The Euler Characteristic

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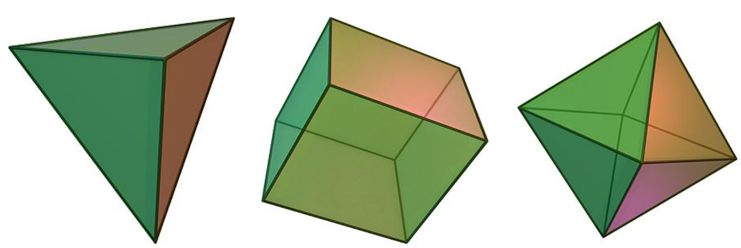
Now that we know how to glue things to create topological objects it is a natural question to ask how one does any kind of mathematics with them. One answer to that is by computing invariants. A topological invariant assigns to a topological space, in our case a surface, an algebraic object such as a number, a polynomial or a vector space. If two surfaces have different topological invariants then they must be topologically inequivalent. However the converse is not always true in that two inequivalent surfaces can have the same topological invariants.

The simplest non-trivial topological invariant is the Euler characteristic. The Euler characteristic assigns to each surface an integer. If two surfaces have different Euler characteristics then they are inequivalent, the converse fails in full generality. However, as we'll see, if two *orientable* surfaces have the same Euler characteristic (so exclude the projective space, the Klein bottle, etc.) then they must be topologically equivalent (or homeomorphic).

Euler characteristic for surfaces is computed using graphs. We begin with the simplest surface, the sphere S^2.

Exercise 1. Let us start with some examples. Consider the following polyhedra (3 dimensional polygons):



tetrahedron cube octahedron

(i) Find the number of vertices v, edges e and faces f for each.

(ii) Compute $v - e + f$ for each.

If you did the computations correctly you'd observe that

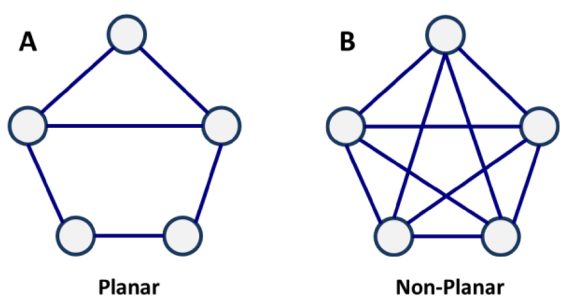
$$v - e + f$$

is the same for all of the three objects. What is common among the three objects? They are all topologically equivalent to the two sphere S^2 which suggests that this number is a topological invariant. Theorem 2 below makes this statement precise.

1. Planar graphs

Before studying the sphere let us try to understand what happens to objects on a plane. We'll start by computing the Euler characteristic of a planar graph.

For us a graph G is a pair of finite sets (V, E) where the vertices, V, are distinct points in the standard plane $\mathbb{R}^2$ and the edges, E, are segments connecting two vertices. We say that a graph is **connected** if each vertex is connected to every other vertex via a sequence of edges. We say that a graph is **planar** if no two edges intersect in a point outside the set of vertices.



A planar graph divides the plane $\mathbb{R}^2$ into **faces**. Denote the set of faces by F. (It is possible for the set F to be empty.) The **Euler characteristic** of a planar graph G is defined to be

$$\chi(G) := |V| - |E| + |F|$$

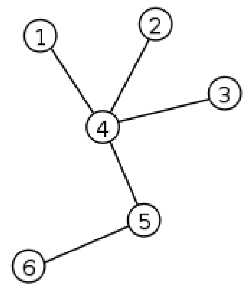
where $|S|$ denotes the size of the set S. For example, for the planar graph **A** above $|V| = 5, |E| = 6, |F| = 2$ and so $\chi(\mathbf{A}) = 5 - 6 + 2 = 1$.

The main theorem we are trying to prove today is the following,

Theorem 1. *The Euler characteristic of a connected planar graph is 1.*

Exercise 3 describes one proof of this theorem using induction on the number of faces. First we prove the theorem in the case when the graph has no faces.

A *connected* graph G without any face, i.e. when F is an empty set or equivalently $|F| = 0$, is called a **tree**. (A graph with no faces but which is not necessarily connected is called a forest!) A vertex in a tree with only one edge attached to it is called a **leaf**.



Tree $(1, 2, 3, 6$ are the leaves)

Exercise 2. For a tree G what is the relationship between $|V|$ and $|E|$? What is $\chi(G)$?

Exercise 3. The following exercise explains how to compute the Euler characteristic for connected planar graphs,

(i) For a connected planar graph G with at least 1 face show that it is possible to delete an edge and obtain a graph G' such that G' has exactly one less face than G.

(ii) What is the relationship between the Euler characteristic of G and G'?

(iii) Induct on $|F|$ to complete the proof of Theorem 1. (What is the base case for induction here?)

