# A Matlab Program to Calculate the Maximum Entropy Distributions

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## INTRODUCTION

Shannon (1948) indicated how maximum entropy (ME) distributions can be derived by a straigtforward application of the calculus of variations technique. He defined the entropy of a probability density function p(x) as

$$H = -\int p(x) \ln p(x) dx \tag{1}$$

Maximizing *H* subject to various side conditions is well–known in the literature as a method for deriving the forms of minimal information prior distributions; e.g. Jaynes (1968) and Zellner (1977). Jaynes (1982) has extensively analyzed examples in the discrete case, while in Lisman and Van Znylen (1972), Rao (1973) and Gokhale (1975), Kagan, Linjik continuous cases are considered. In the last case, the problem, in its general form, is the following

maximize 
$$H = -\int p(x) \ln p(x) dx$$
  
subject to  $\mathbb{E} \{ \phi_n(x) \} = \int \phi_n(x) p(x) dx = \mu_n, \quad n = 0, \dots, N$  (2)

where  $\mu_0=1$ ,  $\phi_0(x)=1$  and  $\phi_n(x), n=0,\ldots,N$  are N known functions, and  $\mu_n, n=0,\ldots,N$  are the given expectation data. The classical solution of this problem is given by

$$p(x) = \exp\left[-\sum_{n=0}^{N} \lambda_n \,\phi_n(x)\right] \tag{3}$$

The (N+1) Lagrangien parameters  $\lambda = [\lambda_0, \dots, \lambda_n]$  are obtained by solving the following set of (N+1) nonlinear equations

$$G_n(\lambda) = \int \phi_n(x) \exp\left[-\sum_{n=0}^N \lambda_n \phi_n(x)\right] dx = \mu_n, \quad n = 0, \dots, N$$
 (4)

The distributions defined by (3) form a great number of known distributions which are obtained by choosing the appropriate N and  $\phi_n(x), n=0,\ldots,N$ . In general  $\phi_n(x)$  are either the powers of x or the logarithm of x. See Mukhrejee and Hurst (1984), Zellner (1988), Mohammad–Djafari (1990) for many other examples and discussions. Special cases have been extensively analyzed and used by many authors. When  $\phi_n(x) = x^n, n = 0,\ldots,N$  then  $\mu_n, n=0,\ldots,N$  are the given N moments of the distribution. See, for example, Zellner (1988) for a numerical implementation in the case N=4.

In this communication we propose three programs written in MATLAB to solve the system of equations (4). The first is a general program where  $\phi_n(x)$  can be any functions. The second is a special case where  $\phi_n(x) = x^n, n = 0, ..., N$ . In this case the  $\mu_n$  are the geometrical moments of p(x). The third is a special case where  $\phi_n(x) = \exp(-jn\omega x), n = 0, ..., N$ . In this case the  $\mu_n$  are the trigonometrical moments (Fourier components) of p(x). We give also some examples to illustrate the usefullness of these programs.

## PRINCIPLE OF THE METHOD

We have seen that the solution of the standard ME problem is given by (3) in which the Lagrange multipliers  $\lambda$  are obtained by solving the nonlinear equations (4). In general, these equations are solved by the standard Newton method which consists of expanding  $G_n(\lambda)$  in Taylor's series around trial values of the *lambda*'s, drop the quadratic and higher order terms, and solve the resulting linear system iteratively. We give here the details of the numerical method that we implemented. When developing the  $G_n(\lambda)$  in equations (4) in first order Taylor's series around the trial  $\lambda^0$ , the resulting linear equations are given by

$$G_n(\lambda) \cong G_n(\lambda^0) + (\lambda - \lambda^0)^t [\operatorname{grad} G_n(\lambda)]_{(\lambda = \lambda^0)} = \mu_n, \quad n = 0, \dots, N$$
 (5)

