How Curved is a Potato?

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0 Introduction

> There is nothing more deceptive than an obvious fact.
>
> Sherlock Holmes

We have an intuitive understanding of what it means for something to be curved. A circle has some curvature where as a line has no curvature. The *larger* the radius of the circle the *smaller* the curvature (after all a line is a circle with infinite radius). Similarly for surfaces, a sphere is curved but a plane is not. The larger the radius of the sphere the smaller the curvature (a plane is a sphere with infinite radius).

Line	No curvature
Circle	Curvature $\sim$? 1/radius
Sine Curve	Varying Curvature
Plane	No curvature
Sphere	Curvature $\sim$? 1/radius
Cylinder	??

But there is a subtle difference between curves and surfaces. It is possible to take a string and form a circle without stretching or compressing it, but it is not possible to take a flat piece of paper and mold it into a sphere without stretching or compressing.[1]

Or is it?

Hmmm.

The curvature of the sphere is in some sense *intrinsic* to the sphere whereas the curvature of a circle isn't. This difference between curves and surfaces was first quantified by Gauss using what's now called Gaussian Curvature and later generalized by Riemann leading to the creation of the field of Riemannian Geometry.

In this class, we'll learn how linear algebra and calculus naturally help us figure out the *correct* definition of Curvature for surfaces - Riemann's generalization of this to higher dimensions lies at the heart of Riemannian Geometry. We'll try to understand the geometric significance of the various kinds of curvature - Gaussian, Mean, and Principal, the statement of Gauss' Theorema Egregium, and what it means for the Gaussian Curvature to be *intrinsic* to the surface.

[1] Neglect the thickness.

1 How Curved are Curves?

> The real problem in speech is not precise language. The problem is clear language.
>
> Richard Feynmann

1.1 Curvature of Curves

Our first goal is to define a quantitative measure of curvature. For example, we would like to determine what the curvature of the sine curve is at various points. For us a **curve** C is the image of a smooth function $c(t) : (a, b) \to \mathbb{R}^2$ or $\mathbb{R}^3$. The function $c(t)$ is called a **parametrization** of the curve C.

Example 1.1.

a) The circle $x^2 + y^2 = r^2$ can be parametrized as

$$c(t) = (r\cos(\omega t), r\sin(\omega t))$$

where ω is any non-zero constant.

b) The sine curve can be parametrized as

$$c(t) = (t, \sin t)$$

c) More generally, the graph $y = f(x)$ can be parametrized as

$$c(t) = (t, f(t))$$

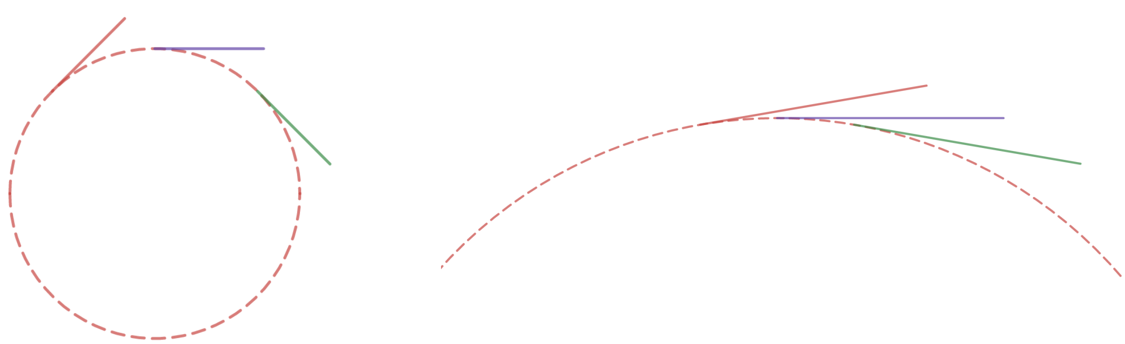


Figure 1: The smaller the radius of the circle the faster the rate of change of the tangent vector.

Recall that $c'(t)$ is the **tangent vector** to the curve C at the point $c(t)$. If a curve has large curvature then the angular component of the tangent vector should change rapidly as we move along the curve, conversely if a curve has small curvature then the angular component should change slowly, which suggests the following definition.

Definition 1.2. The **curvature** κ of a curve C parametrized by $c(t)$ is defined to be the rate of change of the angular component of $c'(t)$.

Recall that every non-zero vector $\vec{v}$ can be written as $\vec{v} = \|\vec{v}\|.u$, where u is the unit vector in the direction of v. The length $\|\vec{v}\|$ is the **radial component** of v and $u = \vec{v}/\|\vec{v}\|$ is the **angular component** of v. We can make the above definition more precise as follows.

Definition 1.3. The **curvature** of a curve C parametrized by $c(t)$ is defined to be

$$\begin{aligned} \kappa &:= \text{ rate of change of the angular component of } c'(t) \\ &= \left\| \frac{d}{dt}\left(\frac{c'(t)}{\|c'(t)\|}\right)\right\| \cdot \frac{1}{\|c'(t)\|} \end{aligned}$$

assuming $c'(t) \neq 0$.

We divide by $\|c'(t)\|$ to normalize the rate at which we're moving along the curve.

Remark 1.4. It's not too hard to find parametrizations with $c'(t) \neq 0$, at least in a small neighborhood of the point under consideration. For example, all the examples in Example 1.1 satisfy this condition (check this).

From this definition we immediately get a way to compute the curvature in the special case when $\|c'(t)\| = 1$ for all t. Such a parametrization is called a **unit speed parametrization**.

