∞- categories , model independent theory

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Lecture 01 :

Goal : develop ∞-categories from first principles.

- "synthetic" not "analytic"
- apply simultaneously to many models
- not at all hand-wavy . axiomatic.
- compatible with other approaches
- more categorically sophisticated, less technical
- work in a 2-category (strict) : ∞-cats, functors, natural transforms

Q. What is an ∞- category?

- generalization of 1- category
- weak ∞- dimensional categories
- ...

- Special case : one example / model for ∞-categories
- Quasi-categories – Joyal "analytic"

Sets : category theory :: Quasi-categories : ∞-categories

Def A quasi-category is a simplicial set A s.t.

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, hook] & A \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd}$$

k-horn $\rightsquigarrow \Lambda^k[n]$

n-simplex $\rightsquigarrow \Delta[n]$

$\forall n \geq 2$
$\forall\ 0 < k < n$
"inner horns can be filled"

Similar to a Kan-complex :

s.t.
$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, hook] & A \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd}$$

$\forall n \geq 1$
$\forall\ 0 \leq k \leq n$

Examples: 1) Kan-complexes are q-category
The "∞-groupoids".

2) $\mathcal{C}$ be a 1-category.
(abuse of notation) $\rightsquigarrow$ $\mathcal{C}$ be its nerve also, $\mathcal{C}[n]$ = n-composable arrows

Prop: Nerves of categories are quasi-categories.

The horns are filled in because of associativity of composition.

$\mathcal{A}$ Nerves of categories are 2-coskeletal.

- Nerve is an embedding

$$\begin{tikzcd} 1\text{-}\mathcal{C}at \arrow[r, hook, bend right] \arrow[r, phantom, "\perp"] & Q\mathcal{C}at \arrow[l, dashed, bend right, "ho"'] \end{tikzcd} \subseteq sSet$$

def^n: A = a quasi-category
homotopy category of A : ob $= A_0$
mor = homotopy classes of parallel 1-simplices

$f \sim g$ iff
$$\begin{tikzcd} & Y \arrow[d, equal] \\ X \arrow[ur, "f"] \arrow[r, "g"'] & Y \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} X \arrow[d, equal] \arrow[dr, "f"] & \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

comp : $h = g \circ f$ iff $\exists$ a 2 simplex
$$\begin{tikzcd} & \cdot \arrow[dr, "g"] & \\ \cdot \arrow[ur, "f"] \arrow[rr, "h"'] & & \cdot \end{tikzcd}$$

$\clubsuit$ If A = nerve of $\mathcal{C}$ then $ho\, A = \mathcal{C}$.

Prop:
$ho : sSet \longrightarrow Cat$ is left adjoint to the nerve.
i.e.
a functor $ho X \longrightarrow \mathcal{C}$ extends uniquely to a simplicial map $X \dashrightarrow \mathcal{C}$.

Features of quasi-categories:

- sSet is cartesian closed
$$\{X \times Y \longrightarrow Z\} \cong \{X \longrightarrow Z^Y\}$$

- $X = \Delta[n]$
$$\{X \longrightarrow Z^Y\} = \text{set of } n\text{-simplices in } Z^Y$$
$$\cong$$
$$\{Y \times \Delta[n] \longrightarrow Z\}$$

- Quasi-categories define an exponential ideal
A quasi-cat, X sSet $\Rightarrow$ A^X q-cat

Def: A map between quasi-cat $A \xrightarrow{f} B$ is an *isofibration*

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, hook] & A \arrow[d, two heads, "f"] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & B \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \bullet \arrow[d, phantom, "\shortparallel"] & \\ \mathbb{1} \arrow[r] \arrow[d] & A \arrow[d, "f"] \\ \mathbb{I} \arrow[r] \arrow[ur, dashed] \arrow[d, phantom, "\shortparallel"] & B \\ \bullet \rightleftarrows \bullet \ \text{iso} & \end{tikzcd}$$

$\forall n \geq 2, \ 0 < k < n$

- $A \twoheadrightarrow \mathbb{1}$ is an isofibration, A is a quasi-category.
- Isofibrations : compose, contain iso, stable under retracts, pullbacks, limits of towers.

Prop: (Joyal) $X \hookrightarrow Y$ is a mono,
$A \xrightarrow{f} B$ is an iso-fib,

$$\begin{tikzcd} A^Y \arrow[rr] \arrow[dr] \arrow[dd] & & B^Y \arrow[dd] \\ & \bullet \arrow[ur] \arrow[dr, phantom, "\lrcorner"] & \\ A^X \arrow[ur] \arrow[rr] & & B^X \end{tikzcd}$$

then all maps in the diagram on the left are isofibrations.

* Very long proof

Def A 1-simplex in a quasi-cat is an iso iff its an iso in its homotopy category.

Prop (Joyal) special outer horn lifting

If $f|_{[0,1]}$ or $g|_{[n-1,n]}$ are isos then lifts exist.

$$\begin{tikzcd} \Lambda^0[n] \arrow[r, "f"] \arrow[d] & A \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd} \qquad \begin{tikzcd} \Lambda^n[n] \arrow[r, "g"] \arrow[d] & A \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd}$$

Cor: A quasi-cat is a Kan-complex iff its homotopy category is a groupoid.

Cor: An arrow in a quasi-cat is an iso iff it's a homotopy coherent iso:

$$\begin{tikzcd}
\mathbb{2} \arrow[r] \arrow[d] & A \\
\mathbb{I} \arrow[ur, dashed] &
\end{tikzcd}$$

Def^n: A equivalence $A \simeq B$ is given by maps $A \xrightarrow{f} B$, $B \xrightarrow{g} A$ s.t.

$$\begin{tikzcd}
 & A \\
A \arrow[ur, "gf"] \arrow[r, "\alpha"] \arrow[dr, equal] & A^{\mathbb{I}} \arrow[u, "ev_1"'] \arrow[d, "ev_0"] \\
 & A
\end{tikzcd}
\qquad
\begin{tikzcd}
 & B \\
B \arrow[ur, "fg"] \arrow[r, "\beta"] \arrow[dr, equal] & B^{\mathbb{I}} \arrow[u, "ev_0"'] \arrow[d, "ev_1"] \\
 & B
\end{tikzcd}$$

Def: A trivial fibration:

$$\begin{tikzcd}
\partial\Delta[n] \arrow[r] \arrow[d, hook] & A \arrow[d, "s"] \\
\Delta[n] \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd}$$

- Trivial fib $\Longleftrightarrow$ isofib + equivalence.