Affineness in Chromatic Homotopy theory

1. TMF

$$\begin{array}{ccccc} \text{Deligne Mumford Stack} & \longrightarrow & \text{Derived stack} & \longrightarrow & \text{Complex Orbifolds} \end{array}$$

$\mathcal{H}$ = upper half plane $\mathcal{H}/Sl_2\mathbb{Z}$ = moduli space of lattices

Λ a lattice, $\mathbb{C}/\Lambda$ - elliptic curve and hence also classifies elliptic curves

Moduli stack of elliptic curves $\mathcal{M}_{ell}$ classifies elliptic curves over arbitrary commutative rings.

points of $\mathcal{M}_{ell}$ = elliptic curves / k
autos of points = autos of elliptic curves

To get a derived stack replace the structure sheaf $\mathcal{O}$ of moduli stack of elliptic curves by a sheaf $\mathcal{O}^{top}$ of E_∞-rings with $\pi_0\mathcal{O}^{top} = \mathcal{O}$

$(\mathcal{M}_{ell}, \mathcal{O}_{top})$ = derived moduli stack of elliptic curves.

Def^n: TMF = $\mathcal{O}^{top}(\mathcal{M}_{ell}) = \Gamma(\mathcal{O}^{top})$ — E_∞ ring spectrum

$\exists$ a map $\mathcal{M}_{ell} \longrightarrow \mathcal{M}_{FG}$ ← stack of a formal group laws
$E \longrightarrow \hat{E}$ $\hat{E}$ is the completion of E at identity
(why is it a fgl)?

$\mathcal{O}^{top}$ is compatible with the map $\mathcal{M}_{ell} \longrightarrow \mathcal{M}_{FG}$:

$$\operatorname{Spec} R \xrightarrow[\text{cover}]{\text{Etale}} \mathcal{M}_{ell} \longrightarrow \mathcal{M}_{FG}$$

$\mathcal{O}^{top}(\operatorname{Spec} R)$ is an even periodic spectrum that is Landweber exact wrt the formal group classified by $\operatorname{Spec} R \longrightarrow \mathcal{M}_{ell} \longrightarrow \mathcal{M}_{FG}$

Abstract version of Main Theorem:

- $\mathcal{X}$ stack (noetherian, separated, Deligne-Mumford stack)
- $\mathcal{X} \longrightarrow \mathcal{M}_{FG}$
- compatible sheaf of E_∞ ring spectra $\mathcal{O}^{top}$ on $\mathcal{X}$

Quasi-affine: $F: \mathcal{X} \longrightarrow \mathcal{M}_{FG}$ is quasi affine if

- F detects automorphisms i.e. $x \in \mathcal{X}(k)$, l auto of x, then $F(l)$ is an auto of $F(x)$ and $F(l)$ identity iff l is.
- $F^*\omega$ is ample on $\mathcal{X}$.

Tame: F is tame if any automorphism of order n is detected if char k/n.

Th^m: Assume F is quasi affine, then $\Gamma\, Q\,Coh(\mathcal{X}; \mathcal{O}^{top}) \longrightarrow \Gamma(\mathcal{O}^{top})$-mod is an equivalence.

Remark: • In classical algebraic geometry this is only true if X is an affine scheme ($\operatorname{Spec} R$).
• The geometry of $(\mathcal{X}, \mathcal{O}^{top})$ is determined by $\Gamma(\mathcal{O}^{top})$ and vice versa.

Th^m: $\mathcal{X} \xrightarrow{F} \mathcal{M}_{FG}$ is quasi affine and let $\begin{matrix} \mathcal{Y} \\ \downarrow \\ \mathcal{X} \end{matrix}$ be a Galois cover for a finite group G, then $\mathcal{O}^{top}(\mathcal{X}) \longrightarrow \mathcal{O}^{top}(\mathcal{Y})^{hG}$ is a faithful G-Galois ext of rings.

