Manifold Calculus and Convex Integration

by

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Abstract

The spaces of embeddings of one manifold inside another, are some of the most fundamental geometric objects studied in algebraic topology, the classical problem of studying knots embedded in $\mathbb{R}^3$ being the most well studied example. However, embedding spaces are extremely difficult to study. Most algebro-geometric techniques either produce a series of simpler spaces approximating them or they stabilize the space of embeddings in a suitable sense and study the stabilized version using stable homotopy theory.

Manifold calculus analyzes embedding spaces as a contravariant functor from some categories of smooth manifolds to topological spaces and constructs *polynomial approximations* to them, analogous to the way polynomial approximations can be constructed to a smooth function using Taylor series. Furthermore, the fundamental theorem of manifold calculus says that these approximations converge to the original functor.

In this thesis, we apply Gromov's theory of convex integration to study embedding spaces of manifolds with tangential structures. We prove that the theory of convex integration is compatible with the theory of manifold calculus, in the sense that the polynomial approximations produced by manifold calculus converge to the structured embedding spaces whenever the theory of convex integration can be applied. This also allows us to construct examples of functors whose polynomial approximations do not converge to the orginal functor.

When N is a symplectic manifold, we prove that the analytic approximation to the Lagrangian embeddings functor $\mathrm{Emb}_{\mathrm{Lag}}(-, N)$ is the totally real embeddings functor $\mathrm{Emb}_{\mathrm{TR}}(-, N)$. When $M \subseteq R^n$ is a parallelizable manifold, we provide a geometric construction for the homotopy fiber of $\mathrm{Emb}(M, R^n) \to \mathrm{Imm}(M, R^n)$. This construction also provides an example of a functor which is itself empty when evaluated on most manifolds but whose analytic approximation is almost always non-empty.

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Chapter 1

Introduction

In this thesis, we use homotopy theoretical techniques to study problems from geometry. More specifically we're interested in applying the theory of manifold calculus to study problems from symplectic geometry and knot theory.

This introductory chapter does not use any advanced homotopy theoretic concepts and should be accessible to anyone with some elementary knowledge of differential geometry and algebraic topology.

1.1 Manifold calculus

1.1.1 Motivation: Embedding spaces

The primary objects of study in manifold calculus are embedding spaces. For smooth manifold M and N denote by $\mathrm{Emb}(M, N)$ the space of smooth embeddings of M inside N[1].

$$\begin{aligned}\operatorname{Emb}(M,N) &:= \{\text{smooth embeddings of } M \text{ inside } N\} \\ &= \left\{\text{smooth } e : M \to N \mid M \xrightarrow{e} e(M) \text{ is a homeomorphism}\right\}.\end{aligned}$$

The classical example of the study of embedding spaces comes from knot theory. A knot is a smooth embedding of the circle S^1 inside $\mathbb{R}^3$. We define two knots to be isotopic if one can be continuously deformed into another. The fundamental question in knot theory is determining when two knots are

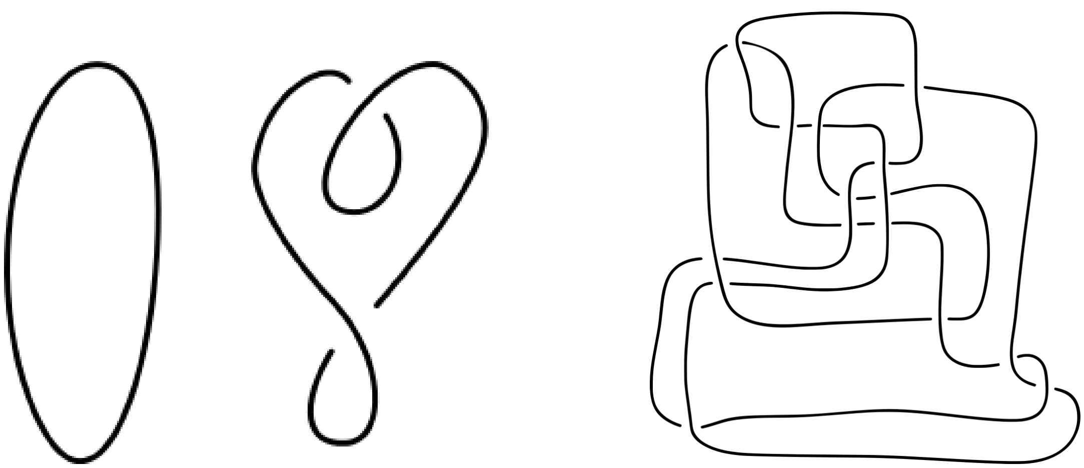

Figure 1.1: These three knots are isotopic and hence lie in the same connected component of $\operatorname{Emb}(S^1, \mathbb{R}^3)$. Images from Wikipedia.

isotopic to each other.

$$\begin{aligned}\text{isotopy classes of knots } &= \pi_0 \operatorname{Emb}(S^1, \mathbb{R}^3) \\ &= \left\{\text{embeddings of } S^1 \text{ inside } \mathbb{R}^3\right\} / \sim_{\text{isotopy}}\end{aligned}$$

[1] $\operatorname{Emb}(M,N)$ is topologized using the weak topology, see Hirsch, 1976, Chapter 2.1.

The space of all knots in $\mathbb{R}^3$ is the space of embeddings $\mathrm{Emb}(S^1, \mathbb{R}^3)$ and two knots are isotopic precisely when they lie in the same connected component. Thus the classical question in knot theory is the study of connected components $\pi_0 \mathrm{Emb}(S^1, \mathbb{R}^3)$.

In the homotopy theoretic approach, we study the entire space $\mathrm{Emb}(S^1, \mathbb{R}^3)$ instead of its connected components. The embedding space carries a much richer structure than the set of its connected components which can be exploited to acquire more information about the space. Several computations that use apply this philosophy using the technique of manifold calculus can be found in the following papers: Sinha, 2006, Sinha, 2009, Dwyer and Hess, 2012, Arone and Turchin, 2014.

1.1.2 Excision and immersions

Manifold calculus borrows the idea of excision from classical algebraic topology and applies it to the study of embedding spaces. One of the first such connection was discovered by Smale-Hirsch in their study of immersions, which was vastly generalized by Gromov et. al. into the theory of h-principles.

Cohomology theories are some of the most computable algebro-topological invariants of spaces. The property that makes cohomology theories useful is *excision*. For a manifold M, excision says that if M can be written as a union of its open submanifolds U and V then the cohomology of M can be (almost) completely determined from the cohomologies of U and V. For example, if $\Omega^*_{dR}(-)$ denotes the chain complex of differential forms on $(-)$ then the

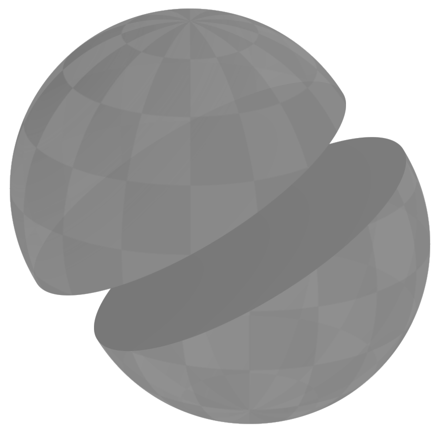

Figure 1.2: Decomposing a two dimensional sphere $M = S^2$. The pieces U and V are the two hemispheres.

following square is a homotopy pullback square,

$$\begin{tikzcd}
\Omega^*_{dR}(M) \arrow[r] \arrow[d] & \Omega^*_{dR}(U) \arrow[d] \\
\Omega^*_{dR}(V) \arrow[r] & \Omega^*_{dR}(U \cap V).
\end{tikzcd}$$

where all the maps are restriction of forms. Smale followed by Hirsch showed that the similar result is true for the space of immersions. An immersion is a weaker notion of an embedding that allows for intersections but no singularities. More precisely, for manifolds M and N, an immersion is a smooth map $i : M \to N$ such that the differential Di has constant rank equal to $\dim M$. Denote the space of immersions by $\mathrm{Imm}(M, N)$.

$$\begin{aligned}
\mathrm{Imm}(M, N) &= \{\text{immersions of } M \text{ inside } N\} \\
&= \{\text{smooth } i : M \to N \mid \text{rank } Di \equiv \dim M\}.
\end{aligned}$$

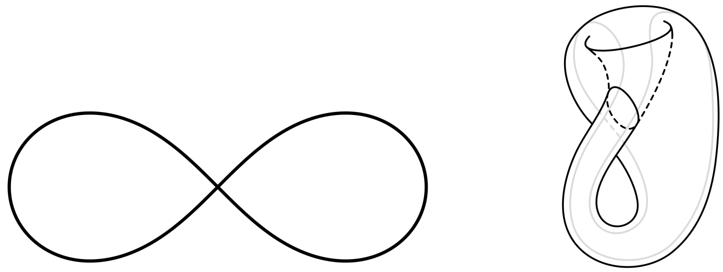

Figure 1.3: Examples of immersions which are not embeddings. Images from Wikipedia.

Following is a theorem first proved in Smale, 1959 for $M = S^2$ and generalized to all manifolds in Hirsch, 1959.

Theorem 1.1 (Smale-Hirsch)**.** *Let M and N be smooth manifolds of dimension m and n respectively with $n - m > 1$. If U, V are open submanifolds of M with $M = U \cup V$, then the following square is a homotopy pullback square*

$$\begin{tikzcd} \mathrm{Imm}(M,N) \arrow[r] \arrow[d] & \mathrm{Imm}(U,N) \arrow[d] \\ \mathrm{Imm}(V,N) \arrow[r] & \mathrm{Imm}(U \cap V, N), \end{tikzcd}$$

where all the maps are restrictions of functions.

For $M = S^2$, we can decompose it as a union of two discs U and V whose intersection is homeomorphic to a cylinder as in Figure 1.2 Using Theorem 1.1 one can show that the space $\mathrm{Imm}(S^2, \mathbb{R}^3)$ is connected. In particular, the standard embedding and the antipodal embedding (one that radially reflects the standard embedding about the origin) can be connected by a path of immersions, this result is known as *sphere eversion*.[2]

[2] At the time this very counter-intuitive result was proven, the only proof was existential

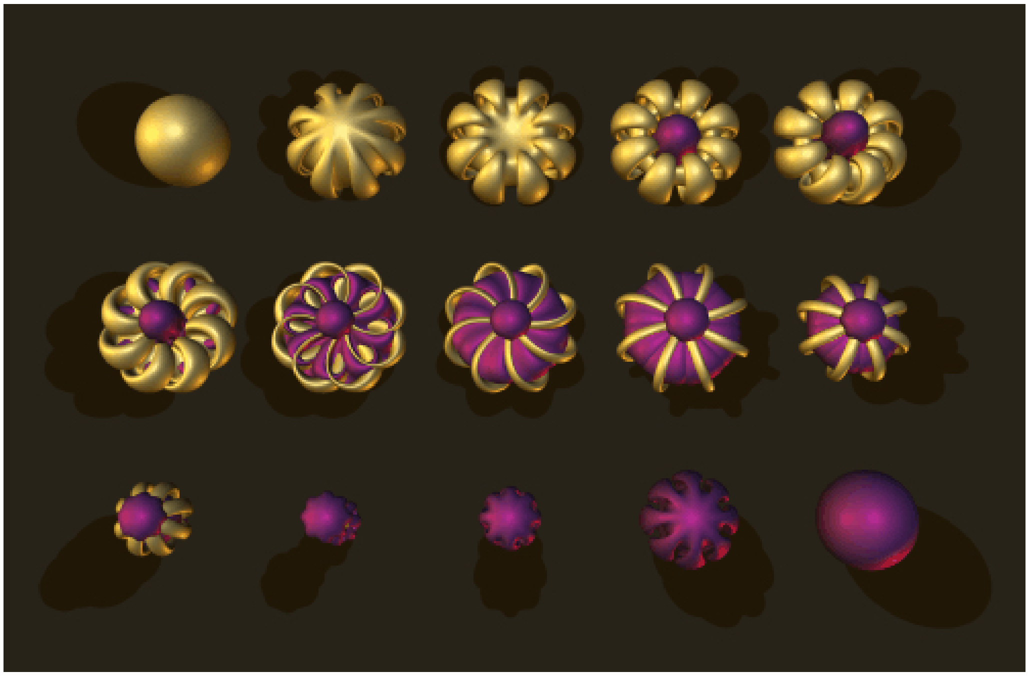

Figure 1.4: Explicit construction of Smale's sphere eversion. Image from Sullivan, 2002, https://youtu.be/w061D9x61NY.

1.1.3 Higher excision and embeddings

An embedding can be thought of as an immersion without any self-intersections. When the manifolds M and N satisfy the dimension restriction $2m < n$, Whitney costructed a self-intersection cancelling trick to costruct an embedding from an immersion, see Milnor, Siebenmann, and J., 1965.

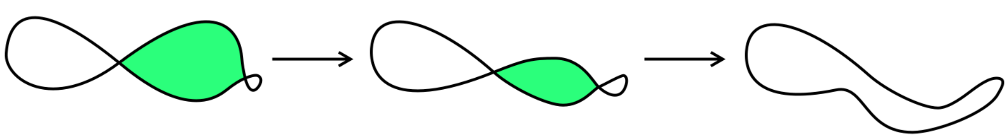

Figure 1.5: Whitney's trick for cancelling self-intersections. Image from Wikipedia.

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and used the computation of homotopy groups. Today several algorithms have been discovered that explicitly construct the sphere eversion and it is possible to visualize such an eversion explicitly.

