Heuristic interpretation of Projective Space $\mathbb{P}^n$:

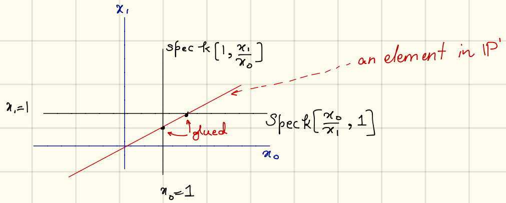
Classically $\mathbb{P}^n_k = \{ \text{lines in } k^{n+1} \}$

$= \{ [x_0 : \dots : x_n] \}$ Covered by $n+1$ opens

Any line with $x_i \neq 0$ has a unique representation:

$$\left[\frac{x_0}{x_i} : \cdots : \frac{x_{i-1}}{x_i} : 1 : \frac{x_{i+1}}{x_i} : \cdots : \frac{x_n}{x_i}\right]$$

$n = 1$:



§4.5 Projective Schemes:

(closed subschemes of $\mathbb{P}^n_?$)

Def: $\mathbb{Z}$-graded rings – $S.$ commutative ring

$S. = \bigoplus_{n \in \mathbb{Z}} S_n$ such that $S_n S_m \subseteq S_{n+m}$

S_0 is a subring of $S.$

& $S.$ is an S_0 algebra and S_n is an S_0-module.

Call S_n – homogenous elements of degree n.

Ideal $I \subseteq S.$ is homogenous if I can be generated by homogenous elements.

Prop: An ideal I is homogenous iff $x_{n_1} + x_{n_2} + \dots + x_{n_k} \in I$ for $x_{n_i} \in S_{n_i}$

$\Rightarrow x_{n_i} \in I$.

Proof:

Cor: 1) I homogenous $\Rightarrow$ $I = \oplus I_n$ for $I_n \subseteq S_n$

$\Rightarrow S/I \cong \oplus S_n / I_n$

$(x_{i_1} + x_{i_2} + \dots + x_{i_n})^k \in I$ $\quad i_1 < i_2 < \dots < i_n$

$\Rightarrow x_{i_1}^k \in I \Rightarrow x_{i_1} \in \sqrt{I}$

2) I homogenous $\Rightarrow$ $\sqrt{I}$ homogenous. $\longleftarrow$ Check this $\rightsquigarrow$ Now induct.

3) $I = \oplus I_n$ is prime iff $I \neq S.$ and I_n prime for all n.

Localization:

4) $T \subseteq S.$ multiplicative set consisting of homogenous elements then $T^{-1}S.$ is also a $\mathbb{Z}$-graded ring.

Defn: $\mathbb{Z}_{\geq 0}$-graded ring, $S.$ with $S_n = 0$ for $n < 0$.

Convention: <u>Graded ring</u> = $\mathbb{Z}_{\geq 0}$ graded ring.

eg: $k[x_1, \dots, x_n]$, each variable x_i has degree 1

$S_+ := \bigoplus_{n > 0} S_n$ irrelevant ideal.

Say $S.$ generated in deg 1 if generated by S_1 as an S_0-algebra.

<u>Proj Construction</u>:

$S.$ graded ring

Set $\operatorname{Proj} S. := \left\{ \begin{array}{l} \text{homogenous prime ideals } \mathfrak{p} \subseteq S. \\ \text{such that } \mathfrak{p} \not\supseteq S_+ \end{array} \right\}$

Scheme structure

$f \in S_+$, homogenous

Prop: $S. \longrightarrow (S.)_f \longleftarrow ((S.)_f)_0$

induces isomorphisms

$\left\{ \begin{array}{l} \mathfrak{q} \in \operatorname{Proj} S. \\ \text{s.t. } f \notin \mathfrak{q} \end{array} \right\} \xleftarrow{1-1} \left\{ \begin{array}{c} \text{homogenous prime} \\ \mathfrak{p} \subseteq (S.)_f \end{array} \right\} \xrightarrow{1-1} \operatorname{Spec}((S.)_f)_0$

Proof: 1) $\{\mathfrak{q} \in \operatorname{Proj} S.\} \xleftarrow{1-1} \operatorname{Spec}(S.)_f$

$\mathfrak{q} \longmapsto \mathfrak{q}(S.)_f$ respects homogenous degrees on both sides.

$\mathfrak{p} \cap S. \longleftarrow \mathfrak{p}$ as as $\deg f \neq 0$, $\mathfrak{p} \cap S. \not\supseteq S_+$.

