— Pramath

- All complexes cohomological
- Double complexes, total complex

Ex: • $A^\bullet \otimes B^\bullet$ is total complex associated to natural complex arising from $\otimes$

• $A^\bullet \to M$, $B^\bullet \to N$ projective (flat) resolutions, then

$$H^{-i}(A^\bullet \otimes B^\bullet) = \mathrm{Tor}_i(M,N)$$

$A^\bullet \otimes B^\bullet \leftarrow 4^{th}$ quadrant double complex

$$0 \leftarrow M \leftarrow A^0 \leftarrow A^{-1} \leftarrow A^{-2}$$
$$0 \leftarrow N \leftarrow B^0 \leftarrow B^{-1} \leftarrow B^{-2}$$

$$\boxed{\begin{array}{c} A^\bullet \otimes N \\ \uparrow \\ \mathrm{Tot}(A^\bullet \otimes B^\bullet) \end{array}} \longrightarrow M \otimes B^\bullet$$

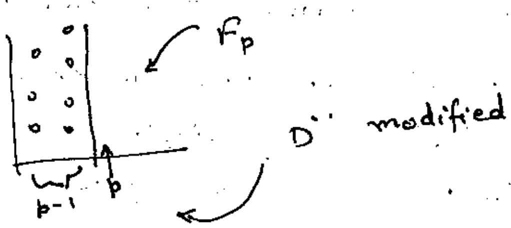
Acyclic $\Rightarrow$ Quasi isomorphism
because double complex
which is the mapping cone is exact

Revisit:

$D^{\cdot\cdot}$ — ~~…~~ 1^{st} quadrant, like commuting double complex.

• assume all columns exact

Let F_p total complex of double complex obtained by replacing first $p-1$ columns of $D^{\cdot\cdot}$ by 0's.



So F_p becomes subcomplex

Lemma: $\mathrm{Tot}(D^{\cdot\cdot}/F_p^{\cdot\cdot})$ is exact for every p.

Proof: (Because columns are exact $D^{\cdot\cdot}/F_p^{\cdot\cdot}$ has exact total complex.)

Note $D^{\cdot\cdot}/F_p^{\cdot\cdot}$ is not a sub complex, but it is a valid image.

Now look at exact seq

$$0 \longrightarrow C_p \longrightarrow D^{\cdot\cdot}/F_p^{\cdot\cdot} \longrightarrow D^{\cdot\cdot}/F_{p-1}^{\cdot\cdot} \longrightarrow 0$$

p^{th} column

$\therefore C_p$ is exact, $D^{\cdot\cdot}/F_{p-1}^{\cdot\cdot}$ is exact by induction

we will get $D^{\cdot\cdot}/F_p^{\cdot\cdot}$ is exact $\square$

$\mathrm{Tot}\,D^{\cdot}$ is exact. because for any i i^{th} entry of $\mathrm{Tot}\,D^{\cdot\cdot}$
lies completely inside D/F_{i+2} which is exact.

- $A^{\cdot}, B^{\cdot}$ complexes.
  ~~$\mathrm{Hom}_{gr}(A^{\cdot},B^{\cdot}) = \{ f : \oplus A^p \to \oplus B^p \mid f$ is zero graded $\}$~~ ✓ correct.
  Need not be chain maps

  $$\mathrm{Hom}^n(A^{\cdot},B^{\cdot}) := \mathrm{Hom}_{gr}(A^{\cdot}, B^{\cdot}[n])$$
  $$:= \prod_{j\in\mathbb{Z}} \mathrm{Hom}(A^j, B^{j+n})$$

- Define:
  $$d : \mathrm{Hom}^n(A^{\cdot},B^{\cdot}) \longrightarrow \mathrm{Hom}^{n+1}(A^{\cdot},B^{\cdot})$$
  $$(f^j : A^j \to B^{j+n})_j \longmapsto (d_B f^j + (-1)^{n+1} f^{j+1} \cdot d_A)$$
  Hom^n becomes a double complex (direct products are used this time)

