The construction of the Adams SS looks very ad hoc, it turns out that it is a very simple SS once recognize the Bar resolution as being some Monadic injective resolution.

1. Adjunctions

Adjunctions are the most basic examples of Monads. Consider an adjunction pair $\mathcal{F}, \mathcal{G}$

$$\mathcal{F} : \mathcal{A} \leftrightarrows \mathcal{B} : \mathcal{G}$$

$$\tau : \hom(\mathcal{F}A, B) \xrightarrow{\simeq} \hom(A, \mathcal{G}B)$$

Using τ one can create unit $\mathbb{1} : id \implies \mathcal{G}\mathcal{F}$ and counit $\mathbb{1}^* : \mathcal{F}\mathcal{G} \implies id$ natural transformations via the following identifications

$$(\mathcal{F}A \xrightarrow{id} \mathcal{F}A) \xrightarrow{\tau} (A \xrightarrow{\mathbb{1}_A} \mathcal{G}\mathcal{F}A)$$

$$(\mathcal{F}\mathcal{G}B \xrightarrow{\mathbb{1}^*_B} B) \xrightarrow{\tau} (\mathcal{G}B \xrightarrow{id} \mathcal{G}B)$$

One can rewrite τ in terms of $\mathbb{1}$ as follows. Given a map $f \in \hom(\mathcal{F}A, B)$ we can construct τf by tracing $id_{\mathcal{F}A}$ in the following map

$$\begin{tikzcd}
id_{\mathcal{F}A} \arrow[d, mapsto, "\tau"] & \hom(\mathcal{F}A, \mathcal{F}A) \arrow[r, "\hom(-,f)"] \arrow[d, mapsto, "\tau"] & \hom(\mathcal{F}A, B) \arrow[d, mapsto, "\tau"] \\
\mathbb{1}_A & \hom(A, \mathcal{G}\mathcal{F}A) \arrow[r, "\hom(-,Gf)"] & \hom(A, \mathcal{G}B)
\end{tikzcd}$$

which gives us the identity

$$\tau(f) = \mathcal{G}f \circ \mathbb{1}_A$$

and similarly there is a dual identity for τ^{-1} in terms of $\mathbb{1}^*$.

2. Monads

Monads and comonads generalize the definition of an adjunction. Rather they rewrite adjunctions as an algebra structure on a category which allows us to do homological algebra.

Definition 2.1. A **monad** is a triple $(\mathcal{C}, \top, *, \mathbb{1})$ where

$\mathcal{C}$ is a category,
$\top : \mathcal{C} \to \mathcal{C}$ is an endofunctor,
$* : \top\top \implies \top$ is a natural transformation,
$\mathbb{1} : id \implies \top$ is a natural transformation,

such that the following diagrams commute

$$\begin{tikzcd} \top\top\top \arrow[r, Rightarrow, "*\top"] \arrow[d, Rightarrow, "\top *"'] & \top\top \arrow[d, Rightarrow, "*"] \\ \top\top \arrow[r, Rightarrow, "*"] & \top \end{tikzcd} \qquad \begin{tikzcd} \top \arrow[r, Rightarrow, "\mathbb{1}"] \arrow[dr, Rightarrow, "id"'] & \top\top \arrow[d, Rightarrow, "*"] \\ & \top \end{tikzcd}$$

I'll suppress the multiplication $*$ though it is by no means obvious or trivial in any way. The categorical dual of a monad is called a **comonad** which I'll denote by $(\mathcal{C}^{op}, \perp, \mathbb{1}^*)$.

We think of the powers of $\top$ as forming a monoid and $\mathcal{A}$ is a module over it, similarly comonads give us a comodule structure.

Adjunctions naturally give rise to a monad structure. With the notation as above, we have a monad

$$(\mathcal{A}, \mathcal{GF}, \mathbb{1})$$

The multiplication $\mathcal{GFGF} \implies \mathcal{GF}$ is given by $\mathcal{G}\mathbb{1}^*_{\mathcal{F}}$ (that is we are applying counit to the middle $\mathcal{FG}$). Both axioms are proven by diagram chasing.

Similarly $(\mathcal{B}, \mathcal{FG}, \mathbb{1}^*)$ has a natural comonad structure.

Of interest to us is the following monad:

$$(Spectra, E \wedge -, \mathbb{1})$$

where E is a multiplicative ring spectrum and $\mathbb{1}$ is it's unit.

3. Injective Resolutions

A monad structure allows us to define the notion of injective resolutions with respect to the monad structure, similarly a comonad structure allows us to define projective resolutions. These resolutions then give rise to the Adams SS.

