Operads

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These are notes from Peter May's "Homology of Iterated Loop spaces"

Contents

1. Definitions

The axioms of **monoid** multiplication can be described as:

$$
\begin{tikzcd} X \arrow[r, bend left, "\bullet 1"] \arrow[r, bend right, "id"'] & X \end{tikzcd}
\qquad
\begin{tikzcd} X \times X \arrow[rr] \arrow[drr] & & X \\ & & X \times X \arrow[u] \end{tikzcd}
\qquad
\begin{tikzcd} X \times X \times X \arrow[r] \arrow[d] & X \times X \arrow[d] \\ X \times X \arrow[r] & X \end{tikzcd}
\tag{1.1}
$$

For an **H-space** we only require these diagrams to commute up to homotopy. In the **loop space** ΩX we have further coherence relations. An A_∞ **operad** is a collection of diagrams generalizing (1.1) describing multiplication in ΩX. Generalizing this, an *operad* is a collection of *diagrams* which can act on objects.

Q: But why?
A: May's Recognition principle in Topology, Kadeishvili's theorem in A_∞ algebras, Fukaya categories, Deformation quantization, Vertex operator algebras.

Definition 1.1. An **operad** $\mathcal{O}$ is a collection of objects $\mathcal{O}_i$ along with the following data and restrictions:
(we think of every *point* of $\mathcal{O}_i$ as an operator with i inputs and 1 output)

(1) *(identity)* $\mathcal{O}_0 = *$ (terminal object in *the* category)
(this operator produces an identity element out of nothing)

(2) *(composition)* A collection of morphisms,

$$
\begin{aligned}
\mathcal{O}_i \times \mathcal{O}_{j_1} \times \cdots \times \mathcal{O}_{j_i} &\to \mathcal{O}_{j_1 + \cdots + j_i} \\
(\alpha_0, \alpha_1, \cdots, \alpha_i) &\mapsto \alpha_0(\alpha_1, \cdots, \alpha_i)
\end{aligned}
$$

(one can combine smaller operators into one big operator)

(3) *(cylinder)* There exists an element $1 \in \mathcal{O}_1$ satisfying,

$$1(\alpha) = \alpha$$
$$\alpha(1, \cdots, 1) = \alpha$$

for any $\alpha \in \mathcal{O}_i$
(the cylinder operator)

(4) *(symmetry)* The permutation groups Σ_i have a *right* action on $\mathcal{O}_i$ satisfying,

$$\alpha_0(\alpha_1\sigma_1, \cdots, \alpha_i\sigma_i) = (\alpha_0(\alpha_1, \cdots, \alpha_i))(\sigma_{j_1} \circ \cdots \circ \sigma_{j_i})$$

where $\sigma_i \in \Sigma_i$ and $\sigma_1 \circ \cdots \circ \sigma_i$ is the concatenated product in $\Sigma_{j_1+\cdots+j_i}$
(this action should be thought of as permuting the input)

Definition 1.2. The operad $\mathcal{O}$ acts on an object X if for each $\alpha \in \mathcal{O}_i$ there exist a morphism

$$\alpha : X^{\times i} \to X$$

satisfying the obvious relations. In this case we say that X is a $\mathcal{O}$ algebra.

A lot of jargon from ring theory is carried over verbatim to operad theory.

2. Topological Examples

From now on we will assume that the ambient category $\mathcal{C}$ is the category of based topological spaces. An unbased space X will be identified with the based space X_+.

2.1. Endomorphism operad.

The most canonical operad that acts on a space X is $\mathcal{E}nd(X)$.

$$\mathcal{E}nd(X)_i = \{\text{continuous maps } f : X^{\times i} \to X\}$$
$$(\alpha_0(\alpha_1, \cdots, \alpha_i))(x_1, \cdots, x_{j_1+\cdots+j_i}) = \alpha_0(\alpha_1(x_1, \cdots, x_{i_1}), \cdots, \alpha_i(x_1, \cdots, x_{j_{i-1}+\cdots+j_i}))$$
$$(\alpha\sigma)(x_1, \cdots, x_i) = \alpha(\sigma(x_1, \cdots, x_i))$$

Any action of an operad $\mathcal{C}$ on X comes from a map $\mathcal{C} \to \mathcal{E}nd(X)$.

2.2. Commutative operad.

To rephrase (1.1) we define the operad $\mathcal{C}omm$ as

$$\mathcal{C}omm_i = * \text{ for all } i$$

If X is an algebra of $\mathcal{C}omm$ then X is a commutative monoid as,

$$id = \mathcal{C}omm_0$$
$$\circ = \mathcal{C}omm_2 : X \times X \to X$$
$$\circ(x, y) = (\circ(1, 2))(x, y) = \circ((1, 2)(x, y)) = \circ(y, x)$$
$$\cdots$$

A more fancy way of describing the same operad is by defining $\mathcal{C}omm_i$ to be the diffeomorphism classes of riemann surfaces with n incoming boundaries and 1 outgoing boundary.

