CS738: Advanced Compiler Optimizations

Liveness based Garbage Collection

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Ideal Garbage Collection

> *. . . garbage collection (GC) is a form of automatic memory management. The garbage collector, or just collector, attempts to reclaim garbage, or memory occupied by objects that are no longer in use by the program. . . .*

From Wikipedia

`https://en.wikipedia.org/wiki/Garbage_collection_(computer_science)`

Real Garbage Collection

> *. . . All garbage collectors use some efficient approximation to liveness. In tracing garbage collection, the approximation is that an object can't be live unless it is reachable. . . .*

From Memory Management Glossary

`www.memorymanagement.org/glossary/g.html#term-garbage-collection`

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  - Programs expected to run faster and with smaller heap.

The language analyzed

- First order eager Scheme-like functional language.
- In Administrative Normal Form (ANF).

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An Example

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(define (append l1 l2)
 (if (null? l1) l2
   (cons (car l1)
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 (let z ←(cons (cons 4 (cons 5 nil))
          (cons 6 nil)) in
 (let y ← (cons 3 nil) in
   (let w ← (append y z) in
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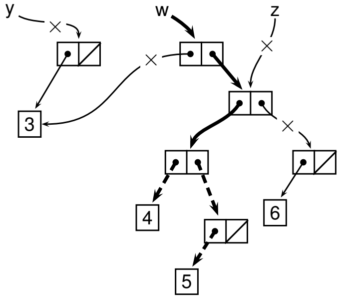

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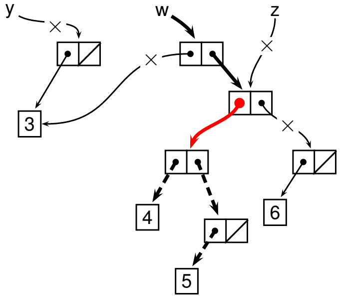


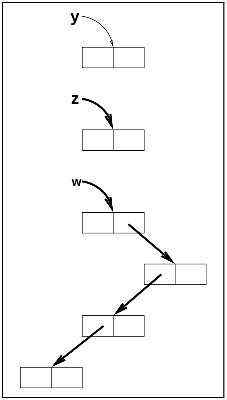
- *Though all cells are reachable at π, a liveness-based GC will retain only the cells pointed by thick arrows.*

Liveness – Basic Concepts and Notations

- *Access paths:* Strings over {**0**, **1**}.

  0 – access **car** field

  1 – access **cdr** field



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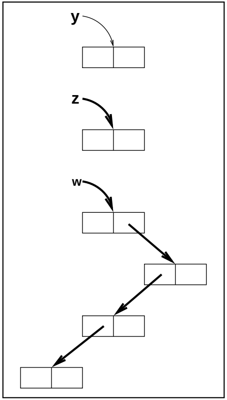

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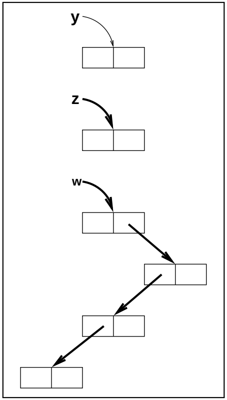
- Denote traversals over the heap graph

- *Liveness environment:* Maps root variables to set of access paths.

$$\mathsf{L}_i \quad : \quad \begin{cases} y \mapsto \emptyset \\ z \mapsto \{\epsilon\} \\ w \mapsto \{\epsilon, 1, 10, 100\} \end{cases}$$



Notation: We write $\mathsf{L}_i(\mathrm{x})$ as $\mathsf{L}_i^{\mathrm{x}}$

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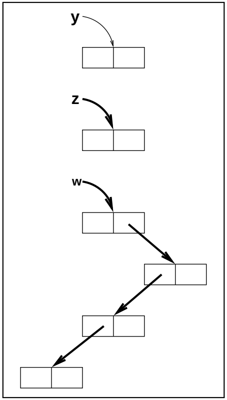
- *Liveness environment:* Alternate representation.

$$\mathsf{L}_i \quad : \quad \begin{cases} \emptyset \ \cup \\ \{z.\epsilon\} \ \cup \\ \{w.\epsilon, w.1, w.10, w.100\} \end{cases}$$