Consider a monad $(\mathcal{C}, \top, \mathbb{1})$.

Definition 3.1. We say that an object $I \in \mathcal{C}$ is $\top$**-injective** if the unit map $\mathbb{1}_I : I \to \top I$ has a retraction that is there is a map $f : \top I \to I$ such that $f \circ \mathbb{1}_I = id_I$.

This is the definition that Weibel uses, however Weibel is mostly concerned with abelian categories and there this seems to be sufficient. I do not think this definition is strong enough for the category of Spectra.

Proposition 3.2 (Injective extension property)**.** *For the monad $(\mathcal{A}, \mathcal{F}\mathcal{G}, \mathbb{1})$ an object I is $\mathcal{F}\mathcal{G}$-injective if and only if for all maps $A_1 \xrightarrow{i} A_2$ such that $\mathcal{G}A_1 \xrightarrow{\mathcal{G}i} \mathcal{G}A_2$ is a retraction, any map $A_1 \to I$ extends to $A_2 \to I$.*

Proof. **if:** We are given morphisms

$$\begin{tikzcd} & I & \\ A_1 \arrow[ur] \arrow[rr, "i"] & & A_2 \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{G}I & \\ \mathcal{G}A_1 \arrow[ur] \arrow[rr, shift left, "\mathcal{G}i"] & & \mathcal{G}A_2 \arrow[ll, shift left] \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, shift left, "\mathbb{1}_I"] & \mathcal{F}\mathcal{G}I \arrow[l, shift left] \end{tikzcd}$$

The second diagram gives us a map $\mathcal{G}A_2 \to \mathcal{G}I$ which by adjunction becomes a map $A_2 \to \mathcal{F}\mathcal{G}I$. Then we use the retraction in the third diagram to get a map $A_2 \to I$. It is easy to check that the triangles commute.

only if: Let $A_1 = I$ and $A_2 = \mathcal{F}\mathcal{G}I$ then we have a map

$$\mathcal{G}A_2 \to \mathcal{G}A_1 = \mathcal{G}\mathcal{F}\mathcal{G}I \to \mathcal{G}I = \mathcal{G}\mathbb{1}_I^*$$

which is a retraction and so $\mathbb{1}_I : I \to \mathcal{F}\mathcal{G}I$ satisfies the required conditions and the identity map $I \to I$ extends to a map $\mathcal{F}\mathcal{G}I \to I$ which is the required retraction making I injective. $\square$

Corollary 3.3. *For any object A, $\mathcal{G}A$ is $\mathcal{F}\mathcal{G}$-injective.*

Proposition 3.4. *For an arbitrary monad $(\mathcal{C}, \top, \mathbb{1})$ an object I is $\top$-injective if there is an object $C \in \mathcal{C}$ and morphisms $r : \top C \to I$ and $i : I \to \top C$ such that $r \circ i = id_C$.*

Proof. **if:** Pick $C = I$.

only if: We are given maps

$$\begin{tikzcd} I \arrow[r, shift left, "i"] & \top A \arrow[l, shift left, "r"] \end{tikzcd}$$

Applying $\top$ to the above map and applying the monad multiplication gives us

$$\begin{tikzcd} \top I \arrow[r, shift left, "\top i"] & \top\top A \arrow[l, shift left, "\top r"] \arrow[r] & \top A \arrow[r, "r"] & I \end{tikzcd}$$

Again it is easy to check that this is indeed a $\square$

Corollary 3.5. *For any object $A \in \mathcal{C}$, $\top A$ is $\top$-injective.*

4. The barr resolution

A monad $(\mathcal{C}, \top, \mathbb{1})$ naturally gives rise to a simplicial object whose homotopy groups then give us the Adams SS when C is the category of *Spectra*.

We associate a **cosimplicial object** to the monad $(\mathcal{C}, \top, \mathbb{1})$ and an object $C \in \mathcal{C}$ which we denote by $\top^* C$. The n^{th} object of this cosimplicial object is given by

$$\top^n := \top^{\circ(n+1)} C$$

The boundary and the degeneracy maps are given by

$$\begin{aligned} \delta^i : \top^n C &\to \top^{n+1} C && \text{for } 0 \le i \le n+1 \\ \top^{\circ(n-i+1)} \circ \{id\} \circ \top^{\circ i} c &\mapsto \top^{\circ(n-i+1)} \circ \{\top\} \circ \top^{\circ i} c \\ \sigma^i : \top^n C &\to \top^{n-1} C && \text{for } 0 \le i \le n-1 \\ \top^{\circ(n-i-1)} \circ \{\top \circ \top\} \circ \top^{\circ i} c &\mapsto \top^{\circ(n-i)} \circ \{\top\} \circ \top^{\circ i} c \end{aligned}$$

where the coboundary maps are given by the counit maps and degeneracy maps are given by the monad multiplication.

