Homework 02
Basepoints and CW structures

Algebraic Topology - Winter 2021

Due: **January 28, 2021, 11:59 pm**

1. Let (X, x) and (Y, y) be pointed topological spaces. Prove that

$$\pi_1(X \times Y, (x, y)) \cong \pi_1(X, x) \times \pi_1(Y, y).$$

2. For a topological space X, define the *fundamental groupoid*, denoted $\Pi_1(X)$, to be the category whose objects are points in X and whose morphisms are path-homotopy classes of paths i.e. for $x, y \in X$ define

$$\Pi_1(x, y) := \{[\gamma] \mid \gamma : [0, 1] \to X, \partial_0\gamma = x, \partial_1\gamma = y\}.$$

   (a) A groupoid is a category in which every morphism is invertible. Show that $\Pi_1(X)$ is a groupoid.

   Let **Cat** denote the category whose objects are categories and whose morphisms are functors.

   (b) Show that Π_1 naturally extends to a functor $\Pi_1 : \mathbf{Top} \to \mathbf{Cat}$.

   Let x be a point in X. We can think of the fundamental "group", $\pi_1(X, x)$, as the full subcategory of $\Pi_1(X)$ with exactly one object x.

   (c) Show that if X is path-connected, then the categories $\Pi_1(X)$ and $\pi_1(X, x)$ are equivalent.

   Thus, the fundamental groupoid is a basepoint-independent variant of the fundamental group.

For the following questions, you do not have to provide completely rigorous proofs. It's enough to draw sufficiently descriptive pictures.

3. Let n be a positive integer.

   (a) Construct a CW structure for S^1 with n 0-cells and n 1-cells.

   (b) Construct a CW structure for S^n with only two cells total.

   (c) Construct a CW structure for S^3 two 0-cells, two 1-cells, two 2-cells, and two 3-cells. (Optional: Can you generalize this to S^n?)

   (d) Construct a CW complex with one 0-cell, one 1-cell, one 2-cell, and one 3-cell that is homotopy equivalent to S^3. (Optional: Can you generalize this to S^{2n-1}?)

   (e) Let v, e, f be positive integers with $v - e + f = 2$. Construct a CW structure for S^2 having v 0-cells, e 1-cells, and f 2-cells.

4. Let X be a topological space that can be given a CW structure with a finite number of cells. Let n_i denote the number of cells in dimension i in this CW structure. The *Euler characteristic* of X is defined as

$$\chi(X) := \sum_{i \in \mathbb{N}} (-1)^i n_i.$$

Assume that the Euler characteristic is well-defined i.e. does not depend on the choice of the CW structure. Further assume that the Euler characteristic is a homotopy invariant.

(a) Compute the Euler characteristic of spheres and genus g surfaces.

(b) Does there exist a CW complex with one 0-cell, one 1-cell, one 2-cell, $\ldots$, one $2n$-cell that is homotopy equivalent to S^{2n}?

(c) What can you say about homeomorphisms between the various spheres and genus g surfaces, using just the Euler characteristic?

Suggested exercises for practice from Hatcher

Pg. 18 2

Pg. 19 9, 15, 16

Pg. 38 1, 2, 3

Pg. 39 10, 11, 15, 17