Monday, Nov 11, 2019

Ch 4 Summary :

3 parts : ① Absolute / local min/max , concavity of $y = f(x)$

- Look at signs of $f'(x)$, $f''(x)$.

② Optimization problems

- Convert word problems to finding min/max.

③ L'Hospital's rule

- Finding limits of a) $\frac{0}{0}, \frac{\infty}{\infty}$

b) $0 \cdot \infty$, 0^{∞}, 1^{∞}, ∞^{0} etc.

c) $\infty - \infty$

$\downarrow$ indeterminate forms

a) = L'Hospital's rule

b) = logarithmic differentiation

c) = algebraic simplification

Review : L'Hospital's Rule :

- $\frac{\infty}{\infty}$, $\frac{0}{0}$ $\leftarrow$ in such cases apply L'Hospital's Rule

eg : $\lim\limits_{x \to \infty} \frac{\ln x}{\sqrt{x}}$ Check : $x \to \infty$ $\ln x \to \infty$, $\sqrt{x} \to \infty$

$= \lim\limits_{x \to \infty} \frac{(\ln x)'}{(\sqrt{x})'}$ by L'Hospital's rule

$= \lim\limits_{x \to \infty} \frac{1/x}{x^{-1/2} \cdot (1/2)}$

$= \lim\limits_{x \to \infty} \frac{x^{-1}}{x^{-1/2} \cdot (1/2)}$

$$= \lim_{x \to \infty} x^{-1} \cdot x^{1/2} \cdot 2$$

$$= \lim_{x \to \infty} x^{-1/2} \cdot 2$$

$$= \lim_{x \to \infty} \frac{1}{\sqrt{x}} \cdot 2$$

as $x \to \infty$
$\sqrt{x} \to \infty$,
so, $\frac{1}{\sqrt{x}} \to 0$.

$$= 0$$

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§ Second Half of the course: Integrals

- Definitions
- Fundamental theorem of calculus
- Computations (u-substition)
- Applications.

• 3.9 Anti-derivatives

Def: A function $F(x)$ is called an anti-derivative of $f(x)$ if

$$F'(x) = f(x).$$

eg: $(\sin x)' = \cos x$

hence

($\sin x$ is the anti-derivative of $\cos x$)

- the anti-derivative of $\cos x$ is $\sin x$.

> we do this later: if $F(x)$ is ~~the~~ an anti-derivative of $f(x)$
>
> then $F(x) = \int f(x)\,dx$.

Q. How to find anti-derivatives?

A. Guess

$$\begin{tikzcd} F(x) \arrow[r, bend left, "\text{derivative}"] & f(x) \arrow[l, bend left, "\text{anti-derivative}"] \end{tikzcd}$$

eg:

	$f(x)$	$F(x)$ antiderivative
a)	1	x $x+1$ $x+c$ for any c.
b)	$\cos x$	$\sin x$ $\sin x + c$, for any c

Theorem / Fact: If $F(x)$ is an anti-derivative of $f(x)$, then so is $F(x)+c$ for any constant c.

So, we always leave an unknown constant when finding anti-derivaties.

- For guessing, you must know ~~anti~~ derivatives of standard functions.

eg : • anti-derivative of $\frac{1}{x} = \ln x + C$

- Usually, the difficulty in finding anti-derivatives is with the constants.

eg. anti-derivative of $\frac{1}{2x}$

Guess : 1) $\ln(2x) + c$

Check : $\ln(2x)' = \frac{1}{2x} \cdot 2$ by chain Rule

$= \frac{1}{x}$ ✗

Guess 2) $\frac{\ln(x)}{2} + C$

Check : $\left(\frac{\ln x}{2}\right)' = \frac{1}{2} \cdot \frac{1}{x}$

$= \frac{1}{2x}$ ✓

Ans : $\frac{\ln x}{2} + c$

eg: anti-derivative of $\cos(2x)$.

