Review :

We defined a lot of functions so far

- polynomials
- exponential $\xrightarrow{\text{inverse}}$ logarithms
- trigonometric $\xrightarrow{\text{inverse}}$ inverse trig

- these satisfy some laws (exponential, log) and identities (trigonometric)

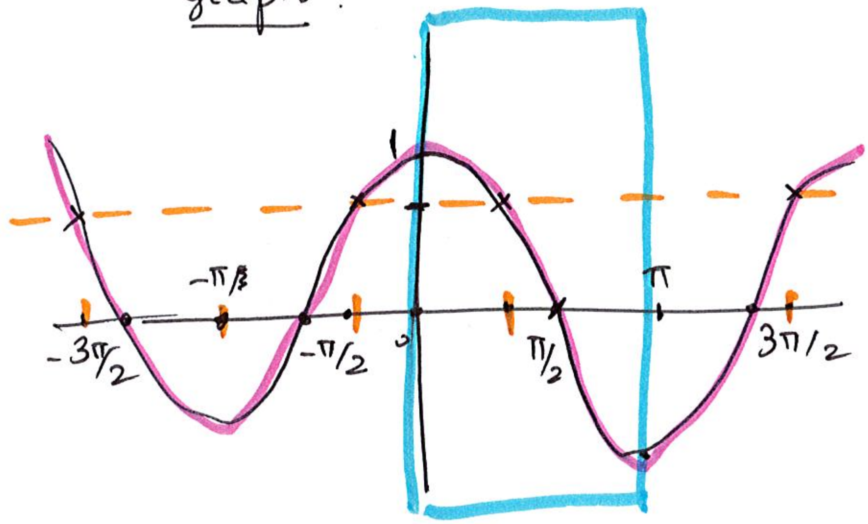
- we also covered one-to-one functions.

Domain & Range of inverse trig :

eg: • arc cos x (which is same as $\cos^{-1} x$)

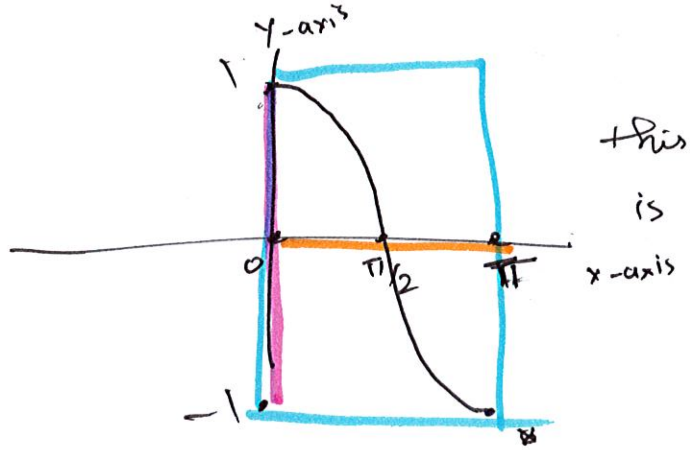
• start with cos x

graph :



not one-to-one as there are multiple x-values for a given y-value.

