H-spaces and Hopf algebras

Given a H-space (X, h) we wish to examine algebraic stuctures on $H^*(X)$ and $H_*(X)$. Our goal is to do this Lie groups, in particular $U(n)$ and $SO(n)$. (X-connected)

→ Hopf algebra structure on $H^*(X,R)$:

We have a natural map $H^*(X,R) \xrightarrow{h^*} H^*(X\times X;R)$ we want to use this to construct a map $H^*(X) \xrightarrow{\Delta} H^*(X)\otimes H^*(X)$. We are missing a map $H^*(X\times X) \longrightarrow H^*(X)\otimes H^*(X)$

There is a natural cross product / map going the other way $H^*(X)\otimes H^*(X) \xrightarrow{\times} H^*(X\times X)$. which has a kernel $\mathrm{Ext}_R^{*-1}(H^*(X), H^*(X))$ and is surjective; so we get

Prop: If $\mathrm{Ext}^1_R(H^*(X,R), H^\square(X,R)) = 0$ then we have a coproduct structure on $H^*(X,R)$. In particular it is enough to require $H^*(X,R)$ to be free. Further this coproduct map is a map of algebras.

The only unproven thing in the proposition is the algebra structure. This follows from the fact that cross product is an algebra map. I do not understand this very well.

Assume from now on that $H^*(X)$ is free over R. Co-efficients are always assumed to be R.

Prop. $\Delta(\alpha) = \alpha\otimes 1 + 1\otimes\alpha + \sum \alpha'\otimes\alpha''$ with $|\alpha'|\geq 1, |\alpha''|\geq 1$ when $|\alpha|>0$.

Proof.

$$H^{p+q}(X) \xrightarrow{h^*} H^{p+q}(X\times X) \longleftarrow H^p(X)\otimes H^q(X)$$

$$\alpha \longmapsto \alpha\circ h$$

$$x\times y \longleftarrow x\otimes y$$

$X\times X \xrightarrow{p_1} X$ first component

$X\times X \xrightarrow{p_2} X$ second component

$$(x\times y) = p_1^* x \cup p_2^* y$$

Now suppose $\Delta(\alpha) = \sum_{p+q=k}\sum_i x_i^p\otimes y_i^q$, $|x_i^p| = p$, $|y_i^q| = q$ then

$$\alpha\circ h = 1\otimes y^k + \sum_{\substack{p>0\\ p+q=k}}\sum_i p_1^*(x_i^p)\cup p_2^*(y_i^q)$$

We wish to understand the case $p=0$, q arbitrary. Because $H^0(X)$ is just R, we get

Because of the isomorphism $C_{p+q}(X) \cong \oplus C_p(X)\otimes C_q(X)$ it suffices to evaluate $\alpha\circ h$ on chains of of the form $\{e\}\times i_*(\tau)$ where $i: X\longrightarrow X\times X$ sends x to (e,x) and $\tau\in C_k(X)$.

$$\alpha\circ h(\{e\}\times i_*(\tau)) = y(\tau) \implies y = \alpha$$

$$\alpha\circ h(\{e\}\times i_*(\tau)) = \alpha(e\cdot\tau) = \alpha(\tau)$$

$\square$

Prop: $H^*(X)$ is a Hopf algebra

Proof: 1. Associativity of co-algebra: want to show

$$(\mathrm{id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id})\Delta$$

Again we note that chains in $X \times X \times X$ are obtained from chains of the form $i_{1*}\tau_1 \times i_{2*}\tau_2 \times i_{3*}\tau_3$, $\quad i_1 : X \longrightarrow X \times X \times X$, $x \longmapsto (x, e, e)$, similarly i_2, i_3 and that $H^*(X \times X \times X) = (H^*(X))^{\otimes 3}$.

Hopefully this gives the result.

2. Compatibility of co-algebra & algebra:

We saw that $\quad \Delta(xy) = \Delta(x)\Delta(y)$

3. Unit, counit

$$R \xrightarrow{\simeq} H^0(X) \hookrightarrow H^*(X), \qquad H^*(X) \longrightarrow H^0(X) \cong R$$

$$\underbrace{1 \cdot x = x \cdot 1 = x}_{\text{unit}} \qquad x \longmapsto \underbrace{1 \otimes x + x \otimes 1 + \sum x' \otimes x''}_{\text{counit}} \longrightarrow 1 \otimes x + 0 + 0 = x$$

$$\begin{tikzcd}
(x_0 + x)(y_0 + y) \arrow[r, mapsto] \arrow[dr] & x_0 y_0 + x_0 y + x y_0 + x y \arrow[d] \\
& x_0 y_0
\end{tikzcd}
\qquad
\begin{tikzcd}
& 1 \arrow[dl] \arrow[dr] & \\
1 \arrow[rr] & & 1 \otimes 1
\end{tikzcd}$$

Pontryagin Product

We do the similar construction in homology. But here we note that there is no requirement for $H_*(X)$ to be free so Pontryagin product always exists, but there is no corresponding coproduct structure on $H_*(X)$ and so this is not very easily computable without $H^*(X)$.

