Dehn Twists

Def: An automorphism of a topological space X is a map $f : X \longrightarrow X$ which is a homeomorphism.

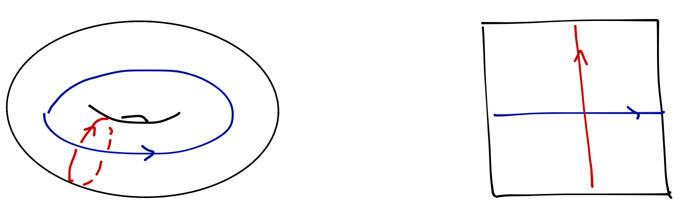
Recall : $f : X \to Y$ is a homeomorphism if f is continuous and has a continuous inverse i.e. there is a map $g : X \to Y$ such that $f \cdot g$ is identity on Y and $f \cdot g$ is an identity on X.

Ques 1) Show that the set of automorphisms of a space X forms a group. (What is product? What about the identity element?)

This is usually denoted $\mathrm{Homeo}(X)$.

Ques: Show that up to deformations a circle has only two automorphisms.

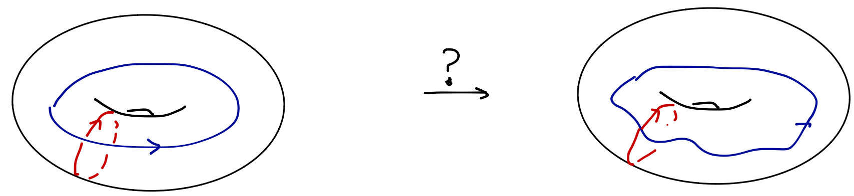
Ques: Show that up to deformorphism any automorphism of the torus is completely determined by what happens to the red & the blue circles.

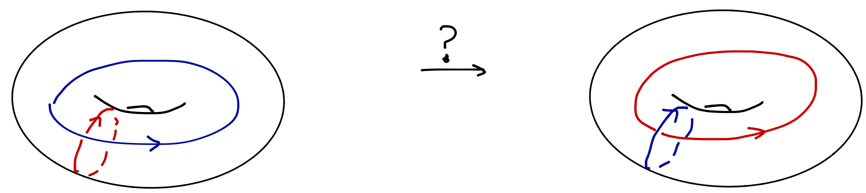


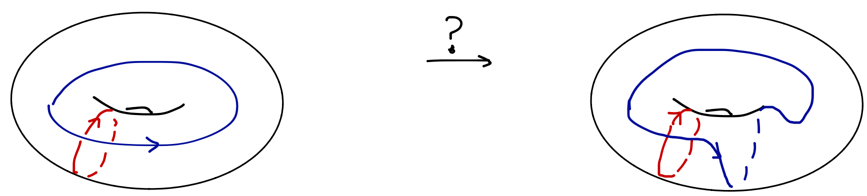
Ques: What are the torus automorphisms corresponding to the reflecting one or both of the red, blue circles?

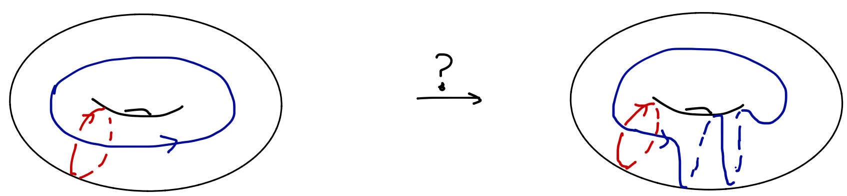
Automorphisms of the Torus:

Ques 2) Is there an automorphism of the torus $f: T \longrightarrow T$ which maps the red and blue circles as shown below:

a) 

b) 

c) 

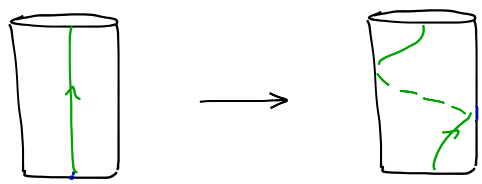
d) 

Ques: what do these automorphisms look like on gluing diagrams?

Dehn Twist

Def : A Dehn Twist is a special automorphism of the cylinder

D :



Let the height of the cylinder be 2π
We rotate the circle at height θ by angle θ

- The Dehn twist has the nice property that the two boundary circles are unchanged.

Ques : What is the inverse of D ?

