Duality :

Def: $(\mathcal{C}, \otimes, 1)$ $\quad \cdot \ \otimes : C \times C \to C$

$\cdot \ 1 \in C \qquad 1 \otimes x = x \otimes 1 = x$

$\cdot$ Associativity,

ex: $(\mathrm{Bord}_n, \amalg, \phi)$, $(\mathrm{Spectra}, \wedge, S)$,

Braiding : $\beta_{x,y}$ $\quad x \otimes y \xrightarrow[\cong]{\beta_{x,y}} y \otimes x$

Commutative : $\quad \beta_{y,x} \circ \beta_{x,y} = 1_{x \otimes y}$

Def^n: $X \in \mathcal{C}$ has a left dual if $\exists y$ s.t.

$ev : y \otimes x \longrightarrow 1$

$coev : 1 \longrightarrow x \otimes y$

$$\begin{tikzcd}
 & x \otimes y \otimes x \arrow[dr, "1 \otimes ev"] & \\
x \arrow[ur, "coev \otimes 1"] \arrow[rr] & & x
\end{tikzcd}
\qquad
\begin{tikzcd}
 & y \otimes x \otimes y \arrow[dr, "ev \otimes 1"] & \\
y \arrow[ur, "1 \otimes coev"] \arrow[rr] & & y
\end{tikzcd}$$

Commute

Adjunction :

$\mathcal{C}(a \otimes x, b) \simeq \mathcal{C}(a, b \otimes y) \qquad y$ left dual of $x \qquad \mathcal{C}(a, x \otimes b) \cong \mathcal{C}(y \otimes a, b)$

y' another dual $\Rightarrow \mathcal{C}(a, y) \cong \mathcal{C}(a \otimes x, 1) \cong \mathcal{C}(a, y')$

Yoneda $\Rightarrow y \cong y'$.

Prop. In symmetric monoidal category, left dualizable $\Leftrightarrow$ right dualizable.

ex:
- in R-mod $\quad$ dualizable $\Leftrightarrow$ f.g. projective
- in Ch_R $\quad$ dualizable $\Leftrightarrow$ f.g. projective for each
- in Spectra $\quad$ dualizable $\Leftrightarrow$ ~~finite~~ ?
- in Bord_n $\quad$ every object is dualizable

Adjunctions:

$(F, G) : \mathcal{C} \longrightarrow \mathcal{D}$

$\psi(1_G) =: \varepsilon : FG \longrightarrow 1$

Bicategory : 2-morphisms

- Category of Categories
- X-top space ob : points of X

  mor : paths in X

  2-mor : homotopies between paths.

Def^n: B bicategory consists of

- objects
- cat $B(x,y)$
- horizontal composition $* = *_{x,y,z} : B(y,z) \times B(x,y) \longrightarrow B(x,z)$

Boring

Handle with caution

& courage.