1 Inverses of functions

Riemann surfaces naturally arise when trying to invert complex functions. As in the real case, the inverse of the function $f(x) = x^2$ is not well-defined. There are multiple ways to remedy this, but the most *geometric* solution is to study the graph $y = x^2$ as a subset of $\mathbb{R}^2$ and use this to construct inverses wherever they make sense.

We do the same for complex functions. Let $p(z)$ be a polynomial of degree n. Finding the inverse function is equivalent to solving

$$w = p(z)$$

to get z in terms of w. This is almost never possible, so instead we study the graph

$$\Gamma_{p(z)} := \{(z, w) : w = p(z)\} \subseteq \mathbb{C}^2$$

Denote by $\pi : \mathbb{C}^2$ the projection onto the w- coordinate axis[1]. $\Gamma_{p(z)}$ is an example of a Riemann surface! Our first goal is to undestand the geometry of this object. We'll do this using the projection π.

1.1 Ramified coverings

Consider the restriction of π to $\Gamma_{p(z)}$.

$$\pi : \Gamma_{p(z)} \to \mathbb{C}$$

Q. 1. Show that this map is surjective.

The inverse set $\pi^{-1}(w)$ (called the *fiber* over w) is the set of roots of $p(z) - w$.

Q. 2. Prove that there are only finitely many values of w for which $\pi^{-1}(w)$ has size $< n$. And that outside this set the fiber has size exactly n.

Definition 1.1. For $w \in \mathbb{C}$, if the polynomial $p(z) - w$ has distinct roots, then w is called *unramified*. This is equivalent to requiring that the fiber $\pi^{-1}(w)$ has size n. If $p(z) - w$ has repeated roots then, if z is a repeated root of $p(z) - w$ of order k we say that the *ramification index* e_P of $P = (z, w)$ is k.

[1] We are working over complex numbers, so everything is twice the dimension of real numbers. So a complex axis is complex dimension 1 but real dimension 2.

> **Q. 3.** For $w_0 \in \mathbb{C}$, let $\pi^{-1}(w_0) = \{P_1, \ldots, P_l\}$, then
>
> $$e_{P_1} + \cdots + e_{P_l} = n$$

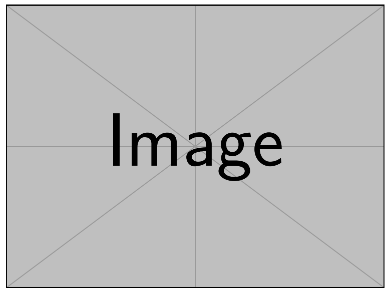

Figure 1: Picture of ramification

A much stronger statement is true on the topological level. Let w be an unramified value. Then $p(z) - w$ has n distinct roots $z_1, \ldots, z_n$. If we perturb w slightly then the corresponding roots z_i will also get perturbed slightly.

For $\epsilon \in \mathbb{R}$ let $B_\epsilon(w) \subset \mathbb{C}$ denote a ball of radius ϵ around w.

Theorem 1.2. *For an unramified value $w \in \mathbb{C}$, let $z_1, \ldots, z_n$ be the distinct roots of $p(z) - w$. Then there exists an ϵ such that $\pi^{-1}(B_\epsilon(w))$ has the following properties.*

1. *$\pi^{-1}(B_\epsilon(w))$ has n-connected components $U_1, \ldots, U_n$,*
2. *$z_i \in U_i$,*
3. *the projections $\pi : U_i \to B_\epsilon(w)$ are a homeomorphism.*

Theorem 1.2 is saying that the map π restricted to the fibers of unramified values is a *covering map*

$$\pi : \pi^{-1}(\{\text{unramified values}\}) \longrightarrow \{\text{unramified values}\}.$$

In Figure 1 the ramified points appear to be singular, but

A graph is always homeomorphic to the base i.e. the

$$\mathbb{C} \to \Gamma_{p(z)} z \mapsto (z, p(z))$$

is always a homeomorphism. So the Riemann surface we have constructed is homeomorphic to $\mathbb{C}$ itself! We'll improve upon this in the following sections.

1.2 Sections as inverse functions

We now have a way of constructing inverse functions, which relies on the following theorem from topology.

A space is said to be *simply-connected* if it is path-connected and every loop in X can be continuously deformed to a point.

Theorem 1.3. *If X is a simply-connected space and $\pi : Y \to X$ is a finite covering of X of degree n then Y is homeomorphic to n-disjoint copies of X and the projection map on each connected component is a homeomorphism.*

$$Y \cong X_1 \sqcup \cdots \sqcup X_n$$

$$\begin{tikzcd} X_i \arrow[d, "\pi"', "\cong"] \\ X \end{tikzcd}$$

This statement is also true when the covering is not finite. In this case, Y is a disjoint union of infinitely many components.

Now we have a meaninful procedure to construct inverse functions of ramified coverings of $\mathbb{C}$. Consider the ramified covering $\pi : \Gamma_{p(z)} \to \mathbb{C}$ with ramified values $w_1, \ldots, w_l$ (these are called the *branch points*).

1. Consider the set of unramified points $\mathbb{C} \setminus \{w_1, \ldots, w_l\}$. This set is never simply-connected (unless $l = 0$).

2. Pick an arbitrary open subset U of $\mathbb{C} \setminus \{w_1, \ldots, w_l\}$ which is simply-connected. This is usually done by removing rays (called *branch cuts*) emanating from the points w_i.

3. The restriction $\pi : p^{-1}(U) \to U$ is a finite covering of a simply-connected connected space U, hence $U \cong U_1 \sqcup \cdots \sqcup U_n$.

4. Pick an aribtrary connected component U_i so that $\pi : U_i \to U$ is a homeomorphism. Because this is a homeomorphism it has an inverse $\pi_i^{-1} : U \to U_i$.

5. The function π_i^{-1} followed by projection onto the z-coordinate is then an inverse of the function p. It is called a *branch* of the inverse function (and so there are n possible branches).

We need to choose *branch cuts* and a *branch* to define an inverse function.

Example 1.4.

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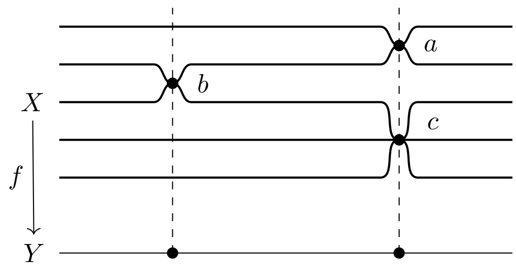

Figure 2: Not sure how to adjust width in this case.

2 Compactification

We saw that the Riemann surface $\Gamma_{p(z)}$ is homeomorphic to $\mathbb{C}$.

We can generalize the previous constructions to studying graphs of

$$p(z) = w^p$$
$$p(z, w) = 0$$

But before we do this, we need to make one more leap which is again inspired from topology.

The surface $\Gamma_{p(z)}$ costructed above is not compact. Non-compact objects are harder[1] to study than compact ones because the behaviour of limits is harder to control for non-compact surfaces and functions on non-compact .

Q. 4. (Optional exercise) Let X be a subset of $\mathbb{C}$.

1. Show that if X is compact then every function $f : X \to \mathbb{R}$ is bounded.
2. If X is open, find an unbounded continuous function $f : X \to \mathbb{C}$.

Furthermore, non-compact subsets come in all shapes and sizes (for example, every open subset of $\mathbb{C}$ is non-compact) but as we'll see in the next section there are various classification theorems for compact ones.

2.1 Behaviour at ∞

[1]In most case, but of course not always.