Milnor-Stasheff notes:

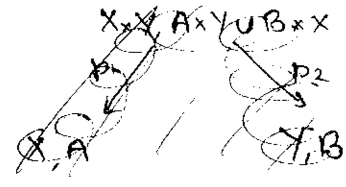
- $H^n(\mathbb{R}^1 \times B, \mathbb{R}^1_0 \times B) \longleftrightarrow H^{n-1}(B)$ e = generator of $H^1(\mathbb{R}^1, \mathbb{R}^1_0)$

  $b \times e \longleftrightarrow b$ is an isomorphism $\mathbb{R}^1_0 = \mathbb{R}^1 - \{0\}$

- $H^n(X,A) \otimes H^m(Y,B) \xrightarrow{\times} H^{n+m}(X \times Y, A \times Y \cup B \times X)$

  $\omega, \tau \longrightarrow p_1^*(\omega) \cup p_2^*(\tau)$

$$\begin{array}{cc} X \times Y, A \times Y & X \times Y, X \times B \\ p_1 \downarrow & p_2 \downarrow \\ \cancel{A \times Y}\ X, A & Y, B \end{array}$$



- $H^n(X,A) \otimes H^n(X,B) \xrightarrow{\cup} H^n(X, A \cup B)$

New Notation: $\mathcal{S}^n\binom{X}{A}$:= n^{th} singular cohomology of pair (X,A).

$\mathcal{S}^{-n}\binom{X}{A}$:= n^{th} singular homology of (X,A).

- How to define $\cup$ on $\binom{X}{A}\binom{Y}{B}$?

  $\omega \in \mathcal{S}^n\binom{X}{A}$, $\tau \in \mathcal{S}^m\binom{Y}{B}$

  Take $n+m$ simplex

  act ω on front face, τ on back face.

  If a simplex lies in $A \cap B$, $\omega\tau$ on it will be 0.

  why will $\omega\tau$ be 0 on $A \cup B$?

  if $\sigma \in A$ or B $\omega\tau(\sigma) = 0$ as $\omega = 0$ or $\tau = 0$

  but σ might not be completely in A or B.

  But $\exists$ a short exact seq:

$$0 \longrightarrow C^*\binom{X}{A \cup B} \longrightarrow C^*\binom{X}{A} \cap C^*\binom{X}{B} \longrightarrow C^*\binom{A \cup B}{A} \cap C^*\binom{A \cup B}{B} \to 0$$

Claim: $C^n\binom{A \cup B}{A} \cap C^n\binom{A \cup B}{B} \xrightarrow{\delta} C^{n+1}(\quad) \cap C^{n+1}(\quad)$

is always acyclic. (i.e. trivial cohomology)

Proof: • $\omega \in \ker \delta \subseteq C^n(\quad)(\quad)$

$\Rightarrow \omega \in C^n(A \cup B)$, $\omega|_A = 0 = \omega|_B$, $\delta\omega = 0$

$\omega \in \operatorname{Im} \delta \subseteq C^n$

$\Rightarrow \omega = \delta\tau \qquad \tau \in C^{n+1}(\quad) \cap C^n(\quad)$

for $\omega \in \operatorname{Ker}\delta$, need to construct $\tau \in C^{n+1}$ s.t. $\delta\tau = \omega$

Do barycentric subdivision of to $A \cup B$

$$C^n_{A,B}\begin{pmatrix} A\cup B \\ A \end{pmatrix} \cap C^n_{A,B}\begin{pmatrix} A\cup B \\ B \end{pmatrix}.$$

ω acts only on chains of A or B so this cochain group itself is trivial.

Now by barycentric subdivision th^m, $C^n = C^n_{A,B}$ $\square$

So when we write ~~coh~~ long-exact sequence

$$0 \longrightarrow \mathcal{S}^*\begin{pmatrix} X \\ A\cup B \end{pmatrix} \longrightarrow H^*\left(C^*\begin{pmatrix} X \\ A \end{pmatrix} \cap C^*\begin{pmatrix} X \\ B \end{pmatrix}\right) \longrightarrow 0$$

So a cochain which is 0 on A or B

i.e. $\omega \in C^n(X)$, $\omega|_A = 0 = \omega|_B$ (is cohomologous)

can we represented by a cochain which is 0 on $A \cup B$

i.e. $\exists \tau \in C^n(X)$, $\tau|_{A\cup B} = 0$, $(\tau - \omega) = \delta\omega'$ for some ω'

$$\mathcal{S}^n(B, \bullet) \otimes \mathcal{S}^1(\mathbb{R}^1, \mathbb{R}^1_0) \xrightarrow{\times} \mathcal{S}^{n+1}(B\times\mathbb{R}^1, B\times\mathbb{R}^1_0)$$

$$b, e \longmapsto b\times e$$

$\mathbb{R}^1_- =$ negative Real axis

$$\begin{array}{ccccccc}
\xrightarrow{\text{injective}} \mathcal{S}^n(B\times\mathbb{R}^1_0) & \longrightarrow & \mathcal{S}^{n+1}\begin{pmatrix} B\times\mathbb{R}^1 \\ B\times\mathbb{R}^1_0 \end{pmatrix} & \longrightarrow & \mathcal{S}^{n+1}(B\times\mathbb{R}^1) & \xrightarrow{\text{injective}} & \mathcal{S}^{n+1}(B\times\mathbb{R}^1_0) \cdots \\
\uparrow \times b & & \uparrow \times b & & \uparrow \times b & & \uparrow \times b \\
\mathcal{S}^0(\mathbb{R}^1_0) & \longrightarrow & \mathcal{S}^{1}\begin{pmatrix} \mathbb{R}^1 \\ \mathbb{R}^1_0 \end{pmatrix} & \longrightarrow & \mathcal{S}^{1}(\mathbb{R}^1) & \longrightarrow & \mathcal{S}^{1}(\mathbb{R}^1_0) \cdots \\
 & & e & \longmapsto & a & &
\end{array}$$

$B\times\mathbb{R}^n_0 \qquad B\times\mathbb{R}^n$

$$\left[B\times\mathbb{R}^n_0\right]^{m-1+n} \xrightarrow{\delta^*} \begin{bmatrix} B\times\mathbb{R}^n_0 \\ B\times\mathbb{R}^n_0\end{bmatrix}^{m+n} \xrightarrow{p^*} \begin{bmatrix} B\times\mathbb{R}^n \\ \end{bmatrix}^{m+n} \xrightarrow{i^*} \left[B\times\mathbb{R}^n_0\right]^{m+n} \longrightarrow$$

$$\begin{bmatrix} B\times\mathbb{R}^n_0 \\ B\times\mathbb{R}^n_0\end{bmatrix} \xrightarrow{\psi_*} [B]^n, \quad [B]^n \underset{i^*}{\overset{p^*}{\rightleftarrows}} [B\times\mathbb{R}^n], \quad [B]^n \xrightarrow{\phi^*} [B\times\mathbb{R}^n_0]$$

ϕ: $B\times\mathbb{R}^n_0 \hookrightarrow B\times\mathbb{R}^n \twoheadrightarrow B$ just the
$b,x \longmapsto b,x \longmapsto b$ usual projection

Interesting: ψ: $B \hookrightarrow B\times\mathbb{R}^n \twoheadrightarrow \begin{matrix}B\times\mathbb{R}^n\\ B\times\mathbb{R}^n_0\end{matrix}$

$b \longmapsto (b, 1) \longrightarrow 0$

note that because of basepoint restrictions cannot map b to $(b,0)$

So we get a ~~split~~ short exact:

$$0 \longrightarrow [B]^{m+n} \xrightarrow{\phi^*} [B\times\mathbb{R}^n_0]^{m+n} \xrightarrow{\delta^*} \begin{bmatrix} B\times\mathbb{R}^n \\ B\times\mathbb{R}^n_0\end{bmatrix}^{m-1+n} \longrightarrow 0$$

$$[B\times\mathbb{R}^n_0]^{m+n} \cong [B]^{m+1} \oplus [B]^m \times [\mathbb{R}^n_0]^n$$

$$\mathcal{S}^m(B) \xrightarrow{\cong} \mathcal{S}^m(B)\otimes \mathcal{S}^n(\mathbb{R}^n_0) \hookrightarrow \mathcal{S}^{m+n}(B\times\mathbb{R}^n_0) \longrightarrow \mathcal{S}^{m-1+n}\begin{pmatrix}B\times\mathbb{R}^n\\ B\times\mathbb{R}^n_0\end{pmatrix} \rightarrow$$

$\omega \qquad \omega \overset{\times}{\otimes} e' \qquad \omega \overset{\times}{\otimes} e'$? ~~need~~ $\omega\times e$ $\omega\times e$

The map δ^* is

Given $\tau \in \mathcal{S}^{m+n}(B\times\mathbb{R}^n_0)$,

Extend τ to $B\times\mathbb{R}^n$, look at $\delta\tau$

Now given $\omega\overset{\times}{\otimes} e'$, we can extend just e' to $\mathbb{R}^n$? Not rigorous

So $\delta^*(\omega\overset{\times}{\otimes} e') = \omega \underset{\times}{\otimes} \delta^* e' \in \mathcal{S}^{m-1+n}\begin{pmatrix}B\times\mathbb{R}^n\\ B\times\mathbb{R}^n_0\end{pmatrix}$

what is $\delta^* e'$? e. $0 \longrightarrow \mathcal{S}^n(\mathbb{R}^n_0) \xrightarrow{\delta^*} \mathcal{S}^{n+1}\begin{pmatrix}\mathbb{R}^n\\ \mathbb{R}^n_0\end{pmatrix} \longrightarrow 0$

$e \longrightarrow e'$

So

$$\mathcal{S}^m(B) \longrightarrow \mathcal{S}^{m+n}\begin{pmatrix}B\times\mathbb{R}^n\\ B\times\mathbb{R}^n_0\end{pmatrix} \quad \text{isomorphism}$$

$$\omega \longmapsto \omega\times e$$

Interesting: Characteristic Classes can be defined over K-Theory also!
Then Thom class of a vector bundle is the vector bundle itself!

Obstruction Theory.

Q. $S^2 \hookrightarrow \mathbb{C}P^\infty$ Given $f: S^2 \to S^2$
Can we extend $\hat{f}: \mathbb{C}P^\infty \to \mathbb{C}P^\infty$

$$\begin{tikzcd} S^1 \arrow[r] & S^\infty \arrow[d] \\ & \mathbb{C}P^\infty \end{tikzcd} \qquad \pi_* \mathbb{C}P^\infty = \begin{cases} \mathbb{Z} & * = 2 \\ 0 & \text{else} \end{cases}$$

$\pi_3(\mathbb{C}P^\infty) = 0 \Rightarrow f: S^2 \to S^2$
Extends to $f: \mathbb{C}P^2 \to \mathbb{C}P^\infty$

Analogosly, we can extend the map f to entire $\mathbb{C}P^\infty$.
Also the extension, $\hat{f}$ is unique upto homotopy. ☒

Q. η- Hopf map

$$\begin{tikzcd} S^3 \arrow[d, "\eta"'] \arrow[dr] & \\ S^2 \arrow[r, "id"] \arrow[d] & S^2 \\ \mathbb{C}P^\infty \arrow[ur, dashed, "?"'] & \end{tikzcd}$$

Can one extend $S^2 \xrightarrow{id} S^2$ to $\mathbb{C}P^2 \to S^2$?