Guess: $\sin(2x)$

Check: $(\sin(2x))' = \cos(2x) \cdot 2$

Guess: $\dfrac{\sin(2x)}{2}$

Check: $\left(\dfrac{\sin(2x)}{2}\right)' = \dfrac{1}{\cancel{2}} \cdot \cos(2x) \cdot \cancel{2}$ ✓

Ans: $\dfrac{\sin(2x)}{2} + C$.

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eg: anti derivative of $x + \sin(2x)$.

~~Guess~~ // Anti-derivative of x

Guess: $\frac{1}{2} x^2$

Check: $\left(\frac{1}{2} x^2\right)' = \frac{1}{2} \cdot 2x$

$= x$ ✓

anti derivative of $\sin(2x)$

Guess: $\frac{1}{2} \cos(2x)$

Check: $\left(\frac{1}{2} \cdot \cos(2x)\right)'$

$= \frac{1}{2} \cdot (-\sin(2x)) \cdot 2$

$= -\sin(2x)$

Guess: $-\frac{1}{2} \cos(2x)$

Check: ✓

Combining the two:

anti-derivative of $x + \sin(2x)$

is $\dfrac{x^2}{2} - \dfrac{1}{2} \cdot \cos(2x) + C$

eg : ~~g(x) =~~

Find anti-derivative of

$$4\sin x + \frac{2x^5 - \sqrt{x}}{x}$$

$$= 4\sin x + \frac{2x^5}{x} - \frac{\sqrt{x}}{x}$$

$$= 4\sin x + 2x^4 - x^{-1/2}$$

anti-derivatives :

Guess : $-4\cos x$

Check: $(-4\cos x)'$

$= -4.(-\sin x)$

$= 4 \cdot \sin x$

Guess: $\frac{2}{5}x^5$

Check: $\left(\frac{2}{5}x^5\right)'$

$= \frac{2}{5}.5.x^4$

$= 2.x^4$

Guess : $x^{1/2}$

increase power by 1 (because of power rule)

Check : $(x^{1/2})'$

$= \frac{1}{2}x^{-1/2}$

Guess: $\frac{x^{1/2}}{1/2}$

Check : $\left(\frac{x^{1/2}}{1/2}\right)'$

$= \frac{\cancel{1}}{\cancel{2}} \cdot \frac{x^{-1/2}}{\cancel{1/2}}$

$= x^{-1/2}$

Ans : $\frac{x^{1/2}}{1/2} = 2x^{1/2}$

Final ans : anti derivative of $4\sin x + 2x^4 - x^{-1/2}$

is

$$\boxed{-4\cos x + \frac{2}{5}x^5 - 2x^{1/2} + c}$$

- Good ~~the~~ anti-derivative to memorize :

$\frac{1}{x} = x^{-1} \longrightarrow \ln x$

$x^n, \longrightarrow \frac{x^{n+1}}{n+1}$

(n is not −1)

eg: $x^{-1/2} \longrightarrow \frac{x^{-1/2+1}}{-1/2+1} = \frac{x^{1/2}}{1/2}$

- We can find the unknown constant "c" if we are given more information about $f(x)$

eg : Given $F'(x) = \frac{1}{x^2+1}$ and $F(0) = 2$

Find $F(x)$.

Ans : $F'(x) = \frac{1}{x^2+1}$

anti-derivative :

$$\boxed{F(x) = \arctan x + C}$$

$F(0) = 2$

$\Rightarrow \arctan(0) + C = 2$ (Plug in $x=0$)

$\Rightarrow 0 + C = 2$

$\Rightarrow \boxed{C = 2}$ $\Rightarrow \boxed{F(x) = \arctan(x) + 2}$ ∎

Note :

Very important :

$f'(x)$	antiderivative $f(x)$
$\frac{1}{x}$	$\ln x + C$
$x^n, n \neq -1$	$\frac{x^{n+1}}{n+1} + C$
$\frac{1}{1+x^2}$	$\arctan x + C$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x + C$

Be careful with the algebra

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eg : Given $f''(x) = \frac{1}{x^2}$

and $f(1) = 3$, $f'(2) = 4$

find $f(x)$.

