Five lectures on Topological Field Theory

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1. Introduction and examples

Topological quantum field theories — TQFTs — arose in physics as the baby (zero-energy) sector of honest quantum field theories, showing an unexpected dependence on the large-scale topology of space-time. The zero-energy part of the Hilbert space of states does not evolve in time, as by definition it is killed by the Hamiltonian; so at first sight its physics appears to be uninteresting. But this argument fails to consider a space-time with interesting topology.

In mathematics, TQFT emerged as an intriguing organizing structure for certain brave new topological or differential invariants of manifolds, which could not be captured by standard techniques of algebraic topology. (We will see a reason for that.)

(1.1) Definition. The original axioms of Atiyah and Witten [W1], inspired by Graeme Segal's axioms of Conformal Field theory, defines an D-dimensional TQFT as a *symmetric monoidal functor*

$$Z : (\mathscr{B}ord_D^{or}, \coprod) \to (Vect, \otimes).$$

Here, $\mathscr{B}ord_D^{or}$ is the category whose objects are compact, boundary-less oriented manifolds of dimension $(D-1)$, and morphisms are oriented n-dimensional bordisms, modulo diffeomorphism relative to the boundary.[1] A bordism is assumed to have an incoming and an outgoing boundary; but, in the oriented world, this choice is also indicated by comparing the boundary orientation on each component with the independent orientation. Thus, an interval with two positive endpoints necessarily has one incoming and one outgoing end.

1.2 Remark. In some precise formulations, the manifolds come embedded in a very large Euclidean space ($\mathbb{R}^\infty$), and bordisms embed in an extra 'time' dimension. We will ignore this structure.

The bordism category has a *symmetric monoidal structure* (an associative and commutative multiplication functor $\coprod : \mathscr{B}ord_D^{or} \times \mathscr{B}ord_D^{or} \to \mathscr{B}ord_D^{or}$) defined by disjoint union. The category of vector spaces has a similar multiplication, defined by the tensor product. Because of the additional linearity properties of $Vect$ and the bi-linearity of $\otimes$, we call this a *tensor structure.*

(1.3) Example: Finite group gauge theory. This theory Z_F, associated to a finite group F, is easy to construct in any dimension. On the downside, it only detects fundamental groups of manifolds.[2]

[1] This rather naive definition is adequate for simple applications. An improved definition takes the topology of the diffeomorphism groups into account to produce a category enriched in topological groupoids, in which the morphisms are the topological groupoids consisting of bordisms and diffeomorphisms relative to the boundary.

[2] A more sophisticated enhancement lurks in the background, the gauge theory of finite homotopy types, which detects higher homotopy information and relates to a categorified notion of the group algebra; see Lecture 3.

Denote by $Bun_F(X)$ the groupoid of principal F-bundles over X. To a closed D-manifold M, Z_F assigns the number of isomorphism classes of principal F-bundles on M, each weighted down by its automorphism group. This number is also $\#Hom(\pi_1(M), F)/\#F$. Thus, for $D = 1$, $Z_F(S^1) = 1$.

For a closed $D-1$-manifold N, $Z_F(N)$ is the space of functions on $Bun_F(N)$.

It should be easy to guess now that $Z_F(M) : Z_F(\partial^- M) \to Z_F(\partial^+ M)$ is the linear map whose matrix entry relating F-bundles $F^- \to \partial^- M$ and $F^+ \to \partial^+ M$ counts the F-bundles on M restricting as specified on the two boundaries. The count is weighted by automorphisms that are trivial on the outgoing boundary $\partial^+ M$. Checking that this gives a TQFT (that is, composition of bordisms maps to composition of linear maps) is an exercise, which is best spelt out in the following

1.4 Remark (Correspondence diagram). The TQFT is secretly defined by a *correspondence of groupoids*, one that we will meet in Lecture 2. Namely, the $Bun_F(M)$ maps by restriction to $Bun_F(\partial^- M)$ and $Bun_F(\partial^+ M)$. Call the maps $p_\pm$. Identify $Z_F(\partial^\pm M)$ with the complex H^0 of the corresponding groupoid; the map $Z_F(M)$ is the 'push-pull' composition $(p_+)_* \circ p_-^*$ on cohomology. Here, we convene that the push-forward $(p_+)_*$ along a groupoid sums over the points, but weighs each point by the inverse order of its automorphism group.

1.5 Remark (Twistings). This finite gauge TQFT does not use orientations, but a *twisted* variant does. Choose a group cohomology class in $\tau \in H^D(BF; \mathbb{C}^\times)$; this defines a class $\tau_P \in H^n(M; \mathbb{C}^\times)$ for every principal F-bundle $P \to M$. In the top dimension, we define $Z_F^\tau(M)$ by weighting P by $\int_M \tau_P$ in the count. It takes a bit more thought to see that τ defines a 1-dimensional character of the automorphism group of each principal bundle on an $D-1$-dimensional manifold N. (Hint: use an automorphism to define a bundle on $S^1 \times N$, and now use τ to extract a complex number for each automorphism.) In defining $Z_F^\tau(N)$, we now *delete all lines corresponding to bundles with non-trivial action of the automorphism actions.* In other words, τ defines a flat line bundle $\mathscr{O}(\tau)$ over $Bun_F(N)$, and $Z_F^\tau(N) = H^0(Bun_F(N; \mathscr{O}(\tau))$.

The correspondence description extends to the twisted case and furnishes a baby example of a path integral in mathematical physics, a beloved (if usually non-rigorous) technique to construct quantum field theories. Recall that in physics, the *partition function* associated to a closed manifold is the integral $\int_{\text{fields } \phi} \exp\{\mathrm{i}S(\phi)\} D\phi$ over the space of fields, with the action $S(\phi)$. Here, the fields are maps to BF, or principal bundles on M, the measure is the inverse number of automorphisms, and the role of the (exponentiated) action is played by τ.

Exercise: Describe explicitly the matrix coefficients of the Z_F^τ defined by a D-manifold with boundary.

(1.6) Baby classification, $D = 1$. The vector spaces $Z(+), Z(-)$ are assigned to the point with the two orientations. The right arc $\supset$ gives a morphism $Z(\supset) : \mathbb{C} = Z(\emptyset) \to Z(+) \otimes Z(-)$, and the left arc a pairing $Z(+) \otimes Z(-) \to \mathbb{C}$. (We will convene that time is flowing right-to-left, to match the composition of operators.) These two maps establish a perfect duality between $Z(+)$ and $Z(-)$, forcing them, in particular, to be finite-dimensional: otherwise, the identity map on $Z(+)$ would not sit in $Z(+) \otimes Z(+)^\vee$. So Z is described by $Z(+) = V$; $Z(S^1) = \dim V$ and all operations involve the standard expansion and contraction tensors in $V \otimes V^\vee$.

Remarkably enough, this baby example contains the germ of Lurie's *Cobordism Hypothesis*, which classifies (fully extended) TQFTs in terms of the datum $Z(+)$, which must satisfy a finiteness hypothesis expressed in terms of dualities. We will spell this out in Lecture 4.

1.7 Remark (The unoriented case). Even the $D = 1$ situation becomes interesting, if we abandon orientations. In that case, we must have $Z(+) = Z(-)$ since there is only one point, and at this stage we have a vector space *with non-degenerate symmetric bilinear form.* (Exercise: prove the symmetry of the form by pictures.) In higher dimension, our choice of oriented, as opposed to *framed* manifolds[3] will enforce a strong restriction on our theories, related to the *Calabi-Yau* condition of complex geometry: a trivialization of the canonical bundle.

(1.8) $D = 2$ and Frobenius algebras. It is not too difficult to classify TQFTs as we defined for $D = 2$; the result, folklore for a while, was written up rigorously by L. Abrams [A].

[3]The distinction is invisible in dimension 1.

1.9 Theorem. *A* 2*-dimensional oriented TQFT is equivalent to the datum of a commutative Frobenius algebra structure on a finite-dimensional vector space* A*, which is none other than* $Z(S^1)$*.*

The notion just introduced describes a unital associative algebra A with a *trace* $\theta : A \to \mathbb{C}$ inducing a non-degenerate pairing $A \times A \to \mathbb{C}$ by $a \times b \mapsto \theta(ab)$. The trace condition, $\theta(ab) = \theta(ba)$, becomes important in the non-commutative case, and will appear for extended TQFTs in 2D. An example of a commutative Frobenius algebra is the cohomology of a closed oriented manifold, with the cup-product and the integration map as a trace. Deforming the cup-product to the *quantum cup-product*, which counts holomorphic curves in a projective manifold, is related to the famous *Gromov-Witten theory.*

The geometric representation of the operations is well-known: the multiplication is represented by the pair of pants, mapping $Z(S^1) \otimes Z(S^1) \to Z(S^1)$; the unit by the disk with outgoing boundary, and the trace by the outgoing disk. The trace pairing, $\theta(ab)$, is implemented by an 'elbow', a bent cylinder with two incoming circles. It is easy to deduce from pictures the commutativity of the multiplication and the non degeneracy of the trace pairing; but the converse direction of the theorem takes a bit more work. Here is the answer in outline: a Frobenius algebra contains a distinguished element, the *Euler class* α, the vector output by a torus with a single outgoing circle. Then, a closed genus g surface computes $\theta(\alpha^g)$, while a surface with p incoming and $q > 0$ outgoing boundaries computes the following map $A^{\otimes p} \to A^{\otimes q}$: the product of the p inputs, times α^g, co-multiplied into the outputs. The reader is invited to check that this formula satisfies the Atiyah-Witten definition 1.1.

The special case when the algebra is semi-simple is worth noting. Then, $A = \bigoplus \mathbb{C}P_i$, for projectors satisfying $P_iP_j = \delta_{ij}P_i$, whose traces $\theta_i := \theta(P_i)$ must be non-zero complex numbers. These numbers, up to order, determine the isomorphism class of the Frobenius algebra. Then, $\alpha = \sum_i \theta_i^{-1}P_i$, and $Z(\Sigma_g) = \sum_i \theta_i^{1-g}$. Whenever some invariant associated to surfaces can be expressed as $\sum_i \theta_i^{1-g}$, one should suspect that it is controlled by a $2D$ TQFT.

(1.10) Finite group gauge theory in $2D$*.* Let us spell out Example 1.3 in 2 dimensions. Isomorphism classes of F-principal bundles on the circle are the conjugacy classes in F. The space of functions thereon has the natural basis of characters. We can compute the multiplication operation, described by the of pants, by the push-pull diagram in Example 1.3 and we arrive convolution of characters, viewed as functions on the group. So the algebra is semi-simple, with projectors given by the class functions $(\#F)^{-1}\dim V \cdot \chi_V$, for the various irreducible representations V. The unit in the algebra is the delta-function at the origin, which is the sum of projectors $(\#F)^{-1}\sum_V \chi_V \cdot \dim V$. The trace, defined by the outgoing disk, is the evaluation at the identity — the holonomy of the unique F-bundle over the circle which extends to the disk — weighted down by the number $\#F$ of automorphisms of the trivial bundle. So the projector traces are $\theta_V = \dim^2 V/(\#F)^2$, and by comparing the two computations for the genus g partition function we get the identities

$$(\#F)^{2g-2}\sum_V (\dim V)^{2-2g} = \frac{\#\{u_1, u_2, \ldots, u_{2g} \in F \mid [u_1, u_2] \cdot [u_3, u_4] \cdot \cdots \cdot [u_{2g-1}, u_{2g}] = 1\}}{\#F}$$

In genera 0 and 1, they are some classical identities form the character theory of finite groups. (Exercise: spell them out.) This example of a TQFT was implicitly known to Frobenius.

(1.11) Finite higher-groupoid theories. The finite gauge theories generalize to capture more of the homotopy of manifolds. For this, we re-think the finite group F in terms of its classifying space BF. Recall that this is the quotient by F of a contractible space with free F-action. (An example you all know is $B\mathbb{Z}/2 \cong \mathbb{RP}^\infty = S^\infty/(\mathbb{Z}/2)$.) This BF is an *Eilenberg-MacLane space*, a space with a single non-vanishing homotopy group: in this case, $\pi_1 = F$. The isomorphism classes of principal bundles on M are the connected components $[M, BF]$ of the space of maps to BF. To follow a physics analogy, we are quantizing BF-valued fields, but the 'path integral' counts components (with automorphisms, and weighted by the action in the twisted case).

Similarly, the space of states for the $(D-1)$-manifold N is the space of locally constant functions on BF-valued fields on N. In the twisted case, we get a (flat) line bundle on this space of maps and we are taking the locally constant sections.

We can now replace BF by a more general target space X. A good notion of finiteness for the resulting theory is the finiteness of the homotopy groups of X. In addition, homotopy groups above π_D

do not affect $[M, X]$ if $\dim M \leq D$, so we will take X to have finite homotopy groups, up to dimension D. We can build a TQFT by declaring $Z_X(N^{D-1})$ to be the vector space of locally constant functions on $\mathrm{Map}(N; X)$, and letting $Z_X(M)$ count the components $[M, X]$ with some weights. The weight is a little tricky to work out: for $m : M \to X$, it is the alternating product of the order of the homotopy groups of the m-component of the mapping space $\mathrm{Map}(M, X)$. Just like gauge theory, this TQFT can be twisted by an 'action' $\tau \in H^D(X; \mathbb{C}^\times)$.

The homotopy groups of $\mathrm{Map}(M; X)$ can be computed using a spectral sequence starting from $H^p(M; \pi_{-q}X)$. The computation, however, involves finer information about M and X, namely the cup-product on M, and the way the homotopy groups of X are layered together (the so-called *Postnikov k-invariants*.) So the TQFT described above is sensitive to more information than the homology of M and the homotopy groups of X. For an easy example,[4] recall that $\pi_2(S^2) \cong \pi_3(S^2) \cong \mathbb{Z}$ and $\pi_4(S^2) \cong \mathbb{Z}/2$. A space with the same homotopy groups in dimension up to 4 is $\mathbb{CP}^\infty \times S^3$. There are plenty of maps from $M = \mathbb{CP}^2$ to the latter space: for example, maps to the factor $\mathbb{CP}^\infty$ are classified by their degree in $H^2(\mathbb{CP}^2)$ of the pull-back generator $\sigma \in H^2(\mathbb{CP}^\infty)$; equivalently, by the induced map $\pi_2\mathbb{CP}^2 \to \pi_2\mathbb{CP}^\infty$ (both groups being equal to $\mathbb{Z}$). None of these interesting maps come from a map into the sphere $S^2 = \mathbb{CP}^1 \subset \mathbb{CP}^\infty$: the obstruction is that the cup-square of the area form σ on S^2 is zero, whereas the cup-square $H^2 \to H^4$ on $\mathbb{CP}^2$ is not zero. So the area form $\sigma \in H^2(S^2)$ pulls back to zero under any map $\mathbb{CP}^2 \to S^2$. We will encounter this quadratic obstruction again, in a different guise, in §3.

(1.12) Yang-Mills theory in $2D$. The final variation on gauge theory, which will occupy next lecture, pertains to a *compact* gauge group G. The story just told, about counting principal bundles, requires interpretation. One way to proceed is to interpret 'principal bundles' as 'flat principal bundles', that is, bundles with flat connections. (For discrete groups, principal bundles have a natural flat structure.)

As before, the moduli space of isomorphism classes of flat bundles on S^1 is identified with the space G/G^{ad} of conjugacy classes, so vector space associated to the circle should be that of *class functions on the group*, again carrying the natural basis of characters. There is also a moduli space $F(\Sigma; G)$ of flat bundles over the closed surface of genus g: it is a compact real-analytic space of real dimension $2(g-1) \cdot \dim(G)$ (if G is a simple group; in general, you must correct by the rank of the center). The variety is usually singular, and can be described with reference to π_1 as

$$\Phi(\Sigma, G) = \left\{(u_1, u_2, \ldots, u_{2g}) \in G^{2g} \,|\, [u_1, u_2] \cdot \cdots \cdot [u_{2g-1}, u_{2g}] = 1\right\} / G \tag{1.13}$$

with the group acting by simultaneous conjugation. Related varieties (of different dimensions) arise when we ask that the commutator should lie instead in some specified conjugacy class; for a generic conjugacy class, the variety will be smooth. These varieties are relevant to the TQFT for surfaces with boundary.

More interestingly, consider a surface with a single boundary circle, which we take as outgoing. As π_1 is free, the moduli space is G^{2g}/G^{ad}. Inspired by the correspondence diagram in Example 1.3, we should study the 'product of commutators' map $G^{2g} \to G$, all equivariant for the conjugation action of G; and the generalizations of Φ that we just mentioned are related to the fibers of this map.

'Counting' bundles is not an option, but an alternative stems from the observation that these moduli spaces have natural volume elements coming from a *symplectic form.* (See next lecture.) If we convene to integrate against this volume instead of counting, we almost obtain a 2D TQFT, whose genus g partition function is the symplectic volume of the space (1.13). 'Almost' means that we really obtain an infinite-dimensional vector space $Z(S^1)$, albeit with a natural basis, the characters of G. The underlying Frobenius algebra structure was computed by Witten [W2]), and the answer is strikingly similar to the finite group story: Witten finds a projector P_V for each irreducible character of G, with trace $\theta_V = \mathrm{vol}(G)^{-2} \dim^2 V$. (The Riemannian volume of G is computed using a conjugation-invariant metric on the Lie algebra $\mathfrak{g}$; the same metric is used to define the symplectic form.) From here, Witten finds the *symplectic volume formula*

$$\int_{\Phi(\Sigma;G)} \exp(\omega) = \#Z(G) \cdot \mathrm{vol}(G)^{2g-2} \sum_V (\dim V)^{2-2g}$$

[4]Albeit with infinite homotopy groups, in order to stay with familiar spaces; one can collapse the homotopy groups mod n to get finite examples

which is a convergent series for simple groups, in genus ≥ 2.

1.14 Remark. The factor $\#Z(G)$, the order of the center of G, may seems out of place. But in fact, central elements define automorphisms of any flat bundle, and generically they are the only automorphisms. So Φ is generically an orbifold with stabilizer $Z(G)$, and we should have divided by that when integrating.

(1.15) Variant of TQFT: Cohomological Field theories. A rather famous class of $2D$ field theories enhances the structure we discussed by remembering that surfaces have diffeomorphisms. They are the *A- and B-models of Mirror symmetry*, which count (pseudo-)holomorphic curves in compact symplectic manifolds, respectively describe operations on coherent sheaves on complex manifolds, and produce invariants valued in characteristic classes of surface bundles.

In the simplest formulation, we retain the vector space $Z(S^1)$ associated to the circle, but asking that the linear maps $Z(S^1)^{\otimes p} \to Z(S^1)^{\otimes q}$ induced by surfaces with p inputs and q outputs 'vary cohomologically' in families. That is, if B is a family of a bundle of such surfaces Σ, we want $Z(\Sigma)$ to define a class in

$$H^*\left(B; \mathrm{Hom}\big(Z(S^1)^{\otimes p}, Z(S^1)^{\otimes q}\big)\right)$$

The H^0 component is the original linear map, so the 'variation on the base' does not refer to a continuous dependence on points $b \in B$, but a coupling to homology classes. This is the only sensible definition in the topological context, where we like to compute homotopy-invariant quantities.

There are many variants of this notion (see [C, T1]), but the one to flag here is the *Cohomological Field theory* defined by Kontsevich and Manin [M], where we allow the surfaces to acquire nodes, as in the Lefschetz fibrations of algebraic geometry. This means that, locally near a singularity, the family of curves is described by the family $((x, y) \to t = xy$, with a local coordinate t on the base, and $t = 0$ describing the nodal locus.