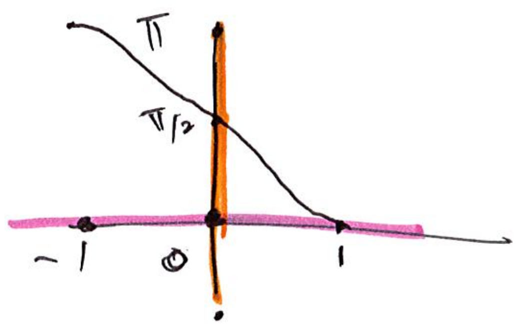
•



this restriction is one-to-one

flip x-y axes

•



range = domain of original = $[0, \pi]$

domain = range of original = $[-1, 1]$

range of arctan x = $(-\pi/2, \pi/2)$

Q. why not $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$?

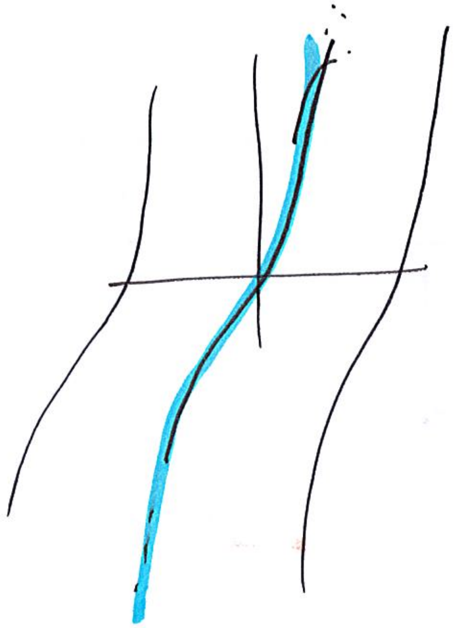
A. $\tan x$ is not defined at $\frac{\pi}{2}, -\frac{\pi}{2}$

i.e. $\frac{\pi}{2}, -\frac{\pi}{2}$ are not in domain of $\tan x$

$\Rightarrow$ they are not in the range of arctan x

Q. Domain of arctan x?

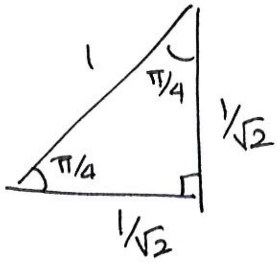
A: Domain of arctan x = range of $\tan x$

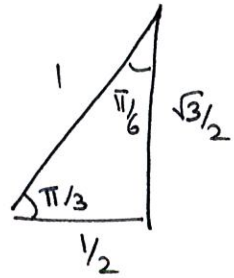


$= (-\infty, \infty)$

$= \mathbb{R}$

~~Basic~~ Trig values for basic angles:





	$\pi/4$	$\pi/3$	$\pi/6$
sin	$1/\sqrt{2}$	$\sqrt{3}/2$	$1/2$
cos	$1/\sqrt{2}$	$1/2$	$\sqrt{3}/2$

Basic trig identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\sin 2\theta = 2\sin\theta \cdot \cos\theta$$

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$= 1 - 2\sin^2\theta$$

double angle formulae

~~Problem Set / suggested exercises:~~

~~This~~ Next week : ∘ Monday , Quiz 01

∘ Section 1.4, 1.5, 2.2, 2.3, 2.5, 2.6, $\underbrace{2.2, 2.3, 2.5, 2.6}_{\text{this week}}$

Appendix D.

∘ ~~Mat~~ Math Help Center, TA office hours } on OWL

Chapter 2 Part Limits 2.2, 2.3, 2.5, 2.6

Limit
↓
Newton / Leibniz
mid 1600
"inuitive definition"

$\xrightarrow{\text{150 years}}$ Cauchy
precise definition

Notation :

$$\lim_{x \to a} f(x) = L$$

a = real number

$f(x)$ = function

L = real number

intuitive meaning

"

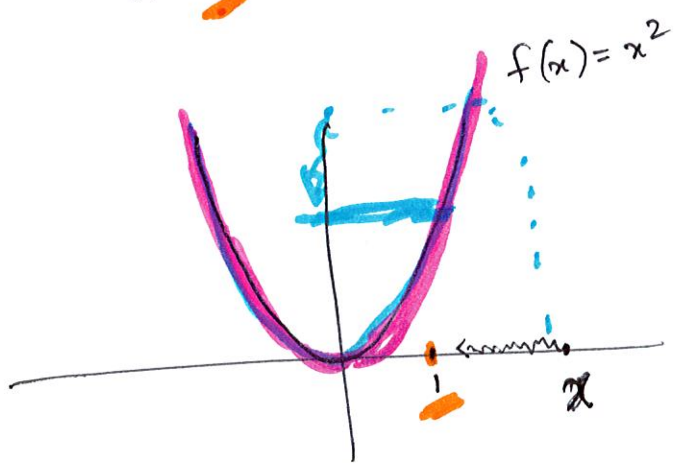
$f(x)$ approaches the number L

as x approaches a.

