Graphs

Undirected graphs

For a set V, let $[V]^k$ denote the set of k-element subsets of V. (Equivalently, $[V]^k$ is the set ofo all k-combinations of V.)

A **(simple, undirected) graph** $G = (V, E)$, consists of a set of **vertices** $V \neq \emptyset$, and a set $E \subseteq [V]^2$ of **edges**.

Every edge $\{u, v\} \in E$ has two distinct vertices $u \neq v$ as **endpoints**, and such vertices u and v are then said to be **adjacent** in the graph G.

The above definitions allow for infinite graphs, where $|V| = \infty$.

Directed graphs

Definitions

Directed graph

A **directed graph**, $G = (V, E)$, consists of a set of vertices $V \neq \emptyset$, and a set $E \subseteq V \times V$ of **directed edges**.

Each directed edge $(u, v) \in E$ has a **start (tail)** vertex u, and an **end (head)** vertex v.

Degrees

The **in-degree** of a vertex v, denoted $deg^{-}(v)$, is the number of edges directed into v. The **out-degree** of v, denoted $deg^{+}(v)$, is the number of edges directed out of v. Note that a loop at a vertex contributes 1 to both in-degree and out-degree.

Theorem

Let $G = (V, E)$ be a directed graph.

Then:

$$|E| = \sum_{v \in V} deg^{-}(v) = \sum_{v \in V} deg^{+}(v)$$

Proof

The first sum counts the number of outgoing edges over all vertices and the second sum counts the number of incoming edges over all vertices. Both sums must be $|E|$.

Complete graphs

A **complete graph on n vertices**, denoted by K_n, is the simple graph that contains exactly one edge between each pair of distinct vertices.

A complete graph K_n has n vertices and $\frac{n(n-1)}{2}$ edges.

Cycles

A **cycle** C_n for $n \geq 3$ consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1, v_2\}$,$\{v_2, v_3\}$,. . .,$\{v_{n-1}, v_n\}$, $\{v_n, v_1\}$.

n-cubes

An **n-dimensional hypercube** or **n-cube**, is a graph with 2^n vertices representing all bit strings of length n, where there is an edge between two vertices iff they differ in exactly one bit position.

For example:

- A vertex 011 on a **3-cube** would have edges connecting to vertices 001, 010 and 111.
- A vertex 110010 on a **6-cube** would have edges connecting to vertices 110011, 110000, 110110, 111010, 100010 and 010010.

Bipartite graphs

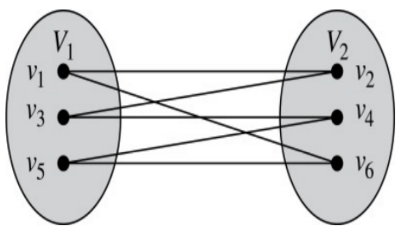
Definition

An equivalent definition of a bipartite graph is one where it is possible to **color** the vertices either red or blue so that no two adjacent vertices are the same color.

Examples

Show that C_6 is bipartite

Partition the vertex set V into $V_1 = \{v_1, v_3, v_5\}$ and $V_2 = \{v_2, v_4, v_6\}$.



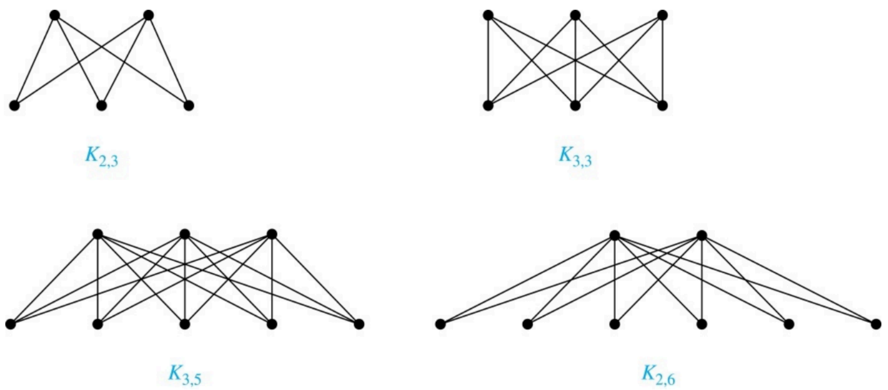
Show that C_3 is not bipartite

If we partition vertices of C_3 into two non-empty sets, one set must contain two vertices. But every vertex is connected to every other. So, the two vertices in the same partition are connected. Hence, C_3 is not bipartite.

