

## Problem Set 1 - The Riemann Sphere

Corrected open mapping theorem statement:

**Theorem 0.1** (Open mapping theorem). *If  $f : U \rightarrow \mathbb{C}$  is a non-constant complex differentiable then for any open set  $V \subseteq U$ , the set  $f(V)$  is an open subset of  $\mathbb{C}$ .*

### Holomorphic functions on $\mathbb{P}^1$

A complex differentiable function  $f : X \rightarrow \mathbb{C}$  is called a *holomorphic* function on  $X$ .

The Riemann sphere  $\mathbb{P}^1$  is the set  $\mathbb{C} \cup \{\infty\}$ . We write  $\mathbb{P}^1$  as the union of two sets

$$U_1 = \mathbb{C} \qquad U_2 = \mathbb{C} \setminus \{0\} \cup \{\infty\}$$

The set  $U_1$  is the standard complex plane, but the set  $U_2$  is not. We can turn  $U_2$  into the complex plane by using the following function

$$\begin{aligned} \varphi_2 : U_2 &\longrightarrow \mathbb{C} \\ z &\mapsto z^{-1} \text{ if } z \neq \infty \\ \infty &\mapsto 0 \end{aligned}$$

Thus we can think of  $\mathbb{P}^1$  as two copies of the complex plane ( $U_1$  and  $\varphi_2(U_2)$ ) glued together.

A function  $f : \mathbb{P}^1 \rightarrow \mathbb{C}$  is defined as a pair of functions

$$f_1 : U_1 \rightarrow \mathbb{C} \qquad f_2 : U_2 \rightarrow \mathbb{C}$$

such that  $f_1$  and  $f_2$  agree on  $U_1 \cap U_2$ .

We can only make sense of the complex differentiable functions when both the source and target are open subsets of  $\mathbb{C}$ . For this reason, we define holomorphic functions on  $\mathbb{P}^1$  as follows.

A function  $f : \mathbb{P}^1 \rightarrow \mathbb{C}$  is holomorphic if

1.  $f|_{U_1}$  is holomorphic,

2.  $f \circ \varphi_2^{-1}|_{\varphi(U_2)}$  is holomorphic.

$$\begin{array}{ccc}
 U_1 & \xrightarrow{f|_{U_1}} & \mathbb{C} \\
 \\ 
 U_2 & \xrightarrow{f} & \mathbb{C} \\
 \varphi_2^{-1} \uparrow \downarrow \varphi_2 & \nearrow f \circ \varphi_2^{-1} & \\
 \mathbb{C} & & 
 \end{array}$$

**Q. 1.** Check that defining a function  $f : \mathbb{P}^1 \rightarrow \mathbb{C}$  is equivalent to defining a pair of functions

$$\begin{aligned}
 f_1 : U_1 &\longrightarrow \mathbb{C} \\
 f_2 \circ \varphi_2^{-1} : \varphi(U_2) &\longrightarrow \mathbb{C}
 \end{aligned}$$

such that

$$f_1(z) = f_2 \circ \varphi_2^{-1}(z^{-1}) \text{ whenever } z \neq 0.$$

Note that the source of both  $f_1$  and  $f_2 \circ \varphi_2^{-1}$  is  $\mathbb{C}$ .

It is kinda hard to come up with examples because of the following theorem.

**Theorem 0.2.** *The only holomorphic functions on  $\mathbb{P}^1$  are the constant functions.*

The proof is in the following exercises.

A subset  $V$  of a topological space  $X$  is compact if it has the following property.

Every infinite sequence has a convergent subsequence i.e. for every infinite sequence of points  $a_1, a_2, \dots$  in  $V$  there exists a subsequence  $a_{n_1}, a_{n_2}, a_{n_3}, \dots$  which converges to a point in  $V$ .

**Q. 2.** Prove that compact subsets  $V$  of a (nice) topological space  $X$  are closed i.e. if a sequence of points  $a_1, a_2, \dots$  in  $V$  converges to point  $a \in X$  then  $a$  is in  $V$ .

