# The Poincaré Conjecture

Patrick Little

Hitchhiker's Guide to Algebraic Topology

# Quick Notes

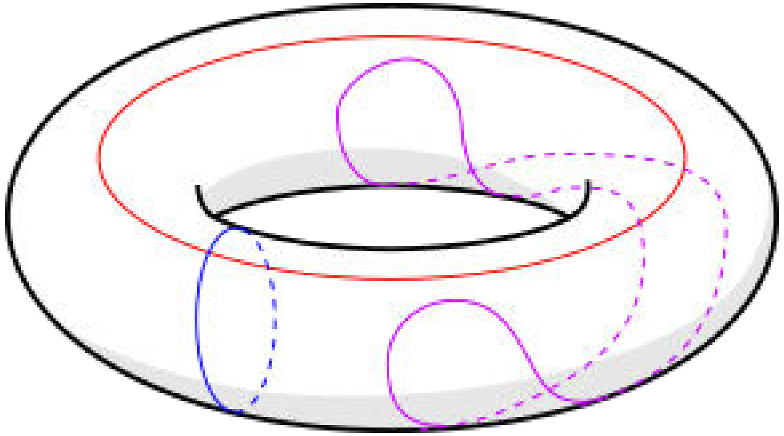
- Math
- Pronunciation
- John Morgan’s lecture, posted to course website

# Poincaré

- Henri Poincaré (1854-1912)

# Conjecture

- Are 3-manifolds equivalent to the 3-sphere? Or, if it looks like a 3-sphere (in terms of homology) is it a 3-sphere?
- No – Poincaré comes up with a new invariant: fundamental group (roughly: what are the unique kinds of loops you can draw on the manifold?) which distinguishes the Poincaré Homology Sphere (with a fundamental group of order 120) from the 3-sphere (trivial fundamental group - all loops can shrink to a point)