- In terms of double complex, write
  $$C^{i,j} = \mathrm{Hom}(A^{-i}, B^j)$$
  So we get a 2^{nd} quadrant double complex
  - Check what explicitly the double complex maps are

$$\begin{tikzcd}
C^{-i,j+1} \arrow[r] & C^{-i+1,j+1} \\
C^{-i,j} \arrow[u, "(-1)^{j-i+1}"] \arrow[r] & C^{-i+1,j} \arrow[u, "(-1)^{j-i+2}"']
\end{tikzcd}$$

gives, $\mathrm{Hom}^{\cdot}(A^{\cdot}, \mathrm{Hom}^{\cdot}(B^{\cdot},C^{\cdot})) \xrightarrow{\sim} \mathrm{Hom}^{\cdot}(A^{\cdot}\otimes B^{\cdot}, C^{\cdot})$

- <u>Balanced Ext:</u>
  $N \to E^{\cdot}$ Injective resolution
  $P^{\cdot} \to M$ projective resolution
  consider, $\mathrm{Hom}^{\cdot}(P^{\cdot}, E^{\cdot})$
  Consider the j^{th} row
  $$\to \mathrm{Hom}(P^{-j}, E^{n+1}) \longrightarrow \mathrm{Hom}(P^{-j}, E^{n+2}) \longrightarrow$$
  Resolves $\mathrm{Hom}(P^{-j}, N)$.
  Similarly the j^{th} column
  Resolves $\mathrm{Hom}(M, E^j)$

Th^m: T left exact, $A^\cdot$ consist of T-acyclic elements.

$0 \to M \to A^\cdot$ exact resolution. then $R^i(TM) = H^i(TA^\cdot)$, $\forall i$.

Proof: 1) split $0 \to M \to A^\cdot$ into short exact sequences

2) Take injective resolution $0 \to M \to E^\cdot$

Have a homotopy unique map $\varphi: E^\cdot \leftarrow A^\cdot$ lifting 1_M

Claim: $TA^\cdot \xrightarrow[T\varphi]{} TE^\cdot$ is ~~an~~ isomorphism – quasi

(Lemma: If $M=0$, then $TA^\cdot$ is exact.)

> The point is that these two ~~#~~ isomorphisms are not the same. They differ ~~upto~~ by a sign of $(-1)^{p(p-1)/2}$!!

$\varphi: A^\cdot \to E^\cdot$ is a quasi-isomorphism

$\Rightarrow$ cone $\varphi^\cdot$ is exact

as $A^\cdot, E^\cdot$ consists of T-acyclic elements,

cone $\varphi^\cdot$ also consists of T-acyclic elements.

Lemma $\Rightarrow$ T cone $\varphi^\cdot$ is exact

But T cone $\varphi^\cdot$ = cone $T\varphi^\cdot$.

$\Rightarrow$ $T\varphi^\cdot$ is a quasi-isomorphism.

<u>Cartan-Eilenberg resolution:</u>

$M^\cdot$ bounded below complex (by 0) Write $B^p = B^p(M^\cdot)$, $Z^p = Z^p(M^\cdot)$

$\to$ Have $0 \to Z^0 \to M^0 \to B^1 \to 0$. Pick injective resolution E_Z^{0*} of Z^0 and E_B^{1*} of B^*. Get a resolution of M^0 – E_M^{0*} by Horseshoe.

$\to$ Next ~~$0 \to B^1 \to M^1 \to H^1(M^*)$~~ Again do Horse-shoe appropriately.

$0 \to B^1 \to Z^1 \to H^1(M^*) \to 0$ Again Horseshoe

$\to$ Next $0 \to Z^1 \to M^1 \to B^2 \to 0$

So we will get various injective resolutions E_M^{2*}.

So we have obtained a double complex E_M^{**} !!

By construction, we have

$$\begin{array}{ccccccc}
0 \to M^0 & \to & M^1 & \to \cdots \to & M^p & \to \cdots \\
\downarrow & & \downarrow & & \downarrow & \\
E_M^{00} & \to & E_M^{10} & \to \cdots \to & E_M^{p0} & \to \cdots \\
\downarrow & & \downarrow & & \downarrow & \\
E_M^{01} & \to & E_M^{11} & \to \cdots \to & E_M^{p1} & \to \cdots \\
\downarrow & & \downarrow & & \downarrow & \\
\vdots & & \vdots & & \vdots &
\end{array}$$

In fact

1. $Z^{pq} := Z^p(E^{*q})$
   Z^{p*} is an injective resolution of $Z^p(M^*)$.
2. $B^{pq} := B^p(E^{*q})$
   B^{p*} is an injective resolution of $B^p(M^\cdot)$
3. $R^{pq} := H^p(E^{*q})$ then
   R^{p*} is an injective resol^n of $H^p(M^\cdot)$