Dehn Twist on a torus

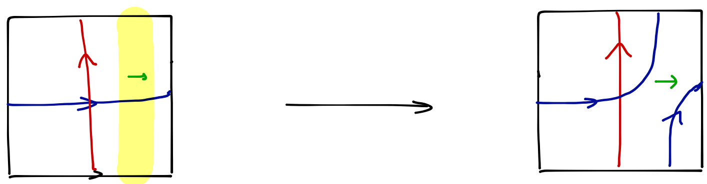
It is possible to use Dehn twists to create automorphisms of the torus via Dehn surgery.

Cut out the cylinder , dehn twist , glue it back

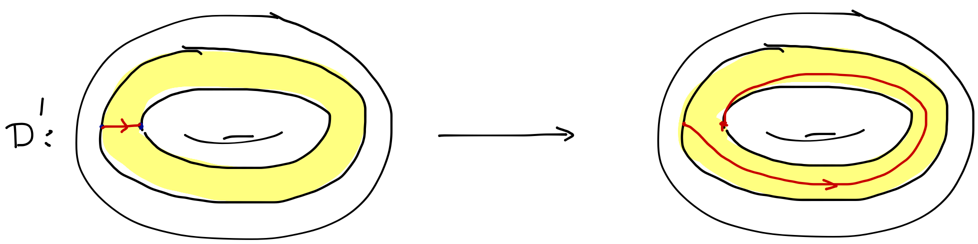
Ques : What is the inverse automorphism ?

Dehn twists on gluing diagrams

If we cut open the cylinder into a square the Dehn twist becomes very interesting

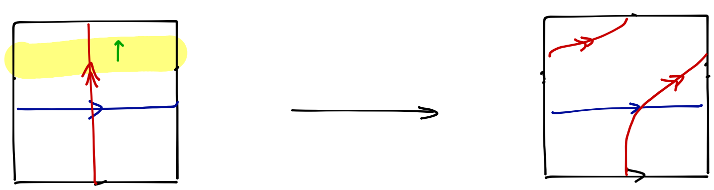


We can perform Dehn surgeries on other cylinders as well



This allows us to create more automorphisms of the torus

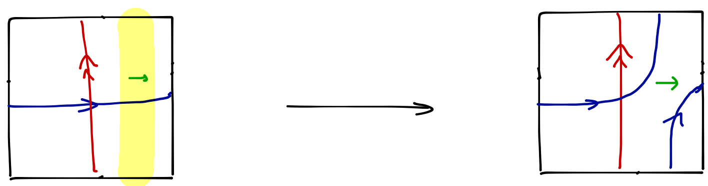
gluing diagram this looks like



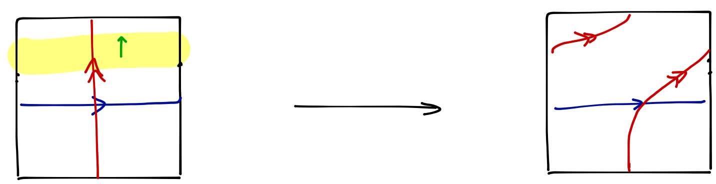
Torus knots:

Let us call the two Dehn twists on the torus D_B (blue) and D_R (red)

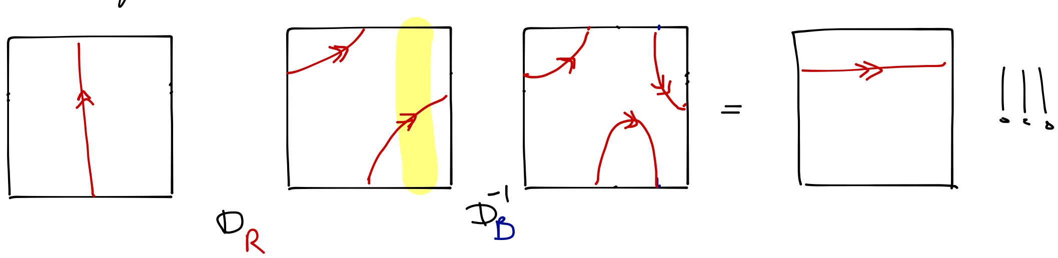
D_B :



D_R :

