Factorization algebras & functorial field theories

Th^m: A factorization algebra on G manifolds determines $_{\text{(twisted)}}$ functorial field theory.

digression: Factorization algebra on G-manifolds:

def^n: Prefactorization algebra :
- $U \longrightarrow A(U)$ ∞-chain complex; here now U is a G-manifold
- $U_1 \hookrightarrow U, \dots, U_k \hookrightarrow U$ embeddings with disjoint images (compatible)

$$A(U_1) \otimes \cdots \otimes A(U_k) \longrightarrow A(U)$$

Factorization algebra:

1) $U = U_1 \amalg \cdots \amalg U_k$
$$\Rightarrow A(U_1) \otimes \cdots \otimes A(U_k) \xrightarrow{\simeq} A(U)$$

2) locality condition : $(U_i \hookrightarrow U)_{i \in I}$ forms a Weiss cover.

$$\operatorname{hocolim}\Big(\underbrace{\oplus A(U_i) \leftleftarrows \oplus A(U_{ij}) \leftleftarrows\!\!\leftarrow \oplus A(U_{ijk}) \cdots}_{\text{simplicial complex}}\Big)$$
$$\simeq A(U)$$

digression: twisted functorial field theories

2-category $\mathcal{G}$-Bord :
- obj : closed $d-1$ manifolds $\times [0,\varepsilon)$ ← to make gluing easy
- mor : G-bordisms
- 2mor : $\mathcal{G}$-diffeo between bordisms

twisted field theories :
$$\begin{tikzcd} \mathcal{G}\text{Bord} \arrow[r, bend left, "T"] \arrow[r, bend right, "T_0 = \text{trivial functor}"'] \arrow[r, phantom, "\Downarrow E"] & T\text{Alg} \end{tikzcd}$$
as in the first lecture

T: $Y \longmapsto T(Y)$

$Y_0 \leftarrow \Sigma \rightarrow Y_1 \longmapsto {}_{T(Y_0)}T(\Sigma)_{T(Y_1)}$

2-mor $\longmapsto$ intertwiners

E is a T-twisted field theories

$E(Y)$ = Right $T(Y)$-module

$E(\Sigma) = E(Y_0) \xrightarrow[T(Y_0)\text{ linear}]{} E(Y_1) \otimes_{T(Y_1)} T(\Sigma)$

Now replace TAlg by dg-TAlg

objects: dg categories = categories enriched over dg-Vect-Chain complexes

mor: dg modules

$\mathcal{A}, \mathcal{B}$ dg-categories

An A-B bimodule is a dg-category $\mathcal{M}$ with

- $\mathrm{ob}\,\mathcal{M} = \mathrm{ob}\,\mathcal{A} \amalg \mathrm{ob}\,\mathcal{B}$
- $\mathcal{A}, \mathcal{B}$ are full subcats of $\mathcal{A}$
- $\mathcal{M}(a,b) = 0 \quad a \in \mathcal{A},\ b \in \mathcal{B}$

2mor: $\quad {}_A\mathcal{M}_B \otimes {}_B N_C =: {}_A(M \otimes N)_C$

$${}_A(\mathcal{M} \otimes N)_C(c,a) :=$$

$$\mathrm{hocolim}\left(\bigoplus_b \mathcal{M}(b,a) \otimes \mathcal{N}(c,b) \leftleftarrows \bigoplus_{b_0 b_1} \mathcal{M}(b_1,a) \otimes \mathcal{B}(b_0,b_1) \otimes \mathcal{N}(c,b_0) \Lleftarrow \cdots \right)$$

Construction of $T_{\mathcal{A}}: G\text{Bord} \longrightarrow \mathrm{dgTAlg}$ associated to localisation algebra $\mathcal{A}$ on G-manifolds

- $Y \times [0,\varepsilon) \longrightarrow T_A(Y) = \begin{cases} \text{beyond} \\ \text{comprehension} \end{cases}$
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