PROBLEM SET 07

Part 1 - Integrals

In all the exercises you can assume that continuous functions are integrable.

Q.1. Divide the interval $[0,1]$ into n equal subintervals. For the resulting partition P_n compute $U(f,P_n)$ and $L(f,P_n)$ for each of the following functions, and determine $\int_0^1 f$ (if it exists)

(1) $f(x) = 2$

(2) $f(x) = x$

(3) $f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{otherwise} \end{cases}$

Q.2. (1) Prove that*

$$\int_{ca}^{cb} f(t)dt = c\int_a^b f(ct)dt$$

(2) Prove that

$$\int_a^{ab} \frac{1}{t}dt = \int_1^b \frac{1}{t}dt$$

(3) Prove that

$$\int_1^a \frac{1}{t}dt + \int_1^b \frac{1}{t}dt = \int_1^{ab} \frac{1}{t}dt$$

(4) Assuming that the area enclosed by the circle $x^2 + y^2 = 1$ is π prove that the area enclosed by the ellipse $x^2/a^2 + y^2/b^2 = 1$ is πab.

Q.3. Evaluate without doing any computations

(1) $\int_{-1}^{1} x^3\sqrt{1-x^2}dx$

(2) $\int_{-1}^{1} (x^5+3)\sqrt{1-x^2}dx$

Q.4. Prove that

$$\int_0^t \frac{\sin t}{t+1} > 0$$

for all $t > 0$.

*Hint: Start with a partition of $[a,b]$ and construct a partition of $[ca,cb]$

Part 2 - Theorems

Q.5. In this exercise we'll (almost) prove that every continuous function is integrable. In order to prove integrability we need a stronger version of continuity.

A function f is said to be **uniformly continuous** on a subset A of the real numbers if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x, y \in A$ whenever $|x - y| < \delta$ we have $|f(x) - f(y)| < \epsilon$.

The difference between uniform continuity and (ordinary) continuity is that in uniform continuity the δ does not depend on x i.e. the same δ should work for all x in A.

(1) Prove that if f is uniformly continuous then it is also continuous.

(2) Prove that the function $f(x) = x$ is uniformly continuous on $\mathbb{R}$.

(3) Prove that the function $f(x) = x^2$ is uniformly continuous on $[0, 1]$ but not on the entire real line $\mathbb{R}$.

(4) Prove that the function $1/x$ is not uniformly continuous on $(0, 1]$.

A deep fact about continuous functions states that if f is continuous on $[a, b]$ then it is uniformly continuous on $[a, b]$. (This is because the set $[a, b]$ is closed and bounded.)

(5) Let f be a continuous (and hence uniformly continuous) function on $[a, b]$. Prove that for every $\epsilon > 0$ there exists a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$.† Conclude that f is integrable on $[a, b]$.

Q.6. Assume that f is a continuous increasing function.

(1) Find

$$\int_{f^{-1}(a)}^{f^{-1}(b)} f^{-1}(x)\, dx$$

in terms of $\int_a^b f$. (Draw a picture) Be careful, there are multiple terms in the final answer.

(2) Suppose that $f(0) = 0$. Prove the **Young's inequality** which states that for all $a, b > 0$ we have

$$ab \leq \int_0^a f(x)dx + \int_0^b f^{-1}(x)dx$$

and equality holds if and only if $b = f(a)$. (Draw a picture.)

† Hint: The size of the intervals in the partition P should be bounded by the δ corresponding to $\epsilon/(b - a)$.

Part 3 - Step functions

The interval $[a, b]$ is the domain for all the functions in this problem set.

Q.7. A function s is called a **step function** if there is a partition $(a = t_0, t_1, \dots, t_n = b)$ such that s is constant on each (t_{i-1}, t_i).

(1) Prove that if s_1 and s_2 are step functions then so is $s_1 + s_2$.

(2) If f is an integrable function then for every $\epsilon > 0$ there exists a step function $f \geq s$ such that

$$\int_a^b f - \int_a^b s < \epsilon$$

(3) If f is an integrable function then for every $\epsilon > 0$ there exists a step function $f \leq s$ such that

$$\int_a^b s - \int_a^b f < \epsilon$$

(4) By drawing pictures show that a step function can be approximated by a continuous, and the approximation can be made as good as we want.

Step functions are a very important class of functions used extensively for approximating arbitrary functions.

Q.8. (1) Suppose f is a continuous function and

$$\int_a^b fg = 0$$

for all continuous functions g. Show that $f(x) = 0$ for all x in $[a, b]$.[‡]

(2) Suppose f is a continuous function and

$$\int_a^b fg = 0$$

for all continuous functions g which satisfy the additional condition $g(a) = 0 = g(b)$. Show that $f(x) = 0$ for all x in $[a, b]$.[§]

[‡] Hint: This has a one line proof. Construct a function g using the function f.

[§] Hint: Assume $f(x) \neq 0$ and come up with a step function s non-zero near x.