Homological Algebra

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R-mod category of left R-modules R= ring with 1
(mod-R) (right) need not be commutative
finite generated unless otherwise stated

<u>claim.</u> set of short exact seq of type

$$0 \longrightarrow B \longrightarrow \cdot \longrightarrow A \longrightarrow 0$$

modulo eq. relation form an abelian group called $Ext^1_R(A,B)$
$\downarrow$
extension of A by B
with 0 being $A \oplus B$.

$$0 \longrightarrow B \xrightarrow{i} A \oplus B \xrightarrow{\pi} A \longrightarrow 0$$

Equivalence:

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r, "f_1"] \arrow[d, equal] & E_1 \arrow[r, "g_1"] \arrow[d, "\phi"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r, "f_2"'] & E_2 \arrow[r, "g_2"'] & A \arrow[r] & 0
\end{tikzcd}$$

$(f_1, g_1)$ , $(f_2, g_2)$ are equivalent if $\exists \phi : E_1 \to E_2$
~~st~~ s.t. the diagrams commute.

1) $\phi$ is injective

$$\begin{tikzcd}
e \arrow[r] \arrow[d] & 0 \\
0 \arrow[ur] &
\end{tikzcd}$$

$\phi(e) = 0 \Rightarrow g_2 \cdot \phi(e) = 0$

$\Rightarrow g_1(e) = 0$

$\Rightarrow e \in \cancel{B}\, f_1(B)$

$\Rightarrow e = 0$

$$\begin{tikzcd}
 & e \arrow[d] \\
e \arrow[ur] \arrow[r] & 0
\end{tikzcd}$$

2) $\phi$ is surjective

$e \in E_2$ $g_2(e)$

Look at $g_1^{-1} g_2(e)$

$$\begin{tikzcd}
e' \in g_1^{-1} g_2(e) \arrow[dr] & \\
e \arrow[r] & g_2(e)
\end{tikzcd}$$

Take some $e' \in g_1^{-1} g_2(e)$

$\phi(e') \in g_2^{-1} g_2(e)$ $\quad \because g_2\phi(e') = \cancel{g_2}\, g_1(e') = g_2(e)$

$\Rightarrow g_2(\phi(e')) - g_2(e) = 0$

$\Rightarrow \phi(e') - e \in f_2(B)$

$\Rightarrow \exists\, b \in B$ s.t. $\phi(e') - e = f_2(b)$

Claim: $e' - f_1(b)$ will work

$\phi(e') - \phi(f_1(a))$

$= \phi(e') - f_2(b)$

$= e$ $\square$

So commutativity of squares imply that

$\phi$ is an isomorphism.

So we have an equivalence relation.

Now, we need to define an ~~additive~~ abelian group structure.

$$[E_1] + [E_2] = \left[\{(e_1, e_2) \mid e_1 \in E_1,\ e_2 \in E_2,\ g_1(e_1) = g_2(e_2)\}\right]$$

$$\begin{tikzcd} E_1 + E_2 \arrow[r, dashed] \arrow[d, dashed] & E_1 \arrow[d, "g_1"] \\ E_2 \arrow[r, "g_2"'] & A \end{tikzcd}$$

1) well defined:

~~$E_1 \sim E_1' \Rightarrow ?\ E_1 + E_2 \sim E_1' + E_2$~~

- $[E_1 + E_2] \in \mathrm{Ext}^1_R(A, B)$

Need to define $B \xrightarrow{f_1+f_2} E_1 + E_2$,

$E_1 + E_2 \xrightarrow{g_1+g_2} A$

$g_1 + g_2\,(e_1, e_2) := g_1(e_1) = g_2(e_2)$ by construction

$f_1 + f_2\ (b) := (f_1(e_1), f_2(e_2))$ works $\because g_1 \circ f_1 = 0 = g_2 \circ f_2$

Exactness easily checked.

- $E_1 \xrightarrow{\phi} E_1'$ ~~iso~~ equivalence

$? \Rightarrow [E_1 + E_2] = [E_1' + E_2]$

$\phi(e_1, e_2) = (\phi(e_1), e_2)$

Easy to check equivalence

2) Zero:

$$E \dotplus A\oplus B \overset{?}{\sim} E$$
$$\|$$
$$\{(e, (a,b)) \mid g(e) = a\}$$

$$\cancel{\phi : E \longrightarrow E + A\oplus B}$$
$$\cancel{e \longmapsto (e, (g(e), 0))}$$

$$\begin{tikzcd} & f(b) \arrow[d, "\phi"] \\ b \arrow[ur, mapsto] \arrow[r, mapsto] & f(b),(0,0) \end{tikzcd}$$

$$\phi : E + A\oplus B \longrightarrow E$$
$$e, (g(e), b) \longmapsto e$$

$$\begin{tikzcd} & f(b),(0,b) \arrow[d] \\ b \arrow[ur] \arrow[r] \arrow[ur, phantom, "?"] & f(b) \end{tikzcd} \qquad \begin{tikzcd} e,(g(e),b) \arrow[r] \arrow[d] \arrow[dr, phantom, "?"] & g(e) \\ e \arrow[ur, "g(e)"'] & \end{tikzcd}$$

3) Inverse:

Given $E_1$, find $E_2$ s.t.

$$E_1 + E_2 \sim \cancel{E_1 \oplus E} \; A\oplus B$$

??