Complete Bipartite Graphs

Definition

A **complete bipartite graph** is a graph that has its vertex set V partitioned into two subsets V_1 of size m and V_2 of size n such that there is an edge from every vertex in V_1 to every vertex in V_2.

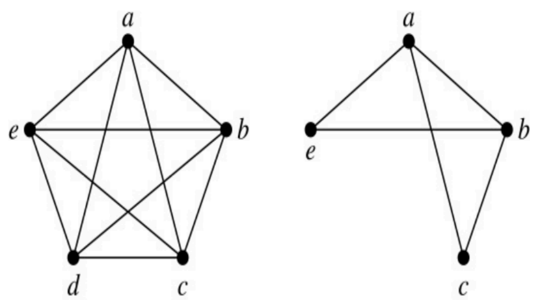


Subgraphs

Definition

A **subgraph** of a graph $G = (V, E)$ is a graph (W, F), where $W \subseteq V$ and $F \subseteq E$. A subgraph H of G is a **proper subgraph** of G if $H \neq G$.

Example



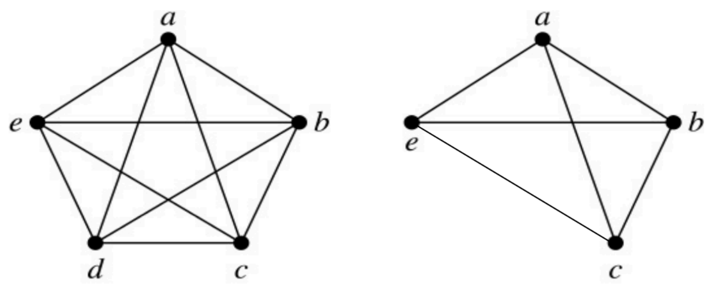
Induced subgraphs

Definition

Let $G = (V, E)$ be a graph. The **subgraph induced** by a subset W of the vertex set V is the graph $H = (W, F)$, whose edge set F contains an edge in E iff both endpoints are in W.

Example

Here is K_5 and its **induced subgraph** induced by $W = \{a, b, c, e\}$.



Bipartite graphs

A **bipartite graph** is a (undirected) graph $G = (V, E)$ whose vertices can be partitioned into two disjoint sets (V_1, V_2), with $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = V$, such that for every edge $e \in E$, $e = \{u, v\}$ such that $u \in V_1$ and $v \in V_2$. In other words, every edge connects a vertex in V_1 with a vertex in V_2.

This is an alternative definition to the *coloring* definition.

Matching in Bipartite Graphs

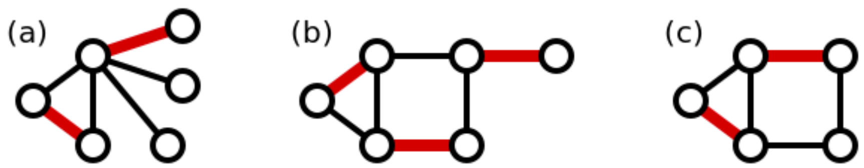
Matching

A **matching** M, in a graph $G = (V, E)$, is a subset of edges, $M \subseteq E$, such that there does not exist two distinct edges in M that are incident on the same vertex. In other words, if $\{u, v\}, \{w, z\} \in M$, then either $\{u, v\} = \{w, z\}$ or $\{u, v\} \cap \{w, z\} = \emptyset$.

i.e. the set of pairwise non-adjacent edges; that is, no two edges share a common vertex.

Maximum matching

A **maximum matching** in graph G is a matching in G with the maximum possible number of edges.



