§ 9.3 Maps as families

$\pi : X \longrightarrow Y$, $y \in Y$

Def: Fiber of π over y is $\operatorname{Spec} K(y) \times_Y X$.

· The underlying set of $\operatorname{Spec} K(y) \times_Y X$ is set theoretic $\pi^{-1}(y)$.

Given $Y' \longrightarrow Y \rightsquigarrow$
$$\begin{tikzcd} Y' \times_Y X \arrow[d] \\ Y' \end{tikzcd}$$
is the base change or pull-back.

Say the <u>general fiber</u> has some property P if $\exists$ dense open $U \subseteq Y$ such that all fibers over pts in U have property P. When Y irreducible, the <u>generic fiber</u> is over the (unique) generic pt on Y.

§9.4 Properties preserved by base change

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d, "\beta'"'] \arrow[dr, phantom, "\square"] & Y \arrow[d, "\beta"] \\ X \arrow[r] & Z \end{tikzcd}$$

If β has property P then so does β' where P is one of:
open embedding, closed embedding, affine, finite, integral, locally finite type, locally finitely presented, quasi-finite, surjective

§ 9.5 Properties not preserved by Base-change

Eg: Suppose E/F finite separable field extension.
Let $E = F(\alpha)$ for some generator α. $p(x) \in F[x]$ minimal poly of α i.e. $E \cong F(x)/(p(x))$
Look at $\operatorname{Spec} F \longrightarrow \operatorname{Spec} E$.
Pick a Galois extension of F containing E.

$$\begin{tikzcd} \operatorname{Spec}(L \otimes_F E) \arrow[r] \arrow[d] & \operatorname{Spec} E \arrow[d] \\ \operatorname{Spec} L \arrow[r] & \operatorname{Spec} F \end{tikzcd}$$

$$L \otimes_F E \cong L[x]/(p(x)) \cong L \oplus L \oplus \cdots \oplus L \Rightarrow \begin{tikzcd} \operatorname{Spec}(\oplus L) \arrow[r] \arrow[d] & \operatorname{Spec} E \arrow[d] \\ \operatorname{Spec} L \arrow[r] & \operatorname{Spec} F \end{tikzcd}$$

Note: π' does not have integral fiber, connected fiber, is not injective

Eg: k char $p>0$, $k(u)$ function field. $\underbrace{k(u)}_{E}/\underbrace{k(u^p)}_{F}$ is a purely inseparable extension.

$E \otimes_F E \cong E[x]/x^p - u^p$

$\cong E[x]/(x-u)^p$

not reduced

$E = k[x]/(x^p - t)$

Def: A geometric point of a scheme X is a morphism of the form $\mathrm{Spec}\,\bar{k} \to X$. $\bar{k}$ algebraically closed.
A geometric point consists of 1) $x \in X$, 2) $\kappa(x) \hookrightarrow \bar{k}$

Def: A geometric fiber is something of the form $\mathrm{Spec}\,\bar{k} \times_Y X$ for some geometric point $\mathrm{Spec}\,\bar{k} \longrightarrow Y$.

Def: Say π has geometrically connected/ irreducible/ integral fiber if all geometric fibers are connected/../.. .
These properties stable under base change.

Fact: The above properties are all OK for geometric base points over a bigger algebraically closed field $\bar{k}$ if they are true for $\overline{\kappa(x)}$.

§ 9.7 Normalization

Def: X integral scheme, a normalization of X is $\tilde{X} \longrightarrow X$ dominant (surjective) such that for any Y normal with a dominant map $Y \longrightarrow X$
normal scheme

$$\exists \quad \begin{tikzcd} Y \arrow[rr, "\exists!"] \arrow[dr] & & \tilde{X} \arrow[dl] \\ & X & \end{tikzcd}$$

Th": Normalization of any scheme X exists.

Proof: If $X = \mathrm{Spec}\,A$, $\tilde{X} = \mathrm{Spec}\,\tilde{A} \leftarrow$ integral closure of A in $\mathrm{Frac}\,A$.
$\tilde{X} \longrightarrow X$ surjective by lying over theorem. These glue.

Important Fact: A Noetherian integral domain, $K = \mathrm{Frac}\,A$, L/K finite field extension, $B =$ integral closure of A in L

a) If A normal and L/K separable $\Rightarrow$ B finitely generated as A-module

b) A f.g. k-algebra $\Rightarrow$ B. f.g. as A-module

Consequence: X is an integral finite type k-scheme then $\tilde{X} \longrightarrow X$ is finite.

§10 Separated, proper morphisms, varieties

§10 Separatedness

Separatedness $\Longleftrightarrow$ Relative Hausdorff

$$\begin{tikzcd} X \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ X \arrow[r, "\pi"] & Y \end{tikzcd} \rightsquigarrow \quad X \xrightarrow{\Delta} X \times X \text{ called the diagonal map.}$$

Prop: Δ is a locally closed embedding.