Classification of surfaces

Theorem 0.1. *Every compact Riemann surface is homeomorphic(=topologically isomorphic) to a surface of genus g.*

Figure 1. Torus, genus 2 surface, higher genus surfaces

Upto isomorphism, there is exactly one Riemann surface of genus 0, namely the Riemann sphere $\mathbb{P}^1$.

The genus can be thought of as the number of "handles" in the surface.

Word of caution: Two Riemann surfaces being homeomorphic does not mean that they are isomorphic as Riemann surfaces. We will show that every elliptic curve is homeomorphic to the genus 1 surface (=torus) but not all elliptic curves are isomorphic to each other as Riemann surfaces.

Euler characteristic

It is possible to compute the genus using triangulations of surfaces. For any triangulation of a genus g surface M with V vertices, E edges, and F faces, we have the following identity:

$$2 - 2g = V - E + F.$$

The important point here is that the right-hand side depends on the triangulation but the left-hand side does not. The right-hand is called the Euler characteristic of M, denoted $\chi(M)$.

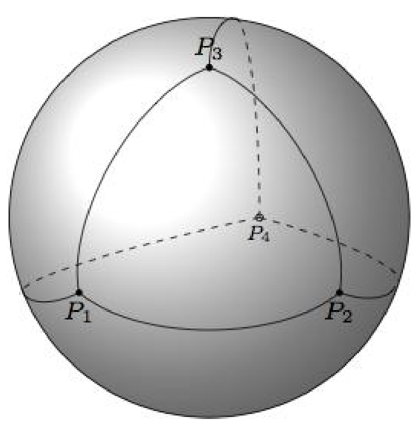
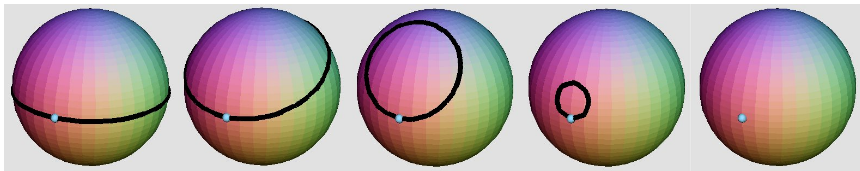


Figure 2. Triangulation of S^2 (genus=0) with $V = 4$, $E = 6$, and $F = 4$ so that $V - E + F = 2 = 2 - 2 \cdot 0$.

Coverings of simply-connected spaces

A topological space Y is said to be *simply-connected* if it is connected and every loop in Y can be continuously contracted to a point.



For example, the Riemann sphere $\mathbb{P}^1$ is simply-connected. If we remove a single point or a single path-connected set from $\mathbb{P}^1$, the resulting space is still simply-connected.

But if Y is the space obtained by removing two or more points from the Riemann sphere, then Y is not simply-connected.

This is the only example that concerns us.

Theorem 0.2. *If $f : X \to Y$ is a continuous $n : 1$ (covering) map and Y is simply-connected then X is topologically isomorphic to n disjoint copies of Y.*

For this theorem we do not allow any ramification points, so f needs to a genuine $n : 1$ (covering) map.