eg: For now, we answer this question using graphs.

eg:

$$\lim_{x \to 1} x^2 = 1$$



Notation :

$$\lim_{x \to a^+} f(x) = L$$

intuitive meaning

"

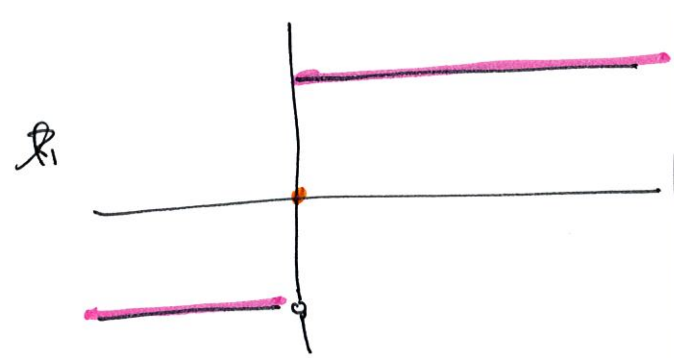
$f(x)$ approaches L

as x approaches a from the right

$\left(\lim_{x \to a^-} \right)$ (from the ~~right~~ left)

eg :

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$



$$\lim_{x \to 0^+} f(x) = 1$$

$$\lim_{x \to 0^-} f(x) = -1$$

$$f(0) = 1$$

- in general, $\lim_{x \to a^\pm} f(x)$ does not have to equal $f(a)$

$\lim_{x \to a} f(x)$ exists if and only if 1) $\lim_{x \to a^+} f(x)$ exists

2) $\lim_{x \to a^-} f(x)$ exists

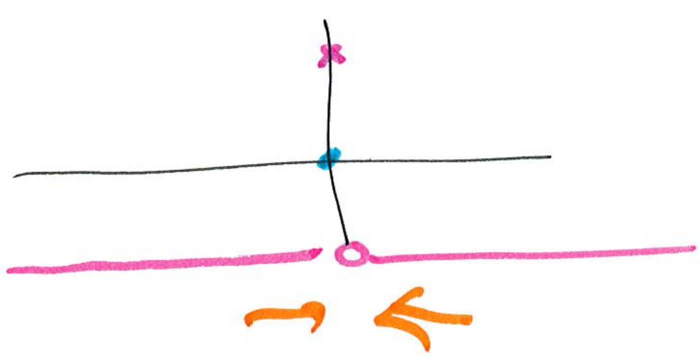
3) they both are the same.

for $f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$

$\lim_{x \to 0} f(x)$ does not exist because condition 3) fails.

Q. $f(x) = \begin{cases} -1 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

$\lim_{x \to 0} f(x) = -1$ as $\lim_{x \to 0^-} f(x) = -1 = \lim_{x \to 0^+} f(x)$



~~eg~~ when either

$$\lim_{x \to a^+} f(x) = \pm\infty$$

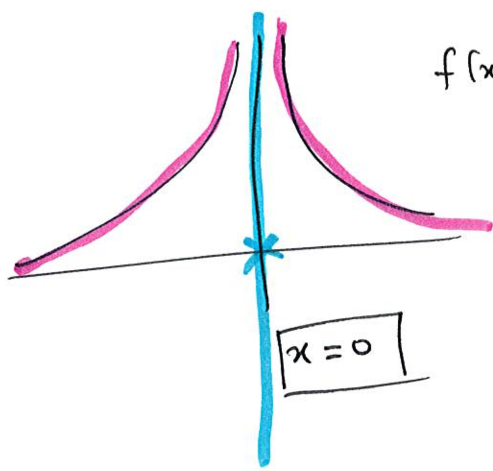
or

$$\lim_{x \to a^-} f(x) = \pm\infty$$

limits can be + or − ∞ for a vertical asymptote.

then $\boxed{x = a}$ is a <u>vertical asymptote</u>

eg:

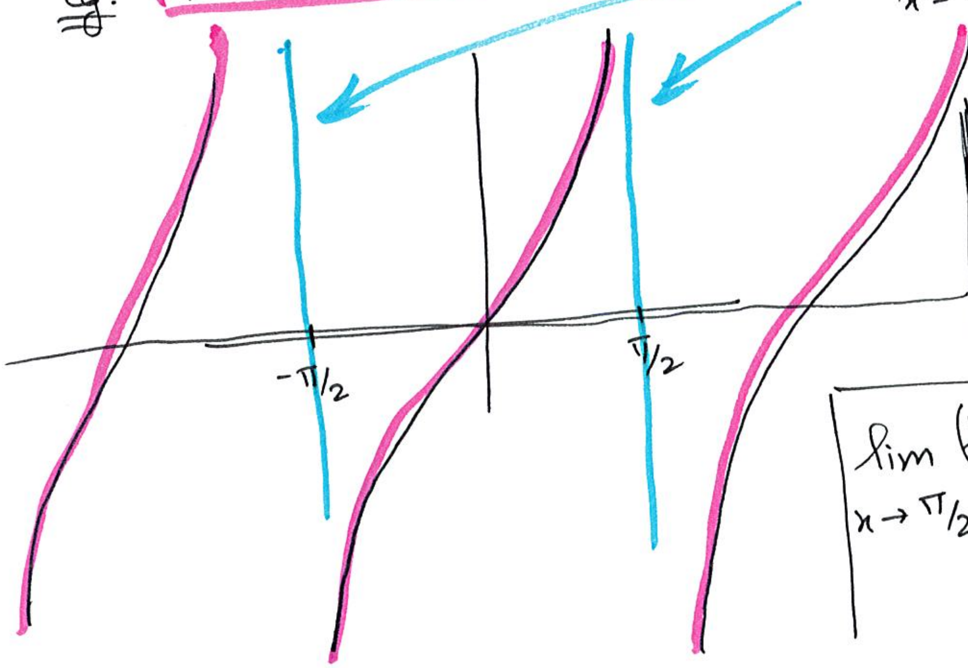


$f(x) = \frac{1}{x^2}$

eg: $\boxed{f(x) = \tan x}$

$x = -\pi/2$

$x = \pi/2$



$$\lim_{x \to \frac{\pi}{2}^+} \tan x = -\infty$$

$$\lim_{x \to \frac{\pi}{2}^-} \tan x = +\infty$$

$\lim_{x \to \pi/2} (\tan x)$ does not exist.

Def: IF at $x = a$

$$\lim_{x \to a} f(x) = f(a)$$

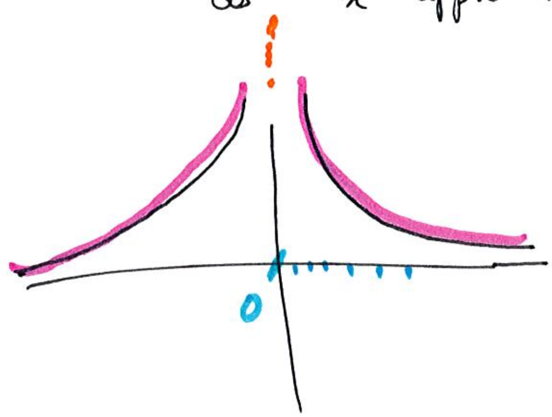
then we say $f(x)$ is continuous at a.