In several papers leading up to Goodwillie and Klein, 2008, Weiss, 1999 this method was vastly generalized into the framework of manifold calculus. For every positive integer k, they construct a *k-excisive* approximation $\mathscr{T}_k \operatorname{Emb}(M, N)$. This space satisfies the property that it can be computed by writing M as a union of k open sets $M = U_1 \cup \cdots \cup U_k$. For example, when $k = 2$ the space $\mathscr{T}_2 \operatorname{Emb}(M, N)$ is *2-excisive*. For A, B, C three disjoint closed subsets of M, let $U = M \setminus A$, $V = M \setminus B$, and $W = M \setminus C$ so that $M = U \cup V \cup W$.

$$\begin{tikzcd}
& \mathscr{T}_2 \operatorname{Emb}(U) \arrow[rr] \arrow[dd] & & \mathscr{T}_2 \operatorname{Emb}(U \cap V) \arrow[dd] \\
\color{red}{\mathscr{T}_2 \operatorname{Emb}(M)} \arrow[ur] \arrow[rr] \arrow[dd] & & \mathscr{T}_2 \operatorname{Emb}(V) \arrow[ur] \arrow[dd] & \\
& \mathscr{T}_2 \operatorname{Emb}(U \cap W) \arrow[rr] & & \mathscr{T}_2 \operatorname{Emb}(U \cap V \cap W) \\
\mathscr{T}_2 \operatorname{Emb}(W) \arrow[ur] \arrow[rr] & & \mathscr{T}_2 \operatorname{Emb}(V \cap W) \arrow[ur] &
\end{tikzcd}$$

Figure 1.6: The value of $\mathscr{T}_2 \operatorname{Emb}(M)$ can be computed by decomposing M as $U \cup V \cup W$ and using the above *homotopy pullback* cube.

This should be interpreted as being analogous to taking the k^{th} Taylor approximation of a polynomial.

Another interpretation of manifold calculus is that the analytic approximation $\mathscr{T}_\infty \operatorname{Emb}(M, N)$ is the space obtained by looking at all the spaces $\operatorname{Emb}(U, N)$ where U is an open subset of M diffeomorphic to a disjoint union of finitely many discs, see Chapter 2.

Analogous to forming the Taylor series by taking the limit of Taylor polynomials, we can then take the (inverse) limit of these k-excisive approximations to obtain the *analytic approximation* $\mathscr{T}_\infty F$ of F.

$$\mathscr{T}_\infty \operatorname{Emb}(M, N) = \operatorname*{holim}_k \mathscr{T}_k \operatorname{Emb}(M, N).$$

We can interpret $\mathscr{T}_\infty F$ as being the invariant obtained by successively covering up a manifold by finer and finer covers and using these covers to reconstruct the invariant on M. The fundamental theorem of manifold calculus then says the following.

Theorem 1.2. *There exists a weak homotopy equivalence*

$$\mathscr{T}_\infty \operatorname{Emb}(M, N) \simeq \operatorname{Emb}(M, N),$$

if $\dim N - \dim M \geq 3$.

1.2 Embeddings of structured manifolds

In geometry, manifolds usually have more structure than just being smooth, like an orientation, a Riemannian metric or a symplectic form. To be able to apply the above techniques to geometry we need a generalization of Theorem 1.2 for structured manifolds. In this thesis, we are particularly interested in applying manifold calculus to symplectic geometry.

1.2.1 Lagrangian embeddings

A *symplectic manifold* is a manifold that generalizes the notion of a phase space from physics. A manifold N^{2m} is said to be symplectic if there is a closed 2-form $\omega \in \Omega^2_{dR}(N)$ such that ω^m is nowhere vanishing. Symplectic manifolds are the right setting in which one can do Hamiltonian and Lagrangian mechanics. Topologically, symplectic manifolds are at the sweet spot between rigid and flexible geometry.

Riemannian manifolds	*Symplectic manifolds*	*Topological manifolds*
Rigid domain	Both	Flexible domain
Analytic invariants	Both kinds of invariants	Topological invariants

A *Lagrangian submanifold* of N is a manifold M^m on which ω vanishes identically. Lagrangian submanifolds model 'space-like' submanifolds of the phase space of a physical system. For example, the trajectories of a Hamiltonian flow of any Hamiltonian function on N will lie along a Lagrangian submanifold. The space of Lagrangian submanifolds inside N is related to several geometric conjectures like the homological mirror symmetry and to physics because of the question of quantization of classical systems.

One such conjecture that has been a motivating example is the following (homotopy theoretical version of) *nearby Lagrangian conjecture* due to Arnol'd. Let M and L be closed simply connected manifolds of the same dimension. The cotangent T^*M carries a canonical symplectic structure.

Conjecture 1.3. *The space of Lagrangian embeddings of L in T^*M is connected if L is diffeomorphic to M, is empty otherwise.*

This conjecture is asking the homotopy type of the space of Lagrangian embeddings $\mathrm{Emb}_{\mathrm{Lag}}(L, T^*M)$ and thus one can ask if it is possible to study $\mathrm{Emb}_{\mathrm{Lag}}(L, T^*M)$ using manifold calculus. More specifically, we're interested in answering whether it is analytic. In Section 4.2 we show that this is not the case by proving the following theorem.

Theorem 1.4. *If $m > 2$, then*

$$\mathscr{T}_\infty \operatorname{Emb}_{\mathrm{Lag}}(M, N) \simeq \operatorname{Emb}_{\mathrm{TR}}(M, N),$$

where $\operatorname{Emb}_{\mathrm{TR}}(M, N)$ *is the space of totally real embeddings of M inside N.*

For definition of totally real embeddings see Section 4.2. We can interpret the above theorem as saying that manifold calculus sees the 'flexible' side of symplectic geometry and does not see the rigid side.

1.2.2 Convex integration & directed embeddings

Convex integration is a pertubative technique developed by Gromov which can be used to construct solutions of differential relations which are also embeddings.

A first order differential relation on the space of smooth maps $f : M \to N$ is a condition on the first differential Df. For example, being an immersion is the condition rank $Di = m$. Being a Lagrangian submanifold can also be expressed as a suitable differential relation. Gromov proves sufficient conditions under which an arbitrary embedding can be perturbed to a solution of the given differential relation.

For smooth manifolds M and N, the *1-jet space* $J^1(M, N)$ of smooth maps from M to N is a vector bundle over $M \times N$ whose fiber over the point $(p, q) \in M \times N$ is the space of linear transformations $\mathcal{L}in(T_pM, T_qN)$. Subsets $\mathcal{R}$ of $J^1(M, N)$ define first order differential relations on the space of smooth maps $f : M \to N$. Using his technique of convex integration, Gromov, 1986 showed that if $\mathcal{R} \subseteq J^1(M, N)$ satisfies certain largeness conditions called *ampleness* (Section 3.2.1), then an arbitrary embedding of e of M inside N can be perturbed to a (sufficiently C^0-close) embedding e' that satisfies the differential relation $\mathcal{R}$. Using this result in Chapter 4 we prove the following theorem.

Theorem 1.5. *If $n - m > 2$ and $\mathcal{R}$ is an ample differential relation, then there is a weak homotopy equivalence*

$$\mathscr{T}_\infty \operatorname{Emb}_{\mathcal{R}}(M, N) \simeq \operatorname{Emb}_{\mathcal{R}}(M, N),$$

where $\operatorname{Emb}_{\mathcal{R}}(M, N)$ denotes the space of embeddings of M inside N that satisfie the differential relation $\mathcal{R}$.

One can think of this as a homotopy principle for embedding spaces. The analytic approximation $\mathscr{T}_\infty \operatorname{Emb}_{\mathcal{R}}(M, N)$ is a purely homotopical construction whereas the space $\operatorname{Emb}_{\mathcal{R}}(M, N)$ is a purely geometric.

1.2.3 Non-analytic embeddings

Using the above theorem we can also construct several exotic embedding functors. Somewhat analogous to the fact that there are non-trivial functions which have trivial Taylor series, we can use convex integration to create functors which are almost always trivial but whose analytic approximation is never so.

Let $M \subseteq \mathbb{R}^n$ be a parallelizable manifold, with a choice of m linearly independent non-vanishing vector fields $X_1, \ldots, X_m$. Denote by $\operatorname{Emb}_{\mathrm{TS}}(-, \mathbb{R}^n)$ the functor on the category of parallelizable manifolds which sends M to the space of embeddings $e : M \hookrightarrow \mathbb{R}^n$ such that $\mathrm{D}e(X_1), \ldots, \mathrm{D}e(X_m)$ are constant non-varying vector fields. Borrowing the terminology from Dwyer and Hess, 2012 we call $\mathrm{Emb}_{\mathrm{TS}}(M, \mathbb{R}^n)$ the space of *tangentially straightened embeddings*. In Theorem 5.7 we show that

Theorem 1.6. *When $n - m > 2$, there is a homotopy equivalence*

$$\mathscr{T}_\infty \mathrm{Emb}_{\mathrm{TS}}(M, \mathbb{R}^n) \simeq \mathrm{hofib}(\mathrm{Emb}(M, \mathbb{R}^n) \to \mathrm{Imm}(M, \mathbb{R}^n))$$

If M is not diffeomorphic to a submanifold of $\mathbb{R}^m$, it is easy to see that no tangentially straightened embedding is possible (see Chapter 5) and hence $\mathrm{Emb}_{\mathrm{TS}}(M, N)$ is empty. However, as $M \subseteq \mathbb{R}^n$ the homotopy fiber

$$\mathrm{hofib}(\mathrm{Emb}(M, \mathbb{R}^n) \to \mathrm{Imm}(M, \mathbb{R}^n))$$

is non-empty. Thus the functor $\mathrm{Emb}_{\mathrm{TS}}(-, \mathbb{R}^n)$ provides an example of a highly *non-analytic* functor.

1.3 Outline

In Chapter 2 we describe the sheaf theoretic formulation of manifold calculus. In Chapter 3 we review the necassary background about convex integration and directed embeddings. In Chapter 4 we prove the main results of this thesis, about the compatibility of the manifold calculus and convex integration and the analytic approximation of Lagrangian embeddings. In Chapter 5 we generalize the previous constructions to manifolds with tangential structures on the source manifold and prove the non-analyticity of tangentially straightened embeddings. In Chapter 6 we state of the of the current work in progress about manifolds with a group acting on them.

Chapter 2

Manifold calculus

In this chapter, we recall the basic definitions of *manifold calculus*, also known as *embedding calculus*. Manifold calculus was first defined in Weiss, 1999 and later reformulated in Brito and Weiss, 2012 using *enriched category theory*. We will review the later formulation.

Fix a positive integer m. Denote by $\mathsf{Man_m}$ the category of smooth manifolds of dimension m with morphisms being codimension zero smooth embeddings. To avoid set-theoretic issues, we'll assume that all the objects of $\mathsf{Man_m}$ are submanifolds of $\mathbb{R}^\infty$. We will also assume that every manifold is diffeomorphic to the interior of a compact manifold with boundary. The category $\mathsf{Man_m}$ is naturally enriched over Spaces, where we endow the space of embeddings with the weak topology (compact-open topology), see Hirsch, 1976, Chapter 2. We are interested in studying *continuous* presheaves (in the enriched sense)

$$\mathsf{Man_m}^{op} \to \mathsf{Spaces}\,.$$

Denote by $\mathrm{PSh}(\mathsf{Man_m})$ the category of such functors and natural transformations beween them. For $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ denote by $\mathsf{Disc_k}$ the full subcategory of $\mathsf{Man_m}$ consisting of manifolds which are diffeomorphic to $\coprod_j \mathbb{R}^m$ for some $0 \leq j < k+1$.

2.1 Sheaves on the category of manifolds

A (Grothendieck) coverage of $\mathsf{Man_m}$ is the smallest additional structure needed to be able to define a sheaf on it.

Definition 2.1. A *coverage* τ on $\mathsf{Man_m}$ is an assignment to each object $M \in \mathsf{Man_m}$ a set of *coverings* $\tau(M)$ the elements of which are collections of submanifolds $\{U_i\}_{i\in I}$ of M such that $M = \cup_{i\in I} U_i$ and each $U_i \in \mathsf{Man_m}$.[1]

Definition 2.2. Given a coverage τ on $\mathsf{Man_m}$, a *homotopy τ-sheaf* is a presheaf $F \in \mathrm{PSh}(\mathsf{Man_m})$ such that for every covering $\{U_i\}_{i\in I} \in \tau(M)$ the natural map

$$F(M) \longrightarrow \operatorname*{holim}_{S\subseteq I, |S|<\infty} F(U_S) \tag{2.3}$$

is a weak homotopy equivalence, where by U_S we mean $\cap_{i\in S} U_i$.

This is equivalent to saying that if F is a homotopy τ-sheaf then for every covering $\{U_i\}_{i\in I} \in \tau(M)$, $F(M)$ is weakly equivalent to

$$\operatorname{holim}\left(\prod_{i\in I} F(U_i) \longrightarrow \prod_{i,j\in I} F(U_i \cap U_j) \longrightarrow \prod_{i,j,k\in I} F(U_i \cap U_j \cap U_k) \longrightarrow \dots \right)$$

The more commonly used notion to define a sheaf is that of a *Grothendieck cover*. A Grothendieck cover puts additional restrictions on the space of coverings. For example, if a coverage τ is a Grothendieck cover then for a covering

[1] Here we are abusing notation and identifying a manifold $U_i \in \mathsf{Man_m}$ with it's image under an embedding $U_i \to M$.

$\{U_i\}_{i\in I} \in M$ we must have $\{U_i \cap X\}_{i\in I} \in M \cap X$ for any manifold $X \in \mathsf{Man_m}$. By enforcing these extra restrictions the notion of a sheaf in a Grothendieck cover τ greatly simplifies and we only need to look at double intersections.

$$F(M) \longrightarrow \operatorname*{holim}_{S\subseteq I, |S|\leq 2} F(U_S).$$

This is more commonly expressed as a (homotopy) equalizer condition

$$F(M) \simeq \operatorname{holim}\left(\prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I} F(U_i \cap U_j)\right)$$

Example 2.4 (trivial coverage)**.** The smallest coverage we can define is simply $\tau(M) := \{\{M\}\}$. For this coverage, every functor $F \in \mathrm{PSh}(\mathsf{Man_m})$ is a homotopy τ-sheaf.

