Recall :

- Slope of tangent to graph of $f(a)$ = limit of slope of secant between $(a, f(a))$ and $(x, f(x))$.

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

- we generalize the above limit to a function, i.e. the derivative of $f(x)$ with respect to x is the function:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

This is denoted $f'(x)$ or $\frac{df(x)}{dx}$.

- We say $f(x)$ is differentiable at $x = a$ if

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$ exists, (and is not $\pm\infty$).

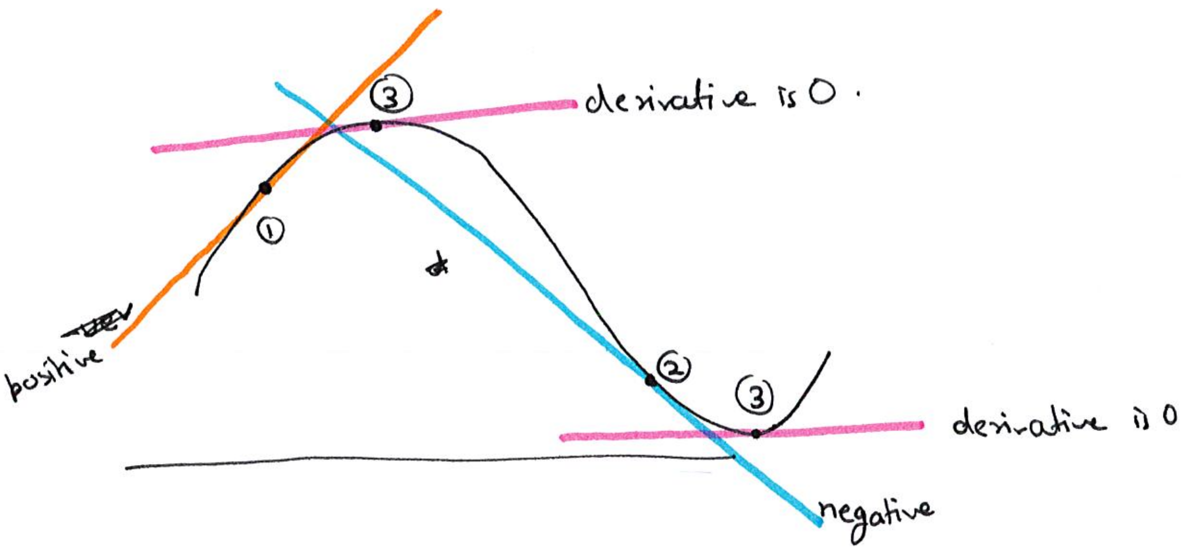
Fact. Differentiable functions are continuous, ~~but~~ but continuous functions are not always differentiable.

eg: $|x|$ is not differentiable at $x = 0$.

- If ① $f(x)$ is increasing near $x=a$ $\iff$ $\boxed{f'(a) > 0}$

② $f(x)$ is decreasing near $x=a$ $\iff$ $\boxed{f'(a) < 0}$

③ $f(x)$ has local minima/ maxima at $x=a$ $\Rightarrow$ $\boxed{f'(a) = 0}$



- we can find derivatives repeatedly

$$f(x) \xrightarrow{\text{derivative}} f'(x) \xrightarrow{\text{derivative}} f''(x) \xrightarrow{\text{derivative}} f'''(x) \rightarrow \cdots$$

$$f'(x) = \frac{d}{dx} f(x), \quad f''(x) = \frac{d^2}{dx^2} f(x), \quad f'''(x) = \frac{d^3}{dx^3} f(x) \quad \cdots$$

Questions :

Q.1) find $\lim_{x \to \infty} e^{-2x} \cdot \cos(2x)$.

Ans : Squeeze theorem .