2.3. Associative operad. Trying to relax (1.1) to define a monoid which is associative but not necessarily commutative is tricky.

$$\mathcal{A}ssoc_i = \Sigma_i$$
$$\alpha_0(\alpha_1, \cdots, \alpha_i) = \alpha_{\alpha_0(1)} \circ \cdots \circ \alpha_{\alpha_0(i)}$$
$$(\alpha)\sigma = \alpha.\sigma$$

For an $\mathcal{A}ssoc$ algebra X, $\mathcal{A}ssoc_i(x_1, \cdots, x_i)$ is nothing but the product of the elements $x_1, \cdots, x_i$. More explicitly if $m = (1, 2) \in \mathcal{A}ssoc_2$ then

$$m(1, m) = (1, 2, 3) = m(m, 1)$$

2.4. Little k cubes operad. Define an operad $\mathcal{C}ube^k$ whose i^{th} space is given by the space of

$$\mathcal{C}ube^k_i = \{\text{rectilinear embeddings } ((-1, 1)^k)^{\times n} \to (-1, 1)^k\}$$

by a **rectilinear** map $(-1, 1)^k \to (-1, 1)^k$ we mean a map of the form $f(\vec{x}) = A\vec{x} + \vec{b}$ where A is a positive scalar matrix (so we are allowed to scale the cubes while keeping the sides parallel to the axes but not rotate or flip them).

This is a **sub-operad** of $End((-1, 1))$ so the composition and the group action are the same as that for $\mathcal{E}nd$.

A map $(-1, 1)^k \to (-1, 1)^k$ can be naturally extended to $(-1, 1)^{k+1} \to (-1, 1)^{k+1}$ by adding a $[1]$ to A and a $[0]$ to $\vec{b}$ (this is like stretching the cubes in one direction to increase their dimension). It is easy to check that this gives us a map of operads

$$\mathcal{C}ube^k \to \mathcal{C}ube^{k+1}$$

hence any $\mathcal{C}ube^{k+1}$ algebra is also a $\mathcal{C}ube^k$ algebra.

2.5. Loop spaces. The operad $\mathcal{C}ube^1$ acts on ΩX naturally via concatenation of loops! So this complicated monster is the generalization of the monoid axioms. Similarly the operads $\mathcal{C}ube^k$ operad acts an $\Omega^k X$ which brings us to our first recognition principle:

Theorem 2.1. *A connected (as a topological space) $\mathcal{C}ube^k$ algebra is weakly equivalent to a k loop space.*

2.6. A_∞ operads and algebras. There is a more general recognition principle than the one stated above. If $\mathcal{O}$ is an operad then so is $\pi_0\mathcal{O}$ where we replace each of the spaces $\mathcal{O}_i$ by $\pi_0\mathcal{O}_i$, for example $\pi_0\mathcal{C}ube^1 = \mathcal{A}ssoc$.

Definition 2.2. An operad $\mathcal{O}$ is called an A_∞ operad if $\pi_0\mathcal{O}$ is isomorphic to $\mathcal{A}ssoc$. An algebra over an A_∞ operad is called an A_∞ algebra.

Theorem 2.3. *A connected A_∞ algebra is weakly equivalent to a loop space.*

2.7. Isometries operad. Let $(U, \langle,\rangle)$ be an inner product space then $\mathcal{I}som$ is a suboperad of $\mathcal{E}nd(X)$ with $\mathcal{I}som_n$ being the isometric embeddings $U^{\times n} \to U$. This operad is non-trivial only when U is infinite dimensional.

The situations where this is used the most is when $U = \mathbb{R}^\infty$ in the non-equivariant case, and in the equivariant case when a finite group G is involved U is infinite copies of the regular representation.

2.8. E_∞ **operads and algebras.** One can look at the direct limit $\lim_{k\to\infty} \mathcal{C}ube^k =: \mathcal{C}ube^\infty$. By sending all the A's to 0 each of the spaces $\mathcal{C}ube_i^\infty$ can be seen to be homotopy equivalent to $Embed(\{*\}^{\times i}, \mathbb{R}^\infty)$ the configuration space of i points in $\mathbb{R}^\infty$ which inductively can be shown to be contractible. Similar arguments hold for $\mathcal{I}som$.

Definition 2.4. An operad $\mathcal{O}$ is an E_∞ operad if the spaces $\mathcal{O}_i$ are all contractible and the Σ_i action is free. An algebra over an E_∞ operad can be delooped

Theorem 2.5. *A connected E_∞ algebra in weakly equivalent to an infinite loop space.*

3. Spectra

Fix an inner product space $(U, \langle,\rangle)$. We will call this the universe and for each universe one can define a spectrum and prespectrum.

For a finite dimensional subspace $V \subset U$ let $S^V \in \mathcal{T}op$ denote the one point compactification of V and let $\Omega^V(X) := \hom(S^V, X)$. Note that we have adjunctions $\hom(S^V X, Y) \cong \hom(X, \Omega^V Y)$.

