CS738: Advanced Compiler Optimizations

Simply Typed Lambda Calculus

Amey Karkare
karkare@cse.iitk.ac.in

http://www.cse.iitk.ac.in/~karkare/cs738
Department of CSE, IIT Kanpur

Reference Book

Types and Programming Languages by Benjamin C. Pierce

Simple Types over Bool

T	:=	– Types
	Bool	– Boolean Type
	$T \to T$	– Function Type

type constructor $\to$ is right-associative, i.e., $T_1 \to T_2 \to T_3$ stands for $T_1 \to (T_2 \to T_3)$

Examples

For each of the type below, write a function (in your favorite programming language) that has the required type:

- Bool $\to$ Bool
- Bool $\to$ Bool $\to$ Bool
- (Bool $\to$ Bool) $\to$ Bool
- (Bool $\to$ Bool) $\to$ Bool $\to$ Bool
- (Bool $\to$ Bool $\to$ Bool) $\to$ Bool
- (Bool $\to$ Bool $\to$ Bool) $\to$ Bool
- ((Bool $\to$ Bool) $\to$ Bool) $\to$ Bool

The Abstract Syntax

Simply Typed λ-terms with conditions and Booleans

$$
\begin{array}{lcll}
\mathsf{t} & := & x & \textit{– Variable} \\
 & | & \lambda x : T.\,\mathsf{t} & \textit{– Abstraction} \\
 & | & \mathsf{t}\;\mathsf{t} & \textit{– Application} \\
 & | & \texttt{true} & \textit{– constant true} \\
 & | & \texttt{false} & \textit{– constant false} \\
 & | & \texttt{if}\;\mathsf{t}\;\texttt{then}\;\mathsf{t}\;\texttt{else}\;\mathsf{t} & \textit{– conditional}
\end{array}
$$

Recap: The Set of Values

$$
\begin{array}{lcll}
v & := & & \textit{– values} \\
 & & \lambda x : T.\,\mathsf{t} & \textit{– Abstraction Value} \\
 & | & \texttt{true} & \textit{– value true} \\
 & | & \texttt{false} & \textit{– value false}
\end{array}
$$

Evaluation

$$\frac{\mathsf{t}_1 \to \mathsf{t}_1'}{\mathsf{t}_1\;\mathsf{t}_2 \to \mathsf{t}_1'\;\mathsf{t}_2} \qquad \text{(E-APP1)}$$

$$\frac{\mathsf{t}_2 \to \mathsf{t}_2'}{v\;\mathsf{t}_2 \to v\;\mathsf{t}_2'} \qquad \text{(E-APP2)}$$

$$(\lambda x : T_1.\,\mathsf{t}_1)v_2 \to [x \mapsto v_2]\mathsf{t}_1 \qquad \text{(E-APPABS)}$$

The Typing Relation

- A *Typing Context* or *Type Environment*, Γ, is a sequence of variables with their types
- $\Gamma, x : T$ denotes extending Γ with a new variable x having type T
  - The name x is assumed to be distinct from any existing names in Γ

The Typing Relation

$$\frac{\Gamma, x : T_1 \vdash \mathrm{t}_2 : T_2}{\Gamma \vdash \lambda x : T_1.\, \mathrm{t}_2 : T_1 \to T_2} \quad \text{(T-ABS)}$$

$$\frac{x : T \in \Gamma}{\Gamma \vdash x : T} \quad \text{(T-VAR)}$$

$$\frac{\Gamma \vdash \mathrm{t}_1 : T_1 \to T_2 \qquad \Gamma \vdash \mathrm{t}_2 : T_1}{\Gamma \vdash \mathrm{t}_1\ \mathrm{t}_2 : T_2} \quad \text{(T-APP)}$$

Inversion of the Typing Relation

- If $\Gamma \vdash x : R$, then $x : R \in \Gamma$.
- If $\Gamma \vdash \lambda x : T_1.\, \mathrm{t}_2 : R$, then $R = T_1 \to R_2$ for some R_2 with $\Gamma, x : T_1 \vdash \mathrm{t}_2 : R_2$.
- If $\Gamma \vdash \mathrm{t}_1\ \mathrm{t}_2 : R$, then $\exists T_1\ s.t.\ \Gamma \vdash \mathrm{t}_1 : T_1 \to R$ and $\Gamma \vdash \mathrm{t}_2 : T_1$.
- If $\Gamma \vdash \mathtt{true} : R$, then $R =$ Bool.
- If $\Gamma \vdash \mathtt{false} : R$, then $R =$ Bool.
- If $\Gamma \vdash \mathtt{if}\ \mathrm{t}_1\ \mathtt{then}\ \mathrm{t}_2\ \mathtt{else}\ \mathrm{t}_3 : R$, then
  - $\Gamma \vdash \mathrm{t}_1 :$ Bool
  - $\Gamma \vdash \mathrm{t}_2 : R$
  - $\Gamma \vdash \mathrm{t}_3 : R$

Exercises

- For each of the term t below, find context Γ and type T such that

$$\Gamma \vdash \mathrm{t} : T$$

  - t is $\lambda x.\ x$
  - t is $(\ (x\ z)\ (y\ z)\ )$
  - t is $\lambda y.\ x$
  - t is $x\ x$

Uniqueness of Types

- In a given type context Γ, A term t, such that the free variables of t are in Γ, has at most one type.
- If t is typeable, then its type is unique.
- Moreover, there is just one derivation of this typing built from the inference rules.

Some Properties

- **Permutation:** If $\Gamma \vdash \mathrm{t} : T$ and Δ is a permutation of Γ, then $\Delta \vdash \mathrm{t} : T$.
  - The derivation with Δ has the same depth as the derivation with Γ.
- **Weakening:** If $\Gamma \vdash \mathrm{t} : T$ and $x \notin \mathrm{domain}(\Gamma)$, then $\Gamma, x : S \vdash \mathrm{t} : T$.
  - The derivation with $\Gamma, x : S$ has the same depth as the derivation with Γ.

Progress

- **Progress:** A well-typed term is not stuck.
  - If $\vdash \mathrm{t} : T$, then t is either a value or there exists some t′ such that $\mathrm{t} \to \mathrm{t}'$.

Preservation

- **Preservation of Types under Substitution:** If $\Gamma, x : S \vdash \mathrm{t} : T$ and $\Gamma \vdash s : S$, then $\Gamma \vdash [x \mapsto s]\mathrm{t} : T$.
- **Preservation:** If a well-typed term takes a step of evaluation, then the resulting term is also well-typed.
  - If $\Gamma \vdash \mathrm{t} : T$ and $\mathrm{t} \to \mathrm{t}'$, then $\Gamma \vdash \mathrm{t}' : T$.