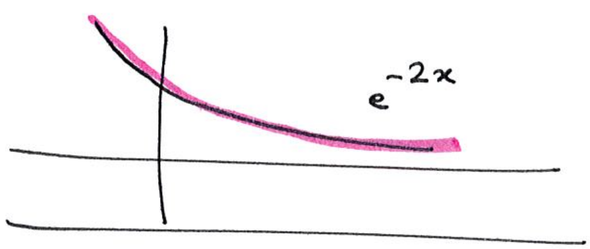
$$-1 \le \cos(2x) \le 1$$

Multiply by e^{-2x}

$$-e^{-2x} \le e^{-2x} \cdot \cos(2x) \le e^{-2x}$$

• $\lim_{x \to \infty} -e^{-2x} = 0$

$\lim_{x \to \infty} e^{-2x} = 0$



because $e^{-2x} = \frac{1}{e^{2x}}$,

& as $x \to \infty$ $e^{2x} \to \infty$

$\Rightarrow \frac{1}{e^{2x}} \to 0$

$$\Rightarrow \lim_{x \to \infty} e^{-2x} \cdot \cos 2x = 0$$

$\boxed{\equiv}$

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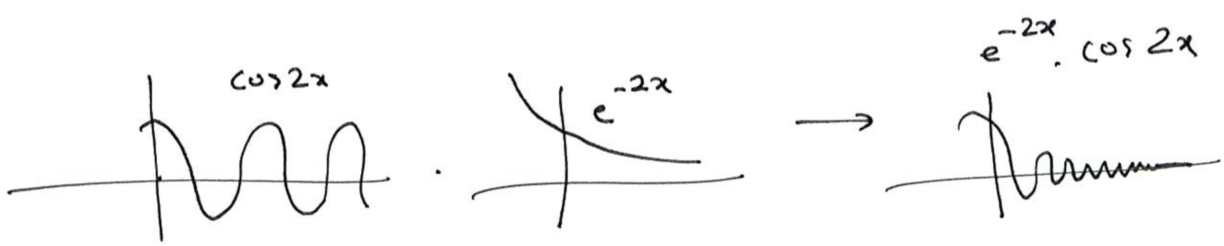
eg: $e^{-2x} \cos 2x$ damped harmonic oscillator



~~Q. Where~~

- $0 \cdot \infty$ is an indeterminate form

eg. Find (1) $\lim_{x\to\infty} x \cdot \frac{1}{x}$.

$\boxed{\infty \cdot 0}$

$$\lim_{x\to\infty} x \cdot \frac{1}{x} = \lim_{x\to\infty} 1 = 1$$

(2) $\lim_{x\to\infty} x^2 \cdot \frac{1}{x}$

$\boxed{\infty \cdot 0}$

$$\lim_{x\to\infty} x^2 \cdot \frac{1}{x} = \lim_{x\to\infty} x = \infty$$

Q. Where is $\sqrt{1-e^x}$ continuous?

A: the standard functions are continuous ~~every~~wherever they are defined

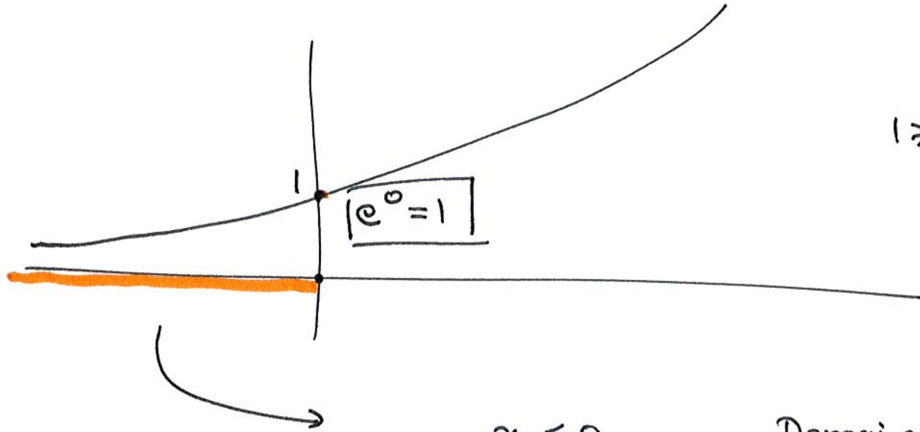
$\Rightarrow$ we only need to find the domain.

- Domain of $\sqrt{1-e^x}$:

Note: $\sqrt{x}$ has domain $x \geq 0$

$\sqrt{1-e^x}$ has domain $1-e^x \geq 0$

$\Rightarrow 1 \geq e^x$



$1 \geq e^x$

$x \leq 0$ Domain: $\{x \leq 0\}$

$= (-\infty, 0]$.

Q. Find ① $\lim_{x \to 7^+} \frac{1}{x^2(x-7)}$. ② $\lim_{x \to 7^-} \frac{1}{x^2(x-7)}$

A: Plugging in $\frac{1}{7^2 \cdot 0} = \frac{1}{0}$ $\Rightarrow$ limit is either $+\infty$, $-\infty$ or d.n.e.