We can form a cochain complex out of this cosimplicial set with the differentials given by the total differential $\sum (-1)^i \delta^i$. This cochain is naturally augemented by the map

$$C \xrightarrow{\mathbb{1}_C} \top^* C$$

Proposition 4.1. *If C is $\top$-injective then the cochain complex $0 \to C \to \top^* C$ is exact.*

Proof. The proof just involves constructing a chain homotopy by hand. □

Corollary 4.2. *For any C the cochain complex $\top^* C$ becomes exact upon applying the functor $\top$.*

Proof. Applying the functor $\top$ the cochain complex becomes the augmented complex corresponding to $\top C$ which is exact. □

By Corrolary 3.5 $\top C$ is $\top$-injective, hence the cosimplicial object $\top^* C$ is in fact a **$\top$-injective resolution** of C.

Definition 4.3. The resolution $C \xrightarrow{\mathbb{1}_C} \top^* C$ is called the (unnormalized) **Bar resolution** of C relative to the functor $\top$.

Let π be a covariant functor $\mathcal{C} \to \mathcal{A}$ such that $\mathcal{A}$ is an abelian category then it is easy to see that $(\mathcal{A}, \pi\top, \pi\mathbb{1})$ would be a monad and

$$\pi C \xrightarrow{\mathbb{1}_C} \pi \top^* C$$

would be an $\pi\top$-injective resolution.

Definition 4.4. The above resolution is called the unnormalized Bar resolution $\beta(\pi, \top)$ of C and the cohomology of the resulting homology is called the monad / cotriple / simplicial cohomology of C with relative to the functor $\top$ with π coefficients and denoted by

$$H^*_\top(X; \pi)$$

How is this a cohomology theory? Are these the left derived functors of π? But then do we require π to be left exact?

5. The Barr Spectral Sequence

Suppose now our category $\mathcal{C}$ is a triangulated category and $\top$ respects the triangulated structure. Our goal is to do homological algebra so this is not too big a restriction. If our category is abelian then we can look at the corresponding homotopy category instead.

In this situation a $\top$-injective resolution naturally leads to an exact couple and hence to spectral sequence.

Proposition 5.1. *For any $C \in \mathcal{C}$ there exists objects $X_i \in \mathcal{C}$ along with maps*

$$\begin{tikzcd} X^0 = \top C \arrow[d] & X^{-1} \arrow[l] \arrow[d] & X^{-2} \arrow[l] \arrow[d] & \cdots \arrow[l] \\ \top^2 C & \Sigma^{-1}\top^3 C & \Sigma^{-2}\top^4 C & \end{tikzcd}$$

such that

$$\begin{tikzcd} X^{-n} \arrow[d] & X^{-n-1} \arrow[l] \\ \Sigma^{-n}\top^{n+2} C & \end{tikzcd}$$

are fibrations.

Proof. Let $I^n = \top^{n+1}C$. We construct X^{-n} inductively.

Base case: We already have $X^0 = I^0$ and a natural map $I^0 \to I^1$.

Induction Hypothesis: We need to make the following stronger induction hypothesis,

1. There exist objects $X^0, X^{-1}, \cdots, X^{-n}$ which fit in required fibration diagram and there exists a map $X^{-n} \to \Sigma^{-n}I^{n+1}$.
2. The composition $X^{-n} \to \Sigma^{-n}I^{n+1} \to \Sigma^{-n}I^{n+2}$ is null.

Then define X^{-n-1} to be the fiber of $X^{-n} \to \Sigma^{-n}I^{n+1}$. The second induction hypothesis can be rewritten as a partial map between the two triangles

$$\begin{tikzcd} \Sigma^{-n-1}I^{n+1} \arrow[r] \arrow[d] & X^{-n-1} \arrow[r] & X^{-n} \arrow[r] \arrow[d] & \Sigma^{-n}I^{n+1} \arrow[d] \\ \Sigma^{-n-1}I^{n+2} \arrow[r, equal] & \Sigma^{-n-1}I^{n+2} \arrow[r] & 0 \arrow[r] & \Sigma^{-n}I^{n+2} \end{tikzcd}$$

and hence there exists a map completing this triangle

$$X^{-n-1} \to \Sigma^{-n-1}I^{n+2}$$

this proves (1). Next we need to show that the composition

$$X^{-n-1} \to \Sigma^{-n-1}I^{n+2} \to \Sigma^{-n-1}I^{n+3}$$

is 0. Notice that this morphism fits into a diagram

$$\begin{tikzcd} I^n \arrow[r] & I^{n+1} \arrow[rr] \arrow[dr] & & I^{n+2} \arrow[r] & I^{n+3} \\ & & \Sigma^{n+1}X^{-n-1} \arrow[ur] & & \end{tikzcd}$$

