Adams Spectral Sequence

We work in the category of bounded below finite type spectra (unbased)
Let $H\mathbb{Z}/2$ be the Eilenberg Maclane Spectrum representing $H^*(-,\mathbb{Z}/2)$,
and denote $H^*(X) = [X; H\mathbb{Z}/2] = H^*(X;\mathbb{Z}/2)$

$\mathcal{A} = H^*(H\mathbb{Z}/2)$ is the Steenrod algebra

We can define for any finite dimensional graded vector space $V/\mathbb{Z}/2$ a generalised Eilenberg Maclane space KV

$$KV := \coprod_{|V_n| = i_n} \coprod_{i_n} \Sigma^n H\mathbb{Z}/2$$

- $\pi_* KV \cong V$
- $H^* KV \cong \mathcal{A} \otimes V^*$ ⟵ There is no analogue of this in the category of topological spaces! Hence we work stably.

Adams tower:

$$Y_0 \longleftarrow Y_1 \longleftarrow Y_2 \longleftarrow \cdots$$

is an Adams tower for Y if

0) There are natural compatible maps $Y \longrightarrow Y_i$

1) the cofibers $\Sigma^{-s}K^s \longrightarrow Y_s \longrightarrow Y_{s-1}$ are all generalized EM spaces, $K^0 = Y_0$

denote by d the composite:

$$K^s \longrightarrow \Sigma^s Y_s \longrightarrow K^{s+1}$$

2) then

$$Y \to K^0 \longrightarrow K^1 \longrightarrow K^2 \longrightarrow \cdots$$

is an Adams Resolution, i.e.

$$0 \leftarrow H^*Y \longleftarrow H^*K^0 \longleftarrow H^*K^1 \longleftarrow \cdots$$

is a free resolution of $\mathcal{A}$-modules.

- An Adams tower can be built out of an Adams resolution.
  Because Y is finite type and conective we can first find an $\mathcal{A}$-resolution for H^*Y and then convert it into spectra.

- This gives us an exact couple: for any X,

$$\begin{tikzcd}
{[\Sigma^{t-s}X, Y_s]} \arrow[rr] & & {[\Sigma^{t-s}X, Y_{s-1}]} \arrow[dl, dashed, "-1"] \\
& {[\Sigma^{t-s}X, \Sigma^{-s}K^s]} \arrow[ul] &
\end{tikzcd}$$

Q. Why the Σ^t?

This gives rise to the Adams SS with

$$E_1^{s,t} = [\Sigma^{t-s}X, \Sigma^{-s}K^s]$$
$$\cong [\Sigma^t X, K^s]$$
$$\cong \mathrm{Hom}_{\mathcal{A}}(H^* K^s, \Sigma^t H^* X)$$

the d^1 is just the d in the Adams' resolution

$$d : K^s \longrightarrow K^{s+1}$$

and so

$$\boxed{E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s}(H^*K, \Sigma^t H^* X)}$$

The SS abuts to $\boxed{E_\infty^{s,t} \Longrightarrow [\Sigma^{t-s}X, Y_2^\wedge]}$

$Y_2^\wedge$ = 2-completion of Y i.e. $Y_2^\wedge$ is a 2-local spectra i.e. $H_*(Y_2^\wedge, \mathbb{Z}_{(2)}) \cong H_*(Y_2^\wedge, \mathbb{Z})$ and $\exists$ a map $f: Y \longrightarrow Y_2^\wedge$ such that for any 2-local space X, any map $Y \longrightarrow X$ factors through f.

in particular $H_*(Y_2^\wedge) \cong H_*(Y)_{(2)}$

The differentials go $d_r : E_r^{s,t} \longrightarrow E_r^{s+r, t+r-1}$

Q. Convergence is best determined for each case independently.

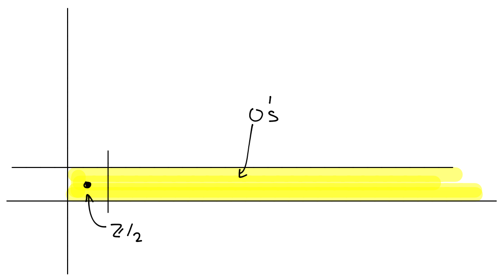
Classical Adams SS:

The classical Adams SS is for $X = Y = S$ the sphere spectrum.

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s}(\mathbb{Z}/2, \Sigma^t \mathbb{Z}/2)$$

Prop: $E_2^{0,t} = \mathrm{Ext}^0_{\mathcal{A}}(\mathbb{Z}/2, \Sigma^t \mathbb{Z}/2)$

$$= \mathrm{Hom}_{\mathcal{A}}(\mathbb{Z}/2, \Sigma^t \mathbb{Z}/2)$$
$$= \begin{cases} \mathbb{Z}/2 & \text{if } t = 0 \\ 0 & \text{else} \end{cases}$$



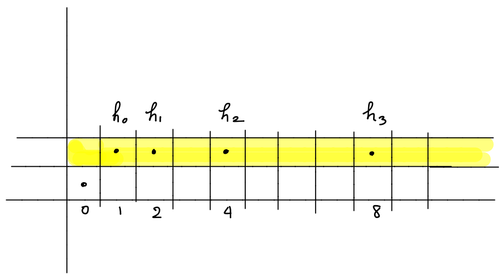
Prop: $E_2^{1,t} = \mathrm{Ext}_{\mathcal{A}}^{1}(\mathbb{Z}/2, \Sigma^t \mathbb{Z}/2)$