- $\lim_{x \to a^+} f(x) = \infty$

arbitrarily

$f(x)$ grows arbitrarily large as x approaches a from the right.

eg:



$f(x) = \frac{1}{x^2}$

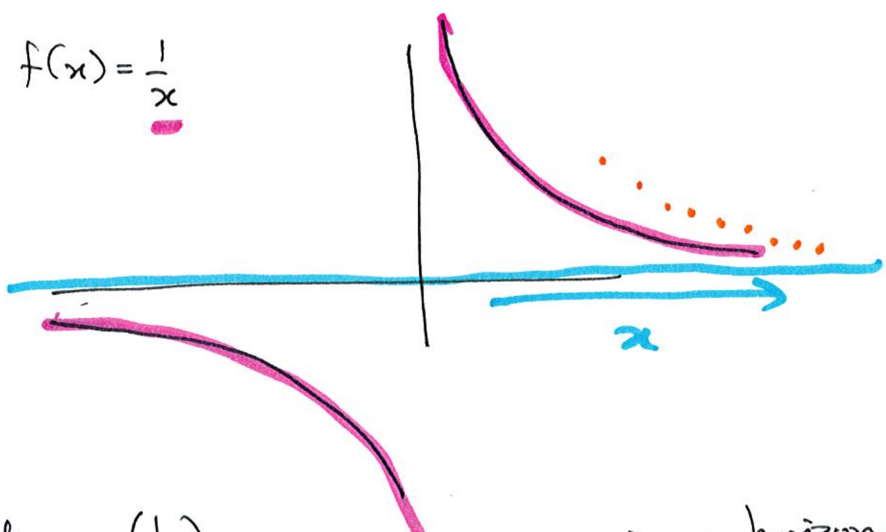
$$\left.\begin{aligned} \lim_{x \to 0^+} \frac{1}{x^2} &= \infty \\ \lim_{x \to 0^-} \frac{1}{x^2} &= \infty \end{aligned}\right\} \Rightarrow \lim_{x \to 0} \frac{1}{x^2} = \infty$$

$$\lim_{x \to \infty} f(x) = L$$

- if $f(x)$ approaches L as x grows arbitrarily large.
- in this case, $\boxed{y = L}$ is a horizontal asymptote.

- Similarly, for $\lim_{x \to -\infty} f(x)$.

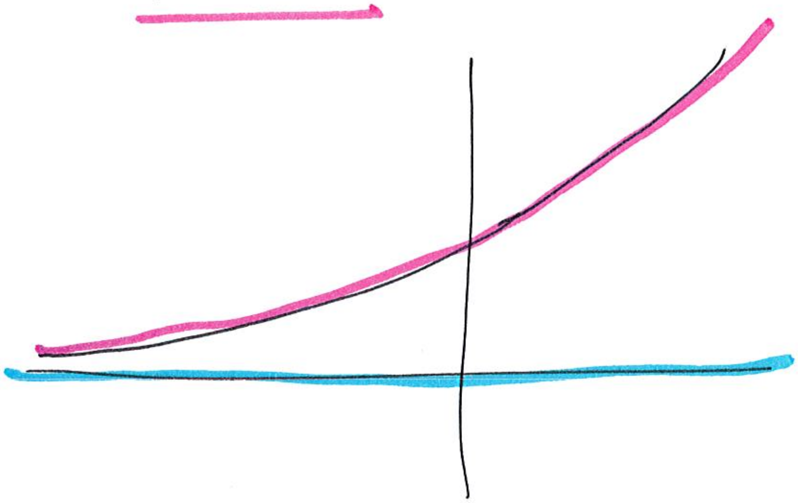
eg: $f(x) = \frac{1}{x}$



$\lim_{x \to \infty} \left(\frac{1}{x}\right) = 0 \leadsto y=0$ is a horizontal asymptote

