Honors Single Variable Calculus

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Contents

1 Proofs

We'll start by learning about the various kinds of proofs that we'll encounter in this class.[1]

Proofs are the heart of mathematics. You must come to terms with proofs – you must be able to read, understand and write them. What is the secret? What magic do you need to know? The short answer is: there is no secret, no mystery, no magic. All that is needed is some common sense and a basic understanding of a few trusted and easy to understand techniques.

The Structure of a Proof

The basic structure of a proof is easy: it is just a series of statements, each one being either

- An assumption or

- A conclusion, clearly following from an assumption or previously proved result

And that is all. Occasionally there will be the clarifying remark, but this is just for the reader and has no logical bearing on the structure of the proof.

A well written proof will flow. That is, the reader should feel as though they are being taken on a ride that takes them directly and inevitably to the desired conclusion without any distractions about irrelevant details.

Each step should be clear or at least clearly justified. A good proof is easy to follow. When you are finished with a proof, apply the above simple test to every sentence: is it clearly

1. an assumption

2. a justified conclusion?

If the sentence fails the test, maybe it doesn't belong in the proof.

Example

In order to write proofs, you must be able to read proofs. See if you can follow the proof below. Don't worry about how you would have (or would not have) come up with the idea for the proof. Read the proof with an eye towards the criteria listed above. Is each sentence clearly an assumption or a conclusion? Does the proof flow? Was the theorem in fact proved?

Theorem 1.1. *The square root of 2 is an irrational number.*

Proof. Let's represent the square root of 2 by s. Then, by definition, s satisfies the equation

$$s^2 = 2.$$

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[1]This section in taken almost verbatim from http://zimmer.csufresno.edu/~larryc/proofs/proofs.html.

If s were a rational number, then we could write $s = p/q$ where p and q are a pair of integers. In fact, by dividing out the common multiple if necessary, we may even assume p and q have no common multiple (other than 1). If we now substitute this into the first equation we obtain, after a little algebra, the equation

$$p^2 = 2q^2.$$

But now, 2 must appear in the prime factorization of the number p^2 (since it appears in the same number $2q^2$). Since 2 itself is a prime number, 2 must then appear in the prime factorization of the number p. But then, $2 \cdot 2$ would appear in the prime factorization of p^2, and hence in $2q^2$. By dividing out a 2, it then appears that 2 is in the prime factorization of q^2. Like before (with p^2) we can now conclude 2 is a prime factor of q. But now we have p and q sharing a prime factor, namely 2. This violates our assumption above (see if you can find it) that p and q have no common multiple other than 1. $\qquad\square$

1.1 Direct Proofs

Most theorems that you want to prove are either explicitly or implicitly in the form

"If P, then Q".

This is the standard form of a theorem (though it can be disguised). A direct proof should be thought of as a flow of implications beginning with P and ending with Q.

$$P \implies \cdots \implies Q$$

Most proofs are (and should be) direct proofs. Always try direct proof first, unless you have a good reason not to. If you find a simple proof, and you are convinced of its correctness, then don't be shy about. Many times proofs are simple and short.

Exercise. 1.2. Prove each of the following.

1. If a divides b and a divides c then a divides $b + c$, where a, b, and c are positive integers.

2. For all real numbers a and b, $a^2 + b^2 \geq 2ab$.

3. If a is a rational number and b is a rational number, then $a + b$ is a rational number.

1.2 Proof by Contradiction

In a proof by contradiction we assume, along with the hypotheses, the logical negation of the result we wish to prove, and then reach some kind of contradiction. That is, if we want to prove "If P, then Q", we assume P and Not Q. The contradiction we arrive at could be some conclusion contradicting one of our assumptions, or something obviously untrue like $1 = 0$. The proof of Theorem 1.1 is an example of this.

Exercise. 1.3. Use the method of Proof by Contradiction to prove each of the following.

1. The cube root of 2 is irrational.

2. If a is a rational number and b is an irrational number, then $a + b$ is an irrational number.

3. There are infinitely many prime numbers.[1]

1.3   Proof by Contrapositive

Proof by contrapositive takes advantage of the logical equivalence between "P implies Q" and "Not Q implies Not P". For example, the assertion "If it is my car, then it is red" is equivalent to "If that car is not red, then it is not mine". So, to prove "If P, then Q" by the method of contrapositive means to prove

"If Not Q, then Not P".

How Is This Different From Proof by Contradiction? The difference between the Contrapositive method and the Contradiction method is subtle. Let's examine how the two methods work when trying to prove "If P, then Q".

Method of Contradiction: Assume P and Not Q and prove some sort of contradiction.

Method of Contrapositive: Assume Not Q and prove Not P.

The method of Contrapositive has the advantage that your goal is clear: Prove Not P. In the method of Contradiction, your goal is to prove a contradiction, but it is not always clear what the contradiction is going to be at the start.

Exercise. 1.4. Use the method of Proof by Contrapositive to prove each of the following.

1. If the product of two integers is even, then at least one of the two must be even.

2. If the product of two integers is odd, then both must be odd.

3. If the product of two real numbers is an irrational number, then at least one of the two must be an irrational number.

1.4   Converse

The converse of an assertion in the form "If P, then Q" is the assertion

"If Q, then P".

A common logical fallacy is to assume that if an assertion is true then so is its converse.

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[1]There are dozens of proofs of this theorem, originally due to Euclid. Feel free to look one up online.

1.4.1 If and Only If

Many theorems are stated in the form "*P*, if, and only if, *Q*". Another way to say the same thing is: "*Q* is necessary, and sufficient for *P*". This means two things:

"If *P*, Then *Q*"   and   "If *Q*, Then *P*".

So to prove an "If, and Only If" theorem, you must prove the theorem and also its *converse*.

Exercise. 1.5. Go back to the problems in Exercises 1.2, 1.3, 1.4 and find the ones which are of the form "If *P*, Then *Q*". For each of these:

- State the converse.

- Prove or disprove the converse (by providing either a proof or a counterexample).

- For the problems where the converse is also true rewrite the assertion as an "If, and Only If" statement.

1.5 Quantifiers

It is extremely important in mathematics to be able to formulate very precise statements. To prevent ambiguous statements and logical fallacies the vocabulary used is very limited and every term has a well-defined meaning. The two concepts we need to get used to are *logical operators* and *quantifiers*.

Logical Operators: Operators allow us to form complex statements by combining simpler ones. The basic logical operators are *and*, *or*, and *not*. Their usage is the same as in everyday language. The more complicated logical operator that we'll be using a lot is *If - then -*. We've already encountered several examples of its usage in the previous sections.

Quantifiers: Quantifiers allow us to make abstract statements which are "universally true" without having to specify a concrete element. There are two commonly used quantifiers:

For all/every   **There exists**

Understanding and formulating complex statements using these requires a lot of practice. You'll get used to these as the course progresses. Here are a few examples,

Example 1.6.

1. *For every* odd integer a, the integer $a + 1$ is even.

2. An integer a is even if and only if *there exists* an integer b such that $a = 2b$.

3. There *do not exist* integers p, q such that $\frac{p}{q} = \sqrt{2}$.

4. *For every* non-zero rational number x *there exists* a rational number y such that $x \cdot y = 1$.

1.5.1 Nesting Quantifiers

When dealing with statements involving multiple quantifiers the order really matters; changing the order changes the meaning of a statement completely. When you're using multiple quantifiers in a single sentence you should always pause and check if you're using the right quantifiers in the right order.

Exercise. 1.7. For each of the following pairs, explain how changing the order of the quantifiers changes the meaning of the statement.[1]

1. (a) *For every* even integer *a there exists* an integer *b* such that $a = 2b$.

    (b) *There exists* an integer *b* such that *for every* even integer *a*, $a = 2b$.

2. (a) *For every* positive integer *n there exists* a positive real number ϵ such that $\epsilon < 1/n$.

    (b) *There exists* a positive real number ϵ such that *for every* positive integer *n*, $\epsilon < 1/n$.[2]

3. (a) *For every* even integer *a*, *for every* odd integer *b*, $a + b$ is odd.

    (b) *For every* odd integer *b*, *for every* even integer *a*, $a + b$ is odd.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let ϵ be a positive real number.

    (a) *For every x, there exists* a $\delta > 0$ such that *for every y*, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.

    (b) *There exists* a $\delta > 0$ such that *for every x, for every y*, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.

1.5.2 Negating Nested Quantifiers

As we saw in Sections 1.2 and 1.3 we often need to negate mathematical statements.

In order to systematically negate complicated logical expressions, we observe that we can negate simple statements in the following manner. (These are called *De Morgan's laws*.)

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[1]It's ok to a bit vague in your answers to this question.

[2]Such an ϵ is called an **infinitesimal**. There are number systems, for example the **hypperreal numbers**, which extend the real numbers by incorporating infinitesimals.

Statement	Negation
For every x the statement P is true.	There exists an x such that the statement P is false.
There exists an x such that the statement P is true.	For every x the statement P is false.
P is true and Q is true	P is false or Q is false.
P is true or Q is true	P is false and Q is false.
If P then Q	P but not Q.

If we have multiple quantifiers in a statement then we start with the outermost quantifier and recursively move inwards.

Example 1.8. The negation of *"For every x, there exits a y such that, the statement P is true"* equals *"There exists an x such that, for every y, the statement P is false."*

Exercise. 1.9. Negate each of the following.

1. *For every* even integer a *there exists* an integer b such that $a = 2b$.

2. *There exists* an integer b such that *for every* even integer a, $a = 2b$.

3. *For every* positive integer n *there exists* a positive real number ϵ such that $\epsilon < 1/n$.

4. *There exists* a positive real number ϵ such that *for every* integer n, $\epsilon < 1/n$.

5. *For every* even integer a, *for every* odd integer b, $a + b$ is odd.

6. *For every* $\epsilon > 0$, *for every* x, *there exists* a $\delta > 0$ such that *for every* y, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.

7. *For every* $\epsilon > 0$, *there exists* a $\delta > 0$ such that *for every* x, *for every* y, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.

Word of caution: When the word "and" is used in a list it is not considered a logical operator (this is a deficiency of the English language). For example, negating the following expression

For every even integer a and *for every* odd integer b, $a + b$ is odd,

does not change the *and* to an *or* (what happens if you do change the *and* to an *or*?).

2 Inequalities

In this section we'll learn how to solve inequalities and in the process figure out how to write proofs of statements involving inequalities.

The biggest different between equalities and inequalities is that, unlike equalities, there are usually infinitely many (if any) solutions to inequalities. When we we're trying to solve an inequality we're not looking for the best solution, we're not even looking for a good solution, we're simply looking for **one** solution. For example, both $x = 1.001$ and $x = 10^{100}$ are equally valid solutions to the inequality $x > 1$.

Exercise. 2.1. Find a positive real solution to each of the following inequalities.

1. $x^3 + x < 1$

2. $(1 + x)^2 - 1 < 1$

3. $1 - (1 - x)^2 < 1$

4. $x^3 + 2x < 1 + x^2$

5. $x^{10} + x > 10$

6. $x^3 - x > 1$

When we try to solve equalities we try to simply the equation until it becomes easy to solve. The same thing is true for inequalities, however, the ways to simplify an inequality are much more subtle.

Example 2.2. If we're trying to solve $P(x) > \epsilon$, where P is some complicated expression that cannot be simplified, then we try to find some simpler $Q(x)$, such that $P(x) > Q(x)$ and try to solve for $Q(x) > \epsilon$ instead. Similarly, if we're trying to solve $P(x) < \epsilon$, then we try to find some simpler $Q(x)$, such that $P(x) < Q(x)$ and try to solve for $Q(x) < \epsilon$ instead.

There is no *canonical way* in which an inequality can be solved. We somehow simplify the inequality and hope that we get a solution.

For solving complicated inequalities, we need a systematic approach towards estimating functions. There are different tricks and identities for estimating different kinds of functions. For now, we'll focus mainly on estimating polynomials.

Exercise. 2.3. Prove the following extremely useful set of inequalities

$$\text{if } 1 \geq x > 0 \quad \text{then } 1 \geq x \geq x^2 \geq x^3 \geq \ldots$$
$$\text{if } 1 \leq x \qquad \text{then } 1 \leq x \leq x^2 \leq x^3 \leq \ldots.$$

Word of Caution: You should be very careful when negative numbers are involved. Multiplying an inequality by a negative number changes it's sign, for example, $2 < 3$ but $-2 > -3$. So,

$$\text{if } 1 \geq x > 0 \quad \text{then } -1 \leq -x \leq -x^2 \leq -x^3 \leq \ldots$$
$$\text{if } 1 \leq x \qquad \text{then } -1 \geq -x \geq -x^2 \geq -x^3 \geq \ldots.$$

2.1 Proofs involving Inequalities

We'll encounter several statements in this class like: *For every $\epsilon > 0$ there exists a positive real number x such that $x^3 + 3x < \epsilon$.* When we want to prove such a statement, we are essentially trying to solve $x^3 + 3x < \epsilon$. (Why?) For example,

Q. 2.4. Prove that for every $\epsilon > 0$ there exists a positive real number x such that $x^3 + 3x < \epsilon$.

We're asking for a solution to the equation $x^3 + 3x < \epsilon$ for an arbitrary positive real number ϵ.

1. The terms x^3 and x have different degrees. We can try to use the inequalities in Exercise 2.3 to simplify their sum, for this we need to know if $x \geq 1$ or $x \leq 1$.

2. We're probably looking for a small x so we could assume $x \leq 1$. If this does not work we'll come back try something else. (Remember that we're trying to find **one** solution.)

3. As $x \leq 1$, $x^3 \leq x$ so that $x^3 + 3x \leq x + 3x = 4x$.

4. As $x^3 + 3x \leq 4x$ and we want to find a solution to $x^3 + 3x < \epsilon$, it suffices to solve $4x < \epsilon$. As ϵ is positive, **one** solution for this is $x = \epsilon/5$.

5. We still need to check if this solution actually works. We can do this by trying to write down a direct proof and making sure that all the implications are logically sound.

6. We required the condition $x \leq 1$ for this solution to work. To ensure this we'll set $x = \min\{1, \epsilon/5\}$.

Now we know what the solution is, but we still need to write down a proof for the original statement. A proof should always start with some assumptions and logically derive the required conclusion. For us the assumption will be $\epsilon > 0$ and $x = \min\{1, \epsilon/5\}$ and the expected conclusion is $x^3 + 3x < \epsilon$.

