Local Co-efficient System :

on B $\quad \mathcal{H} : \pi_1(B) \longrightarrow Aut_R(M)$

eg: $\begin{array}{c} F \longrightarrow E \\ \downarrow \\ B \end{array}$ fibration, we get $\pi_1(B, b_0) \longrightarrow Aut_R(H_n(F, R))$ which is monodromy

Def: Homology with local co-efficients

Let B be CW-complex, $\mathcal{H} : \pi_1(B) \longrightarrow Aut_R(M)$ $\angle$(S, $M \in R$-mod,

define $H_i(B, M)$ as :

Let $\tilde{B} \xrightarrow{\pi} B$ be its universal cover, and hence also a principal $\pi_1(B)$-bdle

$\tilde{B}$ gets induced a CW structure, $\tilde{X}_n = \pi^{-1}(X_n)$

So that $\pi_1(B)$ acts freely on the n-cells of $\tilde{B}$.

So $C_*^{cell}(B; \mathbb{Z})$ is a free $\mathbb{Z}[\pi_1(B)]$-complex (Note we are also saying differential commutes with $\pi_1(B)$ action)

Define :

$$C_*^{cell}(B; M) := C_*^{cell}(B; \mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1(B)]} M$$

$$H_*^{cell}(B; M) := H_*(C_\cdot^{cell}(B; M))$$

Th^m: (Serre SS)

Given fibration $F \to E \to B$, we have SS :

1) $E^2_{p,q} = H_p(B, H_q(F; R)) \implies H_{p+q}(E; R)$

Twisted

2) $E^2_{p,q} = H^p(B, H^q(F; R)) \implies H^{p+q}(E; R)$

We have a multiplicative structure,

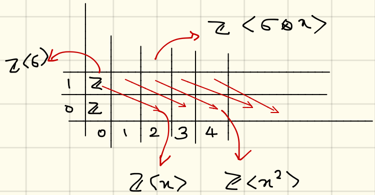
$$\begin{array}{ccc} H^p(B, H^q(F; R)) \otimes_R H^{p'}(B, H^{q'}(F; R)) & \longrightarrow & H^{p+p'}(B \times B; H^q(F; R) \otimes H^{q'}(F'; R)) \\ & & \downarrow \Delta^* \\ & & H^{p+p'}(B; H^{q+q'}(F; R)) \end{array}$$

First quadrant SS, so always converges

Typically, $\pi_1(B)$ acts trivially on $H^*(F,R)$ and either $H^*(B,R)$ or $H^*(F,R)$ will be free / R then $E_2^{p,q} = H^p(B;R) \otimes H^q(F;R)$ as graded rings.

eg: 1) $S^1 \longrightarrow S^{2n+1}$, $S^{2n+1} \to \mathbb{CP}^n$ $\quad n=2 \quad$ $\mathbb{CP}^2$ is simply-connected

$$E_2^{p,q} = H^p(\mathbb{CP}^2; \mathbb{Z}) \otimes H^q(S^1;\mathbb{Z}) \Rightarrow H^{p+q}(S^5;\mathbb{Z})$$



$$d^2(x\otimes\sigma) = d^2(x)\cdot\sigma + (-1)^{|x|} x\cdot d^2\sigma = 0 + x^2$$

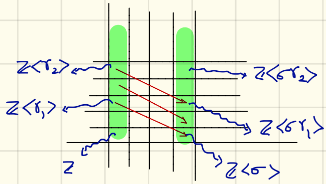
$$\Rightarrow H^\cdot(\mathbb{CP}^2) = \mathbb{Z}[x]/x^3$$

Q. Calculate for $\mathbb{RP}^n$!

Example: $H^*(\Omega S^3, \mathbb{Z})$

$\Omega S^3 \longrightarrow PS^3$, $PS^3 \to S^3$ $\rightsquigarrow$ Serre SS: $E_2^{p,q} = H^p(S^3,\mathbb{Z}) \otimes H^q(\Omega S^3;\mathbb{Z})$

$$\Rightarrow H^*(PS^3;\mathbb{Z}) = \mathbb{Z}$$



Only possible differential is D_3

$E_2 = E_3 \qquad E_4 = E_\infty$

We get $E_3^{0,2n} = \mathbb{Z}\langle\gamma_n\rangle$, $d\gamma_1 = \sigma$, $d\gamma_n = \sigma\gamma_{n-1}$

Claim: $n!\,\gamma_n = \gamma_1^n$ in $H^{2n}(\Omega S^3, \mathbb{Z})$

Follows from the fact that d_3 is an isomorphism.

