Heegard splittings of 3 manifolds

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It is difficult to imagine 3 dimensional manifolds as they require at least 4 dimensions to be embedded in. Instead we describe 3 dimensional manifolds using "gluing diagrams" which are obtained by gluing sides of solid polytopes in 3 dimensions.

1 The three dimensional sphere S^3

The 3 dimensional sphere is defined as

$$S^3 = \{(x, y, z, t) : x^2 + y^2 + z^2 + t^2 = 1\}$$

If we consider the standard sphere S^2 of radius 1 and intersect it by a vertical plane $z = c$ we get an S^1 if $-1 < c < 1$, a point if $c = \pm 1$ and an empty set otherwise. Similarly if we intersect S^3 with $t = c$ we get a sphere S^2 if $-1 < c < 1$, a point if $c = 1$ and an empty set otherwise.

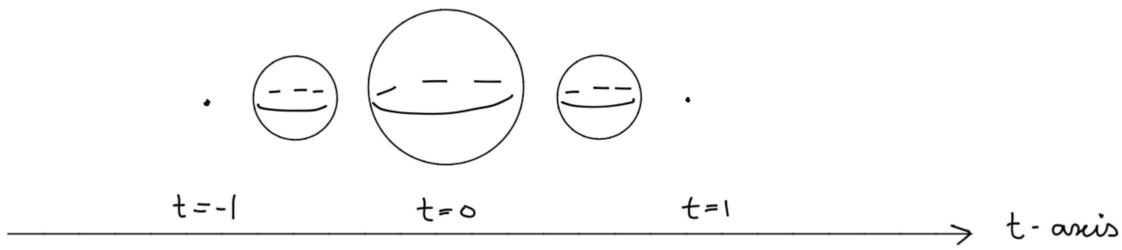

Fig. 1: S^3 as a sequence of spheres

Another way to visualize S^3 is by noticing that if we remove one point from S^3 we get the space $\mathbb{R}^3$. Conversely this implies that S^3 is the one point compactification of $\mathbb{R}^3$ i.e. $S^3 = \mathbb{R}^3 \cup \{*\}$.

1.1 Hemispheres and solid tori

Consider the two hemispheres

$$S^3_+ = \{(x, y, z, t) : x^2 + y^2 + z^2 + t^2 = 1, t \geq 0\}$$
$$S^3_- = \{(x, y, z, t) : x^2 + y^2 + z^2 + t^2 = 1, t \leq 0\}$$

Both S^3_+ and S^3_- are homeomorphic to the three disc

$$D^3 = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1$$

via the projection maps $S^3_+ \to D^3$ and $S^3_- \to D^3$. Hence we can think of S^3 as two 3 dimensional discs being glued together along the boundary.

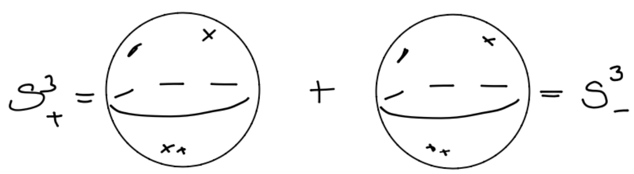

Fig. 2: S^3 as two 3 discs S^3_+, S^3_- glued together

We can perform surgery on S^3_+ and S^3_- by removing and adding a solid cylinder respectively.

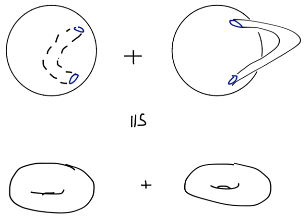

Fig. 3: S^3 as two solid tori glued together

We can also remove and add multiple solid cylinders to get

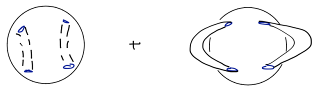

Fig. 4: S^3 as two solid genus 2 surfaces glued together

These are called Heegard splittings of S^3.

1.2 Heegard splittings of 3 manifolds

Let M be a 3 manifold and suppose we have a "gluing diagram" of M consisting of multiple polytopes. However unlike for 2 manifolds, we do not allow different sides of the same polytope to be glued to each other.

For example, S^3 has a gluing diagram consisting of two solid cubes with corresponding sides glued.

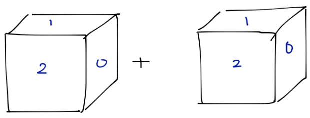

Fig. 5: S^3 as a sum of two solid cubes

Let X be the set of edges of gluing diagram which have been thickened. And let Y be the complement of X in S^3, $Y = S^3 \setminus X$.

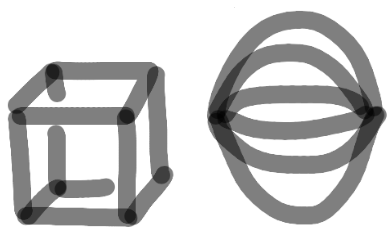

Fig. 6: Heegard splitting of S^3, both X and Y are solid genus 5 surfaces.

It is not hard to see that both X and Y are solid surfaces of the same genus. This shows that every 3 dimensional manifold M which has a "gluing diagram" can be obtained by gluing two solid surfaces of the same genus. This is called the **Heegard splitting** of the manifold M.

2 Exercises

Exercise 2.1. Construct "gluing diagrams" for $T \times S^1$ and $S^2 \times S^1$ using the gluing diagram of T and S^2 and thinking of S^1 as two endpoints of $[0, 1]$ glued together.

Exercise 2.2. Construct Heegard splittings for $T \times S^1$ and $S^2 \times S^1$.

Exercise 2.3. Why does this procedure not work for two dimensional manifolds?