Examples of non-degenerate perturbation theory:

Simple harmonic oscillator:

$$H_0 = \frac{1}{2} m\omega^2 x^2 + \frac{\hat{p}^2}{2m}$$

Assume the spring constant is changed slightly

$$V = \frac{1}{2}\varepsilon m\omega^2 x^2$$

$$\omega \longmapsto \omega\sqrt{1+\varepsilon}$$

$$|0\rangle = |0^{(0)}\rangle + \sum_{k\neq 0} |k^{(0)}\rangle \frac{V_{k0}}{E_0^{(0)} - E_0^{(k)}} + \dots$$

$$\Delta_0 = V_{00} + \sum_{k\neq 0} \frac{|V_{k0}|^2}{E_0^{(0)} - E_k^{(0)}}$$

$$V_{00} = \frac{\epsilon m\omega^2}{2}\langle 0^{(0)}|x^2|0^{(0)}\rangle = \frac{\epsilon\hbar\omega}{4}$$ all other V_{K0} vanish

$$V_{20} = \frac{\epsilon m\omega^2}{2}\langle 2^{(0)}|x^2|0^{(0)}\rangle = \frac{\epsilon\hbar\omega}{\sqrt{2}}$$

$$E_0^{(0)} - E_2^{(0)} = \frac{1}{2}\hbar\omega - \frac{5}{2}\hbar\omega = -2\hbar\omega$$

$$|0\rangle = |0^{(0)}\rangle - \frac{\epsilon}{4\sqrt{2}}|2^{(0)}\rangle + O(\epsilon^2)$$ there is no $|1\rangle$ here because of parity.

$$\Delta_0 = E_0 - E_0^{(0)} = \hbar\omega\left[\frac{\epsilon}{4} - \frac{\epsilon^2}{16} + O(\epsilon^3)\right]$$

- $$\hbar\omega \longmapsto \frac{\hbar\omega}{2}\sqrt{1+\epsilon} = \frac{\hbar\omega}{2}\left[1 + \frac{\epsilon}{2} - \frac{\epsilon^2}{8} + \dots\right]$$

- $$\langle x|0^{(0)}\rangle = \frac{1}{\pi^{1/4}\sqrt{x_0}} e^{-x^2/2x_0^2} \qquad x_0 = \sqrt{\frac{\hbar}{m\omega}}$$

  Introducing the perturbation $x_0 \longmapsto \frac{x_0}{(1+\epsilon)^{1/4}}$

$$\langle x|0^{(0)}\rangle \longrightarrow \frac{1}{\pi^{1/4}\sqrt{x_0}}(1+\epsilon)^{1/8}\exp\left[\frac{-x^2}{2x_0}\cdot(1+\epsilon)^{1/2}\right]$$

Quadratic Stark Effect:

hydrogen atom in an electric field

$$H = \frac{\hat{p}^2}{2m} + V_0(r) \qquad V = -e|\vec{E}|z$$

$$\Delta_K = -e|\vec{E}|z_{KK} + e^2|\vec{E}|^2 \sum \frac{|z_{Kj}|^2}{E_K^{(0)} - E_j^{(0)}}$$

Sheaves on a base / basis (for topology) § 2.7

X - top space $\quad \mathcal{B} = \{B_\alpha\}$ basis for topology on X.

$\mathrm{Open}_{\mathcal{B}}(X)$ – category poset

Full subcategory of $\mathrm{Open}(X)$.

$F_{\mathcal{B}} : \mathrm{Open}_{\mathcal{B}} X \hookrightarrow \mathrm{Open}\, X \xrightarrow{F} \mathrm{Sets} \quad \leftarrow$ Presheaf on $\mathcal{B}$

similarly Sheaf on $\mathcal{B}$

Prop: Sheaves on $X \xrightarrow{R}$ Sheaves on $\mathcal{B}$ is an equivalence of categories.

[i.e. I) R is fully faithful and essential. or equivalently

II) $\exists\, S$ in the other direction which is a "quasi"-inverse.]

