— Tom Mainiero

- X - 3 manifold
- G - compact, 1-connected Lie group
- $\mathrm{conn}_G(X) = \{ (P, \Theta) : \begin{smallmatrix} P \\ \downarrow \\ G \end{smallmatrix}$ principal G-bundle, $\Theta \in \Omega^1(P, \mathfrak{g}) \}$
  trivial

Chern-Simons action:

$$\mathrm{CS} : \mathrm{Conn}_G(X) \longrightarrow \mathbb{R}/\mathbb{Z}$$
$$: (P, \Theta) \longmapsto \int_{X^3} s^* \alpha_{cs}(\Theta)$$

- $\alpha_{cs}(\Theta) = \langle \Theta \wedge F_\Theta \rangle - \frac{1}{6} \langle \Theta \wedge [\Theta, \Theta] \rangle \in \Omega^3(P)$
- $s : X \longrightarrow P$ is a section. why does this exist?

Rmk: α_{cs} depends on choice of $\langle , \rangle$ - ad invariant pairing on $\mathfrak{g}$

$$\updownarrow$$

$$\omega \in H^4(BG; \mathbb{Z}) \quad \text{(level of theory)}$$

In this talk $G = SU(n)$

$$H^4(BG; \mathbb{Z}) \cong \underset{\substack{\cup \\ k}}{\mathbb{Z}} \longleftrightarrow k \,\mathrm{Tr}(ab) = \langle a, b \rangle$$

- CS is invariant under gauge transformations
- Critical (CS) = flat bundles
- when $\partial X^3 \neq \phi$: CS is invariant under g.t. that restrict to $1_{P|_{\partial X}}$ on ∂X.

Claim: Path integral Quantization $\rightsquigarrow$ Canonical Quantization / Holomorphic Quantization

$$\downarrow$$

$$Z : \mathrm{Bord}_{\langle 2,3 \rangle} \longrightarrow \mathrm{Vect}_{\mathbb{C}}$$

1) $Z(X^3 \text{ closed}) \in \mathbb{C}$

2) $Z(\partial X) = \mathcal{H}_{\partial X} \in \mathrm{Vect}_{\mathbb{C}}$

Ref: Quantum field theory and the Jones Polynomial — Witten

Knot Invariants: (eg: Jones Polynomial $G = SU(2)$, $X = S^3$, $L \subseteq S^3$)

Path integral for X with no boundary:

$$Z(X) = \int_{\mathcal{C}_X} e^{ik\, CS(\theta)} \cdot D\theta$$

Groupoid of connections on X.

Rmk:

Measure is not rigorosly defined.

Step 1: Fadeev-Popov Method:

$$\int_{\mathcal{C}_X} e^{ik\, CS(\theta)} \cdot D\theta := \int_{Conn(X)} D\theta \cdot D\Xi \cdot e^{ik\, CS(\theta) + S[g, c, \bar{c}]}$$

$$S: \pi_0 \mathcal{C}_X \longrightarrow Conn_G(X) \quad \text{(Gauge Slice)}$$

$\theta = d + A$

$\{ d + A : d^* A = 0 \}$ ⇝ integrate over these connections

defines im S

$*d*$

Step 2: ∞-dim Stationary phase Approximation

Finite dimensions on $\mathbb{R}^n$, f a Morse function $\mathbb{R}^n \to \mathbb{R}$

signature of Hession of y

$$\int_{\mathbb{R}^n} dx_1 \cdots dx_n \, g(x) \cdot e^{ik f(x)} \underset{k \to \infty}{\sim} \sum_{y \in \mathrm{Crit}(y)} g(y) \cdot e^{ik f(y)} \cdot \frac{c_n \cdot \exp\{ \frac{i\pi}{4} \cdot \underbrace{\sigma_y(f)} \}}{k^{n/2} \sqrt{|\det(\mathrm{Hess}_f(y))|}}$$

For our situation

$$f = CS + S[\ ]$$

$$\text{Critical points} = \text{Critical points of } CS / \text{Gauge group}$$

$$= \text{Moduli Space of flat connections}$$

$$= \mathrm{Hom}(\pi_1(X), G)/G =: \mathcal{M}_X$$

Assume $\dim(\mathcal{M}_X) = 0$, then

$$Z(X) = \sum_{[\theta] \in \mathcal{M}_X} \mu([\theta])$$

Ray Singer Analytic Torsion

$$\mu([\theta]) = e^{ik\, CS([\theta])} RS([\theta]) \cdot e^{i\pi/2\, \eta_y([\theta])}$$ ⇝ eta inversions

RS does not depend upon the metric, but η_g does.

