Cyclic homology

A = associative k-algebra (here we are going to use $M = A$)

Define an action of $\mathbb{Z}/n\mathbb{Z}$ on $A^{\otimes n}$ as

$$t\cdot(a_1,\dots,a_n) := \underset{\text{sign}\,\sigma}{\overset{\shortparallel}{(-1)^{n-1}}}(a_n,a_1,\dots,a_{n-1}) \qquad \text{for } \mathbb{Z}/n\mathbb{Z} = \langle t\rangle$$

$$\sigma = (1\cdots n) \in S_n$$

Def: norm operator $N := 1 + \dots + t^{n-1}$

We get a double complex :

$$\begin{tikzcd}
{} \arrow[d, "b"] & {} \arrow[d, "b'"] & {} \arrow[d, "b"] & \\
A^{\otimes 2} \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "1-t"'] \arrow[d, "b'"] & A^{\otimes 2} \arrow[l, "N"'] \arrow[d, "b"] & {} \arrow[l, "1-t"'] \\
A & A \arrow[l, "1-t"'] & A \arrow[l, "N"'] & {} \arrow[l, "1-t"']
\end{tikzcd}$$

We are mainly interested in this part

what was b'?
Also why is this a double complex?

Lemma: (Connes, Tsygan)

1. $b(1-t) = (1-t)b'$ 2. $b'\cdot N = N\cdot b$ ~~~ Note that the two N's are different

And $Nt^i = N$ for any i

Proof:

Define $\tau : A^{\otimes n+1} \longrightarrow A^{\otimes n}$

then $b = \sum_{i=0}^{n} t^i \tau t^{-i-1}$ and $b' = \sum_{i=0}^{n-1} t^i \tau t^{-i-1}$

2. $b'N = \left(\sum_{i=0}^{n-1} t^i\tau t^{-i-1}\right)(1+t+\dots+t^{n-1}) = \sum_{i=0}^{n-1} t^i \tau N = \left(\sum_{i=0}^{n-1} t^i\right)\tau N = N\tau N$

$Nb = (1+t+\dots+t^{n-1})\left(\sum_{i=0}^{n} t^i\tau t^{-i-1}\right) = \sum_{i=0}^{n} N\tau t^{-i-1} = N\tau\left(\sum_{i=0}^{n} t^{-i-1}\right) = N\tau N$

1. $b(1-t) = \sum_{i=0}^{n} t^i\tau t^{-i-1} - \sum_{i=0}^{n} t^i \tau t^{-i}$ $\qquad (1-t)b' = \sum_{i=0}^{n-1} t^i\tau t^{-i-1} - \sum_{i=0}^{n-1} t^{i+1}\tau t^{-i-1}$

Def: $CC_q(A) := C_q(A) = A^{\otimes q+1}$

$(1-t) : A^{\otimes n+1} \longrightarrow A^{\otimes n+1} \longrightarrow \text{coker} = C_n^\lambda(A)$ $\forall n \geq 0$ } gives a complex by lemma

Cyclic homology : $C_*^\lambda(A)$: $\dots \xrightarrow{b} C_n^\lambda(A) \xrightarrow{b} C_{n-1}^\lambda(A) \xrightarrow{b} \dots \longrightarrow C_0^\lambda(A) = A/1-t$

The homology is $\quad H_n^\lambda(A) := H_n(C_*^\lambda(A))$

There is a natural augementation map

$$\mathrm{Tot}\, CC_*(A) \xrightarrow{p} C_*^\lambda(A)$$

This augementation induces an edge homomorphism of the spectral sequence

$$E^1_{pq} := H_p(\mathbb{Z}/_{q+1}, A^{\otimes q+1}) \Longrightarrow HC_*(A) \qquad \text{associated with } CC_*(A)$$

Th^m: For any k-algebra A with $k \supseteq \mathbb{Q}$ the natural map $p_*: HC_* A \longrightarrow H_*^\lambda(A)$ is an isomorphism (i.e. p is a quasi-isomorphism.)