No.
Because η is not null-homotopic.

$$\begin{tikzcd} S^{2n-1} \arrow[d, "\eta"'] \arrow[dr] & \\ \mathbb{C}P^{n-1} \arrow[r, "id"] \arrow[d] & \mathbb{C}P^{n-1} \\ \mathbb{C}P^n \arrow[ur, dashed, "?"'] & \end{tikzcd}$$

Can one extend $\mathbb{C}P^{n-1} \xrightarrow{id} \mathbb{C}P^{n-1}$ to $\mathbb{C}P^n \longrightarrow \overset{?}{} \mathbb{C}P^{n-1}$

No, because extension is possible iff $S^{2n-1} \xrightarrow{\eta} \mathbb{C}P^{n-1}$ is ~~trivial~~ null homotopic. But if $S^{2n-1} \xrightarrow{\eta} \mathbb{C}P^{n-1}$ were null homotopic then $\mathbb{C}P^n \simeq \mathbb{C}P^{n-1} \vee S^{2n-1}$ which is false (cohomology).

Q. X, Y, Z CW, $Z < X$ Given $Z \to Y$ can we extend it to a map $X \to Y$?

Induction:

$$\begin{tikzcd} \coprod S^n_\alpha \arrow[d, "\phi^n_\alpha"] & \\ X^{(n)} \arrow[r, "f_n"] \arrow[d] & Y \\ X^{(n+1)} & \end{tikzcd}$$

$\phi^n_\alpha : S^n_\alpha \to X^{(n)}$ is the attaching map.

$f_n : X^{(n)} \to Y$ extends iff $f_n \circ \phi^n_\alpha$ are all null homotopic

$f_n \circ \phi^n_\alpha \in \pi_n(Y)$.

Assume $n > 1$,

$$C^{cell}_{n+1}(X, Z) \xrightarrow[(\Delta f_n)]{} \pi_n(Y)$$

cellular chain complex

$$(\Delta f_n)(\sum c_\alpha e^{n+1}_\alpha) = \sum c_\alpha \cdot (f_n \circ \phi^n_\alpha)$$

$$(\Delta f_n) \in \mathrm{Hom}(C^{cell}_{n+1}(X, Z), \pi_n Y)$$
$$\shortparallel$$
$$C^{n+1}_{cell}(X, Z; \pi_n(Y))$$

Claim: (Δf_n) is a ~~cocyc cocha~~ cocycle.

Proof:
$$\delta \Delta f_n (e^{n+2}) = \Delta f_n (\partial e^{n+2})$$
$$= \Delta f_n \Big(\sum_\alpha (\deg \psi_\alpha)\, e^{n+1}_\alpha \Big)$$

$$\psi_\alpha : S^{n+1} \xrightarrow{\phi} \bigvee_\alpha S^{n+1} \to S^{n+1}_\alpha$$

$$= \sum_\alpha (\deg \psi_\alpha) \cdot (f_n \circ \phi^n_\alpha)$$

cohomologous $\sim f_n \Big(\sum_\alpha \deg \psi_\alpha \circ \phi^n_\alpha \Big)$

$= 0$? Work it up.

Require that (X, Z) is simply connected.

So $o_n = [\Delta f_n] \in H^{n+1}(X, Z; \pi_n Y)$

Claim: f_n extends iff $\Delta_n = 0$.

Proof: $\Rightarrow$ trivial

$\Leftarrow$ $\quad \Delta_n = 0 \Leftrightarrow \Delta f_n = \delta\omega \qquad \Delta f_n(e_\alpha^{n+1}) = \delta\omega(e_\alpha^{n+1})$

$\omega \in C^{n+1}_{cell}(X, Z; \pi_n Y) \qquad = \omega(\partial e_\alpha^{n+1})$ ∂ - cellular boundary

$\cancel{= \omega(S^n \xrightarrow[\phi_\alpha^n]{} X^{(n)})}$

Th^m:

Z, X simply connected, $Z < X$;

$Z \xrightarrow{f} Y$

$\exists$ an obstruction $\Delta_n \in H^{n+1}(X, Z, \pi_n Y)$, such that

$\Delta_n = 0 \Leftrightarrow$ map extends to $(n+1)$ skeleton.

extensions $\longleftrightarrow H^{n+1}(X, Z; \pi_n(Y))$ [Note: This obstruction makes sense i.e. Δ_{n+1} is defined only after extending map to $X^{(n)}$]

Principal G-Bundles

G - Topological Group

$$\begin{array}{ll} E & G \circlearrowright E \text{ fibrewise} \\ \downarrow & G \circlearrowright E_x \text{ freely, transitively} \\ X & \end{array}$$

$M^{or} = \{(x, \alpha_x) \mid \alpha_x$ generator of $H^n(M, M-x)\}$

orientation cover

$\downarrow$

M - ~~orientable~~ manifold

$\mathbb{Z}/2$ action on M^{or}: $\tau(x, \alpha_x) = (x, -\alpha_x)$

$\mathbb{Z}/2 = \{1, \tau\}$

$\Rightarrow$ M^{or} is a principal $\mathbb{Z}/2$ bundle.

$Y \downarrow X$ covering space, Galois, ie $\pi_1 Y \triangleleft \pi_1 X$

Deck transformations act on Y ($\pi_1 X / \pi_1 Y$)

$\Rightarrow Y$ principal $\pi_1 X/\pi_1 Y$ bundle.

- $\xi^{(k)} \downarrow X$ $\quad V(\xi) = \{ (x, v_1, \dots, v_k)_x \mid (v_1 \cdots v_k)_x \text{ basis for } \xi_x \} \subseteq X \times \xi \times \cdots \times \xi$, $\quad V(\xi) \downarrow X$

fibre bunde, $\quad$ fibre $= \{ (e_1 \cdots e_k) \mid e_1 \cdots e_k \text{ basis for } \mathbb{R}^k \}$

$= Gl_k(\mathbb{R}^{n})$

$Gl_k(\mathbb{R})$ principal bundle.

- Similarly for a Riemannian vector bundle, we will get a principal ~~G-bun~~ $O(n)$-bundle. For oriented vector bundle $SO(n)$-bundle.

<u>Thm</u> If $f, f' : X \longrightarrow Y$ are such that $f \simeq f'$, then $f^*P \cong f'^*P$

$$\begin{array}{ccccc} f^*P & \longrightarrow & P & \longleftarrow & f'^*P \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longleftarrow & Y \end{array}$$

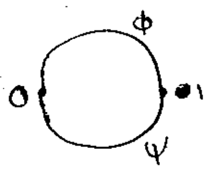
- $P_G(X) =$ Principal G-bundles over $X/\sim \longrightarrow$ G-bundle isomorphism

$f : X \longrightarrow Y \xrightarrow{P_G} f^* : P_G(Y) \xrightarrow{P_G f} P_G(X)$

only depends on $f \in [X, Y]$

<u>Proposition</u>: $G \longrightarrow P \downarrow X$ is trivial iff $\exists s : X \longrightarrow P$ section.

- What is $P_G(S^1)$? ~~Aut(G)~~ $= G$ $\quad \pi_0(G)$



$\xi \downarrow S^1$

We can give trivializations so that

$\phi : \xi|_{\frown} \longrightarrow \frown \times G$

$\psi : \xi|_{\smile} \longrightarrow \smile \times G$

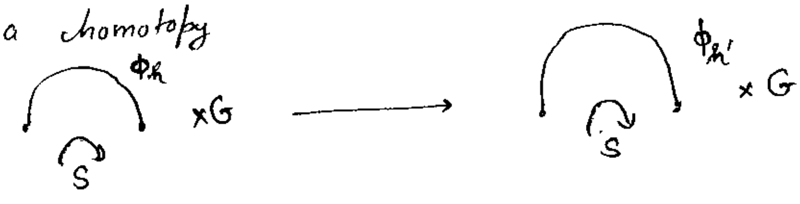
such that $\phi \cdot \psi^{-1}(0) = \mathrm{id}$

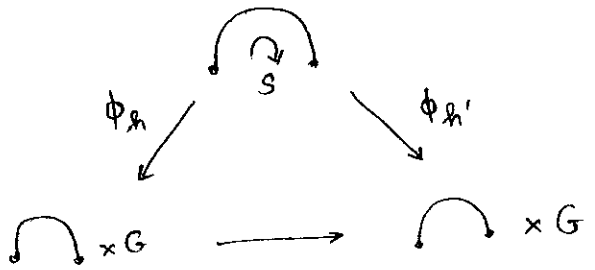
Now $\phi \cdot \psi^{-1}(1) \in \mathrm{Aut}_G G = G$

Call ~~ξ_h~~ the G-bundle corresponding to $\phi \cdot \psi^{-1}(1) = h$ $\quad \xi_h$.

If $\exists$ a path joining h to h'. $\quad \gamma : [0,1] \longrightarrow G$, $\quad (0,1) \longmapsto (h, h')$

we give a homotopy





$$H : \frown \times [0,1] \longrightarrow \frown \times G$$
$$(s, t) \longmapsto$$

<u>Proposition</u>: $\quad P_G(S^1) = \pi_0(G)$

$\cancel{P_G(S^1 \wedge X) = [X, G]_*}$

$P_G(S^1 \wedge X) = [X, G]_*$

<u>Proof</u>:

$S^1 \wedge X = C_1 X \underset{X}{\sqcup} C_2 X$ $\qquad \xi \xrightarrow{\pi} \Sigma X$

Assume trivializations,

$\phi : \pi^{-1}(C_1 X) \longrightarrow C_1 X \times G$

$\psi : \pi^{-1}(C_2 X) \longrightarrow C_2 X \times G$

Further assume that $\qquad x_0 =$ basepoint of X.

$\cancel{\phi|_{x_0} = \psi|_{x_0}}$

$\phi|_{\pi^{-1}(x_0)} = \psi|_{\pi^{-1}(x_0)}$

So on $X = C_1 X \cap C_2 X$

$\psi \cdot \phi^{-1} : X \times G \longrightarrow X \times G$

$(x, g) \longmapsto (x, h(g))$

identity on x_0.

$\psi \circ \phi^{-1} : X \longrightarrow \mathrm{Aut}_G(G) = G$ $\qquad X \times G \xrightarrow{\pi_2} G$

$x \longmapsto \pi_2(\psi \cdot \phi^{-1}(x, e))$

Call this $h \in \cancel{[X,G]_*}\ \mathrm{Hom}_*(X, G)_*$

Then, by this we have a vector bundle ξ_h, corresponding to every element of $\cancel{[X,G]_*}\ \mathrm{Hom}(X, G)_*$.

Next we need to determine when two of these are equivalent.