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$$A \underset{\pi_A}{\overset{i_A}{\rightleftarrows}} C \underset{i_B}{\overset{\pi_B}{\rightleftarrows}} B \qquad \text{when is } C \cong A\oplus B\,?$$

1) $\pi_A \circ i_A = 1_A$

$\pi_B \circ i_B = 1_B$

in an additive category one has a 0 morphism

$\pi_A \circ i_B = 0 = \pi_B \circ i_A$

and $i_A \circ \pi_A + i_B \cdot \pi_B = id$

2) C product and coproduct.

Ex: Prove $Ext^1_R(A,B) = 0 \quad \forall A, B$   f.g. modules only.

$\Leftrightarrow$ every $R$ module is $\underbrace{\text{semi simple}}$

$M = M_1 \oplus M_2 \oplus \cdots \oplus M_n \quad \forall M$

s.t. $M_i$ has no proper $R$-submodule.

$\Rightarrow M_i \cong R/I$ for some ideal $I$ in $R$

Proof:

$\Rightarrow$   $\forall A, B, E$ s.t.

$$0 \to B \to E \to A \to 0$$

$\exists\ \phi$ isomorphism, such that

$$\begin{tikzcd} & & E \arrow[dd, "\phi"] \arrow[dr] & & \\ 0 \arrow[r] & B \arrow[ur] \arrow[dr] & & A \arrow[r] & 0 \\ & & A \oplus B \arrow[ur] & & \end{tikzcd}$$

Take $A = Rx$ for some $x \in M$.

$$0 \to Rx \to M \to M/Rx \to 0$$

$Rx \cong R/\text{Ann } x \quad \Rightarrow \quad M = Rx \oplus M/Rx$

Do this for all generators of $M$ inductively.

So we get $M \cong \oplus R/I$

$\Leftarrow$

$M = M_1 \oplus \cdots \oplus M_k = \oplus R/I_i$

$B = \oplus R/I_{B_i}$

$A = \oplus R/I_{A_i}$   $E = \oplus R/I_{E_i}$

Since a map ~~bet~~ from a simple module to any other module can only be inclusion or 0, we get $E \cong A \oplus B$

i.e. $Ext^1_R(A,B) = 0$.

- Also true for all non-f.g. modules.

See Pg. (4.5)

Exactness of functors:

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$ Assume $R$-commutative

Claim: for $N$, $R$-module

$$0 \to \mathrm{Hom}(N,A) \to \mathrm{Hom}(N,B) \to \mathrm{Hom}(N,C) \to [illegible] \to$$

Proof:

- $0 \to \mathrm{Hom}(N,A) \to \mathrm{Hom}(N,B)$

  $(\phi : N \to A) \mapsto (f\circ\phi : N \to A \to B)$

  $\because$ $f$ is injective,

  $f\circ\phi$ is also injective

  $\Rightarrow$ exactness here.

- $\mathrm{Hom}(N,A) \to \mathrm{Hom}(N,B) \to \mathrm{Hom}(N,C)$

  $\phi \mapsto f\circ\phi$

  $\psi \mapsto g\circ\psi$

  $\Rightarrow \mathrm{im}(\mathrm{Hom}(N,f)) \subseteq \ker \mathrm{Hom}(N,g)$

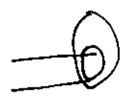
  $\phi \mapsto g\circ f\circ\phi = 0.$

  - $\mathrm{im}\ \mathrm{Hom}(N,f) \supseteq \ker(N,g)$ ? How?

  Need to show $g\circ\psi = 0 \Rightarrow \psi = f\circ\phi$ for some $\phi$

$$\begin{tikzcd} N \arrow[r, "\psi"] \arrow[dr, dashed, "\phi ?"'] & B \arrow[r, "g"] & C \\ & A \arrow[u, "f"'] & \\ & 0 \arrow[u] & \end{tikzcd}$$

is 0 map



B

$g\circ\psi(n) = 0$

$\Rightarrow \psi(n) = f(a)$ for some $a$

set $\phi(n) = (a)$.