Proof of Q. 2.4. Let $\epsilon > 0$ and let $x = \min\{1, \epsilon/5\}$. Note that $x \leq 1$ and hence $x^3 \leq x$. Then,

$$
\begin{aligned}
x^3 + 3x &\leq x + 3x \\
&= 4x \\
&= 4\min\{1, \epsilon/5\} \\
&\leq 4\epsilon/5 \\
&< \epsilon \qquad\qquad\qquad \text{as } \epsilon \text{ is positive}
\end{aligned}
$$

Hence, $x = \min\{1, \epsilon/5\}$ is a solution of $x^3 + 3x < \epsilon$, which proves the proposition. $\square$

First we had to solve the inequality then reverse the steps: start with the solution and write a *direct proof* for the proposition. This is how proofs are usually discovered. You make an "educated guess" and hope that it works. Sometimes it doesn't, so you go back to finding another "educated guess".

Remark 2.5 (A note on quantifiers). For proving the statement *"prove that for every $\epsilon > 0$ there exists a positive real number x such that $x^3 + 3x < \epsilon$"* we had to start with the statement *"let $\epsilon > 0$ and $x = \min\{1, \epsilon/5\}$"*. The variable x depends on the variable ϵ. This is fine because of the order of quantifiers: *"...for every $\epsilon > 0$ there exists a positive real number x ..."*; the quantifier for ϵ comes before the quantifier for x and hence the variable x can depend on the variable ϵ.

If instead, suppose we were trying to prove *"there exists a positive real number x such that for every $\epsilon > 0$, $x^3 + 3x < \epsilon$"*. We're still trying to solve the equation $x^3 + 3x < \epsilon$ but now the quantifier for x comes before the quantifier for ϵ and hence x cannot depend on the variable ϵ but instead should be a constant that universally solves the equation $x^3 + 3x < \epsilon$ for every $\epsilon > 0$. No such x exists (why?) and hence the statement *"there exists a positive real number x such that for every $\epsilon > 0$, $x^3 + 3x < \epsilon$"* is false.

Exercise. 2.6. Prove that for every $\epsilon > 0$ there exists a positive real number x such that ...

1. $x^3 + x < \epsilon$.

2. $(1 + x)^2 - 1 < \epsilon$.

3. $1 - (1 - x)^2 < \epsilon$.

4. $x^3 + 2x < \epsilon + x^2$.

5. $x^{10} + x > \epsilon$.

6. $x^3 - x > \epsilon$. [1]

Finally, we are sometimes required to find not one solution but a range of solutions. We try to come up an "educated guess" by the same method, and many a times this gives us the required range for free with a few changes in the final proof.

Q. 2.7. Prove that for every $\epsilon > 0$, there exists a positive real number δ such that, for all x, if $0 < x < \delta$ then $x^3 + 3x < \epsilon$.

Proof. Let $\epsilon > 0$, let $\delta = \min\{1, \epsilon/5\}$ and let $0 < x < \delta$. Because $\delta \leq 1$, we have $x < 1$ and hence $x^3 < x$. As before,

$$
\begin{aligned}
x^3 + 3x &\leq x + 3x \\
&= 4x \\
&< 4\delta \\
&= 4\min\{1, \epsilon/5\} \\
&\leq 4\epsilon/5 \\
&< \epsilon \qquad\qquad \text{as } \epsilon \text{ is positive}
\end{aligned}
$$

Hence, every $0 < x < \min\{1, \epsilon/5\}$ is a solution of $x^3 + 3x < \epsilon$, which proves the proposition. $\square$

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[1] Hint: Break x^3 as $\frac{x^3}{2} + \frac{x^3}{2}$ and find the conditions on x for which $\left(\frac{x^3}{2} - x\right) > 0$.

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Exercise. 2.8. Prove that for every $\epsilon > 0$, there exists a positive real number δ such that, for all x, if $0 < x < \delta$ then ...

1. $x^3 + x < \epsilon$.

2. $(1 + x)^2 - 1 < \epsilon$.

3. $1 - (1 - x)^2 < \epsilon$.

4. $x^3 + 2x < \epsilon + x^2$.

Prove that for every $\epsilon > 0$, there exists a positive real number δ such that, for all x, if $x > \delta$ then

1. $x^{10} + x > \epsilon$.

2. $x^3 - x > \epsilon$.

Optional Problems

Exercise. 2.9.   1. Prove that for every $\epsilon > 0$, for every real number $x > 0$, there exists a $\delta > 0$ such that, for all real numbers $y > 0$, if $0 < y - x < \delta$ then $y^2 - x^2 < \epsilon$.

2. Prove that the following statement is false: For every $\epsilon > 0$, there exists a $\delta > 0$ such that, for every real number $x > 0$, for all real numbers $y > 0$, if $0 < y - x < \delta$ then $y^2 - x^2 < \epsilon$.

3 Limits & Continuity

We'll now use the proof techniques we've learned so far to study functions.

We'll start with rigorously defining limits. At first this might seem unnecessarily complicated, but having precise definitions will allow us to make more and more sophisticated constructions and later on in the course enable us to prove statements about derivatives and integrals.

Time permitting, towards the end of the semester, we'll construct the Weierstrass function, a function defined on real numbers which is continuous everywhere but differentiable nowhere! But it all starts with the definition of a limit.
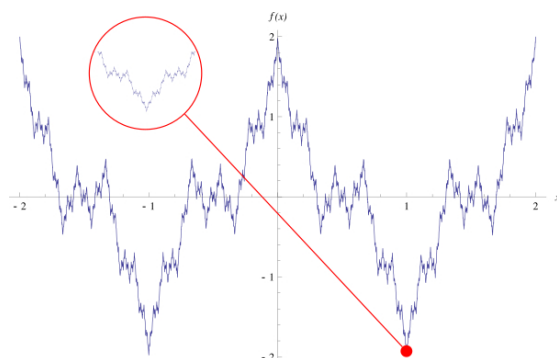
Weierstrass function (Image from Wikipedia)

3.1 Limits

At the heart of all analysis (and hence calculus) lies the notion of a limit. Almost every single concept that we'll define will be a limit of some kind.

Definition 3.1 (Provisional definition of limit). For a function f and a real number a, we say that the function f **approaches a limit** L **at** a, for some real number L, if we can make $f(x)$ as close to L as we like by requiring that x be sufficiently close to, but unequal to, a. This is denoted

$$\lim_{x \to a} f(x) = L$$

There are 3 problems with this definition:

1. The terms *"as close to L as we like"* and *"sufficiently close"* are not precise. (How close is sufficiently close?)

2. The order of implication is unclear.

3. The definition does not give us a way to come up with the limit L.

First, we'll make the terms precise. We can measure the distance between two real numbers x, y by taking their difference $x - y$, however, the difference can be negative so we use the absolute value $|x - y|$ instead. So that

x sufficiently close to, but unequal to, a	translates to	$0 < |x - a| < \delta$
f(x) as close to L as we like	translates to	*for every $\epsilon > 0$, $|f(x) - L| < \epsilon$*

and the definition of limit becomes

> ... f **approaches a limit** L **at** a, for some real number L, if we can make $|f(x) - L| < \epsilon$ for every $\epsilon > 0$ by requiring that $0 < |x - a| < \delta$ for some $\delta > 0$.

Next, to clarify the order of implication we rephrase the statement in the more standard *"If ... then"* form and put all the quantifiers in the front,

> ... f **approaches a limit** L **at** a, for some real number L, if for every $\epsilon > 0$, there exists a $\delta > 0$ such that, for all x, if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

The third problem cannot be fixed! There is no systematic way of finding the limit of a general function, the only thing we can do is *guess* the limit and then use the definition to *prove* that it is indeed the limit. For special functions like polynomials, trig functions, exponential, and logarithms, we can find limits explicitly, which is what makes these functions useful for approximating and estimating more complicated functions.

Definition 3.2 (Formal definition of limit). We say that the function f **approaches a limit** L **at** a, for some real number L, if for every $\epsilon > 0$, there exists a $\delta > 0$ such that, for all x, if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

Exercise. 3.3. Come up with formal definitions for the following:

1. The function f approaches a limit L at a **from the right**, for some real number L, if we can make $f(x)$ as close to L as we like by requiring that x be sufficiently close to, but strictly greater than a. This is denoted

$$\lim_{x \to a^+} f(x) = L.$$

2. The function f approaches a limit L at a **from the left**, for some real number L, if we can make $f(x)$ as close to L as we like by requiring that x be sufficiently close to, but strictly smaller than a. This is denoted

$$\lim_{x \to a^-} f(x) = L.$$

3. The function f **approaches the limit** ∞ **at** a, if $f(x)$ can be made as large as we like by requiring that x be sufficiently close to, but unequal to, a. This is denoted [1]

$$\lim_{x \to a} f(x) = \infty.$$

4. The function f **approaches a limit** L **at** ∞, for some real number L if we can make $f(x)$ as close to L as we like by requiring that x be sufficiently large. This is denoted

$$\lim_{x \to \infty} f(x) = L.$$

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[1] In mathematics ∞ means several things, in fact, there are infinitely many infinities. For us, ∞ is a *placeholder* for a limit of a function that grows very large. ∞ is not a real number and hence cannot be used in equations.

Exercise. 3.4. For each of the following, first guess the limit, then use the formal definition of limit to prove that it is indeed the limit.

1. $\lim\limits_{x \to 1^+} 2x + 1$

2. $\lim\limits_{x \to 1^+} x^2$

3. $\lim\limits_{x \to \infty} x^{10} + x$

4. $\lim\limits_{x \to 0^+} 1/x$

5. $\lim\limits_{x \to 0} f(x)$ where $f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

Optional Problems

Exercise. 3.5. Give examples to show that the following definitions of $\lim\limits_{x \to a} f(x) = L$ are not correct.

1. For every $\delta > 0$, there exists an $\epsilon > 0$ such that, for all x, if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

2. For every $\epsilon > 0$, there exists a $\delta > 0$ such that, for all x, if $|f(x) - L| < \epsilon$ then $0 < |x - a| < \delta$.

Exercise. 3.6. Prove that if $\lim\limits_{x \to a} f(x) = L$ and $L \neq 0$ then $\lim\limits_{x \to a} 1/f(x) = 1/L$.

3.1.1 Triangle Inequality

To prove abstract theorems involving absolute values we will need the following very important inequality called the **triangle inequality**.

$$|x - y| \leq |x| + |y|$$

If you think of the points 0, x, y (on the real axis) as being the three vertices of a (degenerate) triangle, then $|x|$, $|y|$, and $|x - y|$ are the lengths of the three sides and the triangle inequality is saying that: *the sum of the lengths of two sides of a triangle is greater than or equal to the length of the third side.*

There are other forms in which the triangle inequality is commonly used, e.g.

$$|x + y| \leq |x| + |y|$$
$$|x + y| - |y| \leq |x|$$
$$|x| \leq |x - y| + |y|$$

These inequalities are central to a lot of analysis proofs.

Exercise. 3.7.

1. Prove that $|x| + |x - 1| \geq 1$ for any real number x.

2. Prove that for any real number x, at least one of $|x|$ and $|x - 1|$ is $\geq 1/2$.[1]

We'll use these to prove inequalities about the non-existence of limits.

Definition 3.8. If f does not approach the limit L, for any real number or ∞ or $-\infty$, then we say that the limit of f at a **does not exist**.

Exercise. 3.9. Negate the formal definitions of limits and come up with an ϵ, δ definition for the following.

1. f **does not** approach the limit L, where L is a real number, at a.

2. f **does not** approach the limit ∞ at a. (Similarly for $-\infty$.)

Exercise. 3.10. Let $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$

1. Prove that for every real number L, $\lim\limits_{x \to 0} f(x) \neq L$.

2. Prove that $\lim\limits_{x \to 0} f(x) \neq \infty$. (Similarly for $-\infty$.)

Hence the limit of f at 0 does not exist.

Exercise. 3.11.

1. Prove that $\lim\limits_{x \to a} f(x) = L$ if and only if $\lim\limits_{x \to a^+} f(x) = L$ and $\lim\limits_{x \to a^-} f(x) = L$.[2] (Similarly for ∞, $-\infty$.)

2. Prove that $\lim\limits_{x \to 0} \dfrac{1}{x}$ does not exist.

3.2 Non-existence of limits

Let's move backwards and try and understand what it means *intuitively* for a function to not approach a limit L near a. The solution to Exercise 3.9 is the following:

> *(Formal definition)* The function f **does not approach thelimit** L **at** a if there exists an $\epsilon > 0$, such that for all $\delta > 0$, there exists an x such that, $0 < |x - a| < \delta$ and $|f(x) - L| \geq \epsilon$.

Going back to how to we came up with the Formal Definition of Limit 3.2 from the Provisional Definition of Limit 3.1 and replacing the absolute values by the distance between points, we can work backwards and get the following provisional definition:

> *(Provisional definition)* The function f **does not approach the limit** L **at** a if no matter how close we are to a, there is some x for which $f(x)$ is not close to L.

---

[1] Hint: Proof by Contradiction.

[2] Hint: This is very easy, don't overthink! Simply write down the formal definitions.

3.3 Continuity

Definition 3.12. We say that a function f is **continuous** at a if

$$\lim_{x \to a} f(x) = f(a)$$

We say that a function f is **continuous on an interval** (a,b) it f is continuous at every x in (a,b). [1]

Thus continuous functions have nice limits. Further, we have the following theorem which makes continuous functions easy to manipulate.

Theorem 3.13. *If the functions f, g are continuous at $x = a$ then so are the functions*

1. $f + g$

2. $f \cdot g$

3. f/g, *if* $g(a) \neq 0$

The full proof of this theorem is just a tricky application of the triangle inequalities. We'll only prove the first part which is relatively manageable.

Exercise. 3.14. Using the formal definition of limits and the triangle inequality $|x + y| \leq |x| + |y|$, prove that
if the functions f, g are continuous at $x = a$ then so is the function $f + g$.
Is the converse true?

Exercise. 3.15.

