Prop: X irreducible affine, $Y \leq X$ irreducible and $\operatorname{codim}_X Y = 1$, Then,

Y is an irreducible component of $V(f)$ for some $f \in K[X]$

Ex1: X - noetherian topological space

$\Rightarrow X$ satisfies dcc for closed subsets

suppose X has infinite irreducible components $\{X_i\}_{i\in I}$

$$X = \bigcup_{i=1}^{\infty} X_i \cup X'$$

Look at $Y_k = \bigcup_{i=k}^{\infty} X_i \cup X'$

So that $Y_1 \supsetneq Y_2 \supsetneq Y_3 \supsetneq \dots$

is a dc with not stability.

Ex2: $U \subseteq X$ open

if $U = U_1 \cup U_2$, ~~$X = X - U_1 \cup X - U_2$~~

U_1, U_2 closed in U

By subspace topology $\exists X_1, X_2 \in X$ closed s.t.

$U_1 = X_1 \cap U \quad U_2 = X_2 \cap U$

So $\quad X = X_1 \cup X_2 \cup (X \setminus U)$

Ex3: Let $g \in G$ Need to show $g \in UV$.

Thing to use is that any two dense open sets intersect.

look at $g U^{-1} \cap V \neq \{\}$

$\Rightarrow \quad \exists v.\ v = g u^{-1}$

$\Rightarrow \quad g = v u$.

$\Rightarrow \quad G = VU = UV$.

22/01/13 — Krishna I

Dimension Theory:

1. X affine irreducible. $Y < X$

   $\Rightarrow \dim Y < \dim X$

2. $Y < X$, $\dim Y = \dim X - 1$, then

   if Y irreducible, then

   Y is a component of $V(f)$.

All results hold if we drop the condition of affineness.

Proof: ~~$I(Y) \subsetneq k[X]$~~

Pick an element $f \in I(Y)$, $f \neq 0$

$\Rightarrow$ $Y \subseteq V(f)$

Choose irreducible component Z ~~of~~ containing Y in $V(f)$

$$Y \subseteq Z \subseteq V(f) \neq X$$

By previous 1)

$$\dim Z < \dim X$$

$$\Rightarrow \dim Y = \dim Z$$

$$\Rightarrow Y = Z. \quad \text{by 1) again} \qquad \square$$

3. X irreducible affine. $f \neq 0, \neq$ unit then,

Y irreducible component of $V(f)$.

$$\operatorname{codim}_X Y = 1.$$

Proof: [Fact: A is an integral domain which is f.g. k-algebra

$\mathfrak{p} \subseteq A$ prime ideal.

ht of $\mathfrak{p}$:= longest chain of primes in $\mathfrak{p}$

$= \dim A_{\mathfrak{p}}$

Then one has trivially

$$\dim A \geq \operatorname{ht} \mathfrak{p} + \dim (A/\mathfrak{p})$$

For this case

$$\dim A = \operatorname{ht} \mathfrak{p} + \dim (A/\mathfrak{p}).$$]

$Y \subseteq V(f)$ irreducible "component"

$\Rightarrow$ $I(Y)$ ~~irred~~ minimal prime containing (f).

By fact,

$$\underset{\dim X}{\underset{\|}{\dim k[X]}} = \underset{\dim Y}{\underset{\|}{\dim k[X]/I(Y)}} + \underset{\text{ht}}{\cancel{\dim}}\, I(Y)$$

So, remains to show $\operatorname{ht}(I(Y)) = 1$.

Now, we invoke "Krull's principle ideal thm" which says:

$$\cancel{\operatorname{ht}(\mathfrak{p}) = 1}\ \operatorname{ht}(I(Y)) = 1$$

[More generally,

R-Noetherian, $I = (a_1, \dots a_r)$

$I \subseteq \mathfrak{p}$ minimal $\Rightarrow$ $\operatorname{ht}(\mathfrak{p}) \leq r$.]

Corollary: $Y \subseteq X$ irreducible, then

$$\operatorname{codim}_X Y = \operatorname{ht}(I(Y)).$$

4. X irreducible affine,
$f_1 \cdots f_r \in K[X]$. The each irreducible component of $V(f_1 \cdots f_r)$
has codim $\leq r$.

