1 Topological Preliminaries: Solutions

Question 4. g) Good cover for S^2:

For any triangulation of the sphere S^2, the faces of the triangulation form a good cover. This is more generally true for any surface.

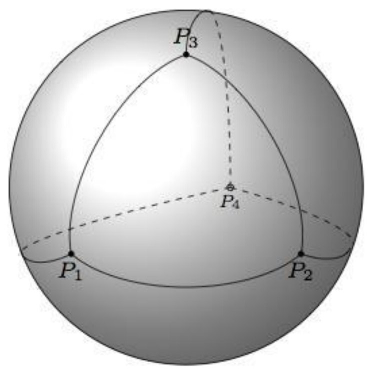

Figure 1: Triangulating a Sphere. (googled image)

Question 5. Good cover for $X \vee Y$:

In general, if X and Y have good covers $\mathcal{U}$ and $\mathcal{U}'$ such that the 'point of gluing' belongs to a unique $U \in \mathcal{U}$ and $U' \in \mathcal{U}'$. Then we can just use the open covers $\mathcal{U}$ and $\mathcal{U}'$ but glue the open sets U and U' at a point. We can easily check that this indeed forms a good cover. The only non-trivial fact needed to be shown is that if U and U' are contractible then so is $U \vee U'$ (exercise).

Question 2. b) If $X \times Y$ is contractible then so is X:

(I think you don't need the contractibility of Y, if you find a flaw in the following proof please let me know.)

Because $X \times Y$ is contractible, there is a point $(x_0, y_0) \in X \times Y$ and a continuous map

$$\Phi : X \times Y \times [0,1] \to X \times Y$$

such that

$$\begin{aligned} \Phi(x, y, 0) &= (x, y) \\ \Phi(x, y, 1) &= (x_0, y_0) \end{aligned}$$

for all $(x, y) \in X \times Y$ i.e. there are "continuously varying paths" connecting each point in $X \times Y$ to (x_0, y_0).

Restrict the map Φ to $X \times \{y_0\}$ so that we get a map

$$\begin{aligned} \Phi|_{y=y_0} : X \times [0,1] &\to X \times Y \\ (x, t) &\mapsto \Phi(x, y_0, t) \end{aligned}$$

This is almost the map we want. Now we compose with the projection onto the first component $\pi_X : X \times Y \to X$ to get a map $\Psi = \pi_X \circ \Phi$

$$\Psi : X \times [0,1] \xrightarrow{\Phi|_{y=y_0}} X \times Y \xrightarrow{\pi_X} X$$

One can check that $\Psi(x, 0) = x$ and $\Psi(x, 1) = x_0$. Further, Ψ is continuous as it is a composition of two continuous functions and hence Ψ witnesses the contractibility of X.

(Try to draw pictures of what this proof is saying geometrically.)