$R$-module map?

- $f\circ[\phi(n_1)+\phi(n_2)] = f(a_1)+f(a_2)$

$= \psi(n_1)+\psi(n_2) = \psi(n_1+n_2)$

$= f\circ\phi(n_1+n_2)$

$f$ injective $\Rightarrow \phi(n_1)+\phi(n_2) = \phi(n_1+n_2)$

- Similarly $f\circ\phi(rn_1) = r\cdot f\cdot\phi(n_1) = f\circ r\phi(n_1)$

$\Rightarrow \phi(rn_1) = r\phi(n_1)$

Example where surjectivity breaks down?

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

$N = \mathbb{Z}/n\mathbb{Z}$

$$\begin{tikzcd} N = \mathbb{Z}/n\mathbb{Z} \arrow[r, "id"] \arrow[dr, dashed, "\times"'] & \mathbb{Z}/n\mathbb{Z} = C \\ & \mathbb{Z} = B \arrow[u] \end{tikzcd}$$ does not lift

- Additive structure on $\mathrm{Ext}^1_R(A,B)$

Def$^n$:

$$0 \to B \xrightarrow{f_1} E_1 \xrightarrow{g} A \to 0$$
$$0 \to B \xrightarrow{f} E_2 \xrightarrow{g} A \to 0$$

Note: Abuse of notation for $f, g$

$[E_1 + E_2]$ :

$$\begin{tikzcd} B \arrow[r] \arrow[d] & E_1 \arrow[d] \\ E_2 \arrow[r] & \text{Pushout} \end{tikzcd} \longleftarrow \begin{tikzcd} \text{Pull back} \arrow[r] \arrow[d] & E_1 \arrow[d] \\ E_2 \arrow[r] & A \end{tikzcd}$$

The image of pullback in the pushout i.e.

$$E_1 + E_2 := \frac{\{(e_1, e_2) \mid g(e_1) = g(e_2)\} \subseteq E_1 \oplus E_2}{(e_1, e_2) \sim (e_1', e_2') \text{ if } \exists\, b \in B \text{ s.t. } e_1 - e_1' = f(b) = -e_2 + e_2'}$$

$f := b \longmapsto [f(b), 0]$

$g := [(e_1, e_2)] \longmapsto g(e_1)$

Well defined:

by naturality of pull back, pushout

Exactness:

$g[e_1, e_2] = 0 \Rightarrow g(e_2) = 0 \Rightarrow \exists\, b_2, b_1 \quad e_2 = f(b_2),\ e_1 = f(b_1)$

So $[e_1, e_2] = [f(b_1), f(b_2)]$

$= [f(b_1), f(b_2)] = [f(b_1 - b_2), 0] \in \operatorname{im} f$

$g \circ f = g\,[f(b), 0] = 0$

Commutativity:

$$E_1+E_2 \sim E_2+E_1$$

$$(e_1,e_2) \longmapsto (e_2,e_1)$$

Zero:

$$E + A\oplus B = \frac{\{(e,(a,b)) \mid a = g(e)\}}{(e,a,b)\sim(e',a',b') \text{ if } e-e' = -f(b)+f(b')}$$

$$\phi: E + A\oplus B \longrightarrow E$$
$$[(e,a,b)] \longmapsto e - f(b)$$

$$\begin{tikzcd} & E+A\oplus B \arrow[d] \\ B \arrow[ur] \arrow[r] & E \end{tikzcd} \qquad \begin{tikzcd} & f(b),(0,0) \arrow[d] \\ b \arrow[ur] \arrow[r] & f(b) \end{tikzcd}$$

$$\begin{tikzcd} E+A\oplus B \arrow[r] \arrow[d] & A \\ E \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} (e,a,b) \arrow[r] \arrow[d] & a = g(e) \\ e-f(b) \arrow[ur] & \end{tikzcd}$$

So $E + A\oplus B \sim E$

Inverse:

Given $0 \longrightarrow B \xrightarrow{f} E \xrightarrow{g} A \longrightarrow 0$