5. X irreducible affine, $0 < \dim_X Y = r$,
Y is an irreducible component of $V(f_1 \cdots f_r)$ for some
some $f_1 \cdots f_r \in K[X]$.

Proof: Fact: R - noetherian,
$\mathfrak{p} < R$ prime, $\operatorname{ht} \mathfrak{p} = r \geq 1$, then,
$\exists\, f_1 \cdots f_r \in \mathfrak{p}$ st. $\mathfrak{p}$ is a minimal prime of
$(f_1 \cdots f_r)$.

$f : X \longrightarrow Y$ morphism of irr affine varieties
$f^* : K[Y] \longrightarrow K[X]$

Def^n: Dominant map:
im f is dense in Y.

In this case, f^* is injective. Also conversely.

eg. • X principal open in Y, f inclusion.

• $\mathbb{A}^2 \longrightarrow \mathbb{A}^2$
$x, y \longmapsto xy, y$ image $=$ Complement of $\{(x,0)\} \cup (0,0)$

• $f : X \to Y$ dominant
$\Rightarrow \dim Y \leq \dim X$
$f^* : K[Y] \hookrightarrow K[X]$
$f^* : K(Y) \hookrightarrow K(X)$.
$\dim X =$ transcendence degree of $K(X)$.
Result follows.

Def^n: $f : X \longrightarrow Y$ X, Y affine.
Finite map:
$K[Y] \xrightarrow{f^*} K[X]$ integral extension.

eg: • $\mathbb{A}^2 \longrightarrow \mathbb{A}^2$
$x, y \longmapsto xy, y$
is $K[T_1, T_2]$ integral over $K[T_1 T_2, T_2]$. No.

Not integral extension

Ex 1: $f: X \to Y$ finite dominant map of irreducible varieties, then $\dim X = \dim Y$.

Prop: $\varphi: X \to Y$ finite, dominant

dominant map between affine varieties means a map dominant on each irreducible component of X.

a) $Z \leq X \Rightarrow \varphi(Z) \leq Y$ and $\varphi|_Z$ is finite

b) X, Y irreducible, $K[Y]$ integrally closed. Given $W \lneq Y$ irreducible. $Z \subseteq \varphi^{-1}(W)$ is a component, then
$$\varphi(Z) = W.$$

Proof:

a) ~~$Z = V(I$~~ $I = I(Z)$ Let.

$S = K[Y] \hookrightarrow K[X] = R$ $\quad S \hookrightarrow R$ integral extension

$\qquad\qquad\qquad\quad \cup$

$\qquad\qquad\qquad\quad I$

$V(I \cap S) \supseteq \varphi(Z)$ $\qquad I \cap S = (\varphi^*)^{-1} I = (\varphi^*)^{-1}(I(Z))$

$f \in \varphi(Z) \Rightarrow f = g \cdot \varphi$

So $S/_{I \cap S} \subseteq R/_I$ integral

$\Rightarrow \exists$ finite $\varphi: Z \to V(I \cap S)$

Enough to show $\varphi|_Z$ surjects on $V(I \cap S)$.

- $\varphi|_Z$ is dominant?
- finite, dominant maps are surjective.

$x \in X$, $\varphi(x) \in Y$

$\Leftrightarrow (\varphi^*)^{-1} m_x = m_y$

Surjection $\Rightarrow \forall y \, \exists x$ s.t. $(\varphi^*)^{-1} m_x = m_y$

This is going up.

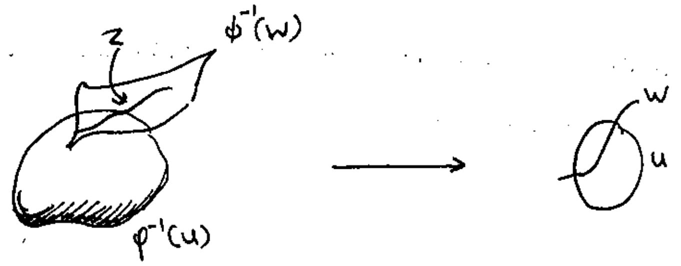
Th^m: $\varphi: X \to Y$ $\quad X, Y$ irred $\quad \varphi$ dominant

$r = \dim X - \dim Y$. Then,

a) $\exists$ non-empty open set $U \subseteq Y$ s.t. $U \subseteq \varphi(X)$.

