Goodwillie Calculus

— Greg Arone

Calculus of homotopy functors :

eg: $X \in Top_*$ $\quad X \longmapsto \underbrace{\pi_*(X)}_{\text{hard to compute}}, \; \underbrace{\tilde{H}_*(X)}_{\text{easier to compute, because of excision}}$

Excision follows from:

$X = X_1 \cup X_2$

$$\begin{tikzcd} C_*(X_1 \cap X_2) \arrow[r] \arrow[d] & C_*(X_1) \arrow[d] \\ C_*(X_2) \arrow[r] & C_*(X) \end{tikzcd}$$

is a homotopy pushout.

Excision – alternate formulation

$C_*: Top_* \longrightarrow Ch$

preserves homotopy pushout squares.

- $\tilde{H}_*(X) \cong \pi_*(\Omega^\infty H\mathbb{Z} \wedge X)$

- Excision – third formulation

$\Omega^\infty(H\mathbb{Z} \wedge -)$ takes homotopy pushouts to homotopy pullbacks

Def: $F : Top_* \longrightarrow Top_*$ is 1-excisive if F takes homotopy pushout squares to homotopy pullback squares.

F ~~red~~ linear if F is 1-excisive + $F(*) \simeq *$.

eg: $\cdot$ $X \longmapsto \Omega^\infty(H\mathbb{Z} \wedge -)$

$\cdot$ 1_{Top^*} identity functor is not 1-excisive

Q. what are the 1-excisive (or linear) functors?

Q. Can a general homotopy functor be approximated by a 1-excisive one?

$$\begin{array}{ccc} X & \longrightarrow & CX \\ \downarrow & & \downarrow \\ CX & \longrightarrow & \Sigma X \end{array}$$

$\uparrow$

Some kind of "final" object in the category of h-pushouts starting at X

F linear $\Rightarrow$

$$\begin{array}{ccc} F(X) & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & F(\Sigma X) \end{array}$$ is h-pullback

$\Rightarrow$ $F(X) \simeq \Omega F(\Sigma X)$

$\simeq \operatorname{hocolim}_{n \to \infty} \Omega^{n} F(\Sigma^n X)$

. we get an equivalence

$$\mathrm{Lin}[Top_*, Top_*] \rightleftarrows \mathrm{Lin}[Top_*, \mathrm{Spectra}]$$

- Lemma: Suppose $F, G \in \mathrm{Lin}[\mathrm{Spaces}, \mathrm{Spectra}]$

  $\alpha : F \to G$ natural transformation s.t.

  $\alpha(S^0) : F(S^0) \to G(S^0)$ is an equivalence, then

  $\alpha(X) : F(X) \to F(X)$ is an equivalence $\forall$ finite CW complexes X.

"Proof": S^n:

$$\begin{tikzcd} S^n \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & S^{n+1} \end{tikzcd} \quad , \quad \text{wedges of } S^n \quad \begin{tikzcd} X \vee Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & * \end{tikzcd}$$

induction for finite CW complexes.

- $f : \mathbb{R} \to \mathbb{R}$ linear $\Rightarrow$ f is determined by $f(1)$.

  continuous

  $f(x+y) = f(x) + f(y)$

- Def: $F : \mathrm{Top} \to \mathrm{Top}$ is finitary if $\forall$ CW complex X,

$$\operatorname*{hocolim}_{\substack{X_\alpha \subseteq X \\ X_\alpha \text{ finite}}} F(X_\alpha) \longrightarrow F(X) \quad \text{is a w.e.}$$

Lemma: continued

If further F, G are finitary then $\alpha(X)$ is an equivalence for all X.

- Let $F \in [\mathrm{Top}_*, \mathrm{Spectra}]$, assume F is $\underbrace{\text{topologically enriched}}$.

  maps $(X, Y) \to$ maps $(F(X), F(Y))$ is continuous

  We have assembly maps

$$F(S^n) \wedge X \longrightarrow F(S^n \wedge X)$$

  which is continuous

$$\begin{tikzcd} \Omega^n(F(S^n)\wedge X) \arrow[r] & \Omega^n F(S^n \wedge X) \\ \operatorname*{hocolim}_{n\to\infty} \Omega^n(F(S^n)\wedge X) \arrow[r] & F(X) \arrow[u, "\cong \text{ if } F \text{ is linear}"'] \end{tikzcd}$$

This is an equivalence of linear functors, at least when evaluated at S^0

<u>Th^m</u>: Every finitary linear functor $F : Top_* \to Top_*$ is naturally equivalent to $\operatorname*{hocolim}_{n\to\infty} \Omega^n(F(S^n)\wedge X)$.

i.e. we have an equivalence of categories

$$\begin{array}{ccc} \text{Lin + Finitary } [Top_*, Top_*] & \xleftrightarrow{\cong} & \text{Spectra} \\ F & \longmapsto & \{F(S^0), F(S^1), \dots\} = F(S) \\ X \mapsto \Omega^\infty(C\wedge X) & \longleftarrow\!\!\shortmid & C \end{array}$$

<u>Note</u>: F is linear + finitary iff $\pi_* F$ is reduced generalized homology theory.

