Honors Single Variable Calculus

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Contents

1 Proofs

We'll start by learning about the various kinds of proofs that we'll encounter in this class.[1]

Proofs are the heart of mathematics. You must come to terms with proofs – you must be able to read, understand and write them. What is the secret? What magic do you need to know? The short answer is: there is no secret, no mystery, no magic. All that is needed is some common sense and a basic understanding of a few trusted and easy to understand techniques.

The Structure of a Proof

The basic structure of a proof is easy: it is just a series of statements, each one being either

- An assumption or

- A conclusion, clearly following from an assumption or previously proved result

And that is all. Occasionally there will be the clarifying remark, but this is just for the reader and has no logical bearing on the structure of the proof.

A well written proof will flow. That is, the reader should feel as though they are being taken on a ride that takes them directly and inevitably to the desired conclusion without any distractions about irrelevant details.

Each step should be clear or at least clearly justified. A good proof is easy to follow. When you are finished with a proof, apply the above simple test to every sentence: is it clearly

1. an assumption

2. a justified conclusion?

If the sentence fails the test, maybe it doesn't belong in the proof.

Example

In order to write proofs, you must be able to read proofs. See if you can follow the proof below. Don't worry about how you would have (or would not have) come up with the idea for the proof. Read the proof with an eye towards the criteria listed above. Is each sentence clearly an assumption or a conclusion? Does the proof flow? Was the theorem in fact proved?

Theorem 1.1. *The square root of 2 is an irrational number.*

Proof. Let's represent the square root of 2 by s. Then, by definition, s satisfies the equation

$$s^2 = 2.$$

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[1]This section in taken almost verbatim from http://zimmer.csufresno.edu/~larryc/proofs/proofs.html.

If s were a rational number, then we could write $s = p/q$ where p and q are a pair of integers. In fact, by dividing out the common multiple if necessary, we may even assume p and q have no common multiple (other than 1). If we now substitute this into the first equation we obtain, after a little algebra, the equation

$$p^2 = 2q^2.$$

But now, 2 must appear in the prime factorization of the number p^2 (since it appears in the same number $2q^2$). Since 2 itself is a prime number, 2 must then appear in the prime factorization of the number p. But then, $2 \cdot 2$ would appear in the prime factorization of p^2, and hence in $2q^2$. By dividing out a 2, it then appears that 2 is in the prime factorization of q^2. Like before (with p^2) we can now conclude 2 is a prime factor of q. But now we have p and q sharing a prime factor, namely 2. This violates our assumption above (see if you can find it) that p and q have no common multiple other than 1. □

1.1 Direct Proofs

Most theorems that you want to prove are either explicitly or implicitly in the form

"If P, then Q".

This is the standard form of a theorem (though it can be disguised). A direct proof should be thought of as a flow of implications beginning with P and ending with Q.

$$P \implies \cdots \implies Q$$

Most proofs are (and should be) direct proofs. Always try direct proof first, unless you have a good reason not to. If you find a simple proof, and you are convinced of its correctness, then don't be shy about. Many times proofs are simple and short.

Exercise. 1.2. Prove each of the following.

1. If a divides b and a divides c then a divides $b + c$, where a, b, and c are positive integers.

2. For all real numbers a and b, $a^2 + b^2 \geq 2ab$.

3. If a is a rational number and b is a rational number, then $a + b$ is a rational number.

1.2 Proof by Contradiction

In a proof by contradiction we assume, along with the hypotheses, the logical negation of the result we wish to prove, and then reach some kind of contradiction. That is, if we want to prove "If P, then Q", we assume P and Not Q. The contradiction we arrive at could be some conclusion contradicting one of our assumptions, or something obviously untrue like $1 = 0$. The proof of Theorem 1.1 is an example of this.

Exercise. 1.3. Use the method of Proof by Contradiction to prove each of the following.

1. The cube root of 2 is irrational.

2. If a is a rational number and b is an irrational number, then $a + b$ is an irrational number.

3. There are infinitely many prime numbers.[1]

1.3 Proof by Contrapositive

Proof by contrapositive takes advantage of the logical equivalence between "P implies Q" and "Not Q implies Not P". For example, the assertion "If it is my car, then it is red" is equivalent to "If that car is not red, then it is not mine". So, to prove "If P, then Q" by the method of contrapositive means to prove

"If Not Q, then Not P".

How Is This Different From Proof by Contradiction? The difference between the Contrapositive method and the Contradiction method is subtle. Let's examine how the two methods work when trying to prove "If P, then Q".

Method of Contradiction: Assume P and Not Q and prove some sort of contradiction.

Method of Contrapositive: Assume Not Q and prove Not P.

The method of Contrapositive has the advantage that your goal is clear: Prove Not P. In the method of Contradiction, your goal is to prove a contradiction, but it is not always clear what the contradiction is going to be at the start.

Exercise. 1.4. Use the method of Proof by Contrapositive to prove each of the following.

1. If the product of two integers is even, then at least one of the two must be even.

2. If the product of two integers is odd, then both must be odd.

3. If the product of two real numbers is an irrational number, then at least one of the two must be an irrational number.

1.4 Converse

The converse of an assertion in the form "If P, then Q" is the assertion

"If Q, then P".

A common logical fallacy is to assume that if an assertion is true then so is its converse.

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[1]There are dozens of proofs of this theorem, originally due to Euclid. Feel free to look one up online.

1.4.1   If and Only If

Many theorems are stated in the form "*P*, if, and only if, *Q*". Another way to say the same thing is: "*Q* is necessary, and sufficient for *P*". This means two things:

$$\text{"If } P, \text{Then } Q\text{"} \qquad \text{and} \qquad \text{"If } Q, \text{Then } P\text{"}.$$

So to prove an "If, and Only If" theorem, you must prove the theorem and also its *converse*.

Exercise. 1.5. Go back to the problems in Exercises 1.2, 1.3, 1.4 and find the ones which are of the form "If *P*, Then *Q*". For each of these:

- State the converse.

- Prove or disprove the converse (by providing either a proof or a counterexample).

- For the problems where the converse is also true rewrite the assertion as an "If, and Only If" statement.

1.5   Quantifiers

It is extremely important in mathematics to be able to formulate very precise statements. To prevent ambiguous statements and logical fallacies the vocabulary used is very limited and every term has a well-defined meaning. The two concepts we need to get used to are *logical operators* and *quantifiers*.

Logical Operators: Operators allow us to form complex statements by combining simpler ones. The basic logical operators are *and*, *or*, and *not*. Their usage is the same as in everyday language. The more complicated logical operator that we'll be using a lot is *If - then -*. We've already encountered several examples of its usage in the previous sections.

Quantifiers: Quantifiers allow us to make abstract statements which are "universally true" without having to specify a concrete element. There are two commonly used quantifiers:

$$\textbf{For all/every} \qquad\qquad \textbf{There exists}$$

Understanding and formulating complex statements using these requires a lot of practice. You'll get used to these as the course progresses. Here are a few examples,

Example 1.6.

1. *For every* odd integer a, the integer $a + 1$ is even.

2. An integer a is even if and only if *there exists* an integer b such that $a = 2b$.

3. There *do not exist* integers p, q such that $\frac{p}{q} = \sqrt{2}$.

4. *For every* non-zero rational number x *there exists* a rational number y such that $x \cdot y = 1$.

1.5.1 Nesting Quantifiers

When dealing with statements involving multiple quantifiers the order really matters; changing the order changes the meaning of a statement completely. When you're using multiple quantifiers in a single sentence you should always pause and check if you're using the right quantifiers in the right order.

Exercise. 1.7. For each of the following pairs, explain how changing the order of the quantifiers changes the meaning of the statement.[1]

1. (a) *For every* even integer *a there exists* an integer *b* such that $a = 2b$.
   (b) *There exists* an integer *b* such that *for every* even integer *a*, $a = 2b$.

