h-Principles
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Lectures delivered by Emmy Murphy

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Disclaimer: Emmy Murphy gave lectures on h-principles at the MSRI Summer School 2019, Cortona, Italy. These are "live-TEXed" notes from these lectures. These notes are not a faithful representation of the course and all the errors the scribe's[1] fault. Special thanks to Javier Martínez-Aguinaga[2] for the beautiful pictures.

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1 Introduction to the h-principle

Idea: Given a PDE on a smooth manifold, which is "underdetermined", we can often compare it to a simpler purely algebraic problem and ask if the algebraic problem has any solution. For example, DE $(y'(t))^2 = -(1+y(t))^2$ has no solutions by purely algebraic considerations. h-principles are a general method for taking algebraic solutions and obtaining solutions to a given PDE.

Definition 1.1. For an immersion $f : S^1 \looparrowright \mathbb{R}^2$, let $\mathrm{rot}(f)$ be the winding number of $df\frac{\partial}{\partial\theta}$ i.e. it is the element in $\pi_1(S^1) = \mathbb{Z}$ represented by $df\frac{\partial}{\partial\theta}$.

Theorem 1.2 (Whitney–Graustein). *If f_0 and f_1 are immersions such that $\mathrm{rot}(f_0) = \mathrm{rot}(f_1)$. Then there exists a continuous family of immersions $f_t : S^1 \looparrowright \mathbb{R}^2$ for $t \in [0,1]$ interpolating between f_0 and f_1.*

Definition 1.3. A formal immersion is a pair (f, F) where $f : S^1 \looparrowright \mathbb{R}^2$ is an immersion, and $F : TS^1 \to T\mathbb{R}^2$ is a bundle map covering F with $F \neq 0$ at any point.

Denote by $\mathrm{Imm}^{\mathrm{f}}$ the space of formal immersions of S^1 inside $\mathbb{R}^2$. It is easy to see that $\pi_0(\mathrm{Imm}^{\mathrm{f}})$ is isomorphic to $\mathbb{Z}$ via the map $(f, F) \mapsto \mathrm{rot}(F)$.

Lemma 1.4. *If $\mathrm{rot}(f_0, F_0) = \mathrm{rot}(f_1, F_1)$ then there exists a continuous family of formal immensions $(f_t, F_t) \in \mathrm{Imm}^{\mathrm{f}}$ interpolating between the two.*

Proof. Let $f_t = (1-t)f_0 + tf_1$ and let F_t be the homotopy between F_0 and F_1. Then (f_t, F_t) interpolates between (f_0, F_0) and (f_1, F_1). □

Using the language of formal immersions, we can rephrase Whitney–Graustein Theorem 1.2 as follows.

Theorem 1.5. *The map $\mathrm{Imm} \to \mathrm{Imm}^{\mathrm{f}}$ is surjective on π_0.*

Proof. Consider two maps f_0 and f_1 with $\mathrm{rot}(f_0) = \mathrm{rot}(f_1)$. Interpolate arbitrarily using a continuous family of functions g_t, with $g_0 = f_0$ and $g_1 = f_1$. Assume that g_t is C^∞-generic, so that we can assume

1. g_t fails to be an immersion precisely at finitely many "times" t's,
2. each of these times, it fails to be an immersion at finitely many points
3. at each of these points of failure, g_t is locally diffeomorphic to the following.[3] For the sake of argument, we will assume that a loop gets created at a singularity. The proof for the other direction is completely symmetric.

[3] To see this, using genericity we can assume that if the first derivative is 0 then then the second derivative is non-zero. The result then follows by looking at the Taylor expansion.

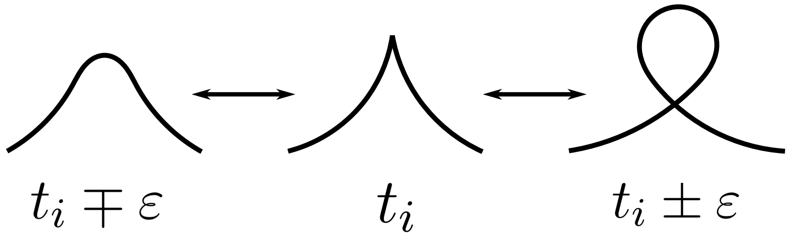

Figure 1: Generic singularities

Then the idea is that whenever a loop is created as in Figure 1, we replace the singularity by an immersion but in the process add an extra small loop to the immersion.

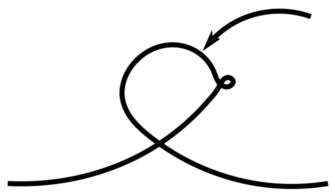

Figure 2: Add a small loop to the big loop in Figure 1.

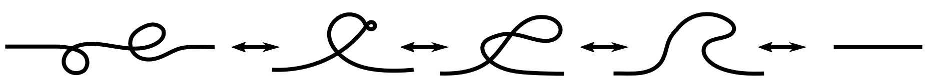

Figure 3: This allows to replace the singularity in Figure 1 by a family of immersions.

Repeatedly applying this process and removing all the singularities, we get a new family $\widetilde{f}_t$ which is always an immersion, but we can no longer expect this family $\widetilde{f}_1 = f_1$ because of the extra loops.

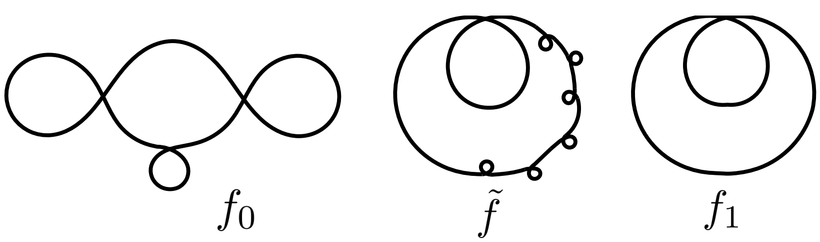

Figure 4: Extra loops on the modified family of immersions $\widetilde{f}_1 \neq f_1$.

So far we have not used the fact that $\mathrm{rot}(f_0) = \mathrm{rot}(f_1)$. If $h_1 : S^1 \to \mathbb{R}^2$ and $h_2 : S^1 \to \mathbb{R}^2$ are two immersions which differ by a single loop, then

$$\mathrm{rot}(h_1) = \mathrm{rot}(h_2) \pm 1$$

where the sign depends on how the extra loop is oriented. Because of the condition $\mathrm{rot}(f_0) = \mathrm{rot}(f_1)$, the number of small loops that get added with one orientation must equal the number of loops added with the other orientation. It does not matter where the loops are as we can always move them around and ensure that the orientations are alternating.

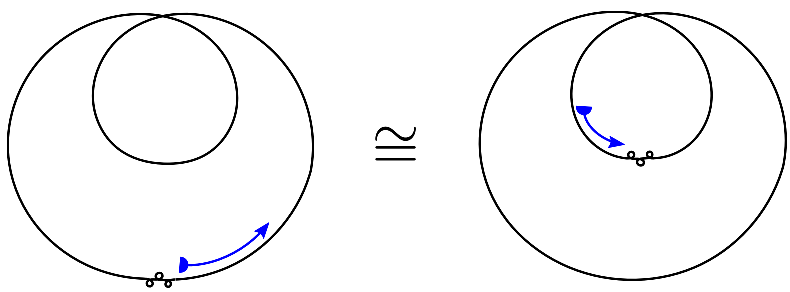

Figure 5: We can move the small loops around the immersion.

We can rearrange the loops so that they are alternating (in orientation) and then cancel the alternating small loops by reversing Figure 4.

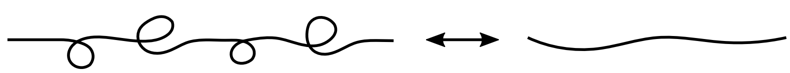

Figure 6: Cancelling the small loops takes us from $\widetilde{f}$ to f_1.

The entire process thus gives us a homotopy of immersions from f_0 to f_1. $\square$

This method is very specific to 1 dimensions and does not generalize very easily to the higher dimensions. In higher dimensions, the Whitney–Graustein Theorem generalizes to the Smale–Hirsch theorem which is an example of an h-principle.

Theorem 1.6 (Smale–Hirsch)**.** *Let M and N be smooth manifolds with $\dim M < \dim N$. Let $\mathrm{Imm}^{\mathrm{f}}(M, N)$ be the space of formal immersions (f, F), where F is a monomorphism of $TM \to TN$ lying over $f : M \to N$.*

$$\begin{tikzcd} TM \arrow[r, hook, "F"] \arrow[d] & TN \arrow[d] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

Then, the inclusion $\mathrm{Imm}(M,N) \to \mathrm{Imm}^{\mathrm{f}}(M,N)$ *is a weak homotopy equivalence.*

1.1 Exercises

1. Show that a generic singularity in a 1-parameter family of immersions is of the form Figure 1.
2. Using the Smale–Hisrch theorem, find the number of non-homotopic ways of immersing a surface Σ in $\mathbb{R}^3$.
3. Draw the 4 distinct immersions $S^1 \times S^1 \looparrowright \mathbb{R}^3$.

Solution to Q.2. We want to find the homotopy classes of maps of the form

$$\begin{tikzcd} T\Sigma \arrow[r, hook, "F"] \arrow[d] & T\mathbb{R}^3 \arrow[d] \\ \Sigma \arrow[r, "f"] & \mathbb{R}^3 \end{tikzcd}$$

Let $\nu = F^*(\mathbb{R}^3/T\Sigma)$ be a line bundle over Σ so that $T\Sigma \oplus \nu \cong \mathbb{R}^3 \times \Sigma$ as bundles over Σ. Any injection $F : T\Sigma \to T\mathbb{R}^3$ extends to a bundle isomorphism $\widetilde{F} : T\Sigma \oplus \nu \to T\mathbb{R}^3$. This is same as giving maps $\Sigma \to GL(3,\mathbb{R})$. In this case, maps are classified by

1. which connected component we are in, and
2. $H^1(\Sigma;\mathbb{Z}/2) \ni \widetilde{F}^*(\text{generator})$ where by generator we mean the generator of $H^1(GL(3,\mathbb{R});\mathbb{Z}/2) \cong \pi_1(GL(3,\mathbb{R})) \cong \mathbb{Z}/2$.

(In general, we need the group $H^k(\Sigma;\pi_k(Gl(n,\mathbb{R})/GL(m,\mathbb{R})))$.)

So the final answer is: $H^1(\Sigma;\mathbb{Z}/2)$.

2 Jet spaces

Idea: Jet spaces are a way to assign extra variables for derivatives. For example, the PDE $y'' = -y$ can be rewritten as $u' = -y$ where u is the derivative of y, see Example 2.12.

Let M be a smooth manifold, let $\pi : X \to M$ be a fiber bundle.

Example 2.1. The most common case that we'll be considering is
$X = N \times M \to M$.

Notation: For a subset S of a topological space X, we will let $\mathcal{O}\mathrm{p}(S)$ denote some open subset of X containing S.

The r^{th} jet space of X, denoted $\mathcal{J}^r(X)$, is intuitively the space of r^{th} derivatives of sections $M \to X$. More precisely, the jet space is a bundle over X

$$\pi : \mathcal{J}^r(X) \longrightarrow M$$

with fiber over $q \in M$ given by the space

$$\mathcal{J}^r(X)_q = \{ \text{ all local sections } \mathcal{O}\mathrm{p}(q) \to X\} \;/\; \sim$$

where the equivalence is given by saying that two sections are the same if they have the same r^{th} order Taylor polynomials at q.

Remark 2.2. This definition is not describing the sections of $\mathcal{J}^r(X)$ but rather just the fibers over each point.

Example 2.3. Let $M = \mathbb{R}$ and let $X = \mathbb{R} \times N$, then 1-jet space of X is given by

$$\mathcal{J}^1(X) = \mathbb{R} \times TN.$$

Example 2.4. If $X = M \times \mathbb{R}$ then $\mathcal{J}^1(X) = T^*M \times \mathbb{R}$. Note that sections of $\mathcal{J}^1(X)$ do not need to be exact i.e. a section of $\mathcal{J}^1(X)$ is a map $M \to \mathbb{R} \times T^*M$ and hence is a pair

$$(z : M \longrightarrow \mathbb{R}\,,\, \beta \in \Omega^1(M))$$

but we do not require $\beta = dz$.

An r-jet at a point $q \in M$ is represented by a local section $U_q \to X$. So to have an r-jet at every point is to have a section from $\mathcal{O}\mathrm{p}(\Delta) \to X$, where $\Delta \subseteq M \times M$ is the diagonal. Two sections are equal at a point $q \in M$ if, fixing q in the second factor, the r^{th} order Taylor expansions in the first M factor are equal. A section $M \to \mathcal{J}^r(X)$ being continuous/ C^k/ smooth etc. means that the section $Op(\Delta) \to X$ is continuous/ C^k/ smooth with respect to the second factor. In particular this defines the bundle structure on $\mathcal{J}^r(X)$ (and not merely the fibers).

Example 2.5. If $X = M \times N \to M$ then

$$\begin{aligned}\mathcal{J}^1(X) &= T^*M \otimes TN \\ &\cong \mathrm{Hom}(TM, TN).\end{aligned}$$

Notation: $\mathcal{J}^r(M, N) := \mathcal{J}^r(N \times M \to M)$.

Example 2.6.

$$\mathcal{J}^2(\mathbb{R}^n, \mathbb{R}) = \underbrace{\mathbb{R}^n}_{M} \times \underbrace{\mathbb{R}}_{\text{0-jets}} \times \underbrace{\mathbb{R}^n}_{\text{1-jets}} \times \underbrace{\mathbb{R}^{n(n+1)/2}}_{\text{2-jets}}$$

More generally,

$$\mathcal{J}^2(\mathbb{R}^n, \mathbb{R}^m) = \underbrace{\mathbb{R}^n}_{M} \times \underbrace{\mathbb{R}^m}_{\text{0-jets}} \times \underbrace{\mathbb{R}^{nm}}_{\text{1-jets}} \times \underbrace{\mathbb{R}^{mn(n+1)/2}}_{\text{2-jets}}$$

For any fibration $X \to M$ we get a tower of fibrations

$$\mathcal{J}^r(X) \longrightarrow \mathcal{J}^{r-1}(X) \longrightarrow \cdots \longrightarrow \mathcal{J}^1(X) \longrightarrow \mathcal{J}^0(X) = X \longrightarrow M$$

where each $\mathcal{J}^r(X) \to \mathcal{J}^{r-1}(X)$ fibration is an affine fibration. This is not a vector bundle but an affine fibration because there is no well-defined 0-section. One way to see this is that, the Hessian (second derivative matrix) is well-defined only when the first derivative vanishes. But even when the first derivatives do not vanish, if two functions have equal first derivative, then (only then) it makes sense to ask whether their second derivatives are equal (i.e. this in coordinate independant).

Given a section $f : M \to X$, we get a section

$$\mathcal{J}^r(f) : M \longrightarrow \mathcal{J}^r(X)$$

given by the k^{th} derivatives of f for all $k \leq r$.

Definition 2.7. If a section $\sigma : M \to \mathcal{J}^r(X)$ is equal to $\mathcal{J}^r(f)$ for some $f : M \to X$, then σ is called *holonomic*.

Note that if σ is holonomic then $\sigma = \mathcal{J}^r(p^0 \circ \sigma)$ where $p^0 : \mathcal{J}^r(X) \to \mathcal{J}^0(X)$ is the standard projection map. But, very importantly, not all sections are holonomic.

Example 2.8. In $\mathcal{J}^1(M, \mathbb{R}) = T^*M \times \mathbb{R}$, a section is a pair (z, β) where $z \in \Omega^0(M)$ and $\beta \in \Omega^1(M)$ and the pair (z, β) is holonomic if $\beta = dz$.

Example 2.9. For $\mathcal{J}^1(\mathbb{R}, N)$, a section is a pair (γ, v) where $\gamma : \mathbb{R} \to N$ is a smooth path and v is a section $\Gamma(\gamma^*TN)$ i.e. a vector field in N along γ. Then (γ, v) is holonomic precisely when $v = D\gamma$.