- Suppose $F : Top_* \to Top_*$ is a homotopy functor. Can it be approximated by a 1-excisive functor?

- Starting with the space X form the homotopy pushout

$$\begin{tikzcd} X \arrow[r] \arrow[d] & CX \arrow[d] \\ CX \arrow[r] & \Sigma X \end{tikzcd} \quad \overset{\text{apply } F}{\rightsquigarrow} \quad \begin{tikzcd} F(X) \arrow[r] \arrow[d] & F(CX) \\ F(CX) & \end{tikzcd}$$

Call the pushout of this diagram $T_1F(X)$

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] & F(CX) \arrow[d] \\ F(CX) \arrow[r] & T_1F(X) \end{tikzcd}$$

- F is 1-excisive $\iff$ $F \simeq T_1F(X)$
- F is reduced $\iff$ $T_1F(X) \simeq \Omega F(\Sigma X)$
- Iterating the procedure the homotopy colimit

$$P_1F := \text{hocolim}\ (F \longrightarrow T_1F \longrightarrow T_1(T_1F) \to \cdots \to T_1^{\,k}F \to \cdots)$$

<u>Th^m</u>: P_1F is always 1-excisive for any homotopy functor F.

- The map $F \to P_1F$ is an equivalence if F is 1-excisive.
- Suppose L is 1-excisive, $F \to L$ is a natural transformation

$$\begin{tikzcd} F \arrow[r] \arrow[d] & L \arrow[d, "\text{is}"] \\ P_1F \arrow[r] \arrow[ur, dashed] & P_1L \end{tikzcd}$$

P_1F is initial in the ?

- <u>~~Connectivity~~ Def</u>: $\alpha : F \to G$ natural transformation. F, G agree to first order if $\exists$ a constant c s.t.

$$F(X) \longrightarrow G(X) \quad \text{is} \quad 2k - c \quad \text{connected}$$

where $k =$ connectivity of X.

[constant $2 = 1+1$
for n^{th} order we have $(n+1)\cdot k - c$]

eg: $X \longrightarrow \Omega^\infty \Sigma^\infty X$ is $2k+1$ connected

Lemma: Suppose F, L agree to first order, L 1-excisive then the induced map $P_1 F \to L$ is an equivalence.

eg: $\Rightarrow$ $P_1(\mathrm{id}_{Top})(X) = \Omega^\infty \Sigma^\infty X$

eg: $\mathrm{Maps}(K, X) \wedge H\mathbb{Z} \longrightarrow \mathrm{Map}(K, X \wedge H\mathbb{Z})$

By Blakers Massey these agree to first order

$\Rightarrow P_1(\mathrm{Maps}(K, -) \wedge H\mathbb{Z})(X) \cong H^*(K; H_*(X))$

A homotopy functor can be approximated by a 1-excisive functor.

1 excisive $\leftrightsquigarrow$ generalized homology, Spectra, $X \mapsto \Omega^\infty(C \wedge X)$.

If F reduced, finitary $\Rightarrow$ $P_1 F \simeq \underset{n \to \infty}{\mathrm{hocolim}}\, \Omega^n (F(S^n \wedge X))$.

§ n-excisive functors:

$P(n)$ = Poset of subsets of $\underline{n} = \{1, 2, \dots, n\}$

a cubical diagram in $\mathcal{C}$ = $\mathcal{X} : P(n) \to \mathcal{C}$.

Def: $\mathcal{X}$ is cocartesian if $\underset{U \subsetneq \underline{n}}{\mathrm{hocolim}}\, \mathcal{X}(U) \longrightarrow \mathcal{X}(\underline{n})$ is a w.e.

$\mathcal{X}$ is cartesian if $\underset{U \neq \phi}{\mathrm{holim}}\, \mathcal{X}(U) \longleftarrow \mathcal{X}(\phi)$ is a w.e.

$\mathcal{X}$ is strongly cocartesian if every 2-dim face of $\mathcal{X}$ is cocartesian.

