- The fact that there are ~~an~~ units, co-units ~~can~~ ~~be~~ for adjoint functors can be obtained from the naturality conditions, looking at image of identity.

- Next we get an adjoint pair Ab Groups $\rightleftarrows$ mod-R

  (forget, $\mathrm{Hom}_{Ab}(R,-)$)

- Using the fact that forget is exact, we will get $\mathrm{Hom}_{A}(R,-)$ takes injectives to injectives.

- Now use $\mathrm{Hom}_{Ab}(R, \mathbb{Q}/\mathbb{Z})$ as the universal injective and repeat the procedure for mod-R as we did for $\mathbb{Z}$-modules.

<u>Th^m</u>: $\mathcal{A} \underset{R}{\overset{L}{\rightleftarrows}} B$ (L, R) adjoint pair, then
- L is ~~be~~ right exact
- R is left exact

<u>claim</u>: In an Abelian category, given $X \xrightarrow{f} Y \xrightarrow{g} Z$

if $\mathrm{Hom}(A,X) \xrightarrow[f_*]{} \mathrm{Hom}(A,Y) \xrightarrow[g_*]{} \mathrm{Hom}(A,Z)$

is exact $\forall A$, then so is $X \xrightarrow[f]{} Y \xrightarrow[g]{} Z$.

- Take $A = X$
  This will give $g \cdot f = 0$
- Why is $\mathrm{im} f = \ker g$?

Not sure if this is true. But this is easily verified for $0 \to X \to Y \to Z$.

Q. Is there any relationship between the derived functors of adjoint pair of functors?

Is true note that one ~~a~~ way is not true.

$X \to Y \to Z$ does not imply

$\mathrm{Hom}^*(A,X) \to \mathrm{Hom}(A,Y) \to \mathrm{Hom}(A,Z)$

The second condition infact implies that we have a ~~map~~ commutative triangle

$$\begin{tikzcd} & X \arrow[d, "g"] \\ \mathrm{Im}\, g \arrow[ur] \arrow[r, hook] & Y \end{tikzcd}$$

- $A \xrightarrow{\Delta} A^I$ $\qquad$ A^I category of functors from I to A

$x \longmapsto$ (~~a~~ constant functor x).

Weibel 2.2.1

What is a projective object in $Ch(\mathcal{A})$?

$$\begin{tikzcd} & P_\bullet \arrow[dl, dashed] \arrow[d] & \\ B_\bullet \arrow[r] & A_\bullet \arrow[r] & 0 \end{tikzcd}$$

- Each gradation is a projective.
- Extra condition we need is commutativity of consecutive squares

$$\begin{tikzcd} & P_n \arrow[dl, dashed, "\hat{\alpha}_n"'] \arrow[d, "\alpha_n"] \\ B_n \arrow[r, "\pi_n"'] & A_n \end{tikzcd}$$

we need $d \cdot \hat{\alpha}_n = \hat{\alpha}_{n-1} \cdot d$

Need to choose special $A_\bullet, B_\bullet$.

~~$B =$ cone $(id_{P_\bullet})$ $\quad$ $A = P[-1]$~~

~~$B_n = P_n \oplus P_{n-1}$ $\quad$ and $\alpha_n = d$~~

~~suppose $\hat{\alpha}_n = \beta_n \oplus \gamma_n$ $\quad$ because $\hat{\alpha}_n \pi_n = \alpha_n$~~

~~we get $\gamma_n = d$ $\quad$ what is β_n?~~

~~$d \cdot \hat{\alpha}_n = \hat{\alpha}_{n-1} \cdot d = \beta_{n-1} d \oplus 0$~~

~~$\| $~~

~~$d(\beta_n \oplus d) = (d\beta_n - d, 0)$~~

~~$\Rightarrow d = d\beta_n + \beta_{n-1} d$~~

~~$\Rightarrow$ split ?~~

$B =$ cone $(id_{P_\bullet})[1]$

$A = P_\bullet$

$\alpha_n = 1$

So $\hat{\alpha}_n = \tilde{\alpha}_n \oplus 1$

$\tilde{\alpha}_n : P_n \longrightarrow P_{n+1}$

$d \cdot \hat{\alpha}_n = \hat{\alpha}_{n-1} d = \tilde{\alpha}_{n-1} d \oplus d$

$\|$

$(d \cdot \tilde{\alpha}_n + 1, d)$

$\Rightarrow 1 = \tilde{\alpha}_{n-1} d - d \tilde{\alpha}_n$

$\Rightarrow$ split exact

Now suppose $P_\bullet$ is split seq of projectives

we need to check $d \cdot \hat{\alpha}_n = \hat{\alpha}_{n-1} d$

Ex 2.2.2) W

ch.$(\mathcal{A})$ has enough projectives

i.e. $A_.$ chain $\Rightarrow$ $\exists$ split chain of projectives $P_.$ along with surjection $P_. \longrightarrow A_. \longrightarrow 0$

For each A_i suppose P_i projective such that $P_i \longrightarrow A_i \longrightarrow 0$.

Look at $\hat{P}_. = P_. \oplus P_.[-1]$. with maps

$$d(a,b) = (b,0)$$

Split? $s(a,b) = (0,a)$

$$ds + sd\,(a,b) = ds(a,b) + s(b,0)$$
$$= d(0,a) + (0,b)$$
$$= (a,0) + (0,b) = (a,b).$$

2.3.4) $\mathcal{A}$ has enough injective $\Rightarrow$ $\mathcal{A}^{op}$ has enough projectives

Result follows.

Claim: Sheaves (X) category of sheaves of abelian groups over X has enough injectives.

