Weibel
Ex. 9.1.3
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$R = k[x_1, \cdots x_m]$ this is commutative

so $R^e = R \otimes R^{op} \cong R \otimes R = k[y_1 \cdots y_m\, z_1 \cdots z_m]$

what is the map $R^e \longrightarrow R$? $y_i \longmapsto x_i$, $z_i \longmapsto x_i$

Obviously $(y_1 - z_1, y_2 - z_2, \cdots, y_m - z_m) R^e \subseteq \ker (R^e \longrightarrow R)$

Other direction is also trivial.

Regular sequence – Do we need k to be a UFD or may be an integral domain atleast?
Not really sure about this.

Kozul Complex:

$x \in R$ central, $\quad K(x) = 0 \longrightarrow R \xrightarrow{x} R \longrightarrow 0$

Why do we need x central? To make this a bimodule map?

$K(x) \otimes K(y) =: K(x, y)$

$$0 \longrightarrow R \longrightarrow R \oplus R \longrightarrow R \longrightarrow 0$$

$h \longmapsto xh - yh$ Note the -ve sign

$(h, k) \longmapsto hx + ky$

Define: $\bar{x} := x_1, \cdots x_n$ all central,

$K(\bar{x}) := K(x_1) \otimes \cdots \otimes K(x_n)$

$H_q(\bar{x}, A) := H_*(K(\bar{x}) \otimes A)$

$H^q(\bar{x}, A) := H_*(\mathrm{Hom}(K(\bar{x}), A))$

$K(\bar{x})_p \cong \Lambda^p R^n$ generated by $e_{x_{i_1}} \wedge \cdots \wedge e_{x_{i_p}}$

$\downarrow$

$K(\bar{x})_{p-1} \cong \Lambda^{p-1} R^n$ $\quad \sum_{j=1}^{p} (-1)^j e_{x_{i_1}} \wedge \cdots \wedge \hat{e}_{x_{i_j}} \wedge \cdots \wedge e_{x_{i_p}}$

K - DG algebra,

$H_*(K) = \{H_p(K)\}$ is an R-graded algebra

i.e. the product structure trickles down to the homologies

$[a] \cdot [b] := [ab]$

Well defined: $da \cdot b = d(ab) \pm \underset{0}{a \cdot db} = d(ab)$

Graded commutativity: $[a] \cdot [b] = (-1)^{|a||b|} [b][a]$

$K(\bar{x})$ - GCDG Only need to check the differentials:

$$d(e_{x_1} \wedge \cdots \wedge e_{x_l} \wedge e_{y_1} \wedge \cdots \wedge e_{y_k}) =: d(e_x \wedge e_y)$$

$$= \sum_{j=1}^{l} (-1)^j x_{i_j} \hat{e}_{x_j} \wedge e_y + \sum_{j=1}^{k} (-1)^{j+l} e_x \wedge \hat{e}_{y_j}$$

$$= d(e_x) \wedge e_y + (-1)^{|x|} e_x \wedge de_y$$

If R-commutative,

$$H_p(\bar{x}, A) \otimes_R H_q(\bar{x}, B) \longrightarrow H_{p+q}(\bar{x}, A \otimes_R B)$$

$$K_p(\bar{x}) \otimes_R A \otimes_R K_q(\bar{x}) \otimes_R B \longrightarrow K_{p+q}(\bar{x}) \otimes_R A \otimes_R B$$

$$x \otimes a \otimes y \otimes b \longmapsto x \wedge y \otimes (a \otimes b)$$ should there be a sign here?

$\bar{x} \in I$, $A = R/I$, $H_*(\bar{x}, A) \overset{?}{=} \Lambda^*(A^n)$? how is the RHS independant of $\bar{x}$?

Suppose $\bar{x} = (x_1)$ $0 \to R \xrightarrow{x_1} R \to 0$ All the differentials are dying.

$\otimes A$ ↓ $0 \to A \xrightarrow{0} A \to 0$ We have trivial chain.

Given $0 \to A \to B \to C \to 0$ SES of R-modules,

$0 \to K_*(\bar{x}, A) \longrightarrow K_*(\bar{x}, B) \longrightarrow K_*(\bar{x}, C) \longrightarrow 0$, because $K(\bar{x})$ is flat

similarly for Hom.