<u>Claim</u>: $\quad \xi_h \cong \xi_{h'} \iff \{h\} \cong \{h'\} \bmod [X, G]_*$

Milnor: Construction of Universal bundles.

$$\Leftarrow \quad H : X \times [0,1] \longrightarrow G$$
$$H_0(x) = h(x)$$
$$H_1(x) = h'(x)$$

$$\begin{tikzcd}
\phi_h, \psi_h : \xi_h \arrow[dr] & & \xi_{h'} \ \phi_{h'}, \psi_{h'} \arrow[dl] \\
& \Sigma X &
\end{tikzcd}$$

$$\phi_h : \pi^{-1}(C_1X) \xrightarrow{\sim} C_1X \times G \qquad \phi_{h'} : \pi^{-1}(C_1X) \xrightarrow{\sim} C_1X \times G$$
$$\psi_h : \pi^{-1}(C_2X) \xrightarrow{\sim} C_2X \times G \qquad \psi_{h'} : \pi^{-1}(C_2X) \xrightarrow{\sim} C_2X \times G$$

Aim is to construct a

$$\begin{tikzcd}
K : \xi_h \arrow[rr] \arrow[dr, "\pi_h"'] & & \xi_{h'} \arrow[dl, "\pi_{h'}"] \\
& \Sigma X &
\end{tikzcd}$$

f is G-isomorphism on each fibre.

Define:

$$K : \pi_h^{-1}(C_1X) \longrightarrow \pi_h^{-1}(C_2X)$$
$$\phi_h^{-1}(x,g) \longmapsto \phi_{h'}^{-1}(x,g)$$

$$\pi_h^{-1}(C_2X) \longrightarrow \pi_h^{-1}(C_2X)$$

$$\begin{tikzcd}
\pi_h^{-1}(C_2X) \arrow[dr, "\pi_h"] \arrow[ddr, bend right, "\phi_h"'] & & \pi_{h'}^{-1}(C_2X) \arrow[dl, "\pi_{h'}"'] \arrow[ddl, bend left, "\phi_{h'}"] \\
& C_2X & \\
& C_2X \times G &
\end{tikzcd}$$

$$C_2X = [0,1] \times X \big/ (x,1) \sim (x',1)$$

$$K \psi_h^{-1}(t,x,g) = \psi_{h'}^{-1}(t,x, H_t(x)^{-1} g)$$

On C_1X $\quad K(\phi_h^{-1}(t,x,g)) = \phi_{h'}^{-1}(t,x,g)$

C_2X $\quad K(\psi_h^{-1}(t,x,g)) = \psi_{h'}^{-1}(t,x, \cancel{H_t(x)^{-1}g}\ H_{1-t}(x)^{-1} \cdot h_0(x)^{-1} g)$

• K - well defined

on $C_1 \times \cap C_2 X$: $\quad K(\phi_h^{-1}(0,x,g)) = \phi_{h'}^{-1}(0,x,g)$ at $t=0$

$K(\psi_h^{-1}(0,x,g)) = \psi_{h'}^{-1}(0,x,h'(x)^{-1} h(x)^{-1}(g))$

$\phi_{h'}^{-1}(0,x,g) \overset{?}{=} \psi_{h'}^{-1}(0,x,h'(x)^{-1}g)$

$\psi_{h'} \cdot \phi_{h'}^{-1}(0,x,g) \overset{?}{=} (0,x,h'(x)^{-1}g)$

$\parallel$

$(0,x,h'(g))$

$\psi_h \cdot \phi_h^{-1}(0,x,g) = (0,x,h(x)^{-1}g)$

$\phi_h^{-1}(0,x,g) = \psi_h^{-1}(0,x,h(x)g)$

$K(\quad) \overset{?}{=} K(\quad)$

$\phi_{h'}^{-1}(0,x,g) \overset{?}{=} \psi_{h'}^{-1}(0,x,h(x)g)$

$h'(x)\, h(x)^{-1}$

$\psi_{h'}\phi_{h'}^{-1}(0,x,g) =$

$\parallel$

$(0,x,h'(x)g)$

so K - defined on X.

at $t=1$, K should be ~~constant ident~~ constand function

$K(\psi_{h'}^{-1}(1,x,g)) = \psi_{h'}^{-1}(1,x, H_{1-t}(x)^{-1} \cdot h(x)^{-1}(g))$ $\quad t=1$

$= \psi_h^{-1}(1,x,g)$

So K - well defined on $\mathcal{E}\Sigma X$.

• K fibrewise isomorphism. Easy.

$$\Rightarrow \quad \begin{tikzcd} \xi_h \arrow[rr, "\cong"] \arrow[dr] & & \xi_{h'} \arrow[dl] \\ & \Sigma X & \end{tikzcd}$$

So far we have obtained a map

$$[X,G]_* \longrightarrow P_G(\Sigma X)$$
$$[h] \longmapsto \xi_h$$

we need to show bijection.

- Surjection is clear
- We will construct an inverse.

$$P_G(\Sigma X) \longrightarrow [X,G]_*$$
$$\phi : \pi^{-1}(C_1X) \longrightarrow C_1X \times G$$
$$\psi : \pi^{-1}(C_2X) \longrightarrow C_2X \times G$$
$$\psi \cdot \phi^{-1} : \underset{X\times G}{\cancel{\pi^{-1}(X)}} \longrightarrow X \times G$$

Give map: $\xi \longmapsto \psi \cdot \phi^{-1}(-, e)$

If well-defined inverse is obvious.
So enough to show, does not depend on choices of Φ, Ψ.
Suppose Φ_1, Ψ_1 are another trivializations.
Need to show: $\quad \psi_1 \cdot \phi_1^{-1}(-,e) \equiv \psi \cdot \phi^{-1}(-,e) \mod [X,G]_*$

$$\phi \cdot \psi^{-1} \cdot \psi_1 \cdot \phi_1^{-1} \ (-,e) \equiv id \qquad \mod [X,G]_*$$

Either need X - path connected or G path connected.
By identifying C_1X, C_2X, we can define the map

$$\phi \cdot \psi^{-1} \cdot \psi_1 \cdot \phi_1^{-1} : CX \times G \longrightarrow CX \times G$$

i.e. $\phi \cdot \psi^{-1} \cdot \psi_1 \cdot \phi_1^{-1}(-,e) : \mathbb{C}X \longrightarrow G$
$\qquad x \longmapsto \Pi_2(\cdots(x,g))$

i.e. on X the map is null homotopic.
i.e. homotopy equivalent to identity.

So to conclude,

$$[X,G]_* \cong P_G(\Sigma X).$$

~~Principal Bun~~

Universal Bundle :

$\begin{matrix} E_Z \\ \downarrow \\ Z \end{matrix}$ Principal G-bundle , Z - CW complex

- $[X, Z] \xrightarrow{\cong} P_G(X)$ $\quad$ Z- <u>Universal G-bundle</u>

  X - CW complex

- $\begin{matrix} E_Z \\ \downarrow \\ Z \end{matrix}$ <u>n-universal</u> if iso. is true for X - C.W complex of dim $\leq n$.

- $\begin{matrix} E_Z \\ \downarrow \\ Z \end{matrix}$ ~~Universal upto~~ unique upto homotopy

  Universal

  i.e. $\begin{matrix} E_{Z_1} \\ \downarrow \\ Z_1 \end{matrix}$ $\begin{matrix} E_{Z_2} \\ \downarrow \\ Z_2 \end{matrix}$ 2 - universal

  $\Rightarrow \exists f : Z_1 \longrightarrow Z_2$ s.t.

  $f^* E_{Z_2} = E_{Z_1}$. $\quad$ f - homotopy equivalence

- Analogously

  $\begin{matrix} E_Z \\ \downarrow \\ Z \end{matrix}$ n-Universal , unique upto n-equivalence.

<u>Theorem</u> :

$\begin{matrix} E_Z \\ \downarrow \\ Z \end{matrix}$ n-universal $\Leftrightarrow$ E_Z $(n-1)$- connected

$\left(\Rightarrow \begin{matrix} E_Z \\ \downarrow \\ Z \end{matrix} \text{ universal } \Leftrightarrow E_Z \text{ contractible} \right)$.

<u>Proof</u>:

- 0- ~~connected~~ universal

  - $[S, Z] = P_G(S)$ $\quad$ S- 0-dim

    $= \{*\}$ $\quad$ $\Rightarrow$ discrete set of pts

    $\Rightarrow$ Every bundle on S trivial

$\Rightarrow$ Z - path connected

π_1 - connected

- 1 universal

~~S~~ $\frown = I$ $\qquad [I, Z] = P_G[I] = *$

$\bigcirc = S^1$ $\qquad [S^1, Z] = \cancel{P_G(S^1)}$

$\pi_1 Z$

$$\begin{array}{ccc} G & \longrightarrow & E_Z \\ & & \downarrow \\ & & Z \end{array}$$

$[S^1, Z] = P_G(S^1) = \pi_0(G)$

$\quad \shortparallel$

$\pi_1(Z)$

$\to \pi_* G \to \pi_* E_Z \to \pi_* Z \to \pi_0(G)$

- $n-1$ dim CW $- X$ $\qquad$ n-Universal

$S^{n-1} \xrightarrow{f} X \qquad Y = X \cup_f e^n$

$$\begin{array}{c} P \\ \downarrow \\ S^{n-1} \end{array}$$ G-bundle

1) Can this be extended to D^n

Iff P is a trivial bundle

i.e. P has a section.

Lemma:

$$\begin{tikzcd} P \arrow[d] \arrow[rrd, phantom] & & E_Z \arrow[d] \\ S^{n-1} \arrow[u, bend left, "s"] \arrow[rr, "f"] \arrow[urr, dashed] & & Z \end{tikzcd}$$

s section exists $\Leftrightarrow f$ lifts to E_Z.

$P = f^* E_Z.$

$$\begin{tikzcd} P \arrow[d] & E_Z \arrow[d] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

$P = f^* E_Z$

Q. When can P be extended to Y?

Iff $\phi^*(P)$ extends to e^n.

Iff $\qquad$ By Lemma

$$\begin{tikzcd} & & E_Z \arrow[d] \\ S^{n-1} \arrow[r, "\phi"] \arrow[urr, dashed] & X \arrow[r, "f"] & Z \end{tikzcd}$$

$f \circ \phi$ lifts

Q. How many extensions exists?

$\pi_{n-1}(G)$.

$Y = X \cup_{\phi} e^n = \text{cone}(\phi)$

$(M_\phi) \qquad (C_\phi)$

$$S^{n-1} \xrightarrow{\phi} X \longrightarrow Y \longrightarrow S^n \longrightarrow \Sigma X \longrightarrow$$

$$[S^{n-1}, Z] \xleftarrow{\phi^*} [X,Z] \longleftarrow [Y,Z] \longleftarrow [S^n, Z] \xleftarrow{(\Sigma\phi)^*} [\Sigma X, Z]$$

$\phi \circ f \longleftarrow\!\shortmid\ f \qquad\qquad [S^n,Z] \cong \pi_n(Z)$

Q. when can $f \in [X,Z]$ be extended to $f \in [Y,Z]$?

- $f \in \mathrm{Im}([Y,Z] \longrightarrow [X,Z])$
  $f \in \ker(\phi^*)$

Q How many extensions?
i.e. How many pre-images?

$g \in [Y,Z]$. $\quad g \longrightarrow f$

$g_1, g_2 \longrightarrow f \quad \Rightarrow \quad g_1 - g_2 = 0$

$\Rightarrow \exists h.\ h \in [S^n, Z], \ h \longrightarrow g_1 - g_2$

So $\exists$ $\pi_n(Z)$ many extensions

$\Rightarrow \quad \pi_n(Z) \cong \pi_{n-1}(G)$

Milnor - Stasheff

$X * Y = \Sigma(X \wedge Y) = S^1 \wedge X \wedge Y$?

