k = comm ring with 1

R = associative unital k-algebra

M = R - bimodule

$n \geq 0 \qquad C_n(R,M) := M \otimes_k R^{\otimes n}$

$b: C_n(R,M) \longrightarrow C_{n-1}(R,M)$

$(m, r_1, \dots, r_n) \longmapsto (mr_1, \dots, r_n) + \sum_{i=1}^{n-1} (-1)^i (m, \dots, r_i r_{i+1}, \dots) + (-1)^n (a_n m, a_1, \dots a_{n-1})$

eg: $b(m,a) = ma - am$

Claim: $b \circ b = 0$ i.e. $\{C_*(M;R), b\}$ is a complex of R-bimodules

Proof: Set $b = \sum_{i=0}^{n} (-1)^i d_i$ where $d_i := (m, a_1, \dots, a_i a_{i+1}, \dots, a_n)$ if $0 < i < n$

$d_0 := (ma_1, \dots, a_n)$

Check: $d_i \circ d_j = d_{j-1} \circ d_i \quad i<j \qquad d_n := (ma_n, \dots, a_{n-1})$

Called pre-simplicial complex

Not yet a simplicial complex as degeneracy maps are missing.

Def^n: Presimplicial module C: collection of modules $\{C_n\}_{n\geq 0}$ and face maps $d_{i(n)}: C_n \longrightarrow C_{n-1} \quad 0 \leq i_{(n)} \leq n$

satisfying $d_{i(n-1)} d_{j(n)} = d_{j-1(n)} d_{i(n)}$ for $0 \leq i < j \leq n$

Lemma: $d := \sum_{i=1}^{n} (-1)^i d_{i(n)}$ then $d \circ d = 0$.

Def^n: $C(R;M) = \{C_*(R,M), b\}$ Hochschild chain complex

$H_n(R,M) := H_n(C_*(R,M))$ n-th H homology

Note: In general these are just k-modules

$$\boxed{H_0(R,M) = M/_{[R,M]} =: M_R \text{ module of co-invariants}}$$

$[R,M]$ = k-module generated by $mr - rm$

There is a left action of $Z(R)$ on $C_n(R,M)$

this induces endo of H-complex $\quad z(m, a_1, \dots, a_n) = (zm, a_1, \dots a_n)$

So $H_n(R,M)$ are modules over $Z(R)$

In particular, if R is commutative then $HH_*(R) := H_*(R,R)$ is an R-module

eg: if $R = M = k$, the H-complex is

$\longrightarrow k \xrightarrow{1} k \xrightarrow{0} k \xrightarrow{1} \cdots \longrightarrow k \xrightarrow{0} k \longrightarrow 0$

$HH_0(k) = k \qquad HH_n(k) = 0 \qquad n \geq 0$

eg: $R = T(V)$ tensor algebra on $V = k$ - vector space

$[R,R]$ = k-module generated by $x_1 \cdots x_{n-1}x_n - x_n x_1 \cdots x_{n-1}$ for $n \geq 2$

$\forall a,b \in R$ $ab = ba$ in $HH_0(T(V)) = R/_{[R,R]} \ncong S(V)$

eg: $x,y \in V$ $xyxy = yxyx \neq x^2y^2$ in $HH_0(T(V))$

Hochschild cohomology

$C^0(R,M) = M$

$C^n(R,M) := \mathrm{Hom}_k(R^{\otimes n}, M)$ $\forall n \geq 1$ (setting $R^{\otimes 0} = k$ we get back $C^0(R,M)$)

$\beta : C^n(R,M) \longrightarrow C^{n+1}(R,M)$

$f \longmapsto \left((a_1 \cdots a_{n+1}) \longmapsto a_1 f(a_2, \cdots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i f(a_1 \cdots a_i a_{i+1} \cdots a_{n+1}) + (-1)^{n+1} f(a_1 \cdots a_n) a_{n+1}\right)$

