PROBLEM SET 08

PART 1 - FUNDAMENTAL THEOREM OF CALCULUS

Q.1. Compute $F'(x)$ for the following functions*

(1) $\int_a^{x^3} \sin^3 t dt$

(2) $\int_x^b \frac{1}{1+t^2+\sin^2 t} dt$

(3) $\int_a^b \frac{x}{1+t^2+\sin^2 t} dt$

(4) $\int_a^x \left(\int_b^y \frac{1}{1+t^2+\sin^2 t} dt \right) dy$

(5) $\int_0^x \frac{1}{1+t^2} dt + \int_0^{1/x} \frac{1}{1+t^2} dt$

(6) $\int_{-\cos x}^{\sin x} \frac{1}{\sqrt{1-t^2}} dt$

Q.2. Find $(f^{-1})'(0)$ if

(1) $f(x) = \int_0^x 1 + \sin(\sin t) dt$

(2) $f(x) = \int_1^x \cos(\cos t) dt$

Q.3. Suppose f is differentiable with $f(0) = 0$ and $0 < f'(x) \leq 1$. Show that for all $x \geq 0$

$$\int_0^x f^3 \leq \left(\int_0^x f \right)^2$$

Q.4. (1) Find $F'(x)$ if $F(x) = \int_0^x x.f(t) dt$ (Be careful: it's not $x.f(x)$).

(2) Prove that†

$$\int_0^x f(t)(x-t) dt = \int_0^x \left(\int_0^u f(t) dt \right) du$$

(3) Prove that

$$\int_0^x f(t)(x-t)^2 dt = 2 \int_0^x \left(\int_0^{u_1} \left(\int_0^{u_2} f(t) dt \right) du_2 \right) du_1$$

Q.5. (1) Suppose $G'(x) = g(x)$ and $F'(x) = f(x)$. Prove that the function $y(x)$ satisfies the differential equation (*a separable differential equation*)

$$g(y).y' = f(x)$$

for all x in some interval, if and only if there is a number c such that

$$G(y) = F(x) + c$$

(2) 'Solve' the following differential equations

(a) $y' = \frac{1+x^2}{1+y}$

(b) $y' = \frac{-1}{1+5y^4}$

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*Hint: Don't forget the chain rule!

†Hint: Differentiate both sides with respect to x.

PART 2 - IMPROPER INTEGRALS

Q.6. The limit $\lim\limits_{x\to\infty} \int\limits_a^x f$, if it exists, is denoted by $\int\limits_a^\infty f$ and called an 'improper integral'. Similarly for the improper integral $\int\limits_{-\infty}^a f$.

(1) Find $\int\limits_1^\infty t^r dt$, when $r < -1$.

(2) Using $\int\limits_1^a t^{-1}\, dt + \int\limits_1^b t^{-1}\, dt = \int\limits_1^{ab} t^{-1}\, dt$, show that $\int\limits_1^\infty t^{-1}\, dt$ does not exist. ‡

Q.7. Assume the following the statement:

> Suppose that $f(x) \geq 0$ for $x \geq 0$ and that $\int\limits_a^\infty f$ exists. If $0 \leq g(x) \leq f(x)$ for all $x \geq 0$ and g is integrable on the interval $[0, N]$ for all $N > 0$ then $\int\limits_0^\infty g$ exists.

- (Optional) Prove the above statement.
- For which of the following functions does the integral $\int\limits_0^\infty f$ exist?§

(1) $\dfrac{1}{1+x^2}$

(2) $\dfrac{1}{\sqrt{1+x^3}}$

(3) $\dfrac{x}{1+x^{3/2}}$

(4) $\dfrac{1}{\sqrt{1+x}}$

Q.8. The improper integral $\int\limits_{-\infty}^{\infty} f$ is defined as $\int\limits_{-\infty}^{0} f + \int\limits_0^\infty f$ if both the integrals exist.

(1) Show that $\int\limits_{-\infty}^{\infty} \dfrac{1}{1+x^2} dx$ exists.

(2) Determine the limit $\lim\limits_{N\to\infty} \int\limits_{-N}^{N} x\, dx$.

(3) Show that $\int\limits_{-\infty}^{\infty} x\, dx$ does not exist.

Q.9. It is possible to have another kind of improper integral, one where the function itself is unbounded but the limits are finite.

For $-1 < r < 0$ draw the graph of x^r and determine $\lim\limits_{\epsilon\to 0^+} \int\limits_\epsilon^a x^r dx$.

(This limit is written as $\int\limits_0^a x^r dx$.)

‡ Hint: What can you say about $\int\limits_1^{2^n} t^{-1}\, dt$?

§ Hint: You might have to break $[0, \infty)$ into multiple intervals and analyze each interval separately.