Exercise 4. What is the Euler characteristic of a planar graph which is not necessarily connected? (We have already encountered this invariant briefly in a previous class.)

2. Euler characteristic of S^2

A **polygonal graph** on a sphere S^2 is a *connected planar* graph $G = (V, E)$ such that V and E are now on S^2 and all the faces are polygons. The tetrahedron, the cube, and the octahedron, provide examples of such polygonal graphs. A polygonal graph is called a **triangulation** if all the faces are triangles.

Theorem 2. *The Euler characteristic of any polygonal graph G on S^2 is 2. Hence we can define the Euler characteristic of S^2 as $\chi(S^2) := \chi(G)$ and we have,*

$$\chi(S^2) = 2$$

Exercise 5. Theorem 2 follows directly from Theorem 1 for planar graphs by the following observation,

(i) Explain how a polygonal graph on S^2 gives rise to a planar graph on $\mathbb{R}^2$.

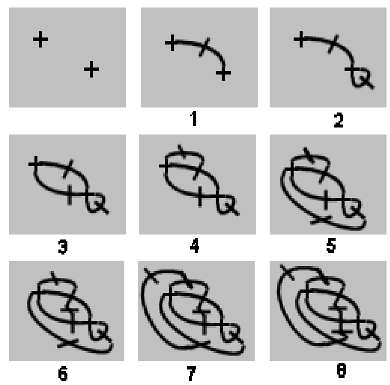
(ii) Draw the planar graphs for the cube, the tetrahedron and the octahedron.

(iii) Show that the Euler characteristic of a graph on S^2 is 2.

As with any other classical theorem there are numerous ways of proving this, see http://www.ics.uci.edu/~eppstein/junkyard/euler/all.html for at least twenty different proofs!!! Check out proof 6 for a particularly slick one.

3. Some applications

3.1. Brussel sprouts. The game of **Brussel sprouts** starts with 2 crosses. Each move involves joining two free ends with a curve not crossing any existing line and then putting a short stroke across the line to create two new free ends. The game ends when no such move is possible.



A game of Brussel sprouts

Exercise 6. Play a few games of Brussel sprouts!

It turns out that every game of Brussel Sprouts always ends in the same number of steps! The following exercises describe a proof of it,

Exercise 7. Let G be the connected planar graph (vertices are the crosses) at the end of the game.

(i) What happens to the number of free ends after each move? How many free ends are there in the end?

(ii) Argue that at each stage of the game every face should have at least one open end on it's boundary. Further, argue that there cannot be two or more open ends on the boundary of a face at the end of the game. Hence every face of G should have exactly 1 open end on it's boundary. The same is true for the *unbounded face.*

(iii) Conclude that G has 7 faces.

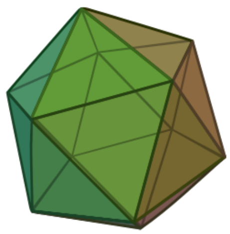
Exercise 8. Assume that the game ends in **n** steps.

(i) How many vertices and edges are added after each move? Argue that $|E| = 2n$ and $|V| = 2 + n$.

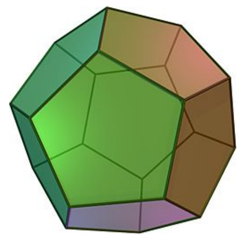
(ii) Use Theorem 1 to find n.

Can you generalize the above proof to k crosses in the beginning? How about playing Brussel sprouts on a Torus? On a Klein bottle??

3.2. Platonic Solids. A platonic solid is a convex polyhedron all of whose faces are regular polygons which are congruent to each other, i.e. all the edges have the same length and all the faces have the same number of sides. We'll assume that the word convex means that the platonic solid is topologically equivalent to S^2. We've seen 3 platonic solids earlier, the cube, the tetrahedron, and the octahedron. There are exactly two more,



icosahedron



dodecahedron

We can use the Euler characteristic to prove that these 5 are the only ones possible.

Consider a platonic solid S and think of it as the graph (V, E, F). Suppose all the faces of S are polygons with **n** number of edges. Further suppose that **p** edges of S intersect at a single vertex.

Exercise 9. Argue that both n and p should be at least 3. What are n and p for the five platonic solids?

Exercise 10.

(i) By counting the total number of edges cleverly, show that $p|V| = 2|E|$ and $n|F| = 2|E|$.

(ii) Use Theorem 2 to conclude

$$\frac{1}{n} - \frac{1}{2} + \frac{1}{p} = \frac{1}{|E|} \tag{3.1}$$

and because $|E| > 0$ this implies $\frac{1}{n} + \frac{1}{p} > \frac{1}{2}$.

(iii) Show that the above inequality cannot hold if n and p are both bigger than 4. Now we have that both n and p are at least 3 and one of them is at the most 4.

(iv) Find (by trial and error) the possible values of $n, p, |E|$ that satisfy (3.1) and relate them to the 5 platonic solids.