How Riemann finally ~~defined~~ understood the logarithms

2-day class: Riemann surfaces

Day 1 : Graphs of $\sqrt{z}$, $\log z$

Day 2 : Elliptic Curves


Real functions: can be continuous everywhere but differentiable nowhere
Complex functions: differentiable in an open disk implies analytic
imgflip.com

Use the chat window without restraint

except when I ask a question

for answering questions we'll be using

CHAT BLASTS

CHAT BLASTS

1. when I ask a question, type your answer in the chat window, but

   do not press enter

2. when I say ~~3-2-1~~ 2-1-0 go everyone presses enter at the same time

Notation:

z, ω = complex numbers

r, θ = real numbers

r = non-negative real number

$\ln r = \log_e r$ as shown by a calculator

$\sqrt{r}$ = positive square root

Polar decomposition:

$$\omega = r e^{i\theta}$$

$$0 \leq \theta < 2\pi$$

Euler's identity:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{i(\theta+2\pi)} = e^{i\theta}$$

Q. Let Z be a complex number. What is log z?

Ans.

↑

Euler's

$$z = re^{i\theta}$$

$$\log z = \log r + \log e^{i\theta}$$

$$\boxed{\log z = \ln r + i\theta}$$

log always means $\log_e$

Notation: log complex
ln real

Q. This has problems. Why?

Problems, problems, problems

- $e^{i\theta} = e^{i(\theta+2\pi)}$

$\log e^{i\theta} = \log e^{i\theta} + 2\pi i$

log is a "multi-valued function"

Didn't we just say $0 \leq \theta < 2\pi$?

$\log z = \ln r + i\theta$

$0 \leq \theta < 2\pi$

This an honest, hardworking function

Don't have
- $\log(1/z) = -\log z$
- $\log(z \cdot w) = \log z + \log w$
- $\log z$ is not continuous if defined this way.

Q. Why is not good enough?

$$\log z = \ln r + i\Theta$$

$0 < \Theta$ slightly $\quad \log z \approx \ln r$

$\Theta < 2\pi$ slightly $\quad \log z \approx \ln r + 2\pi i$

discontinuity at positive reals!

Riemann

How to define square root of complex numbers?

Try: $z = re^{i\theta} \qquad 0 \leq \theta < 2\pi$

$$\boxed{\sqrt{z} := \sqrt{r}\cdot e^{i\theta/2} \qquad 0 \leq \theta < 2\pi}$$

Discontinuous at positive reals

Riemann's idea

The square root might not be well-defined,

let's instead study the "graph of $\sqrt{z}$"

$\downarrow$

flip graph of z^2

$$\text{"}G_z\text{"} = \{ (z, w) : \overbrace{w^2 = z}^{w=\sqrt{z}} \}$$

$z \in \mathbb{C}$, $w \in \mathbb{C}$

Q. Where does this graph live?

$$\boxed{\text{"}G_z\text{"} \subseteq \mathbb{C}^2}$$

$\downarrow$

4-dim space

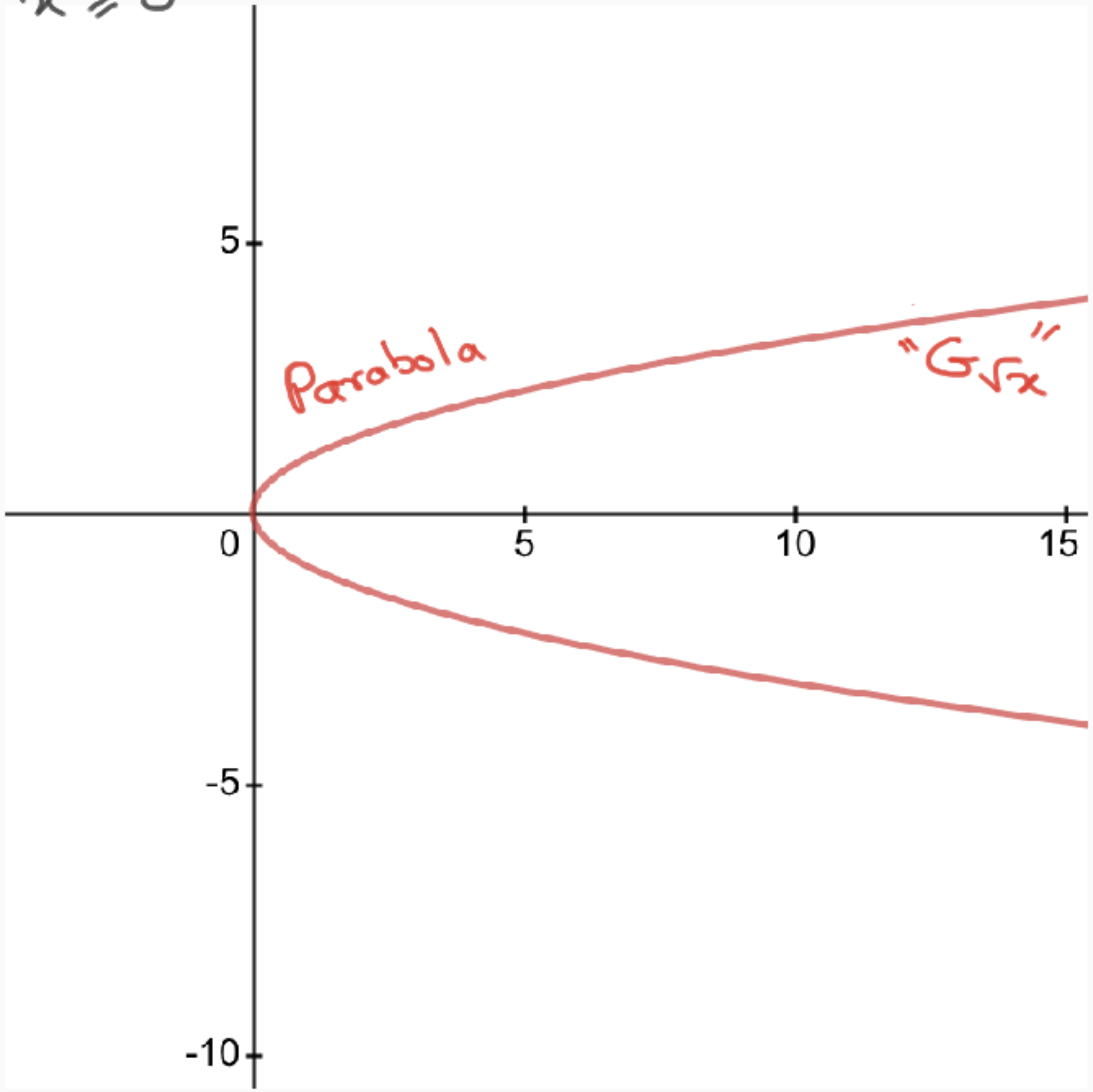
Q. What does this look like?