Proposition 1.5. *For a unit speed parametrization $c(t)$ the curvature is given by*

$$\kappa = \left\| \frac{d}{dt}\left(c'(t)\right)\right\| = \|c''(t)\|$$

Example 1.6. A circle of radius r can be parametrized as

$$c(t) = (r\cos(\omega t), r\sin(\omega t))$$

for some constant ω. For this parametrization

$$\|c'(t)\| = \|(-r\omega\sin(\omega t), r\omega\cos(\omega t))\| = r\omega$$

If we choose $\omega = 1/r$ the parametrization becomes unit speed. In this case the curvature equals

$$\begin{aligned} \kappa &= \|c''(t)\| \\ &= \|(r\cos(t/r), r\sin(t/r))\| \\ &= \|(-1/r\cos(t/r), -1/r\sin(t/r))\| \\ &= \frac{1}{r} \end{aligned}$$

:D

Theorem 1.7. *The circle of radius r has constant curvature of $1/r$.*

It is not easy to find unit speed parametrizations for more complicated curves (try it out). We need a formula which is valid even for non-unit speed parametrizations.

Theorem 1.8. *The* ***curvature*** *of a curve C parametrized by $c(t)$ is given by*

$$\kappa = \frac{\|c'(t) \times c''(t)\|}{\|c'(t)\|^3}$$

assuming $c'(t) \neq 0$.

The proof is in the following exercise.

Question. 1. Let $c(t)$ be a parametrization of C which is not necessarily unit speed. Let $r(t) = \|c'(t)\|$ be the radial component of the velocity, suppose that $r(t)$ is never 0. Let $\theta(t) = c'(t)/\|c'(t)\|$ be the angular component of velocity. Note that $r(t)$ is a scalar valued function and $\theta(t)$ is a vector valued function. The curvature is given by

$$\kappa = \frac{\|\theta'(t)\|}{r(t)}$$

Use the following steps to compute κ.

a) Express $c'(t)$ in terms of $r(t)$ and $\theta(t)$.

b) Show that $\theta'(t) \cdot \theta'(t)$ is a constant and equals 1.

c) Show that $\theta'(t) \cdot \theta''(t) = 0$ and hence $\theta'(t) \perp \theta''(t)$ for all t.

d) Conclude that $\|\theta'(t) \times \theta''(t)\| = \|\theta''(t)\|$.

e) Express $c''(t)$ in terms of $r(t)$, $\theta(t)$, and their derivatives.

f) Prove Theorem 1.8.

Question. 2. Use the parametrization $(r\cos(\omega(t)), r\sin(\omega(t)))$ to compute the curvature of a circle of radius r, where now ω is a function of t.

Question. 3. An easy computation shows that curvature of the graph $y = x^4$ is 0 at $x = 0$. What does this mean geometrically?

Question. 4.

a) Find the curvature of the graph $y = f(x)$ where $f : \mathbb{R} \to \mathbb{R}$ is any smooth function.

b) Guess the relationship between the curvature of the graphs $y = f(x)$ and $y = kf(x/k)$, where k is a positive real number. Prove your guess.

c) Guess the relationship between the curvature of curves parametrized by $c(t)$ and $kc(t)$, where k is a positive real number. Prove your guess.

The formula for curvature in Theorem 1.8 explicitly depends on the parametrization $c(t)$ but the curvature itself does not. Have you seen such phenomenon before? Perhaps in your favorite branch of mathematics?

Remark 1.9 (Generalized Curvature)**.** Often in calculus, not taking absolute value or norm gives us quantities which do not have explicit geometric significance but are easier to manipulate. For example, the integral itself has no geometric significance rather the area under a curve is the *absolute value* of the integral, however, the Fundamental Theorem of Calculus does not contain any absolute values. Analogously, we'll call the vector quantity

$$\vec{\kappa} := \frac{d}{dt}\left(\frac{c'(t)}{\|c'(t)\|}\right) \cdot \frac{1}{\|c'(t)\|}$$

the **Generalized Curvature** for the lack of a better name. The following theorem is the analogue of Fundamental Theorem of Calculus for Curvature.

Question. 5. For a curve C with a parametrization $c(t)$ compute the integral

$$\int_{t_0}^{t_1} \vec{\kappa}\, ds$$

(Hint: Think about what this integral is computing geometrically. It might help to first solve this problem for a unit speed parametrization).

1.2 Curvature and Taylor approximation

Let us now consider curves which are graphs $y = f(x)$. For simplicity we'll assume that 0 is a critical point of $f(x)$ i.e. $f'(0) = 0$, and that $f''(0) \geq 0$. The following theorem is an easy exercise.

Theorem 1.10. *If* 0 *is a the* ***critical point*** *of* $f : \mathbb{R} \to \mathbb{R}$ *then the curvature of the graph* $y = f(x)$ *at* $(0, f(0))$ *equals* $f''(0)$.

Question. 6. Prove this.

Thus the degree 2 Taylor approximation of $f(x)$ at the critical point $x = 0$ is given by

$$\begin{aligned} f(x) &\approx f(0) + f''(0)\frac{x^2}{2} \\ &= f(0) + \kappa\frac{x^2}{2} \end{aligned} \tag{1.1}$$

There is another familiar function which has the same Taylor approximation. The equation of a circle of radius r centered at the origin in $\mathbb{R}^2$ is

$$x^2 + y^2 = r^2$$

Near the point $(0, -r)$ we can express the circle as the graph $y = -\sqrt{r^2 - x^2}$.

Question. 7. Show that the degree 2 Taylor approximation of $-\sqrt{r^2 - x^2}$ at $x = 0$ is given by

$$-\sqrt{r^2 - x^2} \approx -r + \frac{x^2}{2r} \tag{1.2}$$

Notice that $1/r$ which is the coefficient of $x^2/2$ is exactly the curvature of the circle. We can interpret Equations (1.2) and (1.1) as saying that,

Proposition 1.11. *If the curvature of the curve $y = f(x)$ is κ at a critical point p then* ***the circle that best approximates the curve at p has radius $1/\kappa$****.*

Since we can always rotate a curve without changing it's curvature, Proposition 1.11 is true in much more generality and we can drop the condition that p is a critical point. We know from calculus that the first derivative gives us the slope of the tangent line. Here we are saying that the second derivative gives us the *tangent* circle[2].