- $\mathcal{F}_x$ stalk of $\mathcal{F}$ at x, ~~sky~~
  $x_* G$ skyscraper of ht G over x, then

$$\mathrm{Hom}_{Ab}(\mathcal{F}_x, A) \cong \mathrm{Hom}_{\text{sheaves}(X)}(\mathcal{F}, x_* A)$$

$$(\varphi : \mathcal{F}_x \longmapsto A) \longmapsto \left(\mathcal{F}(U) \longrightarrow \mathcal{F}_x \xrightarrow{\varphi} A \text{ if } x \in U;\quad \mathcal{F}(U) \longrightarrow 0 \text{ else} \right)$$

$$\left(\mathcal{F}_x \longmapsto \underset{\underset{A}{\parallel}}{x_* A_x}\right) \longleftarrow (\mathcal{F} \longmapsto x_* A)$$

- ~~sel~~ Pick A to be ~~injective~~ $\mathbb{Q}/\mathbb{Z}$
  Then $x_* \mathbb{Q}/\mathbb{Z}$ will be ~~skys~~ injective

• Intersting adjunction:

Let S be a set considered a category without morphisms

$\mathcal{A}$ be any abelian category.

$\mathcal{A}^S$ be category of functors $S \to \mathcal{A}$.

Then, we have a Δ map $\Delta : \mathcal{A} \longrightarrow \mathcal{A}^S$

$A \longmapsto$ (constant function A)

what are the adjoints, if any, of Δ?

• $\mathrm{Hom}_{\mathcal{A}^I}(\Delta A, \varphi) = \mathrm{Hom}_{\mathcal{A}}(A, ?\varphi)$

$? = \prod_{s \in S} \varphi(s)$

• $\mathrm{Hom}_{\mathcal{A}}(?\varphi, A) = \mathrm{Hom}_{\mathcal{A}^I}(\cancel{\Delta A, \varphi}\ \varphi, \Delta A)$

$? = \bigoplus_{s \in S} \varphi(s)$

So Δ has right adjoint "direct product"

left adjoint "direct sum"

Q. what happens if we replace S by a category?

Q. what are the derived functors of $\prod$? $\oplus$?

replacing S by a filtered system I, we will get direct limits and inverse limits
(co. product) (generalization of product)

Adjoints:

$$\text{G modules} \xleftarrow[\text{Trivial}]{} \text{Ab group}$$

Find left, right adjoints.

- ~~How~~ What about G ~~alge~~ - Lie algebra?

Calculations of Ext groups:

$\mathrm{Ext}^i_S(A,B) := R^i\,\mathrm{Hom}_S(A,-)\,(B)$.

Q. Is there some A s.t. $\mathrm{Ext}^i_S(A,B)=0 \quad \forall B$?

TFAE:
1) $\mathrm{Ext}^i(A,B)=0 \quad \forall A,\ i\geq 1$
2) $\mathrm{Ext}^1(A,B)=0 \quad \forall A$
3) B injective $\Leftrightarrow$ ($\mathrm{Hom}(-,B)$ exact).

2) $\Rightarrow$ 3)
look at the long exact sequence of derived functors

$0 \to A' \to A \to A'' \to 0$

$0 \to R^0F(A') \to R^0F(A) \to R^0F(A'') \to$

Weibel Ex 2.5.2:

TFAE
1) A projective
2) $\mathrm{Hom}(A,-)$ exact
3) $\mathrm{Ext}^i_R(A,B)=0 \quad \forall i>0,\ \forall B$
4) $\mathrm{Ext}^1_R(A,B)=0 \quad \forall B$

Proof:

1) $\Leftrightarrow$ 2) $\Rightarrow$ 3 = 4 Easy

4) $\Rightarrow$ 1)

$\mathrm{Ext}^1_R(A,B)=0 = R^1\,\mathrm{Hom}_R(A,-)\ (B)$.

Given a diagram:

$$\begin{tikzcd} & A \arrow[d] & \\ N \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

let $\ker M \to N$ be $K \to M$

Apply δ-functors of $\mathrm{Hom}(A,-)$ to $0 \to K \to M \to N \to 0$ to get

$$0 \to \mathrm{Hom}(A,K) \to \mathrm{Hom}(A,M) \to \mathrm{Hom}(A,N) \to \underbrace{\mathrm{Ext}^1_R(A,K)}_{=0} \to \cdots$$

$\Rightarrow$ any map $A \to N$ lifts to $A \to M$

$\Rightarrow$ A projective

25.1)

TFAE

i) B injective

ii) $\mathrm{Hom}(-,B)$ exact

iii) $\mathrm{Ext}^i_R(A,B) = 0 \quad \forall A,\ i>0$

iv) $\mathrm{Ext}^1_R(A,B) = 0 \quad \forall A$.

Proof: i) $\Leftrightarrow$ ii) $\Rightarrow$ 3) $\Rightarrow$ iv) easy

iv) $\Rightarrow$ i)

Given diagram:

$$\begin{tikzcd} & B & \\ 0 \arrow[r] & I \arrow[u] \arrow[r] & J \end{tikzcd}$$

we need to extend $I \to B$

Construct pushout P, then

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] & P \arrow[r] & P/B \arrow[r] & 0 \\ 0 \arrow[r] & I \arrow[u] \arrow[r] & J \arrow[u] \arrow[r] & J/I \arrow[u, "\cong"'] \arrow[r] & 0 \end{tikzcd}$$

$\mathrm{Ext}^1_R(A,B) = 0 \Rightarrow$ P splits

So we get ~~canonical~~ projection map

$\pi_B : P \to B$

Claim: $B \xleftarrow{\pi_B} P \leftarrow J$ extends

$$\begin{tikzcd} & B & \\ 0 \arrow[r] & I \arrow[u] \arrow[r] & J \end{tikzcd}$$

Proof:

$\cancel{P = J \oplus B/(-1,f)I \quad (\mapsto) \quad \{(-i, f(i))\} \quad f: I \to B}$