Noting the vectors  $\boldsymbol{\delta}$  and  $\boldsymbol{v}$  by

$$\delta = \lambda - \lambda^0$$

$$\boldsymbol{v} = \left[\mu_0 - G_0(\boldsymbol{\lambda}^0), \dots, \mu_N - G_N(\boldsymbol{\lambda}^0)\right]^t$$

and the matrix G by

$$\mathbf{G} = \left[ g_{nk} \right] = \left[ \frac{\partial G_n(\boldsymbol{\lambda})}{\partial \lambda_k} \right]_{(\boldsymbol{\lambda} = \boldsymbol{\lambda}^0)} n, k = 0, \dots, N$$
 (6)

then equations (5) become

$$G\delta = v \tag{7}$$

This system is solved for  $\delta$  from which we drive  $\lambda = \lambda^0 + \delta$ , which becomes our new initial vector  $\lambda^0$  and the iterations continue until  $\delta$  becomes appropriately small. Note that the matrix G is a symmetric one and we have

$$g_{nk} = g_{kn} = -\int \phi_n(x) \,\phi_k(x) \,\exp\left[-\sum_{n=0}^N \lambda_n \,\phi_n(x)\right] \,\mathrm{d}x \quad n, k = 0, \dots, N$$
 (8)

So in each iteration we have to calculate the N(N-1)/2 integrals in the equation (8). The algorithm of the general Maximum Entropy problem is then as follows:

- 1. Define the range and the discretization step of x (xmin, xmax, dx).
- 2. Write a function to calculate  $\phi_n(x), n = 0, ..., N$  (fin\_x).
- 3. Start the iterative procedure with an initial estimate  $\lambda^0$  (lambda0).
- 4. Calculate the (N+1) integrals in equations (4) and the N(N-1)/2 distinct elements  $g_{nk}$  of the matrix G by calculating the integrals in the equations(8)
  - (Gn, gnk).

5. Solve the equation (7) to find  $\delta$ 

- (delta).
- 6. Calculate  $\lambda = \lambda^0 + \delta$  and go back to step 3 until  $\delta$  becomes negligible.

The calculus of the integrals in equations (4) and (8) can be made by a univariate Simpson's method. We have used a very simplified version of this method.

# Case of geometrical moments

Now consider the special case of moments problem where  $\phi_n(x) = x^n$ , n = 0, ..., N. In this case equations (3), (4) and (8) become

$$p(x) = \exp\left[-\sum_{m=0}^{N} \lambda_m x^m\right] \tag{9}$$

$$G_n(\boldsymbol{\lambda}) = \int x^n \exp\left[-\sum_{m=0}^N \lambda_m x^m\right] dx = \mu_n, \quad n = 0, \dots, N$$
 (10)

$$g_{nk} = g_{kn} = -\int x^n x^k \exp\left[-\sum_{m=0}^N \lambda_m x^m\right] dx = -G_{n+k}(\boldsymbol{\lambda}) \quad n, k = 0, \dots, N \quad (11)$$

This means that the  $[(N+1)\times (N+1)]$  matrix G in equation (7) becomes a symmetric Hankel matrix which is entirely defined by 2N+1 values  $G_n(\lambda), n=0,\ldots,2N$ . So the algorithm in this case is the same as in the precedent one with two simplifications

1. In step 2 we do not need to write a separate function to calculate the functions  $\phi_n(x) = x^n, n = 0, \dots, N$ .

2. In step 4 the number of integral evaluations is reduced, because the elements  $g_{nk}$ of the matrix G are related to the integrals  $G_n(\lambda)$  in equations (10). This matrix is defined entirely by only 2N + 1 components.

