§ 4.1 Structure sheaf of affine scheme

A – commutative ring.

Def^n: $\mathcal{O}_{\mathrm{Spec}\,A}(D(f)) := A_f$ for $f \in A$

$= S^{-1}A$ for $S = \{1, f, f^2, \dots\}$

$= A[f^{-1}]$

$= S^{-1}A$ for $S = \{g \mid f^n \in (g)$ for some $n\}$

· Presheaf: $D(f) \subseteq D(g) \rightsquigarrow A_g \longrightarrow A_f$

$(\Rightarrow V(f) \supseteq V(g)) \Rightarrow f$ inverted $\Rightarrow g$ inverted.

Suppose $D(f) = \bigcup_{i \in I} D(f_i)$

$$\Leftrightarrow V(f) = \bigcap_{i \in I} V(f_i) = V((f_i)_{i \in I})$$

$\Leftrightarrow$ 1) $f^k \in (f_{i_1}, \dots, f_{i_r})$ for some $i_1, i_2, \dots, i_r$.

2) $\sqrt{(f_i,\ i \in I)} = \sqrt{(f)}$ Walog assume $k = 1$

· Identity: Suppose $\alpha \in \mathcal{O}(D(f))$ is such that $\alpha|_{f_i} = 0 \quad \forall i$

$\Rightarrow \alpha \cdot f_i^{k_i} = 0 \quad \forall i$

$\Rightarrow (f)^{\prod k_i} \cdot \alpha = 0$

$\Rightarrow \alpha = 0$ in $\mathcal{O}(D_f)$

· Gluing: Given $\alpha_i \in \mathcal{O}(D_{f_i})$ compatible. We can replace A by A_f and assume $f = 1$.

Quasi-compactness tells us: $A = D_{f_1} \cup \dots \cup D_{f_r}$ for some r.

And compatibility means, $(f_i f_j)^N (f_j \alpha_i - \alpha_j f_i) = 0$

Let $\quad b_i = \alpha_i f_i^N \quad h_i = f_i^{N+1}$

Then the above statement becomes: $b_j h_i = b_i h_j \quad \forall i, j$

$D(f_i) = D(h_i) \Rightarrow 1 = \sum h_i r_i$ for some r_i

Now let $\boxed{\alpha = \sum b_i r_i}$, $(\alpha - b_1) \cdot h_1 = (\sum b_i r_i h_1 - b_1 h_1) = (\sum b_1 r_i h_i - b_1 h_1) = 0$

This then is the required element!

Defⁿ: $\mathcal{O}_{\operatorname{Spec} A}$ extended to sheaf on Spec A is called structure sheaf on Spec A.

Def: M is an A-module,

$$\widetilde{M}(D(f)) := M_f := M \otimes_A A_f$$

is a sheaf of $\mathcal{O}_{\operatorname{Spec} A}$ modules.

Remark: For an arbitrary set $U \subseteq \operatorname{Spec} A$,

in general $\mathcal{O}_{\operatorname{Spec} A}(U) \neq \begin{Bmatrix} a \in A \text{ nowhere} \\ \text{vanishing on } U \end{Bmatrix}^{-1} A$

§ 4.3 Schemes:

Def: An isomorphism of ringed spaces $(X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ consists of

a homeomorphism $\pi: X \longrightarrow Y$

sheaf isomorphisms $\pi_* \mathcal{O}_X \longrightarrow \mathcal{O}_Y$

Defⁿ: An affine scheme is a ringed space isomorphic to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ for some ring A.

Def: A scheme is a ringed space $(X, \mathcal{O}_X)$ such that every point in X has a neighborhood U such that $(U, \mathcal{O}_X|_U)$ is an affine scheme.

- $(D(f), \mathcal{O}_{\operatorname{Spec} A}|_{D(f)}) \cong (\operatorname{Spec} A_f, \mathcal{O}_{\operatorname{Spec} A_f})$ Check

- $\mathcal{O}_X$ scheme, $U \subseteq X$ open $\Rightarrow$ $(U, \mathcal{O}_X|_U)$ is a scheme

  open subscheme

  if $(U, \mathcal{O}_X|_U)$ is affine, call it open affine.

- $\operatorname{Spec} A_1 \amalg \operatorname{Spec} A_2 \amalg \cdots \amalg \operatorname{Spec} A_n \cong \operatorname{Spec}(A_1 \times \cdots \times A_n)$

- Infinite disjoint unions of affine schemes is a scheme, but being non-quasi compact is not an affine scheme.

Let us try to understand modules over $\mathcal{O}_{\mathrm{Spec}\,A}$:

$M \in A\text{-mod} \Rightarrow M \underset{A}{\otimes} \mathcal{O}_{\mathrm{Spec}\,A}$ module over $\mathcal{O}_{\mathrm{Spec}\,A}$

So we have a functor: $A\text{-mod} \longrightarrow \mathcal{O}_{\mathrm{Spec}\,A}\text{-mod}$ $\sim$ ringed spaces with base $\mathrm{Spec}\,A$

Is this exact? Yes because localisation is an exact functor.

If M is free A-module, so would be $\tilde{M}$. ?

What about projectivity? As localization is exact we certainly can lift maps on the level of stalks, but how to ensure compatibility?

· Examples: 1) If $M \subseteq A$ is an ideal,

$$\tilde{M}(D_f) = M \otimes A_f = M[f^{-1}] = M[x] / (xf-1)$$

$$\tilde{M}_{\mathfrak{p}} = M_{\mathfrak{p}} = \begin{cases} \text{subset of } A_{\mathfrak{p}} & \text{if } M \subseteq \mathfrak{p} \\ ?? & \text{else} \end{cases}$$

2) $M = A[x]/g(x)$

$$\tilde{M}(D_f) = A_f[x]/g(x)$$

$$\tilde{M}_{\mathfrak{p}} = A_{\mathfrak{p}}[x]/g(x)$$

3) $M =$ arbitrary R-algebra.

We then have maps
$$\begin{tikzcd} \mathcal{O}_{\mathrm{Spec}\,M} \arrow[r] & \mathcal{O}_{\mathrm{Spec}\,A} \\ ? \arrow[u] \arrow[r] & \tilde{M} \arrow[u] \end{tikzcd}$$

4) $M =$ field of fractions of A

$$\tilde{M}(D_f) = M \otimes A_f = M$$

This gives us the locally constant presheaf M over $\mathcal{O}_{\mathrm{Spec}\,A}$

Examples from the book:

- Quadric Surface : $\mathrm{Spec}\,A$, $A = k[w,x,y,z]/(wz-xy)$
- Two planes meeting at a point.
- $\mathbb{A}^n_k - \{0\}$
- Infinite disjoint union of schemes
- $\mathrm{Spec}\,\bar{\mathbb{Q}} \longrightarrow \mathrm{Spec}\,\mathbb{Q}$.