$\dfrac{X \times CY \cup CX \times Y}{X \times Y}$

Q.5 - A)

$$Gl_n(\mathbb{R}) \longrightarrow V_n(\mathbb{R}^{n+k}) \xrightarrow{q} G_n(\mathbb{R}^{n+k})$$

$f \circ q : V_n(\mathbb{R}^{n+k}) \longrightarrow \mathbb{R}$ C^∞, if $f : G_n(\mathbb{R}^{n+k}) \longrightarrow \mathbb{R}$ C^∞

if $f \circ q$ C^∞. $(f\circ q^{-1})(U)$ open in $V_n(\mathbb{R}^{n+k})$

$q^{-1} \cdot f^{-1}(U)$ open

$\Rightarrow f^{-1}(U)$ open as q is quotient map, hence open

Now remains to check smoothness, i.e.

Claim: $f\circ q$ C^∞ $\Rightarrow$ f C^∞

Locally $G_n(\mathbb{R}^{n+k}) \supseteq U$, $q^{-1}(U) = U\times Gl_n(\mathbb{R})$.

Locally we have a map

$i: U \longrightarrow U\times Gl_n(\mathbb{R})$ which is just a section,

Then

$$f = f\circ q\circ i \quad = C^\infty$$

⊠

Q.5 B)

Problem with direct approach:

The canonical bundle is not defined for arbitrary G-bundles. $G_n(\mathbb{R}^{n+k})$ is a very specific space

$G_n(\mathbb{R}^{n+k}) = \{n \text{ planes in } \mathbb{R}^{n+k}\}$

• C^∞-structure on $G_n(\mathbb{R}^{n+k})$

$x \in G_n(\mathbb{R}^{n+k})$

$U_x = \{y \in G_n(\mathbb{R}^{n+k}) \mid \nexists\, v\in y.\ v\perp x\}$

$x^\perp \in G_k(\mathbb{R}^{n+k}) = \{w \mid w\perp x\}$

$\varphi_x: U_x \longrightarrow \mathbb{R}^{nk} \longleftarrow$ Coordinate Chart

Let $e_1 \cdots e_n$ be a basis for x i.e. $x = \langle e_1 \cdots e_n\rangle$

$f_1 \cdots f_k$ be a basis for $x^\perp$ $x^\perp = \langle f_1 \cdots f_k\rangle$

$y\in U_x$, $y = \langle e_1', e_2', \cdots, e_n'\rangle$

$= \langle e_1 + f_1', e_2+f_2', \cdots, e_n + f_n'\rangle$

Then if

$$\begin{bmatrix} f_1' \\ \vdots \\ f_n' \end{bmatrix} = \begin{bmatrix} & & \\ & y_{ij} & \\ & & \end{bmatrix} \begin{bmatrix} f_1 \\ \vdots \\ f_k \end{bmatrix}$$

$$\varphi_x(y) = [y_{ij}]$$

Then we have

$$\varphi_x^*: T(U_x) \cong \cancel{\varphi_x^*}(T\mathbb{R}^{nk}) \cong \mathbb{R}^{nk}\times\mathbb{R}^{nk} \cong U_x\times\mathbb{R}^{nk}$$

- $$\begin{tikzcd} \gamma \arrow[dr] & & \gamma^{\perp} \arrow[dl] \\ & G_n(\mathbb{R}^{n+k}) & \end{tikzcd}$$

$\gamma = \{(x,\omega) \mid \omega \in x\}$
$\gamma^{\perp} = \{(y,\omega) \mid \omega \perp y\}$

Locally we have:

$x \in G_n(\mathbb{R}^{n+k})$

$x^{\perp}, U_x$ as defined earlier

$$\gamma(U_x) \xrightarrow{\sim} U_x \times \mathbb{R}^n$$
$$(y,\omega) \longmapsto (y, \text{ co-ordinates of } \omega \text{ in the basis } e_1 \dots e_n).$$

$$\gamma^{\perp}(U_x) \xrightarrow{\sim} U_x \times \mathbb{R}^k$$
$$(y,\omega) \longmapsto (y, \text{ co-ordinates of } \omega \text{ in the basis } f_1 \dots f_k)$$

$$\psi : \mathrm{Hom}(\gamma, \gamma^{\perp}) \xrightarrow{\sim} U_x \times \mathbb{R}^{nk}$$
$$(y, \alpha) \longmapsto (y, \alpha \text{ written as a matrix in the basis } e_1 \dots e_n, f_1 \dots f_k)$$

Then we have the following isomorphism:

$$\mathrm{Hom}(\gamma,\gamma^{\perp}) \xrightarrow[\psi]{} U_x \times \mathbb{R}^{nk} \xrightarrow[(\varphi^*)^{-1}]{} T U_x \times \cancel{}_x \, .$$

(We have made a lot of choices of bases. Compatibility is something that needs to be checked).

- $$M^n \hookrightarrow \mathbb{R}^{n+k}$$

$$\bar{g} : M^n \longrightarrow G_n(\mathbb{R}^{n+k}) \qquad \textit{Generalised Gauss map.}$$

$$\bar{g}^* : TM \longrightarrow TG_n(\mathbb{R}^{n+k})$$
$$\cong$$
$$\mathrm{Hom}(\gamma, \gamma^{\perp})$$

At a point $p \in M$,

$$\gamma(\bar{g}(p)) = T_pM, \quad \gamma^{\perp}(\bar{g}(p)) = \nu_p M$$

$$\Rightarrow \quad \begin{tikzcd} \bar{g}^*: TM \arrow[rr] \arrow[dr] & & \mathrm{Hom}(TM, \mathcal{V}) \arrow[dl] \\ & M & \end{tikzcd}$$

$\Rightarrow \quad \bar{g}_p^* \in \quad \mathrm{Hom}(T_pM, \mathrm{Hom}(T_pM, \mathcal{V}_p)) \cong \mathrm{Hom}(T_pM, T_pM^* \otimes \mathcal{V}_p)$

$\cong \mathrm{Hom}(T_pM \otimes T_pM, \mathcal{V}_p)$ ~~$\because$ finite dimensional~~

$(\mathrm{Hom}(A, B^* \otimes C) \underset{?}{\cong} \mathrm{Hom}(A \otimes B, C))$

$\boxtimes$

Q.5 c)

$I_n \subseteq M_n(\mathbb{R}^{n+k}) = \{A \in M_n(\mathbb{R}^{n+k}) \mid A^2 = A, \ \mathrm{trace}(A) = n\}$

Then, we know that Jordan-Canonical form of A is

$A \sim B + C$

$B = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$ only 1's, 0's in diagonal, all other entries 0.

$C = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 0 & 0 & & \\ & & 1 & 0 & \\ & & & \ddots & 0 \end{bmatrix}$ only 1's, 0's in sub-diagonal, all other entries 0.

~~$A^2 = B^2 + BC + C^2 + CB$~~

~~$\Rightarrow C = BC + C^2 + CB$~~

$\because A(A-1) = 0$, size of each Jordan block should be 1.

$\Rightarrow C = 0$

So $\exists P \in Gl_{n+k}(\mathbb{R}^n)$ s.t.

$PAP^{-1} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$ = Projection in the first n-plane.

So $\quad \psi: G_n(\mathbb{R}^{n+k}) \longrightarrow I_n$

$x \longmapsto$ Projection onto the plane x.

Injectivity is clear, surjectivity follows from the above proposition.

There is one gap in proof:

By Jordan decomposition, we get $P \in Gl_n(\mathbb{C})$ s.t.

$$PAP^{-1} = B$$

we need a $P \in Gl_n(\mathbb{R})$.

One needs a stronger version of Jordan canonical decomposition:

- If K is a field, if ~~minimal~~ characteristic polynomial of A splits in K, then $\exists P \in Gl_n(K)$ s.t. $PAP^{-1} =$ Jordan.

Proof of Jordan canonical actually proves this thm.

$$\varphi : V_n(\mathbb{R}^{n+k}) \longrightarrow \Lambda^n(\mathbb{R}^{n+k}) \longrightarrow \mathbb{P}(\Lambda^n(\mathbb{R}^{n+k}))$$

$$(v_1, \dots v_n) \longmapsto (v_1 \wedge v_2 \wedge \dots \wedge v_n) \longrightarrow [(v_1 \wedge \dots \wedge v_n)]$$

If $sp\langle w_1 \cdots w_n \rangle = sp\langle v_1 \cdots v_n \rangle$

$$\begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = A \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \qquad A \in Gl(\langle v_1 \cdots v_n \rangle)$$

Then, $\varphi(w_1 \cdots w_n) = [w_1 \wedge w_2 \wedge \dots \wedge w_n]$

$$= [\det A \, (v_1 \wedge \dots \wedge v_n)]$$

$$= [v_1 \wedge \dots \wedge v_n] = \varphi(v_1 \cdots v_n)$$

By property of quotient topology

$$\tilde{\varphi} : G_n(\mathbb{R}^{n+k}) \longrightarrow \mathbb{P}(\Lambda^n(\mathbb{R}^{n+k}))$$

Injectivity:

$\cancel{\tilde{\varphi}([v_1, \dots v_n]) =}$

Claim: $v_1 \wedge v_2 \wedge \dots \wedge v_n = (w_1 \wedge w_2 \wedge \dots \wedge w_n) \neq 0$

$\Rightarrow sp\langle v_1, \dots, v_n \rangle = sp\langle w_1, \dots, w_n \rangle$.

Proof: If $k=0$,

$v_1 \wedge \dots \wedge v_n = w_1 \wedge \dots \wedge w_n \neq 0$

$\Rightarrow \mathbb{R}^{n+k} = sp\langle v_1 \cdots v_n \rangle = sp\langle w_1 \cdots w_n \rangle$.

$k > 0$

Assume contrary. Let $v' \perp sp\langle v_1 \cdots v_n \rangle$. $w_1 = v' + (\lambda_1 v_1 + \dots \lambda_n v_n)$

We define following multilinear map on $(\mathbb{R}^{n+k})^n$

$$\alpha : \mathbb{R}^{n+k} \times \dots \times \mathbb{R}^{n+k} \longrightarrow \mathbb{R} \cong \Lambda^n(sp\langle v', w_2, \dots, w_n \rangle)$$

$$(u_1, \dots, u_n) \longmapsto \begin{cases} 0 & \text{if } u_1, \dots, u_n \notin sp\langle v', w_2, \dots, w_n \rangle \\ \text{signed volume of } u_1, \dots, u_n & \text{in the space } v', w_2, \dots, w_n \end{cases}$$

$\cancel{\#}$ $\quad \alpha(v_1 \cdots v_n) = 0 \qquad \because \ v' \perp v_i \ \ \forall i$

$$\alpha(w_1 \cdots w_n) = \alpha(v', w_2 \cdots w_n) + \underbrace{\alpha(\lambda_i v_i, w_2, \ldots, w_n)}_{=0}$$

$$= \alpha(v', w_2, \ldots, w_n)$$

$$\neq 0.$$

Being signed volume α is anti-symmetric

$\Rightarrow$

$$\begin{tikzcd} (\mathbb{R}^{n+k})^k \arrow[r, "\alpha"] \arrow[d] & \mathbb{R} \\ \Lambda^k(\mathbb{R}^{n+k}) \arrow[ur, "\exists \hat{\alpha}"'] & \end{tikzcd}$$

$\Rightarrow \quad \hat{\alpha}(v_1 \wedge \cdots \wedge v_n) = 0$

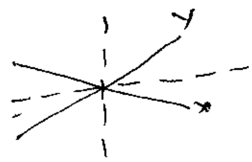
$\quad\quad \hat{\alpha}(w_1 \wedge \cdots \wedge w_n) \neq 0$

$\Rightarrow \quad v_1 \wedge \cdots \wedge v_n \neq w_1 \wedge \cdots \wedge w_n$. Contradiction.