$\Leftrightarrow$ $\mathcal{X}$ is determined by the initial portion

$$\begin{tikzcd} & \mathcal{X}(\phi) \arrow[r] \arrow[dl] \arrow[dr] & \mathcal{X}(\{1\}) \\ \cdots \mathcal{X}(\{3\}) \cdots & & \mathcal{X}(\{2\}) \end{tikzcd}$$

Def: $F: \mathcal{C} \to \mathcal{D}$ is n-excisive if it takes strongly co-cartesian $(n+1)$ diagrams to cartesian cubes.

Eg: 1) $X \mapsto X * X$

Lemma: Takes strongly cocartesian 3-cubes to cocartesian cubes.

Let C be a spectrum.

$$\begin{array}{ccccc} Top_* & \longrightarrow & Spectra & \xrightarrow{\Omega^\infty} & Top_* \\ X & \longmapsto & C \wedge X * X & \longmapsto & \Omega^\infty(C \wedge X * X) \end{array}$$

In spectra cartesian = cocartesian.

2) In general $X \mapsto \Omega^\infty(C \wedge X^{\wedge n})$

3) A holim of n-excisive functors is n-excisive.

4) An n-excisive functor is also $(n+1)$-excisive.

5) $\Omega((\Omega^\infty\Sigma^\infty X \wedge X)/\Delta X)$ is 2-excisive.

6) $F(X) \longrightarrow \Omega^\infty\Sigma^\infty X \xrightarrow{Sq} \Omega^\infty\Sigma^\infty(X \wedge X)$ then $F(X)$ is 2-excisive.

7) An arbitrary hocolim of spectrum valued n-excisive functors is again n-excisive.

eg: C_n be a specturum with Σ_n action.

$$\operatorname{hocolim}_{\Sigma_n}(C_n \wedge X^{\wedge n}) \simeq C_n \wedge X^{\wedge n} \wedge_{S_n} S_{n+} = (C_n \wedge X^{\wedge n})_{h\Sigma_n}$$

is n-excisive. So

$$X \longmapsto \Omega^{\infty}(C_n \wedge X^{\wedge n})_{h\Sigma_n} \quad \text{is } n\text{-excisive.}$$

§ Constructing n-excisive approximations:

Given $X \in Top_*$ construct the strong co-cartesian $(n+1)$ cube $\mathcal{X}$ constructed out of

$$\begin{tikzcd} & X \arrow[dl] \arrow[r] \arrow[d] & CX \\ CX & \cdots \quad CX & \end{tikzcd}$$

More explicitly $\mathcal{X}(U) = X * U$ for $U \subseteq \underline{n+1}$.

Given a functor F define $T_nF = \operatorname{holim}_{\emptyset \neq U \subseteq \underline{n+1}} F(X * U)$

- we have natural transformation $F \longrightarrow T_nF$, which is an equivalence when F is n-excisive.
- Define $P_nF = \operatorname{hocolim}_n (F \to T_nF \to T_n^2F \to \cdots \to T_n^kF \to \cdots)$

Th^m: For every homotopy functor F, P_nF is n-excisive.

Def: F is stably n-excisive with constant c if $\forall$ strongly cocartesian cube $\mathcal{X}$, s.t. the initial map $\mathcal{X}(\phi) \longrightarrow \mathcal{X}(i)$ is k_i connected then the cube $F \circ X$ is $\sum k_i - c$ connected

eg: id_{Top} is stably 1-excisive with $c = 1$. $\hookrightarrow$ Blakers – Massey th^m

id_{Top} is stably n-excisive with $c = n$. $\hookrightarrow$ higher B-M th^m.

Lemma: F stably n-excisive with constant c, then $T_n F$ is n-excisive with ~~a~~ constant $c-1$.

$\Rightarrow$ $T_n^k F$ is n-excisive with constant $c-k$

$\Rightarrow$ $P_n F$ is n-excisive with constant $-\infty$ $\Rightarrow$ n-excisive.

Proof: Say $n=1$, F reduced.

$$\begin{tikzcd} X_0 \arrow[r, "k_1"] \arrow[d, "k_2"'] & X_1 \arrow[d, "r"] \\ X_2 \arrow[r] & X_{12} \end{tikzcd} \longrightarrow \begin{tikzcd} \Sigma X_0 \arrow[r, "k_1+1"] \arrow[d, "k_2+1"'] & \Sigma X_1 \arrow[d] \\ \Sigma X_2 \arrow[r] & \Sigma X_{12} \end{tikzcd} \longrightarrow \begin{tikzcd} F(\Sigma X_0) \arrow[r] \arrow[d] & F(\Sigma X_1) \arrow[d] \\ F(\Sigma X_2) \arrow[r] & F(\Sigma X_{12}) \end{tikzcd}$$

$(k_1 + k_2 + 1)$ - connected

$$\downarrow$$

$$\begin{tikzcd} \Omega F(\Sigma X_0) \arrow[r] \arrow[d] & \Omega(F\Sigma X_1) \arrow[d] \\ \Omega F(\Sigma X_1) \arrow[r] & \Omega(F\Sigma X_{12}) \end{tikzcd}$$

$(k_1 + k_2 - 2)$ - connected

Rmk: Every finitary functor $F: Top_* \to Top_*$ can be written as a homotopy colimit of stably n-excisive functors. (why?)