$\cancel{J \to P}$

$\cancel{j \mapsto [(j,0)]}$

Say: $0 \to B \xrightarrow{s_B} P$ $\quad \pi_B s_B = 1_B$

Then by pushout diagram

$$s_B \cdot f = (J \to P) \circ (I \to J)$$

$$\Rightarrow f = \pi_B s_B f = \pi_B \cdot (J \to P) \cdot (I \to J)$$

Note that in the proof we are using "Pushout" & "Pullback" Theorems which we will prove:

- When is a diagram a pushout / pullback diagram? (in Abelian category)

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "\beta"] \\ C \arrow[r, "\alpha"'] & D \end{tikzcd} \quad ?$$

- $$\begin{tikzcd} B\oplus C \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & 0 \end{tikzcd} \quad , \quad \begin{tikzcd} 0 \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & B\oplus C \end{tikzcd}$$ gives maps $A \longrightarrow B\oplus C$, $B\oplus C \longrightarrow D$

- for pullback:

  $A = \ker(B\oplus C \xrightarrow[\beta\oplus-\alpha]{} D)$ so $0 \to A \overset{f}{\underset{-g}{\rightrightarrows}} B\oplus C \overset{\beta}{\underset{-\alpha}{\rightrightarrows}} D$

  for pushouts:

  $D = \operatorname{coker}(A \xrightarrow[f\oplus-g]{} B\oplus C)$ so

  $A \overset{f}{\underset{-g}{\rightrightarrows}} B\oplus C \overset{\beta}{\underset{-\beta\alpha}{\rightrightarrows}} D \to 0$

  So opposite of a pushout is pullback & vice-versa!!

- Pushouts take epi to epi (similarly for monos, pullbacks)

  - $\operatorname{coker}(f) = \operatorname{coker}(\alpha)\circ\beta$

    Suppose $A \to B \xrightarrow{x} X = 0$

    $$\Rightarrow \begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, "x"] \\ C \arrow[r, "0"'] & X \end{tikzcd} \quad \text{gives} \quad \begin{tikzcd} A \arrow[r] & B \arrow[r, "x"] \arrow[d] & X \\ & D \arrow[ur] & \end{tikzcd}$$

    $$\Rightarrow \begin{tikzcd} A \arrow[r] & B \arrow[d] \\ & D \arrow[r] & X \end{tikzcd} = 0 \quad \Rightarrow \quad \begin{tikzcd} A \arrow[d] & & \\ C \arrow[r] & D \arrow[r] & X \end{tikzcd}$$

    $$\Rightarrow \exists! \quad \begin{tikzcd} A \arrow[r] & B \arrow[r, "x"] \arrow[d] & X \\ & D \arrow[r] & \operatorname{coker}(\alpha) \arrow[u, "!"'] \end{tikzcd}$$

    $\Rightarrow$ $\operatorname{coker}(\alpha)\cdot\beta = \operatorname{coker} f$ ∎

- Pushouts take mono to mono.

  $$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \\ & C & \end{tikzcd} \Rightarrow 0 \to A \rightrightarrows B\oplus C$$

  $\Rightarrow$ This is also a pull back diagram!!

  Result follows by above.

A much simpler proof for 2.5.1)

Need to show $\mathrm{Ext}^1_R(A,B)=0 \ \forall A \Rightarrow B$ injective

Enough to produce a long exact. seq from $0\to A'\to A\to A''\to 0$

$$0\to \mathrm{Hom}(A'',B)\to \mathrm{Hom}(A,B)\to \mathrm{Hom}(A',B)\to \mathrm{Ext}^1\,\cancel{\mathrm{Hom}}(A'',B)\to \mathrm{Ext}^1(A,B)\to$$

To do this look at injective resolution of B and use $0\to A'\to A\to A''\to 0$

$0\to B\to I^\cdot \rightsquigarrow$

$$\begin{tikzcd}
 & & 0 \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}(A'',B) \arrow[r] \arrow[d] & \mathrm{Hom}(A'',I^\cdot) \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}(A,B) \arrow[r] \arrow[d] & \mathrm{Hom}(A,I^\cdot) \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}(A',B) \arrow[r] & \mathrm{Hom}(A',I^\cdot) \arrow[d] \\
 & & 0
\end{tikzcd}$$

Exact

Apply H^* to get long exact sequence and we get result.

Ex: Using method given in the above problem, balance Ext & Tor.

Ex: In a double complex, rows or columns exact $\Rightarrow$ total complex exact.

Ex 2.4.5 Wiebel

To prove: T_* - δ functor, co-effaceable for $n>0$ then T_* is universal.

Proof:

Suppose we are given another δ-functor S_* with map $\varphi_0 : S_0 \to T_0$.

Then, look at $0\to K\to P\to A\to 0$

this map being co-effaceable

Applying long exact seq ~~to~~ using both T_*, S_*

$$\begin{tikzcd}
\arrow[r] & S_1(K) \arrow[r] & S_1(P) \arrow[r] & S_1(A) \arrow[r, "\delta"] \arrow[d, dashed, "\phi_1"] & S_0(K) \arrow[r] \arrow[d, "\phi_0"] & S_0(P) \arrow[r] \arrow[d, "\phi_0"] & S_0(A) \arrow[r] \arrow[d, "\phi_0"] & 0 \\
\arrow[r] & T_1(K) \arrow[r] & T_1(P) \arrow[r, "0"] & T_1(A) \arrow[r, "\delta"] & T_0(K) \arrow[r] & T_0(P) \arrow[r, "0"] & T_0(A) \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
S_1(A) \arrow[r] & S_0(K) \arrow[d] \\
& T_0(K)
\end{tikzcd}$$