**Q. 3.** Prove that every compact subset  $V$  of  $\mathbb{C}$  is bounded i.e. there exists a real number  $M$  such that  $z < M$  for all  $z \in V$ .

**Q. 4.** Use the fact that every infinite sequence has a convergent subsequence to argue that for any continuous function  $g : X \rightarrow Y$  the image of

a compact set is compact.

Assume the following fact:

The Riemann sphere is a compact topological space.

One way to see this is that the Riemann sphere is topologically isomorphic to the sphere  $S^2$  in  $\mathbb{R}^3$  (hence the name) which is a closed and bounded subset of  $\mathbb{R}^3$ . It is not hard to show that such subsets are compact.

**Q. 5.** Argue that the image of any continuous function  $f : \mathbb{P}^1 \rightarrow \mathbb{C}$  is bounded.

**Q. 6.** Using Liouville's theorem, argue that if  $f$  is a holomorphic function on  $\mathbb{P}^1$  then  $f|_{U_1}$  is a constant function.

**Q. 7.** Using continuity, argue that if  $f$  is a holomorphic function on  $\mathbb{P}^1$  then  $f$  is a constant function.

## Meromorphic functions on $\mathbb{C}$

A complex differentiable function  $f : X \rightarrow \mathbb{P}^1$  is called a *meromorphic* function on  $X$ .

We can only make sense of the complex differentiable functions when both the source and target are open subsets of  $\mathbb{C}$ . For this reason, we define meromorphic functions on  $X$  as follows.

A function  $f : X \rightarrow \mathbb{P}^1$  is meromorphic if

1.  $f$  is holomorphic when restricted to  $f^{-1}(U_1)$ ,
2.  $\varphi_2 \circ f$  is holomorphic when restricted to  $f^{-1}(U_2)$ .

$$\begin{array}{ccccc}
 f^{-1}(U_1) & \xrightarrow{f} & U_1 & = & \mathbb{C} \\
 \\ 
 f^{-1}(U_2) & \xrightarrow{f} & U_2 & \xrightarrow{\varphi_2} & \mathbb{C} \\
 & \searrow & \text{ } & \nearrow & \\
 & & \varphi_2 \circ f & & 
 \end{array}$$

It gets tedious to keep track of all the inverses and the sources and targets. We use the following shorthand notation to simplify the clutter. Let  $\varphi_1 : U_1 \rightarrow \mathbb{C}$  be

the identity function,  $\varphi_1(z) = z$ . Then a function  $f : X \rightarrow \mathbb{P}^1$  is meromorphic if the two functions

1.  $\varphi_1 \circ f$ ,
2.  $\varphi_2 \circ f$ ,

are holomorphic wherever they make sense.

$$\begin{array}{ccccc} f^{-1}(U_1) & \xrightarrow{f|_{f^{-1}(U_1)}} & U_1 & \xrightarrow{\varphi_1} & \mathbb{C} \\ & \searrow & & \nearrow & \\ & & \varphi_1 \circ f & & \\ f^{-1}(U_2) & \xrightarrow{f|_{f^{-1}(U_2)}} & U_2 & \xrightarrow{\varphi_2} & \mathbb{C} \\ & \searrow & & \nearrow & \\ & & \varphi_2 \circ f & & \end{array}$$

**Q. 8.** Which of the following functions are meromorphic functions on  $\mathbb{C}$ ?

1.  $f(z) = z$
- 2.

$$f(z) = \begin{cases} z^{-1} & \text{if } z \neq 0 \\ \infty & \text{if } z = 0 \end{cases}$$

- 3.

$$f(z) = \begin{cases} e^{1/z} & \text{if } z \neq 0 \\ \infty & \text{if } z = 0 \end{cases}$$

## Meromorphic functions on $\mathbb{P}^1$

**Q. 9.** Show that a function  $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  is meromorphic if the four functions

1.  $\varphi_1 \circ f \circ \varphi_1^{-1}$ ,
2.  $\varphi_1 \circ f \circ \varphi_2^{-1}$ ,
3.  $\varphi_2 \circ f \circ \varphi_1^{-1}$ ,
4.  $\varphi_2 \circ f \circ \varphi_2^{-1}$ ,

are holomorphic wherever they make sense.