Claim: $\beta^2 = 0$

$H^0(R,M) = \ker(M \longrightarrow C'(M,R) = \mathrm{Hom}_K(R,M))$

$= \{m \in M \mid rm = mr \ \forall r \in R\}$

Note: if R is commutative $H^0(R,M)$ is a symmetric sub-module of M

$H^*(-,-) : \{(R,M)\} \longrightarrow k$-modules contravariant functor

$R \xrightarrow{\alpha} R'$ $M' \xrightarrow{\varphi} M$

$\rightsquigarrow C^*(R',M') \longrightarrow C^*(R,M)$ $\rightsquigarrow H^*(R',M') \longrightarrow H^*(R,M)$

Resolution: — Bar (2-sided) resolution of R

in homological degree n : $C_n^{bar} = R^{\otimes(n+1)}$

$b' : R^{\otimes n+2} \longrightarrow R^{\otimes n+1}$

$b'(a, a_1 \cdots a_{n+1}) = \sum_{i=1}^{n} (-1)^i d_i$ (does not include d_{n+1})

eg: $R^{\otimes 4} \xrightarrow{b'} R^{\otimes 3}$

$(a, a_1 a_2 a_3) \longmapsto (aa_1, a_2, a_3) - (a, a_1 a_2, a_3) + (a, a_1, a_2 a_3)$

$b' \cdot b' = 0$

C_*^{bar} is augmented by $b' = \mu : R \otimes R \longrightarrow R$

Prop: R be a unital k-algebra, then C_*^{bar} is a resolution of the R-bimodule R.

Proof: Define a homotopy

$\forall n \geq 1$ $s : R^{\otimes n} \longrightarrow R^{\otimes n+1}$

$(a_1 \cdots a_n) \longmapsto (1, a_1, \cdots, a_n)$

$s b_i = s \cdot d_{i+1} \quad \forall\, i = 1 \dots n-1, \quad d_0 \cdot s = id$

then $\quad s \cdot b' + b's = 1 \quad$ and so the complex is acyclic.

The map b' is completely determined by

i) b' is left R- module hom

ii) $b' = \mu$ on $R \otimes R$

iii) $b's + sb' = 1$

Prop: If the unital k-algebra R is projective as a k-modul, then

$$H_n(R, M) = \mathrm{Tor}_n^{R^e}(M, R)$$

Proof: $\quad R$ is projective $\Rightarrow R^{\otimes n}$ is projective as k-modules

$\Rightarrow R^{\otimes(n+2)} = R \otimes R^{\otimes n} \otimes R$ is a ~~projective~~ R^e- left module ← Why?

Bar resolution provides a projective resolution of R as left R^e-module

By applying $(M \otimes_{R^e} -)$ we get back the Hochschild complex.

→ $1 \otimes b'$ becomes b under $M \otimes_{R^e} R^{\otimes(n+2)} \cong M \otimes_k R^{\otimes n}$ ← Check

Prop: $\exists$ a subcomplex $D_* \subseteq C_*(R, M)$ which is acyclic (assuming R unital) the projection

$$C_*(R,M) \longrightarrow C_*(R,M)/D_* \text{ is a quasi-isomorphism of complexes.}$$

Claim: R- unital $\Rightarrow \{C_n(R,M)\}_{n \geq 0}$ is an example of simplicial module.

Def: A <u>simplicial module</u> is a pre-simplicial module i.e. $\exists\, d_i : M_i \longrightarrow M_{i-1}$ (face maps) together with a collection of k- homomorphisms $s_i : M_n \longrightarrow M_{n+1}$ together with relations:

$$s_i \cdot s_j = s_{j+1} \cdot s_i \qquad i \leq j$$

$$d_i \cdot s_j = \begin{cases} s_{j-1} \cdot d_i & i < j \\ id & i = j \quad i = j+1 \\ s_j \cdot d_{i-1} & \text{else} \end{cases}$$

Morphism of simplicial modules defined as usual.

eg: $M_n := M \otimes R^{\otimes n}$ with $M = R$-bimodule

$s_j(a_0, \cdots, a_n) = (a_0, \cdots, a_j, \underset{(j+1)^{th} \text{ position}}{1}, a_{j+1}, \cdots, a_n)$ $\quad j = 0, \dots, n$

Let $D_n := \langle s_0 M_{n-1} + \cdots + s_{n-1} M_{n-1} \rangle \subseteq M_n$

$D_* \subseteq M_*$ is then a subcomplex ⟵ Check: What is this in case of $M = R$

Claim: D_* is acyclic.

