CS738: Advanced Compiler Optimizations

Foundations of Data Flow Analysis

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Agenda

- *Intraprocedural* Data Flow Analysis
  - We looked at 4 classic examples
  - Today: Mathematical foundations

Taxonomy of Dataflow Problems

- Categorized along several dimensions

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- Four kinds of dataflow problems, distinguished by
  - the operator used for confluence or divergence
  - data flows backward or forward

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Confluence $\rightarrow$ Direction $\downarrow$	$\bigcup$	$\bigcap$
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Backward		

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Why Data Flow Analysis Works?

- Suitable initial values and boundary conditions
- Suitable domain of values
  - Bounded, Finite
- Suitable meet operator
- Suitable flow functions
  - monotonic, closed under composition
- But what is **SUITABLE** ?

Lattice Theory

Partially Ordered Sets

- Posets

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  S: a set

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  - $x \leq y$ and $y \leq z \Rightarrow x \leq z$ (transitive)

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- We are interested in chains of finite length

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- Any **finite nonempty subset** of a poset has **minimal** and **maximal** elements

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- Any **finite nonempty subset** of a poset has **minimal** and **maximal** elements
- Any **finite nonempty chain** has **unique** minimum and maximum elements

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  - $x \leq y$ if and only if $x \bigwedge y = x$

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- Partial order for semilattice
  - $x \leq y$ if and only if $x \bigwedge y = x$
  - Reflexive, antisymmetric, transitive

Border Elements

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Familiar (Semi)Lattices

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  - if $z \leq x$ and $z \leq y$ then $z \leq g$

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- $x, y \in S$
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- Prove that $x \bigwedge y$ is glb of x and y.

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  - Least upper bound (lub)

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  - except for some proofs

Lattice Diagram

- Graphical view of posets

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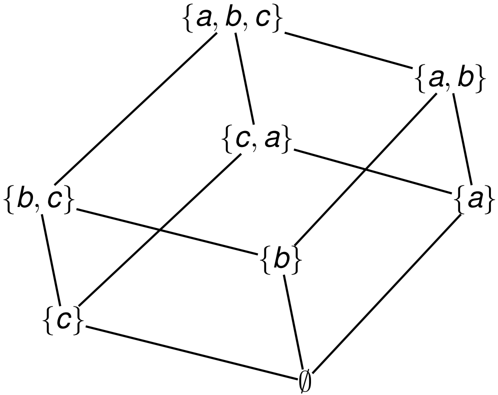
- If $x < y$ then x is depicted lower than y in the diagram
- An edge between x and y (x lower than y) implies $x < y$ and no other element z exists s.t. $x < z < y$ (i.e. transitivity is excluded)

Lattice Diagram



Lattice Diagram for $(\{a, b, c\}, \cap)$

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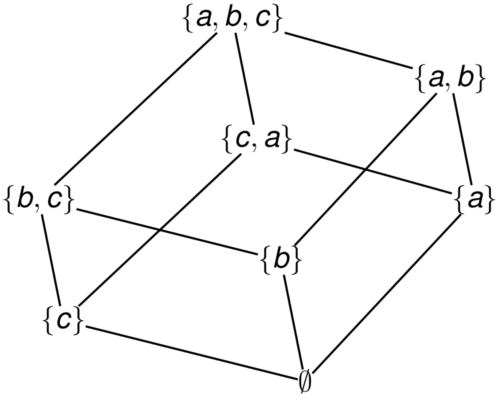

Lattice Diagram for $(\{a, b, c\}, \cap)$

$x \bigwedge y$ = the highest z for which there are paths downward from both x and y.

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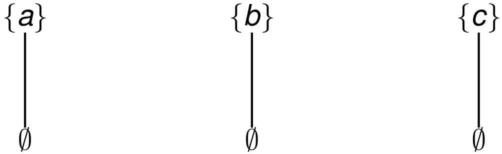
- Superset lattices for singletons



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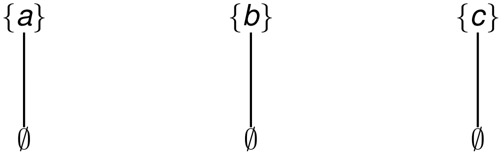

- Combine to form superset lattice for multi-element sets

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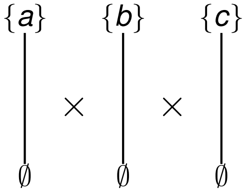
- Can be generalized to product of $N > 2$ lattices
- $(S_1, \bigwedge_1), (S_2, \bigwedge_2), \ldots$ are called component lattices