# Case of trigonometrical moments

Another interesting special case is the case where the data are the Fourier components of p(x)

$$E\{\exp(-jn\omega_0 x)\} = \int \exp(-jn\omega_0 x) p(x) dx = \mu_n, \quad n = 0, \dots, N,$$
 (12)

where  $\mu_n$  may be complex-valued and has the property  $\mu_{-n} = \mu_n$ . This means that we have the following relations

$$\phi_n(x) = \exp(-jn\omega_0 x), \quad n = -N, \dots, 0, \dots N, \tag{13}$$

$$p(x) = \exp\left[-\operatorname{Real}\sum_{n=0}^{N} \lambda_n \exp\left(-jn\omega_0 x\right)\right],$$
 (14)

$$G_n(\lambda) = \int \exp(-jn\omega_0 x) p(x) dx, \quad n = 0, \dots, N,$$
 (15)

$$G_n(\lambda) = \int \exp(-jn\omega_0 x) p(x) dx, \quad n = 0, ..., N,$$

$$g_{nk} = \begin{cases} -G_{n-k}(\lambda) & \text{for } n \ge k, \\ -G_{n+k}(\lambda) & \text{for } n < k \end{cases} \quad n, k = 0, ..., N,$$
(15)

so that all the elements of the matrix G are related to the discrete Fourier transforms of p(x). Note that G is a Hermitian Toeplitz matrix.

#### EXAMPLES AND NUMERICAL EXPERIMENTS

To illustrate the usefullness of the proposed programs we consider first the case of the Gamma distribution

$$p(x;\alpha,\beta) = \frac{\beta^{(1-\alpha)}}{\Gamma(1-\alpha)} x^{\alpha} \exp(-\beta x), \quad x > 0, \alpha < 1, \beta > 0.$$
 (17)

This distribution can be considered as a ME distribution when the constraints are

$$\begin{cases}
\int p(x; \alpha, \beta) dx &= 1 \\
\int x p(x; \alpha, \beta) dx &= \mu_1 \\
\int \ln(x) p(x; \alpha, \beta) dx &= \mu_2
\end{cases}$$

$$\begin{cases}
\text{normalization } \phi_0(x) &= 1, \\
\phi_1(x) &= x, \\
\phi_2(x) &= \ln(x).
\end{cases}$$
(18)

This is easy to verify because the equation (12) can be written as

$$p(x; \alpha, \beta) = \exp[-\lambda_0 - \lambda_1 x - \lambda_2 \ln(x)]$$

with 
$$\lambda_0 = -\ln \frac{\beta^{(1-\alpha)}}{\Gamma(1-\alpha)}$$
,  $\lambda_1 = \beta$  and  $\lambda_2 = -\alpha$ .

Now consider the following problem

Given  $\mu_1$  and  $\mu_2$  determine  $\lambda_0$ ,  $\lambda_1$  and  $\lambda_2$ .

This can be done by the standard ME method. To do this, first we must define the range of x, (xmin, xmax, dx), and write a function fin\_x to calculate the functions  $\phi_0(x) = 1$ ,  $\phi_1(x) = x$  and  $\phi_2(x) = \ln x$  (See the function fin1\_x in Annex). Then we must define an initial estimate  $\lambda^0$  for  $\lambda$  and, finally, let the program works.

The case of the *Gamma* distribution is interesting because there is an analytic relation between  $(\alpha, \beta)$  and the mean  $m = \mathbb{E}\{x\}$  and variance  $\sigma^2 = \mathbb{E}\{(x-m)^2\}$  which is

$$\begin{cases} m = (1 - \alpha)/\beta \\ \sigma^2 = (1 - \alpha)/\beta^2 \end{cases}, \tag{19}$$

or inversely

$$\begin{cases} \alpha = (\sigma^2 - m^2)/\sigma^2 \\ \beta = m/\sigma^2, \end{cases}$$
 (20)

so that we can use these relations to determine m and  $\sigma^2$ . Note also that the corresponding entropy of the final result is a byproduct of the function. Table (1) gives some numerical results obtained by ME\_DENS1 program (See Annex).