Back to the "real" world

$$\text{“}G_{\sqrt{x}}\text{”} = \{ (x,y) : y^2 = x \}$$

Domain of $\sqrt{x}$: $[0, \infty)$

$x \geq 0$



Three observations:

1) If we remove $x = 0$ from domain the "$G_{\sqrt{x}}$" gets disconnected.

2) The two resulting branches are honest graphs

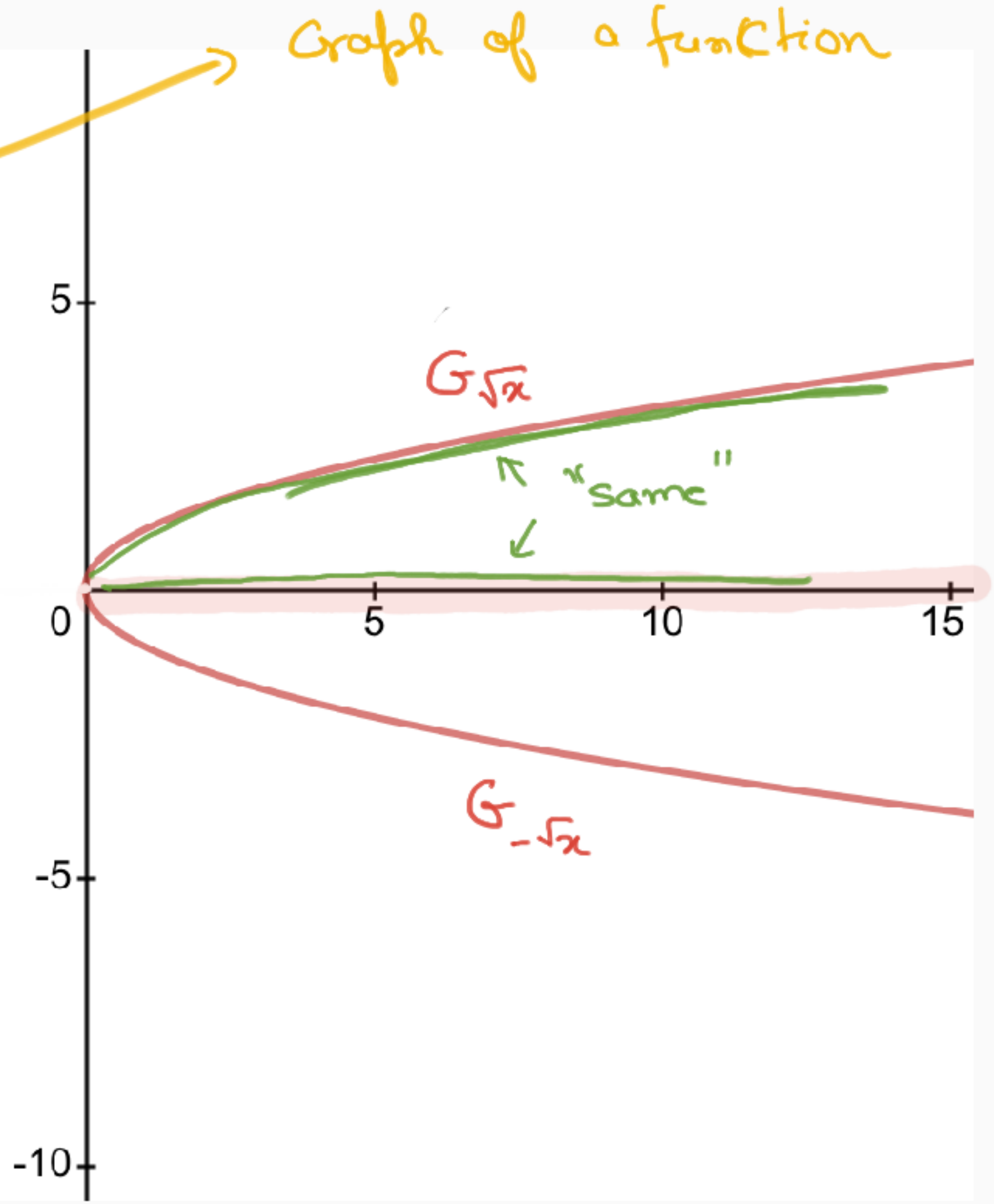
3) Each graph is a "copy" of the new domain

$\boxed{x>0}$

there is a bijection between

$G_{\sqrt{x}}$ and $\{x : x>0\}$

Similarly $G_{-\sqrt{x}}$ and $\{x, x>0\}$



Remove the discontinuity

$$\text{"}G_{\sqrt{z}}\text{"} := \{ (z, w) : w^2 = z \}$$

- Remove the positive real numbers from the domain.

- New domain D

~~Class activity~~

Q. What can we say about the "graph"

"$G_{\sqrt{z}}$" restricted to the new domain D?

$D = \mathbb{C} \setminus \{r\}$

Restricted "graph"

$= \{ (z, w) : \quad w^2 = z \,,$ z is not a positive real number $\}$

1) . "new graph": two disconnected honest graphs

2) $G_{\sqrt{z}} = \{ (z, \omega) : \omega = \sqrt{r} \cdot e^{i\theta/2} \}$

3) $G_{\sqrt{z}} \cong$ new domain D

2) $G_{-\sqrt{z}} = \{ (z, \omega) : \omega = -\sqrt{r} \cdot e^{i\theta/2} \}$

3) $G_{-\sqrt{z}} \cong$ new domain D

Pieces of "$G_{\sqrt{z}}$"

$G_{\sqrt{z}} = \{ (z,\omega) : \omega = re^{i\theta/2} \}$

$G_{-\sqrt{z}}$

$G_{-\sqrt{z}} = \{ (z,\omega) : \omega = -re^{i\theta/2} \}$

We got $G_{\sqrt{z}}$, $G_{-\sqrt{z}}$ by removing some part of "$G_{\sqrt{z}}$".