Inverse is given by

$0 \longrightarrow B \xrightarrow{-f} E \xrightarrow{g} A \longrightarrow 0$ call this $-E$

$$E - E = \frac{\{(e_1,e_2) \mid g(e_1)=g(e_2)\}}{(e_1,e_2)\sim(e_1',e_2') \text{ if } e_1-e_1' = e_2-e_2'}$$

$$= \frac{\{(e_1,e_2) \mid e_1-e_2 \in f(B)\}}{(e_1,e_2)\sim(e_1',e_2') \text{ if } e_1-e_2 = e_1'-e_2'}$$

$$\phi: A\oplus B \longrightarrow E-E \qquad\qquad E-E \longrightarrow A\oplus B$$
$$a,b \longmapsto (e, e+f(b)) \qquad\qquad [e_1,e_2] \longmapsto g(e_1), f^{-1}(e_1-e_2)$$
$$g(e)=a$$

$$\begin{tikzcd} & E-E \arrow[d, "\phi^{-1}"] \\ B \arrow[ur] \arrow[r] & A\oplus B \end{tikzcd} \qquad \begin{tikzcd} & f(b),0 \arrow[d] \\ b \arrow[ur] \arrow[r] & (0,b) \end{tikzcd}$$

$$\begin{tikzcd} & E-E \arrow[d, "\phi^{-1}"] \\ A \arrow[ur] \arrow[r] & A\oplus B \end{tikzcd} \qquad \begin{tikzcd} & e,0 \arrow[d] \\ a \arrow[ur] \arrow[r] & a,0 \end{tikzcd} \qquad g(e)=a$$

So

$E - E = A\oplus B$

Associativity needs to be checked.

⊠

following ~~is true~~ for all R-modules are equivalent

- Every module is sum of simple modules
- Every module is ~~su~~direct sum of simple modules
- Every submodule has a complement.

So, Use the above proposition to show that the statement about semisimplicity is true for non-finitely generated modules too.

Q. Given $R$, 1) for which $A$ do we have $Ext(A,B) = 0 \ \forall B$?

2) similarly which $B$, $Ext(\cancel{B}, A, B) = 0 \ \forall A$?

$0 \to B \to * \to A \to 0$

1) for any $M \to A$ ∃ $A \to M$ (split).

04/01/13

- Chain complex of R-modules – $C_\bullet$

co-chain – $C^\bullet$

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09/01/13

Q. How to find order of an element in $Ext^1_R(A,B)$?

$Ch(\text{mod-}R)$ – category of chain complexes of right R-modules. R – not necessarily commutative

This category has kernels, cokernels, images, short exact sequences

Th$^m$: Given a short exact sequence of chain complexes

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

we get long exact seq of ~~co~~homologies

$$\to H_n(A) \to H_n(B) \to H_n(C) \xrightarrow{\delta} H_{n-1}(A) \to H_{n-1}(B) \to H_{n-1}(C) \to$$

## Snake Lemma

$$\begin{tikzcd} & A' \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & B' \arrow[r] \arrow[d, "g"] \arrow[dr, phantom, "\circlearrowleft"] & C' \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C & \end{tikzcd}$$

exact

exact

$$\ker f \longrightarrow \ker g \longrightarrow \ker h \xrightarrow{\partial} \operatorname{coker} f \longrightarrow \operatorname{coker} g \longrightarrow \operatorname{coker} h$$

Using snake lemma

$$\begin{tikzcd} & A_n/dA_{n+1} \arrow[r] \arrow[d, "d"'] & B_n/dB_{n+1} \arrow[r] \arrow[d, "d"'] & C/dC_{n+1} \arrow[r] \arrow[d, "d"'] & 0 \\ 0 \arrow[r] & \cancel{A_{n-1}/dA_n}\ Z_{n-1}(A_\bullet) \arrow[r] & \cancel{B_{n-1}/dB_n}\ Z_{n-1}(B_\bullet) \arrow[r] & Z_{n-1}(C_\bullet) & \end{tikzcd}$$

we get the long exact seq.

Verifications:

Exactness of

$$A_n/dA_{n+1} \xrightarrow{f} B_n/dB_{n+1} \xrightarrow{g} C_n/dC_{n+1} \longrightarrow 0$$

$\boxed{\xrightarrow{f} \quad \xrightarrow{g}}$

$g \circ f [a] = [g \circ f a]$
$= 0$

$\operatorname{im} f = \ker g$ ?

$g[b] = 0 \Rightarrow gb = dc$

$$\begin{tikzcd} b'_4 \arrow[r] & c_3 \arrow[d] \\ b_1 \arrow[r, mapsto] & dc_2 \end{tikzcd}$$

$gb' = c \Rightarrow gdb' = dgb'$
$= dc$
$= gb$

$\Rightarrow db' - b = fa$

$\Rightarrow [fa] = [b].$

For same reason $f$ is well-defined ← Just need $g$ well defined

$\boxed{B_n/dB_{n+1} \longrightarrow C_n/dC_{n+1} \longrightarrow 0}$

$[b] \longrightarrow [c]$

$b' - b = db''$

$\Rightarrow gb' - gb = \cancel{dgb}\ gdb''$
$= \cancel{gdb''}\ dgb''$

$\Rightarrow [gb'] = [gb].$

Exactness of

$$0 \to Z_n(A_\cdot) \to Z_n(B_\cdot) \to Z_n(C_\cdot)$$

$$0 \to Z_n(A_\cdot) \to Z_n(B_\cdot)$$

$$a \overset{f}{\longmapsto} b$$

$da = 0$

$\Rightarrow db = dfa$

$= fda$

$= 0$

$$Z_n(A_\cdot) \to Z_n(B_\cdot) \to Z_n(C_\cdot)$$

$b \to 0$

$g(a) = b$

why $da = 0$?