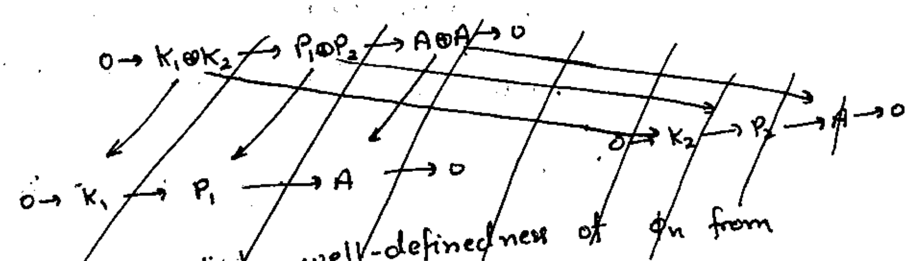
image of this lies in ker $T_0K \to T_0P$
$= \text{im } T_1(A)$
$\cong T_1(A)$

So we ~~get~~ compose $T_1A \xleftarrow{\delta^{-1}} T_0K$ and get ~~T~~ $S_1(A) \to T_1(A)$.

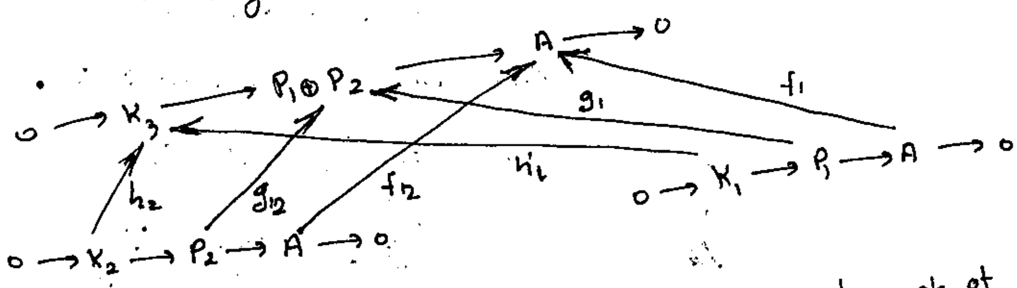
- well defined:

$0 \to K_1 \to P_1 \to A \to 0$ $\qquad$ $0 \to K_2 \to P_2 \to A \to 0$

The problem is there need not be any map $P_1 \to P_2$
so instead look at $P_1 \oplus P_2$



we will conclude well-definedness of ϕ_n from naturality here.



~~Naturality~~ Big diagram will give naturality wrt map of exact sequences.

i.e.

$$\begin{tikzcd}
T_i(A) \arrow[r, "T_i(f_1)"] & T_i(A) \\
S_i(A) \arrow[u, "\phi_i A"] \arrow[r, "S_i(f_1)"'] & S_i(A) \arrow[u, "\tilde{\phi}_i(A)"']
\end{tikzcd}$$

Then because $T_i(\text{id}) = \text{id}$ we are done (as $f_1 = f_2 = \text{id}$)

2.5.4)

$\mathcal{F} \longrightarrow X$ sheaf

Γ : sheaves$(X) \longrightarrow$ Ab Grp   global sections

c : Ab Grp $\longrightarrow$ Sheaves(X) constant sheaves

$$\mathrm{Hom}_{\text{Sheaves}(X)}(cG, \mathcal{F}) \overset{?}{=} \mathrm{Hom}_{Ab}(G, \Gamma\mathcal{F})$$

LHS =

$$\begin{tikzcd} & \mathcal{F}(U) \arrow[d, "\rho_{u,v}"] \\ G \arrow[ur] \arrow[r] & \mathcal{F}(V) \end{tikzcd}$$

diagrams of this form

So an element is completely determined by

$$G \longrightarrow \mathcal{F}(X)$$

= RHS.

$\Rightarrow$ c right exact , Γ - left exact

What are the derived functors?

• c :

$$0 \longrightarrow G \longrightarrow H \longrightarrow K \longrightarrow 0$$

free   free

Apply c,

$$0 \longrightarrow cG \longrightarrow cH \longrightarrow 0$$

what are the cohomologies?

~~Sam~~

Simply $cK = (L_0 c)(K)$

is c left exact too? Yes. What is left adjoint of c?

(• How about Γ? sheaf cohomology.)

$$\mathrm{Hom}_{\text{sheaves}(X)}(\mathcal{F}, cG) = \mathrm{Hom}_X(?\mathcal{F}, G)$$

$$\begin{tikzcd} \mathcal{F}(U) \arrow[d, "\rho_{u,v}"'] \arrow[dr] & \\ \mathcal{F}(V) \arrow[r] & G \end{tikzcd}$$

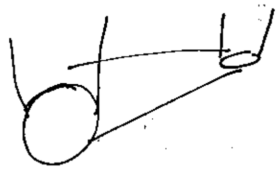
We need a single abelian group to encapsulate the data $\mathcal{F}(U) \longrightarrow \mathcal{F}(V)$.

2.6.2 – Weibel :

- $$\begin{tikzcd} f: X \arrow[r] & Y \\ \mathcal{F} \arrow[u] & f_*\mathcal{F} \arrow[u] \end{tikzcd}$$

$f_*\mathcal{F}$ is a sheaf?

