Sept 11, 2019

Review: § 1.4 Exponential functions, a^x $a = \text{constant}$.

- Def: $2^x = \underbrace{2 \cdots 2}_{x\text{-times}}$

- Laws of exponentiation

  - $2^{x+y} = 2^x \cdot 2^y$
  - $2^{x-y} = 2^x / 2^y$
  - $2^{xy} = (2^x)^y$
  - $2^x \cdot 3^x = (2 \cdot 3)^x$

§ Domain, Range

Domain = possible x-values

Range = possible y-values

ANY QUESTIONS ? about class or homework?

Q.)

$$f(x) = 4\,|x-3| - |x-2|$$

Write f as

$$f(x) = \begin{cases} f_1 & x \le a \\ f_2 & a \le x \le b \\ f_3 & b \le x \end{cases}$$

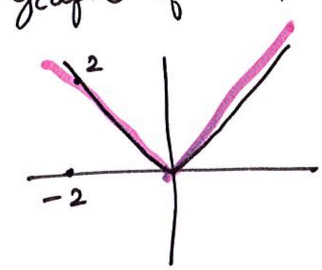
} piecewise function

find $f_1, f_2, f_3;\ a, b$.

KEY : $|x|$ changes behavior at $x = 0$

$$|x| = \begin{cases} x & \text{if } 0 \le x \\ -x & \text{if } x \le 0 \end{cases}$$

graph of $|x|$



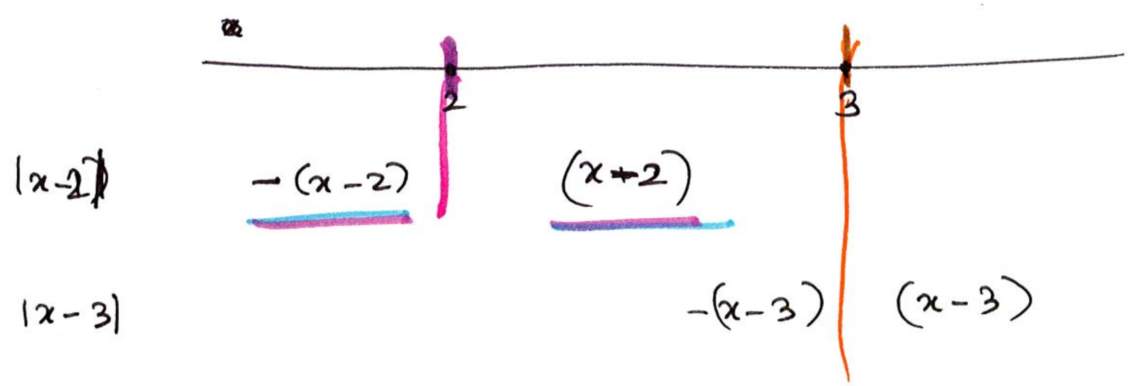
Ans :

$$f(x) = 4 \cdot \underline{|x-3|} - \underline{|x-2|}$$

changes behavior at $x = 3$

changes behavior at $x = 2$

$a = 2$, $b = 3$



$\cancel{x \le 2}$

$$|x-2| = -(x-2) \qquad x \le 2$$

$$|x-3| = -(x-3)$$

$$\Rightarrow f(x) = 4|x-3| - |x-2|$$

$$= \underbrace{4\left(-(x-3)\right) - \left(-(x-2)\right)}_{\shortparallel \atop f_1} \qquad x \le 2$$

$$|x-2| = (x-2) \qquad 2 \le x \le 3$$

$$|x-3| = -(x-3)$$

$$\Rightarrow f(x) = 4|x-3| - |x-2|$$

$$= 4\left(-(x-3)\right) - (x-2) = f_2$$

$$3 \le x$$

$\vdots$

- About Q. 9 — Webwork has "wrong" answer

incomplete

Today: Appendix D, §1.45

- § Appendix D - trigonometric functions

  ↓

  || Aside: Fourier analyis → ~~sig~~ signal progressing

- Radians:

Radians is a measure angle

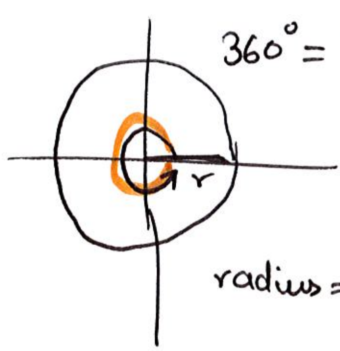
$$\boxed{\pi \text{ radians} = 180^\circ}$$

eg: $\frac{\pi}{2}$ (radians) $= 90^\circ$

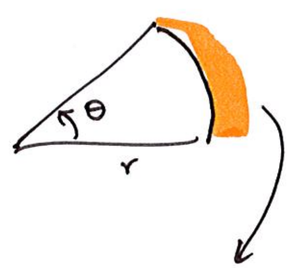
$\frac{\pi}{3} = 60^\circ$

Why use radians?

Reason 1)



Circumference = $2\pi r$



Length of arc

$= \frac{\theta}{2\pi} \cdot 2\pi r$

$= \theta \cdot r$

this formula only works in ~~radi~~ (radians)

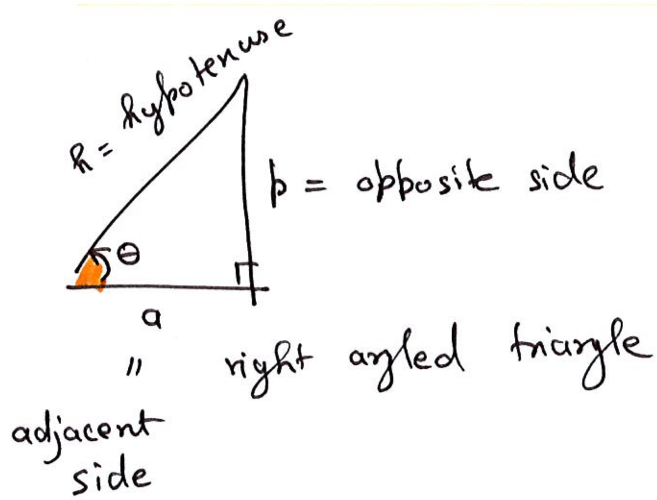
Reason 2)

in radians $\frac{d}{dx} \sin x = \cos x$.

Important angles:

radians	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π	2π
degrees	0°	30°	45°	60°	90°	180°	360°

Classical definitions:



right angled triangle

Def:

$\sin \theta = \frac{p}{h}$

$\cos \theta = \frac{a}{h}$

$\tan \theta = \frac{p}{a}$

$\operatorname{cosec} \theta = \frac{1}{\sin \theta}$

$\sec \theta = \frac{1}{\cos \theta}$

$\cot \theta = \frac{1}{\tan \theta}$

Trig identities :

Pythagorces : $a^2 + b^2 = h^2$

divide both sides by h^2

$$\frac{a^2}{h^2} + \frac{b^2}{h^2} = 1$$

$$\Rightarrow \left(\frac{a}{h}\right)^2 + \left(\frac{b}{h}\right)^2 = 1$$

$$\Rightarrow (\cos\theta)^2 + (\sin\theta)^2 = 1$$

$$\cos^2\theta + \sin^2\theta = 1$$

Divide both sides by $\cos^2\theta$

$$\frac{\cancel{\cos^2\theta}}{\cancel{\cos^2\theta}} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\Rightarrow 1 + \left(\frac{\sin\theta}{\cos\theta}\right)^2 = \left(\frac{1}{\cos\theta}\right)^2$$

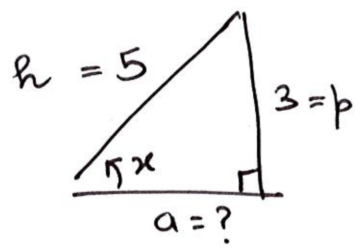
$$\Rightarrow 1 + \tan^2\theta = \sec^2\theta$$

Eg: ① Given $\sin x = \frac{3}{5}$ find $\tan x$. given $0 < x < \frac{\pi}{2}$

Ans: Geometry:

Method 1

Construct a triangle



$\sin x = \frac{3}{5}$

$$a^2 = \sqrt{h^2 - p^2}$$

$$= \sqrt{5^2 - 3^2}$$

$$= \sqrt{25 - 9}$$

$$= \sqrt{16}$$

$$\boxed{a = 4}$$

$$\boxed{\tan x = \frac{p}{a} = \frac{3}{4}}$$

Method 2

Use : $\sin^2 x + \cos^2 x = 1$

$$\Rightarrow \left(\frac{3}{5}\right)^2 + \cos^2 x = 1$$

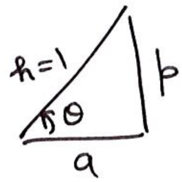
$$\Rightarrow \cos^2 x = 1 - \frac{9}{25} = \frac{16}{25}$$

$$\Rightarrow \cos x = \frac{4}{5} \Rightarrow \tan x = \frac{\sin x}{\cos x}$$

$$= 3/4$$

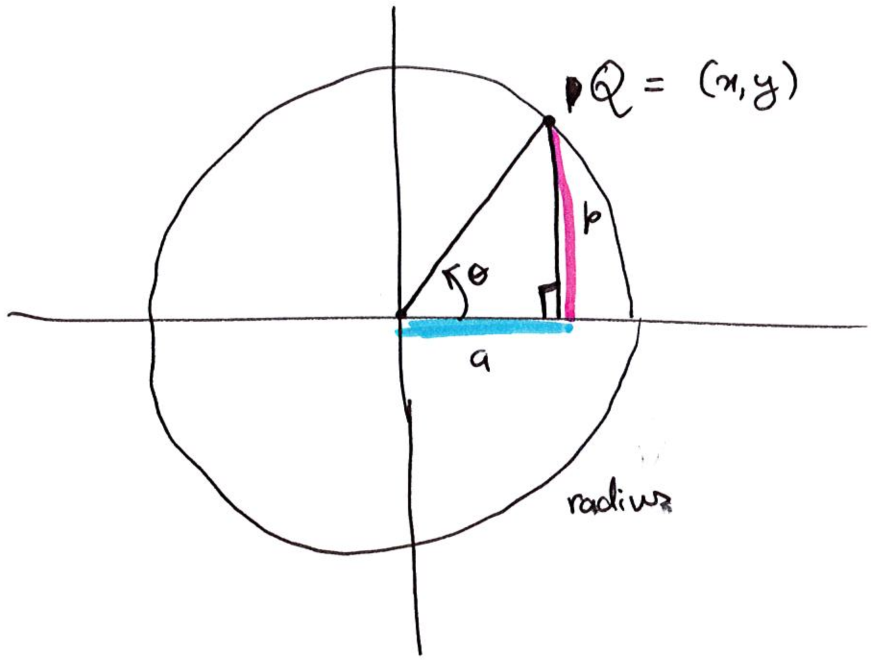
Better definition

if $h=1$,



$\sin\theta = p$

$\cos\theta = a$



Def: let $Q=(x,y)$ be point on unit circle at angle θ with x-axis (measured counter-clockwise)

