~~Problem Set~~ 5 - Affine charts

1 Singularities

Notation: Set of solutions of a bunch of polynomial equations is called a *variety*. Varities in $\mathbb{C}^n$ are called *affine varities* and varities in $\mathbb{P}^n$ are called *projective varieties*.

We'll start with two examples of singularities, one in $\mathbb{C}^2$ and one in $\mathbb{P}^1$.

Example 1.1. The affine variety given by $p(z,w) = z^2 - w^2$ is an intersection of two lines, and hence has a singularity at $(0,0)$. In general, if the lowest degree terms in $p(z,w)$ are of degree ≥ 2 then the corresponding variety has a singularity at $(0,0)$ and hence is not a Riemann surface. As far as varieties in $\mathbb{C}^2$ are concerned, this is the only way singularities arise but in higher dimensions the singularities are much more complicated and hence interesting.

Example 1.2 (Hyperelliptic curve)**.** Consider $p(z,w) = z^2 - q(w)$ with $q(w) = w^4 - w$. The homogenization of p is $\overline{p}(z,w,t) = z^2t^2 - w^4 - wt^3$. The points at ∞ are given by plugging in $t = 0$,

$$\begin{aligned} \overline{p}(z,w,t) &= 0 \\ w^4 &= 0 \end{aligned}$$

So there is exactly one point at ∞, namely $[1:0:0]$.

The projection map $\overline{S_p} \to \mathbb{P}^1$ sending $[z:w:1] \mapsto w$ and $[1:0:0] \mapsto \infty$ is a degree 2 map which ramifies at 5 points: the roots of $q(z)$ and ∞.

Plugging in Riemann-Hurwitz we get:

$$\begin{aligned} \chi(\overline{S_p}) &= 2 \cdot \chi(\mathbb{P}^1) - \sum_5 \text{index} - 1 \\ &= 4 - 5 \\ &= -1 \end{aligned}$$

But there is no surface with Euler characteristic -1. The reason we see this is that $\overline{S_p}$ is not a Riemann surface and has a singularity at ∞.

2 Affine charts on $\mathbb{P}^2$

There is a natural cover of $\mathbb{P}^2$ given by three open sets

$$\begin{aligned}\mathbb{P}^2 &= \{[z:w:t] \mid \text{ not all } z,w,t \text{ zero}\} \\ &= \{[z:w:1]\} \cup \{[z:1:t]\} \cup \{[1:w:t]\} \\ &=: U_t \cup U_w \cup U_z\end{aligned}$$

We can define charts on these as

$$\begin{aligned}\varphi_z : U_z &\longrightarrow \mathbb{C}^2 \\ [1:w:t] &\longmapsto (w,t)\end{aligned}$$

$$\begin{aligned}\varphi_z : U_w &\longrightarrow \mathbb{C}^2 \\ [z:1:t] &\longmapsto (z,t)\end{aligned}$$

$$\begin{aligned}\varphi_z : U_t &\longrightarrow \mathbb{C}^2 \\ [z:w:1] &\longmapsto (z,w)\end{aligned}$$

A projective variety $\overline{S_p}$ cut out by the polynomial $\overline{p}(z,w,t)$ is a Riemann surface if and only if the affine varities $\overline{S_p} \cap U_t$, $\overline{S_p} \cap U_w$, and $\overline{S_p} \cap U_z$ are Riemann surfaces. These are called *affine charts* on $\overline{S_p}$.

The varieties $\overline{S_p} \cap U_t$, $\overline{S_p} \cap U_w$, $\overline{S_p} \cap U_z$ can be described as

$$\overline{S_p} \cap U_t = \{(z,w) \mid \overline{p}(z,w,1) = 0\}$$

$$\overline{S_p} \cap U_w = \{(z,t) \mid \overline{p}(z,1,t) = 0\}$$

$$\overline{S_p} \cap U_z = \{(w,t) \mid \overline{p}(1,w,t) = 0\}$$

In order to check that a projective variety is a Riemann surface, we first break the variety into affine charts, and then check that each of the charts is a Riemann surface using the Jacobian.