Ans:

$$\boxed{f''(x) = \frac{1}{x^2} = x^{-2}}$$

Finding the antiderivative

$$\Rightarrow f'(x) = \frac{x^{-2+1}}{-2+1} + C \quad \text{(Power rule)}$$

$$= \frac{x^{-1}}{-1} + C$$

$$\boxed{f'(x) = -\frac{1}{x} + C}$$

Plugging in $x = 2$

$$\boxed{f'(2) = -\frac{1}{2} + c}$$

$$\Rightarrow \quad 4 = \frac{-1}{2} + c$$

$$\Rightarrow \quad \boxed{c = 4 + \frac{1}{2} = \frac{9}{2}}$$

$$\Rightarrow \quad \boxed{f'(x) = -\frac{1}{x} + \frac{9}{2}}$$

finding the anti-derivative again

$$f(x) = -\ln x + \frac{9}{2} x + c'$$

Plug in $x = 1$

$$f(1) = -\ln 1 + \frac{9}{2} \cdot 1 + c'$$

$$\Rightarrow \quad 3 = 0 + \frac{9}{2} + c'$$

$$\Rightarrow \quad \boxed{c' = 3 - \frac{9}{2}}$$

$$\Rightarrow \quad \boxed{f(x) = -\ln x + \frac{9}{2} x + 3 - \frac{9}{2}}$$

Appendix E :

· Goal: Define integrals eventually.

· Sigma notation : (need this to define integrals)

Given a Sequence of numbes : 1, 2, 3, 4, ..., n.

n is a constant

· we want to find the sum $1 + 2 + 3 + \dots + n$.

· Sigma notation is a way to denote the sum more consisely

$$1 + 2 + \dots + n = \sum_{i=1}^{n} i$$

Summation from $i=1$ to n

$\sum$ = Sigma

- $$\sum_{i=a}^{b} f(i) = f(a) + f(a+1) + \cdots + f(b)$$

start at a $\longrightarrow$ increment by 1 go up to b.

eg: $$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n$$

- ($\sum$ always means increment by 1.)

eg $$\sum_{i=3}^{5} i^2 = 3^2 + 4^2 + 5^2$$

eg $$\sum_{i=1}^{4} 2i = 2 + 4 + 6 + 8$$

$$\sum_{i=2}^{4} (2i+1) = 5 + 7 + 9$$

$5 = 2\cdot 2+1$ $\quad 7 = 2\cdot 3+1$ $\quad 9 = 2\cdot 4+1$

eg: $\displaystyle\sum_{i=3}^{5} (2i-1) = 5+7+9$

To find the endpoint: $2i-1 = 9$
$\Rightarrow 2i = 10$
$\Rightarrow i = 5$

~~eg~~:

- The actual goal is to find these sums
- But they are hard and beyond the scope of this course.

Most problems are simple problems (for us)

eg: find $\displaystyle\sum_{i=5}^{6} \left(\frac{2i+1}{3}\right) =$

Ans: $= \dfrac{2.5+1}{3} + \dfrac{2.6+1}{3}$

$= \dfrac{11}{3} + \dfrac{13}{3}$

$= \dfrac{24}{3}$

$= 8$

eg: Find $\displaystyle\sum_{i=5}^{10} 1$

Ans: $= \underset{5}{1} + \underset{6}{1} + \underset{7}{1} + \underset{8}{1} + \underset{9}{1} + \underset{10}{1} = 6$

eg: Telescoping series

$$\sum_{i=1}^{1000} \left(i^3 - (i+1)^3 \right)$$

Ans: $= (1^3 - 2^3) + (2^3 - 3^3) + (3^3 - 4^3) + \cdots + (1000^3 - (1001)^3)$

$= 1^3 - (1001)^3.$

$$\boxed{\text{Telescoping Sum}: \sum_{i=a}^{b} \left(f(i) - f(i+1) \right)}$$

$= (f(a) - f(a+1)) + (f(a+1) - f(a+2)) + \cdots$

Ans $= f(a) - f(b+1)$