Q. what can "$G_{\sqrt{z}}$" be ?

$$G_{\sqrt{z}} = \{ (z, \sqrt{r} \cdot e^{i\theta/2}) \}$$

$$0 < \theta < 2\pi$$

$$\|$$

$$\{ (z, \sqrt{r}\, e^{i\theta/2}),$$

$$4\pi < \theta < 6\pi \}$$

$$G_{-\sqrt{z}}$$

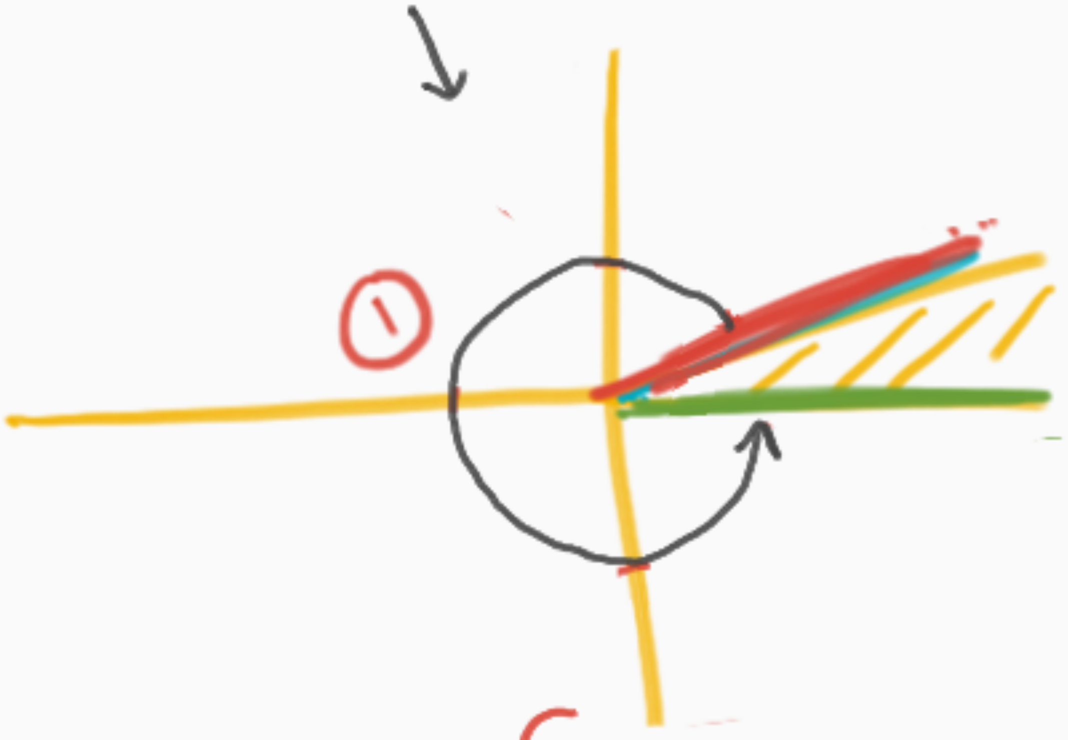
$$= \{ (z, -\sqrt{r}\, e^{i\theta/2}) : 0 < \theta < 2\pi \}$$

$$= \{ (z, \sqrt{r} \cdot e^{i(\theta + 2\pi)/2}) \quad 0 < \theta < 2\pi \}$$

$$= \{ (z, \sqrt{r} \cdot e^{i\theta}) : 2\pi < \theta < 4\pi \}$$

So "$G_{\sqrt{z}}$" must be obtained by gluing the two pieces

$\theta \in [0, 2\pi) = (4\pi, 6\pi)$



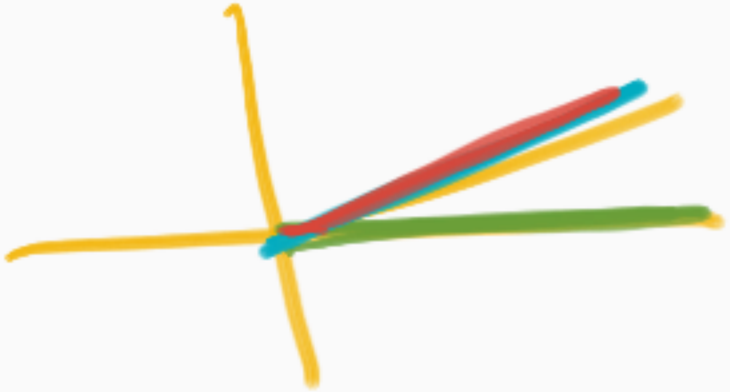
$G_{\sqrt{z}}$

$\theta \in (2\pi, 4\pi)$

$G_{-\sqrt{z}}$

$\parallel$

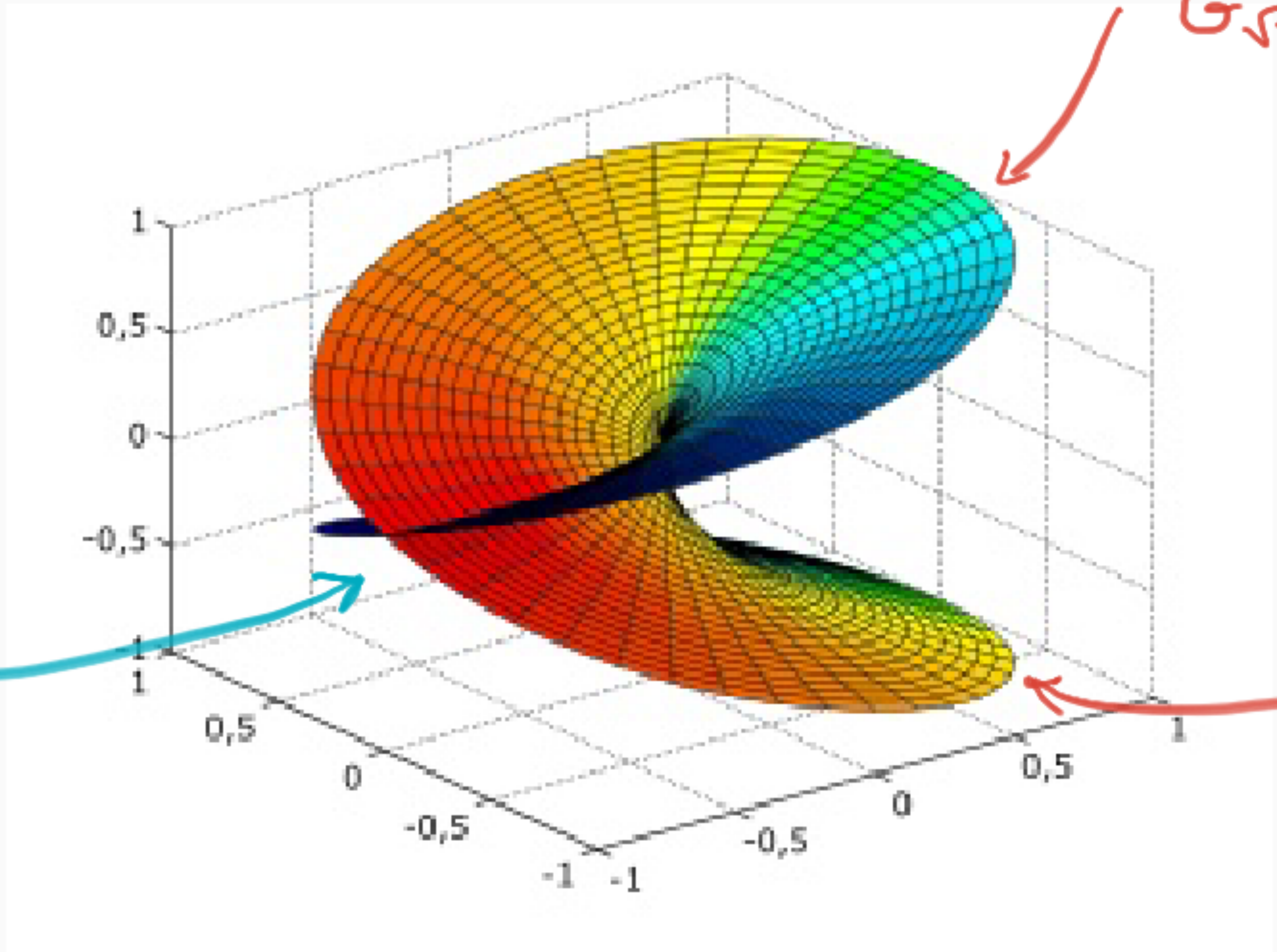
"$G_{\sqrt{z}}$"



