$[\ x_{2n+1}=0\ ]$ $\mathbb{R}P^{2n} \hookrightarrow \mathbb{R}P^{2n+1} \longrightarrow \mathbb{C}P^n$

$[x_0; x_1; \cdots; x_{2n+1}] \longrightarrow [x_0+ix_1, x_1+ix_2, \cdots, x_{2n}+ix_{2n+1}]$

Algebraic Topology

Th^m: (Brower) $\pi_n S^n \longrightarrow \mathbb{Z}$ =: degree is a group isomorphism.

Lemma 1: Given finite sets $U_1, U_2 \in \mathbb{R}^n$, $\exists\ U \subseteq \mathbb{R}^n$ such that $U_1 \subseteq U$, $U_2 \subseteq U^c = \mathbb{R}^n - U$, $U \cong D^n$ $\partial U \cong \partial D^n$. Do Brute Force.

Read Transversality. Write proof of th^m. very interesting
Write CW approx proof

Lemma 2: $f: (D^n, \partial D^n) \longrightarrow (D^n, \partial D^n)$ $\partial D^n = S^{n-1}$
then $\deg f = \deg f|_{\partial D^n}$.

~~Proof~~: Proof:

$$\begin{tikzcd} H_n(D^n, \partial D^n) \arrow[r, "f_*"] \arrow[d] & H_n(D^n, \partial D^n) \arrow[d] \\ H_{n-1}(\partial D^n) \arrow[r, "f|_{\partial D^n *}"] & H_{n-1}(\partial D^n) \end{tikzcd}$$

connecting hom →

Enough to show $\deg f = 0 \Rightarrow f \simeq *$.
Assume true for S^{n-1}. Assume f smooth.
~~A~~ Assume $0, \infty$ regular values of f, $f(\infty) = \infty$,
Let $S = f^{-1}(\infty)$, $T = f^{-1}(0)$
$f: \mathbb{R}^n - S \longrightarrow \mathbb{R}^n$
By lemma 1, choose $U \subseteq \mathbb{R}^n$ s.t. $T \subseteq U$, $S \subseteq U^c$.
By local degree property,
$\deg f = \deg f|_U$.

~~$\partial U = S^{n-1}$~~

~~By lemma 2,~~ ~~im~~

~~$\deg f|_U = \deg f|_{\partial U}$~~

~~$h: \partial U = S^{n-1} \to S^{n-1}$~~

~~$x \to f(x) \to |f(x)|^{-1} \cdot f(x)$~~

∂U $f: S^{n-1} \to \mathbb{R}^n - 0$
$\downarrow$
S^{n-1}

~~$f : \partial U \to \mathbb{R}^n - \{0\}$ $h = \partial U \xrightarrow{f} \mathbb{R}^n - \{0\} \longrightarrow S^{n-1}$ ↖ deformation Retract~~

~~Claim: $\deg f = \deg h$~~

Now we need to modify f so that $f|_{\partial D^n} \subseteq S^{n-1}$

$\bar{U}$ is compact so $f\bar{U}$ is compact.

$\partial\bar{U}$ " $\Rightarrow f\,\partial\bar{U}$ comact!

Suppose $R = \max\{|f(x)| \mid x \in \bar{U}\}$

$R_1 = \min\{|f(x)| \mid x \in \partial\bar{U}\}$

Then f is homotopic to the map

$f' : \mathbb{R}^n - S \longrightarrow \mathbb{R}^n$

$$f'(x) = \begin{cases} f(x) & \text{if } |f(x)| \geq R \\ f(x) \cdot \dfrac{R}{R_1} & \text{if } |f(x)| \leq R_1 \\ \dfrac{f(x)}{|f(x)|} \cdot R & \text{if } R_1 \leq |f(x)| \leq R \end{cases}$$

~~The~~ Call f' f.

Now $f : (\bar{U}, \partial\bar{U}) \longrightarrow (D^n, \partial D^n)$

By lemma 2, [illegible]

$\deg f = \deg f|_{\partial U} = 0$

~~But~~ By induction

$f|_{\partial U} \sim *$

$\Rightarrow$ We can extend f to g on U so that

$g(U) \subseteq \partial D^n$

Now g misses the point 0.

So $g \simeq *$ $f \underset{U}{\simeq} g \Rightarrow f \simeq *$.

Q. $f: X \to Y$ weakly homotopic, $\Rightarrow E^n(X) = E^n(Y)\ \forall E$??

C-W- approximation theorem:

$\forall X$, $\exists Y$ - CW complex such that $Y \to X$ is a weak ~~equ~~ homotopy equivalence.

If X, Y are m-connected, n-connected resp. then, $X \vee Y \to X \times Y$ is an m+n+1 equivalence.

Using this and the fact that $\pi_n(S^n) = \mathbb{Z}$ construct Y.

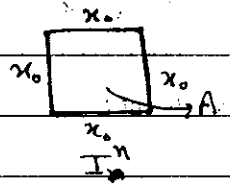
But to show that we have homotopy equivalence we require to invoke homotopy excision formula.

Th^m: Following is exact.

$$\to \pi_2(A) \xrightarrow{i} \pi_2(X) \xrightarrow{\pi} \pi_2(X,A) \xrightarrow{\delta} \pi_1(A) \to \pi_1(X) \to \pi_1(X,A) \to \pi_0(X) \dots$$

- $\pi_n(A) \xrightarrow{i} \pi_n(X) \xrightarrow{\pi} \pi_n(X,A)$

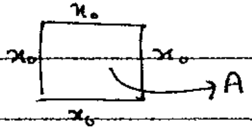
$(\pi \cdot i)$



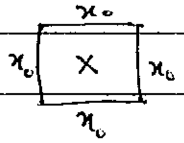
$[I^n, A] = 0$ as I^n is null homotopic

So $\pi \cdot i = 0$

~~ker~~ im i =



Ker π =



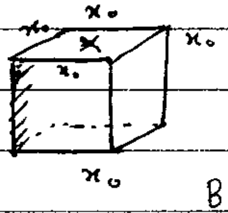
$(I^n, \partial I^n) \to (X, x_0)$

$\downarrow$ ~~homotopic~~

$(I^n, \partial_1 I^n, \partial_2 I^n) \to (X, A, x_0)$

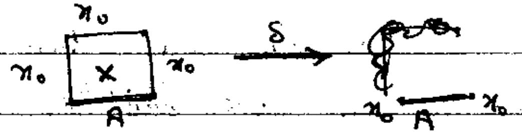
$\downarrow$ homotopic

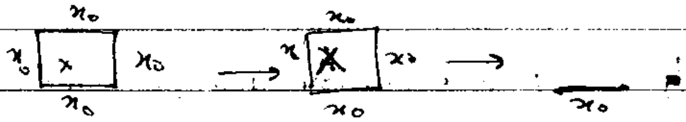
$(I^n, \partial_1 I^n, \partial_2 I^n) \to (x_0, x_0, x_0)$



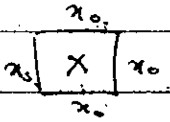
But this homotopy can also be thought as a homotopy between upper face and front face $\Rightarrow$ ker $\pi \subseteq$ im i

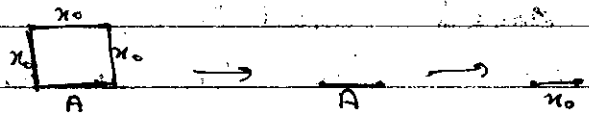
- $\pi_n(X) \xrightarrow{\pi} \pi_n(X,A) \xrightarrow{\delta} \pi_{n-1}(A)$



i[illegible] =


= 0

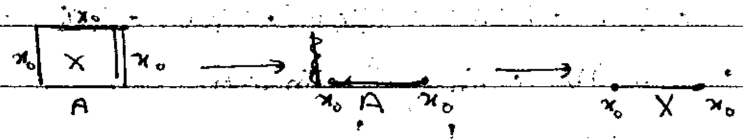
im~~ker~~ π =


ker i =


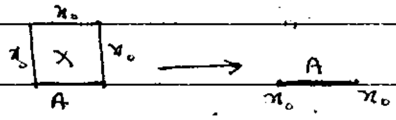
use this homotopy

$(D^n, \partial D^n)$ is a good pair. So extend the homotopy to all of D^n.

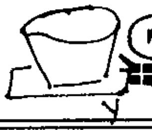
- $\pi_n(X,A) \xrightarrow{\delta} \pi_{n-1}(A) \xrightarrow{i} \pi_{n-1}(X)$

$i\cdot\delta$ =


But fig 1 is homotopy of fig 3 to x_0.

im i =


M_f : Mapping cylinder of f



Th^m (whitehead)

X, Y CW complexes ,

$\exists$ $f: X \to Y$, s.t. $\pi_n(f): \pi_n(X) \xrightarrow{\simeq} \pi_n(Y)$ $\forall n$

Then f is a homotopy equivalance.

Proof: Make f cellular. This is to make M_f CW complex.

Look at $Z = M_f$; $\pi_n(M_f, X) = 0$

$$\begin{tikzcd} \partial D^n \arrow[r, "g"] \arrow[d] & A \arrow[d] \\ D^n \arrow[r, "g"'] & Y \end{tikzcd}$$

→ induces an element of $\pi_n(Y, A)$

$\pi_n(Y,A) = 0$ ⇒ g is homotopic to a map $D^n \to A$ rel ∂D^n

$f: X \hookrightarrow M_f$ need to show deformation retract

$M_f = X \cup (0 \text{ cells}) \cup (1 \text{ cells})$.

Attach 1 cell at a time

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, hook] & X \arrow[d] \\ X \cup e_k \arrow[r, "f"'] \arrow[ur, dashed] & M_f \end{tikzcd}$$

$\pi_{k+1}(M_f, X) = 0$

homotopic rel X to f

Excision

$\pi_k(\mathbb{CP}^n / \mathbb{CP}^{n-1}) = 0$ for $k \le 2n-1$

$\Rightarrow \pi_k(\mathbb{CP}^{n-1}) \hookrightarrow \pi_k(\mathbb{CP}^n)$ $\begin{cases} \text{isomorphism} & q < 2n-1 \\ \text{surjection} & q = 2n-1 \end{cases}$

$\mathbb{CP}^1 = S^2$

$\Rightarrow \pi_2(\mathbb{CP}^n) = \pi_2(\mathbb{CP}^{n-1}) = \dots = \pi_2(\mathbb{CP}^1) = \mathbb{Z}$

$\pi_2(\mathbb{CP}^\infty) = \mathbb{Z}$ ~~To~~ Argue using compact support image

Ex: $X = \varinjlim X_n$ Then $\pi_k(X) \cong \varinjlim_{n} \pi_k(X_n)$

$X_n = S^n$ $X = S^\infty$

$\pi_k(S^\infty) = \varinjlim_{n} \pi_k(S^n) = 0$ $\forall k$

So $* \hookrightarrow S^\infty$ weak homotopy eq.

By whitehead Th^m, S^∞ is contractible.

Homotopy Excision

$X = A \cup B$ A, B subcomplexes

$C = A \cap B$

(A, C) – m connected

(B, C) – n connected

Then $(A, C) \longrightarrow (X, B)$ is an $(m+n)$ equivalence

Long Proof: Step 1: Reduce to the case

~~A = A~~ $A = C \cup e^{m+1}$

$B = C \cup e^{n+1}$

Reduction:

Induction on no. of cells in $A - C$

$X = X' \cup e^M$, $A = A' \cup e^M$ $M \ge m+1$ Use 5-lemma

$$\begin{tikzcd} {} \arrow[r] & \pi_*(A', C) \arrow[r] \arrow[d] & \pi_*(A, C) \arrow[r] \arrow[d] & \pi_*(A, A') \arrow[r] \arrow[d] & {} \\ {} \arrow[r] & \pi_*(X', B) \arrow[r] & \pi_*(X, B) \arrow[r] & \pi_*(X, X') \arrow[r] & {} \end{tikzcd}$$

$B = B' \cup e^N \qquad X = A \cup B'$

$$\begin{tikzcd} \pi_*(X', B') \arrow[r] & \pi_*(X, B) \\ \pi_*(A, C) \arrow[u] & \end{tikzcd}$$

for infinite cells, we direct limit & compact image arg.

Step : $A = C \cup D^{m+1}$

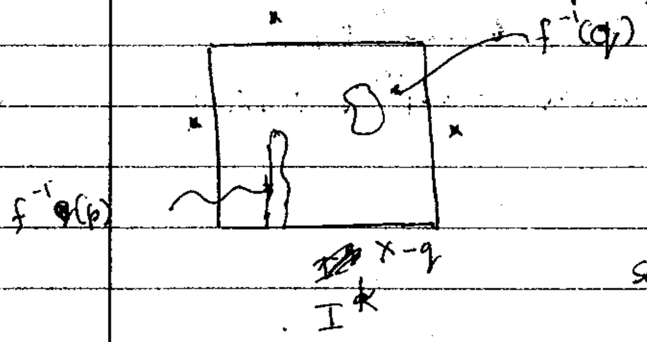
$B = C \cup D^{n+1}$

$X = C \cup D^{m+1} \cup D^{n+1} \qquad p \in D^{m+1} \qquad q \in D^{m+1}$

$$\begin{tikzcd} (A, C) \arrow[r] \arrow[d, phantom, "\wr\wr"] & (X, B) \arrow[d, phantom, "\wr\wr"] \\ (X-p, X-p-q) \arrow[r] & (X, X-q) \end{tikzcd}$$

for $H^k(\ ) = H^k(\ )$ we need to lift a homotopy $[(I^k, \partial I^k), (X, X-q)]$ to $(X-p, X-p-q)$

i.e. can we omit p from the image of I^k



heuristically
$f^{-1}(q)$, $f^{-1}(p)$ are
far apart
So we can

Simplicial Approximation

Simpicial Complex ~~ex~~

$$\begin{tikzcd} K \arrow[r, "f"] & L \end{tikzcd}$$

K: Δ-complex, L: Δ complex

Simplicial map

f simplicial iff f continuous

s.t. $f(\text{int simplex}) \subseteq \text{int}(\text{simplex})$

$f|_{\text{int simplex}} = \text{linear}$

Not all maps ~~bet~~ are simplicial, but $\exists$ a barycentric subdivision such that the map ~~becomes~~ homotopes to a simplicial map

See later

$(K, L) \longrightarrow (X, A)$ $\quad X = A \cup D^{m+1}$

(K,L): Relative Simplicial complex finite

$D_1^{m+1} = \{x \mid |x| \leq 1/2\}$

$\cap$

$\text{int } D^{m+1}$

$D_0^{m+1} = \{x \mid |x| \leq 1/4\}$

Th^m: $\exists$ a barycentric subdivision of (K, L),

$f' : (K, L) \longrightarrow (X, A)$

$f'|_L = f$

f' has the property if

$f'(\text{simplex}) \cap D_0^{m+1} \neq \phi$

(Prove this) $\Rightarrow f'(\text{simplex}) \subseteq D_1^{m+1}$, $f'|_{\text{simplex}}$ is linear.

Q.1 Prove $[\Sigma^2 X, Y]_*$ is abelian.

2 Γ connected graph. Show $\pi_i(\Gamma) = 0$ for $i > 1$

3 Show $[S^1 \vee S^1, S^2]_* = 0$ Calculate $\pi_2(S^2, S^1 \vee S^1)$.

4. X - m connected Y n connected $X \vee Y \hookrightarrow X \times Y$ is $m+n-1$ equivalence

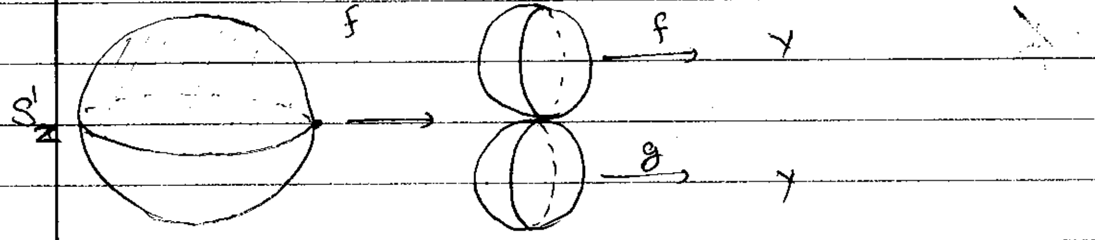
5. Compute $\pi_n(\mathbb{R}P^n, \mathbb{R}P^{n-1})$, $\pi_n(\mathbb{R}P^n / \mathbb{R}P^{n-1})$.

1. $[S^2, Y]_*$ S^1_x, S^1_y, S^1_z denote equators ~~in~~ $x=0, y=0, z=0$.
$\sigma_x^\theta, \sigma_y^\theta, \sigma_z^\theta$ denote rotations about x, y, z axis

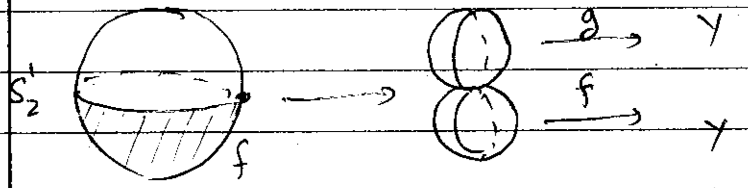
base pt $(1,0,0)$

$$[S^2, Y]_* \cdot [S^2, Y]_* \longrightarrow [S^2, Y]_*$$

$$f \cdot g \longmapsto f \cdot g$$



$$g \cdot f \longmapsto g \cdot f$$



Then ~~H~~ Homotopy on S^2

$$f : S^2 \times I \longrightarrow S^2$$

$$f_t \theta = \sigma_x^{t \cdot 2\pi}$$

composes to give homotopy between fg & gf
so abelian

~~$[S^2, X]_*$~~ $[S^2, X]_* \longrightarrow [S^2 X, Y]_*$

~~$[S^2 \alpha e \longrightarrow$~~ $(S^2$ ~~$\vee S^2$~~$)$

$$[\Sigma^2 X, Y]_* \times [\Sigma^2 X, Y]_* \longrightarrow [\Sigma^2 X, Y]_*$$

$$S^2 \wedge X \longrightarrow (S^2 \vee S^2) \wedge X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

As map is abelian on S^2,
it is abelian for $S^2 \wedge X = \Sigma^2 X$

Q. Universal cover of a CW complex is a CW complex?

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2. Any Universal cover of a graph is a tree.

3. $[S^1 \vee S^1, S^2]_* = [S^1, S^2]_* \oplus [S^1, S^2]_* = 0$

$$\begin{tikzcd} \pi_2(S^1 \vee S^1) \arrow[r] \arrow[d, phantom, "\shortparallel"] & \pi_2(S^2) \arrow[r] & \pi_2(S^2 [illegible] , S^1 \vee S^1) \arrow[r] & \pi_1(S^1 \vee S^1) \arrow[r] & \pi_1(S^2) \\ 0 & & & & \end{tikzcd}$$

$S^1 \vee S^1$ has universal cover cayley graph of F_2

$\Rightarrow \pi_2(S^2, S^1 \vee S^1) = F_2 \oplus \mathbb{Z}$

4. X - m connected . Y - n-connected

Then C-W structure of $X \times Y$ is

$(X \times Y)^{(i)} = \bigcup * \times e^i \cup e'^i \times *$ $\quad i \leq m+n+1$

where e^i is an i cell in Y

e'^i is an i cell in X

So $(X \times Y)^{(i)} = (X \vee Y)^{(i)}$ for $i \leq m+n+1$

$\Rightarrow (X \vee Y \hookrightarrow X \times Y$ is an $m+n+1$ equivalence.

5. $\mathbb{R}P^n/\mathbb{R}P^{n-1} = S^n$ $\quad \pi_n(\mathbb{R}P^n/\mathbb{R}P^{n-1}) = \mathbb{Z}$

$$\begin{tikzcd} \mathbb{Z}/2 = \pi_n(\mathbb{R}P^{n-1}) \arrow[r] & \pi_n(\mathbb{R}P^n) \arrow[r] \arrow[d, phantom, "\shortparallel"] & \pi_n(\mathbb{R}P^n/\mathbb{R}P^{n-1}) \arrow[d] & \\ & \mathbb{Z} & \pi_{n-1}(\mathbb{R}P^{n-1}) \arrow[r] \arrow[d, phantom, "\shortparallel"] & 0 \\ & & \mathbb{Z}. & \end{tikzcd}$$

$\pi_n(\mathbb{R}P^n, \mathbb{R}P^{n-1}) \cong \mathbb{Z} \oplus \mathbb{Z}$

(Need to do more in this case. for $n=2$)

Note: $\mathrm{Hom}(X \times Y, Z) \cong \mathrm{Hom}(X, \mathrm{Hom}(Y, Z))$

in pointed spaces

$[X \wedge Y, Z]_* \cong [X, [Y, Z]_*]_*$

so $[\Sigma^2 X, Z]_* \cong [S^2, [X, Z]_*]_* = \pi_2([X, Z]_*)$

Simplicial Approx Lemma

$X = A \cup e^n$

$$(K, L) \xrightarrow{f} (X, A)$$

finite ↗
simplicial pair

$e_0^n = \{x \in e^n \mid |x| \le 1/4\}$

∃ a subdivision of (K, L) and a homotopy $f \simeq f' : (K, L) \to (X, A)$ relative to $f^{-1}(A)$, such that if for any simplex σ, $f'(\sigma) \cap e_0^n \ne \phi$ then $f'(\sigma) \subset \text{int } e_0^n$ and $f|_\sigma$ is linear.

Proof:

$e_1^n = \{x \mid |x| \le 1/2\}$ $\qquad$ $e_2^n = \{x \mid |x| \le 3/4\}$

$f^{-1}(e_2^n)$ is compact as K, L is finite

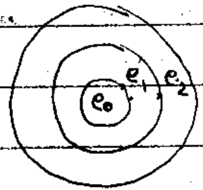
$\Rightarrow f|_{f^{-1}(e_2^n)}$ is uniformly continuous

$\therefore$ Choose: $\delta > 0$ such that

$$|x-y| < \delta \Rightarrow |f(x) - f(y)| < 1/4 \qquad x, y \in f^{-1}(e_2^n)$$

Subdivide (K, L) till diameter of simplices becomes less than δ. we get 3 classes of simplices:

- $C_1 = \{\sigma \mid f(\sigma) \in X - e_1^n\}$
- $C_2 = \{\sigma \mid f(\sigma) \in \text{int } e_1^n\}$
- $C_3 = \{\sigma \mid f(\sigma) \cap \partial e_1^n \ne \phi\}$



$\sigma \in C_3 \Rightarrow \sigma \cap e_0^n = \phi$

Define f' as follows:

$\sigma = [v_0 \cdots v_k]$

if $\sigma \in C_1$ $\quad f'(\sigma) = f(\sigma)$

if $\sigma \in C_2$ $\quad f'(t_0 v_0 + \cdots + t_k v_k) = t_0 f'(v_0) + \cdots + t_k f'(v_k)$

if $\sigma \in C_3$ $\quad$ define inductively on $\dim \sigma$

$\dim = 0$ $\quad f'(\sigma) = f(\sigma)$

suppose defined for $\dim \sigma < k$

$\sigma = [v_0 \cdots v_k]$ b = Barycenter of σ

$f'(b) := f(b)$

$x \in \sigma$, $x \neq b$ suppose line joining b to σ ~~to~~ intersects $\partial\sigma$ in $\lambda(x)$

if $x = tb + (1-t)\lambda(x)$ define,

$$f'(x) := tf'(b) + (1-t)f'(\lambda(x))$$

We do this as there might be simplices ~~inters~~ half in e_1.

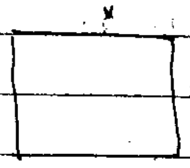
Homotopy: $f' \simeq f$ give Linear Homotopy.

$\square$.

Check Homotopy Excision: $A = C \cup D^{m+1}$, $B = C \cup D^{n+1}$

$$(A, C) \hookrightarrow (A \cup B, B) \quad m+n\text{- equivalence.}$$

$q \leq m+n$



$$\pi_q(A \cup B, B) = \pi_q(C \cup D^{m+1} \cup D^{n+1}, C \cup D^{n+1})$$

$$(I^q, I^{q-1} \times \{0\}, \partial I^{q-1} \times [0,1] \cup I^{q-1} \times \{0\}) \longrightarrow (C \cup D^{m+1} \cup D^{n+1}, C \cup D^{n+1}, *)$$

We want to show that we can remove D^{n+1} from image so that $(C \cup D^{m+1} \cup D^{n+1}, C \cup D^{n+1}) \to (\underset{A}{\underset{\parallel}{C \cup D^{m+1}}}, C)$

It is enough to homotope to a map which misses a point in D^{n+1}.

$$(I^q, \partial I^q) \xrightarrow{\quad} (A \cup B, B)$$

f

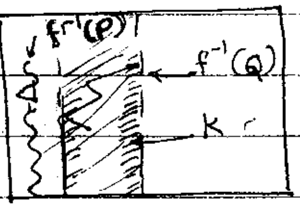
Apply simplicial approximation lemma

if ~~σ~~ σ is s.t. $f(\sigma) \cap e_0^n \neq \phi$

f linear on σ . so dim $f(\sigma) \leq m+1$

$\bigcup_{\dim f(\sigma)\leq m} f(\sigma) \neq e_0^{m+1}$ by lookig at dim

$\exists\ Q \in e_0^{m+1} - \bigcup_{\dim f(\sigma)\leq m} f(\sigma)$



So $\dim f^{-1}(Q) \leq q-m-1$

$\pi : I^q \longrightarrow I^{q-1}$

$(a_1 \cdots a_q) \longrightarrow (a_1, \cdots a_{q-1})$

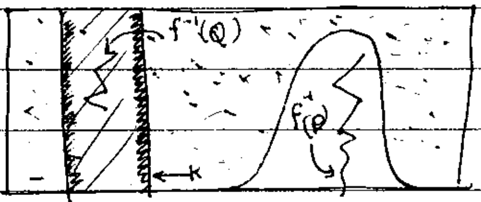
$K = \pi^{-1}(\pi(f^{-1}Q))$ $\dim K \leq q-m \leq n$

So $f(K) \neq D^{n+1}$

$\Rightarrow \exists\ P \in D^{n+1} - f(K)$

$f^{-1}(P) \cap K = \phi$

Now wee need to miss P



$\underset{\text{closed}}{\pi(K)} \cap \underset{\text{closed}}{\pi(f^{-1}(P))} = \phi$

$\exists\ \phi : I^{q-1} \longrightarrow [0,1]$ by ~~Tietze extension th~~ Uryzon Lemma

$\varphi|_{\pi(K)} = 0$

$\varphi|_{\pi f^{-1}(P)} = 1$

$H : I^q \times I \longrightarrow I$

$H(a,t,s) \longmapsto (a, t(1-s\varphi(a)))$

Check : $f \circ H$ is a homotopy ~~which~~ between $f : (I^q, \partial I^q) \longrightarrow (A\cup B, A\cup B - Q)$ to $g : (I^q, \partial I^q) \longrightarrow (A\cup B - P, A\cup B - P - Q)$

$(A\cup B - P, A\cup B - P - Q)$ deformation retracts onto (A, C)

$\Rightarrow \pi_q(A,C) \longrightarrow \pi_q(A\cup B, B)$ is surjective.

Exercise : Prove injectivity.

Freudenthal suspension Th^m

X $(n-1)$ connected.

$$\pi_k(X) \xrightarrow{\Sigma} \pi_{k+1}(\Sigma X)$$

$$f: S^k \to X \longmapsto \Sigma f: S^{k+1} \to \Sigma X$$

Σ isomorphism if $k \leq 2n-2$

Σ surjection if $k = 2n-1$

Th^m : (X,A) X- n connected A- s connected

$\Rightarrow$ $\pi_q(X,A) \longrightarrow \pi_q(X/A, *)$ isomorphism $q \leq n+s$

surjection $q = n+s+1$

Th^m : (Y,X) n-connected via $f: X \longrightarrow Y$

$\Rightarrow$ $[Z,X]_* \xrightarrow{f_*} [Z,Y]_*$ isomorphism $\dim Z < n$

(Z CW-complex) surjection $\dim Z = n$

Cor : X n-connected

$X \longrightarrow *$ $n+1$ connected

$\Rightarrow$ $[Z,X] = *$ $\forall$ CW Z , $\dim Z \leq n$

Cor $X \xrightarrow{f} Y$ ∞- connected i.e. weak equivalence

$\Rightarrow$ $H_n(X) \xrightarrow[H_n(f)]{\cong} H_n(Y)$ $\forall n$

Hurewicz $h : \pi_n(X) \longrightarrow H_n(X)$

$[f] \longmapsto f_*[\alpha]$ where α generator of $H_n(S^n)$

Th^m: X $k-1$ connected, $K \geq 2$. Then h is an isomorphism.

Relative: $h : \pi_n(X, A, x_0) = [I^n, I^{n-1} \times \{0\}, \partial I^{n-1} \times I \cup I^{n-1} \times \{1\}]$

$$\begin{tikzcd}
{[I^n, I^{n-1}\times\{0\}, \partial I^{n-1}\times I \cup I^{n-1}\times\{1\}]} \arrow[d, "f"] \\
{[X, A, x_0]}
\end{tikzcd}$$

$$\pi_n(I^n, \partial I^n) \xmapsto{f_*} H_n(X, A)$$

Th^m: (X, A) $k-1$ connected $K \geq 2$

A simply connected.

$h : \pi_k(X, A) \longrightarrow H_n(X, A)$ is isomorphism.

Proof:

For S^n true

$$\begin{tikzcd}
(X,A) \arrow[d] & \text{- CW pair} \quad (X,A) \ k-1 \text{ connected} \\
(X/A, *) & A \ 1 \text{ connected}
\end{tikzcd}$$

$K+1$ equivalence

$$\begin{tikzcd}
\pi_n(X,A) \arrow[r, "h"] \arrow[d, "\approx"'] & H_K(X,A) \arrow[d, "\approx"] \\
\pi_n(X/A) \arrow[r, "h"'] & H_K(X/A)
\end{tikzcd}$$

$\approx$ because X/A can be reduced to wedge of k spheres.

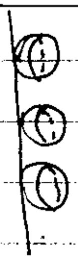
Then for infinite complex use direct limits.

For non CW complexes, use CW approximation.

Note: $\pi_n(\varinjlim_K A_K) = \varinjlim_K (\pi_n(A_K))$ only holds when A_K is CW & Topology is direct limit CW topology

Ex: $S^1 \vee S^2$

Universal cover $\longrightarrow$.



$\pi_1(S^1 \vee S^2) = \mathbb{Z}$

$\pi_2(S^1 \vee S^2) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots$

$\pi_n(S^1 \vee S^n) = \pi_n(\vee S^n)$

Th^m : (Serre) $\pi_{n+k}(S^n) =$ finite for $k > 0$ and ~~at~~ except following
$\pi_{2n-1}(S^{2n}) = \mathbb{Z} \oplus$ finite

Th^m : $A \xrightarrow{f} X$ A;X simply connected
$H_n(A) \xrightarrow{f_*} H_n(X)$ is isomorphism for all n
Then, $\pi_n(A) \xrightarrow{f_*} \pi_n(X)$. is an isomorphism $\forall n$.

Existence of map is important
$S^2 \times S^2$ $\qquad$ $S^2 \vee S^2 \vee S^4$
$H_* (= \mathbb{Z}, 0, \mathbb{Z}^2, 0, \mathbb{Z}$ $\qquad$ $\mathbb{Z}, 0, \mathbb{Z}^2, 0, \mathbb{Z}$
But cohomology ring of first is non-trivial but that of the second is not.

Fibre Bundle

$$\begin{tikzcd} F \arrow[r, hook] & E \arrow[d, "p"] \\ & X \end{tikzcd}$$

Q. Some $p : F \longrightarrow X$ s.t. $p^{-1}(x) = F$ but p not a fibre bundle.
Ans: $X \xrightarrow{\cdot} X$ two different topologies on X.