① $\lim_{x \to 7^+}$ x is to the right of 7.

- $x^2 > 0$

$7 \leftarrow x$

- $(x-7) > 0$

$\Rightarrow \frac{1}{x^2(x-7)} > 0$

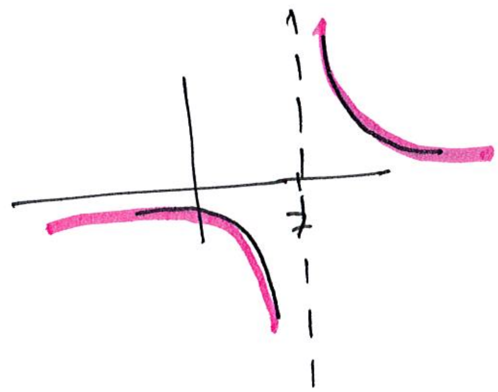
$$\lim_{x \to 7^+} \frac{1}{x^2(x-7)} = \infty$$

② $\lim_{x \to 7^-}$ x is to the left of 7

$$\left.\begin{array}{l} x^2 > 0 \\ x-7 < 0 \end{array}\right] \Rightarrow \frac{1}{x^2(x-7)} < 0$$

$$\Rightarrow \lim_{x \to 7^-} \frac{1}{x^2(x-7)} = -\infty$$

$$\lim_{x \to 7} \frac{1}{x^2(x-7)} = \text{d.n.e}$$



Q. Given $\lim_{x \to 4} \frac{f(x)+5}{x-4} = 12$

what is $f(4)$? $f'(4)$?

A: $\lim_{x \to 4} \frac{f(x)+5}{x-4} = 12$

① Plugging in $x=4$ on LHS

$$\frac{f(4)+5}{4-4} = \frac{f(4)+5}{0}$$

$\Rightarrow$ the above limit exists only if

$f(4)+5=0$

$\Rightarrow \boxed{f(4) = -5}$

(2) $\lim_{x \to 4} \frac{f(x)+5}{x-4} \overset{\text{if}}{=} f'(4)$

$$\| $$

$$\lim_{x \to 4} \frac{f(x)-f(4)}{x-4}$$

compare with

to get $\boxed{f(4) = -5}$

$\boxed{f'(4) = 12}$ by definition of derivative

$\Rightarrow$ • equation of tangent through $(4, f(4))$

$$\| $$

$$(4, -5)$$

$$(y - (-5)) = 12 \cdot (x - 4)$$

(3) ??

Chapter 3 : Finding derivatives using algebraic formulae.

- Using lim definitions to find derivatives is very slow.

- You don't need to memorize the proofs.
- ~~Basic~~

- Basic arithmetic

  - f, g are differentiable functions

(1.) $\cancel{f+g}\left(f(x)+g(x)\right)' =$

$$= \lim_{h\to 0} \frac{f(x+h)+g(x+h)-f(x)-g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)+g(x+h)-g(x)}{h}$$