The connection with complex geometry is quite fundamental: the classifying space for the diffeomorphism group $\mathrm{Diff}(\Sigma)$ of a surface (with marked points, if desired) is homotopy equivalent to the moduli orbifold of complex structures on the same surface. This holds exactly in the *hyperbolic* cases (excluding, that is, genus 0 with one or two marked points, and genus 1 with no marked points). If we impose a *stability* condition on our nodal surfaces — all irreducible components should be hyperbolic — then there are also universal classifying spaces for Lefschetz fibrations, the much-studied *Deligne-Mumford moduli spaces* $\overline{M}_g^n$ of algebraic geometry. The indexes g and n refer to the genus and number of marked points (which may be thought of as tiny boundary circles, in the context of TQFT), and the bar indicates that these spaces are compactifications of the moduli M_g^n of smooth algebraic curves.

The spaces $\overline{M}_g^n$ have a beautiful local structure, with normal-crossing boundary divisors labelled by the manners in which curves can acquire a node: think topologically of pinching a simple closed loop on the surface into a self-intersection. The combinatorics of the strata encodes a sophisticated algebraic structure ('cyclic operad', see [GK]), closely related to that of a CohFT. But, in the simplest definition, a CohFT is determined by a collection of classes in $H^*\left(\overline{M}_g^{p+q}; \mathrm{Hom}\big(Z(S^1)^{\otimes p}, Z(S^1)^{\otimes q}\big)\right)$. These classes are subject to *factorization conditions*, which describe their restriction to boundary strata of $\overline{M}_g^n$, and express, in this context, the conditions inspired by the notion 1.1 that composition of surfaces leads to composition of linear maps.

1.16 Remark. The homotopy equivalence of $B\mathrm{Diff}(\Sigma)$ with the moduli orbifold of smooth Riemann surfaces of topological type Σ is not a mystery. We can realize $BDiff$ as the moduli orbifold of metrics on a surface, modulo diffeomorphisms. But, a complex structure is the same as a metric up to a conformal rescaling, and the space of rescalings is contractible. So the moduli of metric and conformal surfaces are equivalent. The more difficult result lurking in the background is that the components of the group Diff are contractible in the hyperbolic case, so that $B\mathrm{Diff} \sim B\pi_0\mathrm{Diff}$.

The notion of Cohomological Field theory was motivated by the desire to encode the structure of *Gromov-Witten invariants* of a Kähler (or symplectic) manifold X, which count holomorphic curves with prescribed incidence conditions. The Frobenius algebra we get by ignoring the higher classes on the $\overline{M}$ is the famous *quantum cohomology*, a (commutative) deformation of the cup-product on

$H^*(X)$. The Kontsevich-Manin axioms include some constraints specific to the GW situation, most importantly pertaining to the grading. The general strategy of understanding the full structure of GW invariants stumbled upon a fatal obstacle: the cohomology of the spaces $\overline{M}_g^n$ remains unknown to this day. Understanding the structure of CohFTs then is similar to studying modules over an unknown ring! A key motivation for the *extended TQFTs* we will discuss in Lectures 3-5 was Kontsevich's program to classify TQFTs algebraically from minimal data, from which the Gromov-Witten invariants could be reconstructed. This program is still under development, and just beginning to bear fruit.

One great success of the theory concerns the *genus zero* part of the story — interesting enough geometrically, for it counts (trees of) rational curves in algebraic varieties. Note that, while the space of *smooth* genus zero curves with marked points is simple enough (configurations of distinct points in $\mathbb{P}^1$), its Deligne-Mumford compactification $\overline{M}_0^n$ is far from trivial, and its boundary divisors intersect along interesting patterns. The cohomology of this space was completely understood by Keel [K], who gave an explicit presentation as the free $\mathbb{Z}$-algebra generated by (the Poincaré duals of) the boundary divisors, modulo linear and quadratic relations. (The relations are easily derived by studying fibers of the forgetful maps $\overline{M}_0^n \to \overline{M}_0^4 \cong \mathbb{P}^1$; for example, divisors lying in disjoint fibers of that map have zero intersection product.)

A more specialized but important success followed ideas of Givental, who investigated Gromov-Witten theory of manifolds for which the quantum cohomology, the deformed $H^*(X)$, becomes a semi-simple Frobenius algebra. Such is the case for $\mathbb{P}^n$, where the usual cohomology ring becomes $\mathbb{C}[\omega]/\omega^{n+1} = q$, and generally for all toric manifolds. (The parameter q in the theory is used to separate the count of holomorphic curves according to degree.) Based on experimental evidence, Givental conjectured that all GW invariants — which you recall are counting holomorphic curves of various genera and degrees — are uniquely determined from the quantum cohomology Frobenius algebra alone (and the grading information). In other words, there is a unique way to extend a 'naive' semi-simpmle TQFT, as in §1.8 to a full CohFT, and there is an explicit (recursive) formula for this extension. The conjecture was confirmed [T1]; but there seems to be no hope of extending the result without the semi-simplicity assumption, which is very restrictive. ('Most' target manifolds will not meet that; only when faced with an abundance of rational curves that we can hope to deform the nilpotent cup-product on $H^*(X)$ enough to make it semi-simple.) One of Kontsevich's motivations in studying extended TQFTs was to find an analogous structural result for the Gromov-Witten theory of general varieties.

2. Two-dimensional gauge theory

This example has been a favorite ever since Witten [W2] surprised the gauge theory community by computing the theory explicitly, using localization arguments in path integrals. This gives a TQFT computation of all integrals over the moduli space $\Phi(\Sigma; G)$ of flat G-connections on a surface Σ, in terms of an explicitly computed Frobenius algebra. Assume for convenience that G is simple and simply connected; the dimension of $\Phi(\Sigma; G)$ is then $2d = 2\dim G \cdot (g-1)$. We will ignore for a moment the inconvenient truth that Φ usually contains singular points, corresponding to bundles with reducible holonomy representation.

The universal flat bundle over $\Sigma \times \Phi(\Sigma; G)$ is classified by a map u to the classifying space BG (defined up to homotopy). Recall that $H^*(BG; \mathbb{C})$ is a polynomial ring, isomorphic to $(\mathrm{Sym}\mathfrak{g}^*)^G \cong (\mathrm{Sym}\mathfrak{t}^*)^W$ with generators $\phi_2, \ldots, \phi_\ell$ of cohomology degrees[5] $2m_i + 2$, with the exponents m_i of the group. (For example, the exponents are $1, 2, \ldots, n-1$ for $\mathrm{SU}(n)$.) The slant products $\beta \backslash u^*\phi_i$ (integrals of $u^*\phi_i$ over β) with elements β of a basis of the homology of Σ define classes of degrees $2m_i + 2, 2m_i + 1, 2m_i$. Atiyah and Bott [AB] proved that $H^*\Phi(\Sigma; G)$ is generated by these classes, whenever $\Phi(\Sigma; G)$ is smooth.[6]

The classes $[\Sigma] \backslash u^*\phi_i$, of degree $2m_i$, are the most 'interesting' for integration, as they are not easily reduced to elementary computations. As an example, from the *quadratic Casimir* ϕ_2 corresponding to the invariant quadratic form $\langle , \rangle$ on $\mathfrak{g}$, we obtain the symplectic form ω alluded to in Lecture 1.

[5] The symmetric algebra degree is half the cohomology degree.

[6] Smoothness can occur only for variants of $\Phi(\Sigma; G)$; for the singular case, there is a corresponding statement in terms of equivariant cohomology.

(There is a preferred normalization of the quadratic form, called the *basic* one, in which the shortest non-zero $\log(1)$ in $\mathfrak{g}$ has square-length 2.) For two tangent vectors $\delta A, \delta B$ at a principal bundle P, represented by variations in the connection form, we have

$$\omega(\delta A, \delta B) = \int_\Sigma \langle A, B\rangle.$$

Witten's formula reads as follows. Specialize to SU(2), and choose a polynomial Q. For each $k \in \mathbb{Z}_+$, let $\xi(k,t)$ be the formal power series (in t) representing the unique critical point of the function in ξ (depending on additional parameters h, k, t)

$$F(\xi; h, k, t) := \frac{1}{2}(\xi - k)^2 + t \cdot \frac{h}{2\pi^2} \cdot Q(\pi\xi/h) \tag{2.1}$$

in which t is treated as a formal variable, so $\xi(k,t)$ is computed by t-series expansion around the minimum $\xi(k,0) = k$. A special case of Witten's formula then says

$$\int_{F(\Sigma;G)} \exp\left\{h\omega + t\cdot[\Sigma]\backslash Q(u^*\phi_2)\right\} = \#Z(G)\cdot h^{3g-3}\mathrm{vol}(G)^{2g-2}\sum_{k>0}\left[\frac{1 + tQ''(\xi(k,t))/2h}{\xi(t,k)^2}\right]^{g-1}$$

Of course, $\#Z(G) = 2$ for SU(2). The scaling factor h^{3g-3} should really be absorbed in the volume, as we should rescale the basic metric by h. (For SU(2), $\mathrm{vol}(G) = 1/\pi\sqrt{2}$ in the basic metric, $h^{3/2}/\pi\sqrt{2}$ rescaled.) We recognize on the right the genus g partition function for the semi-simple (but infinite-dimensional) Frobenius algebra $\bigoplus \mathbb{C}P_k$, with one projector P_k for each positive integer, of trace

$$\mathrm{vol}_h(G)^{-2}\frac{\xi(t,k)^2}{1 + tQ''(\xi(k,t))/2h} \tag{2.2}$$

They reduce to the values of §1.12 when $h = 1$ and $t = 0$, as they should.

To get the most general integral, choose several polynomials Q_i and couple them to independent formal variables t_i; the $\partial^n/\partial_1 \dots \partial_n$-derivative of the analogous formula computes

$$\int_{F(\Sigma;G)} \exp[h\omega] \wedge ([\Sigma]\backslash Q_1(u^*\phi_2)) \cdots \wedge ([\Sigma]\backslash Q_n(u^*\phi_2)).$$

2.3 Remark. The most general formula for a simple group G incorporates the other generators of $H^*\Phi(\Sigma; G)$. Q will be an invariant polynomial on the Lie algebra $\mathfrak{g}$, restricted to the Cartan subalgebra $\mathfrak{t}$. The role of the integers k is taken over by the highest weights of G (shifted by the Weyl vector ρ). The numerator in the Frobenius structure constants (2.2) is the Hessian determinant of the multi-variable function F of (2.1); $\xi(t,k)^2$ is replaced by the volume of the adjoint orbit of $\xi(t,k)$ in $\mathfrak{g}$.

(2.4) Interpretation in K-theory. Interpreting and understanding Witten's formulas is no easy task. For example, most moduli spaces are singular, and the characteristic classes above do not live on them. (The exception concerns SU(n)-connections with central holonomy prime to n, at some specified point on Σ.) It turns out that the spaces do have preferred (orbifold) desingularizations, and this is one route to the interpretation of the formulas. Here I will discuss an interpretation in terms of (twisted) K-theory, which allows for a topological description and computation of the field theory.[7]

For starters, recall the Hirzebruch-Riemann-Roch theorem for a holomorphic line bundle $E \to X$ over a complex (projective) manifold:

$$\mathrm{Ind}(X; E) := \sum\nolimits_k (-1)^k \dim H^k(X; \mathscr{O}(E)) = \int_X \mathrm{ch}(E)\mathrm{Td}(X) \tag{2.5}$$

where H^k denotes kth cohomology with coefficients in the sheaf of holomorphic sections $\mathscr{O}(E)$ (equal to the Dolbeault cohomology with coefficients in E), and the *Chern character* $\mathrm{ch}(E)$ and *Todd class* $\mathrm{Td}(X)$ are characteristic classes of E and of the tangent bundle of X. Specifically,

$$\mathrm{ch}(E) = \mathrm{Tr}\exp\left(\frac{R_E}{2\pi\mathrm{i}}\right), \qquad \mathrm{Td}(X) = \det\frac{R_{TX}/2\pi\mathrm{i}}{1 - \exp(-R_{TX}/2\pi\mathrm{i})}$$

[7]Based on joint work with Freed and Hopkins [FHT], and with Woodward [TW].

where R denotes the curvature form of some hermitian connection on the respective bundle. An alternative definition, if E is a sum of line bundles L_i, is $\mathrm{ch}(E) = \sum \exp c_1(L_i)$, and a similar (but multiplicative) maneuver works for Td. The *splitting principle* of topological K-theory, the special case of line bundles suffices to define ch and Td, once we take into account the behavior under sums: $\mathrm{ch}(E_1 \oplus E_2) = \mathrm{ch}(E_1) + \mathrm{ch}(E_2)$, $\mathrm{Td}(E_1 \oplus E_2) = \mathrm{Td}(E_1) \cdot \mathrm{Td}(E_2)$.

Actually, the additive map Ind, from holomorphic vector bundles on X to $\mathbb{Z}$, can be extended to all topological bundles, and defines a linear *index* map $\mathrm{Ind} : K(X) \to \mathbb{Z}$ from the Grothendieck group $K(X)$ of complex topological vector bundles. (This last group is called the even topological K-theory of X, and supplies an example of an exotic, or generalized, cohomology theory.) Formula (2.5) gives a factorization of the index through the ring homomorphism $\mathrm{ch} : K(X) \to H^{ev}(X)$, followed by integration against the Todd class.

2.6 Remark. It turns out that ch realizes an isomorphism of algebras $K(X) \otimes \mathbb{Q} \cong H^{ev}(X; \mathbb{Q})$; but viewing the Index map as a trace defines a Frobenius algebra structure on $K(X) \otimes \mathbb{Q}$ different from the one on $H^*(X; \mathbb{Q})$.

(2.7) Integration from the index. Assume for now that E is a line bundle with Chern class ω. There is a way to extract the symplectic volume of X as an asymptotic of the index:

$$\mathrm{Ind}(X; E^{\otimes n}) = n^{\dim X} \int_X \exp(\omega) + O(n^{\dim X - 1}),$$

making the Todd class disappear in the leading term. A similar trick can be used for any vector bundle E, but requires, instead of the tensor power, the use of the *nth Adams operation* ψ^n, defined by $\psi^n L = L^{\otimes n}$ for a line bundle L and imposing additivity: $\psi^n(E \oplus F) = \psi^n(E) + \psi^n(F)$. (From here, the splitting principle pins down $\psi^n : K(X) \to K(X)$ uniquely.) There is an expression for ψ^n in terms of exterior powers, but it involves signs, so it is only valid in the Grothendieck group $K(X)$ and not meaningful in the category of vector bundles. At any rate, if we regard the computation of ψ^n as known, we see by linearity of ψ^n that integration over X can be recovered as an asymptotic of the index:

$$\mathrm{Ind}(X; \psi^n E) = n^{\dim X} \int_X \mathrm{ch}(E) + O(n^{\dim X - 1}).$$

(2.8) Index formulas on $\Phi(\Sigma; G)$. The point of this long preamble is the following: the integration formulas over $\Phi(\Sigma; G)$ can be recovered from index formulas. However, the index formulas, unlike integration, comes from a genuine TQFT, based on a finite-dimensional Frobenius algebra. Moreover, the index formulas can be derived purely geometrically, from correspondence diagrams with moduli spaces of flat connections, as in the case of gauge theory with a finite group.

A word of philosophy may or may not help — the index formulas represent a version of gauge theory for the *loop group* go G. Loop groups have a representation theory reminiscent of that of compact Lie groups, but the theory carries a fundamental discrete parameter, the *level*; and there are only *finitely many* representations at a fixed level. Thus, the theory at a fixed level has some features of the representation theory of *finite* groups.

The easiest story pertains to the K-theory of the analogues of symplectic volumes. For integral h, $\exp(h\omega)$ is the Chern character of a line bundle $\mathscr{O}(h)$ on $\Phi(\Sigma, G)$. Now, a choice of complex structure of Σ gives a complex analytic (and in fact projective algebraic) structure on the variety $\Phi(\Sigma, G)$: this is a deep theorem of Narasimhan and Seshadri, which identifies the latter with the moduli of (poly)stable holomorphic G-bundles on Σ. The line bundles $\mathscr{O}(h)$ turn out to admit holomorphic structures, uniquely so when the group G is simple.[8]

The index of $\mathscr{O}(h)$ has a very nice interpretation, thanks to the

2.9 Theorem (Kumar-Narasimhan). *The higher cohomology $H^{>0}(F(\Sigma, G); \mathscr{O}(h))$ vanishes if $h \geq 0$.*

These indexes thus measure the dimensions of vector spaces. These spaces, the *conformal blocks*, have been much studied. A formula for their dimension was conjectured for SU(2) by E. Verlinde:

$$\dim H^0(F(\Sigma, G); \mathscr{O}(h)) = (2h+4)^{g-1} \sum_{k=1}^{h+1} \left(2 \sin \frac{k\pi}{h+2}\right)^{2-2g}$$

[8] For U(1), $\Phi(\Sigma, G)$ is the Jacobian of Σ, and holomorphic line bundles vary in continuous families.

it was proved, for general G, in the work of numerous authors. Witten's symplectic volume formula can be obtained from the asymptotics of Verlinde's. I will not reproduce the derivation here, see [TW], §5 for the general story. My goal, instead, is to explain why these numbers (and their generalization to indexes of vector bundles) are controlled by a 2-dimensional TQFT.

(2.10) The Verlinde ring. The 2-dimensional TQFT controlling these indexes of line bundles is a semi-simple Frobenius $\mathbb{Z}$-algebra, called the *Verlinde ring* $V(G;h)$. (The extension to vector bundles is less well known; it was indicated in [T2].) I only want to flag here that the projectors in the controlling Frobenius algebra are in natural correspondence with the *projective, positive energy representations* of the smooth loop group LG of G, with projective co-cycle determined by the Chern class h of the line bundle. There is indeed a deeper connection between loop group representations and the Verlinde ring; for example, the Verlinde ring is the Grothendieck K-group of the category of said representations, and there is a *tensor structure* on that category, the *fusion product*, which induces the product in the Verlinde ring; there is a distinguished *vacuum representation* which acts as the unit, and the Frobenius trace of a general representation extracts the multiplicity of the vacuum representation. You notice the formal resemblance with the gauge theory of a finite group, where the 'vacuum representation' is the trivial one. But we digress.

2.11 Remark. From the TQFT point of view, the numbers we associate to surfaces are dimensions of vector spaces H^0. Remembering the spaces themselves suggests the possibility of a $3D$ TQFT, with numbers assigned to 3-manifolds. Indeed, this *Chern-Simons theory* has been constructed rigorously [RT]. The subject leads to deep connections between conformal field theory, loop group representations and 3-dimensional topology.

(2.12) Twisted K-theory, a crash course. We need one more ingredient to describe the TQFT controlling the Verlinde numbers and their generalizations (to be described). Constructing twisted K-theory rigorously, especially the equivariant version, would take a course on its own, but the idea is easy enough. The famous Serre-Swan theorem asserts that vector bundles over a compact Hausdorff space X are precisely the projective modules over the ring $C^0(X)$ of continuous functions. This offers a purely algebraic definition of $K(X)$, as the corresponding Grothendieck group of projectives.

When a compact group G acts on X, we can define the equivariant K-group $K^0_G(X)$ as the Grothendieck group of vector bundles which carry a lifting of the G-action.

