CS738: Advanced Compiler Optimizations

Typed Arithmetic Expressions

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Reference Book

Types and Programming Languages by Benjamin C. Pierce

Recap: Untyped Arithmetic Expression Language

t :=		– *terms*
	`true`	– *constant true*
	`false`	– *constant false*
	`if` t `then` t `else` t	– *conditional*
	0	– *constant zero*
	`succ` t	– *successor*
	`pred` t	– *predecessor*
	`iszero` t	– *zero test*

Recap: The Set of Values

v :=		– *values*
	`true`	– *value true*
	`false`	– *value false*
	0	– *value zero*
	`succ` v	– *successor value*

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Bool	– *Booleans*
Nat	– *Natural Numbers*

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$$\frac{t_1 : \mathrm{Bool} \qquad t_2 : T \qquad t_3 : T}{\texttt{if } t_1 \texttt{ then } t_2 \texttt{ else } t_3 : T}$$

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- A term t is *typable* (or *well typed*) if there is some T such that $t : T$.

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- Moreover, there is just one derivation of this typing built from the inference rules.

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  - If $\vdash t : T$ and $t \to t'$, then $\vdash t' : T$.