~~Thm~~ Homotopy lifting Homotopy lifting for fibre bundles is true.

Fibration: $\begin{matrix} E \\ \downarrow p \\ X \end{matrix}$ fibration if homotopy lifting holds.

$$\begin{tikzcd} A\times\{0\} \arrow[r] \arrow[d, hook] & E \arrow[d] \\ A\times I \arrow[r] \arrow[ru, dashed] & X \end{tikzcd}$$

1. Hurewicz fibration $\forall A$
2. Serke fibration $A = D^n$

$$\begin{tikzcd} F \arrow[r, hook] & E \arrow[d, "p"] \\ & X \end{tikzcd}$$

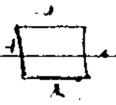
$p^{-1}(*) = F$

Claim: $p : (E, F) \longrightarrow (X, *)$ is an isomorphism.

Surjective:

$$\begin{tikzcd} \partial I^{n-1}\times\{0\} \arrow[r, "\text{trivial}"] \arrow[d] & E \arrow[d] \\ I^n, \partial I^n \arrow[r, "*"] \arrow[ru, dashed] & X, * \end{tikzcd}$$

lift will be an element of $\pi_n(E,F)$.



Injective:

$f \in \pi_n(E,F)$ $\quad H : p(f) \simeq *$

$$\begin{tikzcd} I^n, \partial I^n \arrow[r, "f"] \arrow[d] & E, F \arrow[d, "p"] \\ I\times I^n, \partial I^n \arrow[r, "H"] \arrow[ru, dashed, "\tilde{H}"] & X \end{tikzcd}$$

lift gives $\tilde{H}_t : (I^n, \partial I^n) \to (E, F)$

$\tilde{H}_0 = f$

$\tilde{H}_1 \in F$

X - based space $\Omega X = \text{Maps}_*(S^1, X) \leftarrow$ based loop space

$\pi_n(\Omega X) = \pi_{n+1}(X)$

Path-Loop Fibration $PX = \{\gamma \in X^I \mid \gamma(0) = *\}$. $\Omega X \hookrightarrow PX$

$$\begin{tikzcd} PX \arrow[d, "\pi"] & \pi(\gamma) = \gamma(1) & PX \arrow[d] \\ X & & X \end{tikzcd}$$

PX - contractible.

Proposition X - connected CW-complex. $\begin{tikzcd} E \arrow[d] \\ X \end{tikzcd}$ fibration $\Rightarrow$ all fibres are weakly equivalent.

Pullback
$$\begin{tikzcd} f^*(E) \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$
$f^*(E) = \{(e, y) \mid p(e) = f(y)\}$
$\subseteq E \times Y = E \times_X Y$

Universal Property: $Z \longrightarrow f^*(E)$

$\Leftrightarrow$
$$\begin{tikzcd} Z \arrow[r] \arrow[d] & E \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

Product is a special case of pullback

f^*E is a fibration:
$$\begin{tikzcd} S^n \times I \arrow[r] \arrow[d] & E \times_X Y \arrow[r] \arrow[d] & E \arrow[d] \\ S^n \arrow[r] \arrow[ru, dashed] \arrow[rru, dashed] & Y \arrow[r] & X \end{tikzcd}$$

$$\begin{tikzcd} F \arrow[r, hook] & E \arrow[d] & f^*E \arrow[l] \arrow[d] & F \arrow[l, hook'] \\ & X & Y \arrow[l, "f"'] & \end{tikzcd}$$

Long Exact Sequence:
$$\begin{tikzcd} {} \arrow[r] & \pi_k(F) \arrow[r] \arrow[d, "id"] & \pi_k(f^*E) \arrow[r] \arrow[d] & \pi_k(Y) \arrow[r] \arrow[d, "f"] & {} \\ {} \arrow[r] & \pi_k(F) \arrow[r] & \pi_k(E) \arrow[r] & \pi_k(X) \arrow[r] & {} \end{tikzcd}$$

if "f" is weak equivalence

$\Rightarrow$ $f^*E \longrightarrow E$ is also a weak equivalence

$\gamma : [0,1] \longrightarrow X \qquad \gamma(0) = x \qquad \gamma(1) = y$

$$\begin{tikzcd} F_x \arrow[r] \arrow[d] & \gamma^* E \arrow[r] \arrow[d] & E \arrow[d] \\ \{0\} \arrow[r, hook] & {[0,1]} \arrow[r, "\gamma"'] & X \end{tikzcd}$$

. $\{0\} \hookrightarrow [0,1]$ is a weak equivalence

$\Rightarrow F_x \simeq F_{[0,1]} \simeq F_y$

Eilenberg Maclane

$K(A,n)$ - CW complex, A - finitely generated group abelian $n \geq 1$

$$\pi_i(K(A,n)) = \begin{cases} A & \text{if } i = n \\ 0 & \text{else} \end{cases}$$

$$\overset{\varepsilon_n}{\tilde{H}^n}(K(\mathbb{Z}, n)) \cong \overset{1}{\mathbb{Z}}$$

$\Psi \quad \cancel{k}[X; K(\mathbb{Z},n)]_* \longrightarrow \cancel{H^n(K(\mathbb{Z},n))}\ H^n(X)$

$\qquad f : X \to K(\mathbb{Z},n) \longmapsto f^* \varepsilon_n$

Theorem: $\tilde{H}^n(X) \cong [X, K(\mathbb{Z},n)]_*$ via above map.

(Also true $\tilde{H}^n(\cancel{X}, A) \cong [X, K(\cancel{}A, n)]_*$

Proof:

1. $X = S^n$

   $[S^n, K(\mathbb{Z},n)] \xrightarrow{\cong} \mathbb{Z}$

2. Group Structure on $[X, K(\mathbb{Z},n)]_*$

   $[X, K(\mathbb{Z},n)]_* \cong [X, \Omega^2 K(\mathbb{Z}, n+2)]_*$

   $\qquad \cong [\Sigma^2 X, K(\mathbb{Z}, n+2)]_*$

~~3. Group Homomorphism~~

~~$[X, K(\mathbb{Z},n)]_* = [\Sigma^2 X, K(\mathbb{Z},n+2)]_* \ni f, g$~~

~~$f+g:$ $\cancel{[\Sigma^2 X, K(\mathbb{Z},n+2)] \longrightarrow [\Sigma^2 X \vee \Sigma^2 X, K(\mathbb{Z},n+2)]}$~~

~~$\Sigma^2 X \longrightarrow \Sigma^2 X \vee \Sigma^2 X \xrightarrow{f \vee g} K(\mathbb{Z}, n+2)$~~

~~$\cancel{K(}\tilde{H}^n(X, \mathbb{Z}) \cong \tilde{H}^{n+2}(\Sigma^2 X, \mathbb{Z}) \ni a, b$~~

~~$\Sigma^2 X \longrightarrow \Sigma^2 X \vee \Sigma^2 X \xrightarrow{a \vee b} K(\mathbb{Z}, n+2)$~~

$$\begin{tikzcd} \tilde{H}^{n+2}(\Sigma^2 X, \mathbb{Z}) & \tilde{H}^{2+n}(\Sigma^2 X, \mathbb{Z}) \oplus \tilde{H}^{n+2}(\Sigma^2 X, \mathbb{Z}) \arrow[l] & \tilde{H}^{n+2}(K(\mathbb{Z},n+2), \mathbb{Z}) \arrow[l] \\ & (\alpha & \Psi_*^*(\varepsilon_{n+2}) \arrow[l] & {[K(\mathbb{Z},n+2), K(\mathbb{Z},n+2)]} \arrow[ul, "\Psi"'] \end{tikzcd}$$

4. Cofibration Sequence

$$\begin{tikzcd}
{[A, K(\mathbb{Z},n)]_*} \arrow[d] & {[X, K(\mathbb{Z},n)]} \arrow[l] \arrow[d] & {[X \cup CA, K(\mathbb{Z},n)]_*} \arrow[l] \arrow[d] & \arrow[l] \\
H^n(A,\mathbb{Z}) & H^n(X,\mathbb{Z}) \arrow[l] & H^n(X \cup CA, \mathbb{Z}) \arrow[l] & \arrow[l]
\end{tikzcd}$$

5. X - CW Complex

$X^{(k)}$ - k^{th} skeleton of X

$X^{(k+1)} = X^{(k)} \cup CS^k$ ← attaching maps

Use cofibration sequence and 5-lemma

6. Use CW approximation for arbitrary spaces.

$\square$

3. Group Homomorphism

Mayer Vietoris →

$$\begin{tikzcd}
H^n(X,A) \arrow[d, leftrightarrow, "\cong"'] & {[X; K(\mathbb{Z},n)]_*} \arrow[l] & : \Psi \\
H^{n+2}(\Sigma^2 X, A) & {[\cancel{*}, \Sigma^2 X, K(\mathbb{Z},n+2)]} \arrow[l] \arrow[u, "\cong"] & : \Psi
\end{tikzcd}$$

Claim: $\psi(f+g) = \psi(f) + \psi(g)$

$g, f : \Sigma^2 X \longrightarrow K(\mathbb{Z}, n+2)$

$\widetilde{f+g} : \quad \Sigma^2 X \longrightarrow \Sigma^2 X \vee \Sigma^2 X \xrightarrow{f \vee g} K(\mathbb{Z}, n+2)$

$\psi(f+g) = \cancel{f}_* (\widetilde{f+g})_* \varepsilon_{n+2}$

$$\begin{tikzcd}
\Sigma^2 X \arrow[r] & \Sigma^2 X \vee \Sigma^2 X \arrow[r] & K(\mathbb{Z}, n+2) \\
H^{n+2}(\Sigma^2 X) & H^{n+2}(\Sigma^2 X) \oplus H^{n+2}(\Sigma^2 X) \arrow[l] & H^{n+2} K(\mathbb{Z}, n+2) \arrow[l] \\
f_* \varepsilon_{n+2} + g_* \varepsilon_{n+2} & f_* \varepsilon_{n+2} + g_* \varepsilon_{n+2} \arrow[l] & \varepsilon_{n+2} \arrow[l]
\end{tikzcd}$$

①

Orientable vector bundles

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, hook] & \xi \arrow[d] \\ & X \end{tikzcd}$$

ξ - orientable, if $\exists$ orientation

$\exists$ a class $\alpha_x \in H^n(\xi_x, \xi_{x,0})$ + compatibility

$\xi_{x,0} \to \xi_x - \{0\}$

Orientation $\Rightarrow$ structure group $SO(n)$

Complex $\Rightarrow$ orientable

$$\begin{tikzcd} H^*(\xi, \xi_0) \arrow[r] & H^*(\xi) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & H^*(\xi_0) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & H^{*+1}(\xi, \xi_0) \\ & H^*(X) \arrow[r, "p^*"] & H^*(S(\xi)) & \end{tikzcd}$$

$$\begin{tikzcd} \xi \arrow[d] & \xi_0 \arrow[l, hook'] \arrow[d] \\ X & S(\xi) \arrow[l, "p"] \end{tikzcd} \qquad \begin{tikzcd} S(\xi) \arrow[d, "p"] & S^{n-1} \arrow[l, hook'] \\ X & \end{tikzcd}$$

One-point compactification of each fibre gives a ~~do~~ S^n bundle - S^ξ

$$\begin{tikzcd} S^\xi \arrow[d] \\ X \arrow[u, bend left, "s", "0\text{ section}"'] \arrow[u, bend left=80, "s'"', "\infty\text{ section}"] \end{tikzcd}$$

$H^*(\xi, \xi_0) \cong H^*(S^\xi, \underset{\text{im } X}{\underset{\shortparallel}{S'(X)}})$

Comes because

$H^*(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \cong H^*(S^n, \infty)$

$H^*(\xi, \xi_0) \cong H^*(\xi \cup \overset{S'(X)}{\text{[redacted]}}, \xi_0 \cup \overset{S'(X)}{\text{[redacted]}}) \cong H^*(S^\xi, S'(X))$

$\hookrightarrow$ contractible to $S'(X)$

$$\begin{tikzcd} S^n \arrow[r] & S^\xi \arrow[d, "p"] \\ & X \end{tikzcd}$$

$$\begin{tikzcd} \pi_k(S^n) \arrow[r] & \pi_k(S^\xi) \arrow[r, "p_*"'] & \pi_k(X) \arrow[lld, bend left, rounded corners] \\ \pi_{k-1}(S^n) & & \end{tikzcd}$$

$\Rightarrow$ $p_* \cong$ if $k < n$ $\Rightarrow$ (X, S) n-connected.

$\twoheadrightarrow$ if $k = n$

$\Rightarrow$ $p_* \cong$ on H_* if $k < n$

$\twoheadrightarrow$ on H_* if $k = n$

$\Rightarrow$ $p^* \cong$ on H^* if $k < n$ Universal co-efficient th^m

$\hookrightarrow$ on H^* if $k = n$

$$\begin{tikzcd}
 & & H^*(s'(X)) \arrow[d, phantom, "\cong" rotate=90] & & \\
\arrow[r] & H^*(\xi,\xi_0) \arrow[r] & H^*(S^\xi) \arrow[ur, "s'^*"] \arrow[r] & H^*(X) \arrow[r] & H^{*+1}(\xi,\xi_0) \arrow[r] & {}
\end{tikzcd}$$

s' - section $\Rightarrow$ $p \circ s' = id$

$\Rightarrow$ $s'^* \cdot p^* = id$

$p^* \cong$ $\Rightarrow$ $s'^* \cong$

$p^* \hookrightarrow$ $\Rightarrow$ $s'^* \twoheadrightarrow$] use in the long exact sequence above

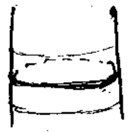
So we get

$$\boxed{\begin{array}{ll} H^*(\xi,\xi_0)=0 & \text{for } k<n \\ H^n(\xi,\xi_0) = \ker(H^n(S^\xi) \to H^n(X)) & \end{array}}$$

Example:

1. $\xi = S^1 \times \mathbb{R}$

$$H^*(\xi,\xi_0) = \begin{cases} \mathbb{Z} & \text{if } * = 2,1 \\ 0 & \text{else} \end{cases}$$



$$\begin{array}{c} H^*(\xi,\xi_0) \\ \cong \\ H^*(\xi,\xi_0') \\ \cong \\ \tilde{H}^*(\xi/\xi_0') \cong \tilde{H}^*(S^2 \vee S^1) \end{array}$$

2. $\xi =$ Mobius bundle

$$H^*(\xi,\xi_0) \cong \tilde{H}^*(\mathbb{R}P^2)$$

$$\cong \begin{array}{ll} \mathbb{Z}/2 & \text{if } * = 2 \\ 0 & \text{else} \end{array}$$

Thom Isomorphism Thm

- $\mathbb{R}^n \to \xi \downarrow X$ orientable $\iff$ $\exists\, \alpha \in H^n(\xi,\xi_0)$ s.t. $\downarrow i_x^*$ $H^n(\xi_x, \xi_{x_0})$ $i^*\alpha = \pm 1$

- for ξ - oriented,

$$H^*(X) \cong H^*(\xi) \xrightarrow{\cup \alpha} H^{*+n}(\xi,\xi_0) \text{ is an isomorphism}$$

For Trivial bundle:

$\xi = B\times\mathbb{R}^n$,

$$\begin{tikzcd} B\times\mathbb{R}^n \arrow[d] \\ B \end{tikzcd}$$

$$H^n(B\times\mathbb{R}^n, B\times\mathbb{R}^n_0) \cong \tilde{H}^n(B_+\wedge S^n)$$
$$\cong \tilde{H}^n(B\times S^n \,/\, (B\times\{\infty\})$$
$$\cong \cancel{H^n(B)\otimes H^n(S^n)} \,/\, \cancel{H^n(B)}$$

$$B\times\{\infty\} \longrightarrow B\times S^n \longrightarrow B\times S^n / B\times\{\infty\}$$

$$\begin{tikzcd} {} & H^n(B\times\{\infty\}) \arrow[l] \arrow[d, phantom, "\cong" rotate=90] & H^n(B\times S^n) \arrow[l] \arrow[d, phantom, "\cong" rotate=90] & H^n(B\times S^n/B\times\{\infty\}) \arrow[l] & {} \arrow[l] \\ & H^n(B) & H^n(B)\otimes H^0(S^n)\oplus H^0(B)\otimes H^n(S^n) & & \end{tikzcd}$$

$$H^n(S^n) = \mathbb{Z}\alpha$$

$$H^n\left(\frac{B\times S^n}{B\times\{\infty\}}\right) = \ker\left(\begin{array}{c} H^0(B)\otimes H^n(S^n) \\ \oplus \\ H^n(B)\otimes H^0(S^n)\end{array} \longrightarrow H^n(B)\right)$$
$$= H^0(B)\otimes H^n(S^n)$$

- Assume true for $U, V, U\cap V$

$$\begin{tikzcd} H^*(\xi U\cup V) \arrow[r, leftrightarrow] \arrow[d, "\cup\alpha_{U\cup V}"] & H^*(U)\oplus H^*(V) \arrow[r, leftrightarrow] \arrow[d, "\cup\alpha_U{,}\alpha_V"] & H^*(U\cap V) \arrow[d, "(\cup\alpha_{U\cap V})"] & \\ H^*(\xi_{U\cup V}, \xi_{U_0\cup V_0}) \arrow[r] & H^*(\xi_U, \xi_{U_0})\oplus H^*(\xi_V, \xi_{V_0}) \arrow[r] & H^*(\xi_{U\cap V}, \xi_{U\cap V_0}) \arrow[r] & {} \end{tikzcd}$$

$$(\alpha_U, \alpha_V) \longmapsto (\alpha_{U\cap V} - \alpha_{U\cap V}) = 0$$

$$\Rightarrow \exists\, \alpha_{U\cup V} \longmapsto (\alpha_U, \alpha_V) \longrightarrow 0$$

- So result is true for X-compact

- for $X = \varinjlim X_{(n)}$ $\quad X_{(n)}$ compact

$$\Rightarrow \begin{tikzcd} \mathrm{Hom}^*(X) \arrow[r] & \varprojlim H^*(X_{(n)}) \arrow[r] & 0 \\ 0 \arrow[r] & \lim{}^1 (H^{*-1}(X_{(n)})) \arrow[ul] & \end{tikzcd}$$

Corresponding th^m for S^{n-1}-bundles

$$\begin{tikzcd} S^{n-1} \arrow[r] & S \arrow[d] \\ & X \end{tikzcd} \quad \text{orientable} \iff \exists\ \alpha \in \begin{tikzcd} H^{n-1}(S) \arrow[d, "i^*"] \\ H^{n-1}(S_x) \end{tikzcd} \qquad i^*(\alpha) = \pm 1$$

$\phi \in H^n(\xi, \xi_0)$ is called the Thom class.

$$\begin{tikzcd} H^n(\xi,\xi_0) \arrow[r] \arrow[dr] & H^n(\xi) \arrow[d, "\cong"] \\ \phi \arrow[r, mapsto] & H^n(X) \end{tikzcd}$$

$\boxed{e(\xi) = \text{Euler class of } \xi.}$ with a chosen orientation.

- Naturality

$$\begin{tikzcd} f^*\xi \arrow[r] \arrow[d] & \xi \arrow[d] \\ Y \arrow[r, "f"] & X \end{tikzcd} \rightsquigarrow$$

$f^*\xi$ is also oriented

$e(f^*\xi) = f^*(e(\xi))$

$\phi(f^*\xi) = f^*\phi(\xi)$

~~$H^n(f^*\xi, f^*\xi_0)$ $H^n(\xi, \xi_0)$~~

$$\begin{tikzcd} \phi(f^*\xi) \arrow[dd] & H^n(f^*\xi, f^*\xi_0) \arrow[d] & H^n(\xi,\xi_0) \arrow[l, "f^*"'] \arrow[d] & \phi(\xi) \arrow[dd] \\ & H^n(f^*\xi) \arrow[d] & H^n(\xi) \arrow[l, "f^*"'] \arrow[d] & \\ e(f^*\xi) & H^n(Y) & H^n(X) \arrow[l, "f^*"'] & e(\xi) \end{tikzcd}$$

$\Rightarrow \quad e(f^*\xi) = f^*e(\xi)$

- If ξ - $\mathbb{R}^{2n+1}$ vector bundle,

$$\begin{tikzcd} \mathbb{R}^{2n+1} \arrow[r] & \xi \arrow[d] \\ & X \end{tikzcd}$$

$-\mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n$ gives

$$\begin{tikzcd} \xi \arrow[rr] \arrow[dr] & & \bar{\xi} \arrow[dl] \\ & X & \end{tikzcd}$$

orientation reversing

- Suppose $s : X \longrightarrow \xi$ is a section s.t. $s(x) \neq 0 \ \forall x$

  Then $e(\xi) = 0$.

- Proof:

$$\begin{tikzcd} H^n(\xi, \xi_0) \arrow[r] \arrow[dr] & H^n(\xi) \arrow[r] \arrow[d, "\cong"] & H^n(\xi_0) \arrow[dl, "s^*"] \\ & H^n(X) & \end{tikzcd}$$

$$\phi \longmapsto e$$

□

$s : X \longrightarrow \xi_0$

$$\begin{tikzcd} \xi_0 \arrow[r, "i"] & \xi \arrow[dl, shift left, "p"] \\ X \arrow[u, "s"] \arrow[ur, shift left, "s"'] & \end{tikzcd}$$

~~this map homotopic to 0-section~~

- <u>Th^m</u> N^n - compact, oriented manifold C^∞

  $e(TN) \in H^n(N)$ $\qquad [N] \in H_n(N) \cong \mathbb{Z}$

  $\quad \| $

  $e(N)$

  Then, $\langle e(N), [N] \rangle = \chi(N)$.

Proof:

$$\Delta : M \longrightarrow M \times M$$
$$x \longmapsto (x, x)$$

$T(M \times M) \cong TM \oplus TM$

$\nu_\Delta \cong$ Normal bundle of N in $N \times N$ $=$ TN

$H^*(\nu_\Delta, \nu_{\Delta 0})$

In general if ν is normal bundle of $i : N \longrightarrow W$, Then $H^*(\nu, \nu_0) \cong H^*(W, W-N)$

$\Rightarrow \phi_\nu \cap [W] = [N]$

Read Milnor, Stasheff

Phodu Proof

(Simpler proof in Bott, Tu)

Milnor-Stasheff notes

- Homology, Cohomology

  $(C^n)\, C_n X$ - (co) - cycles chains

  $(Z^n)\, Z_n X$ - (co) - cycles

  $(B^n)\, B_n X$ - (co) - boundaries

- If $H_{n-1}(X)$ is free, ~~B~~ $H^{n-1} \cong \mathrm{Hom}(H_{n-1}, G)$.

  - $\alpha \in \mathrm{Hom}(H_{n-1}(X), G)$ $\qquad H_n(X) = Z_n(X)/B_n(X)$

$$\alpha : \begin{tikzcd} Z_n(X)/B_n(X) \arrow[r] & G \\ Z_n(X) \arrow[u] & \end{tikzcd}$$

$$C_n(X)/Z_n(X) = B_{n-1}(X) \hookrightarrow Z_{n-1}(X) \hookrightarrow C_n(X)$$

free

$\Rightarrow$ ~~$C_n(X)$~~

$$\begin{tikzcd} 0 \arrow[r] & B_{n-1}(X) \arrow[r] & C_n(X) \arrow[r] \arrow[d, bend left] & Z_n(x) \arrow[r] \arrow[d] & 0 & \\ & G & Z_n/B_n(X) \arrow[l, "\alpha"'] & Z_n(X) \arrow[l] & B_n(X) \arrow[l] & 0 \arrow[l] \end{tikzcd}$$

Following the arrow we get an element of $H^n(X, G)$.

- $\alpha \in H^n(X, G)$ ~~defin~~

  we get $\tilde{\alpha} \in \mathrm{Hom}(H_n, G)$

  $\tilde{\alpha}([\sigma]) = \alpha(\sigma)$.

- ✱ If $\tilde{\alpha} \to 0 \Rightarrow \alpha(\sigma) = 0$ on all cycles

  Need to show $\alpha = \delta\beta \qquad \beta \in H^{n-1}(X, G)$.

  $\alpha : C_n(X) \longrightarrow G \qquad \alpha|Z_n(X) \longrightarrow 0$

  $\alpha = \delta\beta \equiv \alpha$ depends only on boundary

  $\beta : B_{n-1}(X) \longrightarrow G$

  $\beta(\tau) = \alpha(\sigma)$ for some σ, $\partial\sigma = \tau$

  $= \alpha \circ \partial^{-1}\tau$

  $H_{n-1} = Z_{n-1}/B_{n-1}$ - free $\Rightarrow B_{n-1} \overset{\text{splits}}{\hookrightarrow} Z_{n-1} \hookrightarrow C_{n-1}$

  Extend β to C_{n-1}.

  ⊠

In general

$$0 \to \mathrm{Ext}(H_{n-1}(X), G) \longrightarrow H^n(X, G) \longrightarrow \mathrm{Hom}(H_n(X), G) \to 0 .$$

• 9.3-f

$$\begin{tikzcd}\xi \arrow[rr, dash] \arrow[dr, dash] & & \eta \arrow[dl, dash] \\ & B & \end{tikzcd}$$

vector bundles

B ----> Tychonoff i.e. $\forall\, p \in B,\ V \subseteq B,\ V$ ~~open~~ closed, $p \notin V$

$\exists f : B \to [0,1]$

$f^{-1}(0) = p, \quad f^{-1}(1) = V.$

$S(\xi) = \{$continuous $s : B \longrightarrow \xi$, section$\}$

$C^0(B) = \{ f : B \longrightarrow \mathbb{R},$ continuous$\}$

$S(\xi)$ is $C^0(B)$ module

1. $S(\xi \oplus \eta)$

$\Psi : S(\xi) \oplus S(\eta) \xrightarrow{\cong} S(\xi \oplus \eta)$

$\Psi(f\alpha + g\beta)(p) = f(p)\,\alpha(p) + g(p)\,\beta(p)$

Injectivity is clear

Surjectivity

$$\begin{tikzcd}\xi \oplus \eta \arrow[r, "\Delta^*"] \arrow[d, dash] & \xi \times \eta \arrow[d, dash] \\ B \arrow[r, "\Delta"] & B \times B\end{tikzcd}$$

$$\begin{tikzcd} & \xi \oplus \eta \arrow[dl] \arrow[d] \arrow[dr] & \\ \eta \arrow[r] & B & \xi \arrow[l]\end{tikzcd}$$

$(\xi \oplus \eta)_p = (\xi_p, \eta_p)$

$\xi = B \times \mathbb{R}^n$

$\xi = \oplus^n \mathbb{1} \qquad \mathbb{1} = B \times \mathbb{R}$

Enough to show $\mathbb{1} = C^0(B)$ which is by defn

So $\xi = C^0(B)^n$ free

2. If $\xi \oplus \eta = B \times \mathbb{R}^n$

$S(\xi) \oplus S(\eta) = (C^0 B)^n$

So $S(\xi), S(\eta)$ projective

3. $\xi \xrightarrow{\tau} \eta$ τ isomorphism

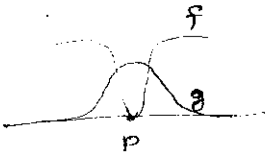
~~$S(\xi) \xleftarrow{\tau^*} S(\eta)$~~ ~~$S(\eta) \xrightarrow{\tau^*} S(\xi)$~~

$S(\xi) \xrightarrow{\tau_*} S(\eta)$ τ_* isomorphism

$f \longmapsto \tau \circ f$

Given $\tau : S(\xi) \longrightarrow S(\eta)$ isomorphism

$m_p = \{ f : B \to \mathbb{R} \mid f(p) = 0 \} \subseteq C^0(B)$



maximal : $g \notin m_p \Rightarrow g(p) \neq 0$

$\exists\, U \ni p$, $U \subseteq B$, open s.t.

$0 \notin g(U)$.

let f be f^n seperating U^c and p, $f \in m_p$

$f^2 + g^2 > 0 \Rightarrow$ unit □

$S(\xi) / m_p S(\xi)$ — module over $C^0(B)/m_p = \mathbb{R}$

Then, $\xi_p \cong S(\xi)/m_p S(\xi)$

$s(p) \longleftarrow [s]$

Define: $\tau^* : \xi \longrightarrow \eta$

$$\begin{tikzcd}
\xi_p \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \eta_p \arrow[d, phantom, "\cong" rotate=90] \\
S(\xi)/m_p S(\xi) & S(\eta)/m_p S(\eta)
\end{tikzcd}$$

Remains to show τ^* is continuous.

This you can do because,
τ^* takes sections to sections.

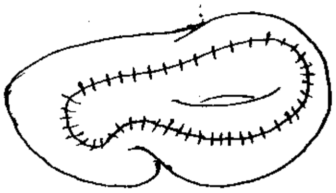
- Tubular Neighborhood Th^{m}:

$M \hookrightarrow A$ submanifold

$$\begin{tikzcd} TM \arrow[r, hook] \arrow[dr] & i^*TA \arrow[r] \arrow[d] & \nu_i \cong i^*TA/_{TM} \\ & M & \end{tikzcd}$$

$\uparrow$ normal bundle of $i: M \hookrightarrow A$

$\exists$ open set $\mathcal{U} \supseteq M$ in A, such that $\mathcal{U} \cong \nu_i$ with M as zero section.



Q. $f: U \longrightarrow V$ smooth

f bijective $\Rightarrow$ f diffeomorphism ?

Claim: $\begin{matrix} TM \\ \downarrow \\ M \end{matrix}$ $\quad \langle e(TM), [M] \rangle = \chi(M)$

Proof: • By Tubular Nbd,

$$\begin{tikzcd} & H^*(A, A-M) \arrow[dl, "\text{Poincare Duality}"] \arrow[r, phantom, "\cong"] & H^*(\nu_i, \nu_i - M) \\ H_{\dim(A)-*}(M) \cong & & \cong \; H^*(\nu_i, \nu_{i0}) \end{tikzcd}$$

$\dim \nu_i = \dim A - \dim M$

(Poincare dual of $i_*[M]$) = orientation of ν_i

$e(\nu_i) = i^*(\text{Poincare Dual}^{-1}(i_*[M]))$

$M \xhookrightarrow{\Delta} M \times M \qquad \nu_\Delta = TM$

- If co-efficient group is $\mathbb{Q}$,

$$H^*(M, \mathbb{Q}) \otimes H^{\dim M - *}(M, \mathbb{Q}) \longrightarrow \mathbb{Q}$$
$$(\alpha, \beta) \longmapsto (\alpha \cup \beta)[M]$$

Poincare duality $\Rightarrow$ non-singular bilinear ~~map~~ graded symmetric form

basis $\quad H^*(M, \mathbb{Q}) \quad \alpha_1 \dots \alpha_r \qquad\qquad (\alpha_i \cup \alpha_j^*)[M] = \delta_{ij}$

$H^{\dim M - *}(M, \mathbb{Q}) \quad \alpha_1^* \dots \alpha_r^*$

- $H^*(M\times M) \cong H^*(M) \otimes H^*(M)$ in $\mathbb{Q}$ coefficients

  $\{\alpha_i\}$ $\qquad$ $\{\alpha_i^*\}$ dual basis

  elements of form $A_{ij}(\alpha_i \otimes \alpha_i^*)$

- τ - orientation class $\nu \in H^*(M\times M, M\times M - \Delta(M)) \longrightarrow H^*(M\times M)$

$\tau \cap [M\times M] \quad \Delta_*[M]$

$\tau = \sum_{i,j} A_{ij}\, \alpha_i^* \otimes \alpha_j \qquad A_{ij} = 0$ if $|\alpha_i| \neq |\alpha_j|$

- $$\begin{aligned} &< (\alpha_i \otimes \alpha_j^*) \cup \tau, [M\times M]> \\ &= < \alpha_i \otimes \alpha_j^*, \tau \cap [M\times M]> \\ &= < \alpha_i \otimes \alpha_j^*, \Delta_*[M]> \\ &= < \Delta^*(\alpha_i \otimes \alpha_j^*), [M]> \\ &= < \alpha_i \cup \alpha_j^*, [M]> \\ &= \delta_{ij} \end{aligned}$$

$$\begin{aligned} &= <(\alpha_i \otimes \alpha_j^*) \cup \sum A_{kl}(\alpha_k^* \otimes \alpha_l), [M\times M]> \\ &= (-1)^{(\,)} A_{ij} < (\alpha_i \cup \alpha_i^*) \otimes (\alpha_j^* \cup \alpha_j), [M] \otimes [M \\ &= (-1)^{|\alpha_i|^2} A_{ij} < \alpha_i \cup \alpha_i^*, [M]> < \alpha_j^* \cup \alpha_j, [M] \\ &= A_{ij}\, (-1)^{n|\alpha_i|} \end{aligned}$$

$$A_{ij} = (-1)^{n|\alpha_i|} \delta_{ij}$$

$$\Rightarrow \quad \tau = \sum_i (-1)^{n|\alpha_i|} \alpha_i^* \otimes \alpha_i$$

$$\begin{aligned} \Rightarrow \quad e(TM) &= \Delta^* \tau \\ &= \sum_i (-1)^{n|\alpha_i|} (\alpha_i^* \cup \alpha_i) \\ <e(TM), [M]> &= \sum_i (-1)^{n|\alpha_i|} <\alpha_i^* \cup \alpha_i, [M]> \\ &= \sum_i (-1)^{|\alpha_i|} \\ &= \chi(M) \end{aligned}$$

⊠

<u>Corollary</u>: Hairy ball theorem.