(by Limit Laws) $$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

$$= f'(x) + g'(x)$$

$$\Rightarrow \boxed{(f+g)' = f' + g'}$$

2. Similarly, $\boxed{(f-g)' = f' - g'}$

3. But $(fg)' \neq f' \cdot g'$ $\leftarrow$ we'll do this later.

4. Scalar multiplications:

~~f~~ Let c be real number

$$(c \cdot f(x))' = \cancel{\lim_{h \to 0}} \quad \cancel{c \cdot f(x+h) - f}$$

$$= \lim_{h \to 0} \frac{c \cdot f(x+h) - c \cdot f(x)}{h}$$

$$= \lim_{h \to 0} c \cdot \frac{f(x+h) - f(x)}{h}$$

as c is a constant

$$= c \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= c f'(x)$$

$$\boxed{(c \cdot f(x))' = c f'(x).}$$

5 $$\left(f(cx)\right)' = \lim_{h\to 0} \frac{f(c(x+h)) - f(cx)}{h}$$

c is a constant

$$= \lim_{h\to 0} \frac{f(cx+ch) - f(cx)}{h}$$

let $k = ch$

as $h \to 0$

$k \to 0$

$$= \lim_{k\to 0} \frac{f(cx+k) - f(cx)}{k/c}$$

$$= \lim_{k\to 0} \frac{f(cx+k) - f(cx)}{k} \cdot c$$

$$= c \lim_{k\to 0} \frac{f(cx+k) - f(cx)}{k}$$

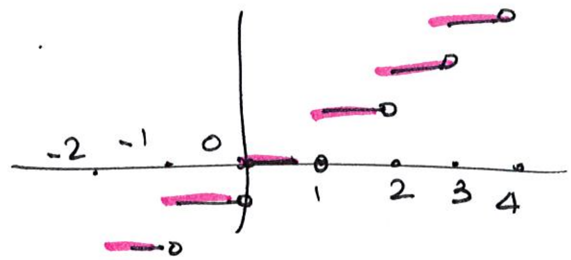
$$\cancel{\frac{d}{dx} f(cx) = c \left.\frac{d}{dx} f(x)\right|_{cx}}$$

$$\boxed{\left(f(cx)\right)' = c \cdot f'(cx)}$$

Question from Webwork:

$f(x) =$ greatest integer less than x

$= \lfloor x \rfloor$



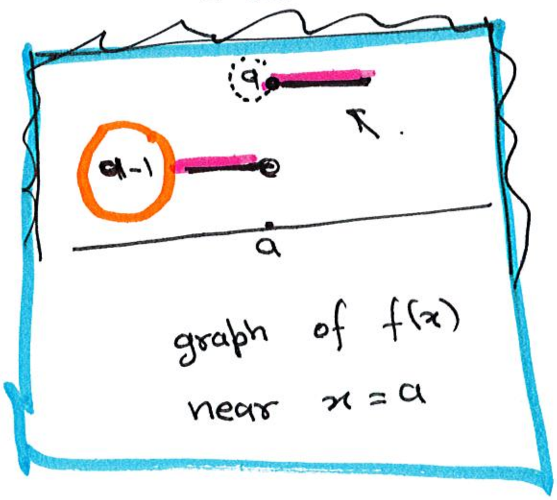
a integer

① $\lim\limits_{x \to a^+} f(x)$

② $\lim\limits_{x \to a^-} f(x)$

① $\lim\limits_{x \to a^+} f(x) = a$

② $\lim\limits_{x \to a^-} f(x) = a - 1$



graph of $f(x)$ near $x = a$

Key: look at the graph of $f(x)$ near a.

- Derivatives of standard functions

① $f(x) = c$ $\qquad$ c is a constant

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{c - c}{h}$$

$$= \lim_{h \to 0} \frac{0}{h}$$

$$\boxed{f'(x) = 0}$$

derivative of constant is 0.

② $f(x) = mx + b$, $\qquad$ m, b are constants

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(m(x+h) + b) - (mx + b)}{h}$$

$$= \lim_{h \to 0} \frac{m \cdot \cancel{h}}{\cancel{h}} = m$$

$$\boxed{f'(x) = m}$$

$(mx + b)' = m$

3 $\quad f(x) = x^2$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$$= \cancel{\lim} \; 2x$$

$$\therefore \quad (x^2)' = 2x$$

More generally,

$$\boxed{(x^n) = n \cdot x^{n-1} \quad \text{when } n \text{ is any real number}}$$

Proof:

- n = integer, use some combinatorics
- n = rational, use chain rule
- n = real, proof is very difficult

$$\boxed{\begin{aligned} &\cdot\ (f+g)' = f'+g' \\ &\cdot\ (f-g)' = f'-g' \\ &\cdot\ (cf)' = c(f') \\ &\cdot\ (x^n)' = n\cdot(x^{n-1}) \end{aligned}}$$

