# PROBLEM SET 10

## Part 1 - Taylor series

**Q.1.** For the polynomial $p(x) = x^3$ compute the Taylor polynomial at $x = 1$ of degree 3. Simplify the Taylor polynomial and verify that it equals the original polynomial.

**Q.2.** Compute the Taylor polynomial of the following functions:

(1) $e^{\sin x}$ at $x = 0$, degree 3 (2) $\dfrac{1}{x+1}$ at $x = 0$, degree $n$

**Q.3.** In this exercise we'll compute the **remainder term** of the Taylor polynomial. Assume that the function $f$ is differentiable enough number of times. Consider the Taylor polynomial of the $f(x)$ at $x = 0$ of degree $n$.

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \qquad a_i = \frac{f^{(i)}(0)}{i!}$$

**Theorem.** The remainder term $R_n(x) = f(x) - P_n(x)$ equals

$$R_n(x) = \int_0^x \frac{f^{(n+1)}(t)}{n!} \cdot (x-t)^n \, dt$$

(1) Prove the theorem directly for $n = 0$. (Note: $0! = 1$.)

(2) Find the Taylor polynomial of $f'(x)$ at $x = a$ of degree $n - 1$.

(3) Assume the following Leibniz's identity:

$$\text{If } g(x) = \int_0^x f(x,t)\, dt \text{ then } g'(x) = f(x,x) + \int_0^x \frac{\partial f(x,t)}{\partial x}\, dt$$

(This very useful trick is called *differentiating under the integral sign*, unfortunately we do not have enough language to prove this.)

Use part (2) and induction on the degree $n$ to prove the Theorem.

**Q.4.** This is a very difficult problem to write down rigorously, for this one it's ok to be vague in your arguments. In this problem $n, k$ are positive integers and all the Taylor series are at $x = 0$. *

(1) If $f(x) = x^n g(x)$. Find the Taylor series of $f(x)$ in terms of Taylor series of $g(x)$ (Don't think too hard). Hence find $f^{(k)}(0)$ in terms of the derivatives of $g$.

(2) If $f(x) = g(x^2)$ find $f^{(k)}(0)$ in terms of the derivatives of $g$. Hence find the Taylor series for $f$ in terms of $g$.

(3) Find the Taylor series for $e^{(x^2)}$.

(4) If $f(x) = g(x^n)$, find $f^{(k)}(0)$ in terms of the derivatives of $g$. Hence find the Taylor series of $f$ in terms of $g$.

(5) Find the Taylor series for $\sin(x^4)$.

* Hint: It is very crucial here that you're only asked to find the derivatives at $x = 0$.

## Part 2 - Numerical Computations

For this week's HW you're allowed (and required) to use wolframalpha (or calculator) to do the computations/simplify large fractions.

A table of Taylor series can be found on Pg. 426 in the book.

**Q.5.** (1) Suppose the coefficients of Taylor series of $f$ are $c_i$ and of $g$ are $d_i$ at $x = a$. Find the Taylor series of $fg$ at $x = a$ in terms of $c_i, d_i$.

(2) Find the general formula for the $n^{th}$ derivative of the product $(fg)^{(n)}(a)$ in terms of the derivatives of $f$, $g$ at $a$.

**Q.6.** Use estimates on the remainder term of the appropriate Taylor series to compute the following quantities within the prescribed error.

(1) $\cos 1$, error $< 10^{-2}$

(2) $\sin 0.01$, error $< 10^{-10}$

(3) $e$, error $< 10^{-2}$

(4) $\log 1.1$, error $< 10^{-4}$

**Q.7.** Show that the remainder terms for the Taylor series of $\log(1 + x)$ and $\arctan(x)$ at $x = 0$ grows with $n$ when $x > 1$. (Hence the standard Taylor series cannot be used to approximate these numbers.)

**Q.8.** Taylor series are useful in theory but not so great for numerical approximations in practice.

(1) Find the value of $\pi/4$ correct up to 1 decimal place using Taylor series of $\arctan(x)$. How many terms of the Taylor series will you need, to determine the value of $\pi/4$ correct up to 2 decimal places?

(2) Prove the following identities:

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\arctan(u) + \arctan(v) = \arctan\left(\frac{u + v}{1 - uv}\right)$$

(3) Show that

$$\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{3}$$

$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}$$

(4) Use the second formula in part (3) to compute $\pi$ correct up to 4 decimal places.

## Part 3 - Integral Computations

**Q.9.** For this week do Q.3, and Q.4 problems - i) to v) on Pg. 378-379 from Ch.19.