Mathematical Induction

Let's begin with an example.

Example: A Sum Formula

Theore. For any positive integer n, $1 + 2 + ... + n = n(n+1)/2$.

Proof. (Proof by Mathematical Induction) Let's let P(n) be the statement "$1 + 2 + ... + n = (n (n+1)/2$." (The idea is that P(n) should be an assertion that for any n is verifiably either true or false.) The proof will now proceed in two steps: the **initial step** and the **inductive step**.

Initial Step. We must verify that P(1) is True. P(1) asserts "$1 = 1(2)/2$", which is clearly true. So we are done with the initial step.

Inductive Step. Here we must prove the following assertion: "If there is a k such that P(k) is true, then (for this same k) P(k+1) is true." Thus, we assume there is a k such that $1 + 2 + ... + k = k (k+1)/2$. (We call this the **inductive assumption**.) We must prove, for this same k, the formula $1 + 2 + ... + k + (k+1) = (k+1)(k+2)/2$.

This is not too hard: $1 + 2 + ... + k + (k+1) = k(k+1)/2 + (k+1) = (k(k+1) + 2 (k+1))/2 = (k+1)(k+2)/2$. The first equality is a consequence of the inductive assumption.

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The Math Induction Strategy

Mathematical Induction works like this: Suppose you want to prove a theorem in the form "For all integers n greater than equal to a, P(n) is true". P(n) must be an assertion that we wish to be true for all n = a, a+1, ...; like a formula. You first verify the **initial step**. That is, you must verify that P(a) is true. Next comes the **inductive step**. Here you must prove "If there is a k, greater than or equal to a, for which P(k) is true, then for this same k, P(k+1) is true."

Since you have verified P(a), it follows from the inductive step that P(a+1) is true, and hence, P(a+2) is true, and hence P(a+3) is true, and so on. In this way the theorem has been proved.

Example: A Recurrence Formula

Math induction is of no use for deriving formulas. But it is a good way to prove the validity of a formula that you might think is true. Recurrence formulas are notoriously difficult to derive, but easy to prove valid once you have them. For example, consider the sequence $a_0, a_1, a_2, ...$ defined by $a_0 = 1/4$ and $a_{n+1} = 2\ a_n(1-a_n)$ for $n \geq 0$.

Theorem. A formula for the sequence a_n defined above, is $a_n = (1 - 1/2^{2^n})/2$ for all n greater than or equal to 0.

Proof. (By Mathematical Induction.)

Initial Step. When n = 0, the formula gives us $(1 - 1/2^{2^n})/2 = (1 - 1/2)/2 = 1/4 = a_0$. So the closed form formula ives us the correct answer when n = 0.

Inductive Step. Our inductive assumption is: Assume there is a k, greater than or equal to zero, such that $a_k = (1 - 1/2^{2^k})/2$. We must prove the formula is true for n = k+1.

First we appeal to the recurrsive definition of $a_{k+1} = 2\ a_k(1-a_k)$. Next, we invoke the inductive assumption, for this k, to get

$a_{k+1} = 2\ (1 - 1/2^{2^k})/2\ (1 - (1 - 1/2^{2^k})/2) = (1 - 1/2^{2^k})(1 + 1/2^{2^k})/2 = (1 - 1/2^{2^{k+1}})/2$. This completes the inductive step.

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Exercises

Prove each of the following by Mathematical Induction.

1. For all positive integers n, $1^2 + 2^2 + ... + n^2 = (n)(n+1)(2n+1)/6$.
2. Define a sequence a_0, a_1, a_2 by the recurrsive formula $a_{n+1} = 2\ a_n - a_n^2$. Then, a closed form formula for a_n is $a_n = 1 - (1 - a_0)^{2^n}$ for all n = 0, 1, 2, ...