Conclusion: $H^*(\Omega S^3;\mathbb{Z}) = \mathbb{Z}\langle \gamma_i \mid |\gamma_i| = 2i,\ \gamma_i\gamma_j = \binom{i+j}{i}\gamma_{i+j}\rangle$

divided power algebra

$H^*(\Omega S^{2k+1};\mathbb{Z}) = \mathbb{Z}\langle\gamma_i \mid |\gamma_i| = 2k\,i,\ \gamma_i\gamma_j = \binom{i+j}{i}\gamma_{i+j}\rangle$

Q. Calculate $H^*(\Omega S^{2n};\mathbb{Z})$.

Existence and Properties of SS:

Assume $F \to E \longrightarrow B$ fiber bundle, B is a CW complex, path connected

Case 1: co-efficients are untwisted i.e. $\pi_1(B)$ acts trivially on $H^*(F;R)$

Fix a basepoint b_e at center of every cell e in the CW complex

Let $F = \pi^{-1}(b_0)$ then using fibration property we may identify $\pi^{-1}(b_e)$ by lifting any path connecting b_0 to b_e

Note that because $\pi_1(B)$ acts trivially on $H^*(F,R)$ this identification is well defined on the homologies.

Follows that

$$H^*(\pi^{-1}(B_n), \pi^{-1}(B_{n-1}); R) \cong \bigoplus_{b_e} H^*(D^n \times F, S^n \times F; R) \quad \text{canonically}$$

B_n – n^{th} skeleton

$$\cong \bigoplus_{b_e} H^n(D^n, S^{n-1}; R) \otimes H^{*-n}(F;R)$$

$$\cong H^n(B_n, B_{n-1}; R) \otimes H^{*-n}(F;R)$$

$$= C^n_{cell}(B;R) \otimes H^{*-n}(F;R)$$

Construction:

Filtration of $B \rightsquigarrow B_n$ – n^{th} skeleton

Filtration of $E \rightsquigarrow E_n = \pi^{-1}(B_n)$

$\Rightarrow S_*(E,R)$ is a filtered chain complex with n^{th} filtration $\underbrace{S_*(\pi^{-1}(B_n);R)}_{F_n}$

gives rise to a spectral sequence:

$$E_1 = H_*(F_p, F_{p-1}) \cong H_*(\pi^{-1}(B_n), \pi^{-1}(B_{n-1}))$$

$$= C_*^{cell}(B) \otimes H_{*-n}(F)$$

$$d_1 : H_*(F_n, F_{n-1}) \longrightarrow H_*(F_{n-1}, F_{n-2})$$

$d_1 = \partial^{cell}(B) \otimes Id_{H_*(F)}$ using the fact that identification was canonical

$$E^2_{p,q} = H_p(B, H_q(F;R))$$

Case 2: $\pi_1(B)$ action non-trivial

$\tilde{B}$ = universal cover of B

Look at $\tilde{E} \longrightarrow \tilde{B}$. As $\tilde{B}$ is simply connected, this is untwisted.

We have a map of filtered complexes: $F_n(S_*(\tilde{E})) \longrightarrow F_n(S_*(E))$

gives a map of Spectral Sequences: $E_1 \longrightarrow E_1$

$$C_*(\tilde{B}) \otimes H_*(F,R) \rightsquigarrow C_*(\tilde{B}) \otimes_{\pi_1(B)} H_*(F,R)$$