Recall: $S \subseteq A$ multiplicative set

$\operatorname{Spec}(S^{-1}A) \subseteq \operatorname{Spec}(A)$ $\{$ideals $\wp \mid \wp \cap S = \phi\}$

- When $S = \{1, f, f^2, \cdots\}$ we denote $S^{-1}A = A_f$

$D(f) := \operatorname{Spec} A_f$

$= \{\wp \text{ prime} \mid f \notin \wp\}$

"distinguished / principal" open subsets.

= locus in Spec A where f does not vanish.

(Such sets form a $\underline{\underline{\text{basis}}}$ for the Zariski topology.)

why?

$D(f) \cap D(g)$
$= \{\wp \mid \wp \not\supseteq \{f, g\}\}$
$= \{\wp \mid \wp \not\ni fg\}$ need prime here
$= D_{fg}$

- $\wp \in \operatorname{Spec} A$ $\quad S := A \setminus \wp$ $\quad S^{-1}A =: A_\wp$

$\operatorname{Spec} A_\wp \cong \{\mathfrak{q} \in \operatorname{Spec} A \mid \mathfrak{q} \subset \wp\}$

$A_\wp$ will be the stalk of $\mathcal{O}_{\operatorname{Spec} A}$ at $\wp$.

$\varphi : A \longrightarrow B$ map of rings

$\varphi^{-1}(\wp\text{rime})$ is a prime

$\Rightarrow \varphi^* : \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$

(This is not true for maximal ideals, so using prime ideals in place of maximal ideals helps.)

Intuition:

$A \longrightarrow B$ is a map of regular functions on $\operatorname{Spec} B \longrightarrow \operatorname{Spec} A$.

This is using a map $C^\infty(X) \longrightarrow C^\infty(Y)$ constructing $Y \longrightarrow X$.

Recall:

$x \in A$ nilpotent if $x^n = 0$ for some n.

$\mathfrak{N}(A) = \{\text{nilpotents in } A\}$

Th^m: $\mathfrak{N}(A) = \bigcap_{\substack{\wp \in A \\ \text{prime}}} \wp$

$\Rightarrow$ If x - nilpotent then x will vanish on every point in Spec A.

§3.4 Closed sets in Spec A.

Let $S \subseteq A$ be any subset

$V(S) := \{ \mathfrak{p} \in \operatorname{Spec} A \mid S \subseteq \mathfrak{p} \}$ vanishing set

If (S) = ideal generated by S then

$V((S)) = V(S)$ so we work with ideals alone.

$=$ solution set to $s = 0$ for $s \in S$.

Eg: $S = (f_1, \dots, f_m) \subseteq k[x_1, \dots, x_n]$, $k = \bar{k}$

maximal ideal $\mathfrak{m} = (x_1 - a_1, \dots, x_n - a_n) \in V(S)$

$\iff f_i(a_1, \dots, a_n) = 0$, $\forall i$.

Def^n: Zariski topology on Spec A: $C \subseteq \operatorname{Spec} A$ is closed iff $C = V(S)$ for some $S \subseteq A$.

Check:

- $\operatorname{Spec} A = V((0))$

  $\phi = V(A)$

- $\bigcap_\alpha V_\alpha = \{ \mathfrak{p} \mid \mathfrak{p} \supseteq I_\alpha \ \forall \alpha \}$

  $= \{ \mathfrak{p} \mid \mathfrak{p} \supseteq \sum_\alpha I_\alpha \}$

  $= V(\sum_\alpha I_\alpha)$

- $V(I) \cup V(J) = \{ \mathfrak{p} \mid \mathfrak{p} \supseteq I \text{ or } \mathfrak{p} \supseteq J \}$

