An Introduction to Seifert Surfaces

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What is a Seifert Surface?

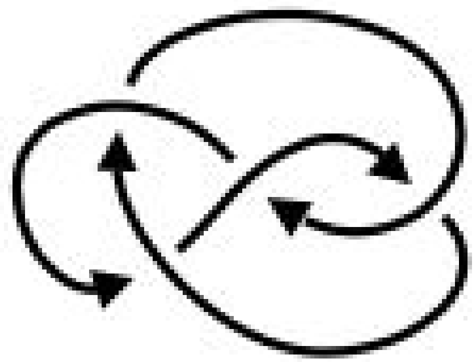
A Seifert surface for a knot K is a compact, orientable, connected surface with boundary equal to K. Any such surface must be topologically equivalent to S^2, T, or $T^{\#n}$, since it has only one boundary component. Therefore any Seifert surface has a well-defined genus. The genus of a knot K, denoted g(K), is the minimum genus of all Seifert surfaces for K.

Herbert Seifert

- German mathematician
- May 27, 1907 - October 1, 1996
- Attended the Dresden Institute of Technology where he received his doctorate in 1930, then he received his second doctorate from the University of Leipzig in 1932
- His thesis outlined what are now called “Seifert fiber spaces” which are ”3-manifold together with a ‘nice’ decomposition as a disjoint union of circles” (Wikipedia)

How to Create a Seifert Surface: Step 1

Choose an orientation for your starting knot.



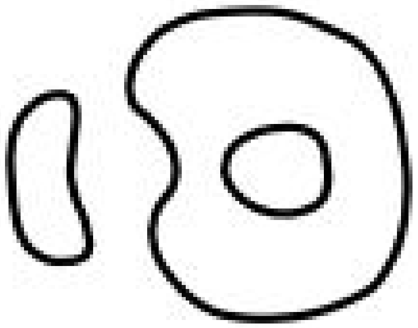
Here is an oriented figure-eight knot.

How to Create a Seifert Surface: Step 2

Trace along the orientation of the knot. At each crossing, go either left or right, whichever direction agrees with the orientation of the knot. When you end up back where you started, repeat from a different starting point on the knot, and do this until you have traced over the entire knot.

How to Create a Seifert Surface: Step 3

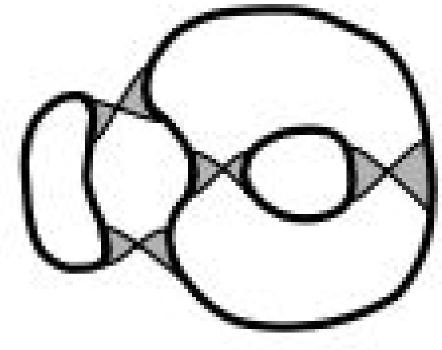
You have now created Seifert circles by tracing the knot. Let each Seifert circle be the boundary for a disk. If there are concentric disks, assign heights to the disks such that the innermost disk is the highest.



Here are the Seifert circles.

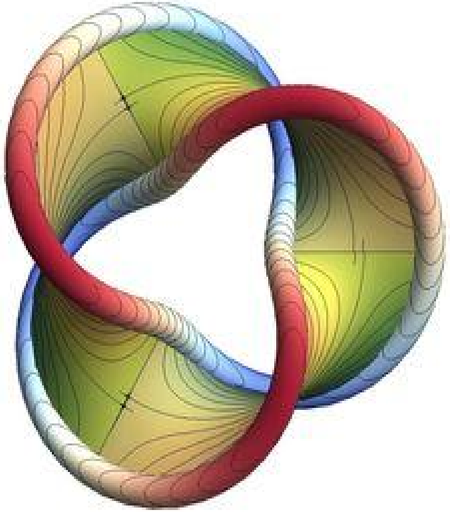
How to Create a Seifert Surface: Step 4

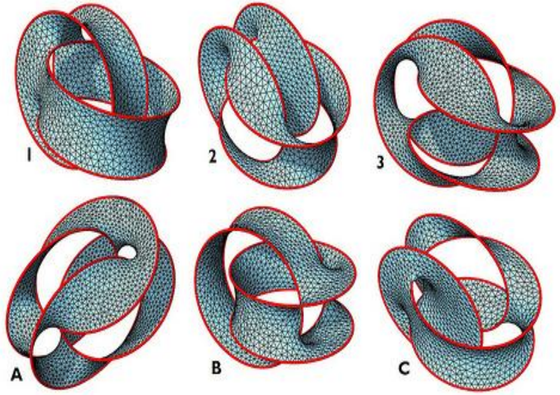
Attach neighboring disks where the original crossings were by gluing a rectangular band with a twist in the direction of the original crossing between the two disks.



Here are the glued rectangular bands.

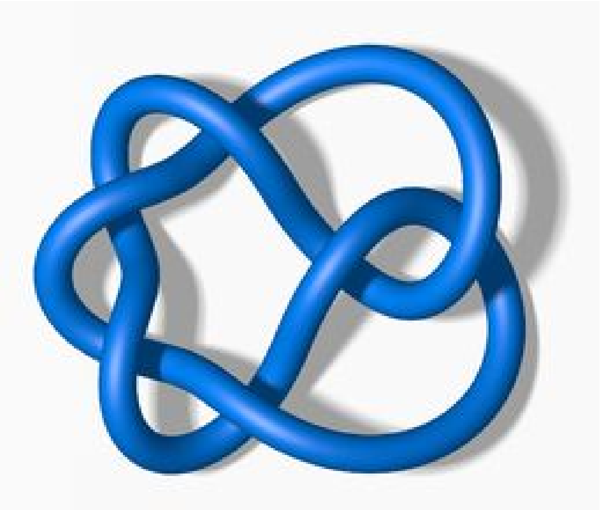
Examples





Problem I

Follow the four steps to create a Siefert surface out of the 6_1 knot, also known as the Stevedore knot. What is this surface topologically equivalent to? Here is a picture of the knot.



Solution to Problem I

Solution should be on the board.

This surface is topologically equivalent to T#T.

Problem II

Follow the four steps to create a Seifert surface for the 8_18 knot. What is this surface topologically equivalent to? Here is the knot.

8_{18}

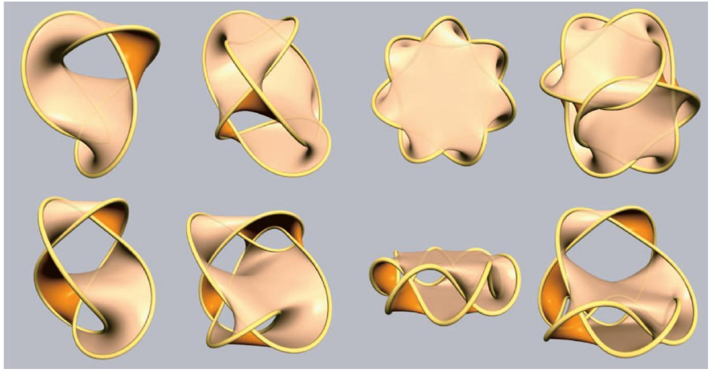
Solution to Problem II

Solution should be on the board.

This surface is topologically equivalent to T#T#T#T#T#T, or $T^{\#6}$.

Other Remarks

- A Seifert surface can always be found for any knot
- Thus, since knots can be cut and linked together, so can Seifert surfaces
- They cannot be topologically equivalent to a projective space or sum of projective spaces because they must be orientable



Sources

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