|         | Table 1. |          |         |        |            |  |  |  |  |
|---------|----------|----------|---------|--------|------------|--|--|--|--|
| $\mu_1$ | $\mu_2$  | $\alpha$ | β       | m      | $\sigma^2$ |  |  |  |  |
| 0.2000  | -3.0000  | 0.2156   | -3.0962 | 0.2533 | 0.0818     |  |  |  |  |
| 0.2000  | -2.0000  | -0.4124  | -6.9968 | 0.2019 | 0.0289     |  |  |  |  |
| 0.3000  | -1.5000  | -0.6969  | -5.3493 | 0.3172 | 0.0593     |  |  |  |  |

The next example is the case of a quartic distribution

$$p(x) = \exp\left[-\sum_{n=0}^{4} \lambda_n x^n\right]. \tag{21}$$

This distribution can be considered as a ME distribution when the constraints are

$$E\{x^n\} = \int x^n p(x) dx = \mu_n, \quad n = 0, \dots, 4 \text{ with } \mu_0 = 1.$$
 (22)

Now consider the following problem : Given  $\mu_n, n=1,\ldots,4$  calculate  $\lambda_n, n=0,\ldots,4$  . This can be done by the ME\_DENS2 program. Table (2) gives some numerical results obtained by this program:

|         | Table 2. |         |         |             |             |             |             |             |  |  |  |
|---------|----------|---------|---------|-------------|-------------|-------------|-------------|-------------|--|--|--|
| $\mu_1$ | $\mu_2$  | $\mu_3$ | $\mu_4$ | $\lambda_0$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ |  |  |  |
| 0       | 0.2      | 0.05    | 0.10    | 0.1992      | 1.7599      | 2.2229      | -3.9375     | 0.4201      |  |  |  |
| 0       | 0.3      | 0.00    | 0.15    | 0.9392      | 0.000       | -3.3414     | 0.0000      | 4.6875      |  |  |  |
| 0       | 0.3      | 0.00    | 0.15    | 0.9392      | 0.000       | -3.3414     | 0.0000      | 4.6875      |  |  |  |

These examples show how to use the proposed programs. A third example is also given in Annex which shows how to use the ME\_DENS3 program which considers the case of trigonometric moments.

#### CONCLUSIONS

In this paper we addressed first the class of ME distributions when the available data are a finite set of expectations  $\mu_n = \mathrm{E}\{\phi_n(x)\}$  of some known functions  $\phi_n(x), n = 0, \ldots, N$ . We proposed then three Matlab programs to solve this problem by a Newton–Raphson method in general case, in case of geometrical moments data where  $\phi_n(x) = x^n$  and in case of trigonometrical moments where  $\phi_n(x) = \exp(-jn\omega_0 x)$ . Finally, we gave some numerical results for some special examples who show how to use the proposed programs.

