Inverses of polynomials

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Today's goal is to build Riemann surfaces on which inverses of polynomials $f(z)$ can be well defined.

1 Riemann surface for $\log z$

If we try to define $\log z$ naively using polar coordinates we get

$$\log(re^{i\theta}) = \log r + i\theta \tag{1.1}$$

We have a discontinuity at $\theta = 0$ and $\theta = 2\pi$ so we need to make a branch cut, say at $\theta = 0$, to get a well defined function. For θ varying in the range $(0, 2\pi)$ we get one branch function. We get other (distinct) branches by adding multiples of 2π to θ.

$$\log(re^{i\theta}) = \log r + i(\theta + 2n\pi) \tag{1.2}$$

To create a Riemann surface on which $\log z$ is well defined we need $\mathbb{Z}$ many copies of $\mathbb{C}$, on which we make branch cuts and glue them together.

2 Winding numbers

In order to generalize the above situation to more complicated functions we analyze what is happening here. For this we need the notions of winding numbers.

Consider a non-constant complex differentiable function $f : \mathbb{C} \to \mathbb{C}$. Fix a point $p \in \mathbb{C}$ be a complex number. Let $\gamma : [0, 2\pi] \to \mathbb{C}$ with $\gamma(0) = \gamma(2\pi)$ be an embedded loop in $\mathbb{C}$ (no self intersections) containing p in it's interior that goes counterclockwise. We say that γ is **sufficiently close** to p if there are no points $p' \neq p$ in the interior of γ such that $f(p) = f(p')$.

Proposition 2.1. *For all points $p \in \mathbb{C}$ there exists a loop γ sufficiently close to p.*

Proof: Consider the function $g(z) = f(z) - f(p)$. The zeroes of this complex differentiable function are exactly the points p' such that $f(p') = f(p)$. Complex differentiable functions have isolated zeroes and hence p has a neighborhood U which does not contain any other zero of g. Any loop γ in U is sufficiently close to p. $\square$

Definition 2.2. Let γ be a loop sufficiently close to p. The composition $f \circ \gamma$ is a loop around $f(p)$ (not necessarily embedded). The **winding number** $w_f(p)$ is the number of counterclockwise revolutions made by $f \circ \gamma$ around $f(p')$ minus the number of clockwise revolutions.

Proposition 2.3. *Without any loss of generality assume that $p = 0$ and $f(p) = 0$. Let $\gamma(t)$ be a loop sufficiently close to p then the winding number is given by*

$$w_f(0) = \frac{\log(f(\gamma(2\pi))) - \log(f(\gamma(0)))}{2\pi i} \tag{2.1}$$

where $\log(z)$ is any branch of the logarithm.

Proof: For every single counterclockwise revolution $\log(z)$ increases by $2\pi i$ and for every single counterclockwise revolution $\log(z)$ decreases by $2\pi i$. The result follows. $\square$

Example 2.4. For the function $f(z) = z^n$ the winding number around 0 is n and the winding number around $p \neq 0$ is 1.

2.1 Points with winding number > 1

For the function z^n the only complex number z for which the winding number is bigger than 1 is $z = 0$ and this is the point where we've to make a branch cut. This suggests that to define inverses of more complicated functions we need to figure out the points at which the winding number is greater than 1.

Let $f(z)$ be a complex differentiable function. Consider a point $p \in \mathbb{C}$ and let $f(z) = \sum_{i=0}^{\infty} a_i(z-p)^i$ be the Taylor series of near p.

Proposition 2.5. *The winding number of $w_f(p)$ is the smallest non-zero integer i such that $a_i \neq 0$.*

Proof: By shifting origin if necessary assume that $p = 0$ and $f(p) = 0$ so that the Taylor expansion looks like

$$f(z) = \sum_{i=n}^{\infty} a_i z^i = z^n \sum_{i=n}^{\infty} a_i z^{i-n} = z^n g(z)$$

for some $n > 0$ where $a_n \neq 0$ and $g(z) = \sum_{i=n}^{\infty} a_i z^{i-n}$. The function $g(z)$ has the property that $g(0) \neq 0$. If we choose a loop γ sufficiently close to 0 such that $g(\gamma(t))$ stays very close to $g(0)$ then $g(z)$ is almost constant non-zero number and hence the winding number $w_f(p) = w_{z^n}(p)$ which equals n. $\square$

With this proposition the points with winding number bigger than 1 are easily determined.