Definition 3.1. A **prespectrum** $\mathbb{E}$ is a collection of objects in $\mathbb{E}_V$ where $V \subset U$ is a finite dimensional (*possibly virtual*) subspace of U and a collection of compatible maps $\mathbb{E}_{W\to V} : \mathbb{E}_W \to \Omega^{V-W}\mathbb{E}_V$ where $W \subseteq V$ is a subspace of V and $V - W$ is the orthogonal complement of W in V. $\mathbb{E}$ is a **spectrum** if $\mathbb{E}_{W\to V}$ are homotopy equivalences.

Given a space $X \in \mathcal{T}op$ we can define the suspension prespectrum $S^\infty X$ by $S^\infty X_V := S^V X$. Given a prespectrum $\mathbb{E}$ one can define the space $\Omega^\infty \mathbb{E} := \varinjlim_V \Omega^V \mathbb{E}_V$. Similarly one can **spectrify** a prespectrum $\mathbb{E}$ into a spectrum $\mathbb{E}'$ by defining $\mathbb{E}'_W := \varinjlim_V \Omega^V \mathbb{E}_{V\oplus W}$. We have natural adjunctions $\mathrm{Hom}(\Sigma^\infty X, \mathbb{E}) = \mathrm{Hom}(X, \Omega^\infty \mathbb{E})$.

To define a spectrum it suffices to define its spaces for a cofinal set of vector subspaces. We'll not make any distinction between a prespectrum and it's spectrification.

All the constructions and concepts that are defined for naive spectrum can be defined analogously for these *genuine* spectrum. The new concepts are the notion of **highly structured ring spectra** and **genuine smash product**.

Definition 3.2. An E_∞ (resp. A_∞) ring spectrum is a prespectrum $\mathbb{E}$ which is an algebra over the $\mathcal{I}som$ (resp. *an* A_∞) operad.

Example 3.3 (Thom Spectra)**.** Given

- $G = (G_n)$ is a sequence of groups
- with natural homomorphisms $G_m \times G_n \to G_{m+n}$
- with representations $\rho_n : G_n \to O(\mathbb{R}^{\gamma(n)})$ and inclusion maps which are compatible

$$\begin{tikzcd} G_n \arrow[r] \arrow[d, hook] & O(\mathbb{R}^{\gamma(n)}) \arrow[d, hook] \\ G_{n+1} \arrow[r] & O(\mathbb{R}^{\gamma(n+1)}) \end{tikzcd}$$

One can define the two E_∞ spectra

$$(BG_+)_{\mathbb{R}^{\gamma(n)}} := BO(\mathbb{R}^{\gamma(n)})_+$$
$$(MG_+)_{\mathbb{R}^{\gamma(n)}} := MO(\mathbb{R}^{\gamma(n)})$$

Because the spaces $\mathbb{R}^{\gamma(n)}$ are cofinal this defines a prespectrum which can then be spectrified. The standard examples are $O(n), SO(n), Spin(n), U(n), SU(n)$.

Example 3.4 (Classifying spaces)**.** If G is a compact Lie group then the Eilenberg Maclane space $K(G,1)_+ = BG_+$ is a spectrum. If one assumes the existence of a classifying spectra BG then the E_∞ structure is induced by the group multiplication $G \times G \to G$ exactly as in the previous example.

Example 3.5 (Sphere Spectrum)**.** The sphere spectrum defined by $\mathbb{S}_V := S^V$ is naturally an E_∞ spectrum as $S^V \wedge S^W \cong S^{V \oplus W}$. More generally any naive spectrum X_n is naturally a genuine spectrum by defining $\mathbb{X}_{\mathbb{R}^n} := X_n$ but the naive ring structure does not necessarily extend to an E_∞ ring structure.

4. Things to do

4.1. Smash product. The naive way to define a smash product of $\mathbb{E}$ and $\mathbb{F}$ would be to say that $(\mathbb{E} \wedge \mathbb{F})_V := \bigvee_W \mathbb{E}_W \wedge \mathbb{F}_{V-W}$. However it is not possible to define the connecting morphisms unambiguously as there are two natural maps $(S^U \wedge \mathbb{E}_V) \wedge \mathbb{F}_W \to (\mathbb{E} \wedge \mathbb{F})_{V \oplus W \oplus U} \leftarrow \mathbb{E}_V \wedge (S^U \wedge \mathbb{F}_W)$. One needs to do considerable kungfu to rectify this. I currently lack the motivation to suffer through this ordeal, the mysteries of this adventure are for a more intrepid reader.

4.2. Steenrod Squares. It is possible to define the Steenrod Squares as being associated to an E_∞ operad in the category of $\mathbb{Z}/2$ augmented DGAs. This point of view is somehow more natural as it associates the cohomology operations to *operads* which are themselves higher cohomology operations and not to Eilenberg Maclane spaces which come into the picture thanks to the very fortuitous Brown Representability. This is something really worth understanding and I'll update this document when I find some time.