$f_*\mathcal{F}(U) := \mathcal{F}(f^{-1}(U))$



Presheaf:
$$\begin{tikzcd} f^{-1}(U) \arrow[r, "f"] & U \\ f^{-1}(V) \arrow[u, hook] \arrow[r, "f"'] & V \arrow[u, hook] \end{tikzcd}$$
presheaf property is satisfied

Sheaf axioms:

1) $\{U_i\}$ cover of Y
$s \in f_*\mathcal{F}(Y)$ s.t. $s|_{U_i} = 0$
$\Leftrightarrow s \in \mathcal{F}(X)$ s.t. $s|_{f^{-1}(U_i)} = 0$
and $f^{-1}(U_i)$ cover X
$\rightarrow s = 0$

2) $\{U_i\}$ cover Y
$s_i \in f_*\mathcal{F}(U_i)$ s.t. $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$
$\Rightarrow s_i \in \mathcal{F}(f^{-1}(U_i))$ s.t. $s_i|_{f^{-1}(U_i \cap U_j)} = s_j|_{f^{-1}(U_i \cap U_j)}$
But $f^{-1}(U_i \cap U_j) = f^{-1}(U_i) \cap f^{-1}(U_j)$.
$\Rightarrow \exists s \in \mathcal{F}(X)$ s.t. $s|_{f^{-1}(U_i)} = s_i$
$\Rightarrow \exists s \in f_*\mathcal{F}(Y)$ s.t. $s|_{U_i} = s_i$

- $$\begin{tikzcd} f: X \arrow[r] & Y \\ f^{-1}\mathcal{F} \arrow[u] & \mathcal{F} \arrow[u] \end{tikzcd}$$

$f^{-1}\mathcal{F}(U) := \varinjlim_{V \supseteq f(U)} \mathcal{F}(V)$

Presheaf:
$$\begin{tikzcd} U \arrow[r] & f(U) \subseteq U' \\ V \arrow[u, hook] \arrow[r] & f(V) \subseteq V' \arrow[u] \end{tikzcd}$$

How to get maps between direct limits?

It is enough to give maps between cofinal objects which respect inclusio poset relations. So for V, U only look at those V', U' s.t. $U' \hookleftarrow V'$. These induce a direct limit map.

- $$\mathrm{Hom}_X(f^{-1}G, \mathcal{F}) \cong \mathrm{Hom}_Y(G, f_*\mathcal{F})$$

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ \mathcal{F} \arrow[u] & G \arrow[u] \end{tikzcd}$$

Presheaves

RHS : $G(U) \longrightarrow f_*\mathcal{F}(U) \cong \mathcal{F}(f^{-1}(U))$

LHS: $\varinjlim_{V \supseteq f(U)} G(V) \longrightarrow \mathcal{F}(U)$

$$V_1 \supseteq V_2 \supseteq f(U) \qquad \begin{tikzcd} G(V_1) \arrow[r] \arrow[d] & \mathcal{F}(U) \\ G(V_2) \arrow[ur] & \end{tikzcd}$$

if $U = f^{-1}(W)$ for some W open
then $\varinjlim_{V \supseteq f(f^{-1}(W))} G(V) = G(W)$ so this is just the map $G(W) \to \mathcal{F}(f^{-1}(W))$

So the two maps are naturally isomorphic.

2.6.3

1)
$$\begin{tikzcd} * \arrow[r] & X \\ cG \arrow[u] & \mathcal{F} \arrow[u] \end{tikzcd}$$

$f(*) = x$

cG - constant ~~pre~~sheaf G

$$f^{-1}\mathcal{F}(*) = \varinjlim_{V \supseteq f(*) = x} \mathcal{F}(V)$$
$$= \mathcal{F}_x$$
$$(f_* cG)(U) = \mathcal{F}(f^{-1}(U))$$
$$= \begin{cases} G & \text{if } x \in U \\ 0 & \text{else} \end{cases}$$
$$\Rightarrow f_* cG = x_* G.$$

2)
$$\begin{tikzcd} X \arrow[r] & * \\ \mathcal{F} \arrow[u] & cG \arrow[u] \end{tikzcd}$$

$$f_*\mathcal{F}(*) = \mathcal{F}(f^{-1}(*))$$
$$= \mathcal{F}(X)$$
$$\Rightarrow f_*\mathcal{F} = \Gamma(\mathcal{F})$$
$$(f^{-1}cG)(U) = \varinjlim_{V \supseteq f(U),\ V = *,\ f(U) = *} cG(V)$$
$$= G.$$
$$f^{-1}cG = cG/X$$

2.6.4). Already did one part.

why are pull backs not exact?

Given exact diagrams

$$\begin{tikzcd} 0 \arrow[r] & A_1 \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & A_3 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & 0 \\ 0 \arrow[r] & C_1 \arrow[r] \arrow[u] & C_2 \arrow[r] \arrow[u] & C_3 \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

we claim

$0 \to P_1 \to P_2 \to P_3$

not surjective is the non-trivial part

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\ A \oplus C \arrow[r] & 0 \arrow[r] & 0 \\ C \arrow[r] \arrow[u] & C \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

$\rightsquigarrow$ on pull backs

$0 \longrightarrow A \times C$

not surjective.

2.6.8 TFAE : i) $\bigoplus_{i \in I} A_i$ exist $\forall I$

ii) $\varinjlim_{I} \varphi \quad \forall \varphi \in A^I$ I ~~filtered~~ diagram

ii) $\Rightarrow$ i) trivially

i) $\Rightarrow$ ii)

How to get $\varinjlim_{I} \varphi$? $\varphi : I \longrightarrow A$

Need some conditions on I : what conditions? None

colim $(\cdot \longrightarrow \odot) = \odot$

Claim: $\varinjlim_{I} \varphi = \operatorname{coker}\left(\bigoplus_{i \to j} \varphi(i) \longrightarrow \bigoplus_{i} \varphi(i) \right)$

This map is

$(x)_{i \to j} \longmapsto (-x)_i + [\varphi(i \to j)(x)]_j$

call this ψ

$(\mathcal{L}, \mathcal{R}, \mathcal{A}, \mathcal{B})$ adjunction

$\varphi : I \to A$ has colim then

$\mathcal{L}(\operatorname{colim} \varphi) = \operatorname{colim}(\mathcal{L} A)$