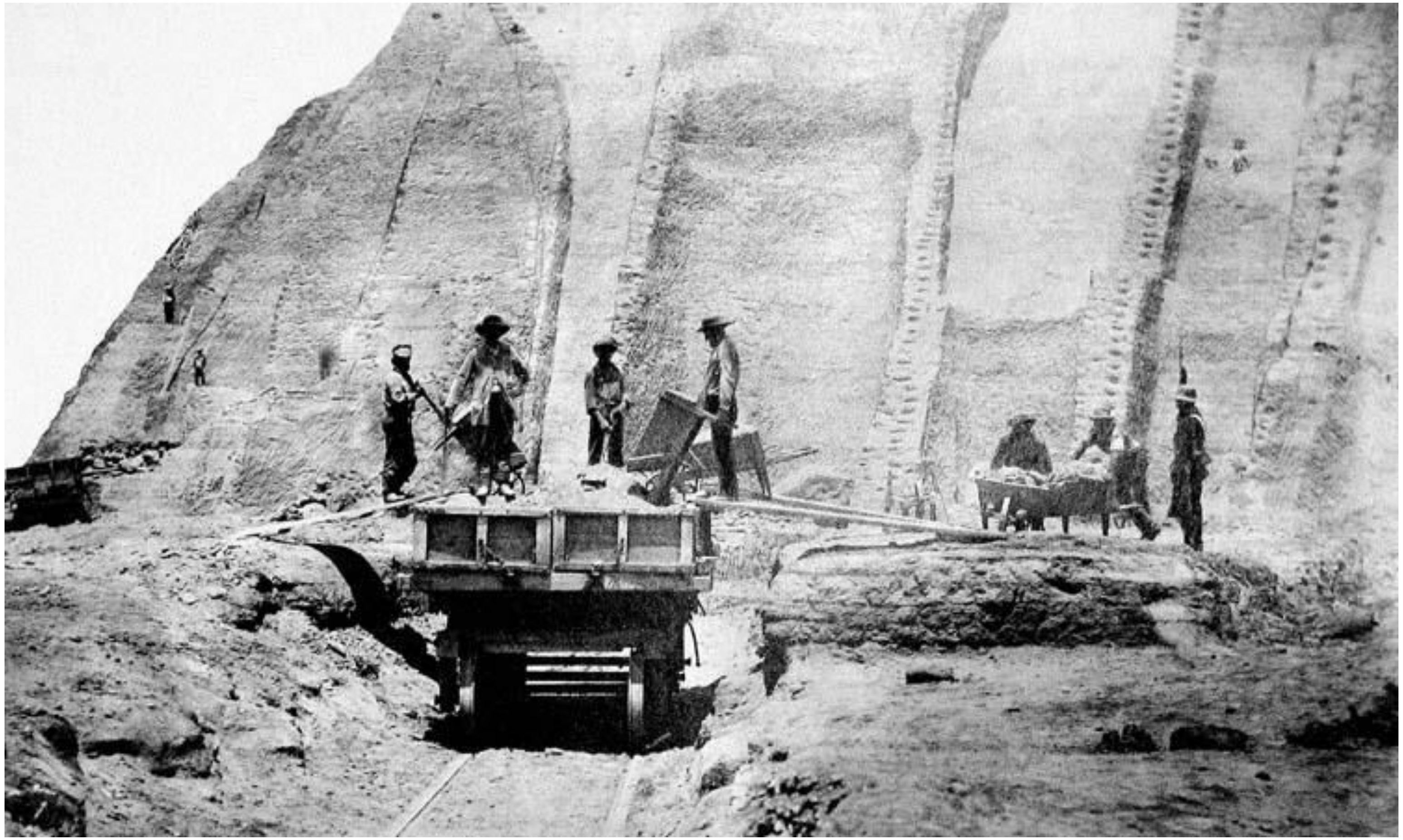


- So, if a compact 3-manifold looks like a sphere and is simply connected – is it equivalent to the 3-sphere?

# But

- “Mais cette question nous entrainerait trop loin”
- (“But this question would lead us too far”)