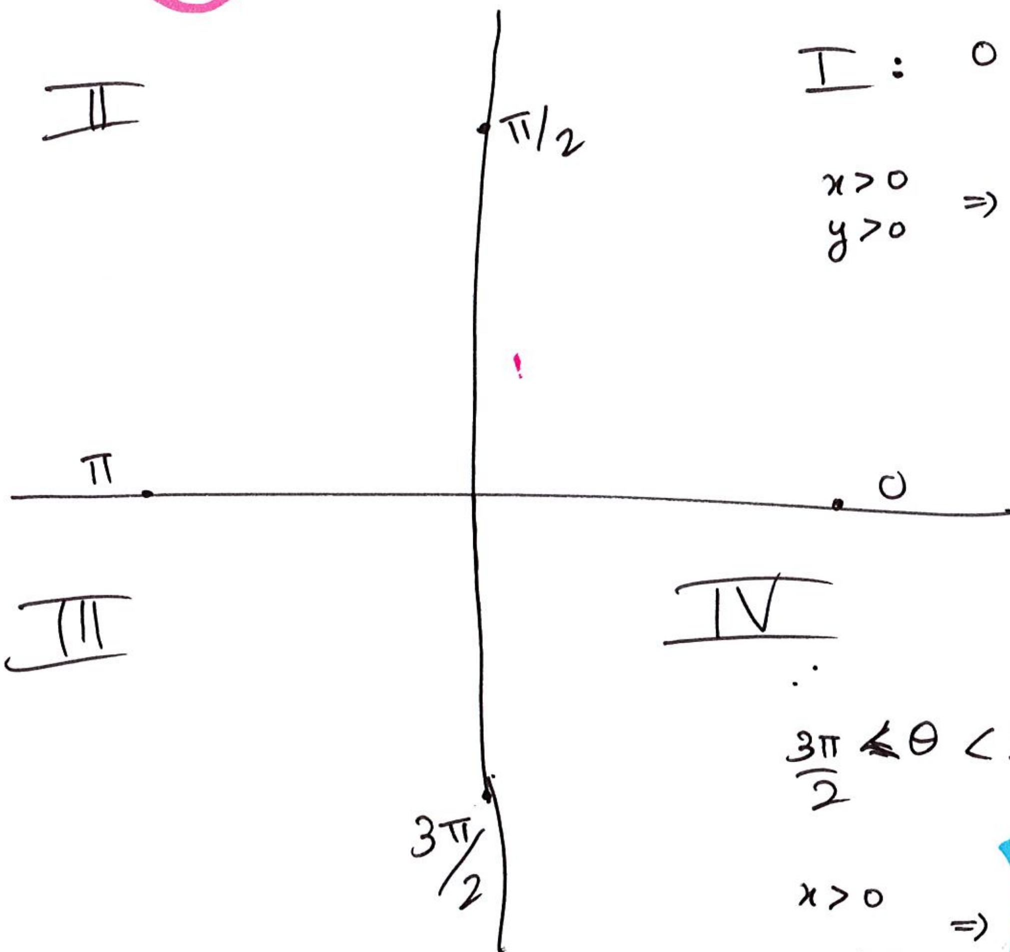
$\sin\theta = y$

$\cos\theta = x$.

- Now, sin, cos are defined for all real numbers.

y-coord x-coord

- Now, sin, cos can take negative values



I : $0 < \theta < \frac{\pi}{2}$

$x > 0$ $\Rightarrow$ $\cos\theta > 0$
$y > 0$ $\sin\theta > 0$

$\tan\theta > 0$

IV

$\frac{3\pi}{2} < \theta < 2\pi$

$x > 0$ $\Rightarrow$ $\cos\theta > 0$
$y < 0$ $\sin\theta < 0$

$\frac{\sin\theta}{\cos\theta} = \tan\theta < 0$

---

eg: Given $\sin x = \frac{-3}{5}$ find $\tan x$,

x is in IV-quadrant.