$\phi(\xi \oplus \xi') = \phi(\xi) \cup \phi(\xi')$

$e(\xi \oplus \xi') = e(\xi) \cup e(\xi')$

- $$\begin{tikzcd} \arrow[r] & H^*(\xi,\xi_0) \arrow[r] \arrow[d, "\cong", "\text{Thom}"'] & H^*(\xi) \arrow[r] \arrow[d, "\cong"] & H^*(\xi_0) \arrow[r] & {} \\ & H^{*-n}(X) \arrow[r, bend left, "\cup e"] & H^*(X) & & \end{tikzcd}$$

$$\longrightarrow H^{*-n}(X) \xrightarrow{\cup e} H^*(X) \longrightarrow H^*(\xi_0) \longrightarrow$$ Gysin Sequence

---

Stiefel Whitney Classes:

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, dashed] & \xi \arrow[d] \\ & B \end{tikzcd}$$

$\omega_i(\xi) \in H^i(B; \mathbb{Z}/2)$

1) $\omega_0(\xi) = 1$, $\omega_i(\xi) = 0$ if $\dim \xi < i$

2) $\omega_i(\xi \oplus \eta) = \sum_{k+l=i} \omega_k(\xi) \cup \omega_l(\eta)$

3) $f^*(\omega_i(\xi)) = \omega_i(f^*\xi)$

4) $\omega_1(\mathcal{L}) \neq 0$ where $\begin{matrix}\mathcal{L}\\ \downarrow \\ \mathbb{R}P^1\end{matrix}$ canonical line bundle

Chern Classes:

$$\begin{tikzcd} \mathbb{C}^n \arrow[r] & \xi \arrow[d] \\ & B \end{tikzcd}$$

$c_i(\xi) \in H^{2i}(B, \mathbb{Z})$

1) $c_0(\xi) = 1$, $c_i(\xi) = 0$ if $2i > \dim \xi$

2) $c_i(\xi \oplus \eta) = \sum_{k+l=i} c_k(\eta) \cup c_l(\eta)$

3) $f^*(c_i(\xi)) = c_i(f^*\xi)$

4) $c_1(\mathcal{L}) =$ where $\begin{matrix}\mathcal{L}\\ \downarrow \\ \mathbb{C}P^1\end{matrix}$ canonical line bundle

Then $\omega = 1 + \omega_1 + \omega_2 + \dots$ Total S-W class

$C = 1 + c_1 + c_2 + \dots$ Total Chern class

- If $\eta \oplus \xi =$ trivial

  $\omega(\eta) = \bar{\omega}(\xi)$ $\quad (\bar{\omega}(\xi) \cdot \omega(\xi) = 1)$

- $\omega(TS^n) = 1$ $\quad \because TS^n \oplus 1 = n+1$

- $$\begin{tikzcd} \mathcal{L} \arrow[r, dashed] \arrow[d] & \mathcal{L}_n \arrow[d] \\ \mathbb{R}P^1 \arrow[r, dashed] & \mathbb{R}P^n \end{tikzcd}$$ so $i^*\omega_1(\mathcal{L}_n) = \omega_1(\mathcal{L}) \neq 0$

  $\Rightarrow \quad \omega_1(\mathcal{L}_n) \neq 0$

- $T\mathbb{R}P^n \cong \mathrm{Hom}(\mathcal{L}, \mathcal{L}^\perp)$

  $\mathrm{Hom}(\mathcal{L},\mathcal{L}^\perp) \oplus \mathrm{Hom}(\mathcal{L},\mathcal{L}) = \mathrm{Hom}(\mathcal{L}, n+1)$

  $\Rightarrow \quad \mathrm{Hom}(\mathcal{L},\mathcal{L}^\perp) \oplus 1 \quad * \quad$ ~~Hom~~ $\oplus^{n+1} \mathrm{Hom}(\mathcal{L}, 1) \cong \mathcal{L}$

  $\Rightarrow \quad \omega(T\mathbb{R}P^n) = \omega(\mathrm{Hom}(\mathcal{L},\mathcal{L}^\perp))$

  $= (\omega(\mathrm{Hom}(\mathcal{L}, 1))^n$

  $= (1+\alpha)^n \qquad \mathbb{Z}\alpha = H^1(\mathbb{R}P^n)$

  $\omega_i(T\mathbb{R}P^n) = \binom{n+1}{i}\alpha^i \quad$ for $\quad i \leq n$

- $T\mathbb{R}P^n \cong n \quad \Rightarrow \quad \omega(T\mathbb{R}P^n) = 1$

  $\Rightarrow \quad \binom{n+1}{i} = 0 \pmod 2 \quad$ for $k < i \leq n$

  $\Rightarrow \quad n = 2^k - 1 \quad$ for some k.

- $\mathbb{R}P^n \overset{i}{\hookrightarrow} \mathbb{R}^{n+k}$ immersion

  Let ν be ~~the~~ normal bundle of i

  $\Rightarrow \quad T\mathbb{R}P^n \oplus \nu = n+k$

  $\Rightarrow \quad \omega(T\mathbb{R}P^n) \cdot \omega(\nu) = 1$

  $\Rightarrow \quad \bar{\omega}_i(T\mathbb{R}P^n) = 0 \quad$ for $i > k$

  for $n = 2^r \quad \binom{n+1}{i} = \binom{2^r+1}{i} = \binom{2^r}{i-1} + \binom{2^r}{i} = 0$ for $i \neq n, 1$

  $\omega(T\mathbb{R}P^{2^r}) = 1 + \alpha + \alpha^{2^r}$

  $\bar{\omega}(T\mathbb{R}P^{2^r}) = 1 + (\alpha + \alpha^{2^r}) + (\alpha + \alpha^{2^r})^2 + (\alpha + \alpha^{2^r})^3 + \dots$

  $= 1 + \alpha + \alpha^2 + \dots + \alpha^{2^r}$

  $\Rightarrow \quad \mathbb{R}P^{2^r}$ can only be immersed in $\mathbb{R}P^{2^{r+1}-1}$

Cohomology, Poincare Duality

$$(X,A)\times(Y\times B) = (X\times Y,\ X\times B \cup Y\times A)$$

- $$\begin{tikzcd} H^m(X,A) \arrow[r, dashed, "\simeq"] & H^{m+n}((X,A)\times(\mathbb{R}^n, \mathbb{R}^n-\{0\})) \\ a \arrow[r, mapsto, dashed] & a\times e \end{tikzcd}$$ $e \in H^n(\mathbb{R}^n, \mathbb{R}^n-\{0\})$

- $K \subset L \subset M$ $\qquad M-L \subseteq M-K$

$$\begin{tikzcd} H_i(M,M-L) \arrow[r, dashed] & H_i(M,M-K) \\ \alpha \arrow[r, mapsto] & \rho_K(\alpha) \end{tikzcd}$$

$$C_i(M-L) \hookrightarrow C_i(M-K)$$

$$C_i(M)/C_i(M-L) \dashrightarrow\!\!\!\rightarrow C_i(M)/C_i(M-K)$$

- <u>Prop</u>: $H_i(M, M-K) = 0$ for $i > \dim M$

  $\alpha \in H_i(M,M-K)$ $\qquad \rho_x : H_i(M,M-K) \longrightarrow H_i(M,M-x)$

  Then,

  $$\alpha = 0 \iff \rho_x(\alpha) = 0 \quad \forall\, x \in K$$

- *Local Orientation*:

  $\mu_x \in H_n(M, M-x)$ s.t. $\forall x\ \exists B \ni x$ satisfying $\exists \alpha$ s.t.

  $$\begin{tikzcd} H_n(M,M-x) & H_n(M,M-B) \arrow[l] \arrow[r, dashed] & H_n(M,M-y) \\ \mu_x & \alpha \arrow[l, mapsto] \arrow[r, mapsto, dashed] & \mu_y \end{tikzcd} \quad \forall y \in B.$$

- *Global Orientation*:

  Given a local orientation, $\forall K \subseteq M$ compact,

  $\exists \mu_K \in H_n(M,M-K)$ s.t. $\rho_x(\mu_K) = \mu_x \quad \forall x \in K$.

- $$H^i_{comp}(M) = \varinjlim_{\substack{K \\ compact}} H^i(M,M-K) \qquad H^i(M,M-K) \longrightarrow H^i(M,M-L)$$

- M oriented

  $$\begin{tikzcd} H^n_{comp}(M) \arrow[r] & \mathbb{Z} \\ a \arrow[r, mapsto] & \langle a', \mu_K \rangle \end{tikzcd} \quad \text{for } a' \in H^n(M,M-K)$$

  Integration /

  $$\begin{tikzcd} H^i(M,M-K) \arrow[r] & H^i_{comp}(M \\ a' \arrow[r, mapsto] & a \end{tikzcd}$$

  Kronecker product with the fundamental class

- $\cap : C^i X \otimes C_n X \dashrightarrow C_{n-i}(X)$

  $\langle a, b \cap \xi \rangle = \langle \cancel{ab}\, a \cup b, \xi \rangle$

- Poincare Duality:

  M compact, oriented

  $$H^i M \xrightarrow{\cong} H_{n-i} M$$
  $$a \longmapsto a \cap \mu_M$$

- $\cap : C^i(X,A) \otimes C_n(X, A \cup B) \dashrightarrow C_{n-1}(X,B)$

  $$D : H^i_{comp.} M \xrightarrow{\cong} H_{n-1} M$$
  $$a \longmapsto a' \cap \mu_K$$

- $H^i_{comp}(M) \xrightarrow{\cong} H_{n-i}(M, \partial M)$ M with boundary

  $H^i_{comp}(M, \partial M) \xrightarrow{\cong} H_{n-i}(\mathring{M})$

- Alexandar duality:

  $K \underset{comp}{\subseteq} S^n$, good

  $$H^i K \cong \varinjlim_{U \supseteq K} H^i(U)$$

  $$H^i(S^n, K) \cong \varinjlim H^i(S^n, U) \cong H^i_{comp}(S^n - K)$$
  $$\cong H_{n-i}(S^n - K)$$

  $$\Rightarrow \tilde{H}^{i-1}(K) \cong \tilde{H}_{n-i}(S^n - K)$$

Ch. 4)

A)

$$\begin{tikzcd} \xi \arrow[d] & \eta \arrow[d] & \hat{\xi} \arrow[r, dashed] \arrow[d] & \xi \arrow[d] & \hat{\eta} \arrow[r, dashed] \arrow[d] & \eta \arrow[d] \\ X & Y & X \times Y \arrow[r, "\pi_1"] & X & X \times Y \arrow[r, "\pi_2"] & Y \end{tikzcd}$$

Then,

$$\begin{tikzcd} \xi \times \eta \arrow[r] \arrow[d] & \hat{\xi} \times \hat{\eta} \arrow[d] \\ X \times Y \arrow[r, "\Delta"] & (X \times Y) \times (X \times Y) \end{tikzcd}$$

i.e. $\xi \times \eta = \hat{\xi} \oplus \hat{\eta}$

So $\omega(\xi \times \eta) = \omega(\hat{\xi} \oplus \hat{\eta}) = \omega(\hat{\xi}) \cdot \omega(\hat{\eta})$

$= \omega(\pi_1^* \xi)\ \omega(\pi_2^* \eta)$

$= \pi_1^* \omega(\xi)\ \pi_2^* \omega(\eta)$

$= \omega(\xi) \times \omega(\eta).$

B) $n+1 = 2^r m$

$$\binom{n+1}{2^r} = \frac{2^r m \cdot (2^r m - 1) \ldots}{2^r \cdot (2^r - 1) \ldots} = \text{odd}$$

$\omega_{2^r}(\mathbb{RP}^n) = \omega_{n+1-2^r}(\mathbb{RP}^n) \neq 0.$

if $T\mathbb{RP}^n = 2^r + \eta$ $\quad$ $\dim \eta = n - 2^r$

then $\omega_{n+1-2^r}(T\mathbb{RP}^n) = 0$

C) $\mathbb{RP}^n$ - $\omega(\mathbb{RP}^n) = (1+a)^{n+1}$ $\quad$ $a \in H^1(\mathbb{RP}^n, \mathbb{Z}_2) \neq 0$

For n-odd we have a non-vanishing vector field.

For n-even $\quad n = 2m$

If $T\mathbb{RP}^{2m} = \eta + \varepsilon$ $\quad$ $\dim \eta = 1$, $\dim \varepsilon = 2m-1$

$\Rightarrow$ $\omega(\eta) = (1+a)$ or 1

$\Rightarrow$ $\varepsilon = (1+a)^{2m}$ or $(1+a)^{2m+1}$

Both not possible as $\dim \varepsilon < 2m$

for $\mathbb{R}P^4$, $\omega = (1+a)^5 = 1+a+a^4$

If $T\mathbb{R}P^4 = \eta + \varepsilon$ $\quad \dim\eta = \dim\varepsilon = 2$

$\Rightarrow \omega(\eta) = 1+a+a^2$ or $1+a$ or $(1+a)^2$ or 1

$\Rightarrow \omega(\varepsilon) = 1+a^2+a^3+a^4$ or $1+a^4$ or $1+a+a^2+a^3$ or $1+a+a^4$

Not possible as $\dim\varepsilon < 3$

for $\mathbb{R}P^6$, $\omega = (1+a)^7 = 1+a+a^2+a^3+a^4+a^5+a^6$.

If $T\mathbb{R}P^6 = \eta + \varepsilon$ $\quad \dim\eta = 2 \quad \dim\varepsilon = 4$

$\Rightarrow \omega(\eta) = 1+a+a^2$ or $(1+a)^2$ or $1+a$ or 1

$\Rightarrow \omega(\varepsilon) = 1+a^3+a^6$ or $1+a+a^4+a^5$ or $1+a^2+a^4+a^6$ or $(1+a)^6$

$\dim\varepsilon < 5$.

D) $M^m \longrightarrow \mathbb{R}^{m+1}$ immersion

$\Rightarrow TM \oplus \eta = m+1 \quad \dim\eta = 1 \quad$ ~~$\omega_1(\eta) = \alpha$~~

$\omega(\eta) = 1$ or $1+$

$\Rightarrow \omega(M) = \frac{1}{\omega(\eta)} = \frac{1}{1+\omega_1(\eta)} = 1 + \omega_1(\eta) + \omega_1(\eta)^2 + \cdots$

$\Rightarrow \omega_i(M) = \omega_1(M)^i$

for $\mathbb{R}P^n$, this implies

$\omega(\mathbb{R}P^n) = 1$ or $\omega(\mathbb{R}P^n) = \frac{1}{1+a}$

$\Rightarrow (1+a)^{n+1} = 1$ or $(1+a)^{n+2} = 1$

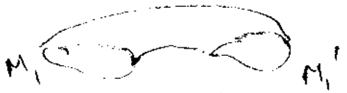
$\Rightarrow \binom{n+1}{i} = 0$ ~~$\forall i >$~~ or $\binom{n+2}{i} = 0 \quad \forall i$ non-trivial

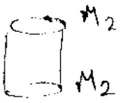
$\Rightarrow n = 2^{r-1}$ or 2^{r-2}

E) $\mathcal{N}_n$ = co-bordism classes of n-manifolds

Additive structure:

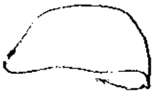
- $[M_1] + [M_2] = [M_1 \sqcup M_2]$





$[M_1] = [M_1'] \Rightarrow [M_1 \sqcup M_2] = [M_1' \sqcup M_2]$

- $[M] + [M] = 0$ ← $\mathbb{Z}/2$ module
- [Boundary] = 0



Characterized by Steifel Whitney nos.

$P^2 \times P^2$ $\quad \omega(P^2 \times P^2) = \omega(P^2) \times \omega(P^2)$

$$= (1+a+a^2) \times (1+a+a^2)$$

$$= 1 + 1\times a + 1\times a^2 + a\times 1 + a\times a + a\times a^2 + a^2\times 1 + a^2\times a + a^2\times a^2$$

$\omega_1^4 = 0$

$$= 1 + (1\times a + a\times 1) + (1\times a^2 + a\times a + a^2\times 1) + (a\times a^2 + a^2\times a) + a^2\times a^2$$

P^4 $\quad \omega(P^4) = (1+a)^5$

$$= 1 + a + a^4$$

$\omega_1^4 = \omega_4$

So $[P^2 \times P^2] \neq [P^4]$

$$\begin{tikzcd} \mathbb{C} \arrow[r] & \mathcal{L} \arrow[d] \\ & \mathbb{C}P^n \end{tikzcd}$$ canonical line bundle

Gysin sequence :

$$H^k(\mathbb{C}P^n) \xrightarrow[-\cup e(L)]{} H^{k+2}(\mathbb{C}P^n) \longrightarrow H^{k+2}(\mathcal{L}_0) \longrightarrow H^{k+1}(\mathbb{C}P^n)$$

? $$\begin{tikzcd} \mathcal{L}_0 \arrow[d] \\ \mathbb{C}P^n \end{tikzcd}$$ $\cong \mathbb{C}^{n+1} - \{0\} \simeq S^{2n+1}$ $\Rightarrow$ $H^*(\mathcal{L}_0) = \begin{cases} 0 & * < 2n+1 \end{cases}$

$H^{k}(\mathbb{C}P^n) \xrightarrow[-e(L)]{} H^{k+2}(\mathbb{C}P^n)$ is an isomorphism for $k \leq 2n+1$

This can only happen if

$$\boxed{e(L) = \pm x.}$$

- $$\begin{tikzcd} \mathcal{L} \arrow[d] \\ \mathbb{C}P^n \end{tikzcd}$$ $e(L)$ $\Rightarrow$ $\{1, -e(L), (-e(L))^2, \dots, (-e(L))^n\}$ is a basis for $H^*(\mathbb{C}P^n)$.

- $$\begin{tikzcd} \xi \arrow[d] & \mathbb{C}^n \arrow[l, hook'] \\ X \end{tikzcd}$$ $x \in X$ $\xi_x \cong \mathbb{C}^n$

  lines in $\xi_x \cong \mathbb{C}P^{n-1}$

$$\begin{tikzcd} \mathbb{C}^n \arrow[r, hook] & \xi \arrow[d, "p"] \\ & X \end{tikzcd} \longrightarrow \begin{tikzcd} \mathbb{C}P^{n-1} \arrow[r, hook] & P(\xi) \arrow[d, "p"] \\ & X \end{tikzcd}$$ Projectivisation of ξ^n.

Form a canonical line bundle over $P(\xi)$

$$\begin{tikzcd} \mathbb{C} \arrow[r] & \mathcal{L}_\xi \arrow[d] \\ & P(\xi) \end{tikzcd}$$ $(\mathcal{L}_\xi)_{(x,\ell)} = \ell$

- $$\begin{tikzcd} \mathcal{L} \arrow[r, hook, "\hat{i}"] \arrow[d] & \mathcal{L}_\xi \arrow[d] \\ \mathbb{C}P^n \arrow[r, hook, "i"] & P(\xi) \end{tikzcd}$$ Now let $y = e(\mathcal{L}_\xi) \in H^2(P(\xi))$

So Pulling back y via i^*, we get

$$e(\mathcal{L}) = i^* e(\mathcal{L}_{P\xi}) = i^* y$$

$$\Rightarrow \quad (1, i^*y, (i^*y)^2, \dots, (i^*y)^{n-1}) \leftarrow \text{basis for } \mathbb{C}P^{n-1}$$

By Leray Hirsch Th^m:

$$\boxed{H^*(P(\xi)) = (p^*H^*(X))\{1, y, y^2, \dots, y^{n-1}\}}$$

$$\begin{tikzcd} P(\xi) \arrow[d, "p"] \\ X \end{tikzcd}$$

Then, $\exists !$ eq^n

$$y^n - c_1 y^{n-1} + c_2 y^{n-2} - + \dots + (-1)^n c_n = 0 \qquad c_i \in H^{2i}(X).$$

Then:

$$c_i = i^{th} \text{ Chern class of } \begin{tikzcd} \xi \arrow[d] \\ X \end{tikzcd}$$

- Need to check axioms:

1. Naturality

$$\begin{tikzcd} & & & L_{f^*\xi} \arrow[r, "f^*"] \arrow[d] & \mathcal{L}_\xi \arrow[d] \\ f^*\xi \arrow[r] \arrow[d] & \xi \arrow[d] \arrow[rr, phantom, "\longrightarrow"] & & f^*P(\xi) \arrow[r, "f^*"] \arrow[d] & P(\xi) \arrow[d] \\ Y \arrow[r, "f"] & X & & Y \arrow[r, "f"] & X \end{tikzcd}$$

2. $c_i(\xi) = 0$ for $i > n$, $c_0(\xi) = 1$

3. $c(\xi \oplus \eta) = c(\xi)\, c(\eta)$

Trick:

$$\mathbb{C}P^n - \mathbb{C}P^i \underset{\substack{\text{def}\\ \text{retract}}}{\simeq} \mathbb{C}P^{n-i-1}$$

LHS $= [x_0 : \dots : x_i : x_{i+1} : \dots : x_n]$ s.t. atleast 1 of $x_{i+1} \cdots x_n \neq 0$

$$\downarrow$$

$[tx_0 : \dots tx_i : x_{i+1} \cdots : x_{n+1}]$ ← def retract

at $t=0$ RHS

at $t=1$ LHS

Now do same thing for $P(V \oplus W)$

$$P(V \oplus W) - P(V) \xrightarrow[\text{def retracts}]{} P(W)$$

Do this fibre wise

$$U = P(\xi \oplus \eta) - P(\xi) \xrightarrow[\text{def}]{\sim} P(\eta)$$

$$V = P(\xi \oplus \eta) - P(\eta) \xrightarrow[\text{def.}]{\sim} P(\xi)$$

$$\begin{tikzcd}
 & {} \arrow[d] & H^*(P(\xi\oplus\eta),V) \arrow[d] & & & H^*(P(\xi\oplus\eta),U) \arrow[r] \arrow[d] & H^*(P(\xi\oplus\eta),U\cup V)=0 \arrow[d] \\
\text{Exact} \arrow[urr, bend left] \arrow[urrrrr, bend left] & \omega_1 \arrow[d] & H^*(P(\xi\oplus\eta)) \arrow[d] & \overset{\smile}{\text{Cup}} & \omega_2 \arrow[d] & H^*(P(\xi\oplus\eta)) \arrow[r] \arrow[d] & H^*(P(\xi\oplus\eta)) \\
 & 0 & H^*(P(\xi)) & & 0 & H^*(P(\eta)) &
\end{tikzcd}$$

Cup product

$$y_{\xi\oplus\eta} = e(\mathcal{L}_{P(\xi\oplus\eta)})$$

$$\omega_1 = y_{\xi\oplus\eta}^n - c_1(\xi)\, y_{\xi\oplus\eta}^{n-1} + \dots + (-1)^n c_n(\xi)$$

$$\omega_2 = y_{\xi\oplus\eta}^m - c_1(\eta)\, y_{\xi\oplus\eta}^{n-1} + \dots + (-1)^m c_m(\xi).$$

As, $\omega_1 \xrightarrow{i^*} 0$, $\omega_2 \xrightarrow{i^*} 0$

$\omega_1 \in H^*(P(\xi\oplus\eta), V)$ $\omega_2 \in H^*(P(\xi\oplus\eta), U)$

$\Rightarrow$ $\omega_1 \cdot \omega_2 = 0$

$\Rightarrow$ whitney product formula.

4) $$\begin{tikzcd} \mathcal{L} \arrow[d] \\ \mathbb{C}P^n \end{tikzcd} \Rightarrow c_1(\mathcal{L}) = -x \qquad \begin{tikzcd} \mathcal{L} \arrow[d] \arrow[r] & P(\mathcal{L}) \cong \mathbb{C}P^n \arrow[d] \\ \mathbb{C}P^n & \mathbb{C}P^n \end{tikzcd}$$

⑪

Milnor-Stasheff notes.

- $H^n(\mathbb{R}^1\times B, \mathbb{R}^1_0\times B) \longleftarrow H^{n-1}(B)$ $\quad e =$ generator of $H^1(\mathbb{R}^1, \mathbb{R}^1_0)$

  $b\times e \longleftarrow b$ $\quad$ is an isomorphism $\quad \mathbb{R}^1_0 = \mathbb{R}^1 - \{0\}$

- $H^n(X,A)\otimes H^m(Y,B) \xrightarrow{\times} H^{n+m}(X\times Y, A\times Y\cup B\times X)$

  $\omega \;,\; \tau \longrightarrow p_1^*(\omega) \overset{}{\underset{\cup}{}} p_2^*(\tau)$

$$\begin{tikzcd} X\times Y, A\times Y \arrow[d, "p_1"'] & X\times Y, X\times B \arrow[d, "p_2"'] \\ \text{~~A\times Y~~}\ X, A & Y, B \end{tikzcd}$$

$$\begin{tikzcd} & X\times Y, A\times Y\cup B\times X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X, A & & Y, B \end{tikzcd}$$

- $H^n(X,A')\otimes H^n(X,B) \xrightarrow{\cup} H^n(X, A\cup B)$

New Notation: $\mathcal{S}^n\binom{X}{A} :=$ n^{th} singular cohomology of pair (X,A).

$\mathcal{S}^{-n}\binom{X}{A} :=$ n^{th} singular homology of (X,A).

- How to define $\cup$ on $\binom{X}{A}\binom{X}{B}$?

  $\omega \in \mathcal{S}^n\binom{X}{A}$, $\tau \in \mathcal{S}^m\binom{Y}{B}$

  Take $n+m$ simplex

  act ω on front face, τ on back face.

  If a simplex lies in $A\cap B$, $\omega\tau$ on it will be 0.

  why will $\omega\tau$ be 0 on $A\cup B$?

  if $\sigma \in A$ or B $\quad \omega\tau(\sigma) = 0$ as $\omega = 0$ or $\tau = 0$

  but σ might not be completely in A or B.

  But $\exists$ a short exact seq:

$$0 \longrightarrow C^*\binom{X}{A\cup B} \longrightarrow C^*\binom{X}{A}\cap C^*\binom{X}{B} \longrightarrow C^*\binom{A\cup B}{A}\cap C^*\binom{A\cup}{B}$$

$$\downarrow 0$$

Claim: $C^n\binom{A\cup B}{A}\cap C^n\binom{A\cup B}{B} \xrightarrow{\delta} C^{n+1}(\quad)\cap C^{n+1}(\quad)$

is always acyclic. (i.e. trivial cohomology)

Proof: $\bullet$ $\omega \in \ker\delta \subseteq C^n(\quad)(\quad)$

$\Rightarrow \omega \in C^n(A\cup B)$, $\omega|_A = 0 = \omega|_B$, $\delta\omega = 0$

$\omega \in \operatorname{Im}\delta \subseteq C^n$

$\Rightarrow \omega = \delta\tau \qquad \tau \in C^{n+1}(\ \ ) \cap C^{n}(\ \ )$

for $\omega \in \operatorname{Ker}\delta$, need to construct $\tau \in C^{n+1}$ s.t. $\delta\tau = \omega$

Do barycentric subdivision of ~~ω~~ to $A \cup B$

$$C^n_{A,B}\begin{pmatrix} A\cup B \\ A \end{pmatrix} \cap C^n_{A,B}\begin{pmatrix} A\cup B \\ B \end{pmatrix}.$$

ω acts only on chains of A or B so this cochain group itself is trivial.

