PROBLEM SET 05

Part 1 - Differentiation

Q.1. Do **Q.1** (parts (i) - (viii)) and **Q.3** from Chapter 10 of the book.

Q.2. If f is three times differentiable and $f'(x) \neq 0$ the **Schwarzian derivative** of f at x is defined to be

$$\mathfrak{D}f(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2$$

Show that $\mathfrak{D}g = 0$ for the function $g(x) = \dfrac{ax+b}{cx+d}$ with $ad - bc \neq 0$.

Q.3. (1) A number a is called a **double root** of a polynomial f if $f(x) = (x-a)^2 g(x)$ for some polynomial g. Prove that a is a double root of f if and only if both f and f' vanish at a.

(2) When does $f(x) = ax^2 + bx + c$ have a double root? What does the condition say geometrically?

Q.4. (1) Try to prove the following formulae using the definition of derivative[1]

$$(\sin x)' = \cos x \qquad\qquad (\cos x)' = -\sin x$$

What identities *should be true* (in terms of limits $\lim\limits_{h\to 0}$) for the above formulae to hold?

(2) Look at the graphs of $\sin x$ and $\cos x$ near $x = 0$, and come up with a heuristic argument as to why these identities might true.

[1] You can use the following trigonometric identities

$$\sin(a+b) = \sin a \cos b + \sin b \cos a$$
$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

Part 2 - Inverse Functions

Q.5. For this problem assume that $x \neq 0$.

(1) Using induction and product rule show that $(x^n)' = nx^{n-1}$ when n is a non-negative integer.

(2) Using the quotient rule show that $(x^n)' = nx^{n-1}$ when n is a negative integer.

(3) Using the fact that $x^{1/n}$ is the inverse of x^n show that $(x^n)' = nx^{n-1}$ when $n = 1/m$ for some non-zero integer m.

(4) Using chain rule show that $(x^n)' = nx^{n-1}$ when n is a rational number.

(5) Can you think of some way of extending this to the case when n is an irrational number?

Q.6. Determine the derivatives of $\sin^{-1} x$ and $\tan^{-1} x$. (Later on we'll use these inverse functions to rigorously define $\sin x$.)

Q.7. Suppose the functions f and g are increasing everywhere i.e. $f(x) > f(y)$ and $g(x) > g(y)$ for all $x > y$.

(1) Which of the functions $f + g$, $f.g$ and $f \circ g$ are necessarily increasing.

(2) Show that f^{-1} is also an increasing function.

(3) Determine $(f \circ g)^{-1}$ in terms of f^{-1} and g^{-1}.

(4) Find g^{-1} in terms of f^{-1} if $g(x) = 1 + f(x)$.

Mathematical Induction

Let's begin with an example.

Example: A Sum Formula

Theore. For any positive integer n, 1 + 2 + ... + n = n(n+1)/2.

Proof. (Proof by Mathematical Induction) Let's let P(n) be the statement "1 + 2 + ... + n = (n (n+1)/2." (The idea is that P(n) should be an assertion that for any n is verifiably either true or false.) The proof will now proceed in two steps: the **initial step** and the **inductive step**.

Initial Step. We must verify that P(1) is True. P(1) asserts "1 = 1(2)/2", which is clearly true. So we are done with the initial step.

Inductive Step. Here we must prove the following assertion: "If there is a k such that P(k) is true, then (for this same k) P(k+1) is true." Thus, we assume there is a k such that 1 + 2 + ... + k = k (k+1)/2. (We call this the **inductive assumption**.) We must prove, for this same k, the formula 1 + 2 + ... + k + (k+1) = (k+1)(k+2)/2.

This is not too hard: 1 + 2 + ... + k + (k+1) = k(k+1)/2 + (k+1) = (k(k+1) + 2 (k+1))/2 = (k+1)(k+2)/2. The first equality is a consequence of the inductive assumption.

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The Math Induction Strategy

Mathematical Induction works like this: Suppose you want to prove a theorem in the form "For all integers n greater than equal to a, P(n) is true". P(n) must be an assertion that we wish to be true for all n = a, a+1, ...; like a formula. You first verify the **initial step**. That is, you must verify that P(a) is true. Next comes the **inductive step**. Here you must prove "If there is a k, greater than or equal to a, for which P(k) is true, then for this same k, P(k+1) is true."

Since you have verified P(a), it follows from the inductive step that P(a+1) is true, and hence, P(a+2) is true, and hence P(a+3) is true, and so on. In this way the theorem has been proved.

Example: A Recurrence Formula

Math induction is of no use for deriving formulas. But it is a good way to prove the validity of a formula that you might think is true. Recurrence formulas are notoriously difficult to derive, but easy to prove valid once you have them. For example, consider the sequence a_0, a_1, a_2, ... defined by $a_0 = 1/4$ and $a_{n+1} = 2\ a_n(1-a_n)$ for $n \geq 0$.

Theorem. A formula for the sequence a_n defined above, is $a_n = (1 - 1/2^{2^n})/2$ for all n greater than or equal to 0.

Proof. (By Mathematical Induction.)

Initial Step. When n = 0, the formula gives us $(1 - 1/2^{2^n})/2 = (1 - 1/2)/2 = 1/4 = a_0$. So the closed form formula ives us the correct answer when n = 0.

Inductive Step. Our inductive assumption is: Assume there is a k, greater than or equal to zero, such that $a_k = (1 - 1/2^{2^k})/2$. We must prove the formula is true for n = k+1.

First we appeal to the recurrsive definition of $a_{k+1} = 2\ a_k(1-a_k)$. Next, we invoke the inductive assumption, for this k, to get

$a_{k+1} = 2\ (1 - 1/2^{2^k})/2\ (1 - (1 - 1/2^{2^k})/2) = (1 - 1/2^{2^k})(1 + 1/2^{2^k})/2 = (1 - 1/2^{2^{k+1}})/2$. This completes the inductive step.

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Exercises

Prove each of the following by Mathematical Induction.

1. For all positive integers n, $1^2 + 2^2 + ... + n^2 = (n)(n+1)(2n+1)/6$.
2. Define a sequence a_0, a_1, a_2 by the recurrsive formula $a_{n+1} = 2\ a_n - a_n^2$. Then, a closed form formula for a_n is $a_n = 1 - (1 - a_0)^{2^n}$ for all n = 0, 1, 2, ...