Crash Course on Representation Theory - Day 2

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Character theory is the part of representation theory that allows one to determine the 'prime' decomposition of representations.

1 Character Theory

Given a linear transformation $A \in GL(V)$ we can define the trace of A to be the sum of diagonal entries of A in any basis, recall that the trace does not depend upon the choice of any basis. For a representation ρ we can post-compose with the trace map, the resulting function is called the **character** of the representation.

$$\chi_\rho : G \xrightarrow{\rho} GL(V) \xrightarrow{\text{trace}} \mathbb{C} \tag{1.1}$$

The trace is a single number associated to a matrix and hence contains very little information about the matrix, but as it turns out, the character of a representation (which is a function $G \to \mathbb{C}$) contains enough information to differentiate between representations. We say that the character is a complete invariant of representations.

1.1 Orthogonality relations

We can use characters to define a hermitian inner product on the space of representations! For representations ρ_1, ρ_2 define an inner product and norm as

$$\langle \rho_1, \rho_2 \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_{\rho_1}(g) \overline{\chi_{\rho_2}(g)} \tag{1.2}$$

$$\|\rho_1\|^2 = \langle \rho_1, \rho_1 \rangle \tag{1.3}$$

The following is an EXTREMELY useful theorem in character theory (which requires nothing more than linear algebra to prove).

Theorem 1.1 (Orthogonality relations). *Let $\rho_1, \rho_2, \ldots, \rho_d$ be the distinct irreducible representations of G. If ρ is an arbitrary representation which has a decomposition $\sigma = \rho_1^{k_1} \oplus \rho_2^{k_2} \oplus \cdots \oplus \rho_d^{k_d}$ then*

$$\langle \rho_i, \sigma \rangle = k_i \tag{1.4}$$

In particular, we have $\|\chi_{\rho_i}\| = 1$ for all i and $\langle \chi_{\rho_1}, \chi_{\rho_2} \rangle = 0$ for $i \neq j$.

1.2 Character Tables

By the cyclicity of trace we have

$$\chi_\rho(h^{-1}gh) = \text{trace}(\rho(h)^{-1}\rho(g)\rho(h)) = \text{trace}(\rho(h)\rho(h)^{-1}\rho(g)) = \text{trace}(\rho(g)) = \chi_\rho(g)$$

and so to determine the character of a representation it is enough to determine the character of one element in each conjugacy class. Such information is typically organized in a table called the **character table**. The character tables from the examples from yesterday's notes are as follows.

$\mathbb{Z}/3$	$\{0\}$	$\{1\}$	$\{2\}$
	1	1	1
	1	ω	ω^2
	1	ω^2	ω

$S_3(= D_6)$	$\{(1)\}$	$\{(1\,2); (2\,3); (1\,3)\}$	$\{(1\,2\,3); (1\,3\,2)\}$
trivial	1	1	1
sign	1	-1	1
2-dim	1	0	$2\cos(2\pi/3)$

Q_8	$\{1\}$	$\{-1\}$	$\{i,-i\}$	$\{j,-j\}$	$\{k,-k\}$
trivial	1	1	1	1	1
sign_i	1	1	1	-1	-1
sign_j	1	1	-1	1	-1
sign_k	1	1	-1	-1	1
2-dim	2	-2	0	0	0

2 Frobenius determinant

The Frobenius determinant can be easily computed using linear algebra. We'll do it using representation theory, this is completely unnecessary.

The first thing to observe is that the Frobenius matrix F_n is constructed out of the regular representation for the group $\mathbb{Z}/n$. Consider the n dimensional vector space with basis $e_0, e_1, \dots, e_{n-1}$ on which $\mathbb{Z}/n$ acts as:

$$\rho(k)(e_j) = e_{(k+j \mod n)} \tag{2.1}$$

The j^{th} column of the matrix $\rho(k)$ has the 1 in the $(k+j \mod n)^{th}$ place and 0 everywhere else. This is exactly the place where the $x_k's$ occur in the Frobenius matrix!!! Hence we get the identity

$$F_n = \sum_{k=0}^{n} x_k \rho(k) = \sum_{k=0}^{n} x_k \rho(1)^k \tag{2.2}$$

For $0 \leq l < n$ let ρ_l be the 1 dimensional irreducible representations of $\mathbb{Z}/n$ with $\rho_l(1) = \omega^l$ where $\omega = e^{2\pi i/n}$. For some constants c_i we must have

$$\rho = \rho_0^{c_0} \oplus \cdots \oplus \rho_{n-1}^{c_{n-1}} \tag{2.3}$$

To compute c_l we compute the inner product $c_l = \langle \rho_l, \rho \rangle = \sum\limits_{k \in \mathbb{Z}/n} \chi_{\rho_l}(k)\overline{\chi_\rho(k)}/n$. The matrices $\rho(k)$ have all the diagonal entries 0 except for $k = 0$ which has all the diagonal entries 1 and hence $\chi_\rho(k) = \begin{cases} n \text{ if k = 0} \\ 0 \text{ otherwise} \end{cases}$ so that $c_l = 1$ for all l and we get

$$\rho = \rho_0 \oplus \rho_1 \oplus \cdots \oplus \rho_{n-1} \tag{2.4}$$

We can choose a basis such that the matrix $\rho(1)$ is a diagonal matrix with entries $\rho_l(1) = \omega^l$, plugging in (2.2)

$$F_n = \sum_{k=0}^{n} \begin{bmatrix} x_k & & & \\ & \omega^k x_k & & \\ & & \ddots & \\ & & & \omega^{(n-1)k} x_k \end{bmatrix} = \begin{bmatrix} \sum\limits_{k=0}^{n} x_k & & & \\ & \sum\limits_{k=0}^{n} \omega^k x_k & & \\ & & \ddots & \\ & & & \sum\limits_{k=0}^{n} \omega^{(n-1)k} x_k \end{bmatrix} \tag{2.5}$$

$$\implies \det F_n = \prod_{l=0}^{n-1} \sum_{k=0}^{n} \omega^{kl} x_k = \prod_{l=0}^{n-1} \left(x_0 + \omega^l x_1 + \cdots + \omega^{(n-1)l} x_{n-1} \right) \tag{2.6}$$

□

This naturally leads to the following question.

Question. Consider the Frobenius matrix F_G whose entries are obtained from the multiplication table for G i.e. the rows and columns are indexed by the entries of G and the g, h^{th} entry is $x_{gh^{-1}}$. What is $\det F_G$?

The above proof generalizes directly to an arbitrary group and the determinant can be computed in terms of the regular representation of G and the irreducible representations of G.