2) $\mathfrak{p} \in (S.)_f$ prime homogenous $\Rightarrow \mathfrak{p} \cap (S_f)_0$ is prime

Now suppose $\mathfrak{q} \subseteq ((S.)_f)_0$ be prime.

$\Rightarrow$ define $\mathfrak{p}_i \subseteq ((S.)_f)_i$ by

$$\mathfrak{p}_i := \left\{ a \in (S.)_f \,\middle|\, \deg a = i, \quad \frac{a^{\deg f}}{f^i} \in \mathfrak{q} \right\}$$

define $\mathfrak{p} = \oplus \mathfrak{p}_i$

This then is the 1-1 correspondance.

Show injectivity of the reverse map.

• Show $\mathfrak{p}$ is prime.

• $a_1, a_2 \in \mathfrak{p}_i$

$\Rightarrow \frac{a_1^{\deg f}}{f^i}, \frac{a_2^{\deg f}}{f^i} \in \mathfrak{q}$

$\Rightarrow \frac{(a_1+a_2)^{2\deg f}}{f^{2i}} \in \mathfrak{q}$ by expanding

$\Rightarrow \frac{(a_1+a_2)^{\deg f}}{f^i} \ni \mathfrak{q}$ by primality.

• $a_i \in \mathfrak{p}_i \Rightarrow \frac{a_i^{\deg f}}{f^i} \in \mathfrak{q}$

$\Rightarrow \frac{(ba_i)^{\deg f}}{f^{i+j}} \in \mathfrak{q}$

$\Rightarrow \mathfrak{q}$ is an ideal.

• $ab \in \mathfrak{q} \Rightarrow \frac{a^{\deg f} \cdot b^{\deg f}}{f^i \cdot f^j} \in \mathfrak{p}$

$\Rightarrow \frac{a^{\deg f}}{f^i} \in \mathfrak{p}$ or $\frac{b^{\deg f}}{f^j} \in \mathfrak{p}$

$\Rightarrow a \in \mathfrak{q}$ or $b \in \mathfrak{q}$

$\Rightarrow \mathfrak{q}$ prime.

Injectivity of reverse map:

If $\mathfrak{q}_1$ is any prime ideal whose 0 component is $\mathfrak{p}$. Then as $f \in (S.)_f$ we must have

$\mathfrak{q} \subseteq \mathfrak{q}_1$

Similarly for the other direction,

as $\frac{1}{f} \in (S.)_f$ we must have

$\mathfrak{q}_1 \subseteq \mathfrak{q}$

Now for $f \in S_+$ homogenous

$D(f) := \{ \mathfrak{p} \in \operatorname{Proj} S. \mid f \notin \mathfrak{p} \}$

These cover $\operatorname{Proj} S.$

Declare these open,

Structure sheaf $\mathcal{O}_{\operatorname{Spec}((S.)_f)_0}$ (via transport of structure from proj.)

Check: Structure sheaves glue.

Check: Closed subsets in $\operatorname{Proj} S.$ are exactly

$V(I) := \{ \mathfrak{p} \in \operatorname{Proj} S. \mid \mathfrak{p} \supseteq I \}$

for some I homogenous

eg: • $S. = A[x_0, x_1, \dots, x_n]$

$D(x_i) = \operatorname{Spec} A\left[\frac{x_0}{x_i} : \dots : \frac{x_{i-1}}{x_i} : \frac{x_{i+1}}{x_i} : \dots : \frac{x_n}{x_i}\right] \cong \mathbb{A}^n_A$

$D(x_0), D(x_1) \cdots D(x_n)$ cover $\operatorname{Proj} S. =: \mathbb{P}^n_A$

• $\mathbb{P}^0_A = \operatorname{Proj} A[x] = D(x) = \operatorname{Spec} A$

The only n for which $\mathbb{P}^n_A$ is affine.

Def: A projective scheme over Spec A is a projective A-scheme is a scheme isomorphic to Proj $S_\cdot$ for some finite generated graded A-algebra (with $S_0 = A$).

Def: A quasi-projective A-scheme is a quasi-compact open subscheme of a proj A-scheme.

eg: $V = (n+1)$ dim vector space over k.

$\mathbb{P}V := \operatorname{Proj} \operatorname{Sym}_k^*(V^\vee)$

dual vector space

Show quasi-compactness.