1. Prove that the constant function $f(x) = c$ is continuous everywhere, where c is some real number.

2. Prove that the function $f(x) = x$ is continuous everywhere.

3. Argue that these two facts along with Theorem 3.13 imply that polynomials are continuous everywhere.

Exercise. 3.16. Let f be a continuous function on $[a,b]$, and let g be a continuous function on $[b,c]$, such that

$$f(b) = g(b).$$

Show that the function h defined as

$$h(x) := \begin{cases} f(x) & \text{if } x \leq b \\ g(x) & \text{if } x > b \end{cases}$$

is continuous at b (i.e. we can *glue* continuous functions).

---

[1]If we use closed intervals $[a,b]$ instead, then we have to use $\lim\limits_{x \to a^+}$ and $\lim\limits_{x \to b^-}$ in the definition of continuity.

4 Completeness of the Real Numbers

Our next goal is to prove the IVT for continuous functions.

Theorem 4.1 (Intermediate Value Theorem)**.** *If f is a continuous function on $[a, b]$ and k is a real number lying between $f(a)$ and $f(b)$ then there exists a real number c lying between a and b such that $f(c) = k$.*

The proof of this trivial looking theorem relies on a defining axiom of the system of real numbers called *completeness* which we'll now try to understand. For this we'll need to rigorously define the simple notions of *min* and *max*.

For a non-empty set S of real numbers, it's maximum element can be defined as follows.

Definition 4.2. max S is the element $a \in S$ such that $a \geq x$ for all elements $x \in S$.

This definition exposes a subtle shortcoming of max S. It doesn't exist even for *nice* sets.

Exercise. 4.3. What is max $(0, 1)$? Make sure that your answer agrees with Definition 4.2.

There is a more intricately defined number, called the **supremum**, that we can associate to a set of real numbers which also captures the *largest-element* property. For sets containing infinitely many elements, the supremum, rather than the maximum gives us the correct *largest-element*.

Definition 4.4. An **upper bound** of a set S is any real number a satisfying $a \geq x$ for all elements $x \in S$. (Note that a does not have to be in S.)

Exercise. 4.5. Upper bounds are not unique. Find all the possible upper bounds on the following sets. (No proofs needed. Make sure you've found *all* the upper bounds.)

1. $[0, 1]$

2. $(0, 1)$

3. $\mathbb{R}$

4. $\{1/n : \text{where } n \text{ is a positive integer}\}$

5. The set of rational numbers x satisfying $x^2 < 2$

6. The set of irrational numbers in interval $[0, 1]$.

Definition 4.6. The **supremum** or the **least upper bound** of a set S of real numbers, denoted sup S, is the smallest real number a such that a is an upper bound of S.

Exercise. 4.7. Find the supremum of the sets in Exercise 4.5.

Definition 4.8. We say that a set S is **bounded from above** if it has at least one upper bound i.e. there exists a real number a such that $a \geq x$ for all elements $x \in S$.

Exercise. 4.9. Which of the sets in Exercise 4.5 are bounded from above?

We can now state the **Completeness of the Real Numbers**:

Theorem 4.10. *If a non-empty set S is bounded from above then S has a supremum.*

Depending on exactly how the real numbers are constructed, this is either a theorem or an axiom. This theorem/axiom is the precise way of saying that there are no holes in the real line. (Read the Wikipedia article titled *Completeness of the real numbers*.) We'll next restate the property of being a supremum using our (favorite) language of ϵ's and δ's.

Exercise. 4.11. Let S be a non-empty set that is bounded from above and let $L = \sup S$.

1. Show that for every $\delta > 0$, there exists an element $x \in S$ such that $L - x < \delta$. [1]

2. Let a be an upper bound of S. Prove that, if for every $\delta > 0$, there exists an element $x \in S$ such that $a - x < \delta$, then $a = L$. [2]

Thus we're saying that elements of S get infinitely close to the supremum.

Optional Problems

Similarly to the supremum we can define the **infimum** of a set S, denoted S, which corresponds to $\min S$.

Exercise. 4.12.

1. Define $\inf S$.

2. Find the infimum of all the sets in Exercise 4.5.

3. Show that

$$\inf S = -\sup(-S)$$

(if they exist), where $-S$ is the set containing elements of the form $-x$ where $x \in S$.

4. State the analogue of Exercise 4.11 for $\inf S$.

---

[1] Hint: Proof by Contradiction.

[2] Hint: Prove by Contrapositive.

4.1 Intermediate Value Theorem

We'll now prove the Intermediate Value Theorem. This is our first encounter with a complicated proof which requires multiple steps. You probably won't see the full picture until you completely write down the proof yourself. It'll help you a lot to draw pictures to keep a track of all the variables.

> *Once you have solved all the exercises in the proof of Theorem 4.14, submit your final solution as one single logically coherent proof of the IVT, also include all the text that is in between the exercises, so that you yourself see how all the pieces fit together. This will also teach you to write complex proofs.*

We'll need the following proposition from the previous section, which says that the elements of a set S are infinitely close to it's supremum.

Proposition 4.13. *For a set S, if $L = \sup S$, then for every $\delta > 0$, there exists an element $x \in S$ such that $L - x < \delta$.*

Theorem 4.14 (Intermediate Value Theorem). *Let f be a continuous function on $[a, b]$ and let k be a real number satisfying*

$$f(a) < k < f(b).$$

Then, there exists a real number c satisfying $a < c < b$ such that

$$f(c) = k.$$

Proof of 4.14. We'll first prove the following slightly easier version.

Proposition 4.15. *If g is a continuous function on $[a, b]$ satisfying*

$$g(a) < 0 < g(b)$$

then, there exists a real number c satisfying $a < c < b$ such that

$$g(c) = 0.$$

Proof of 4.15. Let g, a, b be as in the statement of the Proposition. We'll construct a number c that satisfies $a < c < b$ and $g(c) = 0$.

Let S be a set defined as follows,

$$S = \{x \mid a \leq x \leq b \text{ and } g(x) \leq 0\}.$$

Exercise. 4.16. Argue that S is non-empty and is bounded from above.

Hence, S has a supremum by the *Completeness Property of Real Numbers*. Denote this supremum by c. We'll prove that

    Claim: $g(c) = 0$.

We'll prove this claim by contradiction. Suppose on the contrary that $g(c) \neq 0$. There are two possible cases:

Case 1: $g(c) < 0$

Case 2: $g(c) > 0$

Suppose **Case 1** is true i.e. $g(c) < 0$.

Exercise. 4.17. 1. Use the formal definition of continuity (from right) of g to prove that there exists an $x > c$ such that $f(x) < 0$.[1]

2. Argue that this contradicts one of your assumptions, hence $g(c)$ cannot be less than 0.

Suppose **Case 2** is true i.e. $g(c) > 0$.

Exercise. 4.18. 1. Use the formal definition of continuity (from left) of g to prove that there exists a δ such that, for every x, if $c - x < \delta$ then $g(x) > 0$.[2]

2. For this particular δ, prove that, for every x in S, $c - x > \delta$.

3. Argue that this contradicts Proposition 4.13, hence $g(c)$ cannot be greater than 0.

Thus we've contradicted both the possible cases, which completes the proof by contraction and proves the Claim that $g(c) = 0$.

By construction c satisfies $a < c < b$. So we've constructed a c such that $a < c < b$ and $g(c) = 0$ which completes the proof of Proposition 4.15. □

Exercise. 4.19. Going back to f, use the function $g(x) = f(x) - k$ in Proposition 4.15 to complete the proof of Theorem 4.14.

□

Both the hypotheses are necessary for the IVT to be true. This means that we must be crucially using these hypotheses in our proof somewhere and the proof should fail if we do not have these assumptions. It is clear where we are using continuity, the other one is more subtle.

Exercise. 4.20.

1. Find the *exact* argument in your proof of Proposition 4.15 which fails if we assume $0 < g(a) < g(b)$.

2. Find the *exact* argument in your proof of Proposition 4.15 which fails if we assume $g(a) < g(b) < 0$.

---

[1] Hint: Use $\epsilon = -g(c)/2$.

[2] Hint: Use $\epsilon = g(c)/2$.

5 Differentiation

Remark 5.1. We'll now exclusively work with *continuous* functions. As such, we can compute the various limits in this section without requiring any ϵ, δ arguments. We'll get back to the ϵ, δ's much later. For now, we'll repeated use the following identities,

$$\lim_{x \to a} f(x) + g(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

$$\lim_{x \to a} f(x) \cdot g(x) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

$$\lim_{x \to a} f(x)/g(x) = \lim_{x \to a} f(x)/ \lim_{x \to a} g(x) \qquad \text{assuming } \lim_{x \to a} g(x) \neq 0$$

wherever the appropriate limits exist.

5.1 Derivative

The derivative of a function measures the *instantaneous rate of change* of the function. Several quantities in physics can be naturally expressed as derivatives, which was the original motivation for defining them.

Function	Derivative
Position	Velocity
Velocity	Acceleration
Potential	Force

Definition 5.2. The **derivative** of $f(x)$ with respect to x, at $x = a$, is defined to be the limit

$$f'(a) := \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{5.3}$$

If $f'(a)$ exists then we say that $f(x)$ is **differentiable** at a.

One can rewrite Equation (5.3) as

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{5.4}$$

by making the substitution $h = x - a$.

In this form, it is more explicit that derivative is computing the *instantaneous rate of change*, as the numerator $f(x) - f(a)$ is the change in f and the denominator $x - a$ is the change in x, so their ratio is the rate of change of $f(x)$ with respect to x. Taking the limit $x \to a$ makes this rate of change *instantaneous*. When we want to make the "free variable" more explicit, we write the derivative as

$$\left. \frac{df}{dx} \right|_{x=a}$$

This notation becomes more relevant in multivariable calculus when there are multiple variables with respect to which one can differentiate the function.

Exercise. 5.5. Using the definition, compute the derivatives of the following functions.

1. $f(x) = c$, where c is a real number.
2. $f(x) = x$.
3. $f(x) = x^2$.

Exercise. 5.6. Prove that if $f(x)$ is differentiable at a then $f(x)$ is continuous at $x = a$.

Theorem 5.7. *If f and g are differentiable at a then*

$$(f + g)'(a) = f'(a) + g'(a)$$

$$(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a) \qquad \textbf{Product Rule}$$

$$\left(\frac{1}{g}\right)'(a) = -\frac{g'(a)}{g(a)^2}$$

$$\left(\frac{f}{g}\right)'(a) = \frac{g(a) \cdot f'(a) - g'(a) \cdot f(a)}{g(a)^2} \qquad \textbf{Quotient Rule}$$

$$(f \circ g)'(a) = f'(g(a)) \cdot g'(a) \qquad \textbf{Chain Rule}$$

where we assume that $g(a) \neq 0$ wherever it is in the denominator.

Exercise. 5.8. Prove Theorem 5.7.

The proof of the Product Rule requires a trick: add and subtract $f(a + h)g(a)$ to the numerator.

For proving the Quotient Rule notice that $f/g = f \cdot (1/g)$.

Feel free to read the proof of the Chain Rule from the book.

Because of these rules, once we know the derivatives of standard functions like trig functions, exponential functions etc., derivatives are fairly easy to compute. However, computing these standard derivatives is not easy using simply the definition (give it a try!).

Polynomials are the only functions whose derivatives we can compute right now. We'll get back to computing more complicated derivatives once we learn integration and the Fundamental Theorem of Calculus.

Exercise. 5.9. In this exercise we'll (almost) prove that

$$(x^n)' = nx^{n-1} \tag{5.10}$$

where n is any real number.

Note that Exercise 5.5 proves this for the cases $n = 0, 1, 2$.

1. Use the identity $(x^n) \cdot x = x^{n+1}$ to prove (5.10) for all positive integers.[1]

---

[1]**Optional:** Read about **Mathematical Induction** and use it to write a rigorous proof of this statement.

2. Use the identity $\frac{1}{x^m} = x^{-m}$ to prove (5.10) for all negative integers.

3. Use the $\left(x^{\frac{p}{q}}\right)^q = x^p$ to prove (5.10) for all rational numbers.

This proves (5.10) for all rational numbers. We do not yet know enough to extend this proof to irrationals.

4. Can you think of a way to prove (5.10) for all real numbers? What fact *should* be true for your proof to work?

5.2 Inverse Functions

We say that functions f and g are **inverses** of each other if the following identities hold:

$$f(g(x)) = x$$
$$g(f(x)) = x$$

Exercise. 5.11. Verify that if f and g are inverses of each other and $f(x) = y$ then $x = g(y)$. (Hence the name inverse.)

This exercise is saying that to find the inverse of $f(x)$ we simply solve $f(x) = y$ for x in terms of y, which then gives us g.

Exercise. 5.12.

1. Verify that $f(x) = 2x + 1$ and $g(x) = (x - 1)/2$ are inverses of each other. Draw their graphs.

2. Verify that $f(x) = x^2$ and $g(x) = \sqrt{x}$ are inverses of each other. Draw their graphs.

3. Verify that $f(x) = e^x$ and $g(x) = \ln x$ are inverses of each other. Draw their graphs.

4. What relation do you see between the graphs of inverses?

5. What are the domains of these functions i.e. what is the set of real numbers where the these functions are defined? Why do you think some of the inverses are not defined everywhere?

The inverses of $\sin x$ and $\cos x$ are denoted $\sin^{-1}(x)$ and $\cos^{-1}(x)$, respectively. More generally, the inverse of a function $f(x)$ is denoted by $f^{-1}(x)$ (which is different from $(f(x))^{-1}$). Note that if $f = g^{-1}(x)$ then $g = f^{-1}(x)$.

Exercise. 5.13. Using the Chain Rule show that

$$\left(f^{-1}(a)\right)' = \frac{1}{f'(f^{-1}(a))}$$

Verify this explicitly for $f(x) = 2x + 1$ and for $f(x) = x^2$.

We'll interpret this geometrically in the next section. We'll later on see that the derivatives of $\sin^{-1}(x)$ and $\ln x$ can expressed in terms of polynomials and radicals. This fact along with the above formula will then allow us to compute the derivatives of $\sin x$ and e^x.

5.3 Mean Value Theorems

Derivatives have several physical and geometric interpretations which make then widely applicable in all branches of science. We'll see a few of these interpretations in the next few sections.