Now by barycentric subdivision th^m, $C^n = C^n_{A,B}$ □

So when we write ~~coh~~ long-exact sequence

$$0 \longrightarrow \mathcal{S}^*\begin{pmatrix} X \\ A\cup B \end{pmatrix} \longrightarrow H^*\left(C^*\begin{pmatrix} X \\ A \end{pmatrix} \cap C^*\begin{pmatrix} X \\ B \end{pmatrix}\right) \longrightarrow 0$$

So a cochain which is 0 on A or B

i.e. $\omega \in C^n(X)$, $\omega|_A = 0 = \omega|_B$

can we represented by a cochain (is cohomologous) which is 0 on $A \cup B$

i.e. $\exists\, \tau \in C^n(X)$, $\tau|_{A\cup B} = 0$, $(\tau - \omega) = \delta\omega'$ for some ω'

$$\mathcal{S}^n(B,\bullet) \otimes \mathcal{S}^1(\mathbb{R}^1, \mathbb{R}^1_0) \xrightarrow{\times} \mathcal{S}^{n+1}(B\times\mathbb{R}^1, B\times\mathbb{R}^1_0)$$

$$b, e \longmapsto b \times e$$

$\mathbb{R}^1_- =$ negative Real axis

$$\begin{tikzcd}
{} \arrow[r, "\text{injective}"] & \mathcal{S}^n(B\times\mathbb{R}^1_0) \arrow[r] & \mathcal{S}^{n+1}\begin{pmatrix} B\times\mathbb{R}^1 \\ B\times\mathbb{R}^1_0 \end{pmatrix} \arrow[r] & \mathcal{S}^{n+1}(B\times\mathbb{R}^1) \arrow[r, "\text{injective}"] & \mathcal{S}^{n+1}(B\times\mathbb{R}^1_0) \arrow[r] & {} \\
& \mathcal{S}^{0}(\mathbb{R}^1_0) \arrow[r] \arrow[u, "\times b"'] & \mathcal{S}^{\cancel{n+}1}\begin{pmatrix} \mathbb{R}^1 \\ \mathbb{R}^1_0 \end{pmatrix} \arrow[r] \arrow[u, "\times b"'] & \mathcal{S}^{\cancel{n+}1}(\mathbb{R}^1) \arrow[r] \arrow[u, "\times b"'] & \mathcal{S}^{\cancel{n+}1}(\mathbb{R}^1_0) \cdots \arrow[u, "\times b"'] &
\end{tikzcd}$$

$$e \longmapsto a$$

$B\times\mathbb{R}^n_0$ $\quad$ $B\times\mathbb{R}^n$

$$\begin{tikzcd}
{[B\times\mathbb{R}^n_0]^{m-1+n}} \arrow[r, "\delta^*"] & {\begin{bmatrix}B\times\mathbb{R}^n_0\\ B\times\mathbb{R}^n_0\end{bmatrix}^{m+n}} \arrow[r, "p^*"] \arrow[dr, "\psi^*"'] & {[B\times\mathbb{R}^n]^{m+n}} \arrow[r, "i^*"] \arrow[d, shift left, "i^*"] & {[B\times\mathbb{R}^n_0]^{m+n}} \arrow[r] & {} \\
& & {[B]^n} \arrow[u, shift left, "p^*"] \arrow[ur, "\phi^*"'] & &
\end{tikzcd}$$

$\phi: \; B\times\mathbb{R}^n_0 \hookrightarrow B\times\mathbb{R}^n \twoheadrightarrow B$ $\quad$ just the

$\quad\; b, x \longmapsto b, x \longmapsto b$ $\quad$ usual projection

Interesting: $\psi: \; B \hookrightarrow B\times\mathbb{R}^n \twoheadrightarrow \begin{matrix}B\times\mathbb{R}^n\\ B\times\mathbb{R}^n_0\end{matrix}$

$\quad b \longmapsto (b, 1) \longrightarrow 0$

note that because of basepoint restrictions cannot map b to $(b,0)$

So we get a split short exact:

$$\begin{tikzcd}
0 \arrow[r] & {[B\cancel{\times\mathbb{R}}]^{m+n}} \arrow[r, "\phi^*"] & {[B\times\mathbb{R}^n_0]^{m+n}} \arrow[r, "\delta^*"] \arrow[d, "\cong"] & {\begin{bmatrix}B\times\mathbb{R}^n\\ B\times\mathbb{R}^n_0\end{bmatrix}^{m-1+n}} \arrow[r] & 0 \\
& & {[B]^{m+1} \oplus [B]^m \times [\mathbb{R}^n_0]^n} & &
\end{tikzcd}$$

$$s^m(B) \xrightarrow{\cong} s^m(B)\otimes s^n(\mathbb{R}^n_0) \hookrightarrow s^{m+n}(B\times\mathbb{R}^n_0) \longrightarrow s^{m-1+n}\begin{pmatrix}B\times\mathbb{R}^n\\ B\times\mathbb{R}^n_0\end{pmatrix} \to$$

$$\omega \qquad\qquad \omega \overset{\times}{\otimes} e' \qquad\qquad \omega \overset{\times}{\otimes} e' \qquad\qquad ?$$

~~need~~ $\omega\times e$

$\omega\times e$

The map δ^* is

Given $\tau \in s^{m+n}(B\times\mathbb{R}^n_0)$,

Extend τ to $B\times\mathbb{R}^n$, look at $\delta\tau$

Now given $\omega\overset{\times}{\otimes} e'$, we can extend just e' to $\mathbb{R}^n$? Not rigorous

So $\delta^*(\omega\overset{\times}{\otimes} e') = \omega \underset{\times}{\otimes} \delta^* e' \in s^{m-1+n}\begin{pmatrix}B\times\mathbb{R}^n\\ B\times\mathbb{R}^n_0\end{pmatrix}$

what is $\delta^* e'$? e. $0 \longrightarrow s^n(\mathbb{R}^n_0) \xrightarrow{\delta^*} s^{n+1}\begin{pmatrix}\mathbb{R}^n\\ \mathbb{R}^n_0\end{pmatrix} \longrightarrow 0$

$$e \longrightarrow e'$$

So

$$s^m(B) \longrightarrow s^{m+n}\begin{pmatrix}B\times\mathbb{R}^n\\ B\times\mathbb{R}^n_0\end{pmatrix} \quad \text{isomorphism}$$

$$\omega \longmapsto \omega\times e$$

Interesting: Characteristic Classes can be defined over K-Theory also!
Then Thom class of a vector bundlbe is the vector bundle itself!

Obstruction Theory.

Q. $S^2 \hookrightarrow \mathbb{C}P^\infty$ Given $f: S^2 \to S^2$
Can we extend $\hat{f}: \mathbb{C}P^\infty \to \mathbb{C}P^\infty$

$$\begin{tikzcd} S^1 \arrow[r] & S^\infty \arrow[d] \\ & \mathbb{C}P^\infty \end{tikzcd} \qquad \pi_* \mathbb{C}P^\infty = \begin{cases} \mathbb{Z} & * = 2 \\ 0 & \text{else} \end{cases}$$

$\pi_3(\mathbb{C}P^\infty) = 0 \Rightarrow f: S^2 \to S^2$
Extends to $f: \mathbb{C}P^2 \to \mathbb{C}P^\infty$

Analogosly, we can extend the map f to entire $\mathbb{C}P^\infty$.
Also the extension $\hat{f}$ is unique upto homotopy. ☒

Q. η- Hopf map

$$\begin{tikzcd} S^3 \arrow[d, "\eta"'] \arrow[dr] & \\ S^2 \arrow[r, "id"] \arrow[d] & S^2 \\ \mathbb{C}P^\infty \arrow[ur, dashed, "?"'] & \end{tikzcd}$$

Can one extend $S^2 \xrightarrow{id} S^2$ to $\mathbb{C}P^2 \to S^2$?

No.
Because η is not null-homotopic.

$$\begin{tikzcd} S^{2n-1} \arrow[d, "\eta"'] \arrow[dr] & \\ \mathbb{C}P^{n-1} \arrow[r, "id"] \arrow[d] & \mathbb{C}P^{n-1} \\ \mathbb{C}P^n \arrow[ur, dashed, "?"'] & \end{tikzcd}$$

Can one extend
$\mathbb{C}P^{n-1} \xrightarrow{id} \mathbb{C}P^{n-1}$ to
$\mathbb{C}P^n \longrightarrow ? \ \mathbb{C}P^{n-1}$

No, because extension is possible iff $S^{2n-1} \xrightarrow{\eta} \mathbb{C}P^{n-1}$ is ~~trivial~~ null homotopic. But if $S^{2n-1} \xrightarrow{\eta} \mathbb{C}P^{n-1}$ were null homotopic then $\mathbb{C}P^n \simeq \mathbb{C}P^{n-1} \vee S^{2n-1}$ which is false (cohomology).

Q. X, Y, Z CW, $Z < X$ Given

$$\begin{tikzcd} Z \arrow[r] \arrow[d, hook] & Y \\ {} & \end{tikzcd}$$

can we extend it to a map $X \longrightarrow Y$?

Induction:

$$\begin{tikzcd} \coprod S^n_\alpha \arrow[d, "\phi^n_\alpha"] & \\ X^{(n)} \arrow[r, "f_n"] \arrow[d] & Y \\ X^{(n+1)} & \end{tikzcd}$$

$\phi^n_\alpha : S^n_\alpha \longrightarrow X^{(n)}$ is the attaching map.

f_n : $X^{(n)} \longrightarrow Y$ extends iff $f_n \circ \phi^n_\alpha$ are all null homotopic

$f_n \circ \phi^n_\alpha \in \pi_n(Y)$.

Assume $n > 1$,

$$C^{cell}_{n+1}(X, Z) \xrightarrow[(\Delta f_n)]{} \pi_n(Y)$$

cellular chain complex

$$(\Delta f_n)(\sum c_\alpha e^{n+1}_\alpha) = \sum c_\alpha \cdot (f_n \circ \phi^n_\alpha)$$

$$(\Delta f_n) \in \mathrm{Hom}(C^{cell}_{n+1}(X, Z), \pi_n Y)$$
$$\parallel$$
$$C^{n+1}_{cell}(X, Z; \pi_n(Y))$$

Claim: (Δf_n) is a ~~cocyc~~ ~~cocha~~ cocycle.

Proof: $\delta \Delta f_n(e^{n+2}) = \Delta f_n(\partial e^{n+2})$

$$= \Delta f_n (\sum_\alpha (\deg \psi_\alpha) e^{n+1}_\alpha)$$

$$\begin{tikzcd} \psi_\alpha : S^{n+1} \arrow[r, "\phi"] & \bigvee_\alpha S^{n+1} \arrow[d] \\ & S^{n+1}_\alpha \end{tikzcd}$$

$$= \cancel{f_n} \sum_\alpha (\deg \psi_\alpha) \cdot (f_n \circ \phi^n_\alpha)$$

cohomologous $\sim f_n(\sum_\alpha \deg \psi_\alpha \circ \phi^n_\alpha)$

$= 0$? Work it up.

⊗

Require that (X, Z) is simply connected.

So $o_n = [\Delta f_n] \in H^{n+1}(X, Z; \pi_n Y)$

Claim: f_n extends iff $\Delta_n = 0$.

Proof: $\Rightarrow$ trivial

$\Leftarrow$ $\Delta n = 0 \Leftrightarrow \Delta f_n = \delta\omega$ $\quad \Delta f_n(e_\alpha^{n+1}) = \delta\omega(e_\alpha^{n+1})$

$\omega \in C^{n+1}_{cell}(X, Z; \pi_n Y)$ $\quad = \omega(\partial e_\alpha^{n+1})$ $\quad \partial$ - cellular boundary

~~$= \omega(S^n \xrightarrow[\phi_\alpha^n]{} X^{(n)})$~~

Th^m:

Z, X simply connected, $Z < X$;

$$Z \xrightarrow{f} Y$$

$\exists$ an obstruction $\Delta_n \in H^{n+1}(X, Z, \pi_n Y)$, such that

$\Delta_n = 0 \Leftrightarrow$ map extends to $(n+1)$ skeleton.

extensions $\longleftrightarrow H^{n+1}(X, Z; \pi_n(Y))$ [Note: This obstruction makes sense i.e. Δ_{n+1} is defined only after extending map to $X^{(n)}$]

Principal G - Bundles

G - Topological Group

$$\begin{tikzcd} E \arrow[d] \\ X \end{tikzcd}$$

$G \circlearrowright E$ fibrewise

$G \circlearrowright E_x$ freely, transitively

$M^{or} = \{(x, \alpha_x) \mid \alpha_x$ generator of $H^n(M, M - x)\}$

orientation cover

$$\begin{tikzcd} M^{or} \arrow[d] \\ M \end{tikzcd}$$

M - ~~orientable~~ manifold

$\mathbb{Z}/2$ action on M^{or} : $\tau(x, \alpha_x) = (x, -\alpha_x)$

$\mathbb{Z}/2 = \{1, \tau\}$

$\Rightarrow M^{or}$ is a principal $\mathbb{Z}/2$ bundle.

$$\begin{tikzcd} Y \arrow[d] \\ X \end{tikzcd}$$

Y covering space, Galois, ie $\pi_1 Y \triangleleft \pi_1 X$

Deck transformations act on Y ($\pi_1 X / \pi_1 Y$)

$\Rightarrow Y$ principal $\pi_1 X / \pi_1 Y$ bundle.

⑭

- $$\begin{tikzcd} \xi^{(K)} \arrow[d] \\ X \end{tikzcd} \qquad \begin{tikzcd} V(\xi) \arrow[d] \\ X \end{tikzcd} = \{ (x, v_1, \ldots, v_K)_x \mid (v_1 \cdots v_K)_x \text{ basis for } \xi_x \} \subseteq X \times \xi \times \cdots \times \xi$$

fibre bunde, $\quad$ fibre $= \{ (e_1 \cdots e_K) \mid e_1 \cdots e_K$ basis for $\mathbb{R}^k \}$

$= Gl_K(\mathbb{R}^{*})$

$Gl_K(\mathbb{R})$ principal bundle.

- Similarly for a Riemannian vector bundle, we will get a principal ~~G-bun~~ $O(n)$-bundle. For oriented vector bundle $SO(n)$-bundle.

<u>Th^m</u> If $f, f' : X \longrightarrow Y$ are such that $f \simeq f'$, then $f^*P \cong f'^*P$

$$\begin{tikzcd} f^*P \arrow[r] \arrow[d] & P \arrow[d] & f'^*P \arrow[l] \arrow[d] \\ X \arrow[r] & Y & Y \arrow[l] \end{tikzcd}$$

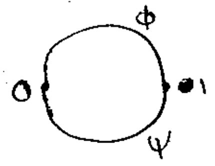
- $P_G(X) =$ Principal G-bundles ~~of~~ over $X / \sim \longrightarrow$ G-bundle isomorphism

$f : X \longrightarrow Y \xrightarrow{P_G} f^* : P_G(Y) \xrightarrow{P_G f} P_G(X)$
only depends on $f \in [X, Y]$

<u>Proposition</u>: $\begin{tikzcd} G \arrow[r] & P \arrow[d] \\ & X \end{tikzcd}$ is trivial iff $\exists s : X \longrightarrow P$ section.

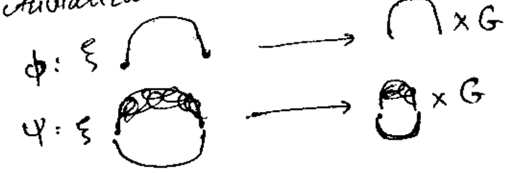
~~Aut_G(G) = G~~ $\pi_0(G)$

- What is $P_G(S^1)$?



$$\begin{tikzcd} \xi \arrow[d] \\ S^1 \end{tikzcd}$$

~~On~~ We can give trivializations so that



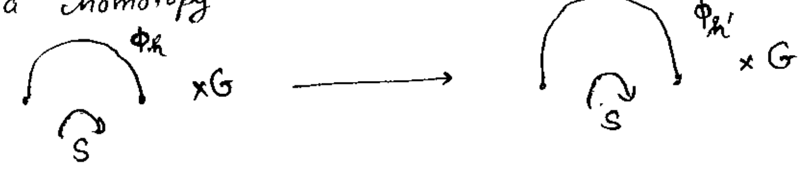
such that $\phi \cdot \psi^{-1}(0) = id$

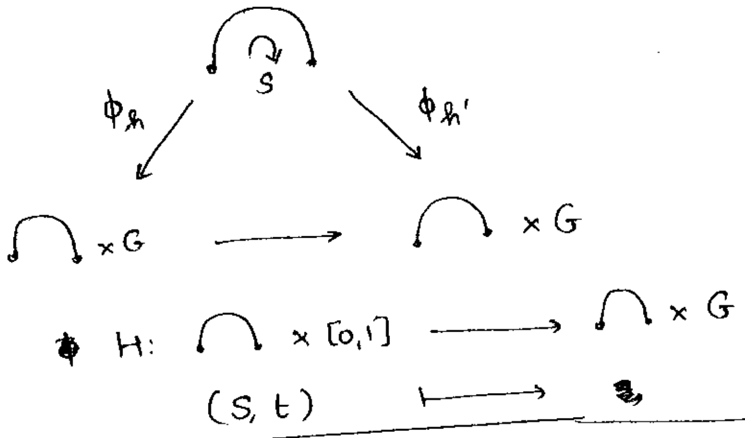
Now $\phi \cdot \psi^{-1}(1) \in Aut_G G = G$

Call ~~ξ_h~~ the ~~P~~ G-bundle corresponding to $\phi \cdot \psi^{-1}(1) = h$ $\quad \xi_h$.

If $\exists$ a path joining h to h'. $\quad \gamma : [0,1] \longrightarrow G$
$(0,1) \longmapsto (h, h')$

we give a homotopy





<u>Proposition</u>:

$$P_G(S^1) = \pi_0(G)$$

~~[illegible]~~

$$P_G(S^1 \wedge X) = [X, G]_*$$

<u>Proof</u>:

$$S^1 \wedge X = C_1X \underset{X}{\sqcup} C_2X$$

$$\begin{tikzcd} \xi \arrow[d, "\pi"] \\ \Sigma X \end{tikzcd}$$

Assume trivializations,

$$\phi : \pi^{-1}(C_1X) \longrightarrow C_1X \times G$$

$$\psi : \pi^{-1}(C_2X) \longrightarrow C_2X \times G$$

Further assume that

x_0 = basepoint of X.

~~$\phi|_{x_0} = \psi|_{x_0}$~~

$$\phi|_{\pi^{-1}(x_0)} = \psi|_{\pi^{-1}(x_0)}$$

So on $X = C_1X \cap C_2X$

$$\psi \cdot \phi^{-1} : X \times G \longrightarrow X \times G$$

$$(x, g) \longmapsto (x, h(g))$$

identity on x_0.

$$\psi \circ \phi^{-1} : X \longrightarrow \mathrm{Aut}_G(G) = G$$

$$x \longmapsto \pi_2(\psi \cdot \phi^{-1}(x, e))$$

$$\begin{tikzcd} X \times G \arrow[d, "\pi_2"] \\ G \end{tikzcd}$$

Call this $h \in$ ~~$[X, G]_*$~~ $\mathrm{Hom}_\bullet(X, G)_*$

Then, by this we have a vector bundle ξ_h, corresponding to every element of ~~$[X, G]_*$~~ $\mathrm{Hom}(X, G)_*$.

Next we need to determine when two of these are equivalent.

<u>Claim</u>: $\xi_h \cong \xi_{h'} \iff \{h\} \cong \{h'$ mod $[X, G]_*$

(15)

$\Leftarrow$ $H : X \times [0,1] \longrightarrow G$

$H_0(x) = h(x)$

$H_1(x) = h'(x)$

Milnor: Construction of Universal bundles.

$$\begin{tikzcd} \phi_h, \psi_h \;\; \xi_h \arrow[dr] & & \xi_{h'} \;\; \phi_{h'}, \psi_{h'} \arrow[dl] \\ & \Sigma X & \end{tikzcd}$$

$\phi_h : \pi^{-1}(C_1X) \xrightarrow{\sim} C_1X \times G$

$\psi_{h'} : \pi^{-1}(C_2X) \xrightarrow{\sim} C_2X \times G$

$\phi_{h'} : \pi^{-1}(C_1X) \xrightarrow{\sim} C_1X \times G$

$\psi_{h'} : \pi^{-1}(C_2X) \xrightarrow{\sim} C_2X \times G$

Aim is to construct a

$$\begin{tikzcd} \xi_h \arrow[rr, "K:"] \arrow[dr, "\pi_h"'] & & \xi_{h'} \arrow[dl, "\pi_{h'}"] \\ & \Sigma X & \end{tikzcd}$$

f is G-isomorphism on each fibre.

Define:

$K : \pi_h^{-1}(C_1X) \longrightarrow \pi_h^{-1}(C_2X)$

$\phi_h^{-1}(x,g) \longmapsto \phi_{h'}^{-1}(x,g)$

$\pi_h^{-1}(C_2X) \longrightarrow \pi_h^{-1}(C_2X)$

$$\begin{tikzcd} \pi_h^{-1}(C_2X) \arrow[dr, "\pi_h"] \arrow[ddr, bend right, "\phi_h"'] & & \pi_{h'}^{-1}(C_2X) \arrow[dl, "\pi_{h'}"'] \arrow[ddl, bend left, "\phi_{h'}"] \\ & C_2X & \\ & C_2X \times G & \end{tikzcd}$$

$C_2X = [0,1] \times X /_{(x,1) \sim (x',1)}$

$K \psi_h^{-1}(t,x,g) = \psi_{h'}^{-1}(t,x, H_t(x)^{-1} g)$

On C_1X $\quad K(\phi_h^{-1}(t,x,g)) = \phi_{h'}^{-1}(t,x,g)$

C_2X $\quad K(\psi_h^{-1}(t,x,g)) = \psi_{h'}^{-1}(t,x, \cancel{H_t(x)^{-1}g)}\; H_{1-t}(x)^{-1} \cdot h_0(x)^{-1} g)$

- K - well defined

on $C_1 \times \cap C_2 X$: $K(\phi_h^{-1}(0,x,g)) = \phi_{h'}^{-1}(0,x,g)$ at $t=0$

$K(\psi_h^{-1}(0,x,g)) = \psi_{h'}^{-1}(0,x,h'(x)\,h(x)^{-1}(g))$

~~$\phi_{h'}^{-1}(0,x,g) \overset{?}{=} \psi_{h'}^{-1}(0,x,h'(x)\cdot g)$~~

~~$\psi_{h'}\cdot\phi_{h'}^{-1}(0,x,g) \overset{?}{=} (0,x,h'(x)g)$~~

~~$\parallel$~~

~~$(0,x,h'(g))$~~

$\psi_h \cdot \phi_h^{-1}(0,x,g) = (0,x,h(x)g)$

$\phi_h^{-1}(0,x,g) = \psi_h^{-1}(0,x,h(x)g)$

$K(\quad\downarrow\quad) \overset{?}{=} K(\quad\downarrow\quad)$

$\phi_{h'}^{-1}(0,x,g) \overset{?}{=} \psi_{h'}^{-1}(0,x,h(x)g)$ $\rightarrow h'(x)\,h(x)^{-1}$

$\psi_{h'}\phi_{h'}^{-1}(0,x,g) =$

$\parallel$

$(0,x,h'(x)g)$ ✓

so K - defined on X.

at $t=1$, K should be ~~constant ident~~ constand function

$K(\psi_{h'}^{-1}(1,x,g)) = \psi_{h'}^{-1}(1,x,H_{1-t}(x)\cdot h(x)^{-1}(g))$ $t=1$

$= \psi_h^{-1}(1,x,g)$

So K-well defined on $\mathcal{E}\Sigma X$.

- K fibrewise isomorphism. Easy.

$$\Rightarrow \begin{tikzcd} \xi_h \arrow[rr, "\cong"] \arrow[dr] & & \xi_{h'} \arrow[dl] \\ & \Sigma X & \end{tikzcd}$$

So far we have obtained a map

$$[X,G]_* \longrightarrow P_G(\Sigma X)$$
$$[h] \longmapsto \xi_h$$

We need to show bijection.

- Surjection is clear
- We will construct an inverse.

$$P_G(\Sigma X) \longrightarrow [X,G]_*$$
$$\phi : \pi^{-1}(C_1X) \longrightarrow C_1X\times G$$
$$\psi : \pi^{-1}(C_2X) \longrightarrow C_2X\times G$$
$$\psi\cdot\phi^{-1} : \cancel{\pi^{-1}(X)}\ X\times G \longrightarrow X\times G$$

Give map: $\xi \longmapsto \psi\cdot\phi^{-1}(-,e)$

If well-defined inverse is obvious.
So enough to show, does not depend on choices of Φ, Ψ.
Suppose Φ_1, Ψ_1 are another trivializations.
Need to show: $\psi_1\cdot\phi_1^{-1}(-,e) \equiv \psi\cdot\phi^{-1}(-,e) \mod [X,G]_*$

$$\phi\cdot\psi^{-1}\cdot\psi_1\cdot\phi_1^{-1}\ (-,e) \equiv \mathrm{id} \qquad \mod [X,G]_*$$

Either need X-path connected or G path connected.
By identifying C_1X, C_2X, we can define the map

$$\phi\cdot\psi^{-1}\cdot\psi_1\cdot\phi_1^{-1} : CX\times G \longrightarrow CX\times G$$

i.e. $\phi\cdot\psi^{-1}\cdot\psi_1\cdot\phi_1^{-1}(-e) : CX \longrightarrow G$
$x \longmapsto \cancel{(x,g)}\ \pi_2(\cdots(x,g))$

i.e. on X the map is null homotopic.
i.e. homotopy equivalent to identity.

So to conclude,

$$[X,G]_* \cong P_G(\Sigma X).$$

~~Principal Bun~~

Universal Bundle :

$\begin{array}{c} E_Z \\ \downarrow \\ Z \end{array}$ Principal G-bundle , Z - CW complex

- $[X, Z] \xrightarrow{\cong} P_G(X)$ Z- <u>Univeral G-bundle</u>

  X - CW complex

- $\begin{array}{c} E_Z \\ \downarrow \\ Z \end{array}$ <u>n-universal</u> if iso. is true for X - C.W complex of dim $\leq n$.

- $\begin{array}{c} E_Z \\ \downarrow \\ Z \end{array}$ ~~Universal upto~~ Universal ~~h~~ unique upto homotopy

  i.e. $\begin{array}{c} E_{Z_1} \\ \downarrow \\ Z_1 \end{array}$ $\begin{array}{c} E_{Z_2} \\ \downarrow \\ Z_2 \end{array}$ 2 - universal

  $\Rightarrow$ $\exists f : Z_1 \longrightarrow Z_2$ s.t.

  $f^* E_{Z_2} = E_{Z_1}$. f- homotopy equivalence

- Analogously

  $\begin{array}{c} E_Z \\ \downarrow \\ Z \end{array}$ n-Universal , unique upto n-equivalence.

<u>Theorem</u> :

$\begin{array}{c} E_Z \\ \downarrow \\ Z \end{array}$ n-universal $\Leftrightarrow$ E_Z $(n-1)$- connected

($\Rightarrow$ $\begin{array}{c} E_Z \\ \downarrow \\ Z \end{array}$ universal $\Leftrightarrow$ E_Z contractible).

<u>Proof:</u>

- 0- ~~connected~~ universal

  - $[S, Z] = P_G(S)$ S- 0-dim

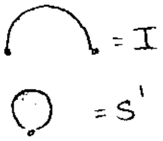
    $= \{*\}$ $\Rightarrow$ discrete set of pts

    $\Rightarrow$ Every bundle on S trivial

  $\Rightarrow$ Z - path connected

  π_1 - connected

- 1 universal



$[I, Z] = P_G[I] = *$

~~$[S^1, Z] = P_G(S^1) = \pi_0(Z)$~~

~~$\| \quad \pi_1 Z$~~

$$\begin{tikzcd} G \arrow[r] & E_z \arrow[d] \\ & Z \end{tikzcd}$$

$[S^1, z] = P_G(S^1) = \pi_0(G)$

$\|$

$\pi_1(Z)$

$\to \pi_{\underset{1}{*}} G \to \pi_{\underset{1}{*}} E_z \to \pi_{\underset{1}{*}} Z \to \pi_0(G)$

- n-1 dim CW - X

n- Universal

$S^{n-1} \xrightarrow{f} X \qquad Y = X \underset{f}{\cup} e^n$

$$\begin{tikzcd} P \arrow[d] \\ S^{n-1} \end{tikzcd}$$

G-bundle

1) Can this be extended to D^n

Iff P is a trivial bundle

i.e. P has a section.

Lemma:

$$\begin{tikzcd} P \arrow[d] & & E_z \arrow[d] \\ S^{n-1} \arrow[u, bend left, "s"] \arrow[urr, dashed] \arrow[rr, "f"] & & Z \end{tikzcd}$$

s section exists $\Leftrightarrow$ f lifts to E_z.

$P = f^* E_z$.

$$\begin{tikzcd} P \arrow[d] & E_z \arrow[d] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

$P = f^* E_z$

Q. When can P be extended to Y?

Iff $\phi^*(P)$ extends to e^n.

iff

$$\begin{tikzcd} & & E_z \arrow[d] \\ S^{n-1}_* \arrow[urr, dashed] \arrow[r, "\phi"] & X \arrow[r, "f"] & Z \end{tikzcd}$$

By Lemma

$f \circ \phi$ lifts

Q. How many extensions exists?

$\pi_{n-1}(G)$.

$Y = X \underset{\phi}{\cup} e^n = \text{cone}(\phi)$

$(M_\phi) \qquad (C_\phi)$

$$\begin{tikzcd} S^{n-1} \arrow[r, "\phi"'] & X \arrow[r] & Y \arrow[r] & S^n \arrow[r] & \Sigma X \arrow[r] & {} \end{tikzcd}$$

$$\begin{tikzcd} {[S^{n-1}, Z]} & {[X, Z]} \arrow[l, "\phi^*"'] & {[Y, Z]} \arrow[l] & {[S^n, Z]} \arrow[l] \arrow[d, phantom, "\cong"] & {[\Sigma X, Z]} \arrow[l, "(\Sigma\phi)^*"'] \\ \phi \circ f & f \arrow[l, mapsto] & & \pi_n(Z) & \end{tikzcd}$$

Q. when can $f \in [X, Z]$ be extended to $f \in [Y, Z]$?

- $f \in \text{Im}([Y,Z] \longrightarrow [X,Z])$

  $f \in \ker(\phi^*)$

Q How many extensions?

i.e. How many pre-images?

$g \in [Y, Z)$. $\quad g \longrightarrow f$

$g_1, g_2 \longrightarrow f \quad \Rightarrow g_1 - g_2 = 0$

$\Rightarrow \exists h.\ h \in [S^n, Z],\ h \longrightarrow g_1 - g_2$

So $\exists$ $\pi_n(Z)$ many extensions

$\Rightarrow \quad \pi_n(Z) \cong \pi_{n-1}(G)$

Milnor - Stasheff

$$\frac{X \times CY \cup CX \times Y}{X \times Y}$$

$X * Y = \Sigma(X \wedge Y)$

$= S^1 \wedge X \wedge Y$?

Q.5 - A) $\quad Gl_n(\mathbb{R}) \longrightarrow V_n(\mathbb{R}^{n+k})$

$$\begin{tikzcd} V_n(\mathbb{R}^{n+k}) \arrow[d, "q"] \\ G_n(\mathbb{R}^{n+k}) \end{tikzcd}$$

$f \circ q : V_n(\mathbb{R}^{n+k}) \longrightarrow \mathbb{R} \quad C^\infty$, if $\quad f : G_n(\mathbb{R}^{n+k}) \longrightarrow \mathbb{R} \quad C^\infty$

if $f \circ q \quad C^\infty$. $\quad (f \circ q)^{-1}(U)$ open in $V_n(\mathbb{R}^{n+k})$

$q^{-1} \cdot f^{-1} \cdot (U)$ open

$\Rightarrow f^{-1}(U)$ open $\quad$ as q is qotient map, hence open

Now remains to check smoothness, i.e.

Claim: $f \circ q \ C^\infty \Rightarrow f \ C^\infty$

Locally $G_n(\mathbb{R}^{n+k}) \supseteq U$, $q^{-1}(U) = U \times Gl_n(\mathbb{R})$.

Locally we have a map

$i: U \longrightarrow U \times Gl_n(\mathbb{R})$ which is just a section,

Then

$$f = f \circ q \circ i = C^\infty$$

⊠

Q.5 B)

Problem with direct approach:

The canonical bundle is not defined for arbitrary G-bundles. $G_n(\mathbb{R}^{n+k})$ is a very specific space

$G_n(\mathbb{R}^{n+k}) = \{ n \text{ planes in } \mathbb{R}^{n+k} \}$

- C^∞-structure on $G_n(\mathbb{R}^{n+k})$

$x \in G_n(\mathbb{R}^{n+k})$

$U_x = \{ y \in G_n(\mathbb{R}^{n+k}) \mid \nexists\, v \in y.\ v \perp x \}$

$x^\perp \in G_k(\mathbb{R}^{n+k}) = \{ w \mid w \perp x \}$

$\varphi_x : U_x \longrightarrow \mathbb{R}^{nk} \longleftarrow$ Cordinate Chart

Let $e_1 \cdots e_n$ be a basis for x i.e. $x = \langle e_1, \cdots e_n \rangle$

$f_1 \cdots f_k$ be a basis for $x^\perp$ $x^\perp = \langle f_1 \cdots f_k \rangle$

$y \in U_x$, $y = \langle e_1', e_2', \cdots, e_n' \rangle$

$= \langle e_1 + f_1', e_2 + f_2', \cdots, e_n + f_n' \rangle$

Then if

$$\begin{bmatrix} f_1' \\ \vdots \\ f_n' \end{bmatrix} = \begin{bmatrix} & & \\ & y_{ij} & \\ & & \end{bmatrix} \begin{bmatrix} f_1 \\ \\ f_k \end{bmatrix}$$

$$\varphi_x(y) = [y_{ij}]$$

Then we have

$$\varphi_x^* : T(U_x) \overset{\sim}{\longleftarrow} \cancel{\varphi_x}^*(T\mathbb{R}^{nk}) \cong \mathbb{R}^{nk} \times \mathbb{R}^{nk} \cong U_x \times \mathbb{R}^{nk}$$

- $$\begin{tikzcd} \gamma \arrow[dr] & & \gamma^{\perp} \arrow[dl] \\ & G_n(\mathbb{R}^{n+k}) & \end{tikzcd}$$

  $\gamma = \{(x, \omega) \mid \omega \in x\}$

  $\gamma^{\perp} = \{(y, \omega) \mid \omega \perp y\}$

  Locally we have:

  $x \in G_n(\mathbb{R}^{n+k})$

  $x^{\perp}, U_x$ as defined earlier

  $\gamma(U_x) \xrightarrow{\sim} U_x \times \mathbb{R}^n$

  $(y, \omega) \longmapsto$ (y, co-ordinates of ω in the basis $e_1 \cdots e_n$).

