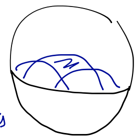
Th^m: Any (closed) 3 manifold has a finite cover.

Proof: Reduce to M^3 hyperbolic.

M^3 hyperbolic $\longrightarrow$ $\mathbb{H}^3$ $\quad |x|<1 \quad$ cover M

$$g_{\mathbb{H}^3} = \frac{4}{1-r^2}\, g_E$$

Euclidean metric



$\mathbb{H}^3$ = interior of the ball

$\pi_1 M \leq \mathrm{Isom}^+(\mathbb{H}^3)$
$\qquad\qquad\quad \| $
$\qquad\quad \mathrm{PSL}_2(\mathbb{C})$

finitely gen.

Any finitely generated group of matrices has many finite quotients

eg: $\mathrm{PSL}_2\mathbb{Z} \longrightarrow \mathrm{PSL}_2(\mathbb{Z}/n)$ with kernel $\Gamma(n)$

congruent subgroup of finite index

Can generalize these maps to arbitrary subgroups.

- M closed 3-manifold $\quad S^3, T^3, S^1\times S^1\times S^1,$  $\times S^1$, $L(p,q)$

Th^m (Thurston-Perelman) Any closed 3 manifold has a geometric decomposition.

Th^m: If $\pi_1 M^3$ is infinite then M has a finite cover $\tilde{M}$ with $H^1(\tilde{M},\mathbb{Z}) \neq 0$ $\quad (\& \ H_2(\tilde{M};\mathbb{Z}) \neq 0)$

Related to $\mathrm{PSl}_2(\mathbb{Z}[i])$

Conjecture: Suppose M is an arithmetic hyperbolic closed 3-manifold M with a tower of regular finite congruence covers (related to $\Gamma(n)$)

$$M = M_0 \longleftarrow M_1 \longleftarrow M_2 \longleftarrow \cdots$$

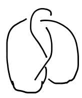
such that $\bigcap \pi_1(M_i) = \{1\}$ then

$$\lim_{n\to\infty} \frac{\log\left|H_1(M_n;\mathbb{Z})_{\text{torsion}}\right|}{\mathrm{Vol}(M_n)} = 1/6\pi$$

equivalently

$$\lim_{n\to\infty} \frac{\log\left|\mathrm{ab}(\pi_1 M_n)_{\text{torsion}}\right|}{[\pi_1 M : \pi_1 M_n]} = \frac{\mathrm{Vol}(M)}{6\pi}$$

Th^m: The hyperbolic structure on M^3 is unique.

eg: $S^3 \setminus$  is a "hyperbolic" manifold & its volume uniquely determines it

"Geometry = Topology" in dim 3.

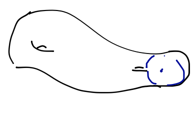
Cor: Can drop the arithmetic hypothesis if we assume $H_1(M_n)$ are completely torsion.

Exhaustive towers: $M=M_0 \longleftarrow M_1 \longleftarrow M_2 \longleftarrow \cdots \quad \bigcap \pi_1(M_n) = \{1\}$, regular finite covers.

Th^m: $\exists$ exhaustive towers of congruence covers where all $H_1(M_i)$ are torsion.

Def^n: $\mathrm{inj}_p(M) := \sup_{\epsilon} \left| B_\epsilon(p) \text{ is isometric to one in } \mathbb{H}^3 \right|$

injective radius



$\mathrm{inj}(M) = \inf_p \mathrm{inj}_p(M) = \frac{1}{2}\cdot$ length of shortest closed geodesic exhaustive tower $\Longleftrightarrow$ inj $M_n \longrightarrow \infty$

Def Say M_n^3 hyperbolic converge to $\mathbb{H}^3$ if $\forall R>0$ $\lim_{n\to\infty} \frac{\mathrm{Vol}\{x\in M_n \mid \mathrm{inj}_x M_n < R\}}{\mathrm{Vol}(M_n)} = 0$

Thm: If hyperbolic $M_n^3 \longrightarrow \mathbb{H}^3$ then $\frac{\mathrm{rank}\, H^1(M_n)}{\mathrm{Vol}(M_n)} \longrightarrow 0 = b_1^2(\mathbb{H}^3)$

$\mathcal{L}^2$ norm of first betti number !?

$\frac{1}{6\pi} = \tau^{(2)}(\mathbb{H}^3)$ $\mathcal{L}^2$-R... ? torsion

Thm: $\exists$ M_n hyperbolic $M_n \longrightarrow \mathbb{H}^3$ where $H_1(M,\mathbb{Z}) = 0$.

Cor: If M_n is an exhaustive tower of covers of hyperbolic 3-manifolds then $\tau(M_n)/\mathrm{Vol}(M_n) \longrightarrow \tau^{(2)}(\mathbb{H}^3) = 1/6\pi$

related to $\log |H_1(M_n;\mathbb{Z})|$ & $\underbrace{H^1(M_n;\mathbb{Z}) \subseteq H^1(M_n;\mathbb{R})}_{\text{index = regulator}}$

- Examples of towers: $\Gamma = PSL_2\,\mathbb{Z}[i] \leq PSL_2\,\mathbb{C}$

  $M = {}_{\Gamma}\backslash \mathbb{H}^3$ exterior of whitehead link ?

  p rational prime $\equiv 3 \mod 4$

$$1 \longrightarrow \Gamma(p_1 \cdots p_n) \longrightarrow \Gamma \longrightarrow \mathbb{Z}[i]/(p_1 \cdots p_n) \longrightarrow 1$$