- Use this to conclude $P_n F$ is n-excisive for every finitary functor.

<u>Second Proof</u>:

Let $\mathcal{X} : P(n+1) \to Top_*$ be a strongly cocartesian cube. Then the map of cubes from $F \circ \mathcal{X} \to T_n F \circ \mathcal{X}$ factors through a homotopy cartesian cube.

Assuming this look at

$$\begin{tikzcd}
F\circ\mathcal{X} \arrow[rr] \arrow[dr] & & T_1F\circ\mathcal{X} \arrow[rr] \arrow[dr] & & T_2F\circ\mathcal{X} \arrow[rr] \arrow[dr] & & \cdots \\
& \mathcal{X}_1 \arrow[ur] & & \mathcal{X}_2 \arrow[ur] & & \mathcal{X}_3 \arrow[ur] &
\end{tikzcd}$$

These are cartesian

$$P_nF\circ\mathcal{X} = \text{hocolim}\, T_nF\circ\mathcal{X} = \text{hocolim}\, \mathcal{X}_n$$

These are all cartesian

- $F \to P_nF$ is the universal n-excisive approximation

- $$\begin{tikzcd}
& \vdots \arrow[d] \\
F \arrow[r] \arrow[dr] \arrow[ddr] & P_nF \arrow[d] \\
& P_{n-1}F \arrow[d] \\
& P_{n-2}F \arrow[d] \\
& \vdots
\end{tikzcd}$$

Taylor tower

If F is stably n-excisive then $F \to P_nF$ is $(n+1)k - c$ connected.

• Let $F: \mathcal{C} \to \mathcal{D}$ be a homotopy functor.

Def: • F, G agree to n^{th} order via $\alpha: F \to G$ if $\forall X$.
$\alpha(X): F(X) \to G(X)$ is $(n+1)\cdot \mathrm{conn}(X) - c$ connected.

• F is ρ-analytic, if $\forall n$ it is stably n-excisive with a constant c_n, where $c_n \leq \rho \cdot n + b$ (b is a constant).

Lemma : F ρ-analytic $\Rightarrow$ $F(X) \to P_n F(X)$ is $(n+1)(k+1-\rho) - \alpha$ connected. (X is k-connected)

Moreover $P_1 F$ is characterized by this property.

eg: id_{Top} is 1-analytic.

Def^n: F is homogenous of degree n if it is n-excisive but $P_{n-1}F$ is trivial.

Q. How to classify homogenous functors?

Th^m (Goodwillie): For any homogenous functor ~~from~~ $Top_* \to Top_*$ takes values in ~~∞~~ ∞-loop spaces $\subseteq$ Spectra. (F reduced)

• $n=1$ (linear functors) : $F(X) \simeq \Omega F(\Sigma X) \cdots \simeq \Omega^k F(\Sigma^k X)$
$\simeq \Omega^\infty F(\Sigma^\infty X)$.

• ~~F~~ ~~ρ-analytic~~

Th^m (continued): Moreover the fibration sequence can be extended

$$D_nF \to P_nF \to P_{n-1}F \to BD_nF$$

(~~what~~ this is saying that the fibration is a principal fibration).

Proof for X analytic: X k-connected, F reduced

$$\begin{tikzcd}
D_nF(X) \arrow[r] \arrow[d, dashed] & P_nF(X) \arrow[r, "nk-c"] \arrow[d, "k-c"'] & P_{n-1}F(X) \arrow[d] \\
\Omega P_{n-1}F(X)/P_nF(X) \arrow[r] & * \arrow[r] & P_{n-1}F/P_nF(X)
\end{tikzcd}$$

By Blakers-Massey this is $(n+1)\cdot k-2c-d$ ~~com~~ connected.

$\Rightarrow$ The map between the fibers is n-connected

$\Rightarrow$ The map is a homotopy equivalence.

1) Once we know this for an analytic functor we can extend this to Finitary functors using the fact that every finitary functor is equivalent to a filtered hocolim of analytic functors.

2) The proof can be extended to all homotopy functors using functor versions of Blakers-Massey theorem.

(Anel - Biedermann - Finster - Joyal)

3) Can prove this completely formally (Goodwillie).