$\left(\mathcal{Y} \to \mathcal{X} \text{ etale}, \quad G \times \mathcal{Y} \xrightarrow{\simeq} \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}\right)$

Def^n: $R \longrightarrow S^{hG}$ of E_∞ rings is called a G-Galois extension if

$$R \xrightarrow{\simeq} S^{hG}, \qquad S \wedge_R S \xrightarrow{\simeq} \prod_G S$$

$$x \wedge y \longmapsto (x\, g(y))_{g \in G}$$

If $R \longrightarrow S^{hG}$ faithful Galois $S_{hG} \xrightarrow{N} S^{hG}$

Th^m: If $Y \to X$ is G-Galois and $\mathcal{X} \longrightarrow \mathcal{M}_{FG}$ tame, $\mathcal{O}^{top}(\mathcal{Y})_{hG} \simeq \mathcal{O}^{top}(\mathcal{Y})^{hG}$

Th^m: If $\mathcal{X} \to \mathcal{M}_{FG}$ is tame, the descent spectral seq

$$H^*(\mathcal{X}, \pi_* \mathcal{O}^{top}) \Rightarrow \pi_* \mathcal{O}^{top}(\mathcal{X}) \qquad (\text{eg}: TMF)$$
collapses at a finite page and has a horizontal vanishing line.

eg: $\mathcal{M}_{ell} \longrightarrow \mathcal{M}_{FG}$ quasi-affine

idea of proof: • $\mathrm{Aut}(E)$ is finite
. $\mathrm{End}(E)$ is an integral domain
. $\mathrm{End}(\hat{E})$ is torsion free.

suppose $f \in E$ has order n
$$\Rightarrow 1 + f + \cdots + f^{n-1} = 0$$
Assume $\hat{f} = 1$,
$$0 = 1 + \hat{f} + \cdots + \hat{f}^{n-1} = n \neq 0$$

$$\Gamma(n) = \left\{ A \in Sl_2\mathbb{Z} \mid A \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod n \right\}$$
$\subseteq Sl_2\mathbb{Z}$ normal
$$Sl_2\mathbb{Z}/\Gamma(n) \cong Sl_2(\mathbb{Z}/n)$$

$$\begin{array}{ccccc} [\mathcal{H}/\Gamma(n)] & \longrightarrow & \mathcal{M}(\Gamma(n)) & \longrightarrow & \mathcal{O}^{top}(\mathcal{M}(\Gamma(n))) \\ \text{Galois} \big\downarrow & & & & \\ [\mathcal{H}/Sl_2\mathbb{Z}] & & & & \end{array}$$

is an $Sl_2(\mathbb{Z}/n)$ Galois extension of $TMF[\frac{1}{n}, \zeta_n]$

$\overline{\mathcal{M}}_{ell}$ = "1 point compactification" of $\mathcal{M}_{ell}$

There are no interesting Galois covers of $\overline{\mathcal{M}}_{ell}$
$$\mathcal{O}^{top}(\overline{\mathcal{M}}_{ell}) = Tmf$$

$$\pi_*(TMF) \otimes \mathbb{Q} = \mathbb{Q}[c_4, c_6, \Delta^{-1}]$$
$$\pi_*(Tmf) \otimes \mathbb{Q} = \mathbb{Q}[c_4, c_6]$$

Shimura covers:

Let D be an indefinite quaternion algebra.
$D/\mathbb{Q}$ central simple $\mathbb{Q}$-algebra of dim 4 such that $D \otimes_{\mathbb{Q}} \mathbb{R} = M_2\mathbb{R}$
$\Lambda \subseteq D$ of maximum order

$$D \longrightarrow D_{\mathbb{R}} \longrightarrow M_2\mathbb{R}$$
$$\Lambda^{N=1} \longrightarrow Sl_2\mathbb{R}$$

$[\mathcal{H}/\Lambda^{N=1}]$ Shimura curve compact

$\rightsquigarrow \mathcal{X}$ integral version