In this section, we'll prove the analogues of Intermediate Value Theorem for differentiable functions.

Definition 5.14. A number c is an **absolute maximum** of a function f in the interval $[a,b]$ if $f(c) \geq f(x)$ for all $x \in [a,b]$.

To prove the Mean Value Theorems we'll need the following **Extreme Value Theorem**. We'll assume it without proof. As for the IVT, the proof of the Extreme Value Theorem fundamentally uses the Completeness of the Real Numbers.

Theorem 5.15 (Extreme Value Theorem). *For every continuous function f and every closed interval $[a,b]$, there exists a real number $c \in [a,b]$ such that c is the absolute maximum of f on $[a,b]$.*

Remark 5.16. This theorem is false if we use open intervals. For example, $f(x) = 1/x$ has no absolute maximum on the interval $(0,1)$ even though it is continuous on it.

Exercise. 5.17. Define an absolute minimum and state the corresponding Extreme Value Theorem.

Note that by our definitions an *absolute* max/min c of f over $[a,b]$ is also a *local* max/min if $a < c < b$. We've already shown that the derivative vanishes at a local max/min, so we have

Theorem 5.18. *Let f be a differentiable function. If c is an absolute max/min for f on the interval $[a,b]$ with $a < c < b$ then $f'(c) = 0$.*
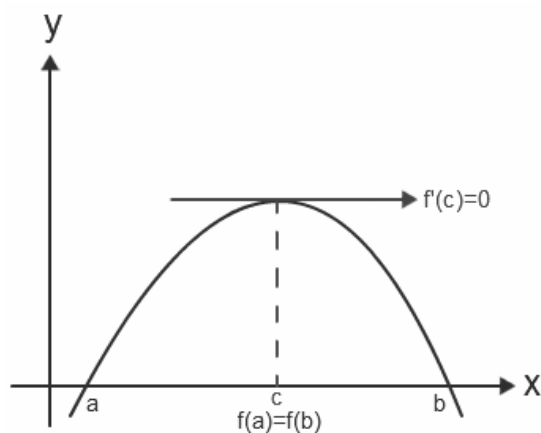
This is all we'll be needing to prove the various Mean Value Theorems.

Theorem 5.19 (Rolle's Theorem). *Let f be a differentiable function. For an interval $[a,b]$, if*

$$f(a) = f(b)$$

then there exists a real number $c \in (a,b)$ such that

$$f'(c) = 0$$

Proof of Rolle's Theorem. This theorem almost immediately follows from Theorem 5.15 and Theorem 5.18. The only issue is that for Theorem 5.18 we need $a < c < b$ but the Extreme Value Theorem only guarantees $a \leq c \leq b$. So we need a small argument to fill this gap.

By the Extremal Value Theorem, f has an absolute maximum c_{max} and an absolute minimum c_{min} on the interval $[a, b]$.

Exercise. 5.20.

1. Argue that if $f(c_{max}) = f(c_{min})$ then f is a constant function on $[a, b]$.

We've already shown that for a constant function, $f'(x)$ equals 0, hence for every $c \in (a, b)$, $f'(c) = 0$ and we're done.
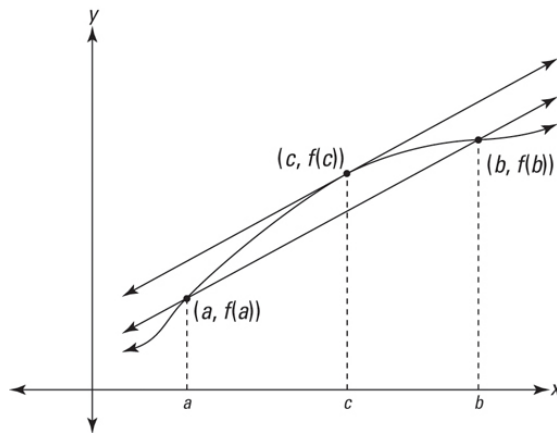
So now we'll assume that $f(c_{max}) \neq f(c_{min})$.

2. Argue that in this case either $a < c_{max} < b$ or $a < c_{min} < b$ (possibly both).

3. Finish the proof of Rolle's Theorem.

$\square$

Theorem 5.21 (Mean Value Theorem). *Let f be a differentiable function. For any interval $[a, b]$ there exists a real number $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

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Proof of Mean Value Theorem. Mean Value Theorem follows from the Rolle's theorem by an algebraic trick. We'll construct a new function $g(x)$ that satisfies the hypotheses of Rolle's theorem.

Exercise. 5.22.

1. Find a real number r such that $f(a) - ra = f(b) - rb$.

For this value of r define

$$g(x) = f(x) - rx$$

By construction, g is differentiable and satisfies $g(a) = g(b)$ and hence satisfies the hypotheses for Rolle's theorem. So, there exists some real number $a < c < b$ such that

$$g'(c) = 0$$

2. Show that for this c, $f'(c) = \dfrac{f(b) - f(a)}{b - a}$, thereby completing the proof of Mean Value Theorem.

$\square$

We'll see later on that the Mean Value Theorem is a key ingredient in the proof of Taylor series approximation. In fact, Taylor series approximations can be thought of as a generalization of MVT for higher order derivatives.

5.4   L'Hopital's Rule

We'll now (almost) provide a proof of L'Hopital's Rule using a stronger version of the Mean Value Theorem.

Theorem 5.23 (Easy L'Hopital's Rule). *Let f, g be differentiable functions. Assume that*

$$f(a) = 0 = g(a) \quad and \quad g'(a) \neq 0,$$

where a is a real number. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

Exercise. 5.24. Notice that $\dfrac{f(x)}{g(x)} = \dfrac{f(x) - f(a)}{g(x) - g(a)}$. Use this to prove the easy version of L'Hopital's Rule. [1]

Theorem 5.25 (Strong Mean Value Theorem). *Let f, g be differentiable functions. Assume that $g(a) \neq g(b)$ for some real numbers a and b. Then there exists a real number $c \in (a, b)$ such that*

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Proof of Strong Mean Value Theorem. The Strong Mean Value Theorem follows from the Rolle's theorem by the same algebraic trick. We'll construct a new function $h(x)$ that satisfies the hypotheses of Rolle's theorem.

Exercise. 5.26.

1. Find a real number r such that $f(a) - rg(a) = f(b) - rg(b)$.

For this value of r define

$$h(x) = f(x) - rg(x)$$

By construction, h is differentiable and satisfies $h(a) = h(b)$ and hence satisfies the hypotheses for Rolle's theorem. So, there exists some real number $a < c < b$ such that

$$h'(c) = 0$$

2. Show that for this c, $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$, thereby completing the proof of the Strong Mean Value Theorem.

$\square$

Exercise. 5.27. Explain how the Mean Value Theorem is a special case of the Strong Mean Value Theorem and how the Rolle's theorem is a special case of the Mean Value Theorem.

Theorem 5.28 (L'Hopital's Rule). *Let f, g be differentiable functions. Assume that*

$$f(a) = 0 = g(a)$$

where a is a real number. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Idea of the proof. For any b, by the Strong Mean Value Theorem, there exists a $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\implies \frac{f'(c)}{g'(c)} = \frac{f(b)}{g(b)} \qquad\qquad (\text{as } f(a) = 0 = g(a))$$

---

[1] Hint: Divide the numerator and denominator by $x - a$ and take $\lim\limits_{x \to a}$.

As c has to lie between a and b, when we take the limit $b \to a$ we must also have $c \to a$. Taking limits of both sides we get the required result.

$$\lim_{c \to a} \frac{f'(c)}{g'(c)} = \lim_{b \to a} \frac{f(b)}{g(b)}$$

$\square$

Remark 5.29. The last argument in the above proof is not rigorous. While the idea is correct, to make it rigorous we need generalized notions of limits called **limit superior** (lim sup) and **limit inferior** (lim inf). This is unfortunately beyond the scope of this class.

There are many more variants of L'Hopital's Rule: for one-sided limits, for $a = \infty$, for $\lim_{x \to a} f(a) = \infty = \lim_{x \to a} g(a)$ etc. They can all be proven using similar techniques.

5.5 First Derivative Test

In this section, we'll prove the First Derivative Test for differentiable functions.

Definition 5.30. We say that a function f is **increasing** on an interval $[a, b]$, if for all x, y in $[a, b]$ if $x < y$ then $f(x) \leq f(y)$. We say that f is **strictly increasing** on $[a, b]$, if for all x, y in $[a, b]$ if $x < y$ then $f(x) < f(y)$.

Exercise. 5.31. Come up with a definition for when a function is **decreasing, strictly decreasing** on an interval.

Exercise. 5.32. Find the interval(s) on which the function x^2 is increasing/decreasing? What about the function x^3?

For differentiable functions, we can determine if a function is increasing or decreasing by calculating it's first derivative.

Exercise. 5.33.

1. **Optional:** Let $g(x)$ be a function such that $\lim_{x \to c} g(x)$ exists for some real number c in the interval (a, b). Prove that if $g(x) \geq 0$ for all x in $[a, b]$, then $\lim_{x \to c} g(x)$ is also ≥ 0.[1]
   (You can assume this part to be true if you choose not to prove it.)

2. Let $f(x)$ be an increasing function on $[a, b]$ and let $c \in (a, b)$. Prove that $g(x) \geq 0$ on the interval $[a, b]$ where $g(x)$ is the function
$$g(x) = \frac{f(x) - f(c)}{x - c}$$

3. Let $f(x)$ be a differentiable function that is increasing on the interval $[a, b]$. Prove that all $c \in (a, b)$, $f'(c) \geq 0$. Similarly, for decreasing functions.

---

[1] Hint: Proof by Contradiction.

Note that this agrees with the interpretation of a derivative as the *instantaneous* rate of change of a function: an increasing function should have a positive instantaneous rate of change and a decreasing function should have a negative instantaneous rate of change. The converse of the above statement is almost true, but for this we need an extra assumption on f and the proof is a bit more intricate.

Proposition 5.34. *If $f'(x) > 0$ for all $c \in (a,b)$ then f is strictly increasing on (a,b). Similarly, for $f'(c) < 0$.*

Exercise. 5.35. Let $f'(x) > 0$ for all $c \in (a,b)$. Prove by Contradiction, that for all $x, y \in (a,b)$, if $x < y$ then $f(x) < f(y)$.[1]

Exercise. 5.36. Use the above Proposition to find the range in which the function x^n is increasing/decreasing, where n is a positive integer.

Definition 5.37. A real number c is said to be a **local maximum** of a function f if there is some interval (a,b) containing c such that if $a < x < b$ then $f(c) \geq f(x)$.

Exercise. 5.38. Come up with a definition for **local minimum**.

Exercise. 5.39. Assume that the function $f(x)$ is differentiable and it's derivative $f'(x)$ is continuous. Using Proposition 5.34 prove that if c is a local maximum of $f(x)$ then $f'(c) = 0$.[2] Similarly for a local minimum.

Remark 5.40. We need $f'(x)$ to be continuous to ensure that if $f'(c) > 0$ then $f'(x) > 0$ for every $x \in (a,b)$, where (a,b) is some interval containing c.

Combining everything in this section we have proven the following Theorem, also called the **First Derivative Test**.

Theorem 5.41 (First Derivative Test)**.** *Let $f(x)$ be a differentiable function whose derivative $f'(x)$ is a continuous function.*
For a real number c,

$$f \text{ has a local max/min at } c \implies f'(c) = 0$$
$$f \text{ is increasing near } c \implies f'(c) \geq 0$$
$$f \text{ is decreasing near } c \implies f'(c) \leq 0$$

Conversely,

$$f'(c) > 0 \implies f \text{ is increasing near } c$$
$$f'(c) < 0 \implies f \text{ is decreasing near } c$$

where by "near c" means for some interval (a,b) containing c.

A real number c satisfying $f'(c) = 0$ is called a **critical point** of f. The First Derivative Test says that a local max/min has to be a critical point, but it does

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[1] Hint: Use the Mean Value Theorem.
[2] Hint: Proof by Contradiction.

not say that a critical point has to be a local max/min. A standard example of this failure is $c = 0$ for the function $f(x) = x^3$.

$$\text{Local min/max} \Rightarrow \text{Critical point}$$
$$\text{Critical point} \nRightarrow \text{Local min/max}$$

There are ways to fix this using higher derivatives. We'll do this later using Taylor Series approximations.

Optional Problems

There are multiple ways of proving Proposition 5.34. However, all the methods require some additional techniques which we have not yet developed.

Exercise. 5.42. Try to write down a rigorous proof of Proposition 5.34. What statement *should* be true for your proof to work?

5.6 Convexity and Concavity

The second derivative of a function $f(x)$ is defined as

$$f''(x) = \left(f'(x)\right)'$$

If the second derivative exists then we say that a function is **twice differentiable**. More generally, the nth derivative $f^{(n)}(x)$ is defined as

$$f^{(n)}(x) = \left(f^{(n-1)}(x)\right)'$$

If all derivatives exist then we say that the function is **smooth** or **infinitely differentiable**. The standard functions like polynomials, trig functions, exponentials, and logarithms are all smooth wherever they're defined.

If a function f is twice differentiable, then the second derivative of f measures it's convexity/concavity.

Definition 5.43. A function f is said to be **convex** (or **concave upwards**) on an interval, if for every real numbers a, b in the interval, the graph of $f(x)$ lies below the line joining $(a, f(a))$ to $(b, f(b))$.
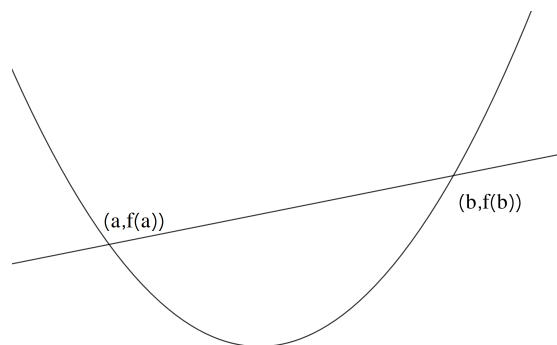
Convex functions play an important role in many areas of mathematics, especially in the study of optimization problems.

The following theorem establishes the connection between convexity and the second derivative.

