PROBLEM SET 13

PART 1 - NOWHERE DIFFERENTIABLE FUNCTION

In this problem set we'll construct a function $f_\infty(x)$ which is continuous everywhere but differentiable nowhere(!) using only the techniques you've learnt in this course.

Q.1. Define the function $\{x\}$ as

$$\{x\} = \text{the distance of } x \text{ from the nearest integer}$$

Let n denote a positive integer. Define the function

$$\begin{aligned} f_n(x) &= \{x\} + \frac{\{2x\}}{2} + \cdots + \frac{\{2^n x\}}{2^n} \\ &= \sum_{i=0}^{n} \frac{\{2^i x\}}{2^i} \end{aligned}$$

Draw the graph of the functions $\{x\}$, $\frac{\{2x\}}{2}$, $\frac{\{2^n x\}}{2^n}$, and $f_1(x)$.

Q.2. Let $x \in [0, 1]$. Determine the relation between

(1) $\{x\}$ and $\{x + 1\}$

(2) $\{x\}$ and $\{1 - x\}$

(3) $\{x\}$ and $\{x + 1/2\}$

(4) $\{x\}$ and $\{2x\}$

Q.3. Using properties of continuous functions argue that $\{x\}$ is a continuous function, and hence so is $f_n(x)$.

Q.4. What are the points at which $f_n(x)$ is not differentiable? Assuming that it makes sense to take limits of functions, what do you think are the points at which the function $\lim\limits_{n\to\infty} f_n(x)$ is not differentiable. (Proof not needed.)

Q.5. Using your favorite test for series convergence, show that for every real number x the series $\sum\limits_{i=0}^{\infty} \frac{\{2^i x\}}{2^i}$ converges.

Hence we can **define** a new function

$$f_\infty(x) = \sum_{i=0}^{\infty} \frac{\{2^i x\}}{2^i}$$

Q.6. Show that $|f_\infty(x) - f_n(x)| \leq 2^{-n}$ for all x.

Q.7. Using the triangle inequality

$$\begin{aligned} |f_\infty(x) - f_\infty(a)| \leq\ & |f_\infty(x) - f_n(x)| \\ & + |f_n(x) - f_n(a)| \\ & + |f_n(a) - f_\infty(a)| \end{aligned}$$

and the $\epsilon - \delta$ definition of continuity, prove that $f_\infty(x)$ is a continuous function.

In order to show that $f_\infty(x)$ is not differentiable anywhere it's helpful to use binary expansions of numbers.

Let x be a real number with binary expansion $x = m + 0.a_1a_2a_3\dots$ where m is an integer and each $a_i = 0$ or 1.

Q.8. Show that the number whose binary expansion consists of all one's $0.111\dots1\dots$ is equal to 1.* What is the corresponding statement for decimal expansions?

Because of this we can assume that there are no *trailing 1's* in the binary expansion of any number.

Q.9. (1) Find $\{x\}$ in terms of the binary expansion of x.

(2) Find $\dfrac{\{2^i x\}}{2^i}$ in terms of the binary expansion of x.

(3) If x has a finite binary expansion $x = m + 0.a_1a_2\cdots a_n$, what is $f_\infty(x)$?

Let b_n be the position of the n^{th} zero after the decimal point in the binary expansion of x. For example,

if $x = 1011.1010101\dots$ then $b_n = 2n$,
if $x = 0.1011011011\dots$ then $b_n = 3n - 1$, etc.

Q.10. (1) Show that

$$\frac{\{2^i(x + 2^{-b_n})\}}{2^i} - \frac{\{2^i x\}}{2^i} = \begin{cases} -2^{-b_n} & \text{if } i < b_n - 1 \\ 0 & \text{if } i \geq b_n \end{cases}$$

(2) Show that $\dfrac{f_\infty(x + 2^{-b_n}) - f_\infty(x)}{2^{-b_n}} < -(b_n - 1) + 2^{-b_n+1}$.

Q.11. Show that $\lim\limits_{n\to\infty} \dfrac{f_\infty(x + 2^{-b_n}) - f_\infty(x)}{2^{-b_n}}$ does not exist. Conclude that $f(x)$ is not differentiable at x.

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*Hint: This is a geometric series.

Part 2 - Trigonometry and Complex Numbers

Complex numbers are numbers of the form $a+i.b$ where a, b are real numbers and $i^2 = -1$. Complex numbers are very useful in calculus, especially for finding integrals, because of the following **Euler's identity**

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Q.12. Verify Euler's identity using the Taylor series for e^x, $\sin x$ and $\cos x$.

Q.13. Using the fact that $e^{i(a+b)} = e^{ia}.e^{ib}$ compute the formulae for $\sin(a+b)$, $\cos(a+b)$, $\sin 2x$, $\cos 2x$, $\sin 3x$, and $\cos 3x$.

Q.14. Show that

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad\qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

(Compare these with the formulae for $\sinh x$ and $\cosh x$.)

Q.15. Use these formulae to compute the following integrals[†]

(1) $e^{ax}\sin bx$

(2) $e^{ax}\cos bx$

(3) $\cos^2 x$

(4) $\sin x \cos 4x$

[†] Hint: You might need to use the identity $\dfrac{1}{a+ib} = \dfrac{a-ib}{a^2+b^2}$