

## Problem Set 3 - Riemann–Hurwitz

For today, let  $X$  and  $Y$  be compact connected Riemann surfaces with atlases  $(U_i, \varphi_i)$  and  $(V_j, \psi_j)$ , respectively, and let  $f : X \rightarrow Y$  be a non-constant complex differentiable function between them.

### 1 Ramified coverings of Riemann surfaces

**Definition 1.1.** Pick a chart  $\varphi_i$  around a point  $z \in X$  and a chart  $\psi_j$  around  $f(z) \in Y$ . Define

$$\text{index}_f(z) := \text{index}_{\psi_j \circ f \circ \varphi_i^{-1}}(\varphi_i(z))$$

**Q. 1.** Let  $g(z)$  be a non-constant holomorphic function on  $U \subseteq \mathbb{C}$  with Taylor expansion

$$g(z) - g(z_0) = a_k(z - z_0)^k + a_{k+1}(z - z_0)^{k+1} + \dots$$

Let  $\psi$  be a biholomorphic function with Taylor expansion

$$\psi(w) - \psi(w_0) = a_1(z - w_0) + a_2(z - w_0)^2 + \dots$$

where  $w_0 = g(z_0)$ . Find the first term in the Taylor expansions of  $\psi \circ g$  at  $z_0$  and argue that  $f$  and  $\psi \circ f$  have the same index at  $z_0$ . Similar statement is true if we pre-compose instead of post-compose with a biholomorphic function.

**Q. 2.** Show that  $\text{index}_f(z)$  does not depend on the choice of charts, and hence is well defined.

**Q. 3.** Using the fact that  $X$  and  $Y$  are compact, show that the ramification and branch loci of  $f$  are finite. <sup>1</sup>

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**Hint:** Use the fact that ramification and branch locus are isolated. <sup>1</sup>

**Definition 1.2.** For a point  $w \in Y$ , define

$$\deg_f(w) = \sum_{z \in f^{-1}(w)} \text{index}_f(z)$$

This definition makes sense as the right-hand side is finite.

Let  $w_0$  be a point in  $Y$ . Suppose  $f^{-1}(w_0) = \{z_1, \dots, z_k\}$  with ramification indices  $\{e_1, \dots, e_k\}$ . We can choose sufficiently small neighborhoods  $W_i \subseteq X$  around each  $z_i$  such that

1.  $f : W_i \rightarrow f(W_i) \subseteq Y$  is a ramified covering of degree  $e_i$  (so that  $f(z) \approx z^{e_i}$ )
2.  $f(W_i) = f(W_j)$  for all  $1 \leq i, j \leq k$ .

Let  $W = f(W_i)$ .

**Q. 4.** Show that

$$\begin{aligned} \deg_f : W &\longrightarrow \mathbb{Z} \\ w &\longmapsto \deg_f w \end{aligned}$$

is a constant function.

Thus, for each point  $w \in Y$ , we can find an open neighborhood  $W$  on which  $\deg_f$  is a constant function. Because  $Y$  is connected this gives us,

**Theorem 1.3.**  $\deg_f$  is constant function on  $Y$ .

This constant is called the *degree/order* of the ramified covering.

**Q. 5.** Find the degree of a non-constant rational function

$$\begin{aligned} f : \mathbb{P}^1 &\longrightarrow \mathbb{P}^1 \\ z &\longmapsto \frac{p(z)}{q(z)} \end{aligned}$$

**Q. 6.** Show that every non-constant meromorphic function  $f : X \rightarrow \mathbb{P}^1$  has the same number of zeroes and poles, counting multiplicities.

## 2 Riemann–Hurwitz formula

*General philosophy:*<sup>1</sup> Algebra and analysis are used for constructing maps and algebraic topology is used for proving non-existence, by providing obstructions to existence of maps.

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<sup>1</sup>Not to be taken seriously.

**Theorem 2.1** (Riemann–Hurwitz). *Let  $X$  and  $Y$  be compact Riemann surfaces and let  $f : X \rightarrow Y$  be a non-constant complex differentiable map which is a ramified covering of order  $N$  with ramification points  $z_1, \dots, z_k$ . Then,*

$$\chi(X) = N \cdot \chi(Y) - \sum_{i=1}^k (\text{index}_f(z_i) - 1)$$

where  $\chi$  is the Euler characteristic.

**Lemma 2.2.** *If we have a covering map  $f : X \rightarrow Y$  of degree  $N$  of compact topological surfaces then  $\chi(X) = N \cdot \chi(Y)$ .*

*Proof.* Put a triangulation on  $Y$  which is fine enough that its lift is a triangulation on  $X$ . If the original triangulation had  $V, E, F$  vertices, edges, and faces, respectively, then the lifted triangulation will have  $NV, NE, NF$  vertices, edges, and faces, respectively. The result follows.  $\square$

*Proof of Theorem 2.1.* Put a triangulation on  $Y$  which is fine enough that its lift is a triangulation on  $X$ . Assume further that all the branch and ramification points are vertices in this triangulation.