To fix this with respect to some trivialization

$$\eta_g(\theta) = \eta_g(0) + \frac{C_2(G)}{\pi} \cdot CS(\theta) \qquad \text{mod } 1$$

$C_2(G)$ ↖ trace of Casimir in adjoint representation.

Atiyah: $\eta_g(0)$ dependance on metric can be replaced by dependance on a canonical 2-frame.

Rmk:

$$Z(x) = \sum e^{i\left(k + \frac{c_2(G)}{2\pi}\right)} \cdot RS(\theta) \cdot (\text{phase})$$

phase: Given by choice of 2-framing

X^3 with boundary

Rmk: with ∂ $\quad e^{ik\, CS[\theta]} \notin \mathbb{C}$

depends on choice of trivialization $\quad S: X \to P$

but in an understood way

$$e^{ik\, CS[\theta]} \in \mathcal{L}_{[\theta|_{\partial X}]}$$

$\mathcal{L}$: Hermitian line

$[\theta|_{\partial X}]$: upto gauge transformations

$$Z_{\partial X} = \left(\begin{array}{c} \mathcal{L} \\ \downarrow \\ \mathcal{M}_{\partial X} \end{array} \right)$$

$\mathcal{L}$: some connections

$\mathcal{M}_{\partial X}$: flat connections

Holomorphic Quantization:

Classical

(M, J, ω) - Kahler $\quad \rightsquigarrow \quad \mathcal{H} = \Gamma_{hol}\left(\begin{array}{c} \mathcal{L} \\ \downarrow \\ M \end{array} \right) \qquad c_1(\mathcal{L}) = [\omega] \in H^2(M; \mathbb{Z})$

$X = Y \times [0,1]$

$\dim \mathcal{M}_{\partial X} < \infty$

Jones polynomial:

• $L \subseteq X$, $L = \bigsqcup C_i$ knots

R_i = representation of G associated to C_i

Wilson-Line Loop:

$$W_{R_i}(C_i) : \mathrm{Conn}(M) \longrightarrow \mathbb{C}$$
$$\Theta \longmapsto \mathrm{Tr}_{R_i}[\mathrm{hol}_{\Theta}(C_i)]$$

$$Z(X, L) := \int_{\mathrm{Conn}(X,L)} D\Theta \cdot e^{ik\, CS[\Theta]} \prod_{i=1}^{r} W_{R_i}(C_i)$$

Claim:

$X = S^3$, $G = SU(2)$, $L = C$, R = fund rep of $SU(2)$

$$Z(S^3, C) \xleftrightarrow[k]{\text{for any}} \text{Jones poly of } C \longleftrightarrow q = q(k)$$

Dan Freed:

$$S = \int dt\, (KE - PE)$$
$$\dim S = [E \cdot T] = mL^2/T$$

Feynman : $\int e^{iS(\phi)} D\phi$ makes no sense hence.

$$\xrightarrow[\text{Correction}]{} \quad \int e^{iS(\phi)/\hbar} D\phi$$

as $\hbar \to 0$ this integral has non zero values only at critical points of S.

Critical points of S = Classical solutions

$$\int \underset{\hbar \to 0}{\sim} \sum_{\phi\ \mathrm{crit}} \frac{e^{iS(\phi_{crit})}}{\sqrt{\det S''(\phi_{crit})}} \left\{ 1 + \hbar(\cdot\cdot) + \hbar^2(\cdot\cdot) + \dots \right\}$$

Chern Simons:

$$\int e^{ik\, CS(\Theta)} d\Theta \qquad k \sim 1/\hbar \qquad \text{we look at } k \to \infty$$