Homework 04
Fundamental group of the circle

Algebraic Topology - Winter 2021

Due: **February 11, 2021, 11:59 pm**

1. Let (X, x_0), (Y, y_0) be path-connected spaces in $\mathbf{Top}_*$. Let $\ell : (X, x_0) \to (Y, y_0)$ be a covering map. For a path γ in Y starting at y_0, denote by $\widetilde{\gamma}$ a lift of γ starting at x_0.

   (a) Lebesgue's number lemma.

   (b) Prove that the "unwinding" map
   $$\begin{aligned} \pi_1(Y, y_0) &\to \ell^{-1}(y_0) \\ [\gamma] &\to \widetilde{\gamma}(1) \end{aligned}$$
   is surjective.

2. Let X, Y be non-empty spaces in $\mathbf{Top}$. A map $\ell : X \to Y$ is said to be a *local homeomorphism* if every $x \in X$ has an open neighborhood $U \subseteq X$ such that $\ell|_U : U \to \ell(U)$ is a homeomorphism, where $\ell(U)$ is open in Y and is given the subspace topology.

   (a) Show that every covering map is a local homeomorphism.

   (b) Show that the converse of the above statement is not true.

3. The real projective space $\mathbb{RP}^n$ is defined as
   $$\mathbb{RP}^n := S^n / (x_0, x_1, \ldots, x_n) \sim (-x_0, -x_1, \ldots, -x_n)$$
   for all $(x_0, x_1, \ldots, x_n) \in S^n$, where n is a positive integer.

   (a) Show that $\mathbb{RP}^1 \cong S^1$.

   (b) Show that the natural quotient map $\ell : S^n \to \mathbb{RP}^n$ is a covering map.

   (c) What is the map $\pi_1(\ell) : \pi_1(S^1) \to \pi_1(\mathbb{RP}^1)$, where we identify both $\pi_1(S^1)$ and $\pi_1(\mathbb{RP}^1)$ with $\mathbb{Z}$?

   (d) Compute $\pi_1(\mathbb{RP}^n)$.

   (e) Show that $\mathbb{RP}^n \ncong S^n$ for $n > 1$.

   (f) In a previous homework, you constructed a CW structure on S^2 with two 0-cells, two 1-cells, and two 2-cells. Using that, construct a CW structure on $\mathbb{RP}^2$ with one 0-cell, one 1-cell, and one 2-cell. Be careful with the gluing maps! (Optional: Generalize this to higher dimensions.)

4. (Optional) For this question, assume that all paths and all homotopies are smooth and all loops are based at $(1,0)$.

   We know that $\mathbb{R}^2 \setminus \{0\}$ deformation retracts onto $S^1 = \{(x,y) : x^2 + y^2 = 1\}$ and hence $\pi_1(\mathbb{R}^2 \setminus \{0\}) \cong \mathbb{Z}$. We can *almost* prove this fact directly using just calculus. The "unwinding" map $\pi_1(\mathbb{R}^2 \setminus \{0\}) \to \mathbb{Z}$ is supposed to send a loop γ to the *number of counterclockwise rotations* made by γ around the origin. We'll construct this map using line integrals.

   We start by defining θ to be the angle that the vector (x,y) makes with the positive x-axis in the counterclockwise direction. For vectors a with positive x-coordinate, this can be defined as

   $$\theta(x,y) := \arctan(y/x).$$

   It is not possible to extend θ continuously to all of $\mathbb{R}^2 \setminus \{0\}$ but its derivatives can be extended!

   (a) Define the following formal expression

   $$d\theta := \frac{\partial \theta}{\partial x} dx + \frac{\partial \theta}{\partial y} dy.$$

   Show that

   $$d\theta = \frac{x dy - y dx}{r^2},$$

   where $r = \sqrt{x^2 + y^2}$. Note that $d\theta$ is well-defined over all of $\mathbb{R}^2 \setminus \{0\}$.

   The line integral $\oint_\gamma d\theta$ measures the change in the angle θ along the path γ. Hence, the *winding number* of γ, which is defined as

   $$w(\gamma) := \frac{1}{2\pi} \oint_\gamma d\theta,$$

   measures the number of counterclockwise rotations made by γ around the origin.

   (b) Show that

   $$w(\gamma_n) = n,$$

   where n is an integer and γ_n is the loop defined as $\gamma_n(t) = (\cos(2\pi n t), \sin(2\pi n t))$.

   (c) Using Stoke's theorem, show that if there is a path homotopy connecting γ to γ' then

   $$w(\gamma) = w(\gamma'),$$

   where γ and γ' are paths in $\mathbb{R}^2 \setminus \{0\}$.

   (d) Using the fact that line integrals are invariant under reparametrization, argue that

   $$w(\gamma \cdot \gamma') = w(\gamma) + w(\gamma'),$$

   where γ and γ' are loops in $\mathbb{R}^2 \setminus \{0\}$.

   (e) Assume that $\pi_1(\mathbb{R}^2 \setminus \{0\})$ is isomorphic to the group of smooth loops modulo smooth homotopies. Then the above arguments *almost* prove that the map $w : \pi_1(\mathbb{R}^2 \setminus \{0\}) \to \mathbb{Z}$ is a group isomorphism. What is missing? It is this last missing part that requires the theory of covering spaces in an essential way.

Suggested exercises for practice from Hatcher

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