Problem Set 1 - The Riemann Sphere

> Corrected open mapping theorem statement:
> **Theorem 0.1** (Open mapping theorem). *If $f : U \to \mathbb{C}$ is a* non-constant *complex differentiable then for any open set $V \subseteq U$, the set $f(V)$ is an open subset of $\mathbb{C}$.*

Holomorphic functions on $\mathbb{P}^1$

A complex differentiable function $f : X \to \mathbb{C}$ is called a *holomorphic* function on X.

The Riemann sphere $\mathbb{P}^1$ is the set $\mathbb{C} \cup \{\infty\}$. We write $\mathbb{P}^1$ as the union of two sets

$$U_1 = \mathbb{C} \qquad\qquad U_2 = \mathbb{C} \setminus \{0\} \cup \{\infty\}$$

The set U_1 is the standard complex plane, but the set U_2 is not. We can turn U_2 into the complex plane by using the following function

$$\begin{aligned} \varphi_2 : U_2 &\longrightarrow \mathbb{C} \\ z &\mapsto z^{-1} \text{ if } z \neq \infty \\ \infty &\mapsto 0 \end{aligned}$$

Thus we can think of $\mathbb{P}^1$ as two copies of the complex plane (U_1 and $\varphi_2(U_2)$) glued together.

A function $f : \mathbb{P}^1 \to \mathbb{C}$ is defined as a pair of functions

$$f_1 : U_1 \to \mathbb{C} \qquad\qquad f_2 : U_2 \to \mathbb{C}$$

such that f_1 and f_2 agree on $U_1 \cap U_2$.

We can only make sense of the complex differentiable functions when both the source and target are open subsets of $\mathbb{C}$. For this reason, we define holomorphic functions on $\mathbb{P}^1$ as follows.

A function $f : \mathbb{P}^1 \to \mathbb{C}$ is holomorphic if

1. $f|_{U_1}$ is holomorphic,
2. $f \circ \varphi_2^{-1}|_{\varphi(U_2)}$ is holomorphic.

$$U_1 \xrightarrow{f|_{U_1}} \mathbb{C}$$

$$\begin{tikzcd} U_2 \arrow[r, "f"] \arrow[d, shift left, "\varphi_2"] & \mathbb{C} \\ \mathbb{C} \arrow[u, shift left, bend left, "\varphi_2^{-1}"] \arrow[ur, dashed, "f\circ\varphi_2^{-1}"'] & \end{tikzcd}$$

> **Q. 1.** Check that defining a function $f : \mathbb{P}^1 \to \mathbb{C}$ is equivalent to defining a pair of functions
>
> $$f_1 : U_1 \longrightarrow \mathbb{C}$$
> $$f_2 \circ \varphi_2^{-1} : \varphi(U_2) \longrightarrow \mathbb{C}$$
>
> such that
>
> $$f_1(z) = f_2 \circ \varphi_2^{-1}(z^{-1}) \text{ whenever } z \neq 0.$$
>
> Note that the source of both f_1 and $f_2 \circ \varphi_2^{-1}$ is $\mathbb{C}$.

It is kinda hard to come up with examples because of the following theorem.

Theorem 0.2. *The only holomorphic functions on $\mathbb{P}^1$ are the constant functions.*

The proof is in the following exercises.

A subset V of a topological space X is compact if it has the following property.

> Every infinite sequence has a convergent subsequence i.e. for every infinite sequence of points $a_1, a_2, \dots$ in V there exists a subsequence $a_{n_1}, a_{n_2}, a_{n_3}, \dots$ which converges to a point in V.

> **Q. 2.** Prove that compact subsets V of a (nice) topological space X are closed i.e. if a sequence of points $a_1, a_2, \dots$ in V converges to point $a \in X$ then a is in V.

> **Q. 3.** Prove that every compact subset V of $\mathbb{C}$ is bounded i.e. there exists a real number M such that $z < M$ for all $z \in V$.

Q. 4. Use the fact that every infinite sequence has a convergent subsequence to argue that for any continuous function $g : X \to Y$ the image of a compact set is compact.

Assume the following fact:

> The Riemann sphere is a compact topological space.

One way to see this is that the Riemann sphere is topologically isomorphic to the sphere S^2 in $\mathbb{R}^3$ (hence the name) which is a closed and bounded subset of $\mathbb{R}^3$. It is not hard to show that such subsets are compact.

Q. 5. Argue that the image of any continuous function $f : \mathbb{P}^1 \to \mathbb{C}$ is bounded.

Q. 6. Using Liouville's theorem, argue that if f is a holomorphic function on $\mathbb{P}^1$ then $f|_{U_1}$ is a constant function.

Q. 7. Using continuity, argue that if f is a holomorphic function on $\mathbb{P}^1$ then f is a constant function.

Meromorphic functions on $\mathbb{C}$

A complex differentiable function $f : X \to \mathbb{P}^1$ is called a *meromorphic* function on X.

We can only make sense of the complex differentiable functions when both the source and target are open subsets of $\mathbb{C}$. For this reason, we define meromorphic functions on X as follows.