2. (a) *For every* positive integer *n there exists* a positive real number ϵ such that $\epsilon < 1/n$.
   (b) *There exists* a positive real number ϵ such that *for every* positive integer *n*, $\epsilon < 1/n$.[2]

3. (a) *For every* even integer *a*, *for every* odd integer *b*, $a + b$ is odd.
   (b) *For every* odd integer *b*, *for every* even integer *a*, $a + b$ is odd.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let ϵ be a positive real number.
   (a) *For every x, there exists* a $\delta > 0$ such that *for every y*, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.
   (b) *There exists* a $\delta > 0$ such that *for every x, for every y*, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.

1.5.2 Negating Nested Quantifiers

As we saw in Sections 1.2 and 1.3 we often need to negate mathematical statements.

In order to systematically negate complicated logical expressions, we observe that we can negate simple statements in the following manner. (These are called *De Morgan's laws*.)

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[1] It's ok to a bit vague in your answers to this question.
[2] Such an ϵ is called an **infinitesimal**. There are number systems, for example the **hypperreal numbers**, which extend the real numbers by incorporating infinitesimals.

Statement	Negation
For every x the statement P is true.	There exists an x such that the statement P is false.
There exists an x such that the statement P is true.	For every x the statement P is false.
P is true and Q is true	P is false or Q is false.
P is true or Q is true	P is false and Q is false.
If P then Q	P but not Q.

If we have multiple quantifiers in a statement then we start with the outermost quantifier and recursively move inwards.

Example 1.8. The negation of *"For every x, there exits a y such that, the statement P is true"* equals *"There exists an x such that, for every y, the statement P is false."*

Exercise. 1.9. Negate each of the following.

1. *For every* even integer a *there exists* an integer b such that $a = 2b$.

2. *There exists* an integer b such that *for every* even integer a, $a = 2b$.

3. *For every* positive integer n *there exists* a positive real number ϵ such that $\epsilon < 1/n$.

4. *There exists* a positive real number ϵ such that *for every* integer n, $\epsilon < 1/n$.

5. *For every* even integer a, *for every* odd integer b, $a + b$ is odd.

6. *For every* $\epsilon > 0$, *for every* x, *there exists* a $\delta > 0$ such that *for every* y, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.

7. *For every* $\epsilon > 0$, *there exists* a $\delta > 0$ such that *for every* x, *for every* y, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.

Word of caution: When the word "and" is used in a list it is not considered a logical operator (this is a deficiency of the English language). For example, negating the following expression

For every even integer a and *for every* odd integer b, $a + b$ is odd,

does not change the *and* to an *or* (what happens if you do change the *and* to an *or*?).

2 Inequalities

In this section we'll learn how to solve inequalities and in the process figure out how to write proofs of statements involving inequalities.

The biggest different between equalities and inequalities is that, unlike equalities, there are usually infinitely many (if any) solutions to inequalities. When we we're trying to solve an inequality we're not looking for the best solution, we're not even looking for a good solution, we're simply looking for **one** solution. For example, both $x = 1.001$ and $x = 10^{100}$ are equally valid solutions to the inequality $x > 1$.

Exercise. 2.1. Find a positive real solution to each of the following inequalities.

1. $x^3 + x < 1$

2. $(1 + x)^2 - 1 < 1$

3. $1 - (1 - x)^2 < 1$

4. $x^3 + 2x < 1 + x^2$

5. $x^{10} + x > 10$

6. $x^3 - x > 1$

When we try to solve equalities we try to simply the equation until it becomes easy to solve. The same thing is true for inequalities, however, the ways to simplify an inequality are much more subtle.

Example 2.2. If we're trying to solve $P(x) > \epsilon$, where P is some complicated expression that cannot be simplified, then we try to find some simpler $Q(x)$, such that $P(x) > Q(x)$ and try to solve for $Q(x) > \epsilon$ instead. Similarly, if we're trying to solve $P(x) < \epsilon$, then we try to find some simpler $Q(x)$, such that $P(x) < Q(x)$ and try to solve for $Q(x) < \epsilon$ instead.

There is no *canonical way* in which an inequality can be solved. We somehow simplify the inequality and hope that we get a solution.

For solving complicated inequalities, we need a systematic approach towards estimating functions. There are different tricks and identities for estimating different kinds of functions. For now, we'll focus mainly on estimating polynomials.

Exercise. 2.3. Prove the following extremely useful set of inequalities

$$\text{if } 1 \geq x > 0 \quad \text{then } 1 \geq x \geq x^2 \geq x^3 \geq \ldots$$
$$\text{if } 1 \leq x \quad \text{then } 1 \leq x \leq x^2 \leq x^3 \leq \ldots.$$

Word of Caution: You should be very careful when negative numbers are involved. Multiplying an inequality by a negative number changes it's sign, for example, $2 < 3$ but $-2 > -3$. So,

$$\text{if } 1 \geq x > 0 \quad \text{then } -1 \leq -x \leq -x^2 \leq -x^3 \leq \ldots$$
$$\text{if } 1 \leq x \quad \text{then } -1 \geq -x \geq -x^2 \geq -x^3 \geq \ldots.$$

2.1 Proofs involving Inequalities

We'll encounter several statements in this class like: *For every $\epsilon > 0$ there exists a positive real number x such that $x^3 + 3x < \epsilon$.* When we want to prove such a statement, we are essentially trying to solve $x^3 + 3x < \epsilon$. (Why?) For example,

Q. 2.4. Prove that for every $\epsilon > 0$ there exists a positive real number x such that $x^3 + 3x < \epsilon$.

We're asking for a solution to the equation $x^3 + 3x < \epsilon$ for an arbitrary positive real number ϵ.

1. The terms x^3 and x have different degrees. We can try to use the inequalities in Exercise 2.3 to simplify their sum, for this we need to know if $x \geq 1$ or $x \leq 1$.

2. We're probably looking for a small x so we could assume $x \leq 1$. If this does not work we'll come back try something else. (Remember that we're trying to find **one** solution.)

3. As $x \leq 1$, $x^3 \leq x$ so that $x^3 + 3x \leq x + 3x = 4x$.

4. As $x^3 + 3x \leq 4x$ and we want to find a solution to $x^3 + 3x < \epsilon$, it suffices to solve $4x < \epsilon$. As ϵ is positive, **one** solution for this is $x = \epsilon/5$.

5. We still need to check if this solution actually works. We can do this by trying to write down a direct proof and making sure that all the implications are logically sound.

6. We required the condition $x \leq 1$ for this solution to work. To ensure this we'll set $x = \min\{1, \epsilon/5\}$.

Now we know what the solution is, but we still need to write down a proof for the original statement. A proof should always start with some assumptions and logically derive the required conclusion. For us the assumption will be $\epsilon > 0$ and $x = \min\{1, \epsilon/5\}$ and the expected conclusion is $x^3 + 3x < \epsilon$.

Proof of Q. 2.4. Let $\epsilon > 0$ and let $x = \min\{1, \epsilon/5\}$. Note that $x \leq 1$ and hence $x^3 \leq x$. Then,

$$
\begin{aligned}
x^3 + 3x &\leq x + 3x \\
&= 4x \\
&= 4\min\{1, \epsilon/5\} \\
&\leq 4\epsilon/5 \\
&< \epsilon \qquad\qquad\qquad \text{as } \epsilon \text{ is positive}
\end{aligned}
$$

Hence, $x = \min\{1, \epsilon/5\}$ is a solution of $x^3 + 3x < \epsilon$, which proves the proposition. $\qquad\square$

First we had to solve the inequality then reverse the steps: start with the solution and write a *direct proof* for the proposition. This is how proofs are usually discovered. You make an "educated guess" and hope that it works. Sometimes it doesn't, so you go back to finding another "educated guess".