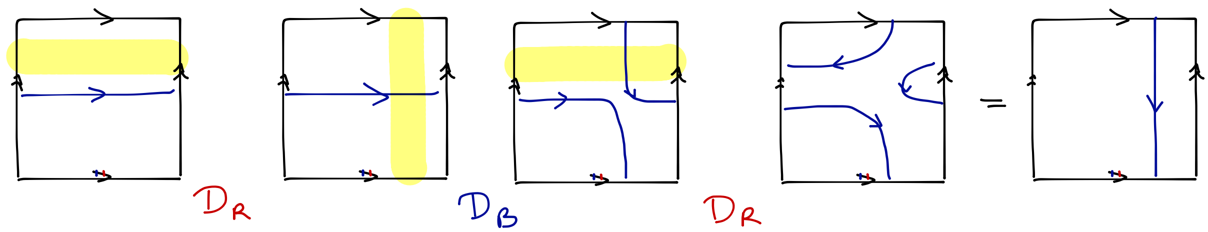


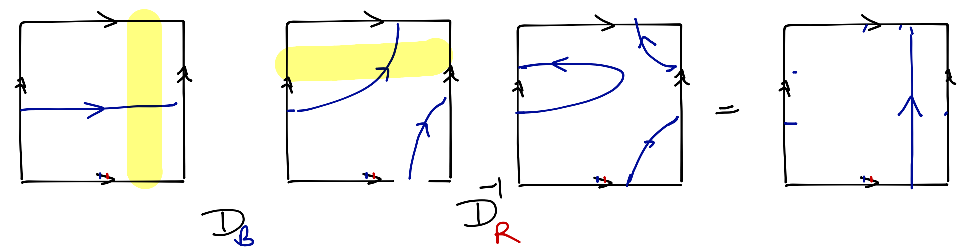
Ques what automorphism is $D_B^{-1} \circ D_R$?
We need to first the red twist then the blue



Ques: what happens to the blue circle?

D_R, D_B infact generate the entire group Homeo(T) (up to deformations)
The proof uses facts about the fundamental $\pi_1(T)$.





$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

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Fig. 1

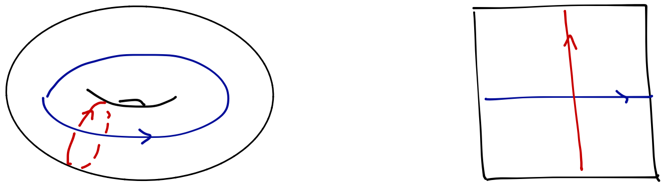


Fig. 2

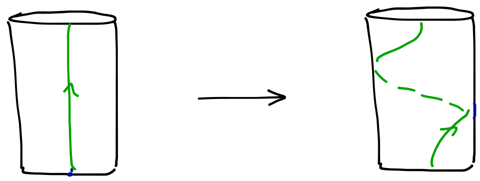


Fig. 3

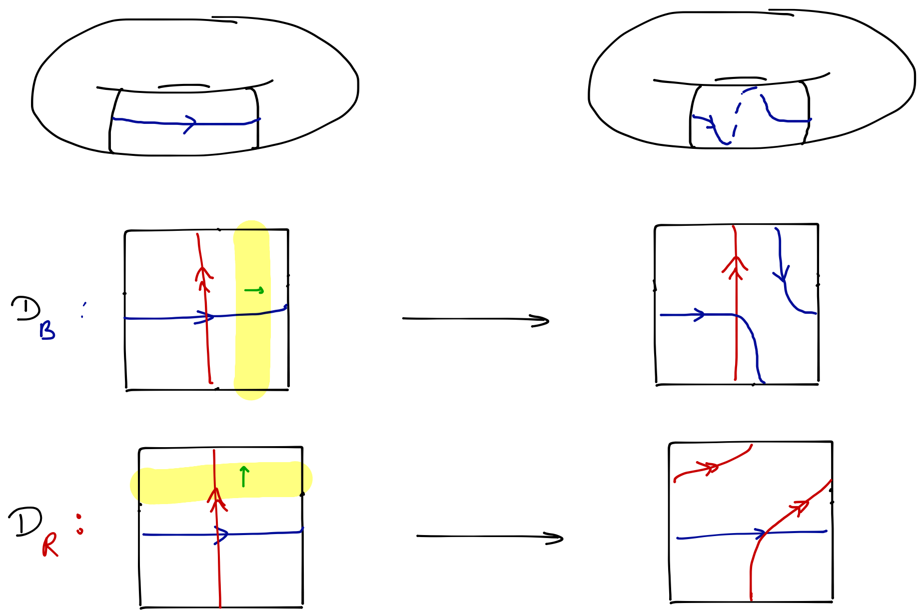
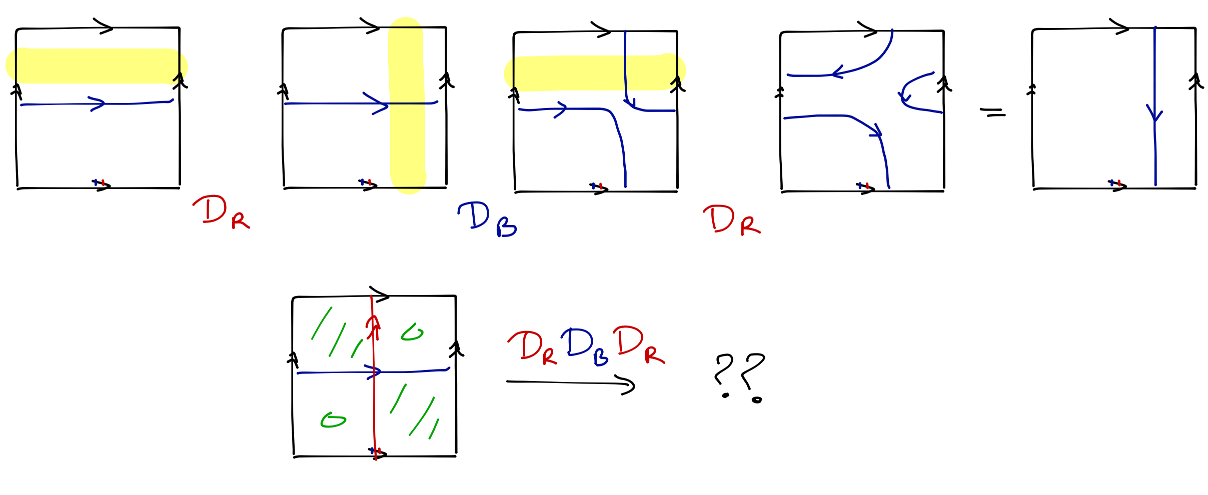


