Cohomology of moduli stack of Riemann Surfaces

Σ = algebraic curve / $\mathbb{C}$
G = complex semisimple Lie group
$\mathrm{Bun}_G(\Sigma)$ = moduli stack of $\underset{\text{algebraic}}{\hat{}}$ G-bundles

Goal: Understand $H^*(\mathrm{Bun}_\Sigma(G))$

$\mathrm{Bun}_G(\Sigma)^{an}$ = moduli stack of holomorphic G-bundles
$\cong$ = " smooth G-bundles equipped with a $\bar{\partial}$-connection

$\left\{\begin{matrix}\text{smooth/topological} \\ \text{G-bundles on} \\ \Sigma\end{matrix}\right\}$ $\quad \mathrm{Bun}_G(\Sigma)^{an} \approx \underbrace{\mathrm{Maps}(X, BG)}$ ↳ understand this space : Goal

Q. What is $\pi_0 \,\mathrm{Map}(\Sigma, BG)$?
Say G = discrete abelian group then this equals $H^1(\Sigma, G) \cong H_1(\Sigma, G)$.

Recall: M = oriented d-manifold : $H^*_c(M;A) \cong H_{d-*}(M;A)$.
eg: $M = \mathbb{R}^d$ we get
$$H^*_c(M;A) \cong \begin{cases} A & *=d \\ 0 & \text{else} \end{cases}$$

$U \in \mathcal{U}(M)$ = open subsets of M
$C^*_c(M;A) \qquad C_*(U;A)$

$$U \longmapsto C_*(U),\ C^*_c(U)$$
$$\mathcal{U}(M) \longrightarrow \text{chain complexes}$$

Prop: These functors are cosheaves over M with values in chain complexes.
i.e. the following is a homotopy pushout square

$$\begin{array}{ccc} C_*(U \cap V) & \longrightarrow & C_*(V) \\ \downarrow & & \downarrow \\ C_*(U) & \longrightarrow & C_*(U \cup V) \end{array}$$

$\mathcal{U}_0(M) \subseteq \mathcal{U}(M)$ open discs in M.

$$\operatorname*{hocolim}_{U \in \mathcal{U}_0(M)} C_*(U) \xrightarrow{\simeq} C_*(M)$$

we have quasi-iso $C_*(U) \xrightarrow{\simeq} C_c^{d-*}(U)$ which induce iso on hocolim.

$$\operatorname*{hocolim}_{U \in \mathcal{U}_0(M)} C_c^*(U) \xrightarrow{\simeq} C_c^*(M)$$

Non-abelian cohomology:

$$H^m(M;A) = \text{homotopy classes of maps } M \longrightarrow K(A;m)$$

Note: If $m=1$, $K(G,1) = BG$ even for non-abelian discrete group G

$$\begin{aligned} H^1(M;G) &= [M, BG] \\ &= \text{iso classes of } G\text{-bundles on } M \\ &= \text{group homs } \pi_1 M \longrightarrow G \,/\, \text{conjugacy}. \end{aligned}$$

Let X be any space. "$H(M;X)$" $:= [M, X]$

Abelian	Non-Abelian
· abelian groups A, degree m	X any space (eg: $X = K(A,m)$)
· $H^m(M,A)$	$[M,X]$
· $H^m_c(M,A)$	$[M,X]_c$ where X is based
· $C^*_c(M,A)$	$\mathrm{Maps}_c(M,X)$ as a topological space π_0
· $C_*(M,A)$	??

$M = \mathbb{R}^d$ $\qquad \mathrm{Maps}_c(\mathbb{R}^d, X) = \Omega^d X$

$$(U \in \mathcal{U}(M)) \longmapsto \mathrm{Maps}_c(U, X)$$

Q. Is this a homotopy cosheaf?

ie is the following a homotopy pushout

$$\begin{tikzcd} \mathrm{Maps}_c(U \cap V, X) \arrow[r] \arrow[d] & \mathrm{Maps}_c(U, X) \arrow[d] \\ \mathrm{Maps}(V, X) \arrow[r] & \mathrm{Maps}(U \cup V, X) \end{tikzcd}$$

Ans: NO. Look at H_0.