There is again an algebraic description. We can form the *crossed product algebra* $G \ltimes C^0(X)$: these are the functions on $G \times X$, with the point wise multiplication on X, the convolution product on G, and using the intertwining action of G on X. For example, if G is finite, an element in $G \ltimes C^0(X)$ can be expressed as a sum $\sum_{g\in G} g \cdot \varphi_g$, with $\varphi_g \in C^0(X)$, and the multiplication is given by

$$\left(\sum\nolimits_{g\in G} g \cdot \varphi_g\right) \cdot \left(\sum\nolimits_{h\in G} h \cdot \psi_h\right) = \sum\nolimits_{k\in G} k \cdot \sum\nolimits_{gh=k} (h^{-1})^* \varphi_g \cdot \psi_h,$$

where for $u \in G$, $(u^*\varphi)(x) := \varphi(u^{-1}(x))$. In other words, we act on the function when moving a group element across.

The equivariant version of the Serre-Swan theorem, in this context, equates $K_G(X)$ with the Grothendieck group of projective modules over $G \ltimes C^0(X)$.

Imagine now a bundle on algebras over X, locally isomorphic to the constant bundle $\mathbb{C}$. Actually, such a bundle would have to be a product bundle, but that is because *isomorphism* is the wrong notion for bundles of algebras, and should be replaced by the notion of *Morita equivalence*. The simplest example is a bundle of matrix algebras. Such a bundle need not be globally trivial, or even Morita equivalent to the trivial bundle. Indeed, given a *projective* vector bundle $\mathbb{P} \to X$, the associated bundle $\mathrm{End}(\mathbb{P})$ of matrix algebras is well-defined: this is because, for a vector space E, $\mathrm{End}(E)$ is canonically defined from the projective space $\mathbb{P}E$ alone. However, $\mathrm{End}(\mathbb{P})$ need not be the endomorphism algebra of a globally defined vector bundle $E \to X$. Indeed, the obstruction to this is precisely that of lifting $\mathbb{P}$ to a vector bundle E, $\mathbb{P} \cong \mathbb{P}E$. From the short exact sequence

$$1 \to \mathrm{GL}(1) \to \mathrm{GL}(n) \to \mathbb{P}\mathrm{GL}(n) \to 1$$

we get the fragment of long exact sequence

$$\cdots \to H^1(X;\mathrm{GL}(1)) \to H^1(X;\mathrm{GL}(n)) \to H^1(X;\mathbb{P}\mathrm{GL}(n)) \to H^2(X;\mathrm{GL}(1))$$

which locates the obstruction in $H^2(X;\mathrm{GL}(1)$. We are in topology and are using continuous coefficients, and the exponential map shows this to be the same as $H^3(X;\mathbb{Z})$. In the literature, this is called the *Dixmier-Douady class* of the *gerbe* defined by our projective bundle.

The classes we get are n-torsion, as the skilled among you will notice by comparison with the sequence

$$1 \to \mu_n \to \mathrm{SL}(n) \to \mathbb{P}\mathrm{GL}(n) \to 1,$$

which places the obstruction in $H^2(X;\mu_n)$. However, there is a good infinite-dimensional version of this construction which employs *projective Hilbert bundles*, where the trick of comparing with $\mathrm{SL}(n)$ fails; and indeed, one can show that *any class in $H^3(X;\mathbb{Z})$ is realized by a Hilbert gerbe*, unique up to a certain equivalence. A technical tweak is that the good analogue of the bundle of matrix algebras, in the Hilbert bundle story, is the bundle of *compact* endomorphisms.

All in all, for each class $[\tau] \in H^3(X;\mathbb{Z})$, we can define a twisted K-group ${}^\tau K(X)$ as the Grothendieck group of projective modules over the sections of the bundle of matrix algebras defined by τ.

Modulo technical difficulties which have been resolved in a number of ways [AS, FHT1], the story extends literally to spaces X with compact group action: a class in $H^3_G(X;\mathbb{Z})$ defines a G-equivariant bundle of matrix algebras over X. (They are infinite-dimensional, unless the class happens to be torsion.) Morally, we define the twisted K-group ${}^\tau K^0_G(X)$ as the Grothendieck group of finitely generated projective modules.[9]

2.13 Remark (Chern character computation of ${}^\tau K$). We mentioned earlier that rational K-theory of a space was rather straightforward to calculate from rational cohomology. Twisted K-theory is also rather easy to compute with rational coefficients, as follows. (Sadly, the equivariant version is more troublesome.)

- There is an odd K-group $K^1(X)$; a cheating definition is $K^1(X) = K^0(S^1 \times X, X)$. This is the definition of K^{-1}, and we are building in the *Bott periodicity theorem* $K^i \cong K^{i+2}$.

- As mentioned, for a reasonable compact space X , the Chern character gives an *isomorphism* $\mathrm{ch} : K^0(X)\otimes\mathbb{Q} \to H^{ev}(X;\mathbb{Q})$. There turns out to be a matching isomorphism $\mathrm{ch} : K^1(X)\otimes\mathbb{Q} \to H^{odd}(X;\mathbb{Q})$. Let now $(C^*(X;\mathbb{Q}),\delta)$ be the algebra of rational cochains on X, with differential δ. (At the price of switching to real coefficients, we can use de Rham's differential forms.) Let also τ be a 3-co-cycle representing the twisting class. It is easy to describe twisted K-theory in this language: the spaces ${}^\tau K^{0|1}$ are isomorphic, via a twisted Chern character, to the odd and even cohomologies of $C^*(X;\mathbb{Q})$ with modified differential $\delta + \tau\wedge$. (Note that $\delta\tau = \tau\wedge\tau = 0$, confirming that $(\delta + \tau\wedge)^2 = 0$; so this is a complex.)

  Alas, this easy model does not help with *equivariant* twisted K-theory; see the more complicated story of the delocalized Chern character in [FHT0].

(2.14) Twisted $K_G(G)$. Let G be a compact, simple, simply connected Lie group. Then, $H^3_G(G;\mathbb{Z})$, the group of twistings of K-theory, is canonically isomorphic to $\mathbb{Z}$. (There is a preferred generator, giving the positive definite basic quadratic form on the Lie algebra.) It turns out that the *levels*, or projective co-cycles of loop group representations are also parametrized by the integers.[10] The key theorem of [FHT] is an isomorphism of Frobenius rings

$${}^\tau K^{\dim G}_G(G) \cong V(G;h)$$

The twisting τ is $h+c$, with the shift c in the level (the *dual Coxeter number*) depending on the group; it is equal to n for $\mathrm{SU}(n)$. Already, $c = 2$ has made an appearance in Verlinde's formula for $\mathrm{SU}(2)$.

[9] Yet again, a technical change is required in the definition of K-theory in the infinite-dimensional case, because compact operators form a *non-unital* algebra: one adjoins a unit and restricts to modules of rank 0 see [AS].

[10] Projective LG-cocycles are always classified topologically by $H^3_G(G;\mathbb{Z})$.

We have alluded to the multiplication on $V(G;h)$; the multiplication on ${}^\tau K_G^{\dim G}(G)$ is the *Pontrjagin product*, induced by the multiplication $G \times G \to G$. Again, the finite group gauge theory should come to mind, if we think of convolution of characters. This time, instead of G-invariant functions on G, we are dealing with G-equivariant vector bundles (or rather, projective 'twisted' by τ). This replacement of complex-valued function with vector bundles — vector-space valued functions — is an instance of categorification, and hints at the fact that the Verlinde theory we describe is a $3D$ theory in disguise.

The Frobenius trace on ${}^\tau K_G^{\dim G}(G)$ is a bit less obvious; see Theorem 2.16.iii below. Before stating that, recall that the stack of flat G-connections on the circle is equivalent to the quotient stack G/G, with G acting on itself by conjugation. A compact oriented surface Σ with incoming boundary $\partial^-\Sigma$ and outgoing one $\partial^+\Sigma$ defines a correspondence of *stacks* of flat connections

$$\begin{tikzcd} & \tilde\Phi(\Sigma;G) \arrow[dl, "\pi_+"'] \arrow[dr, "\pi_-"] & \\ \tilde\Phi(\partial^+\Sigma;G) \cong (G/G)^q & & (G/G)^p \cong \tilde\Phi(\partial^-\Sigma;G) \end{tikzcd}$$

Informally, the stacks $\tilde\Phi$ differ from the underlying moduli spaces in that they remember the automorphisms of bundles. Those of you who dislike stacks must choose a base-point in (every connected component of) Σ and one in each boundary component, and paths from the Σ-base points to each of the others. The stacks are then presented as a product of many copies of G — one parallel transport for each path, and one holonomy for each boundary circle — divided by the simultaneous action of G at all base-points. The drawback of proceeding in this manner is that you must then prove that the operation defined below is independent of the choice of base-points and paths. (This is avoided by defining K-theory and the operations on stacks and representable morphisms.)

(2.15) Exercises.

(i) By starting with a presentation and simplifying as possible, show that the correspondence diagram induced by a pair of pants with two incoming and one outgoing circle is

$$\begin{tikzcd} & (G\times G)/G^{\mathrm{diag}} \arrow[dl, "m"'] \arrow[dr, "\mathrm{Id}/\mathrm{diag}"] & \\ G/G & & G/G \times G/G \end{tikzcd}$$

with all groups acting by conjugation, G^{diag} representing the diagonal subgoup, and m the multiplication map on the group. So the operation is restricting from $G\times G$ to G^{diag}-equivariance, and then multiplication in the group.

(You should have done this exercise for a finite group G, when studying finite gauge theory.)

(ii) Show that the correspondence diagram induced by a cylinder with two outgoing cicles is the anti-diagonal inclusion $G/G \to G/G \times G/G$.

2.16 Theorem ([FHT, TW]). *(i) There is a 2-dimensional TQFT based on the space* ${}^{h+c}K_G^{\dim G}(G)$.

(ii) A surface with p inputs and $q > 0$ outputs induces the linear map

$$(p_+)_* \circ p_-^* : {}^\tau K_G^{\dim G}(G)^{\otimes p} \to {}^\tau K_G^{\dim G}(G)^{\otimes q}.$$

(iii) The bilinear form underlying the Frobenius structure is given by the cylinder with two outgoing ends, and is non-degenerate.

(iv) The partition function for a closed surface computes the index of $\mathscr{O}(h)$ over $F(\Sigma;G)$.

The statement above conceals many fine points discussed in [FHT], such as how to handle the twistings in pullbacks and push-forwards, as well as the K-theory orientations on F, which must be tracked with care when G is not simply connected.

(2.17) Generalizations: Higgs bundles as an example. The theory extends to incorporate the K-theory analogue of the Atiyah-Bott generating cohomology classes on $\tilde{\Phi}(\Sigma; G)$. We refer to [T2, TW] for details; one subtlety *not* present for line bundles appears in statement (iv), where the moduli space of flat connections must be replaced with the moduli stack of *all* holomorphic $G_{\mathbb{C}}$-bundles (including the unstable ones).

Here I just give the formula of K-theory integration in a special case, on the moduli of *Higgs bundles*, famous in other areas of mathematics; I leave the asymptotic derivation of the *integration formula* as an exercise for the reader. The integration formula was found, with physics arguments by Moore, Nekrasov and Shatashvili [MNS].

The moduli stack $\mathfrak{M} := \mathfrak{M}(\Sigma; G)$ of all holomorphic $G_{\mathbb{C}}$-bundles over Σ has the variety $\Phi(\Sigma; G)$ as an associated GIT quotient, meaning

$$\Phi(\Sigma; G) = \mathrm{Proj}\left(\bigoplus\nolimits_h \Gamma(\mathfrak{M}; \mathscr{O}(h))\right)$$

Under certain conditions (after a large $\mathscr{O}(h)$ twist), indexes of vector bundles over $\mathfrak{M}$ and $\Phi(\Sigma; G)$ agree. This often allows us to dispense with the stack in the story of the index and the associated TQFT, just as we did for line bundles.

Now, $\mathfrak{M}$ has a cotangent stack $T^*\mathfrak{M}$; ordinarily, this would be a *differential graded* stack, but in genus 2 or more, the dg structure vanishes and $T^*\mathfrak{M}$ is an ordinary (locally finite Artin) stack, locally presentable as a quotient of a locally complete intersection variety by a reductive group. Using the same line bundle $\mathscr{O}(h)$, we can define an associated moduli space $H(\Sigma; G)$, the moduli of *semi-stable Higgs bundles*, which is a partial compactification of $T^*\Phi(\Sigma; G)$. There is a similiar story equating indexes of bundles over $T^*\mathfrak{M}$ and H, which applies in particular to $\mathscr{O}(h)$ for $h > 0$; so again people who dislike stacks can avoid them in the Higgs story. Non-compactness of H gives infinite answer to index questions; but the $\mathbb{C}^*$-scaling action on the fibers of $T^*\mathfrak{M}$ can be used to render the answers finite. Indexes will not be numbers, but power series in $q \in \mathbb{C}^*$, labelling the dimensions of weight spaces. With these prelimiaries, we are ready for the

2.18 Theorem (following [TW]). *For $G = \mathrm{SU}(2)$,*

$$\mathrm{Ind}(H(\Sigma; G); \mathscr{O}(h)) = (2h+4)^{g-1} \sum\nolimits_{k=1}^{h+1} \left(2\sin\frac{k_q\pi}{h+2}\right)^{2-2g} \cdot \left(1 + \frac{2q}{h+2}\frac{1 - q\cos\frac{2\pi k_q}{h+2}}{(1+q^2) - 2q\cos\frac{2\pi k_q}{h+2}}\right)^{g-1}$$

The points $k_q = k + qk_1 + q^2k^2 + \ldots$, with $k = 1, 2, \ldots, h+1$ are the power series solutions of the equation

$$k_q + \frac{1}{2\pi \mathrm{i}}\log\left(1 - qe^{-2\pi \mathrm{i} k_q/(h+2)}\right) - \frac{1}{2\pi \mathrm{i}}\log\left(1 - qe^{2\pi \mathrm{i} k_q/(h+2)}\right) = k$$

This is a TQFT over the power series ring $\mathbb{C}[\![q]\!]$. At $q = 0$ we get Verlinde's formula, as we should, since q^0 counts the sections which are constant along the fibers of $T^*\mathfrak{M}$ and therefore come from $\mathfrak{M}$. There is a generalization to all compact groups, and vector bundles other than the $\mathscr{O}(h)$ can also be included (but then we must usually take the index over $T^*\mathfrak{M}$ not H).

3. Extended TQFT and Higher Categories

A basic flaw in the Atiyah-Witten definition is the restriction to co-dimension 1 boundaries. While this keeps the story clean and simple, it makes it impossible to compute the TQFT by cutting up the manifold into simple pieces; the cases $D = 1, 2$ were rather special, and the next higher classification theorem, in $D = 3$, due to Reshetikhin-Turaev [RT], requires the use of circles and cutting in co-dimension 2.

Extended TQFTs are, intuitively, functors from the category $\mathscr{B}ord_D^{or}$, enhanced with enough structure to allow the cutting of manifolds into simple pieces with corners of all co-dimensions. This is a rather backwards way of telling the story, and it soon enough becomes clear that we need to replace our two tiers of structure in the bordism category — $(D-1)$-dimensional objects and D-dimensional morphisms — with $(D+1)$ tiers of structure, going down to points.

(3.1) Higher categories. The various ways of encoding such an algebraic structure, with D tiers of morphisms layered over objects, allowing multi-dimensional compositions, are known as *D-categories.* A prime example is the bordism D-category $Bord_D^{or}$, in which (oriented) 0-manifolds are the objects, oriented 1-manifolds with boundary are morphisms between 0-manifolds, 2-manifolds with corners are "2-morphisms" and so forth. It quickly emerges that the algebraic rules of the game are not as clearly set anymore (in fact, many sets of rules are imaginable), so here is a first inductive

3.2 Definition. A *strict D-category* is a category in which all sets $Hom(x, y)$ of morphisms have the structure of $(D-1)$-categories, and the compositions $Hom(x, y) \times Hom(y, z) \to Hom(x, z)$ are strict bi-functors of $(D-1)$-categories. Composition is strictly associative.[11] A *functor* ϕ between D-categories is a functor of underlying categories, such that the induced maps on morphisms $\phi_* : Hom(x, y) \to Hom(\phi x, \phi y)$ are functors of $D-1$-categories.

We are concerned with ($\mathbb{C}$-)linear categories, which at the top two layers reproduce vector spaces and linear maps. It is also customary to require the categories to admit finite direct limits, at least (arbitrary co-limits are sometimes convenient). There is a 'unit object' in the world of D-categories for any D, unit in the sense that everything is a module over it, much like every $\mathbb{C}$-vector space is a module over $\mathbb{C}$. Among linear categories, the unit is the category $Vect$ of finite-dimensional vector spaces. (In a $\mathbb{C}$-linear category, every object can be tensored with a finite-dimensional vector space, and this is functorial both in the object and in the vector space.) Next is the strict 2-categories of linear categories, linear functors and natural transformations, followed by the strict 3-category of strict 2-categories, etc. If direct limits are assumed to exist, we ask for the linear functors to be right exact, that is, to preserve them.

3.3 Remark. Another familiar one-step enhancement of the category of vector spaces of linear maps, instead of the 2-category of linear categories, is the 2-category $\mathscr{A}lg$ of algebras, bi-modules and intertwining maps. One can embed this "fully faithfully" into linear categories by sending every algebra to its category of modules. (The key observation is that, for two algebras A, B, any right-exact functor from A-modules to B-modules is induced by tensoring over A with a $B-A$-bimodule.) This is a *non-strict* 2-category: composition of morphisms — tensoring of bi-modules — is defined only up to natural isomorphism.

(3.4) Strict versus lax categories. Experience with categories should suggest a flaw in the strict definition: when in a category, one should ask for 'well-behaved isomorphism' rather than equality. Good behavior depends on the problem at hand, for instance, functoriality under some operations. For example, it seems wrong to require the composition operation to be associative on the nose — instead we should ask for an *associator* α, a natural isomorphism of the two functors from $Hom(x, y) \times Hom(y, z) \times Hom(z, w)$ to $Hom(x, w)$ obtained by composing in different orders. There is then a natural condition on this associator (Stasheff's pentagon identity), which suffices to allow us to work with 'associativity up to coherent associators' just as we would with strict associativity. Similarly, we could ask for functors between 2-categories to preserve the composition of 1-morphisms only up to coherently chosen 2-morphisms. As we progress in categorical depth, there are more identities that we could relax to 'coherent' isomorphisms. Any systematic listing of the associative and commutative data, and coherence conditions on it, will lead to a theory of *lax* D-categories. Much current work in higher categories is motivated by the search for spelling out methodical, but convenient and practical ways to encode the lax data and its coherence conditions; see [B, L, R] and many others.

Intuition is not completely reliable. It turns out that there is no problem for $D = 2$: any lax 2-category can be 'strictified'; but we run into trouble beyond that, as we will see in an example of groupoids below. Ignoring this trouble for a moment, let us discuss a simple

(3.5) Finite group gauge theory in $2D$. This example will be useful before discussing the general theory of duality in the next lecture. We construct a 2-functor from the unoriented bordism 2-category $\mathscr{B}ord_2$ to $\mathscr{A}lg$.

To the point we associate the group algebra $A := \mathbb{C}\langle F\rangle$, of linear combinations $\sum_{f\in F} a_f \cdot e_f$, with $a_f \in \mathbb{C}$ and multiplication $e_f \cdot e_{f'} = e_{ff'}$ imitating the group one. A-modules are the same as complex

[11] So that 'equal' really means equal and not canonically isomorphic.

representations of F. Note that $A \cong A^{op}$ by the anti-involution $f \leftrightarrow f^{-1}$; so that for instance there is no distinction between $A - A$ bimodules and $A^{\otimes 2}$-modules. (This does not apply to the *twisted* version of gauge theory, see Remark 3.10, and is related to the fact that untwisted gauge theory for a finite group does not require orientations on manifolds.)

