CS738: Advanced Compiler Optimizations

The Untyped Lambda Calculus

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Reference Book

Types and Programming Languages by Benjamin C. Pierce

The Abstract Syntax

t := x – Variable

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\begin{array}{lll}
\mathrm{t} & := & x \qquad\qquad\qquad \text{– Variable} \\
 & & \mid \lambda x.\mathrm{t} \qquad \text{– Abstraction}
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Parenthesis, $(\dots)$, can be used for grouping and scoping.

Conventions

- $\lambda x.\mathrm{t}_1\mathrm{t}_2\mathrm{t}_3$ is an abbreviation for $\lambda x.(\mathrm{t}_1\mathrm{t}_2\mathrm{t}_3)$, i.e., the scope of x is as far to the right as possible until it is

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- Applications associate to the left: $t_1 t_2 t_3$ to be read as $(t_1 t_2)t_3$ and not as $t_1(t_2 t_3)$
- $\lambda xyz.t$ is an abbreviation for $\lambda x \lambda y \lambda z.t$ which in turn is abbreviation for $\lambda x.(\lambda y.(\lambda z.t))$.

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  - Can not change free variables!

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- For example,

$$(\lambda f \lambda x.f\ (f\ x))\ g \xrightarrow{\beta} \lambda x.g\ (g\ x)$$

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- Use α-renaming to avoid variable capture

$$(\lambda x \lambda y.x)(\lambda x.y) \xrightarrow{\alpha} (\lambda u \lambda v.u)(\lambda x.y) \xrightarrow{\beta} \lambda v.\lambda x.y$$

Exercise

- Apply β-reduction as far as possible

1. $(\lambda x\ y\ z.\ x\ z\ (y\ z))\ (\lambda x\ y.\ x)\ (\lambda y.y)$
2. $(\lambda x.\ x\ x)(\lambda x.\ x\ x)$
3. $(\lambda x\ y\ z.\ x\ z\ (y\ z))\ (\lambda x\ y.\ x)\ ((\lambda x.\ x\ x)(\lambda x.\ x\ x))$

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- Multiple ways to apply β-reduction

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- However, if two different reduction sequences terminate then they always terminate in the same term
  - Also called the *Diamond Property*
- Leftmost, outermost reduction will find the normal form if it exists

Programming in λ Calculus

- Where is the other stuff?

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- Constants?

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Abstractions act as functions as well as data!

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  - However, other pairs of objects will work as well
- Lets translate this intuition into λ-expressions

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- Zero = $\lambda m\ w.\ w$

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- Two = $\lambda m\ w.\ m\ (m\ w)$
- $\ldots$
- What about operations?

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- One = $\lambda m\ w.\ m\ w$
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- $\ldots$
- What about operations?
  - add, multiply, subtract, divide, $\ldots$?

Operations on Numbers

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- mult = $\lambda x\ y\ m\ w.\ x\ (y\ m)\ w$

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- mult = $\lambda x\ y\ m\ w.\ x\ (y\ m)\ w$
  - Verify: mult M N = M * N

More Operations

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More Operations

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  - Verify: pred N = N - 1
- nminus = $\lambda x\ y.\ y$ pred x
  - Verify: nminus M N = max(0, M - N) – natural subtraction

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- Intuition: Selection of one out of two (complementary) choices
- True = $\lambda x\ y.\ x$
- False = $\lambda x\ y.\ y$
- Predicate:
  - isZero = $\lambda x.\ x\ (\lambda u.\text{False})\ \text{True}$

Operations on Booleans

- Logical operations

$$\begin{aligned} \mathit{and} &= \lambda p\ q.\ p\ q\ p \\ \mathit{or} &= \lambda p\ q.\ p\ p\ q \\ \mathit{not} &= \lambda p\ t\ f. p\ f\ t \end{aligned}$$

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- The conditional operator *if*

$$if = \lambda c\; e_t\; e_f.\; (c\; e_t\; e_f)$$

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$$

- The conditional operator *if*
  - $\mathit{if}\ c\ e_t\ e_f$ reduces to e_t if c is True, and to e_f if c is False

$$
\mathit{if} = \lambda c\ e_t\ e_f.\ (c\ e_t\ e_f)
$$

More…

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- It is fun to come up with your own definitions for constants and operations over different types
- or to develop understanding for existing definitions.

We are missing something!!

- The machinery described so far does not allow us to define Recursive functions
  - Factorial, Fibonacci, ...
- There is no concept of “named” functions
  - So no way to refer to a function “recursively”!
- Fix-point computation comes to rescue

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- Assume existence of a magic expression, called Y-combinator, that when applied to a λ-expression, gives its fixed point
$$Y\ f = f\ (Y\ f)$$
- Y-combinator gives us a way to apply a function recursively

Recursion Example: Factorial

$$
\begin{aligned}
\mathsf{fact} &= \lambda n.\ \mathit{if}\ (\mathsf{isZero}\ n)\ \mathsf{One}\ (\mathsf{mult}\ n\ (\mathsf{fact}\ (\mathsf{pred}\ n))) \\
&= (\lambda f\ n.\ \mathit{if}\ (\mathsf{isZero}\ n)\ \mathsf{One}\ (\mathsf{mult}\ n\ (f\ (\mathsf{pred}\ n))))\ \mathsf{fact}
\end{aligned}
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$$\text{fact} = g\ \text{fact}$$

- fact is a fixed point of the function

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- Using Y-combinator,

$$\text{fact} = Y\ g$$

Factorial: Verify

$$\text{fact } 2 \quad = \quad (Y\ g)\ 2$$

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&= (\lambda n.\ \mathit{if}\ (\text{isZero}\ n)\ 1\ (\text{mult}\ n\ ((Y\ g)\ (\text{pred}\ n))))\ 2
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&= if\ (\text{isZero } 2)\ 1\ (\text{mult } 2\ ((Y\ g)(\text{pred} 2)))
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&\ \ldots \\
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&\quad \ldots \\
&= (\text{mult}\ 2\ (\text{mult}\ 1\ (\mathit{if}\ (\text{isZero}\ 0)\ 1\ (\ldots)))) \\
&= (\text{mult}\ 2\ (\text{mult}\ 1\ 1)) \\
&= 2
\end{aligned}
$$

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BUT, what about the existence of Y-combinator?

Y-combinators

- Many candidates exist

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- Verify that $(Y\ f)\ =\ f\ (Y\ f)$ for each

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- A cursory look at λ-calculus
- Functions are data, and Data are functions!
- Not covered but important to know: The power of λ calculus is equivalent to that of Turing Machine ("Church Turing Thesis")