$\lim_{x \to -\infty} \left(\frac{1}{x}\right) = 0 \leadsto y=0$ is — " —

eg: $f(x) = 2^x$



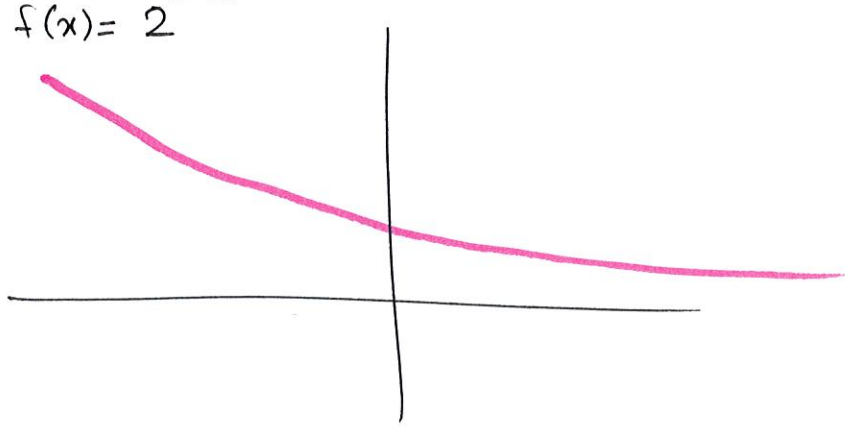
$\lim_{x \to -\infty} 2^x = 0$

$\lim_{x \to \infty} 2^x = \infty$

$\Downarrow$

$y=0$ is a horizontal asymptote

$f(x) = 2^{-x}$



$\lim_{x \to \infty} 2^{-x} = 0$

$\lim_{x \to -\infty} 2^x = \infty$

$\Downarrow$

$y=0$ is a horizontal asymptote

09.75

Notation

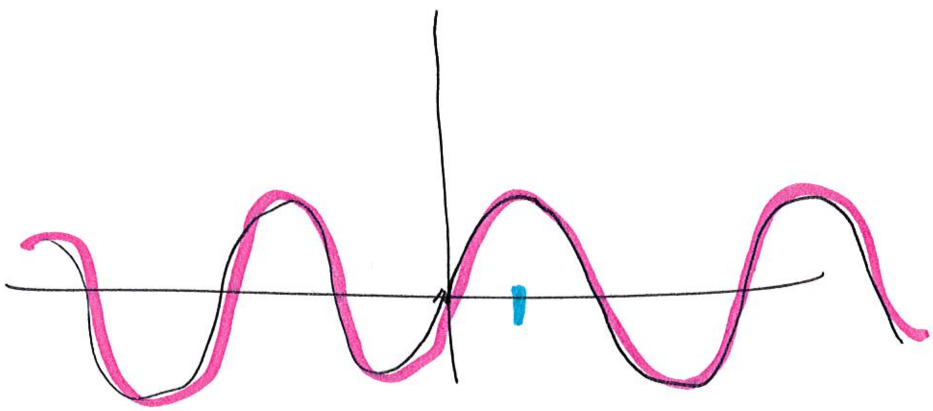
$\lim_{x \to a^+}$ approach from right

$\lim_{x \to -a}$ just a number, it has no directionality

---

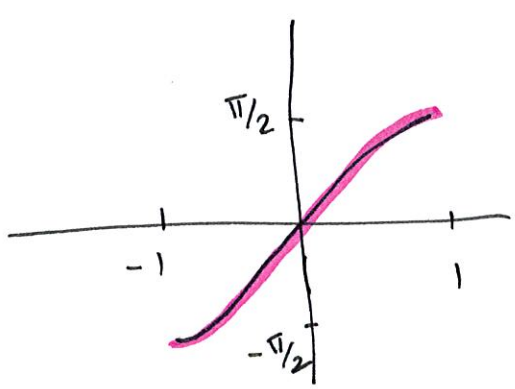
eg :

$\lim_{x \to \infty} \sin x = ?$ limit does not exist.



$\lim_{x \to \pi/2} \sin x = 1$

eg:

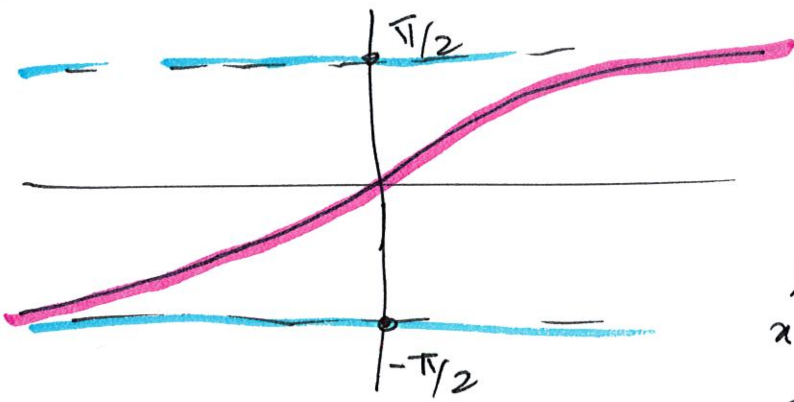


$f(x) = \arcsin x$

- arc sin x is continuous
- if a is in the domain

$\lim_{x \to a^\pm} \arcsin(x) = \arcsin(a)$

⇒ eg



$f(x) = \arctan x$

$$\lim_{x \to \infty} \arctan x = \frac{\pi}{2}$$

$$\lim_{x \to -\infty} \arctan x = -\frac{\pi}{2}$$

in comparison,

$\tan x$ has vertical asymptotes at

$x = \pi/2$, $x = -\pi/2$

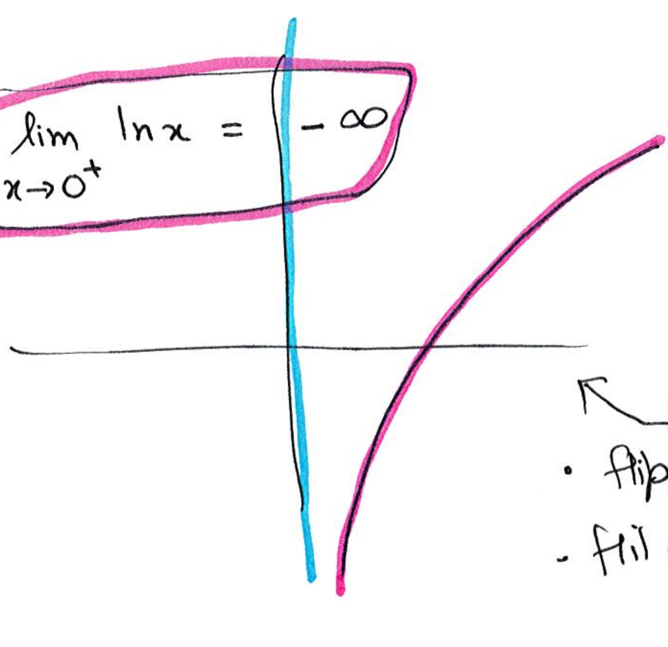
$\Rightarrow$ $y = \pi/2$, $y = -\pi/2$ are horizontal asymptotes