Now to any interval we associate the bi-module A, which is viewed as a left $A^{\otimes 2}$-module, a right one, or an $A-A$ bimodule, according to the endpoints: both outgoing, both incoming or one of each. In other words, the interval can be a bordism from $\emptyset$ to $\{*, *\}$, from $\{*, *\}$ to $\emptyset$, or the identity morphism on $\{*\}$. (As we indicated, we will not need to orient the point $*$.) Note the natural isomorphism

$$A \otimes_A A \cong A, \quad \text{expressing the composition} \quad \mathrm{Id} \circ \mathrm{Id} = \mathrm{Id}. \tag{3.6}$$

To a closed circle, we associate the space $A \otimes_{A \otimes A} A$. This computation is forced upon us by the presentation of the circle as the composition $\subset \circ \supset$, and identifies $Z_F(S^1)$ with the space of class functions on F, recovering the vector space for gauge theory discussed in §1.3.

Having defined the bottom two tiers, consider now a surface with corners. To read it as a 2-morphism, we must supply more data: a labelling of the corners as sources or targets, and a compatible labeling of the edges (1-morphisms) as source or targets of the 2-morphism defined by the surface. The case of twisted gauge theories as in §1.5 is exponentially worse: we need to supply compatible orientations of all those objects. Listing all possibilities and the corresponding maps quickly becomes cumbersome; the good construction is again inspired by the physiscist's path integrals.

Let then Σ be a surface with corners, read as a 2-morphism between an incoming and an outgoing part of its boundary. Those boundary parts, $\partial_\pm\Sigma$, are 1-morphisms with the *same* source and target. That is, the corners on $\partial_-\Sigma$ must be matched with corners on $\partial_+\Sigma$ by some part B of the boundary, standing in for identities on objects.[12] We now specify that the space A associated to a 1-morphism (1-manifold with boundary) should be interpreted as H^0 of the groupoid Bun_F of F-bundles trivialized on the boundary. Similarly, the operation $Z_F(\Sigma)$ is given by $(\pi_+)_* \circ \pi_-^*$ on H^0 in the familiar correspondence diagram

$$\begin{tikzcd}
 & Bun_F(\Sigma, B) \arrow[dl, "\pi_+"'] \arrow[dr, "\pi_-"] & \\
Bun_F(\partial_+\Sigma, B_+) & & Bun_F(\partial^-\Sigma, B-)
\end{tikzcd}$$

(3.7) The correspondence 2*-category.* What we just did was to factor the gauge TQFT from $\mathscr{B}ord_2 \to \mathscr{A}lg$ via a 2-category $\mathscr{C}or_2$ of *correspondences* of finite groupoids. Objects in $\mathscr{C}or_2$ are finite groupoids $X_\bullet$. This notation must be conceived to include the sets X_0 of objects, X_1 of morphisms, the source and target maps $X_1 \rightrightarrows X_0$, the identity section $X_0 \to X_1$, and the composition law. A map of groupoids $X_\bullet \to Y_\bullet$ is a pair of maps on objecs and morphisms, compatible with the structures.

1-morphisms in $\mathscr{C}or_2$ are correspondences between groupoids $Y_\bullet \leftarrow X_\bullet \to Z_\bullet$, and 2-morphisms between $X_\bullet, X'_\bullet$ are again correspondences $X_\bullet \leftarrow C_\bullet \to X'_\bullet$, compatible with the projection to $Y_\bullet \times Z_\bullet$. Composition of 1- and 2-morphisms is the *homotopy fiber product*: this is the naive fiber product obtained after replacing one of the maps with a *fibration* — a morphism where each arrow lifts, once a lifting of the source or target has been found.

(3.8) Example. The point is $* \rightrightarrows *$. The groupoid BF (finite group F) is described as $F \rightrightarrows *$. The morphism from $* \to BF$ is not a fibration, but can be converted to one after replacing $* \rightrightarrows *$ with the equivalent groupoid $F \times F \rightrightarrows F$, representing the translation F-action on F. The homotopy fibre product $* \times_{BF} *$ is then $F^3 \rightrightarrows F$, and is equivalent to the set F (with no non-trivial morphisms). Verify this! □

3.9 Remark. This lax 2-category is strictifiable because every morphism of groupoids can be functorially converted to a fibration.

[12] Confusingly enough, in the literature of open-closed TQFT, those boundary intervals are variously called *free* or *constrained.*

Assigning to a manifold the groupoid of principal F-bundles over it gives a 2-functor from $\mathscr{B}ord_2$ to $\mathscr{C}or_2$. We could think of it as a quantum field theory valued in $\mathscr{C}or_2$. The task is to continue by defining a 2-functor from $\mathscr{C}or_2$ to $\mathscr{A}lg$, which we do as follows.

(i) To each object in $\mathscr{C}or_2$ we assign the *path algebra* of the groupoid.

(ii) To a correspondence $Y_\bullet \leftarrow X_\bullet \rightarrow Z_\bullet$, we assign the sum, over objects in $Y_0 \times Z_0$, of the functions of the homotopy fiber of X there. This is a module over the path algebra of $Y_\bullet \times Z_\bullet$.

(iii) For a correspondence $C_\bullet$ of correspondences, we define a linear map between function spaces by the push-pull construction. (The matrix coefficient relating to functions counts points mapping to both, weighted down by automorphisms relative to the second map; this is already familiar from unextended finite gauge theory.)

There is a more conceptual description of (ii): if $f : X_\bullet \to Y_\bullet$ is a fibration of groupoids, then $f_*\mathbb{C}$, the direct image of the constant sheaf, is a flat vector bundle over $X_\bullet$ — a bundle with a composable lifting of the arrows. Its sections over X_0 therefore give a module for the path algebra of $Y_\bullet$. One should picture here a submersion $f : X \to Y$ of manifolds, for which the cohomologies $R^i f_*\mathbb{C}$ along the fibers are vector bundles with a canonical flat connection.

Example. Show that the path algebra of BF is $\mathbb{C}\langle F\rangle$. Starting from the map $* \to BF$, show that we produce the regular representation of F. For $G \subset F$ and the induced map $BG \to BF$, show that we get the *induced representation*, consisting of functions on F/G.

3.10 Remark (Twisted gauge theory). A class $[\tau] \in H^2(BF; \mathbb{C}^\times)$ defines a central extension of F by $\mathbb{C}^\times$. We think of a central extension as a line bundle over the group, with a compatible multiplication on lines: that is, isomorphisms $\alpha_{f,g} : L_f \otimes L_g \to L_{fg}$, satisfying the coherence expressed by commutativity of the square:

$$\begin{tikzcd} L_f \otimes L_g \otimes L_h \arrow[r, "\mathrm{Id}\otimes\alpha_{g,h}"] \arrow[d, "\alpha f{,}g\otimes\mathrm{Id}"] & L_f \otimes L_{gh} \arrow[d, "\alpha_{f,gh}"] \\ L_{fg} \otimes L_h \arrow[r, "\alpha_{fg,h}"'] & L_{fgh} \end{tikzcd}$$

Of course, all the lines can be identified with $\mathbb{C}$, and then $\alpha_{f,g}$ becomes a $\mathbb{C}^\times$-valued group co-cycle. Changing the isomorphisms $L_h \cong \mathbb{C}$ changes the co-cycle by a co-boundary.

The twisted group algebra ${}^\tau\mathbb{C}\langle F\rangle$ is defined as the space of sections of this line bundle, and it carries an obvious multiplication lifting the product of group elements. Its modules are the τ-projective representations of F. This time however, the opposite algebra is ${}^{-\tau}\mathbb{C}\langle F\rangle$. The twisted gauge theory is defined on oriented manifolds, but does not factor through the unoriented bordism category.

The reader is encouraged to construct the twisted gauge theory for oriented manifolds; the fundamental cycle of a surface relative to its boundary, as well as Stokes' theorem for cohomology in $\mathbb{C}^\times$, should make an appearance in checking compatibility of maps.

(3.11) Inadequacy of strict categories. We mentioned that the strict definition of higher categories, as easy as it seemed, is not appropriate. A precise problem can be identified when restricting to higher groupoids, namely categories where all morphisms are invertible. It is assumed that any sensible theory of higher groupoids is equivalent to the theory of homotopy types in topology; specifically, D-groupoids should correspond to D-types, topological spaces with vanishing homotopy groups above dimension D. (Declaring that homotopy of maps is an equivalence relations is akin to declaring all morphisms to be invertible in a higher category.) So, for any good definition of D-category, restriction to groupoids should produce homotopy D-types. However, the strict inductive definition is faulty beyond $D = 2$:

3.12 Proposition. *A connected homotopy type X can be represented by a strict groupoid if and only if its k-invariants beyond k_2 vanish. That happens if and only if X is the classifying space of a group which is the extension of $\pi_1(X)$ by a topological abelian group.*

In particular, if X is simply connected, it must be equivalent to the classifying space of a topological abelian group; that is, a product of Eilenberg-MacLane spaces.

Most simply connected homotopy types are not of that form: indeed, restricting to such spaces erases nearly all the interesting part of homotopy theory. For example, from the 2-sphere we discussed earlier, we can produce a homotopy 3-type by killing the homotopy groups above 3 and keeping just $\pi_2 = \pi_3 = \mathbb{Z}$. (This can be done by attaching cells of increasing dimensions.) But we have seen that the result differs from the product 3-type $\mathbb{CP}^\infty \times K(\mathbb{Z}; 3)$, having tested maps from $\mathbb{CP}^2$ into the two spaces.

The proof of the theorem imitates the proof of commutativity of π_2 of a topological space: the key lemma is that a space with two strictly commuting associative multiplications is in fact strictly commutative, and the two multiplications agree. In the world of strict 3-groupoids, this proves strict commutativity of the second loop space, and entails the vanishing of the Whitehead bracket. We briefly recall this story.

(3.13) The quadratic map $\pi_2 \to \pi_3$. We give an interpretation of the Postnikov k-invariant $k_3 \in H^4(K(\pi_2(X), \pi_3(X))$ of a space with π_2, π_3 only. The standard argument for commutativity of $\pi_2(X)$ and higher, for any space X, conceals higher operations on homotopy groups, the first of which is the *Whitehead bracket*, a collection of bilinear maps $\pi_m(X) \times \pi_n(X) \to \pi_{m+n-1}(X)$. These are best seen by using the space ΩX of based loops in X and the isomorphism $\pi_m(X) \cong \pi_{m-1}(\Omega X)$. Namely, the commutator map $\Omega X \times \Omega X \to \Omega X$ can be deformed so as to squash the subspace $\Omega X \times \{1\} \cup \{1\} \times \Omega X$ to the identity. Given $\alpha : S^{m-1} \to \Omega X$, $\beta : S^{n-1} \to \Omega X$ representing classes in $\pi_{m,n}(X)$, the composition

$$\alpha \times \beta : S^{m-1} \times S^{n-1} \to \Omega X \times \Omega X \xrightarrow{[,]} \Omega X$$

squashes down to factor through

$$S^{m-1} \times S^{n-1} / (S^{m-1} \times \{*\} \cup \{*\} \times S^{n-1}) \cong S^{m+n-2},$$

and this gives a class in $\pi_{m+n-2}(\Omega X) \cong \pi_{m+n-1}(X)$.

3.14 Remark. When $m = n = 1$, this becomes the commutator bracket on $\pi_1(X)$.

The bilinear Whitehead bracket $\pi_2 \times \pi_2 \to \pi_3$ has a quadratic refinement. This is because one can compute that the generator of $\pi_3(S^2)$, represented by the famous *Hopf fibration*, is *one-half* of the Whitehead square of $1 \in \pi_2(S^2)$. For any space X, one gets a quadratic map $\pi_2(X) \to \pi_3(X)$ by pre-composing any $\alpha : S^2 \to X$ with the Hopf map $S^3 \to S^2$. We have

3.15 Theorem. *A space X with only two non-trivial homotopy groups, $\pi_2(X)$ and $\pi_3(X)$, is completely determined up to homotopy by the two groups and by the quadratic map $\pi_2 \to \pi_3$.*

Homotopy theory tells us that the space is completely determined, up to homotopy, by the *Postnikov invariant* $k_3 \in H^4(K(\pi_2, 2); \pi_3)$. A result of MacLane asserts the correspondence of these classes with quadratic maps $\pi_2 \to \pi_3$; in one direction, the correspondence is induced by the construction we just described.

(3.16) A braided tensor category from X. Now let us produce a more rigid incorporation of k_3 in the form of a *braided tensor category.*

3.17 Definition. A *tensor category T* is a ($\mathbb{C}$-)linear category with a bi-linear multiplication functor $m : T \times T \to T$, containing a unit object 1 with $m(1, x) = x = m(x, 1)$, and an associator $\alpha_{x,y,z} : m(m(x, y), z) \to m(x, m(y, z))$ which allows us to 'move parentheses' in multiplication. The associator satisfies a coherence identity for four objects (Stasheff's *pentagon identity*).[13]

We have taken the identity to be strict, for simplicity. The pentagon identity is a precise way of stating that the choice of order of parenthesis moves is irrelevant. The product $m(x, y)$ is often denoted by $x \otimes y$.

Informally, a braided tensor category is a tensor category with a first order of commutativity. Call τ the transposition automorphism on the square $B \times B$ of a category.

[13] *MacLane's coherence theorem* ensures that we can replace T by an equivalent category in which $\alpha \equiv 1$; but this may require enlarging the category to an equivalent one with many more objects, or possibly breaking some additional structure, such as continuity.

3.18 Definition. A *braided tensor category* B is a tensor category equipped with a *bi*-multiplicative braiding isomorphism of functors $\beta : m \to m \circ \tau$. That means, for each pair x, y, we are given an isomorphism $\beta(x,y) : x \otimes y \to y \otimes x$, functorial in x, y. The category is *symmetric* if $\beta^2 = \mathrm{Id}$.

Bi-multiplicativity means that for any three objects x, y, z, the diagrams commute:

$$\begin{tikzcd} x\otimes y\otimes z \arrow[rr, "\beta(x,y\otimes z)"] \arrow[dr, "\beta(x,y)\otimes\mathrm{Id}_z"'] & & y\otimes z\otimes x \\ & y\otimes x\otimes z \arrow[ur, "\mathrm{Id}_y\otimes\beta(x,z)"'] & \end{tikzcd} \qquad \begin{tikzcd} x\otimes y\otimes z \arrow[rr, "\beta(x\otimes y,z)"] \arrow[dr, "\mathrm{Id}_x\otimes\beta(y,z)"'] & & z\otimes x\otimes y \\ & x\otimes z\otimes y \arrow[ur, "\beta(x,z)\otimes\mathrm{Id}_y"'] & \end{tikzcd}$$

We have omitted the associators α (set them to Id).

3.19 Remark. One way to encode the braided structure is by specifying that the tensor product of n objects in a braided category carries a natural action of the braid group on n strands, with obvious multiplicative compatibilities. In the symmetric case, the action factors through the symmetric group.

(3.20) Examples.

(i) The category of representations of a group, with the tensor product over $\mathbb{C}$, is braided and in fact symmetric.

(ii) The category of modules over a commutative algebra A, with $\otimes_A$, is also symmetric. If A is not commutative, this is generally not a tensor category.

(iii) It is more difficult to produce a non-symmetric braided category, it you have not seen one. A famous example comes from *quantum groups.* Below, I will give a topological example.

Given now X with two homotopy groups π_2, π_3, let B_X be the category of flat vector bundles on the second loop space $\Omega^2 X$, supported on finitely many components. The choice of base-point of X will not matter. We are thus choosing a representation of π_3 for each element of π_2. This category is even abelian and semi-simple, if the homotopy groups are finite.

Let $m : \Omega^2 X \times \Omega^2 X \to \Omega^2 X$ denote the multiplication, and define a tensor structure on B_X by $U \odot V := m_*(U \boxtimes V)$, the fiberwise homology along m.

3.21 Proposition. *The above defines a braided tensor structure on B_X. The braiding is symmetric if $k_3 = 0$.*

Proof. Associativity of $\odot$ should be clear; let us just indicate the braiding. The usual argument for the commutativity of π_2 gives a homotopy between the multiplication m and its transpose $m \circ \tau$, exploiting double loop space structure. Recall the construction: realize $\Omega^2 X$ as maps from the square to X, sending the boundary to the base-point; then two adjacent squares $\square\square$ embedded in a larger square, and representing the product in $\Omega^2 X$, can be continuously moved past each other by clockwise rotation. In this way, we link $\odot$ to $\odot \circ \tau$ by a one-parameter family of maps. The fiber-wise homology bundle along this family carries a flat connection, and following this along the interval gives the braiding isomorphism $\beta(U,V) : U \odot V \to V \odot U$. $\square$

(3.22) Example. Here is B_{S^2} spelt out; recall that $\pi_2(S^2) \cong \pi_3(S^2) = \mathbb{Z}$, and the quadratic map $\mathbb{Z} \to \mathbb{Z}$ is the square. Objects are $\mathbb{Z}$-graded modules over $\mathbb{C}[x^{\pm 1}]$; the grading reflects the points in $\pi_2(S^2)$ of the support of a bundle, while x acts as the generator of π_3. The multiplication is then given by tensoring over $\mathbb{C}[x^{\pm 1}]$. This is naturally symmetric monoidal, but we modify the obvious symmetric braiding as follows: for modules U, V supported in degrees m, n, the modified braiding $\beta(U,V) : U \odot V \to V \odot U$ is the action of x^{mn}.

3.23 Remark. The invariant k_3 is a quadratic refinement of the Whitehead bracket and contains finer information than the braiding. (It is thus possible for B_X to be symmetric even when $k_3 \neq 0$.) Specifically, k_3 gives a *ribbon structure* on B_X, a central automorphism θ_x for each object x satisfying

$$\theta_{x\otimes y} \circ (\theta_x^{-1} \otimes 1) \circ (1 \otimes \theta_y^{-1}) = \beta(y,x) \circ \beta(x,y).$$

("Central" refers to the entire category, and means that θ is an automorphism of the identity functor on B_X.) One interpretation of θ is that it trivializes the square of the braiding *as a braided tensor automorphism* of the category. In the context of TQFT, θ trivializes the *Serre functor*, which is a potential obstruction to defining a TQFT for *oriented* manifolds.

(3.24) Finite homotopy types. We conclude by indicating the construction of the enhanced gauge theory of §1.11 as an extended TQFT, following the example of gauge theory. Except for the language layered with morphisms of all orders, this does not differ from the extended gauge theory in §3.5.

Let X be a space with finite homotopy groups and fix the dimension $D > 0$ of the intendet TQFT. Homotopy groups above D will not play a role and will be truncated. We seek a (lax) functor from $\mathscr{B}ord_D$ to the D-category $\mathscr{C}at_{D-1}$ of lax linear $(D-1)$-categories in two steps:

(i) First we construct the functor $\mathscr{B}ord_D \to \mathscr{C}or_D$ to the D-category of correspondences of topological spaces with finitely many, finite homotopy groups. This is analogous to the classical field theory, over which physicists perform the path integral to quantize.

(ii) Next, we construct the quantization functor $Q : \mathscr{C}or_D \to \mathscr{C}at_{D-1}$

The first part is unambiguous and every manifold M, representing a morphism of whatever layer, gets sent to the space of maps $\mathrm{Map}(M; X)$.

