Combinatorics and counting

Product rule

If A and B are finite sets, then $|A \times B| = |A| \cdot |B|$.

Proof: Obvious, but prove with induction on $|A|$.

General product rule

If $A_1, A_2, \ldots, A_m$ are finite sets, then

$$|A_1 \times A_2 \times \ldots \times A_m| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_m|$$

Proof: Induction on m, using the basic product rule.

Counting subsets

A finite set, S, has $2^{|S|}$ distinct subsets.

Proof: Suppose $S = \{s_1, s_2, \ldots, s_m\}$. There is a one-to-one correspondence (bijection), between subsets of S and bit strings of length $m = |S|$. The bit string of length $|S|$ we associate with a subset $A \subseteq S$ has a 1 in position i if $s_i \in A$, and 0 in position i if $s_i \notin A$, for all $i \in \{1, \ldots, m\}$.

By the product rule, there are $2^{|}S|$ such bit strings.

Counting functions

Number of functions

For all finite sets A and B, the number of distinct functions, $F : A \to B$, mapping A to B is:

$$|B|^{|A|}$$

Proof: Suppose $A = \{a_1, \ldots, a_m\}$. There is a one-to-one correspondence between functions $f : A \to B$ and strings (sequences) of length $m = |A|$ over an alphabet of size $n = |B|$.

By the product rule, there are n^m such strings of length m.

Inclusion-exclusion principle

For any finite sets A and B (not necesarily disjoint):

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Generalized pigeonhole principle

Theorem

If $N \geq 0$ objects are placed in $k \geq 1$ boxes, then at least one box contains at least $\lceil \frac{N}{k} \rceil$ objects.

Proof

Suppose no box has more than $\lceil \frac{N}{k} \rceil - 1$ objects. Sum up the number of objects in the k boxes. It is at most

$$k \cdot \left(\left\lceil \frac{N}{K} \right\rceil - 1 \right) < k \cdot \left(\left(\frac{N}{k} + 1 \right) - 1 \right) = N$$

Thus, there must be fewer than N. Contradiction.

> E.g. "At least d students in this course were born in the same month"
>
> Suppose the number of students registered for the course is 145.
>
> What is the maximum number d for which **it is certain** that the statement is true?

Solution

Since we are assuming there are 145 registered students, $\lceil \frac{145}{12} \rceil = 13$, so by GPP we know that the statement is true for $d = 13$.

Permutations

A **permutation** of a set S is an ordered arrangement of the elements of S. i.e. A sequence containing every element of S exactly once.

k-permutations

A k-permutation of a set S is an ordered arrangement of k distinct elements of S.

If we have a set of size n, the number of k-permutations of the n-set are the different ordered arrangements of a k-element subset of the set with n elements. This is given by:

$$\frac{n!}{(n-k)!}$$

> E.g. how many ways can first and second place be awarded to 10 people?
>
> Here, we have $n = 10$ and $k = 2$.
>
> Order does matter in this case, because someone coming first and another person coming second is not the same, the other way round.
>
> Therefore, we have to find the 2-permutations of 10 elements.

> $\frac{10!}{(10-2)!}$

Combinations

A k-element combination of the set with n elements is a k-element subset, in which the elements are not ordered. The number of k-element combinations of the set with n elements is given by:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

> E.g. How many different 5-card poker hands can be dealt from a deck of 52 cards?
>
> $\binom{52}{5}$

> E.g. How many different 47-card poker hands can be dealt from a deck of 52 cards?
>
> $\binom{52}{47}$

Binomial theorem

$$\forall n \geq 0, \quad (x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$$

k-combinations with repetition

To choose k elements **with repetition allowed** from a set of size n, we can do this in the following number of ways:

$$\binom{n+r-1}{r} = \binom{n+r-1}{n-1}$$

> E.g. How many different solutions in non-negative integers x_1,x_2 and x_3 does the following equation have?
>
> $x_1 + x_2 + x_3 = 11$

Solution: We have to place 11 pebbles into three different bins x_1, x_2 and x_3.

This is the equivalent to choosing an 11-combination (with repetition) from a set of size 3, so the answer is:

$$\binom{11+3-1}{11}$$

Summary

Type	Repetition allowed?	Formula
r-permutations	No	$\frac{n!}{(n-r)!}$
r-combinations	No	$\frac{n!}{(n-r)!r!} = \binom{n}{r}$
r-permutations	Yes	n^r
r-combinations	Yes	$\frac{(n+r-1)!}{r!(n-1)!} = \binom{n+r-1}{r}$

Useful example questions

Question: How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 29$ where x_i ($i = 1,2,3,4,5,6$) is a non-negative integer, such that $x_i > 1$ for $i = 1,2,3,4,5,6$.

Answer: Let $x_i = y_i + 2 \iff y_i = x_i - 2$ for $i = 1,2,3,4,5,6$. Then, we obtain

$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = 29 - 6 \cdot 2 = 17$$

Where y_i is now a non-negative number for $i = 1,2,3,4,5,6$. The number of solutions to this equation can now be reduced to solving a typical stars and bars problem:

$$\binom{17+6-1}{6-1} = \binom{22}{5}$$

Question: How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 29$ where x_i ($i = 1,2,3,4,5,6$) is a non-negative integer, such that $x_1 \leq 5$.

Answer: If x_1 is fixed, then the number of solutions for the other variables is:

$$\binom{5 + (29 - x_1) - 1}{5-1} \quad \text{or} \quad \binom{5 + (29 - x_1) - 1}{(29 - x_1)}$$

Since the solutions are in one-to-one correspondence with the reorderings of $29 - x_1$ stars and $5 - 1 = 4$ bars.

The possible values for x_1 are $0, 1, 2, 3, 4, 5$ and hence, the answer is:

$$\sum_{k=0}^{5} \binom{5 + (29 - k) - 1}{4}$$

Give a formula for the coefficient of x^k in the expansion of $(x + \frac{1}{x})^{100}$, where k is an integer.

Answer: We know that:

$$\text{Coefficient of } x^{n-r}y^r = \binom{n}{r}, \text{ where } y = \frac{1}{x}$$

$$x^{n-r} \cdot \frac{1}{x^r} = x^k$$
$$n - 2r = k$$
$$r = \frac{n-k}{2}$$

Coefficient of x^k in the expansion of $(x + \frac{1}{x})^n$ is $\binom{n}{\frac{n-k}{2}}$.

Formula for the coefficient of x^k in the expansion of $(x + \frac{1}{x})^{100}$ is $\binom{100}{\frac{100-k}{2}}$.

Give a formula for the coefficient of x^k in the expansion of $(x - \frac{1}{x})^{100}$ where k is an integer.

Answer: A general term in this expression is of form:

$$\binom{100}{r} x^{100-r} (-\frac{1}{x})^r = \binom{100}{r} x^{100-2r} (-1)^r$$

For all integers k such that $k = 100 - 2r$. Rearranging for r gives: $r = \frac{100-k}{2}$.

The coefficient is $\binom{100}{\frac{100-k}{2}} (-1)^{\frac{100-k}{2}}$.