Example 2.5 (standard coverage of open covers)**.** The largest coverage τ on M is the one that defines the standard site of open covers. For each $M \in \mathsf{Man_m}$, we let $\tau(M) = \left\{\{U_i\}_{i\in I} : M = \cup_{i\in I} U_i\right\}$ be the collections of open covers of M in the stardard topological sense. This is in fact a Grothendieck site so that a presheaf F is a homotopy τ sheaf for this coverage if for every covering $M = U_1 \cup \cdots \cup U_k$, $F(M)$ equals the following homotopy equalizer of the following diagram.

$$\prod_{1\leq i\leq k} F(U_i) \longrightarrow \prod_{1\leq i,j\leq k} F(U_i \cap U_j)$$

Smale-Hirsch Theorem 1.1 described in the introduction can be rephrased in this language as saying

Theorem 2.6. *Let N be a manifold with dimension $\geq m+1$. The functor* $\mathrm{Imm}(-, N)$ *in* $\mathrm{PSh}(\mathsf{Man_m})$ *is a τ-homotopy sheaf, where τ is the standard coverage of open covers* *on* $\mathsf{Man_m}$.

The theory of manifold calculus then answers the following question.

Q 2.7. *Does there exist a coverage τ such that the embeddings functor* $\mathrm{Emb}(-, N)$ *is a homotopy τ-sheaf?*

The answer to the above question is indeed, yes. In fact, there are three differently defined but equivalent coverages for which the above statement is true.

Definition 2.8. Let $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. For each manifold M, define the coverages $\mathcal{J}_k^h$, $\mathcal{J}_k$, and $\mathcal{J}_k^\circ$ as follows.

$\mathcal{J}_k^h$**:** $\mathcal{J}_k^h(M)$ is a collection of covers $\{M \setminus A_i\}_{0 \leq i < k+1}$ of M where $A_0, \ldots, A_k$ are finitely many disjoint closed subsets of M and j is any postive integer $< k+1$.

$\mathcal{J}_k$**:** $\mathcal{J}_k(M)$ is a collection of covers $\{U_i\}_{i \in I}$ of M satisfying the property that for every finite $S \subseteq M$ with $|S| < k+1$ there exists an i such that $S \subseteq U_i$.

$\mathcal{J}_k^\circ$**:** $\mathcal{J}_k^\circ(M)$ is a collection of covers $\{U_i\}_{i \in I}$ of M satisfying the following two properties,

- the covering $\{U_i\}_{i \in I}$ is in $\mathcal{J}_k(M)$,
- for each finite subset S of I the intersection $\cap_{i \in S} U_i \in \mathsf{Disc_k}$.

For every $M \in \mathsf{Man_m}$, there are natural inclusions

$$\mathcal{J}_k^\circ \subseteq \mathcal{J}_k \supseteq \mathcal{J}_k^h$$

Morever, the homotopy sheaves in each of the three coverages are essentially the same as made precise by the following theorem.

Theorem 2.9. *For a presheaf $F \in \mathrm{PSh}(\mathsf{Man_m})$ the following are equivalent statements*

1. *F is a $\mathcal{J}_k$-sheaf.*
2. *F is a $\mathcal{J}_k^{\circ}$-sheaf.*
3. *F is a $\mathcal{J}_k^{h}$-sheaf and is a good functor.*

where a functor $F \in \mathrm{PSh}(\mathsf{Man_m})$ is said to be good if for every $U_0 \subseteq U_1 \subseteq \cdots \subseteq M$ in $\mathsf{Man_m}$ whose union is M, the natural map

$$F(M) \longrightarrow \operatorname*{holim}_{i} F(U_i)$$

is a weak homotopy equivalence.

The three coverages provide us different ways and tools of interpreting and computing properties of the same class of functors. Let k be a positive integer.

$\mathcal{J}_k^h$: Every covering in $\mathcal{J}_k^h(M)$ has finite cardinality k. This makes $\mathcal{J}_k^h$ extremely useful for doing computations as the homotopy limit diagram in 2.2 becomes a finite (cubical) diagram.

$\mathcal{J}_k$: This is the most canonical of the three coverages. It is the most useful for relating the other coverages and acts as a bridge between them.

$\mathcal{J}_k^{\circ}$**:** Ever open set in a covering in $\mathcal{J}_k^{\circ}(M)$ is a disjoint union of at the most k-discs. This allows us to come up with a description of homotopy sheafs in terms of Kan extensions, Section 2.2.

Example 2.10. When $k = 1$, the coverages $\mathcal{J}_k$ and $\mathcal{J}_k^h$ both equal the largest coverage as explained in Example 2.5. The coverage $\mathcal{J}_k^{\circ}$ is finer and only allows for good open covers of M.

Example 2.11. When $k = 2$ a covering in $\mathcal{J}_k^h$ is exactly the example described in the introduction. For a manifold M, let A, B, C be three disjoint closed subsets of M. We can set $U = M \setminus A$, $V = M \setminus B$, and $W = M \setminus C$ so that $M = U \cup V \cup W$ and the following cube is homotopy pullback cube.

$$\begin{tikzcd}
& F(U) \arrow[rr] \arrow[dd] & & F(U \cap V) \arrow[dd] \\
\mathbf{F}(\mathbf{M}) \arrow[ur] \arrow[rr] \arrow[dd] & & F(V) \arrow[ur] \arrow[dd] & \\
& F(U \cap W) \arrow[rr] & & F(U \cap V \cap W) \\
F(W) \arrow[ur] \arrow[rr] & & F(V \cap W) \arrow[ur] &
\end{tikzcd}$$

In conjuction with the handlebody decomposition of a manifold, this coverage can be used to compute $F(M)$ inductively, see Munson and Volić, 2015, Theorem 10.2.14.

2.2 Higher excision using discs

Given a homotopy presheaf F and a coverage τ, it is possible to 'sheafify' the functor F to obtain a homotopy τ-sheaf $\mathscr{T}_\tau F$.

$$F \to \mathscr{T}_\tau F$$

There is a natural projective model structure on $\mathrm{PSh}(\mathsf{Man_m})$ (see Hirschhorn, 2009) induced by the model structure on Spaces: the fibrant objects are the presheaves which are object-wise fibrant, the weak homotopy equivalences are object-wise weak homotopy equivalences, and the cofibrations are presheaves which satisfy the right lifting property with respect to trivial fibrations. It is then possible to shrink the class of fibrations by declaring homotopy τ-sheaves to be the fibrant objects. This is done by Bousfield localizing the projective model structure, see Brito and Weiss, 2012, Theorem 3.6, to get a new model category $\mathrm{PSh}^\tau(M)$. The sheafification is then just the *fibrant replacement* in this new model category.

Definition 2.12. For a presheaf $F \in \mathrm{PSh}(\mathsf{Man_m})$ define the k^{th} *polynomial approximation* to F to be the the right derived Kan extension of $F|_{\mathsf{Disc_k}}$ along the inclusion $\mathsf{Disc_k} \hookrightarrow \mathsf{Man_m}^G$.

$$\begin{tikzcd}
\mathsf{Disc_k}^{op} \arrow[r, "F|_{\mathsf{Disc_k}^{op}}"] \arrow[d] \arrow[dr, phantom, "\Uparrow"] & \mathsf{Spaces} \\
\mathsf{Man_m}^{op} \arrow[ur, "\mathscr{T}_k F"'] &
\end{tikzcd} \tag{2.13}$$

When $k = \infty$, we say that $\mathscr{T}_\infty(F)$ is the *analytic approximation* to F. More explicitly for $M \in \mathsf{Man_m}$,

$$\mathscr{T}_k F(M) := \mathrm{Hom}_{\mathrm{PSh}(\mathsf{Disc_k})}(Q\,\mathrm{Emb}(-,M), F) \tag{2.14}$$

where $Q\,\mathrm{Emb}(-,M)$ denotes the cofibrant replacement of $\mathrm{Emb}(-,M)$ in $\mathrm{PSh}(\mathsf{Man_m})$.

Because of the inclusions $\mathsf{Disc}_{k-1} \subseteq \mathsf{Disc_k}$ there is a natural tower of functors

$$\begin{tikzcd}
\mathscr{T}_\infty F \arrow[d] & \\
\vdots \arrow[d] & \\
\mathscr{T}_k F \arrow[d] & \\
\mathscr{T}_{k-1} F \arrow[d] & \\
\vdots \arrow[d] & \\
\mathscr{T}_1 F & F \arrow[l] \arrow[uul] \arrow[uuul] \arrow[uuuuul]
\end{tikzcd}$$

which gives us a spectral sequence for computing $\mathscr{T}_\infty F$.

Definition 2.15. We say that functor $F \in \mathrm{PSh}(\mathsf{Man_m})$ is *analytic* if $F(M) \to \mathscr{T}_\infty F(M)$ is a weak homotopy equivalence for all $M \in \mathsf{Man_m}$.

Example 2.16. The following examples of analytic functors will be of use to us in the later sections.

1. By the formal properties of Kan extensions it follows that

$$\mathscr{T}_\infty \mathscr{T}_\infty F \simeq \mathscr{T}_\infty F$$

   for any functor $F \in \mathrm{PSh}(\mathsf{Man_m})$. Hence an analytic approximation $\mathscr{T}_\infty F$ is itself always analytic.

2. The functor $\mathrm{Maps}(-, N)$ of space of smooth maps into N is analytic, Weiss, 1999, Example 2.4.

3. Let $\mathrm{Imm}(M, N)$ denote the space of immersions of M into N. For $n > m$, the functor $\mathrm{Imm}(-, N) \in \mathrm{PSh}(\mathsf{Man_m})$ is analytic, Weiss, 1999, Example 2.3.

Manifold calculus was introduced to study the space of embeddings. One of the deepest theorems in manifold calculus states the following Goodwillie and Weiss, 1999, Goodwillie and Klein, 2008.

Theorem 2.17 (Goodwillie-Klein-Weiss)**.** *If $n - m > 2$, the functor* $\mathrm{Emb}(-, N)$ *in* $\mathrm{PSh}(\mathsf{Man_m})$ *is analytic.*

The main result in this thesis extends this theorem to directed embeddings using h-principles.

Chapter 3

h-principles via convex integration

Homotopy principles or h-principles, as first discovered by Smale-Hirsch and generalized by Gromov and others are a collection of techniques for solving differential relations, which are underdetermined differential equations. In this thesis, we'll focus on convex integration, which is technique for proving h-princples for embedding spaces.

Let M, N be smooth manifolds of dimensions m and n respectively, and let E be a smooth manifold bundle over M with fiber N.

$$\begin{tikzcd} N \arrow[r] & E \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

We are mostly interested in the case when E is the trivial bundle $M \times N$.

3.1 Jet spaces and differential relations

The *1-jet space* $J^1(E)$ of sections of E is a vector bundle over E whose fiber over the point $x \in E$ is the space of linear sections of $D\pi : T_x E \to T_{\pi(x)} M$.

$J^1(E)$ should be thought of as the bundle of 1-germs of sections of π. When $E = M \times N$ and $x = (p, q)$ this space of sections can be identified with the space of linear transformations from T_pM to T_qN. We'll denote the jet space in this case by $J^1(M, N)$

$$\begin{tikzcd} \mathcal{L}in(T_pM, T_qN) \arrow[r] \arrow[d] & J^1(M, N) \arrow[d] \\ \star \arrow[r, "(p,q)"] & M \times N. \end{tikzcd} \tag{3.1}$$

Projecting $J^1(E)$ further down to M we get a manifold bundle $J^1(E) \to M$. A smooth section $f : M \to E$ defines a section $J^1(f)$ of this bundle sending $p \mapsto (f(p), Df|_p)$.

$$\begin{tikzcd} E \arrow[d] \\ M \arrow[u, bend right, "f"'] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} J^1(E) \arrow[d] \\ M \arrow[u, bend right, "J^1(f)=(f,Df)"'] \end{tikzcd}$$

Such sections are called *holonomic sections*. Not every section of $J^1(E) \to M$ is holonomic. An arbitrary section of $J^1(E) \to M$ is called a *formal section*.

Definition 3.2. A *first order differential relation* on $C^\infty(M, N)$ is any subset $\mathcal{R}$ of $J^1(M, N)$. A function $f \in C^\infty(M, N)$ is said to satisfy the differential relation $\mathcal{R}$ if $J^1(f) \in \mathcal{R}$.

A differential relation should be thought of as a generalization of differential equations and is typically obtained by replacing equality signs with inequalities in a differential equation. h-principles are techniques used to identify the homotopy type space of solutions of differential relations for certain (typically) open subsets $\mathcal{R}$ of $J^1(M, N)$.

Definition 3.3. A *formal solution* of a differential relation $\mathcal{R} \subseteq J^1(E)$ is a section $s : M \to J^1(E)$ of the projection $J^1(E) \to M$ whose image is in $\mathcal{R}$. Thus, for each point $p \in M$, a formal solution consists of pairs (f, F) where $f : M \to N$ is a smooth function and $F : TM \to TN$ is a linear transformation lying over f such that image of F is in $\mathcal{R}$.

$$\mathrm{Sol^f}(\mathcal{R}) = \left\{ \begin{tikzcd} \mathcal{R} \arrow[d] \\ M \arrow[u, bend right, "s"'] \end{tikzcd} \right\}$$

$$= \left\{ \begin{tikzcd} TM \arrow[r, "F"] \arrow[d] & TN \arrow[d] \\ M \arrow[r, "f"] & N \end{tikzcd} \quad : \mathrm{Im}(F) \subseteq \mathcal{R} \right\}.$$

A *(holonomic) solution* of the differential relation $\mathcal{R}$ is a formal solution that is also holonomic. Denote by $\mathrm{Sol}(\mathcal{R})$ (resp. $\mathrm{Sol^f}(\mathcal{R})$) the space of solutions (resp. formal solutions) of $\mathcal{R}$.