Proof: Later

Proof. For $X \in \mathrm{Obj}(A)$

suppose we have maps

$\varphi(i) \xrightarrow[f_i]{} X$ compatible with I

i.e. for $\forall\ i \to j$ in I

$$\begin{tikzcd} \varphi(i) \arrow[r, "f_i"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & X \\ \varphi(j) \arrow[ur, "f_j"'] & \end{tikzcd} \equiv \quad f_i(\alpha) = f_j(\varphi(i \to j)\alpha)$$

So we get map $\bigoplus_i f_i : \bigoplus_i \varphi(i) \longrightarrow X$

Enough to show $\bigoplus_i f_i \circ \psi = 0$

$$\begin{array}{c} (x)_{i \to j} \xmapsto{\psi} (-x)_i + \varphi(i \to j)(x)_j \\ \Big\downarrow \oplus f_i \\ f_i(-x) + f_j(\varphi(i \to j)(x)) \\ \parallel \\ -f_i(x) + f_i(x) = 0 \end{array}$$

Incredible $\square$

~~Ate~~ Coming back : left adjoint preserves colimits

$$\mathcal{L}: A \longrightarrow B \qquad R: B \longrightarrow A$$

$$B(\mathcal{L}X, Y) = A(X, RY)$$

Given $\varphi: I \longrightarrow A$

$$\mathcal{L}\varphi: I \longrightarrow A \longrightarrow B$$

Then, by def^n colim is unique obj satisfying

$$B(\varinjlim \mathcal{L}\varphi, X) = B^I(\mathcal{L}\varphi, X^\Delta) \quad \Big\} \text{ only step to think about}$$

$$= A^I(\varphi, RX^\Delta)$$

$$= A(\varinjlim \varphi, RX)$$

$$= B(\mathcal{L}\varinjlim \varphi, X)$$

$$\Rightarrow \quad \varinjlim \mathcal{L}\varphi = \mathcal{L}\varinjlim \varphi$$

- ~~Derived~~ $\mathcal{F}: A \longrightarrow B$ left adjoint ($\Rightarrow$ right exact)

$$L_i\mathcal{F}(\bigoplus_\alpha X_\alpha) = \bigoplus_\alpha L_i\mathcal{F}(X_\alpha)$$

$$X_\alpha \longleftarrow P_{\alpha\cdot} \quad \text{Projective resolution}$$

$$\oplus X_\alpha \longleftarrow \oplus P_{\alpha\cdot} \quad \text{Also projective}$$

$$\oplus \mathcal{F}X_\alpha \longleftarrow \oplus \mathcal{F}P_{\alpha\cdot} \qquad \oplus\mathcal{F} = \mathcal{F}\oplus$$

$$L_i\mathcal{F}(\bigoplus_\alpha X_\alpha) = H_i(\oplus \mathcal{F}P_{\alpha\cdot})$$

$$= \oplus H_i(\mathcal{F}P_{\alpha\cdot})$$

$$= \bigoplus_\alpha L_i\mathcal{F}(X_\alpha)$$

$\rightarrow$ Analogously, R - preserves limits

in particular $\quad R^i\mathcal{F}(\prod_\alpha X_\alpha) = \prod_\alpha (R^i\mathcal{F}(X_\alpha))$

for $\mathcal{F}$ right adjoint.

- $\mathrm{Tor}_*(A, \oplus B_\alpha) = \oplus \mathrm{Tor}(A, B_\alpha)$

2.5.3 Weibel

$x_* A$ – Γ acyclic?

Let $0 \to A \to I^{\cdot}$ be an injective resolution

$\Rightarrow$ $0 \to x_* A \to x_* I^{\cdot}$ is also an injective resolution

Applying Γ,

$$0 \to \underset{\underset{A}{\text{III}}}{\Gamma(x_* A)} \to \underset{\underset{I^{\cdot}}{\text{III}}}{\Gamma(x_* I^{\cdot})} \quad \text{again exact}$$

So Γ acyclic.

~~or we could invoke~~ or we could injectivity of $x_* A$ and hence of

$\prod_{x} x_* A_x$ to get the result.

2.6.1

incl : Sheaves $(X) \longrightarrow$ PreSheaves (X)

right adjoint to sheafification. what are the right derived functors?

2.6.6

~~direct limits in $\mathcal{A}^{op}$ = inverse limits in $\mathcal{A}$.~~

~~Take $I = \mathbb{Z}$ as a poset, $\varphi, \psi \in \mathcal{A}^{\mathbb{Z}}$ defined as~~

~~$\varphi_n = \mathbb{Z}$ $\quad \varphi_{n+1 \to n} = \mathbb{Z} \xrightarrow{2} \mathbb{Z}$~~

~~$\psi_n = \mathbb{Z}/2$ $\quad \psi_{n+1 \to n} = \mathbb{Z}/2 \xrightarrow{\varphi} \mathbb{Z}/2$~~

~~Look at $f \in \mathcal{A}^{\mathbb{Z}}(\varphi, \psi)$ $\quad f_n = \mathbb{Z} \xrightarrow[1 \to 1]{} \mathbb{Z}/2$ is epic~~

~~Then f is a surjection~~

~~But $\varprojlim \varphi = 0$ and $\varprojlim \psi = ($~~

direct limits in $\mathcal{A}^{op}$ = inverse limits in $\mathcal{A}$.