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### **ANNEX A**

```
\texttt{function [lambda,p,entr]=me\_densl(mu,x,lambda0)}
      %ME_DENS1
% [LAMBDA,P,ENTR]=ME_DENS1(MU,X,LAMBDA0)
        This program calculates the Lagrange Multipliers of the ME probability density functions p(x) from the knowledge of the N constraints in the form: E\{fin(x)\}=MU(n) \quad n=0:N \quad \text{with } fio(x)=1, \ MU(0)=1.
9 % MU is a table containing the constraints MU(n),n=1:N.
10 % X is a table defining the range of the variation of x.
11 % LAMBDA0 is a table containing the first estimate of the LAMBDAS.
     (This argument is optional.)

% LAMBDA is a table containing the resulting Lagrange parameters.

P is a table containing the resulting pdf p(x).

ENTR is a table containing the entropy values at each
12
13
14
15
16
17
18
                     iteration.
18 % Author: A. Mohammad-Djafari
19 % Date : 10-01-1991
20 %
     % add mu(0)=1
% x axis
23
                                             if(nargin == 2)
lambda=zeros(size(mu));
      lambda(1)=log(xmax-xmin);
     else
lambda=lambda0(:);
28
29
30
31
32
33
     end
     N=length(lambda);
                                            \ \ \mbox{finl}_{\bf x}({\bf x}) is an external \ \mbox{function which provides fin}({\bf x}).
34
35 iter=0;
36 while 1
37 iter=i
38 disp('
                                                   % start iterations
        iter=iter+1;
      disp('----'); disp(['iter=',num2str(iter)]);
39 %
40
41
      p=exp(-(fin*lambda));
plot(x,p);
                                                 % Calculate p(x)
% plot it
42 %
                                      % Calculate Gn
       G=zeros(N,1);
43
        for n=1:N
G(n)=dx*sum(fin(:,n).*p);
46
      end
47 %
48
49
      50 %
        gnk=zeros(N,N); % Calculate gnk

gnk(1,:)=-G'; gnk(:,1)=-G; % first line and first column

for i=2:N % lower triangle part of the
53
54
55
56
57
58
59
60
         end
        for i=2:N
                                                 % uper triangle part of the % matrix G
         for j=i+1:N
gnk(i,j)=gnk(j,i);
61
         end
62
63
64
        v=mu-G;
                                                    % Calculate v
        delta=gnk\v;
lambda=lambda+delta;
eps=le-6;
                                                    % Calculate delta
% Calculate lambda
% Stopping rules
65
66
67
68
        if(abs(delta./lambda)<eps),
                                                                                  break, end
69
70
71
72
        if(iter>2)
if(abs((entr(iter)-entr(iter-1))/entr(iter))<eps),break, end
        end
     end
73
74
75
76
    p=exp(-(fin*lambda));
plot(x,p);
entr=entr(:);
                                        % Calculate the final p(x) % plot it
77 disp('----' END -----')
```

```
1 function [lambda,p,entr]=me_dens2(mu,x,lambda0)
        [LAMBDA, P, ENTR]=ME DENS2(MU, X.I.AMBDA0)
        This program calculates the Lagrange Multipliers of the ME probability density functions p(x) from the knowledge of the
    % N moment contstraints in the form:  
% E\{x^n\}=mu(n)  n=0:N with mu(0)=1.
                    is a table containing the constraints MU(n), n=1:N.
    % NU Is a table containing the constraints MU(N),n=1:N.
% LAMBDA0 is a table defining the range of the variation of x.
% LAMBDA0 is a table containing the first estimate of the LAMBDAs.
% (This argument is optional.)
% LAMBDA is a table containing the resulting Lagrange parameters.
% P is a table containing the resulting pdf p(x).
% ENTR is a table containing the entropy values at each
% iteration.
10 % X
13
14
15
16
17
18
19
    % Author: A. Mohammad-Djafari
% Date : 10-01-1991
20
     21 mu=mu(:); mu=[1;mu];
22
23
24
                                          % initialize LAMBDA
25 if(nargin == 2)
26
27
28
      lambda=zeros(size(mu));
lambda(1)=log(xmax-xmin);
                                           % This produces a uniform % distribution.
     else
29
      lambda=lambda((:);
     end
N=length(lambda);
30
31
32
33 M=2*N-1;
34 fin=zeros
35 fin(:,1)=
                                              % Calcul de fin(x)=x.^n
     fin=zeros(length(x),M);
                                                       % fi0(x)=1