Now, since we did not commit to a definition of lax higher categories, we cannot actually perform the second step: the details of the construction are model-dependent. One construction, using the notion of *m-algebras* (algebras with layers of compatible multiplications) was indicated in [FHLT]. Another construction can be given in terms of the "Blob complex" of Morrison and Walker [MW]. In both cases, adding the top layer of the theory (linear maps beteween vector spaces, or numbers) is the more delicate step, and uses the finiteness of X. However, let me indicate here the idea common to both: this is to construct the m-linearization functor of a space, which is a linear m-category, in a way that takes homotopy fiber products to tensor products. (A zero-category is a vector space; a linear (-1)-category would be a complex number, I guess.)

The 0-categorification of Y is the vector space of locally constant functions on Y.

The 1-categorification is the category of vector bundles with flat connection.

The 2-categorification is the 2-category of linear category bundles with flat connection (locally constant sheaves of linear categories)

And so forth.

The key observation is that one can define the *cohomology of a space with coefficients in a locally constant sheaf of m-categories* and obtain an m-category. This allows us to define 'wrong-way maps' for bundles of higer categoreis, and thus convert correspondences into functors. The finiteness conditions on X are needed to stay within the world of *dualizable objects* (next lecture), which are needed for the consistency of the TQFT.

The braided tensor category of §3.16 is a model for the 3-categorification of the space X; we will see in the last lecture why a braided tensor category is an object of 3-categorical nature: it has a "3-category of module objects".

4. The cobordism hypothesis in dimensions 1 and 2

The "cobordism hypothesis" is the classification of fully extended D-dimensional TQFTs with values in an arbitrary symmetric monoidal D-category. It was formulated (not completely precisely) by Baez and Dolan [BD]: roughly speaking, such a theory should be completely determined by an object in the D-category, which is assigned to a point; the object must satisfy certain *conditions* and carries additional *structure*. Both of these were made precise, and the conjecture was proved by Lurie [L]. The key insight behind his formulation (distinguishing it from other variants, such as [MW]) was to separate the *conditions* from the *structure*.

This is accomplished by passing to the *framed bordism D-category $\mathscr{B}ord_D^{fr}$*, to be defined below. In this case, the object $Z(*)$ assigned to the point is subject to the *full dualizability* condition, which generalizes the finite-dimensionality of vector spaces in the case $D = 1$. There is no extra structure;

the latter only appears when attempting to factor a framed theory via manifolds with other structure (orientation, spin, or any topologically defined structure on the tangent bundle). Again, speaking loosely, to extend a framed theory as described above from framed manifolds to a category of manifolds whose tangent bundle reduces topologically, from the orthogonal group, to a group $G \to \mathrm{O}(D)$, requires the object $Z(*)$ to be made G-equivariant. More precisely, it requires exhibiting $Z(*)$ as a G-fixed object within the space of all dualizable objects in the target D-category, with respect to the canonical $\mathrm{O}(D)$ action on the space of sub objects.

Let us start with the start of a definition, which conceals incompleteness and many fine points; we will unravel them gradually, beginning in the remark that follows.

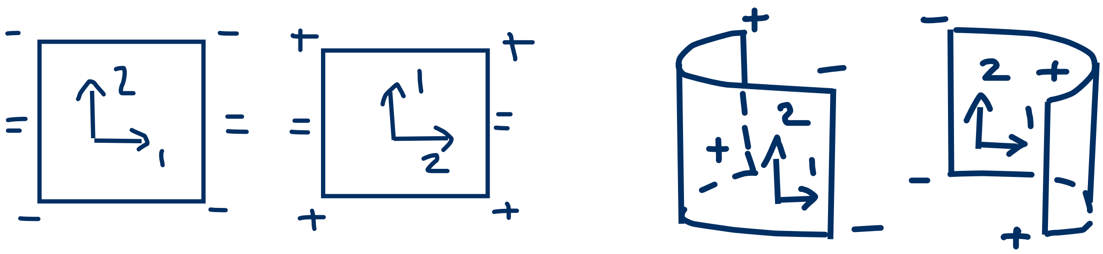
4.1 Definition. A *framing* of an n-dimensional vector bundle on a manifold M is an isomorphism with the trivial vector bundle $M \times \mathbb{R}^n$. A *D-framing* on a smooth k-manifold M (where $k \leq D$) is a framing of the bundle $\mathbb{R}^{D-k} \oplus TM$.

The D-category $\mathscr{B}ord_D^{fr}$ has D-framed points as objects, and its k-morphisms are the compact D-framed manifolds with corners. For $k = D$, we take D-manifolds, modulo boundary-fixing diffeomorphisms and homotopy of the framings.

4.2 Remark. (i) It is important that the extra $\mathbb{R}$'s in the framing *need not* align with the first summands of $\mathbb{R}^D$; otherwise we would merely be defining an ordinary framing of TM. A D-framing is similar to, but slightly more restrictive than, a *stable framing* (which is the notion obtained by letting D arbitrarily large for fixed M). A D-framing becomes stable when $\dim M < D - 1$, because of the stabilization of the homotopy groups of SO.

(ii) On a $(k-1)$-dimensional component B of the boundary of M, $TM|_B$ has an $\mathbb{R}^1$ summand which can be framed by the inward or outward normal to B in M. This choice gives an isomorphism $\mathbb{R}^{D-k} \oplus TM|_B \cong \mathbb{R}^{D-k+1} \oplus TB$. The direction on $\mathbb{R}^1$ is chosen according to whether we read B as the source or the target of the morphism M, and a D-framing of M then restricts to one of B. Armed with a reading of M as a morphism between morphisms (between morphisms ...), we can consistently follow this convention to all co-dimensions to build the D-category of D-framed manifolds.

(iii) A D-framed manifold with corners can be read as a morphism in many ways: its boundaries of various co-dimension must be labelled as sources or targets of morphisms. For example, here are four readings of a square, as the identity 2-morphism of various 1-morphisms:



The 1-morphisms go right-to-left and the 2-morphisms go down. The framing is indicated on the surface, and the sign of the corner points is forced by our orientation convention: the TM on a corner has the basis given by ($2-$*morphism*,$1-$*morphism*) and this is compared with the surface framing.[14] There are many more possibilities, as there is no need for the 2-framing to be constant along the horizontal direction; in fact, a clockwise twist of the framing will play a distinguished role (the Serre automorphism).

(iv) There are (for$D > 0$) exactly two D-framed points $+, -$ in $\mathscr{B}ord_D^{fr}$, up to isomorphism, distinguished by determinant sign of the framing $\mathbb{R}^D \cong \mathbb{R}^D \oplus T(\text{point})$. The isomorphism is realized by any homotopy from one framing to another in the same orientation class, viewed as a D-framed interval (a 1-morphism in $\mathscr{B}ord_D^{fr}$). The opposite homotopy provide the inverse isomorphism (up to invertible 2-morphisms, etc.). The automorphism group of each framed point is $\Omega\mathrm{SO}(D)$.

[14]The vertical lines are the self-identifications of the corner objects.

4.3 Remark. Modding out by diffeomorphisms in the top dimension will look wrong to the experienced among you: this is because diffeomorphism groups have a topology, which is lost in the naive definition. For example, one could never capture the structure of Gromov-Witten theory, which relies on the topology of the $\overline{M}_g^n$, with such a definition. Instead, each top-dimensional manifold should be replaced by a space of morphisms with homotopy type the respective $BDiff$.[15] The preferred way to accomplish the same involves the notion of (∞, D)-category, which has interesting k-morphisms for all values k, but all of them are required to be isomorphisms above D. The homotopy type of $BDiff$ is then captured in the higher automorphism groups of D-morphisms.

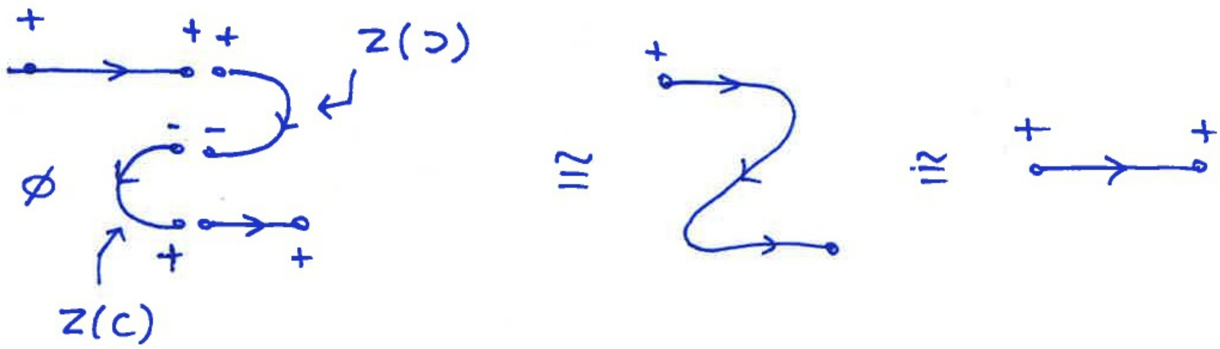
(4.4) Duals and 1D framed TQFTs. Framings and orientations agree for $\mathbb{R}$-bundles, up to a contractible space of choices. Oriented theories, you recall, comprise the vector spaces $Z(+), Z(-)$ assigned to the two oriented points, and the maps

$$\begin{aligned} Z(\subset) &: \mathbb{C} \to Z(-) \otimes Z(+), \\ Z(\supset) &: Z(+) \otimes Z(-) \to \mathbb{C}. \end{aligned} \tag{4.5}$$

These are subject to the conditions that the compositions

$$Z(+) \xrightarrow{\mathrm{Id}\otimes Z(\subset)} Z(+) \otimes Z(-) \otimes Z(+) \xrightarrow{Z(\supset)\otimes \mathrm{Id}} Z(+),$$
$$Z(-) \xrightarrow{Z(\subset)\otimes \mathrm{Id}} Z(-) \otimes Z(+) \otimes Z(-) \xrightarrow{\mathrm{Id}\otimes Z(\supset)} Z(-)$$

are the respective identities. The conditions are seen geometrically by gluing the two half-circles at the respective end: we obtain an 'S' or 'Z' which are diffeomorphic to the identity intervals:



4.6 Definition (Internal duals in a monoidal category)**.** Given a pair of objects and morphisms as in (4.5) in a monoidal category (with the unit object replacing $\mathbb{C}$), $Z(-)$ will be called *right dual* to $Z(+)$, and $Z(+)$ *left dual* to $Z(-)$. $Z(\subset)$ and $Z(\supset)$ are called the *unit* and *trace* of the adjunction.

4.7 Remark. (i) The left dual of a given object, if it exists, is determined up to canonical isomorphism, along with the unit/trace.

(ii) Strenghening the above, a morphism between dualizable objects has an *adjoint morphism* relating the duals in the opposite direction.

(iii) Invertible objects are always dualizable, and their inverses can be taken as their duals.

(iv) In a *symmetric* monoidal category, left and right duality are the same condition; we can then denote the dual of x by $x^\vee$. But we will still run into the left/right distinction later, when considering morphisms in a higher category.

(v) Dualizable vector spaces are precisely the finite-dimensional ones. Finitely generated projective modules over a commutative ring are dualizable, but these are not the only dualizables.

(vi) (Hom definition) An equivalent definition of adjunction is the existence of a bi-functorial isomorphism $\mathrm{Hom}(Z(+) \otimes x, y) \cong \mathrm{Hom}(x, Z(-) \otimes y)$.

[15] Plus the space of framings relative to the boundary.

(4.8) Cobordism hypothesis in 1*D.* Call an object of a symmetric monoidal category *dualizable* if it does have a dual. We have just seen that *functors* $Z : \mathscr{B}ord_1^{fr} \to \mathscr{C}$ *to some symmetric monoidal category are classified, up to isomorphism, by* $Z(+)$ *which is a dualizable object in* $\mathscr{C}$.

It is a worthy exercise to show that any natural transformation of functors $Z \to Z'$ is an *isomorphism*, in particular comes from an isomorphism $f : Z(+) \to Z'(+)$. (Show that the companion morphism $g : Z(-) \to Z'(-)$ must be inverse to the adjoint morphism $f^\vee$; therefore, ${}^\vee g$ will be an inverse of f.)

(4.9) O(1) *action on dualizable objects.* The orthogonal group $\mathrm{O}(D)$ acts on $\mathscr{B}ord_D^{fr}$ by changing the framings.[16] It therefore acts on framed TQFTs, and therefore on dualizable objects in a symmetric tensor category. The action on objects is obvious: it sends each dualizable object to its dual.

Now, when does the theory factor through unoriented manifolds? In that case, $+ = -$, so we may take $Z(+) = Z(-)$. Then, $Z(\supset)$ is a non-degenerate bilinear form on $Z(+)$. The absence of an orientation on $\supset$ forces this to be symmetric, so the theory is determined by a vector space with a non-degenerate quadratic form.

Here is the intrinsic description. The group $\mathbb{Z}/2$ acts on the *groupoid* $G_{>0}Vect$ of finite-dimensional complex vector spaces and *isomorphisms*, sending vector spaces to their duals and linear isomorphisms to their *inverse* duals. This is the action of O(1) on dualizable objects in $Vect$; we see the need to truncate the category to its maximal sub-groupoid. Let us state the following, before explaining it:

4.10 Proposition. *The fixed-point category for this* O(1)*-action is the groupoid of vector spaces with non-degenerate symmetric bilinear form. In particular, unoriented TQFTs are classified by* O(1)*- fixed points among the dualizable objects in* $Vect$. □

First, we say that a (discrete) group G acts on a category $\mathscr{C}$ if, for each element $g \in G$, we are given an autofunctor $F_g : \mathscr{C} \to \mathscr{C}$, and for all g, h we are given an isomorphism of functors $\alpha_{g,h} : F_g \circ F_h \xrightarrow{\sim} F_{gh}$, such that the α's in a triple g, h, k satisfy an obvious coherence relation, so that we can construct a unique isomorphism $F_g \circ F_h \circ F_k \cong F_{ghk}$. It is also convenient to assume that $F_1 = \mathrm{Id}$.

The *fixed-point category* $\mathscr{C}^G$ is, by definition, the one whose objects are tuples $(x, \varphi_{x,g}\,|_{g\in G})$, where the $\varphi_{x,g} \in \mathrm{Isom}(x, F_g(x))$ satisfy an (obvious) coherence constraint with respect to composition, which (ignoring the α's) writes out as $\varphi_{x,gh} = F_g(\varphi_{x,h}) \circ \varphi_{x,g}$. Morphisms in $\mathscr{C}^G$ come from $\mathscr{C}$, but are required to be compatible with the φ.

It is now a good exercise to prove the proposition: there is only one $\varphi : V \to V^\vee$ to specify, for the non-trivial element g of O(1). The condition $g^2 = 1$ implies $(\varphi^\vee)^{-1} \circ \varphi = \mathrm{Id}_V$, the symmetry of φ.

(4.11) Cobordism hypothesis in 2*D.* We need the notion of *adjoints*, the analogue of duals, for morphisms in a higher category. Fortunately, all higher definitions beyond this will be the same. The key example pertains to functors between categories: we say that $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$ form an adjoint pair, with F left and G right adjoint, if we are supplied with a bi-functorial isomorphism $Hom_{\mathscr{C}}(x, Gy) = Hom_{\mathscr{D}}(Fx, y)$. Clearly, the right adjoint of F is unique up to canonical isomorphism, if it exists; we write $G = F^\vee$. Similarly, the left adjoint F is uniquely (up to isomorphism) determined by G, and we write $F = {}^\vee G$.

Taking $y = Fx$ supplies a *unit morphism* $\varepsilon : \mathrm{Id}_{\mathscr{C}} \to G \circ F$ and taking $x = Gy$ gives a *trace* $\theta : F \circ G \to \mathrm{Id}_{\mathscr{D}}$. With the help of ε and θ, we can state the adjunction condition without involving objects in $\mathscr{C}$ and $\mathscr{D}$.

4.12 Definition. A morphism $F : c \to d$ in a 2-category is *left adjoint to* $G : d \to c$ if there exist a *unit* 2*-morphism* $\varepsilon : \mathrm{Id}_c \to G \circ F$ and a *trace* or *evaluation* $\theta : F \circ G \to \mathrm{Id}_d$, such that the following compositions are the respective identity 2-morphisms:

$$F = F \circ \mathrm{Id}_c \xrightarrow{\mathrm{Id}_F \otimes \varepsilon} F \circ G \circ F \xrightarrow{\theta \otimes \mathrm{Id}_F} \mathrm{Id}_d \circ F = F,$$

$$\mathrm{Id}_c \circ G = G \xrightarrow{\varepsilon \otimes \mathrm{Id}_G} G \circ F \circ G \xrightarrow{\mathrm{Id}_G \otimes \theta} G \circ \mathrm{Id}_d = G$$

[16] This is a *homotopical* action: that is, it factors through the *homotopy type* of the group and is not sensitive to the Lie structure; see Remark 4.19.

G is then said to be *right adjoint to F*.

A morphism is *dualizable* if it has right and left adjoints, which in turn have right and left adjoints, ad infinitum.

4.13 Remark. (i) The nonsensical but concise formula "$\theta \circ \varepsilon = 1$" can be made meaningful exactly in the two ways above.

(ii) We can revert to a Hom-definition of adjunction using the induced functors

$$F_* : \mathrm{Hom}_{\mathscr{C}}(\bullet, x) \rightleftarrows \mathrm{Hom}_{\mathscr{C}}(\bullet, y) : G_*,$$

between category-valued functors on $\mathscr{C}$, which we require to be adjoint (functorially in $\bullet$).

(iii) There appear to be infinitely many conditions contained in the definition of dualizability. However, in the application to TQFTs, the conditions will be finite in number. This is because the left and right adjoints must differ by an invertible functor (the *Serre automorphism* in dimension 2), and the checking can stop once we find the functor and check its invertibility: the existence of all further adjoints is guaranteed.

(iv) Note that if ε, θ are isomorphisms, then so are F and G, and then they are mutually inverse.

4.14 Definition. For any 2-category $\mathscr{C}$, let $G_{>1}\mathscr{C}$ ("groupoid above 1-morphisms") be the 2-category retaining all objects and morphisms, but only the *invertible* 2-morphisms.[17] There is a similar definition $G_{>k}\mathscr{C}$ for any D-category with $D > k$.

4.15 Definition. An object x of a symmetric monoidal $(\infty, 2)$-category is *fully dualizable* if:

- It is dualizable in $G_{>1}\mathscr{C}$;
- Its unit and trace for duality are dualizable.

4.16 Remark. We are not asking for futher dualities, because only isomorphisms are present above 2. Insisting on more duality forces everything, including x, to be invertible. (Develop 4.13.iv.)

4.17 Theorem (Classification of framed TQFTs)**.** *A 2D framed, extended TQFT Z with values in a 2-category $\mathscr{C}$ (or $(\infty, 2)$-category) is determined by $x = Z(+)$, which can be any fully dualizable object.*

Natural transformations between two theories Z_x and Z_y form a (2-)groupoid (or ∞-groupoid) which is equivalent to $Hom(x, y)$ in the category $G_{>0}\mathscr{C}$ (obtained from $\mathscr{C}$ by retaining only the invertible 1- and 2-morphisms).

(4.18) The Serre twist. Let x be a fully dualizable object with dual $x^\vee$, and call u, ev the unit, resp. trace of that duality. Let also $S : Z(+) \to Z(+)$ be the *Serre automorphism*, induced by a clockwise twist in the framing along an interval. We will see that, in $\mathscr{B}ord_2^{fr}$,

- $(S \boxtimes \mathrm{Id}_-) \circ u$ is the right adjoint of ev;
- $(S^{-1} \boxtimes \mathrm{Id}_-) \circ u$ is the left adjoint of ev;

The other adjoints are determined from here by algebra, since $S^{-1} = S^\vee$ (left and right adjoint). In particular, this proves half of the cobordism hypothesis, namely that $Z(+)$ must be fully dualizable. The other, more difficult half, is the statement that $\mathscr{B}ord^{fr}$ is the *free* 2-category generated by one fully dualizable object, in other words, there are no extra relations.

