Honors Single Variable Calculus

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1 Proofs

We'll start by learning about the various kinds of proofs that we'll encounter in this class.[1]

Proofs are the heart of mathematics. You must come to terms with proofs – you must be able to read, understand and write them. What is the secret? What magic do you need to know? The short answer is: there is no secret, no mystery, no magic. All that is needed is some common sense and a basic understanding of a few trusted and easy to understand techniques.

The Structure of a Proof

The basic structure of a proof is easy: it is just a series of statements, each one being either

- An assumption or
- A conclusion, clearly following from an assumption or previously proved result

And that is all. Occasionally there will be the clarifying remark, but this is just for the reader and has no logical bearing on the structure of the proof.

A well written proof will flow. That is, the reader should feel as though they are being taken on a ride that takes them directly and inevitably to the desired conclusion without any distractions about irrelevant details.

Each step should be clear or at least clearly justified. A good proof is easy to follow. When you are finished with a proof, apply the above simple test to every sentence: is it clearly

1. an assumption
2. a justified conclusion?

If the sentence fails the test, maybe it doesn't belong in the proof.

Example

In order to write proofs, you must be able to read proofs. See if you can follow the proof below. Don't worry about how you would have (or would not have) come up with the idea for the proof. Read the proof with an eye towards the criteria listed above. Is each sentence clearly an assumption or a conclusion? Does the proof flow? Was the theorem in fact proved?

Theorem 1.1. *The square root of 2 is an irrational number.*

Proof. Let's represent the square root of 2 by s. Then, by definition, s satisfies the equation

$$s^2 = 2.$$

If s were a rational number, then we could write $s = p/q$ where p and q are a pair of integers. In fact, by dividing out the common multiple if necessary, we may even assume p and q have no common multiple (other than 1). If we now substitute this into the first equation we obtain, after a little algebra, the equation

$$p^2 = 2q^2.$$

But now, 2 must appear in the prime factorization of the number p^2 (since it appears in the same number $2q^2$). Since 2 itself is a prime number, 2 must then appear in the prime factorization of the number p. But then, $2 \cdot 2$ would appear in the prime factorization of p^2, and hence in $2q^2$. By dividing out a 2, it then appears that 2 is in the prime factorization of q^2. Like before (with p^2) we can now conclude 2 is a prime factor of q. But now we have p and q sharing a prime factor, namely 2. This violates our assumption above (see if you can find it) that p and q have no common multiple other than 1. □

[1] This section in taken almost verbatim from http://zimmer.csufresno.edu/~larryc/proofs/proofs.html.

1.1 Direct Proofs

Most theorems that you want to prove are either explicitly or implicitly in the form

"If P, then Q".

This is the standard form of a theorem (though it can be disguised). A direct proof should be thought of as a flow of implications beginning with P and ending with Q.

$$P \implies \cdots \implies Q$$

Most proofs are (and should be) direct proofs. Always try direct proof first, unless you have a good reason not to. If you find a simple proof, and you are convinced of its correctness, then don't be shy about. Many times proofs are simple and short.

Exercise. 1.2. Prove each of the following.

1. If a divides b and a divides c then a divides $b+c$, where a, b, and c are positive integers.

2. For real numbers a and b, $a^2 + b^2 \geq 2ab$.

3. If a is a rational number and b is a rational number, then $a+b$ is a rational number.

1.2 Proof by Contradiction

In a proof by contradiction we assume, along with the hypotheses, the logical negation of the result we wish to prove, and then reach some kind of contradiction. That is, if we want to prove "If P, then Q", we assume P and Not Q. The contradiction we arrive at could be some conclusion contradicting one of our assumptions, or something obviously untrue like 1 = 0. The proof of Theorem 1.1 is an example of this.

Exercise. 1.3. Use the method of Proof by Contradiction to prove each of the following.

1. The cube root of 2 is irrational.

2. If a is a rational number and b is an irrational number, then $a+b$ is an irrational number.

3. There are infinitely many prime numbers.[1]

1.3 Proof by Contrapositive

Proof by contrapositive takes advantage of the logical equivalence between "P implies Q" and "Not Q implies Not P". For example, the assertion "If it is my car, then it is red" is equivalent to "If that car is not red, then it is not mine". So, to prove "If P, then Q" by the method of contrapositive means to prove

"If Not Q, then Not P".

How Is This Different From Proof by Contradiction? The difference between the Contrapositive method and the Contradiction method is subtle. Let's examine how the two methods work when trying to prove "If P, then Q".

Method of Contradiction: Assume P and Not Q and prove some sort of contradiction.

Method of Contrapositive: Assume Not Q and prove Not P.

