Model Categories

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§ Axioms & Examples :

$\mathcal{C}$ = category $\quad f : A \to B,\ g : X \to Y$ in mor $\mathcal{C}$

Def f is a retract of g if $\exists$ a commutative diagram

$$\begin{tikzcd} A \arrow[r, "i_1"] \arrow[d, "f"'] & X \arrow[r, "r_1"] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r, "i_2"'] & Y \arrow[r, "r_2"'] & B \end{tikzcd}$$

such that

$$r_1 \circ i_1 = id_A$$
$$r_2 \circ i_2 = id_B$$

Def : f has LLP (left lifting property) w.r.t g iff g has RLP wrt f iff $\exists$ a lift $B \to X$ s.t. following diagrams all commute.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Def : Model structure : triple of classes of morphisms (weak equivalences $\xrightarrow{\sim}$, cofibrations $\rightarrowtail$, fibrations $\twoheadrightarrow$)

satisfying 1) (MC1) 2 out of 3

If any 2 morphisms of the following diag. are weak eq. then so are the other 2.

$$\begin{tikzcd} & \cdot \arrow[dr, "g"] & \\ \cdot \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & & \cdot \end{tikzcd}$$

{MC2} : (Retract axiom)

w.e., fib, cofib are closed under retracts.

MC3 : (Lifting axiom)

cofib satisfy LLP with respect to trivial (or acyclic) fib (= we + fib)

fib " RLP " " cofib (= we + cofib)

i.e.

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook, "\wr"'] & X \arrow[d, two heads, "s"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[r] \arrow[d, hook, "s"'] & X \arrow[d, two heads] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

MC4 : (Factorization)

Any map $f: X \to Y$ in $\mathcal{C}$ can be factored in the following two ways

$$\text{i)} \begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, hook, "\iota"'] & & Y \\ & Z \arrow[ur, two heads] & \end{tikzcd} \qquad \text{ii)} \begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, hook] & & Y \\ & Z \arrow[ur, two heads, "s"'] & \end{tikzcd}$$

Model category : $\mathcal{C}$ contains all finite limits & colimits and has a model structure.

Rmk: $\cdot(\mathcal{C}, (\text{w.e.}, \text{cofib}, \text{fib}))$ is a model category

$\Leftrightarrow (\mathcal{C}^{op}, (\text{w.e.}, \text{fib}, \text{cofib}))$ is a model category.

. it enough to provide either w.e. and cofibs or w.e and fibs and the third class is well defined.

Examples:

1) $\mathcal{C} = Top$ Quillen model structure

w.e = ~~homotopy~~ weak homotopy equivalence

cofib = retracts of inclusions of relative cell complexes.

$i: A \hookrightarrow X$

fib = serre fibrations

2) $\mathcal{C} = sSets$

w.e = homotopy equivalences of geometric realizations

cof = levelwise injections

fib = Kan fibrations $\Leftrightarrow$ rlp wrt inclusions of horns $\Lambda^n_k \to \Delta^n$.

3) $\mathcal{C} = Ch_R^{\geq 0}$

w.e. = quasi-isomorphisms

cofib = levelwise injections with projective kernel

fib = levelwise surjections ~~with~~ for positive ~~surjectives~~ degrees.

§ 2 The Homotopy Category :

Moral Def^n : $\mathcal{C}$ - model category

$Ho(\mathcal{C})$ = homotopy category of $\mathcal{C}$ is the localization wrt weak equivalences.

Def : $\left\{ \underset{\text{initial object}}{\phi} \rightarrow X \quad \text{cofib} \right\} \Leftrightarrow X$ is a cofibrant object

$\left\{ X \rightarrow \underset{\text{terminal object}}{*} \quad \text{fib} \right\} \equiv X$ is a fibrant object

- All objects are weakly equivalent to cofibrant / fibrant objects.

Follows from the factorizations :

$$\begin{tikzcd} \phi \arrow[rr] \arrow[dr, tail] & & X \\ & X_c \arrow[ur, two heads, "\sim"'] & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[rr] \arrow[dr, tail, "\sim"'] & & * \\ & X_f \arrow[ur, two heads] & \end{tikzcd}$$

cofibrant replacement of X

fibrant replacent of X

- similarly we get ~~bifrant~~ bifibrant (= cofibrant + fibrant) replacement

Th^m: $\mathcal{C}_b \hookrightarrow \mathcal{C}$ induces an equivalence of categories

$$\mathcal{C}_b/\sim \xrightarrow{\cong} Ho(\mathcal{C})$$

$$\downarrow$$

objects : ~~ob~~ bifibrant objects of $\mathcal{C}$

mor : $Hom_{\mathcal{C}/\sim}(A,B) = [A,B]$

<u>Rmk</u> : A map $f : X \to Y$ in $\mathcal{C}$ is mapped to an iso in $H_*\mathcal{C}$

$\Leftrightarrow$ $f : X \to Y$ is a w.e.

<u>Def</u> : A Quillen ~~functor~~ adjunction between model categories $\mathcal{C}, \mathcal{D}$ is an ~~functor~~ adjunction

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

~~s.~~ F preserves cofib and trivial cofib., or equivalently

, G preserves fib and trivial fib

<u>Def</u> . This adjunction is Quillen equivalent iff the total derived functors

$$Ho F : Ho(\mathcal{C}) \rightleftarrows Ho(\mathcal{D}) : Ho(G)$$

is a weak equivalence of the Homotopy categories.

eg : $|\,| : sSets \rightleftarrows Top : Sing$

Let $\mathcal{C}_b$ be the full subcategory of $\mathcal{C}$ category consisting of all bifibrant objects.

<u>Def</u>: $A \in Ob(\mathcal{C})$.

Cylinder object $cyl(A)$ = object together with a factorization

$$A \sqcup A \rightarrowtail cyl(A) \xrightarrow{\sim} A$$

such that the composition is identity on each component.

<u>Def</u>: $f, g : A \to B$ in $\mathcal{C}$ are left homotopic $f \overset{l}{\sim} g$ if $\exists$ a cylinder object $cyl(A)$ along with an extension

$$\begin{tikzcd}
A \sqcup A \arrow[r, "f \sqcup g"] \arrow[d, tail] & B \\
cyl(A) \arrow[ur, dashed, "\exists"'] &
\end{tikzcd}$$

<u>Prop</u>: a) Left homotopy is an equivalence on $Hom_{\mathcal{C}}(A,B)$
A - cofibrant

b) If B is fibrant then the composition in $\mathcal{C}$ descends on classes:

$$Hom_{\mathcal{C}}(B,A)/\sim \times Hom_{\mathcal{C}}(A,X)/\sim \longrightarrow Hom_{\mathcal{C}}(B,X)/\sim$$

$Hom_{\mathcal{C}}(B,A)/\sim \; := [B,A]$ homotopy classes of maps