A formal solution (f, F) relaxes the condition that $F = Df$. Hence, the space of formal solutions of a differential relation is much easier to describe than space of its (holonomic) solutions.

Definition 3.4. We say that $\mathcal{R}$ *satisfies h-principle* if the inclusion

$$\mathrm{Sol}(\mathcal{R}) \longrightarrow \mathrm{Sol^f}(\mathcal{R})$$

is a weak homotopy equivalence.

The space of solutions $\mathrm{Sol}(\mathcal{R})$ contains geometric information, whereas the space $\mathrm{Sol^f}(\mathcal{R})$ contains purely topological (homotopy theoretic) information, hence h-principles can be thought of as reducing a geometric problem to homotopy theory. There are several techniques of proving h-principles one of which we'll see below.

Example 3.5 (Immersions)**.** Let $E = M \times N$. Define $\mathcal{R}_{\mathrm{Imm}}$ (locally) to be the collection of 1-jets of immersions $f : U \to N$ i.e. rank of $Df|_p = m$ for all $p \in U$ where U is an open subset of M, so that the space of immersions $\mathrm{Imm}(M, N)$ is the space of solutions of $\mathcal{R}_{\mathrm{Imm}}$.

$$\mathrm{Imm}(M, N) = \{\text{smooth } i : M \to N \mid \text{rank of } Di \equiv \dim M\}$$

$$= \mathrm{Sol}(\mathcal{R}_{\mathrm{Imm}})$$

a formal immersion consists of a smooth function $f \in C^\infty(M, N)$ and a family of linear monomorphisms $F_p \in \mathcal{L}in(T_pM, T_{f(p)}N)$ smoothly varying with $p \in M$. We'll denote this space by $\mathrm{Imm}^{\mathrm{f}}(M, N)$. Smale-Hirsch Theorem 1.1 in the introduction can be restated as an h-principle.

Theorem 3.6 (Smale-Hirsch)**.** *If $m < n$ then $\mathcal{R}_{\mathrm{Imm}}$ satisfies h-principle i.e. the inclusion*

$$\mathrm{Imm}(M, N) \longrightarrow \mathrm{Imm}^{\mathrm{f}}(M, N)$$

is a weak homotopy equivalence.

3.1.1 Directed immersions

Let $\mathrm{Gr}_m(N)$ be the m-plane Grassmannian bundle over N i.e. the fiber over a point $q \in N$ is the space of m-dimensional subspaces of T_qN. Denote by bs the natural projection onto the base,

$$\begin{tikzcd} \mathrm{Gr}_m(\mathbb{R}^n) \arrow[r] & \mathrm{Gr}_m(N) \arrow[d, "\mathrm{bs}"] \\ & N. \end{tikzcd}$$

Example 3.7. Continuing the Example 3.5 of immersions, There is a natural map $\mathrm{Imm}^{\mathrm{f}}(M,N) \to \mathrm{Maps}(M, \mathrm{Gr}_m(N))$ which sends a formal embedding (f, F) to the map $p \mapsto$ image of $F(p)$. This map is a fiber bundle with fiber isomorphic to $\mathrm{GL}_m(\mathbb{R}^m)$, the moduli space of basis of $\mathbb{R}^m$.

$$\begin{tikzcd} \mathrm{GL}_m(\mathbb{R}^m) \arrow[r] & \mathrm{Imm}^{\mathrm{f}}(M,N) \arrow[d] \\ & \mathrm{Maps}(M, \mathrm{Gr}_m(N)) \end{tikzcd} \tag{3.8}$$

We'll use this important observation in the proof of Theorem 4.5.

Instead of arbitrary differential relations, we're interested in the special case of immersions and embeddings. Subsets $\mathcal{A}$ of $\mathrm{Gr}_m(N)$ naturally define a differential relation on $C^\infty(M,N)$ as follows. Let $f : M \to N$ be an immersion. For each $p \in M$, $Df(p)$ is an m dimensional subspace of $T_{f(p)}N$ so that we have a map,

$$\mathrm{Gr}_m(f) : M \to \mathrm{Gr}_m(N).$$

Definition 3.9. For $\mathcal{A} \subseteq \mathrm{Gr}_m(N)$, we say that an immersion (or an embedding) $i : M \to N$ is $\mathcal{A}$-*directed*, if $\mathrm{Gr}_m(i) \subseteq \mathcal{A}$.

Let $\mathcal{R}_{\mathcal{A}}$ be the differential relation on $C^\infty(M,N)$ locally defined by the 1-jets $J^1(i)$ of $\mathcal{A}$-directed immersions $i : M \to N$. Denote by $\mathrm{Imm}^{\mathrm{f}}_{\mathcal{A}}(M,N)$

the space of formal solutions of the differential relation $\mathcal{R}_{\mathcal{A}}$. Note that the space of solutions $\mathrm{Sol}(\mathcal{R}_{\mathcal{A}})$ is exactly $\mathrm{Imm}_{\mathcal{A}}(M, N)$, the space of $\mathcal{A}$-directed immersions.

Example 3.10. As in Example 3.7, we can identify the space of formal $\mathcal{A}$-directed immersions $\mathrm{Imm}^{\mathrm{f}}_{\mathcal{A}}$ as the pullback in the following diagram.

$$\begin{tikzcd} \mathrm{Imm}^{\mathrm{f}}_{\mathcal{A}}(M, N) \arrow[r] \arrow[d] & \mathrm{Maps}(M, \mathcal{A}) \arrow[d] \\ \mathrm{Imm}^{\mathrm{f}}(M, N) \arrow[r] & \mathrm{Maps}(M, \mathrm{Gr}_m(N)) \end{tikzcd}$$

More explicitly, it is the collection of those formal immerions $(f, F) \in \mathrm{Imm}^{\mathrm{f}}(M, N)$ such that $(\text{image of } F) / \mathrm{GL}_m(\mathbb{R})$ is a subset of $\mathcal{A}$.

Definition 3.11. We say that $\mathcal{A} \subseteq \mathrm{Gr}_m(N)$ satisfies the *h*-principle for directed immersions for a manifold M, if the natural inclusion

$$\mathrm{Imm}_{\mathcal{A}}(M, N) \longrightarrow \mathrm{Imm}^{\mathrm{f}}_{\mathcal{A}}(M, N) \tag{3.12}$$

is a weak homotopy equivalence.

3.2 Directed embeddings

The *h*-principle for directed embeddings is a little more complicated to state, as the condition of being an embedding is not given by a differential relation. It is possible to enforce this condition on the mapping space artificially i.e. in the definition of the formal solution (f, F) we require the function f to be an embedding to begin with and the image of F lands in $\mathcal{A}$. However, because f itself is an embedding, the 1-jet of this function $J^1(f)$ also give us something in $J^1(M, N)$ and we need to use both of these pieces of information to define the space of formal embeddings. We do this in 3.19.

3.2.1 Convex integration

Gromov, 1986 developed the techique of *convex integration* to prove h-principles for directed embeddings (among other things). Spring, 1998 and Eliashberg and Mishachev, 2001, Chapter 4, Chapter 19 contains more details and explainations of this theory. We'll only discuss a very simple example which explains the various concepts involved in *convex integration*. Our example is inspired by Borrelli, 2012.

We will study the directed embeddings of $[0, 1]$ inside $\mathbb{R}^2$. The 1-jet space of maps from $[0, 1] \to \mathbb{R}^2$ is isomorphic to $([0, 1] \times \mathbb{R}^2) \times \mathbb{R}^2$. Hence, subsets $\mathcal{A}$ of $\mathbb{R}^2 \setminus \{(0, 0)\}$ define a differential relation $\mathcal{R}_{\mathcal{A}}$. Fix the standard embedding $e(t) = (t, 0)$ along the x-axis. We're interested in answering the following questions.

Q. *Let δ be a positive real number. For which subsets $\mathcal{A}$ of $\mathbb{R}^2 \setminus \{(0, 0)\}$, is it possible to find an embedding $\gamma : [0, 1] \hookrightarrow \mathbb{R}^2$ such that*

1. $\|e - \gamma\|_{C^0} < \delta$,
2. *γ is an $\mathcal{A}$-directed embedding?*

The 1-jet of e is $J^1(e) = e \times \{(1, 0)\}$. If $\mathcal{A}$ contains the vector $(1, 0)$ then e itself is $\mathcal{R}_{\mathcal{A}}$ directed. But if the difference between $\mathcal{A}$ and $(1, 0)$ is large then this is a very non-trivial question.

For $\mathcal{A} = \{(x, y) : x > \delta + 1\}$, the answer to the above question is negative. As any map $\gamma : [0, 1] \to \mathbb{R}^2$ with $\gamma'(t) \in \mathcal{A}$ must have $\gamma(1) = (a, b)$ with $a > \delta + 1$ and hence $\|\gamma - e\|_{C^0} > \delta$.

For $\mathcal{B} = \left\{(x, y) : x^2 + y^2 > (\delta + 1)^2, x > 0\right\}$, quite surprisingly, the answer to the above question is positive even though the distance between $\mathcal{B}$ and $\{(0, 1)\}$ is greater than δ.

Proposition 3.13. *For $\mathcal{B}$ as above, for every positive real number δ there exists a curve γ such that*

1. $\|e - \gamma\|_{C^0} < \delta$,
2. *γ is an $\mathcal{B}$-directed embedding.*

Proof. Construct a loop $\tau = (\tau_x, \tau_y) : [0, 1] \to \mathcal{B}$ with $\tau(0) = \tau(1)$ satisfying

$$\text{average of } \tau := \int_0^1 \tau \, dt = (1, 0) \tag{3.14}$$

Such a loop exists because $(1, 0)$ lies inside the convex hull of $\mathcal{B}$, see Borrelli, 2012. For a positive integer N and $t \in [0, 1/N]$ define

$$\gamma(t) = \int_0^t \tau(sN) \, ds \tag{3.15}$$

Note that $\gamma(1/N) = (1/N, 0)$. We can then extend γ periodically over the entire segment $[0, 1]$ by setting $\gamma(1/N + t) = (1/N + t, 0)$ for $t \leq 1 - 1/N$.

For this γ we get

$$\begin{aligned}
\|\gamma - e\|_{C^0} &= \max_{0\le t\le 1/N} |\gamma(t) - (t,0)| \\
&= \max_{0\le t\le 1/N} \left| \int_0^t \tau(sN)\, ds - (t,0) \right| \\
&\le \sqrt{\frac{1}{N^2} + \max \frac{\tau_y^2}{N^2}}
\end{aligned}$$

By choosing N sufficiently large we can ensure $\|\gamma - e\|_{C^0} < \delta$. Up to smoothing γ is the embedding we are looking for.

To see that γ is an embedding note that $\tau_x' > 0$ and hence the γ_x is an increasing function of t. The γ that we have constructed is C^1 but any C^1 function can be approximated arbitrarily well by a smooth function, and a smooth function that is sufficiently C^1-close to an embedding is itself an embedding, see Hirsch, 1976, Theorem 1.4. Finally, because $\mathcal{B}$ is open, this smoothened embedding can also be chosen to be $\mathcal{B}$-directed. $\square$

The following properties of $\mathcal{B}$ made the above proof work.

1. $(1,0)$ lies in the convex hull of $\mathcal{B}$,
2. $\mathcal{B}$ is an open subset[1],
3. for all $(x,y) \in \mathcal{B}$, $x > 0$.

If we want the above proposition to be true for *every* embedding e then we

[1]It is sometimes possible to get rid of the openness condition and study closed differential relations, for example, see Nash-Kuiper's theorem Eliashberg and Mishachev, 2002, Chapter 21.

should require the convex hull of $\mathcal{B}$ to be the entire space $\mathbb{R}^2$. So we can replace condition 1) by

1. the convex hull of every connected component of $\mathcal{B}$ is the entire space $\mathbb{R}^2$.

It is a most remarkable fact that these conditions are sufficient to generalize the above methods to arbitrary manifolds.

3.2.2 Ample differential equations

To apply the above technique to arbitrary manifold we break up the manifold into sufficiently small cubical charts and runs the convex integration technique on each chart by induction in each coordinate direction. The coordinate independent version of the first and second condition above is called *ampleness* which we describe below.

Consider a subspace $\mathcal{A} \subseteq \mathrm{Gr}_m(N)$ and let $\mathcal{R}_{\mathcal{A}}$ be the associated differential relation (which defines $\mathcal{A}$-directed immersions). For a point $q \in N$, let $\mathcal{A}_q$ denote the fiber of $\mathcal{A}$ over q. Denote by $\mathrm{Gr}_{m-1}(\mathcal{A}_q)$ the subspace of $\mathrm{Gr}_{m-1}(T_qN)$ defined as

$$\mathrm{Gr}_{m-1}(\mathcal{A}_q) := \bigcup_{L \in \mathcal{A}_q} \mathrm{Gr}_{m-1}(L)$$

This is the space of all $m-1$ planes in T_qN which can be extended to get an m-plane in $\mathcal{A}$.

Definition 3.16. The differential relation $\mathcal{R}_{\mathcal{A}}$ as above, is said to be *ample* if $\mathcal{A}$ is an open subset of $\mathrm{Gr}_m(N)$ and for every $q \in N$ and every $S \in \mathrm{Gr}_{m-1}(\mathcal{A}_q)$

the set

$$\Omega_S = \left\{ v \in T_q N : \mathrm{Span}\left\{ S, v \right\} \in \mathcal{A}_q \right\}$$

has the property that the convex hull of the connected component is the entire space $T_q N$.