Remark 2.5 (A note on quantifiers). For proving the statement *"prove that for every $\epsilon > 0$ there exists a positive real number x such that $x^3 + 3x < \epsilon$"* we had to start with the statement *"let $\epsilon > 0$ and $x = \min\{1, \epsilon/5\}$"*. The variable x depends on the variable ϵ. This is fine because of the order of quantifiers: *"...for every $\epsilon > 0$ there exists a positive real number x ..."*; the quantifier for ϵ comes before the quantifier for x and hence the variable x can depend on the variable ϵ.

If instead, suppose we were trying to prove *"there exists a positive real number x such that for every $\epsilon > 0$, $x^3 + 3x < \epsilon$"*. We're still trying to solve the equation $x^3 + 3x < \epsilon$ but now the quantifier for x comes before the quantifier for ϵ and hence x cannot depend on the variable ϵ but instead should be a constant that universally solves the equation $x^3 + 3x < \epsilon$ for every $\epsilon > 0$. No such x exists (why?) and hence the statement *"there exists a positive real number x such that for every $\epsilon > 0$, $x^3 + 3x < \epsilon$"* is false.

Exercise. 2.6. Prove that for every $\epsilon > 0$ there exists a positive real number x such that ...

1. $x^3 + x < \epsilon$.

2. $(1 + x)^2 - 1 < \epsilon$.

3. $1 - (1 - x)^2 < \epsilon$.

4. $x^3 + 2x < \epsilon + x^2$.

5. $x^{10} + x > \epsilon$.

6. $x^3 - x > \epsilon$. [1]

Finally, we are sometimes required to find not one solution but a range of solutions. We try to come up an "educated guess" by the same method, and many a times this gives us the required range for free with a few changes in the final proof.

Q. 2.7. Prove that for every $\epsilon > 0$, there exists a positive real number δ such that, for all x, if $0 < x < \delta$ then $x^3 + 3x < \epsilon$.

Proof. Let $\epsilon > 0$, let $\delta = \min\{1, \epsilon/5\}$ and let $0 < x < \delta$. Because $\delta \leq 1$, we have $x < 1$ and hence $x^3 < x$. As before,

$$
\begin{aligned}
x^3 + 3x &\leq x + 3x \\
&= 4x \\
&< 4\delta \\
&= 4\min\{1, \epsilon/5\} \\
&\leq 4\epsilon/5 \\
&< \epsilon \qquad\qquad\qquad \text{as } \epsilon \text{ is positive}
\end{aligned}
$$

Hence, every $0 < x < \min\{1, \epsilon/5\}$ is a solution of $x^3 + 3x < \epsilon$, which proves the proposition. $\qquad\square$

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[1] Hint: Break x^3 as $\frac{x^3}{2} + \frac{x^3}{2}$ and find the conditions on x for which $\left(\frac{x^3}{2} - x\right) > 0$.

Exercise. 2.8. Prove that for every $\epsilon > 0$, there exists a positive real number δ such that, for all x, if $0 < x < \delta$ then ...

1. $x^3 + x < \epsilon$.

2. $(1 + x)^2 - 1 < \epsilon$.

3. $1 - (1 - x)^2 < \epsilon$.

4. $x^3 + 2x < \epsilon + x^2$.

Prove that for every $\epsilon > 0$, there exists a positive real number δ such that, for all x, if $x > \delta$ then

1. $x^{10} + x > \epsilon$.

2. $x^3 - x > \epsilon$.

Optional Problems

Exercise. 2.9. 1. Prove that for every $\epsilon > 0$, for every real number $x > 0$, there exists a $\delta > 0$ such that, for all real numbers $y > 0$, if $0 < y - x < \delta$ then $y^2 - x^2 < \epsilon$.

2. Prove that the following statement is false: For every $\epsilon > 0$, there exists a $\delta > 0$ such that, for every real number $x > 0$, for all real numbers $y > 0$, if $0 < y - x < \delta$ then $y^2 - x^2 < \epsilon$.

3   Limits & Continuity

We'll now use the proof techniques we've learned so far to study functions.

We'll start with rigorously defining limits. At first this might seem unnecessarily complicated, but having precise definitions will allow us to make more and more sophisticated constructions and later on in the course enable us to prove statements about derivatives and integrals.

Time permitting, towards the end of the semester, we'll construct the Weierstrass function, a function defined on real numbers which is continuous everywhere but differentiable nowhere! But it all starts with the definition of a limit.
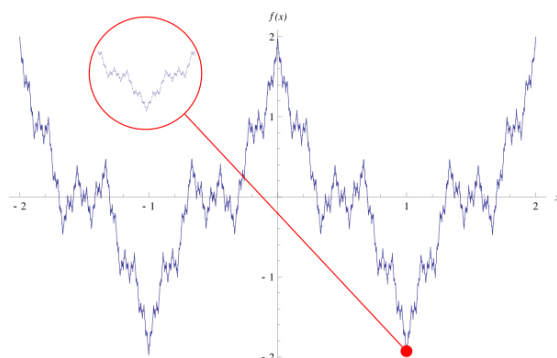
Weierstrass function (Image from Wikipedia)

3.1   Limits

At the heart of all analysis (and hence calculus) lies the notion of a limit. Almost every single concept that we'll define will be a limit of some kind.

Definition 3.1 (Provisional definition of limit)**.** For a function f and a real number a, we say that the function f **approaches a limit L at** a, for some real number L, if we can make $f(x)$ as close to L as we like by requiring that x be sufficiently close to, but unequal to, a. This is denoted

$$\lim_{x \to a} f(x) = L$$

There are 3 problems with this definition:

1. The terms *"as close to L as we like"* and *"sufficiently close"* are not precise. (How close is sufficiently close?)

2. The order of implication is unclear.

3. The definition does not give us a way to come up with the limit L.

First, we'll make the terms precise. We can measure the distance between two real numbers x, y by taking their difference $x - y$, however, the difference can be negative so we use the absolute value $|x - y|$ instead. So that

x sufficiently close to, but unequal to, a	translates to	$0 < |x - a| < \delta$
f(x) as close to L as we like	translates to	*for every $\epsilon > 0$, $|f(x) - L| < \epsilon$*

and the definition of limit becomes

> ... f **approaches a limit** L **at** a, for some real number L, if we can make $|f(x) - L| < \epsilon$ for every $\epsilon > 0$ by requiring that $0 < |x - a| < \delta$ for some $\delta > 0$.

Next, to clarify the order of implication we rephrase the statement in the more standard *"If ... then"* form and put all the quantifiers in the front,

> ... f **approaches a limit** L **at** a, for some real number L, if for every $\epsilon > 0$, there exists a $\delta > 0$ such that, for all x, if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

The third problem cannot be fixed! There is no systematic way of finding the limit of a general function, the only thing we can do is *guess* the limit and then use the definition to *prove* that it is indeed the limit. For special functions like polynomials, trig functions, exponential, and logarithms, we can find limits explicitly, which is what makes these functions useful for approximating and estimating more complicated functions.

Definition 3.2 (Formal definition of limit). We say that the function f **approaches a limit** L **at** a, for some real number L, if for every $\epsilon > 0$, there exists a $\delta > 0$ such that, for all x, if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

Exercise. 3.3. Come up with formal definitions for the following:

1. The function f approaches a limit L at a **from the right**, for some real number L, if we can make $f(x)$ as close to L as we like by requiring that x be sufficiently close to, but strictly greater than a. This is denoted
$$\lim_{x \to a^+} f(x) = L.$$

2. The function f approaches a limit L at a **from the left**, for some real number L, if we can make $f(x)$ as close to L as we like by requiring that x be sufficiently close to, but strictly smaller than a. This is denoted
$$\lim_{x \to a^-} f(x) = L.$$

3. The function f **approaches the limit** ∞ **at** a, if $f(x)$ can be made as large as we like by requiring that x be sufficiently close to, but unequal to, a. This is denoted [1]
$$\lim_{x \to a} f(x) = \infty.$$

4. The function f **approaches a limit** L **at** ∞, for some real number L if we can make $f(x)$ as close to L as we like by requiring that x be sufficiently large. This is denoted
$$\lim_{x \to \infty} f(x) = L.$$

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[1] In mathematics ∞ means several things, in fact, there are infinitely many infinities. For us, ∞ is a *placeholder* for a limit of a function that grows very large. ∞ is not a real number and hence cannot be used in equations.