**Q. 10.** Let  $p(z)$  and  $q(z)$  be polynomials with no common roots. Assume that  $q(z)$  is not the 0 polynomial.

Show that the following function is a meromorphic function on  $\mathbb{P}^1$ .

$$f(z) = \begin{cases} \frac{p(z)}{q(z)} & \text{if } z \neq \infty, q(z) \neq 0, \\ \infty & \text{if } z \neq \infty, q(z) = 0, \\ \lim_{z \rightarrow \infty} \frac{p(z)}{q(z)} & \text{if } z = \infty. \end{cases}$$

Such a function is called a *rational function*. It is common to simply write  $f(z) = \frac{p(z)}{q(z)}$ .

Turns out these are all the meromorphic functions on  $\mathbb{P}^1$ .

**Theorem 0.3.** *Every meromorphic function on  $\mathbb{P}^1$  is a rational function.*

The following exercises provide the proof of this theorem.

Let  $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  be a meromorphic function. Let  $\mathcal{Z} = f^{-1}(0) \cap \mathbb{C}$  and  $\mathcal{P} = f^{-1}(\infty) \cap \mathbb{C}$ .  $\mathcal{Z}$  is called the set of zeroes and  $\mathcal{P}$  is called the set of poles.

**Q. 11.** Using the isolated zeroes property of complex differentiable functions argue that both the sets  $\mathcal{Z}$  and  $\mathcal{P}$  are isolated i.e. for every point  $x \in \mathcal{Z}$  there exists a neighborhood  $U$  of  $x$  such that  $U \cap \mathcal{Z} = \{x\}$ . Similarly, for  $\mathcal{P}$ .

**Q. 12.** Using the fact that every infinite sequence in a compact set has a convergent subsequence, and that  $\mathbb{P}^1$  is compact, argue that both  $\mathcal{Z}$  and  $\mathcal{P}$  are finite sets.

Let  $\mathcal{Z} = \{z_1, \dots, z_m\}$  and  $\mathcal{P} = \{p_1, \dots, p_n\}$ . Assume the following fact for now. We'll prove it in class tomorrow.

The function

$$g(z) = f(z) \cdot \frac{(z - p_1)^{k_1} \dots (z - p_n)^{k_n}}{(z - z_1)^{\ell_1} \dots (z - z_m)^{\ell_m}}$$

is meromorphic and has no zeroes or poles, for some positive integers  $k_1, \dots, k_n$  and  $\ell_1, \dots, \ell_m$ .

**Q. 13.** Check that the open mapping theorem 0.1 extends verbatim to meromorphic functions on  $\mathbb{P}^1$ .

**Q. 14.** Using the open mapping theorem and Q.4 argue that  $g$  is either a constant function or the image of  $g$  is all of  $\mathbb{P}^1$ .<sup>1</sup>

Because the only zero or pole of  $g$  is at  $\infty$  (which can be one or the other) the image of  $g$  cannot be all of  $\mathbb{P}^1$ . Hence,  $g$  is a constant function i.e.  $g(z) = c$  for some  $c \in \mathbb{C}$ . Hence,

$$\begin{aligned} f(z) \cdot \frac{(z - p_1)^{k_1} \dots (z - p_n)^{k_n}}{(z - z_1)^{\ell_1} \dots (z - z_m)^{\ell_m}} &= c \\ \Rightarrow f(z) &= c \frac{(z - z_1)^{\ell_1} \dots (z - z_m)^{\ell_m}}{(z - p_1)^{k_1} \dots (z - p_n)^{k_n}}. \end{aligned}$$

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<sup>1</sup>**Hint:** You will need to use the fact that the only open and closed subsets of  $\mathbb{P}^1$  are the empty set and  $\mathbb{P}^1$  itself.