If the set Ω_S is connected and large enough then the above property will be trivially satisfied.

Lemma 3.17. *If $T_q N \setminus \Omega_S$ lies inside a subspace of $T_q N$ of dimension $\leq n-2$ then Ω_S is connected and its convex hull equals $T_q N$.*

Example 3.18. When $\mathcal{A} = \mathrm{Imm}$, for every $S \in \mathrm{Gr}_{m-1}(T_q N)$ the set Ω_S is the space of all vectors $v \in T_q N$ that are not in S where we're identifying S as an $m-1$ dimensional subspace of $T_q N$ so that $T_q N \setminus \Omega_S = S$. This has dimension $\leq n-2$ if $m \leq n-1$. This is precisely the codimension requirement in the Smale-Hirsch Theorem 1.1.

The third condition is a bit harder to impose as it is not clear how to ensure that a map $M \to N$ is an embedding just by looking at the tangential data. Instead, we look at a fiber bundle of smooth manifolds $E \to M$ and ask for a solution which is C^1-close to a section of this bundle, which will always be an embedding. Combining all of these observations together we get the following definition of the space of formal $\mathcal{A}$-directed embeddings.

Definition 3.19. For $\mathcal{A} \subseteq \mathrm{Gr}(N)$, a formal $\mathcal{A}$-directed embedding consists of a triple (e, F, γ) where

1. e is an embedding $M \to N$,

2. (e, F) is a formal $\mathcal{A}$-directed immersion, and

3. γ is a path in $\mathrm{Gr}_m(N)$ connecting De to F lying over the fixed base manifold $e(M)$.

$$\begin{tikzcd}
\gamma_0 = De \arrow[r, dash] & \gamma_t \arrow[r, dash] \arrow[d, "\mathrm{bs}"] & F = \gamma_1 \\
& e(M) \in \mathrm{Emb}(M, N) &
\end{tikzcd} \tag{3.20}$$

Given the embedding e we look at a tubular neighborhood ν of e and extend the path γ to it. For a sufficiently small t' we replace the differential relation $\mathcal{R}_{\mathcal{A}}$ by a differential relation that finds a solution which is C^1-close to a section of $\nu \to e(M)$ and then repeat the process starting at this new value of t'.

Definition 3.21. We say that $\mathcal{A} \subseteq \mathrm{Gr}_m(N)$ satisfies the h-principle for directed embeddings for a manifold M, if the natural inclusion

$$\mathrm{Emb}_{\mathcal{A}}(M, N) \longrightarrow \mathrm{Emb}^{\mathrm{f}}_{\mathcal{A}}(M, N) \tag{3.22}$$

is a weak homotopy equivalence.

Theorem 3.23 (Gromov, 1986)**.** *With notation as above, if $\mathcal{A}$ is ample then $\mathcal{A}$ satisfies the h-principles for directed immersions and directed embeddings for all matifolds M.*

There is an easy method to come up with a family of differential relation is *ample*; Gromov, 1986, Eliashberg and Mishachev, 2002, Theorems 18.4.2 and 19.4.1).

Definition 3.24. We say that a differential relation $\mathcal{R}$ is the **complement of a thin singularity** if the complement $J^1(M, N) \setminus \mathcal{R}$ is a simplicial complex of $J^1(M, N)$ of codimension ≥ 2.

Theorem 3.25 (Gromov). *For $\mathcal{A} \subseteq \mathrm{Gr}_m(M)$, if $\mathcal{R}_{\mathcal{A}}$ is the complement of a thin singularity then $\mathcal{A}$ satisfies the h-principle for directed immersions and embeddings for manifolds in* $\mathsf{Man_m}$.

This theorem is the primary technique to construct ample differential relations and gives us a large family of relations for which the *h*-principle applies.

Chapter 4

Main Theorems

This chapter contains the main results of this thesis. We show that the theories of manifold calculus and convex integration are 'compatible' with each other. The basic idea is that the existence of an *h*-principle (for directed embeddigs) gives us various homotopy equivalences which fit into homotopy pullback diagrams. Since the analytic approximation is defined as a homotopy limit (a derived right Kan extension), these commute for formal categorical reasons.

4.1 Main theorems

We start with a technical lemma that is repeatedly used in the study of *h*-principles to show that two spaces are weakly equivalent.

Definition 4.1. Let $X \subseteq Y$ be a pair of topological spaces. We say that (Y, X) satisfies the *formal h-principle* if for all good pairs of finite CW complexes $K \subseteq L$ and all maps

$$\phi : (L, K) \to (Y, X)$$

there exists a homotopy

$$\phi_t : (L, K) \times [0,1]_t \to (Y, X)$$

such that

1. $\phi_0 = \phi$,
2. $\phi_t(x) = \phi_0(x)$ for all $x \in K$ and $t \in [0,1]$,
3. $\phi_1(L) \subseteq X$.

$$\begin{tikzcd}
K \arrow[r, "\phi|_K"] \arrow[d] & X \arrow[d] \\
L \arrow[r, "\phi"'] & Y
\end{tikzcd}$$

$$\downarrow$$

$$\begin{tikzcd}
K \times [0,1] \arrow[r, "\phi_t|_K = \phi|_K"] \arrow[d] & X \arrow[d] \\
L \times [0,1] \arrow[r, dashed, "\phi_t"'] & Y
\end{tikzcd}
+
\begin{tikzcd}
 & X \arrow[d] \\
L \times \{1\} \arrow[ur, dashed] \arrow[r, "\phi_1"'] & Y
\end{tikzcd}$$

This is very similar *homotopy lifting property* for fibrations. The difference being that here we do not require $K \hookrightarrow L$ to be a weak homotopy equivalence.

Lemma 4.2. *The pair (Y, X) satisfies the formal h-principle if and only if the inclusion $X \hookrightarrow Y$ is a weak homotopy equivalence.*

Proof. We'll first prove the forward direction.

Suppose (Y, X) satisfy the formal h-principle. Let $K = *$ and $L = S^k$. Then $\phi : (S^k, *) \to (Y, X)$ represents an element $[\phi] \in \pi_k(Y)$. The existence of a ϕ_t such that $\phi_1(L) \subseteq X$ implies that $[\phi] = [\phi_1] \in \pi_1(X)$ which proves that the map $\pi_k(X) \to \pi_k(Y)$ is *surjective*.

If two maps ψ_0, $\psi_1 : S^k \to X$ representing elements in $\pi_k(X)$ become homotopic in Y then the homotopy can be described via a map $\phi : (S^{k+1} \times [0,1], S^k \times \{0,1\} \cup [0,1] \times \{*\}) \to (Y, X)$. Because $(S^k \times [0,1], S^k \times \{0,1\} \cup [0,1] \times \{*\})$ is a good pair we can homotope ϕ to a map ϕ_1 which defines a homotopy between ψ_0, ψ_1 lying entirely in X, which proves that the map $\pi_k(X) \to \pi_k(Y)$ is *injective*.

For the reverse direction, it suffices to consider the case when L is obtained from K by adding a single cell. Assume that $\pi_*(X) \cong \pi_*(Y)$. Let $L = K \cup e^l$ and consider a map $\phi : (L, K) \to (Y, X)$. Then Y/X is null-homotopic and hence the induced map $\phi : L/K \to Y/X$ which represents an element in $\pi_l(Y/X)$ is null-homotopic. This null-homotopy then lifts to a map $\phi_t : L \to Y$. $\square$

We can hence redefine the h-principles as saying that the differential relation $\mathcal{R} \in J^1(M, N)$ satisfies the h-principle if the pair $(\mathrm{Sol}^{\mathrm{f}}, \mathrm{Sol})$ satisfies the formal h-principle. Similarly, the space $\mathcal{A} \in \mathrm{Gr}_m(N)$ satisfies the h-principle for directed immersions (resp. embeddings) if the pair $(\mathrm{Imm}^{\mathrm{f}}_{\mathcal{A}}(M, N), \mathrm{Imm}_{\mathcal{A}}(M, N))$ (resp. $(\mathrm{Emb}^{\mathrm{f}}_{\mathcal{A}}(M, N), \mathrm{Emb}_{\mathcal{A}}(M, N))$ satisfies the formal h-principle.

Theorem 4.3. *If $\mathcal{A} \subseteq \mathrm{Gr}_m(N)$ satisfies the h-principle for directed embeddings for manifolds in $\mathsf{Man_m}$ and $\mathcal{A} \to N$ is a fibration then for every $M \in \mathsf{Man_m}$ the* *following pullback square is in fact a homotopy pullback square,*

$$\begin{tikzcd} \mathrm{Emb}_{\mathcal{A}}(M,N) \arrow[r] \arrow[d] & \mathrm{Maps}(M,\mathcal{A}) \arrow[d] \\ \mathrm{Emb}(M,N) \arrow[r] & \mathrm{Maps}(M,\mathrm{Gr}_m(N)) \end{tikzcd} \tag{4.4}$$

where the horizontal maps are defined as $e \mapsto \mathrm{Gr}_m(e)$ *and the vertical maps are inclusions.*

Proof. Let Θ denote the homotopy pullback of the diagram

$$\begin{tikzcd} & \mathrm{Maps}(M,\mathcal{A}) \arrow[d] \\ \mathrm{Emb}(M,N) \arrow[r] & \mathrm{Maps}(M,\mathrm{Gr}_m(N)) \end{tikzcd}$$

Explicitly Θ is the space of paths $\gamma : [0,1] \to \mathrm{Maps}(M, \mathrm{Gr}_m(N))$ satisfying

1. $\mathrm{bs}\,\gamma(0) \in \mathrm{Emb}(M,N)$ and

2. $\gamma(1) \in \mathrm{Maps}(M,\mathcal{A})$.

Note that there is a natural inclusion $\mathrm{Emb}^f_{\mathcal{A}}(M,N) \to \Theta$ sending a triple (e,F,γ) to γ.

In $\mathrm{Emb}^f_{\mathcal{A}}(M,N)$ the base embedding is fixed whereas in Θ it is allowed to vary. Furthermore, the natural map $\mathrm{Emb}_{\mathcal{A}}(M,N) \to \Theta$ that sends $e : M \hookrightarrow N$ to the path which is constant at $\mathrm{Gr}_m(e)$ factors through $\mathrm{Emb}^f_{\mathcal{A}}(M,N)$.

It suffices to show that the pair $(\Theta, \mathrm{Emb}^f_{\mathcal{A}}(M,N))$ satisfies the formal h-principle. This is a direct consequence of the fact that $\mathcal{A} \to N$ and $\mathrm{Gr}_m(N) \to N$ are fibrations, and hence so are $\mathrm{Maps}(M,\mathcal{A}) \to \mathrm{Maps}(M,N)$ and $\mathrm{Maps}(M,\mathrm{Gr}_m(N)) \to \mathrm{Maps}(M,N)$. The rest of the proof provides the technical details to make this precise. It is a standard homotopy-lifting argument for fibrations, however, as we're dealing with path spaces there is an extra dimension that we need to keep track of.

Suppose we are given a good pair of finite CW complexes (L, K) with a map

$$\phi : (L, K) \to (\Theta, \mathrm{Emb}^f_{\mathcal{A}}(M, N))$$

This is equivalent to a map

$$\phi_s : [0, 1]_s \times L \to \mathrm{Maps}(M, \mathrm{Gr}_m(N))$$

with

$$\mathrm{bs}\,\phi_0 \in \mathrm{Emb}(M, N)$$

$$\phi_1 \in \mathrm{Maps}(M, \mathcal{A})$$

$$\mathrm{bs}\,\phi|_K \text{ is the constant path at an embedding}$$

We need to construct a homotopy $\Phi_{s,t} : [0, 1]_s \times L \times [0, 1]_t \to \mathrm{Maps}(M, \mathrm{Gr}_m(N))$ connecting $\Phi_{s,0} = \phi_s$ to a path lying entirely in $\mathrm{Emb}^f_{\mathcal{A}}(M, N)$.

Define a map $\psi_{s,t} : [0, 1]_s \times L \times [0, 1]_t \to \mathrm{Emb}(M, N)$ as

$$\psi_{s,t} = \begin{cases} \mathrm{bs}\,\phi_s & \text{if } s \leq 1 - t \\ \mathrm{bs}\,\phi_{1-t} & \text{otherwise} \end{cases}$$

so that $\psi_{s,0} = \mathrm{bs}\,\phi_s$ and $\psi_{s,1} = \psi_{0,1} = \mathrm{bs}\,\phi_0$ for all $s \in [0, 1]$, and $\phi_{0,t} = \mathrm{bs}\,\phi_0$ for all $t \in [0, 1]$.

Let $S \subseteq [0, 1]_s \times L \times [0, 1]_t$ be the space $S = \{0\}_s \times L \times [0, 1]_t \cup [0, 1]_s \times$

$L \times \{0\}_t$. Then $S \hookrightarrow [0,1]_s \times L \times [0,1]_t$ is a trivial cofibration.