$\parallel$

Riemann surface for $\sqrt{z}$

$G_{\sqrt{z}}$ branch

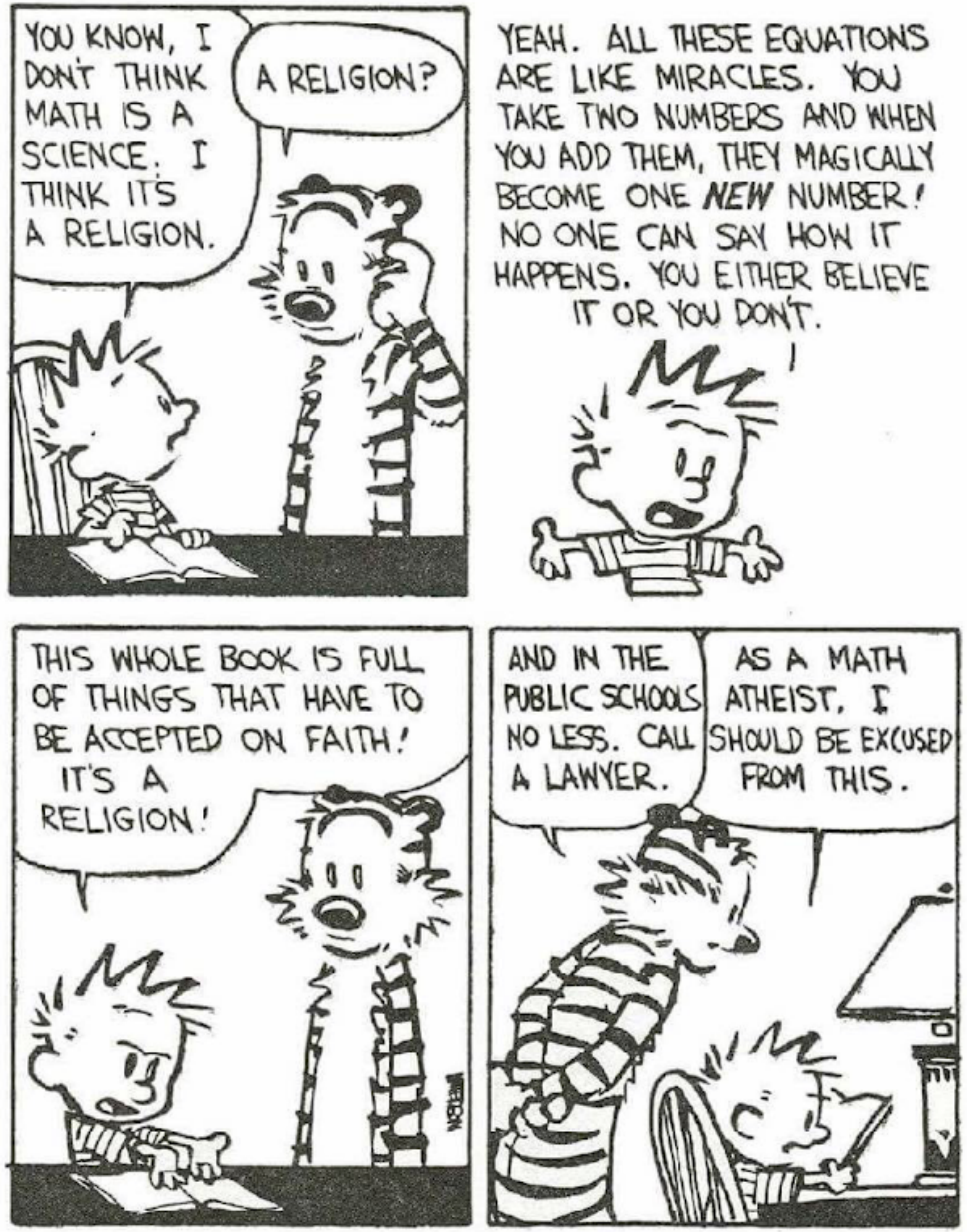


these don't intersect in $\mathbb{C}^2$

$G_{-\sqrt{z}}$ branch

Riemann surface for $\sqrt{z}$

$$\text{“}G_{\sqrt{z}}\text{”} = \{(z, w) : w^2 = z\}$$


YOU KNOW, I DON'T THINK MATH IS A SCIENCE. I THINK IT'S A RELIGION.
A RELIGION?
YEAH. ALL THESE EQUATIONS ARE LIKE MIRACLES. YOU TAKE TWO NUMBERS AND WHEN YOU ADD THEM, THEY MAGICALLY BECOME ONE *NEW* NUMBER! NO ONE CAN SAY HOW IT HAPPENS. YOU EITHER BELIEVE IT OR YOU DON'T.
THIS WHOLE BOOK IS FULL OF THINGS THAT HAVE TO BE ACCEPTED ON FAITH! IT'S A RELIGION!
AND IN THE PUBLIC SCHOOLS NO LESS. CALL A LAWYER.
AS A MATH ATHEIST, I SHOULD BE EXCUSED FROM THIS.