  $\gamma^{\perp}(U_x) \xrightarrow{\sim} U_x \times \mathbb{R}^k$

  $(y, \omega) \longmapsto$ (y, co-ordinates of ω in the basis $f_1 \cdots f_k$)

  $\psi : \mathrm{Hom}(\gamma, \gamma^{\perp}) \xrightarrow{\sim} U_x \times \mathbb{R}^{nk}$

  $(y, \alpha) \longmapsto$ (y, α written as a matrix in the basis $e_1 \cdots e_n, f_1 \cdots f_k$)

  Then we have the following isomorphism:

  $$\mathrm{Hom}(\gamma, \gamma^{\perp}) \xrightarrow[\psi]{} U_x \times \mathbb{R}^{nk} \xrightarrow[(\varphi^*)^{-1}]{} T U_x \times \ _x \, .$$

  (We have made a lot of choices of bases. Compatibility is something that needs to be checked).

- $M^n \hookrightarrow \mathbb{R}^{n+k}$

  $\bar{g} : M^n \longrightarrow G_n(\mathbb{R}^{n+k})$ *Generalised Gauss map.*

  $$\begin{tikzcd} \bar{g}^* : TM \arrow[r] & TG_n(\mathbb{R}^{n+k}) \arrow[d, phantom, "\cong" rotate=90] \\ & \mathrm{Hom}(\gamma, \gamma^{\perp}) \end{tikzcd}$$

  At a point $p \in M$,

  $\gamma(\bar{g}(p)) = T_pM, \quad \gamma^{\perp}(\bar{g}(p)) = \nu_p M$

$$\begin{tikzcd} \tilde{g}^* : TM \arrow[rr] \arrow[dr] & & \mathrm{Hom}(TM, \nu) \arrow[dl] \\ & M & \end{tikzcd}$$

⇒

⇒ $\tilde{g}_p^* \in \mathrm{Hom}(T_pM, \mathrm{Hom}(T_pM, \nu_p)) \cong \mathrm{Hom}(T_pM, T_pM^* \otimes \nu_p)$

$\cong \mathrm{Hom}(T_pM \otimes T_pM, \nu_p)$ ~~∵ finite dimensional~~

$\left(\mathrm{Hom}(A, B^* \otimes C) \underset{?}{\cong} \mathrm{Hom}(A \otimes B, C) \right)$

☒

Q.5 c)

$I_n \subseteq M_n(\mathbb{R}^{n+k}) = \{A \in M_n(\mathbb{R}^{n+k}) \mid A^2 = A,\ \mathrm{trace}(A) = n\}$

Then, we know that Jordan - Canonical form of A is

$A \sim B + C$

$B = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$ only 1's, 0's in diagonal, all other entries 0.

$C = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 0 & 0 & & \\ & & 1 & 0 & \\ & & & \ddots & 0 \end{bmatrix}$ only 1's, 0's in sub-diagonal, all other entries 0.

~~$A^2 = B^2 + BC + C^2 + CB$~~

~~⇒ $C = BC + C^2 + CB$~~

∵ $A(A-1) = 0$, size of each Jordan block should be 1.

⇒ $C = 0$

So $\exists P \in Gl_{n+k}(\mathbb{R}^n)$ s.t.

$PAP^{-1} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$ = Projection in the first n-plane.

- So $\varphi : G_n(\mathbb{R}^{n+k}) \longrightarrow I_n$

$x \longmapsto$ Projection onto the plane x.

Injectivity is clear, surjectivity follows from the above proposition.

There is one gap in proof:

By Jordan decomposition, we get $P \in Gl_n(\mathbb{C})$ st.

$$PAP^{-1} = B$$

we need a $P \in Gl_n(\mathbb{R})$.

One needs a stronger version of Jordan canonical decomposition:

- If K is a field, if ~~minimal~~ characteristic polynomial of A splits in K, then $\exists P \in Gl_n(K)$ s.t. $PAP^{-1} =$ Jordan.

Proof of Jordan canonical actually proves this th^m.

$$\begin{tikzcd}
\varphi : V_n(\mathbb{R}^{n+k}) \arrow[r] & \Lambda^n(\mathbb{R}^{n+k}) \arrow[r] & \mathbb{P}(\Lambda^n(\mathbb{R}^{n+k})) \\
(v_1, \cdots v_n) \arrow[r, mapsto] & (v_1 \wedge v_2 \wedge \cdots \wedge v_n) \arrow[r] & [(v_1 \wedge \cdots \wedge v_n)]
\end{tikzcd}$$

If $sp\langle w_1 \cdots w_n \rangle = sp\langle v_1 \cdots v_n \rangle$

$$\begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = A \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \qquad A \in Gl(\langle v_1 \cdots v_n \rangle)$$

Then, $\varphi(w_1 \cdots w_n) = [w_1 \wedge w_2 \wedge \cdots \wedge w_n]$

$$= [\det A \, (v_1 \wedge \cdots \wedge v_n)]$$

$$= [v_1 \wedge \cdots \wedge v_n] = \varphi(v_1 \cdots v_n)$$

By property of quotient topology

$$\tilde{\varphi} : G_n(\mathbb{R}^{n+k}) \longrightarrow \mathbb{P}(\Lambda^n(\mathbb{R}^{n+k}))$$

Injectivity:

~~$\tilde{\varphi}([v_1 \cdots v_n]) =$~~

Claim: $v_1 \wedge v_2 \wedge \cdots \wedge v_n = (w_1 \wedge w_2 \wedge \cdots \wedge w_n) \neq 0$ ~~$\in \mathbb{R}$~~

$\Rightarrow sp\langle v_1, \cdots, v_n \rangle = sp\langle w_1, \cdots, w_n \rangle$.

Proof: If $k = 0$,

$$v_1 \wedge \cdots \wedge v_n = w_1 \wedge \cdots \wedge w_n \neq 0$$

$$\Rightarrow \mathbb{R}^{n+k} = sp\langle v_1 \cdots v_n \rangle = sp\langle w_1 \cdots w_n \rangle.$$

$k > 0$

Assume contrary. Let $v' \perp sp\langle v_1 \cdots v_n \rangle$. $w_1 = v' + (\lambda_1 v_1 + \cdots \lambda_n v_n)$

We define following multilinear map on ~~A~~$(\mathbb{R}^{n+k})^n$

$$\alpha : \mathbb{R}^{n+k} \times \cdots \times \mathbb{R}^{n+k} \longrightarrow \mathbb{R} \cong \Lambda^n(sp\langle v', w_2, \cdots, w_n \rangle)$$

$$(u_1, \cdots, u_n) \longmapsto \begin{cases} 0 & \text{if } u_1, \cdots, u_n \notin sp\langle v', w_2, \cdots, w_n \rangle \\ \text{signed volume of } u_1, \cdots, u_n & \\ \text{in the space } v', w_2, \cdots, w_n & \end{cases}$$

$\alpha(v_1 \cdots v_n) = 0$ ∵ $v' \perp v_i \ \forall i$

$$\alpha(w_1 \cdots w_n) = \alpha(v', w_2 \cdots w_n) + \underset{\overset{\|}{0}}{\alpha(\lambda_i v_i, w_2, \ldots, w_n)}$$

$$= \alpha(v', w_2, \ldots, w_n)$$

$$\neq 0 .$$

Being signed volume α is anti-symmetric

$$\Rightarrow \begin{tikzcd} (\mathbb{R}^{n+k})^k \arrow[r, "\alpha"] \arrow[d] & \mathbb{R} \\ \Lambda^k(\mathbb{R}^{n+k}) \arrow[ur, bend right, "\exists \hat{\alpha}"'] & \end{tikzcd}$$

$$\Rightarrow \hat{\alpha}(v_1 \wedge \cdots \wedge v_n) = 0$$
$$\hat{\alpha}(w_1 \wedge \cdots \wedge w_n) \neq 0$$

$\Rightarrow v_1 \wedge \cdots \wedge v_n \neq w_1 \wedge \cdots \wedge w_n$. Contradiction.

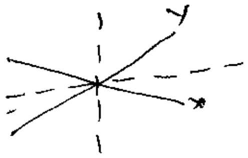
⊠

Q.5 D) $X, Y \in \mathbb{R}^{n+k}$ n-planes. $X \neq Y$

$\phi \in O(\mathbb{R}^{n+k})$

$\phi(X) = Y$

$\phi(Y) = X$



Claim: ϕ exists, ~~unique~~

Proof:

We should get two n-planes P_1, P_2 which are equidistant from X, Y.

Assume $X = W \oplus X_1$ $Y = W \oplus Y_1$

where $W = X \cap Y$

So $X_1 \cap Y_1 = \{0\}$

Enough to find the planes for X_1, Y_1

~~Suppose~~ $X_1 =$ sp $\langle x_1 \cdots x_i \rangle$ orthonormal basis

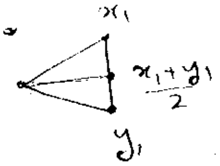
$Y_1 =$ sp $\langle y_1 \cdots y_i \rangle$

s.t. $|y_1| = |x_1| = \cdots = |y_i| = |x_i| = 1$

Then $P_1 =$ sp $\langle \frac{x_1 + y_1}{2}, \ldots, \frac{x_i + y_i}{2} \rangle$

$P_2 =$ sp $\langle \frac{x_1 - y_1}{2}, \ldots, \frac{x_i - y_i}{2} \rangle$

Now we need to show reflecting ~~x_2~~ in P_1 (or P_2) takes X_1 to Y_1 .



$(x_1-y_1)\cdot\left(\frac{x_1+y_1}{2}\right) =$ ~~$x_1^2 - y_1^2$~~ $\frac{|x_1|^2 - |y_1|^2}{2} = 0$

$|x_1 - \frac{x_1+y_1}{2}| =$ ~~$|x_1 - x_1$~~ $|y_1 - \frac{x_1+y_1}{2}|$.

Result follows. $\square$

Next we need ~~do~~ defined an angle $\alpha(X,Y)$ ~~as~~ which is independant of ~~X, Y~~ ϕ.

Again decompose X, Y as $X = W \oplus X_1$, $Y = W \oplus Y_1$

Define :

$$\alpha(X,Y) = \inf_{\substack{x \in X_1 \\ y \in Y_1}} \cos^{-1}\left(\frac{x\cdot y}{\|x\|\|y\|}\right)$$

$$= \min_{\substack{x\in X_1, y\in Y_1 \\ \|x\|=1=\|y\|}} \cos^{-1}(x.y)$$

Infact $\cos^{-1}(x.y)$ is constant for $x \in X_1$, $y \in Y_1$

α is a metric on $G_n(\mathbb{R}^{n+k})$.

Proof:

- $\alpha : G_n(\mathbb{R}^{n+k}) \times G_n(\mathbb{R}^{n+k}) \longrightarrow [0, \pi/2]$ C^∞

Enough to show

$\alpha\circ q : V_n(\mathbb{R}^{n+k}) \times V_n(\mathbb{R}^{n+k}) \longrightarrow [0,\pi/2]$ C^∞

~~Fix one~~ Let $e_1 \cdots e_n e_{n+1} \cdots e_{n+k}$ be standard basis for $\mathbb{R}^{n+k}$

- Enough do show:

$\alpha\circ q((e_1\cdots e_n), -)$ C^∞

$$\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} v_{ij} \end{bmatrix} \begin{bmatrix} e_1 \\ e_{n+k} \end{bmatrix}$$ i.e. $v_i = \sum_j v_{ij} e_j$

$\alpha\circ q\,(v) = \min \cos^{-1}(\alpha_1\cdot$

Technical difficult.

Need different approach

Easier way to define, angle which does not require X_1, Y_1

$$\alpha(X,Y) = \max_{\substack{x \in X \\ y \in Y \\ \|x\| = 1 = \|y\|}} \cos^{-1}(x \cdot y)$$

where max is achieved precisely when $x \in X_1$, $y \in Y_1$

- $\alpha(X,X) = 0$
- $\alpha(X,Y) = \alpha(Y,X)$
- $\alpha(X,Y)$ — C^{∞}

  Proof might be given using Langrange's Multipliers?
- $\alpha(X,Y) + \alpha(Y,Z) \geq \alpha(X,Z)$

Q.5) E)

1) $\xi \oplus \eta = n+k$

$= B \times \mathbb{R}^{n+k}$

$$\begin{tikzcd} \xi \arrow[dr] & & \eta \arrow[dl] \\ & B & \end{tikzcd}$$

$\xi \longrightarrow B \times \mathbb{R}^{n+k} \longrightarrow \mathbb{R}^{n+k}$

$b \in B$, $\xi_b \longmapsto$ n-plane in $\mathbb{R}^{n+k}$

So we get a map

$$\begin{tikzcd} \xi \arrow[r] \arrow[d] & \gamma \arrow[d] \\ B \arrow[r] & G_n(\mathbb{R}^{n+k}) \end{tikzcd}$$

Conversely,

$$\begin{tikzcd} \gamma \arrow[dr] & & \gamma^{\perp} \arrow[dl] \\ & G_n(\mathbb{R}^{n+k}) & \end{tikzcd}$$

$\gamma \oplus \gamma^{\perp} = \mathbb{R}^{n+k}$

Result follows.

2) B normal

- $\exists\, U_i \subseteq B$, finite s.t. $\varphi_i : \xi|_{U_i} \xrightarrow{\sim} B \times \mathbb{R}^n \xrightarrow{\pi} \mathbb{R}^n \qquad 1 \leq i \leq N$

  $f_i : B \longrightarrow \mathbb{R}$

  $f_i(U_i) = (0,1]$

  $f_i(U_i^c) = 0$

Define $\xi \longrightarrow \mathbb{R}^{Nn}$

$x \longmapsto (f_1 \cdot (\pi \cdot \varphi_1)x, f_2 \cdot (\pi \cdot \varphi_2)x, \dots, f_N(\pi \cdot \varphi_N))x)$

where $\varphi_i(x) = 0$ if $x \notin U_i$

This gives the required map.

$$\begin{tikzcd}
\xi \arrow[r] \arrow[d] & \gamma \arrow[d] \\
B \arrow[r] & G_n(\mathbb{R}^{n+k})
\end{tikzcd}$$

$G_n(\mathbb{R}^{n+k})$ is a manifold, compact

$\Rightarrow \exists U_i \subseteq G_n$, finite st. U_i contractible,

$\gamma U_i \cong G_n(\mathbb{R}^{n+k}) \times \mathbb{R}^n$

Result follows from the pullback property of manifold

3) B - paracompact

Use partitions of Unity

4) $\gamma \downarrow \mathbb{R}P^\infty$ $\quad \omega(\gamma) = 1 + x \qquad H^1(\mathbb{R}P^\infty, \mathbb{Z}/2) = \mathbb{Z}/2\, x$

if $\eta \oplus \gamma =$ trivial

$\Rightarrow \omega(\eta) = \frac{1}{1+x} = 1 + x + x^2 + \cdots$

Not possible.

Grassmannian - Cell structures

$G_n(\mathbb{R}^{n+k})$

$\mathbb{R}^0 \subset \mathbb{R}^1 \subset \dots \subset \mathbb{R}^{n+k}$

X - n plane

Schubert symbol of $X = (\sigma_1, \dots, \sigma_n)$

$1 \leq \sigma_1 < \sigma_2 < \dots < \sigma_n \leq n+k$

$\dim(\mathbb{R}^{\sigma_i} \cap X) = i$

$\dim(\mathbb{R}^{\sigma_i - 1} \cap X) = i-1$

$e(\sigma) = \{ X \mid X \text{ nplane, schubert}(X) = \sigma \}$

$e(\sigma) =$ open cell of dim $(\sigma_1 - 1) + (\sigma_2 - 2) + \dots + (\sigma_n - n)$.

$e(\sigma)$ – cells of $G_n(\mathbb{R}^{n+k})$

No. of r-cells in $G_n(\mathbb{R}^{n+k})$ = no. of partitions of r into at most n-integers each $\leq k$.

Q.6-A)

- X - CW, compact

  X = Union of open cells of X, disjoint

  compact $\Rightarrow$ finite

- finite $\Rightarrow$ ~~disjoint union~~ quotient of compact set

  $\Rightarrow$ compact

Q.6-B)

$$i: G_n(\mathbb{R}^{n+k}) \hookrightarrow G_n(\mathbb{R}^{\infty})$$

$$i: G_n(\mathbb{R}^{n+k}) \hookrightarrow G_n(\mathbb{R}^{n+k+1})$$

a p-cell of $G_n(\mathbb{R}^{n+k+1}) = e(\sigma)$ s.t. $\dim e(\sigma) = p$ $\quad p \leq k$

$= \{X\text{-}n\text{planee} \in \mathbb{R}^{n+k+1} \mid \text{schubert}(X) = \sigma\}$

$\mathbb{R}^0 \subset \mathbb{R}^1 \subset \dots \subset \mathbb{R}^{n+k} \subset \mathbb{R}^{n+k+1} \dots$ $\quad (\sigma_1 - 1) + \dots + (\sigma_n - n) = p$

$\dim(X \cap \mathbb{R}^{\sigma_i}) = i$

$\dim(X \cap \mathbb{R}^{\sigma_i - 1}) = i-1$

$p \leq k \Rightarrow \sigma_n - n \leq k \Rightarrow \sigma_n \leq n+k$

Thus $e(\sigma) \subseteq G_n(\mathbb{R}^{n+k}) \subseteq G_n(\mathbb{R}^{n+k+1})$

So k-skeleton of $G_n(\mathbb{R}^{n+k})$ = k-skeleton of $G_n(\mathbb{R}^{n+k+1})$

$\Rightarrow$ i^* isomorphism for $p < k$.

Q.6-C)

$$X \xmapsto{f} \mathbb{R}^1 \oplus X$$

- $f: G_n(\mathbb{R}^{n+k}) \longrightarrow G_{n+1}(\mathbb{R}^{n+k+1})$

  Injectivity is clear. Need to show C^{∞}.

  $\tilde{f}: V_n(\mathbb{R}^{n+k}) \longrightarrow V_{n+1}(\mathbb{R}^{n+k+1})$ $\quad \mathbb{R}^{n+k+1} = \mathbb{R}e_1 \oplus \mathbb{R}^{n+k}$

  $(x_1, \dots x_n) \longmapsto (e_1, x_1, \dots, x_n)$.

- $f^*(\gamma_{n+1}(\mathbb{R}^{n+k}))_{[x]} = \{(e_1, x_1, \dots, x_n) \mid (x_1, \dots x_n) \in [x]\}$

$\Rightarrow f^*(\gamma_{n+1}(\mathbb{R}^{n+k+1})) = e_1 \oplus \gamma_n(\mathbb{R}^{n+k})$

$e(\sigma) \in G_n(\mathbb{R}^{n+k})$

$f(e(\sigma)) = e(1, \sigma_i + 1, \dots, \sigma_n + 1)$

Q.6 - D)

$\omega^{r_1} \dots \omega_n^{r_n}[M]$

$r_1 + 2r_2 + \dots + nr_n = n$ = partition of n using r_i i's.

Q.6 - E)

$$f : G_n(\mathbb{R}^{n+k}) \longrightarrow G_k(\mathbb{R}^{n+k})$$
$$X \longmapsto X^{\perp}$$

Smoothness - difficult to prove, might not be true !

Homeomorphism:

if $X \in e(\sigma)$ $\quad \sigma = (\sigma_1 \dots \sigma_n)$

$1 \leq \sigma_1 < \dots < \sigma_n \leq n+k$

$f(X) \in e(\tau)$ $\quad \tau = (\tau_1 \dots \tau_k)$

$1 \leq \tau_1 < \dots < \tau_k \leq n+k$

$\{\sigma_1 \dots \sigma_n\ \tau_1 \dots \tau_k\} = \{1, 2, \dots, n+k\}$

$$\begin{aligned} \dim(\tau) &= (\tau_1 - 1) + \dots + (\tau_k - k) \\ &= \sum \tau_i - \cancel{\Sigma}\, \frac{k(k+1)}{2} \\ &= \cancel{\Sigma}\, \frac{(n+k)(n+k+1)}{2} - \frac{k(k+1)}{2} - \sum_i \sigma_i \\ &= \frac{n^2 + nk + nk + n}{2} - \sum \sigma_i \\ &= nk - (\dim \sigma) \end{aligned}$$

Clearly NOT ~~a~~ CW complex isomorphism

we will give $G_n(\mathbb{R}^{n+k})$ a different cell-structure

Let

$$A \in Gl_{n+k}(\mathbb{R}) = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \qquad A_{ij} = \delta_{i(n+k-j)}$$

A permutes $e_1\ e_{n+k-1}$, $e_2\ e_{n+k-2}$, ...

Look at the induced map on $G_n(\mathbb{R}^{n+k})$

$$\vec{A}: G_n(\mathbb{R}^{n+k}) \longrightarrow G_n(\mathbb{R}^{n+k})$$

$$X = sp\langle v_1, \dots v_n\rangle \longmapsto \vec{A}X = sp\langle Av_1, \dots, Av_n\rangle.$$

This is clearly a homeomorphism.

Look at cell-structure induced by $\vec{A}$

$$Ae(\sigma) = \{X \mid \vec{A}X \subseteq e(\sigma)\}$$

~~dim (Ae(~~

Now we have

$$f: G_n(\mathbb{R}^{n+k}) \longrightarrow G_k(\mathbb{R}^{n+k})$$

$$X \longmapsto X^{\perp}$$

as before

$$\dim(f\,e(\sigma)) + \dim(e(\sigma)) = nk$$

But $\dim(Ae(\sigma)) = nk - \dim(e(\sigma))$

Reason: $X \in Ae(\sigma)$

$\Rightarrow AX \in e(\sigma)$

$\Rightarrow \dim(AX \cap \mathbb{R}^{n+k-\sigma_i}) = n-i$

$\dim(AX \cap \mathbb{R}^{n+k-\sigma_{i+1}}) = n-i+1$

$\Rightarrow Ae(\sigma) = e(\tau)$

$\tau_{n-i+1} = n+k-\sigma_i+1$

$\Rightarrow \dim(Ae(\sigma)) =$

$= \sum_i \cancel{n}+k-\sigma_i+\cancel{1} - \cancel{n}+i-\cancel{1}$

$= \sum_i k - (\sigma_i - i)$

$= nk - \dim(e(\sigma))$

(Another way of saying the same thing would be to look at the map

$X \longmapsto AX^{\perp}$).

$H^*(G_n, \mathbb{Z}/2)$ – Real Grassmannian

for $\begin{matrix}\xi \\ \downarrow \\ B\end{matrix}$ vectorbundle over paracompact B

$\exists\, f : B \longrightarrow G_n$ st. $\xi = f^*\gamma$

$\Rightarrow \quad \omega(\xi) = f^*\omega(\gamma)$

But for $B = \mathbb{R}P^n$, ξ = orthogonal bundle of the canonical bundle

$$\omega(\xi) = \frac{1}{1+x} = 1 + x + \cdots + x^{n-1}, \quad x \in H^1(\mathbb{R}P^n)$$

Not Useful.

$$B = \underbrace{\mathbb{R}P_1^\infty \times \cdots \times \mathbb{R}P_n^\infty}_{n\text{-times}} \qquad \xi = \gamma_1^1 \times \gamma_2^1 \times \cdots \times \gamma_n^1$$

$\gamma_i^1 \rightarrow$ Canonical line bundle over $\mathbb{R}P_i^\infty$

$\mathbb{R}P^\infty$ – CW complex, hence paracompact

$\omega(\xi) = (1+a_1)(1+a_2)\cdots(1+a_n)$ Always co-eff. group $\mathbb{Z}/2$.

$H^1(\mathbb{R}P_i^\infty) = \mathbb{Z}/2\, a_i$

$w_i(\xi) = i^{th}$ symmetric poly in n-variables.

Claim: $\nexists\, P(z_1, \ldots, z_n) \in \mathbb{Z}/2[z_1, \ldots, z_n]$ s.t.

$$P(\omega_1(\xi), \ldots, \omega_n(\xi)) = 0$$

Proof:

$$\begin{tikzcd} \mathbb{Z}/2(a_1, \ldots, a_n) \arrow[d] \\ \mathbb{Z}/2(\omega_1(\xi), \ldots, \omega_n(\xi)) \end{tikzcd}$$

is finite extension of degree at most $n!$

But since the Galois group is S_n, deg $= n!$

Transcendence degree of $\mathbb{Z}/2(a_1 \cdots a_n) = n$

$\Rightarrow$ " " " $\mathbb{Z}/2(\omega_1 \cdots \omega_n(\xi)) = n$

So result follows. ~~Infact~~ ⊠

Transcendence degree of
So we get that $\exists\ \mathbb{Z}/2[x_1 \cdots x_n] \subseteq H^*(G_n, \mathbb{Z}/2)$

$$x_i \in H^i(G_n, \mathbb{Z}/2)$$

But no. of cells in ~~#~~G_n of dim m = no. of partitions of m in atmost n- ~~nu~~parts.

~~$= \#\{r_1, \dots, r$~~

= no. of partitions of ~~m~~ m ~~of~~ with each partition size atmost n.

$= \#\{r_1, \dots, r_n \mid 1r_1 + 2r_2 + \cdots + nr_n = m\}$

Reason : dim $e(\sigma) = m$

$\Rightarrow (\sigma_1 - 1) + (\sigma_2 - 2) + \cdots + (\sigma_n - n) = m$

Partition of ~~m~~ m in n-parts.

By comparing dimension,

$$H^*(G_n, \mathbb{Z}/2) = \mathbb{Z}/2\,[x_1, \dots, x_n] \qquad |x_i| = i$$

Because each dim- has rank equal to no. of cells of that dim, we must have that all the maps in the cellular complex of G_n are 0 mod 2, so

$$\text{rank } ~~\dim~~ \left(H_i(G_n, \mathbb{Z}/2)\right) = ~~\dim~~ \text{ rank } \left(H^i(G_n, \mathbb{Z}/2)\right)$$

Q.7 A)

$w_r(\gamma^n)$ = r^{th} symmetric polynomial in $a_1 \cdots a_n$

Cup product of a_i's will give this cocycle.
First we find co-cycle representing a_i's.

?? What do we have to find ??

~~Q.7 B)~~

~~$H^*(G_n(\mathbb{R}^{n+k}))$~~

~~$i^* : H^p(G_n) \longrightarrow H^p(G_n(\mathbb{R}^{n+k}))$~~
~~isomorphism for $p < k$.~~

we have

$$\begin{tikzcd} (\mathbb{R}-\{0\})^n \arrow[r] & (\mathbb{R}^\infty - 0)^n \arrow[d] \\ & \mathbb{R}P^\infty \end{tikzcd}$$

But $(\mathbb{R}^{\infty}-0)^n \hookrightarrow \mathbb{R}^{\infty}$ by the map

$$((x_{11},\dots),(x_{21},\dots),\dots,(x_{n1},\dots)) \longmapsto (x_{11},x_{21},\dots,x_{n1},x_{12},x_{22},\dots,x_{n2},\dots)$$

Then $\mathbb{RP}^{\infty}\times\cdots\times\mathbb{RP}^{\infty} \cong$ ~~lines~~ n-tuple of lines in $\mathbb{R}^{\infty}$

~~with~~ a subspace of $\mathbb{R}^{\infty}$

note that these lines will be ~~indep~~ linearly independant;

So the map,

$$\mathbb{RP}^{\infty}\times\cdots\times\mathbb{RP}^{\infty} \longrightarrow G_n \quad \text{is given by}$$

$$(l_1,\dots,l_n) \longmapsto sp\langle l_1,\dots,l_n\rangle$$

$$\begin{tikzcd} G \arrow[r] & E_Z \arrow[d] \\ & Z \end{tikzcd}$$

is n-universal if

$\forall$ CW X $\dim X \le n$,

~~$[X, f$~~ ~~$[Z,X]$~~ $P_G(X) \cong [X,Z]$

~~Proof~~

- $K \subseteq L$, L - CW complex , K subcomplex , $\dim L \le n$

$$\begin{tikzcd} G \arrow[r] & E \arrow[d] \\ & L \end{tikzcd} \qquad \begin{tikzcd} E|_K \arrow[d] \\ K \end{tikzcd} \longleftrightarrow K \xrightarrow{f} Z \qquad E|_K = f^*E_Z$$

q. For any $K \subseteq L$, does $\exists$ $Z \leftarrow L : g$ so that

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d, hook] & Z \\ L \arrow[ur, "g"'] & \end{tikzcd}$$

1) $E = g^*E_Z$

2) g is an extension of f.

Wrong

Answer: Iff E_Z is $(n-1)$ connected.

Proof: Assume we know for $\dim L < n$ $\leftarrow$ Induction

- $\Rightarrow$

~~The statement is true for $K = S^n$,~~

~~Take $K = S^{n-1}$, $L = e^n$~~

~~Let $[f] \in [K, Z] = [S^{n-1}, Z]$~~ $= \pi_{n-1}(Z)$

$K = S^{n-1}$, $L = e^n$

$f : K \longrightarrow E_Z$ be ~~any~~ map

$$\begin{tikzcd} E_Z \arrow[d, "\pi"] \\ Z \end{tikzcd}$$

$[f] \in \pi_{n-1}(E_Z)$

$$\begin{tikzcd} (f\circ\pi)^* E_Z \arrow[r] \arrow[d] & E_Z \arrow[d, "\pi"] \\ S^{n-1} \arrow[r, "f\circ\pi"'] \arrow[ur, "f"] & Z \end{tikzcd}$$

Write proof later

- $K \subseteq L$ subcomplex of L, $\dim L \leq n$

$$\begin{tikzcd} E|_K \arrow[r] \arrow[d] & E_Z \arrow[d] \\ K \arrow[r] & Z \end{tikzcd}$$

Bundle extends to L $\iff$ the map $K \longrightarrow Z$ extends to $L \longrightarrow Z$

$\iff$ E_Z is $(n-1)$ connected.

Theorem:

E_Z n-connected $\Rightarrow$ $\begin{tikzcd} E_Z \arrow[d] \\ Z \end{tikzcd}$ n-universal.

(i.e. $[X, Z] \xrightarrow{\sim} P_G(X)$ for $\dim X \leq n$)

$\begin{tikzcd} E_Z \arrow[d] \\ Z \end{tikzcd}$ n-universal $\Rightarrow$ E_Z $(n-1)$-connected.

Construction of Universal bundles

(Milnor) $X * Y = \Sigma(X \wedge Y)$

$E_G^{(n)} = \underbrace{G * G * \cdots * G}_{n} = \Sigma^n (G \wedge G \wedge \cdots \wedge G)$ ← n-connected why?

$B_G^{(n)} = E_G^{(n)}/G$

$$\begin{tikzcd} G \arrow[r] & E_G^{(n)} \arrow[d] \\ & B_G^{(n)} \end{tikzcd}$$

$E_G = \varinjlim_n E_G^{(n)}$ $\qquad$ $B_G = E_G/G = \varinjlim_n B_G^{(n)}$

X-n connected $\Rightarrow$ $S^1 \wedge X$ n+1-connected

$H^{m+1}(S^1 \wedge X) = H^m(X) = 0$

$m \leq n$

$\Rightarrow H_{m+1}(S^1 \wedge X) = 0$ $\quad m \leq n$

$\Rightarrow S^1 \wedge X$ - n+1 connected

By Hurewitz.