Exercise. 3.4. For each of the following, first guess the limit, then use the formal definition of limit to prove that it is indeed the limit.

1. $\lim\limits_{x \to 1^+} 2x + 1$

2. $\lim\limits_{x \to 1^+} x^2$

3. $\lim\limits_{x \to \infty} x^{10} + x$

4. $\lim\limits_{x \to 0^+} 1/x$

5. $\lim\limits_{x \to 0} f(x)$ where $f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

Optional Problems

Exercise. 3.5. Give examples to show that the following definitions of $\lim\limits_{x \to a} f(x) = L$ are not correct.

1. For every $\delta > 0$, there exists an $\epsilon > 0$ such that, for all x, if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

2. For every $\epsilon > 0$, there exists a $\delta > 0$ such that, for all x, if $|f(x) - L| < \epsilon$ then $0 < |x - a| < \delta$.

Exercise. 3.6. Prove that if $\lim\limits_{x \to a} f(x) = L$ and $L \neq 0$ then $\lim\limits_{x \to a} 1/f(x) = 1/L$.

3.1.1 Triangle Inequality

To prove abstract theorems involving absolute values we will need the following very important inequality called the **triangle inequality**.

$$|x - y| \leq |x| + |y|$$

If you think of the points 0, x, y (on the real axis) as being the three vertices of a (degenerate) triangle, then $|x|$, $|y|$, and $|x - y|$ are the lengths of the three sides and the triangle inequality is saying that: *the sum of the lengths of two sides of a triangle is greater than or equal to the length of the third side.*

There are other forms in which the triangle inequality is commonly used, e.g.

$$|x + y| \leq |x| + |y|$$
$$|x + y| - |y| \leq |x|$$
$$|x| \leq |x - y| + |y|$$

These inequalities are central to a lot of analysis proofs.

Exercise. 3.7.

1. Prove that $|x| + |x - 1| \geq 1$ for any real number x.

2. Prove that for any real number x, at least one of $|x|$ and $|x-1|$ is $\geq 1/2$.[1]

We'll use these to prove inequalities about the non-existence of limits.

Definition 3.8. If f does not approach the limit L, for any real number or ∞ or $-\infty$, then we say that the limit of f at a **does not exist**.

Exercise. 3.9. Negate the formal definitions of limits and come up with an ϵ, δ definition for the following.

1. f **does not** approach the limit L, where L is a real number, at a.

2. f **does not** approach the limit ∞ at a. (Similarly for $-\infty$.)

Exercise. 3.10. Let $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$

1. Prove that for every real number L, $\lim\limits_{x \to 0} f(x) \neq L$.

2. Prove that $\lim\limits_{x \to 0} f(x) \neq \infty$. (Similarly for $-\infty$.)

Hence the limit of f at 0 does not exist.

Exercise. 3.11.

1. Prove that $\lim\limits_{x \to a} f(x) = L$ if and only if $\lim\limits_{x \to a^+} f(x) = L$ and $\lim\limits_{x \to a^-} f(x) = L$.[2] (Similarly for ∞, $-\infty$.)

2. Prove that $\lim\limits_{x \to 0} \dfrac{1}{x}$ does not exist.

3.2 Non-existence of limits

Let's move backwards and try and understand what it means *intuitively* for a function to not approach a limit L near a. The solution to Exercise 3.9 is the following:

> *(Formal definition)* The function f **does not approach thelimit** L **at** a if there exists an $\epsilon > 0$, such that for all $\delta > 0$, there exists an x such that, $0 < |x - a| < \delta$ and $|f(x) - L| \geq \epsilon$.

Going back to how to we came up with the Formal Definition of Limit 3.2 from the Provisional Definition of Limit 3.1 and replacing the absolute values by the distance between points, we can work backwards and get the following provisional definition:

> *(Provisional definition)* The function f **does not approach the limit** L **at** a if no matter how close we are to a, there is some x for which $f(x)$ is not close to L.

---

[1] Hint: Proof by Contradiction.

[2] Hint: This is very easy, don't overthink! Simply write down the formal definitions.

3.3 Continuity

Definition 3.12. We say that a function f is **continuous** at a if

$$\lim_{x \to a} f(x) = f(a)$$

We say that a function f is **continuous on an interval** (a, b) it f is continuous at every x in (a, b). [1]

Thus continuous functions have nice limits. Further, we have the following theorem which makes continuous functions easy to manipulate.

Theorem 3.13. *If the functions f, g are continuous at $x = a$ then so are the functions*

1. $f + g$

2. $f \cdot g$

3. f / g, if $g(a) \neq 0$

The full proof of this theorem is just a tricky application of the triangle inequalities. We'll only prove the first part which is relatively manageable.

Exercise. 3.14. Using the formal definition of limits and the triangle inequality $|x + y| \leq |x| + |y|$, prove that
  if the functions f, g are continuous at $x = a$ then so is the function $f + g$.
Is the converse true?

Exercise. 3.15.

1. Prove that the constant function $f(x) = c$ is continuous everywhere, where c is some real number.

2. Prove that the function $f(x) = x$ is continuous everywhere.

3. Argue that these two facts along with Theorem 3.13 imply that polynomials are continuous everywhere.

Exercise. 3.16. Let f be a continuous function on $[a, b]$, and let g be a continuous function on $[b, c]$, such that

$$f(b) = g(b).$$

Show that the function h defined as

$$h(x) := \begin{cases} f(x) & \text{if } x \leq b \\ g(x) & \text{if } x > b \end{cases}$$

is continuous at b (i.e. we can *glue* continuous functions).

---

[1] If we use closed intervals $[a, b]$ instead, then we have to use $\lim_{x \to a^+}$ and $\lim_{x \to b^-}$ in the definition of continuity.

4 Completeness of the Real Numbers

Our next goal is to prove the IVT for continuous functions.

Theorem 4.1 (Intermediate Value Theorem). *If f is a continuous function on $[a,b]$ and k is a real number lying between $f(a)$ and $f(b)$ then there exists a real number c lying between a and b such that $f(c) = k$.*

The proof of this trivial looking theorem relies on a defining axiom of the system of real numbers called *completeness* which we'll now try to understand. For this we'll need to rigorously define the simple notions of *min* and *max*.

For a non-empty set S of real numbers, it's maximum element can be defined as follows.

Definition 4.2. $\max S$ is the element $a \in S$ such that $a \geq x$ for all elements $x \in S$.

This definition exposes a subtle shortcoming of $\max S$. It doesn't exist even for *nice* sets.

Exercise. 4.3. What is $\max (0, 1)$? Make sure that your answer agrees with Definition 4.2.

There is a more intricately defined number, called the **supremum**, that we can associate to a set of real numbers which also captures the *largest-element* property. For sets containing infinitely many elements, the supremum, rather than the maximum gives us the correct *largest-element*.

Definition 4.4. An **upper bound** of a set S is any real number a satisfying $a \geq x$ for all elements $x \in S$. (Note that a does not have to be in S.)

Exercise. 4.5. Upper bounds are not unique. Find all the possible upper bounds on the following sets. (No proofs needed. Make sure you've found *all* the upper bounds.)

1. $[0,1]$

2. $(0,1)$

3. $\mathbb{R}$

4. $\{1/n : \text{where } n \text{ is a positive integer}\}$

5. The set of rational numbers x satisfying $x^2 < 2$

6. The set of irrational numbers in interval $[0,1]$.

Definition 4.6. The **supremum** or the **least upper bound** of a set S of real numbers, denoted $\sup S$, is the smallest real number a such that a is an upper bound of S.

