Homework 06
Chain Complexes & Singular Homology

Algebraic Topology - Winter 2021

Due: **March 11, 2021, 11:59 pm**

1. For each of the rings below, compute the homology (up to isomorphism) of the following chain complex over R concentrated in degrees 0 and 1,

$$0 \to R^2 \xrightarrow{\partial_1} R^2 \to 0,$$

   where $\partial_1 = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$ in the standard bases.

   (a) $R = \mathbb{Q}$,

   (b) $R = \mathbb{Z}/p\mathbb{Z}$ where p is a prime number,

   (c) $R = \mathbb{Z}$.

   For part (c) express your answers as a (direct sum of) cyclic group(s).

2. Prove that if X is a point then its singular homology is given by

$$H_i(X;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

3. Let n be a non-negative integer. Let $(C_\bullet, d_\bullet)$ be a chain complex over a field $\mathbb{F}$ such that C_i is a finite-dimensional vector space for each $0 \le i \le n$ and is 0 for $i < 0$ or $i > n$.

$$0 \to C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} C_{n-2} \xrightarrow{d_{n-2}} \dots \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \to 0$$

   Show that

$$\sum_{i=0}^{n} (-1)^i \dim H_i(C_\bullet) = \sum_{i=0}^{n} (-1)^i \dim C_i.$$

   Thus, either of the two sides can be used to define the Euler characteristic of $(C_\bullet, d_\bullet)$.