Problem: Cannot add maps.

except when $U \cap V = \phi$ in which case $\mathrm{Maps}_c(U \cup V, X) \cong \mathrm{Maps}_c(U, X) \times \mathrm{Maps}_c(V, X)$.

$U_0(M) \subseteq U_1(M) \subseteq U(M)$

disjoint unions of open discs

$$\begin{tikzcd} \operatorname{hocolim}_{U_0(M)} \mathrm{Maps}_c(U,X) \arrow[r, "\not\simeq"] \arrow[d] & \mathrm{Maps}_c(U,X) \\ \operatorname{hocolim}_{U_1(M)} \mathrm{Maps}_c(U,X) \arrow[ur, "\sim"', "\text{often}"] & \end{tikzcd}$$

Th^m: Non-abelian Poincare Duality:

M manifold of dim d

X pointed space, $(d-1)$ connected

then $\operatorname{hocolim}_{U \in U_1(M)} \mathrm{Maps}_c(U, X) \xrightarrow{\sim} \mathrm{Maps}_c(M, X)$

[LHS = factorization homology

eg: $X = K(A, d)$

take π_* on both sides, then if M is orientable

$\underrightarrow{\operatorname{hocolim}}_{U \in U_1(M)} \mathrm{Map}_c(M, X) \cong \mathrm{Conf}(M, A) \simeq H_*(M, A)$

$\pi_* \mathrm{Map}_c(M, X) \cong H_c^{d-*}(M, A)$

The construction: $(U \in U_0(M)) \longmapsto \mathrm{Maps}_c(U, X) \simeq \Omega^d X$ an E_d-space

is a factorization algebra on M (with values in spaces)

Q. Why is this useful?

$\mathrm{Maps}_c(M, X) \xleftarrow{\simeq} \operatorname{hocolim}_{U \in U_1(M)} \mathrm{Maps}_c(U, X) \simeq \prod \Omega^d X$

Note: Hypothesis satisfies when $M = \Sigma$, $X = BG$

- Want to work over $\mathbb{F}_q$ instead of $\mathbb{C}$.

$\mathrm{Ran}(M) = \left\{ \begin{matrix} \text{non-empty finite} \\ S \subseteq M \end{matrix} \right\} \leftarrow$ topology with basis

special open sets — $\mathrm{Ran}(U_1, \dots, U_n) = \left\{ \begin{matrix} S \subseteq \bigcup U_i \subseteq M, \\ S \cap U_i \neq \emptyset \text{ for each } i \end{matrix} \right.$

U_i pairwise disjoint connected sets

$\mathcal{F}$: special open sets $\longrightarrow$ Spaces

$$\mathcal{F}(\mathrm{Ran}(U_1, \dots, U_n)) = \mathrm{Maps}_c(U_1 \cup \dots \cup U_n, X)$$

$\underline{\text{Th}^m}$ $\mathcal{F}$ is a homotopy cosheaf on $\mathrm{Ran}(M)$.

Assume M connected,

$$\begin{array}{ccc} \underset{\substack{U_1, \dots, U_n \\ \text{disjoint open}}}{\underrightarrow{\mathrm{hocolim}}} \mathcal{F}(\mathrm{Ran}(U_1, \dots, U_n)) & \xrightarrow{\simeq} & \mathcal{F}(\mathrm{Ran}(M)) \\ \text{IS} & & \\ \mathrm{Maps}_c(U_1 \cup \dots \cup U_n, X) & \longrightarrow & \mathrm{Maps}_c(M, X) \end{array}$$

Why $(d-1)$ connected.

On the level of π_0 this can be shown by triangulating M and using the simplices. The obstruction for gluing lies precisely in the first $d-1$ homotopy groups of X.