Kunneth formula:

C_* - chain complex of R-mods, $\Rightarrow$ $0 \to H_0(x, H_q(C)) \longrightarrow H_q(K(x) \otimes_R C) \longrightarrow H_1(x, H_{q-1}(C)) \longrightarrow 0$

$x \in R$ central element

If x_1 is a unit, Pick $C = K(x_2, \dots, x_n, A)$

we get $0 \to \underset{\overset{\|}{0}}{H_0(x_1, H_q C)} \longrightarrow H_q(K(\bar{x}), A) \longrightarrow \underset{\overset{\|}{0}}{H_1(x_1, H_{q-1}(C))} \to 0$

$\Downarrow$

0

If $(x_1 \cdots x_n)$ regular seq $H_q(K(\bar{x}), A) = 0$ $q > 0$ $= A/xA$ for $q = 0$

Proof again by Kunneth: true for $n=1$.

Assume true for $(x_1, \dots, x_{n-1})$

Look at $C = K(x_1 \cdots x_{n-1}) \otimes A$, $x = x_n$

Kunneth: $0 \to H_0(x_n, H_q(C)) \longrightarrow H_q(K(x_n) \otimes C.) \to H_1(x_n, H_{q-1}(C.)) \to 0$

$q=0$, $\quad H_0(x_n, H_0(C)) = A/_{xA} = H_0(K(\bar{x}), C)$

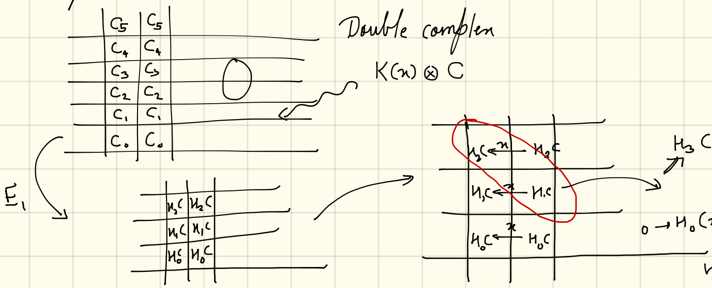
$q=1$, $\quad H_1(x_n, H_0 C) = 0$

similarly for $q>1$.

This is a really useful lemma:

Kunneth: $\quad 0 \to H_0(x, H_q(C)) \longrightarrow H_q(K(x) \otimes C) \longrightarrow H_1(x, H_{q-1}(C)) \longrightarrow 0$

Lets try a spectral sequence:



$H_3(K(x) \otimes C)$

$\wr$

$0 \to H_0(x, H_2 C.)$

$\downarrow$

$H_3(K(x) \otimes C.)$

$\downarrow$

$H_1(x, H_1 C.) \to 0$

Q. How about Hom? Is there a spectral seq?

So when we are given a regular sequence $x_1 \cdots x_n$, we get a resolution of $R/(x_1 \cdots x_n)$

$K.(\vec{x}) \longrightarrow R/_{(x_1 \cdots x_n)} \to 0$ so that

$$H_q(K(\bar{x}), A) \cong \mathrm{Tor}_q^R(R/_{(x_1 \cdots x_n)}, A)$$

$$H^q(K(\bar{x}), A) \cong \mathrm{Ext}_R^q(R/_{(x_1 \cdots x_n)}, A)$$

Back to $R = k[x_1, \dots x_n]$. Is R flat as $R \otimes R^{op}$ module? We know $R \cong R^e/(y_i - z_i)$ which is a regular sequence. So for what ideals I is R/I a flat R-module? How about projective?

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We don't need flat as an R^e module, flat as k-module is enough. See Cor 9.1.5

Polynomial rings$/k$ are free$/k$ so the Kozul resolution is a good resolution for computing $HH_*(R, M)$

$$H_*(R, M) \cong H_*(K(y_i - z_i), M) \cong \mathrm{Tor}_*^{R^e}(R^e/(y_i - z_i), M)$$

When $M = R$,

all the differentials in the Kozul complex die, hence we get

$$H_*(R, R) \cong H_*(K(y_i - z_i), R^e/(y_i - z_i)) = \Lambda^*(R^e)$$