PROBLEM SET 09

Part 1 - Trigonometric functions

You should probably make a table of trig identities before you start this problem sheet.

Q.1. (1) Using the angle sum formula for sin and cos prove that for $m, n > 0$ integers

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases}$$

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0$$

(2) Show that the minimum value of

$$g(a) = \int_{-\pi}^{\pi} \left(f(x) - a \cos nx\right)^2 dx$$

occurs when $a = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$.

(3) Repeat part (2) for $g(a) = \int_{-\pi}^{\pi} \left(f(x) - a \sin mx\right)^2 dx$.

Q.2. (1) Prove the following identities[*]

$$1/2 + \cos x + \cos 2x + \cdots + \cos nx = \frac{\sin\left(n + \frac{1}{2}\right)x}{2 \sin\left(x/2\right)}$$

(2) Use these to find $\int_0^a \cos x \, dx$ (for $a \in [0, \pi/2]$) directly from the definition of integral.

Q.3. (1) Show that $\lim_{\lambda \to \infty} \int_a^b \sin \lambda x \, dx = 0$. What does this mean geometrically?

(2) Show that if s is a step function then $\lim_{\lambda \to \infty} \int_a^b s(x) \sin \lambda x \, dx = 0$.

(3) Using Q.7 from Problem Set 07 show that if f is an integrable function then

$$\lim_{\lambda \to \infty} \int_a^b f(x) \sin \lambda x \, dx = 0$$

This theorem is called the **Riemann Lebesgue lemma**.

These trig identities (and many more) naturally show up while studying Fourier series.

[*] Hint: Use the identity $\sin(k + 1/2)x - \sin(k - 1/2)x = 2sin(x/2)\cos(kx)$

Part 2 - Logarithms and exponentials

Q.4. Use the identity $(\log f)' = f'/f$ to find f' for the following functions. This trick is sometimes called **logarithmic differentiation**.

(1) $f(x) = \dfrac{(3-x)^{1/3}x^2}{(1-x)(3+x)^{2/3}}$

(2) $f(x) = (\sin x)^{\cos x} + (\cos x)^{\sin x}$

Q.5. The **hyperbolic** functions are defined as

$$\sinh x := \frac{e^x - e^{-x}}{2} \qquad \cosh x := \frac{e^x + e^{-x}}{2}$$

(1) Simplify each of the following:

$$\cosh^2 x - \sinh^2 x,$$
$$\sinh' x, \quad \cosh' x,$$
$$\sinh x.\cosh y + \sinh y.\cosh x,$$
$$\cosh x.\cosh y + \sinh x.\sinh y.$$

(2) Determine $(\sinh^{-1})'x$, $(\cosh^{-1})'x$.

(3) Find an explicit formula for $\sinh^{-1} x$ and $\cosh^{-1} x$.[†]

(4) Compute $\int_a^b \frac{1}{\sqrt{x^2+1}}\, dx$ and $\int_a^b \frac{1}{\sqrt{x^2-1}}\, dx$.

Q.6. (1) Prove that if r is a root of the polynomial equation

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0 \tag{1}$$

then the function $y(x) = e^{rx}$ satisfies the differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 \tag{2}$$

(2) Prove that if r is a double root of the polynomial equation (1) then $y = x.e^{rx}$ is also a solution of the differential equation (2).[‡]

(3) Prove that if y_1 and y_2 satisfy (2) then so does $c_1y_1 + c_2y_2$ where c_1, c_2 are arbitrary real numbers.

(4) Find solutions of the differential equation $y''' - y' = 0$. (Be careful!)

The differential equation (2) is called a **constant coefficient linear differential equation** and this is the standard way to solve it.

Q.7. (1) Sketch the graph of $\dfrac{\log x}{x}$ for $x > 0$.

(2) Determine, with proof, which one is larger: e^π or π^e?

Q.8. Find the following limits:

(1) $\lim\limits_{x\to 0} \dfrac{\log(1+ax)}{x}$

(2) $\lim\limits_{x\to\infty} x.\log(1+a/x)$

(3) $\lim\limits_{x\to\infty} (1+1/x)^x$

(4) $\lim\limits_{x\to\infty} (1+a/x)^x$

[†] Hint: Substitute $y = e^x$ in the right hand side of $\sinh x = \dfrac{e^x - e^{-x}}{2}$.

[‡] Hint: Recall that if r is a double root of a polynomial $f(x)$ then r is also a root of $f'(x)$.

Part 3 - Computations

From now on the Friday HWs will be about computing integrals. I'll usually assign the problems from the book.

Q.9. For this week do Q.1 and Q.2 on Pg. 377-378 from Ch.19.
These might look like a lot of problems but they all have very short solutions, usually one trick will give you the answer.