Applying $\top$ we know that kernel $\top I^n \to \top I^{n+1}$ = cokernel $\top I^{n+2} \to \top I^{n+3}$ and so

$$\top \Sigma^{n+1} X^{-n-1}$$

is the kernel of $\top I^{n+2} \to \top I^{n+3}$. This then should imply that $X^{-n-1} \to \Sigma^{-n-1} I^{n+2} \to \Sigma^{-n-1} I^{n+3}$ is trivial but for that I think we need a stronger injectivity condition. □

Corollary 5.2. *This gives rise to an exact couple,*

$$\begin{tikzcd}
\pi \coprod X^i \arrow[rr] & & \pi \coprod X^i \arrow[dl] \\
& \pi \coprod \top^i C \arrow[ul] &
\end{tikzcd}$$

(with appropriate bigradings) and hence there is a spectral sequence starting at

$$E_2 = H^*_{\top}(X; \pi)$$

(The convergence is conditional and I do not understand it.)

6. Hopf Algebroid

The final ingredient for the Adams SS is the Hopf algebroid.

Definition 6.1. A **Hopf Algebroid** is a pair of graded unital rings (A, Γ) such that A is graded commutative and Γ is a - possibly non-commutative - Hopf algebra over (the "field") A such that Γ is flat as a A module.

Thus Hopf algebras are a special case of Hopf algebroids when A is a field. This further implies that the pair $(\hom_{Rings}(A, X), \hom_{Rings}(\Gamma, X))$ is a groupoid for any ring X.

The complete data of a Hopf algebroid consists of the following maps

$$A \underset{\eta_R}{\overset{\eta_L}{\rightrightarrows}} \Gamma \longrightarrow \Gamma \otimes_A \Gamma$$

we want the map η_L to be flat.

Definition 6.2. A **comodule** over (A, Γ) is an A module M along with a coaction map given by

$$M \to M \otimes_A \Gamma$$

satisfying the dual of the usual module conditions.

It is easy to see that $\otimes_A \Gamma$ induces a monadic structure on the category of comodules. To this monad we can apply the functor $\otimes_A M$. The corresponding simplicial homology groups are usually denoted by

$$Ext_{(A,\Gamma)}(X, M)$$

7. The E^2 page for the Adams SS

Just like with the Serre SS if we put conditions on the exact couple then the E^2 page for the Bar SS simplifies.

Now we specialize to the case when our category is the category of *Spectra* and E is a commutative ring spectra and the monad is given by wedging with E. Let X be a spectra and π_* is the functor $[S, -]$. Then as above we have the Bar SS whose E^2 is given by

$$H^*_E(E \wedge X, \pi_*)$$

This is the **Adams SS**.

The pair $(\pi_*(E), \pi_*(E \wedge E)) = (E_*, E_*E)$ is called a **Hopf algebroid**. I think this is just saying that E_*E is a Hopf algebra with the base field E_*. The algebroid part is simply to emphasize the fact that these aren't sets but abstract objects (groupoids).

There are two natural maps $E_* \to E_*E$ including E into the left one or the right one. We are concerned with the case when these maps are flat (because E is commutative flatness of one implies the flatness of the other).

Theorem 7.1. *When the map* $\mathbb{1}_l = \mathbb{1} \wedge E : \pi_*(E) \to \pi_*(E \wedge E)$ *is a flat the* E^2 *simplifies as*

$$Ext_{E_*E}(E_*, E_*X)$$

Proof. The flatness condition implies the following,

$$\pi_*(E \wedge E \wedge X) = E_*E \otimes_{E_*} E_*X$$

I do not understand the proof of this.

Hence the Bar resolution becomes,

$$0 \to E_*X \to E_*E \otimes_{E_*} E_*X \to E_*E \otimes_{E_*} E_*E \otimes_{E_*} E_*X \to \cdots$$

This is an injective resolution of E_* being tensored with E_*X in the category of E_*E comodules and hence it's cohomology is the Ext groups. *Again I do not understand the proof of this.*

$\square$