     fin(:,1)=ones(size(x));
36
37
38
     for n=2:M
     fin(:,n)=x.*fin(:,n-1);
end
39
40
    %
iter=0;
41
42
43
     while 1
iter=iter+1;
                                               % start iterations
      disp('----'); disp(['iter=',num2str(iter)]);
44 %
45
46
47 %
     G=zeros(M,1);
48
                                                % Calculate Gn
       for n=1:M
G(n)=dx*sum(fin(:,n).*p);
51
      end
52 %
53
54
       55 %
56
57
58
       gnk=zeros(N,N);
                                              % Calculate gnk
% Matrix G is a Hankel matrix
       for i=1:N
gnk(:,i)=-G(i:N+i-1);
59
60
61
62
      end
                                               % Calculate v
% Calculate delta
        v=mu-G(1:N);
       delta=qnk\v;
63
64
65
66
       lambda=lambda+delta;
                                                % Calculate lambda
       eps=le-6;
if(abs(delta./lambda)<eps),
                                                % Stopping rules
                                                                            break, end
       if(iter>2)
         if(abs((entr(iter)-entr(iter-1))/entr(iter))<eps),break, end
67
68
69
     end
70
end
 1 %ME2
 2 % This script shows how to use the function ME_DENS2
3 % in the case of the quartic distribution. (see Example 2.)
4 xmin=-1; xmax=1; dx=0.01; % define the x axis
 5 x=[xmin:dx:xmax]';
 5 x=|xmin:dx:xmax|';
6 mu=[0.1,.3,0.1,.15]';
7 [lambda,p,entr]=me_dens2(mu,x);
8 disp([mu;lambda;entr(length(entr))]')
                                                   % define the mu values
```

```
1 function [lambda,p,entr]=me_dens3(mu,x,lambda0)
       [LAMBDA.P.ENTR]=ME DENS3(MU.X.LAMBDA0)
        This program calculates the Lagrange Multipliers of the ME probability density functions p(x) from the knowledge of the
    % Fourier moments values :   
% E\{\exp[-j \ n \ w0 \ x]\}=mu(n) \ n=0:N \ with \ mu(0)=1.
                  is a table containing the constraints MU(n),n=1:N.
9 % MU is a table containing the constraints MU(n),n=1:N.

10 % X is a table defining the range of the variation of x.

11 % LAMBDA0 is a table containing the first estimate of the LAMBDAS.

2 % (This argument is optional.)

13 % LAMBDA is a table containing the resulting Lagrange parameters.

14 % P is a table containing the resulting pdf p(x).

15 % ENTR is a table containing the entropy values at each

16 % iteration.
17
mu=mu(:);mu=[1;mu];
                                          % add mu(0)=1
% x axis
21
     22
23
24
25
                                                   % This produces a uniform
26
27
28
      lambda(1)=log(xmax-xmin);
                                            % distribution.
    else
lambda=lambda0(:);
29
30
31
     N=length(lambda);
     M=2*N-1;
                                            % Calculate fin(x)=exp[-jnw0x]
32
33
34
35
                                             % fin3_x(x) is an external
% function which provides fin(x).
     fin=fin3_x(x,M);
     iter=0;
36
37
38
                                            % start iterations
     while 1
     iter=iter+1;
disp('-----'); disp(['iter=',num2str(iter)]);
39
40
                                             % Calculate p(x)
     41
42
43
44
45
46
47
      G=zeros(M,1);
                                            % Calculate Gn
      for n=1:M
G(n)=dx*sum(fin(:,n).*p);
      %plot([real(G(1:N)),real(mu),imag(G(1:N)),imag(mu)])
48
     51
52
53
54
      %
gnk=zeros(N,N); % Calculate gnk
      for n=1:N $ watculate gnk $ matrix gnk is a Hermitian for k=1:n $ Toeplitz matrix. gnk(n,k)=-G(n-k+1); $ Lower triangle part end
      for n=1:N
55
56
57
58
      end
59
60
61
62
      for n=1:N
       for k=n+1:N

gnk(n,k)=-conj(G(k-n+1)); % Upper triangle part
       end
      end
63
64
65
66
67
68
69
                                             % Calculate v
                                             % Calculate delta
% Calculate lambda
% Stopping rules
      break, end
70
71
72
73
74
      if(iter>2)
       if(abs((entr(iter)-entr(iter-1))/entr(iter))<eps),break, end
     end
end
                                             % Calculate p(x)
75
76
77
    plot(x,p);
entr=entr(:);
78 disp('----' END -----')
```