Proof: Because of the spectral sequence, suffices to show $H_p(\mathbb{Z}/_{q+1}, A^{\otimes q+1}) = 0$ for $p > 0$

$\rightarrow$ if $n \in \mathbb{Z}$ is such that $n \cdot 1_K$ is not a zero divisor, then we get a resolution

2-periodic:

$$\begin{tikzcd}
\longrightarrow k[\mathbb{Z}/n\mathbb{Z}] \arrow[r, "(1-t)"] & k[\mathbb{Z}/n\mathbb{Z}] \arrow[r, "N"] \arrow[l, bend left, "h"] & k[\mathbb{Z}/n\mathbb{Z}] \arrow[r] \arrow[l, bend left, "h'"] & \cdots \arrow[r, "N"] & k[\mathbb{Z}/n\mathbb{Z}] \arrow[r, "1-t"] & k[\mathbb{Z}/n\mathbb{Z}] \arrow[r, "\circ"] \arrow[l, bend left, "h"] & k \arrow[r] & 0
\end{tikzcd}$$

$$k[\mathbb{Z}/n\mathbb{Z}] \cong k[t]/_{t^n-1}$$

where $\quad h := -\frac{1}{n}\sum_{i=1}^{n-1} i t^i \qquad h' := \frac{1}{n}\,\mathrm{id} \quad$ are the homotopies

satisfying

$$h(1-t) + N h' = \mathrm{id} = h' N + (1-t) h$$

$$\overset{\shortparallel}{-\frac{1}{n}\sum_{i=1}^{n-1} i t^i \cdot (1-t) + \sum_{i=0}^{n-1} \frac{t^i}{n}}$$

Thus for any $\mathbb{Z}/n\mathbb{Z}$-module M (in particular $M = A^{\otimes n}$) the complex

$\rightarrow M \xrightarrow{1-t} M \xrightarrow{N} M \longrightarrow \cdots \quad$ is acyclic except for H_0.

Prop: $C_{\cdot\cdot} \longrightarrow C'_{\cdot\cdot}$ be a map of double complexes which is a quasi-iso when restricted to each column. Then the map of total complex is a quasi-isomorphism. (This will prove the theorem for us.)

Lemma (killing contractible complexes): Let

$$\cdots \longrightarrow A_n \oplus A_n' \xrightarrow[d = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}]{} A_{n-1} \oplus A_{n-1}' \xrightarrow{d}$$

be a double complex and is contractible and (A_*, δ) is a complex with contracting homotopy $h: A_n' \longrightarrow A_{n+1}'$, then the following inclusion of complexes is a quasi-isomorphism:

$$(\mathrm{id}, -h\gamma) : (A_*, \alpha - \beta h \gamma) \hookrightarrow (A_* \oplus A_*', d)$$

Fact: if A unital, the first b' column of $CC_*(A)$ can be considered as a quotient of $\mathrm{Tot}\, CC_*(A)$ with contracting homotopy

$$h = -s : A^{\otimes n} \longrightarrow A^{\otimes n+1}$$
$$(a_1 \cdots a_n) \longmapsto (-1, a_1, \dots, a_n)$$

that satisfies $\quad d_i \cdot h = h \cdot d_{i-1} \quad i = 1, \dots, n-1 \quad$ and $\quad -d \circ s = \mathrm{id}$

The kernel of this map is deduced from $CC_*(A)$ as follows:

1) delete the first b'-column
2) add a map $\quad B : CC_{2,q}(A) = A^{\otimes q+1} \longrightarrow CC_{0,q+1}(A) = A^{\otimes q+2}$ described as follows: $\quad d = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \quad \alpha = b, \ \beta = (1-t), \ \gamma = N, \ \delta = -b'$

$$\begin{tikzcd}
{} \arrow[d, "b"] & {} \arrow[d, "b"] & {} \arrow[d, "b"] \\
A^{\otimes 2} \arrow[d] & A^{\otimes 2} \arrow[d, "b"] & A^{\otimes 2} \arrow[d, "b"] \\
A^{\otimes 1} & A \arrow[ul, "B"'] & A \arrow[ul, "B"'] \\
(0) & (2) & (4)
\end{tikzcd}
\quad \overset{\text{Reindexing}}{\rightsquigarrow} \quad
\begin{tikzcd}
A^{\otimes 3} \arrow[d, "b"] & A^{\otimes 2} \arrow[l, "B"'] \arrow[d, "b"] & A \arrow[l, "B"'] \\
A^{\otimes 2} \arrow[d, "b"] & A \arrow[l, "B"] \\
A
\end{tikzcd}$$