Vector Bundles:

$G \longrightarrow O(k)$ continuous group homomorphism

Then $\mathrm{Vect}_k^G(X) \xrightarrow{\sim} P_G(X)$

$\Rightarrow \mathrm{Vect}_k^G(X) \xrightarrow{\sim} [X, BG]$

$$\begin{tikzcd} G \arrow[r] & EG \arrow[d] \\ & BG \end{tikzcd} \qquad \begin{tikzcd} \mathbb{R}^k \times_G G \arrow[r] & \mathbb{R}^k \times_G EG \arrow[d] \\ & BG \end{tikzcd}$$

$\mathbb{R}^k \times_G EG = \{(x,y) \in \mathbb{R}^k \times EG\}/\sim$

$(x, gy) \sim (g^{-1}x, y)$

$\mathbb{R}^k \times_G G = \{(x,y) \in \mathbb{R}^k \times G\}/\sim$

$(x, gy) \sim (g^{-1}x, y)$

$\Rightarrow (x, gy) \sim (y^{-1}g^{-1}x, 1)$

$= \mathbb{R}^k$

So we get a vector bundle

$$\begin{tikzcd} \mathbb{R}^k \times_G EG \arrow[d] \\ BG \end{tikzcd}$$ ~~corresponding~~

Aside:

$G = \mathbb{Z}/2$

$$\begin{tikzcd} E = S^{n-1} \arrow[d] \\ B = \mathbb{R}P^{n-1} \end{tikzcd}$$

$\mathbb{Z}/2 \circlearrowright S^{n-1}$, $\mathbb{Z}/2 \circlearrowright \mathbb{R}^n$

antipode action

$$\begin{tikzcd} \mathbb{Z}/2 \arrow[r] & S^{n-1} \arrow[d] & \arrow[r] & & \mathbb{R}^n \times_{\mathbb{Z}/2} \mathbb{Z}/2 \arrow[r] & \mathbb{R}^n \times_{\mathbb{Z}/2} S^{n-1} \arrow[d] \\ & \mathbb{R}P^{n-1} & & & & \mathbb{R}P^{n-1} \end{tikzcd}$$

$\mathbb{R} \times_{\mathbb{Z}/2} S^{n-1} = \{(t,x)\}/\sim$ $\qquad \mathbb{R} \times_{\mathbb{Z}/2} \mathbb{Z}/2 \cong \mathbb{R}$

$(t,x) \sim (-t,-x)$

~~claim~~

$$\begin{tikzcd} \mathbb{R} \times_{\mathbb{Z}/2} S^{n-1} \arrow[rr] \arrow[dr] & & \gamma_1 \arrow[dl] \\ & \mathbb{R}P^{n-1} & \end{tikzcd}$$ isomorphism

$(t,x) \longmapsto (tx)$

Similarly

$$\begin{tikzcd} S^1 \arrow[r] & S^{2n+1} \arrow[d] \\ & \mathbb{C}P^n \end{tikzcd} \qquad \begin{tikzcd} \gamma \arrow[d] \\ \mathbb{C}P^n \end{tikzcd} \qquad \gamma \cong \mathbb{R} \times_{S^1} S^{2n+1}$$

$G = O(n), U(n)$ $\quad$ $\mathrm{Vect}_k^{\mathbb{R}}(X) \cong [X, BO(n)]$ $\quad$ $\mathrm{Vect}_k^{\mathbb{C}}(X) \cong [X, BU(n)]$

$BO(n) = Gr_n(\mathbb{R}^\infty)$ $\quad$ $BU(n) = Gr_n(\mathbb{C}^\infty)$

Proof of Previous Th^m :

Induction. $n=0$ → Trivial

Assume the statement to be true for $n-1$.

- E_z - n-connected. ⇒ $(n-1)$-connected

  E_z - $(n-1)$ ~~con~~ universal

  $$\Rightarrow \begin{tikzcd} E_z \arrow[d] \\ Z \end{tikzcd}$$

  X = n-dimensional C-W-complex, $\begin{tikzcd} E \arrow[d] & G \arrow[l] \\ X \end{tikzcd}$

  Y = $(n-1)$ skeleton of X.

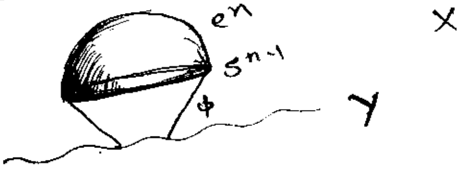
$$\begin{tikzcd} E|_Y \arrow[r] \arrow[d] & E_z \arrow[d] \\ Y \arrow[r, "f"] & Z \end{tikzcd}$$

By induction hypothesis $\exists\, f : Y \longrightarrow Z$ s.t $f^* E_Z = E|_Y$.

We need to extend f to X so that $f^* E_Z = E$.

We do it 1-cell at a time.

$X = Y \cup \{e^n\}$



WALOG ~~assume~~

$S^{n-1} \xrightarrow{\phi} Y$

$$\begin{tikzcd} E_{S^{n-1}} \arrow[r] \arrow[d] & E|_Y \arrow[r] \arrow[d] & E_Z \arrow[d] \\ S^{n-1} \arrow[r, "\phi"'] \arrow[u, dashed, bend left] \arrow[urr, dashed] & Y \arrow[r, "f"'] & Z \end{tikzcd}$$

$E_{S^{n-1}} = \phi^* E|_Y$

we also have

$E_{S^{n-1}} = i^*(\phi^* E)$
$= i^*(E_{e^n})$

$$\begin{tikzcd} E_{S^{n-1}} \arrow[r] \arrow[d] & E_{e^n} \arrow[r] \arrow[d] & E \arrow[d] \\ S^{n-1} \arrow[r, hook, "i"'] & e^n \arrow[r, "\phi"'] & X \end{tikzcd}$$

⇒ $E_{S^{n-1}}$ - trivial G-bundle

⇒ ∃ section $S^{n-1} \longrightarrow E_{S^{n-1}}$

Pushing forward we get a map

$S^{n-1} \xrightarrow{\widehat{f \circ \phi}} E_Z$

Because $\pi_{n-1}(E_Z) = 0$

$\Rightarrow$ we can extend $\widehat{\phi \circ f}$ to e^n $\widehat{\phi \circ f} : e^n \longrightarrow E_Z$

Push this down to get a map $g : e^n \longrightarrow Z$

$$\begin{tikzcd}
 & & E_Z \arrow[d] \arrow[d, bend left=60, dashed, "g"] \\
e^n \arrow[urr, dashed] & S^{n-1} \arrow[l] \arrow[ur] \arrow[r, "\phi\circ f"'] & Z
\end{tikzcd}$$

Define :

$$\hat{f} : X \longrightarrow Z$$
$$\hat{f}|_Y = f$$
$$\hat{f}|_{e^n} = g \,.$$

So we have obtained:

$$[X, Z] \longrightarrow P_G(X) \quad \text{surjective.}$$

<u>Injectivity:</u>

$$E \cong f^*E_Z \underset{\varphi}{\cong} g^*E_Z \qquad Z \longleftarrow X : f, g$$

$$\begin{tikzcd}
f^*E_Z \arrow[rr, "\varphi"] \arrow[dr] & & g^*E_Z \arrow[dl] \\
 & X &
\end{tikzcd}$$

Construct a vector bundle over $X \times I$

$$\begin{tikzcd}
p^*(E) \arrow[r] \arrow[d] & E \arrow[d] \\
X \times I \arrow[r, "p"] & X
\end{tikzcd}$$

~~$X\times\{0\} \cup$~~ $p^*E|_{X \times \{0\}} \cong E$

Construction of Universal Vector Bundles (contd.)

- If G discrete group, ~~$\pi_n G =$~~ $\pi_n G = \begin{cases} G & n = 0 \\ 0 & n > 0 \end{cases}$

$$\Rightarrow \pi_n BG = \begin{cases} G & n = 1 \\ 0 & \text{else} \end{cases} \qquad \Rightarrow \quad BG = K(G, 1)$$

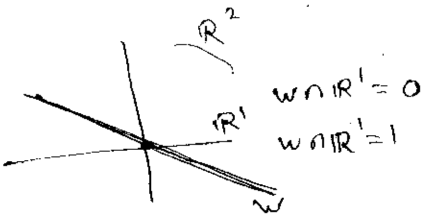
- $BU(n) = Gr_n(\mathbb{C}^\infty)$ $\qquad BO(n) = Gr_n(\mathbb{R}^\infty)$

$BU(1) = \mathbb{CP}^\infty$ $\qquad BO(1) = \mathbb{RP}^\infty$

- CW- complex structure on $Gr_n(\mathbb{R}^\infty)$ / $Gr_n(\mathbb{C}^\infty)$

Schubert Cells

(27)



$W \cap \mathbb{R}^1 = 0$
$W \cap \mathbb{R}^1 = 1$

$W \in Gr_n(\mathbb{R}^\infty)$ $\quad \mathbb{R}^k = \{(x_1 \cdots x_k, 0 \cdots 0)\}$

$W \cap \mathbb{R}^1 \subseteq W \cap \mathbb{R}^2 \subseteq \cdots \subseteq W \cap \mathbb{R}^k \subseteq W \cap \mathbb{R}^{k+1} \subseteq \cdots$

$\dim(W \cap \mathbb{R}^{k+1}) \leq \dim(W \cap \mathbb{R}^k) + 1$

$W \rightsquigarrow (\sigma_1 \cdots \sigma_n)$ $\quad 0 < \sigma_1 < \cdots < \sigma_n$. $\dim(W \cap \mathbb{R}^{\sigma_i}) = i$

$\mathrm{seq}(W)$ $\qquad \dim(W \cap \mathbb{R}^{\sigma_i - 1}) = i-1$

$A(\sigma_1 \cdots \sigma_n) = \{W \in Gr_n^\infty \mid \mathrm{seq}(W)_i \leq \sigma_i \;\forall i\}$ ← will be skeleton of ~~$A(\sigma)$~~ $Gr_n(\mathbb{R}^\infty)$

Orthonormal Basis of W as Row vectors:

for $W \in$ ~~Int~~ $(A(\sigma_1 \cdots \sigma_n))$

$$\begin{array}{c|cccccccccccc} & & & & \sigma_1 & & \sigma_2 & \sigma_3 & & \sigma_n & & \\ \hline & * & * \cdots & * & 1 & 0 & 0 & \cdots & & 0 & 0 & \cdots \\ & * & * \cdots & * & 0 & * & 1 & 0 & \cdots & 0 & 0 & 0 \cdots \\ & * & * \cdots & & 0 & & 0 & * & 1 & & & \\ & & & & \vdots & & & & & & & \\ & * & \cdots & & 0 & & 0 & * & 0 & 1 & 0 & 0 \cdots \end{array}$$

Guess:
$\dim A(\sigma_i)$
$= (\sigma_1 - 1) + (\sigma_2 - 2) + \cdots + (\sigma_n - n)$

But ~~Interior~~ $A(\sigma_i)$ needs to be replaced by ~~interior~~ $B(\sigma_i)$

- ~~Interior~~ $B(\sigma_i) = \{W \mid \mathrm{seq}\, W_i = \sigma_i \;\forall i\}$ ← cell of $G_n(\mathbb{R}^\infty)$

To each $W \in B(\sigma_i)$, we can associate an orthonormal basis $v_1, \ldots, v_n$ s.t. $v_i \in \mathbb{R}^{(\mathrm{seq}\, W)_i}$, $v_i \notin \mathbb{R}^{(\mathrm{seq}\, W)_i - 1}$, $(v_i)_{\mathrm{seq}\, w_i} > 0$

This association gives a continuous map, $B(\sigma_i) \longrightarrow V_n(\mathbb{R}^\infty)$.

$B(\sigma_1 \cdots \sigma_n)$ homeomorphic to its image

$\dim B(\sigma_1 \cdots \sigma_n) = (\sigma_1 - 1) + (\sigma_2 - 2) + \cdots + (\sigma_n - n)$

For $\mathbb{C}^\infty$ $\quad = 2(\sigma_1 - 1) + 2(\sigma_2 - 2) + \cdots + 2(\sigma_n - n)$.

$1\ \mathrm{eq}^n - n+1$ $\qquad x_1 \cdots x$
$1 \quad - n+2$

$- n+k$

$a_1 = b_n - b_{n-1}$
$a_2 = b_{n-1} - b_{n-2}$
$\vdots$

$1 \cdot a_1 + 2 \cdot a_2 + \cdots + n a_n$

Q. Is G_n^{n+k} k-universal?

Q. what is $H^*(G_n(\mathbb{R}^\infty), \mathbb{Z})$?

- Griffits, Morgan
- Felix

$$\begin{tikzcd} P(E) \arrow[r, hook] \arrow[d, "p"] & (B) \arrow[r, two heads] & B/P(E) \\ B & & \end{tikzcd}$$

Any vector bundle on B which is trivial on $P(E)$ is trivial.

$H(B/P(E)) \longleftrightarrow H(B)$ $\qquad M/P(E) = B$

M = Mapping cylinder of p $\quad \boxed{C_p P(E)}$

$M \longrightarrow BSO(3)$
$S^3 \longrightarrow BSO(3)$

$S^3 \longrightarrow P$

$\mathbb{RP}^3$
$0 \quad \mathbb{Z}/2 \quad 0$
$0 \quad 1 \quad 2 \quad 3$

Steifel- Whitney Classes

$w(T\mathbb{R}P^n) = (1+x)^{n+1}$. $\mathbb{Z}/2\langle x\rangle = H^1(\mathbb{R}P^n)$

Open Problem: what is the smallest n_k s.t. $\mathbb{R}P^k$ immerses in $\mathbb{R}^{n+k}$?

Immersion $\Rightarrow$ normal bundle ν $w(\nu) = \frac{1}{(1+x)^{n+1}} = \sum \binom{-n-1}{k} x^k$

Th^m: $\begin{matrix}\xi \\ \downarrow \\ X\end{matrix}$ vector bundle, orientable $\Leftrightarrow w_1(\xi)=0$.

Proof:

$$\begin{tikzcd}
\xi \arrow[r, "f^*"] \arrow[d] & \gamma \arrow[d] \\
X \arrow[r, "f"] \arrow[dr, dashed, "?"'] & BO(n) \\
 & BSO(n) \arrow[u]
\end{tikzcd}$$

$SO(n)$ connected component of $O(n)$

$\pi_* SO(n) = \begin{cases} \pi_* O(n) & \text{if } * > 0 \\ 0 & \text{else}\end{cases}$

$\pi_* BSO(n) = \begin{cases} \pi_{*-1} O(n) & \text{if } * > 1 \\ 0 & \text{else}\end{cases}$

$= \begin{cases} \pi_* BO(n) & \text{if } * > 1 \\ 0 & \text{else}\end{cases}$

Then $BSO(n)$ is just the universal coversal of $BO(n)$

$\pi_1(BSO(n)) = \pi_1(O(n)) = \mathbb{Z}/2$ $\longleftarrow$ For group G, $\pi_0 G$ is also a group $= G/$connected component of id.

?? $p^*: H^*(BO(n);\mathbb{Z}/2) \longrightarrow H^*(BSO(n);\mathbb{Z}/2)$

is 0 when $* = 1$?

$\therefore$ $\pi_1(BSO(n)) = 0$, $H_1(BSO(n)) = 0 \Rightarrow H^1(BSO(n);\mathbb{Z}/2) = 0$.

- if ξ - orientable

$\Rightarrow \exists \tilde{f}: X \longrightarrow BSO(n)$

$\Rightarrow w_1(\xi) = f^* w_1(\gamma) = (p\circ \tilde{f})^* w_1(\gamma)$

$= \tilde{f}^* \cdot \underbrace{p^* \cdot w_1(\gamma)}_{0} = 0$ $\therefore w_1(\gamma) \in H^1(BO(n);\mathbb{Z}/2)$

- if $w_1(\xi) = 0$

$\Rightarrow f^* w_1(\gamma) = 0$

$\Rightarrow f^*: H^1(BO(n)) \longrightarrow H^1(X)$ 0 map $\therefore H^*(BO(n))$ generated by $w_1, w_2, \dots, w_n(\gamma)$.

$\Rightarrow f: H_1(X) \longrightarrow H_1(BO(n))$ 0 map

$\Rightarrow f: \pi_1(X) \longrightarrow \pi_1(BO(n))$ 0 map

$\Rightarrow f$ lifts to $BSO(n)$. orientable

⊠

Th^m: $M^n - C^\infty$, $M^n \hookrightarrow \mathbb{R}^{n+1}$ embedded in $\mathbb{R}^{n+1}$ $\Rightarrow$ M^n orientable.

Proof:

Normal bundle ν 1-dim

$TM \oplus \nu = T\mathbb{R}^{n+1}$ = trivial

No need

$\Rightarrow \omega(TM) \cdot \omega(\nu) = 1$

$\Rightarrow \omega(TM) = 1 + \omega_1(\nu) + \omega_1(\nu)^2 + \cdots + \cancel{\omega_n(\nu)}$

$\omega_i(TM) = (\omega_1(\nu))^i = (\omega_1(TM))^i$

Prove directly that $\exists$ an outward normal vector field?

Need to use a seperation th^m:

M divides $\mathbb{R}^n$ in two parts. → Jordan curve for dim n.

For this Use Alexandar Duality.

Th^m: For all n - odd, Steifel Whitney numbers of $\mathbb{R}P^n$ are 0.

Proof: $\omega_i(T\mathbb{R}P^n) = \binom{n+1}{i} x^i$

$\omega_1^{i_1} \cdots \omega_n^{i_n} = \binom{n+1}{1}^{i_1} \cdot \binom{n+1}{2}^{i_2} \cdots \binom{n+1}{n}^{i_n} \cdot x^n$

Claim: $1 \cdot i_1 + 2i_2 + \cdots + n i_n = n$

$\Rightarrow \binom{n+1}{1}^{i_1} \cdots \binom{n+1}{n}^{i_n} = 0$ mod 2

Steenrod Square

$\exists$ operations $Sq^k : H^n(X) \longrightarrow H^{n+k}(X)$ satisfying

1. $Sq^k(f^*\omega) = f^* Sq(\omega)$
2. $Sq^k(\omega) = 0$ if $k > |\omega|$
   $= \omega^2$ if $k = |\omega|$
   $= \omega$ if $k = 0$
3. $Sq^k(\alpha \cup \omega) = \sum_{i+j=k} Sq^i(\alpha) \cup Sq^j(\omega)$
4. $Sq^k(\Sigma \omega) = \Sigma Sq^k(\omega)$ $\quad H^n(X) \xrightarrow[\cong]{\Sigma} H^{n+1}(\Sigma X)$
5. Adem's Relations.

eg: $\Sigma \mathbb{C}P^2$ $\quad S^3 \vee S^5$ both have trivial rings

$H^*(\Sigma \mathbb{C}P^2) = \mathbb{Z}\{1, y_3, y_5\}$ $\quad \mathbb{Z}/_2$ co-eff.

$H^*(\mathbb{C}P^2) = \mathbb{Z}\{1, x, x^2\}$

$y_3 = \Sigma x$, $y_5 = \Sigma x^2$ $\Rightarrow$ $Sq^2(y_3) = Sq^2(\Sigma x) = \Sigma Sq^2(x) = y_5$

$H^*(S^3 \vee S^5) = \mathbb{Z}\{1, z_3, z_5\}$

$$S^3 \vee S^5 \longrightarrow S^3$$

$$H^*(S^3 \vee S^5) \xleftarrow{f} H^*(S^3) \quad \text{injection}$$

$$z_3 \longleftarrow 1$$

$Sq_2(z_3) = f^*(Sq_2(1)) = 0$

So steenrod squares differentiate $\Sigma \mathbb{C}P^2$, $S^3 \vee S^5$

$\mathbb{C}P^2$ $\quad S^3 \xrightarrow[\eta]{} S^2$ hopf map

$[\eta] \in \pi_3(S^2)$ not null homotopic $\therefore \mathbb{C}P^2 \not\simeq S^2 \vee S^4$

$\Sigma \mathbb{C}P^2$ $\quad S^4 \xrightarrow[\Sigma\eta]{} S^3$

$[\Sigma\eta] \in \pi_4(S^3)$ not null homotopic $\therefore \Sigma \mathbb{C}P^2 \not\simeq S^3 \vee S^5$.

Q. $[\Sigma^k \eta] \in \pi_{k+3}(S^{k+2})$ not null homotopic.

$\mathbb{R}P^\infty$. $\quad H^*(\mathbb{R}P^\infty; \mathbb{Z}/_2) = \mathbb{Z}/_2[x]$

$$Sq^r(x^k) = Sq^{r-1}(x^{k-1})\, Sq^1(x) + Sq^r(x^{k-1})\, x$$
$$= Sq^{r-1}(x^{k-1})\, x^2 + Sq^r(x^{k-1})\, x$$

Guess: $Sq^k(x^k) = x^{2k}$

$$Sq^{k-1}(x^k) = Sq^{k-2}(x^{k-1}) x^2 + Sq^{k-1}(x^{k-1}) \cdot x$$
$$= Sq^{k-3}(x^{k-2}) x^3 + Sq^{k-2}(x^{k-2}) \cdot x^3$$
$$+ Sq^{k-1}(x^{k-1}) x$$
$$= (k) x^{2k}$$

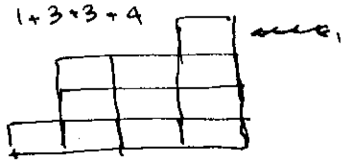
$$Sq^n(x^k) = \binom{k}{n}$$

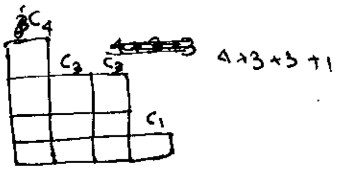
Q.7) A) for G_n - Schubert symbols $(\sigma_1, \dots, \sigma_n)$ correspond to cells of dim $(\sigma_1 - 1) + (\sigma_2 - 2) + \dots + (\sigma_n - n)$ where $1 \leq \sigma_1 < \sigma_2 < \dots < \sigma_n$

So, if the Chern classes are $c_i \in H^{2i}(G_n)$, (similary $\omega_i \in H^i(G_n(\mathbb{R})$

The cup product $\overset{\text{str.}}{\wedge}$ is given by,

$$c_i(\sigma_1, \dots, \sigma_n) \longmapsto \cancel{c_1^{(\sigma_n - \sigma_{n-1})} \cdot c_2^{(\sigma_{n-1} - \sigma_{n-2})} \cdot \dots \cdot c_n^{\sigma_2 - \sigma}}$$

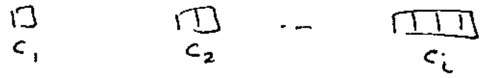




$n = 4$

$\sigma = (1+1, 2+3, 3+3, 4+4) \longrightarrow c_1^1 \, c_2^0 \, c_3^2 \, c_4^1$

$(\sigma_1, \dots, \sigma_n) \longmapsto c_1^{\#1\text{'s in dual partion}} \dots c_i^{\#i\text{'s in dual partition}} \dots$



Q.7 B), ??

Q.7) c)

$$\begin{tikzcd} \xi^m \arrow[dr] & & \eta^n \arrow[dl] \\ & B & \end{tikzcd}$$

$\omega_1(\xi^m \otimes \eta^n) = ?$

we will use splitting principal:

1) $m = n = 1$, $\exists f, g : B \longrightarrow \mathbb{R}P^\infty$ s.t. $\xi = f^*\gamma$, $\eta = g^*\gamma$

canonical line bundle

$$\begin{tikzcd} \xi \arrow[r] \arrow[d] & \gamma \arrow[d] & \eta \arrow[l] \arrow[d] \\ B \arrow[r, "f"] & \mathbb{R}P^\infty & B \arrow[l, "g"'] \end{tikzcd}$$

$\omega_1(\xi \otimes \eta) = \omega_1(f^*\gamma \otimes g^*\gamma)$ $\omega_1(\xi) = f^*x$, $\omega_1(\eta) = g^*x$

$\underset{G_1}{\overset{\shortparallel}{\mathbb{R}P^\infty}} = K(\mathbb{Z}/2, 1) \Rightarrow H^1(B, \mathbb{Z}/2) \cong [B, \mathbb{R}P^\infty]_*$

$\leftarrow\!\!\mapsto [f]$

f^*x ↪ generator of $H^1(\mathbb{R}P^\infty, \mathbb{Z}/2)$

$x : S^1 \hookrightarrow \mathbb{R}P$

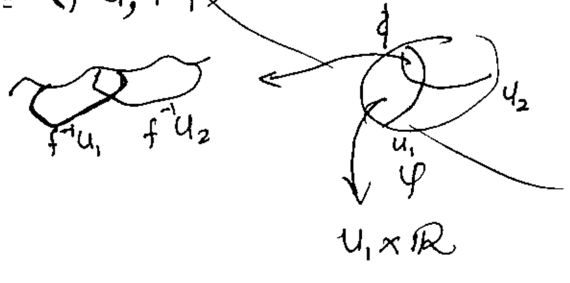
$\mathbb{R}P^\infty \times \mathbb{R}P^\infty \xrightarrow[\pi_2]{\pi_1} \mathbb{R}P^\infty$

$\tilde{K}(x) = [X, BO]$

$\text{Vect}^1_{\mathbb{R}}(SX) = [X, Gl_1(\mathbb{R})] = [X, \mathbb{R} - \{0\}]$

charts of $f^*\gamma = (f^{-1}U, f^*\varphi) = (f^{-1}U, \varphi \circ f)$

I know the transition f's for $\xi \otimes \eta$. what are the charts?



How are $E_1 \otimes E_2$, $E_1 \oplus E_2$ related?

$\downarrow$ $\begin{bmatrix} fg \end{bmatrix}$ $\quad$ $\downarrow$ $\begin{bmatrix} f & \\ & g \end{bmatrix}$

$\begin{bmatrix} -1 & \\ & +1 \end{bmatrix}$ $\quad$ $\begin{matrix} +1 & \\ & -1 \end{matrix}$

For $\mathbb{R}P^\infty$, $\mathbb{C}P^\infty$

$$\begin{tikzcd} i^*E \arrow[d] & E \arrow[d] \\ B^{(1)} \arrow[r, hook, "i"] & B \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "f"] & \mathbb{R}P^\infty \\ & \mathbb{R}P^1 \arrow[u] \end{tikzcd}$$

$\omega_1(i^*E) = i^*\omega_1(E)$? ✓

$i^*: H^1(B) \longrightarrow H^1(B^{(1)})$

is ~~surjective~~ ? ✓
injective

---

1) $m = n = 1$

Claim: $\omega_1(\xi \otimes \eta) = \omega_1(\xi) \oplus \omega_1(\eta)$

Proof: Let $B^{(1)}$ be 1 skeleton of B. $i: B^{(1)} \hookrightarrow B$

$$\cdots \longleftarrow H^1(B^{(1)}) \xleftarrow{i^*} H^1(B) \longleftarrow H^1(B/B^{(1)}) \longleftarrow \cdots$$

$H^1(B/B^{(1)}) = 0$ $\because$ No 1-skeleton

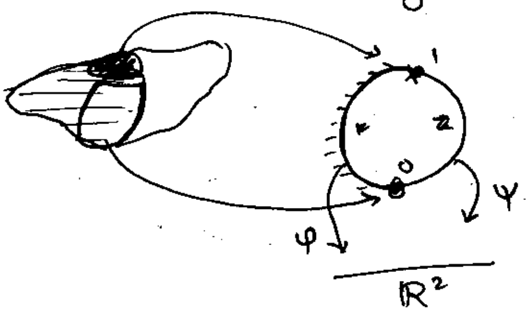
$\Rightarrow$ $i^*: H^1(B) \longrightarrow H^1(B^{(1)})$ injective

Enough to show

$$\omega_1(i^*\xi \otimes i^*\eta) = \omega_1(i^*\xi) \oplus \omega_1(i^*\eta)$$

So Assume B has only 1 skeleton

Then $\exists$ $f, g: B \longrightarrow \underset{\displaystyle S^1}{\underset{\shortparallel}{\mathbb{R}P^1}}$ s.t. $\xi = f^*\gamma$
$\eta = g^*\gamma$



Charts on B:

E_1 : ~~$(f^{-1}U_1, \varphi \circ f)$, $(f^{-1}U_2, \psi \circ f)$~~

Transition f^n: ~~ξ φ~~

Next we $\boxed{\text{combine } f, g}$ as

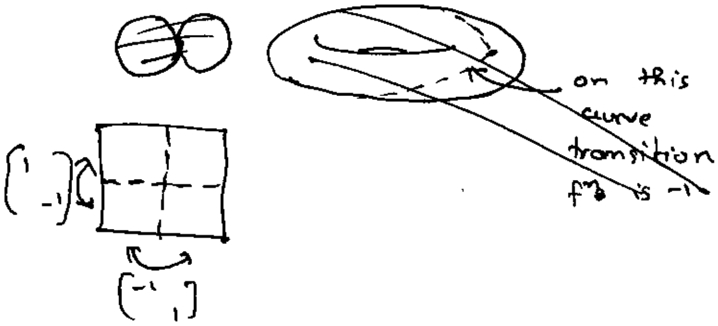
$$\begin{tikzcd} B \arrow[r, "f"] & S^1_0 \times S^1 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ S^1 & & S^1 \end{tikzcd}$$

$\omega_1(f^*\gamma \otimes g^*\gamma)$

$= \omega_1(F^*(\pi_1^*\gamma \otimes \pi_2^*\gamma))$

$= F^*\omega_1(\pi_1^*\gamma \otimes \pi_2^*\gamma)$

γ now is the mobius strip.



$H^1(S^1) = \mathbb{Z}/_2 \qquad H^1(S^1\times S^1) = \mathbb{Z}/_2 \oplus \mathbb{Z}/_2$

~~So just need to check~~

$S^1 \xrightarrow{i_1} S^1\times S^1 \xrightarrow{\pi_1} S^1$

$\pi_1 \circ i_1 = \text{id}$

$F = i_1 \quad B = S^1 \quad f = \text{id} \quad g = 0$

$\omega_1(f^*\gamma \otimes g^*\gamma) = \omega_1(f^*\gamma) = i_1^*\omega_1(\pi_1^*\gamma \otimes \pi_2^*\gamma)$

$\Rightarrow \omega_1(\pi_1^*\gamma \otimes \pi_2^*\gamma) = \mathbb{Z}/_2 \oplus \mathbb{Z}/_2 = \pi_1^*\omega_1(\gamma) \oplus \pi_2^*\omega_2(\gamma)$

$\Rightarrow \boxed{\omega_1(\xi\otimes\eta) = \omega_1(\xi) \oplus \omega_1(\eta)}$

ii) For general m, n

$$\begin{tikzcd} \xi \arrow[d] & \eta \arrow[dl] & p^*\xi \arrow[d] & p^*\eta \arrow[dl] & q^*p^*\xi \arrow[d] & q^*p^*\eta \arrow[dl] \\ B & & F(\xi) \arrow[ll, "p"] & & F(p^*\eta) \arrow[ll, "q"] & \end{tikzcd}$$

Both $q^*p^*\xi, q^*p^*\eta$ are sum of line bundles

and p^*, q^* injective on cohomologies

So enough to show for sum of line bundles

$\omega(\oplus \xi_i \otimes \oplus \eta_j) = \omega(\bigoplus_{i,j} (\xi_i \otimes \eta_j))$

$= \bigcup_{i,j} \omega(\xi_i \otimes \eta_j) =$

$= \bigcup_{i,j} (1 + \omega_1(\xi_i) + \omega_1(\eta_j))$

$=$

Steifel whitney classes :

- Thom isomorphism: $\mathbb{Z}/2$

$\exists u \in H^n(E, E_0)$, $\phi: H^{i}(B) \xrightarrow{\sim} H^i(E) \xrightarrow[\cup u]{\sim} H^{i+n}(E, E_0)$

$u|_{\text{Fibre}}$ = generator of $H^n(E, E_0; \mathbb{Z}/2)$.