<u>Corollary</u> : There is an equivalence of homotopy categories

$$\begin{array}{ccc} n\text{-homogenous } [Top_*, Top_*] & \longleftrightarrow & n\text{-homogenous } [Top_*, Spectra] \\ \text{functors} & & \text{functors} \end{array}$$

<u>Prop</u> : There is an equivalence

$$\begin{array}{ccc} n\text{-homogenous } [Top_*, Spectra] & \longleftrightarrow & \text{Symmetric } [Top_*^{\bullet}, Spectra] \\ & & \text{multi-linear} \\ & & \text{functors} \end{array}$$

$$\left\{ \begin{array}{c} F(X_1, \cdots X_n) \xrightarrow{\cong} F(X_{\sigma(1)}, \cdots, X_{\sigma(n)}), \\ \text{linear in each variable} \end{array} \right\}$$

$$\left(X \longmapsto F(X, \dots, X)_{h\Sigma_n} \right) \longleftarrow F$$

For this to be a homotopy functor this functor should take values in Spectra.

$$G \longmapsto cr_n G(X_1, \dots, X_n) = \text{total hofiber of}$$

$$\begin{bmatrix} G(X_1 \vee \cdots \vee X_n) & \longrightarrow & \\ \downarrow & & \ddots \\ & & \end{bmatrix}$$

$$G\left(\bigvee_{i \in S} X_i\right) = G(S)$$

$\underline{\underline{\text{Th}^m}}$: A finitary homogenous functor of deg n is equivalent to one of ~~(C, is a spectrum)~~ the form $X \longmapsto (C_n \wedge X^{\wedge n})_{h\Sigma_n}$

C_n is a spectrum with action of Σ_n.

- Going back to the Taylor tower

$$\begin{array}{ccccc} & & \vdots & & \\ F(X) & \longrightarrow & P_nF(X) & \longleftarrow & D_nF(X) \simeq \Omega^\infty (C_n \wedge X^{\wedge n})_{h\Sigma_n} \\ & & \downarrow & & \\ & & P_{n-1}F(X) & & \end{array}$$

Let $\boxed{\partial_n F := C_n}$ call this the derivative of F at a point.

$$\partial F = (\partial_1 F, \partial_2 F, \ldots, \partial_n F, \ldots)$$

$\underline{\underline{\text{Th}^m}}$ (Goodwillie)

$\partial_n F$ is the stabilization of the n^{th} cross effect.

- $cr_n F(X_1, \ldots, X_n) = \underset{\text{hofib}}{\text{total}} \left[S \mapsto F\left(\bigvee_{i \in S} F(X_i) \right) \right]$

If $X_i = *$ then $cr_n F(X_1 \ldots X_n) \simeq *$.

- We have structure maps

$$\Sigma\, cr_n F(X_1, \ldots, X_n) \longrightarrow cr_n F(X_1, \ldots, \Sigma X_i, \ldots X_n)$$

- Then $\partial_n F$ is the spectrum

$$(\partial_n F)_{nd} = \Omega^{(n-1)d} cr_n (S^d, \ldots, S^d) \qquad \Sigma_n \text{ also acts on } \Omega^{(n-1)\cdot d}$$

eg: 1) $F : \text{Spectra} \longrightarrow \text{Spectra}$

$$\underline{\underline{X}} \longrightarrow \Sigma^\infty \Omega^\infty \underline{\underline{X}}$$

$$cr_n(\Sigma^\infty\Omega^\infty)(x_1, \cdots, x_n) = \underset{\text{n fiber}}{\text{total}} \left[S \mapsto \Sigma^\infty \Omega^\infty \left[\bigvee_{i \in S} X_i \right] \right]$$

$$\left[\begin{array}{l} \Sigma^\infty \Omega^\infty (\bigvee_{i \in S} X_i) \\ = \Sigma^\infty (\prod_{i \in S} \Omega^\infty X_i) \end{array} \right]$$

$$\simeq \Sigma^\infty (\Omega^\infty X_1 \wedge \cdots \wedge \Omega^\infty X_n)$$

$$\partial_n (\Sigma^\infty \Omega^\infty)_{nd} = \underset{d \to \infty}{\text{hocolim}} \; \Omega^{nd} \Sigma^\infty (\Omega^\infty \Sigma^\infty S^d \wedge \cdots \Omega^\infty \Sigma^\infty S^d)$$

$$\Big\uparrow \sim nd-c \text{ connected}$$

$$\underset{d \to \infty}{\text{hocolim}} \; \Omega^{nd} \Sigma^d (\Sigma S^d \wedge \cdots \wedge S^d)$$

