# Homology continued

$X$ - cell complex

$$C_*(X): \quad 0 \leftarrow C_0(X) \xleftarrow{\partial_1} C_1(X) \xleftarrow{\partial_2} C_2(X) \xleftarrow{\partial_3} C_3(X) \leftarrow \cdots$$

$C_i(X)$ = free vector space over the $i$-cells
= formal $R$-linear combinations of $i$-cells

$$\partial_n [0\, 1 \cdots n] = \sum_{i=0}^{n} (-1)^i \, [0\, 1 \cdots \hat{i} \cdots n]$$

$$H_i(X) = \ker \partial_i / \operatorname{im} \partial_{i+1}$$

Definitions : 1) Vectors in $C_i(X)$ are called $i$-chains

2) Vectors in $\ker \partial_i$ are called $i$-cycles
These are the chains which have no boundary.

3) Vectors in $\operatorname{im} \partial_{i+1}$ are called $i$-boundaries.

We think of homology as cycles modulo boundaries.

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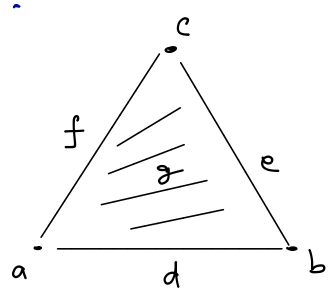
## Computations :




Simplices :

$a, b, c$ — 0-simplices

$d = [a\, b]$, $e = [b\, c]$, $f = [a\, c]$ — 1-simplices

$g = [a\, b\, c]$ — 2-simplex

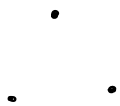
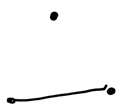
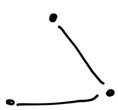
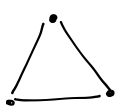
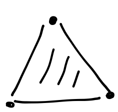
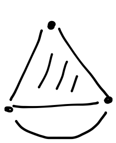
3)  $\quad C_*(X) = \quad 0 \leftarrow \mathbb{R}\langle a, b, c\rangle \xleftarrow{\partial} \mathbb{R}\langle d, e\rangle \leftarrow 0$

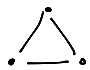
$\partial d = b - a$
$\partial e = c - b$

$$\partial = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Rank $\partial = 2$
$\Rightarrow \ker \partial = 0$

## Computations :

| $X$ | $C_*(X)$ | $H_0(X)$ | $H_1(X)$ | $H_2(X)$ |
|---|---|---|---|---|
| 1)  | $0 \leftarrow \mathbb{R}^3 \leftarrow 0 \leftarrow 0 \leftarrow 0$ | $\mathbb{R}^3$ | $0$ | $0$ |
| 2)  | $0 \leftarrow \mathbb{R}^3 \xleftarrow[\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}]{} \mathbb{R} \leftarrow 0 \leftarrow 0$ | $\mathbb{R}^2$ | $0$ | $0$ |
| 3)  | $0 \leftarrow \mathbb{R}^3 \xleftarrow[\begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}]{} \mathbb{R}^2 \leftarrow 0 \leftarrow 0$ | $\mathbb{R}$ | $0$ | $0$ |
| 4)  | $0 \leftarrow \mathbb{R}^3 \xleftarrow[\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}]{} \mathbb{R}^3 \leftarrow 0 \leftarrow 0$ | $\mathbb{R}$ | $\mathbb{R}$ | $0$ |
| 5)  | $0 \leftarrow \mathbb{R}^3 \xleftarrow[\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}]{} \mathbb{R}^3 \xleftarrow[\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}]{} \mathbb{R} \leftarrow 0$ | $\mathbb{R}$ | $0$ | $0$ |
| 6)  | $0 \leftarrow \mathbb{R}^3 \xleftarrow[\begin{bmatrix} 1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & -1 \end{bmatrix}]{} \mathbb{R}^4 \xleftarrow[\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}]{} \mathbb{R} \leftarrow 0$ | $\mathbb{R}$ | $\mathbb{R}$ | $0$ |

4)  $C_*(X) = \quad 0 \leftarrow \mathbb{R}\langle a,b,c\rangle \xleftarrow{\partial} \mathbb{R}\langle d,e,f\rangle \longleftarrow 0$

$$\begin{aligned} \partial d &= b-a \\ \partial e &= c-b \\ \partial f &= c-a \end{aligned} \qquad \partial = \begin{bmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad \begin{aligned} \operatorname{Im} \partial &\cong \mathbb{R}^2 \\ \ker \partial &\cong \mathbb{R}^1 \end{aligned}$$