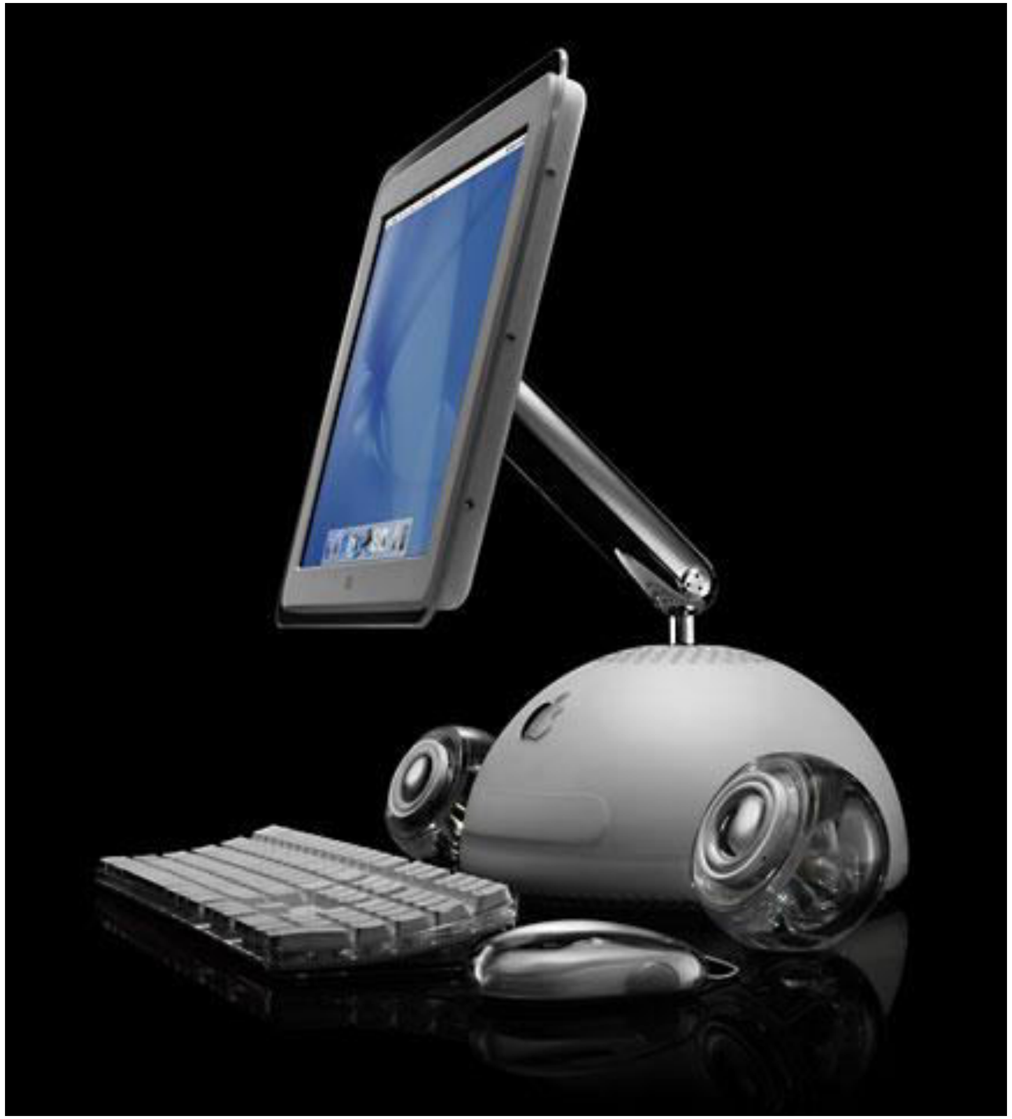
# Quite far: 1904-



# Developments

- (Dimensions 1 and 2 are trivial, apparently)
- 1960- Stallings, Smale prove demensions 7+
- 1961- Smale proves 5 and 6 (Fields medal, 1966)
- 1970- Pocket calculator invented
- 1981- Hamilton introduces Ricci flow
- 1982- Freedman proves 4 (Fields medal, 1986)
- 1982- Thurston poses geometrization conjecture (includes Poincaré)
- 2000- Poincaré Conjecture made one of 7 Millennium Prize Problems by the Clay Mathematics Institute, given $1 million bounty

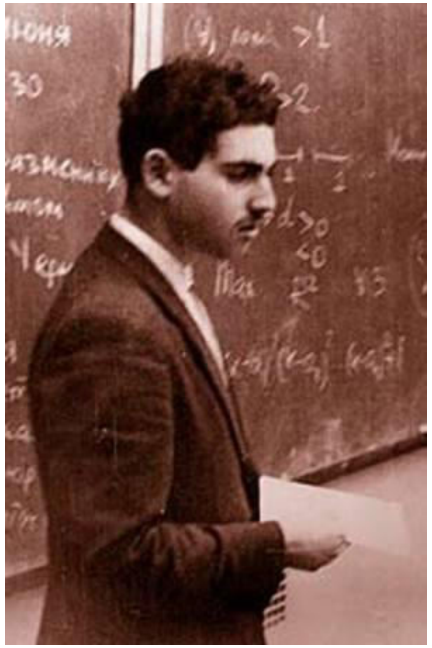
# To 2002



# Enter Perelman

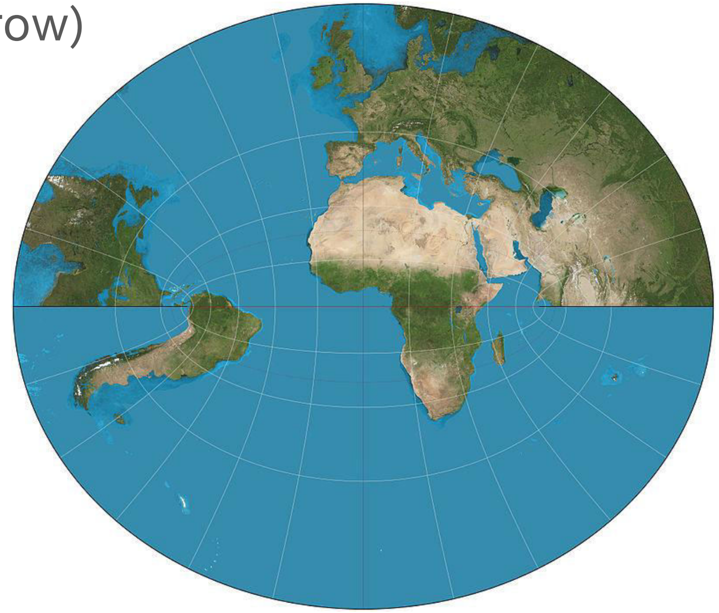
# Grigori Perelman

- Born June 13, 1966 in Leningrad (St. Petersburg)
- Perfect score and gold medal in 1982 IMO
- Honesty- anecdote that he refused to take of his excessively warm winter cap because he had promised his mother he wouldn't



