Topological Automorphic Forms – Minnesota

Want to understand $\mathcal{S}$-stable homotopy category

Using cells :

1. $\mathrm{Ext}_{\pi_*^S}(\pi_* X, \pi_* Y) \Longrightarrow [X, Y]$
2. $\mathrm{Tor}_{\pi_*^S}(\pi_* X, \pi_* Y) \Longrightarrow \pi_*(X \wedge Y)$
3. $H_*(X, \pi_*^S) \Longrightarrow \pi_*^S(X)$

Disadvantages :

1. Need $\pi_* X$, $\pi_*^S = \pi_* S$
2. π_*^S is an unfriendly ring
3. Deceptive ?

"$\mathcal{S}$ is like $Ch(\pi_*^S\text{-mod})$"

Adams Spectral Sequence :

1. $\mathrm{Ext}_{A^*}(H^* Y, H^* X) \Rightarrow [X, Y]$
2. $H^*(X) \hat{\otimes} H^*(Y)$

"$\mathcal{S}$ is like $Ch(A\text{-mod})$"

Good : 1. Organize things
2. Allow computation

Bad : 1. Hides information
2. Don't see things
3. Deceptive ?

Adams Novikov Spectral Sequence :

$\mathrm{Ext}_{MU_*MU}(MU_* X, MU_* Y) \Rightarrow [X, Y]$

Formal group laws :

FGL/R is a power series, $F(x,y) =: x +_F y = \sum a_{ij} x^i y^j$ such that

i) $x +_F 0 = x$
ii) $x +_F y = y +_F x$
iii) $(x +_F y) +_F z = x +_F (y +_F z)$

Chern classes : $c_1(\mathcal{L}) +_F c_1(\mathcal{L}') = c_1(\mathcal{L} \otimes \mathcal{L}')$

MU_* carries universal FGL, MU_*MU parametrizes (strict) isos of FGl's

Given R spectrum with appropriate multiplications :

1. MU_*R caries a FGL
2. operations on MU_*R change of co-ordinates on the FGL.

Sometimes we can write down rings + FGL's (or diagrams of them) and realize them by spectra.

eg: $H\mathbb{Z} \longleftrightarrow F(x,y) = x+y$

$HBU \longleftrightarrow F(x,y) = x+y-\beta xy$

Over finite fields, $\mathbb{F}_q$, there are a lot of FGL's 1) height $n \in \mathbb{Z}_{>0}$ 2) Cohomology ring

↳ These give rise to Morava K-theories

Universal deformations of FGL's over finite fields $W(\mathbb{F}_q)[[u_1, \dots u_{p-1}]]$

↳ Morava E-theories / Lubin Tate spectra, coherently commutative multiplication

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Automorphic forms:

Eg: Quadratics:

R ring, quadratic: x^2+bx+c,

translation: $x \mapsto x+t \rightsquigarrow b \mapsto b+2t$, $c \mapsto c+bt+t^2$

if $\frac{1}{2} \in R$, we can get a complete invariant (a only) $b^2-4c = \Delta$

if $2=0$ in R, then there is a new invariant b.

$\mathbb{Z}[b,c]$ parametrizes quadratics $\circlearrowleft$ action by translations Γ

$\mathbb{Z}[b,c]^{\Gamma} = \mathbb{Z}[\Delta]$

$$
\begin{tikzcd}
\mathbb{Z}[b,c] \arrow[r] \arrow[d, squiggly] & \mathbb{Z}/2[b,c] \arrow[r] \arrow[d, squiggly] & 0 \\
\mathbb{Z}[\Delta] \arrow[r] & \mathbb{Z}/2[b] \arrow[r] & H^1
\end{tikzcd}
$$

There is higher cohomology for these quadratics $-$ $\mathbb{Z}[\Delta,\eta]/2\eta \rightsquigarrow E_2^{ANSS}$ for $\pi_* KO$

Given a quadratic, $\exists$ fgl $x +_F y = \dfrac{x+y+bxy}{1-cxy}$

Eg: Elliptic curves$/R$

$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$

$y \mapsto y + rx + s$

$x \mapsto x + t$

$\circlearrowleft \mathbb{Z}[a_1, a_2 \cdots a_6]$

Invariants:

$\mathbb{Z}[c_4, c_6, \frac{c_4^3 - c_6^2}{1728}]$ Δ + higher cohomology mod 2: a_1, mod 3: $(a_2 - a_1^2)$

Elliptic curves have multiplication with identity $u = -y/x$

expand multiplication of curve in terms of u

cohomology: E_2^{ANSS} for computing π_* tmf ex: Formal group laws MU_*, change of variables using power series

$\Rightarrow E_2^{ANSS}$ for computing π_*^S.

Quadratics : Detect information from heights ≤ 1

Elliptic curves : height 2

K3 surfaces : heights ≤ 10

? : higher heights $\rightsquigarrow y^{p-1} = x^p - x \rightsquigarrow$ Jacobians, Abelian varieties/schemes