Theorem 5.44. *Let f be a twice differential function.*

If f is convex on an interval then $f''(x) \geq 0$ on that interval. Conversely, if for every real number x in some interval, $f''(x) > 0$, then f is convex on that interval.

Remark 5.45. To avoid clutter, we'll often drop the terms *on an interval, in some interval* etc. in the following proof. Just keep in mind that there is an ambient interval and all our variables belong to that interval.

Once you have solved all the exercises below, submit your final solutions as one single logically coherent proof. Also include all the text that is in between the exercises, so that you yourself see how all the pieces fit together. As before, this is to practice writing complex proofs.

Proof. Let f be a twice differential function.

Exercise. 5.46. For real numbers a, b denote by $m_{a,b}$ the slope of the line joining the points $(a, f(a))$ and $(b, f(b))$. Find a formula for $m_{a,b}$.
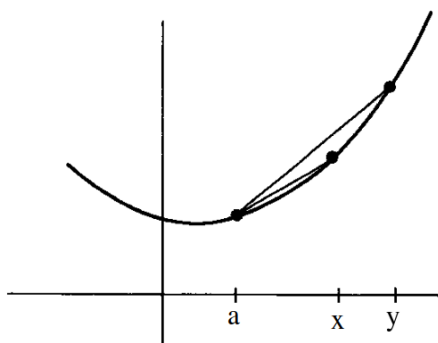
We'll first prove the forward direction.

Claim: *If f is convex on an interval then $f''(x) \geq 0$ on that interval.*

We'll prove that $f'(x)$ is an increasing function which will then imply that $f''(x) \geq 0$ (as the derivative of an increasing function is ≥ 0).

Let f be a convex function. Let $a < b$ be two real numbers.

Exercise. 5.47. Let $a < x < y$ be real numbers.

1. *Geometrically* argue that $m_{a,x} < m_{a,y}$.

2. Using the formula for $m_{a,x}$, $m_{a,y}$ and the above inequality argue that

$$f'(a) \leq m_{a,y}$$

for real numbers $a < y$.

Exercise. 5.48. Let $x < y < b$ be real numbers.

1. *Geometrically* argue that $m_{x,b} < m_{y,b}$.

2. Using the formula for $m_{y,b}$, $m_{x,b}$ and the above inequality argue that

$$m_{x,b} \leq f'(b)$$

   for real numbers $x < b$.

Exercise. 5.49. Combine the results from the previous two exercises to conclude that $f'(a) \leq f'(b)$.

Hence $f'(x)$ is an increasing function and hence $f''(x) \geq 0$.

We'll next prove the other direction.
Claim: *If for every real number x in some interval, $f''(x) > 0$, then f is convex on that interval.*

We'll prove this by *contradiction*. Let f be a function such that $f''(x) > 0$ on some interval. This implies that $f'(x)$ is an increasing function on that interval. This is the statement that we'll contradict.

Assume on the contrary that f is not convex. This means that for some real numbers $a < b$, there exists a number c with $a < c < b$ such that the point $(c, f(c))$ lies above the line joining $(a, f(a))$ and $(b, f(b))$.

Exercise. 5.50. Draw a picture.

Applying the Mean Value Theorem to the intervals $[a, c]$ and $[c, b]$ gives us real numbers $d_1 \in (a, c)$ and $d_2 \in (c, b)$ such that

$$f'(d_1) = m_{a,c} \text{ and } f'(d_2) = m_{c,b}$$

Exercise. 5.51.

1. Draw a picture.

2. *Geometrically* argue that $f'(d_1) > f'(d_2)$.

3. Why is this a contradiction?

This completes a proof of the other direction, and hence of the main theorem.

$\square$

5.6.1 Second Derivative Test

Let $f(x)$ be a twice differentiable function. If a real number c is a local minimum of f then f is convex on some interval containing c. A similar statement is true when c is a local maximum. Combining this observation with the previous Theorem and the First Derivative Test, we get the **Second Derivative Test**.

Theorem 5.52 (Second Derivative Test). *Let $f(x)$ be a twice differentiable function whose second derivative $f''(x)$ is a continuous function.*

$$f \text{ has a local min at } c \implies f'(c) = 0 \text{ and } f''(c) \geq 0$$
$$f \text{ has a local max at } c \implies f'(c) = 0 \text{ and } f''(c) \leq 0$$

Conversely,

$$f'(c) = 0 \text{ and } f''(c) > 0 \implies f \text{ has a local min at } c$$
$$f'(c) = 0 \text{ and } f''(c) < 0 \implies f \text{ has a local max at } c$$

We need continuity of $f''(x)$ so that $f''(c) > 0$ implies $f''(x) > 0$ on some interval containing c.

Optional Problems

Exercise. 5.53.

1. Find an example of a function which is convex but $f''(x)$ is not always > 0.

2. Why does the proof of Theorem 5.44 only proves that *if f is convex then* $f''(x) \geq 0$ (instead of $f''(x) > 0$)?

Exercise. 5.54. Several times in this section we used *geometric* arguments. This is less preferable than algebraic arguments as geometric arguments are not rigorous and hence susceptible to being incorrect.

1. Convert the definition of a convex function into an algebraic equation involving inequalities.

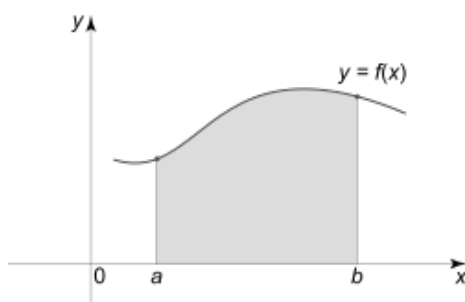2. Use this equation to rigorously prove that, say, $m_{a,x} < m_{a,y}$ if $a < x < y$.

Exercise. 5.55. Let f be a convex function on an interval containing the real numbers a, b. Show that for all $t \in [0,1]$,

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b)$$

This inequality is called **Jensen's inequality** for convex functions.

6 Integrals

6.1 Integrals as Areas

Signed area under the curve is given by $\int_a^b f(t)\,dt$

The integral of a function captures the concept of an area.

$$\int_a^b f(t)dt = \text{ signed area between the curve } y = f(x) \text{ and the x-axis}$$

where by signed area we mean that the integral is negative if $f(x) < 0$ and hence equals negative of the area in this case (as area is always positive). This is a good *interpretation* but not a good *definition* as we have not rigorously defined what area is. Instead we'll define the integral in terms of a limit and then define the (signed) area as the integral.

Exercise. 6.1. Express the following areas as (sums of) integrals. For each problem, draw pictures to justify your answer.

1. Area between the line $x + y = 1$, the x-axis and the y-axis.

2. Area enclosed between the graph $y = \sin x$ and the x-axis for $x \in [0, 2\pi]$.

3. Area enclosed between the parabola $y = x^2$ and the line $y = 1$.

4. Area of a circle of radius 1.

Several physical quantities can be expressed as integrals, for example, if t denotes time and $f(t)$ is the speed of a particle then the integral measures the distance traveled, if t denotes length and $f(t)$ is the linear density of a one dimensional object then the integral measures the total mass, etc.

6.2 Definition of Integral

We need several preliminary definitions to define the integral.

Definition 6.2.

1. For an interval $[a, b]$ a **partition** P is a sequence of real numbers

$$a = x_0 < x_1 < \cdots < x_n = b$$

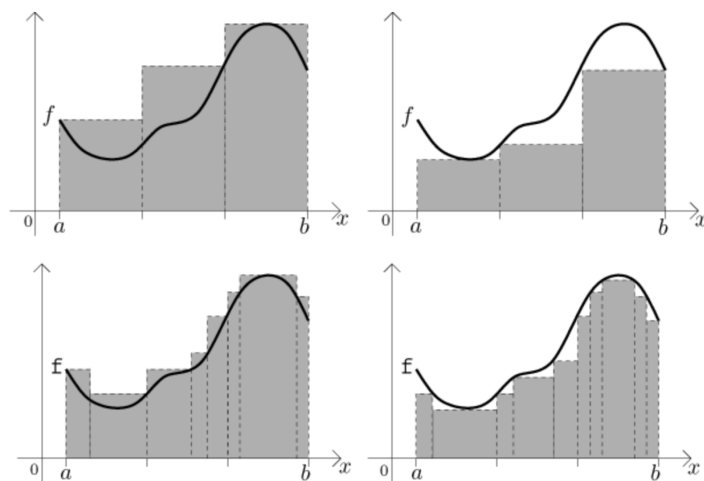2. A **uniform partition** P_n of $[a, b]$ is the *uniform length* partition

$$x_0 = a$$

$$x_1 = a + \frac{b-a}{n}$$

$$\vdots$$

$$x_i = a + i \cdot \frac{b-a}{n}$$

$$\vdots$$

$$x_n = b$$

3. For a partition P define

$$M_i = \sup\{f(x) : x_i \leq x \leq x_{i+1}\}$$
$$m_i = \inf\{f(x) : x_i \leq x \leq x_{i+1}\}$$

Define the **upper Riemann sum** $U(f, P)$ and the **lower Riemann sum** $L(f, P)$ as

$$U(f, P) = \sum_{i=0}^{n-1} M_i \cdot (x_{i+1} - x_i)$$

$$L(f, P) = \sum_{i=0}^{n-1} m_i \cdot (x_{i+1} - x_i)$$

For finer partitions, the Riemann sums provide better approximations of the integral, hence in the limiting case as $n \to \infty$ we obtain the exact integral.

Definition 6.3. We say that a function $f(x)$ is (Riemann) **integrable** on an interval $[a, b]$ if there exists a real number I such that

$$\lim_{n \to \infty} U(f, P_n) = I = \lim_{n \to \infty} L(f, P_n)$$

In this case, we define the integral of f on $[a,b]$ as

$$\int_a^b f(t)\, dt = I$$

Thus when a function $f(x)$ is integrable it suffices to compute either $\lim_{n\to\infty} U(f, P_n)$ or $\lim_{n\to\infty} L(f, P_n)$ to compute the integral.

We'll assume the following theorem without proof. The proof requires the notion of **uniform continuity** which is beyond the scope of this class.

Theorem 6.4. *If f is a continuous function on the interval $[a,b]$ then f is integrable on $[a,b]$.*

We'll do a few examples to better understand the definition of integral.

Exercise. 6.5. Let $f(x) = -x$, let $a = 0$ and $b = 1$.

1. Use basic geometry to find $\int_0^1 f(t)\, dt$.

2. Using pictures describe and compute $L(f, P_1)$, $L(f, P_2)$, and $L(f, P_3)$.

3. Compute $L(f, P_n)$ where n is a positive integer. You'll need to use

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

4. Find $\lim_{n\to\infty} L(f, P_n)$, which equals $\int_0^1 f(t)\, dt$ by Theorem 6.4, and verify that it agrees with part 1.

Exercise. 6.6. Let $f(x) = x^2$, let $a = 0$ and $b = 1$.

1. Compute $U(f, P_n)$ where n is a positive integer. You'll need to use the identity

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

2. Find $\lim_{n\to\infty} U(f, P_n)$, which equals $\int_0^1 f(t)\, dt$ by Theorem 6.4.

Exercise. 6.7. Let $a = 0$, $b = 1$, and let

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

1. For n a positive integer, compute $U(f, P_n)$ and $L(f, P_n)$.

2. Find $\lim_{n\to\infty} U(f, P_n)$ and $\lim_{n\to\infty} L(f, P_n)$ and show that the $f(x)$ is not integrable.

6.2.1 Indefinite Integrals

We'll assume the following theorem without proof.

Theorem 6.8. *For real numbers $a < b < c$, if f is integrable on $[a, c]$ then*

$$\int_a^c f(t)\, dt = \int_a^b f(t)\, dt + \int_b^c f(t)\, dt$$

Exercise. 6.9. Draw a picture to explain the statement of the above theorem.

So far, we've only defined $\int_a^b f(t)\, dt$ for $a < b$. We now extend this to all real numbers a, b by defining

$$\int_a^a f(t)\, dt = 0 \qquad \text{and} \qquad \int_a^b f(t)\, dt = -\int_b^a f(t)\, dt \quad \text{if } a > b$$

With this extended notation one can check that Theorem 6.8 is true for all real numbers a, b, c. For example, for $a < b < c$

$$\int_a^c f(t)\, dt = \int_a^b f(t)\, dt + \int_b^c f(t)\, dt$$

$$\implies \qquad \int_a^c f(t)\, dt = \int_a^b f(t)\, dt - \int_c^b f(t)\, dt$$

$$\implies \qquad \int_a^c f(t)\, dt + \int_c^b f(t)\, dt = \int_a^b f(t)\, dt$$

Definition 6.10. For an integrable function f and a real number a, an **antiderivative** is a function on the real numbers defined as

$$F_a(x) = \int_a^x f(t)\, dt$$

Exercise. 6.11. For real numbers a, b show that

$$F_a(x) - F_b(x)$$

is a constant that does not depend on x.

Thus for any integrable function f any two antiderivatives differ by a constant. This allows us to *define* the **indefinite integral** denoted

$$\int f(x)\, dx$$

as $\int_a^x f(t)\, dt$ for some real number a. Changing the constant a changes $F_a(x)$ by a constant, hence the indefinite integral is only defined up to a constant.

7 Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus is a remarkable theorem that connects the notions of derivatives and integrals.

If we denote by $\mathcal{D}$ the *operator* that takes a function and produces it's derivative i.e. $\mathcal{D}(f) = f'$ and by $\mathcal{I}$ the *operator* that takes a function and produces (one of) it's antiderivative i.e. $\mathcal{I}(f) = \int_a^x f(t)\ dt$, for some real number a, then the Fundamental Theorem of Calculus says that

1. $\mathcal{D}(\mathcal{I}(f)) = f$, and

2. $\mathcal{I}(\mathcal{D}(f)) = f +$ a constant.

so that Integral and Derivative are (almost) inverse operators.[1] This is why $\int_a^x f(t)\ dt$ is called (an) antiderivative.

Theorem 7.1 (Fundamental Theorem of Calculus).