$$B(A)_{p,q} := \begin{cases} A^{\otimes(q-p+1)} & q \geq p \\ 0 & \text{else} \end{cases}$$

and $\quad Bb + bB = 0$

Explicitly:
$$B : A^{\otimes(n+1)} \longrightarrow A^{\otimes(n+2)}$$
$$(a_0 \cdots a_n) \longmapsto \sum_{i=0}^{n} (1, a_i, \dots, a_n, a_0, \dots, a_{i-1}) \cdot (-1)^{ni} +$$
$$\sum_{i=1}^{n+1} (-1)^{n(i-1)} (a_{i-1}, 1, a_i, \dots, a_n, a_0, \dots, a_{n-2})$$
$$/\!/$$
$$(1-t) \circ s \circ N$$

this induces $\quad B_* : HH_n(A) \longrightarrow HH_{n+1}(A)$

The (b, B)-complex can be further simplified by replacing the H_* complex by the normalisation: $\quad \bar{A} := A/k \cdot 1_A = \mathrm{coker}\,(\eta : k \to A)$

Consider:

$$\begin{tikzcd}
A \otimes \bar{A}^{\otimes 2} \arrow[d, "b"] & A \otimes \bar{A} \arrow[l, "\bar{B}"'] \arrow[d, "b"] & \arrow[l] \\
A \otimes \bar{A} \arrow[d] & A \arrow[l, "\bar{B}"'] \\
A
\end{tikzcd}$$

where $\quad \bar{B}(a_0 \cdots a_n) = \sum_{i=0}^{n} (-1)^n (1, a_i, \dots, a_n, a_0, \dots, a_{i-1})$

Conclusion: if A is unital

$$(\mathrm{Tot}\, \bar{B}(A), b + \bar{B}) \xleftarrow{\sim} (\mathrm{Tot}\, B(A), b + B) \xrightarrow{\sim} (\mathrm{Tot}\, CC_*(A), d) \xrightarrow[\sim]{\text{if } \mathbb{Q} \subseteq k} (C^\lambda(A)_*, b)$$

Eg: $A = k$ commutative ring with 1, then looking at the $\overline{B}$-complex we get

$$HC_{2n}(k) = k \quad , \quad HC_{2n+1}(k) = 0$$

with generator $1 \in k \cong \overline{B}(k)_{n,n}$

In $CC_{**}(k)$ the generator of $HC_{2n}(k)$ is the cycle

$$u_n := ((-1)^n (2(n-1))!, \ldots, -4, 2, 1) \in (\mathrm{Tot}\, CC_{**}(k))_n$$

For $H^{\lambda}_{2n}(k)$ the generator is the class of $(1, 1, \ldots, 1)$ and the map p sends $u^n \longmapsto (2(n-1))!\,(1, \ldots, 1)$ for $n > 0$.

In general, for non-commutative k-algebras $HC_0(A) = HH_0(A) = A/[A,A]$

Prop: For any commutative unital k-algebra A,

$$HC_1(A) \cong \Omega^1_{A|k} / dA \qquad \left(HH_1(A) \xleftarrow{\varepsilon_1} \Omega^1_{A|k} \text{ as direct summand} \right)$$

Proof: From $B(A)$.. we see that, $HC_1(A) \cong A \otimes A / \sim$

image of b: $ab \otimes c - a \otimes bc + ca \otimes b = 0 \qquad a, b, c \in A$

image of B: $1 \otimes a + a \otimes 1 \qquad a \in A$

$$A \otimes A / b + B \longrightarrow \Omega^1_{A|k} / dA$$

$$[(a \otimes b)] \longmapsto [a\, db]$$

Well defined and isomorphism follows from staring at the relations.