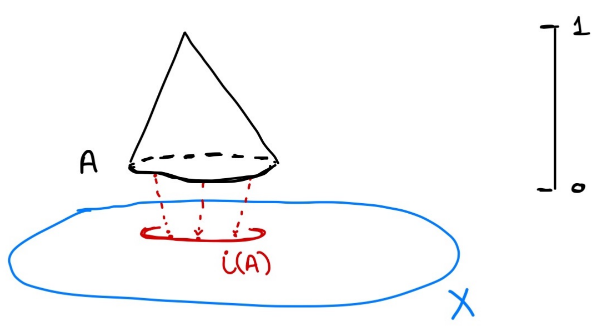
Homework 03
Computations using the Seifert–van Kampen Theorem

Algebraic Topology - Winter 2021

Due: **February 04, 2021, 11:59 pm**

1. Let X be a non-empty space in **Top**.
   (a) We defined the cone over X as
   $$\mathrm{cone}(X) := X \times [0,1]/(x,1) \sim (x',1)$$
   for all $x, x' \in X$. Show that cone(X) is contractible.
   (b) We defined the suspension of X as
   $$SX := X \times [-1,1]/((x,-1) \sim (x',-1), (x,1) \sim (x',1))$$
   for all $x, x' \in X$. Show that SX is path-connected.
   (c) Show that if X is path-connected then SX is simply-connected.
   (d) Show that $S^{n+1} \cong S(S^n)$, where S^n is the n dimensional sphere
   $$S^n = \{(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} : x_0^2 + x_1^2 + \cdots + x_n^2 = 1\}.$$
   (e) Conclude that S^n is simply-connected for $n > 1$.
2. Let A, X be path-connected spaces in **Top** and let $i : A \to X$ be a continuous map between them. Define the mapping cone of i as
   $$\mathrm{cone}(i) := A \times [0,1] \sqcup X/((a,1) \sim (a',1), (a,0) \sim i(a))$$



Compute $\pi_1(\mathrm{cone}(i))$ in terms of $\pi_1(A)$, $\pi_1(X)$, and $\pi_1(i)$. In the special case when i is an inclusion of CW complexes, one can show that cone$(i) \simeq X/A$. This gives us a way to compute $\pi_1(X/A)$ for CW complexes.

You do not have to provide completely rigorous proofs for the following question. For all the technical details see Hatcher, Pg. 50.

3. Let X be a path-connected CW complex with a finite number of cells. Let $\{\mathbb{D}^n_\alpha\}_{\alpha\in\text{cell}(n)}$ be the set of n-cells of X. Let $\text{gl}^n_\alpha : \partial\mathbb{D}^n_\alpha \to X^{n-1}$ be the gluing maps, where X^{n-1} is the $(n-1)^{th}$ skeleton of X.

   (a) Argue that the space obtained by attaching a single cell $\mathbb{D}^n_\alpha$ to X^{n-1} using the gluing map $\text{gl}^n_\alpha : \partial\mathbb{D}^n_\alpha \to X^{n-1}$ is homeomorphic to $\text{cone}(\text{gl}^n_\alpha)$.

   Thus CW complexes are essentially sequences of mapping cones. Use your answers for Q.1 and Q.2 to prove the following.

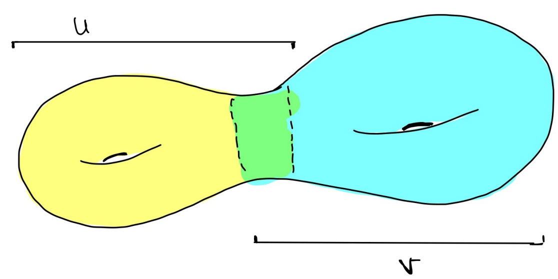
   (b) The map $\pi_1(X^2) \to \pi_1(X)$ induced by the inclusion of the 2-skeleton is an isomorphism.

   (c) The map $\pi_1(X^1) \to \pi_1(X^2)$, induced by the inclusion $X^1 \hookrightarrow X^2$, is a surjection.

   (d) The kernel of the above map is the normal subgroup generated by the elements $\pi_1(\text{gl}^2_\alpha)([\gamma_\alpha])$ where $\alpha \in \text{cell}_2(X)$ and $[\gamma_\alpha]$ is the generator of $\pi_1(\partial\mathbb{D}^2_\alpha)$.

   Thus the 1-skeleton gives us the generators and the 2-cells give us the relations of $\pi_1(X)$.

4. (a) Compute the fundamental group of a genus 2 surface by applying the Seifert–van Kampen to the following open sets:



   (b) Compute the fundamental group of genus g surfaces using gluing diagrams.

   It is highly non-trivial to determine directly the isomorphism classes of these fundamental groups (give it a try!). One trick is to instead study their abelianizations $\pi_1(X)/[\pi_1(X), \pi_1(X)]$.[1]

   (c) Compute $\pi_1(X)/[\pi_1(X), \pi_1(X)]$ for the various spheres and genus g surfaces.

   (d) What can you say about homeomorphisms between the various spheres and genus g surfaces using just $\pi_1(X)/[\pi_1(X), \pi_1(X)]$?

Suggested exercises for practice from Hatcher

Pg. 19 19, 21, 22

Pg. 53 3, 4, 5, 7, 8, 9, 11

[1] We'll later on show that abelianization of the fundamental group is isomorphic to the first homology group.