## Problem Set 1 - The Riemann Sphere

Corrected open mapping theorem statement:

**Theorem 0.1** (Open mapping theorem). If  $f: U \to \mathbb{C}$  is a non-constant complex differentiable then for any open set  $V \subseteq U$ , the set f(V) is an open subset of  $\mathbb{C}$ .

## Holomorphic functions on $\mathbb{P}^1$

A complex differentiable function  $f: X \to \mathbb{C}$  is called a *holomorphic* function on X.

The Riemann sphere  $\mathbb{P}^1$  is the set  $\mathbb{C} \cup \{\infty\}$ . We write  $\mathbb{P}^1$  as the union of two sets

$$U_1 = \mathbb{C}$$
  $U_2 = \mathbb{C} \setminus \{0\} \cup \{\infty\}$ 

The set  $U_1$  is the standard complex plane, but the set  $U_2$  is not. We can turn  $U_2$  into the complex plane by using the following function

$$\begin{split} \varphi_2: U_2 &\longrightarrow \mathbb{C} \\ z &\mapsto z^{-1} \text{ if } z \neq \infty \\ \infty &\mapsto 0 \end{split}$$

Thus we can think of  $\mathbb{P}^1$  as two copies of the complex plane  $(U_1$  and  $\varphi_2(U_2))$  glued together.

A function  $f:\mathbb{P}^1\to\mathbb{C}$  is defined as a pair of functions

$$f_1: U_1 \to \mathbb{C}$$
  $f_2: U_2 \to \mathbb{C}$ 

such that  $f_1$  and  $f_2$  agree on  $U_1 \cap U_2$ .

We can only make sense of the complex differentiable functions when both the source and target are open subsets of  $\mathbb{C}$ . For this reason, we define holomorphic functions on  $\mathbb{P}^1$  as follows.

A function  $f: \mathbb{P}^1 \to \mathbb{C}$  is holomorphic if

1.  $f|_{U_1}$  is holomorphic,

2.  $f \circ \varphi_2^{-1}|_{\varphi(U_2)}$  is holomorphic.

$$U_1 \xrightarrow{f|_{U_1}} \mathbb{C}$$

$$\begin{array}{c} U_2 \stackrel{f}{\longrightarrow} \mathbb{C} \\ \varphi_2^{-1} \bigwedge^{\uparrow} \varphi_2 & f \circ \varphi_2^{-1} \end{array}$$

**Q. 1.** Check that defining a function  $f:\mathbb{P}^1\to\mathbb{C}$  is equivalent to defining a pair of functions

$$f_1: U_1 \longrightarrow \mathbb{C}$$
  
 $f_2 \circ \varphi_2^{-1}: \varphi(U_2) \longrightarrow \mathbb{C}$ 

such that

$$f_1(z) = f_2 \circ \varphi_2^{-1}(z^{-1})$$
 whenever  $z \neq 0$ .

Note that the source of both  $f_1$  and  $f_2 \circ \varphi_2^{-1}$  is  $\mathbb{C}$ .

It is kinda hard to come up with examples because of the following theorem.

**Theorem 0.2.** The only holomorphic functions on  $\mathbb{P}^1$  are the constant functions. The proof is in the following exercises.

A subset V of a topological space X is compact if it has the following property.

Every infinite sequence has a convergent subsequence i.e. for every infinite sequence of points  $a_1, a_2, \ldots$  in V there exists a subsequence  $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$  which converges to a point in V.

**Q. 2.** Prove that compact subsets V of a (nice) topological space X are closed i.e. if a sequence of points  $a_1, a_2, \ldots$  in V converges to point  $a \in X$  then a is in V.

**Q. 3.** Prove that every compact subset V of  $\mathbb C$  is bounded i.e. there exists a real number M such that z < M for all  $z \in V$ .

**Q. 4.** Use the fact that every infinite sequence has a convergent subsequence to argue that for any continuous function  $g: X \to Y$  the image of

a compact set is compact.

Assume the following fact:

The Riemann sphere is a compact topological space.

One way to see this is that the Riemann sphere is topologically isomorphic to the sphere  $S^2$  in  $\mathbb{R}^3$  (hence the name) which is a closed and bounded subset of  $\mathbb{R}^3$ . It is not hard to show that such subsets are compact.

**Q. 5.** Argue that the image of any continuous function  $f: \mathbb{P}^1 \to \mathbb{C}$  is bounded.

**Q. 6.** Using Liouville's theorem, argue that if f is a holomorphic function on  $\mathbb{P}^1$  then  $f|_{U_1}$  is a constant function.

**Q. 7.** Using continuity, argue that if f is a holomorphic function on  $\mathbb{P}^1$  then f is a constant function.

## Meromorphic functions on $\mathbb C$

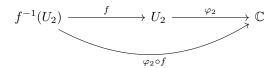
A complex differentiable function  $f: X \to \mathbb{P}^1$  is called a *meromorphic* function on X.

We can only make sense of the complex differentiable functions when both the source and target are open subsets of  $\mathbb{C}$ . For this reason, we define meromorphic functions on X as follows.