Exercise. 4.7. Find the supremum of the sets in Exercise 4.5.

Definition 4.8. We say that a set S is **bounded from above** if it has at least one upper bound i.e. there exists a real number a such that $a \geq x$ for all elements $x \in S$.

Exercise. 4.9. Which of the sets in Exercise 4.5 are bounded from above?

We can now state the **Completeness of the Real Numbers**:

Theorem 4.10. *If a non-empty set S is bounded from above then S has a supremum.*

Depending on exactly how the real numbers are constructed, this is either a theorem or an axiom. This theorem/axiom is the precise way of saying that there are no holes in the real line. (Read the Wikipedia article titled *Completeness of the real numbers*.) We'll next restate the property of being a supremum using our (favorite) language of ϵ's and δ's.

Exercise. 4.11. Let S be a non-empty set that is bounded from above and let $L = \sup S$.

1. Show that for every $\delta > 0$, there exists an element $x \in S$ such that $L - x < \delta$. [1]

2. Let a be an upper bound of S. Prove that, if for every $\delta > 0$, there exists an element $x \in S$ such that $a - x < \delta$, then $a = L$. [2]

Thus we're saying that elements of S get infinitely close to the supremum.

Optional Problems

Similarly to the supremum we can define the **infimum** of a set S, denoted S, which corresponds to $\min S$.

Exercise. 4.12.

1. Define $\inf S$.

2. Find the infimum of all the sets in Exercise 4.5.

3. Show that

$$\inf S = -\sup(-S)$$

(if they exist), where $-S$ is the set containing elements of the form $-x$ where $x \in S$.

4. State the analogue of Exercise 4.11 for $\inf S$.

---

[1] Hint: Proof by Contradiction.

[2] Hint: Prove by Contrapositive.

4.1 Intermediate Value Theorem

We'll now prove the Intermediate Value Theorem. This is our first encounter with a complicated proof which requires multiple steps. You probably won't see the full picture until you completely write down the proof yourself. It'll help you a lot to draw pictures to keep a track of all the variables.

Once you have solved all the exercises in the proof of Theorem 4.14, submit your final solution as one single logically coherent proof of the IVT, also include all the text that is in between the exercises, so that you yourself see how all the pieces fit together. This will also teach you to write complex proofs.

We'll need the following proposition from the previous section, which says that the elements of a set S are infinitely close to it's supremum.

Proposition 4.13. *For a set S, if $L = \sup S$, then for every $\delta > 0$, there exists an element $x \in S$ such that $L - x < \delta$.*

Theorem 4.14 (Intermediate Value Theorem)**.** *Let f be a continuous function on $[a, b]$ and let k be a real number satisfying*

$$f(a) < k < f(b).$$

Then, there exists a real number c satisfying $a < c < b$ such that

$$f(c) = k.$$

Proof of 4.14. We'll first prove the following slightly easier version.

Proposition 4.15. *If g is a continuous function on $[a, b]$ satisfying*

$$g(a) < 0 < g(b)$$

then, there exists a real number c satisfying $a < c < b$ such that

$$g(c) = 0.$$

Proof of 4.15. Let g, a, b be as in the statement of the Proposition. We'll construct a number c that satisfies $a < c < b$ and $g(c) = 0$.

Let S be a set defined as follows,

$$S = \{x \mid a \leq x \leq b \text{ and } g(x) \leq 0\}.$$

Exercise. 4.16. Argue that S is non-empty and is bounded from above.

Hence, S has a supremum by the *Completeness Property of Real Numbers*. Denote this supremum by c. We'll prove that

Claim: $g(c) = 0$.

We'll prove this claim by contradiction. Suppose on the contrary that $g(c) \neq 0$. There are two possible cases:

Case 1: $g(c) < 0$

Case 2: $g(c) > 0$

Suppose **Case 1** is true i.e. $g(c) < 0$.

Exercise. 4.17.   1. Use the formal definition of continuity (from right) of g to prove that there exists an $x > c$ such that $f(x) < 0$.[1]

2. Argue that this contradicts one of your assumptions, hence $g(c)$ cannot be less than 0.

Suppose **Case 2** is true i.e. $g(c) > 0$.

Exercise. 4.18.   1. Use the formal definition of continuity (from left) of g to prove that there exists a δ such that, for every x, if $c - x < \delta$ then $g(x) > 0$.[2]

2. For this particular δ, prove that, for every x in S, $c - x > \delta$.

3. Argue that this contradicts Proposition 4.13, hence $g(c)$ cannot be greater than 0.

   Thus we've contradicted both the possible cases, which completes the proof by contraction and proves the Claim that $g(c) = 0$.

   By construction c satisfies $a < c < b$. So we've constructed a c such that $a < c < b$ and $g(c) = 0$ which completes the proof of Proposition 4.15.   □

Exercise. 4.19. Going back to f, use the function $g(x) = f(x) - k$ in Proposition 4.15 to complete the proof of Theorem 4.14.

□

   Both the hypotheses are necessary for the IVT to be true. This means that we must be crucially using these hypotheses in our proof somewhere and the proof should fail if we do not have these assumptions. It is clear where we are using continuity, the other one is more subtle.

Exercise. 4.20.

1. Find the *exact* argument in your proof of Proposition 4.15 which fails if we assume $0 < g(a) < g(b)$.

2. Find the *exact* argument in your proof of Proposition 4.15 which fails if we assume $g(a) < g(b) < 0$.

---

[1] Hint: Use $\epsilon = -g(c)/2$.

[2] Hint: Use $\epsilon = g(c)/2$.

5 Differentiation

Remark 5.1. We'll now exclusively work with *continuous* functions. As such, we can compute the various limits in this section without requiring any ϵ, δ arguments. We'll get back to the ϵ, δ's much later. For now, we'll repeated use the following identities,

$$\lim_{x \to a} f(x) + g(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

$$\lim_{x \to a} f(x) \cdot g(x) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

$$\lim_{x \to a} f(x)/g(x) = \lim_{x \to a} f(x) / \lim_{x \to a} g(x) \qquad \text{assuming } \lim_{x \to a} g(x) \neq 0$$

wherever the appropriate limits exist.

5.1 Derivative

The derivative of a function measures the *instantaneous rate of change* of the function. Several quantities in physics can be naturally expressed as derivatives, which was the original motivation for defining them.

Function	Derivative
Position	Velocity
Velocity	Acceleration
Potential	Force

Definition 5.2. The **derivative** of $f(x)$ with respect to x, at $x = a$, is defined to be the limit

$$f'(a) := \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \tag{5.3}$$

If $f'(a)$ exists then we say that $f(x)$ is **differentiable** at a.

One can rewrite Equation (5.3) as

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{5.4}$$

by making the substitution $h = x - a$.

In this form, it is more explicit that derivative is computing the *instantaneous rate of change*, as the numerator $f(x) - f(a)$ is the change in f and the denominator $x - a$ is the change in x, so their ratio is the rate of change of $f(x)$ with respect to x. Taking the limit $x \to a$ makes this rate of change *instantaneous*. When we want to make the "free variable" more explicit, we write the derivative as

$$\frac{df}{dx}\bigg|_{x=a}$$

This notation becomes more relevant in multivariable calculus when there are multiple variables with respect to which one can differentiate the function.

Exercise. 5.5. Using the definition, compute the derivatives of the following functions.

1. $f(x) = c$, where c is a real number.

2. $f(x) = x$.

3. $f(x) = x^2$.

Exercise. 5.6. Prove that if $f(x)$ is differentiable at a then $f(x)$ is continuous at $x = a$.