4.19 Remark. An equivalent description of S starts from the observation that $Aut(+) \sim \Omega\mathrm{SO}(2) \sim \mathbb{Z}$, and S is the negative generator. The $\mathbb{Z}$-action on a fully dualizable object captures the topological action of $\mathrm{SO}(2)$ on *the space of all fully dualizables*: because $\mathrm{SO}(2)$ is connected, $gx \cong x$ for any fully dualizable x; a choice of isomorphism arises from any choice of path from 1 to g, and is not sensitive to deformation of the path. However, attempting to straighten out these isomorphisms coherently to trivialize the action on the orbit of x is obstructed precisely by the automorphism S, arising from the non-trivial loop in $\mathrm{SO}(2)$.

[17] If we start with an $(\infty, 2)$-category, we get an $(\infty, 1)$-category this way.

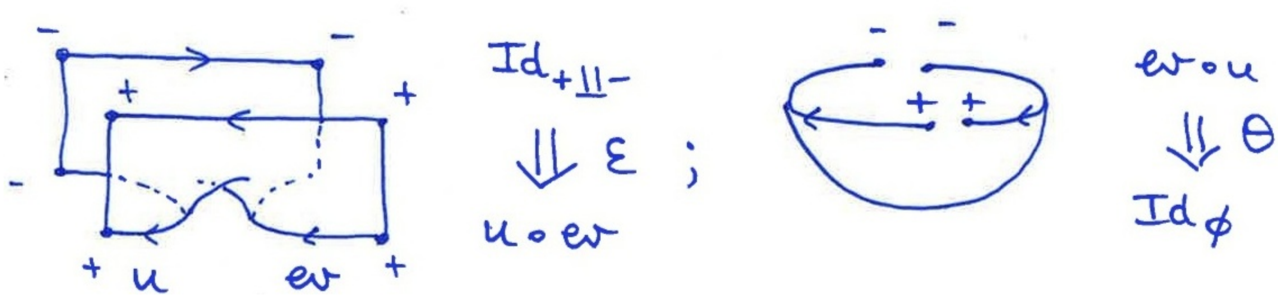
(4.20) Oriented and r-spin theories. armed with the Serre automorphism, we will now state the classification of oriented and Spin TQFTs. More precisely, we seek the condition under which a framed theory factors through the oriented bordism category. In fact, for each r, we can enquire about TQFTs for surfaces and circles with *r-Spin structure*, with is a chosen rth root of the tangent bundle (assumed to be oriented). For $r = 1$, we have oriented surfaces, and for $r = 2$, traditional Spin structures.

4.21 Theorem. *Factorizations of a framed theory $Z : \mathscr{B}ord_2^{fr} \to \mathscr{C}$ through the r-Spin category $\mathscr{B}ord_2^{r-\mathrm{Spin}}$ correspond to isomorphisms between S^r and the identity automorphism of $Z(+)$.*

(4.22) Example: semi-simple TQFTs. For $\mathbb{C}$-linear TQFTs, taking values in the 2-category of linear categories, a given trivialization of the Serre functor can be rescaled by any $\lambda \in \mathbb{C}^\times$. If so, a computation shows that the invariant $Z(\Sigma_g)$ for a closed genus g surface gets rescaled by λ^{2-2g}. More precisely, the Frobenius trace θ on $Z(S^1)$ rescales by λ^2, while the vector in $Z(S^1)$ defined by the torus with one outgoing boundary circle (sometimes called the *Euler class*) rescales by λ^{-2}. Recall now that theories defined from *semi-simple* Frobenius algebras led to the closed surface invariant $Z(\Sigma_g) = \sum \theta_i^{1-g}$. Such theories can be interpreted as sums of 1-dimensional extended TQFTs, each generated by the algebra $\mathbb{C}$ with Serre functor $\theta_i^{1/2}$. In other words, the closed surface invariants are entirely traceable to a choice of Serre automorphism, which we can scale independently for each summand. However, extending the theory down to dimension zero requires us to choose *square roots* of the Frobenius structure constants θ_i.

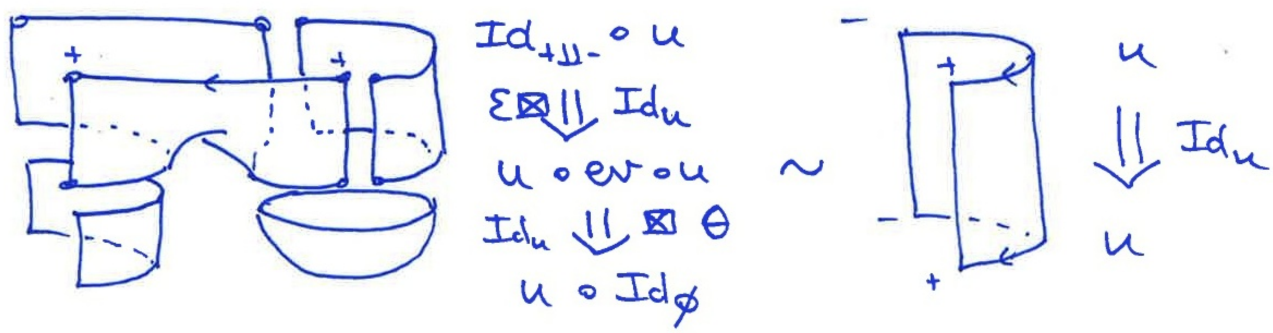
(4.23) Adjunction in pictures: oriented handles. We now spell out the 2-duality data and relations geometrically, in terms of standard handles and handle cancellations. We have already done this for $D = 1$, when converting the 'Z' to the interval, but it may have been too obvious to notice.

There are fewer and simpler pictures if we assume that $S = \mathrm{Id}_+$, when the TQFT factors through oriented surfaces. In that case, u and ev are each other's left and right adjoints, and the unit ε and trace θ for adjunction are the standard handles in the topology of surfaces:

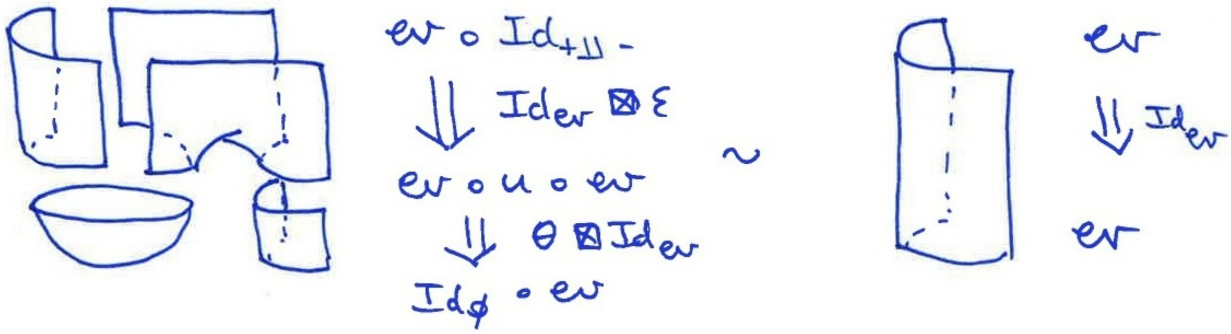


The (horizontal) 1-morphisms are read right-to-left, and the surface 2-morphisms are read downwards. The arrows indicate the orientations of the 1-morphism intervals; the orientation of the surface is deduced form the rule that the 2-morphism direction comes first in the framing, at the boundary.

The identities '$\theta \circ \varepsilon = 1$" are the standard handle cancellation relations familiar form Morse theory; here are the cancellations of the 1-handle (saddle) by a 2-handle:



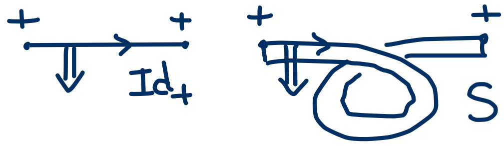
The relation $(\mathrm{Id}_u \boxtimes \theta) \circ (\varepsilon \boxtimes \mathrm{Id}_u) = \mathrm{Id}_u$



The relation $(\theta \boxtimes \mathrm{Id}_{ev}) \cdot (\mathrm{Id}_{ev} \boxtimes \varepsilon) = \mathrm{Id}_{ev}$

The opposite adjunction (u, ev) involves the same pictures, but read from bottom to top (the labels ε and θ are also swapped). This time, ε is a 0-handle, and θ is a cancelling 1-handle.

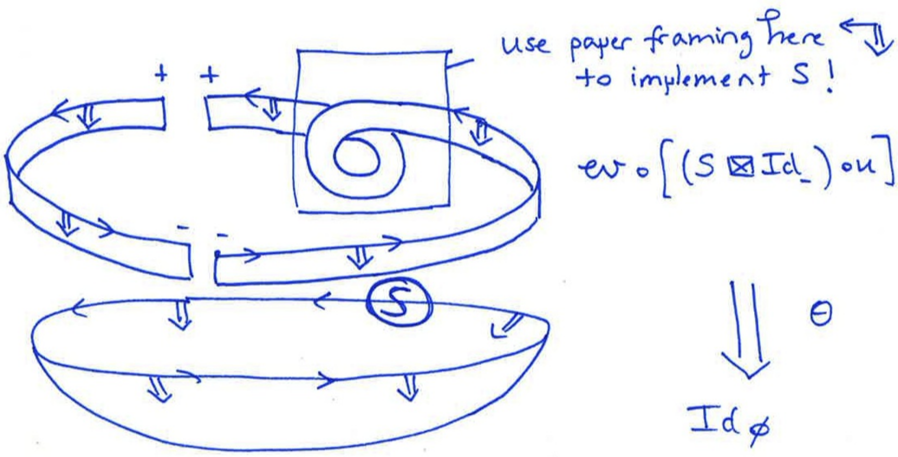
(4.24) Framed handles. A few attempts should convince you that it is not possible to place 2-framings on the 1-dimensional handles $\supset$ and $\subset$ to make *both* adjunctions (ev, u) and (u, ev) work. In fact, the most natural framings to choose in defining u and ev — the product (tangent, normal) on a narrow strip — do not permit either adjunction: this is because the radial product 2-framing at the boundary circle of a disk does not extend over the interior. Instead, we must use the Serre functor S defined by a framing twist:

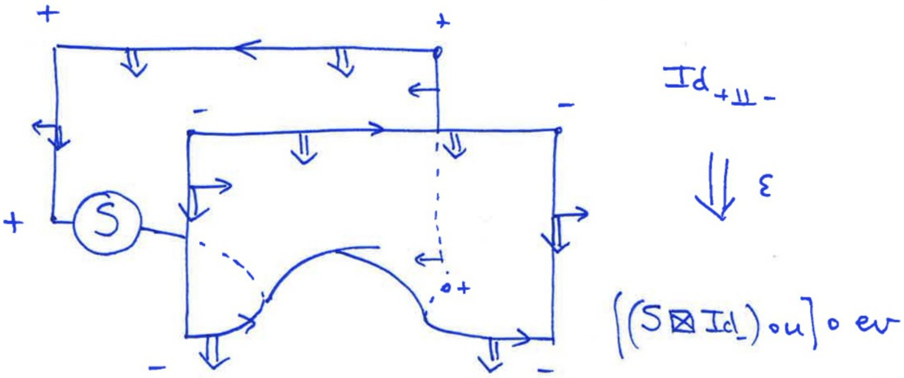


Serre twist on + given by page framing

Here, we read 1-morphisms left-to right and 2-morphisms down (so that the product framing on Id_+ matches the page framing). On the ribbon drawn, the same page framing, a positive twist compared to the product framing, defines the functor S.

Now we can extend the boundary framings to the disk and saddle to exhibit the adjunction pair $(ev, (S \boxtimes \mathrm{Id}_-) \circ u)$:





The same handle cancellation identities apply to give $\theta \circ \varepsilon = 1$. The other adjunction, $((S^{-1} \boxtimes \mathrm{Id}_-) \circ u, ev)$, is obtained by reading the pictures up. However, we must also flip the sign of the vertical copy of $\mathbb{R}$ on the boundaries to match our source-target conventions, and that switches the framing twist in the picture.

(4.25) Adjunction: algebraic conditions. To spell out the adjunction conditions, we specialize to the case of the 2-category $\mathscr{A}lg$ of algebras, bimodules and intertwiners. Here, $\mathrm{Hom}(A, B)$ is the category of $B - A$ bimodules, and composition of morphisms

$$\mathrm{Hom}(A, B) \otimes \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$$

is given by the tensor product of bi-modules over B.[18] Taking this example literally is too restrictive: one can prove that a fully dualizable algebra must be semi-simple. (More generally, a compactly generated abelian category is fully dualizable within linear categories iff it it semi-simple, with finitely many simple isomorphism classes.) This is still instructive, as we will identify the automorphism S canonically and relate a trivialization of S to a *Frobenius* or *Calabi-Yau* structure on the algebra. However, to get more interesting examples of TQFTs, one must use differential graded algebras and their derived categories of modules; we review below the example of coherent sheaves on a projective manifold.

Every $A \in \mathscr{A}lg$ is dualizable, with (left and right) dual the opposite algebra A°. Indeed, let $A^e := A \otimes A^\circ$; we can then take for u and ev the space A, viewed as a $A^e - \mathbb{C}$, respectively a $\mathbb{C} - A^e$ bimodule. From this, we compute $Z_A(S^1) = A \otimes_{A^e} A$, known as the *zeroth Hochschild homology* of A.

The Hom definition is more convenient for computing adjoints. The left adjoint ${}^\vee u$ of u is a $\mathbb{C} - A^e$ bimodule satisfying

$$\mathrm{Hom}_{A^e}(M, A \otimes_{\mathbb{C}} N) = \mathrm{Hom}_{\mathbb{C}}({}^\vee u \otimes_{A^e} M, N) \tag{4.26}$$

for any A^e-module M and $\mathbb{C}$-vector space N. (Actually, we should ask that this should hold for any algebra B, and a right B-module structure on N; but the latter will come for free.) Taking first $M = A^e$ and $N = \mathbb{C}$, but then enforcing it to any N, leads to the conditions

$$A = \mathrm{Hom}_{\mathbb{C}}({}^\vee u, \mathbb{C}), \quad \dim_{\mathbb{C}} {}^\vee u < \infty;$$

so that $\dim A < \infty$ and ${}^\vee u = A^\vee := \mathrm{Hom}(A, \mathbb{C})$, the dual vector space to A. In addition, to get (4.26) for all M requires left exactness of $M \mapsto A^\vee \otimes_{A^e} M$, that is, A^e-flatness of $A^\vee$.
Exercise: Show that, as a consequence, the category of finite-dimensional A-modules is semi-simple, and therefore A is semi-simple. (Deduce the exactness of Hom in the category of finite-dimensional A-modules.)

To continue, since $ev^\vee = (S \boxtimes \mathrm{Id}) \circ u$, we must have $ev = {}^\vee u \circ (S^{-1} \boxtimes \mathrm{Id})$, and we conclude that S is canonically the $A - A$ bimodule $A^\vee$. For the left adjoint ${}^\vee ev$,

$$\mathrm{Hom}_{A^e}({}^\vee ev \otimes_{\mathbb{C}} M, N) = \mathrm{Hom}_{\mathbb{C}}(M, A \otimes_{A^e} N) \tag{4.27}$$

[18] The conversion to the 2-category of linear categories, or differential-grades categories, is not difficult. The delicate step involves the definition of the monoidal structure, the tensor product of categories; see [G] for of abelian categories. In general, the categories of modules over algebras A and B tensor together to the category of $A \otimes B$-modules.

from which, when taking $M = \mathbb{C}, N = A^e$, we obtain $A = \mathrm{Hom}_{A^e}({}^\vee ev, A^e)$. Moreover, to recover the isomorphism (4.27) in general, we need the flatness and finite presentation of A over A^e. Existence of the remaining adjoints offer no further information or constraints:

$$u^\vee = \mathrm{Hom}_{A^e}(A, A^e) = {}^\vee ev, \qquad ev^\vee = A^\vee = {}^\vee u.$$

4.28 Remark. (i) It is not an accident that S is the *vector space* dual of A; see the next section.

(ii) Working with a differential graded algebra instead and using quasi-isomorphisms in lieu of equalities, finite-dimensionality of A becomes finiteness of the homology of A, and is usually called *compactness*; while finiteness of A over A^e is *smoothness*. This is because they match the respective properties in the case of complex varieties. With that language, fully dualizable dga's are the compact, smooth ones; this may have been first flagged in the work of Kontsevich and Soibelman [KS].

(4.29) Oriented TQFTs from Frobenius algebras. A trivialization of the Serre functor is an A^e-module isomorphism $A \cong A^\vee$. The image of 1 under this identification defines a linear map $t : a \to \mathbb{C}$, which induces a symmetric non-degenerate trace $a \times b \mapsto t(ab) = t(ba)$. This makes A into a *non-commutative Frobenius algebra.* Using t, we can identify the dual space to

$$Z_A(S^1) = ev \circ u = \mathrm{Hom}(A \otimes_{A^e} A; \mathbb{C})$$

with $Z(A) = \mathrm{Hom}_{A^e}(A, A^\vee) \cong \mathrm{Hom}_{A^e}(A, A)$, the center of A. This is also a semi-simple commutative algebra, which acquires a Frobenius structure from the co-unit $\theta : Z_A(S^1) \to \mathbb{C}$ of adjunction. The inclusion $Z(A) \to A$ is an algebra homomorphism, while the projection $A \to Z_A(S^1$ can be used to factor the trace t on A via θ. These examples were discussed in great detail by Moore and Segal [MS], forming an inspiration for much subsequent work [C, KS].

In conclusion, 2-dimensional, extended, oriented TQFTs correspond precisely to semi-simple, but not necessarily commutative, Frobenius algebras.

(4.30) Finite gauge theory revisited. Following [MS], let A be the group ring $\mathbb{C}\langle F\rangle$ of a finite group; we choose the trace $t : \mathbb{C}\langle F\rangle \to \mathbb{C}$ to pick out the coefficient of 1. Explicitly, the pairing is then

$$t(\varphi \cdot \psi) = \sum\nolimits_{f \in F} \varphi(f)\psi(f^{-1}).$$

I claim that Z is the *F-gauge theory for surfaces with corners* (which in fact is an an unoriented TQFT). The theory is obvious on 1-manifolds: send the interval to $A = \mathbb{C}\langle F\rangle$ and the circle to $Z(A)$, the class functions on F. It is again defined on surfaces by 'counting F-bundles, weighted down by automorphisms'; the difference from the $(1, 2)$-theory of Lecture 1 is that bundles are now *trivialized* at the corners (and on the identity segments bounding a 2-morphism).
Exercise: Make the definition above precise, and check that it gives a functor from $\mathscr{B}ord_2^{\mathrm{O}(2)}$ to $\mathscr{A}lg$.