b) We can choose U to have the following property:

if $W \lneq Y$ irred s.t. $W \cap U \neq \emptyset$ and $Z \lneq \varphi^{-1}(W)$ component s.t. $Z \cap \varphi^{-1}(U) \neq \emptyset$ then
$$\dim Z = \dim W + r.$$



In particular:

$W = \{y\}$, $y \in U$, $Z \subseteq \varphi^{-1}(y)$ irreducible component

$\Rightarrow \dim Z = r$

eg: • $\varphi: \mathbb{A}^2 \longrightarrow \mathbb{A}^2$

$(x,y) \longmapsto (xy, y)$

$r = 0$

$U = \{(x,y) \mid y \neq 0\}$

[if $y \in \varphi(X) \setminus U$ then we always have $\dim \varphi^{-1}(y) \geq r$].

Def^n: X variety,

$A \subseteq X$ "locally closed" if $A = U \cap V$ U-open, V closed

$\equiv$ A open in $\bar{A}$. (Take $V = \bar{A}$)

"constructible" finite union of locally closed sets

Prop: A is constructible. Then,

$\exists U \subseteq A$ s.t. U open, dense in $\bar{A}$.

(if A locally closed, take $U = A$.)

Th^m: Image of a constructible set under a morphism is also constructible.

(Chevalley)

Proof: of Prop

$Y = A_1 \cup A_2 \cup \cdots \cup A_K$ (Y is the A in the proposition.

$\bar{Y} = \bar{A}_1 \cup \bar{A}_2 \cup \cdots \cup \bar{A}_K$

if $\bar{Y}$ is irreducible, $\bar{Y} = \bar{A}_i$ for some i

in this case A_i will work

if $Y = Y_1 \cup Y_2 \cup \cdots \cup Y_r$ irreducible component of Y.

$Y_i \rightarrow$ irreducible ~~closed~~ constructible set

$\bar{Y} = \bar{Y}_1 \cup \bar{Y}_2 \cup \cdots \cup \bar{Y}_r$ Choose U_i dense, open in $\bar{Y}_i$.

$V = U_1 \cup \cdots \cup U_r \subseteq Y$

V – will do the job.

Lemma: H constructible subgroup of G, then $\overline{H} = H$.

Proof: H constructible $\Rightarrow \exists U \in H$. U dense, open in $\overline{H}$.

$\Rightarrow U \cdot U = \overline{H} \subseteq H$

$\Rightarrow H = \overline{H}$.

Lemma: A, B closed subgroups of G. $B \leq N_G A = \{x \in G \mid xAx^{-1} \subseteq A\}$

Then AB closed subgroup of G.

Proof: $B \subseteq N_G(A) \Rightarrow AB$ subgroup of G.

AB is image of $A \times B \longrightarrow G$, $x, y \longmapsto xy$

By Chevalley's th^m AB is constructible in G.

By previous lemma AB closed.

Prop: $\varphi : G \longrightarrow G'$ morphism of alg. groups, then

i) $\ker \varphi$ is closed subgroup of G^0

ii) $\operatorname{im} \varphi$ " " G'

iii) $\varphi(G^0) = (\operatorname{im} \varphi)^0$

iv) $\dim(\ker \varphi) + \dim(\operatorname{im} \varphi) = \dim G$

Proof.

i) trivial

ii) Chevalley's th^m

iii) Use $\infty > [G : G^0] \geq [\varphi(G) : \varphi(G^0)]$

iv) $\exists$ non-empty open $U \subseteq \varphi(G)$ such that

$y \in U \Rightarrow$ ~~$\dim \varphi^{-1}(y)$~~

$W \subseteq \varphi^{-1}(y)$ irreducible component

$\Rightarrow \dim W = \dim G - \dim(\operatorname{im} \varphi)$

$\quad \overset{\shortparallel}{\dim \varphi^{-1}(y)}$

$\varphi^{-1}(y) = x \ker \varphi \qquad \varphi(x) = y$

$\Rightarrow \dim(\ker \varphi) + \dim(\operatorname{im} \varphi) = \dim G$

• for $Gl(n, k) \longrightarrow k^*$

~~dim~~ $\ker = Sl(n, k)$

$\Rightarrow \dim Sl(n, k) = n^2 - 1$.