Definition 2.10. A *partial differential relation* on a bundle $X \to M$ of order r is a subset $\mathcal{R} \subseteq \mathcal{J}^r(M)$. A *formal solution* of $\mathcal{R}$ is a section σ of $\mathcal{J}^r(X)$ with image in $\mathcal{R}$. Denote the space of all formal solutions by $\mathrm{Sol}^{\mathrm{f}}(\mathcal{R})$. A (genuine) *solution* of $\mathcal{R}$ is a holonomic section $\sigma = \mathcal{J}^r(f) : M \to \mathcal{J}^r(X)$ such that σ has image in $\mathcal{R}$. Denote the space of all solutions by $\mathrm{Sol}(\mathcal{R})$.

Example 2.11. A PDE is an algebraic equation on the jet bundle $\mathcal{J}^r(X)$ and $\mathcal{R} = H^{-1}(0)$ where H is an algebraic equation involving the k^{th} derivatives for $k \leq r$.

Example 2.12. For $\mathcal{J}^2(\mathbb{R}, \mathbb{R}) = \mathbb{R}^4 = (t, y, u, w)$ where $t \in \mathbb{R}$, y is the 0-jet, u is the formal $\dfrac{dy}{dt}$ and w is the formal $\dfrac{d^2y}{dt^2}$, the differential equation $y'' = -y$ is simply the set $\mathcal{R} = \{(t, y, u, w) \mid w = -y\}$.

Definition 2.13. Let $\mathcal{R}$ be a PDR. We say that, for $\mathcal{R}$,

1. An *h-principle* holds if any formal solution can be homotoped to a genuine solution.
2. A *parametric h-principle* holds if the inclusion

$$\mathrm{Sol}(\mathcal{R}) \subseteq \mathrm{Sol}^{\mathrm{f}}(\mathcal{R})$$

   is a weak homotopy equivalence.
3. A *relative h-principle* holds if given any closed set $A \subseteq M$ and a formal solution $\sigma : M \to \mathcal{J}^r(X)$ which is holonomic on $\mathcal{O}p(A)$, we can deform σ to a genuine solution rel A.
4. A C^0*-dense h-principle* holds if for any $\sigma \in \mathrm{Sol}^{\mathrm{f}}(\mathcal{R})$ can be homotoped to a genuine solution $\mathcal{J}^r(f)$ so that $\left\| p^0(\sigma) - f \right\|_{C^0} < \varepsilon$, for any $\varepsilon > 0$.

Theorem 2.14 (Smale-Hirsch)**.** *Given M and N manifolds and $\mathcal{R}$ = "immersions" $\subseteq \mathcal{J}^1(M, N)$, if $\dim M < \dim N$ then a (parametric, relative, C^0-dense) h-principle holds for $\mathcal{R}$.*

Definition 2.15. A fiber bundle $X \to M$ is *natural* if there is a homomorphism

$$\mathrm{Diff}(M) \longrightarrow \mathrm{Diff}_M(X)$$

commuting with the projection map $X \to M$, where $\mathrm{Diff}_M(X)$ is the group of automorphisms φ of X that commute with the bundle map $X \to M$,

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr] & & X \arrow[dl] \\ & M. & \end{tikzcd}$$

Example 2.16. The trivial bundle $N \times M \to M$ is natural, with $\mathrm{Diff}(M) \longrightarrow \mathrm{Diff}_M(X)$ sending $\varphi \mapsto \mathrm{Id} \times \varphi$.

Example 2.17. TM and T^*M are natural as bundles over M by taking the differential.

Example 2.18. If $X \to M$ is natural, then so is $\mathcal{J}^r(X) \to M$.

Theorem 2.19 (Gromov)**.** *Suppose $X \to M$ is natural, and M is an open manifold. Let $\mathcal{R}$ be any PDR which is* $\mathrm{Diff}(M)$ *invariant and open. Then, a (parametric, relative) h-principle holds for $\mathcal{R}$.*

$\mathrm{Diff}(M)$ invariant simply means that we can express the relation in a coordinate independent format. C^0-density holds in a small neighborhood of the core of M but fails to hold over all of M as the isotopy that shrinks M to a small neighborhood of its core itself not C^0-close.

2.1 Exercises

1. $\mathcal{J}^r(\mathbb{R}^m, \mathbb{R}^n) \cong \mathbb{R}^{??}$.
2. (Jet transversality) Let $\pi : X \to M$ be a fibration and let Σ be a submanifold of $\mathcal{J}^r(X)$. Show that for a generic section $f : M \to X$ we have $\mathcal{J}^r(f) \pitchfork \Sigma$.
3. Suppose $\dim M = \dim N$ and suppose M is closed. Let $\mathcal{R}_{imm} \subseteq \mathcal{J}^1(M, N)$ be the PDR "immersions". Show that, in general, an h-principle fails for $\mathcal{R}_{imm}$.
4. Let $\mathcal{J}^1(\mathbb{R}_x, \mathbb{R}_y) = \mathcal{R}_x \times \mathcal{R}_y \times \mathcal{R}_u$ where u is the "formal derivative" dy/dx. Let $\sigma : \mathcal{R} \to \mathcal{J}^1(\mathcal{R}, \mathcal{R})$ be a section defined as $\sigma(x) = (y(x), u(x)) = (0, 1)$. Show that there does not exist a holonomic section $\mathcal{J}^1(f) : \mathcal{R}_x \to \mathcal{J}^1(\mathbb{R}, \mathbb{R})$ with $\|\sigma - \mathcal{J}^1(f)\|_{C_0}$ small.
5. Let $\pi : X \to M$ be a fibration. Show that there are local trivializations

$$\begin{tikzcd} \mathcal{J}^r(M) \arrow[r, "\pi"] & M \\ \mathbb{R}^k \times U \arrow[u, hook] \arrow[r, "\pi"] & U \arrow[u, hook] \end{tikzcd}$$

so that every constant section $U \to \mathbb{R}^k \times U$ is holonomic.

3 Holonomic approximation

Theorem 3.1 (Holonomic Approximation, Eliasherg–Michachev)**.** *Let $X \to M$ be a manifold bundle and let $A \subseteq M$ be a submanifold (or stratified by submanifolds) so that* $\operatorname{codim} A \geq 1$. *Let*

$$F : \mathcal{O}\mathrm{p}(A) \to \mathcal{J}^r(X)$$

be a formal section. Then, for all $\varepsilon, \delta > 0$, there exists an isotopy

$$\varphi_t : M \to M$$

with φ_t uniformly δ-close to identity and a holonomic section

$$\widetilde{F} : \mathcal{O}\mathrm{p}(\varphi_1(A)) \to \mathcal{J}^r(X)$$

such that $\left\| F - \widetilde{F} \right\|_{C^0} < \varepsilon$. (The relative and parametric versions are also true.)

Example 3.2. Consider $\mathcal{J}^1(\mathbb{R}^2, \mathbb{R})$ and a section $\mathbb{R}^2 \to \mathcal{J}^1(\mathbb{R}^2, \mathbb{R})$ defined as $F(x, y) = (x, 0, 0)$. Let A be the x-axis in $\mathbb{R}^2$. It is not possible to approximate F by a holonomic section in a neighborhood of A, essentially by the Mean Value Theorem, as, if $F(x, y)$ is δ-close to $(x, 0)$ then $\partial_x F(x, y)$ is δ-close to 1 and hence cannot be ε-close to 0.

But we can perturb A to $A' = \left\{ \left(x, \dfrac{\sin(2\pi N x)}{N} \right) \right\}$ via a linear isotopy. Suppose the curve A' has length ℓ. Then on A' we can define the function $f(x) = x/\ell$, where we are identifying A' with $[0, \ell]$. The new function f has a smaller gradient which can be made as close to 0 as want by increasing N.

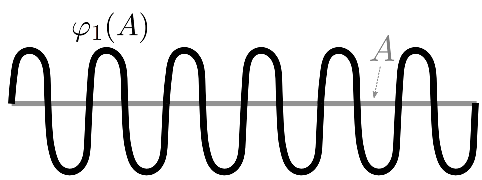

Figure 7: C^0-close perturbation of A

We are avoiding having problems with the Mean Value theorem because of the fact that A' has much longer length than A.

Proof of the h-principle for open $\mathrm{Diff}(M)$*-invariant PDRs on open manifolds.* If M is an open manifold, then there exists $A \subseteq M$ of codimension ≥ 1 such that there exists an isotopy ψ_t of M relative to A, where $\psi_0 = \mathrm{Id}$ and $\psi_1(M) = \mathcal{O}\mathrm{p}(A)$.

Let $F : M \to \mathcal{J}^r(X)$ be a formal solution of $\mathcal{R}$. Let $\widetilde{F} : \mathcal{O}\mathrm{p}(\varphi_1(A)) \to \mathcal{J}^r(X)$ be a holonomic approximation of F. $\widetilde{A}$ has image inside $\mathcal{R}$, because $\mathcal{R}$ is open. Let $G = \widetilde{F} \circ \varphi_1 \circ \psi_1$. The function G is defined on all of M. G is also a solution of the PDR, by $\mathrm{Diff}(M)$-invariant. There are homotopies (φ_t, ψ_t) which take G to $\widetilde{F} \circ \varphi_0 \circ \psi_0$ via formal solutions to F on A. □

Proof of Holonomic Approximation 3.1. Since holonomic approximation holds in the relative version, it suffices to prove it for $M = [0,1]^k \subseteq \mathbb{R}^n$ for $n > k$. Because we can "triangulate" our manifold and break into a union of "cubes". Think induction on cells using dimension. It suffice to prove holonomic approximation for $F : \mathcal{O}\mathrm{p}([0,1]^k) \to \mathcal{J}^r(\mathbb{R}^n, \mathbb{R})$. We do this by induction on the "directions" of $[0,1]^k$.

Definition 3.3. Let $F : M \to \mathcal{J}^r(X)$ be a section, and let $N \subseteq M$ be a submanifold. We say that F is holonomic along N, if there exists a holonomic section $\widetilde{F} : \mathcal{O}\mathrm{p}(N) \to \mathcal{J}^r(X)$ such that $\widetilde{F}|_N = F|_N$.

Example 3.4. For the previous example, with $\mathcal{J}^1(\mathbb{R}^2, \mathbb{R})$ with $F(x,y) = (x, 0, 0)$ is not holonomic, but it IS holonomic along the set $\{x = c\}$ where c is a constant, by setting $\widetilde{F}(x,y) = (c, 0, 0)$

Definition 3.5. If $F : M \to \mathcal{J}^r(X)$ and $p : M \to B$ is a fiber bundle, we say that F is fiberwise holonomic if it is holonomic along all fibers $p^{-1}(b)$ for $b \in B$.

Example 3.6. For example, if $p : M \to M$ is the identity map then every section is a fiberwise holonomic section.

If F is fiberwise holonomic, then for every $b \in B$ we get $\widetilde{F}_b : \mathcal{O}\mathrm{p}(\pi^{-1}(b)) \to \mathcal{J}^r(X)$ so that $\widetilde{F}_b(q) = F(q)$ is holonomic whenever $q \in \pi^{-1}(b)$. (Think of $\widetilde{F}$ as Taylor polynomials.) But it is not true that $\widetilde{F}_b(q) = F(q)$ when $p(q) \neq b$, unless we are holonomic. $\widetilde{F}$ is now a smooth function $\mathcal{O}\mathrm{p}(\text{graph of } p) \to \mathcal{J}^r(X)$ where graph of p is a subset of $M \times B$. F is holonomic, if $\widetilde{F}_b(q) = \widetilde{F}_{b'}(q)$ for all b, b' wherever defined.

Inductive step: Let $p : [0,1]^k \to [0,1]^{k-\ell}$ be the coordinate projection. If $F : \mathcal{O}\mathrm{p}([0,1]^k) \to \mathcal{J}^r(\mathbb{R}^n, \mathbb{R})$ is fiberwise holonomic, then there exists φ_t, $\widetilde{F}$ so that $\widetilde{F} : \mathcal{O}\mathrm{p}(\varphi_1([0,1]^k)) \to \mathcal{J}^r(\mathbb{R}^n, \mathbb{R})$ and $\widetilde{F}$ is fiberwise holonomic with respect to $[0,1]^k \to [0,1]^{k-(\ell+1)}$.

Let N be a very large integer, and let $\widetilde{F} : \mathcal{O}\mathrm{p}(\gamma_p) \to \mathcal{J}^r(\mathbb{R}^n, \mathbb{R})$, only look at $\widetilde{F}_i/N$ where $\widetilde{F}_i/N$ is defined on $\mathcal{O}\mathrm{p}([0,1]^k \times \{i/N\})$ and choose N large enough so that these domains cover the entire cube $[0,1]^k$.

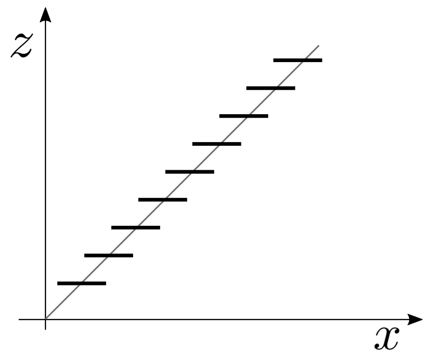

Figure 8: For example, if $F(x,y) = (x,0,0)$

Our goal is to glue these together to be holonomic, but still uniformly close. The only way to glue is using cutoff functions, but picking naive cutoff functions will result in large derivatives. As $k < n$, there is an additional direction in which we can perturb.

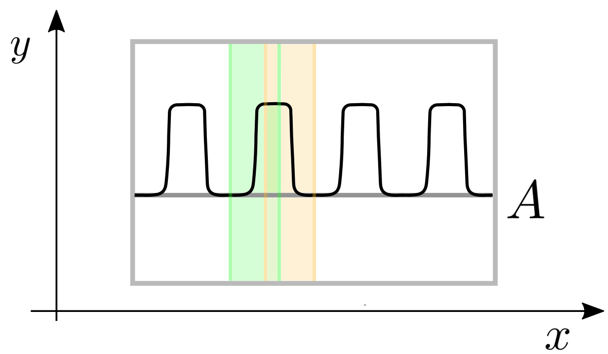

Figure 9: The function $F(x,y,z) = (x,0,0)$ glued using cutoff functions.

Choose a cutoff in the x_n-direction where x_n transverse to $[0,1]^k$. Let $\widehat{F}_{i/N} = \mathcal{J}^r(f_{i/N})$ and $\widehat{F}_{i+1/N} = \mathcal{J}^r(f_{i+1/N})$ where $\widetilde{f} = \chi(x_n) \cdot f_{i/N} + (1 - \chi(x_n)) \cdot f_{i+1/N}$ defined on $\mathcal{O}\mathrm{p}([0,1]^{k-l} \times \{i/N\})$. All derivatives of χ are bounded by δ^{-r}. $\square$

3.1 Exercises

1. (a) Let $\begin{array}{ccc} TM & \xrightarrow{F} & TN \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & N \end{array}$ be a formal immersion. Define a vector bundle ("formal normal bundle") $p : E(\nu) \to M$ with total with dimension

equal to $\dim N$ which has a formal immersion

$$\begin{tikzcd} T(E(\nu)) \arrow[r, "\widehat{F}"] \arrow[d] & TN \arrow[d] \\ E(\nu) \arrow[r, "f \circ p"] & N \end{tikzcd} .$$

(b) Using either Gromov's h-principle or holonomic approximation theorem, prove Smale–Hirsch Theorem 2.14.

2. (a) Let M be any open smooth manifold. Show that M admits a Riemannian metric g, such that all sectional curvatures are in $(c - \varepsilon, c + \varepsilon)$ where $c \in \mathbb{R}$ and $\varepsilon > 0$.

(b) Let M be an orientable open 3-manifold. Show that there is an immersion $M \looparrowright \mathbb{R}^3$. (**Hint:** Wu formula.)

(c) Let $A \subseteq M$ be a submanifold of codim ≥ 1. Let $\beta \in \Omega^k(M)$ and let $a \in H^k(A; \mathbb{R})$. Show that there exists an isotopy φ_t and $\alpha \in \Omega^k(\mathcal{O}\mathrm{p}(\varphi_1(A)))$ such that $\left\| \beta|_{\mathcal{O}\mathrm{p}(\varphi_1(A))} - \alpha \right\|_{C^0}$ is small and $d\alpha = 0$ and $[\alpha] = a \in H^k(A)$.

