

## CLASSIFICATION OF SURFACES

**Theorem 0.1.** *Every compact Riemann surface is homeomorphic(=topologically isomorphic) to a surface of genus  $g$ .*



FIGURE 1. Torus, genus 2 surface, higher genus surfaces

*Upto isomorphism, there is exactly one Riemann surface of genus 0, namely the Riemann sphere  $\mathbb{P}^1$ .*

The genus can be thought of as the number of “handles” in the surface.

**Word of caution:** Two Riemann surfaces being homeomorphic does not mean that they are isomorphic as Riemann surfaces. We will show that every elliptic curve is homeomorphic to the genus 1 surface (=torus) but not all elliptic curves are isomorphic to each other as Riemann surfaces.

## EULER CHARACTERISTIC

It is possible to compute the genus using triangulations of surfaces. For any triangulation of a genus  $g$  surface  $M$  with  $V$  vertices,  $E$  edges, and  $F$  faces, we have the following identity:

$$2 - 2g = V - E + F.$$

The important point here is that the right-hand side depends on the triangulation but the left-hand side does not. The right-hand side is called the Euler characteristic of  $M$ , denoted  $\chi(M)$ .

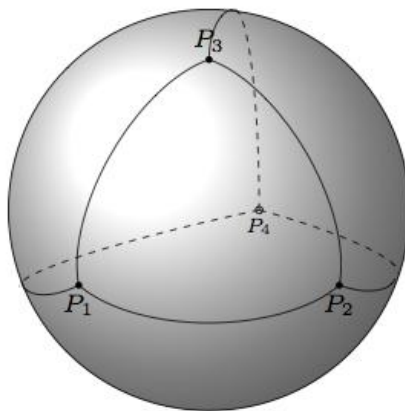
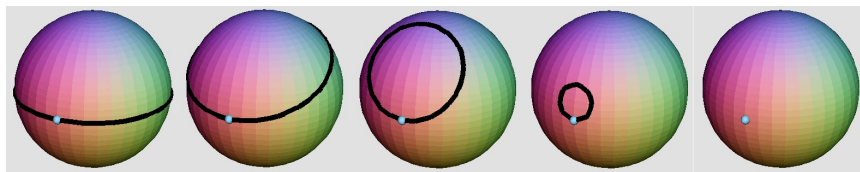


FIGURE 2. Triangulation of  $S^2$  (genus=0) with  $V = 4$ ,  $E = 6$ , and  $F = 4$  so that  $V - E + F = 2 = 2 - 2 \cdot 0$ .

## COVERINGS OF SIMPLY-CONNECTED SPACES

A topological space  $Y$  is said to be *simply-connected* if it is connected and every loop in  $Y$  can be continuously contracted to a point.



For example, the Riemann sphere  $\mathbb{P}^1$  is simply-connected. If we remove a single point or a single path-connected set from  $\mathbb{P}^1$ , the resulting space is still simply-connected.

But if  $Y$  is the space obtained by removing two or more points from the Riemann sphere, then  $Y$  is not simply-connected.

This is the only example that concerns us.

**Theorem 0.2.** *If  $f : X \rightarrow Y$  is a continuous  $n : 1$  (covering) map and  $Y$  is simply-connected then  $X$  is topologically isomorphic to  $n$  disjoint copies of  $Y$ .*

For this theorem we do not allow any ramification points, so  $f$  needs to be a genuine  $n : 1$  (covering) map.