Perfect/complete matchings

For a graph $G = (V, E)$, we say that a subset of edges, $W \subseteq E$, **covers** a subset of vertices, $A \subseteq V$, if for all vertices $u \in A$ there exists an edge $e \in W$, such that e is incident on u, i.e., such that $e = \{u, v\}$, for some vertex v.

In a bipartite graph $G = (V, E)$ with bipartition (V_1, V_2), a **complete matching** with respect to V_1, is a matching $M' \subseteq E$ that covers V_1, and a **perfect matching** is a matching, $M^* \subseteq E$, that covers V.

Figure (b) above is an example of a perfect matching.

Hall's Marriage Theorem

Theorem

For a bipartite graph $G = (V, E)$, with bipartition (V_1, V_2), there exists a matching $M \subseteq E$ that covers V_1 **iff** $\forall S \subseteq V_1, |S| \leq |N(S)|$.

Proof

Slides 5-8 on Lecture 19.

Corollary

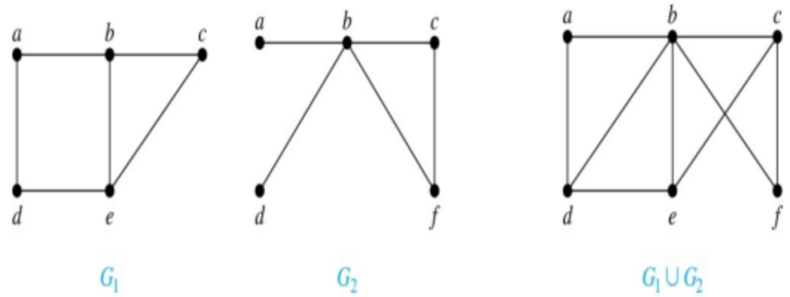
A bipartite graph $G = (V, E)$ with bipartition (V_1, V_2) has a **perfect** matching iff $|V_1| = |V_2|$ and $\forall S \subseteq V_1, |S| \leq |N_G(S)|$.

Union of graphs

Definition

The **union** of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of G_1 and G_2 is denoted by $G_1 \cup G_2$.

Examples



Adjacency lists

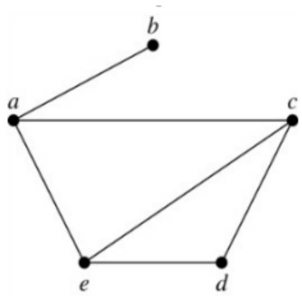
Definition

An **adjacency list** represents a graph (with no multiple edges) by specifying the vertices that are adjacent to each vertex.

Example

Vertex	Adjacent vertices
a	b, c, e
b	a
c	a, d, e
d	c, e
e	a, c, d

This adjacency list corresponds to the graph:



Adjacency matrices

Definition

Suppose that $G = (V, E)$ is a simple graph where $|V| = n$. Arbitrarily list the vertices of G as $v_1, v_2, \ldots, v_n$.

The **adjacency matrix**, $\mathbf{A}$, of G, **with respect to this listing of vertices**, is the $n \times n \quad (0-1)$ matrix with its $(i, j)^{th}$ entry = 1 when v_i and v_j are adjacent, and $= 0$ when they are not adjacent.

In other words: $A = [a_{ij}]$ and:

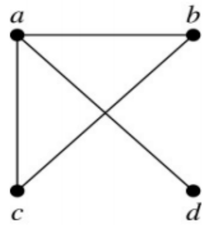
$$a_{ij} = \begin{cases} 1, & \text{if } \{v_i, v_j\} \text{ is an edge of } G \\ 0, & \text{otherwise} \end{cases}$$

Examples

Example 1

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

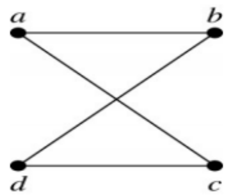
Corresponds to the graph:



Example 2

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Corresponds to the graph:



Adjacency matrices (continued)

- The adjacency matrix of an undirected graph is **symmetric**:

$$a_{ij} = a_{ji}, \forall i, j$$

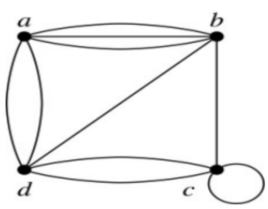
Also, since there are no loops, each diagonal entry is zero: $a_{ii} = 0, \forall i$.