$db = 0$, $gb = 0$

$f(da) = d(fa) = db = 0$

$\Rightarrow da = 0$ ∵ $f$ injective.

Proof of snake lemma:

$$\begin{tikzcd}
& A' \arrow[r, "f"] \arrow[d, "d"] & B' \arrow[r, "g"] \arrow[d, "d"] & C' \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & A \arrow[r, "f"'] & B \arrow[r, "g"'] & C &
\end{tikzcd}$$

$$\ker d_1 \to \ker d_2 \to \ker d_3$$

$gb = 0$

$db = 0$

$fa = b$

$fda = dfa$

$= db = 0$

$\Rightarrow da = 0$ ∵ $f$ injective below

$$\operatorname{coker} d_1 \xrightarrow{f} \operatorname{coker} d_2 \xrightarrow{g} \operatorname{coker} d_3$$

$f$ well defined:

$f[a] := [fa]$

$a' = a - da''$

$fa' - fa = fda''$

$= dfa''$

$\Rightarrow [fa'] = [fa]$

$g$ well defined for same reason

$gb = dc$

$gb' = c$

$$\begin{tikzcd}
b' \arrow[r] & c \arrow[d, no head] \\
b \arrow[r] & dc
\end{tikzcd}$$

$g\,db' = dgb'$

$= dc$

$= gb$

$\Rightarrow b - db' = fa$

$\Rightarrow f[a] = [b]$

$$\ker d_3 \xrightarrow{\partial} \operatorname{coker} d_1$$

$\partial$-well defined:

$$\begin{tikzcd}
& b \arrow[r] \arrow[d] & c \arrow[d] \\
a \arrow[r] & db & 0
\end{tikzcd}$$

$\partial c = [a]$

$a' - a = da''$

$\Rightarrow f[a'] = f[a]$.

So well defined

Exactness:

$\ker d_2 \xrightarrow{g} \ker d_3 \xrightarrow{\partial} \operatorname{coker} d_1$

- $b \longmapsto gb$

$db = 0$

$$\begin{tikzcd} & b \arrow[r] \arrow[d] & gb \\ 0 \arrow[r] & 0 & \end{tikzcd}$$

$\partial \cdot g = 0$

- $\partial c = [0]$

$\partial c = da$

$$\begin{tikzcd} a \arrow[r] \arrow[d] & b', b \arrow[r] \arrow[d] & c \arrow[d] \\ da \arrow[r] & db & 0 \end{tikzcd}$$

$b' = fa$

$db' = f da$
$\quad = db$

$(b - b')$

$g(b - b') = gb$
$\quad = c$

$d(b - b') = 0$

$\ker d_3 \xrightarrow{\partial} \operatorname{coker} d_1 \xrightarrow{f} \operatorname{coker} d_2$

- 

$$\begin{tikzcd} & b \arrow[r] \arrow[d] & c \arrow[d] \\ a \arrow[r] & db & 0 \end{tikzcd}$$

$\partial c = [a]$

$f[a] = [fa] = [db] = 0$
$\quad \parallel$
$f \circ \partial c$

- $f[a] = 0 \Rightarrow fa = db$

$$\begin{tikzcd} & b \arrow[r] \arrow[d] & gb \arrow[d] \\ a \arrow[r] & db & 0 \end{tikzcd}$$

$dgb = gdb = gfa = 0$

$\Rightarrow \partial gb = [a]$

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$H_n : Ch(\text{mod-}R) \longrightarrow \text{mod-}R$ functor

Th$^m$: $S$ = cat of short exact sequences in $Ch(\text{mod-}R)$

$\mathcal{L}$ = cat of long exact seq. in mod-$R$

Then $\exists$ a functor

$S \rightsquigarrow \mathcal{L}$.

Ex: 3x3 lemma

In 5-lemma:

$$\begin{tikzcd} & A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C & \end{tikzcd}$$

if 1) $0 \to A' \to B'$ then $0 \to \ker \to \ker$

2) $B \to C \to 0$ then $\mathrm{coker} \to \mathrm{coker} \to 0$.

