Manifold Calculus and the h-principle
Talk at Workshop on Functor Calculus OSU

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Contents

Abstract

in this project we use manifold calculus and h-principle to study a problem in symplectic geometry.

$$\text{manifold calculus} + h\text{-principle} \rightsquigarrow \text{symplectic geometry}$$

1 Motivation

a symplectic manifold	is a pair (N, ω) where N is a smooth manifold of dimension $2m$ and $\omega \in \Omega^2_{DeRham}(N)$ satisfying 1. $d\omega = 0$ 2. ω^m is nowhere vanishing ω defines an isomorphism $\omega : TM \to T^*M$

Example 1.1. $\begin{array}{c|l} N & = \mathbb{R}^{2m} \\ \omega & = \sum_{i=1}^m dp_i \wedge dq_i \end{array}$

Example 1.2. $\begin{array}{c|l} N & = T^*M \\ \omega & \text{(locally)}= \sum_{i=1}^m dp_i \wedge dq_i \end{array}$

where q_i are the base coordinates and p_i are the tangential coordianates. in classical mechanics, T^*M is the phase space, q_i are the position coordinates and p_i are the momentum coordinates.

Darboux's theorem says that *every* symplectic manifold is locally symplectomorphic to R^{2m} with the standard symplectic form, and so symplectic manifolds have no *local invariants* but they do have very non-trivial *global invariants*; see [CdS01]. many global invariants are constructed out of Lagrangian submanifolds.

a Lagrangian submanifold	is a submanifold $M \subseteq N$ of dimension m such that $\omega\|_M \equiv 0$ [1]
$\mathrm{Emb}_{\mathrm{Lag}}(M, N)$	is the space of Lagrangian embeddings $M \hookrightarrow N$

The *Nearby Lagrangian Conjecture* (still open) due to Arnol'd has been a guiding question for several recent advances in symplectic geometry. The current state of the art results about the Nearby Lagrangian Conjecture rely on a combination of homotopy theoretic and Floer theoretic techniques.

Conjecture 1.3 (Arnold's Nearby Lagrangian Conjecture)**.** *If L and M are simply connected closed manifolds of* $\dim m$*, then the space*

$$\mathrm{Emb}_{\mathrm{Lag}}(L, T^*M) \text{ is } \begin{cases} \text{connected} & \text{if } L \cong M, \\ \text{empty} & \text{otherwise.} \end{cases}$$

the goal of this project is to apply homotopy theoretical methods (manifold calculus) to study $\mathrm{Emb}_{\mathrm{Lag}}(M, N)$.

[1] basic example is the position space inside a phase space, in classical mechanics.

2 Statement of the Theorem

Man	the **Spaces**-enriched category of smooth manifolds of a fixed dimension m, with morphisms being the embedding spaces
$\mathbf{Disc}_\infty$	the full subcategory of **Man** consisting of manifolds which are diffeomorphic to disjoint union of $\mathbb{R}^m$
$\mathrm{Psh}(\mathbf{Man})$	space valued presheaves on **Man**, functors $\mathbf{Man}^{op} \to \mathbf{Spaces}$
$T_\infty F$	right derived Kan extension of $F \in \mathrm{Psh}(\mathbf{Man})$ along the inclusion $\mathbf{Disc}_\infty{}^{op} \hookrightarrow \mathbf{Man}^{op}$

$$\begin{tikzcd} \mathbf{Disc}_\infty{}^{op} \arrow[rr, "F|_{\mathbf{Disc}_\infty{}^{op}}"] \arrow[d] \arrow[drr, phantom, "\Uparrow"] & & \mathbf{Spaces} \\ \mathbf{Man}^{op} \arrow[urr, "T_\infty F"'] & & \end{tikzcd}$$

we say that $T_\infty F$ is the **analytic approximation** of F.

See [Wei99], [BdBW12], [MV15].

Theorem 2.1 (N., [Nak17]). *let (N, ω) be a symplectic manifold with dimensions $n = 2m$. for $m > 2$ there is a homotopy equivalence*

$$T_\infty \, \mathrm{Emb}_{\mathrm{Lag}}(-, N) \simeq \mathrm{Emb}_{\mathrm{TR}}(-, N)$$

where

$\mathrm{Emb}_{\mathrm{Lag}}(-, N)$ *is the space of Lagrangian embeddings and*
$\mathrm{Emb}_{\mathrm{TR}}(-, N)$ *is the space of totally real embeddings.*

2.1 Totally Real Submanifolds

a compatible almost complex structure on (N, ω)	is a vector space automorphism $J : TN \to TN$ with $J^2 = -1$ such that 1. $\omega(J-, -)$ defines a Riemannian metric on N, 2. $\omega(J-, J-) = \omega(-, -)$. there exists a unique compatible almost complex structure up to homotopy, we'll assume to have chosen one
a totally real manifold	is a submanifold $M \subseteq N$ of dimension m such that $TM \oplus J(TM) \cong TN\|_M$ [2]
$\mathrm{Emb}_{\mathrm{TR}}(M, N)$	is the space of totally real embeddings $M \hookrightarrow N$ by compatibility, every Lagrangian submanifold is also a totally real submanifold, $\mathrm{Emb}_{\mathrm{Lag}}(M, N) \subseteq \mathrm{Emb}_{\mathrm{TR}}(M, N)$