Cor: (Gluing) $X = \bigcup U_\alpha$, F_α a sheaf on U_α satisfy:

1) $\exists\, \phi_{\alpha,\beta} : F_\alpha|_{\alpha\cap\beta} \xrightarrow{\cong} F_\beta|_{\alpha\cap\beta} \quad \forall \alpha,\beta$

2) $\phi_{\beta,\gamma} \circ \phi_{\alpha,\beta} = \phi_{\alpha,\gamma} \qquad \phi_{\alpha,\alpha} = 1$ (cocycle)

Then there is a sheaf F over X such that $F|_\alpha \cong F_\alpha$ unique upto a canonical isomorphism.

(This is saying that $U \longrightarrow \{\text{sheaves on } U\}$ is a stack.)

§ 3. Towards affine schemes:

Def^n: A ringed space (X, R): X - topological space

R – sheaf of rings

$\mathrm{Spec}\, A = \{\text{prime ideals in } A\}$, $\mathfrak{p} \subsetneq A$ $\quad \rightsquigarrow$ Think of A as "ring of regular functions" on $\mathrm{Spec}\, A$.

Value of $a \in A$ at the point $\mathfrak{p}$ is

$\mathfrak{p} \longmapsto a + \mathfrak{p} \in A/\mathfrak{p} \subseteq \underbrace{\kappa(\mathfrak{p})}$

field of fractions of $A/\mathfrak{p}$

eg: $\cdot\ \mathbb{A}^1_A := \mathrm{Spec}\, A[x] \qquad A = k$ algebraically closed field.

$\quad\quad = \{(x-a) \mid a \in k\} \amalg \{(0)\}$

$\quad\quad = (k + \text{a point})$ as a set.

(generic point)

$\cdot\ \mathrm{Spec}\, \mathbb{Z} = \{(2), (3), (5), \dots\} \amalg \{(0)\}$

$\cdot$ Convention: $\mathrm{Spec}\, \mathbb{Z} = \phi$

- $\mathbb{A}^1_{\mathbb{R}} = \operatorname{Spec} \mathbb{R}[x] = \{(x-a) \mid a \in \mathbb{R}\} \amalg \{x^2+ax+b \mid a^2 < 4b\} \amalg \{0\}$

$$\mathbb{R}[x]/x-a \xrightarrow{\cong} \mathbb{R} \qquad\qquad \mathbb{R}[x]/x^2+ax+b \xrightarrow{\cong} \mathbb{C}$$
$$x \longmapsto a \qquad\qquad x \longmapsto \text{a root of } x^2+ax+b$$

- $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \dots, x_n]$

Weak nullstellensatz: If $k \to K$ is an extension and K is finitely generated as a k-algebra, then K is a finite dim'l$/k$.

$\Rightarrow$ For $\mathfrak{m}$ maximal in $k[x_1, \dots, x_n]$,
$k[x_1, \dots, x_n]/\mathfrak{m}$ is a finite extension of k

$\Rightarrow$ If $k = \bar{k}$, $k[x_1, \dots, x_n]/\mathfrak{m} \xrightarrow{\cong} k$

$\Rightarrow \mathfrak{m} = (x_1-a_1, x_2-a_2, \dots, x_n-a_n)$ for $a_i \in k$.

Other ideals: (0), $(\underset{\text{irreducible}}{f(x_1, \dots, x_n)})$

- A a ring, I an ideal of A then

$$\left\{\begin{matrix}\text{prime ideals of}\\ A/I\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{prime ideals of } A\\ \text{containing } I\end{matrix}\right\}$$

$\Longrightarrow \operatorname{Spec} A/I \subseteq \operatorname{Spec} A$

These are going to be the closed subsets of Spec A.

- Let $S \subseteq A$ be a multiplicative subset. We have a canonical map:

$$A \longrightarrow S^{-1}A$$

$$\left\{\begin{matrix}\text{prime ideals in}\\ S^{-1}A\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{prime ideals } I \text{ such that}\\ I \cap S = \{0\}\end{matrix}\right\}$$

$$\operatorname{Spec} S^{-1}A \subseteq \operatorname{Spec} A$$