Recap: It is not possible to define a continuous, single-valued function $\sqrt{z}$ on the entire complex plane

Riemann's idea: instead study the "graph"

$$\text{"}G_{\sqrt{z}}\text{"} := \{ (z, \omega) : \cancel{\omega = \sqrt{z}} \quad \omega^2 = z \}$$

Q. What does this object that lives in $\mathbb{C}^2$ look like?

- If we "cut" the domain along the non-negative real numbers

$$D = \mathbb{C} \setminus \{r : r \geq 0\}$$

- The "graph" "$G_{\sqrt{z}}$" breaks up into two honest graphs

$$G_{\sqrt{z}} = \{(z, \sqrt{r}e^{i\theta/2})\} \qquad G_{-\sqrt{z}} = \{(z, -\sqrt{r}e^{i\theta/2})\}$$

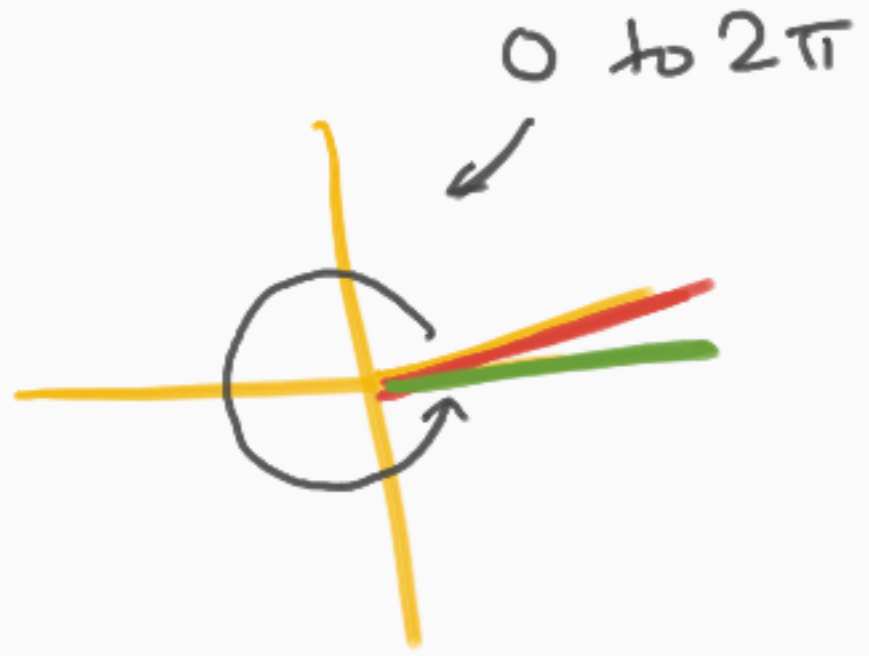
- Both of these are copies of D

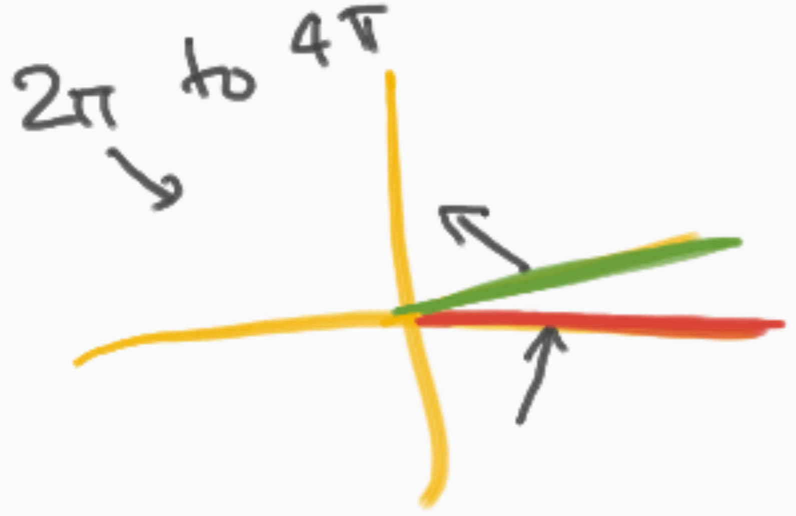
Q. If these are the two pieces then what does the total space look like?

Trick: $G_{\sqrt{z}} = \{ (z, \sqrt{r}.e^{i\theta/2}) : 0 < \theta < 2\pi \}$

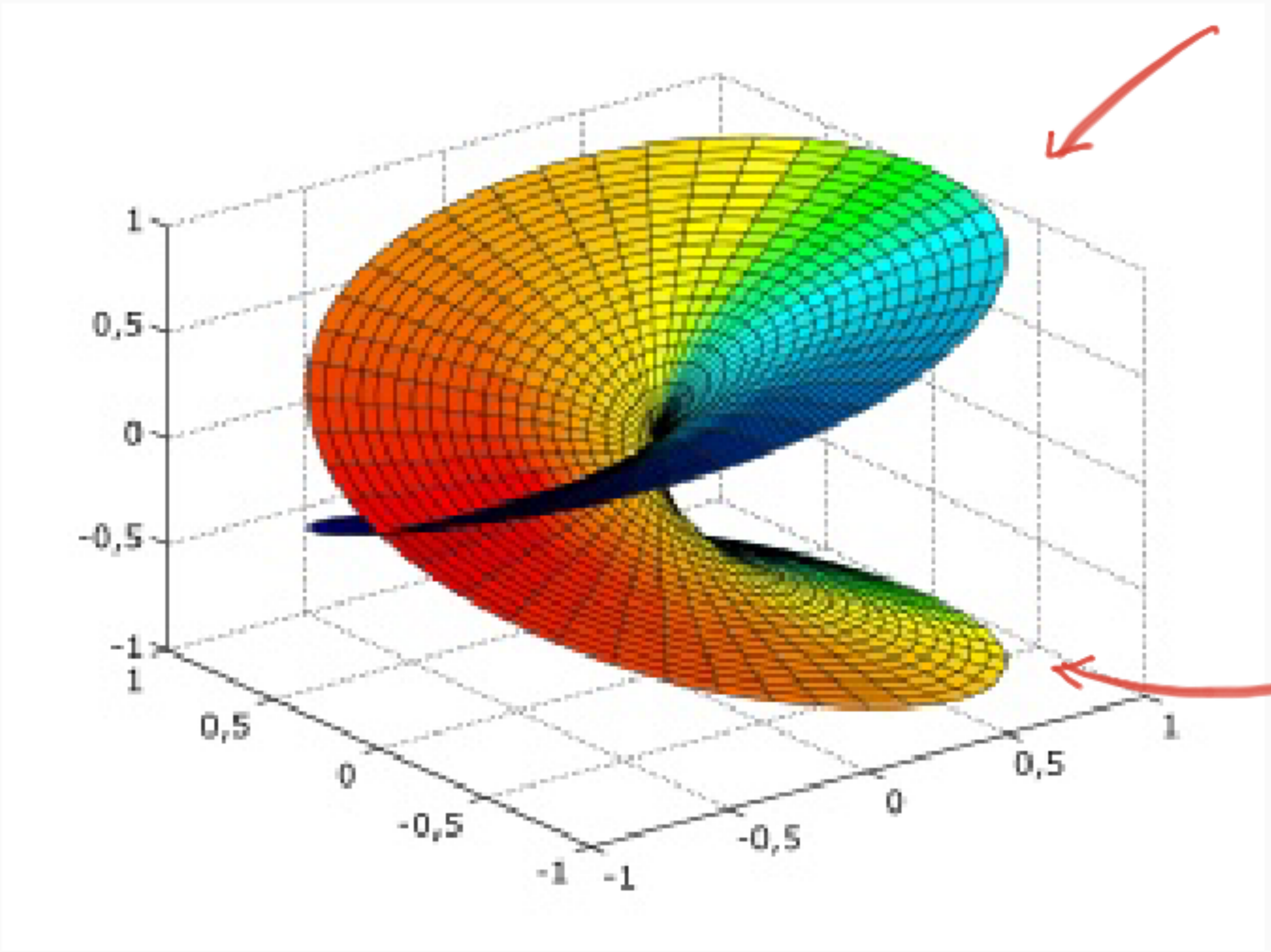
$G_{-\sqrt{z}} = \{ (z, -\sqrt{r}e^{i\theta}) : 0 < \theta < 2\pi \}$