Take $I = \mathbb{Z}$ as a poset, $\varphi, \psi \in \mathcal{A}^{\mathbb{Z}}$ defined as

$\varphi_n = \mathbb{Z}$ $\quad \varphi_{n+1 \to n} = \mathbb{Z} \xrightarrow{1} \mathbb{Z}$

$\psi_n = \mathbb{Z}/2^n$ $\quad \psi_{n+1 \to n} = \mathbb{Z}/2^{n+1} \xrightarrow[1 \mapsto 1]{} \mathbb{Z}/2^n$

Take $f \in \mathcal{A}^{\mathbb{Z}}(\varphi, \psi)$ defined as $f_n = \mathbb{Z} \xrightarrow[1 \mapsto 1]{} \mathbb{Z}/2^n$

f is surjective

But $\varprojlim \varphi = \mathbb{Z}$ $\quad \varprojlim \psi$ = 2-adic integers

and we do not have a surjection $\varprojlim \varphi \longrightarrow \varprojlim \psi$.

2.6.6) f^{-1}: Sheaves $(Y) \longrightarrow$ Sheaves (X) $f: X \longrightarrow Y$

we know that f^{-1} is a left adjoint hence right exact
Only need to show left exactness.

$$\begin{tikzcd} & Y & \\ 0 \arrow[r] & \mathcal{F} \arrow[u] \arrow[r] & G \arrow[ul] \end{tikzcd}$$

then in category of presheaves

$$(f^{-1})\mathcal{F}(V) = \varinjlim_{U \supseteq f(V)} \mathcal{F}(U)$$

$$(f^{-1})G(V) = \quad ''$$

$\because$ $\varinjlim$ is exact in R-mod, we have in Presheaves

$$\begin{tikzcd} 0 \arrow[r] & f^{-1}\mathcal{F} \arrow[r] \arrow[dr] & f^{-1}G \arrow[d] \\ & & X \end{tikzcd}$$

But sheafification is exact(?) so true for sheaves too.

2.6.16)

($\varinjlim$; Δ) adjoint pair and Δ is exact
$\Rightarrow$ $\varinjlim$ takes projectives to projectives.
So the above ~~setut~~ result will follow if we can prove

$$\varinjlim H_* = H_* \varinjlim$$

This will follow from exactness of $\varinjlim$.

Def^n: Flat Module : ${}_R B$ flat $\equiv$ $\otimes_R B$ exact

$\equiv$ $\mathrm{Tor}_n^R(A, B) = 0$ $\forall A, n$

$\equiv$ $\mathrm{Tor}_1^R(A, B) = 0$ $\forall A$

eg:
- B projective $\Rightarrow$ B flat
- S central multiplicatively closed set in R,
  Then $S^{-1}R$ flat.
- $0 \longrightarrow B_1 \longrightarrow B_2 \longrightarrow B_3 \longrightarrow 0$ ⊛
  B_1, B_3 flat $\Rightarrow$ B_2 flat
  B_2, B_3 flat $\Rightarrow$ B_1 flat

Def^n: Given ${}_RB$, $B^* := \mathrm{Hom}_{\mathbb{Z}}({}_RB_{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z})$ ~~R~~ mod-R

Note: • $B \longrightarrow B^*$ is an exact functor

Because $\mathbb{Q}/\mathbb{Z}$ is ~~m~~ injective $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ exact

• $B \to C$ monic $\iff$ $B^* \leftarrow C^*$ epic

Here we are using $A \neq 0 \iff A^* \neq 0$.

TFAE:
1) B-flat in R-mod
2) B^*- injective in mod-R
3) $I \otimes_R B \simeq IB$ naturally, for all right ideals I
4) $\mathrm{Tor}\,(R/I, B) = 0$.

Proof:

$$0 \to I \to R \to R/I \to 0$$

$$\begin{tikzcd}
& 0 \arrow[r] & IB \arrow[r] & B \arrow[r] & B/IB \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Tor}_1(R/I, B) \arrow[r] & I \otimes B \arrow[r] \arrow[u] & R \otimes B \arrow[r] \arrow[u, "\mathrm{IS}"] & R/I \otimes B \arrow[r] \arrow[u, "\mathrm{IS}"] & 0
\end{tikzcd}$$

gives 3) $\iff$ 4)

2) $\Leftrightarrow$ 1)

$A' \hookrightarrow A$

$$\begin{tikzcd}
0 & \mathrm{Hom}(A', B^*) \arrow[l] \arrow[d, "\wr"] & \mathrm{Hom}(A, B^*) \arrow[l] \arrow[d, "\wr"] \\
0 & \mathrm{Hom}(A' \otimes B, \mathbb{Q}/\mathbb{Z}) \arrow[l] \arrow[d, equal] & \mathrm{Hom}(A \otimes B, \mathbb{Q}/\mathbb{Z}) \arrow[l] \arrow[d, equal] \\
0 & (A' \otimes B)^* \arrow[l] & (A \otimes B)^* \arrow[l]
\end{tikzcd}$$

B^* injective $\Longleftrightarrow$

$\Updownarrow$

B flat $\iff$ $0 \to A' \otimes B \longrightarrow A \otimes B$

2) $\iff$ 4)

Put $A' = I$, $A = R$.

To get $0 \to I \otimes B \longrightarrow$ ~~B~~ $R \otimes B$

which is lower row in $3 \iff 4$

Example of non-projective flat module — $\mathbb{Q}$?

$$\begin{tikzcd}
& Q \arrow[d] & \\
M \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

Th^m: M - finitely presented then $\longrightarrow$ $R^k \to R^m \to M \to 0$
M flat $\Rightarrow$ M projective

[Ex. Let F be covariant right exact.
$M_\cdot \to A \to 0$ be resolution s.t. $L_i F(M_j) = 0 \;\forall j \;\forall i > 0$
i.e. $M_\cdot$ - F acyclic resolution
Then, $L_i F A = H_i(FM_\cdot)$]