Question. 8. Think about how you might generalize the above methods to surfaces.

[2]The technical term is an *osculating* circle.

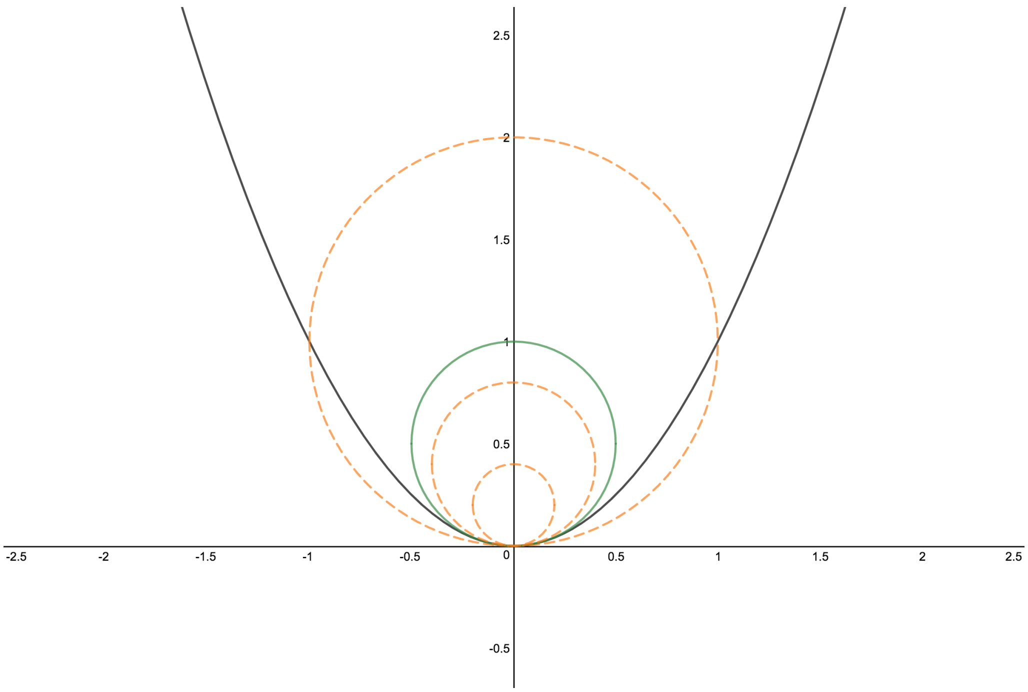


Figure 2: A circle of radius $1/2$ best approximates the parabola $y = x^2$ at $x = 0$.

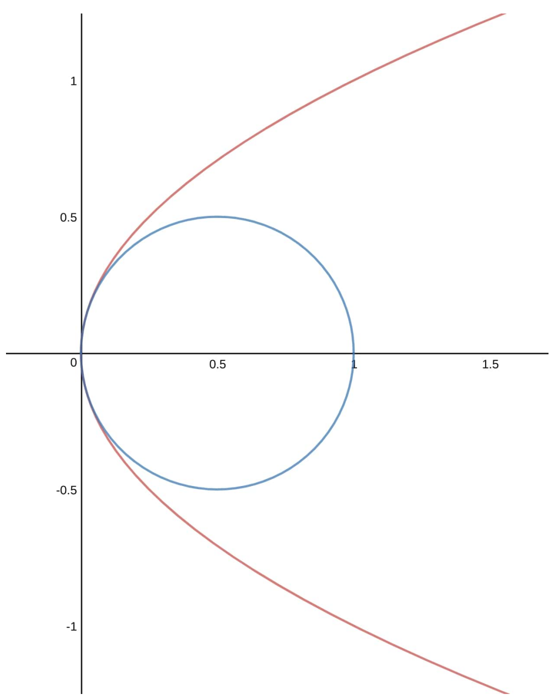


Figure 3: Even if our curve is not a graph the same interpretation remains true.

2 How Curved is a Potato?

> The heart of mathematics consists of concrete examples and concrete problems. Big general theories are usually afterthoughts based on small but profound insights; the insights themselves come from concrete special cases.
>
> Paul Halmos

Surfaces are much more varied than curves. The standard examples of surfaces are a plane, a cylinder, a saddle, a sphere, a torus etc. It is hard to come up with a single number that completely describes how curved these objects are. So the better question is what's the right amount of information we need to quantify curvature of these objects?

2.1 Defining Curvature for Graphs

For us a surface S is just a connected smooth 2-dimensional subset of $\mathbb{R}^3$. We want our surfaces to be nice, with no sharp edges and no self-intersections.[3]

It is unclear that we can use an approximating sphere or something similar to define curvature. But we know that for curves the curvature measures the *quadratic behavior* of the curve. As with curves, we'll start by analyzing graphs of functions $f(x,y) : \mathbb{R}^2 \to \mathbb{R}$

$$S = \{(x,y,z) : z = f(x,y)\}$$

We'll further assume that $p = (0,0)$ is a critical point of $f(x,y)$ i.e.

$$f_x(p) = 0 = f_y(p)$$

Our experience with curves suggests that we should define the curvature of S at p to be the triple

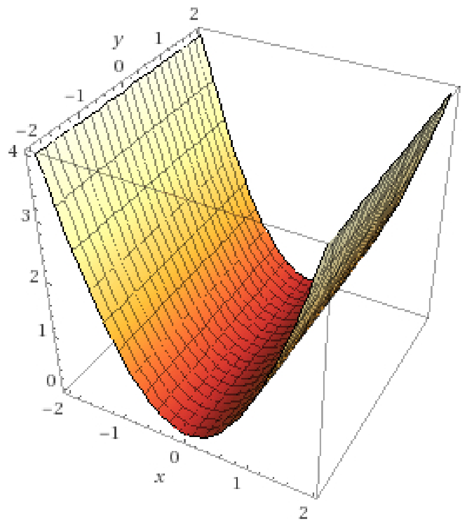
$$\kappa \stackrel{?}{=} (f_{xx}(p), f_{xy}(p), f_{yy}(p))$$

but there is a catch.