$= \{ (z, \sqrt{r}e^{i\theta}) : 2\pi < \theta < 4\pi \}$





To obtain "$G_{\sqrt{z}}$" glue $G_{\sqrt{z}}$ and $G_{-\sqrt{z}}$ using the color coding.



$G_{\sqrt{z}}$

The two branches

$G_{\sqrt{-z}}$

Riemann Surface for $\sqrt{z}$

"$G_{\sqrt{z}}$"

Class activity:

- Find the Riemann surface for

$$\text{"}G_{\log z}\text{"} := \{(z,\omega) : \cancel{\omega = \log z}\ e^{\omega} = z\}$$

Steps (same as for $\sqrt{z}$):

1) Cut the domain $D := \mathbb{C} \setminus \{r : r \geq 0\}$

2) Find the branches (these will be copies of D)

3) Figure out how the branches glue together

☆ branches of $\log z$ are more interesting

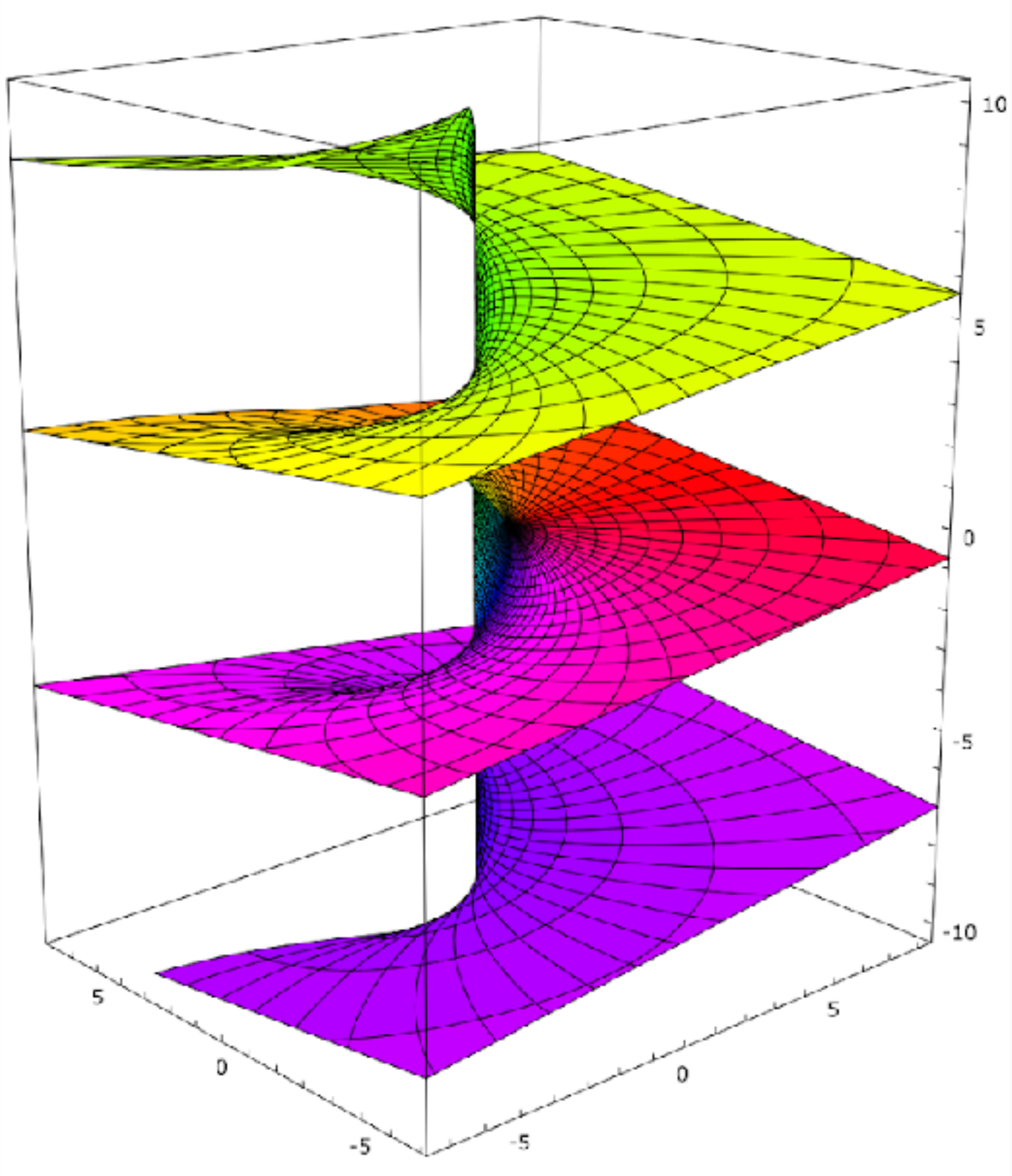
Class activity:

Branches of $\log z$: ∞, one for each integer

$$G_n = \{(z, \ln r + n\cdot 2\pi i + i\theta) \quad n \in \mathbb{Z}, \quad 0 \le \theta < 2\pi\}$$

Trick:

$$G_n = \{(z, \ln r + i\theta) : 2\pi n \le \theta < 2\pi(n+1)\}$$



Riemann surface of log (z)

Leveling up

Riemann's next idea:

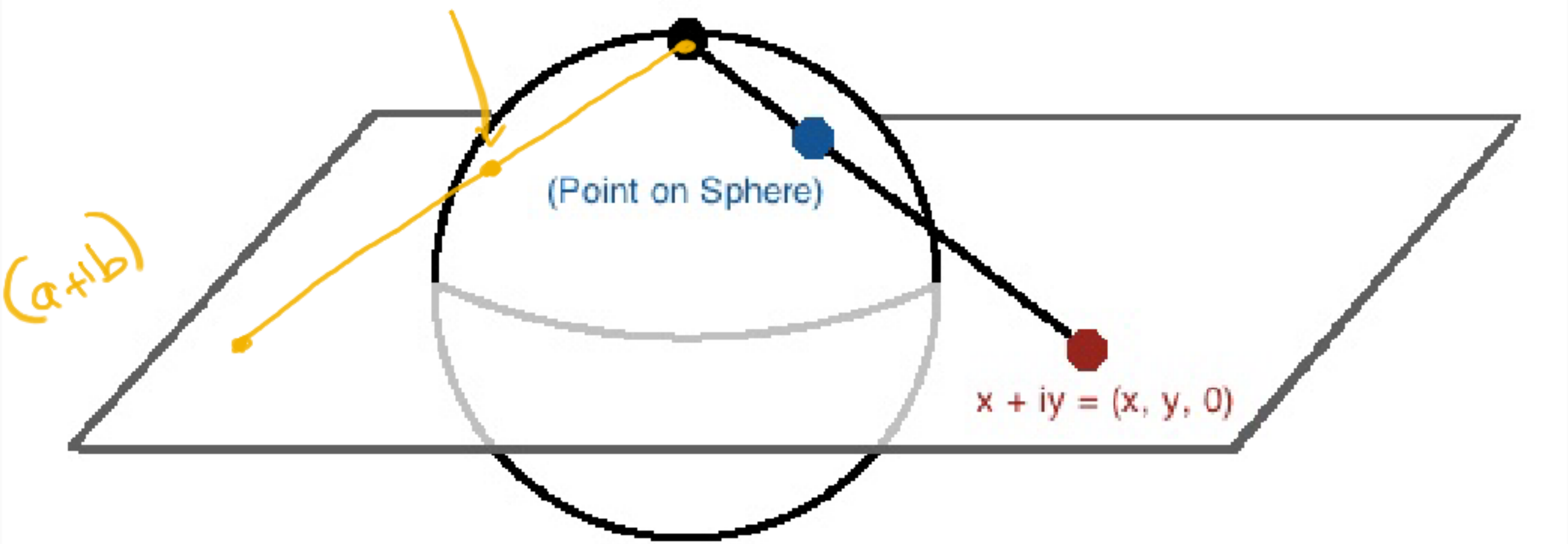
- $\mathbb{C}$ is too big a space
- make it smaller by adding a point
  ↳ one point compactification

$$\underbrace{\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}}$$

Q. What "shape" does this have?

A. Sphere, by the stereographic projection identify this point of sphere with $(a+ib)$

extra point ∞



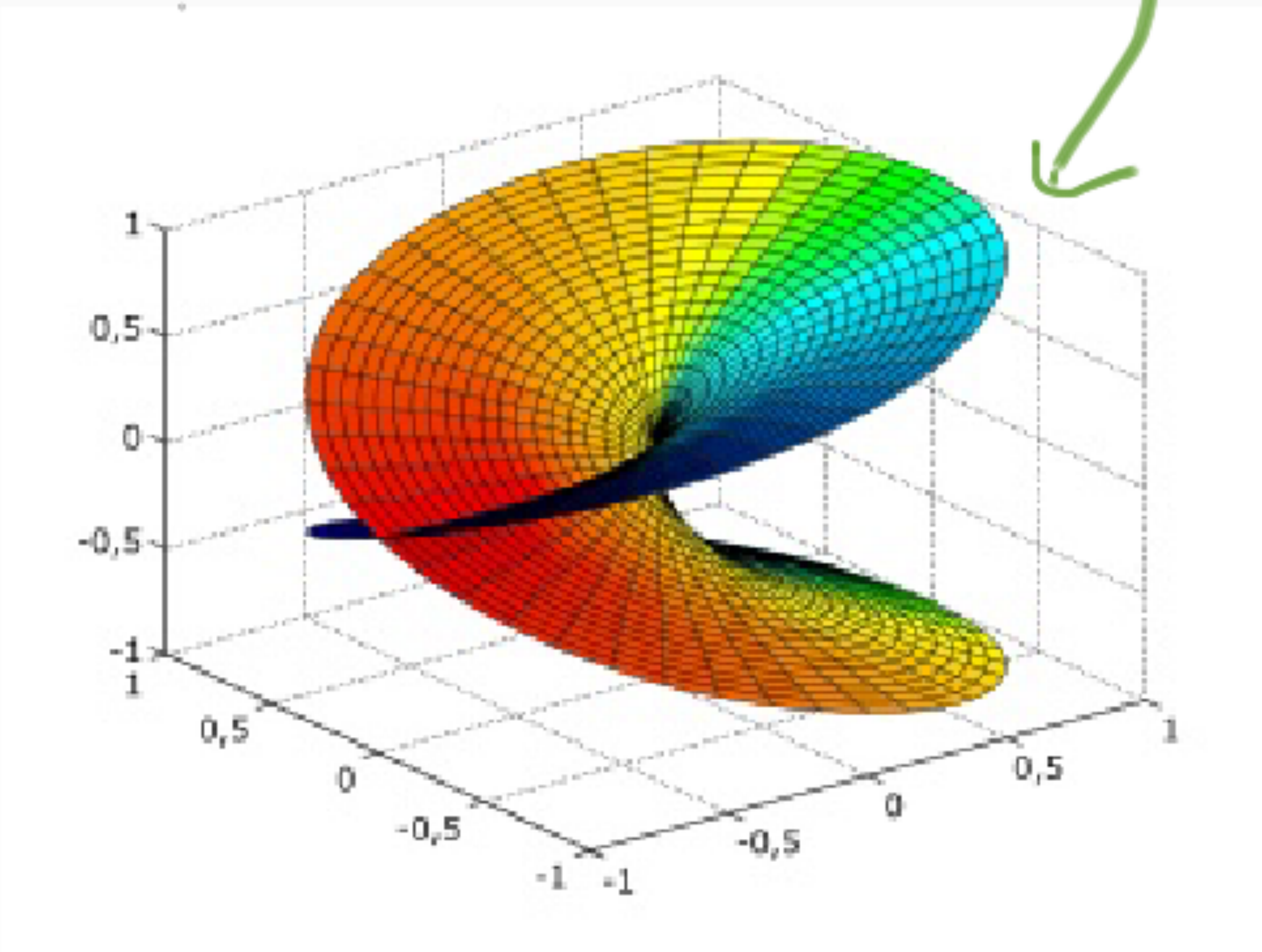
Riemann sphere

$$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

$$\text{"}\underbrace{\hat{G}_{\sqrt{z}}}\text{"} := \text{"}\underbrace{G_{\sqrt{z}}}\text{"} \cup \{(\infty,\infty)\}$$

what does this look like?

