Fermat's Theorem for Polynomials

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Theorem 0.1 (Riemann-Hurwitz). *Given an N sheeted branched covering map of compact Riemann surfaces $\pi : S' \to S$ we have the identity*

$$g(S') = N(g(S) - 1) + 1 + \sum_{P \in S'} \frac{e_P - 1}{2} \tag{0.1}$$

where $g(S)$ and $g(S')$ denote the genus of S and S', the sum is over the ramified points P and e_P denotes the ramification degree of P.

Corollary 0.2. *If there is a branched covering $\pi : S' \to S$ of compact Riemann surfaces then $g(S') \geq g(S)$. In particular there is no branched covering $\pi : \widehat{\mathbb{C}} \to S$ unless $S = \widehat{\mathbb{C}}$.*

Proof: N is at least 1 and the sum $\sum_{P \in S'} \dfrac{e_P - 1}{2}$ is always non-negative, the result follows. □

Theorem 0.3. *Every non-constant complex differentiable map between compact Riemann surfaces is a branched covering.*

For $d > 2$ denote by S_d the compactified Riemann surface defined by the equation

$$S_d = \text{ compacification of } \{(z, w) : z^d + w^d = 1\} \tag{0.2}$$

Lemma 0.4. *Genus of S_d is $\binom{d-1}{2}$, hence for $d > 2$ the genus is at least 1.*

Proof: The projection onto the z coordinate and the equation $w^d = 1 - z^d$ gives us a natural d sheeted branched covering $S_d \to \widehat{C}$. This covering is branched exactly over the roots of unity where all the d sheets ramify. Further the monodromy at each branch point is exactly $\mathbb{Z}/d$ and there are d branch points so the point at ∞ is not a branch point (see Figure). Thus there are d many branch points with ramification degree d. Plugging into the Riemann-Hurwitz formula we get

$$g(S_d) = d(0 - 1) + 1 + \frac{d(d-1)}{2} \tag{0.3}$$

$$= \frac{(d-1)(d-2)}{2} = \binom{d-1}{2} \tag{0.4}$$

□

Theorem 0.5. *For $d > 2$ are no non-constant solutions of the equation*

$$x(t)^d + y(t)^d = z(t)^d \tag{0.5}$$

where $x(t), y(t), z(t)$ are polynomials with complex coefficients.

Proof: Suppose $x(t)$, $y(t)$, $z(t)$ are non-constant polynomials satisfying the equation (0.5). Without any loss of generality assume that $x(t)$, $y(t)$, $z(t)$ have no common factors. We can define a complex differentiable map

$$\pi : \widehat{\mathbb{C}} \to S_d \tag{0.6}$$

$$p \mapsto \left(\frac{x(p)}{z(p)}, \frac{y(p)}{z(p)} \right) \tag{0.7}$$

where p is either a 0 of $z(t)$ or $p = \infty$ then p maps to $\lim\limits_{p \to \infty} \left(\dfrac{x(p)}{z(p)}, \dfrac{y(p)}{z(p)} \right)$. Because the Riemann surface is compact all limits exist and the above map is well defined. As it is defined using polynomials it is also complex differentiable and hence is a branched covering. Together with Lemma 0.4 this contradicts Corollary 0.2. □