Proof: Consider the following increasing filtration of D_n: Is there an s_0?

$n, p \geq 0 \quad F_p D_n := \{$ linear span of $s_0, \dots, s_p : D_{n-1} \longrightarrow D_n \}$

Claim: $F_{n-1} D_n = F_n D_n = \cdots = D_n$ ⟵ stabilization of filtration

This filtration determines a spectral sequence where E^1-term is the homology of $Gr_* D_*$ $\left(E^1_{p,q} := H_p(Gr_q D_*) \right)$

whose abutment is $\Downarrow$

the homology of D_* $\quad E^\infty_{p,q} = F_p H_{p+q}(D_*) / F_{p-1} H_{p+q}(D_*) \subseteq E^\infty = H_{p+q}(D_*)$

Lemma: $\forall p \geq 0$ the complex $Gr_p D_*$ is acyclic.

Proof: For $n \leq p$; $Gr_p^F D_n = 0$

If $n \geq p$, $(-1)^p s_p$ induces a chain homotopy from id to 0.

i.e. $(d \cdot s_p + s_p \cdot d) s_p \equiv (-1)^p s_p \mod F_{p-1}$

$\boxed{\text{Check everything on this page.}}$

If $i < p$, $d_i \cdot s_p = s_{p-1} \cdot d_i = 0 \mod F_{p-1}$

$i = p$, $d_p \cdot s_p$ and $d_{p+1} \cdot s_p$ cancel each other

$i > p$, $d_i \cdot s_p = s_p \cdot d_{i-1}$ so $d_i \cdot s_p \cdot s_p$ cancel with $s_p \cdot d_{i-1} \cdot s_p$ for $p+2 < i$

hence the only term left out is $s_p \cdot d + d \cdot s_p$ is

$$(-1)^{p+2} \cdot d_{p+2} \cdot s_p \cdot s_p = (-1)^p \cdot s_p \cdot \underbrace{d_{p+1} \cdot s_p}_{id} = (-1)^p s_p$$

Th^m: In a first quadrant SS abutting to $A_* (= H_*(D_*))$ with $E^2_{p,q} = 0 \;\; \forall p > 0, q > 0$ then there is a LES

$$\rightarrow E^2_{n+1,0} \longrightarrow E^2_{0,n} \longrightarrow A_n = H_n(D_*) \longrightarrow E^2_{n,0} \longrightarrow E^2_{0,n-1} \longrightarrow \cdots$$

in this case $E^2_{n,0} = 0 \;\; \forall n > 0$.

→ s_i's are numbered wrong, it goes from s_0 to s_n

→ Why is $R \otimes R \otimes \cdots \otimes R$ a projective R^e module? What is the R^e module structure on $R_k^{\otimes(n+2)}$?

$$(\alpha, \beta)(s_1, r_1, \dots, r_n, s_2) = (\alpha s_1, r_1, \dots, r_n, s_2 \beta)$$

Projectivity:

R^e-modules
$$\begin{tikzcd} & R^{\otimes(n+2)} \arrow[d] & \\ M \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

for each R we have
$$\begin{tikzcd} & R_i \arrow[dl, dashed] \arrow[d] & \\ M \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$
as k-modules

as R is a projective k-mod

$\longrightarrow$ eg: $V = k$-module (remember k is a commutative ring & not a field)

Weibel Prop. 9.1.6 Pg. 303

$T = T(V)$ tensor algebra / V

$= k \oplus V \oplus V\otimes V \oplus V\otimes V\otimes V \oplus \dots$

$M = T$-T bi-module

Q. What is $H_i(T,M)$?

Claim: $0 \to H_1(T,M) \longrightarrow M\otimes V \xrightarrow{b} M \longrightarrow H_0(T,M) \longrightarrow 0$, rest all 0.