3 Proof

following are the steps to prove 2.1:

1. $\mathrm{Imm}_{\mathrm{Lag}}(M, N) \xrightarrow{\simeq} \mathrm{Imm}_{\mathrm{TR}}(M, N)$, see [EM01].
   - immersions are linear, in the sense of manifold calculus
   - so the homotopy type is completely determined by the tangential data
   - on the level of tangent spaces the result follows by the homotopy equivalence $U(m)/O(m) \xrightarrow{\simeq} GL(m, \mathbb{C})/GL(m, \mathbb{R})$, see [Arn67].
2. $T_\infty \mathrm{Emb}_{\mathrm{Lag}}(M, N) \xrightarrow{\simeq} T_\infty \mathrm{Emb}_{\mathrm{TR}}(M, N)$

[2] the basic example is $\mathbb{R}^m \subseteq \mathbb{C}^m$.

- this is because of the following homotopy pullback diagram

$$\begin{tikzcd}
T_\infty \operatorname{Emb}_{\mathrm{Lag}}(M,N) \arrow[r] \arrow[d] & T_\infty \operatorname{Emb}_{\mathrm{TR}}(M,N) \arrow[d] \\
\operatorname{Imm}_{\mathrm{Lag}}(M,N) \arrow[r] & \operatorname{Imm}_{\mathrm{TR}}(M,N)
\end{tikzcd}$$

3. $\operatorname{Emb}_{\mathrm{TR}}(M,N) \xrightarrow{\simeq} T_\infty \operatorname{Emb}_{\mathrm{TR}}(M,N)$
   - this last step in the proof uses Gromov's h-principle for directed embeddings. See [Gro86], [Spr98].

4 Gromov's h-principle

$\mathrm{Gr}_m(N) \to N$	is the m-plane Grassmannian bundle over N
$\mathrm{Lag} \subseteq \mathrm{Gr}_m(N)$	subbundle corresponding to the Lagrangian subspaces of TN
$\mathrm{TR} \subseteq \mathrm{Gr}_m(N)$	subbundle corresponding to the totally real subspaces of TN
	by compatibility, $\mathrm{Lag} \subseteq \mathrm{TR}$.

an embedding $e : M \hookrightarrow N$ induces a map

$$\mathrm{Gr}_m(e) : M \to \mathrm{Gr}_m(TN)$$

when e is a totally real (resp. Lagrangian) embedding the image of $\mathrm{Gr}_m(e)$ lies in TR (resp. Lag).

Gromov's idea (originally by Smale) was to consider the space of **formal totally real sections** consisting of triples (e, s, γ)

e	an embedding $M \to N$
s	a section of TR over $e(M)$
γ	a path of sections of $\mathrm{Gr}_m(TN)$ over $e(M)$ from $De(TM)$ to s

a formal section (e, s, γ) is called **holonomic** if $s = De(TM)$ and γ is the constant path.

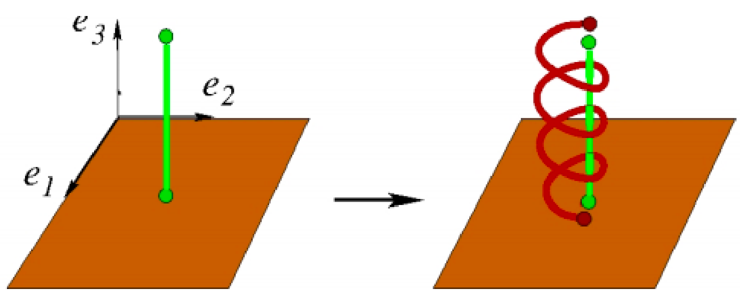

Figure 1: an example of h-principle. in the above image e is a vertical embedding of a segment and E is a horizontal vector field. image taken from http://math.univ-lyon1.fr/~borrelli/Diablerets/1-diablerets.pdf.

Gromov showed using his technique of convex integration that the space of formal totally real sections is homotopy equivalent to the space of holonomic totally real sections.

$$\{\text{ formal totally real sections }\} \simeq \{\text{ holonomic totally real sections }\}$$

this is *false* for lagrangian sections. this is because the space TR is *large* but Lag is *small*. more specifically, the complement of TR inside $\mathrm{Gr}_m(N)$ is a variety of codimension ≥ 2 (it is the complement of a thin singularity).

5 Future Work

1. how to incorporate Floer theoretic data in the Goodwillie-Weiss tower?
2. using manifold calculus to study foliations (which are studying using h-principle)
3. extending the above techniques to manifolds with a group action (current work in progress).

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