Def^n: $M \subseteq G$.

$A(M) :=$ closed subgroup generated by M

$=$ Smallest closed subgroup containing M

$= \bigcap_{H \leq G} H$ (H closed subgroup of G containing M)

Prop: $f_i : X_i \longrightarrow G$, $i \in I$ variety morphism, X_i irreducible

$e \in Y_i = f(X_i)$. Let $M = \bigcup_{i \in I} Y_i$

Then, $A(M)$ is connected subgroup of G.
irreducible

Proof: Consider

$$f_i^{-1} : X_i \longrightarrow G$$
$$x \longmapsto f_i(x)^{-1}$$

Include f_i^{-1} also in our family of morphisms.

Then M does not change.

Given $a = (a_1, \dots a_n)$ finite set of indices in I.

Let $Y_a = Y_{a_1} \cdots Y_{a_n}$.

Y_a constructible in G, and also irreducible

$\Rightarrow \overline{Y_a}$ irreducible.

$\Rightarrow e \in Y_a \subseteq \overline{Y_a} \subseteq G^\circ$

Pick a maximal $\overline{Y_a}$.

<u>claim</u>: b, c seq in I, then

$$\overline{Y_b}\,\overline{Y_c} \subseteq \overline{Y_{bc}}$$

$$\overline{Y_b} Y_c \subseteq \overline{Y_{bc}}, \quad Y_b \overline{Y_c} \subseteq \overline{Y_{bc}}$$

$$\overline{Y_b}\,\overline{Y_c} \subseteq \overline{Y_{bc}}.$$

$\overline{Y_a} \subseteq \overline{Y_a}\,\overline{Y_b} \subseteq \overline{Y_{ab}}$ $\forall b$

$\Rightarrow \overline{Y_a} = \overline{Y_{ab}} = \overline{Y_a}\,\overline{Y_b}$ $\forall b$

$\Rightarrow \overline{Y_a} = \overline{Y_a}\,\overline{Y_a}$, $\overline{Y_a} = \overline{Y_a}\,\overline{Y_a}^{-1}$

Because $\overline{Y_a}$ contains e $\quad \overline{Y_a} = \overline{Y_a}^{-1} \Rightarrow$ closed under -ve

and also closed under composition.

$\overline{Y_a}\,\overline{Y_b} = \overline{Y_a} \Rightarrow \overline{Y_b} \subseteq \overline{Y_a}$ $\because e \in \overline{Y_a}$

$\Rightarrow \boxed{A(M) = \overline{Y_a}}$

Each $\overline{Y_a}$ was connected, so $A(M)$ is also conn/irred.
irreducible

Cor: $\{Y_i\}$ family of closed connected subgroups generating G.
(irreducible)
Then G is connected.

Cor: $Sl(n, K)$ connected. $\left[\, U_{ij} = \begin{bmatrix} 1 & & c \\ & 1 & \\ & & 1 \end{bmatrix} \quad (i,j)^{th} \text{ entry} \right]$

Ex 2: Prove that $Sl(n,k)$ connected.

Ex 1: $f: X \longrightarrow Y$ finite dominant

$f^*: K[Y] \longrightarrow K[X]$ injective, integral extension.

Then result follows by going up and property of compatibility of extensions of prime ideals.

Ex 2: ~~X_i = { diagonal matrix with i^{th} entry arbitrary~~

$D =$ diagonal matrices in $Sl(n,k)$

$D_{ij} =$ matrices in $Sl(n,k)$ with only off diagonal entry non-zero being in the $(i,j)^{th}$ position and all diagonal entries being 1

$$\begin{bmatrix} 1 & & \\ & 1 & c \\ & & \ddots \\ & & & 1 \end{bmatrix} \quad (i,j)^{th} \text{ position}$$

Then one has

$$\cancel{D \cong \underbrace{G_m \times \cdots \times G_m}_{n \text{ times}} \cap \text{ and } Sl(n,k)}$$

$$M = D \bigcup_{i,j} D_{ij}$$

And $Sl(n,k) = A(M)$. Using previous lemma, it suffices to prove each of D, D_{ij} irreducible.