- Adjacency matrices can also be used to represent graphs with loops and multi-edges.
- When multiple edges connect vertices v_i and v_j, (or if multiple loops present at the same vertex), the $(i, j)^{th}$ entry equals the number of edges connecting the pair of vertices.

Example

$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

Corresponds to the graph:



- Adjacency matrices can represent directed graphs in exactly the same way.

  The matrix $\mathbf{A}$ for a directed graph $G = (V, E)$ has a 1 in its $(i, j)^{th}$ position if there is an edge from v_i to v_j, where $v_1, v_2, \ldots, v_n$ is a list of the vertices.

  In other words:

$$\begin{aligned} a_{ij} &= 1 \qquad \text{if } (i, j) \in E \\ a_{ij} &= 0 \qquad \text{if } (i, j) \notin E \end{aligned}$$

  Note: The adjacency matrix for a directed graph does not have to be symmetric.

- A **sparse** graph has few edges relative to the number of possible edges.

  Sparse graphs are more efficient to represent using an adjacency list than an adjacency matrix. But for a **dense** graph, an adjacency matrix is often preferable.

Isomorphism of graphs

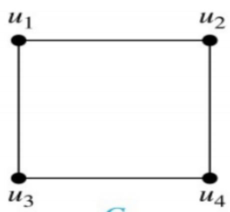
Definition

Two (undirected) graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **isomorphic** if there is a bijection, $f : V_1 \mapsto V_2$, with the property that for all vertices $a, b \in V_1$:

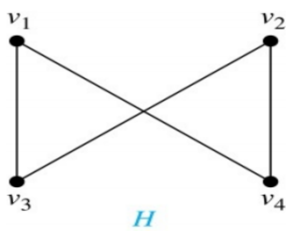
$$\{a, b\} \in E_1 \iff \{f(a), f(b)\} \in E_2$$

Such a function f is called an **isomorphism**.

Intuitively, isomorphic graphs are **the same** except for the renamed vertices.



G



H

It is difficult to determine whether two graphs are isomorphic by brute force: there are $\mathbf{n!}$ bijections between vertices of n-vertex graphs.

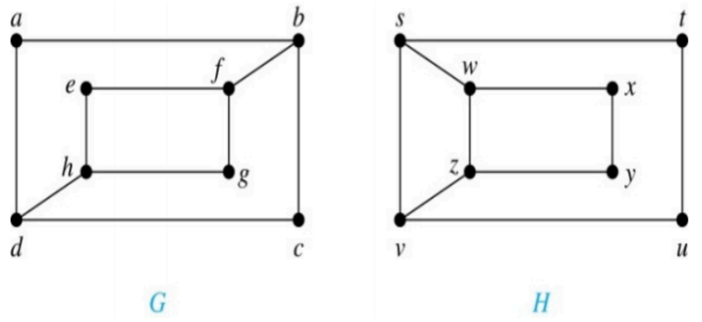
To show that two graphs are not isomorphic, we can find a property that only one of the two graphs has. Such a property is called **graph invariant**:

e.g.

- Number of vertices of given degree
- Degree sequence (list of the degrees)
- Number of edges

- Number of cycles

Example



These graphs are not isomorphic. This is because $deg(a) = 2$ in G, a must correspond to t, u, x or y, since these are the vertices of degree 2 in H.

But each of these vertices is adjacent to another vertex of degree 2 in H, which is not true for a in G. So, G and H can not be isomorphic.

Paths (in undirected graphs)

Informally, a **path** is a sequence of edges connecting vertices.