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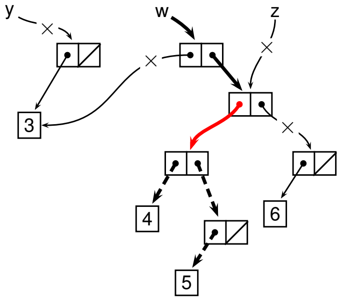
Demand

(car (cdr `w`**))**



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- We assume the demand on the main expression to be $(0+1)^*$, which we call σ_{all}.
- The demands on each function body, σ_f, have to be computed.

Liveness analysis – The big picture

```
πmain: (let z ← ... in
         (let y ← ... in
  π9:  (let w ← (append y z) in
  π10: (let a ← (cdr w) in
  π11: (let b ← (car a) in
  π12: (return b))))))
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Liveness environments:

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\begin{aligned}
L_1 &= \ldots \\
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Function summaries:

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$\mathcal{L}app(a, \sigma)$ – Liveness environment generated by an *application* *a*, given a demand σ.

$\mathcal{L}exp(e, \sigma)$ – Liveness environment before an *expression* *e*, given a demand σ.

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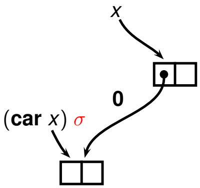
Notice the similarity with:

$$live_{in}(B) = live_{out}(B) \setminus kill(B) \cup gen(B)$$

in classical dataflow analysis for imperative languages.

Liveness analysis of Primitive Applications



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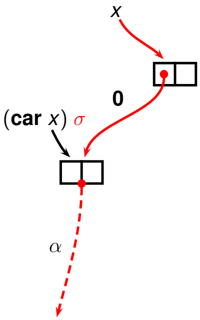

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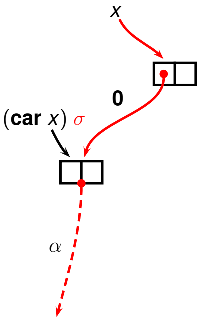

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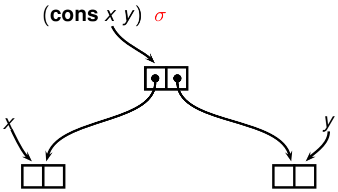

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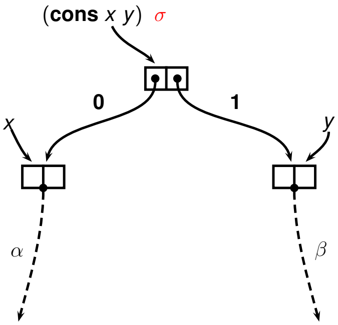

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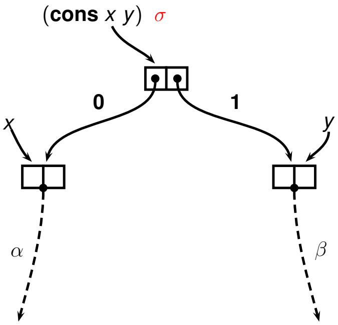

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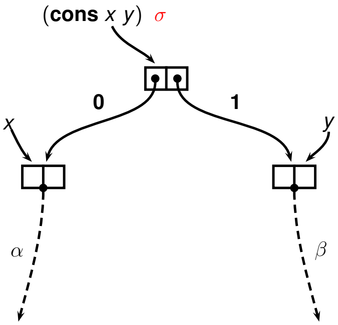

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  $\bar{\mathbf{1}}$ – Removal of a leading **1**

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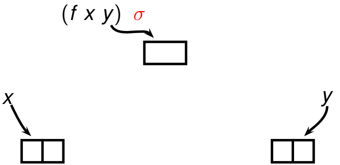
Liveness Analysis of Function Applications



$\mathcal{L}app((f\ x\ y), \sigma) =$

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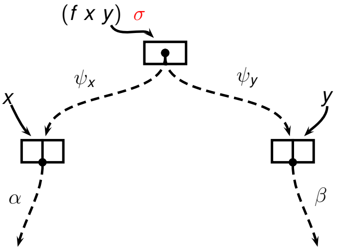

$$\mathcal{L}app((f\ x\ y), \sigma) = \ x.\overline{\psi_x}\sigma \cup \ y.\overline{\psi_y}\sigma$$

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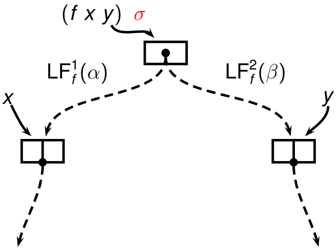

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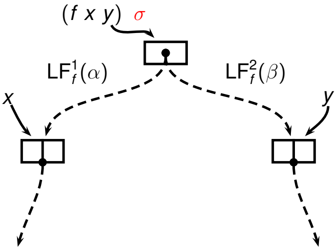