Another shorter proof:

$$Gl(n,k) \longrightarrow Sl(n,k)$$

$$[x_{ij}] \longmapsto \begin{bmatrix} x_{11}/\det & x_{12} & \cdots & x_{1n} \\ x_{21}/\det & x_{22} & & \vdots \\ \vdots & \vdots & & \\ x_{n1}/\det & x_{n2} & & x_{nn} \end{bmatrix}$$

This is surjective ~~and~~ as restriction of this map to $Sl(n,k)$ is identity.

But $Gl(n,k)$ open in $\mathbb{C}^{n^2}$ $\Rightarrow$ irreducible

$\Rightarrow Sl(n,k)$ irreducible.

Remark: Geometrically a map being finite ~~is the same~~ ~~as~~ implies inverse of each point consists of finitely many points

$X \xrightarrow{f} Y$ finite

$\Rightarrow K[Y] \xrightarrow{f^*} K[X]$ integral ~~extension~~

a point in Y corresponds to a maximal ideal m_y.

$f(x) = y$ then we must have

$(f^*)^{-1} m_x = m_y$

$$K[Y] \longrightarrow \underbrace{f^* K[Y] \hookrightarrow K[X]}_{\text{integral extension}}$$

Now $m_{x_1} \cap f^* K[Y] = m_{x_2} \cap f^* K[Y]$

is possible for only finitely many x_i's.

So far we donot have any problem. But going ~~from~~ from $f^* K[Y]$ to $K[Y]$ we might get a higher dimensional fiber.

So just finite is not enough we also require $K[Y] \longrightarrow f^* K[Y]$ has only finite maximal ideals lying over ~~$K[Y]$~~ m in $K[Y]$.

This is certainly true for dominant maps.

Check: $Sl(n,k)$ generated by $u_{i,j}$

- Group action:

$G \circlearrowright S$ $\quad$ $G\times S \longrightarrow S$ $\quad$ $G \to \mathrm{Aut}\, S$

$g,s \longmapsto g\cdot s$ $\quad$ group homomorphism

- $e.s = s \quad \forall s \in S$
- $(g_1 g_2)\cdot s = g_1\cdot(g_2\cdot s) \quad \forall s, g_1, g_2$

Orbit: ~~Gs~~ Gs

Stabilizer: $\{g \mid gs = s\}$ (isotropy) G_s

Transitive action, $G/G_s \longleftrightarrow Gs$

$G \circlearrowright G$ left multiplication, conjugation

- G - algebraic group, X - variety

$g\cdot$ is a morphism of varieties

Def^n $\quad Y, Z \subseteq X$, $G \circlearrowright X$, $\mathrm{Tran}_G(Y, Z) = \{g \in G \mid g\cdot Y \in Z\}$

- ~~Stab~~$_x$ $G_s = \mathrm{Tran}_G(\{x\}, \{x\})$

Def^n $\quad$ Centralizer of Y

$$C_G(Y) := \bigcap_{y \in Y} \underset{G_y}{\underset{\shortparallel}{\mathrm{Stab}(y)}}$$

Prop: $\quad G \circlearrowright X \quad Y, Z \subseteq X \quad Z \leq X$

a) $\mathrm{Tran}_G(Y, Z)$ is closed in G

b) Stabilizers, centralizers are all closed

c) Fixed point set of $g \in G$ is closed

$X^g = \{x \mid g x = x\} \quad X^G = \bigcap_{g \in G} X^g$

d) G connected $\Rightarrow$ G stabilizes each connected component of X.

Proof:

a) $x \in X$

$$G \longrightarrow G\times X \xrightarrow{\varphi} X \qquad G \xrightarrow{\varphi_x} X$$
$$g \longmapsto (g, x) \longmapsto g\cdot x \qquad g \longmapsto g\cdot x$$

$\varphi_x^{-1}(Z)$ closed in G. $\forall x \in X$.

$\mathrm{Trans}_G(Y,Z) = \bigcap_{y\in Y} \varphi_y^{-1}(Z)$

a) $\Rightarrow$ b)

c) $X \xrightarrow{\varphi_g} X\times X$ $\qquad X^g = \varphi_g^{-1}(\underbrace{\text{diagonal of } X\times X}_{\text{closed}})$

$x \longmapsto (x, g.x)$

Remark: Varieties we defined so far ~~have~~ do not have diagonal closed in general. These are called pre-varieties. Prevarieties having closed diagonal are called varieties.

