Manifold Calculus and Convex Integration

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Introduction

Goal

Study *embedding spaces* in *symplectic geometry*
using *manifold calculus* and *convex integration*.

Embedding spaces	space of embeddings of one manifold inside another
Symplectic geometry	connections to mathematical physics, algebraic geometry, and algebraic topology, several open conjectures
Manifold calculus	relatively new homotopy theoretic technique for studying embedding spaces
Convex integration	perturbative technique for solving differential relations

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Examples of embeddings: Knots

Knot	Embedding of S^1 inside $\mathbb{R}^3$
Classical Knot theory	Study of path connected components of $\mathrm{Emb}(S^1, \mathbb{R}^3)$

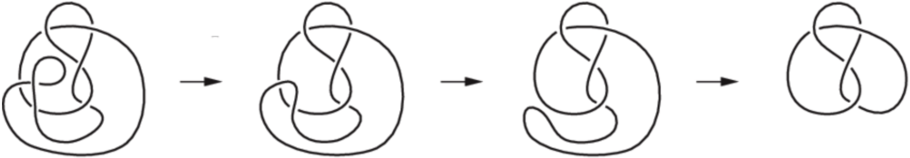

Figure: Example of a path inside $\mathrm{Emb}(S^1, \mathbb{R}^3)$.

Homotopy-theoretic point of view

Knot — Embedding of S^1 inside $\mathbb{R}^3$

~~Classical~~ Knot theory — Study of ~~path connected components of~~ $\mathrm{Emb}(S^1, \mathbb{R}^3)$

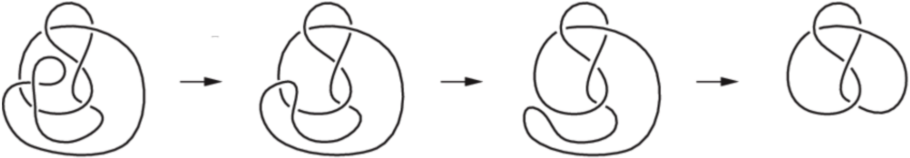

Figure: Example of a path inside $\mathrm{Emb}(S^1, \mathbb{R}^3)$.

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Symplectic manifolds

We are interested in studying certain embedding spaces arising in symplectic geometry.

Definition

A *symplectic manifold* is a pair (N, ω) where

- N is a smooth manifold of dimension $n = 2m$,
- ω is a closed 2 form,
- ω^m is nowhere vanishing.

Example: $N = \mathbb{R}^{2m}$, $\omega = dx_1 dy_1 + \cdots + dx_m dy_m$.
Locally every symplectic manifold is of this form.

Symplectic manifolds

- Example: the cotangent bundle T^*M of a smooth manifold M is naturally a symplectic manifold.
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M smooth manifold	Position space
T^*M cotangent bundle	Phase space

  Generalizes the notion of phase space.
- Natural framework for Hamiltonian and Lagrangian mechanics.

Lagrangian submanifolds

Definition

A submanifold M of N is called *Lagrangian* if $\dim M = \dim N/2$ and ω vanishes on M.

Example: M is a Lagrangian submanifold of T^*M.

- Lagrangian submanifolds are the main objects of study in symplectic geometry.
- They naturally arise when studying Hamiltonian flows and quantization of classical systems.
- There are several conjectures about the space of Lagrangian submanifolds.

Nearby Lagrangian conjecture

Arnold's nearby Lagrangian conjecture

Let M and L be closed simply connected manifolds of the same dimension. The space of Lagrangian embeddings of L inside T^*M is connected if L is diffeomorphic to M, is empty otherwise.

$$\pi_0 \operatorname{Emb}_{Lag}(L, T^*M) \simeq \begin{cases} * & \text{if } L \cong M \\ \phi & \text{otherwise} \end{cases}$$

Goal

Study *Lagrangian embeddings* using *manifold calculus* and *convex integration*.

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Manifold calculus

$\mathrm{Man_m} :=$ category of smooth manifolds of a *fixed dimension m*, with morphisms being open embeddings, (enriched over topological spaces).

$\mathrm{Emb}(M, N) :=$ space of embeddings of M inside N.

$\mathrm{Emb}(-, N)$ defines a functor

$$\begin{aligned} \mathrm{Emb}(-, N) : \mathrm{Man_m}^{op} &\to \mathrm{Top} \\ M &\mapsto \mathrm{Emb}(M, N) \end{aligned}$$

Manifold calculus

More generally,

Objects of interest in manifold calculus

Functors on $\mathrm{Man_m}^{op}$ valued in topological spaces

$$F : \mathrm{Man_m}^{op} \to \mathrm{Top}\,.$$

We analyze $F(M)$ by evaluating $F(-)$ on all the discs inside of M ...