Theorem 5.7. *If f and g are differentiable at a then*

$$(f + g)'(a) = f'(a) + g'(a)$$

$$(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a) \qquad \textbf{Product Rule}$$

$$\left(\frac{1}{g}\right)'(a) = -\frac{g'(a)}{g(a)^2}$$

$$\left(\frac{f}{g}\right)'(a) = \frac{g(a) \cdot f'(a) - g'(a) \cdot f(a)}{g(a)^2} \qquad \textbf{Quotient Rule}$$

$$(f \circ g)'(a) = f'(g(a)) \cdot g'(a) \qquad \textbf{Chain Rule}$$

where we assume that $g(a) \neq 0$ wherever it is in the denominator.

Exercise. 5.8. Prove Theorem 5.7.

The proof of the Product Rule requires a trick: add and subtract $f(a + h)g(a)$ to the numerator.

For proving the Quotient Rule notice that $f/g = f \cdot (1/g)$.

Feel free to read the proof of the Chain Rule from the book.

Because of these rules, once we know the derivatives of standard functions like trig functions, exponential functions etc., derivatives are fairly easy to compute. However, computing these standard derivatives is not easy using simply the definition (give it a try!).

Polynomials are the only functions whose derivatives we can compute right now. We'll get back to computing more complicated derivatives once we learn integration and the Fundamental Theorem of Calculus.

Exercise. 5.9. In this exercise we'll (almost) prove that

$$(x^n)' = nx^{n-1} \tag{5.10}$$

where n is any real number.

Note that Exercise 5.5 proves this for the cases $n = 0, 1, 2$.

1. Use the identity $(x^n) \cdot x = x^{n+1}$ to prove (5.10) for all positive integers.[1]

---

[1]**Optional:** Read about **Mathematical Induction** and use it to write a rigorous proof of this statement.

2. Use the identity $\frac{1}{x^m} = x^{-m}$ to prove (5.10) for all negative integers.

3. Use the $\left(x^{\frac{p}{q}}\right)^q = x^p$ to prove (5.10) for all rational numbers.

This proves (5.10) for all rational numbers. We do not yet know enough to extend this proof to irrationals.

4. Can you think of a way to prove (5.10) for all real numbers? What fact *should* be true for your proof to work?

5.2 Inverse Functions

We say that functions f and g are **inverses** of each other if the following identities hold:

$$f(g(x)) = x$$
$$g(f(x)) = x$$

Exercise. 5.11. Verify that if f and g are inverses of each other and $f(x) = y$ then $x = g(y)$. (Hence the name inverse.)

This exercise is saying that to find the inverse of $f(x)$ we simply solve $f(x) = y$ for x in terms of y, which then gives us g.

Exercise. 5.12.

1. Verify that $f(x) = 2x + 1$ and $g(x) = (x - 1)/2$ are inverses of each other. Draw their graphs.

2. Verify that $f(x) = x^2$ and $g(x) = \sqrt{x}$ are inverses of each other. Draw their graphs.

3. Verify that $f(x) = e^x$ and $g(x) = \ln x$ are inverses of each other. Draw their graphs.

4. What relation do you see between the graphs of inverses?

5. What are the domains of these functions i.e. what is the set of real numbers where the these functions are defined? Why do you think some of the inverses are not defined everywhere?

The inverses of $\sin x$ and $\cos x$ are denoted $\sin^{-1}(x)$ and $\cos^{-1}(x)$, respectively. More generally, the inverse of a function $f(x)$ is denoted by $f^{-1}(x)$ (which is different from $(f(x))^{-1}$). Note that if $f = g^{-1}(x)$ then $g = f^{-1}(x)$.

Exercise. 5.13. Using the Chain Rule show that

$$\left(f^{-1}(a)\right)' = \frac{1}{f'(f^{-1}(a))}$$

Verify this explicitly for $f(x) = 2x + 1$ and for $f(x) = x^2$.

We'll interpret this geometrically in the next section. We'll later on see that the derivatives of $\sin^{-1}(x)$ and $\ln x$ can expressed in terms of polynomials and radicals. This fact along with the above formula will then allow us to compute the derivatives of $\sin x$ and e^x.

5.3 First Derivative Test

Derivatives have several physical and geometric interpretations which make then widely applicable in all branches of science. We'll see a few of these interpretations in the next few sections.

Definition 5.14. We say that a function f is **increasing** on an interval $[a, b]$, if for all x, y in $[a, b]$ if $x < y$ then $f(x) \leq f(y)$.

Exercise. 5.15. Come up with a definition for when a function is **decreasing** on an interval.

Exercise. 5.16. Find the interval(s) on which the function x^2 is increasing/decreasing? What about the function x^3?

For differentiable functions, we can determine if a function is increasing or decreasing by calculating it's first derivative.

Exercise. 5.17.

1. **Optional:** Let $g(x)$ be a function such that $\lim_{x \to c} g(x)$ exists for some real number c in the interval (a, b). Prove that if $g(x) \geq 0$ for all x in $[a, b]$, then $\lim_{x \to c} g(x)$ is also ≥ 0.[1]
   (You can assume this part to be true if you choose not to prove it.)

2. Let $f(x)$ be an increasing function on $[a, b]$ and let $c \in (a, b)$. Prove that $g(x) \geq 0$ on the interval $[a, b]$ where $g(x)$ is the function

$$g(x) = \frac{f(x) - f(c)}{x - c}$$

3. Let $f(x)$ be a differentiable function that is increasing on the interval $[a, b]$. Prove that all $c \in (a, b)$, $f'(c) \geq 0$. Similarly, for decreasing functions.

Note that this agrees with the interpretation of a derivative as the *instantaneous* rate of change of a function: an increasing function should have a positive instantaneous rate of change and a decreasing function should have a negative instantaneous rate of change. The converse of the above statement is almost true, but for this we need an extra assumption on f and the proof is a bit more intricate. You can assume the following Proposition without proof.

Proposition 5.18. *Let $f(x)$ be a differentiable function whose derivative $f'(x)$ is a continuous function.*
If $f'(c) > 0$ then there exists an interval $[a, b]$ containing c such that f is increasing on $[a, b]$. Similarly, for $f'(c) < 0$.

Exercise. 5.19. Use the above Proposition to find the range in which the function x^n is increasing/decreasing, where n is a positive integer.

Remark 5.20. We have shown that $f(x)$ differentiable implies that $f(x)$ is continuous, this is not the same thing as saying that $f'(x)$ is continuous. All we can say is that $f'(x)$ exists. There are functions whose derivative exists but the derivative itself is not continuous, we might see examples of this later.

---

[1] Hint: Proof by Contradiction.

Definition 5.21. A real number c is said to be a **local maximum** of a function f if there is some interval (a, b) containing c such that if $a \leq x \leq b$ then $f(c) \geq f(x)$.

Exercise. 5.22. Come up with a definition for **local minimum**.

Exercise. 5.23. Assume that the function $f(x)$ is differentiable and it's derivative $f'(x)$ is continuous. Using Proposition 5.18 prove that if c is a local maximum of $f(x)$ then $f'(c) = 0$.[1] Similarly for a local minimum.

Combining everything in this section we have proven the following Theorem, also called the **First Derivative Test**.

Theorem 5.24 (First Derivative Test). *Let $f(x)$ be a differentiable function whose derivative $f'(x)$ is a continuous function.*

For a real number c,

$$f \text{ has a local max/min at } c \implies f'(c) = 0$$
$$f \text{ is increasing near } c \implies f'(c) \geq 0$$
$$f \text{ is decreasing near } c \implies f'(c) \leq 0$$

Conversely,

$$f'(c) > 0 \implies f \text{ is increasing near } c$$
$$f'(c) < 0 \implies f \text{ is decreasing near } c$$

where by "near c" means for some interval $[a, b]$ containing c.

A real number c satisfying $f'(c) = 0$ is called a **critical point** of f. The First Derivative Test says that a local max/min has to be a critical point, but it does not say that a critical point has to be a local max/min. A standard example of this failure is $c = 0$ for the function $f(x) = x^3$.

$$\text{Local min/max} \Rightarrow \text{Critical point}$$
$$\text{Critical point} \nRightarrow \text{Local min/max}$$

There are ways to fix this using higher derivatives. We'll do this later using Taylor Series approximations.