Definition

For an undirected graph $G = (V, E)$, an integer $n \geq 0$, and vertices $u, v \in V$, a **path (or walk) of length n from u to v in G is a sequence:**

$$x_0, e_1, x_1, e_2, \ldots, x_{n-1}, e_n, x_n$$

of interleaved vertices $x_j \in V$ and edges $e_1 \in E$, such that $x_0 = u$ and $x_n = v$, and such that $e_i = \{x_{i-1}, x_i\} \in E \quad \forall i \in \{1, \ldots, n\}$.

Such a path **starts** at u and **ends** at v. A path of length $n \geq 1$ is called a **circuit** (or **cycle**) if $n \geq 1$ and the path starts and ends at the same vertex, i.e. $u = v$.

A path or circuit is called **simple** if it does not contain the same edge more than once.

More

Don't confuse a simple undirected graph with a simple path.

There can be a simple path in a non-simple graph, and a non-simple path in a simple graph.

A path is **tidy** when no vertex is repeated.

A path is **simple** when no edge is repeated.

Connectedness

Definition

An undirected graph $G = (V, E)$ is called **connected** if there is a path between every pair of distinct vertices. It is called **disconnected** otherwise.

Proposition

There is always a simple, and tidy, path between any pair of vertices u, v of a connected undirected graph G.

Proof

Read slide 7

Connected components

A **connected component** $H = (V', E')$ of a graph $G = (V, E)$ is a maximal connected subgraph of G, meaning H is connected and $V' \subseteq V$ and $E' \subseteq E$ but H is not a proper subgraph of a larger connected subgraph R of G.

Strongly connected graphs

A directed graph $G = (V, E)$ is **strongly connected** if for every pair of vertices in u and v in V, there is a directed path from u to v, and a directed path from v to u.

It is **weakly connected** if there is a path between every pair of vertices in V in the underlying undirected graph (meaning we ignore the direction of edges).

A **strongly connected component** of a directed graph G, is a maximal strongly connected subgraph H of G which is not contained in a larger strongly connected subgraph of G.

Euler/Hamiltonian paths and circuits

Definitions

An **Euler path** in a multigraph G is a simple path that contains every edge of G. (so every edge occurs exactly once in the path)

An **Euler circuit** in a multigraph G is a simple circuit that contains every edge of G. (so every edge occurs exactly once in the circuit)

A **Hamiltonian path** is a multigraph G is a simple path that passes through every vertex (not necessarily each edge), exactly once.

A **Hamiltonian circuit** is a multigraph G is a simple circuit that passes through every vertex (not necessarily each edge), exactly once.

Graph colouring

Definition

Suppose we have k disinct colours with which to colour the vertices of a graph. Let $[k] = \{1, \ldots, k\}$. For an undirected graph, $G = (V, E)$, an admissable vertex **k-colouring** of G is a function $c : V \to [k]$, such that $\forall u, v \in V$, $\{u, v\} \in E \implies c(u) \neq c(v)$.

For an integer $k \geq 1$, we say an undirected graph $G = (V, E)$, is k-colourable if there exists a **k-colouring** of G.

The **chromatic number** of G, denoted $\chi(G)$, is the smallest positive integer k, such that G is k-colourable.

Observations

- Any graph G with n vertices is n-colourable.
- The **n-clique**, K_n i.e., the complete graph on n vertices, has chromatic number $\chi(K_n) = n$. All its vertices must get assigned different colours in any admissable colouring.
- The **clique number**, $\omega(G)$, of graph G is the maximum positive integer $r \geq 1$, such that K_r is a subgraph of G.
- For all graphs G, $\omega(G) \leq \chi(G) : \text{G has a } r\text{-clique} \implies \text{G is not } (r-1)\text{-colourable}$.
- In general, $\omega(G) \neq \chi(G)$.

  > e.g. C_5 has $\omega(C_5) = 2$ and $\chi(C_5) = 3$. Note that in this case, $\omega(C_5) < \chi(C_5)$.

- Any bipartite graph is 2-colourable. This is an alternative definition of being bipartite.
- Generally, a graph G is k-colourable precisely if it is k-partite, meaning its vertices can be partitioned into k disjoint sets such that all edges of the graph are between nodes in different parts.