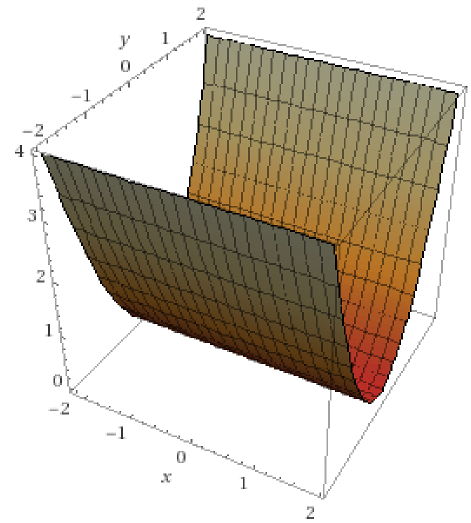
2.2 The Catch

We can rotate $\mathbb{R}^3$ about the $z-$axis and the new surface S' will still satisfy the properties mentioned above. The new surface S' will be described by a completely different function $g(x,y)$ and the second derivatives of $g(x,y)$ will be different from those of $f(x,y)$. However, the curvatures of S and S' should be the same. This problem did not arise for curves because there is no way to rotate $\mathbb{R}^2$ about the y-axis.

[3] A surface is a connected smooth 2-dimensional submanifold of $\mathbb{R}^n$ without a boundary.



(a) $f(x, y) = x^2$ has second derivatives $(2, 0, 0)$



(b) $f(x, y) = y^2$ has second derivatives $(0, 0, 2)$

Figure 4: The two surfaces are related to each other by a rotation about the z-axis but they have very different second derivatives.

Question. 1. Rotate the parabolic cylinder $z = x^2$ about the z-axis by an angle θ in the counter-clockwise direction. Describe the new surface as the graph of some function $f_\theta(x, y)$. Find the second derivatives of $f_\theta(x, y)$.

2.3 Rotations and Second Derivatives

We need to understand how the second derivatives change when we rotate the xy-plane about the z-axis. There are two tricks to do this efficiently.[4]

Trick #1 Instead of analyzing the second derivatives we analyze the degree 2 Taylor polynomial

$$2f(x, y) \approx 2f(p) + f_{xx} \cdot x^2 + 2f_{xy} \cdot xy + f_{yy} \cdot y^2$$

Trick #2 We rewrite the degree 2 Taylor polynomial in matrix form

$$2f(x, y) \approx 2f(p) + \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

The rotation or the reflection of the xy-plane about the z-axis is a linear transformation and hence can be described by some orthogonal matrix A

$$\begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} x' \\ y' \end{bmatrix}$$

[4]In order to avoid clutter we'll stop writing the (p). All the derivatives are being taken at the critical point $p = (0, 0)$.

Plugging this in the right hand side of the Taylor polynomial we get

$$\begin{aligned} 2f(x,y) &\approx 2f(p) + f_{xx}x^2 + 2f_{xy}xy + f_{yy}y^2 \\ &= 2f(p) + \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= 2f(p) + \begin{bmatrix} x' & y' \end{bmatrix} A^T \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} A \begin{bmatrix} x' \\ y' \end{bmatrix} \end{aligned} \tag{2.1}$$

Notice that the right hand side is still quadratic, and not surprisingly this is **the Taylor approximation of f in terms of (x', y')**.

Question. 2. Let A denote reflection of the xy-plane about the line $x = y$.

a) Express A as a 2×2 matrix.

b) Expand the right hand side of (2.1) for this A and verify that this is indeed the degree 2 Taylor approximation in the new variables.

Question. 3. Suppose that $A = \begin{bmatrix} m & 0 \\ 0 & n \end{bmatrix}$ where m, n are non-zero real numbers. This linear transformation corresponds to scaling the x, y axes by m, n respectively. Expand the right hand side of (2.1) for this A and verify that this is indeed the degree 2 Taylor approximation in the new variables.

Definition 2.1. The matrix $\begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}$ is called the **Hessian** of f, denoted $\mathrm{Hess}(f)(x,y)$.

Theorem 2.2. *At a critical point p, if the coordinates change according to the linear transformation*

$$\begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} x' \\ y' \end{bmatrix}$$

then the Hessian $\mathrm{Hess}(f)$ *changes as*

$$\begin{bmatrix} f_{x'x'} & f_{x'y'} \\ f_{x'y'} & f_{y'y'} \end{bmatrix} = A^T \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} A \tag{2.2}$$

We say that $\mathrm{Hess}(f)$ *is a 2-tensor at a critical point.*

Remark 2.3. The assumption that we're at a critical point i.e. $f_x = 0 = f_y$ is extremely crucial here and cannot be dropped. If, for example, we further had $f_x = 0 = f_y$ and $f_{xx} = 0 = f_{xy} = f_{yy}$ then we will get a similar change of coordinate rule for the third derivatives, which will be a 3-tensor.

