PROBLEM SET 02 - LIMITS

Part 1

The hardest thing about an $\epsilon - \delta$ proof is realizing that the order in which we come up with the proof is different from the order in which we write it down, in some ways we need to start at the end. This week's goal is to make friends with these $\epsilon - \delta$ proofs and hopefully also get good at them.

Q.1. Explain in your own words how the provisional definition of limit is equivalent to the more rigorous $\epsilon - \delta$ definition.

Provisional definition: The function f approaches a limit L near a i.e. $\lim_{x \to a} f(x) = L$, if we can make $f(x)$ as close to L as we like by requiring that x be sufficiently close to, but unequal to, a.

$\epsilon - \delta$ **definition:** The function f approaches a limit L near a i.e. $\lim_{x \to a} f(x) = L$, if for every $\epsilon > 0$ there is some $\delta > 0$ such that, for all x,

$$\text{if } 0 < |x - a| < \delta, \text{ then } |f(x) - L| < \epsilon.$$

Q.2. For each of the following functions f and real numbers a,

- Guess the limit $L = \lim_{x \to a} f(x)$.
- Find a δ corresponding to $\epsilon = 1$ in the $\epsilon - \delta$ definition of limit.
- Find a δ corresponding to an arbitrary real number ϵ and use this to **prove** that L is indeed the limit.

(1) $f(x) = 7x + 2, \quad a = 0$

(2) $f(x) = |x|, \quad a = 0$

(3) $f(x) = x^2, \quad a = 0$

(4) $f(x) = x^2, \quad a = 1$

Q.3. (1) Use the $\epsilon - \delta$ notation give a rigorous definition of the following statement,

The function f *does not approach* the limit L at a.

(2) Using the $\epsilon - \delta$ notation prove that the limit $\lim_{x \to 0} x^2 \neq 1$.

(3) For the function $f(x) = \dfrac{1}{x}$ prove that $\lim_{x \to 0} f(x) \neq L$ for any real number L. In this case we say that the **limit does not exist**.

Part 2

Q.4. Show that $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^+} f(x) = L$ and $\lim_{x\to a^-} f(x) = L$.

Q.5. For the function

$$f(x) = \begin{cases} x^2 . \sin \dfrac{1}{x} & \text{if } x < 0 \\ 0 & \text{otherwise} \end{cases}$$

Determine, with proof, the limits $\lim_{x\to 0^+} f(x)$, $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0} f(x)$.

Q.6. Determine, with proof, $\lim_{x\to\infty} 1/x$.

Q.7. (1) Show that for every $\epsilon_1, \epsilon_2 \in \mathbb{R}$ the following inequality holds

$$|\epsilon_1 - \epsilon_2| \leq |\epsilon_1| + |\epsilon_2|$$

(2) Using the $\epsilon - \delta$ definition of limit prove that if $\lim_{x\to a} f(x) = l \neq 0$ and $\lim_{x\to a} g(x) = m \neq 0$ then

(a) $\lim_{x\to a} (f(x) - g(x)) = l - m$

(b) $\lim_{x\to a} (f(x).g(x)) = lm$

Q.8. We say that $\lim_{x\to a} f(x) = \infty$ if for all N there exists a $\delta > 0$ such that for all x, if $0 < |x - a| < \delta$ then $f(x) > N$.

(1) Show that $\lim_{x\to 0} 1/x^2 = \infty$.

(2) Analogously define $\lim_{x\to a^+} f(x) = \infty$ and show that $\lim_{x\to 0^+} 1/x = \infty$.

(3) Prove that $\lim_{x\to 0} 1/x \neq \infty$.

Q.9. Give examples to show that the following definitions of $\lim_{x\to a} f(x) = L$ are not correct. (Hint: Think graphically)

(1) For every $\delta > 0$ there exists an $\epsilon > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

(2) For every $\epsilon > 0$ there exists a $\delta > 0$ such that, for all x, if $|f(x) - L| < \epsilon$ then $0 < |x - a| < \delta$.