$SO(n)$

$SO(n) \circlearrowright \mathbb{R}^n, S^{n-1}$ and stabilizer of a pt. is S^{n-1} so that we have filtration

$$\begin{tikzcd} SO(n) \arrow[r] & SO(n+1) \arrow[d] \\ & S^n \end{tikzcd}$$

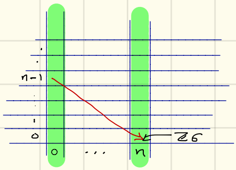
We only look at $\mathbb{Z}/2$ cohomology

Claim: Cohomology of $SO(n+1)$ is generated by elements of degree $\leq n$ and has the same Betti numbers as $S^1 \times S^2 \times \dots \times S^n$. In $H^n(SO(n+1))$ there is at the most one element which is not a multiple of lower elements and this is image of $H^n(S^n) \longrightarrow H^n(SO(n+1))$ which is always an inclusion

Proof: We have the identifications $SO(2) \cong S^1$, $SO(3) \cong \mathbb{RP}^3$, $SO(4) \cong SO(3) \times S^3$

Next we proceed by induction. Assume truth for $SO(n)$.

For $SO(n+1)$ we have the Serre Spectral Sequence:



The only differentials are d^n

Note that the result would follow if we can prove that all the d^n's are 0.

Now for $SO(n)$ all generators are in $H^i(SO(n))$, $i<n$. By Leibnitz rule it suffices to show that differentials d^n are 0 here. d^n is trivially 0 for $i<n-1$. So we are reduced to show

Claim: $d^n : E^{0,n-1} \cong H^{n-1}(SO(n)) \longrightarrow E^{n,0} \cong H^n(S^n)$ is 0.

Proof: For this we notice that we have an isometric immersion $STS^n \hookrightarrow S^n \times \mathbb{R}^{n+1}$ and so we can lift the action of $SO(n+1)$ to STS^n, the stabilize of a point would then be $SO(n-1)$ so that $STS^n \cong SO(n+1)/SO(n-1)$!!

We can rewrite the fiber bundle $S^{n-1} \to STS^n \to S^n$ as $SO(n)/SO(n-1) \longrightarrow SO(n+1)/SO(n-1) \longrightarrow S^n$

So that we have a map of bundles

$$\begin{tikzcd} SO(n) \arrow[r] \arrow[d] & S^{n-1} \arrow[d] \\ SO(n+1) \arrow[r] \arrow[d] & STS^n \arrow[d] \\ S^n \arrow[r, equal] & S^n \end{tikzcd}$$

So now we intend to study the second bundle.

Unit Sphere bundle in TS^n

We have the sphere bundle $\quad S^{n-1} \longrightarrow STS^n \longrightarrow S^n$

The Serre SS gives:

$$\begin{array}{c|cccccc}
n-1 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & 0 & 0 \\
\vdots & & & & & & \\
0 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & 0 & 0 \\
\hline
 & 0 & \cdots & n & & &
\end{array}$$

These are the only surviving differentials

The next page is E_∞, then

$$0 \to E_\infty^{i,0} \longrightarrow H^i(STS^n) \longrightarrow E_\infty^{0,i} \longrightarrow 0$$

Also this differential gives:

$$0 \to E_\infty^{i,n-1} \longrightarrow \underset{\substack{\| \\ H^i(S^n, H^{n-1}(S^{n-1}))}}{E_2^{i,n-1}} \xrightarrow{d} \underset{\substack{\| \\ H^{n+i}(S^n, H^0(S^{n-1}))}}{E_2^{n+i,0}} \longrightarrow E_\infty^{n+i,0} \to 0$$

In general this gives rise to a Gysin Seq.
But we don't need this for our special case

There is only 1 unknown map $\quad d: \underset{\substack{\cong \\ H^{n-1}(S^{n-1})}}{E^{0,n-1}} \longrightarrow \underset{\substack{\cong \\ H^n(S^n)}}{E^{n,0}}$ $\quad$ this is a transgression so that

$$\begin{tikzcd}
S^{n-1} \arrow[r] & STS^n \arrow[r] & (STS^n, S^{n-1}) \arrow[d] \\
 & & (S^n, *)
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
H^n(STS^n, S^{n-1}) & H^{n-1}(S^{n-1}) \arrow[l, "\delta"'] \\
\tilde{H}^n(S^n) \arrow[u, "\pi^*"] &
\end{tikzcd}
\qquad d = (\pi^*)^{-1} \cdot \delta$$

Claim: d is multiplication by the Euler number.

