Code Generation: Aho Johnson Algorithm

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- Does not use algebraic properties of operators.
- Generates optimal code, where, once again, the cost measure is the number of instructions in the code.
- Complexity is linear in the size of the expression tree.

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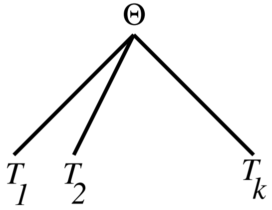
  1. A single vertex labeled by a name from Σ is an expression tree.
  2. If T_1, T_2, ..., T_k are expression trees whose leaves all have distinct labels and θ is a k-ary operator in Θ, then

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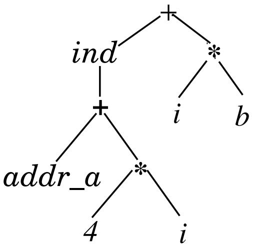
Example

- An example of an expression tree is



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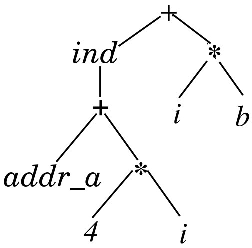

- **Notation:** If T is an expression tree, and S is a subtree of T, then T/S is the the tree obtained by replacing S in T by a single leaf labeled by a distinct name from Σ.

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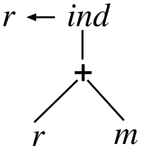
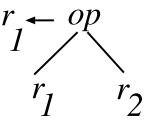
   b. $m \leftarrow r$, a store instruction.

Example Of A Machine

$r \leftarrow c$	**{MOV #c, r}**
$r \leftarrow m$	**{MOV m, r}**
$m \leftarrow r$	**{MOV r, m}**
	**{MOV m(r), r}**
	**{op $\mathbf{r_2}$, $\mathbf{r_1}$}**

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  1. We want to specify what it means to say that a *program* P *computes an expression tree* T. This is when the value of the program $v(P)$ is the same as T.
  2. We also want to talk of *equivalence* of two programs P_1 and P_2. This is true when $v(P_1) = v(P_2)$.

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- If I_q is $z \leftarrow E$, then the value of P is $v_q(z)$.

EXAMPLE

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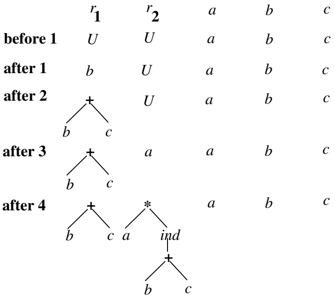
- the values of r_1, r_2, a, b and c at different time instants are:



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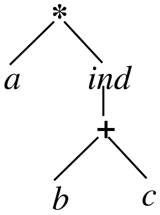

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- NOTE: We shall assume that our programs do not have any useless instructions.

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  b. This register/memory location is not redefined by the instructions between I_t and I_s.
- The relation between I_s and I_t is expressed by saying that I_s is the *last use* of I_t, and is denoted by $s = U_p(t)$.

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- Why is this result important? This is because our algorithm considers programs which are in strong normal form only. The above result assures us that by doing so, we shall not miss out an optimal solution.
- To show the above result, we shall have to consider the kinds of rearrangements which retain program equivalence.

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- π induces a rearranged program $Q = J_1, J_2, \dots, J_q$ with I_i in P becoming $J_{\pi(i)}$ in Q.
- Then Q is equivalent to P if $\pi(U_P(t)) = U_Q(\pi(t))$.

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- To see why the statement of the theorem is true, reason as follows.

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c. Repeat this argument, till you come across an instruction with all constants on the right hand side.

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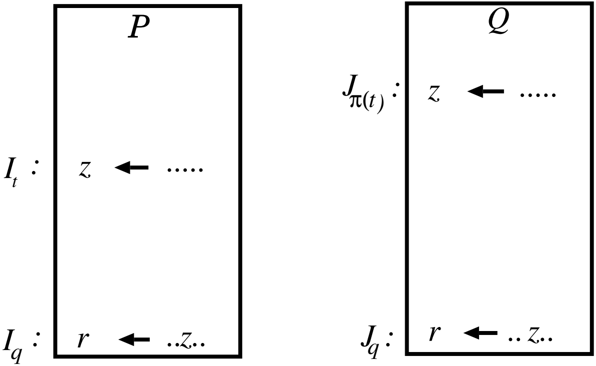

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$r_3 \leftarrow c$	$r_2 \leftarrow c$
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- In the example above, the first program has width 2 but uses 3 registers. By suitable renaming, the number of registers in the second program has been brought down to 2.

LEMMA

Let P be a program of width w, and let R be a set of w distinct registers. Then, by renaming the registers used by P, we may construct an equivalent program P', with the same length as P, which uses only registers in R.

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   a. There is no question of choice for the registers on the RHS of an instruction. These had been decided at the point of their definitions (consistent relabeling).
   b. There is no question of choice for the register r in the instruction $r \leftarrow E$, where E has some register operands. r has to be one of the registers occurring in E.
   c. The only instructions involving a choice of registers are instructions of the form $r \leftarrow E$, where E has no register operands.

PROOF OUTLINE

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4. Therefore a register can always be found for r in the rearranged program P'.