A function  $f: X \to \mathbb{P}^1$  is meromorphic if

- 1. f is holomorphic when restricted to  $f^{-1}(U_1)$ ,
- 2.  $\varphi_2 \circ f$  is holomorphic when restricted to  $f^{-1}(U_2)$ .

$$f^{-1}(U_1) \xrightarrow{f} U_1 = \mathbb{C}$$

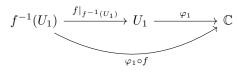


It gets tedious to keep track of all the inverses and the sources and targets. We use the following shorthand notation to simplify the clutter. Let  $\varphi_1: U_1 \to \mathbb{C}$  be

the identity function,  $\varphi_1(z) = z$ . Then a function  $f: X \to \mathbb{P}^1$  is meromorphic if the two functions

- 1.  $\varphi_1 \circ f$ ,
- $2. \varphi_2 \circ f,$

are holomorphic wherever they make sense.



$$f^{-1}(U_2) \xrightarrow{f|_{f^{-1}(U_2)}} U_2 \xrightarrow{\varphi_2} \mathbb{C}$$

**Q. 8.** Which of the following functions are meromorphic functions on  $\mathbb{C}$ ?

- 1. f(z) = z
- 2.

$$f(z) = \begin{cases} z^{-1} & \text{if } z \neq 0\\ \infty & \text{if } z = 0 \end{cases}$$

3.

$$f(z) = \begin{cases} e^{1/z} & \text{if } z \neq 0\\ \infty & \text{if } z = 0 \end{cases}$$

## Meromorphic functions on $\mathbb{P}^1$

**Q. 9.** Show that a function  $f: \mathbb{P}^1 \to \mathbb{P}^1$  is meromorphic if the four

- 1.  $\varphi_1 \circ f \circ \varphi_1^{-1}$ , 2.  $\varphi_1 \circ f \circ \varphi_2^{-1}$ , 3.  $\varphi_2 \circ f \circ \varphi_1^{-1}$ , 4.  $\varphi_2 \circ f \circ \varphi_2^{-1}$ ,

are holomorphic wherever they make sense.

**Q. 10.** Let p(z) and q(z) be polynomials with no common roots. Assume that q(z) is not the 0 polynomial.

Show that the following function is a meromorphic function on  $\mathbb{P}^1$ .

$$f(z) = \begin{cases} \frac{p(z)}{q(z)} & \text{if } z \neq \infty, q(z) \neq 0, \\ \infty & \text{if } z \neq \infty, q(z) = 0, \\ \lim_{z \to \infty} \frac{p(z)}{q(z)} & \text{if } z = \infty. \end{cases}$$

Such a function is called a rational function. It is common to simply write  $f(z) = \frac{p(z)}{q(z)}$ .

Turns out these are all the meromorphic functions on  $\mathbb{P}^1$ .

**Theorem 0.3.** Every meromorphic function on  $\mathbb{P}^1$  is a rational function.

The following exercises provide the proof of this theorem.

Let  $f: \mathbb{P}^1 \to \mathbb{P}^1$  be a meromorphic function. Let  $\mathcal{Z} = f^{-1}(0) \cap \mathbb{C}$  and  $\mathcal{P} = f^{-1}(\infty) \cap \mathbb{C}$ .  $\mathcal{Z}$  is called the set of zeroes and  $\mathcal{P}$  is called the set of poles.

**Q. 11.** Using the isolated zeroes property of complex differentiable functions argue that both the sets  $\mathcal{Z}$  and  $\mathcal{P}$  are isolated i.e. for every point  $x \in \mathcal{Z}$  there exists a neighborhood U of x such that  $U \cap \mathcal{Z} = \{x\}$ . Similarly, for  $\mathcal{P}$ .

**Q. 12.** Using the fact that every infinite sequence in a compact set has a convergent subsequence, and that  $\mathbb{P}^1$  is compact, argue that both  $\mathcal{Z}$  and  $\mathcal{P}$  are finite sets.

Let  $\mathcal{Z} = \{z_1, \dots, z_m\}$  and  $\mathcal{P} = \{p_1, \dots, p_n\}$ . Assume the following fact for now. We'll prove it in class tomorrow.

The function

$$g(z) = f(z) \cdot \frac{(z - p_1)^{k_1} \dots (z - p_n)^{k_n}}{(z - z_1)^{\ell_1} \dots (z - z_m)^{\ell_m}}$$

is meromorphic and has no zeroes or poles, for some positive integers  $k_1, \ldots, k_n$  and  $\ell_1, \ldots, \ell_m$ .

**Q. 13.** Check that the open mapping theorem 0.1 extends verbatim to meromorphic functions on  $\mathbb{P}^1$ .

**Q. 14.** Using the open mapping theorem and Q.4 argue that g is either a constant function or the image of g is all of  $\mathbb{P}^1$ .

Because the only zero or pole of g is at  $\infty$  (which can be one or the other) the image of g cannot be all of  $\mathbb{P}^1$ . Hence, g is a constant function i.e. g(z) = c for some  $c \in \mathbb{C}$ . Hence,

$$f(z) \cdot \frac{(z - p_1)^{k_1} \dots (z - p_n)^{k_n}}{(z - z_1)^{\ell_1} \dots (z - z_m)^{\ell_m}} = c$$

$$\Longrightarrow \qquad f(z) = c \frac{(z - z_1)^{\ell_1} \dots (z - z_m)^{\ell_m}}{(z - p_1)^{k_1} \dots (z - p_n)^{k_n}}.$$

**Hint**: You will need to use the fact that the only open and closed subsets of  $\mathbb{P}^1$  are the empty set and  $\mathbb{P}^1$  itself.