2.4 The Curvatures

The Curvature(s) should be *invariants* of the matrix $\text{Hess}(f)$ which remain unchanged when we apply the transformation (2.2) where A is either a rotation or a reflection. When A is one of these we further have $A^T = A^{-1}$ so that

$$\begin{bmatrix} f_{x'x'} & f_{x'y'} \\ f_{x'y'} & f_{y'y'} \end{bmatrix} = A^{-1} \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} A$$

We say that $\text{Hess}(f)(x, y)$ and $\text{Hess}(f)(x', y')$ are **similar** or **conjugate** to each other i.e. they represent the same linear transformation. If two matrices represent the same linear transformation then they have the same *determinant* and *trace*.

Question. 4. Let A be an $n \times n$ matrix and let P be an invertible $n \times n$ matrix. Prove that

$$\det A = \det(PAP^{-1}) \quad \text{and} \quad \operatorname{tr} A = \operatorname{tr}(PAP^{-1})$$

We now have well defined notions of curvature which remain unchanged when we rotate or reflect the xy-plane.

Definition 2.4. If S is the graph of a smooth function $f(x, y)$ satisfying $f_x(p) = 0 = f_y(p)$ then

a) The **Mean Curvature** of S at p is defined to be

$$H = \operatorname{tr}(\text{Hess}(f))/2 = (f_{xx} + f_{yy})/2$$

b) The **Gaussian Curvature** of S at p is defined to be **determinant**

$$K = \det(\text{Hess}(f)) = f_{xx} f_{yy} - f_{xy}^2$$

Question. 5. Go back to your computation in Question 1 and verify that the Gaussian and Mean curvatures for $z = f_\theta(x, y)$ do not depend on θ.

Question. 6. For each of the following surfaces, find the second order Taylor polynomial, the Hessian, and the Mean and Gaussian curvatures at $(0, 0)$.

The Perfect Potato Chip: $z = x^2 - y^2$

Cylindrical Potato: $z = -\sqrt{r^2 - x^2}$

Spherical Potato: $z = -\sqrt{r^2 - x^2 - y^2}$

Parabolic Cylinder: $z = x^2$

(You can use the estimate $-\sqrt{r^2 - \alpha} \approx -r + \frac{\alpha}{2r}$. This is called the **binomial approximation**.)

Remark 2.5. Although we're only analyzing critical points of graphs of functions, we can always rotate $\mathbb{R}^3$ (which should not change curvatures) so that the surface looks like a graph near the point of interest, and the point becomes a critical point. As such, the above method defines curvature in all generality.

2.5 Appendix: Taylor approximation

Taylor polynomials are a way to approximate functions by polynomials.

Definition 2.6. The degree n **Taylor approximation** of a function $f : \mathbb{R} \to \mathbb{R}$ at a point $x = a$ is defined to be

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

where $f^{(n)}(a)$ denotes the n^{th} derivative of f at a.

We're only interested in the degree 2 approximation at $x = 0$ i.e.

$$f(x) \approx f(0) + f'(0)x + f''(0)\frac{x^2}{2}$$

This generalizes to functions in multiple variables easily.

Definition 2.7. The degree 2 or **second order Taylor approximation** of a function $f : \mathbb{R}^2 \to \mathbb{R}$ at the point $p = (0,0)$ is defined to be

$$f(x,y) \approx f(p) + f_x(p) \cdot x + f_y(p) \cdot y + f_{xx}(p) \cdot \frac{x^2}{2} + f_{xy}(p) \cdot xy + f_{yy}(p) \cdot \frac{y^2}{2}$$

where $f_*(p)$ denotes the partial derivatives.[5]

Question. 7. If $f(x,y)$ is polynomial in 2 variables x, y find it's the second order Taylor approximation at $(0,0)$.

Question. 8. Let $T_2(f)$ denote the degree 2 Taylor approximation of f at $(0,0)$. Let P denote the vector space of smooth functions $\mathbb{R}^2 \to \mathbb{R}$ and let P_2 denote the vector space of polynomials of degree ≤ 2 with real coefficients.

a) Show that T_2 defines a linear transformation $P \to P_2$.

b) Further show that $T_2(T_2(f)) = T_2(f)$.

Linear transformations L which satisfy $L \circ L = L$ are called **projections**, and thus taking the degree 2 Taylor approximation is like projecting onto the space of degree 2 polynomials.

[5] f_{xy} has coefficient 1 instead of $1/2$ as it is secretly a sum of two terms f_{xy} and f_{yx} which happen to be equal for all twice differential functions.

2.6 Appendix: Orthogonal Transformations

Question. 9.

a) Prove that every rotation of $\mathbb{R}^2$ about the origin is given by a linear transformation of the form

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

b) Prove that every reflection of $\mathbb{R}^2$ about a line passing through the origin is given by a linear transformation of the form

$$\begin{bmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

c) Prove that both reflections and rotations satisfy

$$A^{-1} = A^T$$

d) Show that if A is a 2×2 matrix satisfying $A^{-1} = A^T$ then A is either a rotation or a reflection.

In higher dimensions it is difficult to give explicit descriptions of rotations and reflections, instead we define **orthogonal matrices** to be the ones that satisfy $A^{-1} = A^T$. It is easy to show that orthogonal matrices preserve distances and angles and hence are the correct generalizations of rotations and reflections.

3 Principal Curvatures

> If you are receptive and humble, mathematics will lead you by the hand.
>
> Paul Dirac

For the graph $z = f(x, y)$ the Gaussian and Mean curvatures are the determinant and trace of $\mathrm{Hess}(f)$ respectively, and are invariant under rotation of the xy-plane. We can ask: how much can we simplify $\mathrm{Hess}(f)$ by replacing it with a suitable $P^{-1}\,\mathrm{Hess}(f)P$?
Answer: A Lot.