Define the map $\Phi_{s,t}$ on S as

$$\Phi_{0,t} := \phi_0$$

$$\Phi_{s,0} := \phi_s$$

Using the homotopy lifting property of the fibration pair

$$(\mathrm{Maps}(M, \mathrm{Gr}_m(N)), \mathrm{Maps}(M, \mathcal{A})) \to \mathrm{Maps}(M, N)$$

we can extend $\Phi_{s,t}$ to the entire space $[0,1]_s \times L \times [0,1]_t$.

$$\begin{tikzcd}
S \arrow[r, "\Phi_{s,t}"] \arrow[d] & (\mathrm{Maps}(M, \mathrm{Gr}_m(N)), \mathrm{Maps}(M, \mathcal{A})) \arrow[d, "\mathrm{bs}"] \\
{[0,1]_s \times L \times [0,1]_t} \arrow[r, "\psi_{s,t}"] \arrow[ur, dashed] & \mathrm{Maps}(M, N)
\end{tikzcd}$$

As $\mathrm{bs}\,\Phi_{s,1} = \psi_{s,1} = \psi_{0,1}$ for all $s \in [0,1]$, $\Phi_{s,1}$ is a map $L \to \mathrm{Emb}_{\mathcal{A}}{}^f(M, N)$. Further as K is a CW subcomplex of L we can require the lift to be constant along K and so $\Phi_{s,t}$ is the homotopy that witnesses the formal h-principle. $\square$

We can interpret the above theorem as saying that when $\mathcal{A}$ satisifes the h-principle for directed embeddings and $\mathcal{A} \to N$ is a fibration, the 'difference' (as measured by the homotopy fiber) between the space of all embeddings $\mathrm{Emb}(M, N)$ and $\mathrm{Emb}_{\mathcal{A}}(M, N)$ is purely tangential, and hence 'linear' in the sense of manifold calculus.

Theorem 4.5. *If $\mathcal{A} \to N$ is a fibration that satisfies the h-principles for directed immersions and directed embeddings for all manifolds in $\mathsf{Man_m}$ then the following* *pullback square is in fact a homotopy pullback square for all $M \in \mathsf{Man_m}$,*

$$\begin{tikzcd} \mathrm{Emb}_{\mathcal{A}}(M,N) \arrow[r] \arrow[d] & \mathrm{Imm}_{\mathcal{A}}(M,N) \arrow[d] \\ \mathrm{Emb}(M,N) \arrow[r] & \mathrm{Imm}(M,N) \end{tikzcd} \tag{4.6}$$

where all the maps are inclusions.

Proof. The homotopy pullback square (4.4) factors as

$$\begin{tikzcd} \mathrm{Emb}_{\mathcal{A}}(M,N) \arrow[r] \arrow[d] & \mathrm{Imm}_{\mathcal{A}}(M,N) \arrow[r] \arrow[d] & \mathrm{Imm}^{\mathrm{f}}_{\mathcal{A}}(M,N) \arrow[r] \arrow[d] & \mathrm{Maps}(M,\mathcal{A}) \arrow[d] \\ \mathrm{Emb}(M,N) \arrow[r] & \mathrm{Imm}(M,N) \arrow[r] & \mathrm{Imm}^{\mathrm{f}}(M,N) \arrow[r] & \mathrm{Maps}(M,\mathrm{Gr}_m(N)) \end{tikzcd}$$

where the maps $\mathrm{Imm}(M,N) \to \mathrm{Maps}(M,\mathrm{Gr}_m(N))$ and $\mathrm{Imm}_{\mathcal{A}}(M,N) \to \mathrm{Maps}(M,\mathcal{A})$ are defined as in Example 3.10.

The rightmost square

$$\begin{tikzcd} \mathrm{Imm}^{\mathrm{f}}_{\mathcal{A}}(M,N) \arrow[r] \arrow[d] & \mathrm{Maps}(M,\mathcal{A}) \arrow[d] \\ \mathrm{Imm}^{\mathrm{f}}(M,N) \arrow[r] & \mathrm{Maps}(M,\mathrm{Gr}_m(N)) \end{tikzcd}$$

is a pullback square by Example 3.10.

The middle square

$$\begin{tikzcd} \mathrm{Imm}_{\mathcal{A}}(M,N) \arrow[r] \arrow[d] & \mathrm{Imm}^{\mathrm{f}}_{\mathcal{A}}(M,N) \arrow[d] \\ \mathrm{Imm}(M,N) \arrow[r] & \mathrm{Imm}^{\mathrm{f}}(M,N) \end{tikzcd}$$

is a pullback square by definition. As $\mathcal{A}$ satisfies the h-principle for directed immersions, the horizontal arrows are weak homotopy equivalences, hence it is a homotopy pullback square.

We've already show in Theorem 4.3 that the larger square is a homotopy pullback square, hence the remaining leftmost square must also be one by the 2-out-of-3 property of homotopy pullbacks. □

With the above setup the analyticity of $\mathrm{Emb}_{\mathcal{A}}(-, N)$ follows from formal properties of right Kan extensions.

Lemma 4.7. *Given a small I-shaped diagram of analytic functors $F : I \to \mathrm{PSh}(\mathsf{Man_m})$, the homotopy limit $\mathrm{holim}_{i \in I} F_i$ is also analytic.*

Proof. For a small diagram of analytic functors $F : I \to \mathrm{PSh}(\mathsf{Man_m})$ we have,

$$(\mathcal{T}_\infty \operatorname*{holim}_I F_i)(M) = \mathrm{Hom}_{\mathsf{Disc}_\infty}(Q\,\mathrm{Emb}(-, M), \operatorname*{holim}_I F_i) \tag{4.8}$$

$$\simeq \operatorname*{holim}_I \mathrm{Hom}_{\mathsf{Disc}_\infty}(Q\,\mathrm{Emb}(-, M), F_i) \tag{4.9}$$

$$= \operatorname*{holim}_I (\mathcal{T}_\infty F_i)(M) \tag{4.10}$$

$$\simeq \operatorname*{holim}_I F_i(M) \tag{4.11}$$

where the equalities are by the definition of $\mathcal{T}_\infty$, the homotopy equivalence in (4.9) follows from the universal property of enriched holim and the homotopy equivalence in (4.11) follows from the analyticity of F. □

Theorem 4.12. *Let $n - m > 2$.*

1. *If $\mathcal{A} \subseteq \mathrm{Gr}_m(N)$ is a fibration over N that satisfies the h-principles for directed immersions and directed embeddings for all manifolds in $\mathsf{Man_m}$ then the functor*

$\mathrm{Emb}_{\mathcal{A}}(-,N)$ *in* $\mathrm{PSh}(\mathsf{Man_m})$ *is analytic i.e. the natural map*

$$\mathrm{Emb}_{\mathcal{A}}(M,N) \xrightarrow{\simeq} \mathcal{T}_\infty \mathrm{Emb}_{\mathcal{A}}(M,N)$$

is a homotopy equivalence for all manifolds $M \in \mathsf{Man_m}$.

2. *If further* $\mathcal{A}' \subseteq \mathcal{A}$ *is a fibration over* N *such that* $\mathcal{A}'$ *is homotopy equivalent to* $\mathcal{A}$, *then there is a homotopy equivalence*

$$\mathscr{T}_\infty \mathrm{Emb}_{\mathcal{A}'}(M,N) \simeq \mathrm{Emb}_{\mathcal{A}}(M,N)$$

for all manifolds $M \in \mathsf{Man_m}$.

Proof. For $n-m>2$, as mentioned in Example 2.16 and Theorem 2.17 the three functors

$$\mathrm{Maps}(-,\mathcal{A}),\ \mathrm{Maps}(-,\mathrm{Gr}_{\mathcal{A}}(N)),\ \mathrm{Emb}(-,N)$$

are analytic. By applying Lemma 4.7 to the homotopy pullback square from Theorem 4.3

$$\begin{tikzcd}
\mathrm{Emb}_{\mathcal{A}}(-,N) \arrow[r] \arrow[d] & \mathrm{Maps}(-,\mathcal{A}) \arrow[d] \\
\mathrm{Emb}(-,N) \arrow[r] & \mathrm{Maps}(-,\mathrm{Gr}_m(N))
\end{tikzcd}$$

we get the analyticity of $\mathrm{Emb}_{\mathcal{A}}(-,N)$.

As $\mathcal{A}' \simeq \mathcal{A}$, the inclusion of spaces $\mathrm{Emb}_{\mathcal{A}'}(M,N) \hookrightarrow \mathrm{Emb}_{\mathcal{A}}(M,N)$ is a homotopy equivalence when restricted to $M \in \mathsf{Disc}_\infty$. As $\mathscr{T}_\infty$ is defined to be the Kan extension along the inclusion $\mathsf{Disc}_\infty \hookrightarrow \mathsf{Man_m}$, there is a natural homotopy equivalence $\mathscr{T}_\infty \mathrm{Emb}_{\mathcal{A}'}(M,N) \xrightarrow{\simeq} \mathscr{T}_\infty \mathrm{Emb}_{\mathcal{A}}(M,N)$. The second part of the theorem follows from the analyticity of $\mathrm{Emb}_{\mathcal{A}}(-, N)$. $\square$

4.2 Application: Symplectic geometry

In this section, we apply the above framework to the Lagrangian embeddings functor. We'll assume that $n = 2m$.

We'll start by recalling some definitions from symplectic geometry. A good reference for all the basic definitions and results about symplectic and almost complex manifolds in this section is Silva, 2001.

A *symplectic manifold* is a pair (N, ω) there N is a smooth manifold and ω is a closed non-degenerate differential 2-form ω on N. Existence of a symplectic form on N forces it to be even dimensional, $n = 2m$. A submanifold L of N is called *Lagrangian* if $\dim L = m$ and ω vanishes when restricted to L.

4.2.1 Lagrangian embeddings

Let $\mathrm{Lag} \subseteq \mathrm{Gr}_m(N)$ be the subfibration over N whose fiber over each point $p \in N$ is the space of m dimensional subspaces of $T_p(N)$ on which ω vanishes. The spaces $\mathrm{Emb}_{\mathrm{Lag}}(M, N)$ of Lag-directed embeddings of M inside N, equals the space of Lagrangian embeddings. The space Lag does not satisfy the h-principles for directed immersions and embeddings. Instead, in this case we'll construct a subspace of $\mathrm{Gr}_m(N)$ containing Lag which is homotopy equivalent to it and which satisfies both the h-principles.

An *almost complex structure* J on a smooth manifold N is a linear isomorphism $J_p : T_pN \to T_pN$ for each $p \in N$, varying smoothly with p, satisfying $J^2 = -1$.

This is equivalent to requiring that N has even dimensions and it's structure group can be reduced to $\mathrm{GL}_m(\mathbb{C})$ i.e. the tangent bundle TN is naturally an m dimensional complex vector bundle.

For a point p in an almost complex manifold N, a real subspace V of T_pN is called *totally real* if $\dim V = m$ and $V \oplus JV = T_pN$. A submanifold N' of N is called totally real if T_pN' is a totally real subspace of T_pN for all points $p \in N'$.

Definition 4.13. Let $\mathrm{TR} \subseteq \mathrm{Gr}_m(N)$ be the subfibration whose fiber over $p \in N$ is the space of m dimensional totally real subspaces of $T_p(N)$.

The space $\mathrm{Emb}_{\mathrm{TR}}(M, N)$ of TR-directed embeddings of M inside N equals the space of totally real embeddings.

Definition 4.14. An almost complex structure J on a symplectic manifold N is said to be *compatible* with the symplectic structure if the following two conditions are satisfied,

1. $\omega(-, J-)$ defines a Riemannian metric on N,
2. $\omega(J-, J-) = \omega(-,-)$.

On every symplectic manifold N there exists a compatible almost complex structure which is unique up to homotopy. For the rest of this section we'll assume that (N, ω) is a symplectic manifold with a compatible almost complex structure J.

Compatibility of J with ω implies that all Lagrangian submanifolds are also totally real, hence there is a natural inclusion $\mathrm{Lag} \subseteq \mathrm{TR}$.

Proposition 4.15. *The inclusion* Lag $\hookrightarrow$ TR *is a homotopy equivalence.*

Proof. $\mathrm{GL}_m(\mathbb{C})$ acts transitively on the space of all totally real m dimensional subspaces of $\mathbb{C}^m$. The stabilizer of each subspace is $\mathrm{GL}_m(\mathbb{R})$ and hence the fiber of the fiber bundle $\mathrm{TR} \to N$ is diffeomorphic to $\mathrm{GL}_m(\mathbb{C}) / \mathrm{GL}_m(\mathbb{R})$. By the polar decomposition, the inclusions of the unitary group $\mathrm{U}(m) \subseteq \mathrm{GL}_m(\mathbb{C})$ and the orthogonal group $\mathrm{O}(m) \subseteq \mathrm{GL}_m(\mathbb{R})$ induce a homotopy equivalence

$$\mathrm{U}(m)/\mathrm{O}(m) \xrightarrow{\simeq} \mathrm{GL}_m(\mathbb{C}) / \mathrm{GL}_m(\mathbb{R})$$

The fiber of the bundle $\mathrm{Lag} \to N$, which is also called the *Lagrangian Grassmannian*, is known to be diffeomorphic to $\mathrm{U}(m)/\mathrm{O}(m)$; see Arnold, 1967. □

Proposition 4.16. *The differential $\mathcal{R}_{TR} \subseteq J^1(M, N)$ is an ample differential relation.*

Proof. Fix a point $q \in N$. Let S be an $m-1$ dimensional subspace of N which can be extended to an m dimensional totally real subspace of N so that $S \in \mathrm{Gr}_{m-1}(TR_q)$, as defined in 3.16. Because N is an almost complex manifold there is a natural hermitian inner product on T_qN and

$$\Omega_S = \left\{ v \in T_qN : v \in (S \oplus JS)^{\perp} \right\}.$$

So that $T_qN \setminus \Omega_S \cong S \oplus JS$ which has codimension at least 2. Hence, $\mathcal{R}_{TR}$ is ample. □

Combining this with Proposition 4.15 and Theorem 4.12 gives us the following result.