CONTIGUITY AND STRONG CONTIGUITY

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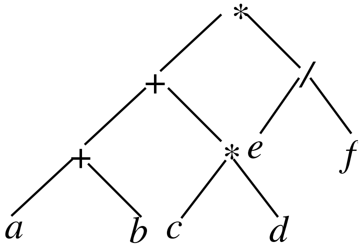
- EXAMPLE: All the three programs P_1, P_2, and P_3 compute the expression tree shown below:

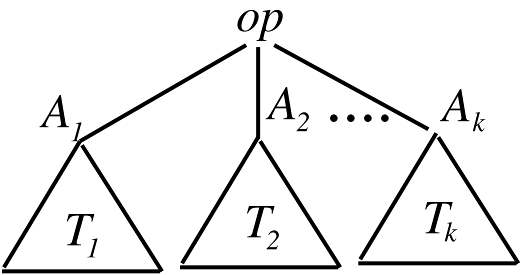


$\underline{P_1}$	$\underline{P_2}$	$\underline{P_3}$
$r_1 \leftarrow a$	$r_1 \leftarrow a$	$r_1 \leftarrow a$
$r_2 \leftarrow b$	$r_2 \leftarrow b$	$r_2 \leftarrow b$
$r_3 \leftarrow c$	$r_3 \leftarrow c$	$r_1 \leftarrow r_1 + r_2$
$r_4 \leftarrow d$	$r_4 \leftarrow d$	$r_2 \leftarrow c$
$r_5 \leftarrow e$	$r_1 \leftarrow r_1 + r_2$	$r_3 \leftarrow d$
$r_6 \leftarrow f$	$r_3 \leftarrow r_3 * r_4$	$r_2 \leftarrow r_2 * r_3$
$r_5 \leftarrow r_5 / r_6$	$r_1 \leftarrow r_1 + r_3$	$r_1 \leftarrow r_1 + r_2$
$r_3 \leftarrow r_3 * r_4$	$r_2 \leftarrow e$	$r_2 \leftarrow e$
$r_1 \leftarrow r_1 + r_2$	$r_3 \leftarrow f$	$r_3 \leftarrow f$
$r_1 \leftarrow r_1 + r_3$	$r_2 \leftarrow r_2 / r_3$	$r_2 \leftarrow r_2 / r_3$
$r_1 \leftarrow r_1 * r_5$	$r_1 \leftarrow r_1 * r_2$	$r_1 \leftarrow r_1 * r_2$

$\underline{P_1}$	$\underline{P_2}$	$\underline{P_3}$
$r_1 \leftarrow a$	$r_1 \leftarrow a$	$r_1 \leftarrow a$
$r_2 \leftarrow b$	$r_2 \leftarrow b$	$r_2 \leftarrow b$
$r_3 \leftarrow c$	$r_3 \leftarrow c$	$r_1 \leftarrow r_1 + r_2$
$r_4 \leftarrow d$	$r_4 \leftarrow d$	$r_2 \leftarrow c$
$r_5 \leftarrow e$	$r_1 \leftarrow r_1 + r_2$	$r_3 \leftarrow d$
$r_6 \leftarrow f$	$r_3 \leftarrow r_3 * r_4$	$r_2 \leftarrow r_2 * r_3$
$r_5 \leftarrow r_5 / r_6$	$r_1 \leftarrow r_1 + r_3$	$r_1 \leftarrow r_1 + r_2$
$r_3 \leftarrow r_3 * r_4$	$r_2 \leftarrow e$	$r_2 \leftarrow e$
$r_1 \leftarrow r_1 + r_2$	$r_3 \leftarrow f$	$r_3 \leftarrow f$
$r_1 \leftarrow r_1 + r_3$	$r_2 \leftarrow r_2 / r_3$	$r_2 \leftarrow r_2 / r_3$
$r_1 \leftarrow r_1 * r_5$	$r_1 \leftarrow r_1 * r_2$	$r_1 \leftarrow r_1 * r_2$

The program P_2 has a width less than P_1, whereas P_3 has the least width of all three programs. P_2 is a *contiguous* program whereas P_3 is a *strongly contiguous* program.

CONTIGUITY AND STRONG CONTIGUITY



CONTIGUITY AND STRONG CONTIGUITY

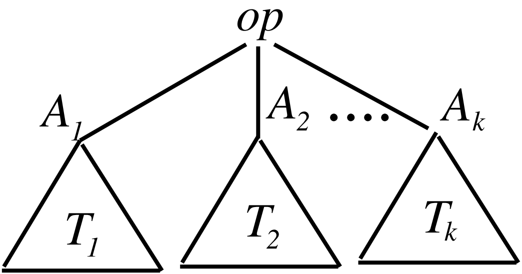
THEOREM: Let $P = I_1, I_2, \ldots, I_q$ be a program of width w with no stores. I_q uses k registers whose values at time $q-1$ are $A_1, \ldots, A_k$. Then there exists an equivalent program $Q = J_1, J_2, \ldots, J_q$, and a permutation π on $\{1, \ldots, k\}$ such that

i. Q has width at most w.

ii. Q can be written as $P_1 \ldots P_k J_q$ where $v(P_i) = A_\pi(i)$ for $1 \leq i \leq k$, and the width of P_i, by itself, is at most $w - i + 1$.

CONTIGUITY AND STRONG CONTIGUITY

Consider an evaluation of the expression tree:.



This tree can be evaluated in the order mentioned below:

CONTIGUOUS AND STRONG CONTIGUOUS EVALUATION

1. Q computes the entire subtree T_1 first using P_1. In the process all the w registers could be used.
2. After computing T_1 all registers except one are freed. Therefore T_2 is free to use $w - 1$ registers and its width is at most $w - 1$. T_2 is computed by P_2.
3. T_3 is similarly computed by P_3, whose width is $w - 2$.

Of course $A_1, \ldots, A_3$ need not necessarily be computed in this order. This is what brings the permutation π in the statement of the theorem. A program in the form $P_1 \ldots P_k J_q$ is said to be in

contiguous form. If each of the P_is is, in turn, contiguous, then the program is said to be in *strong contiguous form*. THEOREM: Every program *without stores* can be transformed into strongly