Optional Problems

There are multiple ways of proving Proposition 5.18. However, all the methods require some additional techniques which we have not yet developed.

Exercise. 5.25. Try to write down a rigorous proof of Proposition 5.18. What statement *should* be true for your proof to work?

---

[1] Hint: Proof by Contradiction.

5.4 Mean Value Theorems

In this section we'll prove the analogues of Intermediate Value Theorem for differentiable functions.

Definition 5.26. A number c is an **absolute maximum** of a function f in the interval $[a, b]$ if $f(c) \geq f(x)$ for all $x \in [a, b]$.

To prove the Mean Value Theorems we'll need the following **Extreme Value Theorem**. We'll assume it without proof. As for the IVT, the proof of the Extreme Value Theorem fundamentally uses the Completeness of the Real Numbers.

Theorem 5.27 (Extreme Value Theorem). *For every continuous function f and every closed interval $[a, b]$, there exists a real number $c \in [a, b]$ such that c is the absolute maximum of f on $[a, b]$.*

Remark 5.28. This theorem is false if we use open intervals. For example, $f(x) = 1/x$ has no absolute maximum on the interval $(0, 1)$ even though it is continuous on it.

Exercise. 5.29. Define an absolute minimum and state the corresponding Extreme Value Theorem.

Note that by our definitions an *absolute* max/min c of f over $[a, b]$ is also a *local* max/min if $a < c < b$. We've already shown that the derivative vanishes at a local max/min, so we have

Theorem 5.30. *Let f be a differentiable function. If c is an absolute max/min for f on the interval $[a, b]$ with $a < c < b$ then $f'(c) = 0$.*
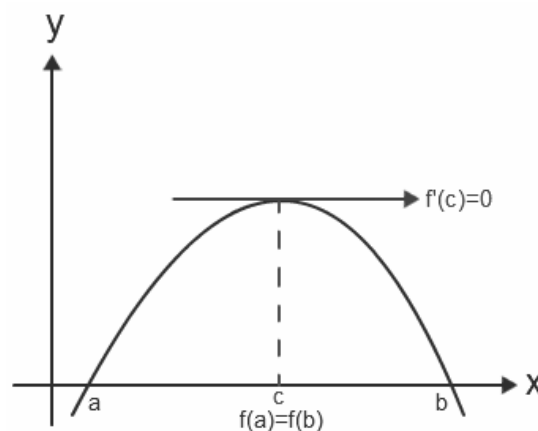
This is all we'll be needing to prove the various Mean Value Theorems.

Theorem 5.31 (Rolle's Theorem). *Let f be a differentiable function. For an interval $[a, b]$, if*

$$f(a) = f(b)$$

then there exists a real number $c \in (a, b)$ such that

$$f'(c) = 0$$

Proof of Rolle's Theorem. This theorem almost immediately follows from Theorem 5.27 and Theorem 5.30. The only issue is that for Theorem 5.30 we need $a < c < b$ but the Extreme Value Theorem only guarantees $a \leq c \leq b$. So we need a small argument to fill this gap.

By the Extremal Value Theorem, f has an absolute maximum $c_{\max}$ and an absolute minimum $c_{\min}$ on the interval $[a, b]$.

Exercise. 5.32.

1. Argue that if $f(c_{\max}) = f(c_{\min})$ then f is a constant function on $[a, b]$.

We've already shown that for a constant function, $f'(x)$ equals 0, hence for every $c \in (a, b)$, $f'(c) = 0$ and we're done.
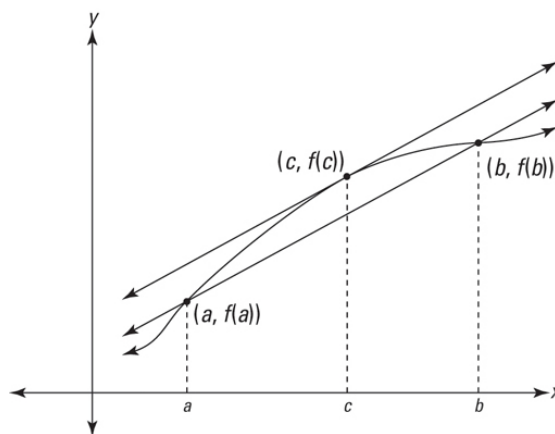
So now we'll assume that $f(c_{\max}) \neq f(c_{\min})$.

2. Argue that in this case either $a < c_{\max} < b$ or $a < c_{\min} < b$ (possibly both).

3. Finish the proof of Rolle's Theorem.

$\square$

Theorem 5.33 (Mean Value Theorem). *Let f be a differentiable function. For any interval $[a, b]$ there exists a real number $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof of Mean Value Theorem. Mean Value Theorem follows from the Rolle's theorem by an algebraic trick. We'll construct a new function $g(x)$ that satisfies the hypotheses of Rolle's theorem.

Exercise. 5.34.

1. Find a real number r such that $f(a) - ra = f(b) - rb$.

For this value of r define

$$g(x) = f(x) - rx$$

By construction, g is differentiable and satisfies $g(a) = g(b)$ and hence satisfies the hypotheses for Rolle's theorem. So, there exists some real number $a < c < b$ such that

$$g'(c) = 0$$

2. Show that for this c, $f'(c) = \dfrac{f(b) - f(a)}{b - a}$, thereby completing the proof of Mean Value Theorem.

$\square$

We'll see later on that the Mean Value Theorem is a key ingredient in the proof of Taylor series approximation. In fact, Taylor series approximations can be thought of as a generalization of MVT for higher order derivatives.

5.4.1 L'Hopital's Rule

We'll now (almost) provide a proof of L'Hopital's Rule using a stronger version of the Mean Value Theorem.

Theorem 5.35 (Easy L'Hopital's Rule). *Let f, g be differentiable functions. Assume that*

$$f(a) = 0 = g(a) \quad and \quad g'(a) \neq 0,$$

where a is a real number. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

Exercise. 5.36. Notice that $\dfrac{f(x)}{g(x)} = \dfrac{f(x) - f(a)}{g(x) - g(a)}$. Use this to prove the easy version of L'Hopital's Rule. [1]

Theorem 5.37 (Strong Mean Value Theorem). *Let f, g be differentiable functions. Assume that $g(a) \neq g(b)$ for some real numbers a and b. Then there exists a real number $c \in (a, b)$ such that*

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Proof of Strong Mean Value Theorem. The Strong Mean Value Theorem follows from the Rolle's theorem by the same algebraic trick. We'll construct a new function $h(x)$ that satisfies the hypotheses of Rolle's theorem.

Exercise. 5.38.

1. Find a real number r such that $f(a) - rg(a) = f(b) - rg(b)$.

---

[1] Hint: Divide the numerator and denominator by $x - a$ and take $\lim_{x \to a}$.

For this value of r define

$$h(x) = f(x) - rg(x)$$

By construction, h is differentiable and satisfies $h(a) = h(b)$ and hence satisfies the hypotheses for Rolle's theorem. So, there exists some real number $a < c < b$ such that

$$h'(c) = 0$$

2. Show that for this c, $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$, thereby completing the proof of the Strong Mean Value Theorem.

$\square$

Exercise. 5.39. Explain how the Mean Value Theorem is a special case of the Strong Mean Value Theorem and how the Rolle's theorem is a special case of the Mean Value Theorem.

