Equivariant homotopy theory – Mike Hopkins

Fan fiction for equivariant homotopy theory :)

Space form problem: Which manifolds have sphere as a universal covering?

Ans : Madsen, Thomas, Wall.

– Thom isomorphism, poincare duality – in equivariant homology

Equivariant Moore Space Problem: No. (Carlson) — Uses $H^*(BG;\mathbb{Z})$, Steenrod algebra

$X \circlearrowleft G$. What can we say about $X^G = \{x \in X \mid \forall g \in G,\ gx = x\}$.

– Fixed point formulas

eg: $G = T = S^1 \times \cdots \times S^1 \circlearrowleft X$ then $H^*_G(X)$ module over $H^*_G(pt) = \mathbb{Q}[x_1, \cdots x_r]$

$$\mathbb{Q}(x_1, \dots x_r) \otimes_{\mathbb{Q}[x_1, \cdots x_r]} H^*_G(X) \approx \mathbb{Q}(x_1 \dots x_r) \otimes H^*(X^G)$$

Read: Atiyah-Bott The moment map and equivariant cohomology.

A theorem of Archimedes.

$X^{\circlearrowleft G}$ $Y^{\circlearrowleft G}$ Want to study $\mathrm{Map}^G(X, Y) = \{ f: X \to Y \mid f(gx) = g f(x) \}$

Borel equivariant $\rightsquigarrow \mathrm{Map}^G_\infty(X, Y) = \{ \quad \mid \quad \approx_\infty \quad \} = \mathrm{Map}^G(X \times EG, Y)$

G - finite group

G-CW-complexes, built from $G/H \times D^n$

X G-CW-complex $\Rightarrow$ So is X^G + X is built from X^G by attaching $G/H \times D^n$, $H \subsetneq G$.

Products: $G/H \times D^m \times G/H' \times D^n = (G/H \times G/H') \times D^{m+n}$ $\rightsquigarrow$ issues when dealing with compact groups

$\mathcal{T}^G$ = G-CW-complexes (homotopy theory)

Stabilize:

Naive: Work with spectrum objects $[X, Y]_{st} = \lim_{n \to \infty} [\Sigma^n X, \Sigma^n Y]$ X finite CW complex.

$\mathcal{T}^G \longrightarrow$ universal stabilization $\vee X_\alpha \longrightarrow \prod X_\alpha$, cofibrations ~ fibrations

Gives a stable theory, but Spanier Whitehead Duality does not work.

As we will not be able to embed $X^{\circlearrowleft G}$ in a sphere. – S^V one point compactification of V - representation

$\{X,Y\}^G := \lim_{V\to\infty} [\Sigma^V X, \Sigma^V Y]$ $\quad \mathcal{T}^G \longrightarrow \mathcal{S}^G$ G-spectra

Now we do have SW duality.

- Finite G sets are self duals

$$\left.\begin{array}{ll} S_+ \wedge X \xrightarrow{\sim} Map(S,X) & \\ S \text{ finite G set} \quad \bigvee_{s\in S} X \longrightarrow \prod_{s\in S} X & \end{array}\right\} \text{equivariant additivity}$$

Q. What is $\{S^0, S^0\}^G$? $= \lim_{V\to\infty} {}^G[S^V, S^V] \longrightarrow Map(S^{V^H}, S^{V^H}) \cong \mathbb{Z}$

$$\{S^0, S^0\}^G \longrightarrow \prod_{H\leq G} \mathbb{Z}$$

Category: $Burn_G$ Objects: finite G-sets

Morphisms: $S \leftarrow U \rightarrow T$?? group completion.

For defining coproduct as an adjoint to diagonal $\Delta: C \longrightarrow C^J$. But difficult to do this in equivariant case.

$Burn_G(pt, pt) = A(G) =$ grothendieck group of finite G-sets.

Th^m: Tom Dieck Segal $\quad Burn_G \longrightarrow ho\mathcal{S}^G$ is fully faithful. i.e.

$S \longmapsto S_+ \qquad Burn_G(S,T) \cong \{S_+, T_+\}^G$

Sketch of Proof:

→ suffices to do $S = T = pt$

→ Need to show $A(G) \cong \{S^0, S^0\}^G$

$$\begin{tikzcd} A(G) \arrow[r, "\cong"] \arrow[dr] & \{S^0,S^0\}^G \arrow[d] \\ & \prod_{H\leq G}\mathbb{Z} \end{tikzcd} \qquad T \longmapsto \# T^H$$

$T =$ finite G-set, $T^G = \phi$, $T^H \neq \phi$ $\forall H \subsetneq T$

$T_+ \longrightarrow S^0 \longrightarrow \tilde{T}$

$\tilde{T}^G = S^0 \qquad \tilde{T}^H = \vee S^1$

$$\begin{tikzcd} \{S^0, T_+\}^G \arrow[r] & \{S^0, S^0\}^G \arrow[r] & \{S^0, \tilde{T}\}^G \\ Burn_G(pt, T) \arrow[u] \arrow[r] & A(G) \arrow[u] & \end{tikzcd}$$

$\{S^0, \tilde{T}\}^G = \lim_{V\to\infty} \{S^V, \tilde{T}\wedge S^V\}^G$

$S^{V^G} \longrightarrow \tilde{T}\wedge S^V$

Attach another cell here

$G/H \times S^n \longrightarrow \tilde{T}\wedge S^V$

$\Longleftrightarrow$ $S^n \longrightarrow (\tilde{T}\wedge S^V)^H = \vee S^1 \wedge S^{V^H}$ ⟵ increase in connectivity

$n \leq \dim V^H$

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finite G-sets ⟷ equivariant additivity

$[S_+, X]^G = Burn_G \longrightarrow$ Abelian groups

Mackey Functor

Q. Describe $H^*_G(S^V, M)$

Mackey functor