$$\Big\uparrow \text{SI}$$

$$\Sigma^\infty S^0$$

$X \longrightarrow \Omega^\infty \Sigma^\infty X$

$\downarrow$

K-conn

$\Rightarrow$ the map $2k-1$ connected by Freudenthal

with trivial action of Σ_n

Conclusion: $\partial_n(\Sigma^\infty \Omega^\infty) \simeq S^0$

So we have a tower of spectra:

$$\begin{array}{ccc} & & \vdots \\ & & \downarrow \\ \Sigma^\infty \Omega^\infty \underline{X} & \longrightarrow & P_n \Sigma^\infty \Omega^\infty X \longleftarrow \left(X^{\wedge n}_{h\Sigma_n} \right) \\ & & \downarrow \\ & & P_{n-1} \Sigma^\infty \Omega^\infty X \\ & & \downarrow \\ & & \vdots \end{array}$$

Converges for 0-connected spectra.

- If X is a suspension spectrum then the tower splits.

- $\partial_* \Sigma^\infty \Omega^\infty = (\underline{\underline{S}}^0, \underline{\underline{S}}^0, \cdots, \underline{\underline{S}}^0, \cdots)$

  this is the commutative co-operad.

  $\Uparrow$

  $\Sigma^\infty \Omega^\infty$ is a comonad in spectra.

  And the Spanier-Whitehead dual of $\partial_* \Sigma^\infty \Omega^\infty$ is an operad.

- $F : Top_* \longrightarrow Top_* \xrightarrow{\Sigma^\infty} Spectra$

  Consider $\Sigma^\infty \circ F : Top_* \longrightarrow Spectra.$

  $\Sigma^\infty F$ is a right comodule over $\Sigma^\infty \Omega^\infty$.

  i.e. $\exists$ $\Sigma^\infty F \longrightarrow \Sigma^\infty \Omega^\infty (\Sigma^\infty F)$.

$\underline{\underline{\text{Th}^m}}$: $\partial_* \Sigma^\infty F$ is a right comodule over $\partial_* (\underset{\text{comm}}{\underset{\parallel}{\Sigma^\infty \Omega^\infty}})$ and

Arone Ching $\partial_* F = \text{cobar}(\mathbb{1}_*, \underline{\underline{\text{com}}}, \partial_* \Sigma^\infty F)$.

- Recall the composition product of symmetric sequences:

  $M_* = \{M_n\}$, $\Sigma_n \curvearrowright M_n$, $P_* = \{P_n\}$ $\Sigma_n \curvearrowright P_n$.

  $$(M \circ P)_n = \bigvee_i \left[\bigvee_{\underline{n} \twoheadrightarrow \underline{i}} \left(M_i \wedge P_{n_1} \wedge \cdots \wedge P_{n_i} \right)_{S_i} \right] \left[\begin{array}{l} n_1, \cdots, n_i \text{ preimages} \\ \text{of } i \end{array} \right.$$

This a (non-symmetric) monoidal product

operads = monoids wrt $\circ$.

$$\partial_* F = \mathbb{1}_* \overset{comm}{\circ} \partial_\circ F = \mathrm{tot}\left(\begin{array}{c} \mathbb{1}_* \circ \partial_* \Sigma^\infty F \\ \downdownarrows\uparrow \\ \mathbb{1}_* \circ comm \circ \partial_* \Sigma^\infty F \\ \downarrow\downarrow\downarrow\uparrow \\ \vdots \\ \mathbb{1}_* \circ comm^{\circ k} \circ \partial_* \Sigma^\infty F \\ \vdots \end{array} \right)$$

where $\mathbb{1}_* = (S^0, *, *, * \cdots)$ = unit for $\circ$

Ex: $F = id_{Top}$

$X \longmapsto \Sigma^\infty X$

$$\partial_* id_{Top_*} = \mathbb{1}_* \overset{comm}{\circ} \mathbb{1}_*$$

= Koszul dual of the commutative co-operad.

= "Spectrum" version of Lie operad.

Review of Symmetric Sequences, Operads

- $(\mathcal{C}, \wedge, \mathbb{S})$ closed symmetric monoidal category

  assume $\mathcal{C}$ has an initial object & final object $*$.

- Σ category of finite sets and bijections.

- Symmetric sequence: $P: \Sigma \longrightarrow \mathcal{C}$

  $\Leftrightarrow$ sequence of objects $P_0, P_1, P_2, \ldots$

  with an action of Σ_n on P_n.