- $\boxed{w_i^{\xi}(E) = \phi^{-1} \cdot Sq^i \cdot \phi(1)}$

$\Rightarrow \quad w_i(E) \cup u = Sq^i(u)$

Q.8. A) Wu's formula :

$$Sq^k(w_m) = \sum_{i=0}^{k} \binom{k-m}{i} w_{k-i} w_{m+i}$$

eg: $\xi = L_1 \oplus L_2 \oplus L_3$

$w_2(\xi) = w_1 L_1 \cup w L_2 + w_1 L_2 \cup w_1 L_3 \oplus w_1 L_1 \cup w_1 L_3$

$Sq^1(w_2) = w_1 L_1 \cup w_1 L_1 \cup w_1 L_2 + \cdots$

RHS $= w_1(\xi)\, w_2(\xi) + w_0 w_3$

$= (w_1 L_1 + w_1 L_2 + w_1 L_3)(w_1 L_1 \cup w_1 L_2 + \cdots + \cdots)$

$+ w_1 L_1 \cup w_1 L_2 \cup w_1 L_3$

Proof:

splitting principle

$\xi = L \oplus \eta$ L - Line bundle

$w_m(\xi) = w_m(\eta) + w_1(L) \cup w_{m-1}(\eta)$

Assume inductively formulae for η

$Sq^k(w_1(L) \cup w_{m-1}(\eta)) = w_1(L) \cup Sq^k w_{m-1}(\eta)$

$+ Sq^1 w_1(L) \cup Sq^{k-1} w_{m-1}(\eta)$

$+ \underset{0}{Sq^2 (w_1(L))} \cup Sq^{k-2} w_{m-1}(\eta)$

By Universal Property, we can assume ξ, L, η bundles over $\mathbb{RP}^\infty$.

- if L = trivial, we are done as $w_1(L) = 0$
- $L = \gamma_1$ $w_1(L) = x \in H^1(\mathbb{RP}^\infty, \mathbb{Z}/2)$

$Sq^1(x) = x^2$ [illegible]

~~$Sq^2(x^2) = x^4 = Sq^0(x^4) \cdot Sq^2(x) + Sq^1(x)\, Sq^1(x) + Sq^2(x)\, Sq^0(x)$~~

Adem's Relations:

$$Sq^i Sq^j = \sum_{k=0}^{[i/2]} Sq^{i+j-k} Sq^k \cdot \binom{j-k-1}{i-2k}$$

$i,j>0$

$i \leq 2j$

$$Sq^1 Sq^1 = Sq^2 \cdot Sq^0 \binom{1-0-1}{1-0}$$

$i=j=1$

$$= Sq^2$$

$$Sq^1 Sq^1(x) = Sq^1(x^2) = Sq^0(x) \cdot \cancel{Sq^1(x)} + \cancel{Sq^1(x) \cdot Sq^0(x)} = 0$$

$Sq^2(x) = 0$

$$Sq^k(x \cdot \omega_{m-1}(\eta)) = x \cdot Sq^k \omega_{m-1}(\eta) + x^2 \cdot Sq^{k-1} \omega_{m-1}(\eta)$$

$$\text{LHS} = Sq^k(\omega_1 L \cup \omega_{m-1}(\eta)) = \omega_1 L \cup Sq^k \omega_{m-1}(\eta) + \omega_1 L \cup \omega_1 L \cup Sq^{k-1} \omega_{n-1}(\eta)$$

$+ Sq^k \omega_m \eta$

$$= \omega_1 L \cup \left[\omega_k \omega_{m-1} + \binom{k-m+1}{1} \omega_{k-1} \omega_m + \cdots \right]\eta$$

$$Sq^k \omega_m(\eta) + \; + \omega_1 L \cup \omega_1 L \cup \left[\omega_{k-1}\omega_{m-1} + \binom{k-m}{1} \omega_{k-2} \omega_m + \cdots \right]\eta$$

$$\text{RHS} = \omega_k \omega_m(\xi) + \binom{k-m}{1} \omega_{k-1} \omega_{m+1} + \cdots$$

$$= \binom{k-m}{i} \omega_{k-i} \omega_{m+i} (L \oplus \eta) = \binom{k-m}{i} \omega_{k-i} \cancel{[\omega_{m+i}(\eta)} + \omega_1(L)\, \omega_{m+i-1}$$

$$= \cancel{Sq^k \omega_m(\eta)} + \cancel{\binom{k-m}{i}} \omega_m \omega_{k-i}\, \omega_1(L)\, \omega$$

$$= \binom{k-m}{i} \left\{ [\omega_{k-i}(\eta) + \omega_1(L) \cdot \omega_{k-i-1}(\eta)] + [\omega_{m+i}(\eta) + \omega_1(L) + \omega_{m+i-1}(\eta) \right.$$

$$= \binom{k-m}{i} [\omega_{k-i} \cdot \omega_{m+i}\, \eta + \cancel{\binom{k-m}{i}}\, \omega_1(L)\, \omega_{k-i-1}(\eta) \cdot \omega_{m+i}(\eta)$$

$$+ \; \omega_1(L)\, \omega_{k-i}(\eta)\, \omega_{m+i-1}(\eta)$$

$$+ \; \omega_1(L)\, \omega_1(L)\, \omega_{k-i-1}(\eta)\, \omega_{m+i-1}(\eta)]$$

$$\text{LHS} - \text{RHS} = \omega_1 L \cup \left[\binom{k-m}{i-1} \omega_{k-i} \omega_{m+i-1} - \binom{k-m}{i} \omega_{k-i-1} \cdot \omega_{m+i} \right)\eta$$

$$= 0$$

Base case - Line Bundle.

Q. 8.B)

~~Sol~~ $n =$ smallest no. s.t. $w_n \neq 0$

$\Rightarrow k+m=n \quad , \; k, m > 0$

$$\underset{\underset{0}{\parallel}}{Sq^k(w_m)} = \binom{k-m}{k} \; \cancel{Sq_0 Sq_n} \, w_0 w_n$$

$\Rightarrow$ ~~$\binom{k-m}{k}$ even~~ ~~$\rightarrow$ always 0~~ ~~take $m=0$~~

$$\Rightarrow \binom{k-m}{k} = \text{even} \quad \forall \quad k, m > 0 \; , \; k+m=n$$

$$\Rightarrow \binom{2k-n}{k} = \text{even}$$

if $n = 2^a b \qquad b$ odd, $b > 1$,

Take $k = 2^a$

$$\Rightarrow 0 = \binom{2^{a+1} - 2^a b}{2^a} \quad \cancel{2^a(2-b) \cdot 2}$$

$$= \frac{(2^{a+1} - 2^a b)(2^{a+1} - 2^a b - 1) \cdots (2^{a+1} - 2^a b - 2^a + 1)}{2^a \cdot (2^a - 1) \cdot \quad \cdots \quad \cdot 1}$$

~~Power of $2 \times 2^{a+1} - 2^a b$~~

$$= \frac{(2^a b - 2^{a+1})(2^a b + 1 - 2^{a+1}) \cdots (2^a b + 2^a - 1 - 2^{a+1})}{2^a !}$$

$$= \binom{2^a b - 2^a - 1}{2^a}$$

$$\text{Power of } 2 = \left[\frac{2^a b - 2^a - 1}{2}\right] + \cdots + \left[2^a b - \right.$$

$$- \left[\frac{\cancel{2^a b -} 2^a}{2}\right] + \cdots$$

$$- \left[\frac{2^a b - 2^{a+1} - 1}{2}\right] - \left[\frac{2^a b - 2^{a+1} - 1}{2^2}\right] + \cdots$$

$= 0 \qquad \because \nexists$ multiple of 2^{a+1} between $2^a b - 2^a - 1 \; , \; (2^a b - 2^a - 1) - 2^a$

which means $\binom{2k-n}{k}$ odd

So $b = 1 \; , \; n = 2^a$

Thom iso for oriented vector bundles:

$$\phi : H^k(B) \xrightarrow{\sim} H^k(E, \mathbb{Z}) \xrightarrow[\cup u]{\sim} H^{n+k}(E, E_0)$$

$$u|_{\text{fibre}} = \text{generator of } H^n(F, F_0)$$

euler class :
$$H^{n+k}(E, E_0) \xrightarrow{p^*} H^{n+k}(E) \overset{\sim}{\to} H^{n+k}(B) \quad p : E \longrightarrow E/E_0$$
$$u \longmapsto e \in H^{n+k}(B)$$

$$\begin{tikzcd}
H^{\ell}(B) \arrow[r, "\sim"] \arrow[dr, mapsto] & H^{\ell}(E) \arrow[r, "\cup u"] & H^{n+\ell}(E, E_0) \arrow[d] \\
 & e \quad H^n(B) \arrow[d] & H^n(E) \arrow[l, "\sim"'] \\
 & H^n(E) \arrow[r, "\cup u"'] & H^{n+n}(E, E_0) \\
 & & \phi(e)
\end{tikzcd}$$

e is just the Thom class thought of as an element of $H^n(E)$.

$$\Phi(e) = e|_E \cup u = p^* u \cup u$$
$$= u \cup u$$

---

in $\mathbb{Z}/_2$ $e = w_n$

Q.9 - A)

we know that $w_i(G_n(\mathbb{R}^\infty))$ generate $H^*(G_n(\mathbb{R}^\infty))$ — ✱

$$\begin{tikzcd}
\gamma^n \oplus \gamma^n \arrow[d] \\
G_n(\mathbb{R}^\infty)
\end{tikzcd}$$

$E \oplus E$ is orientable for any bundle E.

we choose ~~bas~~ orientation as (v, v) for each fibre :

- $w_{2n}(\gamma^n \oplus \gamma^n) = w_n(\gamma^n)\, w_n(\gamma^n) \neq 0$ by (✱)

$\Rightarrow e \neq 0$

- Let f denote orientation as above

n-odd f' be opposite orientation, i.e. having basis of the form $(v, -v)$

Then $e(E,f) = -e(E,f')$

Because reversing orientation reverses Euler class

$e(E,f) = e(E,f')$

$$\begin{tikzcd}
E' \arrow[r, "F"] \arrow[d] & E \arrow[d] \\
G_n(\mathbb{R}^\infty) \arrow[r, "id"'] & G_n(\mathbb{R}^\infty)
\end{tikzcd}$$

~~$F(b,v) =$~~ $F(b,v,w) = F(b,v,-w)$

F covers identity while sending (E,f) to $(E,-f)$

$\Rightarrow$ $2e(E,f) = 0.$

Q.9-B) $\mathbb{C}^n$ bundle

$$\begin{tikzcd}
\xi^n \arrow[d, "\pi"] \\
G_n(\mathbb{C}^\infty)
\end{tikzcd}$$

$i: \mathbb{R}^\infty \hookrightarrow \mathbb{C}^\infty$ inclusion

$i: G_n(\mathbb{R}^\infty) \hookrightarrow G_n(\mathbb{C}^\infty)$

$i^*(\xi^n) = ?$ $\quad i^*(\xi^n) = \{(b,v) \mid b \in \mathbb{R}^\infty,\ v \in \xi^n$
$i(b) = \pi(v)\}$

b is an n-plane in $\mathbb{R}^\infty$

$i(b)$ is an n-plane in $\mathbb{C}^\infty$ – $\{v_1 + iv_2 \mid v_i \in b\}$

v is a vector in $\mathbb{C}^\infty$

$\pi(v) = i(b) \Rightarrow \{v = w_1 + iw_2 \mid \pi(w_i) = b$ (i.e. $w_i \in b\}$

$i^*\xi^n = v^n \oplus v^n$

Q.9-c) $TS^n \to S^n$ $\quad A \subseteq S^n \times S^n = \{(x,-x) \mid x \in S^n\}$

- $TS^n \cong S^n \times S^n - A$

Let ρ_x denote stereographic projection from $x \in S^n$.

$\rho_x : S^n - \{x\} \xrightarrow{\cong} TS^n_{-x}$

~~Then~~ $\phi: S^n \times S^n - A \longrightarrow TS^n$

$(x,y) \longmapsto \rho_x(y)$

isomorphism. easy.

- $E = TS^n \cong S^n \times S^n - A$

$E_0 = S^n \subseteq TS^n \cong \{(x,y) \mid x \neq y\} = S^n \times S^n - A - D\}$

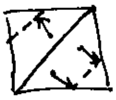
$H^*(E, E_0) \cong H^*(S^n \times S^n - A,$ ~~$S^n \times S^n - A - D\}$~~ $S^n \times S^n - A - D\}$

$\cong H^*(S^n \times S^n, S^n \times S^n - D)$

From the long exact seq^n of triple $(S^n \times S^n, S^n \times S^n - D, A)$ and deformation retract of $S^n \times S^n - D$ onto A

$H : S^n \times S^n - D \times I \longrightarrow A$

$p, t \longmapsto tp + (1-t)q$



where q is the point on the anti-diagonal closest to p.

Note that q is well-defined because D has been removed.

$\cong H^*(S^n \times S^n, A)$

$$\begin{tikzcd}
H^*(A) & H^*(S^n \times S^n) \arrow[l] & H^*(S^n \times S^n, A) \arrow[l] \\
H^{*-1}(A) \arrow[urr, bend left]
\end{tikzcd}$$

- Euler class : $E \cong TS^n$ $\qquad n$-even

$e(\tau) = \phi^{-1}(u \cup u)$

$\phi : H^*(S^n) \xrightarrow{\simeq} H^*(TS^n) \xrightarrow{\cup u} H^{n+*}(TS^n, TS^n - S^n)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{ıı}S$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad H^{*+n}(S^n \times S^n, A)$

$u \in H^n(TS^n, TS^n - S^n) \cong H^n(S^n \times S^n, A)$

u restricted to each fiber must be generator.

Need to trace each fiber

$F \hookrightarrow E \qquad F = T_x S^n$

$$\begin{tikzcd}
(F, F_0) \arrow[r, hook] \arrow[d, phantom, "\cong" rotate=90] & (E, E_0) \arrow[d, phantom, "\cong" rotate=90] \\
((-x)\times S^n, (-x,x)) \arrow[r, hook] & (S^n\times S^n, A)
\end{tikzcd}$$

Now $0 \longrightarrow H^n(S^n\times S^n, A) \longrightarrow H^n(S^n\times S^n) \longrightarrow H^n(A) \longrightarrow 0$

$$(a) \longmapsto (a,+a) \quad (a,b) \longmapsto (a-b)$$

This comes by looking a
each projection $S^n\times S^n \longrightarrow S^n$

$$\begin{tikzcd}
(-x\times S^n, (-x,x)) \arrow[r, hook] \arrow[dr, "id"] \arrow[drr, bend right=40, "trivial"'] & S^n\times S^n, A \arrow[d] \arrow[dr] & \\
& -x\times S^n, -x,x & (S^n\times x, (-x,x))
\end{tikzcd}$$

$\Rightarrow$ u = generator of $H^n(S^n\times S^n, A)$

$\phi(x) = \pi^*(x) \cup u$ $\qquad \pi: \underset{-D}{S^n\times S^n} \longrightarrow S^n \cong A$

what is $\phi^{-1}(u \cup u)$?

For what x, $\pi^*(x) \cup u = u \cup u$?

$\pi^*(x)$ is in $H^n(S^n\times S^n)$, u is in $H^n(S^n\times S^n, A)$

$u \cup u \in H^{2n}(S^n\times S^n, A) \cong \mathbb{Z}\oplus\mathbb{Z}$

$$\begin{tikzcd}
H^*(S^n\times S^n, A) \arrow[r, hook] & \tilde{H}^*(S^n\times S^n)
\end{tikzcd}$$

ring homo?

$\mathbb{Z}$ $\quad a \longmapsto a \quad \mathbb{Z}$ $\qquad * = 2n$

$\mathbb{Z}$ $\quad a \longmapsto (a,a) \quad \mathbb{Z}\oplus\mathbb{Z}$ $\qquad * = n$

$\downarrow$

cup product

$(a,b) \cup (c,d) = ad + bc$ $\qquad \because$ n-even

$\Rightarrow$ $a \cup a \longmapsto (a,a) \cup (a,a) = 2a \cup a$

$\times$ So, $\pi^*(x) \cup u = 2 \times$ generator of $H^{2n}(S^n\times S^n, A) \cong \tilde{H}^{2n}(S^n\times S^n)$

we know that ϕ is an isomorphism

So $\phi^{-1}(u \cup u) = 2$ generator of $H^n(S^n)$

⇒ Note: $u \cup u$ will be 0 for n-odd

also the map on cohomolgies will be different.

- Suppose $TS^n = V \oplus W$, $\begin{matrix} V \\ \downarrow \\ S^n \end{matrix}$, $\begin{matrix} W \\ \downarrow \\ S^n \end{matrix}$

$e(TS^n) = e(V) \cdot e(W)$

But cohomology ring of S^n is trivial

⇒ V, or W has dim 0.

---

11. Computations in Smooth Manifold — Tough

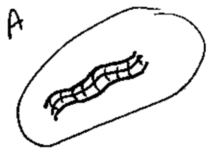
- $M \hookrightarrow A$ manifolds, M closed embedded

$\begin{matrix} \nu \\ \downarrow \\ M \end{matrix}$ normal bundle

$H^*(\nu, \nu_0) \cong H^*(A, A-M)$

Follows from Excision and Tubular Nbd. Th^m.

- The Thom class of ν, in $(A, A-M)$ is called fundamental class of M in A — denoted by u.



$$\begin{tikzcd}
u \arrow[rr, mapsto] & & e \\
H^n(\nu,\nu_0) \arrow[r] \arrow[d, leftrightarrow] \arrow[dr, phantom, "\circlearrowright"] & H^n(\nu) \arrow[r] \arrow[d, leftrightarrow] \arrow[dr, phantom, "\circlearrowright"] & H^n(M) \arrow[d, leftrightarrow, "\cong"] \\
H^n(A, A-M) \arrow[r] & H^n(A) \arrow[r] & H^n(M) \\
u \arrow[rr, mapsto] & & e
\end{tikzcd}$$

By def^n of euler class

Similarly for $\mathbb{Z}/2$ coeff.

image of u in $H^n(A) =: u'$ dual class of M in A

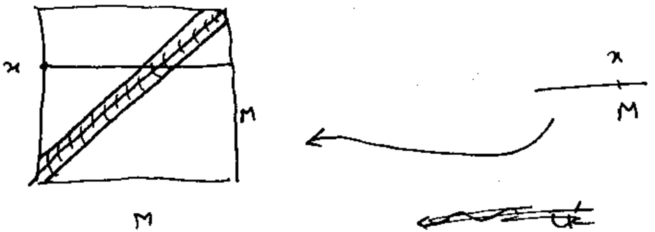
Cor: if $\bar{w}_k(TM) \neq 0$ Then M cannot be embedded in $\mathbb{R}^{n+k}$

$\bar{w} = \frac{1}{w}$

- $\Delta : M \hookrightarrow M \times M$ $\qquad \nu$ of M in $M \times M \cong TM$, u'- fundamental class of M in $M \times M$

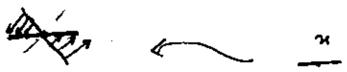
$$j_x : (M, M - x) \longrightarrow (M \times M, M \times M - \Delta)$$
$$y \longmapsto (x, y)$$

$j_x^*(u') = u_x$ $\qquad u_x$- orientation of M at x



$$\begin{tikzcd} M \arrow[r, "\Delta"] & M \times M \\ & M \arrow[u, "j_x"'] \end{tikzcd}$$

$$\begin{tikzcd} e & u' \arrow[l] \arrow[d] \\ & ? \end{tikzcd}$$



Reason: locally j_x is homotopic to diagonal

Now diagonal map maps thom class to u_x

By homotopy, j_x maps u' to u_x ∵ normal bundle $\cong TM$.

- $u' \in H^n(M \times M, M \times M - \Delta) \longrightarrow H^n(M \times M)$

  $u' \longmapsto u''$ diagonal cohomology class of M

- $a \in H^*(M)$

  $(1 \times a) \cup u'' = (a \times 1) \cup u''$

$$\begin{tikzcd} M \times M \arrow[r, "p_1"] \arrow[d, "p_2"] & M \\ M & \end{tikzcd}$$

$1 \times a = p_1^*(a)$

$a \times 1 = p_2^*(a)$

$(M \times M, M \times M - \Delta) \underset{\text{excision}}{\cong} (N_\varepsilon, N_\varepsilon - M)$, N_ε - tube

Inside N_ε - $p_1 \simeq p_2$
rotation by 90°

so $\quad p_1^*(a) = p_2^*(a)$ in $H^*(N_\varepsilon, N_\varepsilon - M)$

- $/ : H^{p+q}(X\times Y) \otimes H_q(Y) \longrightarrow H^p(X)$ "field" co-efficients

$$\begin{tikzcd}
H^{p+q}(X\times Y) \otimes H_q(Y) \arrow[d, phantom, "\cong" rotate=90] \arrow[r] & H^p(X) \\
H^*(X)\otimes H^*(Y) \otimes H_*(Y) \arrow[r] & H^*(X) \\
\alpha,\ \beta,\ \mu \arrow[r, mapsto] & \alpha\langle \beta,\mu\rangle
\end{tikzcd}$$

$$(\alpha\times\beta)/_\mu = \alpha\langle\beta,\mu\rangle$$

$$[(\alpha\times 1)\cup\beta]/_\mu = \alpha\cup(\beta/_\mu)$$

- M compact, ~~β~~ $[M]\in H_n M$ $\qquad u''\in H^n(M\times M, M\times M-\Delta)$

$$u''/_{[M]} = 1$$

$$x \hookrightarrow M$$

$$\begin{tikzcd}
H^*_n(M, M-x) & H^*_n(M) \arrow[l, leftrightarrow] \\
{[M_x]} & {[M]} \arrow[l]
\end{tikzcd}$$

- In field co-efficients

1) $u'' = \sum_{\sigma} (-1)^{|\sigma|}\,\sigma\times\tilde{\sigma}$ $\qquad \{\sigma\}$ generate $H^*(M)$ as a vector space

$\langle\tilde{\sigma}, \sigma\cap[M]\rangle = 1$
dual basis

2) $\langle e(TM), [M]\rangle = \chi(M)$

$$e(TM) = \Delta^*\mu''$$

- Wu's formula

$$\omega_i = \phi^{-1} Sq^i\,\phi(1)$$
$$= \phi^{-1} Sq^i(u)$$

$\phi: H^*(E,E_0) \overset{\cup u}{\longleftrightarrow} H^{*-n}(M)$
Thom iso.

$\Rightarrow \pi^*\omega_i \cup u = Sq^i(u)$ $\qquad \pi^*: E\longrightarrow M$

$E = TM$

$$\begin{tikzcd}
(E,E_0) \arrow[r, phantom, "\cong"] \arrow[dr, "\pi"'] & (M\times M, M\times M-\Delta) \arrow[d, "\pi_1\cong\pi_2"] \\
& M
\end{tikzcd}$$

~~Apply~~ By naturality,

$$\pi_1^*\omega_i \cup u' = Sq^i(u')$$
(or π_2^*)

$\pi_1^* \alpha = \alpha \times 1$ So, $(\omega_i \times 1) \cup u' = Sq^i u'$

Again by naturality, $H^*(M \times M, M \times M - \Delta) \longrightarrow H^*(M \times M)$

$$(\omega_i \times 1) \cup u'' = Sq^i(u'')$$

Applying $/[M]$,

$$(\omega_i \times 1) \cup u'' /_{[M]} = Sq^i(u'')/_{[M]}$$

$$\|$$

$$\omega_i \cup u''/_{[M]}$$

$$\|$$

$$\omega_i$$

$$\boxed{\omega_i = Sq^i(u'')/[M]}$$

- $$x \longmapsto \langle Sq^i(x), [M] \rangle \qquad x \in H^{n-i}(M)$$

By Poincare duality: (as we are in $\mathbb{Z}/2$)

$\exists\, v_i$ s.t. $\quad v_i \in H^i(M)$

$$x \cup v_i = Sq^i(x)$$

Then

$$\omega_K = \sum_{i+j=k} Sq^i(v_j)$$

$$\omega = Sq(v)$$

Q.11 - A)

$$\mathbb{P}^n \hookrightarrow \mathbb{P}^{n+1} \longrightarrow S^{n+1}$$

$$\Rightarrow \quad H^*(\mathbb{P}^{n+1})/_{H^{n+1}(\mathbb{P}^{n+1})} \cong H^*(\mathbb{P}^n) \quad \text{in } \mathbb{Z}/2$$

$$\cong \mathbb{Z}/_2[x]/_{x^{n+1}} \quad \text{by induction}$$

By duality,

if $H^{n+1}(\mathbb{P}^{n+1}) = \mathbb{Z}/2\, y$

$$\langle y, [\mathbb{P}^{n+1}] \rangle = 1$$

$$\|$$

$$\langle x' \cup x^{n}, [\mathbb{P}^{n+1}] \rangle$$

$\because$ dual of $H^1 \in H^{n}$, $\Rightarrow$ x' dual to x^n

$$\Rightarrow x' \cup x^n = y$$

$$\Rightarrow y = x^{n+1}$$

Base case. $\mathbb{P}^1 = S^1$.

Q.9-B]

$$\cap[M] : H^k(M) \xrightarrow{\cong} H_{n-k}(M)$$

$$[\alpha] \longmapsto [\alpha \cap [M]]$$

$$u'' \in H^n(M \times M)$$

$$u''/ \ : \ H_{n-k}(M) \longrightarrow H^k(M)$$

1) $(u''/) \circ (\cap[M]) \circ (\alpha) = u''/\alpha \cap [M] \overset{?}{=} \alpha \cdot (-1)^{|\alpha||M|}$

$$u'' = \sum_{\alpha} (-1)^{|\alpha|} \cdot \alpha \times \tilde{\alpha}$$

$$u''/\alpha \cap [M] = \sum_{\beta} (-1)^{|\beta|} \cdot \beta \times \tilde{\beta}/\alpha \cap [M]$$

$$= \sum_{\beta} (-1)^{|\beta|} \cdot \beta \langle \tilde{\beta}, \alpha \cap [M] \rangle$$

$$= \sum_{\beta} (-1)^{|\beta|} \cdot \beta \langle \tilde{\beta} \cup \alpha, [M] \rangle$$

Take α to be a basis element

$$= (-1)^{|\alpha|} \alpha \langle \tilde{\alpha} \cup \alpha, [M] \rangle$$

$$= (-1)^{|\alpha| + (|\alpha| + |M|)(|\alpha|)} \cdot \alpha$$

$$= (-1)^{|\alpha| \cdot |M|} \cdot \alpha$$

2) $(\cap[M]) \cdot (u''/\ ) \circ \mu = u''/_{\mu} \cap [M] \overset{?}{=} \mu \cdot (-1)^{|M| \cdot |M|}$

$$= \sum_{\beta} (-1)^{\beta} \beta \times \tilde{\beta}/_{\mu} \cap [M]$$

$$= \sum_{\beta} (-1)^{\beta} \beta \langle \tilde{\beta}, \mu \rangle \cap [M]$$

Take μ to be a basis dual to α i.e. $\langle \alpha, \mu \rangle = 1$

$$= (-1)^{|\tilde{\alpha}|} \tilde{\alpha} \cap [M] \qquad \langle \alpha, \tilde{\alpha} \cap [M] \rangle = \langle \alpha \cup \tilde{\alpha}, [M] \rangle = 1$$

$$= (-1)^{|\tilde{\alpha}|} \mu$$

Sign problem

Q.~~8~~ 11 C)

~~$A^p \hookrightarrow M^m$ $k = m - p$~~

$M^m \hookrightarrow A^p$ $k = p - m$

$\cap$ ~~μ_A~~ $[A] : H^k(A) \longrightarrow H_m(A)$

$u' \in$ ~~$H^k(A)$~~

$u' \in H^k(A, A-M)$

Q.11 D)

Wu's classes : v_k satisfy $v_k \in H^k(M)$

$v_k \cup x = Sq^k(x)$ for $x \in H^{n-k}(M)$

For 3-manifold:

$1 = v_0, v_1, v_2, v_3$

$v_3 \cup x = Sq^3(x) = 0$ ~~$x \in H^0$~~

$\Rightarrow v_3 = 0$

$v_2 \cup x = Sq^2(x)$ $x \in$ ~~Sq^1~~ H^1

$\Rightarrow v_2 = 0$

So $1, v_1$ $\forall\, x \in H^2(M)$

v_1 satisfies $\boxed{v_1 \cup x = Sq^1(x)}$

$w_1 = v_1$

if M is orientable, $w_1 = 0 \Rightarrow v_1 = 0$.

(The statement cannot be true for M non-orientable).

Q.11 E)

$Sq : H^*(M) \longrightarrow H^*(M)$

$x \longmapsto x + Sq^1(x) + \dots + Sq^i(x) + \dots$

why automorphism?

- Injectivity, Ring homo-morphism is clear
- Surjectivity:

Can we invert $(1 + Sq^1 + \dots)$

~~$(1 - (Sq^1 + \dots) + (Sq^1 + \dots)^2 - (\ )^3 + \dots)$~~

Yes. Because they are ring operators

Do it by hand.

~~$\langle v \cdot x, [M] \rangle = \langle Sq\, x, [M] \rangle$~~

~~$Sq\, v = w \Rightarrow v = \bar{Sq}\, w$~~

~~$\Rightarrow \langle \bar{Sq}\, w \cdot x, [M] \rangle = \langle Sq\, x, [M] \rangle$~~

~~$y = Sq\, x \Rightarrow x = \bar{Sq}\, y$~~

~~$\Rightarrow \langle \bar{Sq}(w \cdot y), [M] \rangle = \langle y, [M] \rangle$~~

~~$\Rightarrow \langle w \cdot y, [M] \rangle = \langle Sq\, y, [M] \rangle$!! WTF.~~

Problem: $\langle x, [M] \rangle = \langle (z), [M] \rangle$
$\not\Leftrightarrow \langle Sq\, x, [M] \rangle = \langle Sq(z), [M] \rangle$

How does $\bar{Sq}$ look?

$\bar{Sq}(x) = x + Sq^1(x) + \cdots$

$Sq(Sq^1(x)) = Sq^1(x) + Sq^1 Sq^1(x) + \cdots$

So $\bar{Sq}(x) = x - Sq^1(x) - Sq^2(x) + Sq^1 Sq^1(x) - \cdots$

So Call $\bar{Sq}^i(x) =$ component of $Sq(x)$ in $H^{|x|+i}(M)$.

we need to show, for $x \in H^{n-k}(M)$

$\bar{w}_k \cup x = \bar{Sq}^k(x)$.

i.e. in degree n,

$\bar{w} \cdot x \overset{?}{=} \bar{Sq}(x)$

Now, $\exists z$ s.t. $x = Sq(v \cdot z)$ $\quad (z = \bar{v} \cdot \bar{Sq}(x))$

$\Rightarrow$ To show in deg n, ..

$\bar{w} \cdot Sq(v \cdot z) \overset{?}{=} \bar{Sq}(\cdot Sq(v \cdot x))$

But $\bar{w} = Sq \cdot \bar{v}$

$\Rightarrow Sq(\bar{v}) \cdot Sq(v \cdot z) \overset{?}{=} \bar{Sq}(Sq(v \cdot z))$ in deg n

$\Rightarrow$ $Sq(z) \overset{?}{=} v.z$ in deg n

But this is simply original Wu's formula.

