§ Constructing homotopy colimits

· Geometric Realisation :

Given $X : \Delta^{op} \longrightarrow Top$ a simplicial space define its geometric realisation as

$$|X| := coeq\left(\coprod_{[n]\to[k]} X_k \times \Delta^n \rightrightarrows \coprod_n X_n \times \Delta^n \right)$$

Top map: $X([n]\longrightarrow[k]) : X_k \longrightarrow X_n$
Bottom map: $\Delta([n]\longrightarrow[k]) : \Delta^n \longrightarrow \Delta^k$

$$\cong \coprod_n X_n \times \Delta^n \Big/ \begin{array}{l}(\partial_i x, t) \sim (x, d^i t) \\ (s_i x, t) \sim (x, s^i t)\end{array}$$

$\{ \Delta^n = \text{standard } n\text{-simplex} \}$

Q. Given objectwise weak equivalence $X \longrightarrow Y$ when is $|X| \simeq |Y|$?
A. When X, Y are Reedy cofibrant (as defined below)

Def^n: n^{th} Latching object of X:
$$L_n X := \bigcup_{i=1}^{n-1} s_i(X_{n-1})$$

eg: $L_0 X = \phi$, $L_1 X = X_0$, $L_2 X = X_1 \coprod_{X_0} X_1$

We have a natural inclusion: $L_n X \longrightarrow X_n$

We say that X is Reedy cofibrant if this map is a cofibration.

· We also have a natural map $|X| \longrightarrow colim_{\Delta^{op}} X \left(\cong coeq\left(X_1 \underset{\partial_0}{\overset{\partial_1}{\rightrightarrows}} X_0\right)\right)$ which map a point in $\overset{X_n \times \{*\}}{X_n \times \Delta^n}$ to any of its vertices. Such a map is well-defined as we're taking the colimit over Δ^{op}.

· Homotopy colimits:

Def: Given a small diagram $\mathcal{D} : I \longrightarrow Top$ a simplicial replacement of $\mathcal{D}$ is defined as:

$$srep(\mathcal{D}) := \coprod_{i_0 \in I} \mathcal{D}(i_0) \leftleftarrows \coprod_{i_0 \leftarrow i_1} \mathcal{D}(i_1) \leftleftarrows \coprod_{i_0 \leftarrow i_1 \leftarrow i_2} \mathcal{D}(i_2) \cdots$$

Define $\quad hocolim_I \mathcal{D} := |srep(\mathcal{D})|$

· We have a natural map

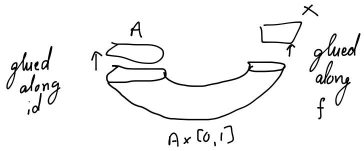
$$hocolim_I \mathcal{D} \longrightarrow colim_{\Delta^{op}}\, srep\mathcal{D}$$
$$\cong coeq\left(\coprod_{i_1 \to i_0} \mathcal{D}(i_1 \to i_0) \longrightarrow \coprod_{i_0} \mathcal{D}(i_0) \right) = colim_I \mathcal{D}$$

Th^m: $srep(\mathcal{D})$ is always Reedy cofibrant,
so if $\mathcal{D}, \mathcal{D}' : I \longrightarrow Top$ are objectwise equivalent then so are $hocolim_I \mathcal{D}$, $hocolim_I \mathcal{D}'$.

The above theorem is false if we replace Top with an arbitrary simplicially enriched model category. In that case, we further require $\mathcal{D}, \mathcal{D}'$ to be diagrams of cofibrant objects.

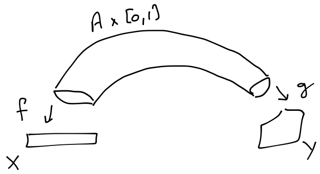
eg: $\mathcal{D} = \quad X \xrightarrow{f} Y$ $\qquad\qquad$ degenerate

In this case the hocolim $\mathcal{D}$ is the mapping cylinder



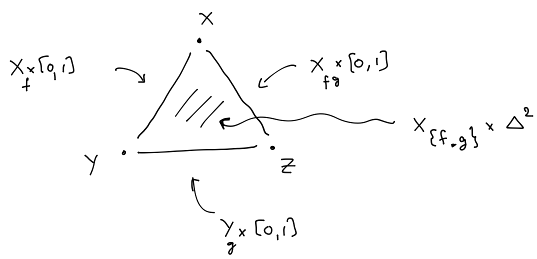
$= \operatorname{hocolim} \mathcal{D}$

eg: $\mathcal{D} = \begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle g}\downarrow & & \\ Y & & \end{array}$ $\quad \operatorname{srep}(\mathcal{D}) = \begin{array}{c} A \\ \sqcup \\ X \\ \sqcup \\ Y \end{array} \leftleftarrows \begin{array}{c} A_f \\ \sqcup \\ A_g \end{array}$ all other simplices degenerate



$= \operatorname{hocolim} \mathcal{D}$

eg: $\mathcal{D} = X \xrightarrow{f} Y \xrightarrow{g} Z$ $\quad \operatorname{srep}(\mathcal{D}) = \begin{array}{c} X \\ \sqcup \\ Y \\ \sqcup \\ Z \end{array} \leftleftarrows \begin{array}{c} X_f \\ \sqcup \\ Y_g \\ \sqcup \\ X_{gf} \end{array} \overset{\longleftarrow}{\leftleftarrows} X_{\{f,g\}}$ + degenerate simplices



$= \operatorname{hocolim} \mathcal{D}$

- Alternative formula for hocolim:

given $\mathcal{D} : I \to Top$ we have

$$\operatorname{hocolim}_I \mathcal{D} \cong \operatorname{coeq}\left(\coprod_{i \to j} \mathcal{D}(i) \times B\big((j \downarrow I)^{op}\big) \rightrightarrows \coprod_i \mathcal{D}(i) \times B\big((i \downarrow I)^{op}\big) \right)$$ where B denotes nerve

- $j \downarrow I$ is the category of objects $\left\{ \begin{array}{c} j \\ \downarrow \\ i \end{array} \right\}$ and morphisms $\left\{ \begin{array}{ccc} & j & \\ \swarrow & & \searrow \\ i & \longrightarrow & k \end{array} \right\}$

- The nerve of $B((j \downarrow I)^{op})$ is the simplicial set

$$\coprod_{j \to i} \Delta^0 \leftleftarrows \coprod_{\substack{j \\ \swarrow \ \searrow \\ i \to k}} \Delta^1 \overset{\longleftarrow}{\leftleftarrows} \coprod_{\substack{j \\ \swarrow \downarrow \searrow \\ i_0 \to i_1 \to i_2}} \Delta^2 \quad \cdots$$

the boundary maps are

$$\left(\begin{array}{c} j \\ \swarrow \downarrow \searrow \\ i_0 \to i_1 \to \cdots \to i_n \end{array} \right) \begin{array}{l} \xrightarrow{\partial_0} i_0 \to i_1 \to \cdots \to i_{n-1} \\ \xrightarrow{\partial_1} i_0 \to i_1 \to \cdots \to (i_{n-2} \to i_{n-1} \to i_n) \\ \quad \vdots \\ \xrightarrow{\partial_n} i_1 \to i_2 \to \cdots \to i_n \end{array}$$

this is where $B(\ \ ^{op})$ & B differ