PROBLEM SET 04

Part 1 - Limits and Continuity

Q.1. Proofs are not required for this problem. Describe your answers as precisely as possible and provide some explanation.

(1) Find a function which is discontinuous at $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ but continuous at all other points.

(2) Find a function which is discontinuous at $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ and 0 but continuous at all other points.

Q.2. For a non-empty set A let $-A$ denote the set of all $-x$ for x in A.

(1) Prove that if x is a lower bound of A then $-x$ is an upper bound of $-A$.

(2) Prove that if x is the greatest lower bound of A then $-x$ is the lowest upper bound of $-A$.

(3) Similarly prove that $\sup(-A) = -\inf A$.

Q.3. We say that a subset A of $\mathbb{R}$ is **open** if for every number x in A the interval $(x - r, x + r)$ is a subset of A for some $r > 0$.

(1) Determine, with proof, which of the sets $[0, 1]$, $(0, 1)$, $(0, 1]$, and $\mathbb{R}$ are open?

(2) Is the empty set open?

For a set A and a function $f : \mathbb{R} \to \mathbb{R}$ define $f^{-1}(A)$ to be the set of real numbers x which are mapped to A by f. The following is a very fundamental theorem about continuity, we'll verify it for a few functions.

Theorem. f is continuous iff $f^{-1}(A)$ is open for every open set A.

(3) For the function $f(x) = x^2$ find the set $f^{-1}(A)$ when A is one of the following sets: $(0, 1)$, $(-1, 0)$, $(-1, 1)$, $\mathbb{R}$, and the empty set.

(4) For $f(x) = x^2$ find a set A such that A is not open but $f^{-1}(A)$ is open. How does this not contradict the **Theorem**?

(5) For the function

$$g(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Find a set A such that A is open but $g^{-1}(A)$ is not open.

(6) For the function

$$h(x) = \begin{cases} 1/x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Find a set A such that A is open but $h^{-1}(A)$ is not open.

(7) (Optional) Prove the **Theorem**. The proof is easy but requires you to reason *very* precisely. It is good exercise to do if you have time.

Part 2 - Differentiation

Q.4. (1) Using the definition prove that if $f(x) = 1/x$ then $f'(a) = -1/a^2$ for $a \neq 0$.

(2) Prove that the tangent line to the graph of f at $(a, f(a))$ does not intersect the graph of f at any other point.

Q.5. (1) Using the definition prove that if $f(x) = 1/x^2$ then $f'(a) = -2/a^3$ for $a \neq 0$.

(2) Prove that the tangent line to the graph of f at $(a, f(a))$ intersects the graph of f at one other point.

Q.6. Suppose the function f is differentiable at a and let $c, d \neq 0$ be constants. Determine in terms of $f'(a)$ the following limits.

(1)

$$\lim_{h\to 0} \frac{f(a+ch) - f(a)}{h}$$

(2)

$$\lim_{h\to 0} \frac{f(a+ch) - f(a+dh)}{h}$$

Q.7. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be **even** if $f(-x) = f(x)$ for all x. f is said to be an **odd** function if $f(-x) = -f(x)$ for all x.

(1) Show that if f is an even function then $f'(-a) = -f'(a)$. (Draw a picture.)

(2) Show that if f is an odd function then $f'(-a) = f'(a)$. (Draw a picture.)

Q.8. Suppose that $f(x) \leq g(x) \leq h(x)$ for all x and that $\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$. Prove that $\lim_{x\to a} g(x) = L$. (This is usually called the **Squeeze theorem**.)

Q.9. (1) Suppose that $f(a) = g(a) = h(a)$, and that $f(x) \leq g(x) \leq h(x)$ for all x and that $f'(a) = h'(a)$. Prove that g is differentiable at a, and that $f'(a) = g'(a) = h'(a)$.

(2) Show that the conclusion does not follow if we omit the hypothesis $f(a) = g(a) = h(a)$.

Part 3 - Differentiation

Q.10. Using the definition find $f'(a)$ for $f(x) = \sqrt{x}$ and $x > 0$.

Q.11. Find $f'(x)$ if $f(x) = |x|^3$. Find $f''(x)$. Does $f'''(x)$ exist for all x?

Q.12. (1) Prove that the following function is differentiable at 0.

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{otherwise} \end{cases}$$

(2) More generally prove that if a function $f(x)$ satisfies $|f(x)| < x^2$ then $f(x)$ is differentiable at 0.

Q.13. Suppose that $f(a) = g(a)$ and that the left-hand derivative of f at a equals the right hand derivative of g at a. Define

$$h(x) = \begin{cases} f(x) & \text{if } x \leq a \\ g(x) & \text{if } x > a \end{cases}$$

Prove that h is differentiable at a.