– Val

Extended TQFT:

dim	assignment
n	number
$n-1$	vector space
$n-2$	2 vector space

Higher Algebras:

$V_0 := \mathbb{C}$

$V_1 :=$ { finite dim inner product spaces over $\mathbb{C}$ }

$V_2 :=$ { " over V_1 }

Def^n (2 vector space) : $W \in V_2$, is a category with functors

$$W \times W \to W$$
$$W \times V_1 \to W \qquad + \text{ many more conditions}$$
$$W \times \overline{W} \to V_1$$

Can similarly define V_K.

Invariant Sections / limits

- C finite groupoid $\quad \pi_0, \pi_1$ are finite
- $F : C \to V_1$

Def: $\mathcal{V}_F = \{ \{ s_x \in F(x) \}_{x \in C} \mid \forall f \in \hom_C(x,y) \quad F(f) s_x = s_y \}$

Remark: If C is connected, $\forall x$. $\dim F(x) = 1$

$\forall f \in \hom(x,y) \quad F(f) = id$

then $\dim \mathcal{V}_F = 1$.

$\to \mathcal{V}_F$ is the limit of F.

- If $\forall x \in C \quad F(x) = D \in V_1$, $\quad \mathcal{V}_F = \left\{ \begin{matrix} \text{functors} \\ C \to D \end{matrix} \right\}$ $\quad$ (only identity morphism in D)

Classical Action:

- G finite group
- $\mathbb{T} = U(1) \subseteq \mathbb{C}$
- $\alpha \in C^{n}(BG; \mathbb{T})$ cocycle (twisting)
- All manifolds compact, oriented
- For X-manifold, C_X groupoid of G-bundles over X.
- Action Z

1) $\dim X = n \qquad \partial X = \phi \qquad P \in C_X \quad P: X \longrightarrow BG$

$$Z(P) = \langle P^{*}\alpha , [X] \rangle$$

2) $\dim X = n-1, \quad \partial X = \phi \quad , p \in C_X : X \longrightarrow BG$

Define a category D_X^P : Ob : $\left\{ (x, f) \;\middle|\; \begin{array}{ll} x \in C_{n-1}(X) & \text{s.t } [x] = [X] \\ f : X \to BG & \text{representing } P \end{array} \right\}$

$\mathrm{hom}((x_1, f_1), (x_2, f_2)) = \left\{ \begin{array}{l} \text{homotopy classes of homotopies} \\ \text{between } f_1, f_2 \end{array} \right\}$

Define : $D_X^P \longrightarrow \mathcal{V}_1 \quad : F_X^P$

$(x, f) \longmapsto \mathbb{C}$

$h : (x_1, f_1) \longrightarrow (x_2, f_2) \longmapsto \langle h^{*}\alpha, \omega \rangle$

where $\omega \in C_n(X \times [0,1])$, $[\omega] \in H_n(X \times [0,1], \partial(X \times [0,1]))$

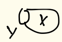
$\partial\omega = x_1 - x_2$

$$Z(P) := \mathcal{V}_{F^P} \in \mathcal{V}_1$$

3) $\dim X = n-2, \quad \partial X = \phi, \quad p \in C_X$

$$Z(P) \in \mathcal{V}_2$$

4) $\dim X = n, \quad \partial X = Y, \quad P \in C_X, \quad \partial P = Q \in C_Y$



$$Z(P) \in Z(Q)$$

Choose $x \in C_n(X)$, $[x] \in H_n(X, \partial X)$ fundamental class

Then $(\partial Y, f_P|_Y)$ trivializes $Z(Q)$.

$$Z(P) := \langle f^{*}\alpha , x \rangle \in \mathbb{C} \cong_{(\partial x_1, f_P|_Y)} Z(Q)$$

Assertions: functoriality, orientation, additivity, Gluing

$$(P \to P') \;\downarrow\; Z(P) \to Z(P')$$

$$Z_{-X}(P) = Z_X(P)$$

$$Z_{X_1 \sqcup X_2}(P_1 \sqcup P_2) = Z_{X_1}(P_1) \otimes Z_{X_2}(P_2)$$

$$\operatorname{Tr}\left(Z_{X_1}(P_1) \underset{Y}{,} Z_{X_2}(P_2)\right) = Z_{X_1 \underset{Y}{\sqcup} X_2}(P_1 \sqcup P_2)$$

Quantizations:

- Define TQFT by "summing" classical action
- $\dim X = n-k$, $\partial X = \phi$, $P \in C_X$, $[P] \subset C_X$

$$Z([P]) = e_{i??}\, Z|_{[P]} \in \mathcal{V}_k$$

- Quantized action E:

$$E(X) = \bigoplus_{[P] \in C_X} \frac{1}{\# \operatorname{Aut} P} Z([P])$$

- $\partial X = Y$ $\qquad E(X) \in E(Y)$

for $Q \in C_Y$ $\quad C_X(\Theta) = \{(P, \Theta) \mid P \in C_X, \Theta \cdot \partial P = Q\}$

$$Z : C_X(Q) \longrightarrow C_X(Q)$$

for $P \in C_X(Q)$ $\qquad Z([P]) = \lim Z|_{[P]} \in Z(Q)$

$$J(Q) = \sum_{[P] \in C_X(Q)} \frac{1}{\# \operatorname{Aut} P} Z[P] \in Z(Q)$$

Claim: $J(Q)$ is invariant i.e. $J([Q]) \in Z([Q])$

$$E(X) := \bigoplus_{[Q] \in \overline{C}_Y} \frac{1}{\operatorname{Aut} Q} J(Q)$$

Examples:

$n=2 \qquad d=0$,

- $C_{S^1} = G /\!/ G$
- $E(S^1) = \bigoplus_{[g] \in G/\!/G} \frac{1}{|C(g)|} \cdot \mathbb{C}$ = class functions

- $C_{pt} \cong G$
- $E(pt) = \frac{1}{|G|} \mathcal{V}^G$ where $\mathcal{V}^G$ = finite dimensional unitary representations of G.

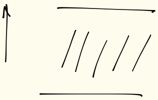
$= \{$ functors from $C_{pt} \to \mathcal{V}_1 \}$

dim $n=3$:

- $E(S^1)$ is a modular tensor category

Dan Freed:

Physics: time

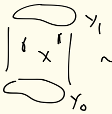


t_1 $\qquad \rightsquigarrow \qquad$ $\mathcal{H}_{t_1}$

$\uparrow U_X$

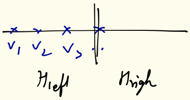
t_0 $\qquad\qquad$ $\mathcal{H}_{t_0}$

Topology:



$\rightsquigarrow$ $F(Y_1) \xleftarrow{\ F(X)\ } F(Y_0)$

$$F(Y_1)$$
$$\uparrow F(X)$$
$$F(Y_0)$$



$\mathcal{H} = \otimes V_i$

$\mathcal{H} = \mathcal{H}_{left} \otimes \mathcal{H}_{right}$

$\rightsquigarrow$ v_1 v_2 $\qquad$ $v_1 \cdot v_2$

$C_{left} \otimes C^{\vee}_{right}$ $\qquad$ categories instead of Hilbert spaces