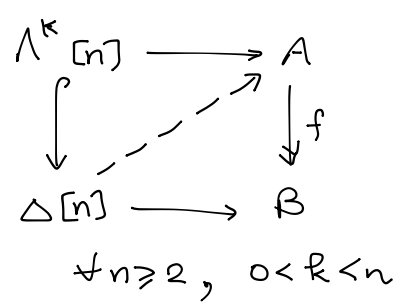
$$\{X \rightarrow Z^Y\} = \text{set of } n\text{-simplices in } Z^Y$$

$$\{Y \times \Delta[n] \rightarrow Z\}$$

• Quasi-categories define an exponential ideal

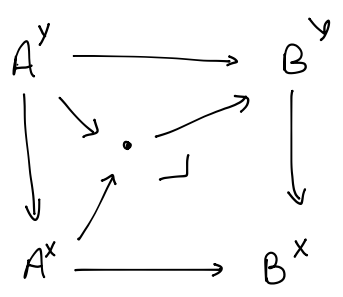
$$A \text{ quasi-cat, } X \text{ sSet} \Rightarrow A^X \text{ q-cat}$$

Def: A map between quasi-cat  $A \xrightarrow{f} B$  is an *isofibration*



- $A \twoheadrightarrow 1$  is an isofibration,  $A$  is a quasi-category.
- Isofibrations: compose, contain iso, stable under retracts, pullbacks, limits of towers.

Prop: (Joyal)  $X \hookrightarrow Y$  is a mono,  $A \xrightarrow{f} B$  is an iso-fib,



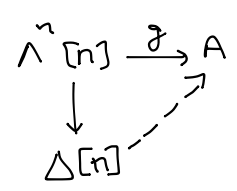
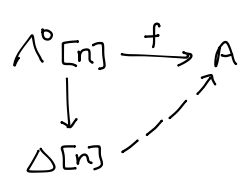
then all maps in the diagram on the left are isofibrations.

\* Very long proof

Def A 1-simplen in a quasi-cat is an iso iff its an iso in its homotopy category.

Prop (Joyal) special outer horn lifting

If  $f|_{[0,1]}$  or  $g|_{[n-1,n]}$  are iso then lifts exist.



Cor: A quasi-cat is a Kan-complex iff its homotopy category is a groupoid.

Cor: An arrow in a quasi-cat is an iso iff it's a homotopy coherent iso:

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & A \\ \downarrow & \nearrow & \\ \mathcal{I} & & \end{array}$$

Def<sup>n</sup>: A equivalence  $A \simeq B$  is given by maps  $A \xrightarrow{f} B$ ,  $B \xrightarrow{g} A$  s.t.

$$\begin{array}{ccc} & & A \\ & \nearrow g^f & \uparrow \omega_1 \\ A & \xrightarrow{f} & A \\ & \searrow & \downarrow \omega_0 \\ & & A \end{array}$$

$$\begin{array}{ccc} & & B \\ & \nearrow f^g & \uparrow \omega_0 \\ B & \xrightarrow{g} & B \\ & \searrow & \downarrow \omega_1 \\ & & B \end{array}$$

Def: A trivial fibration:

$$\begin{array}{ccc} \partial \Delta[n] & \longrightarrow & A \\ \downarrow & \nearrow & \downarrow s \\ \Delta[n] & \longrightarrow & B \end{array}$$

• Trivial fib  $\Leftrightarrow$  isofib + equivalence.