Form 1 *Let f be a continuous function. For a real number a, let $F_a(x) = \int\limits_a^x f(t)\, dt$.*

Then $F_a(x)$ is differentiable with

$$F_a'(x) = f(x).$$

Form 2 *Let f be a differentiable function with f' continuous. Then for any real numbers a and x,*

$$\int\limits_a^x f'(t)\, dt = f(x) - f(a).$$

The idea of the proof of Form 1 is very straightforward. By definition of derivative,

$$
\begin{aligned}
F_a'(x) &= \lim_{h \to 0} \frac{F_a(x+h) - F_a(x)}{h} \\
&= \lim_{h \to 0} \frac{\int_a^{x+h} f(t)\, dt - \int_a^x f(t)\, dt}{h} \qquad (7.2)\\
&= \lim_{h \to 0} \frac{\int_x^{x+h} f(t)\, dt}{h} \qquad \text{by Exercise 6.11}
\end{aligned}
$$

For small h this last integral can be approximated by the area of a rectangle with height $f(x)$ and base h.

$$\int\limits_x^{x+h} f(t)\, dt \approx f(x) \cdot h$$

$$\implies \frac{\int_x^{x+h} f(t)\, dt}{h} \approx f(x)$$

---

[1] Compare this to the definition of inverse functions: $f(g(x)) = x$ and $g(f(x)) = x$.
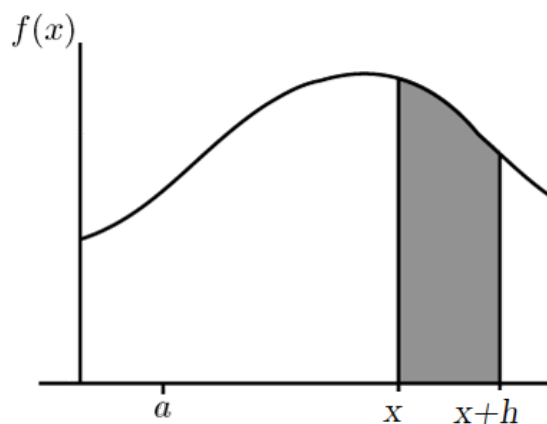
The area $\int_x^{x+h} f(t)\,dt$ can be approximated by a rectangle with height $f(x)$ and base h.

We'll make this last statement precise using ϵ, δ arguments. We'll start with a small Proposition.

Proposition 7.3. *If for all y in the interval $(x, x+h)$,*

$$m < f(y) < M$$

for some real numbers m, M, then

$$mh < \int_a^b f(t)\,dt < Mh$$

Exercise. 7.4.

1. Prove the above Proposition using a *geometric* argument.

2. **Optional:** Prove the above Proposition rigorously using the definition of integral involving Riemann sums.

Now we get back to the proof of the Form 1 of the Fundamental Theorem of Calculus.

Proof of Form 1 of the Fundamental Theorem of Calculus. Let $f(x)$ be a continuous function, let a be a real number. We want to show that

$$F_a'(x) = f(x)$$

By Equations (7.2) this is equivalent to proving that

$$\lim_{h \to 0} \frac{\int_x^{x+h} f(t)\,dt}{h} = f(x)$$

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We'll only prove this for the right hand limit, the left hand limit is proved similarly. We want to show that

$$\lim_{h \to 0^+} \frac{\int_x^{x+h} f(t)\, dt}{h} = f(x) \tag{7.5}$$

Exercise. 7.6. Convert Equation (7.5) into an ϵ, δ statement using the definition of limit.

Exercise. 7.7. Recall what it means for $f(x)$ to be continuous from the right, using the ϵ, δ definition of continuity.

Exercise. 7.8. Using this δ and Proposition 7.3 prove the statement in Exercise 7.6, thereby completing the proof of the Fundamental Theorem of Calculus.

☐

Exercise. 7.9 (Optional). Prove that $\lim_{h \to 0^-} \dfrac{\int_x^{x+h} f(t)\, dt}{h} = f(x)$. (Be careful of the signs.)

The Form 2 of the fundamental theorem follows directly from Form 1 using a small Proposition about derivatives.

Proposition 7.10. *If $g(x)$ is a differentiable function such that $g'(x) = 0$ for all real numbers x, then $g(x)$ is a constant function i.e. $g(x) = g(y)$ for all real numbers x, y.*

Exercise. 7.11. Prove the above Proposition by Contradiction[1].

Proof of Form 2 of the Fundamental Theorem of Calculus. Let f be a diffentiable function with f' continuous, and let a be a real number. We want to show that

$$\int_a^x f'(t)\, dt = f(x) - f(a)$$

Consider the function

$$g(x) = f(x) - \int_a^x f'(t)\, dt.$$

Exercise. 7.12. Using Form 1 of the Fundamental Theorem and Proposition 7.10 prove that $g(x)$ is a constant function.

Exercise. 7.13. As g is a constant function, for all real numbers x, $g(x) = g(a)$. Use this to prove Form 2 of the Fundamental Theorem of Calculus.

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[1] Hint: Use the Mean Value Theorem

Remark 7.14. For real numbers b, c,

$$\int_b^c f(t)\, dt = \int_b^a f(t)\, dt + \int_a^c f(t)\, dt$$

$$= -\int_a^b f(t)\, dt + \int_a^c f(t)\, dt$$

$$= F_a(c) - F_a(b)$$

where $F_a(x)$ is *an* antiderivative of $f(x)$ i.e. it is a function whose derivative is $f(x)$. This is the form is which the Fundamental Theorem of Calculus is most often used. Which a we use to find the antiderivative does not matter, so the integral can be written in the following forms.

$$\int_b^c f(t)\, dt = F_a(c) - F_a(b)$$

$$= F_a(x)\big|_b^c$$

$$= \int f(x)dx\bigg|_b^c$$

$$= \int f(x)dx\bigg|_{x=b}^{x=c}$$

It is acceptable to use any of these forms as they all mean the same thing.

7.1 Understanding the Fundamental Theorem of Calculus

The Fundamental Theorem has a LOT of applications. We'll spend the next several sections just understanding this theorem.

Exercise. 7.15. Let a be a real number. Find derivatives of the following functions using the Fundamental Theorem and simple algebraic manipulations.

1. $\int_x^a f(t)\,dt$.

2. $\int_a^{x^2} f(t)\,dt$.

3. $\int_a^{g(x)} f(t)\,dt$, where $g(x)$ is a differentiable function.

4. $\int_x^{x^2} f(t)\,dt$.

5. $\int_{h(x)}^{g(x)} f(t)\,dt$, where $g(x), h(x)$ are differentiable functions.

6. $\int_a^x x \cdot f(t)\,dt$. (Be careful!)

Exercise. 7.16. Prove that if

$$f'(t) > 0$$

for all $t \in [a, b]$, then f is increasing on the interval $[a, b]$ i.e. for all x, y in the interval $[a, b]$ if $x < y$ then $f(x) < f(y)$.[1]

Theorem 7.17 (u-substitution). *Let f, g be differentiable functions. Then,*

$$\int_a^x f(g(t))g'(t)\,dt = \int_{g(a)}^{g(x)} f(u)\,du$$

Exercise. 7.18 (Proof of u-substitution). Consider the function

$$h(x) = \int_a^x f(g(t))g'(t)\,dt - \int_{g(a)}^{g(x)} f(u)\,du$$

1. Show that $h'(x) = 0$.

2. Show that $h(a) = 0$.

3. Argue that these two statements imply that $h(x) = 0$ for all x, thereby proving u-substitution.

---

[1] Hint: Compute the integral $\int_a^x f'(t)\,dt$.

Theorem 7.19 (Integration by parts). *Let f, g be differentiable functions. Then,*

$$\int_a^x f(t)g'(t)\,dt = f(x)g(x) - \int_a^x f'(t)g(t)\,dt + c \qquad (7.20)$$

for some constant c.

Exercise. 7.21 (Proof of Integration by parts). Consider the function

$$h(x) = \int_a^x f(t)g'(t)\,dt - \left(f(x)g(x) - \int_a^x f'(t)g(t)\,dt \right)$$

Show that $h'(x) = 0$. Argue that this proves Integration by parts.

Exercise. 7.22. Find the contant c in Equation (7.20).

Because of the constant c that shows up in Equation (7.20) integration by parts is often expressed using indefinite integrals as,

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx + c$$

Remark 7.23. These two theorems

- u-substitution

- integration by parts

provide the main techniques for computing integrals. In the later sections we'll do several problems to practice these.

8 Trig, Exp, and Log functions

We'll use the Fundamental Theorem of Calculus to compute the derivatives of trigonometric functions, exponentials, and logarithms.

## 8.1 Trigonometric Functions

In this section, we'll prove that

$$(\cos x)' = -\sin x$$
$$(\sin x)' = \cos x$$

We'll only show this for $0 < x < \pi/2$, however, the statements are true for all real numbers x. We'll do this by first computing $\left(\cos^{-1} x\right)'$ using basic geometry and the Fundamental Theorem and then using the formula for the derivative of inverse functions.

Exercise. 8.1. Consider the unit circle $x^2 + y^2 = 1$. In the following figure, the shaded region is a sector of angle θ (in radians), for $0 < \theta < \pi/2$, so that the corresponding point on the circle is $(\cos\theta, \sin\theta)$.
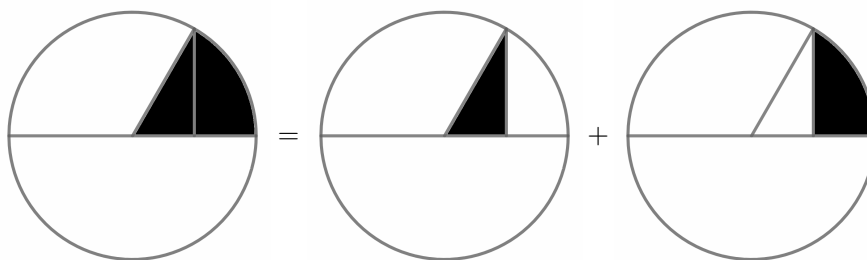
1. Argue that the above decomposition of areas can be expressed algebraically as

$$\frac{\theta}{2} = \frac{x\sqrt{1-x^2}}{2} + \int_x^1 \sqrt{1-t^2}\, dt$$

   which can further be simplified to

$$\frac{\cos^{-1} x}{2} = \frac{x\sqrt{1-x^2}}{2} + \int_x^1 \sqrt{1-t^2}\, dt$$

2. Differentiate both sides to show that

$$\left(\cos^{-1} x\right)' = -\frac{1}{\sqrt{1-x^2}}$$

3. Use the formula for the derivative of inverse function[1] to show that

$$(\cos\theta)' = -\sin\theta$$

---

[1] $\left(f^{-1}(a)\right)' = \dfrac{1}{f'(f^{-1}(a))}$

Exercise. 8.2. Differentiate both sides of the trig identity

$$\sin^2 x + \cos^2 x = 1$$

and use your computation of $(\cos x)'$ to show that

$$(\sin x)' = \cos x$$

Exercise. 8.3.   1. Use the formula for the derivative of inverse function to compute the derivative of $\sin^{-1} x$.

   2. What is the relationship between the derivatives of $\sin^{-1} x$ and $\cos^{-1} x$? Why do you think this is the case?

Exercise. 8.4. Compute the derivatives of $\tan x$ and $\tan^{-1} x$.

8.2   Exponential Functions and Logarithms

Exponential functions are a bit tricky to define from first principles as every natural definition of e^x uses either limits of sequences or differential equations. We'll instead give a more *ad hoc* definition of logarithms as in the book and *verify* that it satisfies the properties that logarithms are supposed to satisfy.

Definition 8.5. Define the **natural logarithm** to be the integral

$$\ln x = \int_1^x \frac{1}{t}\, dt$$

for $x > 0$.

Remark 8.6. As $\ln x$ is *defined* to be the antiderivative of $1/x$,

$$(\ln x)' = \frac{1}{x}$$

Exercise. 8.7.   1. Draw the graph of $\dfrac{1}{x}$ for $x > 0$.

   2. *Geometrically* argue that

$$\ln x \text{ is } \begin{cases} \text{positive} & \text{if } x > 1 \\ 0 & \text{if } x = 1 \\ \text{negative} & \text{if } x < 1 \end{cases}$$

   3. Using u-substitution[1] show that

$$\int_1^y \frac{1}{t}\, dt = \int_x^{xy} \frac{1}{t}\, dt$$

   for real numbers $x, y > 0$. (You should think about what this means geometrically.)

---

[1] Hint: In the u-substitution formula $\int_a^b f(g(t))\, g'(t)\, dt = \int_{g(a)}^{g(b)} f(u)\, du$ use $g(t) = x \cdot t$.

4. Show that this implies that

$$\ln x + \ln y = \ln xy \qquad (8.8)$$

This last identity (8.8) is the fundamental identity of logarithms.

Definition 8.9. Define $\exp(x)$ to be the inverse function of $\ln x$ i.e.

$$\exp(\ln x) = x$$
$$\ln(\exp x) = x.$$

Definition 8.10. Define e to be the value of $\exp(x)$ at $x = 1$.

Exercise. 8.11.   1. Show that $\exp(0) = 1$.

2. Show that Equation (8.8) implies that

$$\exp(a + b) = \exp(a) \cdot \exp(b)$$

3. Use this to argue that

$$\exp(n) = e^n$$

where n is a positive integer.

4. **(Optional)** Extend the above statement first to negative integers and then to rational numbers.
   It follows then from continuity arguments that

$$\exp(x) = e^x$$

for all real numbers x.

Exercise. 8.12. Use the formula for the derivative of inverse function to show that

$$(e^x)' = e^x$$

This completes the computation of derivatives of all the standard functions.

Exercise. 8.13. The Fundamental Theorem of Calculus says that if $f'(x) = g(x)$ then $f(x) = \int g(x)\, dx + c$, where $\int g(x)\, dx$ stands for the indefinite integral. Go back to your derivative computations in this section and rewrite them as indefinite integrals.

We'll next use the u-substitution, and Integration by Parts to compute integrals of functions which can be written in terms of these standard functions.

8.3 Trigonometric Identities

We'll need several trigonometric identities for computing integrals. While it is possible to derive these identities using Euclidean geometry, there is a much faster trick to derive these using *Euler's identity*.

