1 Introduction

We know from real analysis that is difficult to define inverses of functions, mainly because many functions of interest are not bijective. But inverses *are* very important.

Function	Inverse
x^2	$\pm\sqrt{x}$
x^n	$x^{1/n}$
e^x	$\ln x$

We use various tricks to define the inverse. For example, for definiting the inverse of $f(x) = x^2$ we can

1. Restrict the domain of the inverse to non-negative integers and *choose* $\sqrt{x}$ to always be positive.

2. Study the graph $y = x^2$ instead, thereby bypassing the need to define an inverse explicitly.

Both of these methods generalize to (nice) complex functions, with the first giving rise to the notion of branch cuts and branch points and the second giving us Riemann surfaces.

2 Inverses of functions

Riemann surfaces naturally arise when trying to invert complex functions. As in the real case, the inverse of the function $f(x) = x^2$ is not well-defined. There are multiple ways to remedy this, but the most *geometric* solution is to study the graph $y = x^2$ as a subset of $\mathbb{R}^2$ and use this to construct inverses wherever they make sense.

We do the same for complex functions. Let $p(z)$ be a polynomial of degree n. Finding the inverse function is equivalent to solving

$$w = p(z)$$

to get z in terms of w. This is almost never possible, so instead we study the graph

$$\Gamma_{p(z)} := \{(z, w) : w = p(z)\} \subseteq \mathbb{C}^2$$

Denote by $\pi : \mathbb{C}^2$ the projection onto the w- coordinate axis[1]. $\Gamma_{p(z)}$ is an example of a Riemann surface! Our first goal is to undestand the geometry of this object. We'll do this using the projection π.

2.1 Ramified coverings

Consider the restriction of π to $\Gamma_{p(z)}$.

$$\pi : \Gamma_{p(z)} \to \mathbb{C}$$

Q. 1. Show that this map is surjective.

The inverse set $\pi^{-1}(w)$ (called the *fiber* over w) is the set of roots of $p(z) - w$.

Q. 2. Prove that there are only finitely many values of w for which $\pi^{-1}(w)$ has size $< n$. And that outside this set the fiber has size exactly n.

Definition 2.1. For $w \in \mathbb{C}$, if the polynomial $p(z) - w$ has distinct roots, then w is called *unramified*. This is equivalent to requiring that the fiber $\pi^{-1}(w)$ has size n. If $p(z) - w$ has repeated roots then, if z is a repeated root of $p(z) - w$ of order k we say that the *ramification index* e_P of $P = (z, w)$ is k.

[1] We are working over complex numbers, so everything is twice the dimension of real numbers. So a complex axis is complex dimension 1 but real dimension 2.

Q. 3. For $w_0 \in \mathbb{C}$, let $\pi^{-1}(w_0) = \{P_1, \dots, P_l\}$, then

$$e_{P_1} + \cdots + e_{P_l} = n$$

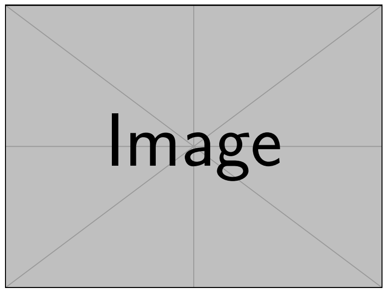

Figure 1: Picture of ramification

A much stronger statement is true on the topological level. Let w be an unramified value. Then $p(z) - w$ has n distinct roots $z_1, \dots, z_n$. If we perturb w slightly then the corresponding roots z_i will also get perturbed slightly.

For $\epsilon \in \mathbb{R}$ let $B_\epsilon(w) \subset \mathbb{C}$ denote a ball of radius ϵ around w.

