CS738: Advanced Compiler Optimizations

Simply Typed Lambda Calculus

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Reference Book

Types and Programming Languages by Benjamin C. Pierce

Simple Types over Bool

$T \quad :=$ – Types

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type constructor $\to$ is right-associative, i.e., $T_1 \to T_2 \to T_3$ stands for $T_1 \to (T_2 \to T_3)$

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Simply Typed λ-terms with conditions and Booleans

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	$\vert$	`if` t `then` t `else` t	– *conditional*

Recap: The Set of Values

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$$(\lambda x : T_1.\, \mathsf{t}_1)\, v_2 \rightarrow [x \mapsto v_2]\mathsf{t}_1 \qquad \text{(E-AppAbs)}$$

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  - The name x is assumed to be distinct from any existing names in Γ

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- If $\Gamma \vdash \mathrm{t}_1\ \mathrm{t}_2 : R$, then $\exists T_1\ s.t.\ \Gamma \vdash \mathrm{t}_1 : T_1 \to R$ and $\Gamma \vdash \mathrm{t}_2 : T_1$.

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- Moreover, there is just one derivation of this typing built from the inference rules.

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