5)  $C_*(X) = \quad 0 \leftarrow \mathbb{R}\langle a,b,c\rangle \xleftarrow{\partial_1} \mathbb{R}\langle d,e,f\rangle \xleftarrow{\partial_2} \mathbb{R}\langle g\rangle \longleftarrow 0$

$$\begin{aligned} \partial_1 d &= b-a \\ \partial_1 e &= c-b \\ \partial_1 f &= c-a \end{aligned} \qquad \partial_1 = \begin{bmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad \begin{aligned} \operatorname{Im} \partial_1 &\cong \mathbb{R}^2 \\ \ker \partial_1 &\cong \mathbb{R}^1 \end{aligned}$$

$$\begin{aligned} \partial_2 g &= \partial_2 [abc] \\ &= [bc] - [ac] + [ab] \\ &= e - f + d \end{aligned} \qquad \partial_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \qquad \begin{aligned} \operatorname{im} \partial_2 &\cong \mathbb{R}^1 \\ \ker \partial_2 &= 0 \end{aligned}$$

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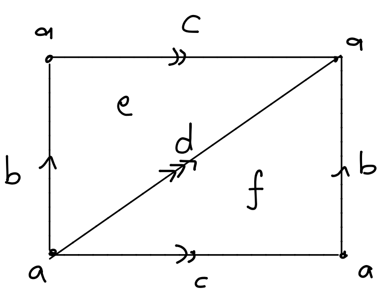
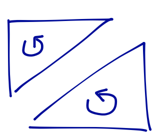
$T^2$ : 

$$\begin{aligned} C_0(T) &= \mathbb{R}\langle a\rangle \\ C_1(T) &= \mathbb{R}\langle b,c,d\rangle \\ C_2(T) &= \mathbb{R}\langle e,f\rangle \end{aligned}$$

$$\partial_1 b = \partial_1 c = \partial_1 d = a - a = 0$$

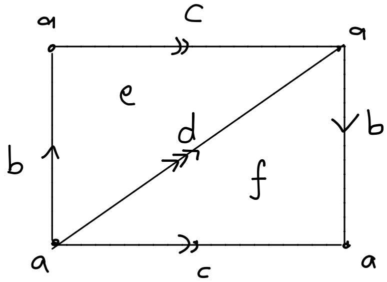
$$\begin{aligned} \partial_2 e &= d - c - b \\ \partial_2 f &= c + b - d \end{aligned}$$

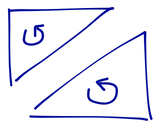
$$\partial_2 = \begin{bmatrix} -1 & 1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \begin{aligned} \operatorname{im} \partial_2 &\cong \mathbb{R}^1 \\ \ker \partial_2 &\cong \mathbb{R}^1 \end{aligned}$$

$$0 \leftarrow \mathbb{R} \xleftarrow{0} \mathbb{R}^3 \longleftarrow \mathbb{R}^2 \longleftarrow 0$$

$$\begin{aligned} H_0(T) &\cong \mathbb{R} \\ H_1(T) &\cong \mathbb{R}^3/\mathbb{R}^1 \cong \mathbb{R}^2 \\ H_2(T) &\cong \mathbb{R}^1/0 \cong \mathbb{R}^1 \end{aligned}$$

$K:$



$\partial_1 b = \partial_1 c = \partial_1 d = a - a = 0$

$\partial_2 e = d - c - b$

$\partial_2 f = c - b - d$

$$\partial_2 = \begin{bmatrix} -1 & -1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \operatorname{im} \partial_2 \cong \mathbb{R}^2 \qquad \ker \partial_2 \cong 0$$

$$0 \longleftarrow \mathbb{R} \xleftarrow{0} \mathbb{R}^3 \xleftarrow{\partial_2} \mathbb{R}^2 \longleftarrow 0$$

$H_0(K) \cong \mathbb{R}^1$

$H_1(K) \cong \mathbb{R}^3 / \mathbb{R}^2 \cong \mathbb{R}^1$

$H_2(K) \cong 0$

Fun Facts about homology:

- $\dim H_0(X)$ = number of connected components of $X$
- If $X$ is a graph $\dim H_1(X)$ = no. of edges need to remove from $X$ to make it a tree.
- If $X$ is a $g$-holed torus $H_0(X) \cong \mathbb{R}$, $H_1(X) \cong \mathbb{R}^{2g}$, $H_2(X) \cong \mathbb{R}$
- If $X$ is a non-orientable surface $H_2(X) = 0$.
- Another chain complex associated to $X$ is the following:

$$C_i(X) = \text{free } \mathbb{R}\text{-vector space over } \operatorname{Maps}(\Delta^i, X).$$

The resulting chain complex is called the singular chain complex and resulting homology is called singular homology. Singular homology is isomorphic to cellular homology.