So,

$$\langle \bar{Sq}^i(x), [M] \rangle = \langle \bar{\omega}^i \cup x, [M] \rangle.$$

for $i = n$, $x = 1$

$$\Rightarrow \langle \bar{Sq}^n(1), [M] \rangle = \langle \bar{\omega}^n, [M] \rangle$$

$$\overset{\shortparallel}{0}$$

$\Rightarrow$ $\bar{\omega}^n = 0$ $\quad\because \bar{Sq}(1) = 1 \Rightarrow \bar{Sq}^n(1) = 0$

for $i = n-1$,

$$\langle \bar{Sq}^{n-1}(x), [M] \rangle = \langle \bar{\omega}^{n-1}x, [M] \rangle$$

- or simply $\bar{Sq}^{n-1}(x) = \bar{\omega}^{n-1}x$ for $x \in H^1(M)$

Need to show $\bar{Sq}^{n-1} = 0$

$Sq(x) = x + Sq^1(x) = x + x^2$ for $x \in H^1$ ~~$+ Sq^2(x) + \dots$~~

~~$\bar{Sq}(x) = \dots$~~ [illegible]

~~$+ Sq^1 Sq^1(x) + Sq^1 Sq^2(x) + \dots$~~

~~$Sq(Sq^1(x)) = Sq^1(x) + Sq^1 Sq^1(x) + Sq^1(Sq^2(x)) + \dots$~~

~~$+ Sq^2(Sq^1(x))$~~

~~$Sq(x+x^2) =$~~

$Sq(x) = x + x^2 \Rightarrow Sq(x^i) = (x+x^2)^i$

$= x^i(1+x)^i$

So Sq of what $= x$?

$$Sq(x + a_2x^2 + \dots + a_ix^i + \dots) = x$$

$$\Rightarrow (x+x^2) + a_2(x+x^2)^2 + \dots + a_i(x+x^2)^i + \dots = x$$

By trial and error one gets

$a_{2^i} = 1$, $a_i = 0$ if i not a power of 2

This is because $(x+x^2)^2 = x^2 + x^4$

$(x+x^2)^4 = x^4 + x^8$

$(x+x^2)^{2^i} = x^{2^i} + x^{2^{i+1}}$

☆ So $\bar{S}q(x) = x + x^2 + \dots + x^{2^i} + \dots$ ☆

So if n is not a power of 2

$\bar{S}q^{n-1}(x) = 0$ as there is no n degree term in $\bar{S}q(x)$

$\Rightarrow \bar{\omega}^{n-1}(x) = 0$.

11. F)

$$Sq^i : H_k(X) \longrightarrow H_{k-i}(X)$$

$$\langle x, Sq^i(\beta)\rangle = \langle \bar{S}q^i(x), \beta\rangle \qquad |x| + i = |\beta|$$

- $Sq^i(\alpha \cap \beta) \overset{?}{=} \sum Sq^k(a) \cap Sq^l(\beta) \longrightarrow -|a| + |\beta| + k - l$

$\quad$ ~~$-|a|+|\beta|$~~ $\boxed{k - l = -i}$

$\text{deg} = -i + \text{deg}\,\alpha + \text{deg}\,\beta$

$= -i + |a| + |\beta|$

$$\langle x, Sq^i(a \cap \beta)\rangle = \langle \bar{S}q^i(x), \alpha \cap \beta\rangle$$
$$= \langle \bar{S}q^i(x) \cup a, \beta\rangle$$

$$\sum_{k-l=i} \langle x, Sq^k(a) \cap Sq^l(\beta)\rangle = \sum_{k-l=-i} \langle x \cup Sq^k(a), Sq^l(\beta)\rangle$$
$$= \sum_{k-l=-i} \langle \bar{S}q^l(x)\, \bar{S}q^l(Sq^k(a)), \beta\rangle$$

$$\bar{S}q^i(x) \cup a \overset{?}{=} \sum_{k-l=-i} \bar{S}q^l(x) \cdot \bar{S}q^l(Sq^k(a))$$

$$= \sum_{k=l-i} \bar{S}q^l(x \cdot Sq^k(a)) = \sum_{\substack{m+n=l \\ k=l-i}} \bar{S}q^m(x) \cdot \bar{S}q^n Sq^k($$

~~If true for all i, we will get~~

~~$\bar{S}q(x)\,a = \sum_{k \le l} \bar{S}q^l(x \cdot Sq^k(a))$~~

Applying $Sq \longrightarrow$

$$x \cdot Sq(a) = \sum_{k \le l} Sq\left(\bar{Sq}^{l}(x \cdot Sq^{k}(a))\right)$$

- $\langle x, Sq(\beta) \rangle = \langle \bar{Sq}(x), \beta \rangle$

$\langle x, Sq(a \cap \beta) \rangle = \langle \bar{Sq}(x) \cup a, \beta \rangle$

$\langle x, Sq(a) \cap Sq(\beta) \rangle = \langle x \cup Sq(a), Sq(\beta) \rangle$

$= \langle \bar{Sq}(x \cup Sq(a)), \beta \rangle$

$= \langle \bar{Sq}(x) \cup a, \beta \rangle$

$\Rightarrow \quad Sq(a \cap \beta) = Sq(a) \cap Sq(\beta)$

- $Sq(u''/\beta)$

Slant product:

co-efficients in a field

$/ : \ H^{p+q}(X \times Y) \otimes H_q(Y) \longrightarrow H^p(X)$

$\quad \cong$

$H^{p+i}(X) \otimes H^{q-i}(Y)$

$(a, b)/\mu \longmapsto a \langle b, \mu \rangle.$

- $(a \times b)/\mu = a \langle b, \mu \rangle$

$(a \times 1) \cup p/\mu = a \cup (p/\beta)$

No idea how to do this problem.

What does slant product do?

- ~~$Sq(x)$~~ $\langle x, \bar{\omega} \cap \mu \rangle = \langle x \cup \bar{\omega}, \mu \rangle$

$= \langle \bar{Sq}(x), \mu \rangle$ with sign issues

$= \langle x, Sq(\mu) \rangle$

but we are using
$\mathbb{Z}/2$ co-eff
So no problem

$\langle x, v \cap \mu \rangle = \langle x \cup v, \mu \rangle$

$= \langle Sq(x), \mu \rangle = \langle x, \bar{Sq}(\mu) \rangle.$

~~12 obst~~

12. obstructions

Need to read Local co-efficients

- Skifel Manifold Bundle

$$\begin{tikzcd} E \arrow[d, "\pi"] \\ B \end{tikzcd}$$ $\mathbb{R}^n$- vector bundle

$$\begin{tikzcd} V_k(\mathbb{R}^n) \arrow[r, hook] & V_k(E) \arrow[d] \\ & B \end{tikzcd}$$

$V(E)$ - k-frames in each fibre. orthonormal
$= \{(x, (v_1, \dots, v_k)) \mid \pi(v_i) = x,\ v_i$'s linearly ind$\}$

$V_k(\mathbb{R}^n)$ - $n-k-1$ connected ??

(problem is finding cross-section over $n-k+1$ skeleton of B.)
Requires local coefficients.

~~Alternately we can~~ look at ~~only orthogonal~~ normal frames.
~~This is the same manifold bundle because GL_k deformation retracts onto O_k.~~

- $V_k(\mathbb{R}^n)$ - $(n-k-1)$ connected
  $V_k(\mathbb{C}^n)$ - $2(n-k)$ connected

Proof: $\ell < k \leq n$

$$\exists \quad \begin{tikzcd} V_{k-\ell}(\mathbb{R}^{n-\ell}) \arrow[r] & V_k(\mathbb{R}^n) \arrow[d, "p"] \\ & V_\ell(\mathbb{R}^n) \end{tikzcd}$$

$$p : ((v_1, v_2, \dots, v_k), x) = ((v_1, \dots, v_\ell), x)$$

Restricting to $\ell = 1$

$$V_{k-1}(\mathbb{R}^{n-1}) \longrightarrow V_k(\mathbb{R}^n) \longrightarrow S^{n-1}$$

$$\pi_i(S^{n-1}) \longrightarrow$$

$$\rightarrow \pi_i(V_{k-1}\mathbb{R}^{n-1}) \longrightarrow \pi_i(V_k\mathbb{R}^n) \longrightarrow \pi_i(S^{n-1}) \longrightarrow$$

$i < n-1$ $\quad \pi_i(V_{k-1}\mathbb{R}^{n-1}) = \pi_i(V_k\mathbb{R}^n)$

for $i < n-k$ $\quad \pi_i(V_k\mathbb{R}^n) = \pi_i(V_{k-1}\mathbb{R}^{n-1})$

$$\vdots$$

$$= \pi_i(V_1(\mathbb{R}^{n-k+1}))$$

$$= \pi_i(S^{n-k}) = 0$$

For $\mathbb{C}^n$,

$$V_{k-1}(\mathbb{C}^{n-1}) \longrightarrow V_k(\mathbb{C}^n) \longrightarrow S^{2n-1}$$

- Gysin Sequence :

$$\begin{tikzcd} {} \arrow[r] & H^i(E,E_0) \arrow[r] & H^i(E) \arrow[r] \arrow[dd, dash, "\cong"] & H^i(E_0) \arrow[r] & H^{i+1}(E,E_0) \arrow[r] & {} \\ & H^{i-n}(E) \arrow[u, "\cup u", "\cong"'] \arrow[d, dash, "\cong"] \arrow[r, phantom, "\circlearrowright"] & {} & & & \\ & H^{i-n}(B) \arrow[r] & H^i(B) & & & \end{tikzcd}$$

$$\begin{tikzcd} E \arrow[d, "\pi"] \\ B \end{tikzcd}$$ oriented

~~u: Thom class~~

u : Thom class
$\in H^n(E,E_0)$

Follow left + top $\to$ $x \longmapsto \pi^* x \longmapsto \pi^* x \cup u$
$\downarrow$
$\pi^* x \cup u|_E$

Follow ~~right~~ bottom + right $\to$ $x \longmapsto ? \longmapsto \pi^* ?$

$$\pi^* ? = \pi^* x \cup u|_E$$
$$= \pi^*(x \cup e)$$

e- euler class

$$\Rightarrow \quad ? = x \cup e$$

$$\to H^{i-n}(B) \xrightarrow{\cup e} H^i(B) \longrightarrow H^i(E_0) \longrightarrow H^{i+1-n}(B) \longrightarrow$$

for un-oriented, $e \longmapsto w_n$

For $\tilde{B} \longrightarrow B$ .. two cover

$$\begin{tikzcd} \tilde{B} \times \mathbb{R}/\mathbb{Z}/2 \arrow[d] \\ B \end{tikzcd}$$

Line bundle $\leftarrow$

$\mathbb{Z}/2 \circlearrowright \mathbb{R}$ by antipode

$\mathbb{Z}/2 \circlearrowright \tilde{B}$ via Deck

~~is a~~

Here $E = \tilde{B} \times \mathbb{R}/\mathbb{Z}/2$

$E_0 \cong \tilde{B}$
$\downarrow$
This is because

$$E_0 \cong \tilde{B} \times \mathbb{R}-\{0\}/\mathbb{Z}/2$$
$$\cong \tilde{B} \times S^0/\mathbb{Z}/2$$
$$\cong \tilde{B}$$

- Oriented Grasmanian

$\tilde{G}_n(\mathbb{R}^{n+k})$ two covering of $G_n(\mathbb{R}^{n+k})$

being a top cover
use above lemma

$$\begin{tikzcd} \tilde{\gamma}^n \arrow[r] \arrow[d] & \gamma^n \arrow[d] \\ \tilde{G}_n(\mathbb{R}^{n+k}) \arrow[r] & G_n(\mathbb{R}^{n+k}) \end{tikzcd}$$

$\nearrow$
universal property wrt oriented bundles

Two cover $\Rightarrow$ $\xrightarrow{} H^i(B) \xrightarrow[\cdot \cup w_1]{} H^{i+1}(B) \longrightarrow H^{i+1}(\tilde{B}) \longrightarrow H^{i+1}(B) \xrightarrow[\cup w_1]{}$

$\mathbb{Z}/2$ co-eff

$B = G_n(\mathbb{R}^{n+k})$

$\tilde{B} = \tilde{G}_n(\mathbb{R}^{n+k})$

$w_1 \in H^1(G_n(\mathbb{R}^{n+k})) = \mathbb{Z}/2\, w_1(\gamma^n)$

$w_1 = 0 \Rightarrow$

$0 \longrightarrow H^{i+1}(B) \longrightarrow H^{i+1}(\tilde{B}) \longrightarrow H^{i+1}(B) \longrightarrow 0$

at $i = -1$ we get

$H^{i+1}(\tilde{B}) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \Rightarrow \tilde{G}_n(\mathbb{R}^{n+k})$ not connected

Why is $\tilde{G}$ connected ? ~~since G is connected, enough to show ∃ path taking~~ ~~$(p, [p]) \longrightarrow (p, -[p])$~~

$\hookrightarrow$ orientation of p.

$$\begin{tikzcd} \tilde{\xi}_n \quad Gl^+(\mathbb{R}^n) \arrow[r] & V_n(\mathbb{R}^{n+k}) \arrow[d] \\ & \tilde{Gl}_n(\mathbb{R}^{n+k}) \end{tikzcd}$$

$$\begin{tikzcd} \pi_0(Gl^+ \mathbb{R}^k) \arrow[r] & \pi_0(V_n) \arrow[r] & \pi_0(\tilde{Gl}_n) \arrow[r] & 0 \\ \pi_1(\tilde{Gl}_n) \arrow[u, bend left] & \pi_1(V_n) \arrow[l] & \pi_1(Gl^+(\mathbb{R}^k)) \arrow[l] & \arrow[l] \end{tikzcd}$$

So $\pi_0(V_n) = 0 \Rightarrow \pi_0(\tilde{Gl}_n) = 0$

or Simply, V_n-connected $\Rightarrow \tilde{Gl}_n$ connected

$\Rightarrow w_1(\tilde{B}) \neq 0 \Rightarrow w_1(\tilde{B}) = w_1(\tilde{\gamma}^n)$

But $H^*(B)$ generated by $w_1(\gamma^n), \dots, w_n(\gamma^n)$

$\Rightarrow 0 \longrightarrow H^i(B) \xrightarrow[\cup w_1]{} H^{i+1}(B) \longrightarrow H^{i+1}(\tilde{B}) \longrightarrow 0$

$H^{i+1}(\tilde{B})$ generated by images of $w_2, \dots, w_n(\gamma^n)$

$$\begin{tikzcd} H^{i}(E) \arrow[r] \arrow[d, phantom, "\wr"] & H^i(E_0) \arrow[d, phantom, "\wr"] \\ H^i(B) \arrow[r, "?"] & H^{i+1}(\tilde{B}) \end{tikzcd}$$

$\leadsto$ But image of $w_2(\gamma^n) = w_2(\tilde{\gamma}^n)$

So $H^*(\tilde{G}(\mathbb{R}^{n+k}))$ generated by

$w_2(\tilde{\gamma}^n), \dots, w_n(\tilde{\gamma}^n)$ and

$w_1(\tilde{\gamma}^n) = 0$.

$$\begin{tikzcd} E_0 \arrow[r, hook] \arrow[d] & E \arrow[d] \\ \tilde{B} \arrow[r, "?"] & B \end{tikzcd}$$

? is the projection map

12 - A) $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$ orientable $\iff$
$$\begin{tikzcd} E \arrow[r] \arrow[d] & \tilde{\gamma} \arrow[d] \\ B \arrow[r, "\exists f"] & \tilde{G} \end{tikzcd}$$

Since $w_1(\tilde{\gamma}) = 0$, Result follows.

(This implies

$\phi : H^k(B) \longrightarrow H^{k+n}(E, E_0)$

$x \longmapsto \pi^* x \cup u$

- $\phi^{-1} Sq^1 \phi(1) = 0$

$\Rightarrow \phi^{-1} Sq^1 u = 0$

ϕ being an isomorphism, we get

$\boxed{Sq^1 u = 0}$

Sq^1 is the Bockstein, $0 \to \mathbb{Z}/2 \xrightarrow[2]{} \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$

what does this mean?

- Also $w_1(E \oplus F) = w_1(E) \oplus w_1(F)$

$\Rightarrow$ $E \oplus F$ orientable iff both or none of E, F orientable

- For a manifold, by Wu's formula

$w_1 = v_1 = 0$

$\Rightarrow$ $Sq^1(x) = 0$ $\forall x \in H^{n-1}(M)$)

12-B) By 11-) $w_i(M) =$

$M^{(i)}$ - i^{th} cw skeleton of M.

$M^{(0)}$ - Assume single point.

- $TM|_{M^{(1)}}$ is trivial

Look at a
$$\begin{tikzcd} S^1 \arrow[r, hook, "i"] & M^{(1)} \arrow[r] & M \\ i^* TM \arrow[u] \arrow[r] & TM \arrow[u] \arrow[r] & TM \arrow[u] \end{tikzcd}$$

Then, $i^*(\underset{0}{\underset{\shortparallel}{w_1}} TM) = w_1(i_* TM)$

on S^1 a 3- bundle can ~~have~~ be

Mobius $\oplus$ Trivial or Trivial.

In first case $w_1 \neq 0$.

- $TM|_{M^{(2)}}$ is trivial

Let (D^2, ϕ) be a two cell in M, $\phi : \partial D^2 \longrightarrow M^{(1)}$ attaching map.

~~Because~~, $TM|_{M^{(1)}}$ is trivial,

So ~~mon~~ boundary of D^2 we have a trivial bundle i.e. this is bunelle on S^2

we need to check whe~~te~~ther bundle is trivial on D^2 rel ∂D^2 given that w_i's are all 0.

~~Es²~~ $\mathrm{Vect}^3_{\mathbb{R}}(S^2) = [S^1, SO(3)] = \pi_1(SO(3)) = \pi_1(\mathbb{R}P^3)$

$= \mathbb{Z}/2$

So, $\exists$ only 2 non-eq. $\mathbb{R}^3$ bundles on S^2.

Enough to show that $\exists$ bundle on S^2 with $w_2 \neq 0$.

Look at $\tilde{G}_3(\mathbb{R}^\infty)$ – cohomology generated by $\tilde{w}_2, \tilde{w}_3$

So $\pi_2(\tilde{G}_3) \otimes \mathbb{Z}/2 = \mathbb{Z}/2$, $(\because \pi_1 = 0)$

$\Rightarrow$ $S^3 \longrightarrow \tilde{G}_3$ iso. on $\pi_2 \otimes \mathbb{Z}/2$

Pull back $\tilde{\gamma}_3$. This will have $w_2 \neq 0$. $\square$

So we get that $TM|_{M^2} = 0$

- TM~~Pa~~ is trivial

$(D^3, \phi) \hookrightarrow M$

Bundle trivial on $\partial D^3 \Rightarrow$ bundle on S^3

~~Es³~~, $\mathrm{Vect}^3_{\mathbb{R}}(S^3) = \pi_2(\mathbb{R}P^3) = \pi_2(S^3) = 0$

$\Rightarrow$ Bundle trivial on D^3

$\boxtimes$

12-c) $\longrightarrow H^{i-1}(B) \xrightarrow{\cup w_1} H^i(B) \longrightarrow H^i(\tilde{B}) \longrightarrow H^{i+1}(B)$

$\tilde{B} = S^n \Rightarrow$ $H^{i+1}(B) \cong H^{i-1}(B) \cup w_1$.

~~$TM \downarrow M$~~

12.D)

$$\begin{tikzcd}\tilde{G}_n(\mathbb{R}^{n+k}) \arrow[d] \\ G_n(\mathbb{R}^{n+k})\end{tikzcd}$$

two covering $\Rightarrow$ $\tilde{G}_n$ is C^∞,

$\pi_1(G_n \mathbb{R}^{n+k}) = \mathbb{Z}/2$

$\Rightarrow \pi_1(\tilde{G}_n ~~\mathbb{R}~~) = 0$

$\Rightarrow$ $\tilde{G}_n$ orientable

$\tilde{G}_n$ quotient of $(G_n \times \mathbb{Z}/2)^{\alpha}$.

$\Rightarrow \tilde{G}_n$ compact. $\quad$ ↖ compact by Tychonoff

---

$$\phi : [b_1 \cdots b_n] \longmapsto \frac{b_1 \wedge b_2 \wedge \cdots \wedge b_n}{|b_1 \wedge \cdots \wedge b_n|}$$

Well defined:

$[c_1 \cdots c_n] = [b_1 \cdots b_n]$

Let $c_i = \sum \alpha_{ij} b_j$,

$\Rightarrow \quad c_1 \wedge \cdots c_n = \{\det \alpha\} \cdot b_1 \wedge \cdots \wedge b_n$

$$\Rightarrow \frac{c_1 \wedge \cdots \wedge c_n}{|c_1 \cdots \wedge c_n|} = \frac{b_1 \wedge \cdots \wedge b_n}{|b_1 \wedge \cdots \wedge b_n|} \cdot \text{sign} \det \alpha$$

But orientation $\Rightarrow$ sign det $\alpha = \pm 1$.

Injectivity:

easy

Smooth:

Near $x_0 \in \tilde{G}_n(\mathbb{R}^{n+k})$, with basis $e_1 \cdots e_n$,

choose $f_1 \cdots f_k$ ~~in~~ basis for $x_0^{\perp}$.

Then basis for x near x_0, chart is given by :

Choose a basis of x of the form

$v_i = e_i + f_i'$ where $f_i' \in x^{\perp}$.

then

$$\left[f_i'\right] \left[\quad T \quad\right] \left[\begin{matrix} e_1 \\ \vdots \\ e_n \end{matrix}\right][e_i] \quad T \in \mathrm{Hom}(x_0, x_0^{\perp})$$

i.e. $f_i' = Te_i$

Then, co-ordinates of x are ~~values~~ entries of T

So,

$$\Phi[v_1, \ldots, v_n] = \frac{(e_1+f_1') \wedge (\ldots) \wedge (e_n+f_n')}{|\qquad|}$$

in local co-ordinates

~~$= \wedge(e_i + T_{i1}e_1 + T_{i2}e_2 + \cdots + T_{in}e_n)$~~

~~$\wedge(e_i+$~~

$$= \frac{(e_1+Te_1) \wedge \cdots \wedge (e_n+Te_n)}{|\qquad|}.$$

$$= \frac{(e_1 + T_1^1 f_1 + \cdots + T_1^n f_n) \wedge \cdots \wedge (e_n + T_n^1 f_1 + \cdots + T_n^n f_n)}{|\qquad|}$$

which is smooth in T_i^j ?.

Q. 13 - A)

$J: E(\xi) \longrightarrow E(\xi)$ $\quad \xi$ - $2n$ dim $\mathbb{R}$ vector bundle

~~satisfies~~

$$\begin{tikzcd} \xi \arrow[d] \\ B \end{tikzcd} \quad U \subseteq B \quad \pi^{-1}(U) \cong U \times \mathbb{R}^{2n}$$

Then ~~$\pi^{-1}(U) \cong U \times \mathbb{C}^n$ via~~

Choose a basis for $\mathbb{R}^{2n}$:

$e_1, Je_1, \ldots, e_n, Je_n$

$$\begin{tikzcd}
\mathbb{R}^{2n} \arrow[r, "d"] & \mathbb{C}^n \\
a_1e_1+\dots+a_ne_n+b_1Je_1+\dots+b_nJe_n \arrow[r, mapsto] \arrow[d, "T"'] & (a_1+ib_1,\dots,a_n+ib_n) \arrow[d, "\tilde{T}"] \\
\mathbb{R}^{2n} \arrow[r, "\phi"] & \mathbb{C}^n
\end{tikzcd}$$

e_i'

is $\tilde{T}$ $\mathbb{C}$-linear ?

$\tilde{T}(a_1+ib_1,\dots,a_n+ib_n) =$ ~~$\phi(Ta_1, TJb_1, \dots)$~~

~~$= \phi(Ta_1, J\cdot Tb.$~~

$= \phi(Ta_1e_1 + T\cdot Jb_1e_1 + \dots + a_nTe_n + b_nTJe_n)$

$=$

$J\cdot Te_1 = T\cdot Je_1$

$\tilde{T}(a_i+ib_i) = \phi\cdot T\cdot\phi^{-1}(a_i+ib_i)$

$= \phi T(a_ie_i + b_iJe_i)$

$= a_i\phi T(e_i) + b_i\,\phi\cdot T\cdot J\cdot e_i$

$= a_i\cdot(\phi\cdot T)e_i + b_iJ(\phi\cdot T)e_i$

a_1, rest 0

$\tilde{T}(a_1) = a_1(\phi\cdot T)e_1$

$\tilde{T}(ia_1) = a_1 J\cdot(\phi\cdot T)e_1$

$= i\,a_1(\phi\cdot T)e_1 = i\,\tilde{T}(a_1)$

b_1, rest 0

$\tilde{T}(ib_1) = ib_1(\phi\cdot T)e_1$

$\tilde{T}(ib_1) = -b_1\cdot(\phi\cdot T)e_1$

$= i\,\tilde{T}(b_1)$

13 - B)

M - complex manifold

U, V charts
$\phi \;\; \psi$

$$\begin{tikzcd}
\varphi\circ\phi^{-1} : \phi(U\cap V) \arrow[r] & \cancel{z}\,\psi(U\cap V) \\
\mathbb{C}^n \arrow[u, phantom, sloped, "\subset"] & \mathbb{C}^n \arrow[u, phantom, sloped, "\subset"]
\end{tikzcd}$$

holomorphic.

on TM - charts

$\pi^{-1}U, \; \pi^{-1}V$
$\tilde{\phi}, \; \tilde{\psi}$

$$\tilde{\psi}\cdot\tilde{\phi}^{-1} : \tilde{\phi}(U\cap V) \longrightarrow \tilde{\psi}(U\cap V)$$

$$z_i \longmapsto \psi\cdot\phi_i^{-1}(z)$$

$$\frac{\partial}{\partial z_i} \longmapsto \left[D\psi\circ\phi_{e}^{-1}\left(\frac{\partial}{\partial z}\right)\right]_i$$

$f: M \to N$ holomorphic

$Df: TM \longrightarrow TN$
locally is (f, Df), hence holomorphic.

13. c)

$$\begin{tikzcd}
f: M \arrow[r] & \mathbb{C}
\end{tikzcd}$$

holo ↙ ↳ compact

f atains maxima say at p

By maximum modulus around p, f constant.

13. D)

$P^n(\mathbb{C})$ - $\mathbb{C}P^n$

Chart: $(z_0, \cdots, z_n) \longmapsto \left(\frac{z_1}{z_0}, \cdots, \frac{z_n}{z_0}\right)$ at $z_0 \neq 0$.

$G_n(\mathbb{C}^{n+k})$

Chart: for $x_0 \in G_n(\mathbb{C}^{n+k})$, $\mathbb{C}\langle e_1 \cdots e_n\rangle = x_0$, $\mathbb{C}\langle f_1 \cdots f_k\rangle = x_0^{\perp}$

$x \longmapsto$ if x = spanned by $e_1 + f_1', \cdots, e_n + f_n'$

and $f_i' = Te_i$

T.

Holomorphic?

$x_0, x_1 \in G_n(~~\mathbb{R}~~\mathbb{C}^{n+k})$

bases for x:

$e_{01}+f_{01}', \dots, e_{0n}+f_{0n}'$

$e_{11}+f_{11}', \dots, e_{1n}+f_{1n}'$

~~e~~ $f_{0i} = T_0 e_{0i}$

$f_{1i} = T_1 e_{1i}$

Then, $T_0 \longrightarrow T_1$ holo?

$B[e_{01} \cdots e_{1n}\; f_{01} \cdots f_{0k}] = [e_{\varphi 1} \cdots e_{\varphi n}\; f_{\varphi 1} \cdots f_{\varphi k}]$

$e_{1i}+f_{1i}' = e_{1i} + T_1 e_{1i}$

~~$e_{1i}+T_1$~~ $= Be_{0i} + T_1 Be_{0i}$

How to find T_1 in terms of T_0?

T_0, T_1 can be thought as in Hom (x, V)

Then we have the identities

$e_i = Be_{0i}$

f_{1i} ~~e_{1i}~~ $= B$ ~~e_{0i}~~ f_{0i}

$f_{0i}' = Te_{0i}$, $f_{1i}' = Te_{1i}$

B change of basis e_0, f_0 $\downarrow$ $e_1\; f_1$ in $V(n+k)$.

$$\langle e_{0i} + f_{0i}' \rangle = \langle e_{1i} + f_{1i}' \rangle$$

$$\langle (1+T_0)\, e_{0i} \rangle \qquad \langle (1+T_1)\, e_{1i} \rangle$$

$$(1+T_0)\langle e_{0i} \rangle \qquad (1+T_1) B \langle e_{0i} \rangle$$

$$\Rightarrow \quad T_1 = (1+T_0)\cdot B^{-1} - 1$$

which is holomorphic?

13-E)

$$\begin{tikzcd} \gamma_n^1 \arrow[d] \\ \mathbb{C}P^n \end{tikzcd}$$

- Need to show γ_n^1 ~~ho~~ does not have any ~~ex~~ holo. cross section. Need not be non-vanishing.

$$\begin{tikzcd} c: \mathbb{C}P^n \arrow[r] & \gamma_n^1 \subseteq \mathbb{C}P^n \times \mathbb{C}^{n+1} \arrow[d] \\ & \mathbb{C}^{n+1} \end{tikzcd}$$

cross section

compose →

c holomorphic, $\mathbb{C}P^n$ - compact ⇒ constant

But the only point common to all lines is 0.

- ~~$\mathbb{C}P^n$~~ $\mathrm{Hom}_{\mathbb{C}}(\gamma_n^1, \mathbb{C})$

section : $p_i: \mathbb{C}P^n \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\gamma_n^1, \mathbb{C})$

projection onto the i^{th} co-ordinate

$c_i([z_0 : \cdots z_n], (z_0, \cdots, z_n)) = z_i$

$\mathbb{C}$- linearly independant ? — $\sum \alpha_i c_i = 0$

$\Rightarrow \sum_i \alpha_i z_i = 0$. □

13.F) M – TM – $\mathrm{Hom}_{\mathbb{R}} TM =: T^*M$

$$T^*M \otimes \mathbb{C} \cong \underset{\substack{\| \\ \mathrm{Hom}_{\mathbb{C}}(T_{\bullet}M, \mathbb{C})}}{T^*_{\mathbb{C}}M} \oplus \overline{T^*_{\mathbb{C}}M} = \overline{\mathrm{Hom}_{\mathbb{C}}(TM, \mathbb{C})}$$

- $\mathrm{Hom}_{\mathbb{R}}(\xi, \mathbb{C}) = \mathrm{Hom}_{\mathbb{R}}(~~\xi~~ \mathbb{C}^n, \mathbb{C})$ fibrewise
  - Holo part of $f =$ ~~$f+f$~~ $\frac{f + i f(-i)}{2}$
  - Anti holo of $f = \frac{f - i(f \circ -i)}{2}$.

ex : $x \longmapsto \frac{x+iy}{2} + \frac{x-iy}{2}$

$\| $

$z/2 + \bar{z}/2$

- dz_i - holo? $dz_i = dx_i + i\, dy_i$

$dz_i\left(\frac{\partial}{\partial x_j}\right) = \delta_{ij}$ $\qquad dz_i\left(\frac{\partial}{\partial y_j}\right) = i\, \delta_{ij}$

Change of variable :

$z_i \longrightarrow w_i$

$dz_i \longrightarrow dw_i$

Jacobian : $\left[\frac{\partial w_i}{\partial z_i}\right]$ holomorphic

Chern - Classes

$$\begin{tikzcd} \xi \arrow[d, "\pi"'] \\ B \end{tikzcd}$$

complex n- vector bundle

$c_n(\xi) = e(\xi)$ ← euler class.

Construct a vector bundle on "$E(\xi)_0$". fibre at a point $v \in \pi^{-1}(x)$ is $v^{\perp}$ where some hermitian inner product is given

Call this ξ_0.

$$\begin{tikzcd} \xi_0 \arrow[d] \\ E(\xi)_0 \end{tikzcd}$$

$n-1$ dim complex vector bundle

Then $c_{n-1}(\xi_0) := c_{n-1}(\xi_0) = e(\xi_0)$ pushed forward to $H^{2n-2}(B)$. via

i.e. $H^i(E(\xi)_0) \longleftrightarrow H^i(B)$

This can be done as in the Gysin seq

$H^i(E, E_0) \cong H^{i 2n}(B) = 0$ for $i < 2n$.

- Grassmannian

$$H^*(G_n(\mathbb{C}^\infty)) = \mathbb{Z}[c_1(\gamma^n), \dots, c_n(\gamma^n)]$$

- $c(\omega \oplus \phi) = c(\omega) \cdot c(\phi)$

- $c(\bar{\omega}) = 1 - c_1(\omega) + c_2(\omega) - \cdots$

- $\omega_1(\gamma^1_n) =$ - generator of $H^2(\mathbb{C}P^n) = -x$

- $c_n(T\mathbb{C}P^n) = e(T\mathbb{C}P^n) = (n+1)(-x)^n$.