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Ab~~elian~~ category:

- Hom$(B,A)$ is an abelian group
- distributes ~~over composition~~

$$C \xrightarrow{} B \overset{g_1}{\underset{g_2}{\rightrightarrows}} A \xrightarrow{f} B$$

$$f(g_1+g_2) = fg_1 + fg_2$$

$$(g_1+g_2)f = g_1f + g_2f$$

one point category is a ring

- Additive Functor

$\mathcal{F} : C \longrightarrow D$

$\mathrm{Hom}(B,A) \longrightarrow \mathrm{Hom}(\mathcal{F}A, \mathcal{F}B)$

map of abelian categories

eg: 1) $\mathcal{C} \xrightarrow{\mathrm{Hom}(A,-)} \mathrm{Ab}$

$C = \text{mod-}R$

$$\begin{tikzcd} X \arrow[r, mapsto] \arrow[d, "f"] & \mathrm{Hom}(A,X) \arrow[d, "\mathrm{Hom}(A,f)"] \\ Y \arrow[r, mapsto] & \mathrm{Hom}(A,Y) \end{tikzcd}$$

$$f \longmapsto (\varphi \mapsto f\circ\varphi)$$

$$f+g \longmapsto (\varphi \mapsto (f+g)\circ\varphi)$$

$$\qquad\qquad \| $$

$$(\quad f\circ\varphi + g\circ\varphi \quad)$$

$$\| $$

$$(\varphi \longmapsto f\circ\varphi) + (g \mapsto g\circ\varphi)$$

Additive category:

- Ab~~elian~~ Cat +
- $\exists$ 0-object (initial + terminal)
- finite products exist

Ex: finite products = finite co products

Proofs

Product:

$$\begin{tikzcd} A\times B \arrow[r, "\pi"] \arrow[dr, "p"'] & A \\ & B \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "1"] \arrow[dr, "0"'] & A \\ & B \end{tikzcd} \quad \text{gives} \quad \begin{tikzcd} A \arrow[r, "i"] \arrow[rr, dashed, bend left, "1"] \arrow[drr, dashed, bend right, "0"'] & A\times B \arrow[r] \arrow[dr] & A \\ & & B \end{tikzcd}$$

similarly

$$\begin{tikzcd} B \arrow[r, "0"] \arrow[dr, "1"'] & A \\ & B \end{tikzcd} \qquad B \xrightarrow{j} A\times B$$

Co-product property:

$$\begin{tikzcd} C & A \arrow[l, "f"'] \\ & B \arrow[ul, "g"] \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[dr, "i"] & & B \arrow[dl, "j"'] \\ & A\times B \arrow[dl, "\pi"'] \arrow[dr, "p"] & \\ C & A \arrow[l, "f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

$f\cdot\pi + g\cdot p$

$$\begin{tikzcd} & B \arrow[dl, "g"'] \arrow[dr, "j"] & \\ C & & A\times B \arrow[ll, "g\cdot p + f\cdot\pi"'] \\ & A \arrow[ul, "f"] \arrow[ur, "i"'] & \end{tikzcd}$$

2) $\mathcal{C} \xrightarrow{M \otimes_R -} \mathcal{C}$

$X \longmapsto M \otimes_R X$

Example of non-additive functor:

3) $\mathcal{C} \longrightarrow \mathcal{C}$

$x \longmapsto x \otimes x$

Coproduct: $A \xrightarrow[i]{} A \oplus B \xleftarrow[\pi]{} B$

$$\begin{tikzcd} A \arrow[r, "1"] & A \\ & B \arrow[u, "0"'] \end{tikzcd} \longmapsto A \oplus B \xrightarrow{\pi} A$$

$$\begin{tikzcd} A \arrow[r, "0"] & A \\ & B \arrow[u, "1"'] \end{tikzcd} \longmapsto A \oplus B \xrightarrow{p} B$$

Product property:

$B \xleftarrow{g} C \xrightarrow{f} A$

$$\begin{tikzcd} & & A \\ C \arrow[r, "i \cdot f + j \cdot g"] & A \oplus B \arrow[ur, "\pi"] \arrow[dr, "p"'] & \\ & & B \end{tikzcd}$$

---

Abelian category:

In addintion to being additive

- monics $\quad Y \xrightarrow{f} I \xrightarrow{i} X \qquad i f = 0 \Rightarrow f = 0$
- epics $\quad X \xrightarrow{\pi} I \xrightarrow{f} Y \qquad f \cdot \pi = 0 \Rightarrow f = 0$
- kernels
- cokernels
- ~~cokernel~~ of ~~monic~~ ~~is~~
  a monic is kernel of its cokernel
  a epic is cokernel of its kernel.

$\longrightarrow$ Prove kernel is monic.