The Hessian $\mathrm{Hess}(f)$ is a symmetric 2×2 matrix, hence by the **Spectral Theorem**, there exist orthonormal vectors (eigenvectors) v_1, v_2 and constants (eigenvalues) κ_1, κ_2 such that for the matrix P whose columns are v_1, v_2 we have

$$P^{-1}\,\mathrm{Hess}(f)P = \begin{bmatrix} \kappa_1 & \\ & \kappa_2 \end{bmatrix} \tag{3.1}$$

The set $\{\kappa_1, \kappa_2\}$ is uniquely determined by $\mathrm{Hess}(f)$.

Definition 3.1. The eigenvalues κ_1, κ_2 of A are called the **principal curvatures** and the eigenvectors v_1, v_2 are called the **principal directions** at p.

Remark 3.2. In the case $\kappa_1 = \kappa_2$ all directions are principal.

Question. 1. Show that the various curvatures are related as follows:

$$\begin{aligned} H &= (\kappa_1 + \kappa_2)/2 \\ K &= \kappa_1 \kappa_2 \\ &\kappa_1, \kappa_2 \text{ are the roots of } \kappa^2 - 2H\kappa + K \end{aligned} \tag{3.2}$$

Question. 2. Find the Principal Curvatures and Principal Directions of the curves you analyzed yesterday

The Perfect Potato Chip: $z = x^2 - y^2$

Cylindrical Potato: $z = -\sqrt{r^2 - x^2}$

Spherical Potato: $z = -\sqrt{r^2 - x^2 - y^2}$

Parabolic Cylinder: $z = x^2$

(You can use the estimate $-\sqrt{r^2 - \alpha} \approx -r + \frac{\alpha}{2r}$.)

This is all the algebra we'll be needing. We'll now start analyzing the curvatures geometrically.

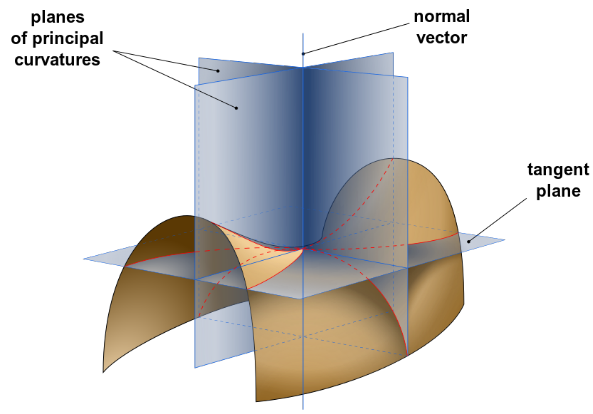


Figure 5: Principal Directions and Normal Curvatures. Image from Wikipedia.

3.1 Curves on a Potato

Without any loss of generality assume that the x, y axes are the principal directions with principal curvatures κ_1, κ_2 respectively. This is equivalent to saying that the degree 2 Taylor approximation of f is

$$f(x,y) \approx f(0,0) + \kappa_1 \cdot \frac{x^2}{2} + \kappa_2 \cdot \frac{y^2}{2}$$

Let us look at the *curves on the surface* passing through $(0,0)$ and compute their curvature in terms of κ_1, κ_2. We'll consider curves of the form

$$c_\theta(t) = (t\cos\theta, t\sin\theta, f(t\cos\theta, t\sin\theta))$$

Note that upon projecting onto the xy-plane $c_\theta(t)$ projects onto the straight line $y = \tan\theta \cdot x$. Applying the formula for curvature we get:

$$\begin{aligned} c'_\theta(t) &= (\cos\theta, \sin\theta, f'(t\cos\theta, t\sin\theta) \\ c''_\theta(t) &= (0, 0, f''(t\cos\theta, t\sin\theta)) \end{aligned}$$

Because $c(0) = (0,0)$ is the critical point, the tangent plane to S at $(0,0)$ is horizontal and hence we must have $f'(t\cos\theta, t\sin\theta)|_{t=0} = 0$ so that

$$c'_\theta(0) = (\cos\theta, \sin\theta, 0)$$

which is unit length. By explicitly computing the cross product we get

$$\begin{aligned} \kappa &= |c''_\theta \times c'_\theta|_{t=0} \\ &= |f''(t\cos\theta, t\sin\theta)|_{t=0} \end{aligned}$$

We'll remove the absolute value $|-|$ and compute the **signed curvature**

$$\kappa = f''(t\cos\theta, t\sin\theta)|_{t=0}$$

Definition 3.3. The signed curvature of $c_\theta(t)$ at $t = 0$ is called the **normal curvature** at p along the direction $(\cos\theta, \sin\theta)$ (we will abbreviate this as just the direction θ).

We can compute this using the Taylor approximation

$$\begin{aligned} & f(t\cos\theta, t\sin\theta) \approx f(0,0) + \kappa_1 \cdot \frac{(t\cos\theta)^2}{2} + \kappa_2 \cdot \frac{(t\sin\theta)^2}{2} + \cdots \\ \Rightarrow \qquad & f''(t\cos\theta, t\sin\theta) \approx \kappa_1 \cdot \cos^2\theta + \kappa_2 \cdot \sin^2\theta + \cdots \end{aligned}$$

When we plug in $t = 0$ the higher degree terms in the Taylor approximation vanish and we get

$$\kappa = f''(t\cos\theta, t\sin\theta)|_{t=0} = \kappa_1 \cdot \cos^2\theta + \kappa_2 \cdot \sin^2\theta$$

Proposition 3.4. *With the notation as above, the normal curvature at the point p in the direction θ equals*

$$\kappa = \kappa_1 \cdot \cos^2\theta + \kappa_2 \cdot \sin^2\theta$$

Theorem 3.5. *Without any loss of generality assume that $\kappa_1 \geq \kappa_2$. Then the maximum and minimum normal curvatures at the point p are κ_1 and κ_2 respectively, in the corresponding principal directions.*

Proof is in the following exercise.