same computations as before

but $\tan x < 0$

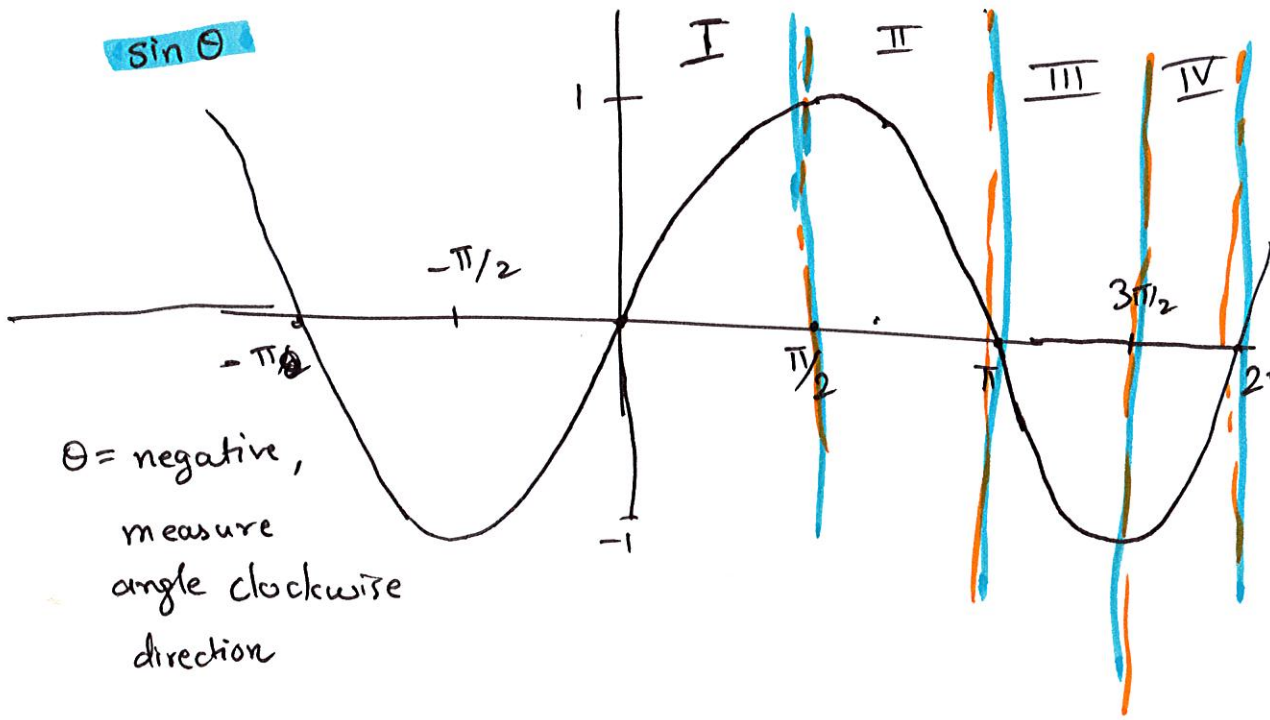
$\Rightarrow \tan x = -\frac{3}{4}$

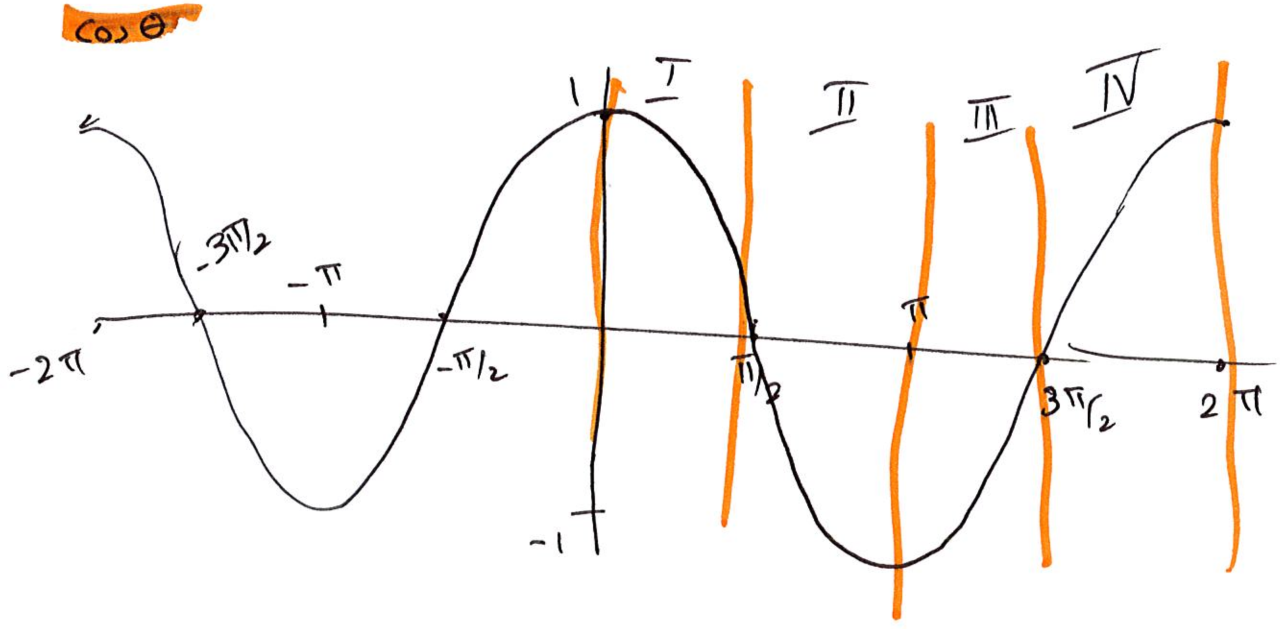
- Identities don't change.
  The quadrant / range on θ tells you the signs of sin, cos.