(d) Let ξ be a k-plane distribution on N (i.e. a k-plane $\xi_p \subseteq T_pN$ at each point $p \in N$, smoothly varying in p). Let M be a manifold with $\dim M < \dim N - k$. Prove an h-principle for immersions $f : M \looparrowright N$, satisfying $Df(TM) \cap \xi = \{0\}$ everywhere. (**Hint:** First prove this when M is open.)

(e) Let N be a complex manifold with $\dim_{\mathbb{R}} N = 2n$ and suppose M is an open smooth manifold with $\dim M \geq n$. Prove an h-principle for immersions $f : M \looparrowright N$ with the property that $Df(TM) \subseteq TN$ contains no $\mathbb{C}$-lines.

(f) Same, but let M be closed and $\dim M \leq n - 1$.

4 Convex integration

Gromov's h-principle can only be applied to open manifolds or to manifolds with positive codimension. Convex integration can be applied to closed manifolds, but with stronger hypotheses on $\mathcal{R}$.

Example 4.1. Consider maps $\mathbb{R} \to \mathbb{R}^2$ with the property that $\|\gamma'\| = 1$. Suppose we have a function $f : \mathbb{R} \to \mathbb{R}^2$ such that $\|f'\| \leq 1$. We can C^0 perturb f to have norm 1. We can achieve this by going around in circles rapidly.

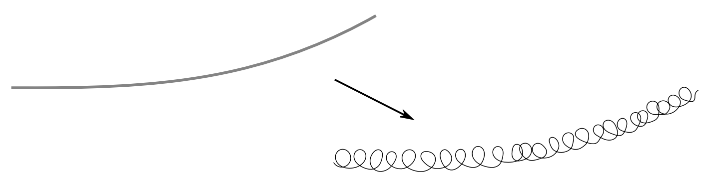

Figure 10: C^0-approximation of a map $f : \mathbb{R} \to \mathbb{R}^2$ with norm $\|f'\| \leq 1$ by one with norm $\|f'\| = 1$.

Suppose we have $\mathcal{R} \subseteq \mathcal{J}^1(\mathbb{R}, N)$ in each fiber $(t, x) \in \mathbb{R} \times N$ with $\mathcal{R}_{(t,x)} = \mathcal{R} \cap TN_{(t,x)} = \pi^{-1}(t, x)$ where

$$\begin{aligned} \pi : \mathcal{J}^1(\mathbb{R}, N) &\longrightarrow \mathcal{J}^0(\mathbb{R}, N), \\ \mathbb{R} \times TN &\longmapsto \mathbb{R} \times N. \end{aligned}$$

If $F_{(t,x)} \in \mathcal{R}_{(t,x)}$ then define $\operatorname{Conv}_{F(t,x)} \mathcal{R}_{(t,x)}$ to be the convex hull of the connected component of $\mathcal{R}_{(t,x)}$ containing $F_{(t,x)}$.

Definition 4.2. A formal solution (f, F) of $\mathcal{R} \subseteq \mathcal{J}^1(\mathbb{R}, N)$ is called *short* if f' is in $\operatorname{Conv}_{F(t)} \mathcal{R}_{(t,f(t))}$.

Theorem 4.3 (Gromov, 1-dim convex integration)**.** *Given an open set $\mathcal{R} \subseteq \mathcal{J}^1(\mathbb{R}, \mathbb{R}^n)$, any formal short solution is homotopic (among formal short solutions) and C^0 close to a genuine solution. Also, the relative version holds.*

Proof. First, subdivide $\mathbb{R}$ into intervals such that (f, F) lives in single chart of N. Only need to consider $\mathcal{R} \subseteq \mathcal{J}^1(\mathbb{R}, \mathbb{R}^n)$. Further suffices to consider the case when $f \equiv 0 \in \mathbb{R}^n$, (if necessary by shifting the function f) so that $\operatorname{Conv}_{F(t,x)} \mathcal{R}_{(t,z)} \ni 0$ for all (t, z). By shriking, if necessary, we can assume that $\mathcal{R}_{(t,z)}$ is a fixed $R \subseteq T\mathbb{R}^n_{(t,z)}$.

Claim: We can find a loop $\gamma : [0, 1] \to R$ so that $\int_0^1 \gamma dt = 0$ with basepoint $= F_{(0,0)}$.

Since $0 \in \operatorname{Conv}_{F(t,z)} R$, we can find points $a_1, \ldots, a_k$ in the connected component of $\mathrm{Conv}_{F(t,z)} R$ containing 0 such that

$$\begin{aligned} c_1 a_1 + \cdots + c_k a_k &= 0, \\ c_1 + \cdots + c_k &= 1, \end{aligned}$$

for some positive constants $c_j \in (0, 1]$.[4]

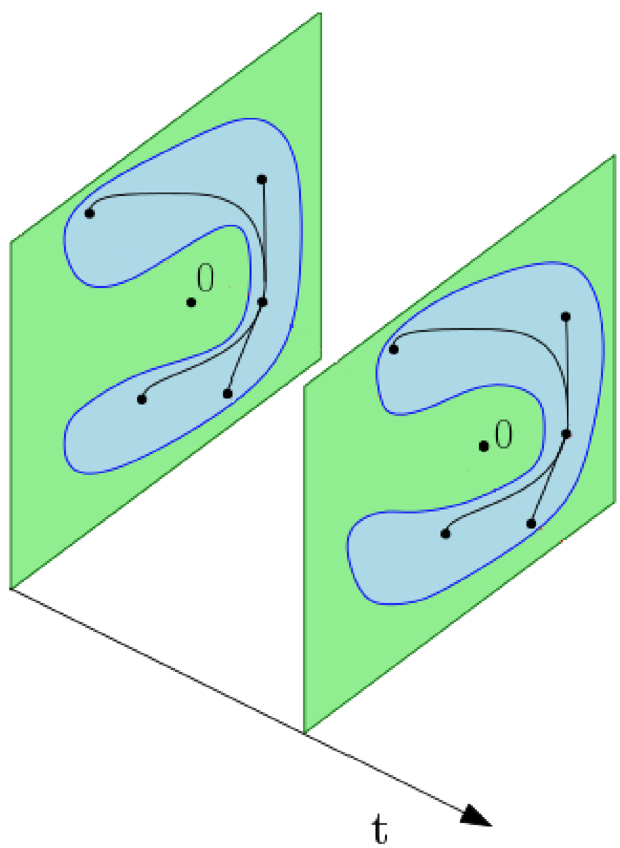

Figure 11: Loops in the convex hull averaging to 0. Note that even if the relation may slightly change in a small neighborhood, the path can be chosen as constant due to the openness condition.

Let γ be a loop which is constant at a_j and stays there for time c_j, appropriately smoothened. Let $\gamma^{(n)}$ be the N-fold concatenation of γ, appropriately time-

[4]Note to self: this is where we need convexity.

rescaled so that $\gamma^{(N)}$ is a map $[0,1] \to R$. Then for any $t \in [0,1]$,

$$\left\| \int_0^t \gamma^N \, dt \right\| \leq \frac{1}{N} \sup_t \int_0^t \gamma \, dt,$$

which can be made less than any $\varepsilon > 0$. Now, let $\widetilde{f}(t) = \int_0^t \gamma^{(N)} \, dt$ which is the required solution. □

4.1 Parametric 1-dimensional convex integration

The above proof can be easily made parametric, but more is true.

Given a family of short formal solutions $(f_\eta, F_\eta) : \mathbb{R} \to \mathcal{J}^1(\mathbb{R}, N)$ for $\eta \in D^k$, we can find a family of genuine solutions $\widetilde{f}_\eta$ which additionally satisfy

$$\frac{\partial \widetilde{f}_\eta}{\partial \eta} \approx_{C^0} \frac{\partial f_\eta}{\partial \eta} \tag{4.1}$$

Indeed, since $\int_0^1 \gamma_\eta \, dt = 0$,

$$\begin{aligned} & \int_0^1 \frac{\partial \gamma_\eta}{\partial \eta} \, dt = \frac{\partial}{\partial \eta} \int_0^1 \gamma_\eta \, dt = 0 \\ \implies \quad & \left\| \int_0^t \frac{\partial \gamma_\eta^{(N)}}{\partial \eta} \, dt \right\| < \frac{1}{N} < \varepsilon \\ \implies \quad & \frac{\partial \widetilde{f}_\eta}{\partial \eta} < \frac{1}{N} < \varepsilon. \end{aligned}$$

Definition 4.4. Let $R \subseteq V$ be a subset of an affine space. We say that R is *ample* if the convex hull of any connected component of R is V.

Note that the empty set $R = \varnothing$ is ample. Convex integration is not very useful for linear PDRs as the corresponding set in the jet space is not ample.

Definition 4.5. Let $R \subseteq \mathrm{Mat}_{n \times k}$ with $k \leq n$. We say that R is *ample*, if, after fixing any $(k-1)$ columns $A \in \mathrm{Mat}_{n \times (k-1)}$, the collection of vectors $v \in \mathbb{R}^n$ such that $[A; v] \in R$ is ample as a subset of $\mathbb{R}^n$.

Remark. Ampleness for subsets of $\mathrm{Mat}_{n \times k}$ is not the same concept as ampleness for vector spaces, i.e. if we think of $\mathrm{Mat}_{n \times k}$ as a vector space and forget that it consists of matrices, different sets will be ample.

Example 4.6. The set of R consisting of "full rank matrices" is ample if $k < n$ but not if k=n.

Example 4.7. If $R \subseteq \mathrm{Hom}(\mathbb{R}^k, \mathbb{C}^n)$ with $k \leq n$, the set of complex linearly independence, R is ample.

Theorem 4.8 (Convex integration)**.** *Let $\mathcal{R} \subseteq \mathcal{J}^1(X)$ be an open and ample differential relation i.e. $\mathcal{R}_{(q,x)} = \pi^{-1}(q,x)$ is ample, where $\pi : \mathcal{J}^1(X) \to \mathcal{J}^0(X)$. Then a full h-principle holds for $\mathcal{R}$ (parametric, relative, C^0-dense).*

Proof. We'll prove the relative h-principle directly. Because the h-principle is relative, it suffices to prove the h-principle for $\mathcal{R} \subseteq \mathcal{J}^1(\mathbb{R}^m, \mathbb{R}^n)$, in which case a formal solution is $(f, F_1, \ldots, F_m)$. Apply 1-dimensional convex integration parametrically. We get $(\widetilde{f}, \frac{\partial \widetilde{f}}{\partial q_1}, F_2, \ldots, F_n)$. Do the same for F_2 to get $(\widetilde{\widetilde{f}}, \frac{\partial \widetilde{f}}{\partial q_1}, \frac{\partial \widetilde{\widetilde{f}}}{\partial q_2}, \ldots, F_n)$. This might not be holomonic in the q_1 direction. But by observation (4.1) we have $\frac{\partial \widetilde{f}}{\partial q_1} \approx^{C^0} \frac{\partial \widetilde{\widetilde{f}}}{\partial q_1}$ and so $(\widetilde{\widetilde{f}}, \frac{\partial \widetilde{\widetilde{f}}}{\partial q_1}, \frac{\partial \widetilde{\widetilde{f}}}{\partial q_2}, \ldots, F_n)$ is holomonic in the first two directions. □

This gives us solutions to open sets, which are PDRs but not PDEs. To get solutions to PDEs we find an approximate solution being approximated by ε, and by taking a limit of solutions as $\varepsilon \to 0$ we can try to get a solution to the original PDE. But the limits might not exist, and when they exist they might have low regularities.

4.2 Exercises

1. Let N be a complex manifold with real dimension $2n$ and let M be a smooth manifold of dimension $\leq n$. An immersion $f : M \looparrowright N$ is called *totally real* if

$$Df(TM) \cap JDf(TM) = \{0\}$$

where J is multiplication by i.

   (a) Prove a full h-principle for totally real immersions.

   (b) Let $g : M \hookrightarrow N$ be a smooth embedding and for $s \in [0,1]$ let

$$\begin{array}{ccc} TM & \xrightarrow{G_s} & TN \\ \downarrow & & \downarrow \\ M & \xrightarrow{g} & N \end{array}$$

   be a homotopy of bundle monomorphisms, with $G_0 = dg$ and $G_1(TM) \subseteq TN$ totally real. Show that g is isotopic to $\widetilde{g} : M \hookrightarrow N$, a totally real embedding.(**Hint:** Consider $\mathcal{R}_{\text{totally real}} \subseteq \mathcal{J}^1(\nu(g(M)))$ and note than any section is automatically an embedding.)

   (c) Prove that there exists a totally real embedding $S^3 \hookrightarrow \mathbb{C}^3$.

2. (a) Let $\mathcal{R} \subseteq \mathcal{J}^1(S^1, \mathbb{R}^2)$ be the relation

$$\mathcal{R} = \left\{ v \in T\mathbb{R}^2 \mid v \cdot \partial/\partial y > -\varepsilon\,|v| \right\}$$

for some fixed $\varepsilon > 0$. Prove an h-principle for $\mathcal{R}$.

(b) Given the formal solution $(f, F) = (e^{i\theta}, \partial/\partial y)$ find a genuine solution $\widetilde{f}$ which is C^0-close to f.

3. Suppose $\dim M = 2n + 1$. Prove an h-principle for 1-forms $\lambda \in \Omega^1(M)$ satisfying $\dim \ker(d\lambda) = 1$, where $d\lambda$ is thought of as a map $d\lambda : TM \to T^*M$.

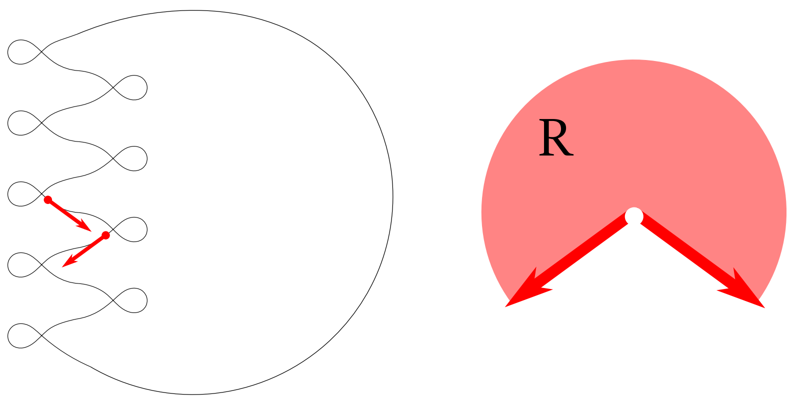

Figure 12: **Solution to Q.2 b)**

5 Nash-Kuiper embedding theorem

For the sake of convenience, assume that the source manifold V is *compact*. We also need (dimension of target) - (dimension of source) > 1.

Theorem 5.1 (Nash-Kuiper embedding theorem). *Let $f : (V, g) \to (W, h)$ be a* short *C^∞ map, i.e. $\|Df(v)\|_h < \|v\|_g$ for all $v \in TV$. Then f can be C^0-approximated by a C^1-map $\overline{f}$ such that $\left\|d\overline{f}(v)\right\|_h = \|v\|_g$ for all $v \in TV$.*

This statement cannot be true for C^2-embeddings because of curvature considerations. But curvature obsructions only exist when f is C^2.

To say f is short is to say that $f^*h < g$ i.e. $\Delta = g - f^*h$ is *positive definite*.

Definition 5.2. Given two metrics $\widetilde{g}$ and g on V, define

$$
\begin{aligned}
r(\widetilde{g}, g) : TV \setminus V &\longrightarrow \mathbb{R}_{\geq 0} \\
r(\widetilde{g}, g)(v) &:= \frac{\|v\|_{\widetilde{g}}}{\|v\|_g}
\end{aligned}
$$

Note that this still makes sense when $\widetilde{g}$ is only semi-definite.

Definition 5.3. Given two maps $f, \widetilde{f} : (V, g) \to (\mathbb{R}^q, h)$ define the distance between them as

$$
\begin{aligned}
d(f, \widetilde{f}) : TV \setminus V &\longrightarrow [0, \infty) \\
d(f, \widetilde{f})(v) &:= \frac{\left\|Df(v) - D\widetilde{f}(v)\right\|_h}{\|v\|_g}
\end{aligned}
$$

We will only be considering distance between maps which are C^0-close and we only use the distance to make sense of maps getting closer to each other, so we can use the above definition to talk about a converging sequence of maps $f_i : (V, g) \to (W, h)$.

