Homework 01
Fundamentals of Fundamental Groups

Algebraic Topology - Winter 2021

Due: **January 21, 2021, 11:59 pm**

In the following questions, all maps are continuous and all homotopies are basepoint preserving.

1. Show that *path-homotopy* is an equivalence relation.
2. Let (X, x) be a pointed topological space.
   (a) Let c_x denote the constant path at x and let γ be a loop in X based at x. Explicitly construct a reparametrization $\varphi : [0,1] \to [0,1]$ such that $\gamma \cdot c_x = \gamma \circ \varphi$.
   (b) Let γ_1, γ_2, γ_3 be loops in X based at x. Explicitly construct a reparametrization $\varphi : [0,1] \to [0,1]$ such that $(\gamma_1 \cdot \gamma_2) \cdot \gamma_3 = (\gamma_1 \cdot (\gamma_2 \cdot \gamma_3)) \circ \varphi$.
   (c) *Justify* your answers by drawing pictures.
3. Let X be a topological space and let A be a subspace of X. Let a_0 be a point in A. We say that a map $r : X \to A$ is a *retraction* if $r(a) = a$ for all $a \in A$.
   (a) What can you say about $\pi_1(r) : \pi_1(X, a_0) \to \pi_1(A, a_0)$ if r is a retraction?

   We can post-compose r with the inclusion of A in X to get a map $r' : X \to X$. We say that r is a *deformation retraction* if it is a retraction and r' is homotopic to the identity map.

   (b) What can you say about $\pi_1(r) : \pi_1(X, a_0) \to \pi_1(A, a_0)$ if r is a deformation retraction?
   (c) Come up with a concrete non-trivial example of a retraction which is not a deformation retraction, where non-trivial means that $A \neq \{a_0\}$. You do not have to provide proofs for this part.
4. Show that homotopy equivalences satisfy the *2-out-of-3 property*:[1] given spaces (X, x), (Y, y), and (Z, z) and maps $f : (X, x) \to (Y, y)$, and $g : (Y, y) \to (Z, z)$, if any two of f, g, and $g \circ f$ are homotopy equivalences, then so is the third.

$$\begin{tikzcd} (X,x) \arrow[r, "f"'] \arrow[rr, bend left, "g\circ f"] & (Y,y) \arrow[r, "g"'] & (Z,z) \end{tikzcd}$$

[1] See also `https://ncatlab.org/nlab/show/category+with+weak+equivalences#definition`.