PROBLEM SET 06

Part 1 - Min-Max

Q.1. If $a_1 < a_2 < \cdots < a_n$ are real numbers,

(1) Find the minimum value of the function $f(x) = \sum_{i=1}^{n} (x - a_i)^2$.

(2) Now find the minimum value of $f(x) = \sum_{i=1}^{n} |x - a_i|$. [*]

(3) If $a > 0$, find the maximum value of the function

$$f(x) = \frac{1}{1+|x|} + \frac{1}{1+|x-a|}$$

Q.2. (1) Prove that if $f'(x) \geq M$ for all $x \in [a,b]$ then $f(b) \geq f(a) + M(b-a)$.

(2) Prove that if $f'(x) \leq M$ for all $x \in [a,b]$ then $f(b) \leq f(a) + M(b-a)$.

(3) Formulate a similar theorem when $|f'(x)| \leq M$ for all x in $[a,b]$.

Q.3. Suppose that $f'(x) > M > 0$ for all x in $[0,1]$. Show that there is an interval of length $\frac{1}{4}$ on which $|f| \geq M/4$. [†]

Q.4. Sketch the following functions and find their local maxima and minima:

(1) $\dfrac{x+1}{x^2+1}$

(2) $x + \dfrac{1}{x}$

(3) $\dfrac{x^2}{x^2-1}$

(4) $\dfrac{1}{1+x^2}$

Q.5. (Optional) Show that if f is twice differentiable with $f(0) = 0$ and $f(1) = 1$ and $f'(0) = f'(1) = 0$ then $|f''(x)| \geq 4$ for some x in $[0,1]$.

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[*] Hint: Think graphically.

[†] Hint: Compare $f(x)$ with the line of slope M passing through $f(0)$.

Part 2 - Applications

Q.6. (1) What is the relationship between the critical points of f and f^2?

(2) Consider the straight line described by the equation $Ax + By + C = 0$. Show that the distance from the origin to this line is $\dfrac{C}{\sqrt{A^2 + B^2}}$.

Q.7. Show that the sum of a positive number and it's reciprocal is at least 2.

Q.8. Prove that if $\dfrac{a_0}{1} + \dfrac{a_1}{2} + \dfrac{a_2}{3} + \cdots + \dfrac{a_n}{n+1} = 0$ then $a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n = 0$ for some x in $[0, 1]$.

Q.9. Prove that the function $x^2 = \cos x$ has precisely 2 solutions. (Draw a picture.)

Q.10. Prove that if $n > 1$ and $x > 0$ then $(1 + x)^n > 1 + nx$.

Q.11. Suppose that f is continuous and differentiable on $[0, 1]$ such that $f(x)$ is in $[0, 1]$ for each x, and that $f'(x) \neq 1$ for all x in $[0, 1]$. Show that there is exactly one number x in $[0, 1]$ such that $f(x) = x$.

Q.12. Suppose f and g are two differentiable functions which satisfy $f'g - g'f = 0$. Prove that if a and b are adjacent zeroes of f, and $g(a) \neq 0$ and $g(b) \neq 0$ then $g(x) = 0$ for some x between a and b. [‡]

‡ Hint: Proof by contradiction.

Part 3 - Applications

Q.13. Prove that if $f(0) = 0$ and $f'(x)$ is increasing then the function $g(x) = f(x)/x$ is increasing on $(0, \infty)$.

This is a deceptively hard problem, try to work it out yourself, in case you get stuck use the following steps.

(1) Write $g(x)$ as the slope of an appropriate secant line. Then use the Mean Value Theorem to relate $g(x)$ to the slope of some tangent.

(2) Find $g'(x)$ and rewrite it in terms of $f'(x)$ and $g(x)$.

(3) Use parts (1) and (2) to write $g'(x)$ entirely in terms of f' and use this to conclude that $g'(x) > 0$ for all $x > 0$.

Q.14. (1) What is wrong with the following application of l'Hospital's rule:

$$\lim_{x\to 1} \frac{x^3 + x - 2}{x^2 - 3x + 2} = \lim_{x\to 1} \frac{3x^2 + 1}{2x - 3} = \lim_{x\to 1} \frac{6x}{2} = 3$$

What is the correct limit?

(2) Find the following limits

(a) $\displaystyle\lim_{x\to 0} \frac{\tan x}{x}$ (b) $\displaystyle\lim_{x\to 0} \frac{\cos^2 x - 1}{x^2}$

Q.15. l'Hospital's rule is used in various forms all of which are closely related to each other.

l'Hospital's rule: If $\displaystyle\lim_{x\to 0+} f(x) = \lim_{x\to 0+} g(x)$ and both are equal to either 0 or ∞, and if $\displaystyle\lim_{x\to 0+} \frac{f'(x)}{g'(x)} = l$ then $\displaystyle\lim_{x\to 0+} \frac{f(x)}{g(x)} = l$.

Using only algebraic manipulations (no complicated proofs) derive the following versions from the above theorem.

(1) If $\displaystyle\lim_{x\to 0-} f(x) = \lim_{x\to 0-} g(x) = 0$ or ∞ and if $\displaystyle\lim_{x\to 0-} \frac{f'(x)}{g'(x)} = l$ then $\displaystyle\lim_{x\to 0-} \frac{f(x)}{g(x)} = l$.

(2) If $\displaystyle\lim_{x\to \infty} f(x) = \lim_{x\to \infty} g(x) = 0$ or ∞ and if $\displaystyle\lim_{x\to \infty} \frac{f'(x)}{g'(x)} = l$ then $\displaystyle\lim_{x\to \infty} \frac{f(x)}{g(x)} = l$.