$K \xrightarrow{i} X \xrightarrow{f} Y \quad \rightsquigarrow \quad Z \xrightarrow{g} K \xrightarrow{i} X \xrightarrow{f} Y$

$$\begin{tikzcd} Z \arrow[r, shift left, "0"] \arrow[r, shift right, "g"'] \arrow[rr, bend right] & K \arrow[r, "i"] & X \end{tikzcd}$$

$i \cdot g = 0 \Rightarrow$
$= i \cdot 0$
$\Rightarrow \boxed{g = 0}$

$\longrightarrow$ cokernel is epic

$X \xrightarrow{f} Y \xrightarrow{\pi} C \quad \rightsquigarrow$

$$\begin{tikzcd} Y \arrow[r, "\pi"] \arrow[rr, bend right, "0"'] & C \arrow[r, "g"] & Z \end{tikzcd}$$

$g \cdot \pi = 0 = 0 \cdot \pi$
$\Rightarrow \boxed{g = 0}$

$\longrightarrow$ $K \xrightarrow{i} X \xrightarrow{f} Y \xrightarrow{\pi} C$

$$\begin{tikzcd} K \arrow[r, "i"] & X \arrow[r, "\pi'"] \arrow[d, dashed, "f'"'] \arrow[drr, "f"] & I & & \\ & I' \arrow[rr, "i'"'] & & Y \arrow[r, "\pi"] & C \end{tikzcd}$$

$f$ factors through $c'$
$f = i' f'$
$f'$ factors through $\pi'$
$f' = f'' \pi'$
So $\quad f = i' \cdot f'' \cdot \pi'$

Need to show $f''$ is an isomorphism

what is the kernel of $f''$?

$$\begin{tikzcd} K \arrow[r, "i"] & X \arrow[r, "\pi'"] & I \arrow[d, "f''"] & Z \arrow[l, "g"'] \\ & & I' \arrow[r, "i'"'] & Y \arrow[r, "\pi"'] & C \end{tikzcd}$$

Suppose $Z \xrightarrow{g} I \xrightarrow{f''} I''$

$f''\cdot g = 0$

$\Rightarrow i'\cdot f''\cdot g = 0$

$\Rightarrow$

Q. Does monic + epic $\Rightarrow$ isomorphism?

Q. ~~How to~~ Consider this special chain complex:

$$C_n := B_n \oplus H_n \oplus B_{n-1} \qquad d(a,b,c) = (c,0,0)$$

$$\downarrow$$

$$C_{n-1} := B_{n-1} \oplus H_{n-1} \oplus B_{n-2}$$

Complex: $d^2(a,b,c) = d(c,0,0) = (0,0,0)$

~~co~~homology: $Z_n = B_n \oplus H_n \oplus 0 \qquad B_n = B_n \oplus 0 \oplus 0$

$\Rightarrow H_n(C_\cdot) = H_n$

for vector space chain complex:

$C_n$ - dim $f(n)$. vector space over $k$

$H_n(C_\cdot)$ - dim $g(n)$ vector space over $k$.

$$\begin{tikzcd} B_n \arrow[r, hook] & Z_n \arrow[r, hook] \arrow[dr, two heads] & C_n \\ & & H_n \end{tikzcd}$$

$B_{n-1} = C_n / Z_n$

Example of a non-abelian category:

Filtrations of abelian groups, morphisms respecting the filtration

$$\begin{tikzcd} 0 \arrow[r, hook] & H_1 \arrow[r, hook] \arrow[d] & H_2 \arrow[r, hook] \arrow[d] & \cdots \arrow[r, hook] & H_n \arrow[d, "f"] \\ 0 \arrow[r, hook] & G_1 \arrow[r, hook] & G_2 \arrow[r, hook] & \cdots \arrow[r, hook] & G_n \end{tikzcd}$$

Kernels: $0 \hookrightarrow \ker f|_{H_1} \hookrightarrow \ker f|_{H_2} \hookrightarrow \cdots \hookrightarrow \ker f$

$F_\cdot \xrightarrow[g_*]{} H_\cdot \xrightarrow[f_*]{} G_\cdot \qquad f_* g_* = 0$

$\Rightarrow g\cdot f = 0 \quad \Rightarrow \quad \exists\, h : F \longrightarrow \ker f$

Cokernels: These are more subtle

$0 \hookrightarrow \operatorname{im} H_1$ in $\operatorname{coker} f \hookrightarrow \operatorname{im} H_2$ in $\operatorname{coker} f \hookrightarrow \cdots \hookrightarrow \operatorname{coker} f$

$H_\cdot \xrightarrow{f_*} G_\cdot \xrightarrow{g_*} F_\cdot$

$\boxed{\begin{array}{l} 4182766512 - \text{JHU} \\ 4182759372 - \text{IN} \end{array}}$