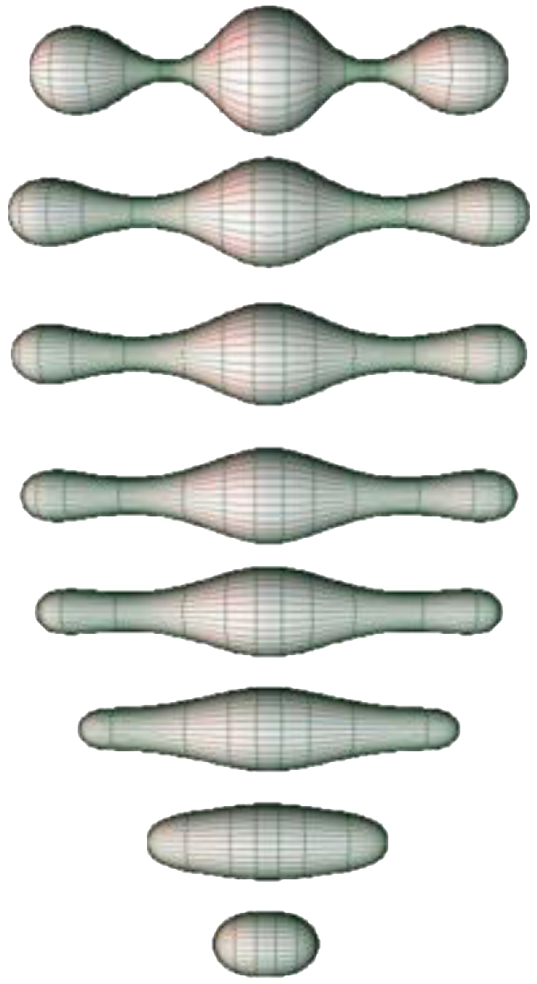
# Grigori Perelman

- Did post-doctoral work at NYU, Stony Brook, fellowship at Berkeley
- Offered positions at Stanford, Princeton
- Instead went back to St. Petersburg at the Steklov Institute, reportedly for $100 a month
- (Steklov, of course, the great-great-grand-student of astronomer Joseph von Littrow)



# Richard S. Hamilton

# Ricci Flow



Proof

# 1 Ricci flow as a gradient flow

**1.1.** Consider the functional $\mathcal{F} = \int_M (R + |\nabla f|^2)e^{-f}dV$ for a riemannian metric $g_{ij}$ and a function $f$ on a closed manifold $M$. Its first variation can be expressed as follows:

$$\delta\mathcal{F}(v_{ij}, h) = \int_M e^{-f}[-\triangle v + \nabla_i\nabla_j v_{ij} - R_{ij}v_{ij}$$

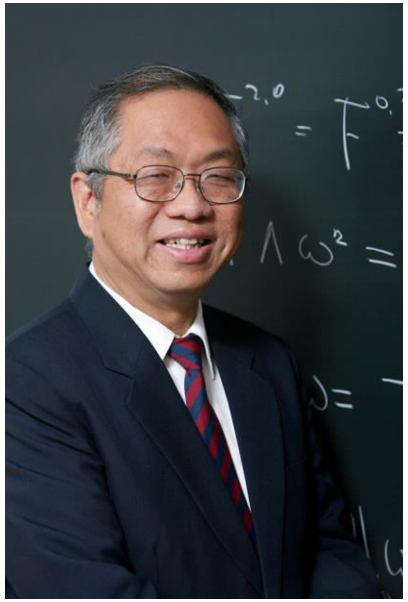
$$-v_{ij}\nabla_i f\nabla_j f + 2 < \nabla f, \nabla h > +(R + |\nabla f|^2)(v/2 - h)]$$

$$= \int_M e^{-f}[-v_{ij}(R_{ij} + \nabla_i\nabla_j f) + (v/2 - h)(2\triangle f - |\nabla f|^2 + R)],$$