Ex 1: Prove affine varieties have closed diagonals.

Note: Topology on $X\times X$ is not product topology but rather Zariski topology.

d) G connected $\Rightarrow$ orbits connected

$H = \mathrm{Trans}_G(X', X')$ $\qquad X'$ - connected component of X°

by a) H closed. $\Rightarrow$ H closed subgroup of G.

$G \circlearrowleft$ set of connected components ?.

H is a stabilizer of this action.

$\Rightarrow$ $G/_H \equiv$ set of connected components

$\Rightarrow$ $[G:H] < \infty$

$\Rightarrow$ $H \geqslant G^\circ = G$. $\quad \because G$ connected.

~~Prop~~ Cor: G algebraic, $H \leqslant G$. $\Rightarrow$ $C_G(H)$, $N_G(H)$ are closed.

Need to notice that $\mathrm{Trans}_G(H,H) = N_G H$.

Prop: $G \circlearrowleft X$. Each orbit is locally closed smooth subset of X, whose boundary is union of orbits of strictly lower dim.
$(\bar{A} \setminus A)$ In particular orbits of minimal dim are closed.

Proof: $Y = G\cdot y \quad y \in X$

$$\begin{array}{ll} G \xrightarrow{\varphi_y} X & Y = \varphi_y(G) \Rightarrow Y \text{ constructible} \\ g \longmapsto g\cdot y & \Rightarrow \exists\, U \subseteq Y,\ U \text{ open dense in } \overline{Y}. \\ & \Rightarrow G\cdot U = Y \\ & g\cdot U \xrightarrow{\sim} U \quad \forall g \in G \end{array}$$

$\Rightarrow$ boundary of Y closed $\qquad \Rightarrow Y = \bigcup_{g\in G} g\cdot U =$ open in $\overline{Y}$

$\Rightarrow \dim$ boundary $< \dim \overline{Y} = \dim Y$ $\qquad$ = locally closed.

See smoothness

. To show that ∂Y is union of orbits it suffices to show

$G\cdot\overline{Y} = \overline{Y}$

$\varphi_g : X \longrightarrow X$ $\qquad$ we have $\varphi_g(Y) = Y$

$\Rightarrow \varphi_g(\overline{Y}) = \overline{Y} \Rightarrow G\cdot\overline{Y} = \overline{Y}.$

. Now split $\overline{Y}$ as union of orbits.

=

- $Gl(2,k) \circlearrowleft Gl(2,k)$ by conjugation

$\rightarrow$ orbits of diagonal matrices $D(2,k)$

$B = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ $\qquad a \neq b :\ M = PBP^{-1}$ iff M has eigenvalues a, b
trace $M = a+b$
det $M = ab$

$\Rightarrow$ orbit B closed.

$a = b \Rightarrow$ orbit B singleton.

Ex 2: Do this for $Gl(n,k)$, $D(n,k)$.

$\rightarrow$ Now take $B = \begin{bmatrix} a & 1 \\ & a \end{bmatrix}$

orbit of B = Matrices with $\underset{\underset{Y}{\parallel}}{\text{char}}$ poly $(t-1)^2 \setminus \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \right\}$

To show orbit of B not closed one needs to show

31/01/13

— Kannan

Def^n: $\rho : G \longrightarrow Gl(V)$

rational representation if ρ morphism of algebraic groups

eg: $K^* \longrightarrow K^*$

$x \longmapsto x^n$ $\quad n \in \mathbb{Z}$

rational not polynomial!