Suppose the triangulation on  $X$  has  $V, E, F$  vertices, edges, and faces, respectively. If there were no ramification points the triangulation on  $Y$  would have  $NV, NE, NF$  vertices, edges, and faces, respectively.

But now consider a ramified point  $z \in X$  with ramification degree  $e$  and let  $w = f(z) \in Y$  be the corresponding branch point. Suppose there are  $k$  triangles with vertex  $w$ .

**Q. 7.** Show that the triangles around  $w$  have a total of  $1 + k$  vertices,  $2k$  edges, and  $k$  faces.

**Q. 8.** Show that in the lifted triangulation, the triangles around  $z$  have a total of  $1 + k \cdot e$  vertices,  $2k \cdot e$  edges, and  $k \cdot e$  faces.

If there was no ramification at  $z$  then  $f$  should have been an  $e : 1$  mapping and hence we should have had  $(1 + k) \cdot e$  vertices,  $2k \cdot e$  edges, and  $k \cdot e$  faces. Hence, a ramification of index  $e$  at  $z$  results in a drop in the Euler characteristic by

$$((1 + k) \cdot e - 2k \cdot e + k \cdot e) - ((1 + k \cdot e) - 2k \cdot e + k \cdot e) = e - 1$$

The result follows.  $\square$

**Q. 9.** Explicitly lift the following triangulation for the function  $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ ,  $f(z) = z^2$  and verify the proof of the Riemann–Hurwitz

formula.

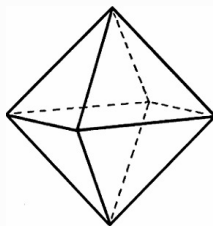


Figure 1: Triangulation of  $\mathbb{P}^1$ : the north pole is  $\infty$ , the south pole is 0, “the square equator” is the unit circle in  $\mathbb{C}$ .

Using  $\chi(X) = 2 - 2g(X)$ , we can rewrite Theorem 2.1 as

$$g(X) - 1 = N \cdot (g(Y) - 1) + \sum_{i=1}^k (\text{index}_f(z_i) - 1) \cdot 1/2$$

where  $g$  is the genus.

**Corollary 2.3.** *For a compact Riemann surface  $Y$ , there are no non-constant differentiable functions  $f : \mathbb{P}^1 \rightarrow Y$  if  $Y \not\cong \mathbb{P}^1$ .*

**Corollary 2.4.** *If  $X$  and  $Y$  are complex tori (genus=1) then any non-constant complex differentiable map  $f : X \rightarrow Y$  has no ramification points, i.e. the only maps between complex tori are (genuine) covering maps.*

**Corollary 2.5.** *If  $X$  and  $Y$  are compact Riemann surfaces and there is a non-constant complex differentiable map  $f : X \rightarrow Y$  which is not an isomorphism, then  $g(X) \geq g(Y)$ .*

**Q. 10.** Prove the above corollaries using Theorem 2.1.

### 3 Elliptic curves

*Analogy:* We can construct 1-dimensional real manifolds by looking at solutions to equations  $f(x, y) = 0$  inside  $\mathbb{R}^2$ .

We can do a similar thing for complex manifolds. Consider

$$S_p = \{(z, w) : p(z, w) = 0\} \subseteq \mathbb{C}^2$$

Under certain restrictions on  $p$ , this defines a Riemann surface. In particular, this is true when  $p(z, w) = z^2 - q(w)$  where  $q(w)$  is a degree three polynomial

with distinct roots. Further, it is possible to compactify this object by adding a single point at  $\infty$  and the resulting object is called an *elliptic curve*.

$$\mathcal{E}ll_q = \{(z, w) : z^2 = q(w)\} \cup \{\infty\}$$

There is a natural map

$$\begin{aligned}\mathcal{E}ll_q &\longrightarrow \mathbb{P}^1 \\ (z, w) &\longmapsto w \\ \infty &\longmapsto \infty\end{aligned}$$

Turns out this is a complex differentiable map of degree 2 which has exactly 4 distinct ramification points, the three roots of  $q$  and the point at infinity. Plugging in the Riemann–Hurwitz formula we get

$$\begin{aligned}\chi(\mathcal{E}ll_q) &= 2\chi(\mathbb{P}^1) + \sum_{4 \text{ points}} (2 - 1) \\ &= 4 - 4 \\ &= 0\end{aligned}$$

Hence,  $\mathcal{E}ll_q$  is homeomorphic to a torus.

Almost nothing in this section generalizes arbitrarily. Not all compact Riemann surfaces can be embedded in  $\mathbb{P}^2$ , not all non-compact Riemann surfaces can be compactified by adding a single point at infinity.

But things DO generalize with some effort. All compact Riemann surfaces can be embedded in  $\mathbb{P}^3$ , many non-compact Riemann surfaces of interest can be compactified by adding multiple points at infinity. It is a very non-trivial theorem in complex analysis that every Riemann surface admits a non-constant meromorphic function.

It is a remarkable accident that things work out to be so nice for elliptic curves.

We will make all of this rigorous (as much as possible) in the next two classes.