Corollary 2.6. *$w_f(p) > 1$ iff $f'(p) = 0$ i.e. the points with winding number bigger than 1 are exactly the critical points of f.*

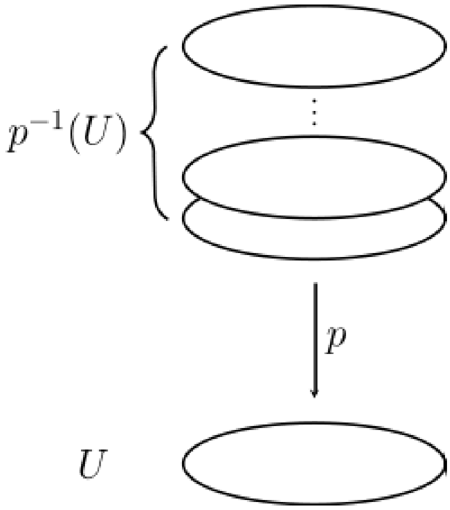
3 Covering spaces

Consider the function $f(z) = z^n$. For any non-zero p the set $f^{-1}(p)$ contains exactly n elements $p_1, \cdots, p_n$. More is true, there exists a neighborhood $U \ni p$ such that $f^{-1}(U)$ is isomorphic to n non-intersecting neighborhoods of the n points $U_1 \ni p_1, \cdots, U_n \ni p_n$. Furthermore the restriction of the map $f|_{U_i}$ is an isomorphism[1]. These properties define what is called a covering space.

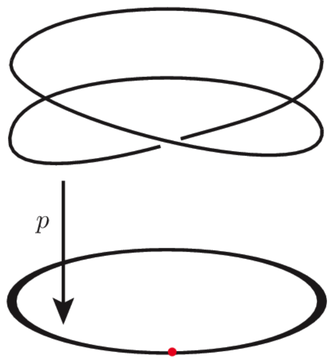
Definition 3.1. A map $p : X \to Y$ is a **covering map** if for every point $y \in Y$ there exists a neighborhood $U \ni y$ such that

1. $p^{-1}(U)$ is a disjoint union of subsets $\{V_i\}_{i \in I}$ of X for some indexing set I
2. The restriction $p|_{V_i} : V_i \to U$ is an isomorphism for each $i \in I$.

X is called a **covering space** of Y.



Covering Space



Branched Covering Space

So the map $z^n : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ is a covering map. A map which is a covering map at all points except some isolate ones is called a **branched covering**. So $z^n : \mathbb{C} \to \mathbb{C}$ is a branched covering.

More generally every complex differentiable function $f(z) : \mathbb{C} \to \mathbb{C}$ is a branched covering and it's restriction to the complement of critical points of $f(z)$ is a covering map.

[1] homeomorphism

3.1 Monodromy

Consider again the function $z^n : \mathbb{C} \to \mathbb{C}$. Consider a loop around the origin in the **target** complex plane passing through $z = 1$. If we traverse the loop once and try to lift the loop back up to the domain z then the loop does not remain a loop but becomes a path connecting 1 to the root of unity $e^{2\pi i/n}$. In this we say that the covering z^n has non-trivial **monodromy**. Monodromy is the mathematical reason why need to glue complex planes to create Riemann surfaces.[2]

3.2 Constructing the Riemann surface

So the strategy to define the inverse of a polynomial $f(z)$ is the following:

1. Find the critical points of $f(z)$
2. On the non-critical points $f(z)$ is a covering map, find the number of *sheets* over each point using winding numbers
3. Make branch cuts at each critical point and glue the sheets of complex plane to get a Riemann surface.

4 Exercises

Exercise 4.1. Show that the winding number of $f(z) = z^n$ at $p \in \mathbb{C}$ is equal to

$$w_f(p) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-p)^{n+1}} dz \tag{4.1}$$

γ is a loop sufficiently close to p.

Prove this formula for an arbitrary complex differentiable function $f(z)$.

Exercise 4.2. Fill in the gaps in the proof of Proposition 2.5.

Exercise 4.3. Construct Riemann surfaces for the following polynomials:

1. $f(z) = z^2 - 2z$
2. $f(z) = z^3 - 3z$

[2] There is in fact a monodromy group that is acting upon the various sheets via permuting them.