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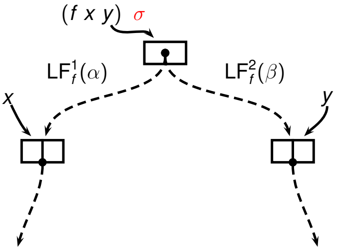

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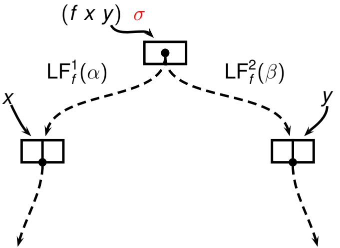

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  - How to handle recursive calls? Use LF_f with appropriate demand !!

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Liveness environments:

$$\mathsf{L}_1^{\mathtt{l1}} = \{\epsilon\} \cup \mathbf{0}\bar{\mathbf{0}}\sigma_{\textbf{append}} \cup \mathbf{1}\mathsf{LF}^1_{\textbf{append}}(\bar{\mathbf{1}}\sigma_{\textbf{append}})$$

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Obtaining a closed form solution for LF

- Function summaries will always have the form:

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- Substitute the assumed form in the equation:

$$\mathsf{I}^1_{\textbf{append}} \cup \mathsf{D}^1_{\textbf{append}}\sigma = \{\epsilon\} \ \cup \ \mathbf{0}\overline{\mathbf{0}}\sigma \cup \mathbf{1}(\mathsf{I}^1_{\textbf{append}} \cup \mathsf{D}^1_{\textbf{append}}\overline{\mathbf{1}}\sigma)$$

- Equating the terms without and with σ, we get:

$$\begin{aligned}\mathsf{I}^1_{\textbf{append}} &= \{\epsilon\} \ \cup \ \mathbf{1}\mathsf{I}^1_{\textbf{append}} \\ \mathsf{D}^1_{\textbf{append}} &= \mathbf{0}\overline{\mathbf{0}} \cup \mathbf{1}\mathsf{D}^1_{\textbf{append}}\overline{\mathbf{1}}\end{aligned}$$

Summary of Analysis Results

Liveness at program points:

$$\mathsf{L}_1^{\mathtt{l1}} = \{\epsilon\} \cup \mathbf{0}\bar{\mathbf{0}}\sigma \cup \mathbf{1}(\mathsf{l}^1_{\textbf{append}} \cup \mathsf{D}^1_{\textbf{append}}\bar{\mathbf{1}}\sigma_{\textbf{append}})$$

$$\mathsf{L}_1^{\mathtt{l2}} = \{\epsilon\} \cup \mathsf{l}^2_{\textbf{append}} \cup \mathsf{D}^2_{\textbf{append}}\bar{\mathbf{1}}\sigma_{\textbf{append}}$$

$$\mathsf{L}_5^{\mathtt{l1}} = \{\epsilon\} \cup \mathbf{0}\bar{\mathbf{0}}\sigma_{\textbf{append}}$$

$$\mathsf{L}_5^{\mathtt{tl}} = \mathsf{l}^1_{\textbf{append}} \cup \mathsf{D}^1_{\textbf{append}}\bar{\mathbf{1}}\sigma_{\textbf{append}}$$

$$\mathsf{L}_5^{\mathtt{l2}} = \mathsf{l}^2_{\textbf{append}} \cup \mathsf{D}^2_{\textbf{append}}\bar{\mathbf{1}}\sigma_{\textbf{append}}$$

$\ldots$

Demand summaries:

$$\sigma_{\textbf{append}} = \{\epsilon,\ \mathbf{1}\} \cup \bar{\mathbf{1}}\sigma_{\textbf{append}} \cup \mathbf{10}\sigma_{all}$$

Function summaries:

$$\mathsf{l}^1_{\textbf{append}} = \{\epsilon\} \cup \mathbf{1}\mathsf{l}^1_{\textbf{append}}$$

