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M – n dim closed oriented manifold

$LM = C^\infty(S^1 \to M)$

$e : LM \longrightarrow M$, $c \mapsto c(0)$

String topology $\rightsquigarrow$ algebraic structure on $H_*(LM)$

Th^m: $(\mathbb{H}_*(LM), \cdot, B)$ – BV algebra.

Construct 2-TQFT S_M : $S_M(\bigcirc) = \mathbb{H}_*(LM)$

$S_M(\;) = \{\cdot : \mathbb{H}_*(LM)^{\otimes 2} \longrightarrow \mathbb{H}_*(LM)\}$

almost correct as $\mathbb{H}_*(LM) \neq$ finite dimensional

Loop product:

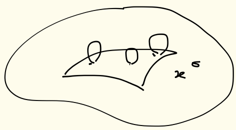
$\cdot : C_i(LM) \otimes C_j(LM) \longrightarrow C_{i+j-n}(LM)$

$\cdot$ = amalgamation of $\cap$ on $H_*(M)$ and Pontryagin product on $H_*(\Omega M)$

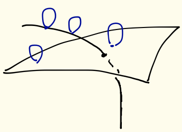
i- simplices in LM:

$$\Delta^i \xrightarrow{x} LM \overset{e}{\dashrightarrow} M, \qquad x^\sigma = e \circ x$$

x^σ support simplex



$x : \Delta^i \longrightarrow LM$ $\qquad y : \Delta^j \longrightarrow LM$ $\qquad x^\sigma \pitchfork y^\sigma$



$x \cdot y$ = family of loops parametrized by $x^\sigma \cap y^\sigma$ where the loop at every intersection pt = concatenation of loops at x and y.

Prop: • descends to homology & defines

$$H_i(LM) \times H_j(LM) \longrightarrow H_{i+j-n}(LM)$$

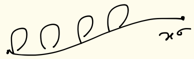
Def: $\mathbb{H}_*(LM) = H_{*+n}(LM)$

Prop. • $\mathbb{H}_* \otimes \mathbb{H}_* \longrightarrow \mathbb{H}_*$ in deg 0, associative, graded commutative.

$\rho : LM \times S^1 \longrightarrow LM$

$(c \cdot \theta)(t) = c(t+\theta)$

$x : \Delta^i \longrightarrow LM$



Bx — family of loops parametrized by $\Delta^i \times S^1$

B descends to $\mathbb{H}_*(LM)$ and gives a degree 1 operator.

$B(x) = \rho_*(x \times [S^1]) \implies B^2 = 0$

Thm: Chas-Sullivan

$H_*(LM, \cdot, B)$ is a BV-algebra;

1) $\cdot$ graded commutative, associative

2) $B^2 = 0$

3) B has order 2 wrt $\cdot$

$B \neq$ derivation $\quad B(x \cdot y) - Bx \cdot y - (-1)^{|x|} x \cdot By = \{x, y\} = (-1)^{|x|}$

Prop:

$$H_{*+n}(M) \xrightarrow{\text{constant loop}} \mathbb{H}_*(LM) \xrightarrow{\text{Umkehr}} H_*(\Omega_{m_0} M)$$

$\cap \qquad \cdot \qquad \times$

Map of algebras

$\rightsquigarrow$

$$\begin{tikzcd} \Omega_{m_0} M \arrow[r, hook] \arrow[d, "e"'] & LM \arrow[d, "e"] \\ m_0 \arrow[r, hook] & M \end{tikzcd}$$

finite codimension embedding

gives $\nu_{\Omega_{m_0} \hookrightarrow LM}$ is Tubular nbd $(\Omega_{m_0} \subseteq LM)$

Pontryagin-Thom construction $\rightsquigarrow$

eg: $\cdot$ G = Lie $\quad LG = G \times \Omega_* G$

$\cdot$ $M = S^{2n-1}$ $\quad \mathbb{H}_*(LS^1) = k[t, t^{-1}] \otimes \Lambda a_{-1}, \quad |t| = 0$

$\mathbb{H}_*(LS^{2n-1}) = k[u] \otimes \Lambda a \qquad |a| = -(2n-1), \ |u| = 2n-2$

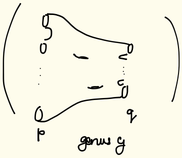
Thm (Cohen-Jones):

$\mathbb{H}_*(LM) \overset{\text{gr. rings}}{\simeq} HH^*(C^*(M)) \qquad M$ simply connected.

S_M: 2-"TQFT"

$S_M(\bigcirc) = \mathbb{H}_*(LM)$

$S_M($  $) = \cdot$

$S_M($  $) \longrightarrow \begin{array}{c} \mathbb{H}_*^{\otimes p} \\ \big\downarrow ?? \\ \mathbb{H}_*^{\otimes q} \end{array}$

$\Sigma_{g,p,q}$

$$LM \times LM \longleftarrow \text{maps}(8 \to M) \xrightarrow{\text{out}} LM$$

codim n embedding

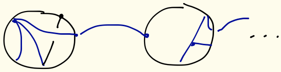
Do Pontryagin Thom again.

We get a map

$$\begin{tikzcd} H_*(LM \times LM) \arrow[r, "\text{collapse}"] & H_*(\text{Thom}(e^* \nu_{M \subset M \times M})) \arrow[d, "\cong"] & \\ & H_{*-n}(8, M) \arrow[r, "\text{out}_*"] & H_*(LM) \end{tikzcd}$$

$\Sigma_{g,p,q}$ $\quad p \geq 1$

p- circles + trees $\qquad$ Sullivan Chord diagram



- at each vertex a cyclic ordering of incoming edges

$$\begin{tikzcd} (LM)^p \arrow[d] & \text{maps}_*(\text{ constant on trees } \to M) \arrow[l] \arrow[r, "\text{out}"] \arrow[d] & (LM)^q \\ M^{\#\text{circular vertices}} & M^{\#\text{trees}} \arrow[l] & \end{tikzcd}$$

Codim of $M^{\# \text{trees}} \hookrightarrow M^{\# \text{circular vertices}}$

$= n \, (\# \text{ circ. vertices} - \# \text{ trees}) = -\chi \cdot n$

$\chi(\text{Sullivan} - \text{chord diagram}) = \chi_{\Sigma_{g,p,q}}$
$\quad\|$
$\chi(\text{diag. obtained by contracting trees})$

Given a surface $\Sigma_{g,p,q}$ $\rightsquigarrow$ category of fat graphs $\mathcal{F}at_n(g)$

$|\mathcal{F}at_n(g)| \simeq$

$\{\text{top space of metric fat graphs}\}$
$\quad\|$
classifying space of mapping class group of $\Sigma_{g,p,q}$