$y = \sin\theta$

Graphs:

$x = \cos\theta$





sin θ

Range : $[-1,1]$

domain : $\mathbb{R}$

cos θ

Range : $[-1,1]$

Domain : $\mathbb{R}$

---

other trig identities:

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$= 2\cos^2\theta - 1$$

Necessary to memorize

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

---

§ 1.5 Inverse functions

$y = f(x)$ $\rightsquigarrow$ we want to write x in terms of y

$\downarrow$

$f^{-1}(y) = x$

eg: 1 $y = x^2$ $\rightsquigarrow$ $\pm\sqrt{y} = x$

convention

$f(x) = x^2$

we say ~~$f^{-1}(x) = +\sqrt{}$~~

$f^{-1}(y) = +\sqrt{y}$

• we say $\underbrace{f^{-1}(y)}_{\sqrt{y}}$ is the inverse function of $\underbrace{f(x)}_{x^2}$

in this example

2 $y = x^3+1$ $\rightsquigarrow$ $y-1 = x^3$ (subtract 1 from both sides)

$\Rightarrow \sqrt[3]{y-1} = x$

the inverse function of

$f(x) = x^3+1$

is

$f^{-1}(y) = \sqrt[3]{y-1}$.

3

eg (3) ~~$y = 2^x$~~

$y = 2^{x} \rightsquigarrow x = \log_2 y$

Define $\log_2 y$ to be the inverse function of 2^x.

eg:

$4 = 2^2 \rightsquigarrow 2 = \log_2 4$

eg: Solve $e^{3x+1} = 7$ (for x).

· take $\log_e$ of both sides

$$3x + 1 = \log_e 7$$

$$\Rightarrow \quad x = \frac{\log_e 7 - 1}{3}$$

Notation: $\log_e = \ln$ = natural logarithm

= inverse function of e^x

- $f(x) = y$

  Apply f^{-1} to both sides

  $x = f^{-1}(y)$

in the above eg.

$e^{3x+1} = 7$

Apply ~~to~~ $\ln$ to b.s.

$3x+1 = \ln 7$

---

eg : $\sin x = y \rightsquigarrow$ ~~inverse~~ $x = \arcsin y$

Def : $\arcsin y$ is the inverse function of $\sin x$.

eg : $\sin \frac{\pi}{2} = 1 \rightsquigarrow$ $\boxed{\frac{\pi}{2} = \arcsin 1}$

apply arc sin to both sides

- Sometimes $\arcsin x$ is written as $\sin^{-1} x$

this is horrible notation. ⍨

because $\arcsin x \neq \frac{1}{\sin x}$

$\sin^2 x = (\sin x)^2$

$\sin^{-1} x \neq \frac{1}{\sin x}$

$\quad \|$

$\arcsin x$

eg: $y = \frac{1}{x}$ $\rightsquigarrow$ $\frac{1}{y} = x$

$= f(x)$ $\qquad f^{-1}(y) =$

inverse function

of $f(x) = \frac{1}{x}$ is $f^{-1}(y) = \frac{1}{y}$

Logarithms:

Laws of exponentiation	Logarithm Laws
• $2^x \cdot 2^y = 2^{x+y}$	~~$\log_2 x + \log$~~ $\log_2(xy) = \log_2 x + \log_2 y$
• $2^x / 2^y = 2^{x-y}$	$\log_2(x/y) = \log_2 x - \log_2 y$
• $2^{xy} = (2^x)^y$	$\log_2(x^y) = y \cdot \log_2 x$
• $2^x \cdot 3^x = (2 \cdot 3)^x$	Base change: $\log_2 x = \frac{\log_3 x}{\log_3 2}$

eg: Base change identity

$$\frac{\log_{10} 3}{\log_{10} 2} = \log_2 3 = \frac{\log_e 3}{\log_e 2} = \frac{\ln 3}{\ln 2}$$

Webwork only understans ln

$$\log_b a = \frac{\log_c a}{\log_c b}$$

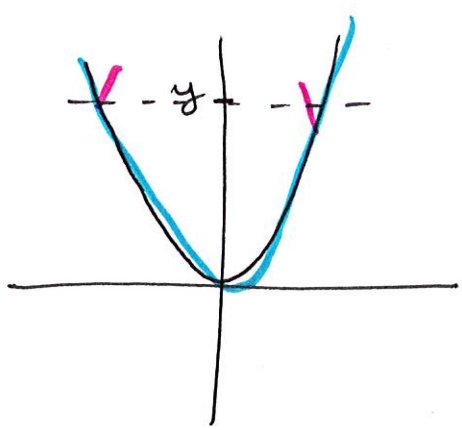
$$= \frac{\ln a}{\ln b} \quad \text{if } c = e$$

Later: $\frac{d}{dx} \ln x = \frac{1}{x}$ but $\frac{d}{dx} \log_2 x = \frac{1}{x} \cdot \frac{1}{\ln 2}$

---

§ Domain shrinking.