$g_* \cdot f_* = 0 \quad \Rightarrow \quad g\cdot f = 0$

$\Rightarrow \exists\, h : F \longrightarrow \operatorname{coker} f.$

Why is the category not abelian?

f is ~~inclusion in the first~~ component

$H = 0 \to \dots$

$G = 0 \to \dots$

Ker: $0 \hookrightarrow 0 \hookrightarrow 0$   coker: $0 \hookrightarrow 0 \hookrightarrow 0$

$\Rightarrow$ f monic + epic. So why is f ~~not~~ an isomorphism?

$$\begin{array}{ccccc} 0 & \to & G & \to & H \\ & & \downarrow & & \downarrow \\ 0 & \to & H & \to & H \end{array}$$

Take a subgroup $H < G$ such that there is no map $H \twoheadrightarrow G$
Then look at the sequences

$$\begin{array}{ccccc} 0 & \hookrightarrow & H & \hookrightarrow & G \\ & & \downarrow i & & \downarrow \mathrm{is} \\ 0 & \hookrightarrow & G & \hookrightarrow & G \end{array} \qquad \begin{array}{c} =: C_1 \\ \downarrow i_* \\ =: C_2 \end{array}$$

Then kernel = 0   cokernel = 0.

Why is $i_*$ not an isomorphism?

As there is no map from G to ~~H~~.

Even this is not needed as commutativity of diagram is enough

Q. Do Abelian categories have pullbacks and pushouts?

$A \longrightarrow B \longleftarrow C$   Pullback should be an ~~...~~ subset? of $A \oplus C$

Seems unlikely that all abelian categories have pullbacks and pushout. ~~Counterexample???~~ Not true.

2. Gelfand-Manin:

→ Let $\mathcal{A}$ be an abelian category. Aim is to treat objects of $\mathcal{A}$ as abelian groups. Define:

for $Y \in Ob(\mathcal{A})$

elements of $Y = \{(X, h) \mid X \in Ob(\mathcal{A}),\ h: X \longrightarrow Y\} / \sim$

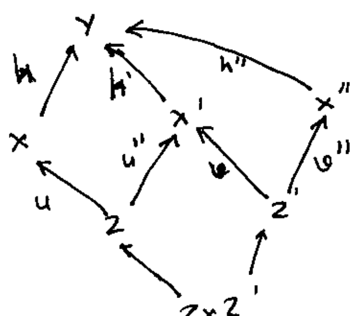
$(X, h) \sim (X', h')$ if $\exists\, Z, u, u'$

$u: Z \to X,\ u': Z \to X'$ s.t.

$hu = h'u'$.

Check: Equivalence Relation:

Transitivity



Take the pull back of

$$\begin{array}{ccc} \bullet & \longrightarrow & Z' \\ \downarrow & & \downarrow \\ Z'' & \longrightarrow & X' \end{array}$$

## Existence of Pull back and Pushout in abelian category:

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd} \qquad \begin{tikzcd} X \times Y \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"] & X \arrow[r, "f"] & Z \\ Y \arrow[r, "g"] & Z & \end{tikzcd}$$

Then: Pull back: $X \times_Z Y = \boxed{\ker[f \cdot \pi_X - g \cdot \pi_Y]}$

$$\begin{tikzcd} X \times_Z Y \arrow[d] \\ X \times Y \arrow[d, "f \times g"] \\ Z \end{tikzcd}$$

Given $$\begin{tikzcd} T \arrow[r, "\alpha"] \arrow[d, "\beta"'] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

we have $T \xrightarrow[\alpha \times \beta]{} X \times Y$

$$(f\pi_X - g\pi_Y) \cdot (\alpha \times \beta) = (f\alpha - g\beta)$$
$$= 0.$$

**Pushout:**

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] & X \\ Y & \end{tikzcd}$$

Pushout: $X \oplus_Z Y :=$ $\boxed{\text{coker } [\, i_X f - i_Y g \,].}$

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[r, "i_X"'] & X \oplus Y \\ Y \arrow[d, "i_Y"] & & \\ X \oplus Y & & \end{tikzcd}$$

$$Z \xrightarrow[f \oplus g]{} X \oplus Y \longrightarrow X \oplus_Z Y$$

Given $T$ :
$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "\alpha"] \\ Y \arrow[r, "\beta"'] & T \end{tikzcd}$$

we have $X \oplus Y \xrightarrow[\alpha \oplus \beta]{} T$

$$(\alpha \oplus \beta) \cdot (i_X f - i_Y g) = \alpha \cdot f - \beta g$$
$$= 0.$$

□