$$\mathsf{D}^1_{\textbf{append}} = \mathbf{0}\bar{\mathbf{0}} \cup \mathbf{1}\mathsf{D}^1_{\textbf{append}}\bar{\mathbf{1}}$$

$$\mathsf{l}^2_{\textbf{append}} = \mathsf{l}^2_{\textbf{append}}$$

$$\mathsf{D}^2_{\textbf{append}} = \{\epsilon\} \cup \mathsf{D}^2_{\textbf{append}}\bar{\mathbf{0}}$$

Solution of the equations

View the equations as grammar rules:

$$
\begin{aligned}
\mathsf{L}_1^{\mathtt{1}\,\mathtt{1}} &\rightarrow \epsilon \mid \mathbf{0}\overline{\mathbf{0}}\sigma \mid \mathbf{1}(\mathsf{I}^1_{\mathbf{append}} \mid \mathsf{D}^1_{\mathbf{append}}\overline{\mathbf{1}}\sigma_{\mathbf{append}}) \\
\mathsf{I}^1_{\mathbf{append}} &\rightarrow \epsilon \mid \mathbf{1}\mathsf{I}^1_{\mathbf{append}} \\
\mathsf{D}^1_{\mathbf{append}} &\rightarrow \mathbf{0}\overline{\mathbf{0}} \mid \mathbf{1}\mathsf{D}^1_{\mathbf{append}}\overline{\mathbf{1}}
\end{aligned}
$$

The solution of $\mathsf{L}_1^{\mathtt{1}\,\mathtt{1}}$ is the language $\mathscr{L}(\mathsf{L}_1^{\mathtt{1}\,\mathtt{1}})$ generated by it.

Working of Liveness-based GC (Mark phase)

- GC invoked at a program point π
- GC traverses a path α starting from a root variable x.
- GC consults L_π^x:
  - Does $\alpha \in \mathscr{L}(\mathsf{L}_\pi^x)$?
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- GC consults L_π^x:
  - Does $\alpha \in \mathscr{L}(\mathsf{L}_\pi^x)$?
  - If yes, then mark the current cell
- Note that α is a *forward*-only access path
  - consisting only of edges **0** and **1**, but not $\bar{\mathbf{0}}$ or $\bar{\mathbf{1}}$
  - But $\mathscr{L}(\mathsf{L}_\pi^x)$ has access paths marked with $\bar{\mathbf{0}}/\bar{\mathbf{1}}$ for **0**/**1** removal arising from the **cons** rule.

$\overline{\mathbf{0}}/\overline{\mathbf{1}}$ handling

- $\mathbf{0}$ removal from a set of access paths:

$$\alpha_1 \overline{\mathbf{0}}\mathbf{0} \alpha_2 \hookrightarrow \alpha_1 \alpha_2$$
$$\alpha_1 \overline{\mathbf{0}}\mathbf{1} \alpha_2 \hookrightarrow \text{ drop } \alpha_1 \overline{\mathbf{0}}\mathbf{1} \alpha_2 \text{ from the set}$$

- $\mathbf{1}$ removal from a set of access paths:

$$\alpha_1 \overline{\mathbf{1}}\mathbf{1} \alpha_2 \hookrightarrow \alpha_1 \alpha_2$$
$$\alpha_1 \overline{\mathbf{1}}\mathbf{0} \alpha_2 \hookrightarrow \text{ drop } \alpha_1 \overline{\mathbf{1}}\mathbf{0} \alpha_2 \text{ from the set}$$

GC decision problem

- Deciding the membership in a CFG augmented with a fixed set of unrestricted productions.

$$\bar{\mathbf{0}}\mathbf{0} \rightarrow \epsilon$$
$$\bar{\mathbf{1}}\mathbf{1} \rightarrow \epsilon$$

- The problem shown to be undecidable[1].