Fig. 4



Mapping class groups

Apurva Nakade

July 7, 2017

1 Automorphisms

Definition 1.1. An **automorphism** or a **self-homeomorphism** of a manifold X is a map $f : X \to X$ which is a homeomorphism. The set of automorphisms forms a group under composition, denoted $\text{Homeo}(X)$.

This group is usually too big to get a good handle on, so instead we study automorphisms up to deformations i.e. we consider two automorphisms to be the same if one automorphism can be continuously deformed into another.

Definition 1.2. The group $\text{Homeo}(X)/deformations$ is called the **mapping class group**, denoted $\text{MCG}(X)$.

Example 1.3. Every automorphism of $\mathbb{R}^1$ is a strictly increasing or a strictly decreasing function $f : \mathbb{R}^1 \to \mathbb{R}^1$. It is possible to deform a strictly increasing function f_1 to another strictly increasing function f_2 via the path of maps $t.f_1 + (1-t).f_2$ for $t \in [0, 1]$, similarly for decreasing functions. And composition of two decreasing functions is an increasing function. Together these imply that $\text{MCG}(\mathbb{R}^1) \cong \mathbb{Z}/2$.

The main object of interest for us is the mapping class group of the torus $\text{MCG}(T)$ i.e. automorphisms of the torus *up to deformations*. Let us fix two non-parallel circles on the torus and call these the **principal circles**. While there are various choices for these all of which work we'll pick the simples ones and call them the **red** and the **blue** circles. See **Fig.1**.

2 Dehn Twists

One way to construct non-trivial automorphisms of the torus is via Dehn twists.

Definition 2.1. A **Dehn twist**, denoted D, is a special automorphism of the cylinder which twists the cylinder as in **Fig.2**.

Dehn twist has the nice property that the two boundary circles are unchanged. We can use Dehn twists to create non-trivial automorphisms of the torus by cutting out a cylinder, performing a Dehn twist, and glueing it back. This is an example of surgery on the torus! Dehn twists look even more interesting on gluing diagrams. See **Fig.3**.

We can perform Dehn twists on other cylinders sitting inside a torus which allows us to create more automorphisms of the torus. A theorem of Dehn-Lickorish says that for genus g surfaces the mapping class group is generated by a small set of Dehn twists. Dehn twists on punctured discs give rise to Braid groups establishing further connections between topology and group theory.

3 Exercises

Exercise 3.1. Use the following exercises to show that the mapping class group of $S^1 = \{(x, y) : x^2+y^2 = 1\}$ is $\mathbb{Z}/2$.

1. Show that every automorphism can be continuously deformed to one that fixes the point $(1, 0)$.
2. Find two automorphisms of S^1 which fix $(1, 0)$ which cannot be deformed into each other via automorphisms.
3. Show that $\text{MCG}(S^1) \cong \mathbb{Z}/2$.