$\boxtimes$

Q. 5 D)

$X, Y \in \mathbb{R}^{n+k}$ n-planes. $X \neq Y$



$\cancel{}$ $\phi \in O(\mathbb{R}^{n+k})$

$\phi(X) = Y$

$\phi(Y) = X$

Claim: ϕ exists, ~~unique~~

Proof:

We should get two n-planes P_1, P_2 which are equidistant from X, Y.

Assume $X = W \oplus X_1 \qquad Y = W \oplus Y_1$

where $W = X \cap Y$

So $X_1 \cap Y_1 = \{0\}$

Enough to find the planes for X_1, Y_1

~~Suppose~~ $X_1 = \cancel{\{x_1 \ldots\}}\ sp\langle x_1 \cdots x_i \rangle$ orthonormal basis

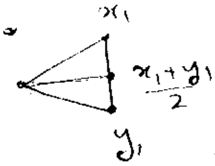
$Y_1 = sp\langle y_1 \cdots y_i \rangle$

s.t. $|y_1| = |x_1| = \cdots = |y_i| = |x_i| = 1$

Then $P_1 = sp\left\langle \frac{x_1+y_1}{2}, \ldots, \frac{x_i+y_i}{2} \right\rangle$

$P_2 = sp\left\langle \frac{x_1-y_1}{2}, \ldots, \frac{x_i-y_i}{2} \right\rangle$

Now we need to show reflecting ~~x_2~~ in P_1 (or P_2) takes x_1 to y_1.



$$(x_1-y_1)\cdot\left(\frac{x_1+y_1}{2}\right) = \cancel{x_1^2 - y_1^2}\ \frac{|x_1|^2-|y_1|^2}{2} = 0$$

$$\left|x_1 - \frac{x_1+y_1}{2}\right| = \left|y_1 - \frac{x_1+y_1}{2}\right|.$$

Result follows. $\square$

Next we need to defined an angle $\alpha(X,Y)$ which is independant of ϕ.

Again decompose X, Y as $X = W \oplus X_1$, $Y = W \oplus Y_1$.

Define:

$$\alpha(X,Y) = \inf_{\substack{x\in X_1\\ y\in Y_1}} \cos^{-1}\left(\frac{x\cdot y}{\|x\|\|y\|}\right)$$

$$= \min_{\substack{x\in X_1, y\in Y_1\\ \|x\|=1=\|y\|}} \cos^{-1}(x\cdot y)$$

Infact $\cos^{-1}(x\cdot y)$ is constant for $x\in X_1$, $y\in Y_1$

α is a metric on $G_n(\mathbb{R}^{n+k})$.

Proof:

- $\alpha: G_n(\mathbb{R}^{n+k})\times G_n(\mathbb{R}^{n+k}) \longrightarrow [0,\pi/2]$ C^∞

Enough to show

$\alpha\circ q: V_n(\mathbb{R}^{n+k})\times V_n(\mathbb{R}^{n+k}) \longrightarrow [0,\pi/2]$ C^∞

~~Fix one~~ Let $e_1 \dots e_n e_{n+1} \dots e_{n+k}$ be standard basis for $\mathbb{R}^{n+k}$

- Enough to show:

$\alpha\circ q((e_1\dots e_n), -)$ C^∞

$$\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} v_{ij} \end{bmatrix} \begin{bmatrix} e_1 \\ e_{n+k} \end{bmatrix} \quad \text{i.e. } v_i = \sum_j v_{ij} e_j$$

$\alpha\circ q\,(v) = \min \cos^{-1}(\alpha_1\cdot$

Technical difficult.

Need different approach

Easier way to define, angle which does not require X_1, Y_1

$$\alpha(X,Y) = \max_{\substack{x \in X \\ y \in Y \\ \|x\| = 1 = \|y\|}} \cos^{-1}(x \cdot y)$$

where max is achieved precisely when $x \in X_1$, $y \in Y_1$

- $\alpha(X,X) = 0$
- $\alpha(X,Y) = \alpha(Y,X)$
- $\alpha(X,Y)$ — C^{∞}

  Proof might be given using Langrange's Multipliers?
- $\alpha(X,Y) + \alpha(Y,Z) \geq \alpha(X,Z)$

Q.5) E)

1) $\xi \oplus \eta = n+k$

$= B \times \mathbb{R}^{n+k}$

$$\begin{tikzcd} \xi \arrow[dr] & & \eta \arrow[dl] \\ & B & \end{tikzcd}$$

$\hat{\xi} \longrightarrow B \times \mathbb{R}^{n+k} \longrightarrow \mathbb{R}^{n+k}$

$b \in B$, $\xi_b \longmapsto$ n-plane in $\mathbb{R}^{n+k}$

so we get a map

$$\begin{tikzcd} \xi \arrow[r] \arrow[d] & \gamma \arrow[d] \\ B \arrow[r] & G_n(\mathbb{R}^{n+k}) \end{tikzcd}$$

Conversely,

$$\begin{tikzcd} \gamma \arrow[dr] & & \gamma^{\perp} \arrow[dl] \\ & G_n(\mathbb{R}^{n+k}) & \end{tikzcd}$$

$\gamma \oplus \gamma^{\perp} = \mathbb{R}^{n+k}$

Result follows.

2) B normal

- $\exists U_i \subseteq B$, finite s.t. $\varphi_i : \xi|_{U_i} \xrightarrow{\sim} B \times \mathbb{R}^n \xrightarrow{\pi} \mathbb{R}^n$ $\quad 1 \leq i \leq N$

  $f_i : B \longrightarrow \mathbb{R}$

  $f_i(U_i) = (0,1]$

  $f_i(U_i^c) = 0$

Define $\xi \longrightarrow \mathbb{R}^{Nn}$

$x \longmapsto (f_1 \cdot (\pi\cdot\varphi_1)x, f_2\cdot(\pi\cdot\varphi_2)x, \ldots, f_N(\pi\cdot\varphi_N))x)$

where $\varphi_i(x) = 0$ if $x \notin U_i$

This gives the required map.

$$\begin{tikzcd} \xi \arrow[r] \arrow[d] & \gamma \arrow[d] \\ B \arrow[r] & G_n(\mathbb{R}^{n+k}) \end{tikzcd}$$

$G_n(\mathbb{R}^{n+k})$ is a manifold, compact

$\Rightarrow \exists U_i \subseteq G_n$, finite st. U_i contractible,

$\gamma U_i \cong G_n(\mathbb{R}^{n+k}) \times \mathbb{R}^n$

Result follows from the pullback property of manifold

3) B - paracompact

Use partitions of Unity

4) $\gamma \downarrow \mathbb{R}P^\infty$ $\quad \omega(\gamma) = 1 + x \quad H^1(\mathbb{R}P^\infty, \mathbb{Z}/2) = \mathbb{Z}/2\, x$

if $\eta \oplus \gamma =$ trivial

$\Rightarrow \omega(\eta) = \frac{1}{1+x} = 1 + x + x^2 + \cdots$

Not possible.

Grassmannian - Cell structures

$G_n(\mathbb{R}^{n+k})$

$\mathbb{R}^0 \subset \mathbb{R}^1 \subset \cdots \subset \mathbb{R}^{n+k}$

X - n plane

Schubert symbol of $X = (\sigma_1, \ldots, \sigma_n)$

$1 \leq \sigma_1 < \sigma_2 < \cdots < \sigma_n \leq n+k$

$\dim(\mathbb{R}^{\sigma_i} \cap X) = i$

$\dim(\mathbb{R}^{\sigma_i - 1} \cap X) = i - 1$

$e(\sigma) = \{X \mid X \text{ nplane, schubert}(X) = \sigma\}$

$e(\sigma) =$ open cell of dim $(\sigma_1 - 1) + (\sigma_2 - 2) + \cdots + (\sigma_n - n)$.

$e(\sigma)$ – cells of $G_n(\mathbb{R}^{n+k})$

No. of r-cells in $G_n(\mathbb{R}^{n+k})$ = no. of partitions of r into at most n-integers each ~~of~~ $\leq k$.

Q.6-A)

- X – CW, compact
  X = Union of open cells of X, disjoint
  compact ⇒ finite
- finite ⇒ ~~disjoint union~~ quotient of compact set
  ⇒ compact

Q.6-B)

$i: G_n(\mathbb{R}^{n+k}) \hookrightarrow G_n(\mathbb{R}^{\infty})$

$i: G_n(\mathbb{R}^{n+k}) \hookrightarrow G_n(\mathbb{R}^{n+k+1})$

a p-cell of $G_n(\mathbb{R}^{n+k+1}) = e(\sigma)$ s.t. $\dim e(\sigma) = p$ $\quad p \leq k$

$= \{X\text{-n planes} \in \mathbb{R}^{n+k+1} \mid \text{schubert}(X) = \sigma\}$

$\mathbb{R}^0 \subset \mathbb{R}^1 \subset \dots \subset \mathbb{R}^{n+k} \subset \mathbb{R}^{n+k+1} \dots$ $\quad (\sigma_1 - 1) + \dots + (\sigma_n - n) = p$

$\dim(X \cap \mathbb{R}^{\sigma_i}) = i$

$\dim(X \cap \mathbb{R}^{\sigma_i - 1}) = i-1$

$p \leq k \Rightarrow \sigma_n - n \leq k \Rightarrow \sigma_n \leq n+k$

Thus $e(\sigma) \subseteq G_n(\mathbb{R}^{n+k}) \subseteq G_n(\mathbb{R}^{n+k+1})$

So k-skeleton of $G_n(\mathbb{R}^{n+k})$ = k-skeleton of $G_n(\mathbb{R}^{n+k+1})$

⇒ i^* isomorphism for $p < k$.

Q.6-C) $\quad X \xmapsto{f} \mathbb{R}^1 \oplus X$

- $f: G_n(\mathbb{R}^{n+k}) \longrightarrow G_{n+1}(\mathbb{R}^{n+k+1})$
  Injectivity is clear. Need to show C^∞.
  $\tilde{f}: V_n(\mathbb{R}^{n+k}) \longrightarrow V_{n+1}(\mathbb{R}^{n+k+1})$ $\quad \mathbb{R}^{n+k+1} = \mathbb{R}e_1 \oplus \mathbb{R}^{n+k}$
  $(x_1, \dots x_n) \longmapsto (e_1, x_1, \dots, x_n)$.

- $f^*(\gamma_{n+1}(\mathbb{R}^{n+k}))_{[x]} = \{(e_1, x_1, \dots, x_n) \mid (x_1, \dots x_n) \in [x]\}$

⇒ $f^*(\gamma_{n+1}(\mathbb{R}^{n+k+1})) = e_1 \oplus \gamma_n(\mathbb{R}^{n+k})$

$e(\sigma) \in G_n(\mathbb{R}^{n+k})$

$f(e(\sigma)) = e(1, \sigma_{i+1}, \dots, \sigma_{n+1})$

Q.6-D)

$w_1^{r_1} \dots w_n^{r_n} [M]$

$r_1 + 2r_2 + \dots + nr_n = n$ = partition of n using r_i i's.