  - Reduction from Halting problem.

[1] Prasanna, Sanyal, and Karkare. *Liveness-Based Garbage Collection for Lazy Languages*, ISMM 2016.

Practical $\overline{\mathbf{0}}/\overline{\mathbf{1}}$ simplification

- The simplification is possible to do on a finite state automaton.
- Over-approximate the CFG by an automaton (Mohri-Nederhoff transformation).
- Perform **0**/**1** removal on the automaton.

Example

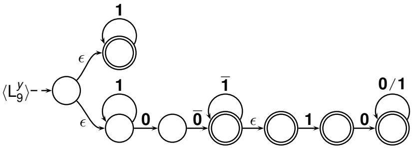
Grammar for L_9^{y}

$$
\begin{aligned}
L_9^{y} &\rightarrow I^1_{\textbf{append}} \mid D^1_{\textbf{append}}(\epsilon \mid \mathbf{1} \mid \mathbf{10}\sigma_{all}) \\
I^1_{\textbf{append}} &\rightarrow \epsilon \mid \mathbf{1} I^1_{\textbf{append}} \\
D^1_{\textbf{append}} &\rightarrow \mathbf{0}\overline{\mathbf{0}} \mid \mathbf{1} D^1_{\textbf{append}} \overline{\mathbf{1}} \\
\sigma_{all} &\rightarrow \epsilon \mid \mathbf{0}\sigma_{all} \mid \mathbf{1}\sigma_{all}
\end{aligned}
$$

After Mohri-Nederhoff transformation

$$
\begin{aligned}
L_9^{y} &\rightarrow I^1_{\textbf{append}} \mid D^1_{\textbf{append}}(\epsilon \mid \mathbf{1} \mid \mathbf{10}\sigma_{all}) \\
I^1_{\textbf{append}} &\rightarrow \epsilon \mid \mathbf{1} I^1_{\textbf{append}} \\
D^1_{\textbf{append}} &\rightarrow \mathbf{0}\overline{\mathbf{0}}\widehat{D}^1_{\textbf{append}} \mid \mathbf{1} D^1_{\textbf{append}} \\
\widehat{D}^1_{\textbf{append}} &\rightarrow \overline{\mathbf{1}}\widehat{D}^1_{\textbf{append}} \mid \epsilon \\
\sigma_{all} &\rightarrow \epsilon \mid \mathbf{0}\sigma_{all} \mid \mathbf{1}\sigma_{all}
\end{aligned}
$$

Automaton for L_9^y



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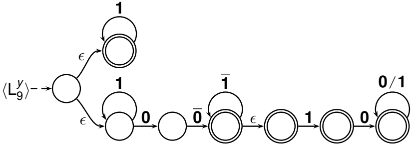

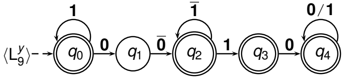

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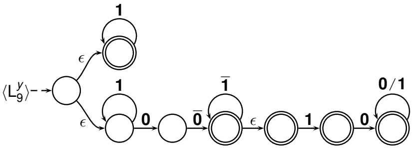

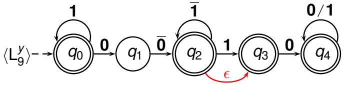

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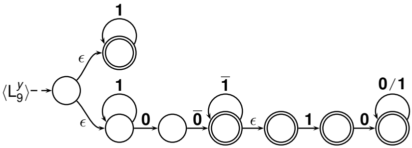

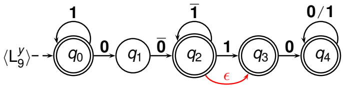

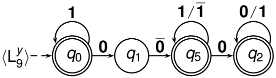

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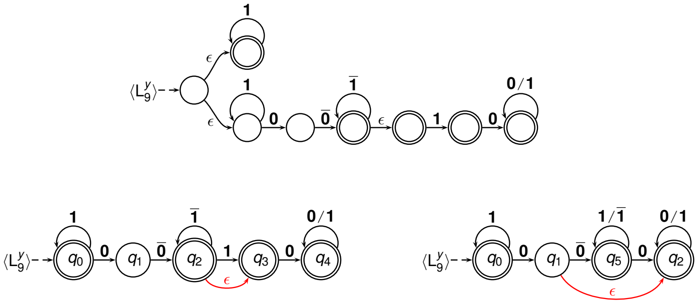

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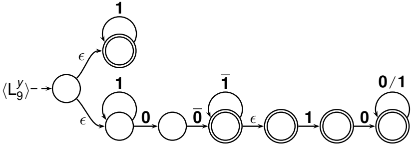

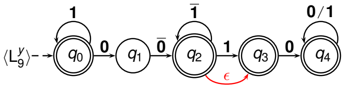

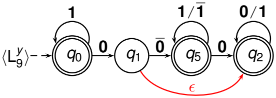

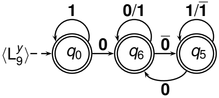

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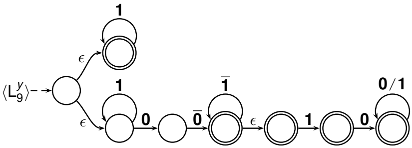

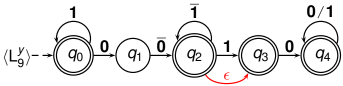