Gluing discs

$\mathcal{D}\mathrm{isc}_\infty \subseteq \mathrm{Man_m}$: full subcategory of $\mathrm{Man_m}$ whose objects are manifolds diffeomorphic to disjoint union of finitely many open discs

Manifold Calculus (Goodwillie-Weiss, Boavida-Weiss)

Try to recover F from its restriction to $\mathcal{D}\mathrm{isc}_\infty$.

Analytic approximation

Definition (Analytic approximation of $F : \mathsf{Man_m}^{op} \to \mathsf{Top}$)

$$T_\infty F := \text{ right derived Kan extension of } F \text{ along } \mathcal{D}\mathsf{isc}_\infty{}^{op} \hookrightarrow \mathsf{Man_m}^{op}$$

$$\begin{tikzcd}
\mathcal{D}\mathsf{isc}_\infty{}^{op} \arrow[rr, "F|_{\mathcal{D}\mathsf{isc}_\infty{}^{op}}"] \arrow[d] & \arrow[d, phantom, "\Uparrow"] & \mathsf{Top} \\
\mathsf{Man_m}^{op} \arrow[urr, "T_\infty F"'] & &
\end{tikzcd}$$

Analyticity of the embeddings functor

Definition (Analytic functor)

F is **analytic** if $F(M) \simeq T_\infty F(M)$ for all M.

- $T_\infty F(M)$ is the space "reconstructed" out of the discs inside M.
- Smooth $\mathrm{Maps}(-, N)$ is always analytic.
- $\mathrm{Imm}(-, N)$ is analytic, if $n - m > 0$.

Theorem (Goodwillie-Weiss, Goodwillie-Klein)

If $n - m > 2$, then the functor $\mathsf{Emb}(-, N)$ *is analytic.*

Why analytic functors?

- $T_\infty F(M)$ can be approximated by a series of spaces

$$T_1F(M) \longleftarrow T_2F(M) \longleftarrow T_3F(M) \longleftarrow \dots$$

- When F is analytic, these spaces provide approximations for $F(M)$ itself.
- The (homotopy) fibers of $T_kF(M) \leftarrow T_{k+1}F(M)$ have relatively simple descriptions.

Lagrangian embeddings

Question Theorem (N.)

What can we say about Lagrangian embeddings When $n - m > 2$ there is a homotopy equivalence,

$$T_\infty \operatorname{Emb}_{Lag}(-, N) \simeq ?\, \operatorname{Emb}_{TR}(-, N)$$

where $\operatorname{Emb}_{TR}(M, N)$ is the space of totally real embeddings.

- All Lagrangian submanifolds are totally real submanifolds.
- The space of totally real submanifolds is much larger than the space of Lagrangian submanifolds.
- T_∞ 'expands' the space of Lagrangian submanifolds.

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Directed embeddings

Theorem (N.)

When $n - m > 2$ there is a homotopy equivalence,

$$T_\infty \operatorname{Emb}_{A}(-, N) \simeq \operatorname{Emb}_{A}(-, N)$$

where A satisfies h-principle for directed embeddings.

- A is a condition on the submanifolds - like being a Lagrangian submanifold, or being a totally real submanifold.
- A is a collection of m dimensional subspaces of TN.
- An embedding $e : M \hookrightarrow N$ is in $\operatorname{Emb}_{A}(M, N)$ if the image of De lands inside A.

h-principle for directed embeddings

We say that A satisfies h-principle for directed embeddings if the following holds.

An arbitrary embedding e + tangential homotopy connecting De to A can be perturbed to an embedding e' in $\mathrm{Emb}_A(M, N)$.

This is done using Gromov's technique of *convex integration*.

$$\begin{array}{c} e \in \mathrm{Emb}(M, N) + \text{ tangential homotopy } De \leftrightarrow A \\ \Big\downarrow \text{Convex integration} \\ e' \in \mathrm{Emb}_A(M, N) \end{array}$$

Proof sketch

- This perturbation (using convex integration) commutes with T_∞

$$\implies T_\infty \operatorname{Emb}_A(M, N) \simeq \operatorname{Emb}_A(M, N).$$

- TR satisfies h-principle for directed embeddings

$$\implies T_\infty \operatorname{Emb}_{TR}(M, N) \simeq \operatorname{Emb}_{TR}(M, N).$$

- $\operatorname{Emb}_{Lag}(-, N)$ and $\operatorname{Emb}_{TR}(-, N)$ agree on discs

$$\implies T_\infty \operatorname{Emb}_{Lag}(M, N) \simeq T_\infty \operatorname{Emb}_{TR}(M, N)$$

Future directions

- Manifold calculus for manifolds with a group action (work in progress).
- Other applications of convex integration to homotopy theory.
- Modify manifold calculus to incorporate more information about symplectic geometry.

Thank you!