n- Groupoids

Category of fibrant objects:

$\mathcal{V}$ a small category.

$\mathcal{W} \subseteq \mathcal{V}$ subcategory of weak equivalences

- contains iso
- 2-out-of-3

Dwyer-Kan : Can construct simplicial localization of $\mathcal{V}$ respect to $\mathcal{W}$. (components of this is the derived cat.)

$\mathcal{V}_\bullet \longrightarrow \mathcal{V}$ simplicial resolution

$Ob(\mathcal{V}_n) = Ob(\mathcal{V}) \qquad n \geq 0$

$\mathcal{V}_n$ is a free category for $n \geq 0$

$\hookrightarrow$ n^{th} level = words in $\mathcal{V}$ with parantheses of depth n.

$\mathcal{V}_\bullet[W^{-1}] = \mathcal{W}_\bullet^{-1}\mathcal{V}_\bullet$ $\leftsquigarrow$ Goal: understand this localization

2-out-of-6

$W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$ If gf and $hg \in \mathcal{W}$ then f, g, h in $\mathcal{W}$

$\underline{Th^m}$ if 2 out of 6 is satisfied then every weak equiv goes to iso in $\mathcal{V}_\bullet[\mathcal{W}^{-1}]$. (Called Saturation property)

Q. Check this for $\pi_i(-)$.

categories of fibrations

models category of manifolds

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

strictly

- $\mathcal{F} \subseteq \mathcal{V}$ subcategory. All pullbacks in $\mathcal{F}$ exist
- $\exists$ a terminal object.
- all iso $\in \mathcal{F}$

CFO

A category of fibrant objects satisfies following axioms: (has we & fibrations)

1) Every object is fibrant (i.e. $X \longrightarrow e$ is a fibration)
2) Pullback of a trivial fibration is a trivial fibration (= w.e. + fib)
3) Every morphism factors into a w.e. followed by a fibration.

- These axioms are weaker than model category axioms.

eg: Kan Complexes, manifolds

Vertices of Simplicial Localization for CFO

Generalised morphisms

$$\begin{tikzcd} & P \arrow[dl, "\sim"'] \arrow[dr, "\text{morphism}"] & \\ X & & Y \end{tikzcd} \in \mathrm{Hom}(X, Y)$$

$N.\mathcal{V}.[\mathcal{W}^{-1}]$ is a quasi-category

1-simplices

N_0 objects of $\mathcal{V}$

N_1 $x \xleftarrow{} P \to y$

N_2

$$\begin{tikzcd} & & x_1 & & \\ & x_{01} \arrow[ur] \arrow[ddl, "\sim"'] & & x_{12} \arrow[ul, "\sim"'] \arrow[ddr] & \\ & & x_{012} \arrow[ul, "\sim"'] \arrow[ur] \arrow[d, "\sim"] & & \\ x_0 & & x_{02} \arrow[ll, "\sim"] \arrow[rr] & & x_2 \end{tikzcd}$$

Brown's lemma :

W.E. in a CFO have the following form : section of trivial fib. followed by a fib

$$\begin{tikzcd} & P \arrow[dl, bend right, "\sim"'] \arrow[dr, "\sim"] & \\ X \arrow[ur] & & Y \end{tikzcd}$$

- This lemma allows us to work exclusively with fibrations.

- Fibrations of s.Sets : $X \to Y$
  - for $n > 0$, $0 \le i \le n$

$$X_n \longrightarrow \mathrm{Hom}(\Delta^n, X) \times_{\mathrm{Hom}(\Lambda^n_i, Y)} Y_n \quad \text{is surjective}$$

i^{th} horn

- Trivial Fibration of Kan complexes
  - for $n \ge 0$

$$X_n \longrightarrow \mathrm{Hom}(\Delta^n, X) \times_{\mathrm{Hom}(\partial\Delta^n, Y)} Y_n \quad \text{is surjective}$$