A function $f : X \to \mathbb{P}^1$ is meromorphic if

1. f is holomorphic when restricted to $f^{-1}(U_1)$,
2. $\varphi_2 \circ f$ is holomorphic when restricted to $f^{-1}(U_2)$.

$$f^{-1}(U_1) \xrightarrow{f} U_1 = \mathbb{C}$$

$$\begin{tikzcd}
f^{-1}(U_2) \arrow[r, "f"] \arrow[rr, bend right, "\varphi_2 \circ f"'] & U_2 \arrow[r, "\varphi_2"] & \mathbb{C}
\end{tikzcd}$$

It gets tedious to keep track of all the inverses and the sources and targets. We use the following shorthand notation to simplify the clutter. Let $\varphi_1 : U_1 \to \mathbb{C}$ be the identity function, $\varphi_1(z) = z$. Then a function $f : X \to \mathbb{P}^1$ is meromorphic if the two functions

1. $\varphi_1 \circ f$,
2. $\varphi_2 \circ f$,

are holomorphic wherever they make sense.

$$\begin{tikzcd}
f^{-1}(U_1) \arrow[r, "f|_{f^{-1}(U_1)}"] \arrow[rr, bend right, "\varphi_1 \circ f"'] & U_1 \arrow[r, "\varphi_1"] & \mathbb{C}
\end{tikzcd}$$

$$\begin{tikzcd}
f^{-1}(U_2) \arrow[r, "f|_{f^{-1}(U_2)}"] \arrow[rr, bend right, "\varphi_2 \circ f"'] & U_2 \arrow[r, "\varphi_2"] & \mathbb{C}
\end{tikzcd}$$

Q. 8. Which of the following functions are meromorphic functions on $\mathbb{C}$?

1. $f(z) = z$
2. $$f(z) = \begin{cases} z^{-1} & \text{if } z \neq 0 \\ \infty & \text{if } z = 0 \end{cases}$$
3. $$f(z) = \begin{cases} e^{1/z} & \text{if } z \neq 0 \\ \infty & \text{if } z = 0 \end{cases}$$

Meromorphic functions on $\mathbb{P}^1$

Q. 9. Show that a function $f : \mathbb{P}^1 \to \mathbb{P}^1$ is meromorphic if the four functions

1. $\varphi_1 \circ f \circ \varphi_1^{-1}$,
2. $\varphi_1 \circ f \circ \varphi_2^{-1}$,
3. $\varphi_2 \circ f \circ \varphi_1^{-1}$,
4. $\varphi_2 \circ f \circ \varphi_2^{-1}$,

are holomorphic wherever they make sense.

Q. 10. Let $p(z)$ and $q(z)$ be polynomials with no common roots. Assume that $q(z)$ is not the 0 polynomial.
Show that the following function is a meromorphic function on $\mathbb{P}^1$.

$$f(z) = \begin{cases} \dfrac{p(z)}{q(z)} & \text{if } z \neq \infty, q(z) \neq 0, \\ \infty & \text{if } z \neq \infty, q(z) = 0, \\ \lim\limits_{z \to \infty} \dfrac{p(z)}{q(z)} & \text{if } z = \infty. \end{cases}$$

Such a function is called a *rational function*. It is common to simply write $f(z) = \dfrac{p(z)}{q(z)}$.

Turns out these are all the meromorphic functions on $\mathbb{P}^1$.

Theorem 0.3. *Every meromorphic function on $\mathbb{P}^1$ is a rational function.*

The following exercises provide the proof of this theorem.

Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a meromorphic function. Let $\mathcal{Z} = f^{-1}(0) \cap \mathbb{C}$ and $\mathcal{P} = f^{-1}(\infty) \cap \mathbb{C}$. $\mathcal{Z}$ is called the set of zeroes and $\mathcal{P}$ is called the set of poles.

Q. 11. Using the isolated zeroes property of complex differentiable functions argue that both the sets $\mathcal{Z}$ and $\mathcal{P}$ are isolated i.e. for every point $x \in \mathcal{Z}$ there exists a neighborhood U of x such that $U \cap \mathcal{Z} = \{x\}$. Similarly, for $\mathcal{P}$.

Q. 12. Using the fact that every infinite sequence in a compact set has a convergent subsequence, and that $\mathbb{P}^1$ is compact, argue that both $\mathcal{Z}$ and $\mathcal{P}$ are finite sets.

Let $\mathcal{Z} = \{z_1, \ldots, z_m\}$ and $\mathcal{P} = \{p_1, \ldots, p_n\}$. Assume the following fact for now. We'll prove it in class tomorrow.

> The function
>
> $$g(z) = f(z) \cdot \frac{(z - p_1)^{k_1} \ldots (z - p_n)^{k_n}}{(z - z_1)^{\ell_1} \ldots (z - z_m)^{\ell_m}}$$
>
> is meromorphic and has no zeroes or poles, for some positive integers $k_1, \ldots, k_n$ and $\ell_1, \ldots, \ell_m$.

Q. 13. Check that the open mapping theorem 0.1 extends verbatim to meromorphic functions on $\mathbb{P}^1$.

> **Q. 14.** Using the open mapping theorem and Q.4 argue that g is either a constant function or the image of g is all of $\mathbb{P}^1$. [1]

Because g has no zeroes or poles the image of g cannot be all of $\mathbb{P}^1$. Hence, g is a constant function i.e. $g(z) = c$ for some $c \in \mathbb{C}$. Hence,

$$
\begin{aligned}
& f(z) \cdot \frac{(z - p_1)^{k_1} \ldots (z - p_n)^{k_n}}{(z - z_1)^{\ell_1} \ldots (z - z_m)^{\ell_m}} = c \\
\implies \quad & f(z) = c \frac{(z - z_1)^{\ell_1} \ldots (z - z_m)^{\ell_m}}{(z - p_1)^{k_1} \ldots (z - p_n)^{k_n}}.
\end{aligned}
$$

[1] **Hint:** You will need to use the fact that the only open and closed subsets of $\mathbb{P}^1$ are the empty set and $\mathbb{P}^1$ itself.