Exercise 3.2. Find $\text{MCG}(X)$ when X is one of the following spaces

1. Two parallel lines in $\mathbb{R}^2$
2. Two intersecting lines in $\mathbb{R}^2$

3. Union of two intersecting circles
4. The 2 dimensional unit disk $D^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$
   This one is non-trivial. Read the wikipedia page on Alexander's trick.

Exercise 3.3. Based on the above exercise what is the relationship between $\mathrm{MCG}(X \sqcup X)$ and $\mathrm{MCG}(X)$, where $X \sqcup X$ denotes the disjoint union of two copies of X.

Exercise 3.4. Describe the homeomorphisms which are inverses of D_R and D_B in the mapping class group of the torus.

Exercise 3.5. Perform a Dehn twist on the cylinder around the equator on S^2. What is the corresponding element in $\mathrm{MCG}(S^2)$?

Exercise 3.6.

1. Verify that $D_B D_R D_B$ and $D_R D_B D_R$ are equal in $\mathrm{MCG}(T)$ by checking what they do to the principal circles.
2. Consider the 2×2 matrices $M_B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $M_R = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$. Verify that $M_B M_R M_B = M_R M_B M_R$.
3. Assuming that $\mathrm{MCG}(T)$ is generated by D_R and D_B show that this defines a homomorphism from $\mathrm{MCG}(T)$ to the group $SL_2(\mathbb{Z})$ of 2×2 matrices with integer coefficients and determinant 1.
4. Describe the action of the matrices M_R and M_B on the plane and relate it to the Dehn twists D_R and D_B.

Exercise 3.7. $D_B D_R D_B$ is NOT a reflection! In **Fig. 4** describe what Dehn twists do to the shaded regions and figure out what $D_B D_R D_B$ really is.

Fig. 1

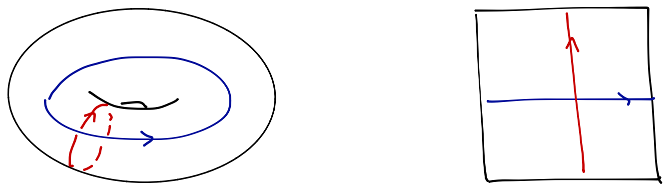


Fig. 2

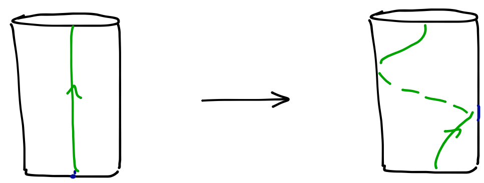


Fig. 3

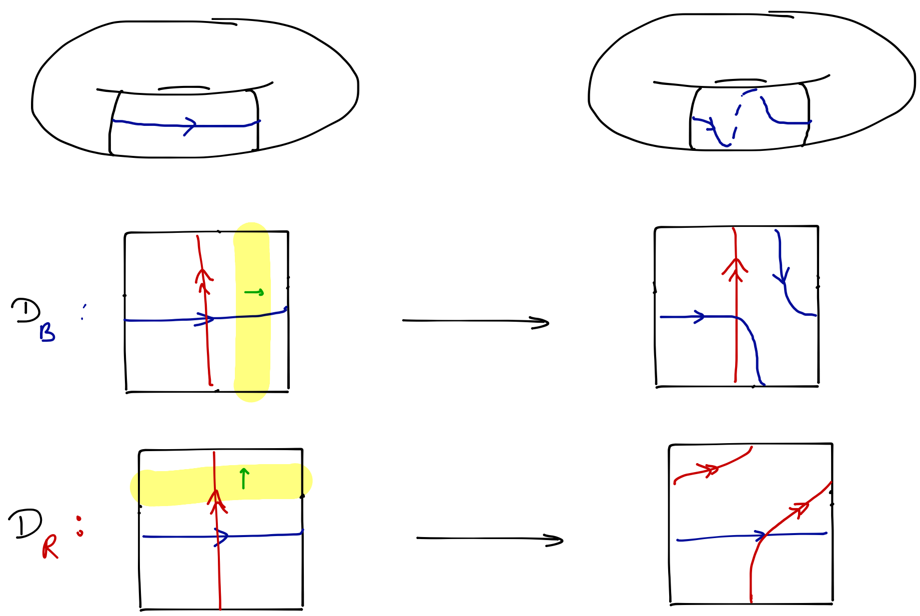


Fig. 4