

Facts about quasi-categories :

• Category of $q\text{Cats}$ + maps being simplicial sets is enriched over $q\text{Cats}$.

i.e. the maps $\{A \rightarrow B\}$ between $q\text{Cats}$ coincides with the set of vertices of the $q\text{Cat}$ $B^A =: \text{Fun}(A, B)$.

• \exists special maps : isofib, equiv, trivial fib, isofib.

\equiv There are many strict 1-cats of ∞ -cats + functors between them that admit these same functors.

Def An ∞ -cosmos \mathcal{K} is a category that is enriched over $q\text{Cats}$

- objects A, B are ∞ -categories
- functor-spaces $\text{Fun}(A, B)$ s.t. vertices are the maps in the underlying 1-cat \mathcal{K} .
- specific classes of isofib. satisfying closedness wrt ...
- completeness wrt certain limits
- equivalences have sections.

Recall $f: A \rightarrow B$ in \mathcal{K} is equiv if $\forall x \in \mathcal{K}$

$\text{Fun}(x, A) \xrightarrow{\cong} \text{Fun}(x, B)$ is a trivial fib [equiv + isofib]

or equiv. $A \xrightarrow{f} B$ is an equivalence iff

$$\exists g: B \rightarrow A$$

$$\begin{array}{ccc} & A & \\ \nearrow & & \nwarrow \\ A & \xrightarrow{f} & A^{\text{II}} \\ \searrow & & \downarrow \\ & B & \end{array}$$

$$\begin{array}{ccc} & B & \\ \nearrow & & \nwarrow \\ B & \xrightarrow{f} & B^{\text{II}} \\ \searrow & & \downarrow \\ & B & \end{array}$$

2-categories

eg: 2-category Cat : categories
 functors
 natural transforms

$$A, B, C$$

$$f: A \longrightarrow B$$

$$A \begin{array}{c} \xrightarrow{f} \\ \Downarrow \alpha \\ \xrightarrow{g} \end{array} B$$

\equiv Any ∞ -cosmos \mathcal{K} has an associated homotopy 2-category (or strict 2-cat) analogous to Cat .

Def The homotopy 2-category $\mathfrak{h}\mathcal{K}$ of \mathcal{K} :

- objects are same as \mathcal{K}
- 1-cells $\text{---}''\text{---}$
- 2-cells $A \begin{array}{c} \xrightarrow{\quad} \\ \Downarrow \alpha \\ \xrightarrow{\quad} \end{array} B$ are homotopy classes of 1-simplices in $\text{Fun}(A, B)$.

$$\text{i.e. } \begin{array}{c} \text{Fun}^{\mathfrak{h}}(A, B) \\ (\text{in } \mathfrak{h}\mathcal{K}) \end{array} = \text{ho} \begin{array}{c} (\text{Fun}(A, B)) \\ (\text{in } \mathcal{K}) \end{array}$$

Rmk (on 2-cells) A natural transformation $A \begin{array}{c} \xrightarrow{f} \\ \alpha \Downarrow \\ \xrightarrow{g} \end{array} B$ in $\mathfrak{h}\mathcal{K}$ is represented by $\mathcal{D} \xrightarrow{\quad} \text{Fun}(A, B)$.

- A natural iso (an invertible 2-cell)

$$A \begin{array}{c} \xrightarrow{f} \\ \cong \Downarrow \alpha \\ \xrightarrow{g} \end{array} B \text{ in } \mathfrak{h}\mathcal{K} \text{ is represented by}$$

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\alpha} & \text{Fun}(A, B) \\ \downarrow & \nearrow \alpha & \\ \mathbb{I} & & \end{array}$$

Prmk: Cat is an ∞ -cosmos. Its homotopy 2-category is itself.

Prop: $\mathcal{H}\mathcal{K}$ has a terminal object, in a 2-categorical sense.

Proof: \mathcal{K} has a terminal object 1 in a simplicially enriched sense (axiom)
i.e. $\text{Fun}(X, 1) \cong 1$

Apply $\eta: \text{Cat} \xrightarrow{h_0} \text{Cat}$ which sends $\text{Fun}(A, B) \longrightarrow \text{Fun}^{\mathcal{H}}(A, B)$

$$\begin{array}{ccc} \text{Fun}(X, 1) & \xrightarrow{\quad} & \text{Fun}^{\mathcal{H}}(X, 1) \\ \parallel & & \parallel \\ 1 & \xrightarrow{\quad} & 1 \in \text{Cat} \end{array}$$

Prop: $\mathcal{H}\mathcal{K}$ has products in 2-categorical sense.

Proof: follows from the fact that h_0 preserves products.

- Any 1-category has a notion of iso between objects.
- Any 2-category " " equiv "

Def: Equiv:

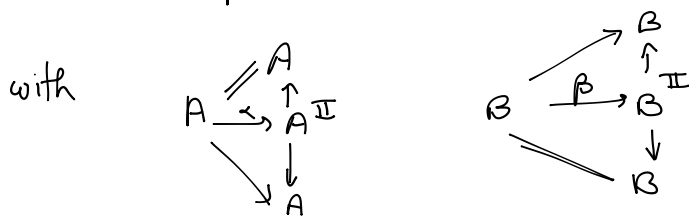
- obj A, B
- 1-cells $A \xrightarrow{f} B, B \xrightarrow{g} A$
- invertible 2-cells,

$$\begin{array}{c} \overline{\overline{A}} \\ \cong \downarrow \alpha \\ A \end{array} \xrightarrow{gf} A$$

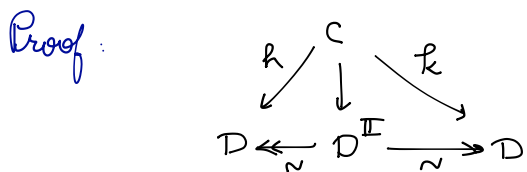
$$\begin{array}{c} \overline{\overline{B}} \\ \cong \downarrow \beta \\ B \end{array} \xrightarrow{gf} B$$

Prop: $A \xrightarrow{f} B$ is an equiv in \mathcal{K} iff it's an equiv in $\mathcal{H}\mathcal{K}$.

Proof. $\Rightarrow A \xrightarrow{f} B$ equiv in \mathcal{K} means $\exists B \xrightarrow{g} B$



\Leftarrow Claim: $C \begin{matrix} \xrightarrow{h} \\ \cong \Downarrow \\ \xrightarrow{k} \end{matrix} D$ then h equiv $\Rightarrow k$ equiv $\in hK$



Now, if f, g are equiv in hK

$\Rightarrow gf, fg \cong id$ in hK

Then by a 2-out-of-6 property for equiv in \mathcal{K} we're done.

Lem: 2 functors preserve equivalences.

Lem: Any simplicially enriched functor $\mathcal{K} \xrightarrow{F} \mathcal{L}$ gives a 2-functor $hK \rightarrow hL$.

Cor: $A \xleftarrow{\sim} A', B \xrightarrow{\sim} B'$ in an ∞ -cosmos. Then

$$\text{Fun}(A, B) \xrightarrow{\sim} \text{Fun}(A', B') \in q\text{Cat}.$$