PROBLEM SET 10

PART 1 - TAYLOR SERIES

Q.1. For the polynomial $p(x) = x^3$ compute the Taylor polynomial at $x = 1$ of degree 3. Simplify the Taylor polynomial and verify that it equals the original polynomial.

Q.2. Compute the Taylor polynomial of the following functions:

(1) $e^{\sin x}$ at $x = 0$, degree 3

(2) $\dfrac{1}{x+1}$ at $x = 0$, degree n

Q.3. In this exercise we'll compute the **remainder term** of the Taylor polynomial. Assume that the function f is differentiable enough number of times. Consider the Taylor polynomial of the $f(x)$ at $x = 0$ of degree n.

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \qquad a_i = \frac{f^{(i)}(0)}{i!}$$

Theorem. The remainder term $R_n(x) = f(x) - P_n(x)$ equals

$$R_n(x) = \int_0^x \frac{f^{(n+1)}(t)}{n!} \cdot (x-t)^n \, dt$$

(1) Prove the theorem directly for $n = 0$. (Note: $0! = 1$.)

(2) Find the Taylor polynomial of $f'(x)$ at $x = a$ of degree $n - 1$.

(3) Assume the following Leibniz's identity:

$$\text{If } g(x) = \int_0^x f(x,t)\, dt \text{ then } g'(x) = f(x,x) + \int_0^x \frac{\partial f(x,t)}{\partial x}\, dt$$

(This very useful trick is called *differentiating under the integral sign*, unfortunately we do not have enough language to prove this.)

Use part (2) and induction on the degree n to prove the Theorem.

Q.4. This is a very difficult problem to write down rigorously, for this one it's ok to be vague in your arguments. In this problem n, k are positive integers and all the Taylor series are at $x = 0$. *

(1) If $f(x) = x^n g(x)$. Find the Taylor series of $f(x)$ in terms of Taylor series of $g(x)$ (Don't think too hard). Hence find $f^{(k)}(0)$ in terms of the derivatives of g.

(2) If $f(x) = g(x^2)$ find $f^{(k)}(0)$ in terms of the derivatives of g. Hence find the Taylor series for f in terms of g.

(3) Find the Taylor series for $e^{(x^2)}$.

(4) If $f(x) = g(x^n)$, find $f^{(k)}(0)$ in terms of the derivatives of g. Hence find the Taylor series of f in terms of g.

(5) Find the Taylor series for $\sin(x^4)$.

* Hint: It is very crucial here that you're only asked to find the derivatives at $x = 0$.

Part 2 - Numerical Computations

For this week's HW you're allowed (and required) to use wolframalpha (or calculator) to do the computations/simplify large fractions.

A table of Taylor series can be found on Pg. 426 in the book.

Q.5. (1) Suppose the coefficients of Taylor series of f are c_i and of g are d_i at $x = a$. Find the Taylor series of fg at $x = a$ in terms of c_i, d_i.

(2) Find the general formula for the n^{th} derivative of the product $(fg)^{(n)}(a)$ in terms of the derivatives of f, g at a.

Q.6. Use estimates on the remainder term of the appropriate Taylor series to compute the following quantities within the prescribed error.

(1) $\cos 1$, error $< 10^{-2}$

(2) $\sin 0.01$, error $< 10^{-10}$

(3) e, error $< 10^{-2}$

(4) $\log 1.1$, error $< 10^{-4}$

Q.7. Show that the remainder terms for the Taylor series of $\log(1 + x)$ and $\arctan(x)$ at $x = 0$ grows with n when $x > 1$. (Hence the standard Taylor series cannot be used to approximate these numbers.)

Q.8. Taylor series are useful in theory but not so great for numerical approximations in practice.

(1) Find the value of $\pi/4$ correct up to 1 decimal place using Taylor series of $\arctan(x)$. How many terms of the Taylor series will you need, to determine the value of $\pi/4$ correct up to 2 decimal places?

(2) Prove the following identities:

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$
$$\arctan(u) + \arctan(v) = \arctan\left(\frac{u + v}{1 - uv}\right)$$

(3) Show that

$$\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{3}$$
$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}$$

(4) Use the second formula in part (3) to compute π correct up to 4 decimal places.

Part 3 - Integral Computations

Q.9. For this week do Q.3, and Q.4 problems - i) to v) on Pg. 378-379 from Ch.19.