  $= \{ \mathfrak{p} \mid \mathfrak{p} \supseteq IJ \}$ ⟵ here we need primeness

  $= V(IJ)$

Recall: $I \subseteq A$ an ideal,

radical $(I) =: \sqrt{I} := \{ x \in A \mid x^n \in I \text{ for some } n \}$

$= \phi^{-1}(\mathcal{N}(A/I))$

$V(I) = V(\sqrt{I})$

$\sqrt{I} = \bigcap_{\substack{\mathfrak{p} \supseteq I \\ \text{prime}}} \mathfrak{p}$

eg: $\mathbb{A}^1_k$, $k = \bar{k}$

$\mathrm{Spec}\, k[x] = \{(x-a) \mid a \in A\} \cup \{(0)\} \cong k_+$

$$V(f) = \begin{cases} \phi & \text{if } f \in k^* \\ \mathbb{A}^1_k & \text{if } f = 0 \\ \text{finitely many max ideals} & \text{else} \end{cases}$$

"Almost" cofinite topology

$(0) \subseteq$ every non-empty open

Lemma: $\phi^*: \mathrm{Spec}\, B \longrightarrow \mathrm{Spec}\, A$ induced by $\phi: A \to B$ is continuous.

Proof: Let $V(I) \subseteq \mathrm{Spec}\, A$. $\phi^{*-1}(V(I)) = ?$

Claim: $\phi^{*-1}(V(I)) = V(\phi(I))$

$\subseteq$: $\mathfrak{p} \in \phi^{*-1}(V(I)) \Rightarrow \phi^{-1}(\mathfrak{p}) \in V(I) \Rightarrow$ If $f \in I$, $f \in \phi^{-1}(\mathfrak{p}) \Rightarrow$ If $f \in I$, $\phi(f) \subseteq \mathfrak{p}$

$\Rightarrow \phi(I) \subseteq \mathfrak{p}$

$\Rightarrow \mathfrak{p} \subseteq V(\phi(I))$

$\supseteq$: $\mathfrak{p} \in V(\phi(I)) \Rightarrow$ If $f \in \phi(I)$, $f \in \mathfrak{p}$

$\Rightarrow$ If $f \in I$, $f \in \phi^{-1}(\mathfrak{p}) = \phi^*(\mathfrak{p})$

$\Rightarrow \phi^*(\mathfrak{p}) \in V(I) \Rightarrow \mathfrak{p} \in \phi^{*-1}(V(I))$

Why am I doing this to myself?

Not every continuous map $\mathrm{Spec}\, A \longrightarrow \mathrm{Spec}\, B$ comes from a ring homomorphism.

eg: Any bijection $k \longrightarrow k$ gives a bijection between $\mathbb{A}^1_k \longrightarrow \mathbb{A}^1_k$ $(k = \bar{k})$

§3.5 Open sets in Spec A

$f \in A$

$D(f) := \{\mathfrak{p} \in \mathrm{Spec}\, A \mid f \notin \mathfrak{p}\}$

$= \mathrm{Spec}\, A_f$

Lemma: $\{D(f)\}$ forms a basis.

Proof: Suppose $S = \mathrm{Spec}\, A \setminus V(I)$

Then $S = \bigcup_{f \notin I} D(f)$ and $D(f) \cap D(g) = D(fg)$.

- $\bigcup_\alpha D(f_\alpha) = \mathrm{Spec}\, A$

$\Longleftrightarrow$

$\exists\, \alpha_1, \ldots, \alpha_n$ s.t. $1 \in (f_{\alpha_1}, \ldots, f_{\alpha_n})$

$\phi: k[x] \longrightarrow k[x]$

$x \longmapsto f(x)$

what is ϕ^*?

$\mathrm{spec}\, k[x] = \{(g(x)) \mid g \text{ irred}\}$

$\phi^*(g) = \ker [k[x] \to k[x] \longrightarrow k[x]/g(x)]$

$h(x) \longmapsto h(f(x)) \longmapsto h \circ f \pmod g$

$= \{h(x) \in k[x] \mid g(x) \mid h(f(x))\}$

if $g = (x-a)$

$= \{h(x) \in k[x] \mid h(f(a)) = 0\}$

$= (x - f(a))$

Cor: Spec A is quasi-compact i.e. every open cover has a finite subcover.

- $D(f) \subseteq D(g) \iff f^n \in (g)$ for some n

$\Updownarrow \qquad \iff f \in \sqrt{g}$

$V(g) \subseteq V(f)$

- $D(f) = \phi \iff f \in \eta(A)$.