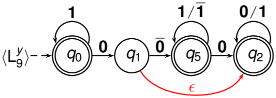

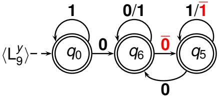

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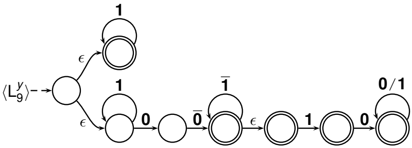

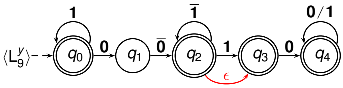

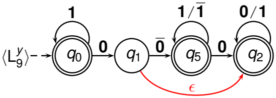

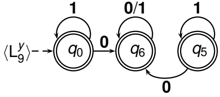

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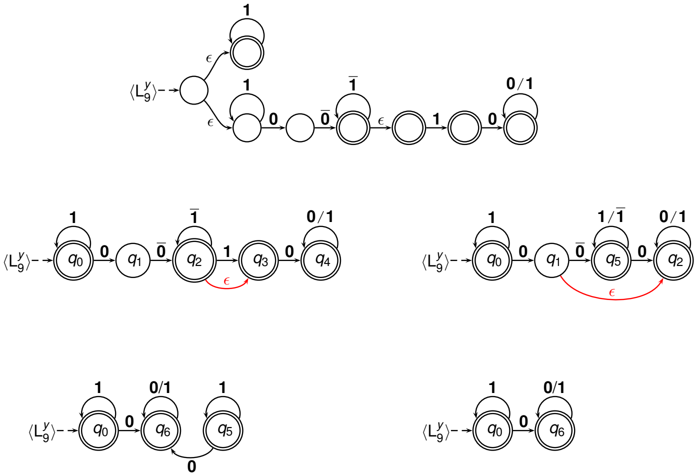
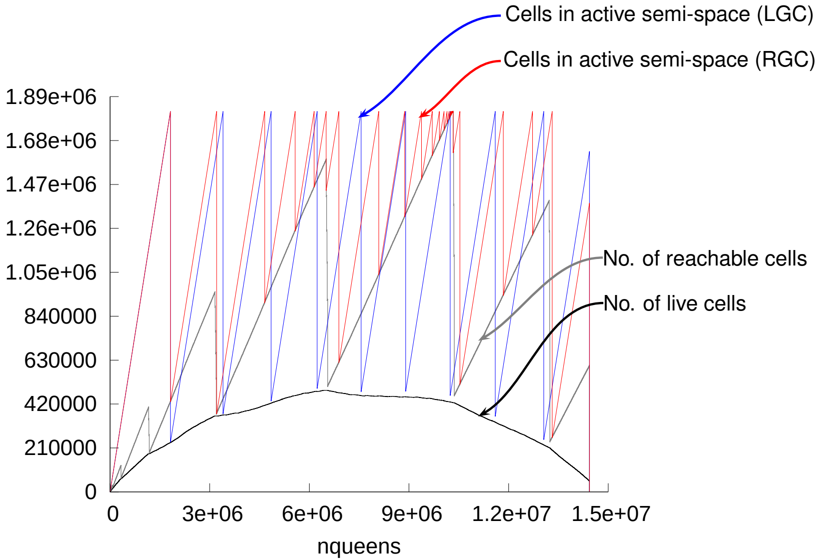


Experimental Setup

- Built a prototype consisting of:
  - An ANF-scheme interpreter
  - Liveness analyzer
  - A single-generation copying collector.
- The collector optionally uses liveness
  - Marks a link during GC only if it is live.
- Benchmark programs are mostly from the no-fib suite.

GC behavior as a graph



Results as Tables

Analysis Performance:

Program	sudoku	lcss	gc_bench	knightstour	treejoin	nqueens	lambda
Time (msec)	120.95	2.19	0.32	3.05	2.61	0.71	20.51
DFA size	4251	726	258	922	737	241	732
Precision(%)	**87.5**	**98.8**	**99.9**	**94.3**	**99.6**	**98.8**	**83.8**

Results as Tables

Garbage collection performance

Program	# Collected cells per GC		#GCs		MinHeap (#cells)		GC time (sec)	
	RGC	LGC	RGC	LGC	RGC	LGC	RGC	LGC
sudoku	**490**	**1306**	22	9	1704	589	.028	.122
lcss	**46522**	**51101**	8	7	52301	1701	.045	.144
gc_bench	**129179**	**131067**	9	9	131071	6	.086	.075
nperm	**47586**	**174478**	14	4	202597	37507	1.406	.9
fibheap	**249502**	**251525**	1	1	254520	13558	.006	.014
knightstour	**2593**	**314564**	1161	10	508225	307092	464.902	14.124
treejoin	**288666**	**519943**	2	1	525488	7150	.356	.217
nqueens	**283822**	**1423226**	46	9	1819579	501093	70.314	24.811
lambda	**205**	**556**	23	8	966	721	.093	2.49

- LGC collects more garbage than RGC.

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- # collections of LGC no higher than RGC. Often, smaller.

Results as Tables

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- Programs require smaller heaps to execute with LGC.

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- GC time is smaller for LGC in some cases…

Results as Tables

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- An evaluation strategy in which evaluation of an expression is postponed until its value is needed

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- An evaluation strategy in which evaluation of an expression is postponed until its value is needed
  - Binding of a variable to an expression **does not force evaluation** of the expression
- Every expression is evaluated at most once

Laziness: Example

