Homework 05
Galois Correspondence of Covering Space

Algebraic Topology - Winter 2021

Due: **March 04, 2021, 11:59 pm**

1. Consider the following commuting diagram between path-connected spaces.

$$\begin{tikzcd} X \arrow[rr, "\ell_1"] \arrow[dr, "\ell_3"'] & & Y \arrow[dl, "\ell_2"] \\ & Z. & \end{tikzcd}$$

   (a) Show that if ℓ_2 and ℓ_3 are covering maps then so is ℓ_1.

   (b) Find some non-trivial conditions for ℓ_1 and ℓ_2 under which if ℓ_1 and ℓ_2 are covering maps then so is ℓ_3. Why does *your* proof not work for arbitrary ℓ_1 and ℓ_2.

2. Classify connected covers, up to isomorphism of covers, of each of the following spaces:

   (a) $\mathbb{RP}^2$,

   (b) $\mathbb{RP}^2 \times \mathbb{RP}^2$,

   (c) the Mobius strip,

   (d) (optional) $\mathbb{RP}^2 \vee \mathbb{RP}^2$.

3. Let n be a positive integer greater than 1. Denote by $\bigvee_n S^1$ the wedge of n circles at a point.

   (a) Construct a covering map $\ell : X \to \bigvee_2 S^1$ where $X \simeq \bigvee_n S^1$.

   (b) Conclude that there exists an inclusion $\mathbb{F}_n \hookrightarrow \mathbb{F}_2$ where $\mathbb{F}_k$ denotes the free group on k generators.[1]

   (c) Determine generators and index of $\mathrm{im}(\pi_1(\ell))$ inside $\mathbb{F}_2$.

   (d) (optional) For what positive integers m does there exist an inclusion $\mathbb{F}_m \hookrightarrow \mathbb{F}_n$.

   (e) (optional) Show that every subgroup of a free group is free.

[1] See also the ping-pong lemma.

Suggested exercises for practice from Hatcher

Pg. 79 1, 2, 4, 8, 10

Pg. 80 11, 12, 13, 14

Pg. 81 21, 22