Theorem 4.17. *Let $n - m > 2$, $n = 2m$ and let N be a symplectic manifold with a compatible almost complex structure. Then the analytic approximation of $\mathrm{Emb}_{\mathrm{Lag}}(M, N)$ is homotopy equivalent to $\mathrm{Emb}_{\mathrm{TR}}(M, N)$ via a zig-zag of maps,*

$$\mathrm{Emb}_{\mathrm{TR}}(M, N) \xrightarrow{\simeq} \mathscr{T}_\infty \mathrm{Emb}_{\mathrm{TR}}(M, N) \xleftarrow{\simeq} \mathscr{T}_\infty \mathrm{Emb}_{\mathrm{Lag}}(M, N)$$

In general, we do not expect $\mathrm{Emb}_{\mathrm{Lag}}(-, N)$ to be analytic. For example, there are no simply connected Lagrangian submanifolds of $(\mathbb{C}^n, \omega)$, where ω is the standard symplectic structure, see Gromov, 1985, but S^3 can be embedded in $\mathbb{C}^3$ as a totally real manifold, Gromov, 1986. Thus $\mathrm{Emb}_{\mathrm{Lag}}(S^3, \mathbb{C}^3)$ is empty but $\mathscr{T}_\infty \mathrm{Emb}_{\mathrm{Lag}}(S^3, \mathbb{C}^3) \simeq \mathscr{T}_\infty \mathrm{Emb}_{\mathrm{TR}}(S^3, \mathbb{C}^3)$ is not.

Corollary 4.18. $\mathrm{Emb}_{\mathrm{Lag}}(-, \mathbb{C}^3)$ *is not analytic on the category of 3 dimensional smooth manifolds.*

Remark 4.19. The space that is of interest to us is the embeddings space $\mathrm{Emb}_{\mathrm{Lag}}(M, T^*M)$. In this case, it is not clear how much of the symplectic information is captured by $\mathrm{Emb}_{\mathrm{TR}}(M, N)$ and hence by $\mathscr{T}_\infty \mathrm{Emb}_{\mathrm{Lag}}(M, N)$.

Remark 4.20. Computations of $\pi_0(\mathrm{Emb}_{\mathrm{TR}}(M, \mathbb{C}^m))$ first appear in Audin, 1988. $\mathbb{C}^n$ is naturally an almost complex manifold. By Theorem 4.3 the following square is a homotopy pullback square.

$$\begin{tikzcd}
\mathrm{Emb}_{\mathrm{TR}}(M, \mathbb{C}^n) \arrow[r] \arrow[d] & \mathrm{Maps}(M, \mathrm{TR}) \arrow[d] \\
\mathrm{Emb}(M, \mathbb{C}^n) \arrow[r] & \mathrm{Maps}(M, \mathrm{Gr}_m(\mathbb{C}^n))
\end{tikzcd}$$

This pullback square was used in Borrelli, 2002 (without actually using manifold calculus) to compute $\pi_1(\mathrm{Emb}_{\mathrm{TR}}(M, \mathbb{C}^m))$.

Chapter 5

Tangential structures

We will now discuss extend manifold calculus to manifolds with tangential structures on them. The basic observation that allows one to use to manifold calculus to study manifolds with a tangential structure is the following: for a fixed manifold M and a presheaf $F \in \mathrm{PSh}(\mathsf{Man_m})$ the homotopy type of $\mathscr{T}_k F$ depends only upon the slice category $\mathsf{Disc_k} \downarrow M$.

5.1 Tangential structures

Let B be a topological space with a continuous map $B \xrightarrow{\theta} B\,\mathrm{GL}(n)$. A *manifold with a θ- structure* is a manifold M along with a map $M \to B$ such that the composite $M \to B \xrightarrow{\theta} B\,\mathrm{GL}(n)$ is a tangent space classifier.

$$\begin{tikzcd} & B \arrow[d, "\theta"] \\ M \arrow[ur] \arrow[r, "TM"'] & B\,\mathrm{GL}(m) \end{tikzcd}$$

We are only interested in the case when there is a family of groups $G_i \subseteq \mathrm{GL}(i)$ for every $i \in \mathbb{Z}_{\geq 0}$ and $B = BG(m)$ and the map $B \to B\,\mathrm{GL}(m)$ being the naturally induced map. We let $G = G(m)$ to simplify the notation (the dimenison m can be inferred from the dimension of the manifold M). In this case, we'll abuse notation and call such manifolds, manifolds with a tangential BG-structure. Having a tangential BG-structure on M is equivalent to saying that the structure group of M is reduced to G. Let $\mathsf{Man}_{\mathrm{m}}^{\mathrm{BG}}$ be the category of m dimensional manifolds with a BG-tangential structure and morphims being open embeddings that respect the BG-tangential structure.

Example 5.1.
1. For $G = \mathrm{GL}^{+}(m)$, $\mathsf{Man}_{\mathrm{m}}^{\mathrm{BG}}$ is the category of oriented manifolds.
2. For $G = O(m)$, $\mathsf{Man}_{\mathrm{m}}^{\mathrm{BG}}$ is the category of Riemannian manifolds.
3. For $G = \{e\}$ the trivial group, M is the category of framed parallelizable manifolds.

We will study this last example in Section 5.2.

For $M \in \mathsf{Man}_{\mathrm{m}}^{\mathrm{BG}}$ there is an isomorphism of categories

$$\mathsf{Man}_{\mathrm{m}} \downarrow M \cong \mathsf{Man}_{\mathrm{m}}^{\mathrm{BG}} \downarrow M$$

as the BG-structure uniquely (and naturally) restricts to every submanifold U of M. Hence, the theory of manifold calculus in Chapter 2 carries over naturally to $\mathsf{Man}_{\mathrm{m}}^{\mathrm{BG}}$.

As before, let $\mathrm{GL}_m(N)$ denote the m-frame bundle over N and let $\mathrm{Gr}_m(N)$ be the m-plane Grassmannian bundle over N.

Definition 5.2. For a subgroup G of $\mathrm{GL}_m(\mathbb{R}^n)$ define the ***m*-plane *G*-structured**

Grassmannian bundle over N to be the quotient space

$$\mathrm{Gr}_m^G(N) := \mathrm{GL}_m(N)/G$$

We'll identify subsets of $\mathrm{Gr}_m^G(N)$ with G-invariant subsets of $\mathrm{GL}_m(N)$.

$$\begin{tikzcd} \mathrm{GL}_m(\mathbb{R}^n)/G \arrow[r] & \mathrm{Gr}_m^G(N) \arrow[d, "\mathrm{bs}"'] \\ & N \end{tikzcd}$$

Example 5.3.

1. When $G = \mathrm{GL}_m(\mathbb{R})$, $\mathrm{Gr}_m^G(N)$ is the standard Grassmannian $\mathrm{Gr}_m(N)$.
2. When $G = \mathrm{GL}_m^+(\mathbb{R})$, $\mathrm{Gr}_m^G(N)$ is the oriented Grassmannian.
3. When G is the trivial group, $\mathrm{Gr}_m^G(N)$ equals the frame bundle $\mathrm{GL}_m(M)$ itself.

When M is a manifold with a BG-tangential structure a subset $\mathcal{A}$ of $\mathrm{Gr}_m^G(N)$ defines a differential relation $\mathcal{R}_\mathcal{A}$ as follows. The map $M \to BG$ defines a principal G-bundle P over M which is a subbundle of $B\,\mathrm{GL}_m(N)$ which is a pullback of the canonical principal G bundle $EG \to BG$.

$$\begin{tikzcd} P \arrow[r] \arrow[d] & EG \arrow[d] \\ M \arrow[r] & BG \end{tikzcd}$$

An immersion $f : M \to N$ defines a $\mathrm{GL}_m(\mathbb{R}^m)$ equivariant map on the frame bundles $\mathrm{GL}_m(f) : \mathrm{GL}_m(M) \to \mathrm{GL}_m(N)$ which we restrict to P to get a G-equivariant map $\mathrm{GL}_m(f)|_P : P \to \mathrm{GL}_m(N)$. Quotienting out the G action we get a map,

$$\mathrm{Gr}_m^G(f) : M \to \mathrm{Gr}_m^G(N).$$

Definition 5.4. For a manifold M with a B-tangential structure and a subset $\mathcal{A}$ of $\mathrm{Gr}(N)$, we say that an immersion (or an embedding) $i : M \to N$ is *$\mathcal{A}$-directed* if $\mathrm{Gr}_m^G(i) \subseteq \mathcal{A}$.

The differential relations defined using subsets of $\mathrm{GL}_m^G(N)$ are *finer* than the differential relations defined using subsets of $\mathrm{GL}_m(N)$ and one can create interesting functors using this refinement, we'll see an example of this below.

Consider a subspace $\mathcal{A} \subseteq \mathrm{Gr}_m^G(N)$ and let $\mathcal{R}_{\mathcal{A}}$ be the associated differential relation (which defines $\mathcal{A}$-directed immerions). For a point $q \in N$, let $\mathcal{A}_q$ denote the fiber of $\mathcal{A}$ over q. Denote by $\mathrm{Gr}_{m-1}^G(\mathcal{A}_q)$ the subspace of $\mathrm{Gr}_{m-1}^G(T_qN) = \mathrm{Gr}_{m-1}(T_qN)/G(m-1)$ defined as

$$\mathrm{Gr}_{m-1}^G(\mathcal{A}_q) := \bigcup_{L \in \mathcal{A}_q} \mathrm{Gr}_{m-1}^G(L)$$

This is the space of all $m-1$ dimensional $G(m-1)$-equivariant frames in T_qN which can be extended to get an m-plane in $\mathcal{A}$.

Definition 5.5. The differential relation $\mathcal{R}_{\mathcal{A}}$ as above, is said to be *ample* if $\mathcal{A}$ is an open subset of $\mathrm{Gr}_m^G(N)$ and for every $q \in N$ and every $S \in \mathrm{Gr}_{m-1}^G(\mathcal{A}_q)$ the set

$$\Omega_S^G = \left\{ v \in T_qN : \mathrm{Span}\,\{S, v\} \in \mathcal{A}_q \right\}$$

has the property that the convex hull of the connected component is the entire space T_qN.

If the set Ω_S^G is connected and large enough then the above property will be trivially satisfied.

Proposition 5.6. *With the notation as above, if $T_qN \setminus \Omega_S^G$ lies inside a subspace of T_qN of dimension $\leq n-2$ then Ω_S is connected and its convex hull equals T_qN. Hence, the associated differential relation $\mathcal{R}_{\mathcal{A}}$ satisfies the h-princple for directed immersions and embeddings for manifolds in $\mathsf{Man}_{\mathsf{m}}^{\mathsf{BG}}$.*

5.2 Application: Parallelizable manifolds

Let $G = \{e\}$ be the trivial group, so that we're looking at m-dimensional framed parallelizable manifolds. We'll let our target manifold N equal $\mathbb{R}^n$, so we're studying embeddings of parallelizable manifolds inside Euclidean spaces.

We are only interested in the m dimensional manifolds M which are subsets of $\mathbb{R}^n$. For such an M we can define

$$\overline{\text{Emb}}(M, N) := \text{hofib}\,(\text{Emb}(M, N) \to \text{Imm}(M, N))$$

as the homotopy fiber taken over the connected component of $\text{Imm}(M, N)$ containing the inclusion $M \subseteq \mathbb{R}^n$.

As G is trivial, $\text{Gr}_m^G(N)$ equals the frame bundle $\text{GL}_m(N) = \mathbb{R}^n \times \text{GL}_m(\mathbb{R}^n)$, where we are abusing notation and letting $\text{GL}_m(\mathbb{R}^n)$ denote the space of m linearly independent vectors in the *vector space* $\mathbb{R}^n$. Let $\{\vec{e}_i\}_{i=1}^n$ be the standard basis for $\mathbb{R}^n$. Let

$$\text{TS} := \mathbb{R}^n \times \{(\vec{e}_1, \cdots, \vec{e}_m)\} \subseteq \text{Gr}_m^G(N)$$

The space $\mathrm{Emb}_{\mathrm{TS}}(M, N)$ can be thought of as the space of **tangentially straightened** embeddings of M in N. We're borrowing the terminology from Dwyer and Hess, 2012 where it is used in a slightly different context.

Theorem 5.7. *When $n - m > 2$ and M is an m dimensional submanifold of $N = \mathbb{R}^n$, there is a natural homotopy equivalence*

$$\mathscr{T}_\infty \mathrm{Emb}_{\mathrm{TS}}(M, N) \simeq \overline{\mathrm{Emb}}(M, N)$$

Proof. Note that TS $\to N$ is a fibration. We'll construct a fibration $\mathcal{A} \subseteq \mathrm{Gr}_m^G(N)$ over N containing TS and homotopy equivalent to it and satisfying the h-principles for directed immersions and embeddings for parallelizable manifolds. Once we have this we'll get a homotopy equivalence,

$$\mathscr{T}_\infty \mathrm{Emb}_{\mathrm{TS}}(M, N) \xrightarrow{\simeq} \mathscr{T}_\infty \mathrm{Emb}_{\mathcal{A}}(M, N) \xleftarrow{\simeq} \mathrm{Emb}_{\mathcal{A}}(M, N)$$

by Theorem 4.3 and Theorem 4.12. And by Theorem 4.5 the square

$$\begin{tikzcd}
\mathrm{Emb}_{\mathcal{A}}(M, N) \arrow[r] \arrow[d] & \mathrm{Imm}_{\mathcal{A}}(M, N) \arrow[d] \\
\mathrm{Emb}(M, N) \arrow[r] & \mathrm{Imm}(M, N)
\end{tikzcd}$$

will be a homotopy pullback square. The proof will then be completed once we show that $\mathrm{Imm}_{\mathcal{A}}(M, N) \simeq *$.

