Chromatic Homotopy Theory

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Motivation :

X - finite CW complex, what are $\pi_n(X)$?

Say $X = S^n$

- eg: $S^n \xrightarrow{p} S^n$
- $S^3 \xrightarrow{\eta} S^2$

  $S^7 \longrightarrow S^4$ Hopf inv 1

  $S^{15} \longrightarrow S^8$

- Image of J

$$O(n) \hookrightarrow H(n) \hookrightarrow \Omega^n S^n$$

Homotopy equivalences of S^n

$$\pi_i(O(n)) \longrightarrow \pi_i \Omega^n S^n = \pi_{n+i} S^n$$

Take limits : $\pi_i(O) \xrightarrow{J} \pi_i^s$

Adams computed the image of J.

- Need a way of producing more elements in $\pi_* S^n$.

Def^n: $\Sigma^d X \xrightarrow{f} X$ self map -

Self maps can be composed :

$$f^2 = \Sigma^{2d} X \xrightarrow{\Sigma^d f} \Sigma^d X \xrightarrow{f} X$$

f is nilpotent if $f^t \simeq 0$ for some t

else f is called periodic. (why periodic?)

- $S^k \xrightarrow{p} S^k$ is a periodic map

  cofiber $= V(0)_k$ mod p Moore space

Th^m: Adams - Toda

For $k \gg 0$ $\exists\, \alpha : \Sigma^q V(0)_k \longrightarrow V(0)_k$ $q = 8$ for $p = 2$

$q = 2p-2$ for p odd

such that i) α is periodic

ii) induces multiplication by "Bott class" in complex K-theory

(i.e. $K_* \alpha = \eta^i \wedge -$ for some i)

Note that ii) $\Rightarrow$ i)

Now consider :

$$\underset{\text{bottom cell}}{S^{k+qt}} \longrightarrow \Sigma^{qt} V(0)_k \xrightarrow{\alpha^t} V(0)_k \longrightarrow \underset{\text{top cell}}{S^{k+1}}$$

(Adams) This map is not null homotopic and hence gives $\alpha_t \in \pi^S_{qt-1}$

Now let $V(1)_k :=$ cofiber of α

Th^m (Smith - Toda)

$\exists\, \beta : \Sigma^{2p^2-2} V(1)_k \longrightarrow V(1)_k$ such that

- β is periodic
- MU detects β (and so does $K(2)$ - Morava K-theory)

Again, $$\underset{\text{bottom cell}}{S^{k+2(p^2-1)t}} \longrightarrow \Sigma^{2(p^2-1)t} V(1)_k \xrightarrow{\beta^t} V(1)_k \longrightarrow \underset{\text{top cell}}{S^{k+2p}}$$

gives us $\beta_t \in \pi^S_{2(p^2-1)t-2p}$

- Can repeat again

$V(2)_k =$ cofiber of β

$\gamma : \Sigma^{2p^3-2} V(2)_k \longrightarrow V(2)_k$

detected by MU_* (or $K(3)$)

We get $\gamma_t \in S^{k+2(p^3-1)t} \longrightarrow S^{k+(p+2)q+3}$

Th^m (- - -) : $\gamma_t \neq$ null homotopic.

Difficult proof

- Can continue this forever.

$\longrightarrow$ Need homology theories to detect these

$\longrightarrow$ A way of generating self maps

(But we do not get homotopy groups any more, or rather we do not know of the maps we get are null-homotopic)

Th^m (Devinatz, Hopkins, Smith)

Nilpotence: $\exists$ homology theory MU s.t.

$f : \Sigma^d X \longrightarrow X$ is nilpotent if and only if some iterate of $MU_*(f)$ is trivial, for finite CW complex X.

Cor: Homotopy groups of speres are torsion.

Now work p-locally:

Th^m (Johson-Wilson)

There $\exists$ sequence of cohomology theories $K(n)$ - Morava K-theory such that

- $K(0)_*(X) = H_*(X;\mathbb{Q})$
- $K(1)_*$ is one of the $p-1$ isomorphic summands of mod p complex K-theory
- $K(0)_* = \mathbb{Q}$ $\quad K(n)_* = \mathbb{Z}/p\,[v_n^{\pm 1}]$ $\quad |v_n| = 2(p^n-1)$

Note that $K(n)_*$ is a field and so the $K(n)_*(X)$ is a vector space, and so

- $K(n)_*(X \times Y) \cong K(n)_*(X) \otimes K(n)_*(Y)$
  Kunneth
- $\tilde{K}(n+1)_*(X) = 0 \Rightarrow \tilde{K}(n)_*(X) = 0.$

Def^n: X - finite CW complex
X - has type n if $K(n)_* X \neq 0$ but $K(n-1)_*(X) = 0$.
(A non-contractible space has finite height)

Prop: $\Sigma^d X \xrightarrow{f} X$
If X has type n, cofiber(f) has type $n+1$.

Periodicity th^m:
X - finite CW complex of type n then
$\exists\, v_n : \Sigma^{d+i} X \longrightarrow \Sigma^i X$ for some i, d
such that $K(n)_* f$ is an iso and $K(m)_* f = 0$ for $m > n$.

Use this to produce elements in π_i^S

Periodic self map $\rightsquigarrow$ family of elements in π_*^S

Need to do more work than before.
X - finite CW bottom cell in dim k, top cell in dim $k+e$.

$i_0 : S^k \longrightarrow X \qquad j_0 : X \longrightarrow S^{k+e}$

Let $f : \Sigma^d X \longrightarrow X$
Consider
$$S^{k+td} \xrightarrow{i_0} \Sigma^{td} X \xrightarrow{f^t} X \xrightarrow{j_0} S^{k+e}$$

This map may be null. Let $k < r \leq s < k+e$

$$X_r^s = \text{cofiber of } X^{r-1} \longrightarrow X^s$$

if $j \cdot f_e \simeq *$
we get f_{e-1}
and so on.

$$\begin{tikzcd}
\Sigma^{td} X \arrow[r, "f_e"] \arrow[dr, dashed, "f_{e-1}"'] \arrow[ddr, dashed, bend right] & X_k^{k+e} \arrow[r, "j"] & X_{k+e}^{k+e} \\
 & X_k^{k+e-1} \arrow[u] \arrow[r, "j"] & X_{k+e-1}^{k+e-1} \\
 & \vdots \arrow[u] & \\
 & X_k^k \arrow[u] \arrow[r, "j"] & X_k^k
\end{tikzcd}$$

This process has to stop as f_e is not null homotopic, and so some e_1
$$j \cdot f_{e_1} \simeq *$$

$$\begin{tikzcd}
\Sigma^{td} X_k^k \arrow[r, "i"] & \Sigma^{td} X_k^{k+e} \arrow[r] \arrow[d] & X_{k+e_1}^{k+e_1} \\
\Sigma^{td} X_{k+1}^{k+1} \arrow[r, "i"] & \Sigma^{td} X_{k+1}^{k+e} \arrow[ur, dashed] \arrow[d] & \\
 & \vdots &
\end{tikzcd}$$

by similar reasoning some bottom cell survives.

And so this gives us an element of π_*^S.

Turns out every stable homotopy group element occurs this way.

Th^m: $\exists$ a chromatic filtration on $\pi_*^S X$.