Q.6-E)

$$f: G_n(\mathbb{R}^{n+k}) \longrightarrow G_k(\mathbb{R}^{n+k})$$
$$X \longmapsto X^{\perp}$$

Smooth ness - difficult to prove, might not be true !

Homeomorphism:

if $X \in e(\sigma)$ $\quad \sigma = (\sigma_1 \dots \sigma_n)$

$1 \leq \sigma_1 < \dots < \sigma_n \leq n+k$

$f(x) \in e(\tau)$ $\quad \tau = (\tau_1 \dots \tau_k)$

$1 \leq \tau_1 < \dots < \tau_k \leq n+k$

$\{\sigma_1 \dots \sigma_n \; \tau_1 \dots \tau_k\} = \{1, 2, \dots, n+k\}$

$$\dim(\tau) = (\tau_1 - 1) + \dots + (\tau_k - k)$$
$$= \sum \tau_i - \frac{k(k+1)}{2}$$
$$= \frac{(n+k)(n+k+1)}{2} - \frac{k(k+1)}{2} - \sum \sigma_i$$
$$= \frac{n^2 + nk + nk + n}{2} - \sum \sigma_i$$
$$= nk - (\dim \sigma)$$

Clearly NOT a CW complex isomorphism

we will give $G_n(\mathbb{R}^{n+k})$ a different cell-structure

Let

$$A \in Gl_{n+k}(\mathbb{R}) = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \qquad A_{ij} = \delta_{i(n+k-j)}$$

A permutes $e_1\, e_{n+k-1}$, $e_2\, e_{n+k-2}$, ...

Look at the induced map on $G_n(\mathbb{R}^{n+k})$

$$\hat{A}: G_n(\mathbb{R}^{n+k}) \longrightarrow G_n(\mathbb{R}^{n+k})$$
$$X = sp\langle v_1,\dots v_n\rangle \longmapsto \hat{A}X = sp\langle Av_1,\dots,Av_n\rangle.$$

This is clearly a homeomorphism.

Look at cell-structure induced by $\hat{A}$

$$Ae(\sigma) = \{X \mid \hat{A}X \subseteq e(\sigma)\}$$

~~dim (Ae(~~

Now we have

$$f: G_n(\mathbb{R}^{n+k}) \longrightarrow G_k(\mathbb{R}^{n+k})$$
$$X \longmapsto X^{\perp}$$

as before

$$\dim(f\,e(\sigma)) + \dim(\cancel{\phi}\, e(\sigma)) = nk$$

But $\quad \dim(Ae(\sigma)) = nk - \dim(e(\sigma))$

Reason: $\quad X \in Ae(\sigma)$

$$\Rightarrow AX \in e(\sigma)$$
$$\Rightarrow \dim(AX \cap \mathbb{R}^{n+k-\sigma_i}) = n-i$$
$$\dim(AX \cap \mathbb{R}^{n+k-\sigma_i+1}) = n-i+1$$
$$\Rightarrow Ae(\sigma) = e(\tau)$$
$$\tau_{n-i+1} = n+k-\sigma_i+1$$
$$\Rightarrow \dim(Ae(\sigma)) = \cancel{\sum(n+k-\sigma_i+1)}$$
$$= \sum_i \cancel{n}+k-\sigma_i+\cancel{1} - \cancel{n}+i-\cancel{1}$$
$$= \sum_i k - (\sigma_i - i)$$
$$= nk - \dim(e(\sigma))$$

(Another way of saying the same thing would be to look at the map

$$X \longmapsto AX^{\perp}$$).

$\boxtimes$

$H^*(G_n, \mathbb{Z}/2)$ – Real Grassmannian

for $\begin{matrix} \xi \\ \downarrow \\ B \end{matrix}$ vectorbundle over paracompact B

$\exists\, f : B \longrightarrow G_n$ s.t. $\xi = f^*\gamma$

$\Rightarrow \quad \omega(\xi) = f^*\omega(\gamma)$

But for $B = \mathbb{R}P^n$, ξ = orthogonal bundle of the canonical bundle

$$\omega(\xi) = \frac{1}{1+x} = 1 + x + \dots + x^{n-1}, \quad x \in H^1(\mathbb{R}P^n)$$

Not Useful.

$B = \underbrace{\mathbb{R}P_1^\infty \times \cdots \times \mathbb{R}P_n^\infty}_{n\text{-times}}$ $\quad \xi = \gamma_1^1 \times \gamma_2^1 \times \cdots \times \gamma_n^1$

$\gamma_i^1 \to$ Canonical line bundle over $\mathbb{R}P_i^\infty$

$\mathbb{R}P^\infty$ – CW complex, hence paracompact

$\omega(\xi) = (1+a_1)(1+a_2)\cdots(1+a_n)$ $\quad$ Always co.eff. group $\mathbb{Z}/2$.

$H^1(\mathbb{R}P_i^\infty) = \mathbb{Z}/2\, a_i$

$w_i(\xi) = i^{th}$ symmetric poly in n-variables.

Claim: $\nexists\ P(z_1, \dots, z_n) \in \mathbb{Z}/2\,[z_1, \dots, z_n]$ s.t.

$$P(\omega_1(\xi), \dots, \omega_n(\xi)) = 0$$

Proof:

$$\begin{array}{c} \mathbb{Z}/2\,(a_1, \dots, a_n) \\ \Big\downarrow \\ \mathbb{Z}/2\,(\omega_1(\xi), \dots, \omega_n(\xi)) \end{array}$$

is finite extension of degree at most $n!$

But since the Galois group is S_n, deg $= n!$

Transcendence degree of $\mathbb{Z}/2(a_1 \cdots a_n) = n$

$\Rightarrow$ " " " $\mathbb{Z}/2(\omega_1 \cdots \omega_n(\xi)) = n$

So result follows. ~~Infact~~ $\boxtimes$

Transcendence degree of

So we get that $\exists\ \mathbb{Z}/2[x_1 \cdots x_n] \subseteq H^*(G_n, \mathbb{Z}/2)$

$x_i \in H^i(G_n, \mathbb{Z}/2)$

But no. of cells in G_n of dim m = no. of partitions of m in atmost n- parts.

~~$= \# \{r_1, \ldots, r$~~ = no. of partitions of m with each partition size atmost n.

$= \# \{r_1, \ldots, r_n \mid 1r_1 + 2r_2 + \cdots + nr_n = m\}$

Reason : dim $e(\sigma) = m$

$\Rightarrow (\sigma_1 - 1) + (\sigma_2 - 2) + \cdots + (\sigma_n - n) = m$

Partition of m in n- parts.

By comparing dimension,

$$H^*(G_n, \mathbb{Z}/2) = \mathbb{Z}/2\, [x_1, \ldots, x_n] \qquad |x_i| = i$$

Because each dim- has rank equal to no. of cells of that dim, we must have that all the maps in the cellular complex of G_n are 0 mod 2, so

$$\text{rank}\left(H_i(G_n, \mathbb{Z}/2)\right) = \text{rank}\left(H^i(G_n, \mathbb{Z}/2)\right)$$

Q. 7 A)

$\omega_r(\gamma^n)$ = r^{th} symmetric polynomial in $a_1 \cdots a_n$

Cup product of a_i's will give this cocycle.

First we find co-cycle representing a_i's.

?? What do we have to find ??

Q. 7 B)

~~$H^*(G_n(\mathbb{R}^{n+k}))$~~

~~$i^* : H^p(G_n) \longrightarrow H^p(G_n(\mathbb{R}^{n+k}))$ isomorphism for $p < k$.~~

we have $(\mathbb{R} - \{0\})^n \longrightarrow (\mathbb{R}^\infty - 0)^n$

$\downarrow$

$\mathbb{R}P^\infty$

But $(\mathbb{R}^\infty - 0)^n \hookrightarrow \mathbb{R}^\infty$ by the map

$$((x_{11},\dots),(x_{21},\dots),\dots,(x_{n1},\dots)) \longmapsto (x_{11},x_{21},\dots,x_{n1},x_{12},x_{22},\dots,x_{n2},\dots)$$

Then $\mathbb{R}P^\infty \times \cdots \times \mathbb{R}P^\infty \cong$ ~~lines~~ n-tuple of lines in $\mathbb{R}^\infty$
~~with~~ a subspace of $\mathbb{R}^\infty$

note that these lines will be ~~indep~~ linearly independant.

So the map,

$$\mathbb{R}P^\infty \times \cdots \times \mathbb{R}P^\infty \longrightarrow G_n \quad \text{is given by}$$
$$(l_1,\dots,l_n) \longmapsto \mathrm{sp}\langle l_1,\dots,l_n\rangle$$

$$\begin{matrix} G \to E_Z \\ \downarrow \\ Z \end{matrix}$$ is <u>n-universal</u> if

$\forall$ CW X $\dim X \le n$,

~~$[X, f$ $E_Z, X]$~~ $P_G(X) \cong [X,Z]$

~~Proof~~

- $K \subseteq L$, L - CW complex, K subcomplex, $\dim L \le n$

$$\begin{matrix} G \to E \\ \downarrow \\ L \end{matrix} \qquad \begin{matrix} E|_K \\ \downarrow \\ K \end{matrix} \longleftrightarrow K \xrightarrow{f} Z \qquad E|_K = f^* E_Z$$

q. For any $K \subseteq L$, does $\exists$ $Z \leftarrow L : g$ so that

1) $E = g^* E_Z$

2) g is an extension of f.

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d, hook] & Z \\ L \arrow[ur, "g"'] & \end{tikzcd}$$

Wrong

<u>Answer</u>: Iff E_Z is $(n-1)$ connected.

<u>Proof</u>: Assume we know for ~~$\dim$~~ $\dim L < n$ $\leftarrow$ Induction

- $\Rightarrow$

~~The statement is true for $K = S^n$,~~

~~Take $K = S^{n-1}$, $L = e^n$~~

~~Let $[f] \in [K, Z] = [S^{n-1}, Z] = \pi_{n-1}(Z)$~~

$K = S^{n-1}$, $L = e^n$

$f : K \longrightarrow E_Z$ be ~~any~~ map

$$\begin{tikzcd} E_Z \arrow[d, "\pi"] \\ Z \end{tikzcd}$$

$[f] \in \pi_{n-1}(E_Z)$

$$\begin{tikzcd} (f\circ\pi)^* E_Z \arrow[r] \arrow[d] & E_Z \arrow[d, "\pi"] \\ S^{n-1} \arrow[r, "f\circ\pi"'] \arrow[ur, "f"] & Z \end{tikzcd}$$

Write proof later

- $K \subseteq L$ subcomplex of L, $\dim L \leq n$

$$\begin{tikzcd} E|_K \arrow[r] \arrow[d] & E_Z \arrow[d] \\ K \arrow[r] & Z \end{tikzcd}$$

Bundle extends to L $\iff$ the map $K \longrightarrow Z$ extends to $L \longrightarrow Z$

$\iff$ E_Z is $(n-1)$ connected.

Theorem:

E_Z n-connected $\Rightarrow$ $\begin{smallmatrix} E_Z \\ \downarrow \\ Z \end{smallmatrix}$ n-universal.