Question. 3. Assume $\kappa_1 \geq \kappa_2$. Prove that $\kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta$ attains it maximum at $\theta = 0, \pi$ and minimum at $\theta = \pi/2, 3\pi/2$ and the maximum value is κ_1 and the minimum value is κ_2.

Question. 4. There is another sense in which the Mean Curvature is the mean: it is the mean of all normal curvatures.

a) Plot $r = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta$ in polar coordinates. Interpret this geometrically.

b) Show that

$$H = \frac{1}{2\pi} \int_0^{2\pi} \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta \, d\theta$$

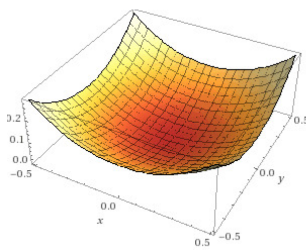
Remark 3.6. Because curvature does not change when we rotate $\mathbb{R}^3$ we can drop the condition that p is a critical point from the above theorem. It is worth rephrasing the theorem to state this.

Theorem 3.7. *At every point p on a surface S there are two perpendicular directions (principal directions) along which the normal curvatures attain the maximum and minimum (principal curvatures).*

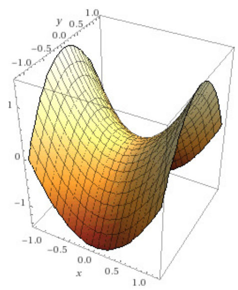
3.2 Classification of Points

The signs of the principal curvatures allow us to classify the points on the surface.

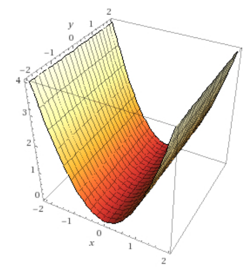
	the surface looks like a ...	such points are called ...
κ_1, κ_2 both positive/negative	ellipsoid	elliptic
κ_1, κ_2 have opposite signs	hyperboloid	hyperbolic
$\kappa_1 \kappa_2 = 0$	cylinder/plane	parabolic



(a) $K > 0$



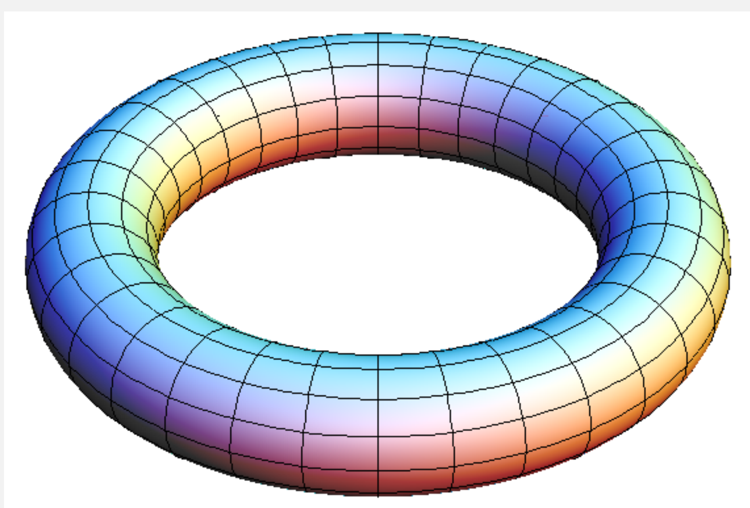
(b) $K < 0$



(c) $K = 0$

Note that because $K = \kappa_1 \kappa_2$ the above three classifications are essentially the classifications based on Gaussian curvature. This already suggests that the Gaussian curvature sees some *intrisic* curvature of surfaces.

Question. 5. Classify the points on a torus as elliptic, hyperbolic, or parabolic.



3.3 Appendix: Spectral Theorem

A basis $v_1, v_2, \cdots, v_n$ of $\mathbb{R}^n$ is said to be **orthonormal** if $\|v_i\| = 1$ for each i and $v_i \cdot v_j = 0$ for $i \neq j$.

Theorem 3.8 (Spectral Theorem for Symmetric Matrices)**.** *Let A be a **symmetric** $n \times n$ matrix with real entries. Then there exists an orthonormal basis $v_1, v_2, \cdots, v_n$ basis of $\mathbb{R}^n$ and real numbers κ_i such that*

$$Av_i = \kappa_i v_i$$

Equivalently, if P is the matrix with columns v_i then

$$P^{-1}AP = \begin{bmatrix} \kappa_1 & & & \\ & \kappa_2 & & \\ & & \ddots & \\ & & & \kappa_n \end{bmatrix}$$

*The κ_i (**eigenvalues**) are unique up to permutations and the orthonormal basis $v_1, v_2, \cdots, v_n$ (**eigenvectors**) and the matrix P are 'essentially' unique.*

The following exercise outlines the proof for $n = 2$.

Question. 6. Consider the symmetric matrix $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$. Assume that A is not a scalar matrix i.e. $A \neq \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = aI$ (this case is trivial).

a) Show that the equation

$$\det(A - \kappa I) = \det \begin{bmatrix} a - \kappa & b \\ b & c - \kappa \end{bmatrix} = 0$$

has 2 distinct real solutions κ_1, κ_2.

b) Argue that there exist 2 linearly independent unit length vectors v_1, v_2 satisfying $Av_i = \kappa_i v_i$ for $i = 1, 2$.[a]

c) Prove that $v_1 \perp v_2$.[b]

Let P be the matrix with column vectors v_1, v_2.

d) Prove that $P^T = P^{-1}$.

e) By an explicit computation show $P^T AP = \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix}$

[a] Hint: Look at the kernel of $A - \kappa_i I$.

[b] Hint: Find $v_1^T A v_2$ in 2 different ways.