Product Lattice: Example



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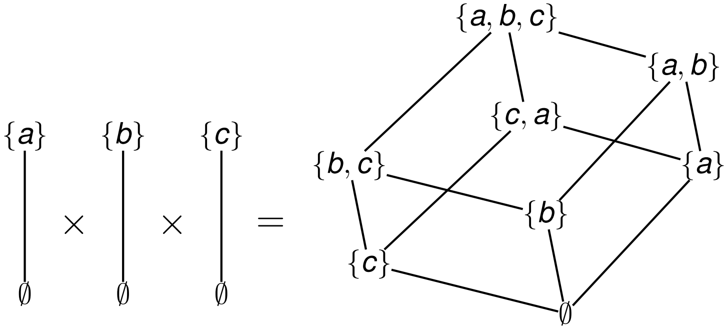

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- Height of the semilattice = $K - 1$

Data Flow Analysis Framework

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Data Flow Analysis Framework

- $(D, S, \bigwedge, F)$
- D: direction – Forward or Backward
- $(S, \bigwedge)$: Semilattice – Domain and meet
- F: family of transfer functions of type $S \rightarrow S$ (see next slide)

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  - Closed under composition:

$$f, g \in F, \quad f \circ g \quad \Rightarrow \quad h \in F$$

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- Composition preserves monotonicity
  - If f and g are monotonic, $h = f \circ g$, then h is also monotonic

Monotone Frameworks

- $(D, S, \bigwedge, F)$ is monotone if the family F consists of monotonic functions only

$$f \in F, \quad \forall x, y \in S \qquad x \leq y \Rightarrow f(x) \leq f(y)$$

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- Proof? : QQ in class

Knaster-Tarski Fixed Point Theorem

- Let f be a monotonic function on a complete lattice $(S, \bigwedge, \bigvee)$. Define

  Then,

Knaster-Tarski Fixed Point Theorem

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  - $\text{red}(f) = \{v \mid v \in S, f(v) \leq v\}$, pre fix-points

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  Then,
  - $\bigwedge \text{red}(f) \in \text{fix}(f)$. Further, $\bigwedge \text{red}(f) = \bigwedge \text{fix}(f)$

Knaster-Tarski Fixed Point Theorem

- Let f be a monotonic function on a complete lattice $(S, \bigwedge, \bigvee)$. Define
  - $\mathrm{red}(f) = \{v \mid v \in S, f(v) \le v\}$, pre fix-points
  - $\mathrm{ext}(f) = \{v \mid v \in S, f(v) \ge v\}$, post fix-points
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  - $\bigvee \mathrm{ext}(f) \in \mathrm{fix}(f)$. Further, $\bigvee \mathrm{ext}(f) = \bigvee \mathrm{fix}(f)$

Knaster-Tarski Fixed Point Theorem

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  - $\mathrm{red}(f) = \{v \mid v \in S, f(v) \leq v\}$, pre fix-points
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  - $\mathrm{fix}(f) = \{v \mid v \in S, f(v) = v\}$, fix-points

  Then,
  - $\bigwedge \mathrm{red}(f) \in \mathrm{fix}(f)$. Further, $\bigwedge \mathrm{red}(f) = \bigwedge \mathrm{fix}(f)$
  - $\bigvee \mathrm{ext}(f) \in \mathrm{fix}(f)$. Further, $\bigvee \mathrm{ext}(f) = \bigvee \mathrm{fix}(f)$
  - $\mathrm{fix}(f)$ is a complete lattice

Application of Fixed Point Theorem

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- Notation: $f^0(x) = x, f^{i+1}(x) = f(f^i(x)), \forall i \geq 0$
- The greatest fixed point of f is

$$f^k(\top), \text{ where } f^{k+1}(\top) = f^k(\top)$$

Fixed Point Algorithm

```
// monotonic function f on a meet semilattice
```

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```
// monotonic function f on a meet semilattice
x := ⊤;
```

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```
// monotonic function f on a meet semilattice
x := ⊤;
while (x ≠ f(x)) x := f(x);
```

Fixed Point Algorithm

```
// monotonic function f on a meet semilattice
x := ⊤;
while (x ≠ f(x)) x := f(x);
return x;
```