(i.e. $[X, Z] \xrightarrow{\sim} P_G(X)$ for $\dim X \leq n$)

$\begin{smallmatrix} E_Z \\ \downarrow \\ Z \end{smallmatrix}$ n-universal $\Rightarrow$ E_Z $(n-1)$-connected.

Construction of Universal bundles

(Milnor) $X * Y = \Sigma(X \wedge Y)$

$E_G^{(n)} = \underbrace{G * G * \cdots * G}_{n} = \Sigma^n (G \wedge G \wedge \cdots \wedge G) \leftarrow$ n-connected why?

$B_G^{(n)} = E_G^{(n)}/G$ $\qquad G \longrightarrow \begin{smallmatrix} E_G^{(n)} \\ \downarrow \\ B_G^{(n)} \end{smallmatrix}$

$\boxed{\begin{aligned} &X\text{-}n\text{ connected} \Rightarrow S^1 \wedge X \ \ n+1\text{-connected} \\ &\tilde{H}^{m+1}(S^1 \wedge X) = \tilde{H}^m(X) = 0 \quad m \leq n \end{aligned}}$

$E_G = \varinjlim_n E_G^{(n)}$ $\qquad B_G = E_G/G \cong \varinjlim_n B_G^{(n)}$

$\Rightarrow H_{m+1}(S^1 \wedge X) = 0 \quad m \leq n$

$\Rightarrow S^1 \wedge X$ - $n+1$ connected By Hurewitz.

Vector Bundles:

$G \longrightarrow O(k)$ continuous group homomorphism

$$\begin{array}{ccc} G & \longrightarrow & EG \\ & & \downarrow \\ & & BG \end{array} \qquad \begin{array}{ccc} \mathbb{R}^k \times_G G & \longrightarrow & \mathbb{R}^k \times_G EG \\ & & \downarrow \\ & & BG \end{array}$$

Then $\operatorname{Vect}^G_k(X) \xrightarrow{\sim} P_G(X)$

$\Rightarrow \operatorname{Vect}^G_k(X) \xrightarrow{\sim} [X, BG]$

$\mathbb{R}^k \times_G EG = \{(x,y) \in \mathbb{R}^k \times EG\}/\sim$

$(x, gy) \sim (g^{-1}x, y)$

So we get a vector bundle

$$\begin{array}{c} \mathbb{R}^k \times_G EG \\ \downarrow \\ BG \end{array}$$

~~corresponding~~

$\mathbb{R}^k \times_G G = \{(x,y) \in \mathbb{R}^k \times G\}/\sim$

$(x, gy) \sim (g^{-1}x, y)$

$\Rightarrow (x, gy) \sim (y^{-1}g^{-1}x, 1)$

$= \mathbb{R}^k$

Aside:

$G = \mathbb{Z}/2$ $\qquad \begin{array}{c} E = S^{n-1} \\ \downarrow \\ B = \mathbb{R}P^{n-1} \end{array}$ $\qquad \mathbb{Z}/2 \circlearrowright S^{n-1}$ antipode action, $\quad \mathbb{Z}/2 \circlearrowright \mathbb{R}^n$

$$\begin{array}{ccc} \mathbb{Z}/2 & \longrightarrow & S^{n-1} \\ & & \downarrow \\ & & \mathbb{R}P^{n-1} \end{array} \longrightarrow \mathbb{R}^n \times_{\mathbb{Z}/2} \mathbb{Z}/2 \longrightarrow \begin{array}{c} \mathbb{R}^n \times_{\mathbb{Z}/2} S^{n-1} \\ \downarrow \\ \mathbb{R}P^{n-1} \end{array}$$

$\mathbb{R} \times_{\mathbb{Z}/2} S^{n-1} = \{(t,x)\}/\sim$ $\qquad \mathbb{R} \times_{\mathbb{Z}/2} \mathbb{Z}/2 \cong \mathbb{R}$

$(t,x) \sim (-t,-x)$

~~Claim~~

$$\begin{tikzcd} \mathbb{R} \times_{\mathbb{Z}/2} S^{n-1} \arrow[rr] \arrow[dr] & & \gamma_1 \arrow[dl] \\ & \mathbb{R}P^{n-1} & \end{tikzcd}$$

isomorphism

$(t,x) \longmapsto (tx)$

Similarly

$$\begin{array}{ccc} S^1 & \longrightarrow & S^{2n+1} \\ & & \downarrow \\ & & \mathbb{C}P^n \end{array} \qquad \begin{array}{c} \gamma \\ \downarrow \\ \mathbb{C}P^n \end{array} \qquad \gamma \cong \mathbb{R} \times_{S^1} S^{2n+1}$$

$G = O(n), U(n)$ $\quad$ $Vect_k^{\mathbb{R}}(X) \cong [X, BO(n)]$ $\quad$ $Vect_k^{\mathbb{C}}(X) \cong [X, BU(n)]$

$BO(n) = Gr_n(\mathbb{R}^\infty)$ $\quad$ $BU(n) = Gr_n(\mathbb{C}^\infty)$

Proof of Previous Th^m:

Induction. $n=0 \to$ Trivial

Assume the statement to be true for $n-1$.

- E_Z - n-connected $\Rightarrow$ $(n-1)$-connected

$\Rightarrow$ $\begin{matrix} E_Z \\ \downarrow \\ Z \end{matrix}$ - $(n-1)$ ~~con~~ universal

X = n-dimensional C-W-complex, $\quad \begin{matrix} E \leftarrow G \\ \downarrow \\ X \end{matrix}$

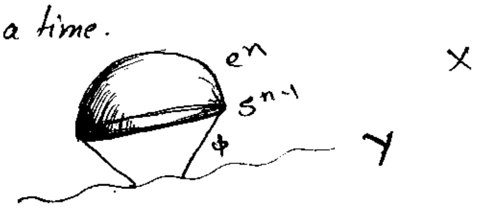
Y = $(n-1)$ skeleton of X.

$$\begin{tikzcd} E_Y \arrow[r] \arrow[d] & E_Z \arrow[d] \\ Y \arrow[r, "f"] & Z \end{tikzcd}$$

By induction hypothesis $\exists f : Y \longrightarrow Z$ s.t. $f^* E_Z = E|_Y$.

We need to extend f to X so that $f^* E_Z = E$.

We do it 1-cell at a time.



$X = Y \cup \{e^n\}$

WALOG ~~assume~~

$S^{n-1} \xrightarrow{\phi} Y$

$$\begin{tikzcd} E_{S^{n-1}} \arrow[r] \arrow[d] & E|_Y \arrow[r] \arrow[d] & E_Z \arrow[d] \\ S^{n-1} \arrow[u, bend left, dashed] \arrow[r, "\phi"] \arrow[urr, dashed] & Y \arrow[r, "f"] & Z \end{tikzcd}$$

$E_{S^{n-1}} = \phi^* E|_Y$

we also have

$E_{S^{n-1}} = i^*(\phi^* E)$
$= i^*(E_{e^n})$

$$\begin{tikzcd} E_{S^{n-1}} \arrow[r] \arrow[d] & E_{e^n} \arrow[r] \arrow[d] & E \arrow[d] \\ S^{n-1} \arrow[r, hook, "i"] & e^n \arrow[r, "\phi"] & X \end{tikzcd}$$

$\Rightarrow E_{S^{n-1}}$ - trivial G-bundle

$\Rightarrow \exists$ section $S^{n-1} \longrightarrow E_{S^{n-1}}$

Pushing forward we get a map

$S^{n-1} \xrightarrow{\widehat{f \circ \phi}} E_Z$

Because $\pi_{n-1}(E_z) = 0$

$\Rightarrow$ we can extend $\widehat{\phi\circ f}$ to e^n $\widehat{\phi\circ f} : e^n \longrightarrow E_z$

Push this down to get a map $g : e^n \longrightarrow Z$

$$\begin{tikzcd}
 & & E_z \arrow[d] \arrow[d, bend left=60, dashed, "g"] \\
e^n \arrow[urr, dashed] & S^{n-1} \arrow[l] \arrow[ur] \arrow[r, "\phi\circ f"'] & Z
\end{tikzcd}$$

Define :

$$\hat{f} : X \longrightarrow Z$$
$$\hat{f}|_Y = f$$
$$\hat{f}|_{e^n} = g .$$

So we have obtained:

$$[X, Z] \longrightarrow P_G(X) \quad \text{surjective.}$$

<u>Injectivity</u>:

$$E \cong f^* E_z \underset{\varphi}{\cong} g^* E_z \qquad Z \xleftarrow{\ } X : f, g$$

Construct a vector bundle over $X \times I$

$$\begin{tikzcd}
p^*(E) \arrow[r] \arrow[d] & E \arrow[d] \\
X \times I \arrow[r, "p"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
f^*E_z \arrow[rr, "\varphi"] \arrow[dr] & & g^*E_z \arrow[dl] \\
 & X &
\end{tikzcd}$$

$$\cancel{X\times\{0\}\cup} \quad p^*E|_{X\times\{0\}} \cong E$$

Construction of Universal Vector Bundles (contd.)

- If G discrete group, $\cancel{\pi_n G =}$ $\pi_n G = \begin{cases} G & n = 0 \\ 0 & n > 0 \end{cases}$

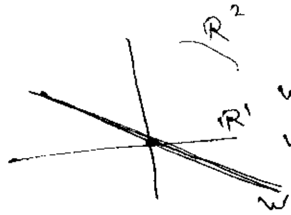
  $\Rightarrow \pi_n BG = \begin{cases} G & n = 1 \\ 0 & \text{else} \end{cases} \qquad \Rightarrow \quad BG = K(G, 1)$

- $BU(n) = Gr_n(\mathbb{C}^\infty)$ $\qquad BO(n) = Gr_n(\mathbb{R}^\infty)$

  $BU(1) = \mathbb{C}P^\infty$ $\qquad BO(1) = \mathbb{R}P^\infty$

- CW-complex structure on $Gr_n(\mathbb{R}^\infty) \,/\, Gr_n(\mathbb{C}^\infty)$

  Schubert Cells



$W \in Gr_n(\mathbb{R}^\infty)$ $\qquad \mathbb{R}^k = \{(x_1, \dots x_k, 0 \dots 0)\}$

$W \cap \mathbb{R}^1 \subseteq W \cap \mathbb{R}^2 \subseteq \dots \subseteq W \cap \mathbb{R}^k \subseteq W \cap \mathbb{R}^{k+1} \subseteq \dots$

$\dim(W \cap \mathbb{R}^{k+1}) \leq \dim(W \cap \mathbb{R}^k) + 1$

$W \rightsquigarrow (\sigma_1 \cdots \sigma_n)$ $\quad 0 < \sigma_1 < \dots < \sigma_n$. $\dim(W \cap \mathbb{R}^{\sigma_i}) = i$

$seq(W)$ $\qquad \dim(W \cap \mathbb{R}^{\sigma_i - 1}) = i-1$

$A(\sigma_1 \cdots \sigma_n) = \{W \in Gr_n^\infty \mid seq(W)_i \leq \sigma_i \ \forall i\}$ $\longleftarrow$ will be skeleton of ~~A(σ)~~ $Gr_n(\mathbb{R}^\infty)$

Orthonormal Basis of W as Row vectors:

for $W \in$ ~~Int~~ $(B(\sigma_1 \cdots \sigma_n))$

$$\begin{array}{c} \quad\quad\quad \sigma_1 \quad \sigma_2 \quad \sigma_3 \quad \sigma_n \quad 0\,0 \cdots \\ \begin{bmatrix} * * \cdots * 1 & 0 \, 0 \cdots & 0 & \cdots \\ * * \cdots * & 0 \;\; * \; 1 \; 0 \cdots & 0 & 0\,0 \cdots \\ * * \cdots & 0 \quad\; 0 \; * \; 1 & & \\ & \vdots & & \\ * \cdots & 0 \quad 0 \; * \; 0 & 1 & 0\,0 \cdots \end{bmatrix} \end{array}$$

Guess:
$\dim A(\sigma_i)$
$= (\sigma_1 - 1) + (\sigma_2 - 2) + \dots + (\sigma_n - n)$
But ~~Interior~~ $A(\sigma_i)$ needs to be replaced by ~~interior~~ $B(\sigma_i)$

- ~~Interior~~ $B(\sigma_i) = \{W \mid seq\, W_i = \sigma_i \ \forall i\}$ $\longleftarrow$ cell of $G_n(\mathbb{R}^\infty)$ ~~we have a basis of W as above~~

To each $W \in B(\sigma_i)$, we can associate an orthonormal basis $v_1, \dots, v_n$ s.t. $v_i \in \mathbb{R}^{(seq\, W)_i}$, $v_i \notin \mathbb{R}^{(seq\, W)_i - 1}$, $(v_i)_{seq\, W_i} > 0$

This association gives a continuous map, $B(\sigma_i) \longrightarrow V_n(\mathbb{R}^\infty)$.