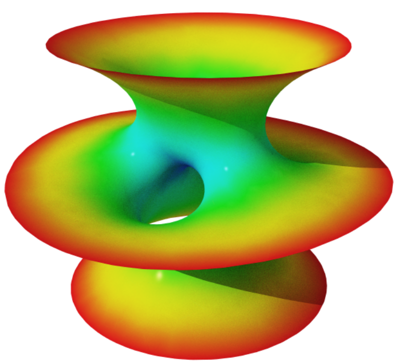
4 Geometric Meaning of Curvature

4.1 Mean Curvature

The Mean Curvature shows up in physics while studying soap films. At a point on a soap film the difference between the pressure on two sides is proportional to the mean curvature of the surface at that point. This is called the **Young-Laplace equation**.

$$\Delta(\text{pressure}) \propto H$$

If the soap film does not bound a volume, for example, if it is bounded by a curve then the pressure on both the sides is the same and hence the mean curvature at every point must be 0. Such surfaces are called **minimal surfaces**, minimal because these surfaces also have the minimal surface area of all the surfaces bounding the curve. The study of minimal surfaces is a very active area of research in geometry.



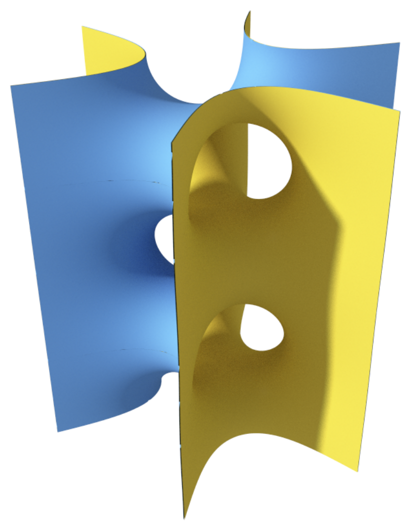


Figure 7: Examples of minimal surfaces. The Mean Curvature at *every* non-boundary point is 0 and hence at every point the surface looks like the perfect potato chip. Images from Wikipedia.

4.2 Gaussian Curvature

The Gaussian Curvature is much more subtle and has several interpretations.

Definition 4.1. For a surface S the **geodesic distance** $d_S(p,q)$ between two points $p,q \in S$ is defined to be the shortest length of the curve on the surface S that connects p to q.

For example, on a plane the geodesic distance is simply the Euclidean distance. On a sphere, the geodesic distance between two points is the length of the arc of the great circle connecting them.

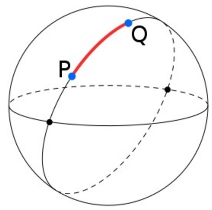


Figure 8: The geodesic distance between two points on a sphere is the length of the arc of the great circle connecting them. Image from Wikipedia.

Definition 4.2. The **geodesic ball** of radius r centered at a point $p \in S$ is the set of points which are at a geodesic distance of at the most r from p

$$B_r(p) := \{x \in S : d_S(x, p) \leq r\}$$

We'll assume the following theorem without proof.

Theorem 4.3. *The Gaussian curvature of S at p equals*

$$K = 3 \lim_{r \to 0} \frac{2\pi r - \text{length of } \partial B_S(p, r)}{\pi r^3}$$

Corollary 4.4. *The Gaussian curvature can be computed by measuring distances on the surface (without knowing anything about the ambient space).*

This leads directly to the next theorem.

4.2.1 Theorema Egregium

We can take a sheet of paper and roll into a cylinder without stretching or compressing the sheet.[6] Such a map is called an **isometry**.

Definition 4.5. A smooth map $\phi : S \to S'$ is called an **isometry** if

$$d_S(p, q) = d_{S'}(\phi(p), \phi(q))$$

for any two points $p, q \in S$.

The map that sends a plane to a cylinder is an example of such an isometry.

Theorem 4.6 (Theorema Egregium)**.** *If there exists an isometry $\phi : S \to S'$ between two surfaces then the Gaussian curvature of S at $p \in S$ equals the Gaussian curvature of S' at $\phi(p)$.*

Proof. This is a direct consequence of Corollary 4.4. It is possible to measure the Gaussian curvature using only the geodesic distances and geodesic distances are preserved under isometry. □

This theorem is interpreted as saying that the Gaussian curvature is *intrinsic* to a surface.

[6] Neglect the *thickness* of the paper.

4.2.2 Gauss Map

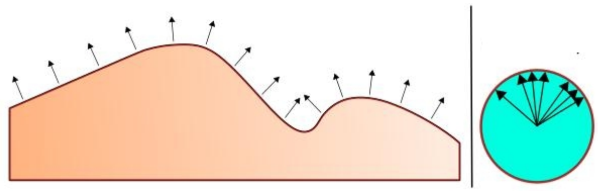


Figure 9: The unit normal vector $\vec{n}$ defines a map from the surface to the unit sphere $\vec{n} : S \to S^2$. Image from Wikipedia.

Let $\vec{n}$ denote a continuously varying unit normal vector field on the surface S. There are two possibly choices for $\vec{n}$, we just pick one. We can think of $\vec{n}$ as a map, called the **Gauss map** from S to the unit sphere S^2. Then,

$$K = \lim_{r \to 0} \frac{\text{area of } \vec{n}(B_r(p))}{\text{area of } B_r(p)}$$

4.2.3 Gauss-Bonnet theorem

Gaussian curvature has a topological significance as well. If S is a closed surface then

Theorem 4.7. *The total Gaussian Curvature*

$$\int_S K \, dA = 2\pi\chi(S)$$

where $\chi(S)$ denotes the Euler characteristic of the surface.

4.3 Final Remarks

The methods that we've described only define the various curvatures for *embedded* surfaces. In general, because the Gaussian curvature can be computed by measuring Geodesic distances it is possible to define the Gaussian curvature (but not the principal and mean curvatures) for manifolds with a metric (also called Riemannian manifold).