Theorem 5.40 (L'Hopital's Rule). *Let f, g be differentiable functions. Assume that*

$$f(a) = 0 = g(a)$$

where a is a real number. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Idea of the proof. For any b, by the Strong Mean Value Theorem, there exists a $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$
$$\implies \frac{f'(c)}{g'(c)} = \frac{f(b)}{g(b)} \qquad\qquad (\text{as } f(a) = 0 = g(a))$$

As c has to lie between a and b, when we take the limit $b \to a$ we must also have $c \to a$. Taking limits of both sides we get the required result.

$$\lim_{c \to a} \frac{f'(c)}{g'(c)} = \lim_{b \to a} \frac{f(b)}{g(b)}$$

$\square$

Remark 5.41. The last argument in the above proof is not rigorous. While the idea is correct, to make it rigorous we need generalized notions of limits called **limit superior** (lim sup) and **limit inferior** (lim inf). This is unfortunately beyond the scope of this class.

There are many more variants of L'Hopital's Rule: for one-sided limits, for $a = \infty$, for $\lim_{x \to a} f(a) = \infty = \lim_{x \to a} g(a)$ etc. They can all be proven using similar techniques.

5.5 Convexity and Concavity

The second derivative of a function $f(x)$ is defined as

$$f''(x) = \left(f'(x)\right)'$$

If the second derivative exists then we say that a function is **twice differentiable**. More generally, the nth derivative $f^{(n)}(x)$ is defined as

$$f^{(n)}(x) = \left(f^{(n-1)}(x)\right)'$$

If all derivatives exist then we say that the function is **smooth** or **infinitely differentiable**. The standard functions like polynomials, trig functions, exponentials, and logarithms are all smooth wherever they're defined.

If a function f is twice differentiable, then the second derivative of f measures it's convexity/concavity.

Definition 5.42. A function f is said to be **convex** (or **concave upwards**) on an interval, if for every real numbers a, b in the interval, the graph of $f(x)$ lies below the line joining $(a, f(a))$ to $(b, f(b))$.
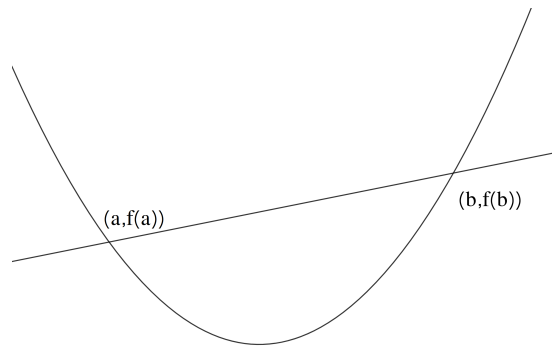
Convex functions play an important role in many areas of mathematics, especially in the study of optimization problems.

The following theorem establishes the connection between convexity and second derivative.

Theorem 5.43. *Let f be a twice differential function.*
If f is convex on an interval then $f''(x) \geq 0$ on that interval. Conversely, if for every real number x in some interval, $f''(x) > 0$, then f is convex on that interval.

Remark 5.44. To avoid clutter, we'll often drop the terms *on an interval, in some interval* etc. in the following proof. Just keep in mind that there is an ambient interval and all our variables belong to that interval.

> *Once you have solved all the exercises below, submit your final solutions as one single logically coherent proof. Also include all the text that is in between the exercises, so that you yourself see how all the pieces fit together. As before, this is to practice writing complex proofs.*

Proof. Let f be a twice differential function.

Exercise. 5.45. For real numbers a, b denote by $m_{a,b}$ the slope of the line joining the points $(a, f(a))$ and $(b, f(b))$. Find a formula for $m_{a,b}$.
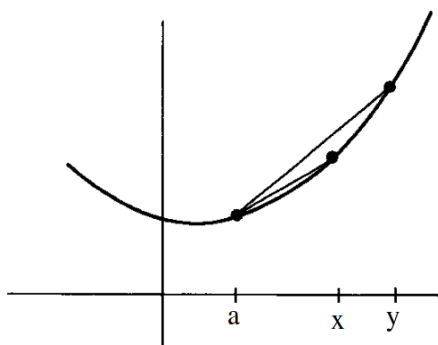
We'll first prove the forward direction.
Claim: *If f is convex on an interval then $f''(x) \geq 0$ on that interval.*
 We'll prove that $f'(x)$ is an increasing function which will then imply that $f''(x) \geq 0$ (as the derivative of an increasing function is ≥ 0).

 Let f be a convex function. Let $a < b$ be two real numbers.

Exercise. 5.46. Let $a < x < y$ be real numbers.

1. *Geometrically* argue that $m_{a,x} < m_{a,y}$.

2. Using the formula for m_{**} and the above inequality argue that

$$f'(a) \leq m_{a,z}$$

 for real numbers $a < z$.

Exercise. 5.47. Let $x < y < b$ be real numbers.

1. *Geometrically* argue that $m_{y,b} < m_{x,b}$.

2. Using the formula for m_{**} and the above inequality argue that

$$m_{z,b} \leq f'(b)$$

 for real numbers $z < b$.

Exercise. 5.48. Combine the results from the previous two exercises to conclude that $f'(a) \leq f'(b)$.

 Hence $f'(x)$ is an increasing function and hence $f''(x) \geq 0$.

Now we'll prove the other direction.
Claim: *If for every real number x in some interval, $f''(x) > 0$, then f is convex on that interval.*

We'll prove this by *contradiction*. Let f be a function such that $f''(x) > 0$ on some interval. This implies that $f'(x)$ is an increasing function on that interval. This is the statement that we'll contradict.

Assume on the contrary that f is not convex. This means that for some real numbers $a < b$, there exists a number c with $a < c < b$ such that the point $(c, f(c))$ lies above the line joining $(a, f(a))$ and $(b, f(b))$.

Exercise. 5.49. Draw a picture.

Applying the Mean Value Theorem to the intervals $[a, c]$ and $[c, b]$ gives us real numbers $d_1 \in (a, c)$ and $d_2 \in (c, b)$ such that

$$f'(d_1) = m_{a,c} \text{ and } f'(d_2) = m_{c,b}$$

Exercise. 5.50.

1. Draw a picture.

2. *Geometrically* argue that $f'(d_1) > f'(d_2)$.

3. Why is this a contradiction?

This completes a proof of the other direction, and hence of the main theorem.

$\square$

5.5.1 Second Derivative Test

Let $f(x)$ be a twice differentiable function. If a real number c is a local minimum of f then f is convex on some interval containing c. A similar statement is true when c is a local maximum. Combining this observation with the previous Theorem and the First Derivative Test, we get the **Second Derivative Test**.

Theorem 5.51 (Second Derivative Test). *Let $f(x)$ be a twice differentiable function whose second derivative $f''(x)$ is a continuous function.*

$$f \text{ has a local min at } c \implies f'(c) = 0 \text{ and } f''(c) \geq 0$$
$$f \text{ has a local max at } c \implies f'(c) = 0 \text{ and } f''(c) \leq 0$$

Conversely,

$$f'(c) = 0 \text{ and } f''(c) > 0 \implies f \text{ has a local min at } c$$
$$f'(c) = 0 \text{ and } f''(c) < 0 \implies f \text{ has a local max at } c$$

We need continuity of $f''(x)$ so that $f''(c) > 0$ implies $f''(x) > 0$ on some interval containing c.

Optional Problems

Exercise. 5.52.

1. Find an example of a function which is convex but $f''(x)$ is not always > 0.

2. Why does the proof of Theorem 5.43 only proves that *if f is convex then* $f''(x) \geq 0$ (instead of $f''(x) > 0$)?

Exercise. 5.53. Several times in this section we used *geometric* arguments. This is less preferable than algebraic arguments as geometric arguments are not rigorous and hence susceptible to being incorrect.

1. Convert the definition of a convex function into an algebraic equation involving inequalities.

2. Use this equation to rigorously prove that, say, $m_{a,x} < m_{a,y}$ if $a < x < y$.

Exercise. 5.54. Let f be a convex function on an interval containing the real numbers a, b. Show that for all $t \in [0,1]$,

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b)$$

This inequality is called **Jensen's inequality** for convex functions.