Limits :

Def^n An ∞-cat has all limits of shape J if

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ A^J \arrow[ur, dashed, "\exists"] \arrow[r, "id"'] \arrow[ur, phantom, shift right=3ex, "\Downarrow\lambda"] & A^J \end{tikzcd}$$

an absolute right lifting of id through Δ.

A particular limit of a J-shaped diagram $1 \xrightarrow{d} A^J$

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ 1 \arrow[ur, dashed, "\ell"] \arrow[r] \arrow[ur, phantom, shift right=3ex, "\Downarrow\lambda"] & A^J \end{tikzcd}$$

an absolute right lifting.

Universal property of (ℓ, λ) :

Given

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow\chi"] & A \arrow[d, "\Delta"] \\ 1 \arrow[r, "d"] & A^J \end{tikzcd} \quad = \quad \begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, shift left=3ex, "\exists\Downarrow"] & A \arrow[d, "\Delta"] \\ 1 \arrow[ur, dashed] \arrow[r] \arrow[ur, phantom, shift right=3ex, "\Downarrow\lambda"] & A^J \end{tikzcd}$$

Goal: Right adjoints preserve limits.

Lemma: Let

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[ur, "\ell"] \arrow[r, "g"'] \arrow[ur, phantom, shift right=3ex, "\Downarrow\lambda"] & A \end{tikzcd}$$

be absolute right lifting diagram

Given

$$\begin{tikzcd} & D \arrow[d] \\ & B \arrow[d] \\ C \arrow[uur, bend left] \arrow[ur, "\ell"] \arrow[r] \arrow[uur, phantom, shift right=1ex, "\Downarrow K"] \arrow[ur, phantom, shift right=3ex, "\Downarrow\lambda"] & A \end{tikzcd}$$

K is an absolute right lifting iff $\lambda \cdot K$ is.

(This generalizes pullbacks)

Digression: right liftings are right adjoints

$$\begin{tikzcd} C^B \arrow[r, "f_*"] & C^A \arrow[l, dashed, bend left, "\text{lift by } f"] \end{tikzcd}$$

and λ is a counit.

Th^m: Right adjoints preserve limits.

Proof: Given a limit diagram

$$\begin{tikzcd} & A \arrow[d, "\Delta"] \\ 1 \arrow[ur, dashed, "\lim d"] \arrow[r, "d"'] \arrow[ur, phantom, shift left=3, "\Downarrow\rho"] & A^J \end{tikzcd}$$

and

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "u"] \end{tikzcd} \qquad fu \overset{\varepsilon}{\Rightarrow} id_A$$

Claim we need to show

$$\begin{tikzcd} & A \arrow[r, "u"] \arrow[d] & B \arrow[d] \\ 1 \arrow[ur, "\lim d"] \arrow[r, "d"'] & A^J \arrow[r, "u^J"'] & B^J \end{tikzcd}$$

Proof: $f \dashv u$ with counit $\varepsilon : fu \Rightarrow id_A$

$\Rightarrow f^J \dashv u^J$ with counit ε^J

i.e. we've a right lifting diagram

$$\begin{tikzcd} & B^J \arrow[d, "f^J"] \\ A^J \arrow[ur, "u^J"] \arrow[r, equal] \arrow[ur, phantom, shift right=3, "\Downarrow\varepsilon^J"] & A^J \end{tikzcd}$$

To prove this consider

$$\begin{tikzcd} & A \arrow[r] \arrow[d, "\Delta"] & B \arrow[d, "\Delta"] \\ 1 \arrow[ur, dashed, "\lim d"] \arrow[r, "d"'] \arrow[ur, phantom, shift left=3, "\Downarrow\rho"] & A^J \arrow[r, "u^J"] \arrow[dr, equal] \arrow[dr, phantom, shift left=3, "\Downarrow\varepsilon^J"] & B^J \arrow[d, "f^J"] \\ & & A^J \end{tikzcd}$$

and simplify this too

$$\begin{tikzcd} & B \arrow[d, "f"] \\ & A \arrow[d] \\ 1 \arrow[uur, bend left, ""] \arrow[ur, ""] \arrow[r, "d"'] \arrow[ur, phantom, shift left=4, "\Downarrow\varepsilon"] \arrow[ur, phantom, shift right=2, "\Downarrow\rho"] & A^J \end{tikzcd}$$

and both ε, ρ are also right liftings.

Initial Functor

Idea Consider $1 \xrightarrow{d} A^J$, $J \in qCat$, J has an initial element $1 \xrightarrow{i} J$
then $\lim d = d(i) \in A$.

Def $I \xrightarrow{k} J \in qCat$ is initial (or in $\mathcal{K}$ if $\mathcal{K}$ is cart. closed)
is initial if J-indexed limits exist iff restricted I-indexed limits exist + limits coincide.

Prop: If $I \xrightarrow{K} J$ has a right adjoint then K is initial.