λ- operations and Eulerian Idempotents in $\mathbb{Q}[S_n]$

A- $\mathbb{Q}$ module, ($k=\mathbb{Q}$), $H := T'(A) = \bigoplus_{i\geq 0} H_i$ graded cotensor Hopf Algebra

$H_0 = k,\ H_1 = A,\ H_n = A^{\otimes n}\ \forall n\geq 1$ with

- $\Delta(a_1 \cdots a_n) = \sum_{i=0}^{n} (a_1 \cdots a_i) \otimes (a_{i+1} \cdots a_n)$ "cut - product"
- $\mu((a_1 \cdots a_p)\otimes(a_{p+1}\cdots a_{p+q})) := \sum_{S_n \ni \sigma = (p,q)\text{ shuffle}} \mathrm{sign}(\sigma).\ \sigma(a_1 \cdots a_{p+q})$ "signed shuffle prod"

- $(T'(A), \mu, u, \Delta, c) =: \mathcal{H}$ is a commutative graded Hopf Algebra

Def: $Id^{*k} := \lambda^k : T'(A) \longrightarrow T'(A)$, $\lambda_n^k := \lambda^k|_{H_n = A^{\otimes n}}$ $n \geq k$ ⟵ λ- operations

Recall: $f * g := \mu \circ (f \otimes g) \circ \Delta$, then

$$\lambda_n^k (a_1 \cdots a_n) = \sum_{\sigma \in S_n} \underset{\mathbb{Z}}{\underset{\cap}{a(\sigma)}}\, \sigma(a_1 \cdots a_n) \quad \text{so that}$$

$\lambda_n^k := \sum_{\sigma\in S_n} a(\sigma)\,\sigma$ are elements of $\mathbb{Z}[S_n]$.

Note: $\lambda_n^1 = 1 \in \mathbb{Z}[S_n]$ is the unit element

Rmk: $\bar{\lambda}_n^k := (-1)^{k-1} \lambda_n^k$, $\psi_n^k := k\cdot \lambda_n^k$ (Adams' Operations)

related to λ-rings which has its roots in exterior- power function.

The composition of endomorphism in $\mathrm{End}_k(H_n)$ corresponds to the product in the group ring $\mathbb{Z}[S_n]$: the formula $Id^{*k} \cdot Id^{*k'} = Id^{*kk'}$ becomes $\lambda_n^k \cdot \lambda_n^{k'} = \lambda_n^{kk'} \in \mathbb{Z}[S_n]$

Similarly one defines:

$e_n^{(i)} \in \mathbb{Q}[S_n]$ $n \geq 1$ ⟵ Eulerian idempotents

$\lambda_n^k = k\, e_n^{(1)} + \cdots + k^n e_n^{(n)}$ $\forall n \geq 1$

Eulerian Decomposition of S_n:

$\sigma \in S_n$, if $\sigma(i) > \sigma(i+1)$ then σ is said to have a descent at i.

eg: $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$ has descent only at 2.

Let $S_{n,k} := \{\sigma \in S_n \mid \sigma \text{ has } (k-1) \text{ descents}\}$ ⟶ Eulerian partitions of S_n

$S_{n,1} = \{id\}$ $S_{n,n} = \left\{ \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix} =: \omega_n \right\}$

Fact: $S_{n,k} \cdot \omega_n = S_{n,n-k+1}$

$\alpha_{n,k} := |S_{n,k}|$ are called the Eulerian numbers

$\ell_n^k := \sum_{\sigma \in S_{n,k}} \mathrm{sign}(\sigma).\sigma \in \mathbb{Z}[S_n]$ Eulerian elements

$l_0^0 := 1$, $l_n^k := 0 \;\forall k \notin \{1 \dots n\}$; $l_n^1 = \mathrm{id} \in \mathbb{Z}[S_n]$, $l_n^n = (-1)^{n(n-1)/2} \omega_n$

Prop: 1) $\lambda_n^k = \sum_{i=0}^{k} \binom{n+i}{n} l_n^{k-i}$ in $\mathbb{Z}[S_n]$

2) $l_n^k = \sum_{i=0}^{k} (-1)^i \binom{n+i}{n+1-i} \lambda_n^k$ in $\mathbb{Z}[S_n]$

3) $e_n^{(i)} = \sum_{i=1}^{n} a_n^{i,j} e_n^j$ in $\mathbb{Q}[S_n]$

Stirling numbers, defined as $\sum_{i=1}^{n} a_n^{i,j} X^i = \binom{X-j+n}{n}$

Cor: $e_n^{(n)} = \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) . \sigma = \frac{1}{n!} \varepsilon_n \in \mathbb{Q}[S_n]$