$B(\sigma_1 \cdots \sigma_n)$ homeomorphic to its image

$\dim B(\sigma_1 \cdots \sigma_n) = (\sigma_1 - 1) + (\sigma_2 - 2) + \dots + (\sigma_n - n)$

For $\mathbb{C}^\infty$ $\qquad = 2(\sigma_1 - 1) + 2(\sigma_2 - 2) + \dots + 2(\sigma_n - n)$.

Q. Is G_n^{n+k} k-universal?

Q. what is $H^*(G_n(\mathbb{R}^\infty), \mathbb{Z})$?

- Griffits, Morgan
- Felix

$1 \, eq^n - n+1$ $\qquad x_1 \cdots x$
$1 \; - n+2$
$- n+k$

$a_1 = b_n - b_{n-1}$
$a_2 = b_{n-1} - b_{n-2}$
$\vdots$
$1 \cdot a_1 + 2 \cdot a_2 + \dots + n a_n$

$P(E) \hookrightarrow (B) \twoheadrightarrow B/P(E)$
$\downarrow p$
B

Any vector bundle on B which is trivial on $P(E)$ is trivial.

$H(B/P(E)) \longleftrightarrow H(B)$ $\qquad M/P(E) = B$

M = Mapping cylinder of p

$\boxed{C, P(E)}$

$M \longrightarrow BSO(3)$

$S^3 \longrightarrow BSO(3)$

$S^3 \longrightarrow P$

$\mathbb{R}P^3$
$0 \quad \mathbb{Z}/2 \quad 0$
$0 \quad 1 \quad 2 \quad 3$

Steifel-Whitney Classes

$w(T\mathbb{RP}^n) = (1+x)^{n+1}$. $\quad \mathbb{Z}/2\langle x\rangle = H^1(\mathbb{RP}^n)$

Open Problem: what is the smallest n_k s.t. $\mathbb{RP}^k$ immerses in $\mathbb{R}^{n_k+k}$?

Immersion $\Rightarrow$ normal bundle ν $\quad w(\nu) = \frac{1}{(1+x)^{n+1}} = \sum \binom{-n-1}{k} x^k$

Th^m: $\xi \downarrow X$ vector bundle, orientable $\Leftrightarrow w_1(\xi)=0$.

Proof:

$$\begin{tikzcd}
\xi \arrow[r, "f^*"] \arrow[d] & \gamma \arrow[d] \\
X \arrow[r, "f"] \arrow[dr, dashed, "?"'] & BO(n) \\
 & BSO(n) \arrow[u]
\end{tikzcd}$$

$SO(n)$ connected component of $O(n)$

$\pi_* SO(n) = \begin{cases} \pi_* O(n) & \text{if } * > 0 \\ 0 & \text{else} \end{cases}$

$\pi_* BSO(n) = \begin{cases} \pi_{*-1} O(n) & \text{if } * > 1 \\ 0 & \text{else} \end{cases}$

$= \begin{cases} \pi_* BO(n) & \text{if } * > 1 \\ 0 & \text{else} \end{cases}$

Then $BSO(n)$ is just the universal coversal of $BO(n)$

$\pi_1(BSO(n)) = \pi_1(O(n)) = \mathbb{Z}/2 \longleftarrow$ For group G, $\pi_0 G$ is also a group $= G/$connected component of id.

$p^*: H^*(BO(n);\mathbb{Z}/2) \longrightarrow H^*(BSO(n);\mathbb{Z}/2)$

?? is 0 when $* = 1$?

$\because \pi_1(BSO(n)) = 0$, $H_1(BSO(n)) = 0 \Rightarrow H^1(BSO(n);\mathbb{Z}/2) = 0$.

- if ξ - orientable

$\Rightarrow \exists \tilde{f}: X \longrightarrow BSO(n)$

$\Rightarrow w_1(\xi) = f^* w_1(\gamma) = (p \circ \tilde{f})^* w_1(\gamma)$

$= \tilde{f}^* \cdot \underbrace{p^* w_1(\gamma)}_{0} = 0 \quad \because w_1(\gamma) \in H^1(BO(n);\mathbb{Z}/2)$

- if $w_1(\xi) = 0$

$\Rightarrow f^* w_1(\gamma) = 0$

$\Rightarrow f^*: H^1(BO(n)) \longrightarrow H^1(X)$ $\quad$ 0 map $\quad \because H^*(BO(n))$ generated by $w_1, w_2, \dots, w_n(\gamma)$.

$\Rightarrow f: H_1(X) \longrightarrow H_1(BO(n))$ $\quad$ 0 map

$\Rightarrow f: \pi_1(X) \longrightarrow \pi_1(BO(n))$ $\quad$ 0 map

$\Rightarrow f$ lifts to $BSO(n)$. orientable

$\blacksquare$

Th^m: $M^n - C^\infty$, $M^n \hookrightarrow \mathbb{R}^{n+1}$ embedded in $\mathbb{R}^{n+1}$ $\Rightarrow$ M^n orientable.

Proof:

Normal bundle ν 1-dim

$TM \oplus \nu = T\mathbb{R}^{n+1}$ = trivial

$\Rightarrow$ $w(TM) \cdot w(\nu) = 1$

$\Rightarrow$ $w(TM) = 1 + w_1(\nu) + w_1(\nu)^2 + \cdots + \cancel{\ldots}$

$w_i(TM) = (w_1(\nu))^i = (w_1(TM))^i$

No need

Prove directly that $\exists$ an outward normal vector field?

Need to used a seperation th^m:

M divides $\mathbb{R}^n$ in two parts. $\rightarrow$ Jordan curve for dim n.

For this Use Alexandar Duality.

Th^m: For all n - odd, Steifel Whitney numbers of $\mathbb{R}P^n$ are 0.

Proof: $w_i(T\mathbb{R}P^n) = \binom{n+1}{i} x^i$

$w_1^{i_1} \cdots w_n^{i_n} = \binom{n+1}{1}^{i_1} \cdot \binom{n+1}{2}^{i_2} \cdots \binom{n+1}{n}^{i_n} \cdot x^n$

Claim: $1 \cdot i_1 + 2i_2 + \cdots + n i_n = n$

$\Rightarrow$ $\binom{n+1}{1}^{i_1} \cdots \binom{n+1}{n}^{i_n} = 0$ mod 2

Steenrod Square

$\exists$ operations $Sq^k : H^n(X) \longrightarrow H^{n+k}(X)$ satisfying

1. $Sq^k(f^* \omega) = f^* Sq(\omega)$
2. $Sq^k(\omega) = 0$ if $k > |\omega|$
   $= \omega^2$ if $k = |\omega|$
   $= \omega$ if $k = 0$
3. $Sq^k(\alpha \cup \omega) = \sum_{i+j=k} Sq^i(\alpha) \cup Sq^j(\omega)$
4. $Sq^k(\Sigma \omega) = \Sigma\, Sq^k(\omega)$ $\quad H^n(X) \xrightarrow[\cong]{\Sigma} H^{n+1}(\Sigma X)$
5. Adem's Relations.

eg: $\Sigma \mathbb{C}P^2$ $\quad S^3 \vee S^5$ both have trivial rings

$H^*(\Sigma \mathbb{C}P^2) = \mathbb{Z}\{1, y_3, y_5\}$ $\quad \mathbb{Z}/_2$ co-eff.

$H^*(\mathbb{C}P^2) = \mathbb{Z}\{1, x, x^2\}$

$y_3 = \Sigma x$, $\quad y_5 = \Sigma x^2 \Rightarrow Sq^2(y_3) = Sq^2(\Sigma x) = \Sigma Sq^2(x) = y_5$

$H^*(S^3 \vee S^5) = \mathbb{Z}\{1, z_3, z_5\}$

$S^3 \vee S^5 \longrightarrow S^3$

$H^*(S^3 \vee S^5) \xleftarrow{f} H^*(S^3)$ $\quad$ injection

$z_3 \longleftarrow 1$

$Sq_2(z_3) = f^*(Sq_2(1)) = 0$

So ~~to~~ steenrod squares differentiate $\Sigma \mathbb{C}P^2$, $S^3 \vee S^5$

$\mathbb{C}P^2$ $\quad S^3 \xrightarrow[\eta]{} S^2$ hopf map

$[\eta] \in \pi_3(S^2)$ not null homotopic $\therefore \mathbb{C}P^2 \not\simeq S^2 \vee S^4$

$\Sigma \mathbb{C}P^2$ $\quad S^4 \xrightarrow[\Sigma\eta]{} S^3$

$[\Sigma\eta] \in \pi_4(S^3)$ not null homotopic $\therefore \Sigma \mathbb{C}P^2 \not\simeq S^3 \vee S^5$.

Q. $[\Sigma^k \eta] \in \pi_{k+3}(S^{k+2})$ not null homotopic.

$\mathbb{R}P^\infty$. $\quad H^*(\mathbb{R}P^\infty; \mathbb{Z}_2) = \mathbb{Z}/_2[x]$

$Sq^r(x^k) = Sq^{r-1}(x^{k-1})\, Sq^1(x) + Sq^r(x^{k-1})\, x$

$= Sq^{r-1}(x^{k-1})\, x^2 + Sq^r(x^{k-1})\, x$

Guess: $Sq^k(x^k) = x^{2k}$

$Sq^{k-1}(x^k) = Sq^{k-2}(x^{k-1})x^2 + Sq^{k-1}(x^{k-1}).x$

$= Sq^{k-3}(x^{k-2})x^3 + Sq^{k-2}(x^{k-2}).x^3$

$+ Sq^{k-1}(x^{k-1})x$

$= (k)\, x^{2k}$

$Sq^n(x^k) = \binom{k}{n}$