Power rule

eg: By power rule

① $(x^2)' = 2\cdot x^{2-1}$

$= 2\cdot x$

② $(\sqrt{x})' = (x^{1/2})'$

$= \frac{1}{2}\cdot x^{(1/2-1)}$

$= \frac{1}{2}\cdot x^{-1/2}$

$= \frac{1}{2}\cdot\frac{1}{\sqrt{x}}$

③ $\left(\frac{1}{\sqrt{x}}\right)' = (x^{-1/2})'$ [$n = -\frac{1}{2}$

$= -\frac{1}{2}\cdot x^{(-1/2-1)}$

$= -\frac{1}{2}\cdot x^{-3/2}$

$$= \cancel{\frac{1}{x^{3/2}}}$$

(4) $$\left(x + \frac{1}{x}\right)' = (x)' + \left(\frac{1}{x}\right)'$$

$$= (x^{1})' + (x^{-1})'$$

$$= 1 \cdot x^{0} + (-1) \cdot x^{-2}$$

$$= 1 - x^{-2}$$

$$= 1 - \frac{1}{x^{2}} \qquad \blacksquare$$

$$= \frac{x^{2} - 1}{x^{2}}$$

(5) $$\left(x^{2} - 2\sqrt{x}\right)' = (x^{2})' - (2\sqrt{x})'$$

$$\boxed{\sqrt{x} = x^{1/2}}$$

$$= (x^{2})' - 2(\sqrt{x})'$$

$$= 2 \cdot x^{1} - 2 \cdot \frac{1}{2} \cdot x^{-1/2}$$

$$= 2 \cdot x - x^{-1/2}$$

$$= 2x - \frac{1}{\sqrt{x}} \qquad \blacksquare$$

$$= \frac{2x\sqrt{x} - 1}{\sqrt{x}}$$

- Exponential functions :

$$f(x) = e^x$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$

$$= \lim_{h \to 0} \frac{e^x \cdot e^h - e^x}{h}$$

$$= \lim_{h \to 0} e^x \cdot \left(\frac{e^h - 1}{h} \right)$$

$$= e^x \lim_{h \to 0} \frac{e^h - 1}{h}$$

- e is defined to be the number for which

$$\boxed{\lim_{h \to 0} \frac{e^h - 1}{h} = 1}$$

← You should know this for the exams.

Back to the derivative $= e^x$

$$\boxed{(e^x)' = e^x}$$

Q. Find

$$\lim_{h\to 0} \frac{e^{2h}-1}{h}.$$

Let $k = 2h$.

$h\to 0$, $k \to 0$

$$\lim_{h\to 0} \frac{e^{2h}-1}{h} = \lim_{k\to 0} \frac{e^{k}-1}{k/2} \quad \left(\text{as } h = k/2\right)$$

$$= \lim_{k\to 0} \frac{e^{k}-1}{k} \cdot 2$$

$$= 2 \cdot \left(\lim_{k\to 0} \frac{e^{k}-1}{k} \right)$$

$$\lim_{h\to 0} \frac{e^{2h}-1}{h} = 2 \cdot 1 = 2$$

Q. Find $\displaystyle\lim_{h\to 0} \frac{e^{h^2}-1}{h^2}$

A

Let $k = h^2$.

⋮

$\displaystyle\lim_{h\to 0^+} \frac{e^{\sqrt{h}}-1}{\sqrt{h}}$

Let $k = \sqrt{h}$

Q. $(e^x + x^2)' = (e^x)' + (x^2)'$

$$= e^x + 2x.$$

- We did: $\boxed{g(x) = f(cx) \quad \text{then} \quad g'(x) = c.f'(cx)}$ $\quad (f(cx))' = c \cdot f'(cx)$

eg: $(e^{2x})'$ $\quad f(x) = e^x$

$$g(x) = e^{2x} = f(2x)$$

$$g'(x) = 2 \cdot f'(2x)$$

$$= 2 \cdot e^{2x}$$

as $f(x) = e^x$
$f'(x) = e^{2x}$

$$(e^{2x})' = 2 \cdot e^{2x}$$

Q. Find $(2^x)'$.

A. $2 = e^{\ln 2}$ $\quad$ (e^x & $\ln x$ are inverses of each other)

$$\Rightarrow 2^x = (e^{\ln 2})^x$$

$$= e^{(\ln 2) \cdot x}$$

$$f(x) = e^x.$$

$$g(x) = e^{(\ln 2)\cdot x}$$

$$\left[\cancel{f'}\,(e^x)' = e^x \right]$$

$$g'(x) = (\ln 2)\cdot f'((\ln 2)x)$$

$$= \ln 2 \cdot e^{(\ln 2)\cdot x}$$

$$= \ln 2 \cdot \left(e^{\ln 2}\right)^x$$

$$\boxed{(2^x)' = \ln 2 \cdot 2^x}$$

- $(f+g)' = f' + g'$

$(f-g)' = f' - g'$

$(c\cdot f)' = c(f')$

$(x^n)' = n\cdot x^{n-1}$

$(e^x)' = e^x$

$(a^x)' = \ln a \cdot a^x$

$$\boxed{\lim_{h\to 0} \frac{e^h - 1}{h} = 1}$$