Cauchy convergence: Suppose $f_i : (V, g) \to (W, h)$ is a sequence of C^∞ maps, such that $d(f_i, f_{i+1}) < c_i$ with $\sum c_i < \infty$. Suppose $f_i \underset{C^0}{\longrightarrow} \overline{f}$. Then $\overline{f}$ is C^1 with $f_i \underset{C^1}{\longrightarrow} \overline{f}$. (This is because of the way the distance function $d(f, \widetilde{f})$ is defined.)

A semi-metric g on $\mathbb{R}^n$ is *primitive* if

$$g_x(v, v) = a(x) \cdot (\ell(v))^2$$

where $\ell : \mathbb{R}^n \to \mathbb{R}$ is a linear functional and $0 \leq a \in C_c^\infty(\mathbb{R}^n)$ is a non-negatively valued compactly supported smooth function. A semi-metric on V is *primitive* if it is supported on a chart, and primitive on that chart.[5]

[5]This is where we need compactness. If V is not compact, then we allow a primitive function to be non-zero on multiple charts, as long as the charts are non-overlapping.

Proposition 5.4. *Any metric on a compact manifold V can be written as the sum of finitely many primitive semi-metrics.*

Proof. Take an open cover, take a partition of unity and multiply the two to get a locally (compactly) supported metric. On a single chart, any semi-metric can be decomposed as a sum of primitive metrics as follows: Pick a basis of orthogonal/null vectors $e_1(x), \dots, e_n(x)$. Define $a_i(x) = g_x(e_i, e_i)$. Let $v = \sum_i \ell_i(v) e_i$, then

$$\begin{aligned} g_x(v,v) &= g_x(\sum_i \ell_i(v)e_i, \sum_i \ell_i(v)e_i) \\ &= \sum_i g_x(\ell_i(v)e_i, \ell_i(v)e_i) \\ &= \sum_i \ell_i(v)^2 g_x(e_i, e_i) \\ &= \sum_i a_i(x)\ell_i(v)^2. \end{aligned}$$

□

Important: Further when we write $g = \sum_{i=1}^N \alpha_i$ with α_i primitive, we can ensure that N does not depend on g. Essentially $N = \dim V \cdot$ (number of charts on V).

Definition 5.5. A map $f : (V, g) \to (W, h)$ is called ε-isometric if

$$(1 - \varepsilon)g < f^* h < (1 + \varepsilon)g$$

By standard theory of convex integration, we immediately get an h-principle for ε-isometric immersions. The relation of being ε-isometric is itself is not ample, but instead we need to invoke the notion of "short" solutions for higher dimensional manifolds. Essentially, not every map can be perturbed to an ε-isometric immersion but a *short* map can be.

Theorem 5.6 (Approximation lemma)**.** *Suppose $f : (V, h) \to (W, h)$ is a short map. For constants $\delta, \varepsilon > 0$, we can C^0-approximate f by an ε-isometric immersion $\widetilde{f}$ so that*

$$d(f, \widetilde{f}) < Nr(\Delta, g) + \delta$$

where $\Delta = g - f^* h$, the function r is as defined above and N is the same N as in $g = \sum_{i=1}^N \alpha_i$ with α_i primitive.

Proof of Nash-Kuiper using the Approximation lemma. Assume, after invoking Smale-Hirsch if necessary, that f is an immerison.

Choose a sequence of positive real numbers $\delta_i > 0$ with $\sum \delta_i < \infty$, and a sequence ρ_i with the following properties

1. $\rho_i \to 1$,

2. ρ_i is increasing,

3. $\sqrt{\rho_1} + \sqrt{\rho_2 - \rho_1} + \sqrt{\rho_3 - \rho_2} + \cdots < \infty$.

Let $g_i = f^*h + \rho_i \Delta$ (interpolation between $g = g_1$ and f^*h). Note that f is short for all g_i. Think of f as a map $f = f_0 : (V, g_1) \to (W, h)$.

We will use the Approximation lemma to construct an f_1 satisfying

$$\begin{aligned} d(f_0, f_1) &< Nr(\rho_1 \Delta, g_1) + \delta_1 \\ &\leq Nk\sqrt{\rho_1} r(\Delta, g) + \delta_1 \end{aligned}$$

where k is a constant such that $k^2 f^* h > g$. (This is why we needed h to be an immersion.)

Apply Approximation lemma 5.6 to $f_1 : (V, g_2) \to (W, h)$ (for $\varepsilon < \rho_1 - \rho_2$, f_1 short) to get an f_2 satisfying

$$\begin{aligned} d(f_1, f_2) &< Nr((\rho_2 - \rho_1)\Delta, g_2) + \delta_2 \\ &< Nk\sqrt{\rho_2 - \rho_1} r(\Delta, g_2) + \delta_2. \end{aligned}$$

We can also ensure that $f_2 : (V, g_3) \to (W, h)$ is short, as $f_2^* h \approx_\varepsilon g_2 < g_3$. Continuing this way, we get a sequence of functions f_i such that

$$\begin{gathered} f_i \xrightarrow[C^0]{} \overline{f} \\ f_i \xrightarrow[C^1]{} \overline{f} \end{gathered}$$

The second statement follows from the Cauchy Criterion for convergence. □

Proof of the Approximation lemma. Start with the 1-dimensional case.

Claim: A short function $f : [0,1] \to (\mathbb{R}^q, h)$ can be C^0-approximated by an ε-isometry $\widetilde{f}$ such that $d(f, \widetilde{f}) < r(\Delta, g) + \delta$.

Proof of Claim. We have a short map f i.e. at each point $\left\| df \frac{\partial}{\partial t} \right\| < 1$.

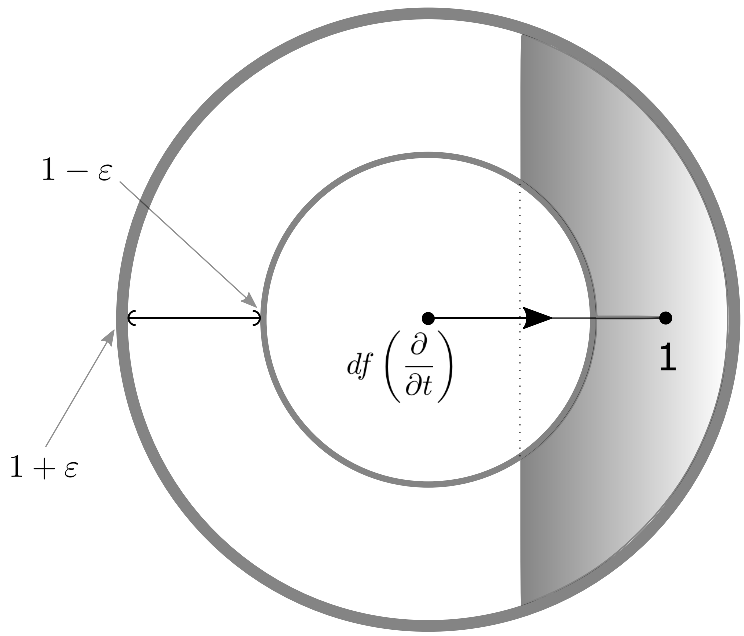

Figure 13: The shaded region describes the relation $\mathcal{R}$.

Consider now the vectors v that satisfy

$$|1 - \|v\|| < \varepsilon,$$
$$\langle v, df\frac{\partial}{\partial t}\rangle > \left\| df\frac{\partial}{\partial t} \right\|^2 - \delta.$$

Such v define a new differential relation $\mathcal{R}$ and $df\frac{\partial}{\partial t}$ is still in the convex hull for this relation $\mathcal{R}$, and so we can still find a genuine solution to $\mathcal{R}$ close to f using convex integration.

This gives us an $\widetilde{f} : [0,1] \to (\mathbb{R}^q, h)$. It remains to show that $d(f, \widetilde{f}) < r(\Delta, g) + \delta$. Expanding this out, we get

$$\left\| df\frac{\partial}{\partial t} - d\widetilde{f}\frac{\partial}{\partial t} \right\|_h < \left\| 1 - df\frac{\partial}{\partial t} \right\|_h + \delta.$$

This follows from triangle inequality. □

Everything in the above proof holds parametrically where, as before, we ensure $\frac{\partial \widetilde{f}}{\partial \eta} \approx_{C^0} \frac{\partial f}{\partial \eta}$. We decompose Δ into a sum of N primitive semi-metrics, and let $\Delta = \sum_j a_j \ell_j^{\otimes 2}$ and apply the above proof to a parametric family for each $a_j \ell_j^{\otimes 2}$. □

6 Directed immersions/embeddings

Consider smooth manifolds M^m, N^n. Define a fiber bundle $\mathrm{Gr}_m(TN) \to N$ to be the "bundle of m-planes in TN" with fiber over each point being isomorphic to the m-plane Grassmannian $\mathrm{Gr}_m(\mathbb{R}^n)$ = the space of m-planes in $\mathbb{R}^n$, which is isomorphic to $O(n)/O(n-m) \times O(m)$. Given a subset $A \subseteq \mathrm{Gr}_m(TN)$, this defines a relation $\mathcal{R}_A \subseteq \mathcal{J}^1(M,N) \to N \times M$ as

$$\mathcal{R}_A = \{F \in \mathrm{Hom}(TM, TN) \mid F(TM) \in A\}$$

Example 6.1. "Totally real" $= A \subseteq \mathrm{Gr}_m(\mathbb{C}^n)$.

In this context, ampleness is given by the following proposition.

Proposition 6.2. *Let $q \in N$ and let $P \in \mathrm{Gr}_{m-1}(T_qN)$. Define,*

$$\Omega(P,q) := \left\{ v \in T_qN/P \text{ such that } P \oplus \mathbb{R}v \in A \right\}.$$

If $\Omega(P,q)$ is ample for all (P,q) then $\mathcal{R}_A$ is ample.

Q. Let M be a closed manifold of $\dim M < n = \dim_{\mathbb{C}} N$. Prove an h-principle for totally real immersion $M \looparrowright N$.

Let us start with a formal solution on M. M need not be open, so we might not be able to apply h-principle to such an M. We need to replace M by an open manifold. One candidate for this is $M \times (-\varepsilon, \varepsilon)$.

Let $\widetilde{M} = M \times (-\varepsilon, \varepsilon)_s$ and this is open, so we use an h-principle for $\widetilde{M}$ (restriction of a totally real embedding to a submanifold is still totally real).

Suppose $\dim M = n - 1$. By assumption, we have a formal solution with $F(TM)$ containing no complex lines i.e. $F(TM) \cap JF(TM) = \{0\}$. Can we get a formal solution in $\widetilde{M}$?

To define

$$\begin{array}{ccc} T\widetilde{M} = TM \oplus \mathbb{R}_{\partial s} & \xrightarrow{\widetilde{F}} & TN \\ \downarrow & & \downarrow \\ \widetilde{M} & \xrightarrow{\quad\widetilde{f}\quad} & N \end{array}$$

we need $\widetilde{F}(\partial_s)$ to be $\mathbb{C}$-linearly independent from $F(TM) \oplus JF(TM)$ i.e. a non-zero section of $TN/F(TM) \oplus JF(TM)$. But such a solution might not exist. So even though we have an h-principle, a formal solution might not exist.

First, let $M_0 = M \setminus$ closed disk. Then, we can apply the h-principle to M_0 to get a genuine solution $\widetilde{f}_0 : M_0 \to N$. $(\widetilde{f}_0, d\widetilde{f}_0)$ is homotopic to $(f|_{M_0}, df|_{M_0})$ through formal solutions. Hence, $\widetilde{f}_0$ extends to a formal solution (g, G) on all of M with $g|_{M_0} = \widetilde{f}_0$ and $G|_{M_0} = d\widetilde{f}_0$ and $(g, G) \sim (f, F)$.

We apply the previous strategy to the closed ball, relative to the boundary. Let $\widetilde{B} =$ closed disk $\times (-\varepsilon, \varepsilon)$. Then, there is a formal solution

$$\begin{array}{ccc} T\widetilde{B} & \xrightarrow{\widetilde{G}} & TN \\ \downarrow & & \downarrow \\ \widetilde{B} & \xrightarrow{\widetilde{g}} & N \end{array}$$

extending (g, G) i.e. $TN/G(TB) \oplus JG(TB)$ is a trivial bundle over $B \times (-\varepsilon, \varepsilon)$ and hence has a non-zero section. Hence, we can apply the relative h-principle to $\widetilde{B}$ relative to the boundary.

7 Symplectic geometry

7.1 Motivation from classical mechanics

Let $q \in \mathbb{R}^n$ denote the position, and let F be a force field on $\mathbb{R}^n$ given by the gradient $-\nabla u$ of a potential function $u : \mathbb{R}^n \to \mathbb{R}$. The variable $p = \dot{q}/m \in \mathbb{R}^n$ is the "momentum of q". The Hamiltonian/energy is defined as the function

$$H(q, p) = u(q) + \frac{1}{2m} \|p\|^2 .$$

Then

$$\begin{aligned} \frac{\partial H}{\partial p} &= \frac{1}{m} p = \dot{q}, \\ -\frac{\partial H}{\partial q} &= -\nabla u = \dot{p}. \end{aligned}$$

The set of vectors $z = q + \sqrt{-1}p \in \mathbb{C}^n$ is called the phase space. Denote by $J : T\mathbb{C}^n \to T\mathbb{C}^n$ multiplication by $\sqrt{-1}$. Then the above two equations can be expressed more compactly as

$$\dot{z} = -J\nabla H$$

This gives us **conservation of energy** by noticing that $-J\nabla H \perp \nabla H$. We can replace H with any function, which will then be the energy function. The vector field corresponding to H is

$$X_H = -J\nabla H.$$

Q. What geometry do we need to define X_H from H?
We want $X_H = -J\nabla H$, which gives us $JX_H = \nabla H$. For any vector $v \in T\mathbb{C}^n$, we get

$$\langle JX_H, v \rangle = dH(v).$$

Let $\omega(w, v) = \langle Jw, v \rangle$. Then, we only need to know ω to define the vector field X_H.

Q. What properties should ω have? We need ω to be a section of

$$\begin{tikzcd} TM \otimes TM \arrow[d, shift right] \\ \mathbb{R} \arrow[u, dashed, shift right, bend right, "\omega"'] \end{tikzcd}$$

1. ω should be non-degenerate. That is, if we define $\omega^* : TM \to T^*M$, where $\omega^*(X_H) = dH$, then we want ω^* to be an isomorphism.

2. ω should be alternating i.e. $\omega(u, v) = -\omega(v, u)$. To see this note that

$$X_H(H) = dH(X_H) = \omega(X_H, X_H)$$

For conservation of energy, we want $X_H(H) = 0$ and hence $\omega(X_H, X_H) = 0$, which is equivalent to saying that ω is alternating. So, ω is a 2-form on M.

3. $d\omega = 0$ as a 3-form on M. We want the Lie derivative of ω with respect to a Hamiltonian vector field to vanish i.e. $0 = \mathscr{L}_{X_H}(\omega)$ as "classical mechanics should be invariant under flowing along a Hamiltonian". Using Cartan's formula for the Lie derivative we get

$$\mathscr{L}_X(\omega) = (d\omega)(X_H, -) + d(\omega(X_H, -)) = d\omega(X_H, -)$$

Because X_H can be any vector field, the only way this is zero if $d\omega = 0$.

7.2 Symplectic manifolds

Definition 7.1. On a smooth manifold M, a 2-form $\omega \in \Omega^2(M)$ is *symplectic* if it is

1. non-degenerate,
2. closed.

Basic linear algebra tells us that $\dim M$ is even. Further, ω non-degenerate implies that $\omega^{\wedge n} = \omega \wedge \cdots \wedge \omega \in \Omega^{2n}(M)$ is non-vanishing everywhere, where $2n = \dim M$.

Example 7.2. Consider $\mathbb{C}^n$, equipped with standard Hermitian metric

$$h(v, w) = \langle v, w \rangle + \sqrt{-1}\, \omega(v, w).$$

Then ω is symplectic. In standard coordinates, $\omega = \sum_{i=1}^n dx_j \wedge dy_j$.

Example 7.3. Any Kahler manifold is a symplectic manifold. Moreover, if $M \subseteq \mathbb{C}^N$ or $M \subseteq \mathbb{CP}^N$ holomorphically, then $\omega_{std}|_M$ is symplectic. Because $d\omega_{std}|_M = 0$, and $\omega_{std}(v, Jv) > 0$ if $v \neq 0$ giving us non-degeneracy.