Grothendieck - Serre Spectral Sequence:

Th^m: $G:\bar{A} \longrightarrow \bar{B}$, $F:\bar{B} \to \bar{C}$ be left exact functors between abelian categories with enough injectives. Suppose G takes injectives to F-acyclic modules. Then for every object $M \in \bar{A}$, we have a spectral sequence -

$$E_2^M = R^q F\, R^p G M \Longrightarrow R^{p+q}(FG)M$$

Examples: 1) I, J be two ideals in noetherian commutative ring R

$\Gamma_I : R\text{-Mod} \longrightarrow R\text{-mod}$ $\qquad$ $\Gamma_J : R\text{-Mod} \longrightarrow R\text{-mod}$

$\Gamma_I(M) = \{m \in M \mid I^n m = 0 \text{ for some } n \geq 0\}$

Interpretation:

let $\mathcal{F} = \tilde{M}$ be quasi-coherent sheaf on $X = \operatorname{Spec} R$ corresponding to M. Then an element $m \in M$ can be regarded as a section of $\mathcal{F}$ over X.

$M = \Gamma(X, \mathcal{F})$.

$I^n m = 0 \Leftrightarrow \frac{m}{1} = 0$ in $M_{\mathfrak{p}}$ for an $\mathfrak{p} \not\supset I$

Conversely,

$\frac{m}{1} = 0$ in $M_{\mathfrak{p}}$ for every $\mathfrak{p} \notin V(I)$ and hence

$I^n m = 0$ for some $n \geq 1$. (Need Noetherian here)

one can check : $\Gamma_I \circ \Gamma_J = \Gamma_{I+J}$

Then $R^p\Gamma_I \cdot R^q\Gamma_J(M) \Longrightarrow R^{p+q}_{\Phi}\Gamma_{I+J}(M)$

$X \xrightarrow{f} Y$

$\mathcal{F}$

$H^p(Y, R^q f_* \mathcal{F})$

$\Downarrow$

$H^{p+q}(X, \mathcal{F})$

Leray Hirsch

To conclude, we have quasi-isomorphisms

$$\begin{tikzcd} & \mathrm{Hom}^i(M, E^\cdot) \\ & C^{\cdot\cdot} \arrow[u] \arrow[r] & \mathrm{Hom}^\cdot(P^\cdot, N) \end{tikzcd}$$

Gives balancing of Ext.

Yoneda and Ext:

- $\mathrm{Ext}^1(A,B) \longleftrightarrow$ $0 \to B \to M \to A \to 0$ exact
- Suppose (E) $0 \to B \to F^1 \to F^2 \to \dots \to F^n \to A \to 0$ is exact

Let $0 \to B \to I^\cdot$ be an injective resolution

So, by comparison we can lift identity on B (upto homotopy unique)

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & F^1 \arrow[r] \arrow[d] & F^2 \arrow[r] \arrow[d] & \dots \arrow[r] & F^n \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\theta"] & 0 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & \dots \arrow[r] & I^{n-1} \arrow[r] & I^n \arrow[r] & I^{n+1} & \end{tikzcd}$$

commutativity here means $\theta(A)$ is a cocycle

Can think of θ as a map
$\theta : A[-n] \longrightarrow I^\cdot$ map of complexes

Ex: Show 1) $Z^0(\mathrm{Hom}^\cdot(C^\cdot, D^\cdot)) = \mathrm{Hom}_{\text{complex}}(C^\cdot, D^\cdot)$

2) $H^0(\mathrm{Hom}^\cdot(C^\cdot, D^\cdot)) = \mathrm{Hom}_{\text{complex}}(C^\cdot, D^\cdot)/\text{homotopy}$

Apply to θ to get

$\theta : A[-n] \longrightarrow I^\cdot$ which is same as

$\theta : A \longrightarrow I^\cdot[n]$

Gives us an element of $Z^n \mathrm{Hom}^\cdot(A, I^\cdot)$ and have an element of $H^n(\mathrm{Hom}^\cdot(A, I^\cdot)) = \mathrm{Ext}^n(A,B)$. corresponding to E. say ξ_E

Fact: $E \longrightarrow \xi_E$ is well-defined.

Yoneda Th^m: This is infact bijective.