→ Dwyer – Operads

1) Operads and actions
2) Four operads
3) Applications – Homotopy theory
   – Manifold theory
   – Algebra
4) Mashup

Def: Monoid $\xleftrightarrow{\text{model}}$ $Map(X,X)$

$M \times M \longrightarrow M$ $\qquad (f,g) \longmapsto f\circ g$

commutative
associative

Operad $\xleftrightarrow{\text{model}}$ $Maps(X^r, X)$, $r \geq 0$

$P(n) \times P(r_1) \cdots P(r_n)$ $\qquad f, g_1, \ldots, g_r \longmapsto f(g_1 \cdots g_r)$

$\downarrow$

$P(r_1 + \cdots + r_n)$

$\Sigma_r$ acts on $P(r)$ $\qquad \Sigma_r$ acts on $Map(X^r, X)$

+ compatibility

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P action on X (P-alg): Operad map $P \longrightarrow \{Map(X^r, X), r \geq 0\}$

$$\coprod_{r\geq 0} P(r) \times_{\Sigma_r} X^r \longrightarrow X$$ (unital, associative)

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- Symmetric Seq $\equiv \{S(r), r \geq 0, \Sigma_r$ action on $S(r)\}$
- Monoidal structure on symmetric seq — $(S \cdot T)(n) = \left( \coprod_{\substack{k, l_1 \ldots l_k, \\ l_1 + \cdots + l_k = r}} S(k) \times T(l_1) \times \cdots \times T(l_k) \right) / \sim$

- Operad $P$ = monoid for $\cdot$ $\qquad$ $P \cdot P \longrightarrow P$ assoc. unital

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Action of $P$ on a symmetric seq $S$: $\qquad$ left module: $P \cdot S \longrightarrow S$

right module $\quad S \cdot P \longrightarrow S$

algebra $X \quad P \cdot X\langle 0 \rangle \longrightarrow X\langle 0 \rangle$

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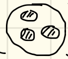
## 2) Family of four

Commutative operad $\qquad C(r) = * \quad \forall r$

Associative operad $\qquad A(r) = \Sigma_r \quad \forall r$

$D_n$ = little $n$-disk operad $\qquad D_n(r) = \{$  $\}$ + operad structure by inclusion

Lie operad (in Spectra) $\qquad \mathcal{L}_r \simeq \bigvee_{(r-1)!} S^{1-r}$

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Koszul duality: $\quad P \longleftrightarrow P^{\vee}$

P acts on Things $\quad$ $P^{\vee}$ acts on cochains of Things

$A \longleftrightarrow A$

$D_n \longleftrightarrow D_n$

$C \longleftrightarrow \mathcal{L} \quad \longleftarrow$ Comes up in Rational homotopy Theory

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B-V Tensor product: $\quad P \boxtimes Q$ = P-algebra in Q-algebra or Q in P

$D_n \simeq D_1 \boxtimes^h D_1 \boxtimes^h \cdots \boxtimes^h D_1 \longleftarrow$ derived $\boxtimes$ product

$A \sim \pi_0 D_1 \qquad C \sim \operatorname{colim} D_n = D_\infty$

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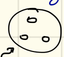
## 3)a Homotopy theory

$D_n =$  $\rightsquigarrow \dfrac{\text{Big Disk}}{\partial} \longrightarrow$ Big disk / complement of small disk

a point in

$S^n \longrightarrow \bigvee_r S^n$

$X$ a pointed space: $\quad X \wedge S^n \longrightarrow \bigvee_r (X \wedge S^n)$

$X \wedge S^n$ is analogue of $D(n)$ in $\mathrm{Top}_*^{op}$

$\Omega^n X \times \cdots \times \Omega^n X \longrightarrow \Omega^n X$

Thm (Hopkins) - $\quad$ $n$-fold suspensions $\longleftrightarrow$ algebra over $D(n)$ in $\mathrm{Top}_*^{op}$

(May) - $\quad$ $n$-fold Loop space $\longleftrightarrow$ algebra over $D(n)$.

## 3)b Manifolds

$N$ - framed manifold, $n$ - fixed integer

$N^{\Delta}(r)$ = space of embeddings $\coprod_r D^n \hookrightarrow N$ tangentially straightened

$N^{\Delta}$ is a right module over $D(r)$

Th$^m$: (Weiss-B) If $n = \dim N$, $\dim M \geq n+2$, then

$$\mathrm{Emb}^{TS}(N, M) \sim \mathrm{Map}^h_{Rt-D_n}(N^{\Delta}, M^{\Delta}) \quad \leftsquigarrow \text{ Useful for computational purposes.}$$

Th$^m$ (?) space of loops in $\mathbb{R}^n$, $n>3$ $\sim \Omega^2 \mathrm{Map}^h_{operad}(D_1, D_n)$

? Approximation to long knots in $\mathbb{R}^3$ $\sim \Omega^2 \mathrm{Map}^h_{op}(D_1, D_3)$

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## 3c) Algebra

$F_2 \sim \pi_1(\mathbb{C} \setminus \{0,1\})$

$\mathbb{A}^1 \setminus \{0,1\}$ defined over $\mathbb{Q}$

$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on $\pi_1^{\text{étale}}(\mathbb{A}^1_{\mathbb{C}} \setminus \{0,1\}) \simeq F_2^{\wedge}$

Grothendieck : $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow GT^{\wedge} \hookrightarrow \mathrm{Aut}(F_2^{\wedge})$

? : $GT_{\mathbb{Q}} \hookrightarrow \mathrm{Aut}(F_{2,\mathbb{Q}})$

? : $GT_{\mathbb{Q}} \xrightarrow{\simeq} \pi_*(\mathrm{Aut}(D_{2\,\mathbb{Q}}))$ rationalisation of $D_2$ operad

Question : $GT^{\wedge} \xrightarrow[?]{\simeq} \pi_0(\mathrm{Aut}\, D_2^{\wedge})$

Question : $\mathrm{Top}(n) \xrightarrow[?]{\simeq} \mathrm{Aut}^h(D_n)$

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Mashup :

Geometric Topology — GT — Galois Theory

Grothendieck Tiechmuller

} Operads