Look at the resolution $0 \to \bar{\mathcal{A}} \longrightarrow \mathcal{A} \longrightarrow \mathbb{Z}/2 \longrightarrow 0$

Augmentation ideal

Apply $\mathrm{Hom}(-, \Sigma^t \mathbb{Z}/2)$ we see that

$$\mathrm{Ext}^1_{\mathcal{A}}(\mathbb{Z}/2, \Sigma^t \mathbb{Z}/2) \cong \mathrm{Hom}_{\mathcal{A}}(\bar{\mathcal{A}}, \Sigma^t \mathbb{Z}/2)$$



Now if $Sq^I \in \mathcal{A}$, $I = (i_1, i_2, \ldots, i_k)$ and $i_1 + \cdots + i_k = t$

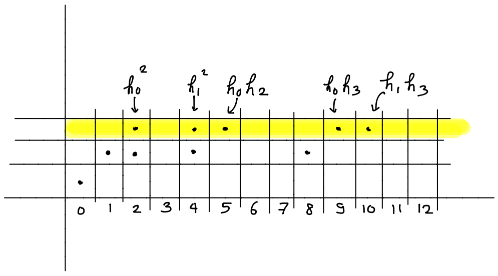
then unless $k = 1$ we must have $Sq^{i_1} = 0 \Rightarrow Sq^I = 0$

so $\mathrm{Hom}_{\mathcal{A}}(\bar{\mathcal{A}}, \Sigma^t \mathbb{Z}/2)$ = no. of linearly independent indecomposables in degree t

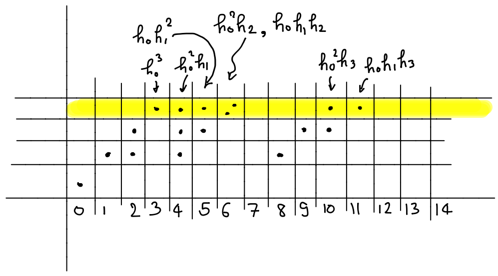
$$= \begin{cases} \mathbb{Z}/2 & \text{if } t = 2^i \\ 0 & \text{else} \end{cases}$$

Generators are denoted by h_i

• Th^m $\quad Ext^2(\mathbb{Z}/2, \Sigma^* \mathbb{Z}/2)$ is generated by $h_i h_j = h_j h_i$, $h_i h_{i+1} = 0$

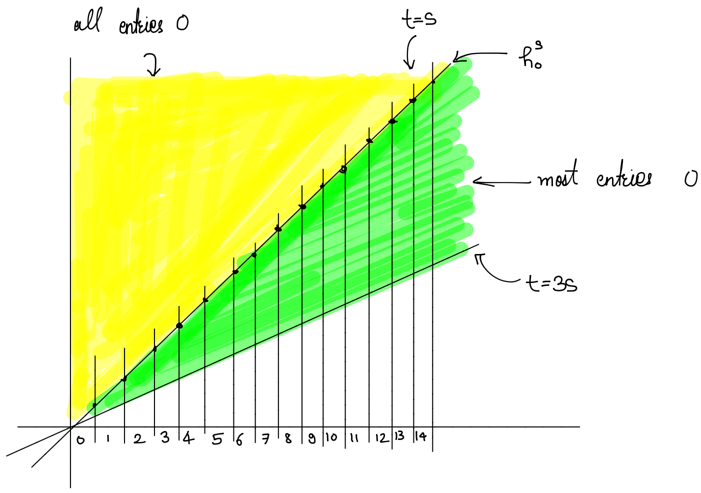


• Th^m $\quad Ext^3(\mathbb{Z}/2, \Sigma^* \mathbb{Z}/2)$ is generated by $h_i h_j h_j$ with $h_i h_{i+2}^2 = 0$, $h_i^2 h_{i+2} = h_{i+1}^2$



• Th^m
Vanishing results

$$E_2^{s,t} = \begin{cases} 0 & \text{if } \quad t-s<0 \\ \mathbb{Z}/2\, h_0^s & \text{if } \quad t=s \\ 0 & \text{if } \quad 0<t-s<2s+\epsilon \end{cases} \quad \text{and} \quad \epsilon = \begin{cases} 1 & s \equiv 0,1 \mod 4 \\ 2 & s \equiv 2 \mod 4 \\ 3 & s \equiv 3 \mod 4 \end{cases}$$



As a corollary the classical Adams SS converges.

- h_0, h_1, h_2, h_3 survive the SS and detect the Hopf invariant 1-maps η, ν, σ
- $d_2 h_i = h_{i-1} h_0^2 \quad \Rightarrow \quad h_i$, $i>3$ do not survive the SS

  This somehow solves the Hopf invariant one problem.
- h_i^2 , $i>6$ are Kervaire invariant classes
- $h_0 h_2$, $h_0 h_3$, $h_2 h_4$, h_i^2 $i \leq 5$, $h_j h_1$ $j \geq 3$ survive the SS.

Adams Novikov Spectral Sequence

If we try to replace $H\mathbb{Z}$ by a spectrum E, we get a similar spectral sequence as the Adams :

ANSS: $\exists$ a SS with

$$E_2^{s,t} = \mathrm{Ext}^s_{E_*E}(\Sigma^t E_*X, E_*Y)$$

abutting to

$$[\Sigma^{t-s}X, L_E Y]$$

where
- E^* has a graded cup product
- $\eta_L, \eta_R : E_* \longrightarrow E_*E$ makes E_*E a flat E_* module.

Note that we are using homology and not cohomology groups and hence we need to use comodules. The definitions are what one expects, its better to do examples than fuss about the definitions.