Cones over diagrams

$A = \infty$-category , I, J simplicial sets

Def: Fat join of I, J : $I \diamond J$

$$\begin{tikzcd} I \times J \sqcup I \sqcup J \arrow[r] \arrow[d] & I \sqcup J \arrow[d] \\ I \times 2 \times J \arrow[r] & I \diamond J \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Note: $$\begin{tikzcd} I \sqcup J \arrow[r] \arrow[d] & 1 + 1 \arrow[d] \\ I \diamond J \arrow[r] & 2 \end{tikzcd}$$

Lem: $\mathrm{Hom}_{A^J}(\Delta, A^J) \cong A^{\Delta[0] \diamond J}$

Def: Join of I, J :

- $(I * J)_0 = I_0 \sqcup J_0$
- $(I * J)_n = I_n \sqcup J_n \sqcup \left(\coprod_{0 \leq k \leq n-1} I_{n-k-1} \times J_k \right)$

- $\Delta[k] * \Delta[l] = \Delta(k+l)$

Fact:

$$\begin{tikzcd} A^{I * J} \arrow[rr, "\simeq"] \arrow[dr] & & A^{I \diamond J} \arrow[dl] \\ & A & \end{tikzcd}$$

Def: $J * \Delta^0 =: J^{\triangleright}$ $\qquad\qquad$ $\Delta^0 * J =: J^{\triangleleft}$

then

$$\begin{tikzcd} & A^{J^{\triangleleft}} \arrow[dl] \arrow[dr] & \\ A^J \arrow[dr, equal] & & A \arrow[ll, phantom, "\overset{\phi}{\Leftarrow}"] \arrow[dl, "\Delta"] \\ & A^J & \end{tikzcd}$$

is a comma square.

Prop: $D \xrightarrow{d} A$ has a limit iff

$$\begin{tikzcd} & A^{J^{\triangleleft}} \arrow[d, "\mathrm{res}"] \\ D \arrow[ur, dashed, "\mathrm{ran}"] \arrow[r] & A \arrow[ul, phantom, "\Downarrow \cong"] \end{tikzcd}$$