Example 7.4. Let Q be any smooth manifold, and let $M = T^*Q$. There is a tautological 1-form $\lambda_{std} \in \Omega^1(T^*Q)$ defined by the property that:

1. for any section $\sigma : Q \to T^*Q$, we have $\sigma^*(\lambda_{std}) = \sigma$,
2. $\lambda_{std} = 0$ along the fibers.

Let $(q_1, \ldots, q_n)$ be local coordinates on Q and let $p_j = dq_j$ be coordinates on fibers of T^*Q, then locally

$$\lambda_{std} = \sum_j p_j dq_j.$$

Let $\omega_{std} = d\lambda_{std} \in \Omega^2(T^*Q)$, then ω is symplectic, which in local coordinates is given by $\omega_{std} = \sum_j dp_j \wedge dq_j$.

Example 7.5. Let Q be equipped with a metric. Consider the norm function $H(q,p) = \|p\|^2/2$. If $\gamma : \mathbb{R} \to T^*Q$ satisfies $\dot{\gamma}(t) = X_H(\gamma(t))$ then γ is a geodesic (with respect to parallel transport). Hence, X_H is a vector field on T^*Q giving rise to (co)-geodesic flow. (**Exercise.**)

More generally, if $H = \frac{1}{2}\|p\|^2 + U(q)$ then the resulting trajectories satisfy

$$\frac{\partial}{\partial t}[\dot{\gamma}] = -\nabla\gamma.$$

Q. What are the symmetries of (M, ω)?

Definition 7.6. The group of *symplectomorphisms* is

$$\mathrm{Symp}(M,\omega) = \{\varphi \in \mathrm{Diff}(M) \mid \varphi^*\omega = \omega\}.$$

If $\varphi_t \in \mathrm{Diff}(M)$ with $\varphi_0 = \mathrm{Id}$ what does $\dot{\varphi}_t$ need to satisfy to ensure that $\varphi_t \in \mathrm{Symp}(M,\omega)$? This is same as requiring that

$$\begin{aligned} 0 &= \mathscr{L}_{\dot{\varphi}_t}\omega \\ &= \varphi_t^*((d\omega)(\dot{\varphi}_t, -) + d(\omega(\dot{\varphi}_t, -))). \end{aligned}$$

So $\varphi_t \in \mathrm{Symp}(M,\omega)$ if and only if $\omega(\dot{\varphi}_t, -) \in \Omega^1(M)$ is closed for all t.

Remark 7.7. One can show that $\mathrm{Symp}(M,\omega) \subsetneq \mathrm{Diff}(M)$ is both infinite dimensional and infinite codimensional.

Q. We do not know what $\pi_0(\mathrm{Symp}_c(\mathbb{C}^n, \omega_{std}))$ is. **Big open problem.**

We do know that for $n = 1, 2$ this group is trivial i.e. all compactly supported symplectomorphisms are path-connected to the identity morphism.

If $H(x,t) : M \times \mathbb{R} \to \mathbb{R}$ is smooth, then we have a (time-dependent) vector field X_{H_t} defined as $\omega(X_H, -) = -dH_t$.[6]

Definition 7.8. The symplectomorphism $\varphi_t \in \mathrm{Symp}(M,\omega)$ given by $\dot{\varphi}_t = X_{H_t}$ is called a *Hamiltonian flow* and X_{H_t} is called a *Hamiltonian vector field*. $\mathrm{Ham}(M,\omega) \subseteq \mathrm{Symp}(M,\omega)$ is called the *Hamiltonian diffeomorphism group*.

Exercise. Write a Hamiltonian function for $\varphi_{H_t} \circ \varphi_{G_t}$.

Remark 7.9. This is where symplectic geometry differs from Riemannian geometry, in that there are a lot of symmetries of symplectic manifolds whereas symmetry groups of Riemannian manifolds are usually small, finite dimensional. In this sense, symplectic geometry is closer to topology than geometry, the symplectic manifolds are a bit *squishy*.

[6]We need "-" signs in symplectic geometry, but there is no universally accepted conventions about where to put the "-" signs.

Theorem 7.10 (Moser's theorem)**.** *If (M, ω_t) is a family of symplectic structures with the cohomology class $[\omega_t] \in H^2(M)$ fixed and ω_t is compactly supported, then there exists a family of diffeomorphisms $\varphi_t \in \mathrm{Diff}(M)$ such that $\varphi_t^*(\omega_0) = \omega_t$.*

Corollary 7.11 (Darboux's theorem)**.** *Any symplectic manifold is locally symplectomorphic to a neighborhood of 0 inside $(\mathbb{C}^n, \omega_{std})$.*

Theorem 7.12 (Gromov)**.** *Let M be any open manifold. Symplectic structures on M satisfy a complete h-principle in any fixed cohomology class $[\omega] \in H^2(M)$.*

Proof. Assume that we are working with the trivial class $[0] \in H^2(M)$ (if not we can always shift the class to 0 by subtracting).

$\mathcal{R}_{non-deg} \subseteq \mathcal{J}^1(T^*M)$ defined by $d\lambda$ non-degenerate, for a 1-form λ, is an *open* PDR on M. Given a formal solution (i.e. non-degenerate $\eta \in \Omega^2(M)$) we can find a λ defined near codim ≥ 1 skeleton of M such that $d\lambda \approx_{C^0} \eta$, hence $d\lambda$ is non-degenerate and hence symplectic on $\mathcal{O}\mathrm{p}($ deformed skeleton$)$. But the deformed skeleton is diffeomorphic to M. ☐

7.3 Exercises

1. Complete the proof of Moser's theorem 7.10.
2. (Generalized Weinstein neighborhood theorem) Let (M_1, ω_1), (M_2, ω_2) be symplectic and let $N_1 \subseteq M_1$ and $N_2 \subseteq M_2$ be smooth submanifolds. Given a pair of maps (f, F),

$$\begin{tikzcd} TM_1|_{N_1} \arrow[r, "F"] \arrow[d] & TM_2|_{N_2} \arrow[d] \\ N_1 \arrow[r, "f"] & N_2 \end{tikzcd}$$

   where f is a diffeomorphism, F is a bundle isomorphism satisfying $F^*\omega_2 = \omega_1$ and

$$F|_{TN_1} = Df : TN_1 \to TN_2 \subseteq TM_2|_{TN_2}.$$

   Show that there exist neighborhoods $U_1 \supseteq N_1$ and $U_2 \supseteq N_2$ and $\varphi : U_1 \to U_2$ a symplectomorphism, so that $\varphi(N_1) = N_2$, $\varphi|_{N_1} = f$, and $D\varphi|_{TU|_{N_1}} = F$.
3. (a) Show that the previous problem implies Darboux's theorem.

   (b) Let $L \subseteq (M, \omega)$ be Lagrangian. Show that L has a neighborhood symplectomorphic to a neighborhood of the zero section in T^*L.
4. Let $\varphi_t \in \mathrm{Ham}(T^*S^1)$ be a Hamiltonian flow and let $Z \subseteq T^*S^1$ be the zero section $Z = \{q \in S^1, p = 0\}$. Show that $\varphi_1(Z) \cap Z \neq \varnothing$.[7]

[7] $L \subseteq (M, \omega)$ is called *isotropic* if $\omega|_L \cong 0$. This implies that $\dim L \leq \dim M/2$. If $\dim L = \dim M/2$, then L is called *Lagrangian*.

8 Symplectic embeddings & immersions

Let (M, ω) be a symplectic manifold.

Definition 8.1. An embedding or an immersion of the smooth manifold $f : N \to M$ is called *symplectic* if $f^*\omega$ is symplectic.

It suffices to say that $f^*\omega$ is non-degenerate as a closedness condition is always true.

Definition 8.2. If additionally, (N, ω_N) is symplectic, we say that $f : N \to M$ is *isosymplectic* if $f^*\omega_M = \omega_N$.

In general, there is no h-principle for symplectic embeddings, no matter how large the codimension is, but there is an h-principle for isosymplectic embeddings.

Definition 8.3. Let (N, ω_N) and (M, ω_M) be symplectic. A *formal isosymplectic immersion* is a pair (f, F),

$$\begin{array}{ccc} TN & \xrightarrow{F} & TM \\ \downarrow & & \downarrow \\ N & \xrightarrow{f} & M \end{array}$$

so that f is smooth and $F^*\omega_M = \omega_N$[8] and

$$f^*[\omega_M] = [\omega_N] \in H^2(N; \mathbb{R}).$$

Similarly, a *formal isosymplectic embedding* is a pair (f, F_s) for $s \in [0,1]$,

$$\begin{array}{ccc} TN & \xrightarrow{F_s} & TM \\ \downarrow & & \downarrow \\ N & \xrightarrow{f} & M \end{array}$$

such that

1. f is a smooth embedding,
2. $F_0 = Df$,
3. $F_1^*\omega_M = \omega_N$,
4. F_s is a bundle monomorphism for all $s \in [0,1]$,
5. $f^*[\omega_M] = [\omega_N] \in H^2(N; \mathbb{R})$.

Theorem 8.4. *Let (N, ω_N) and (M, ω_M) be symplectic. Suppose either*

- *N is open and $\dim N \leq \dim M - 2$, or*
- *N is closed and $\dim N \leq \dim M - 4$.*

[8]This implies that F is a monomorphism as both ω_M and ω_N are non-degenerate.

There is a full h-principle for isosymplectic embeddings and immersions.

Definition 8.5. A submanifold $L \hookrightarrow (M, \omega_N)$ is called *isotropic* if $\omega_M|_L \equiv 0$ if $\dim L < \dim M/2$.

Corollary 8.6. *There is a full h-principle for isotropic embeddings and immersions.*

Proof. Apply Theorem 8.4 to T^*L. $\square$

Remark 8.7 (Embeddings of $S^3 \to (\mathbb{C}^3, \omega_{std})$)**.** When ω is Kahler, being Lagrangian means that $TL \perp JTL$. The space of formal Lagrangian embeddings is homotopy equivalent to the space of formal totally real embeddings (i.e. $TL \cap JTL = \{0\}$). It is easy to see that there exists a formal Lagrangian embedding $S^3 \to (\mathbb{C}^3, \omega_{std})$, but there does not exist a Lagrangian embedding of $S^3 \to \mathbb{C}^3$. And so, h-principle for Lagrangian embeddings fails.

Proof of Theorem 8.4. We will work in the open case: N is open, and $\dim N \geq \dim M + 2$. First, let's work with symplectic embeddings (not isosymplectic). There is an h-principle for this. Now we have $\widetilde{f} : N \to M$ so that $\widetilde{f}^*\omega_M$ is symplectic, and $\widetilde{f}^*\omega_M$ is formally homotopic to ω_N, i.e. there exists $\eta_t \in \Omega^2(N)$ such that η_t is non-degenerate and $\eta_0 = \widetilde{f}^*\omega_M$ and $\eta_1 = \omega_N$. Also, $[\widetilde{f}^*\omega_M] = [\omega_N]$.

Using the parametric h-principle for symplectic structures on N, we can find a family $\omega_t \in \Omega^2 N$ such that

1. ω_t is symplectic,
2. $\omega_0 = \widetilde{f}^*\omega_M$
3. $\omega_1 = \omega_N$ with $[\omega_t]$ constant.

We want to find an isotopy $g_t : N \to M$ such that $g_0 = \widetilde{f}$ and $g_1^*\omega_M = \omega_1$ (more generally, $g_t^*\omega_M = \omega_t$). This is where we need an extra dimension.

The Weinstein Neighborhood Theorem (Exercises 7.3 2) tells us that $\mathcal{O}p(\widetilde{f}^*(N))$ is symplectomorphic to a symplectic vector bundle

$$\begin{array}{ccc} \mathbb{C}^{m-n} & \longrightarrow & (E, \omega_E) \\ & & \big\downarrow \pi \\ & & N \end{array}$$

with $\omega_E = \pi^*\widetilde{f}^*\omega_M \oplus \omega_{fiber}$ where $\omega|_{fiber} = \omega_{std} = \sum dx \wedge dy$. Note that $\mathcal{O}p_M(\widetilde{f}^*(\omega_N))$ is symplectomorphic to $\mathcal{O}p_E(Z)$ where Z is the zero section of E.

First, $\omega_t = \omega_0 + d\lambda_t$, for $\lambda_t \in \Omega^1 N$. For now, make the following assumptions:

$$\lambda_t = t \cdot r\, ds, \qquad r, s \in C^\infty(N, \mathbb{R}) \tag{8.1}$$

$$E \text{ is trivial as a vector bundle} \tag{8.2}$$

$$\omega_E = \omega_M = \omega_0 + d(\textstyle\sum x_j dy_j) \tag{8.3}$$

$$\|r\|, \|s\| \leq \text{ normal radius of } E \tag{8.4}$$

Let $g_t(p) = (p, \sqrt{t} \cdot (r(p) + \sqrt{-1}s(p)))$ i.e. a section of E (into $\mathbb{C} \times \{0\} \subseteq \mathbb{C}^{m-n}$). Then $g_t^* \sum x_j dy_j = \lambda_t$ so $g_t^* \omega_M = \omega_t$ which gives us $g_1^* \omega_M = \omega_N$, which completes the proof.

It remains to justify the four assumptions. Rationale behind Assumption (8.4): for any R, ε, there exists an immersion

$$i : D^2(R) \looparrowright D^2(\varepsilon)$$

such that $i^*(\omega_{area}) = \omega_{area}$ (see Figure 14).

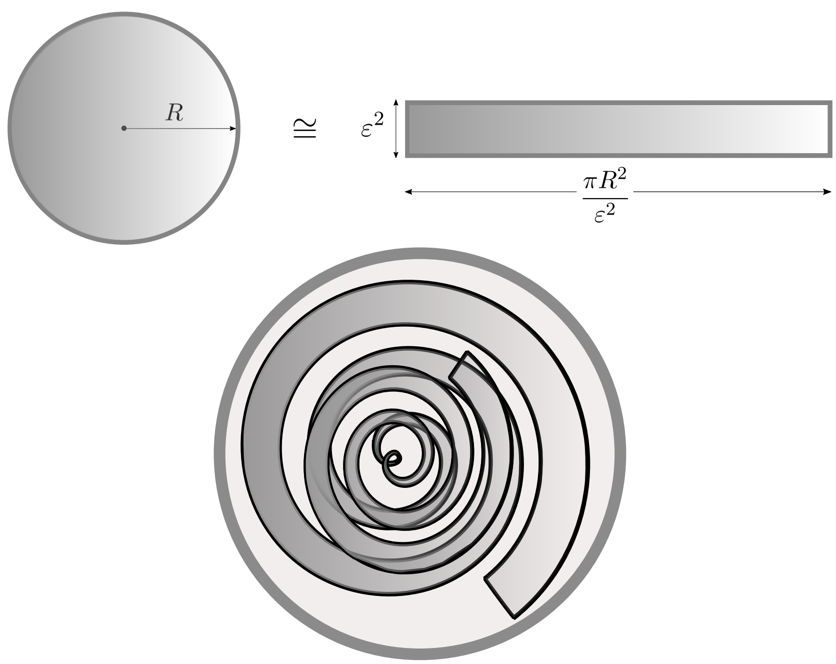

Figure 14: First map the disk of radius R to a band with width ε^2, and then wrap the band around inside a disk of radius ε. All of these maps are area preserving immersions.

Rationale behind Assumptions (8.1), (8.2): $\lambda_t = t \cdot r\, ds$. First, an arbitrary λ_t can be written as a locally finite sum of primitive $r\, ds$ such that r, s can be supported on a single chart with compact closure on N. Piecewise linear in time can be arranged in a C^1_t-time way. $\square$

Along the way, we proved an h-principle for symplectic embeddings/immersions of open manifolds. But no such h-principle exists for compact manifolds. We know this to be false by explicit examples: $\mathbb{CP}^2\#\mathbb{CP}^2\#\mathbb{CP}^2$ cannot be given a symplectic structure (but it has a "formal symplectic structure").

8.1 Lagrangian submanifolds

If a submanifold $L \subseteq (M, \omega)$ satisfies $\omega|_L \equiv 0$ then $\dim L \leq \dim M/2$. For such an L, if $\dim L = \dim M/2$ then L is called *Lagrangian*.