$(m,v) \longmapsto mv - vm$

Complex: $0 \longleftarrow M \longleftarrow M\otimes T \longleftarrow M\otimes T\otimes T \longleftarrow M\otimes T\otimes T\otimes T \longleftarrow \dots$

$mt - tm \longleftarrow\!\shortmid\, m,t$

$H_0 = M/[M,T]$ so need to say $\operatorname{im} b = [M,T]$ $\rightsquigarrow$ why are the two equal?

$\operatorname{im} b = [M,V]$

$$m\otimes v_1\otimes v_2 - v_1\otimes v_2\otimes m = m\otimes v_1\otimes v_2 - v_2\otimes m\otimes v_1 + v_2\otimes m\otimes v_1 + v_1\otimes v_2\otimes m$$

$$= (m\otimes v_1)\otimes v_2 - v_2\otimes(m\otimes v_1) + (v_2\otimes m)\otimes v_1 - v_1\otimes(v_2\otimes m)$$

$$\in [M,V]$$

By using cyclic permutations we see that $[M,T] = [M,V]$!

$H_1 = \ker(mt - tm) / \operatorname{im}(mt_1\otimes t_2 - m\otimes t_1t_2 + t_2m\otimes t_1)$

Need to say $H_1 = \ker(mv - vm)$

As seen above $[M,T] = [M,V]$ so infact need to say

$\cancel{\operatorname{im}(mt_1\otimes t_2 - m\otimes t_1t_2 + t_2m\otimes t_1) = 0}$ this does not feel right

does not mean kernels are the same

Now clearly $\ker b \subseteq \ker \partial$

Think this will work: by quotienting $\operatorname{im}(\ )$ we are setting

$m\otimes t_1\otimes t_2 = mt_1\otimes t_2 + t_2m\otimes t_1$

so that we end back in $[M,V]$.

$H_2 = 0 \iff \ker(M\otimes T\otimes T \longrightarrow M\otimes T) = \operatorname{im}(M\otimes T\otimes T\otimes T \longrightarrow M\otimes T\otimes T)$

$m \otimes t_1 \otimes t_2 \longmapsto mt_1 \otimes t_2 - m \otimes t_1 t_2 + t_2 m \otimes t_1$ Finding ker is a difficult thing

Finding im is simpler.

$m \otimes t_1 \otimes t_2 \otimes t_3$

$\downarrow$

$mt_1 \otimes t_2 \otimes t_3 - m \otimes t_1 t_2 \otimes t_3 + m \otimes t_1 \otimes t_2 t_3 - t_3 m \otimes t_1 \otimes t_2$

Q. Is $T(V)$ flat as a k-module?

This does not look do-able

Need to find → either a chain homotopy
or some simplification.

Book uses notion of a relative ext/tor → Read about it.

Here the trick: $T(V)$ has a resolution called cotriple resolution

$$0 \to T \otimes V \otimes T \longrightarrow T \otimes T \longrightarrow T \to 0$$

$a \otimes b \longmapsto ab$

$a \otimes b \otimes c \longmapsto ab \otimes c - a \otimes bc$

Can use this to compute the Hochschild homology

Tensor over T^e by M to get $0 \to M \otimes V \longrightarrow M \longrightarrow 0$

HH complex

Taking cohomology gives us back $0 \to H_1 \longrightarrow M \otimes V \to M \longrightarrow H_0 \longrightarrow 0$

Now for $M = T$, $T \otimes V \cong T$

$$HH_1(T) = \ker(tv - vt) = k \oplus (V^{\otimes j})_\sigma$$

$$HH_0(T) = \bigoplus_{j=1}^{\infty} (V^{\otimes j})^\sigma$$

Note error in book HH_1, HH_0 are interchanged.