```
(define (length l)
  (if (null? l)
    return 0
    return (+ 1 (length (cdr l)))))
```

```
(define (main)
  (let a ← ( a BIG closure ) in
    (let b ← (+ a 1) in
      (let c ← (cons b nil) in
        (let w ← (length c) in
          (return w))))))
```

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  - For precision: need to update the liveness information as execution progresses

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  - If (**car** x) is not evaluated at all, it does not generate any liveness for x
- Require a new terminal **2** with following semantics

$$\mathbf{2}\sigma \hookrightarrow \begin{cases} \emptyset & \text{if } \sigma = \emptyset \\ \{\epsilon\} & \text{otherwise} \end{cases}$$

$$\mathcal{L}app((\textbf{car }\mathrm{x}), \sigma) = \mathrm{x}.\{\mathbf{2}, \mathbf{0}\}\sigma$$

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    - All path problem, requires doing intersection of demands
    - $\Rightarrow$ intersection of CFGs $\Rightarrow$ under-approximation

Conclusions

- Proposed a liveness-based GC scheme.
- Not covered in this talk:
  - The soundness of liveness analysis.
  - Details of undecidability proof.
  - Details of handling lazy languages.
- A prototype implementation to demonstrate:
  - the precision of the analysis.
  - reduced heap requirement.
  - reduced GC time for a majority of programs.
- Unfinished agenda:
  - Improving GC time for a larger fraction of programs.
  - Extending scope of the method.