Proof: What is the Euler class?

$S^{n-1} \to E \xrightarrow{p} B$ then let $\mathbb{D}^n \to \tilde{E} \xrightarrow{\tilde{p}} B$ be the corresponding disk bundle. Then, Thom isomorphism states $\exists$ a class $u \in H^n(\tilde{E}, E)$ such that $\quad H^{n+k}(\tilde{E}, E) \xleftarrow[\cup u]{\cong} \tilde{p}^* H^k(B)$.

u is called the Thom class. The Euler class lives in $H^n(B) \cong H^*(\tilde{E})$ is simply the restriction of the Thom class to $H^n(\tilde{E})$, $\quad e \in H^n(B) \cong H^n(\tilde{E})$, $\quad e = u|_{\tilde{E}}$.

To compute d we need to compute π^*, δ. For the unit sphere bundle we would get $u \in H^n(DTS^n, STS^n)$, The only non-zero cohomolo ies (DTS^n, STS^n) are in H^n and H^{2n}.

We have long exact seq

$$\to \underset{\substack{\| \\ 0}}{H^{n-1}(DTS^n)} \to \underset{\substack{\| \\ ?}}{H^{n-1}(STS^n)} \to \underset{\substack{\cong \\ \mathbb{Z}u}}{H^n(DTS^n, STS^n)} \longrightarrow \underset{\substack{\cong \\ \mathbb{Z}}}{H^n(DTS^n)} \longrightarrow \underset{\substack{\| \\ ?}}{H^n(STS^n)} \longrightarrow \underset{\substack{\| \\ 0}}{H^{n+1}(DTS^n)}$$

It all depends on this map. But this is precisely the map that sends u to e

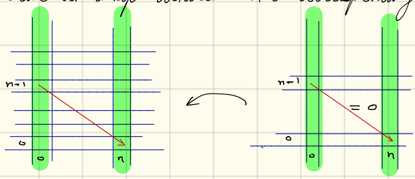
So that $H^{n-1}(STS^{n-1})$ is kernel of $\mathbb{Z}u \longrightarrow \mathbb{Z}e$ and H^n is the cokernel.
$u \mapsto e$

Looking back at the spectral sequence, this would imply d is multiplication by e.
There was no need of invoking transgressions :D.

The Euler number is 0 or 2 so that in $\mathbb{Z}/_2$ it is always 0, implying that in $\mathbb{Z}/_2$ coefficients $d \equiv 0$.

Back to SO(n)

We have a map between the corresponding spectral sequences



So that we have a commutative diagram

$$\begin{tikzcd} H^{n-1}(SO(n-1)) \arrow[d] & H^{n-1}(S^{n-1}) \arrow[l] \arrow[d, "0"] \\ H^n(S^n) & H^n(S^n) \arrow[l, "\cong"'] \end{tikzcd}$$

Because we have assumed that the only generator here is induced by the inclusion map
$d : H^{n-1}(SO(n)) \longrightarrow H^n(S^n)$ is 0 as was required to be proved.

Q. We have not answered the question when is $H^n(S^n)$ a new generator in $H^*(SO(n+1))$ and when it is a power. Can we do this using Serre SS? Turns out by looking at the cell structure we get that If n is even then we get a new generator (in deg $n-1$) and when n is odd, $n-1 = 2^k m$ $H^n(S^n)$ includes as a $2^{k\text{th}}$ power the generator corresponding to $H^{k+1}(S^{k+1})$. !!

Thm: $H^*(SO(n); \mathbb{F}_2)$ has a fundamental system of generators x_i in degree $0<i<n$

Pf. $H^*(SO(2)) = \langle 1, x \rangle$ $\quad H^*(SO(3)) = \langle 1, x, x^2, x \cdot x^2 \rangle$ $\quad SO(4) = \langle 1, x_1, x^2, x x^2, x_3, x_1 x_3, x_1 x_2 x_3, x_1^2 x_2 x_3 \rangle$
Rest follows just by looking at the spectral sequence.

This property says something about the Koszul Resolution