Compactifying "$G_{\sqrt{z}}$".



Branch cuts of "$G_{\sqrt{z}}$"

$G_{\sqrt{z}}$

copies of domain obtained from $\mathbb{C}$

$G_{-\sqrt{z}}$

What do these look like in $\hat{G}_{\sqrt{z}}$?

$\hat{G}_{\sqrt{z}}$

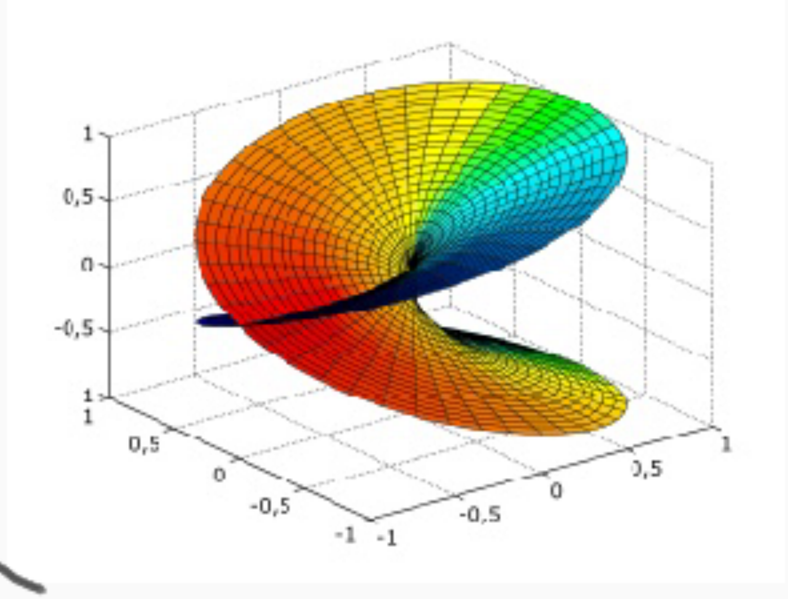
copies of domains obtained from $\hat{\mathbb{C}}$

$\hat{G}_{-\sqrt{z}}$

~~Leveling up again:~~

Hence,

"$\underbrace{\hat{G}_{\sqrt{z}}}_{\text{compact Riemann surface}}$" is a sphere!!



$$G = \{(z, w) : w^2 = (z-a)(z-b)(z-c)\}$$

like "graph" of $\sqrt{(z-a)(z-b)(z-c)} = G_e$

a, b, c distince

$$\hat{G}_e = G_e \cup \{(\infty, \infty)\}$$

elliptic curve

Q. What does $\hat{G}_e$ look like?

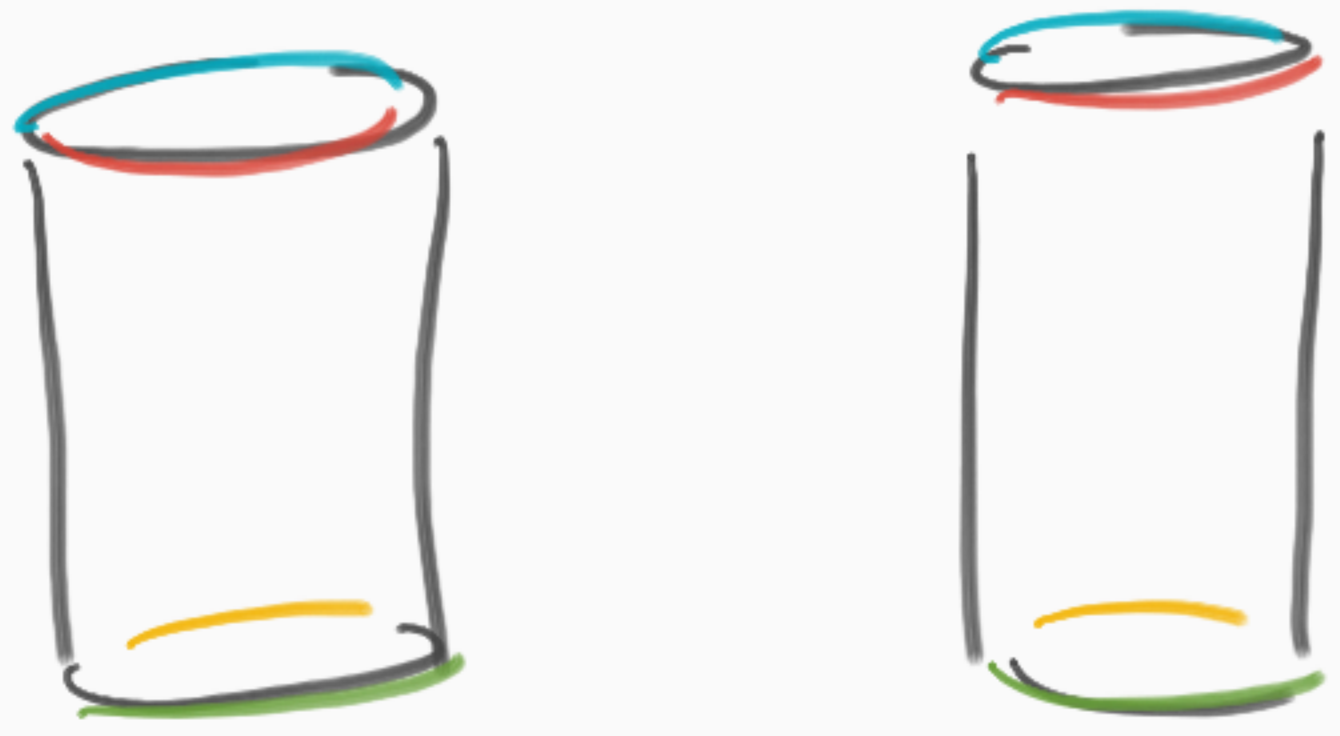
the "function" " $\sqrt{(z-a)(z-b)(z-c)}$ " still has two branches

$\sqrt{(z-a)(z-b)(z-c)}$

$-\sqrt{(z-a)(z-b)(z-c)}$

Need 2 branch cuts

Glue along the colors



$\parallel$ $\nparallel$

Sphere with 2 cuts

- An elliptic curve is a torus:  = 2 cylinders glued together.