Example 8.8. If $M = T^*Q$ and λ is the tautological 1-form then $\omega = d\lambda$ is symplectic. For a 1-form $\sigma \in \Omega^1(Q)$ we have $\sigma^*\lambda = \sigma$. If $L = \sigma(Q) \subseteq T^*Q$ is the graph of σ, then $\omega|_L = \sigma^*\omega = d\sigma$. Hence, L is Lagrangian if and only if σ is closed.

In an intuitive sense, a Lagrangian submanifold is a submanifold for which we don't have Heisenberg uncertainty. For $M = T^*Q$, $L = \{p = 0\}$ (you can know the position to arbitrary accuracy) or $L = T^qQ$ for $q \in Q$ (you can know the momentum to arbitrary accuracy). Or if $L = \{p_1 = 0, q_2 = 0\}$.

Consider $f \in \mathrm{Diff}(M)$. Consider a symplectic manifold $(M \times M, \omega_1 \oplus (-\omega_1))$. Consider the graph of f,

$$\Gamma_f = \{(x, f(x)) \mid x \in M\}$$

Then f is a symplectomorphism if and only if Γ_f is Lagrangian. Often dynamics of symplectic manifolds is interesting.

Q. If $\varphi \in \mathrm{Ham}(M, \omega)$, what are the fixed points of φ?

Conjecture 8.9 (Arnol'd conjecture, Floer theorem)**.** *For a compact manifold M, if all fixed points of φ are non-degenerate, then the number of fixed points of φ is at least* $\dim H_*(M; \mathbb{F})$ *where $\mathbb{F}$ is any field.*

In comparison, the Lefschetz fixed point theorem says that the number of fixed points of φ is at least $\chi(M)$. The bound given by Arnol'd conjecture is in general much bigger than $\chi(M)$, thus in general Hamiltonian symplectomorphisms have many more fixed points than arbitrary diffeomorphisms.

Notice that the fixed points of φ are exactly $\Gamma_\varphi \cap \Delta \subseteq M \times M$ where $\Delta = \Gamma_{\mathrm{Id}}$. The fact that Hamiltonian diffeomorphisms have many fixed points is saying that that Lagrangians are forced to intesect in more points that topology would predict. This is in some sense saying that there are formal Lagrangians which cannot be approximated by genuine Lagrangians, and hence h-principles cannot hold for Lagrangian embeddings. (h-principles are saying that the geometry and topology are very close to each other, which is not the case for Lagrangian embeddings.)

Let $f_t : L \to (M, \omega)$ be a family of Lagrangian embeddings. Using Cartan's formula we can see that $d(\omega(\dot{f}_t, -)|_L) \cong 0$ i.e. $[\omega(\dot{f}_t, -)]$ defines an element in $H^1(L)$. Call this the *flux* of f_t.

Proposition 8.10. *With the above notation, if the flux of f_t is identically 0, then there exists a Hamiltonian isotopy $\varphi_t : M \to M$ such that $\varphi_t \circ f_0 = f_t$.*

Proof. We can assume that locally, $\mathcal{O}p(f_0(L)) \cong \mathcal{O}p_{T^*L}(Z)$, $f_t(L) = \sigma_t(L)$, where Z is the 0-section of T^*L, for some 1-form $\sigma_t \in \Omega^1(L)$ and $d\sigma_t = 0$ for all t. Flux of $\sigma_t = [\sigma_t] \in H^1(L)$. If this is 0, then $\sigma_t = dH_t$ for $H_t \in C^\infty(T^*L)$. This then gives us that X_{H_t} is just fiberwise translation by dH_t. □

Consider the case when $L = S^1$. If $\varphi_t(Z)$ is a Hamiltonian flow, then the flux is 0, where Z is the zero section of T^*S^1. For $[\gamma] \in H_1(L)$,

$$\int_0^t \mathrm{flux}(f_t(L))[\gamma] = \int_{\gamma \times [0,1]} \omega$$

which implies that

$$\varphi_1(Z) \cap Z \geq 2.$$

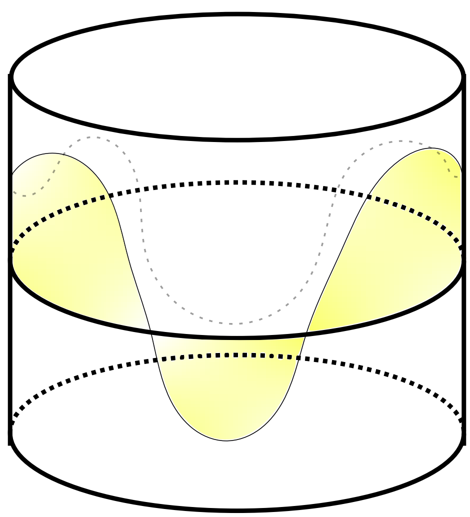

Figure 15: Hamiltonian flow of the zero-section on T^*S^1.

For a general T^*Q, let $Z = \{p = 0\}$ and consider the graph of an exact 1-form: $L = \{p = \sigma\}$ where $\sigma \in \Omega^1 Q$ is exact, $\sigma = dH$. (Both Q and L compact.) Then $L \cap Z =$ critical points of H. If all the critical points of H are non-degenerate, then by Morse theory the number of critical points is at least $H_*(L)$.

Aside: Morse theory says that we can determine the homology of a manifold L by looking at the critical points of a Morse function $H : L \to \mathbb{R}$.

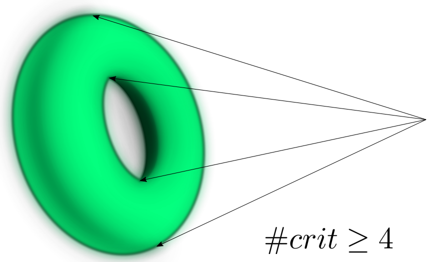

Figure 16: The height function on a generic torus is Morse.

Theorem 8.11 (Lagrangian Arnol'd's conjecture, Floer's theorem)**.** *If $L \subseteq T^*Q$ is any Lagrangian which is Hamiltonian isotopic to the zero-section Z and $L \pitchfork Z$, then the size of $L \cap Z$ is at least* $\dim H_*(Q)$.

Note that there exits a *formal* Lagrangian isotopy $(f_t, F_{s,t}) : S^3 \hookrightarrow T^*S^3$ so that $f_1(S^3) \cap Z = \varnothing$, where Z is the zero-section. But Floer's theorem prevents such a formal Lagrangian isotopy from being a genuine Lagrangian isotopy.

If U is an open subset of M, and V is an open subset with $\overline{V} \subseteq U$. If $\varphi_t : V \to U$ is a Hamiltonian isotopy, then there exists a global Hamiltonian diffeomorphism on M, $\Phi_t : M \to M$ such that $\Phi_t|_V = \varphi_t$ i.e. Hamiltonian diffeomorphisms admit cutoffs (cutoff H). This is false about symplectic flows as closed 1-forms cannot be extended to closed 1-forms, but exact 1-forms can be extended to exact 1-forms.

Hamiltonian flows can be extremely non-intuitive. Consider the (open) manifold $(0,2) \times (0,1) \subseteq \mathbb{R}^2$ with the standard symplectic structure and let V be the subsets $V = (\varepsilon, 1-\varepsilon) \times (\varepsilon, 1-\varepsilon)$ and $V' = (1+\varepsilon, 2-\varepsilon) \times (\varepsilon, 1-\varepsilon)$. For every $\varepsilon > 0$, there exists a Hamiltonian flow of the ambient manifold that takes V to V', an example of which is seen in the following Figure 17.

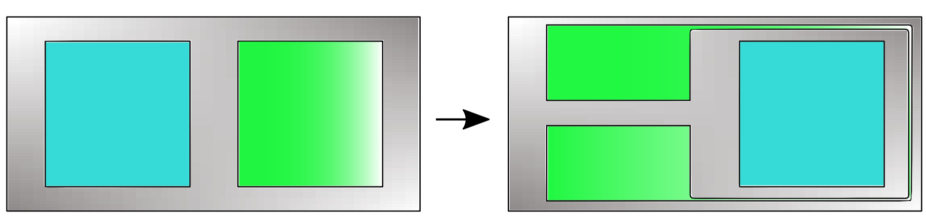

Figure 17: A Hamiltonian flow that takes V to V' (blue regions)

9 Contact geometry

Contact geometry is the "odd dimensional counterpart" of symplectic geometry.

Definition 9.1. Let M^{2n+1} be a smooth manifold. A *contact structure* on M is a hyperplane field $\xi^{2n} \subseteq TM$, which is maximally non-integrable i.e. there does not exist manifolds of dimension $\geq n+1$ tangent to ξ even up to 2^{nd} order at any point.

Maximal non-integrability is equivalent to saying that if $\xi = \ker \alpha$ for a 1-form $\alpha \in \Omega^1(M)$ then $\alpha \wedge (d\alpha)^n \neq 0$. $d\alpha$ is a linear symplectic structure on ξ, but if we replace α with $\widetilde{\alpha} = e^f \alpha$, then $d\widetilde{\alpha}|_\xi = e^f d\widetilde{\alpha}|_\xi$. Hence, $d\alpha$ is well defined up to conformal scaling. If $\Lambda \subseteq M$ is a manifold with $T\Lambda \subseteq \xi$, then $\alpha|_\Lambda = 0$. $d\alpha|_\Lambda = 0$ implies that $T\Lambda \subseteq \xi$ is isotropic and hence $\dim \Lambda \leq n$.

Example 9.2. $\mathbb{R}^{2n+1}, \xi_{std} = \ker(dz - \sum_{j=1}^n y_j dx_j)$.

Proposition 9.3 ("Moser's theorem", Gray stability)**.** *Given a family of contact structures ξ_t which are constant outside a compact set, there exists a family of diffeomorphisms $\varphi_t \in \mathrm{Diff}(M)$ with $D\varphi_t(\xi_0) = \xi_t$.*

Note that there is no stability theorem for 1-forms but there is one for plane fields, and hence we are interested in studying plane fields instead of 1-forms.

Definition 9.4. A smooth map $\varphi : (M_1, \xi_1) \to (M_2, \xi_2)$ is called a *contactomorphism* if $D\varphi(\xi_1) = \xi_2$, i.e. if $\xi_j = \ker \alpha_j$ then $\varphi^*\alpha_2 = e^f \alpha_1$ for some $f \in C^\infty(M)$.

The Lie algebra of the contactomorphism group $\mathfrak{con}(M, \omega)$ is isomorphic to $C^\infty(M)$. To see this, note that given a family of contactomorphisms φ_t the vector field $\dot{\varphi}_t = X$ satisfies the equation $(d\alpha)(X, -)|_\xi = -dH|_\xi$, where $H = \alpha(X)$ and $\xi = \ker \alpha$.

Definition 9.5. The *Reeb vector field* associated to α is the unique vector field $\mathcal{R}_\alpha$ satisfying $(d\alpha)(\mathcal{R}_\alpha, -) \cong 0$ and $\alpha(\mathcal{R}_\alpha) = 1$.

The Reeb vector field depends on the choice of α and not just the hyperplane field ξ. For a vector field X, if $H = \alpha(X)$ then $(d\alpha)(X, -) = -dH + \mathcal{R}_\alpha(H)\alpha$.

Proposition 9.6 (Generalized Weinstein neighborhood theorem)**.** *For a smooth submanifold $N \subseteq (M, \xi)$, the contactomorphism type of a tubular neighborhood $\mathcal{O}\mathrm{p}(N)$ of N is completely determined by $\xi|_N$.*

In particular, if $\Lambda \subseteq M$ is *Legendrian* (isotropic and $\dim \Lambda = n$) then an open neighborhood $\mathcal{O}\mathrm{p}(\Lambda)$ is contactomorphic to $\mathcal{O}(Z)$ where $Z \subseteq \mathbb{R} \times T^*\Lambda$ and $\xi = \ker(dz - \sum p_j dq_j)$.

Other isotropic submanifolds Λ have neighborhoods given by fiber bundles

$$\begin{tikzcd} \mathbb{C}^k \arrow[r] & \mathcal{O}p(\Lambda) \arrow[d] \\ & \mathbb{R} \times T^*\Lambda \end{tikzcd}$$

where $\alpha = dz - \sum p_j dq_j = \sum y_i dz_i$.

If N is contact as a submanifold i.e. $\xi \cap TN$ is contact, then

$$\begin{tikzcd} \mathbb{C}^k \arrow[r] & \mathcal{O}p(\Lambda) \arrow[d] \\ & \mathbb{R} \times T^*\Lambda \end{tikzcd}$$

and $\alpha = \alpha|_N + \sum y_j dx_j$.

Theorem 9.7 (Darboux's theorem)**.** *Any (M, ξ) is locally contactomorphic to $(\mathbb{R}^{2n+1}, \xi_{std})$.*

9.1 Direct relations to symplectic geometry

Let (N, ω) be a symplectic manifold with $\omega = d\lambda$. Then $M = \mathbb{R} \times N$ with $\alpha = dz - \lambda$ is contact.

Conversely, if $(N, \omega = d\lambda)$ is symplectic, and X_λ is a vector field with $\omega(X_\lambda, -) = \lambda$. If $M \subseteq N$ is a codimension 1 submanifold, and $M \pitchfork X_\lambda$ then $(M, \ker(\lambda|_M))$ is contact.

Example 9.8. If $N = T^*Q$ and $X_\lambda = \sum_j p_j \partial p_j$, the manifold $M = S^1(T^*Q)$ is contact.

Example 9.9. If N is holomorphically embedded in $\mathbb{C}^{\dim N}$ with $\omega = d(\sum x_j dy_j - y_j dx_j)/2$ then the intersection of N with any sphere having center at the origin is contact. Every simply-connected 5 dimensional manifold can be obtained in this manner, and hence can be given a contact structure.

Example 9.10. Consider the submanifold $M \subseteq N$ given by $M = \{H = 0\}$ for some $H : N \to \mathbb{R}$. Let $\alpha = \lambda|_M \in \Omega^1(M)$. The Reeb vector field $\mathcal{R}_\alpha$ and the Hamiltonian X_H on M are propotional to each other.

Theorem 9.11 (Eliashberg–Mishachev)**.** *Suppose (M, ξ_M) and (P, ξ_P) are contact with*

1. codim ≥ 4 *or,*
2. codim ≥ 2 *and P open,*

then there is a full h-principle for isocontact embeddings.

9.2 Contact structures on 3-manifolds

Any 2-plane distribution is either contact or integrable at every point.

Definition 9.12. On a $2n+1$ co-orientable manifold, a *formal contact structure* is a pair (β, σ) with $\beta \in \Omega^1(M)$ and $\sigma \in \Omega^2(M)$ such that $\beta \wedge \sigma^{\wedge n} > 0$.

On a 3-manifold M, the two form σ carries no information, as if we fix a β then $\beta \wedge \sigma > 0$ is choosing a volume form on M but the space of such choices is contractible. Hence, a formal conact structure is simply a 2-plane field (which is the same data as a non-vanishing 1-form).

Q. h-principle questions for π_0(contact structures on a 3-manifold M):

1. Given a 2-plane field $\eta \in TM$, is η homotopic to a contact structure?
2. If contact structures ξ_0 and ξ_1 are homotopic through 2-plane fields, are they homotopic as contact structures.

If M is open, both answers are yes - there is a full h-principle, because being contact is a $\mathrm{Diff}(M)$-invariant open relation.

When M is closed, we have different answers to the two questions. A theorem of Lutz–Martinet says that answer to the first question is *yes*, a theorem of Bennequin says that the answer to the second question is *no* (counterexample on S^3). Bennequim shows the following: on $\mathbb{R}^3$ there are two contact structures

$$\begin{aligned} \alpha_{std} &= dz + r^2 d\theta & \text{(cylindrical coordinates)} \\ &= dz - y_j dx_j + x_j dy_j \\ &\cong dz - y dx \end{aligned}$$

and let

$$\alpha_{ot} = \cos(r) dz + r \sin(r) d\theta \qquad \text{(over-twisted)}$$

are not contactomorphic. These two structures can be suitably extended to S^3, and on S^3 these two contact structures are homotopic (via formal contact structures) but not contactomorphic. In $\mathbb{R}^3_{ot}$, let $D^2_{ot} = \{z = 0, r \leq \pi\}$. Then D^2_{ot} has the property that $\pi|_{\partial D^2_{ot}}$ is tangent to the disc i.e. $\pi|_{\partial D^2_{ot}} = TD^2_{ot}|_{\partial D^2_{ot}}$. And there does not exst such a disk in $\mathbb{R}^3_{std}$ up to contactomorphism.