Theorem 2.2. *For an unramified value $w \in \mathbb{C}$, let $z_1, \dots, z_n$ be the distinct roots of $p(z) - w$. Then there exists an ϵ such that $\pi^{-1}(B_\epsilon(w))$ has the following properties.*

1. *$\pi^{-1}(B_\epsilon(w))$ has n-connected components $U_1, \dots, U_n$,*
2. *$z_i \in U_i$,*
3. *the projections $\pi : U_i \to B_\epsilon(w)$ are a homeomorphism.*

Theorem 2.2 is saying that the map π restricted to the fibers of unramified values is a *covering map*

$$\pi : \pi^{-1}(\{\text{unramified values}\}) \longrightarrow \{\text{unramified values}\}.$$

In Figure 1 the ramified points appear to be singular, but

A graph is always homeomorphic to the base i.e. the

$$\mathbb{C} \to \Gamma_{p(z)} z \mapsto (z, p(z))$$

is always a homeomorphism. So the Riemann surface we have constructed is homeomorphic to $\mathbb{C}$ itself! We'll improve upon this in the following sections.

2.2 Sections as inverse functions

We now have a way of constructing inverse functions, which relies on the following theorem from topology.

A space is said to be *simply-connected* if it is path-connected and every loop in X can be continuously deformed to a point.

Theorem 2.3. *If X is a simply-connected space and $\pi : Y \to X$ is a finite covering of X of degree n then Y is homeomorphic to n-disjoint copies of X and the projection map on each connected component is a homeomorphism.*

$$Y \cong X_1 \sqcup \cdots \sqcup X_n$$

$$\begin{array}{c} X_i \\ {\scriptstyle\pi}\big\downarrow{\scriptstyle\cong} \\ X \end{array}$$

This statement is also true when the covering is not finite. In this case, Y is a disjoint union of infinitely many components.

Now we have a meaninful procedure to construct inverse functions of ramified coverings of $\mathbb{C}$. Consider the ramified covering $\pi : \Gamma_{p(z)} \to \mathbb{C}$ with ramified values $w_1, \dots, w_l$ (these are called the *branch points*).

1. Consider the set of unramified points $\mathbb{C} \setminus \{w_1, \dots, w_l\}$. This set is never simply-connected (unless $l = 0$).

2. Pick an arbitrary open subset U of $\mathbb{C} \setminus \{w_1, \dots, w_l\}$ which is simply-connected. This is usually done by removing rays (called *branch cuts*) emanating from the points w_i.

3. The restriction $\pi : p^{-1}(U) \to U$ is a finite covering of a simply-connected connected space U, hence $U \cong U_1 \sqcup \cdots \sqcup U_n$.

4. Pick an aribtrary connected component U_i so that $\pi : U_i \to U$ is a homeomorphism. Because this is a homeomorphism it has an inverse $\pi_i^{-1} : U \to U_i$.

5. The function π_i^{-1} followed by projection onto the z-coordinate is then an inverse of the function p. It is called a *branch* of the inverse function (and so there are n possible branches).

We need to choose *branch cuts* and a *branch* to define an inverse function.

Example 2.4.

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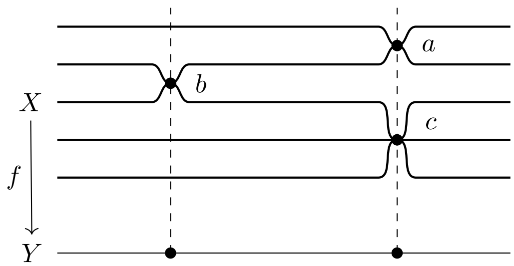

Figure 2: Not sure how to adjust width in this case.

3 Compactification

We saw that the Riemann surface $\Gamma_{p(z)}$ is homeomorphic to $\mathbb{C}$.

We can generalize the previous constructions to studying graphs of

$$\begin{aligned} p(z) &= w^p \\ p(z, w) &= 0 \end{aligned}$$

But before we do this, we need to make one more leap which is again inspired from topology.

The surface $\Gamma_{p(z)}$ costructed above is not compact. Non-compact objects are harder[1] to study than compact ones because the behaviour of limits is harder to control for non-compact surfaces and functions on non-compact .

Q. 4. (Optional exercise) Let X be a subset of $\mathbb{C}$.

1. Show that if X is compact then every function $f : X \to \mathbb{R}$ is bounded.

2. If X is open, find an unbounded continuous function $f : X \to \mathbb{C}$.

Furthermore, non-compact subsets come in all shapes and sizes (for example, every open subset of $\mathbb{C}$ is non-compact) but as we'll see in the next section there are various classification theorems for compact ones.

3.1 Behaviour at ∞

[1] In most case, but of course not always.