$\mathcal{P}, \mathcal{Q} : \Sigma \to \mathcal{C}$, $\quad P(0) = * = Q(0)$

$$(P \circ Q)_n = \bigvee_i \Big(\bigvee_{\alpha : \underline{n} \to \underline{i}} P_i \wedge Q_{\alpha^{-1}(n_1)} \wedge \cdots \wedge Q_{\alpha^{-1}(n_i)} \Big)_{\Sigma_i}$$

unit $\quad \mathbb{1}_* = (*, \mathbb{S}, *, *, \ldots)$

- An operad $\mathcal{O}$ is a monoid w.r.t. composition product.

- Left, right, bi-modules over $\mathcal{O}$ are symmetric sequences ~~equ~~ P equipped with maps: $\quad P \circ \mathcal{O} \to P$ (right), $\quad \mathcal{O} \circ P \to P$ (left)

$\underline{n} \to \underline{i} \qquad P_i \wedge \mathcal{O}_{n_1} \wedge \cdots \wedge \mathcal{O}_{n_i} \longrightarrow P_n$ (right)

$\qquad\qquad \mathcal{O}_i \wedge P_{n_1} \wedge \cdots \wedge P_{n_i} \longrightarrow P_n$ (left)

Given right, left $\mathcal{O}$ modules R, L we can form a bar construction:

- $\mathrm{Bar}(R, \mathcal{O}, L) = R \circ L \leftleftarrows R \circ \mathcal{O} \circ L \leftleftarrows R \circ \mathcal{O} \circ \mathcal{O} \circ L \cdots \quad R \circ \mathcal{O}^{\circ n} \circ L \cdots$

- $\mathcal{O}$ = reduced if $\quad \mathcal{O}_0 = * \quad \mathcal{O}_1 = S^1$

There is a map $\mathcal{O} \longrightarrow \mathbb{1}$. this makes $\mathbb{1}$ an $\mathcal{O}$-bimodule.

$\mathrm{Bar}(\mathbb{1}, \mathcal{O}, \mathbb{1}) = \mathrm{Bar}(\mathcal{O})$

<u>Th</u>^{m}: ~~The~~ $B(\mathcal{O})$ is a co-operad for any reduced operad.

(Ginzburg, Kapranov,
Fresse, Ching)

<u>eg</u>: $\quad \mathrm{Comm} = (*, S^0, S^0, \cdots)$

$B(\mathrm{comm})_n = \mathrm{comm}^{\circ n}$

$(\mathrm{comm}^{\circ 2})_n = \bigvee_i \left[\text{surjections } \underline{n} \to \underline{i} / \Sigma_n \right]$

$= \text{partions of } n_+$

$(\mathrm{comm}^{\circ 3})_n = \{ (\lambda_1 \le \lambda_2) : \text{Part of } \underline{n},\ \lambda_1 \text{ a refinement of } \lambda_2 \}$

Let P_n be the poset of partitions of $\underline{n}$ ordered by refinement.

$P_1 = \bullet \qquad P_2 = \begin{tikzcd} (12) \\ (1)(2) \arrow[u] \end{tikzcd} \qquad P_3 = \begin{tikzcd} & (123) & \\ (12)(3) \arrow[ur] & (13)(2) \arrow[u] & (13)(2) \arrow[ul] \\ & (1)\,(2)\,(3) \arrow[ul] \arrow[u] \arrow[ur] & \end{tikzcd}$

we ~~see~~ Geomeric realizations: 

Def: $T_n = \dfrac{|P_n|}{|\partial P_n|}$, $\partial P_n = \left\{ \begin{array}{l} \text{simplicial subset of } P_n \text{ consisting of} \\ \text{all simplices that do not contain} \\ \text{both the initial and the final object} \\ \text{as vertices.} \end{array}\right.$

lemma: • $B(\mathrm{Comm}_n) \cong T_n \circlearrowleft \Sigma_n$

• $T_n \simeq \bigvee_{(n-1)!} S^{n-1}$, Σ_{n-1} equivariantly

$$\bigvee_{(n-1)!} S^{n-1} = \Sigma_{n-1\,+} \wedge S^{n-1}$$

By Ching, $T_1, T_2, \dots, T_n$ is a co-operad in Top_*.

• Cooperad : $\exists\ T_n \longrightarrow T_i \wedge T_{n_1} \wedge \dots \wedge T_{n_i}$ $\forall\ \alpha: \underline{n} \to \underline{i}$, $n_i = \alpha^{-1}(i)$

associative, unital.

• $T_n^{\vee} = F(T_n, \mathbb{S})$ is an operad.

• The groups $\{H^{n-1}(T_n)\}$ form an operad in Ab

• $\{H^{n-1}(T_n) \otimes \text{sign representation}\} \cong$ Lie operad

Back to Calculus of Functors:

$F : Top_* \to Top_*$ reduced,

$\Sigma^\infty \circ F : Top_* \to Top_* \longrightarrow$ Spectra.