- V a representation of G. Then
$U \leq V$ sub representation
$\exists$ W subrep s.t. $U = V \oplus W$

⋮

05/02/13

$G \curvearrowright X$ morphically

Prop: i) orbits are locally closed

ii) Boundary union of orbits of lower dim

iii) $\Rightarrow$ orbits of minimal dimension are closed

$G \curvearrowright X \Rightarrow G \curvearrowright K[X]$

$g(f)(x) = f(g^{-1}x)$

denote this action by $\tau_g : K[X] \longrightarrow K[X]$

Then we have

$\tau : G \longrightarrow Gl(K[X])$

Prop: $G \curvearrowright X$ $\quad F \subseteq K[X]$ finite dim subspace

$\Rightarrow$ 1. $\exists$ a finite dim subspace $E \subseteq K[X]$ s.t.
$F \subseteq E$ and $\tau_g E = E \ \forall g \in G$

2. F stable under all τ_g
$\iff \phi^* F \subseteq K[G] \otimes_K F$

$G \times X \longrightarrow X$

ϕ^*: $K[X] \longrightarrow K[G \times X] \cong K[G] \otimes_K K[X]$

Proof: 1. Assume F = span of $f \in K[X]$

$$\phi^*(f) = \sum_{i=1}^{n} f_i \otimes g_i \qquad f_i \in K[X], \quad g_i \in K[G]$$

We will prove

$E = \langle T_g f \mid g \in G \rangle$ is finite dim.

and clearly E is G-stable

For this one shows that E is spanned by the f_i's and hence is finite dimensional.

2. $\Leftarrow$

$\phi^* F \subseteq K[G] \otimes F$

$\Rightarrow f \in F \Rightarrow \phi^*(f) = \sum_i f_i \otimes g_i \qquad f_i \in F$

E as above $= \langle T_g f \mid g \in G \rangle$

$= \langle f_i \rangle \subseteq F$

$\Rightarrow E = F \Rightarrow F$ G-stable

$\Rightarrow$

F stable under T_g's

$\{f_i\}$ basis for F. Extend to a basis $\{f_i\} \cup \{f_j'\}$ for $K[X]$.

$f \in F$

$$\phi^* f = \sum r_i f_i + \sum s_j f_j' \qquad r_i, s_j \in K[G]$$

$$T_g f = \sum r_i(g^{-1}) f_i + \sum s_j(g^{-1}) f_j'$$

$T_g f \in F \Rightarrow s_j(g^{-1}) = 0 \ \forall g \in G$

$\Rightarrow s_j = 0$

$\Rightarrow \phi^* f \subseteq K[X] \otimes_K G$

Th^m: G affine alg. group. $\Rightarrow$ G isomorphic to a closed subgroup of $Gl(n,k)$.

Proof: $f_1 \cdots f_n$ algebra generators of $K[G]$.

F = K-span of $\{f_1, \cdots f_n\}$

To be continued...

- X - variety
  $\Rightarrow$ $\mathrm{Sing}(X)$ is proper closed

  $p \in U \subseteq X$ U open affine, then

  $\mathcal{O}_{U,p} \cong \mathcal{O}_{X,p}$. $\Rightarrow$ we can assume X affine.

- p singular

  $\Rightarrow$ $\underbrace{J(p)}_{\text{Jacobian at } p}$ has rank $< n-r$

  because $\mathfrak{m} \subseteq \mathcal{O}_{X,p}$ maximal

  $\Rightarrow$ $\dim \mathfrak{m}/\mathfrak{m}^2 + \mathrm{rank}\, J(p) = n$

  & $\dim \mathfrak{m}/\mathfrak{m}^2 \geq r$

  $\Rightarrow$ $\mathrm{Sing}(X) = \{p \mid \mathrm{rank}(J(p)) < n-r\}$

  = vanishing of all $(n-r)\times(n-r)$ minors and all larger minors of J

  = closed.

- $X = V(f) \subseteq \mathbb{A}^n$ (irreducible-f) - hypersurface

  if $\mathrm{sing}(X) = X$, then we get

  $$\frac{\partial f}{\partial T_i} \in I(X) = (f).$$

  But this is not possible by degree considerations. (in char 0)

  unless $\frac{\partial f}{\partial T_i} = 0 \ \forall i$ $\Rightarrow$ f constant.

  char p: then f consists of p-powers

  but then f cannot be irreducible as we will have f is a p^{th} power.

- Any variety is "birational" to a hyper surface

  i.e. an open subset of X is isomorphic to an open subset of hyper surface

- G algebraic group. what is $\mathrm{Sing}(G)$?

  ~~Since $\exists g \in \mathrm{Sing}(G)$ such that~~

  Since $\mathrm{Sing}(G)$ proper closed in G, $\exists g \in G$ non-singular.

  Translate this g to generate the entire G.

- Same argument says that orbits are smooth.