where $\delta g_{ij} = v_{ij}$, $\delta f = h$, $v = g^{ij}v_{ij}$. Notice that $v/2 - h$ vanishes identically iff the measure $dm = e^{-f}dV$ is kept fixed. Therefore, the symmetric tensor $-(R_{ij}+\nabla_i\nabla_j f)$ is the $L^2$ gradient of the functional $\mathcal{F}^m = \int_M (R + |\nabla f|^2)dm$, where now $f$ denotes $\log(dV/dm)$. Thus given a measure $m$ , we may consider the gradient flow $(g_{ij})_t = -2(R_{ij} + \nabla_i\nabla_j f)$ for $\mathcal{F}^m$. For general $m$ this flow may not exist even for short time; however, when it exists, it is just the Ricci flow, modified by a diffeomorphism. The remarkable fact here is that different choices of $m$ lead to the same flow, up to a diffeomorphism; that is, the choice of $m$ is analogous to the choice of gauge.

# Scandal

- Enter Shing-Tung Yau of Harvard, a Fields medalist and an outspoken and occasionally controversial figure in Chinese and American mathematics
- Hamilton, on Yau: “He’s a great figure. He’s Shakespearean, larger than life. His virtues are larger than life, and his vices are larger than life.”
- Yau had shown that negative Ricci curvature ”happens” in closed 3-manifolds

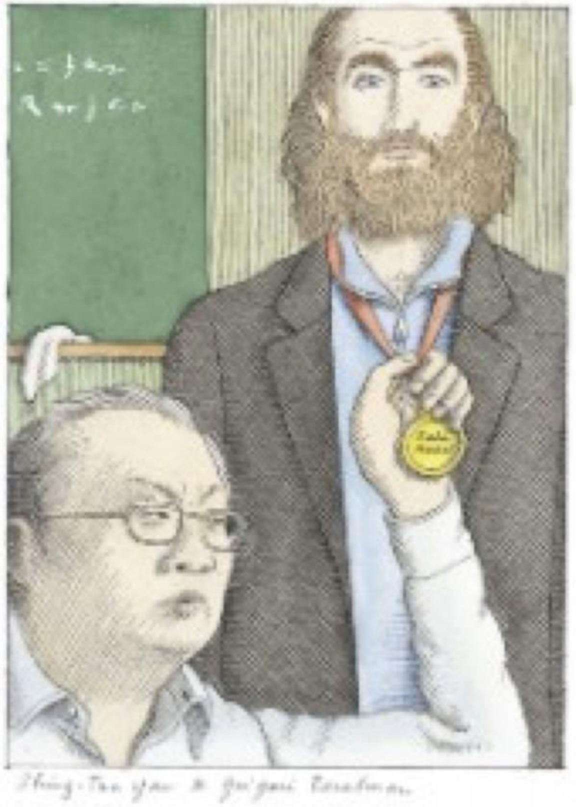


# Scandal

- Yau: “In Perelman’s work, many key ideas of the proofs are sketched or outlined, but complete details of the proofs are often missing”
- Yau promoted the first fully fleshed-out proof by Huai-Dong Cao and Xi-Ping Zhu, which seemed to imply the proof was their achievement, aided by theories of Hamilton and Perelman

# Backlash

- Some considered this attribution unfair
- Morgan: “There was no mystery they suddenly resolved [in Perelman’s proof]”



# Twist

- Cao and Zhu had connected the dots with work that was suspiciously similar to work Bruce Kleiner and John Lott had published
- Ultimately walked back statements, giving proper citations to Kleiner and Lott and emphasizing Perelman’s role

# No

- 2005- Perelman quits job in Russia, perhaps math entirely
- 2006- Science names his proof the "Breakthrough of the Year"
- 2006- Perelman declines Fields Medal and ~ $14,000
- 2010- Perelman declines Millenium Prize and $1,000,000
- Perelman: "I don't want to be on display like an animal in a zoo. I'm not a hero of mathematics. I'm not even that successful"

# End

- As bodybuilding.com user Larfleeze put it in 2015,

  “lol dude does not give one single flying ****, mirin”