8.3.1 Complex Numbers

First we need some basic algebraic facts about complex numbers. **Complex numbers** are numbers of the form

$$z = a + bi$$

where a and b are real numbers and i is a *formal* variable that satisfies

$$i^2 = -1.$$

The number a is called the **real part** of z, denoted $\operatorname{Re}(z)$, and the number b is called it's **imaginary part**, denoted $\operatorname{Im}(z)$.

A real number r can be thought of the complex number $r + 0i$. Thus the set of complex numbers contains the set of real numbers.

As with real numbers, we can add, subtract, multiply, and divide complex numbers.

Example 8.14. Multiplying complex numbers:

$$
\begin{aligned}
(a + bi)(c - di) &= a(c - di) + bi(c - di) \\
&= ac - adi + bci + bd && \text{as } i^2 = -1 \\
&= (ac + bd) + i(-ad + bc)
\end{aligned}
$$

Exercise. 8.15. Show that

$$(a + bi)(a - bi) = a^2 + b^2.$$

The complex number $\bar{z} = a - bi$ is called the **complex conjugate** of $z = a + bi$. By the above exercise, we get a real number after multiplying a complex number by it's conjugate which is an extremely useful fact.

Example 8.16. Complex conjugates are useful when dividing complex numbers. To simplify a fraction, we multiply and divide by the complex conjugate of the denominator.

$$
\begin{aligned}
\frac{a + bi}{c + di} &= \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} \\
&= \frac{(a + bi)(c - di)}{c^2 + d^2} && \text{by Exercise (8.15)} \\
&= \frac{(ac + bd) + i(-ad + bc)}{c^2 + d^2} \\
&= \frac{ac + bd}{c^2 + d^2} + i \cdot \frac{-ad + bc}{c^2 + d^2}
\end{aligned}
$$

Exercise. 8.17. Find the real and imaginary parts of the following complex numbers:

1. $(2 - 3i)(i)$

2. $(2 - 3i)(1 + i)$

3. $\dfrac{1}{i}$

4. $\dfrac{1}{1+i}$

5. $\dfrac{2 - 3i}{i}$

6. $\dfrac{2 - 3i}{1 + i}$

8.3.2 Euler's Identity

It is possible to extend trigonometric and exponential functions (but not logarithms) to complex numbers. These functions have the same derivatives and integrals as in the real case. The following **Euler's Identity** establishes a deep connection between trigonometric and exponential functions.

Theorem 8.18 (Euler's identity).

$$e^{i\theta} = \cos\theta + i\sin\theta$$

for all real numbers θ.

The proof of this theorem requires us to extend the entire theory of calculus to complex numbers (called complex analysis) and is beyond the scope of this class. We'll simply use it as a fast method of (re)deriving several trig identities, and later on for doing integrals computations.

Example 8.19. We know that

$$e^{i\theta} \cdot e^{-i\theta} = e^0$$

Simplifying the left hand side we get,

$$
\begin{aligned}
\Rightarrow \quad & (\cos\theta + i\sin\theta) \cdot (\cos(-\theta) + i\sin(-\theta)) = e^0 \\
\Rightarrow \quad & (\cos\theta + i\sin\theta) \cdot (\cos\theta - i\sin\theta) = 1 \\
\Rightarrow \quad & \cos^2\theta + \sin^2\theta = 1 \qquad \text{by (8.15)}
\end{aligned}
$$

which is the fundamental identity of trigonometric functions. (In the above derivation we used the fact that $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$.)

Exercise. 8.20. Using Euler's identity and

$$\left(e^{i\theta}\right)^2 = e^{2i\theta}$$

find the formulae for $\cos 2\theta$ and $\sin 2\theta$. (These are called the **double angle formulae**.)

Exercise. 8.21.   1. Using Euler's identity and

$$e^{ix} \cdot e^{iy} = e^{i(x+y)}$$

find the formulae for $\cos(x + y)$ and $\sin(x + y)$.

2. Using Euler's identity and

$$e^{ix} \cdot e^{-iy} = e^{i(x-y)}$$

find the formulae for $\cos(x - y)$ and $\sin(x - y)$.

Exercise. 8.22. Using the previous exercise, show that

$$2 \cos x \cos y = \cos(x - y) + \cos(x + y)$$
$$2 \sin x \sin y = \cos(x - y) - \cos(x + y)$$
$$2 \sin x \cos y = \sin(x - y) + \sin(x + y)$$

We'll now apply these identities to compute some definite integrals. In later sections, we'll use these to compute indefinite integrals such as $\int e^{ax} \sin bx \, dx$.

Exercise. 8.23. Let n be an integer.

1. Draw graphs of

$$\sin x, \sin(-x), \sin 2x, \sin(-2x),$$
$$\cos x, \cos(-x), \cos 2x, \cos(-2x),$$

for $-\pi \le x \le \pi$.

2. Geometrically argue that

$$\int_{-\pi}^{\pi} \cos(nt) \, dt = \begin{cases} 0 & \text{if } n \ne 0 \\ 2\pi & \text{if } n = 0 \end{cases}$$

3. Geometrically argue that

$$\int_{-\pi}^{\pi} \sin(nt) \, dt = 0$$

4. Why do we need to make two cases for $\cos x$ but not for $\sin x$?

Exercise. 8.24. Let m and n be positive integers. Using the previous two exercises, show that

$$\int_{-\pi}^{\pi} \cos(nt) \cos(mt) \, dt = \begin{cases} 0 & \text{if } m \ne n \\ \pi & \text{if } m = n \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(nt) \sin(mt) \, dt = \begin{cases} 0 & \text{if } m \ne n \\ \pi & \text{if } m = n \end{cases}$$

$$\int_{-\pi}^{\pi} \cos(nt) \sin(mt) \, dt = 0$$

These integrals are the foundational identities for Fourier analysis, which is one of the most powerful mathematical theories emerging out of elementary calculus.

9 Computing Indefinite Integrals

In this section, we're interested in computing *indefinite integrals* or *antiderivatives*. Indefinite integrals cannot be computed for all functions, for example, there is no closed form for the **Gaussian Integral**

$$\int \exp\left(-\frac{x^2}{2}\right) dx$$

which is widely used in statistics, and the only way to compute it is using numerical methods.

There is no general method of computing indefinite integrals. Instead, in this section we'll learn a few common tricks which you can combine in various ways to solve complicated problems. There are two basic tools for computing indefinite integrals

1. u-substitution and

2. integration by parts.

These are often combined with various algebraic and trig identities.

One thing that makes the computation of indefinite integrals far more complicated than that of derivatives, in practice, is that similar looking functions can have wildly different integrals. For example,

$$\int 1/x \, dx = \ln x + c$$

$$\int 1/x^2 \, dx = -1/x + c$$

$$\int 1/(1+x^2) \, dx = \tan^{-1} x + c$$

$$\int 1/(1-x^2) \, dx = 1/2 \cdot \ln((1+x)/(1-x)) + c$$

$$\int 1/\sqrt{1-x^2} \, dx = \sin^{-1} x + c$$

So just by looking at a problem you cannot easily *guess* what it's integral is going to look like.

9.1 u-substitution

In terms of indefinite integrals, u-substitution takes the form

$$\int f(g(x)) g'(x) \, dx = \int f(u) \, du + \text{constant} \tag{9.1}$$

Ideally we want the integral on the right-hand side to be simpler than the one on the left-hand side. We interpret this identity as having obtained the right-hand from the left-hand side by making the "substitution"

$$u = g(x)$$
$$du = g'(x) \, dx$$

This is just a *mnemonic* and doesn't have a concrete mathematical meaning (at least not a simple one).

Remark 9.2. After finding the integral $\int f(u)\, du$ you should ALWAYS plug back $u = g(x)$ to get your final answer in terms of the original variable.

Exercise. 9.3. Find the following indefinite integrals using u-substitution(s).

1. $\int xe^{x^2}\, dx$

2. $\int \dfrac{1}{x \ln x}\, dx$

3. $\int \dfrac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$

4. $\int \dfrac{x}{4x^2 + 5}\, dx$

5. $\int \dfrac{1}{x^2 + 2}\, dx$

6. $\int \sec^2 x \tan^5 x\, dx$

7. $\int \tan x\, dx$

8. $\int \dfrac{e^x}{e^{2x} + 2e^x + 1}\, dx$

9.2 Integration by Parts

Integration by parts is *sometimes* used to integrate products of functions (you should always try u-substitution first, unless the use of integration by parts is extremely obvious). There are various ways of memorizing integration by parts, you should pick one that you find easy to remember.

One possible way to express it is as follows.

$$\int fg = fG - \int f'G + \text{constant} \tag{9.4}$$

where G is any antiderivative of g. In most problems, integration by parts is useful only if the function $f'G$ is simpler that the function fg.

Example 9.5. For computing $\int xe^x\, dx$ by parts, we have (at least) two possible choices:

1. $f = x$ and $g = e^x$. In this case,

$$f'G = e^x$$

2. $f = e^x$ and $g = x$. In this case,

$$f'G = e^x \cdot x^2/2$$

In the first case, the function $f'G$ is simpler than fg and hence this is the decomposition we should use for applying integration by parts.

Exercise. 9.6. Find the following indefinite integrals using integration by parts.

1. $\int xe^x\, dx$

2. $\int x^2 e^x\, dx$

3. $\int x \sin x\, dx$

4. $\int \ln x\, dx$ [1]

5. $\int x \ln x\, dx$

6. $\int x(\ln x)^2\, dx$

---

[1] Hint: Use $f = \ln x$ and $g = 1$.

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9.3   Trigonometric Integrals

There are a LOT of tricks for computing integrals of functions involving trigono-metric functions. We'll only compute integrals of two kinds of functions:

$$\sin^m x \cos^n x \qquad \text{and} \qquad e^{ax} \sin bx$$

Let m and n be non-negative integers. For integrating $\sin^m x \cos^n x$ if **either** m **or** n **is odd** then we get the answer by a simple u-substitution. For example,

$$\sin^{2k+1} x \cdot \cos^n x = \sin^{2k} x \cdot \sin x \cdot \cos^n x$$
$$= \left(\sin^2 x\right)^k \cdot \sin x \cdot \cos^n x$$
$$= \left(1 - \cos^2 x\right)^k \cdot \sin x \cdot \cos^n x$$

after which we can integrate using the u-substitution $u = \cos x$.

Exercise. 9.7. Compute the following indefinite integrals.

1. $\int \sin^3 x \cos^2 x \, dx$
2. $\int \sin^2 x \cos^5 x \, dx$

Let m and n be non-negative integers. For integrating $\sin^m x \cos^n x$ if **both** m **and** n **are even** then we use the double angle formulae,

$$\sin^{2k} x = \left(\sin^2 x\right)^k \qquad\qquad \cos^{2l} x = \left(\cos^2 x\right)^l$$

$$= \left(\frac{1 - \cos(2x)}{2}\right)^k \qquad\qquad = \left(\frac{1 + \cos(2x)}{2}\right)^l$$

Exercise. 9.8. Compute the following indefinite integrals.

1. $\int \sin^2 x \, dx$
2. $\int \sin^2 x \cos^2 x \, dx$

Let a, b be real numbers. Using the Euler's identity

$$e^{ibx} = \cos bx + i \sin bx$$

we get

$$\int e^{ax} \cos bx \, dx = \text{real part of } \int e^{ax} e^{ibx} \, dx$$
$$\int e^{ax} \sin bx \, dx = \text{imaginary part of } \int e^{ax} e^{ibx} \, dx$$

Exercise. 9.9. Compute the following indefinite integrals.

1. $\int e^{ax} \cos bx \, dx$
2. $\int e^{ax} \sin bx \, dx$

9.4 Trigonometric Substitutions

Trigonometric substitutions are extremely useful when *eliminating radicals* (among other things) because of the identities

$$\sin^2 x + \cos^2 x = 1$$
$$\sec^2 x = 1 + \tan^2 x \qquad (9.10)$$

Example 9.11. If there is a term

$$\sqrt{1 + x^2}$$

in our integral, then we can substitute $x = \tan u$ so that

$$\sqrt{1 + x^2} = \sqrt{1 + \tan^2 u}$$
$$= \sec u \qquad \text{by (9.10)}$$

We'll need a few preliminary computations.

Exercise. 9.12.   1. Compute $(\sec x)'$.

2. Show using the fundamental theorem of calculus that

$$\int \sec x \, dx = \ln(\sec x + \tan x) + c$$
$$\int \csc x \, dx = -\ln(\csc x + \cot x) + c$$

3. **Optional:** Compute the integrals $\int \sec x \, dx$ and $\int \csc x \, dx$ directly (without using the fundamental theorem).

Exercise. 9.13. Compute the following integrals using trig substitutions.

1. $\int \dfrac{1}{\sqrt{1 - x^2}} \, dx$

2. $\int \dfrac{1}{\sqrt{1 + x^2}} \, dx$

3. $\int \dfrac{1}{\sqrt{x^2 - 1}} \, dx$

4. $\int \dfrac{1}{x\sqrt{x^2 - 1}} \, dx$

5. $\int \dfrac{1}{x\sqrt{1 - x^2}} \, dx$

6. $\int \dfrac{1}{x\sqrt{1 + x^2}} \, dx$

7. $\int \sqrt{1 - x^2} \, dx$

8. $\int x^3 \sqrt{1 - x^2} \, dx$

Exercise. 9.14. (Optional)

1. Compute the integral

$$\int \sec^3 x \, dx$$

2. Compute the integral

$$\int \sqrt{1 + x^2} \, dx$$

9.5 Partial Fractions

Partial fractions is a technique used to compute integrals of the form

$$\int \frac{P(x)}{Q(x)} \, dx$$

where $P(x)$ and $Q(x)$ are polynomials. The higher the degree of the denominator $Q(x)$ the harder it is to compute the integral.

Linear Polynomials

When the denominator $Q(x)$ is linear the integral can be computed easily using u-substitution.

Exercise. 9.15. Compute the following integrals

1. $\int \frac{x^2 + 1}{x} \, dx$

2. $\int \frac{x^2 + 1}{x + 1} \, dx$

3. $\int \frac{x}{2x - 3} \, dx$

4. $\int \frac{x^2}{2x + 2} \, dx$

Quadratic with Repeated Roots

When the denominator is a quadratic $Q(x) = ax^2 + bx + c$ there are 3 different methods for finding the integral, depending on what the roots of $Q(x)$ are.

$$x^2 + ax + b \text{ has } \begin{cases} \text{repeated roots} \\ \text{complex roots} \\ \text{real non-repeated roots} \end{cases}$$

The first case of **repeated roots** is the easiest. In this case, our goal is to find a simple u-substitution to reduce the problem to

$$\int \frac{R(u)}{u^2} \, du$$

where $R(u)$ is some polynomial.