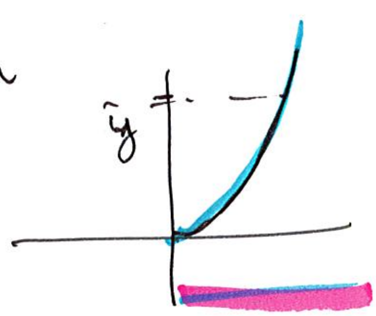
eg: $y = x^2 \rightsquigarrow \pm\sqrt{y} = x$



shrink domain

new domain:

$x \geq 0$



the inverse function of $y = x^2$ on $x \geq 0$

is $f^{-1}(y) = \sqrt{y}$

- $f(x) = y \xrightarrow[f^{-1} \text{ to both sides}]{\text{apply}} x = f^{-1}(y)$

but this makes sense only if y is in the range of $f(x)$

$$\boxed{\begin{array}{c}\text{domain of } f^{-1}(y) \\ \| \\ \text{range of } f(x)\end{array}}$$

- Recall: Range = possible y-values

Back to example:

$f(x) = x^2$

this has a problem,

shrink domain

$f(x) = x^2, \ x \geq 0 \rightsquigarrow \sqrt{y} = x, \ y \geq 0$

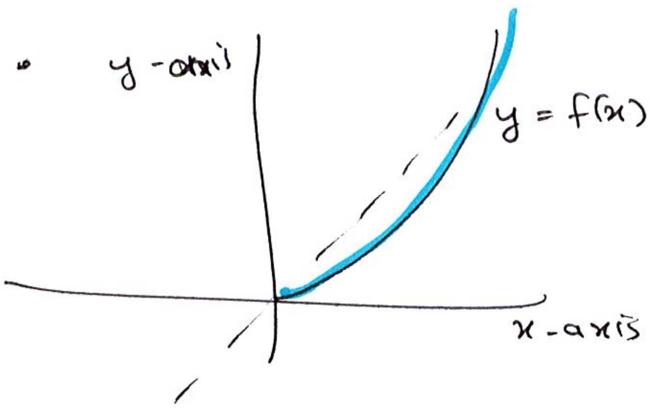
domain of $\sqrt{y}$

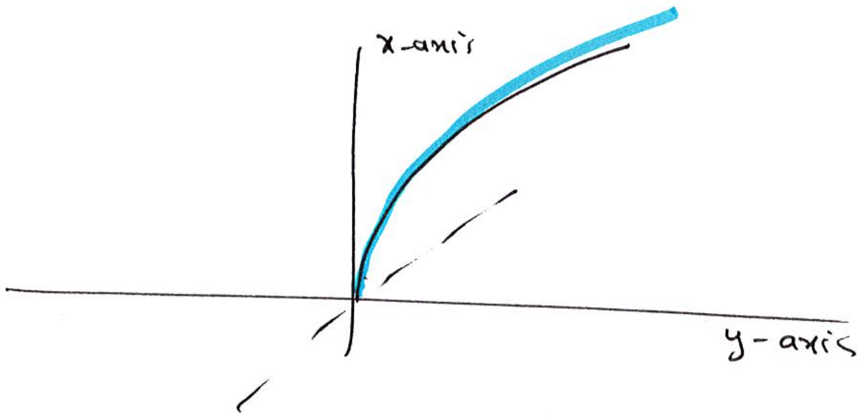
$=$ range of x^2

$= [0, \infty)$

- Same happens for domain.

