## Quasi coherent sheaves §13.2

$M$ – $A$ module

$\tilde{M}$ – sheaf on Spec $A$, $\tilde{M}(D_f) = M_f \ \forall f \in A$

**Def**: An $\mathcal{O}_X$ module $\mathcal{F}$ is quasi-coherent if for every Spec $A \overset{\text{open}}{\subseteq} X$,

$$\mathcal{F}\big|_{\operatorname{Spec} A} \cong \tilde{M}$$

**Th^m**: Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Quasi-coherence is true if it is true on an affine cover.

Long abstract proof.

**Examples**: 1) $\mathcal{O}_X$ is quasi-coherent

2) $\mathcal{O}_X^n$

3) $\mathcal{F}$ such that $\mathcal{F}|_{U_i} \simeq \mathcal{O}_X^n|_{U_i}$ ⟵ Locally free of rank $n$

⇝ Global sections of such a $\mathcal{F}$ are projective modules over $\Gamma(X, \mathcal{O}_X)$. And all projective modules occur this way.

4) $X = \mathbb{A}^1_k = \operatorname{Spec} k[t]$

$(0) \overset{i_0}{\hookrightarrow} \mathbb{A}^1_k$, $\mathcal{F} = (i_0)_* k(t)$ skyscraper sheaf at $0$

NOT quasi-coherent

**Fact**: Sheaf of $\mathcal{O}_X$-module $\mathcal{F}$ is quasi-coherent

$\Leftrightarrow \forall \operatorname{Spec} A_f \subsetneq \operatorname{Spec} A \overset{\text{open}}{\subset} X$

$$\Gamma(\mathcal{F}, \operatorname{Spec} A)_f \longrightarrow \Gamma(\mathcal{F}, \operatorname{Spec} A_f)$$

this map is an isomorphism.

**Prop**: $X$ qcqs (= coverable by finitely many open affine, intersection of any two of these is coverable by finitely many affines) $\mathcal{F}$ q coherent on $X$, then for $f \in \Gamma(X, \mathcal{O}_X)$
$\Gamma(X, \mathcal{F})_f \longrightarrow \Gamma(X_f, \mathcal{F})$ is an isomorphism.

**Prop**: $\pi: X \longrightarrow Y$ qcqs morphism, qcoherent on $X$ then $\pi_* \mathcal{F}$ is a quasi-coh. $\mathcal{O}_Y$ module.

## §13.4 Q. coherent sheaves form an Abelian category

ker, images, quotients exist.

## § 13.5 Module like constructions

· Tensor products

· $Y \overset{i}{\hookrightarrow} X$ a closed embedding corresponding to ideal sheaf $\mathscr{I}$

$\iff$ $\mathscr{I}$ is a q-coherent sheaf on $\mathcal{O}_X$

## § 13.6 Finite type & coherent sheaves

Def: A-mod M is

1) fin. generated if $\exists\, A^p \longrightarrow M \longrightarrow 0$ (or finite type)

2) fin-presented if $\exists\, A^m \longrightarrow A^p \longrightarrow M \longrightarrow 0$

3) coherent if fin. gen & ker $A^p \longrightarrow M$ is finitely generated for all p.

Prop: A noetherian $\Rightarrow$ all 3 equivalent

Coherent A-mods form abelian subcategory of A-mods

Def: Extend naturally to modules over $\mathcal{O}_X$.

· Enough to check these properties on some open affine cover.

## § 13.7 Good properties of finite type and coherent sheaves

$\mathcal{F}, \mathcal{G} \in \mathcal{O}_X$-mod

$\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) = \operatorname{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ sheaf of $\mathcal{O}_X$-modules.

· Not in general q-coherent even when $\mathcal{F}, \mathcal{G}$ are! :(

Easy: · $\mathcal{F}$ is finitely presented then this holds.

· $\mathcal{F}, \mathcal{G}, \mathcal{O}_X$ coherent $\Rightarrow$ so is $\mathcal{H}om(\mathcal{F}, \mathcal{G})$.

Rem: If $X = \operatorname{Spec} A$, then the category of quasi-coherent X modules is equivalent to A-mod.