3 Jacobian

Theorem 3.1. *The affine variety $S = \{(z,w) : p(z,w) = 0\} \subseteq \mathbb{C}^2$ is a Riemann surface if the Jacobian, defined as*

$$J(z,w) := \begin{bmatrix} \frac{\partial p}{\partial z} & \frac{\partial p}{\partial w} \end{bmatrix}$$

does not vanish, at all points (z,w) in S.

Proof. The reason is essentially that if at (z, w), we have $\dfrac{\partial p}{\partial z} \neq 0$ then projection onto the w coordinate locally defines a chart around (z, w). Similarly, if at (z, w), we have $\dfrac{\partial p}{\partial w} \neq 0$ then projection onto the z coordinate locally defines a chart around (z, w). If both are non-zero then both charts are valid and the deritvatives of transition functions are given by the rational functions

$$\left(\frac{\partial p}{\partial w}\right) \cdot \left(\frac{\partial p}{\partial z}\right)^{-1}$$

which are complex differentiable as the denominator is non-zero. $\square$

Example 3.2. The polynomial $z^2 - q(w)$ has Jacobian $\begin{bmatrix} 2z & -q'(w) \end{bmatrix}$. The Jacobian vanishes 0 precisely when $z = 0$ and $q'(w) = 0$. For this to be true for a point on the curve, $z = 0 \implies q(w) = 0$. Both $q(w) = 0$ and $q'(w) = 0$ implies that w is a repeated root of q. Thus the corresponding variety is a Riemann surface if $q(w)$ has no repeated roots.

Example 3.3 (Fermat)**.** For the projective variety cut out by $\overline{p}(z, w, t) = z^p + w^p - t^p$, the three affine charts are given by

$$\begin{array}{cc} z^p + w^p - 1 & \begin{bmatrix} pz^{p-1} & pw^{p-1} \end{bmatrix} \\ z^p + 1 - t^p & \begin{bmatrix} pz^{p-1} & pt^{p-1} \end{bmatrix} \\ 1 + w^p - t^p & \begin{bmatrix} pw^{p-1} & pt^{p-1} \end{bmatrix} \end{array}$$

The Jacobians vanish at $(0, 0)$ but these points are not on the affine varities.

Example 3.4 (Elliptic curves)**.** The equation $z^2 = w^3 + w$ has homogenization $\overline{p}(z, w, t) = z^2 t - w^3 - wt^2$. In the three charts this polynomial and the Jacobians become

$$\begin{array}{cc} z^2 - w^3 - w & \begin{bmatrix} 2z & -3w^2 - 1 \end{bmatrix} \\ z^2 t - 1 - t^2 & \begin{bmatrix} 2zt & z^2 - 2t \end{bmatrix} \\ t - w^3 - wt^2 & \begin{bmatrix} 1 - 2wt & -3w^2 - t^2 \end{bmatrix} \end{array}$$

It is easy to see that all the Jacobians do not vanish anywhere on the varieties.

Example 3.5 (Hyperelliptic curves)**.** The equation $z^2 = w^4 + w$ has homogenization $\overline{p}(z, w, t) = z^2 t^2 - w^3 - wt^3$. In the three charts this polynomial and the Jacobians become

$$\begin{array}{cc} z^2 - w^4 - w & \begin{bmatrix} 2z & -4w^3 - 1 \end{bmatrix} \\ z^2 t^2 - 1 - t^3 & \begin{bmatrix} 2zt^2 & 2z^2 t - 3t^2 \end{bmatrix} \\ t^2 - w^3 - wt^3 & \begin{bmatrix} 2t - 3wt^2 & -3w^2 - t^2 \end{bmatrix} \end{array}$$

In this case, the third Jacobian vanishes at the point $(0, 0)$ which is on the curve, and hence our original projective variety is singular at the point at ∞.

It is possible to remove singularities of hyperelliptic curves ($z^2 = q(w)$ with $\deg q > 3$) by putting charts at ∞ artificially. The resulting Riemann surface has genus $\left\lfloor \dfrac{\deg q - 1}{2} \right\rfloor$. See https://en.wikipedia.org/wiki/Hyperelliptic_curve#Genus_of_the_curve.