$\operatorname{Spec} \mathbb{Z} = \{(p)\} \cup \{(0)\}$

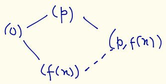
An element $n \in \mathbb{Z}$ value 0 on (p) if $p \mid n$, $n \in \mathbb{Z}$ takes value 0 on (0) iff $n = 0$.

$(0) \subseteq (p)$ $\qquad (0)$ is not closed.

$V((n)) = \{(p) \mid p \mid n\} \qquad V((0)) = \operatorname{Spec} \mathbb{Z} \qquad V((1)) = \phi$

$D_f = \{(p) \mid p \nmid n\}$

$\operatorname{Spec} \mathbb{Z}[x] = \{(0)\} \cup \{(p)\} \cup \{(f(x))\} \cup \{(p, f(x))\}$



$g(x) \in \mathbb{Z}[x]$, what is $V(g(x))$? $\qquad \mathbb{Z}[x]$ is a UFD

$g(x) = n \cdot g_1(x) \cdot g_2(x) \cdots g_k(x)$

$0 \in V(I)$ if $g \equiv 0$

$p \in V(I)$ if $p \mid n$

$f(x) \in V(I)$ if $f = g_i$ for some i

$(p, f(x)) \in V(I)$ if $g \in (p, f(x))$.

$g \longmapsto \mathbb{Z}[x]/(p, f(x)) = \mathbb{Z}/p\,[\alpha] \qquad \alpha$ root of $f(x)$ mod p.

One of the g_i's belong to $(p, f(x))$

Now let us look at $\mathbb{Z}/p\,[x]$: $\quad \operatorname{Spec} \mathbb{Z}/p = \{(0)\}$

$\operatorname{Spec} \mathbb{Z}/p\,[x] = \{(f(x))\}$

More generally $\quad \operatorname{Spec} k[x] = \{(f(x)) \mid f(x)$ irreducible mod $k\}$

Let $l \leftarrow k$ be a finite algebraic extension,

$\operatorname{Spec} l[x] = \{(f(x)) \mid f(x)$ irreducible mod $l\}$

How do these two relate?

$$\operatorname{Spec} \mathbb{R}[x] \xleftarrow{i^*} \operatorname{Spec} \mathbb{C}[x]$$

$\{(0)\} \cup \{(x-a)\} \cup \{(x-ia)(x+ia)\}$ $\qquad$ $\{(0)\} \cup \{(x-a)\}$

What is $i^*((x-a-ib))$? $= i^{-1}((x-a-ib).p(x)) = (x-(a+ib))(x-(a-ib))$

$$\operatorname{Spec} k[x] \xleftarrow{i^*} \operatorname{Spec} l[x]$$
$$? \longleftarrow (f)$$

$i^*((f)) = i^{-1}(f \cdot g)$

Better thing to do is to lift all the way to the universal cover.

$$\operatorname{Spec} \mathbb{Q}[x] \longleftarrow \operatorname{Spec} \mathbb{Q}(\sqrt[3]{2})[x]$$
$$(x^3-2) \longleftarrow ((x-\sqrt[3]{2}))$$

No I think lifting to the Galois cover is enough.

So assume $l \hookleftarrow k$ Galois, then

$$\operatorname{Spec} k[x] \longleftarrow \operatorname{Spec} l[x]$$
$$\left(\sqrt{\prod_{\sigma\in G} \sigma(f)}\right) \longleftarrow (f)$$

G - Galois group

What would be the preimage of some $(f) \in \operatorname{Spec} k[x]$?

$$\underset{k}{f} \longleftarrow \underset{l}{f = f_1 f_2 \cdots f_m} \qquad (f) \longleftarrow f_1, \dots, f_m$$

What would be m?

All the f_i's would have the same degree if the extension is Galois. Else one should check by hand.

Q. Note that ϕ inj $\not\Rightarrow \phi^*$ surjective. But ϕ surjective $\Rightarrow \phi^*$ injective.

What about $0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0 \overset{\phi^*}{\rightsquigarrow}$??

Q. Can we abelianize the category of all Spec and make sense of an exact sequence in this category? Does then Spec have higher derived functors?