- Proof:

$$B^* \otimes_R M \xrightarrow{\sigma} \mathrm{Hom}_R(M, B)^*$$

$B, M \in R$-mod

$$f \otimes m \longmapsto \varphi \longmapsto f(\varphi(m))$$

[Note: σ iso if $M = R$, but not if $M = R^\infty$

$$\mathrm{Hom}(\oplus_\alpha M_\alpha, N) = \prod_\alpha \mathrm{Hom}(M_\alpha, N)$$

$$\mathrm{Hom}(M, \prod_\alpha N_\alpha) = \prod_\alpha \mathrm{Hom}(M, N_\alpha)$$

Q. what is $\mathrm{Hom}(\prod_\alpha M_\alpha, N)$? $\mathrm{Hom}(M, \oplus_\alpha N_\alpha)$?]

By above note, σ is also an isomorphism for M finitely presented.

$$R^k \to R^m \to M \to 0$$

$$\begin{tikzcd}
B^* \otimes R^k \arrow[r] \arrow[d, "\sigma", "\wr"'] & B^* \otimes R^m \arrow[r] \arrow[d, "\sigma", "\wr"'] & B^* \otimes M \arrow[r] \arrow[d, "\sigma"] & 0 \\
\mathrm{Hom}(R^k, B)^* \arrow[r] & \mathrm{Hom}(R^m, B)^* \arrow[r] & \mathrm{Hom}(M, B)^* \arrow[r] & 0
\end{tikzcd}$$

- Use σ to change from flatness to projectivity.

- Flat base change for Tor

$R \to T$ ring homomorphism, T flat in R-mod

$$\mathrm{Tor}_n^R(A_R, {}_T C) = \mathrm{Tor}_n^T(A_R \otimes_R T, {}_T C)$$

Here we are using P projective in mod-R
$\Rightarrow P \otimes_R T$ projective in mod-T

- R-commutative $\Rightarrow$ $\mathrm{Tor}_*^R(A,B)$ is an R-module.

$$T\otimes \mathrm{Tor}_*^R(A,B) \simeq \mathrm{Tor}_*^T(T\otimes A,\ T\otimes B)$$

Excercise:

R-commutative $A,B \in R\text{-mod}$

Then,

$$\mathrm{Tor}_n^R(A,B)=0$$

$$\Leftrightarrow \mathrm{Tor}_n^{R_p}(A,B)=0 \quad \forall\ p \text{ prime}$$

$$\Leftrightarrow \mathrm{Tor}_n^{R_m}(A,B)=0 \quad \forall\ m \text{ maximal}$$

For this excersise all we require is that R_p ~~= $p\,R$~~ is flat over R.

- Any abelian group is direct limit of its finitely generated subsets.

Proof: Let $G = \varinjlim_{H<A} H$ Claim: $G \xrightarrow{\sim} A$ isomorphism

direct limit over finitely generated ~~abeli~~ subgroups of A

- inclusion $H \hookrightarrow A$ gives map $G \to A$

- $H_1 \to H_1+H_2$, $H_2 \to H_1+H_2$

- injection:

  $g \mapsto 0$ in A, $g\in G$

  By direct limit condition $\exists\ h\in A$ s.t.

  $h \mapsto g$ ~~and $h\mapsto g$ in~~

  $[h]=g$

  So that $g\mapsto 0 \Rightarrow h\mapsto 0$ but $H\hookrightarrow A \Rightarrow h=0$

- Surjection:

  Take $H=\mathbb{Z}a$ for $a\in A$

  ☐

• Flat base change for Tor:

$R \longrightarrow T$ ring homo., T- flat as an R-module.

Claim: $\mathrm{Tor}_n^R(A_R, {}_TC) = \mathrm{Tor}_n^T(A_R \underset{R}{\otimes} T, {}_TC)$

Proof: $\text{LHS} = L_i(- \underset{R}{\otimes} {}_TC)(A_R)$

$= H_i(P_\bullet \otimes C)$ $\qquad P_\bullet \longrightarrow A$ projective resolution of A

$\text{RHS} = L_i(- \underset{T}{\otimes} {}_TC)(A_R \otimes T)$

$\approx$ LHS = RHS

If $\quad P_\bullet \otimes T \longrightarrow A \otimes T$ is a projective resolution of $A \otimes T$.

Resolution:

Because T is flat.

Projectivity:

P projective $\Rightarrow$ $P \otimes T$ projective?

$$\begin{tikzcd}
P \arrow[r] \arrow[d, dashed] & A \underset{R}{\otimes} T \arrow[d] & \\
{}_TN \arrow[r] & {}_TM \arrow[r] & 0
\end{tikzcd}$$

Need to say $P_R \longrightarrow {}_TN$ $\quad$ R-module map

gives a unique $P \underset{R}{\otimes} {}_RT \longrightarrow {}_TN$ $\quad$ T-module map

• Now look at $\mathrm{Tor}_n^T(T \otimes A, T \otimes B)$

$= \mathrm{Tor}_n^R(A_R, T \otimes B)$

$= L_i(A \underset{R}{\otimes} -)(T \otimes B)$

$= L_i(- \underset{R}{\otimes}(T \otimes B))(A)$ $\qquad$ Here we are again using the fact that $T\otimes$ is exact.

$= T \otimes \mathrm{Tor}_n^R(A, B)$

• Why is $S^{-1}R$ flat?

$0 \longrightarrow {}_RM \xrightarrow{f} {}_RN \longrightarrow {}_RP \longrightarrow 0$ $\qquad$ We know $S^{-1}R \otimes$ is right exact need to only check for left exactness

$0 \longrightarrow S^{-1}R \otimes M \longrightarrow S^{-1}R \otimes N$ $\qquad$ if $\frac{rf(m)}{s} = 0 \Rightarrow \frac{f(rm)}{s} = 0$

$\Rightarrow \exists s.\ f(srm) = 0$

$\Rightarrow \exists s.\ srm = 0$

$\Rightarrow \exists s.\ \frac{rm}{s} = 0.$

$\boxtimes$