Exercise. 9.16. For each of the following problems, verify that the denominator has repeated roots. Then compute the integrals.

1. $\int \frac{x}{4x^2 - 4x + 1} \, dx$

2. $\int \frac{x^2}{x^2 + 2x + 1} \, dx$

Quadratic with Complex Roots

When the denominator $Q(x) = ax^2 + bx + c$ has complex roots our goal is to find a u-substitution to reduce the problem to the integrals

$$\int \frac{1}{u^2 + 1} \, du \qquad \text{and} \qquad \int \frac{u}{u^2 + 1} \, du$$

Exercise. 9.17. For each of the following problems, verify that the denominator has complex roots. Then compute the integrals.

1. $\int \dfrac{1}{x^2 + 4}\, dx$

3. $\int \dfrac{3x}{x^2 + 2x + 2}\, dx$

2. $\int \dfrac{3}{x^2 + 2x + 2}\, dx$

4. $\int \dfrac{x}{4x^2 - 4x + 3}\, dx$

Exercise. 9.18. If the degree of the numerator is ≥ 2 then we first have to do a long division to simplify the numerator. Compute the following integrals.

1. $\int \dfrac{x^2}{x^2 + 4}\, dx$

2. $\int \dfrac{3x^3}{x^2 + 2x + 2}\, dx$

Quadratic with Real Non-repeated Roots

Finally, when the denominator $Q(x)$ has real non-repeated roots we need to use the method of partial fractions.

In this method we first need to find a factorization of $Q(x)$ as a product of linear terms, say $Q(x) = Q_1(x) \cdot Q_2(x)$ where both $Q_1(x)$ and $Q_2(x)$ are linear. If the degree of the numerator $P(x)$ is ≤ 1 then we can always write

$$\frac{P(x)}{Q(x)} = \frac{A}{Q_1(x)} + \frac{B}{Q_2(x)}$$

for some constants A and B. This is called the **partial fraction decomposition** of $\frac{P(x)}{Q(x)}$. We find A and B by multiplying both sides by $Q(x)$ and comparing the coefficients on the left and right hand sides.

Exercise. 9.19. For each of the following problems, verify that the denominator has real non-repeated roots. Then compute the integrals.

1. $\int \dfrac{1}{x^2 - 1}\, dx$

3. $\int \dfrac{5}{x^2 - 2x}\, dx$

2. $\int \dfrac{4x}{x^2 - 4}\, dx$

4. $\int \dfrac{3x}{x^2 + x - 2}\, dx$

Exercise. 9.20. As before if the degree of the numerator is ≥ 2 then we first need to do long division to simplify the numerator. Compute the following integrals.

1. $\int \dfrac{3x^2}{x^2 + x - 2}\, dx$

3. $\int \dfrac{4x^3 - 3x + 5}{x^2 - 2x}\, dx$

2. $\int \dfrac{x^2 - 1}{x^2 - 4}\, dx$

Degree ≥ 3

When degree of the denominator $Q(x)$ is ≥ 3, one can show the existence of a partial fraction decomposition using *abstract algebra*. But the details are much more complicated and hard to do by hand.

Instead, feel free to use the internet to compute partial fraction decompositions. For example, go to `http://www.wolframalpha.com` and input

```
partial fractions (x^2 - 2)/(x+1)^3
```

for computing the partial fraction decomposition of $\dfrac{x^2 - 2}{(x+1)^3}$.

Once you have the partial fraction decomposition you can use u-substitution to compute the integral.

Exercise. 9.21. Compute the following integrals using partial fraction decompositions.

1. $\int \dfrac{x^2 - 2}{(x+1)^3}\, dx$

2. $\int \dfrac{8}{3x^3 + 7x^2 + 4x}\, dx$

3. $\int \dfrac{x^3 + 8}{(x^2 - 1)(x - 2)}\, dx$

9.6   Practice Problems

We've learned several techniques for computing indefinite integrals. Of these

1. Basic u-substitution

2. Trigonometric substitutions (+ trig identities)

are the tricky ones as there are a lot of possible substitutions to choose from. The other three

3. Integration by Parts

4. Trigonometric integrals

5. Partial fractions

are much easier to use.

The following problems will require you to use all of the above techniques. You should not expect to *see* the solution right away, instead, systemically try different things until you reduce the problem to something that looks familiar.

Exercise. 9.22.   Compute the following integrals.

1. $\int \dfrac{e^x}{(e^x - 1)(e^x - 3)}\, dx$

2. $\int \dfrac{1}{1 + e^x}\, dx$

3. $\int e^{\sqrt{x}}\, dx$

4. $\int \sin \sqrt{x}\, dx$

5. $\int \dfrac{\sin^3 x}{\cos^2 x}\, dx$

6. $\int \dfrac{1 - \sin x}{\cos^2 x}\, dx$

7. $\int \dfrac{1}{1 + \sin x}\, dx$

8. $\int \sqrt{1 + \cos 2x}\, dx$

9. $\int \sec^3 x \tan x\, dx$

10. $\int x \tan^{-1} x\, dx$

11. $\int x^2 \tan^{-1} x\, dx$

12. $\int \tan^{-1} \sqrt{x}\, dx$

13. $\int \dfrac{x}{\sqrt{2 + 2x + x^2}}\, dx$

14. $\int \dfrac{1}{\sqrt{2x - x^2}}\, dx$

15. $\int \ln(1 + x^2)\, dx$

16. $\int \tan^2 x\, dx$

10 Applications of Integrals

10.1 Improper Integrals

Improper integrals compute areas of unbounded regions. These integrals are a source of several false paradoxes. As we'll see below, counter-intuitive as it may seem, unbounded regions *can* have finite areas.

We'll study two types of improper integrals: depending on whether the corresponding region is unbounded along the *x*-axis or along the *y*-axis. The idea is to *view an unbounded region as a limit of bounded regions*.

Definition 10.1. Improper integrals (of the first kind) are integrals where one (or both) of the bounds is $\pm\infty$.[1] Let f be a continuous function. Define,

$$\int_a^\infty f(t)\, dt = \lim_{x\to\infty} \int_a^x f(t)\, dt$$

$$\int_{-\infty}^a f(t)\, dt = \lim_{x\to-\infty} \int_x^a f(t)\, dt$$

$$\int_{-\infty}^\infty f(t)\, dt = \lim_{x\to\infty} \int_0^x f(t)\, dt + \lim_{x\to-\infty} \int_x^0 f(t)\, dt$$

where a is a real number.

Exercise. 10.2. For the following problems, draw the region whose area the integral is computing (you can use a calculator for graphing) and compute the integral.

1. $\int_1^\infty \dfrac{1}{t}\, dt$

2. $\int_1^\infty \dfrac{1}{t^2}\, dt$

3. $\int_0^\infty \sin t\, dt$

4. $\int_0^\infty 2te^{-t^2}\, dt$

5. $\int_1^\infty \dfrac{9}{(1-3t)^4}\, dt$

6. $\int_{-\infty}^\infty \dfrac{6t^3}{(t^4+1)^2}\, dt$

Exercise. 10.3.   1. The choice of the number 0 in the definition of $\int_{-\infty}^\infty$ is not important. We can replace 0 with any real number and the answer would not change. Compute

$$\lim_{x\to\infty} \int_1^x \frac{6t^3}{(t^4+1)^2}\, dt + \lim_{x\to-\infty} \int_x^1 \frac{6t^3}{(t^4+1)^2}\, dt$$

and compare it with Problem 6 of the previous exercise. Why does the answer not change?

---

[1]Note that ∞ is not a number here either, but represents a limit.

2. On the other hand, consider the following change. Compute

(a) $\lim\limits_{x\to\infty} \int_{-x}^{x} \sin t \, dt$, and

(b) $\lim\limits_{x\to\infty} \int_{0}^{x} \sin t \, dt + \lim\limits_{x\to-\infty} \int_{x}^{0} \sin t \, dt$

Why are the two answers different?

Thus, when limits are involved, our intuition does not always provide us with the correct answer.

Definition 10.4. Improper integrals (of the second kind) are integrals where the function being integrated has a discontinuity in the interval of integration. Let $[a, b]$ be an interval and let $c \in [a, b]$ be a point such that $f(x)$ is discontinuous at c but continuous on the rest of interval. Define,

$$\text{if } a < c < b, \qquad \int_{a}^{b} f(t) \, dt = \lim_{x\to c^-} \int_{a}^{x} f(t) \, dt + \lim_{x\to c^+} \int_{x}^{b} f(t) \, dt$$

$$\text{if } c = a, \qquad \int_{c}^{b} f(t) \, dt = \lim_{x\to c^+} \int_{x}^{b} f(t) \, dt$$

$$\text{if } c = b, \qquad \int_{a}^{c} f(t) \, dt = \lim_{x\to c^-} \int_{a}^{x} f(t) \, dt$$

Exercise. 10.5. For the following problems, draw the region whose area the integral is computing (you can use a calculator for graphing if you like) and compute the integral.

1. $\int_{0}^{3} \dfrac{1}{\sqrt{3-t}} \, dt$

2. $\int_{2}^{3} \dfrac{4t}{\sqrt[3]{t^2 - 4}} \, dt$

We can compute integrals of more general unbounded regions using the same ideas.

10.2 Arc Length

Recall that the integral $\int_{a}^{b} f(t) \, dt$ is defined as the limit of the Riemann sum

$$\sum_{i=0}^{n-1} m_i (x_{i+1} - x_i)$$

where x_i's form a partition of the interval $[a, b]$ and m_i is the min (inf) of the $f(x)$ values in the interval $[x_i, x_{i+1}]$. One way to *represent* this sum is as the sum of several quantities of the form

$$y \, \Delta x$$

where Δx stands for the length of the interval $x_{i+1} - x_i$ which can be thought of as **the change in** x, and y stands for **the minimum value of** $f(x)$ for $x \in [x_i, x_{i+1}]$. So that

$$\lim_{\Delta x \to 0} \left(\sum y \, \Delta x \right) = \int y \, dx \qquad (10.6)$$

The reason for doing this, is that this allows us to interpret several *naturally occurring* quantities as integrals (which is why the Riemann sum formulation of integral is much more useful than the area formulation). One such example is an **arc length**.

Arc length is just a fancy name for the length of a curve. For example, the arc length of a circle of radius r is $2\pi r$. We are interested in computing the arc length of the graph of a function $y = f(x)$.
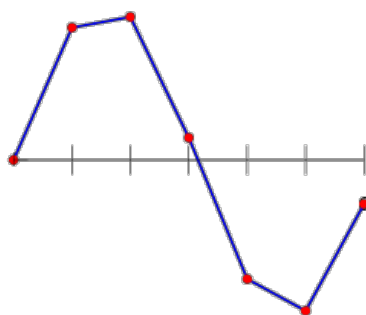
Approximating the graph $y = f(x)$ by line segments

We do this by partitioning the interval $[a, b]$ and approximating the graph $y = f(x)$ over each partition by a straight line. The length of each small straight line segment equals

$$\sqrt{\Delta x^2 + \Delta y^2}$$

and hence the arc length equals

$$\lim_{\Delta x \to 0} \left(\sum \sqrt{\Delta x^2 + \Delta y^2} \right)$$

This looks very close to (10.6). Next we need to separate a Δx term.

$$\lim_{\Delta x \to 0} \left(\sum \sqrt{1 + \left(\frac{\Delta y}{\Delta x} \right)^2} \, \Delta x \right)$$

This looks exactly like (10.6) with y replaced with $\sqrt{1 + \left(\frac{\Delta y}{\Delta x} \right)^2}$. Finally we notice that

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

which gives us the formula for the arc length.

Theorem 10.7. *Let f be a differentiable function on the interval $[a, b]$. The arc length of the graph $y = f(x)$ over the interval $[a, b]$ is given by the integral*

$$\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

While the above derivation is not a *proof*, it can be converted into a proof using sup's and inf's, using the same method we used to prove the Fundamental Theorem of Calculus.

Exercise. 10.8. Using the arc length formula in Theorem 10.7 compute the following:

1. Circumference of a circle of radius r.

2. Arc length of the parabola $y = x^2$ from 0 to 1.

3. Arc length of line segment $y = mx$ from a to b, where m, a, b are real numbers. Explain the result geometrically.

Exercise. 10.9. Here we encounter the first example of a *naturally occurring* unsolvable integrals. Find the expression for the arc length of the sine curve $y = \sin x$ from 0 to a. This integral, called an **elliptic integral**, cannot be algebraically computed.[1]

Exercise. 10.10. In this problem, you'll need to use the fact that the (curved) surface area of a cylinder is $2\pi r h$ and the volume is $\pi r^2 h$, where r is the radius and h is the height of the cylinder.
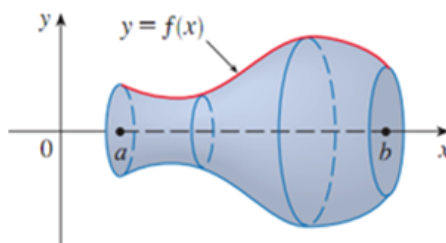
1. Derive the formula for the area of the surface obtained by revolving the curve $y = f(x)$ over the interval $[a, b]$ about the x-axis. Such a surface is called a **surface of revolution**.

2. Derive the formula for the volume of the solid obtained by revolving the region under the curve $y = f(x)$ over the interval $[a, b]$ about the x-axis. Such a solid is called a **solid of revolution**.

We can do both of these constructions by revolving the curve about the y-axis instead of the x-axis. In this case, we simply switch x and y and use the function $x = f^{-1}(y)$ instead of $y = f(x)$.

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[1]Even though elliptic integrals cannot be simplified algebraically, they have deep connections to number theory. The impossibility of algebraically evaluating elliptic integrals presents the first obstacle towards a complete understanding of prime numbers!
See: https://en.wikipedia.org/wiki/Elliptic_function