Definition 9.13. If (M^3, ξ) is contact we say that ξ is *overtwisted* if there exists a $D^2 \subseteq M$ such that $\xi|_{\partial D^2} = TD^2|_{\partial D^2}$.

Theorem 9.14 (Eliashberg, 89)**.** *Every homotopy class of 2-plane fields on M is realized by a unique overtwisted contact structure. Further, if ξ_0 and ξ_1 are overtwisted and are formally homotopic to each other then they are also homotopic to each other via overtwisted contact structures.*

Overtwisted contact structures have very non-intuitive properties. For example, it is possible for two non-isomorphic contact structures (on the same manifold) to become isomorphic to each other by replacing an arbitrarily small ball with an overtwisted disk.

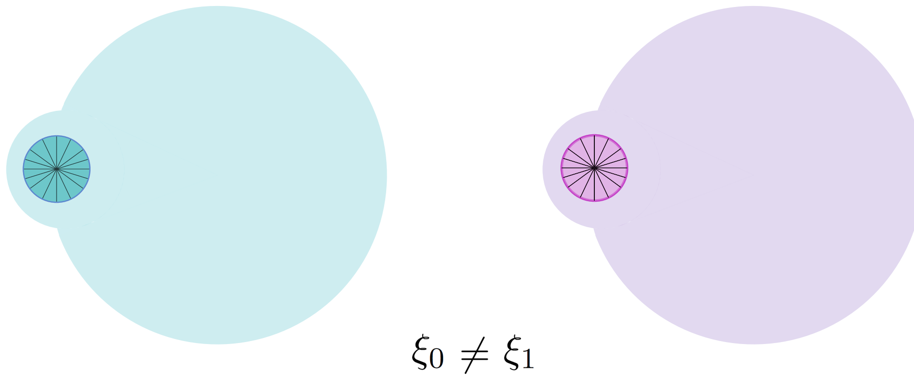

Figure 18: Two non-isomorphic contact structures $\xi_0 \neq \xi_1$ can become isomorphic by replacing a small region in each with an overtwisted disk.

If $(M, \xi) = \partial(N^4, \omega)$ implies that ξ is not overtwisted.

Conjecture 9.15 (Weinstein conjecture)**.** *There always exists a closed Reeb orbit on a contact manifold.*

This is know to be true for 3-manifolds (Seiberg-Witten theory). The case for overtwisted contact structures has a slightly easier proof, in a sense, we understand the dynamics of overtwisted structures better. In higher dimensions the Weinstein conjecture is open.

Theorem 9.16 (Lutz–Martinet)**.** *Any 2-plane field on M^3 is homotopic to a unique overtwisted contact structure.*

Proof. We will prove existence. Start with a 2-plane field η. Take a very fine triangulation of M^3, with the property that

1. the 1-skeleton and the 2-skeleton are transverse to η,
2. η is almost constant on any 3-simplex.

It is possible to make η contact on 2-skeleton by a C^0-small homotopy, retaining the two conditions.

[We will be repeatedly using the fact that any 2-plane field can be locally written as $\eta = \ker(dz - f(x,y,z)dx)$, and η is contact if and only if $\frac{\partial f}{\partial y} > 0$.]

Start with points: and perturb η so that it is twisting to the right.

1-skeleton: do the same thing, some normal direction (think of the edge as a z-axis.)

2-skeleton: we can do the same thing, because we can take $\partial/\partial y$ to be transversal to a given face Δ^2.

The new 2-plane field η is contact near the 2-skeleton and is "almost horizontal" on any 3-simplex Δ^3, i.e. two of the vertices v_0 and v_1 have the property that the entire 3-simplex lies above/below v_0/v_1 respectively, and for the other two, the simplex is cut in half. Consider the 2-plane field $\eta \cap \Delta^3$. We work with a smooth 2-sphere inside the Δ^3 very close to $\partial\Delta^3$.

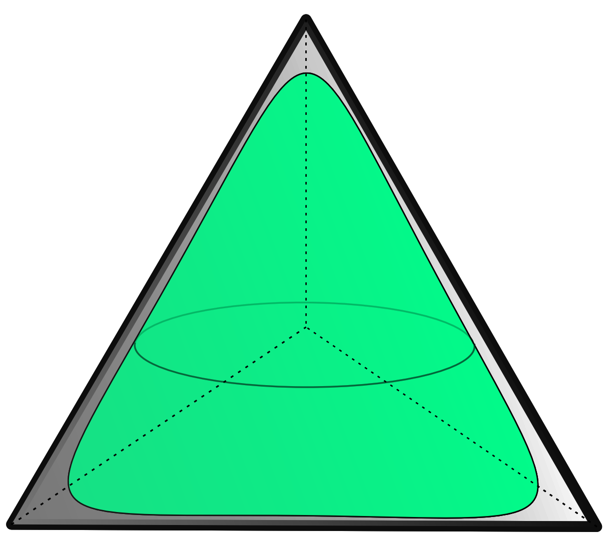

Figure 19: Spheres very close to a $\partial\Delta^3$ in a triangulation.

Starting with triangulation, we have made η contact near 2-skeleton and there are finitely many disjoint balls $B_i^3 \subseteq M$

- η is contact outside of $\coprod_i B_i^3$,
- $\mathscr{F} = \eta \cap T(\partial B_i^3)$ is almost horizontal.

For $S^2 = \partial B_i^3 = \{(z, \theta)\}$ with cylindrical coordiantes, the foliation $\mathscr{F} = \eta \cap T(\partial B_i^3)$ is given by

$$\mathscr{F} = \left\{ \frac{\partial z}{\partial \theta} = H(z, \theta) \text{ for some function } H \right\}$$

with $H > 0$ near the poles $z = \pm 1$. The important point here is that $\mathscr{F}$ is never tangent to the vertical direction $\partial/\partial\theta$.

Q. How to realize $\mathscr{F}$ as a characteristic foliation, for contact structure on B^3?

Easy case: If $H > 0$ on $z \in (-1, 1)$. Define,

$$S_H^2 = \left\{ (r, \theta, z \in \mathbb{R}^3 \mid r^2 = H(z, \theta)) \right\}.$$

Note that this only makes sense if $H(z, \theta) > 0$. Then define a contact structure on the three ball given by

$$\alpha = dz + r^2 d\theta, \qquad \xi = \ker \alpha.$$

This is just *one* example of a contact structure. So if $H \not> 0$, then we just choose a different contact structure.

General case: If H is not necessarily positive, we choose

$$\alpha_{ot} = \cos r dz + r \sin r d\theta, \qquad \xi = \ker \alpha_{ot}.$$

The function $\frac{dz}{dr} = \frac{r \sin r}{\cos r} = r \tan r$ takes all real values for $r \in (3\pi/2, 5\pi/2)$ i.e. for any real number a, there is a unique solution to $r \tan r = a$. We can then define,

$$S_H^2 = \left\{ (r, \theta, z) \in \mathbb{R}^3 \mid r \tan r = H(z, \theta) \right\}.$$

This gives us the required foliation, but it is no longer a sphere!

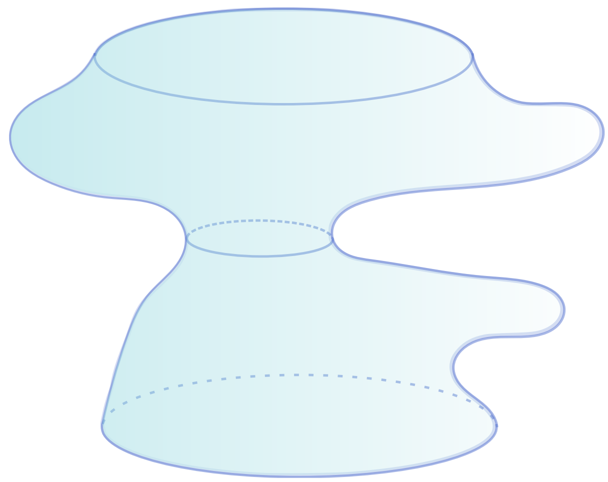

Figure 20: Image of S^2_H in the overtwisted case.

Lemma 9.17 (Transverse Curve Lemma)**.** *Let $\gamma \in (Y^3, \xi)$ be a smooth curve. Then we can C^0-approximate (relative to endpoints) γ with a curve $\widetilde{\gamma}$ such that $T\widetilde{\gamma} \pitchfork \xi$ positively everywhere.*

Finishing the Lutz–Martinet proof: For each i, take a smooth curve $\gamma_i \subseteq M \setminus \coprod_i B^3_i$ connecting north pole to the south pole, mutually disjoint from each other. Using Lemma 9.17, we can approximate them by transverse curves.

Consider the solid torus $(D^2 \times S^1)_i = B^3_i \cup \mathcal{O}\mathrm{p}(\widetilde{\gamma}_i)$ whose boundary is parametrized as

$$\partial(D^2 \times S^1)_i = \mathbb{T}^2 = \{(\theta, z) : z \in [-2, 2]\} / (2 \sim -2)$$

On the $[-1, 1]$ part, the foliation is given by

$$\mathscr{F} = \eta \cap T\mathbb{T}^2 = \left\{ \frac{\partial z}{\partial \theta} = H(z, \theta) \right\}.$$

Then we can extend H as

$$\begin{aligned} \widetilde{H}(z, \theta) &= H(z, \theta) && \text{for } z \in (-1+\varepsilon, 1+\varepsilon) \\ \widetilde{}(z, \theta) &= \delta && \text{constant for } |z| > 1. \end{aligned}$$

By the standard neighborhood theorem for transverse curves:

$$\mathcal{O}\mathrm{p}(\widetilde{\gamma}_i) = \left\{ \alpha = dz + r^2 d\theta \right\}, \qquad \widetilde{\gamma}_i = \{r = 0\}.$$

Now define

$$\mathbb{T}^2_H = \left\{ \widetilde{H}(z, \theta) = r \tan r \mid r \in (3\pi/2, 5\pi/2) \right\}$$

inside $\alpha_{ot} = \cos r dz + r \sin r d\theta$ on $\mathbb{R}^2_{(r,\theta)} \times S^1_z$.

We can glue this torus in M to get the required contact structure. One needs to check that the contact structure we obtain by this surgery is homotopic through 2-plane fields to the original plane field η. □

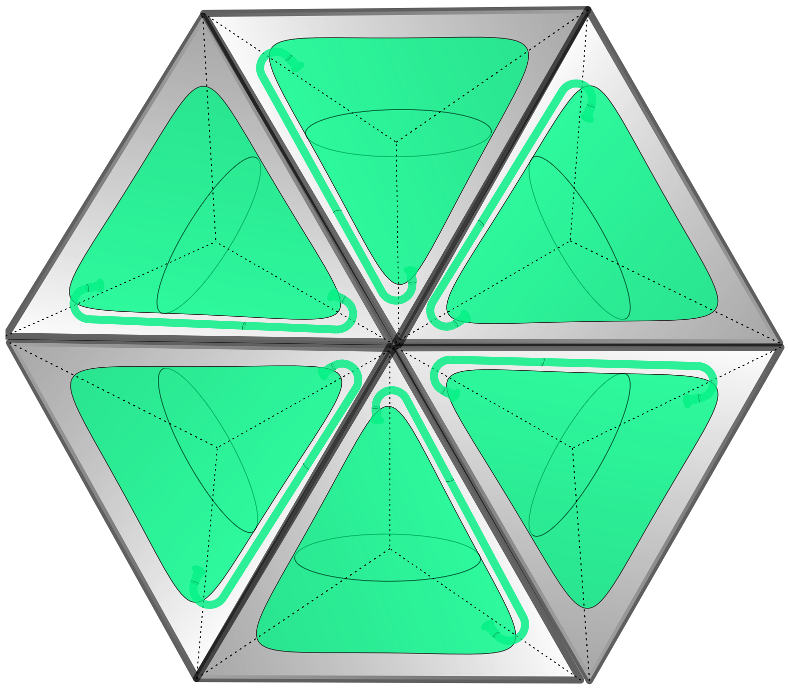

Figure 21: Lutz tubes.

This theorem was very recently proven for higher dimensions, but the proof is much harder because it is not easy to find explicit models.

Proof idea for Lemma 9.17. Consider $\gamma(t) = (r = 0, z = -t)$ in $(\mathbb{R}^3, \alpha = dz + r^2 d\theta)$. The curve $\gamma(t)$ is transverse, but it is *negatively* transverse. We want to approximate this with a *positively* transverse curve i.e. a curve $\widetilde{\gamma}(t)$ satisfying $\alpha(\dot{\widetilde{\gamma}}(t)) > 0$. Consider

$$\widetilde{\gamma}(t) = \left(z = -t, r = \delta, \theta = \frac{t}{\delta^2} \right)$$

for some small $\delta > 0$. This is the worst case scenario, and for other γ's a similar argument works. □

Parametric case: Consider two contact structures ξ_0 and ξ_1, and a family of 2-plane fields η_t interpolating between them.

Step 1: We can find a parametric family of good triangulations, in an appropriate sense.

Step 2: We can find a parametric family $B_i^3(t) \hookrightarrow M$ such that for all t,

1. $B_i^3(t)$ are all disjoint,
2. η_t is contact outside of $\coprod_i B_i^3(t)$, and
3. η_t is almost horizontal in $B_i^3(t)$.

Step 3: Find families $\gamma_i(t)$ connecting north and south poles, approximate by smooth family of curves which are transverse to the contact structure.

Step 4: Find a parametric family of $(D^2 \times S^1)_i(t)$ with $\mathscr{T}_t = \left\{ \frac{\partial z}{\partial \theta} = H_t(z, \theta) \right\}$. Fill the contact structures by $\{r \tan r = H_t(z, \theta)\} = \mathbb{T}^2_{H,t}$ for all t.

Given η_t we found a way to build ξ_t which is contact for all t. The issue is that at $t = 0, 1$ where η is already contact, we are changing the already existing contact structure.

Lutz: we can always replace $dz + r^2 d\theta$ by $\cos r dz + r \sin r d\theta$ and the sphere $r^2 = 1$ looks like a tube with $r \tan r = 1$. And so given any two contact structures which are homotopic via 2-plane fields, it is possible to add some Lutz tubes to both the structures to make them.

Theorem 9.18. *Given two overtwisted contact structures ξ_0 and ξ_1 and a homotopy η via 2-plane fields between them, there exists a homotopy ξ_t of overtwisted contact structures connecting ξ_0 to ξ_1.*

Proof. Consider overtwisted contact structures ξ_0 and ξ_1, and let η_t a homotopy via 2-plane fields between them. Up to isotopy, we can assume that $D^2_{ot} \subseteq (M, \xi_0)$ is equal to $D^2_{ot} \subseteq (M, \xi_1)$. Also, we can assume that each η_t is contact near D^2_{ot} and constantly equal to ξ_0 and ξ_1.

Redo everything, relative to their fixed set $A \subseteq M$, with $\text{int}(A) \subseteq D^2_{ot}$, and η_t is constant on A.

This gives us a parametric family $B^3_i(t) \subseteq M$ such that η_t is contact inside $\coprod_i B^3_i(t)$, almost horizontal foliations $\mathscr{F}_t$ on ∂B^3_i.

Contact structure on A for all η_t looks as follows:

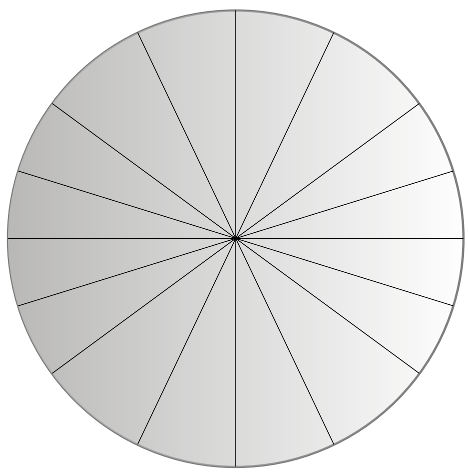

Figure 22: $TD^2_{ot} \cap \eta_t$

Exercise: If there exists a D^2 with $TD^2|_{\partial D^2} = \xi|_{\partial D^2}$, then there exists a D^2_{ot} with foliation diffeomorphic to this one.