eg: $f(x) = \ln x$

vertical asymptote

$x = 0$

$$\boxed{\lim_{x \to 0^+} \ln x = -\infty}$$

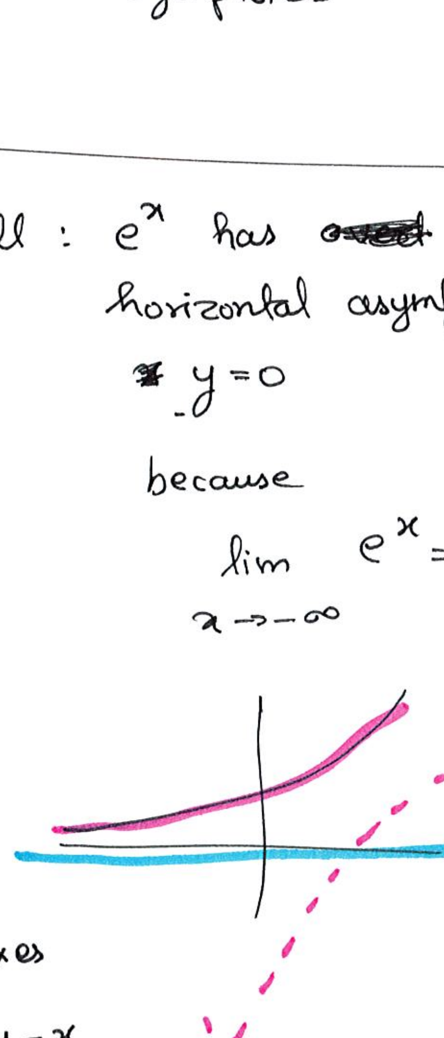


- flip x-y axes
- flip along $y = x$

recall : e^x has horizontal asymptote at $y = 0$

because

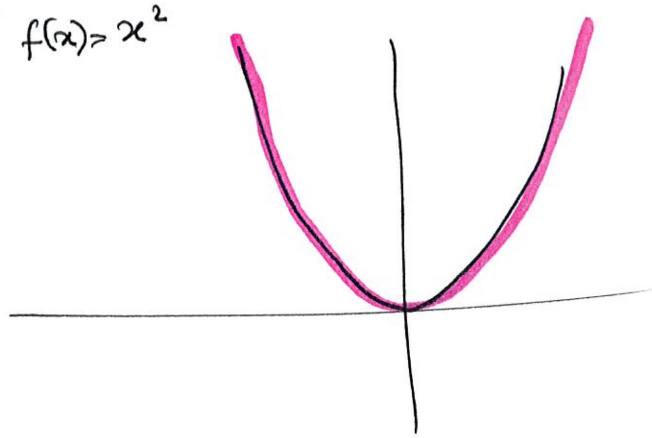
$$\lim_{x \to -\infty} e^x = 0$$



· ~~f~~ For standard functions : you need to know their graphs to find the limits.

· ~~poly~~ polynomials, e^x, $\ln x$, sin, cos, tan.

eg: $f(x) = x^2$



· $\lim_{x \to 2} x^2 = 2^2 = 4$

· $\lim_{x \to -\infty} x^2 = \infty$ . . .

· these standard functions are continuous wherever they are defined.

· $\lim_{x \to a} f(x) = f(a)$ if $f(x)$ is poly, sin, cos, tan, e^x, $\ln x$ etc.

eg: $\frac{e^x}{x^2+1}$ cannot find graph $\ddot{\frown}$

· § Algebraic methods of finding limits

2.3 Limit Laws : Pg 95.

can do basic arithmetic on limits

(1) $\lim_{x \to a}$ a is a real number

(2) limits of $f(x)$, $g(x)$ must exist

(3) ~~lim~~ for $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$

IF $\lim_{x \to a} g(x) \neq 0$

eg: $\frac{e^x}{x^2+1}$ $\quad$ $f(x) = e^x$, $g(x) = x^2+1$

$$\lim_{x \to 2} \frac{e^x}{x^2+1} = \frac{\lim_{x \to 2} e^x}{\lim_{x \to 2} x^2+1}$$

as both limits exist, and denominator is non-zero.

$$= \frac{e^2}{2^2+1}$$

as e^x, x^2+1 are continuous functions

$$\lim_{x \to 2} \frac{e^x}{x^2+1} = \frac{e^2}{5}$$

eg:

$$\lim_{x \to 0} \frac{(x+1)^2 - 1}{x}$$

- There are many tricks to do this
- Cannot apply basic ~~a~~ arithmetic limit laws directly

this is wrong

$$= \frac{\lim_{x \to 0} (x+1)^2 - 1}{\lim_{x \to 0} x}$$

$$= \frac{(0+1)^2 - 1}{0}$$

$$= \frac{0}{0}$$

Does not exist ?

Nope.

$$\lim_{x \to 0} \frac{(x+1)^2 - 1}{x}$$

$$= \lim_{x \to 0} \frac{x^2 + 2x + \cancel{1} - \cancel{1}}{x}$$

$$= \lim_{x \to 0} \frac{x^2 + 2x}{x}$$

$$= \lim_{x \to 0} \frac{\cancel{x}(x+2)}{\cancel{x}} = \lim_{x \to 0} x+2 = 2$$

$$\lim_{x \to 0} \frac{(x+1)^2 - 1}{x}$$

$$= 2$$