$\partial^n(\Sigma^\infty F) := \partial_n(\Sigma^\infty F)^\vee$

<u>Th</u>^{m}: $\partial^n(\Sigma^\infty F)$ is a left module over $\partial^*(\Sigma^\infty \Omega^\infty) \simeq$ comm.

$\partial^* F \cong \mathrm{Bar}(\mathbb{1}, \mathrm{comm}, \partial^* \Sigma^\infty F)$.

<u>Ex</u>: $F = \mathbb{1}_{Top}$ ← id_{Top} , $\partial^* \Sigma^\infty_\infty = \mathbb{1}$

$\Sigma^\infty F(x) = \Sigma^\infty x$

$\mathcal{B}(\Theta) = \{\Sigma^\infty T_n\}_{n=1}^\infty = \partial^* \mathbb{1}_{Top}$

$\partial_* \mathbb{1}_{Top} = F(T_n, \Sigma^\infty) \simeq \bigvee_{(n-1)!} S^{n-1}$

So the Taylor tower of $id : Top_* \to Top_*$ has the following description:

$$\begin{tikzcd}
X \arrow[r] \arrow[dr] \arrow[dddr] & P_n\, id(X) \arrow[d] & \Omega^\infty\left(\mathrm{Map}(T_n, \Sigma^\infty X^{\wedge n})_{h\Sigma_n}\right) \arrow[l] \\
& P_{n-1}\, id(X) \arrow[d] & \\
& \vdots \arrow[d] & \\
& P_1\, id(X) = \Omega^\infty \Sigma^\infty X &
\end{tikzcd}$$

$\underline{\underline{\text{Th}^m}}$: $F : Top_* \longrightarrow Top_*$ then $\partial_* F$ is a bimodule over $\partial_* id$.

If $G : Top_* \longrightarrow Top_*$ then
$$\partial_*(FG) \simeq Bar(\partial_* F, \partial_* id, \partial_* G) = \partial_* F \underset{Lie}{\circ} \partial_* G.$$

Turns out that when $X = S^d$ the identity functor has additional features not predicted by the general theory.

- Let $X = S^{2\ell+1}$

$$X \longrightarrow P_1 id(X) = \Omega^\infty \Sigma^\infty X$$

is a rational equivalence,

$\Leftrightarrow$ v_0-periodic equivalence.

- $Maps(T_n, \Sigma^\infty X^{\wedge n})_{h\Sigma_n} \simeq *$ unless $n = p^k$ for some prime p.

- If $n = p^k$ then this spectrum is p-local and rationally trivial.

- when $n = p^k$, then we can replace the action of Σ_{p^k} on T_{p^k} with action of a smaller group on a smaller space

B_k = Tits building for $Gl_k(\mathbb{F}_p)$

$Aff_k(\mathbb{F}_p) = Gl_k(\mathbb{F}_p) \ltimes \mathbb{F}_p^{\,k}$ (affine transformations)

$$B_k \hookrightarrow T_k$$

$\underline{\text{Th}^m}$: The inclusion $B_k \hookrightarrow T_{p^k}$ induces a p-local equivalence

$$\mathrm{Map}\left(T_{p^k}, \Sigma^\infty X^{\wedge p^k}\right)_{h\Sigma_n} \longrightarrow \mathrm{Maps}\left(B_k, \Sigma^\infty X^{\wedge p^k}\right)_{h\mathrm{Aff}_n(\mathbb{F}_p)}$$

$$\wr\!\wr$$

(*) $$\mathrm{Maps}\left(B_k, \Sigma^\infty X^{\wedge p^k}_{h(\mathbb{Z}/p)^k}\right)_{hGL_n(\mathbb{F}_p)}$$

(only true when $X = S^{2l+1}$).

$\boxed{\text{Steinberg Representation of } GL_n(\mathbb{F}_p).}$

- Mitchell: discovered a type n spectrum for each n.

$$St_k\left(\Sigma^\infty O(p^k)\Big/(\mathbb{Z}/p)^k\right)_{hO(p^k)}$$

St_k = steinberg idempotent inside $Gl_k(\mathbb{F}_p)$.

This turns out is equivalent to (*).

$\underline{\underline{\text{Th}^m}}$: For $l < k$, $\mathrm{Maps}\left(T_{p^k}, \Sigma^\infty X^{\wedge p^k}\right)_{hS_{p^k}}$ has trivial v_l periodic homotopy.

$\Rightarrow$ in v_l periodic homotopy the tower is finite.

$\underline{\underline{\text{Th}^m}}$: The tower converges in v_l-periodic homotopy when X is a sphere.