Lemma 9.19. *It is possible to cancel an elliptic and a hyperbolic point in a foliated disk via a C^0-small isotopy in the contact manifold. (**Hint:** Use Poincaré–Bendixson theorem)*

Then $\mathcal{O}p(D^2_{ot}) = \{|z| \leq \varepsilon, r \leq \pi + \varepsilon\}$ inside $\mathbb{R}^3$ with $\alpha_{OT} = \cos r dz + r \sin r d\theta$, where $D^2_{ot} = \{z < 0, r \leq \pi\}$. This implies that there are infinitely many overtwisted disks $\{z = \text{ constant}\}$. Choose one for each B^3_i.

Let $S^2_{ot} = \partial\, \mathcal{O}\mathrm{p}(D^2_{ot})_i$. An example of such a sphere is $z^2/\varepsilon^2 + r^2/(\pi+\varepsilon) = 1$. The foliation $\mathscr{F}_{ot} = \eta_t \cup TS^2_{ot}$ looks as follows:

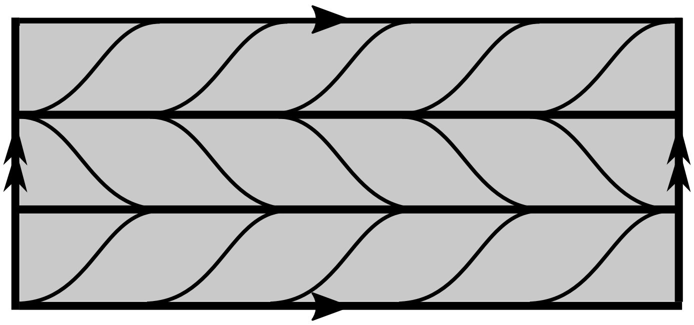

Figure 23: Foliation of an overtwisted sphere.

the two closed orbits correspond to $r = \pm\pi$ and the flow lines are given by $\frac{\partial z}{\partial \theta} = r \tan r$.

Trying to fill $S^2_i = \partial B^3_i$ (arbitrary spheres with horizontal dynamics): Take a family of transverse curves γ_i connecting the south pole of S^2_i with the north pole of S^2_{ot}, and take a small tube around γ_i and think of the resulting surgered object as a single sphere $\widehat{S}^2_i$.

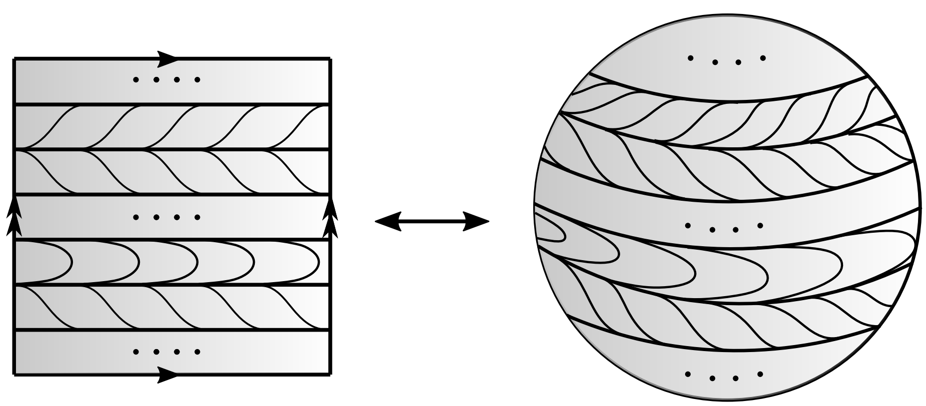

Figure 24: We can represent a sphere by a planar diagram, where the top and bottom edges are understood to be points.

Embed $\widehat{S}^2$ inside $\mathbb{R}^3_{ot}$ with $\alpha = \cos r dz + r \sin r d\theta$ as follows:

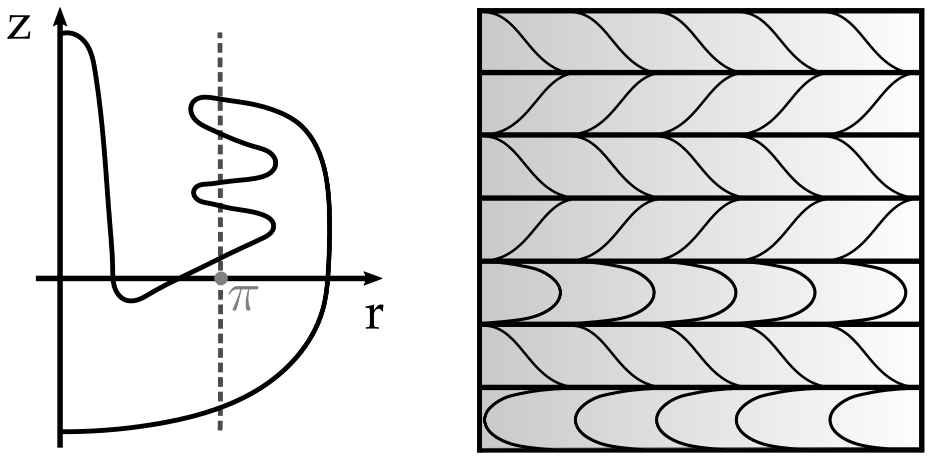

Figure 25: Any foliation sphere $\widehat{S}^2$ can be realized as the boundary of a solid sphere in $\mathbb{R}^3_{ot}$.

Fill $\widehat{S}^2_i$ with the interior of this embedding. This gives a contact structure for any fixed t. At $t = 0$ or $t = 1$, the ball B^3_i are already contact with the standard contact structures. We need to check that gluing an overtwisted disk to a standard ball is contactomorphic to an overtwisted disk. $\square$

10 Summary of modern results

Theorem 10.1 (Borman–Eliashberg–Murphy)**.** *In any odd dimension, every formal contact structure is homotopic to a unique overtwisted contact structure.*

Definition 10.2. $(M, \xi = \ker\alpha)$ is *overtwisted* if it contains the submanifold $\mathbb{R}^3_{ot} \times \mathbb{C}^n$, with the contact form $\alpha = \alpha^3_{ot} + \sum y_i dx_i$.

For M^5, two formal contact structures are homotopic if and only if $c_1(\eta_1) = c_1(\eta_2) \in H^2(M)$. This is because $SO(6)/U(3) \cong \mathbb{CP}^3 \approx \mathbb{CP}^\infty$. On S^5 there is a unique formal contact structure. There exists a connected sum of contact manifolds. Let Ξ be the set of isotopy classes of contact structures on S^5. This is a monoid under connected sum. The identity is

$$\begin{aligned} \xi_{std} &= \partial(\mathbb{C}^3, \omega_{std}) \\ &= TS^5 \cap JTS^5. \end{aligned}$$

The unique overtwisted contact structure ξ_{ot} is the 0 element as

$$\xi_{ot} \# \xi = \xi_{ot}.$$

Theorem 10.3. *Ξ is infinite and infinitely generated as a monoid. Further, there exists a submonoid $\overline{\Xi} \subset \Xi$ containing $\mathbb{Q}$.*

Open questions:

1. Do there exist ξ_1 and ξ_2 such that $\xi_1 \# \xi_2 = \xi_{std}$?
2. Do there exist non-overtwisted ξ_1 and ξ_2 such that $\xi_1 \# \xi_2 = \xi_{ot}$?

10.1 Legendrian embeddings

3-dimensions: there does not exist existence or uniqueness results

5-dimensions: there exist existence and non-uniqueness results

5-dimensions: there exist existence and uniqueness results for "loose" Legendrians

Theorem 10.4 (Weinstein handlebody theorem)**.** *If $(Y^{2n-1}, \xi) = \partial(M^{2n}, \omega = d\lambda)$, $\Lambda^k \subseteq Y$ is isotropic with $k \leq n-1$ then we can construct a new manifold $M(\Lambda)$ by attaching a handlebody.*

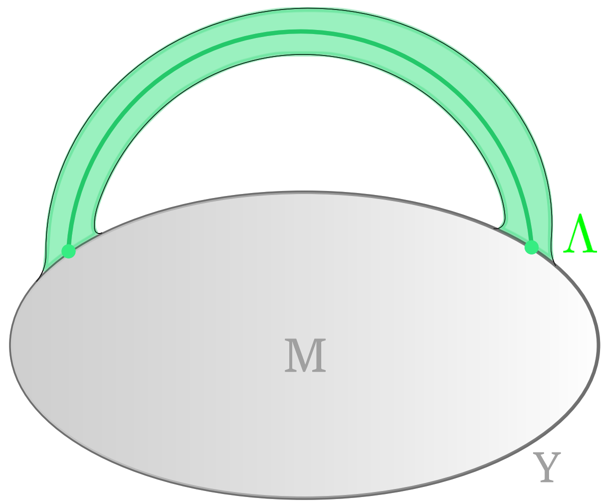

Figure 26: Weinstein handle attachment.

Theorem 10.5 (Existence of isotopies)**.** *If M is smooth* $\dim M = 2n \geq 6$, *M is homotopy equivalent to an n-dimensional CW complex and M has a formal symplectic structure then M has a symplectic structure $\omega = d\lambda$ which is constant at the boundary.*

This theorem is false in 4-dimensions. $S^2 \times \mathbb{R}^2$ is not exact symplectic but it has a formal symplectic structure. Further, such a symplectic structure will not in general be unique.

But, there DOES exist a uniqueness result among *flexible Weinstein manifolds*. Weinstein manifolds are built from Legendrians, and flexible Weinstein manifolds are built from loose legendrians and isotropics of dimension $< n - 1$.

There exists many many non-isomorphic symplectic structures on B^{2n} which are contact at infinity (Seidel-Smith).

Further, Mclean shows that given a group presentation $G = \langle g_1, \ldots, g_k \mid r_1, \ldots, r_\ell \rangle$, it is possible to construct a symplectic form ω_G on B^{2n} such that $\omega_G \cong \omega_{std}$ implies that G is the trivial group. Thus the question of determining the isomorphism type of a symplectic ball is atleast as hard as the word problem for finitely presented groups. All of these examples have $\partial(B^{2n}, \omega_G) \not\cong (S^{2n-1}, \xi_{std})$.

Open question. Suppose $\partial(B^{2n}, \omega) \cong (S^{2n-1}, \xi_{std})$, does this imply that $\omega = \omega_{std}$?

Theorem 10.6. *If (M, ω) is flexible, and (M_2, ω_2) satisfies $\partial M \cong \partial M_2$ as contact manifolds, then $H_*(M) \cong H_*(M_2)$.*

10.2 Lagrangian embeddings

Theorem 10.7 (Eliashberg–Murphy)**.** *If $\Lambda \subseteq (Y, \xi)$ is a loose Legendrian and $(Y, \xi) = \partial^{concave}(M, \omega)$ then there is an existence h-principle for Lagrangian embeddings $L \subseteq M$ with $\partial L = \Lambda$.*

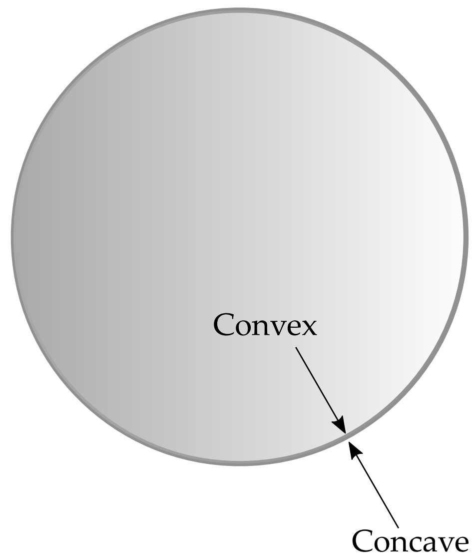

Figure 27: Concave boundary: $(S^{2n-1}, \xi_{std}) = \partial^{concave}(\mathbb{C}^n \setminus B^{2n})$.

Given $B^{2n} \subseteq \mathbb{C}^n$, there exists a Lagrangian disk $D^n \subseteq \mathbb{C}^n \setminus B^{2n}$ such that $\partial D^{2n} \subseteq \partial B^{2n}$ with $\partial D^n \perp T(\partial B^{2n})$.

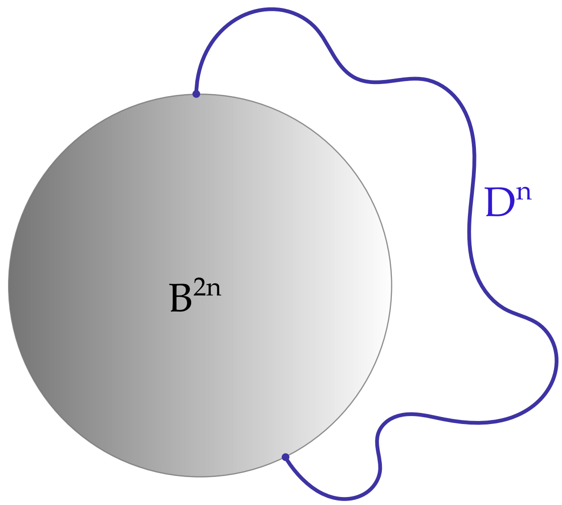

Figure 28: $D^n \subseteq \mathbb{C}^n \setminus B^{2n}$ such that $\partial D^{2n} \subseteq \partial B^{2n}$ with $\partial D^n \perp T(\partial B^{2n})$.

But it is known that this cannot exist if $n = 2$.

Theorem 10.8 (Eliashberg–Murphy)**.** *If L is any $n = 2k$ manifold such that $TL \otimes \mathbb{C} \cong_{\mathbb{C}} L \times \mathbb{C}^n$, and $\chi(L) = -2$ then there exits an exact Lagrangian immersion $L \looparrowright \mathbb{C}^{2k}$ such that it has only a single double point. This is also true for $n = 2$.*

Theorem 10.9 (Ekholm–Smith)**.** *If $L \looparrowright \mathbb{C}^{2k}$ has a single double point and $\chi(L) \neq -2$ then L is diffeomorphic to S^{2k} if $2k > 4$. (If $2k = 4$ we only get a homeomorphism.)*

Corollary 10.10. *There exits a Lagrangian $S^1 \times S^2 \hookrightarrow \mathbb{C}^3$ with 0 Maslov number.*

Corollary 10.11. *If M is a flexible Weinstein manifold, then there is an h-principle for symplectic embeddings of $M \hookrightarrow (X, \omega)$.*

Example 10.12. For $M = \left\{ x \cdot y^p = 1 + \sum z_j^2 \right\} \subseteq \mathbb{C}^{n+1}$ is flexible for all $p > 1$. (If $p = 1$, $M = T^*S^n$.)

Theorem 10.13. *If M is flexible, and X is Weinstein then there exists an h-principle for Weinstein embeddings $M \subseteq X$.*

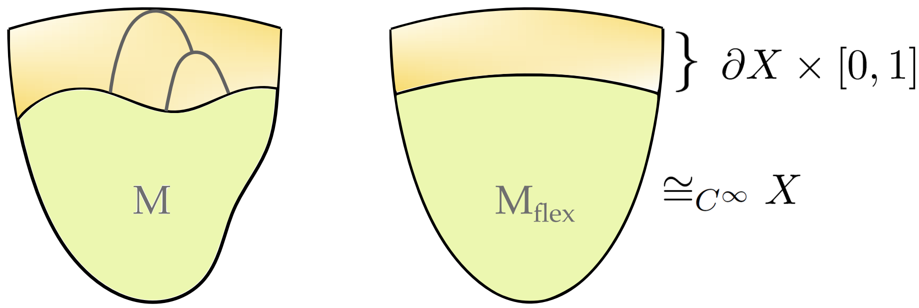

Figure 29: h-Principle for Weinstein embeddings.

Theorem 10.14. *Let M be a smooth cobordism with* $\dim M = 2n > 4$*, formally symplectic. Put an overtwisted contact structure on $\partial_- M$ and arbitrary contact structure on $\partial_+ M$. Then there exists a Liouville cobordism structure $\omega = d\lambda$ on M.*

Corollary 10.15. *If M is a compact manifold with $H^1(M; \mathbb{Z}) \neq 0$, and M is formally symplectic, then there exists locally conformal symplectic structures on M i.e. there exists a non-degenerate ω on M and $\eta \in \Omega^1(M)$ closed such that $d\omega = \eta \wedge \omega$.*