4.31 Remark. The trace t has a natural extension to the entire category $\mathrm{Rep}(F)$ of F-representations: define $t_V = (\#F)^{-1} \cdot \mathrm{Tr}_V : \mathrm{End}^G(V) \to \mathbb{C}$, for any object V in the representation category. The *trace* property is $t(\varphi \circ \psi) = t(\psi \circ \varphi)$, for any pair of morphisms with opposite sources and targets. This defines a non-degenerate pairing between $\mathrm{Hom}^G(V, W)$ and $\mathrm{Hom}^G(W, V)$ for all V, W, and provides an example of a *Calabi-Yau structure* on $\mathrm{Rep}(F)$: a trivialization of the Serre functor on the category, which we examine next in the context of varieties. It also furnishes an example of *open-closed theory*, with $\mathrm{Rep}(F)$ as *category of branes.*

(4.32) The Serre functor on a scheme. In studying derived categories of coherent sheaves on a projective manifold X, Bondal and Orlov flagged the role of the *Serre automorphism*, characterized in any linear category $\mathscr{C}$ by the condition that we should have a bi-functorial isomorphism

$$\mathrm{Hom}(x, S(y)) = \mathrm{Hom}(y, x)^\vee.$$

If it exists, $S : \mathscr{C} \to \mathscr{C}$ is unique up to canonical isomorphism. If S is invertible, it forces the Hom spaces to be isomorphic to their double duals, hence finite-dimensional; this is a strong condition. If we also assume *compact generation* of $\mathscr{C}$ (a milder finiteness condition), then it can be shown that $\mathscr{C}$ is a 2-dualizable object in the 2-category of linear categories, so defines a framed 2-dimensional TQFT. Moreover, the Serre functor defined above agrees with the one defined by the framing twist.

In the case of the derived category, taking for Hom the zeroth group $\mathrm{Ext}^0(\mathscr{F}^\bullet, \mathscr{G}^\bullet)$ between complexes of sheaves, the classical duality theorem of Serre identifies S with the functor of tensoring with $K_X[\dim X]$, the canonical bundle shifted into degree $(-\dim X)$. The following is a remarkable application of the cobordism hypothesis; the Calabi-Yau case had been proved, in slightly different variant, by Costello [C] and Kontsevich.

4.33 Theorem. *The derived category $D^bCoh(X)$ of (bounded complexes of) coherent sheaves on a projective manifold X is fully dualizable. The Serre functor is the clockwise framing twist. Defining an* oriented *TQFT requires a* Calabi-Yau *structure on X: a trivialization of the canonical bundle.*

This is a vast generalization of the semi-simple case, obtained when X is a finite set.

(4.34) Vector spaces associated to the circle. Let us understand $Z(S^1)$ in the case $\mathscr{C} = Coh(X)$. When $D = 1$, the circle computes a number, the dimension of the vector space. So we are generalizing the notion of dimension to linear categories, and obtaining a vector space for an answer.

In the framed context, there are many circles S^1_n, one for each integer, counting the winding number of the 2-framing around the circle. So our TQFT will have a space of states, $Z(S^1_n)$, for each $n \in \mathbb{Z}$. Let us use 0 for the framing of the circle which bounds the standard unit disk in $\mathbb{R}^2$. If we construct S^1_n by attaching two half-circles together, these circles differ by inserting powers of the Serre automorphism before the gluing. With our convention, $Z(S^1_0) = ev \circ (S^{-1} \boxtimes \mathrm{Id}_-) \circ u$, and $Z(S^1_1) = ev \circ u$, the direct analogue of the dimension.

4.35 Theorem. *When $Z(+) = D^bCoh(X)$, $Z(S^1_n) = H^*(X; K^{\otimes n} \otimes \Lambda^*(T_X))$.*

4.36 Remark. For $n = 0$, this is the Hochschild cohomology of the manifold X. For $n = 1$, it is the Hochschild homology. Cohomology has a natural ring structure, and homology is a module over it (as are all other spaces). The algebra is the obvious one described by the TQFT, as the pair of pants defines a distinguished morphism $S^1_0 \coprod S^1_0 \to S^1_0$: picture two smaller disks embedded in a larger one, all with the standard framing in $\mathbb{R}^2$.

There is no Frobenius algebra structure away from the Calabi-Yau case, because the trace, the outgoing framed disk, comes from the map $S^1_2 \to \emptyset$. There is however a pairing $S^1_1 \coprod S^1_1 \to S^1_2$, and from there to $\emptyset$, giving a perfect bilinear pairing on Hochschild homology.

5. Cobordism hypothesis in general dimension

The required definitions and the statement of the Cobordism Hypothesis in any dimension should come as less of a surprise now. Let $\mathscr{C}$ be a symmetric monoidal (∞, D)-category, and recall the definition 4.14 of the groupoid truncations $G_{>k}\mathscr{C}$, which retain only the weakly invertible morphisms of level greater than k (invertible up to a coherent system of higher morphisms).

5.1 Definition (Lurie)**.** An object x in $\mathscr{C}$ will be called *fully dualizable* if:

- it is dualizable in $G_{>1}\mathscr{C}$,
- Its unit and trace for duality are dualizable in $G_{>2}\mathscr{C}$,
- The unit and trace for dualities established above are dualizable in $G_{>3}\mathscr{C}$,
- and so forth until we reach D-morphisms, for which we are *no longer* requiring dualizability.

The reason for stopping at D is the following: the morphisms implementing the D-dimensional dualities would land in the groupoid part of the (∞, D)-category, and would therefore be invertible. This would force the duality-implementing D-morphisms themselves to be invertible, and downward induction would force all morphisms, along with x itself, to be invertible. While invertible objects construct valid examples of field theories, they leave out most *interesting* examples.

5.2 Theorem (Lurie, [L]). *(i) A functor $Z : \mathscr{B}ord_D^{fr} \to \mathscr{C}$ is determined, up to isomorphism, by a fully dualizable object, namely the image $Z(+)$ of the positively framed point.*

(ii) More precisely, the (∞, D)-category of such functors and natural transformations is a groupoid, and is equivalent to the full sub-groupoid of fully dualizable objects in the groupoid $G_{>0}\mathscr{C}$.

(iii) The orthogonal group $\mathrm{O}(D)$ acts on such functors Z, and therefore on the space of fully dualizable objects in $\mathscr{C}$, by a change of D-framing on the point.

(iv) Given a "structure group" $G \to \mathrm{O}(D)$ for D-manifolds, a factorization of Z

$$\begin{tikzcd} & \mathscr{B}ord_D^G \arrow[dr] & \\ \mathscr{B}ord_D^{fr} \arrow[ur] \arrow[rr, "Z"] & & \mathscr{C} \end{tikzcd}$$

via a the bordism category of manifolds with G-structure on the tangent bundle corresponds to a G-fixed point structure on the generating object $Z(+)$.

5.3 Remark. (i) Fixing $Z(+)$ under the G-action is a combination of data and conditions. Thus, for $D = 1$ and $G = \mathrm{O}(1)$, we need a bilinear form on the vector space $Z(+)$, which is induced by an isomorphism with its dual $Z(-)$. Pictures show that the form must be symmetric and non-degenerate.

(ii) When $G \to \mathrm{SO}(D)$, a G-fixed point structure is a trivialization of the $\Omega\mathrm{SO}(D)$-action on $Z(+)$. Thus, for $D = 2$ and $G = \mathrm{SO}(2)$, the action of $\Omega\mathrm{SO}(2)$ on $Z(+)$ is generated by the Serre automorphism. The fixed point information is an isomorphism $S \cong \mathrm{Id}_{Z(+)}$.

(5.4) Reduction to co-dimension 2. The number of conditions for full dualizability of an object seems to proliferate exponentially with the dimension; if true, this would make any exploitation of the theorem impossible in high dimension. I shall briefly explain below why this is not the case; in fact, if we ignore framings and focus on oriented theories, we have essentially seen the maximal complexity at $D = 2$. This simplification stems from three observations:

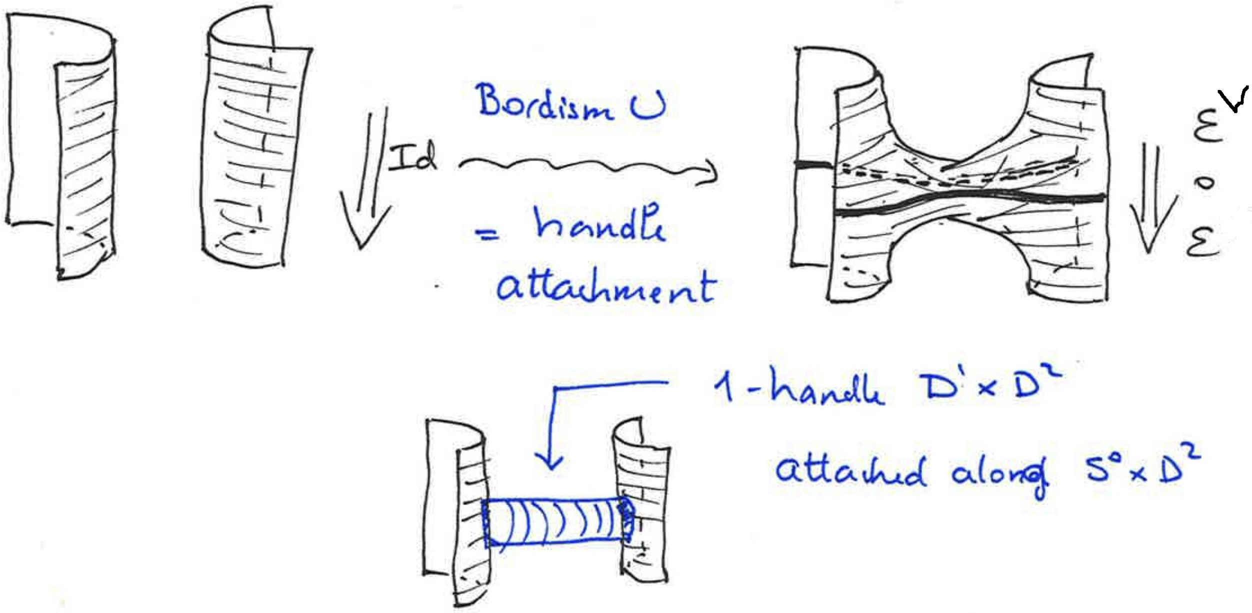
- All the unit-trace morphisms to be supplied correspond to cubes in the bordism category
- All relations take the geometric form of gluing two cubes along a common face;
- All boundary structure in co-dimension greater than 2 may be ignored.

As a result, we may assume that our cubes of morphisms are the standard handles $D^p \times D^q$ of surgery theory, with boundary decomposed as $D^p \times S^{q-1} \coprod_{S^{p-1}\times S^{q-1}} S^{p-1} \times D^q$. That is, we are dealing with a $(p+q)-$morphism relating two $p+q-1-$morphisms, both going from $\emptyset$ to the $(p+q-2)-$endomorphism $S^{p-1} \times S^{q-1}$ of the empty set: thus, a $3-$layered object. The duality relations "$\theta \circ \varepsilon = 1$" to be checked will be the *standard handle cancellation relations* of surgery theory.

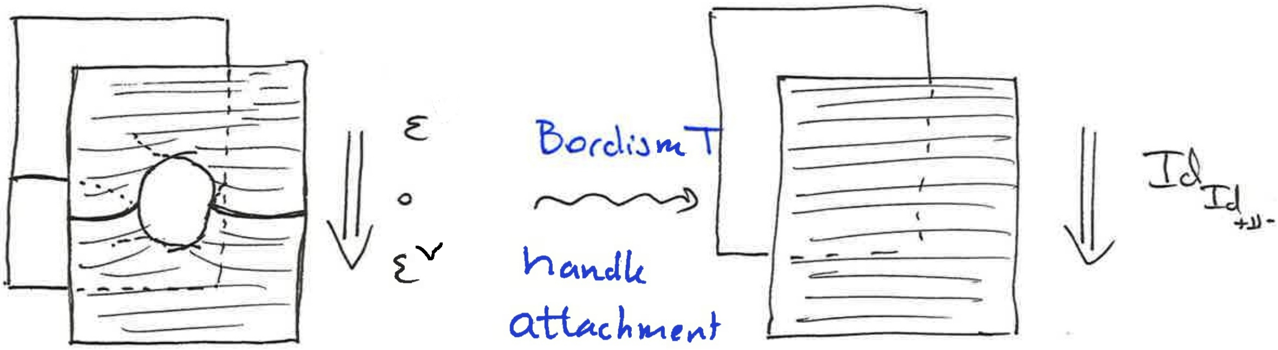
The impact of these observations is that, to demonstrate full dualizability of an object, it suffices to provide the k-morphisms associated to all handles of all dimensions through k, and check the handle cancellation relations.

Framed theories requires framed handles, in which case a presentation implicitly involves coming to grips with the homotopy of $\mathrm{SO}(D)$; this adds to the complexity of framed, as opposed to oriented theories, in higher dimension.

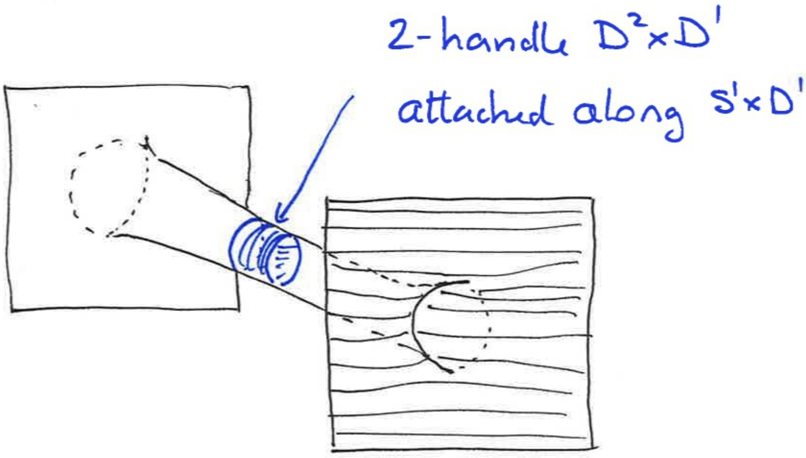
(5.5) 3D example. Let us illustrate these ideas in dimension 3, ignoring framings and retaining only orientations. The oriented category is especially easy, because the adjoint of a morphism is always represented by the opposite bordism, meaning the same manifold with opposite orientation, read backwards as a bordims. (The frame-reversing convention requires more care.) Recall from the previous lecture that $\varepsilon : \mathrm{Id} \to u \circ ev$, the unit of adjunction in $\mathrm{Hom}(+ \coprod -, + \coprod -)$, was represented by the 'arch', an upside-down saddle; I will exhibit the left adjoint ${}^\vee\varepsilon$ as the standard saddle. The bordism $U : \mathrm{Id} \to \varepsilon \circ {}^\vee\varepsilon$ is the standard 1-handle attachment familiar from Morse theory:



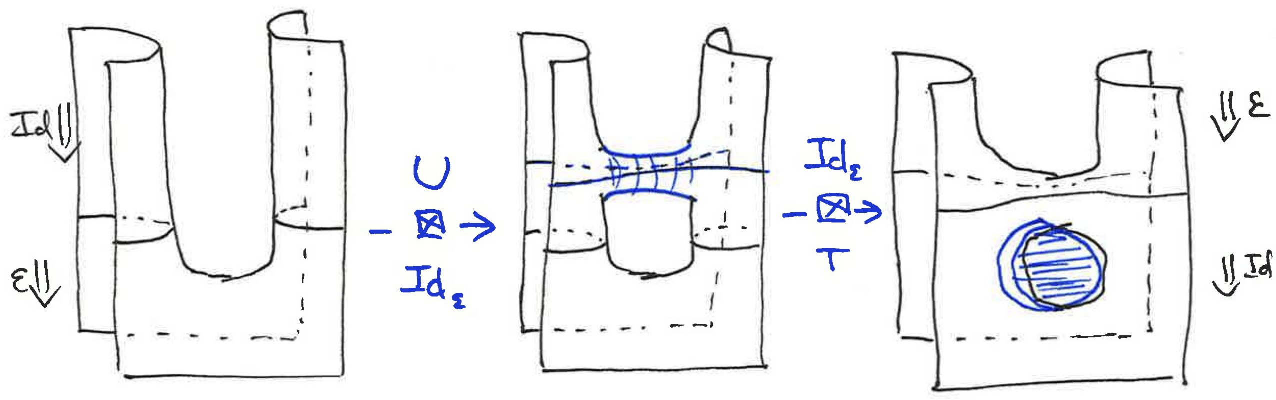
(In the oriented case, ${}^\vee\varepsilon = \varepsilon^\vee$; reading the pictures upwards gives the opposite adjunction.) The trace morphism $T : {}^\vee\varepsilon \circ \varepsilon \to \mathrm{Id}$ is a 2-handle attachment,



and may require a stretch of the connecting tube to see:

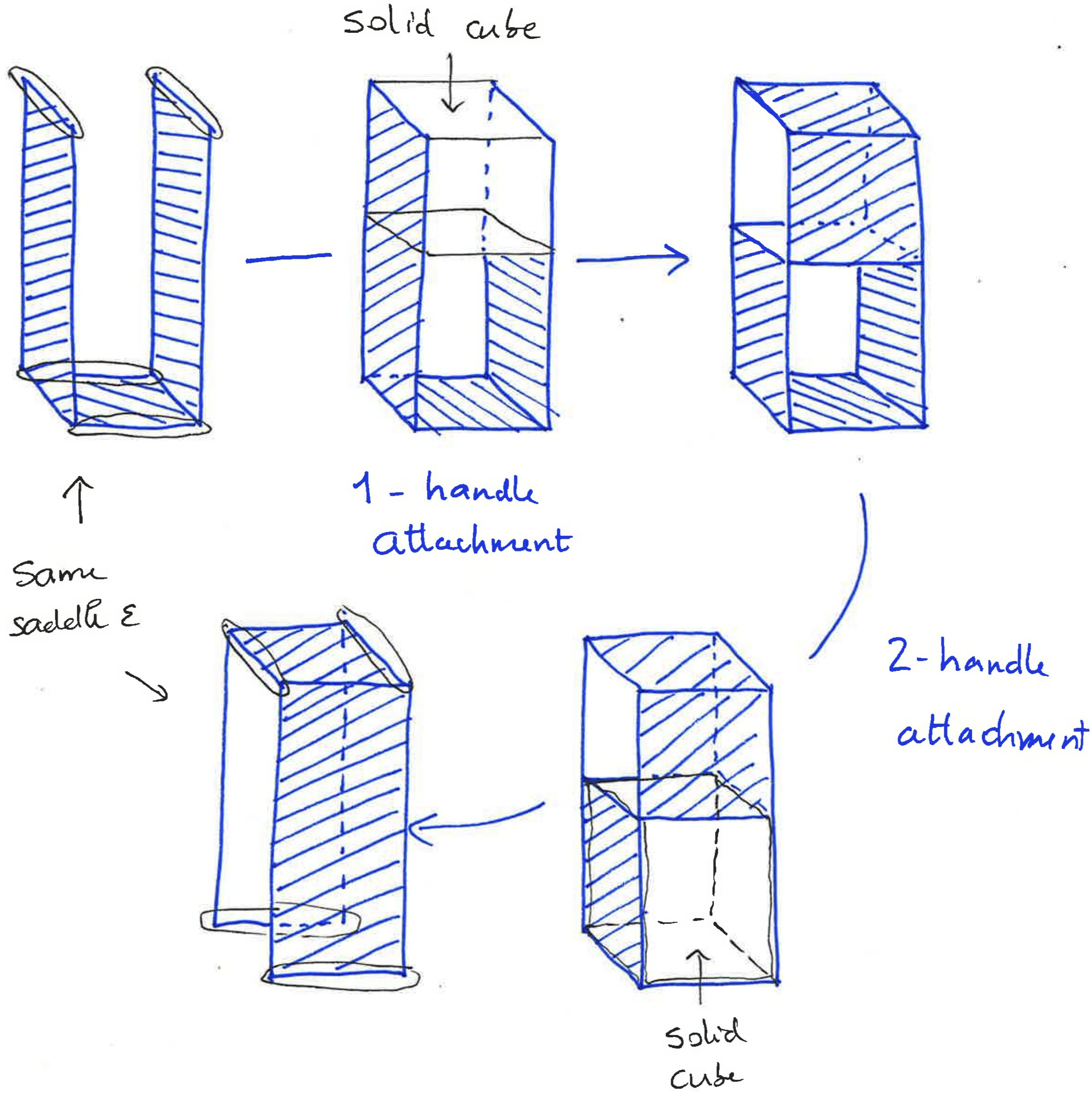


Below is one of the relations "$T \circ U = 1$", namely $(\mathrm{Id}_{{}^\vee\varepsilon} \boxtimes T) \circ (U \boxtimes \mathrm{Id}_{{}^\vee\varepsilon}) = \mathrm{Id}_{{}^\vee\varepsilon}$:



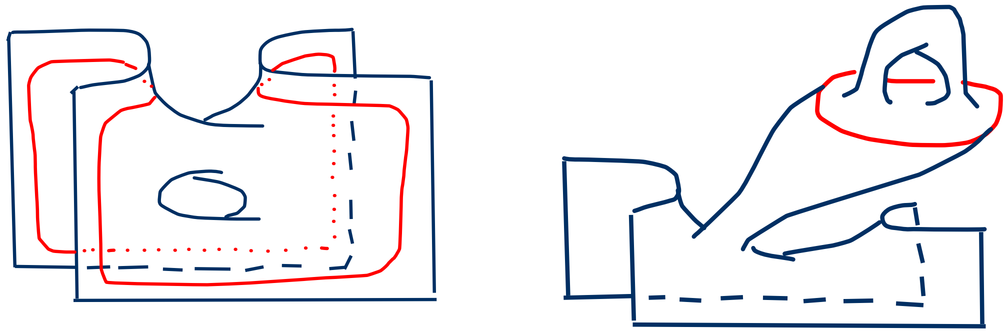
This is the standard cancellation of a 1-handle by a 2-handle.

