PROBLEM SET 01

Part 1

The hardest step in writing an $\epsilon - \delta$ proof is the first step, once you know how to start the proof the rest is mechanical.

Q.1. Explain in your own words how the provisional definition is equivalent to the more rigorous $\epsilon - \delta$ definition of limit.

Provisional definition: The function f approaches a limit L near a i.e. $\lim_{x \to a} f(x) = L$, if we can make $f(x)$ as close to L as we like by requiring that x be sufficiently close to, but unequal to, a.

$\epsilon - \delta$ **definition:** The function f approaches a limit L near a i.e. $\lim_{x \to a} f(x) = L$, if for every $\epsilon > 0$ there is some $\delta > 0$ such that, for all x, if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

Q.2. For each of the following functions f and real numbers a,

- Guess the limit $L = \lim_{x \to a} f(x)$.
- Find a δ corresponding to $\epsilon = 0.1$ in the $\epsilon - \delta$ definition of limit.
- Find a δ corresponding to an arbitrary real number ϵ and **prove** that L is indeed the limit.

(1) $f(x) = x + 2, \quad a = 0$

(2) $f(x) = 7x + 2, \quad a = 0$

(3) $f(x) = x^2 - 2, \quad a = 0$

(4) $f(x) = 1/x, \quad a = 1$

Part 2

It is tedious to use $\epsilon - \delta$ proofs in practice, instead we use Theorem 2 and it's analogues.

Q.3. (1) Show that for every $\epsilon_1, \epsilon_2 \in \mathbb{R}$ the following holds

$$|\epsilon_1 - \epsilon_2| \le |\epsilon_1| + |\epsilon_2|$$

(2) For which values of ϵ_1, ϵ_2 does equality hold?

(3) Using the $\epsilon - \delta$ definition of limit to prove that if $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$ then

$$\lim_{x \to a} (f(x) - g(x)) = l - m$$

Q.4. For the function

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{otherwise} \end{cases}$$

determine, with proof, the limits $\lim_{x \to 0^+}$, $\lim_{x \to 0^-}$. How will you answers change if we replace $x < 0$ by $x \le 0$ in the definition of $f(x)$?

Q.5. (1) Determine, with proof, $\lim_{x \to \infty} 1/x$.

(2) Prove that for no real number l do we have $\lim_{x \to 0^+} 1/x = l$.

Q.6. Give examples to show that the following definitions of $\lim_{x \to a} f(x) = L$ are not correct i.e. find functions f which are continuous but do not satisfy the following conditions. (Hint: Think graphically)

- For all $\delta > 0$ there exists an $\epsilon > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.
- For every $\epsilon > 0$ there is some $\delta > 0$ such that, for all x, if $|f(x) - L| < \epsilon$ then $0 < |x - a| < \delta$.

Trigonometric functions, exponential functions and logarithms are continuous wherever they are defined. We will assume this fact without proof for now and come back to it later.

PART 3

To understand continuity it is equally important to understand discontinuity.

Q.7. (1) Let n be a positive integer. Use the $\epsilon - \delta$ definition to prove that the function $f(x) = x^n$ is continuous at 0.

(2) Use Theorem 2 to prove that every polynomial $p(x)$ is continuous at 0.

(3) Prove that a function $f(x)$ is continuous at a real number a if and only if the function $g(x) = f(x + a)$ is continuous at 0.

(4) Prove that x^n is continuous at every real number a.

(5) Use Theorem 2 to prove that every polynomial $p(x)$ is continuous at every real number a.

(6) What are the real numbers at which the ratio of two polynomials $\dfrac{p(x)}{q(x)}$ is continuous?

Q.8. Prove that the function

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{otherwise} \end{cases}$$

is discontinuous everywhere. What can you say about the continuity of the function $g(x) = x.f(x)$?