Define $\mathcal{A}_m(\mathbb{R}^n) \subseteq \mathrm{GL}_m(\mathbb{R}^n)$ to be the set of of m-frames $(\vec{v}_1, \ldots, \vec{v}_m)$ satisfying the condition

$$\vec{v}_i \neq -k\vec{e}_i$$

for any $k \in \mathbb{R}$, $k \geq 0$ and all $1 \leq i \leq m$. Let

$$\mathcal{A} := \mathbb{R}^n \times \mathcal{A}_m(\mathbb{R}^n) \subseteq \mathrm{Gr}_m^G(N)$$

It is clear than $TS \subseteq \mathcal{A}$ and that $\mathrm{bs} : \mathcal{A} \to N$ is a fibration.

When $m = 1$, the space

$$\mathcal{A}_1(\mathbb{R}^n) = \{v \in \mathbb{R}^n : v \neq -ke_1 \text{ for any } k \in \mathbb{R}, k > 0\}$$

is diffeomorphic to $(S^{n-1} \setminus \{*\}) \times \mathbb{R}$ which is contractible. For $m > 1$ there is a natural forgetful map $f : \mathcal{A}_m(\mathbb{R}^n) \to \mathcal{A}_1(\mathbb{R}^n)$ sending $(\vec{v}_1, \ldots, \vec{v}_m)$ to $\vec{v}_1$. The map f is a fibration with the fiber being homotopy equivalent to $\mathcal{A}_{m-1}(\mathbb{R}^{n-1})$.

$$\begin{tikzcd}
\mathcal{A}_{m-1}(\mathbb{R}^{n-1}) \arrow[r] & \mathcal{A}_m(\mathbb{R}^n) \arrow[d] \\
& \mathcal{A}_1(\mathbb{R}^n)
\end{tikzcd}$$

The base $\mathcal{A}_1(\mathbb{R}^n)$ is contractible and hence $\mathcal{A}_m(\mathbb{R}^n) \simeq \mathcal{A}_{m-1}(\mathbb{R}^{n-1})$. Repeatedly applying this argument we get $\mathcal{A}_m(\mathbb{R}^n) \simeq \mathcal{A}_1(\mathbb{R}^{n-m+1})$ which is contractible.

Let $\mathcal{R}_{\mathcal{A}} \subseteq J^1(M, N)$ be the differential relation defined by $\mathcal{A}$. We need to find the codimension of the complement $J^1(M, N) \setminus \mathcal{R}_{\mathcal{A}}$. The complement is a union of m spaces each diffeomorphic to $\mathcal{R}_{\mathcal{A}'}$ where $\mathcal{A}' = \mathbb{R}^n \times \mathcal{A}'_m(\mathbb{R}^n)$ and $\mathcal{A}'_m(\mathbb{R}^n)$ is the set of m frames $(\vec{v}_1, \ldots, \vec{v}_m)$ satisfying

$$\vec{v}_1 = -k\vec{e}_1$$

for some $k \in \mathbb{R}$, $k \geq 0$. It is easy to see that $\mathcal{A}'_m(\mathbb{R}^n)$ is diffeomorphic to

$\mathbb{R} \times F_{m-1}(\mathbb{R}^{n-1})$. This has codimension at least $n - m$ inside $\mathrm{GL}_m(\mathbb{R}^n)$, which is > 2. Thus $\mathcal{R}_{\mathcal{A}}$ is the complement of a thin singularity and by Theorem 3.25, $\mathcal{A}$ satisfies the h-principles for immersions and embeddings.

Finally, there is a homotopy equivalence $\mathrm{Imm}_{\mathcal{A}}(M, N) \simeq \mathrm{Imm}^f_{\mathcal{A}}(M, N)$ as $\mathcal{A}$ satisfies the homotopy principle for directed immersions. As $G = \{e\}$, we can identify $\mathrm{Imm}^f_{\mathcal{A}}(M, N)$ with $\mathrm{Maps}(M, \mathcal{A})$. Contractibility of $\mathcal{A}$ implies that $\mathrm{Imm}_{\mathcal{A}}(M, N) \simeq *$, which completes the proof. $\square$

Unless M is diffeomorphic to an open subset of $\mathbb{R}^m$ the space $\mathrm{Emb}_{\mathrm{TS}}(M, \mathbb{R}^n)$ is empty. However, as $M \subseteq \mathbb{R}^n$ the homotopy fiber of $\mathrm{Emb}(M, \mathbb{R}^n) \to \mathrm{Imm}(M, \mathbb{R}^n)$ is non-empty and hence so is the analytic approximation $\mathscr{T}_\infty \mathrm{Emb}_{\mathrm{TS}}(M, \mathbb{R}^n)$. Thus, the functor $\mathrm{Emb}_{\mathrm{TS}}$ is highly *non-analytic*.

Chapter 6

Future Directions

We state some work in progress and some future projects that we intend to study in the future.

6.1 G-manifolds

Let $\mathsf{Man}_{\mathsf{m}}^{\mathsf{G}}$ be the category whose objects are manifolds of dimension m with a finite group G acting on them. There are two possible ways to define the space of morphisms:

1. (naive setting)[1] The space of morphisms between M_1, M_2 is the space of G-equivariant embeddings $\mathrm{Emb}^G(M_1, M_2)$. In this case the category is enriched over Spaces.

2. (genuine setting) The space of morphisms between M_1, M_2 is the space of all embeddings $\mathrm{Emb}(M_1, M_2)$. In this case, the category is enriched over Spaces with a G action.

[1] We're borrowing this terminology from its stardand usage in equivariant homotopy theory.

The second (genuine) setting is the more interesting situation, but equivariant homotopy theory needs a lot of machinery to setup, especially since we are dealing with manifolds with group actions. In this thesis, we'll study the first (naive) setting. For us, the space of morphisms of $\mathsf{Man}^{\mathrm{G}}_{\mathrm{m}}$ will be the space of G-equivariant embeddings $\mathrm{Emb}^{\mathrm{G}}(-,-)$.

As in the non-equivariant setting, we need to study the category of discs inside $\mathsf{Man}^{\mathrm{G}}_{\mathrm{m}}$. If a point p in a G-manifold M let G_p denote the isotropy subgroup of p i.e. elements of G that fix p. The tangent space T_pM is a G_p representation. We need the category $\mathsf{Disc}^{\mathrm{G}}_{\infty}$ to capture all possible neighbourhoods of points. This motivates the following definition.

Definition 6.1. Define the category $\mathsf{Disc}^{\mathrm{G}}_{\infty}$ to be the category of G-manifolds which are G-equivariantly diffeomorphic to finite direct sums of the following manifolds,

$$\mathrm{ind}^{G}_{H}(\rho) = G \times_H \rho$$

where ρ is a representation of a subgroup H of G.

Definition 6.2. For a valued presheaf $F : (\mathsf{Man}^{\mathrm{G}}_{\mathrm{m}})^{op} \to \mathsf{Spaces}$, define its analytic approximation $\mathscr{T}_\infty F$ to be the (derived) right Kan extension of F along the inclusion $\mathsf{Disc}^{\mathrm{G}}_{\infty} \hookrightarrow \mathsf{Man}^{\mathrm{G}}_{\mathrm{m}}$.

$$\begin{tikzcd} \mathsf{Disc}^{\mathrm{G}}_{\infty} \arrow[r, "F|_{\mathsf{Disc}^{\mathrm{G}}_{\infty}}"] \arrow[d] & \mathsf{Spaces} \\ \mathsf{Man}^{\mathrm{G}}_{\mathrm{m}} \arrow[ur, dashed, "\mathscr{T}_\infty F"'] & \end{tikzcd} \tag{6.3}$$

6.1.1 Equivariant embeddings

We are primarily interested in studying the functor $\mathrm{Emb}^{\mathrm{G}}(-, N)$ where N is a G-manifold.

Conjecture 6.4. *With the notation as above, if* $\dim N^H - \dim M^H \geq 3$ *for all subgroups* $H \leq G$ *the* $\mathrm{Emb}^{\mathrm{G}}(-, N)$ *is an analytic functor.*

We provide some partial results toward proving this conjecture using techniques similar to the ones in Chapter 4.

From now on, we'll let $G = C_p$ the cyclic group of prime order.

In order to study the functor $\mathrm{Emb}^{\mathrm{G}}(-, N)$ we break the manifold M into strata

$$\begin{aligned} M^G &= \text{ points of } M \text{ with stabilizer group } G \\ &= \text{ the } G \text{ fixed points of } M \\ M^{\mathbb{1}} &= \text{ points of } M \text{ with the trivial stabilizer group} \\ &= \text{ the free orbits of } M \end{aligned}$$

Remark 6.5. This is a non-standard notation. In the standard notation, M^H denotes the space of points fixed by H but here we mean it to be the space of points with stabilizer group H. We are using this notation to avoid clutter.

M^G and $M^{\mathbb{1}}$ are G-manifolds with G-action being trivial on M^G and free on $M^{\mathbb{1}}$. Neither of these two spaces contain interesting G-structures, instead all the information is contained in the gluing map $M^G \hookrightarrow \overline{M^{\mathbb{1}}}$. [2] Embeddings respect this stratification. If $e \in \mathrm{Emb}^G(M,N)$ then $e(M^G) \subseteq N^G$ and $e(M^{\mathbb{1}}) \subseteq N^{\mathbb{1}}$. In order to obtain information about the gluing data we will work with tubular neighorhoods of M^G and N^G.

Let $\nu(M^G)$ be a tubular neighborhood of M^G. There is a natural restriction map

$$\mathrm{Emb}^G(M,N) \to \mathrm{Emb}^G(\nu(M^G),N) \tag{6.6}$$

This is a fibration by *isotopy extension theorem*, Palais, 1960, Lima, 1963. The fiber over a point $e \in \mathrm{Emb}^G(\nu(M^G),N)$ consists of those embeddings $M \hookrightarrow N$ which agree over $\nu(M^G)$ with e. Denote this space by $\mathrm{Emb}^G_\partial(M,N)$.

$$\begin{tikzcd} \mathrm{Emb}^G_\partial(M,N) \arrow[r] & \mathrm{Emb}^G(M,N) \arrow[d] \\ & \mathrm{Emb}^G(\nu(M^G),N) \end{tikzcd} \tag{6.7}$$

As $\mathscr{T}_\infty$ is a right Kan extension, by arguments similar to the ones in Chapter 4 $\mathscr{T}_\infty$ preserves homotopy colimits in the equivariant setting as well.

The primary difficulty is the following: the functors $\mathrm{Emb}^G_\partial(-,N)$ and $\mathrm{Emb}^G(\nu(-^G),N)$ are not canonically defined on the category $\mathsf{Man}^G_\mathrm{m}$. They are however well-defined up to homotopy. The strategy then is to study the categories $\mathsf{Man}^G_\mathrm{m} \downarrow \nu(M^G)$ and a suitable category of manifolds with boundary of $\mathsf{Man}^G_\mathrm{m} \downarrow (M \setminus \nu(M^G))$ and relating the two using double categories, as done in Weiss, 1999. Assuming that this can be done in a compatible way, we get a

[2] When $G \neq C_p$, we need to decompose M into more than two strata corresponding to the subgroups of G, and these strata do not intersect nicely. While the current strategy is still doable a lot more care technical details need to be checked.

map of fibrations.

$$\begin{array}{ccccc} \mathrm{Emb}^G_\partial(M,N) & \longrightarrow & \mathrm{Emb}^G(M,N) & \longrightarrow & \mathrm{Emb}^G(\nu(M^G),N) \\ \downarrow & & \downarrow & & \downarrow \\ \mathscr{T}_\infty \mathrm{Emb}^G_\partial(M,N) & \longrightarrow & \mathscr{T}_\infty \mathrm{Emb}^G(M,N) & \longrightarrow & \mathscr{T}_\infty \mathrm{Emb}^G(\nu(M^G),N). \end{array} \tag{6.8}$$

To show that the middle map is a weak homotopy equivalence it suffices to show that the left and right maps are as well.

$\mathrm{Emb}^G(\nu(M^G),N)$ **:** because embeddings respect the isotropy subgroups we can identify this space with

$$\mathrm{Emb}^G_{\mathrm{Bun}}(\nu(M^G),N = \{ \text{ vector bundle maps } TM|_{M^G} \to TM|_{N^G}$$
$$\text{lying over a smooth map } M^G \to N^G\}$$

This space is a fiber bundle over $\mathrm{Emb}(M^G,N^G)$ which we know to be an analytic functor. The fiber is the space of monomorphisms and hence is a linear functor, and hence also an analytic functor. By invoking the 2-out-3 property again, we get the analyticity of $\mathrm{Emb}^G(\nu(-^G),N)$.

$\mathrm{Emb}^G_\partial(M,N)$ **:** this functor has a fixed value on the space M^G hence this is really an embedding functor on $\mathrm{Emb}^G_\partial(M^{\mathbb{1}},N^{\mathbb{1}})$ with some boundary restrictions. The G-action on the space $M^{\mathbb{1}}$ is free, hence $\mathrm{Emb}^G_\partial(M^{\mathbb{1}},N^{\mathbb{1}})$ is a principle G-bundle over $\mathrm{Emb}_\partial(M^{\mathbb{1}}/G,N^{\mathbb{1}}/G)$. This is also known to be analytic by Goodwillie and Weiss, 1999.

6.2 Symplectic tower

Our results about the analytic approximation of the Lagrangian embeddings functor suggests that manifold calculus only sees the flexible side of symplectic geometry. In order to study the rigid side then one would need to construct a finer tower which sees Floer theoretic data like J-holomorphic curves. It in unclear if such a thing is possible given that there has been very little success in combining Floer theory and homotopy theory.

6.3 Reformulating manifold calculus using convex integration

This thesis proves that the technique of convex integration is compatible with manifold calculus, at least for the case of directed embeddings. This is not that surprising given that both manifold calculus and convex integrations are local-to-global techniques. However, convex integration is a perturbative technique and is much easier to generalize and implement in practice.

In the future, we intend on working on a reformulation of manifold calculus that uses ideas from convex integration.