Hopf Algebra

Def: $\mathcal{H} := (H, \mu, u, \Delta, c)$ is a Hopf Algebra if (H, μ, u) is a k-algebra,
$\mu : H \otimes H \longrightarrow H$, $u : k \longrightarrow H$ AND
(H, Δ, c) is a k-module which is a k-coalgebra i.e.
$\Delta : H \longrightarrow H \otimes H$ and $c : H \longrightarrow k$ satisfying,
k-linear

a) coassociativity : $(id \otimes \Delta)\Delta = (\Delta \otimes id) \cdot \Delta$
b) co-unit : $(id \otimes c) \cdot \Delta = id = (c \otimes id) \cdot \Delta$

Further H satisfies :
a) μ is a coalgebra map (equivalently Δ is a algebra map)
b) u is a co-algebra map
c) c is an algebra map
d) $c \cdot u : k \longrightarrow k$ is id_k

Tensor co-algebra (cotensor algebra)

$V = k$- module

$T(V) := k \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \cdots$ has 2 distinct natural algebra structures

a) $(v_1 \otimes \cdots \otimes v_n) \otimes (\omega_1 \otimes \cdots \otimes \omega_m) = (v_1 \otimes \cdots \otimes v_n \otimes \omega_1 \otimes \cdots \otimes \omega_m)$ ⟵ Tensor Algebra

$(T(V), \Delta)$ coalgebra

$$\Delta(v_1 \otimes \cdots \otimes v_n) := \sum_{p=0}^{n} (v_1 \otimes \cdots \otimes v_p) \otimes (v_{p+1}, \ldots, v_n)$$

Both non-commutative

b) $(T'(V), \Delta)$ as above

$\mu : T'(V) \otimes T'(V) \longrightarrow T'(V)$

$$(v_1 \otimes \cdots \otimes v_p) \otimes (v_{p+1} \otimes \cdots \otimes v_{p+q}) \longmapsto \sum_{\sigma\text{-}(p,q)\text{ shuffle}} \sigma(v_1 \otimes \cdots \otimes v_{p+q})$$

σ-(p,q) shuffle $\in S_{p+q}$ satisfy
$\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(p+q)$

Now we get a commutative Hopf Algebra

Q: Why do we have commutativity ?

Def: H - commutative Hopf algebra, $f, g : H \longrightarrow H$ k-linear maps

Convolution: $f * g := \mu \cdot (f \otimes g) \cdot \Delta$

Note: If f is an algebra morphism then $f \circ (g * h) = (f \circ g) * (f \circ h)$

Remark: H commutative and f, g algebra morphism $\Rightarrow f * g$ is algebra morphism.
In particular, if $f = id$ then f^{*n} is an algebra morphism

From this we get, $Id^{*n} \circ Id^{*n'} = Id^{*(nn')}$

Note: $*$ is associative with $u \circ c$ being the neutral element.
hence $(End_k H, +, *)$ becomes a k-algebra.

Assume H graded $H = \bigoplus_{i \geq 0} H_i$, $\mu(H_i \otimes H_j) \subseteq H_{i+j}$, $1_H \subseteq H_0$
Graded twisting map $T(a \otimes b) := (-1)^{|a||b|} b \otimes a$, commutative if $\mu \cdot T = \mu$
Augementation: $\varepsilon : H \longrightarrow k = H_0$
Connectedness: $k \cong H_0$ via map $\lambda \longmapsto \lambda \cdot 1_H$

$H_0 = k$, consider only deg 0 maps
Suppose $H \xrightarrow{f} H$ $\quad f(\underset{k = H_0}{\underset{\cap}{1}}) = 0$ then $f^{*k}\big|_{H_n} = 0$ if $n < k$ — $(*)$

Look at the series:

$$e'(f) := \log(uc + f)$$
$$:= f - \frac{f^{*2}}{2} + \dots + (-1)^{i+1} \frac{f^{*i}}{i} + \dots$$

In view of $(*)$,

$$e'(f)|_{H_n} := f - \frac{f^{*2}}{2} + \dots + (-1)^{n+1} \frac{f^{*n}}{n}$$

$$e^i(f) := \frac{(e^{(1)} f)^{*i}}{i!} \qquad e^{(0)}(f) := uc$$

well defined as a polynomial

$e_n^{(i)}(f) \in End_k(H_n)$ $\qquad e_0^{(1)}(f) = 0$ and $e_n^{(i)}(f) = 0$ for $i > n$

$\exp(f) := uc + f + f^{*2}/2! + \cdots + f^{*n}/n!$

$\Rightarrow \quad (uc+f)^{*n} = \exp(n \log(uc+f)) = uc + \sum_{i\geq 1} k^i e^{(i)}(f)$

Set $\quad f = Id - uc, \qquad e^{(i)} := e^{(i)}(Id - uc)$

$$Id^{*n}\Big|_{H_m} = \sum_{i=1}^{m} n^i e_m^{(i)} \qquad \text{for } m > 0$$

Prop: Commutative graded Hopf algebra $H/k \supset \mathbb{Q}$ we have

a) $Id\,|_{H_n} = e_n^{(1)} + \cdots + e_n^{(n)}$

b) $e_n^{(i)} \cdot e_n^{(j)} = \delta_{ij}\, e_n^{(i)}$

Prop: For any CG Hopf algebra $/k \supseteq \mathbb{Q}$ the elements $e_n^{(i)} = e_n^{(i)}(Id - uc) \in End_k(H_n), \forall n \geq 1$

a) $Id|_{H_n} = e_n^{(1)} + \cdots + e_n^{(n)}$

b) $e_n^{(i)} \cdot e_n^{(j)} = 0$ if $i \neq j$ and $e_n^{(i)} \cdot e_n^{(i)} = e_n^{(i)}$

Proof: a) follows from $(Id^{*k})|_{H_n} = \sum_{i=1}^{n} k^i e_n^{(i)}$

b) Fix $n \geq 1$. Consider the n equations,

$$(Id^{*k})|_{H_n} = \sum_{i=1}^{n} k^i e_n^{(i)} \qquad \text{for } k = 1, 2, \ldots, n$$

We have matrix of co-efficients which will be Vandermonde which is invertible

So that $e_n^{(i)}$ are completely determined by $Id^{*k}|_{H_n}$

$(Id^{*k}) \circ (Id^{*k'}) = Id^{*kk'} \Rightarrow \exists!$ formula of the form $e_n^{(i)} \cdot e_n^{(j)} = \sum_{m=1}^{n} a_{ij}^m e_n^{(m)}$

expanding we get $\displaystyle\sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} k^i (k')^j a_{ijm} = (k k')^m$

One can show that this then implies b)