Rewriting the diagrams in cubular form illustrates an inductive formulation, which allows one to reduce the duality morphisms to handles in any dimension.



The 1-morphisms for which the blue surfaces are 2-morphisms are circled in black in the initial and final picture: they are $u \circ ev$ at the top and Id at bottom, both from $+\coprod -$ to itself. The initial and final angular saddles represent the same 2-morphism ε in the bordism category. The two cubes to be inserted represent a 1-handle attachment and its a 2-handle cancellation. The generalization to any dimension arises by replacing the top square, $I \times I$, by $I^p \times I^q$.

The cubes we just drew have co-dimension 3 vertices, so the reduction to manifolds with corners only of co-dimension 2 requires a further argument. Full execution is a bit involved, but the idea is simple enough, and relies on inductive application of the cobordism hypothesis. The picture below suggests the argument for dimension 3, assuming that 2-duality has been checked: that is, we have established the existence and consistency of the TQFT up to dimension 2:



We draw one undivided circle along the boundary of the surface, which encloses all the handle attachment and cancellation steps. This step separates the subdivision or recombination of a boundary with zero-dimensional corners into the $2D$ part of the theory, and shows that the operations and relations in the $3D$ theory can be defined and checked on surfaces with circle boundaries only.

5.6 Remark. Recall from Morse theory that any manifold can be built from the empty set by attaching handles, and all operations involve manifolds with co-dimension 2 corners only. At first sight, this might suggest a classification of TQFTs which go only 2 layers deep, and avoid categories beyond 2 altogether. In fact, a partial result in this direction does exist in 3 dimension, the *Reshetikhin-Turaev theorem* [RT], classifying 2-deep $3D$ TQFTs with a semi-simple category associated to the circle. However, going beyond dimension 3 is hopeless: the spheres used for handle attachment can be embedded in complicated ways. For example, the entire complexity of simply connected, differentiable 4-manifolds is concealed in the theory of links in 3-manifolds, used for attaching 2-handles. Nevertheless, the observation is useful: if we can cut up the boundaries themselves, and their further boundaries, into handles, then this 'codimension 2' intuition becomes correct, and develops into the recursive procedure for checking dualizability using only standard handles and cancellations.

(5.7) Tensor and module categories. We now move to a 3-dimensional illustration of the cobordism hypothesis, which allows the construction of interesting $3D$ TQFTs. A key example is provided by recent work by Douglas, Schommer-Pries and Snyder.[19].

We defined a tensor category $\mathscr{T}$ as a linear category with bilinear monoidal structure.[20] Recall a minor complication: associativity of the tensor product is moderated by an associator

$$\alpha_{x,y,x} : (x \otimes y) \otimes z \xrightarrow{\sim} x \otimes (y \otimes z),$$

subject to a Stasheff's pentagon identity. We are in luck, though, because *Mac Lane's coherence theorem* allows us to replace T by an equivalent category with strict associativity.

A *module category* $\mathscr{M}$ over $\mathscr{T}$ is a linear category with a bilinear multiplication functor $\mathscr{T} \times \mathscr{M} \to \mathscr{M}$, satisfying the obvious constraints that define an action of $\mathscr{T}$ on $\mathscr{M}$. (Again, it turns out that we can modify $\mathscr{M}$ to obtain a strict module category, where the coherence isomorphisms are identities [G].)

(5.8) Tensor product of categories. [EGNO, G] The definition for abelian categories (due to Deligne) is borrowed from algebra: for a right $\mathscr{T}$-module $\mathscr{M}$ and a left one $\mathscr{N}$, $\mathscr{M} \boxtimes_{\mathscr{T}} \mathscr{N}$ is the category co-representing the functor which sends a linear category $\mathscr{C}$ to the category of $\mathscr{T}$-bilinear, right exact

[19]The author is most grateful for their explanations, long before the paper was publicly available

[20]Some sources, such as [EGNO], impose additional conditions.

bi-functors $\mathscr{M} \times \mathscr{N} \to \mathscr{C}$. One can then show the tensor product of abelian categories is co-represented by an abelian category [G].

5.9 Remark. The tensor product of abelian categories over $Vect$ is already a bit subtle: for instance, $\mathscr{M} \boxtimes_{Vect} \mathscr{N}$ has more objects than just the direct sums $m \boxtimes n$, with $m \in \mathscr{M}$ and $n \in \mathscr{N}$; we must close this under quotients (co-kernels). The resulting category is abelian. If $\mathscr{M}$ and $\mathscr{N}$ are the categories of A- and B-modules, then $\mathscr{M} \boxtimes \mathscr{N}$ is the category of $A \otimes B$-modules.

The upshot of the definitions should be the construction of a (lax) symmetric monoidal 3-category, with tensor categories as object, bi-module categories as morphisms, tensor product as composition, bi-module functors as 2-morphisms and finally natural transformations between the latter as 3-morphisms. Some ingredients for the construction are spelt out, for instance, in [G]; full details of the construction are found in the recent paper [DSS].

5.10 Definition. A tensor category is *rigid* if every object has left and right duals.

Recall that we can make the left and right duals into functors on $\mathscr{T}$; they are then mutually inverse. Rigid tensor categories have some particularly good properties [EGNO]; in particular, if $\mathscr{T}$ is abelian, then the multiplication $\otimes : \mathscr{T} \times \mathscr{T} \to \mathscr{T}$ is bi-exact, as are the internal dualization functors.

(5.11) 2-dualizability. Every tensor category $\mathscr{T}$ is 1-dualizable, with dual the *multiplicatively opposite* category $\mathscr{T}^{mo}$, the same category with opposite tensor product. The argument for this repeats the one for algebras, and u, ev are represented by $\mathscr{T}$ as $\mathscr{T}^e$-module. We can also mimic the algebra computations for 2-duality from Lecture 4. We need $\mathscr{T}$ to be dualizable as a linear category, and dualizable as a $\mathscr{T}^e$-module. The Serre functor is the dual of $\mathscr{T}$ as linear category, which we denote $\mathscr{T}^\vee$. 2-dualizability requires this to be an invertible bimodule, and the inverse must the bimodule category of (right exact) functors $Hom_{\mathscr{T}^e}(\mathscr{T}; \mathscr{T}^e)$. Furthermore, tensoring with $\mathscr{T}$ over $\mathscr{T}^e$ should be an *exact* operation on categories (commute with finite filtered limits and colimits). This amounts to asking for the exactness of the functor of tensoring over $\mathscr{T}$.

We can identify $\mathscr{T}^\vee$ more precisely. If all objects in $\mathscr{T}$ are compact — such is the case, for instance, for the category of finitely generated modules over a Noetherian ring — then $\mathscr{T}^\vee$ can be identified with the *linearly opposite* category $\mathscr{T}^\circ$ as follows: an object $a^\circ \in \mathscr{T}^\circ$ is sent to the co-represented functor $x \mapsto Hom(a, x)$. We shall spell out the bimodule structure of $\mathscr{T}^\vee$ following Theorem 5.19 below, in the case of rigid categories.

(5.12) Drinfeld center and Hocschild homology. Associated to closed 1-manifolds are linear categories, and those associated to the framed circles S^1_n, $n > 0$, are the tensor products $\mathscr{T} \otimes_{\mathscr{T}^e} (\mathscr{T}^\vee)^{\otimes_{\mathscr{T}}(n-1)}$, with the $(n-1)$st power $(\mathscr{T}^\vee)^{\otimes_{\mathscr{T}}(n-1)} := \mathscr{T}^\vee \otimes_{\mathscr{T}} \cdots \otimes_{\mathscr{T}} \mathscr{T}^\vee$ of the Serre bi-module $\mathscr{T}^\vee$. The description for $n \le 0$ involves the inverse bi-module of $\mathscr{T}^\vee$, but an alternative presentation exploits the dualizbility of $\mathscr{T}$ over $\mathscr{T}^e$ to rewrite the tensor product with inverse powers as a Homs. This presents the categories for $n \le 0$ as $\mathrm{Hom}_{\mathscr{T}^e}((\mathscr{T}^\vee)^{\otimes_{\mathscr{T}}(-n)}, \mathscr{T})$. Here, Hom stands the category of linear, right exact functors compatible with the $\mathscr{T}^e$-action. (Compatibility with the $\mathscr{T}$-action carries data, not just conditions, as we will see in a moment.)

The category $\mathrm{Hom}_{\mathscr{T}^e}(\mathscr{T}, \mathscr{T})$ for the 0-framed circle plays a distinguished role, as it has a natural tensor structure under composition. It is equivalent to the *Drinfeld center* of $\mathscr{T}$: this is the category of pairs (x, β) where $x \in \mathscr{T}$ and β is a *half-brading with* x, a multiplicative isomorphism between the functors $x\otimes$ and $\otimes x$ of left and right tensoring with x. The reader can probably guess what we mean by multiplicativity of β, based on the condition for a braiding (Lecture 3); it is the commutativity of the triangle

$$
\begin{tikzcd}
 & y \otimes x \otimes z \arrow[dr, "\mathrm{Id}_y \otimes \beta(z)"] & \\
x \otimes y \otimes z \arrow[ur, "\beta(y)\otimes \mathrm{Id}_z"] \arrow[rr, "\beta(y\otimes z)"] & & y \otimes z \otimes x
\end{tikzcd}
$$

The Drinfeld center comes with a natural *braiding* $\beta(x, y) : x \otimes y \xrightarrow{\sim} y \otimes x$, from the half-braiding carried by the object x. This braiding is usually not symmetric.

5.13 Remark. (i) In practice, the braiding tries to be 'maximally non-symmetric'. For example, if $\mathscr{T}$ is the tensor category $(\mathrm{Rep}(F), \otimes)$ of representations of a finite abelian group F under the tensor product over $\mathbb{C}$, its center is the tensor category of reps on $F \times F^\vee$, but with a Heisenberg-like braiding defined from the natural pairing of $F, F^\vee$.

(ii) Specifically, calling $n = \#F$, a distinguished class in $H^4(B^2F \times B^2F^\vee; \mu_n)$ is induced by the Pontrjagin pairing into $\mu_n \subset \mathbb{C}^\times$. Treating this as a k_3-invariant builds for us a space with $\pi_2 = F \times F^\vee$ and $\pi_3 = \mu_n$. Hence, we construct a braided tensor category as in Proposition 3.21. This tensor category splits into Fourier components according to the characters of μ_n. The summand corresponding to the standard character is isomorphic to the Drinfeld center of $(\mathrm{Rep}(F), \otimes)$.

(iii) More generally, any finite group G has a 'categorized group ring' which is the category of vector bundles on G, with convolution as the tensor structure. The Drinfeld center of this is the tensor category of G-equivariant vector bundles on G, with the convolution structure and an interesting braiding. This is the categorized analogue of class functions on G. Example (ii) has $G = F^\vee$.

5.14 Remark. In the 3-dualizable case, the square of the Serre functor is the identity because $\pi_1\mathrm{SO}(3) = \mathbb{Z}/2$; so there will be only two categories, going with the two 3-framings on the circle, S^1_{even} and S^1_{odd}. They are $Z(\mathscr{T}) = \mathrm{Hom}_{\mathscr{T}^e}(\mathscr{T}, \mathscr{T})$ and $\mathscr{T} \otimes_{\mathscr{T}} \mathscr{T}$.

(5.15) Fusion categories. There is no complete classification of 3-dimensional extended TQFTs based on tensor categories resembling the one in $2D$, by semi-simple (not necessarily commutative) algebras. There are some constraints; for instance, $\mathscr{T}$ is abelian, then full dualizability requires the Drinfeld center (and its twisted form, Hochschild homology) to be *semi-simple*. This is because we can *dimensionally reduce* the $3D$ TQFT $Z_{\mathscr{T}}$ generated by $\mathscr{T}$ along S^1, to produce a 2-dimensional theory, which must be generated by $Z(\mathscr{T})$. (The dimensionally reduced theory, on a manifold M, is the original theory evaluated on $M \times S^1$.) This implies that $Z(\mathscr{T})$ is semi-simple.

5.16 Remark. Semi-simplicity of the center is automatic for rigid categories, but generally a strong constraint. *Exercise:* compute the Drinfeld center of the "2×2 upper triangular matrix algebra over $Vect$", the category $Vect^{\oplus 3}$, with tensor product imitating the multiplication of matrices $\left[\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right]$.

However, a beautiful class of extended $3D$ TQFTs comes form the following result of Douglas, Schommer-Pries and Snyder. It is based on the notion of *fusion category* that has been the subject of much research (Etingof, Ocneanu *et al*). The wording similarity with the 2-dimensional story is a bit deceptive, as the structure of a fusion category is substantially more involved than that of a complex semi-simple algebra.

5.17 Definition. A *fusion category* is a semi-simple rigid tensor category with finitely many simple isomorphism classes.

Examples:

(i) The 'categorified group ring' of a finite group G, 5.13.iv. This generates the $3D$ gauge theory with finite group F. The tensor structure, and the resulting TQFT, can be twisted by a co-cycle $\tau \in H^3(BF; \mathbb{C}^\times)$ — a higher analogue of a central extension of the group by $\mathbb{C}^\times$, which in $2D$ generates a twisted version of gauge theory. In the tensor category, the cocycle appears as an associator.

(ii) The semi-simplified category of representations of a *quantum group* at a root of unity (the category of finite-dimensional representations modded out by representations of quantum dimension zero). The associated 3-dimensional field theory computes *Turaev-Viro theory*, whose closed 3-manifold invariants are the square norms of the famous Chern-Simons invariants.[21]

The conditions are strong enough to imply the good behavior of the tensoring operation:

[21] There is at present no construction of the Chern-Simons invariants for general non-abelian G, although a there is a promising program by Bartels, Henriques and Douglas. The generating tensor cateogory analogue is far from discrete, though, and is described in terms of von Neumann algebras. For torus groups, a C^* Hopf-like algebra appears, [FHLT].

5.18 Theorem (Etingof-Nykshich-Ostrik)**.** *For any right and left semi-simple module categories $\mathscr{M}, \mathscr{N}$ over a fusion category $\mathscr{F}$, the tensor product $\mathscr{M} \boxtimes_{\mathscr{F}} \mathscr{N}$ is semi-simple. It is also exact in $\mathscr{M}$ and $\mathscr{N}$.*

So we can define a 'small' 3-category of fusion categories, semi-simple bimodule categories, functors and natural transformations. In the setting of fusion categories, we also have a duality which allows us interpret tensor products as functor categories. In particular, this establishes the semi-simplicity of the Drinfeld center and its twisted version.

5.19 Theorem (Douglas, Schommer-Pries, Snyder, [DSS])**.** *Every fusion category $\mathscr{F}$ is a fully dualizable objects of the 3-category $\mathscr{T}cat$. The category $Z_{\mathscr{F}}(S^1_0)$ is the Drinfeld center of $\mathscr{F}$, with its natural tensor structure. The Serre functor is the double dual functor in $\mathscr{F}$.*

Because $\pi_1\mathrm{SO}(3) = \mathbb{Z}/2$ (and no longer $\mathbb{Z} = \pi_1\mathrm{SO}(2)$), the Serre automorphism of any 3-dualizable object must square to 1. As a consequence, the authors get an enlightening proof of a recent result result about fusion categories [ENO], in turn based on earlier work of Radford's [Rad].

5.20 Corollary (Etingof-Nykshich-Ostrik)**.** *In any fusion category, there is a canonical isomorphism of the quadruple dual functor with the identity functor.*

(5.21) The Serre automorphism $\mathscr{T}^\vee$. Let us spell out its identification of Serre with the double right dual functor, in the rigid case. This does not rely on semi-simplicity of $\mathscr{T}$. Returning to the identification $\mathscr{T}^\circ = \mathscr{T}^\vee$ of §5.11, we have

$$x \otimes a^\circ \otimes y = \left({}^\vee y \otimes a \otimes x^\vee\right)^\circ$$

from the Hom-duality properties of $x^\vee, {}^\vee y$. Now let us use the left dual to identify $\mathscr{T}$ with $\mathscr{T}^\circ$: $a \mapsto ({}^\vee a)^\circ$. The relation becomes

$$x \otimes ({}^\vee a)^\circ \otimes y = \left({}^\vee y \otimes {}^\vee a \otimes x^\vee\right)^\circ = \left[\left({}^\vee(x^{\vee\vee} \otimes a \otimes y\right)\right]^\circ$$

The result is identifying the Serre bi-module $\mathscr{T}^\vee$ with $\mathscr{T}$, but with the left tensor action twisted by double right dualization. This is the bimodule implementation of the double right dual functor.

5.22 Remark. (i) If S is not isomorphic to Id, then the TQFT can be defined for Spin surfaces. A rigid category is called *pivotal* if the double dual is isomorphic to the identity. In the fusion case, a pivotal structure allows us to pass from Spin surfaces to oriented surfaces. To go on to 3-manifolds and pass from the framed to the oriented setting, we need what is called a *spherical* structure, a pivotal structure in which the trivialization of the doulbe dual squares to the canonical trivialization of the quadruple dual.

(ii) The condition that Serre should be a tensor functor, rather than a bimodule, is closely related to the existence of internal duals: a $\mathscr{F} - \mathscr{F}$ bimodule $\mathscr{S}$ is implemented by a tensor automorphism iff $\mathscr{S}$ is equivalent to $\mathscr{F}$ as a left and as a right $\mathscr{F}$-module separately. The tensor automorphism it implements is the composition of these separate 'straightening' isomorphisms. In the rigid case, both straightening isomorphisms between $\mathscr{F}$ and $\mathscr{F}^\vee$ are the internal duality functors.

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