Descent

Invariants of E_∞-rings

R be an E_∞ rings

$\longrightarrow$ $\pi_* R$ is a graded commutative ring

But π_* is a very crude invariant

If R is an E_∞ ring, there is a good theory of R-modules

$\mathrm{Mod}(R)$ = stable, symmetric monoidal model/∞-category

Def: $\mathrm{Pic}(R)$ = group of iso classes of R-modules which are $\otimes$-invertible

$= \left\{ \begin{array}{l} M \in \mathrm{Mod}(R) \text{ such that} \\ \exists N . \quad M \otimes_R N \simeq R \end{array} \right.$

Ex: G finite p-group, k- field of char p

$\exists$ a stable module category $\mathrm{St\,Mod}(G) \simeq \mathrm{Mod}(k^{tG})$

Picard group = "endo trivial module" - only depends on E_2 structure

Def: A full subcategory $\mathcal{C}' \subseteq \mathcal{C}$ is thick,

1) $\mathcal{C}'$ is a triangulated subcategory

2) $\mathcal{C}'$ is closed under retracts, i.e. $X \oplus Y \in \mathcal{C}' \Rightarrow X, Y \in \mathcal{C}'$

Q. What are thick subcategories of $\mathrm{Mod}^{\omega}(R)$? "finitely generated" modules

Th^m - (Hopkins-Neeman)

Let R be a discrete commutative, noetherian ring $\Rightarrow \exists$ a bijection

$$\left\{ \begin{array}{l} \text{thick subcat} \\ \text{of } \mathrm{Mod}^{\omega}(R) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{subsets of } \mathrm{spec}\, R \\ \text{which are specialization closed} \end{array} \right\}$$

eg: $R = \mathbb{Z}$, given a set S of prime numbers,

$C_S \subsetneq \mathrm{Mod}^{\omega}(\mathbb{Z})$ consists of objects such that $\pi_*(M)$ is S-torsion.

Th^m: For $S^0_{(p)}$ the thick subcats are stratified (?) by $(M_{FG})_{(p)}$.

If $M \in \mathrm{Mod}(R)$, $\pi_* M$ is a graded $\pi_* R$ module.

Given $M, N \in \mathrm{Mod}(R)$, $\exists$ SS such that
$$\mathrm{Ext}^{s,t}_{\pi_* R}(\pi_* M, \pi_* N) \Longrightarrow \pi_{t-s}(\mathrm{Hom}(M,N))$$

Suppose $\pi_* R$ has finite homological dimension this is finite at E_2.

eg: $R = KU$ $\quad \pi_* KU$ = Laurant series

every perfect KU module $M \simeq \oplus$ copies KU, KU/p^i, ΣKU, $\Sigma KU/p^i$

Th^m: Assume R is even periodic $\cdot$ $\pi_1 R = 0$, $\pi_2 R$ is a unit. $\pi_0 R$ is regular Noetherian.

1) (Backer-Richter) $\mathrm{Pic}(R) \simeq \mathrm{Pic}^{\text{algebraic}}(\pi_0 R) \times \mathbb{Z}/2$ — $\mathbb{Z}/2$ because ΣR is also invertible but not detected by algebra

2) The thick subcategory of $\mathrm{Mod}^\omega(R)$ are stratified by $\mathrm{Spec}\, \pi_0 R$
$\Longleftrightarrow$ $\left\{ \begin{array}{l} \text{subsets of } \mathrm{Spec}\, \pi_0 R \\ \text{closed under specializations} \end{array} \right.$

But $\pi_0 tmf$ is not reg noetherian as a lot of stable stems lie in it.

R classical commutative ring, R' a faithfully flat R-algebra.

Th^m (Grothendieck) $\mathrm{Mod}\, R \simeq \left\{ \begin{array}{l} R'\text{-module} + \\ \text{descent data} \end{array} \right.$

$$M \longrightarrow M \otimes_R R', \quad R \to R' \rightrightarrows R' \otimes_R R'$$

Form a cosimplicial R-algebra, cobar complex
$$R \to R' \rightrightarrows R' \otimes R' \substack{\rightarrow\\\rightarrow\\\rightarrow} R' \otimes R' \otimes R' \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots$$

Descent data of R'-modules of descent data is
$$\mathrm{Tot}\left(\mathrm{Mod}_{R'} \rightrightarrows \mathrm{Mod}_{R' \otimes R'} \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots\right)$$

R is limit of the above complex. Descent theory says that same holds if we replace rings by their category of modules.

Def^n: R is an E_∞ ring and R' is an E_∞-module if

1) $\pi_0 R \to \pi_0 R'$ is faithfully flat
2) $\pi_* R \otimes_{\pi_0 R} \pi_0 R' \xrightarrow{\simeq} \pi_* R'$

Th^m (Lurie) In this case $\exists$ a natural equivalence
$\mathrm{Mod}(R) \simeq \mathrm{Tot}(\mathrm{Mod}(R'^{\otimes}))^{\pi_0 R}$ = local data

π_*(Faithfully flat algebra) map injectively into $\pi_* R$.

Let $R \to R'$ be a map of E_∞ rings
Def^n: φ is descendable if the thick $\otimes$ ideal R' generates in $\mathrm{Mod}(R)$ is all of $\mathrm{Mod}(R)$.

$R \xrightarrow{\varphi} R'$ is descendable if $\exists\, N \in \mathbb{Z}_{>0}$ such that if
$M_1 \xrightarrow{f_1} M_2 \longrightarrow \cdots \xrightarrow{f_N} M_N$ in mod R with $f_i \otimes R' = 0 \Rightarrow f_N \circ \cdots \circ f_1 = 0$ in $\mathrm{Mod}(R)$

Th^m (M.): Conclusion of f.f. descent holds for a descendable data.

eg:
- $L_n S^0 \longrightarrow E_n$ is descendable.
- G - finite group, char $k = p$ $\qquad k^{BG} \longrightarrow \prod_{A \subseteq G} k^{BA}$
  elementary abelian p subgroups of G
- $TMF[\frac{1}{n}]_{, n \in \mathbb{Z}} \longrightarrow TMF(n)$, if $n \geq 3$, $TMF(n)$ is even periodic with regular π_0
  ↳ This is "Galois" extension with group $Gl_2(\mathbb{Z}/n)$ and descendable.

Cor: $\mathrm{Mod}\, TMF[\frac{1}{N}] \simeq \mathrm{Mod}(TMF(N))$

Th^m (M.) - The thick subcategories of $\mathrm{Mod}^\omega(TMF)$ are stratified by M_{ell}.
$\simeq$ $\left\{ \begin{array}{l} \text{subsets of space } |M_{ell}| \\ \text{closed under specialization} \end{array} \right.$

Th^m:
1) $\mathrm{Pic}(TMF) \simeq \mathbb{Z}/576$
2) $\mathrm{Pic}(Tmf) \simeq \mathbb{Z} \oplus \mathbb{Z}/24$