# Relations

## Relations

Let $S$ be a set. A **relation** on $S$ is a subset $R$ of $S \times S$.

$R$ consists of ordered pairs $(s, t)$ with $s, t \in S$. For those ordered pairs $(s, t) \in R$, we write $s \sim t$ or $sRt$ and say $s$ is related to $t$.

> Example: Let $S = \mathbb{R}$ and define $a \sim b \iff a < b$. Here:
>
> $$R = \{(s, t) \in \mathbb{R} \times \mathbb{R} \mid s < t\}$$
>
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>
> Example: Let $S = \mathbb{Z}$ and let $m$ be a positive integer. Define $a \sim b \iff a \equiv b \pmod{m}$, then:
>
> $$R = \{(s, t) \in \mathbb{Z} \times \mathbb{Z} \mid s \equiv t \pmod{m}\}$$

## Equivalence relations

Let $S$ be a set, and let $\sim$ be a relation on $S$. Then $\sim$ is an equivalence relation if the following three properties hold $\forall a, b, c \in S$:

- **Reflexivity**: $a \sim a$
- **Symmetry**: $a \sim b \implies b \sim a$
- **Transitivity**: $(a \sim b) \wedge (b \sim c) \implies a \sim c$

> Consider the two examples of relations in the previous section:
>
> - $R = \{(s, t) \in \mathbb{R} \times \mathbb{R} \mid s < t\}$
>   - **Reflexivity**: If $(x, x) \in R$, then for reflexivity we need $x < x$. However, this is clearly not the case, so this relation is **not reflexive**.
>   - **Symmetry**: If $(x, y) \in R$, then for symmetry we need $(x < y \implies y < x)$. Again, this is clearly not the case, so this relation is **not symmetric**.
>   - **Transitivity**: If $(x, y), (y, z) \in R$, then for transitivity we need $(x, z) \in R \iff x < z$. This is clearly the case, as $x < y < z \iff x < z$, so this relation **is transitive**.
>   - The relation is **not an equivalence relation** because not all three properties hold.
> - $R = \{(s, t) \in \mathbb{Z} \times \mathbb{Z} \mid s \equiv t \pmod{m}\}$
>   - **Reflexivity**: If $(x, x) \in R$, then for reflexivity, we need $x \equiv x \pmod{m}$, which is clearly the case, so this relation **is reflexive**.
>   - **Symmetry**: If $(x, y) \in R$, then for symmetry, we need $x - y \equiv 0 \pmod{m} \implies y - x \equiv 0 \pmod{m}$. Note that $x - y = -(y - x)$,

> so this relation **is symmetric**.
>
> - **Transitivity**: If $(x,y),(y,z) \in R$, then for transitivity we need $(x,z) \in R \iff x \equiv z \pmod m$. We have $x - y \equiv 0 \pmod m$ and $y - z \equiv 0 \pmod m$. Therefore, $(x-y)+(y-z) \equiv 0 \pmod m$, giving us $x \equiv z \pmod m$, so this relation **is transitive**.
> - The relation **is an equivalence relation** because all three properties hold.

# Equivalence classes

Let $S$ be a set and $\sim$ an equivalence relation on $S$. For $a \in S$, define

$$cl(a) = \{s \mid s \in S, s \sim a\}$$

Thus, $cl(a)$ is the set of things that are related to $a$. The subset $cl(a)$ of $S$ is called an equivalence class of $\sim$. The equivalence classes of $\sim$ are the subsets $cl(a)$ as $a$ ranges over the elements of $S$.

> Example: Consider the equivalence relation
>
> $$R = \{(s,t) \in \mathbb{Z} \times \mathbb{Z} \mid s \equiv t \pmod m\}$$
>
> Some various equivalence classes are:
>
> - $cl(0) = \{s \in \mathbb{Z} \mid s \equiv 0 \pmod m\}$
> - $cl(1) = \{s \in \mathbb{Z} \mid s \equiv 1 \pmod m\}$
>
> $$\dots$$
>
> - $cl(m-1) = \{s \in \mathbb{Z} \mid s \equiv m-1 \pmod m\}$
>
> We claim that these are all the equivalence classes. For if $n$ is any integer, then $\exists q, r \in \mathbb{Z} : n = qm + r$ with $0 \leq r < m$. Then $n \equiv r \pmod m$, so $n \in cl(r)$, which is one of the classes listed above.