-



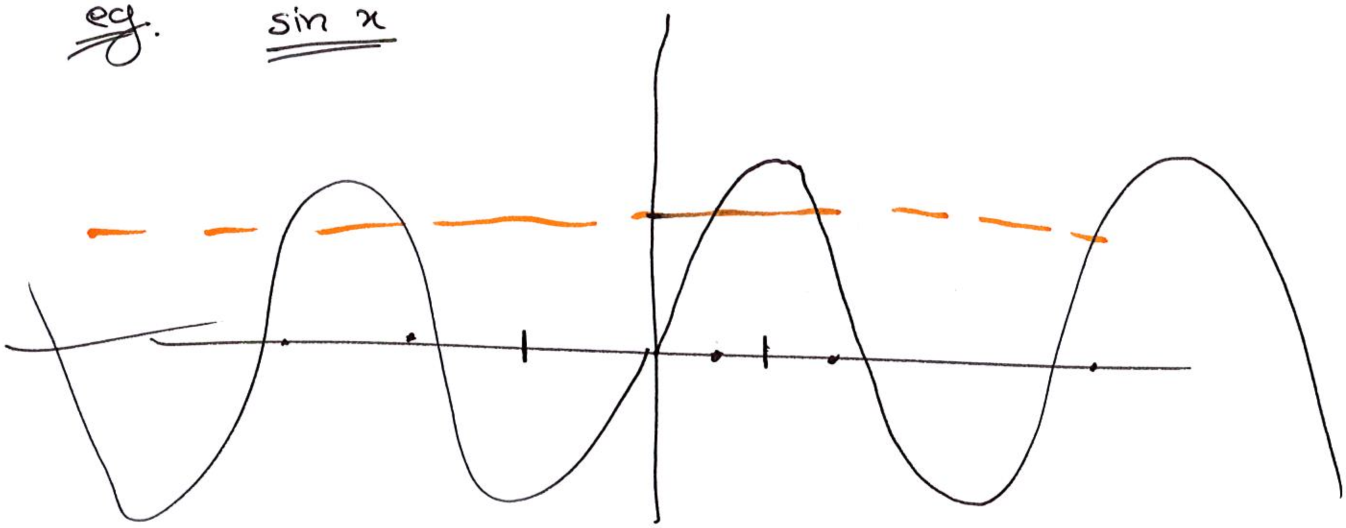


- The graph of $f^{-1}(y)$ is the graph of $f(x)$ with the x-y axes flipped.

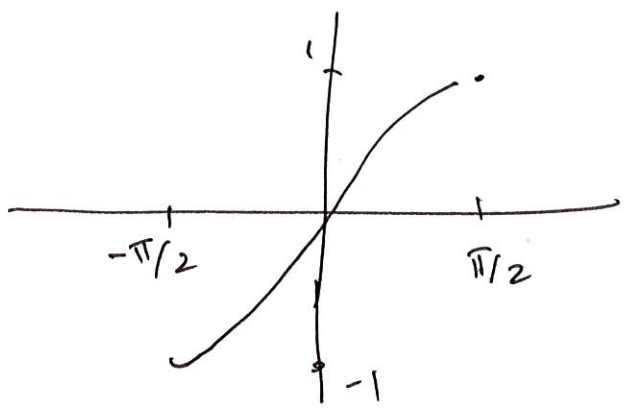
Def: a function $y=f(x)$ is called one-to-one if for every y in the range there is exactly one x in the domain with $f(x)=y$

- If f is one-to-one then only we can define $f^{-1}(x)$. else shrink domain & make it one-to-one.

eg. sin x



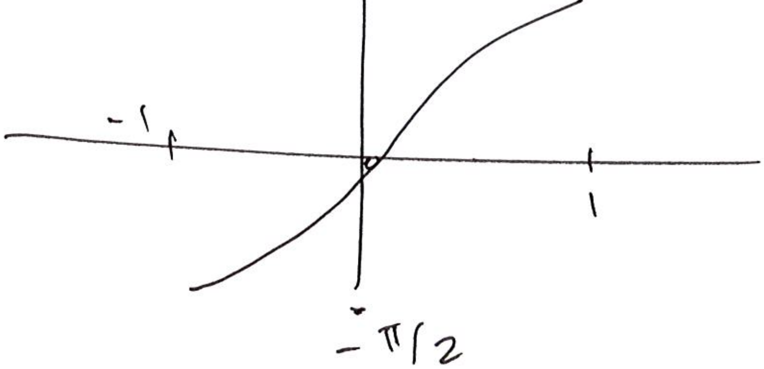
new domain



domain : $[-\pi/2, \pi/2]$

range : $[-1, 1]$

arcsin x



domain : $[-1, 1]$

range : $[-\pi/2, \pi/2]$