The method of Contrapositive has the advantage that your goal is clear: Prove Not P. In the method of Contradiction, your goal is to prove a contradiction, but it is not always clear what the contradiction is going to be at the start.

[1] There are dozens of proofs of this theorem, originally due to Euclid. Feel free to look one up online.

Exercise. 1.4. Use the method of Proof by Contrapositive to prove each of the following.

1. If the product of two integers is even, then at least one of the two must be even.

2. If the product of two integers is odd, then both must be odd.

3. If the product of two real numbers is an irrational number, then at least one of the two must be an irrational number.

1.4 Converse

The converse of an assertion in the form "If P, then Q" is the assertion

"If Q, then P".

A common logical fallacy is to assume that if an assertion is true then so is it's converse.

1.4.1 If and Only If

Many theorems are stated in the form "P, if, and only if, Q". Another way to say the same thing is: "Q is necessary, and sufficient for P". This means two things:

"If P, Then Q" and "If Q, Then P".

So to prove an "If, and Only If" theorem, you must prove the theorem and also it's *converse*.

Exercise. 1.5. Go through the problems of the previous sections and find the ones which are of the form "If P, Then Q". For each of these:

1. State the converse.

2. Prove or disprove the converse (by providing either a proof or a counterexample).

3. For the problems where the converse is also true rewrite the assertion as an "If, and Only If" statement.

2 For all and There exists

Very often in mathematics we are required to make statements which are non-constructive (we'll do this a lot throughout the course). This is done using two *quantifiers*:

For all/every **There exists**

They mean exactly what you expect them to mean. However, understanding and formulating complex logical expressions out of these takes some getting used to. Here are a few examples,

Example 2.1.

1. *For every* odd integer a, the integer $a + 1$ is even.

2. An integer a is even if and only if *there exists* an integer b such that $a = 2b$.

3. There *do not exist* integers p, q such that $\frac{p}{q} = \sqrt{2}$.

4. *For every* non-zero rational number x *there exists* a rational number y such that $x \cdot y = 1$.

2.1 Order Matters

When parsing logical expressions the order really matters and changing it changes the meaning of the statement completely.

Exercise. 2.2. Explain the difference between the meanings of the following pairs of sentences.

1. (a) *For every* even integer a *there exists* an integer b such that $a = 2b$.
   (b) *There exists* an integer b such that *for every* even integer a, $a = 2b$.
2. (a) *For every* positive integer n *there exists* a positive real number ϵ such that $\epsilon < 1/n$.
   (b) *There exists* a positive real number ϵ such that *for every* integer n, $\epsilon < 1/n$.[1]
3. (a) *For every* even integer a and *for every* odd integer b, $a + b$ is odd.
   (b) *For every* odd integer b and *for every* even integer a, $a + b$ is odd.
4. * Let $f : \mathbb{R} \to \mathbb{R}$ be a function.
   (a) *For every* x and *every* $\epsilon > 0$ *there exists* a $\delta > 0$ such that *for every* y, $|x - y| < \delta$ implies that $|f(x) - f(y)| < \epsilon$.
   (b) *For every* $\epsilon > 0$ *there exists* $\delta > 0$ such that *for every* y, $|x - y| < \delta$ implies that $|f(x) - f(y)| < \epsilon$.

2.2 Negating Expressions

To negate a nested logical expression you start at the outermost expression and recursively move inwards using the following rules called **De Morgan's laws**,

Not (For every x ...)	= There exists an x such that *not* (...)
Not (There exists an x such that ...)	= For every x *not* (...)
Not (P or Q)	= *Not* P and *not* Q
Not (P and Q)	= *Not* P or *not* Q

Exercise. 2.3. Negate each of the following.

1. *For every* even integer a *there exists* an integer b such that $a = 2b$.
2. *There exists* an integer b such that *for every* even integer a, $a = 2b$.
3. *For every* positive integer n *there exists* a positive real number ϵ such that $\epsilon < 1/n$.
4. *There exists* a positive real number ϵ such that *for every* integer n, $\epsilon < 1/n$.
5. *For every* even integer a and *for every* odd integer b, $a + b$ is odd.
6. *For every* x and *every* $\epsilon > 0$ *there exists* a $\delta > 0$ such that *for every* y, $|x - y| < \delta$ implies that $|f(x) - f(y)| < \epsilon$.
7. *For every* $\epsilon > 0$ *there exists* $\delta > 0$ such that *for every* y, $|x - y| < \delta$ implies that $|f(x) - f(y)| < \epsilon$.

[1] Such an ϵ is called an **infinitesimal**. There are number systems, for example the **hypperreal numbers**, which extend the real numbers by incorporating infinitesimals.