Classical Actions

– John Campbell

2 views of Classical physics – Hamiltonian , Lagrangian

"Local" "Global"

Diff Eq Action

Actions: $\mathcal{L}(x, \dot{x}, t) \rightarrow$ Lagrangian

Principle of least action:

for a physical system $\exists$ an $\mathcal{L}$ such that a particle follows a path which minimizes action. (comment: local minimum)

$$S(\bar{x}) = \int_{t_0}^{t_1} \mathcal{L}(x, \dot{x}, t)\, dt$$

(Q. Why only $\dot{x}$?)

Finding minimal paths:

Let $\bar{x}$ be least action path. δx a perturbation, $\delta x(t_0) = \delta x(t_1) = 0$

$$\delta S = S[\bar{x} + \delta x] - S[\bar{x}]$$

$\nwarrow$ We want δS 0 to first order.

$$S(\bar{x} + \delta x)$$

$$= \int_{t_0}^{t_1} \mathcal{L}(\bar{x} + \delta x, \dot{\bar{x}} + \delta \dot{x}, t)\, dt$$

} Not rigorous

$$= \int_{t_0}^{t_1} \mathcal{L}(\bar{x}, \dot{x}, dt) + \delta \dot{x} \frac{\partial \mathcal{L}}{\partial x} + \delta x \cdot \frac{\partial \mathcal{L}}{\partial x} \Big)\, dt$$

$$= S[\bar{x}] + \int_{t_0}^{t_1} \left(\delta \dot{x} \frac{\partial \mathcal{L}}{\partial \dot{x}} + \delta x \cdot \frac{\partial \mathcal{L}}{\partial x} \right) dt$$

$$= S[\bar{x}] + \underbrace{\left(\delta x \cdot \frac{\partial \mathcal{L}}{\partial \dot{x}} \right)\Big|_{t_0}^{t_1}}_{0} - \int_{t_0}^{t_1} \delta x \left(\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} \right) dt$$

Why?

Hence we need $\quad \dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \dfrac{\partial \mathcal{L}}{\partial x} = 0$

Ex: $\mathcal{L} = \underbrace{\frac{m}{2}\dot{x}^2}_{} - V(x,t) \rightsquigarrow \frac{d}{dt}(m\dot{x}) - \frac{\partial V}{\partial t} = 0 \equiv ma = F$

Lagrangian for particle in a potential.

Quantum:

Feynman: There is a Lagrangian formulation of quantum mechanics.

Instead of least action, add all paths with appropriate phase.

"$\int_{t_0}^{t_1} e^{t/\hbar\, S(x)} \mathcal{D}x$" — measure over all paths with $x(t_0), x(t_1)$ fixed

amplitude of finding the particle at $x(t_1)$

Physical theory:

1. Space-time ⟶ manifolds
2. Fields ⟶ local structures
3. Action ⟶ some function on fields

(1–3: Field theory)

4. Measure on paths/fields ⟶ Quantum field theory

Ex: G - finite group Chern-Simons Theory for finite groups

Spacetime: $(d+1)$ dim manifold with d dim boundary

Fields: principal G-bundles

Action: $\alpha \in H^{d+1}(BG, \mathbb{R}/\mathbb{Z})$ consider $\hat{\alpha} \in C^{d+1}(BG, \mathbb{R}/\mathbb{Z})$

X - closed oriented $(d+1)$ manifold

Let $P \to X$ be G-bundle,

$S_X(P) = \hat{\alpha}(P_*[X])$

Quantize: Sum over all principal bundles ⟶ form a groupoid $\mathcal{C}_X$

Measure on a groupoid: $Y \in \mathcal{G}$ $\quad \mu(Y) = \frac{1}{|\mathrm{Aut}(Y)|}$

$$Z_X = \sum_{P \in \mathcal{C}_X} \mu([P]) \cdot e^{2\pi i S_X([P])}$$

$$= \sum \frac{1}{|\mathrm{Aut}(P)|} e^{2\pi i S_X([P])}$$

Ex: $\alpha = 0$ then $S_X = 0$

$$Z_X = \sum \frac{1}{|\mathrm{Aut}(P)|} \Rightarrow |\mathcal{C}_X| = \mathrm{Hom}(\pi_1 X; G)/G$$ ~ conjugation action.

Extension to Manifolds with boundary:

Imp. Construction

Def^n: $\mathcal{C}$ = groupoid (connected), $\mathcal{L}$ = Category of metrized $\mathbb{C}$, Morphisms unitary morphisms.

$F : \mathcal{C} \longrightarrow \mathcal{L}$ functor

"Space of invariant sections" $:= \lim\limits_{\mathcal{C}} F$ Check this

Remark: If F has no $\underbrace{\text{holonomy}}_{??}$ limit is a line, else it does not exist.
Check this

Q. How to integrate element $\alpha \in H^{d+1}(BG; \mathbb{R}/\mathbb{Z})$ over d-dim?

Define $\mathcal{C}_Y$ → category with objects – $y \in C_d(Y)$ representing $[Y] \in H_d(Y)$

Y – n dim oriented mor – $y \to y'$ $y \equiv y + \partial a$

$F : \mathcal{C}_Y \to \mathcal{L}$ $F(y) = \mathbb{C}$

$F(y \to y') = e^{2\pi i \, \hat{\alpha}(a)}$

Define: $I_{Y,\alpha}$ = Integration line

$:= \lim\limits_{\mathcal{C}_Y} F$

Freed & Quinn : Chern Simons theory with finite gauge groups. ⟵ Reference.

$\hat{\alpha} \in C^{d+1}(BG; \mathbb{R}/\mathbb{Z})$

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G · finite groups

$Q \longrightarrow \mathcal{L}_Q \qquad Q \longrightarrow Y$

$P \longmapsto e^{2\pi i S_X(P)} \qquad P \longrightarrow X$

1) Functorial

2) Orientation $\quad L_{\bar{Q}} = \overline{L_Q}$

3) Additive : $\quad L_{Q \amalg Q'} = L_Q \otimes L_{Q'}$

Gluing: $\quad Y \hookrightarrow X$ codim 1 submanifold

$X^{cut} = X$ cut along Y

$\partial X^{cut} = \partial X \amalg Y \amalg -Y$

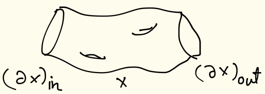
$P^{cut} =$ Bundle over X^{cut}, $Q = P|_Y$

$$e^{2\pi i S_X[P]} = \mathrm{Tr}_Q\left(e^{2\pi i S_{X^{cut}}[P^{cut}]}\right)$$

Epilogue:

$$S_{[t_0,t_1] \amalg [t_1,t_2]} = S_{[t_0,t_1]} + S_{[t_1,t_2]}$$

$n = d+1$



$\rightsquigarrow$ Fields : · orientation

· principal G-bundle

Theory: $\quad I_\alpha\left(Y^d, \text{orientation}, \begin{smallmatrix} Q \\ \downarrow \\ Y \end{smallmatrix}\right) \longmapsto$ complex line

$$\begin{array}{ccc} L_0 & \xrightarrow{e^{2\pi i S}} & L_1 \\ \| & & \| \\ I_\alpha((\partial X)_0) & & I_\alpha((\partial X)_1) \end{array}$$