A more serious problem is that this product need not be graded commutative.

$$H_p(X) \otimes H_q(X) \longrightarrow H_{p+q}(X \times X) \xrightarrow{h_*} H_{p+q}(X) \quad : P$$

$$\begin{array}{lllll}
\sigma : \Delta_p \to X & \longmapsto & \Delta_p \times \Delta_q \longrightarrow X \times X & \longmapsto & h(\sigma(x), \tau(y)) \\
\tau : \Delta_q \to X & & (x, y) \longmapsto (\sigma(x), \tau(y)) & &
\end{array}$$

Now on assume $H_*(X)$ free over R. Then we have:

$$\begin{tikzcd}
H^k(X) \arrow[r, "h^*"] \arrow[d, leftrightsquigarrow, "\text{duals}"'] & H^k(X \times X) \arrow[r] \arrow[d, leftrightsquigarrow, "\text{dual}"] & \bigoplus H^i(X) \otimes H^{k-i}(X) \arrow[d, leftrightsquigarrow, "\text{dual}"] \\
H_k(X) & H_k(X \times X) \arrow[l, "h_*"'] \arrow[r] & \bigoplus H_i(X) \otimes H_{k-i}(X)
\end{tikzcd}$$

Note: Require $H_k(X)$ to be finitely generated

so that Δ and pontryagin product are duals of each other !! ☺

So that $P(x,y) = (x\otimes y)\circ\Delta : H^*(X) \longrightarrow R$, $\Delta(x) = x\circ P : H_*(X)\otimes H_*(X) \longrightarrow R$

$x,y\in H_*(X) \cong (H^*(X))^*$ $\qquad$ $x\in H^*(X) \cong (H^*(X))^*$

Def. For a graded, commutative algebra A/R a set of elements $\{x_i\}$ is said to be a fundamental generating set if the monomials $\{x_I\}$ form a basis over which A is free as an R-module

Prop. If $H^*(X)$ has fundamental generating set $\{x_i\}_{i\in I}$ and each x_i is primitive ($\Delta x_i = 1\otimes x_i + x_i\otimes 1$) then $(H_*(X),P)$ is an exterior algebra over $\{x_i^*\}$.

Proof: $\Delta(x_{i_1}\cdots x_{i_k}) = \Delta(x_{i_1})\Delta(x_{i_2})\cdots\Delta(x_{i_k}) = (1\otimes x_{i_1} + x_{i_1}\otimes 1)(x_{i_2}\otimes 1 + 1\otimes x_{i_2})\cdots(x_{i_k}\otimes 1 + 1\otimes x_{i_k})$

$= \sum_{(I,J)} (-1)^{IJ} x_I\otimes x_J$ where (I,J) varies over partitions of $(i_1,i_2,\dots,i_k)$

$P(x_i^*,x_j^*)(x_{i_1}\cdots x_{i_k}) = (x_i^*,x_j^*)\Delta(x_{i_1}\cdots x_{i_k}) = \sum_{(I,J)} (-1)^{IJ} x_i^* x_I \otimes x_j^* x_J$

which is non-zero iff

$x_I = c x_i$ and $x_J = d x_j$ for some I,J and $c,d\in R$

So that

$P(x_i^*,x_i^*) = 0 \quad \forall i$

$P(x_i^*,x_j^*)(x_i x_j) = (x_i^*\otimes x_j^*)\Delta(x_i)\Delta(x_j) = (x_i^*\otimes x_j^*)(1\otimes x_i + x_i\otimes 1)(1\otimes x_j + x_j\otimes 1)$

$= (x_i^*\otimes x_j^*)(1\otimes x_i x_j + (-1)^{|x_i||x_j|} x_j\otimes x_i + x_i\otimes x_j + (-1)^{|x_j|} x_i x_j\otimes 1)$

$= 1$

$P(x_i^*,x_j^*)(x_j x_i) = (x_i^*\otimes x_j^*)\Delta(x_j)\Delta(x_i) = (x_i^*\otimes x_j^*)(1\otimes x_j + x_j\otimes 1)(1\otimes x_i + x_i\otimes 1)$

$= (-1)^{|x_i||x_j|}$

So that we get $P(x_i^*,x_j^*) = (-1)^{|x_i||x_j|} P(x_j^*,x_i^*)$ $\square$

Prop: $H^*(U(n),\mathbb{Z})$ and $H^*(SO(n),\mathbb{Z}/2)$ have primitive fundamental set of generators. And $H_*(U(n),\mathbb{Z}) \cong \Lambda_{\mathbb{Z}}(x_1\cdots x_n)$, $|x_i| = 2i-1$ and $H_*(SO(n);\mathbb{Z}/2) \cong \Lambda_{\mathbb{Z}/2}(y_1\cdots y_{n-1})$ $|y_i| = i$

Proof: That we have a fundamental set of generators follows by looking at the Serre SS for

$U(n-1)\longrightarrow U(n)\longrightarrow S^{2n-1}$ and $SO(n-1)\longrightarrow SO(n)\longrightarrow S^{n-1}$

Remains to show primitive

Primitivity follows from looking at the commutative diagrams:

$$\begin{tikzcd}
H^*(U(n),\mathbb{Z}) \arrow[d, "\Delta"'] & H^*(U(n+1),\mathbb{Z}) \arrow[l] \arrow[d, "\Delta"] & & 0 & x_n \arrow[l] \arrow[d, mapsto] \\
H^*(U(n),\mathbb{Z})\otimes H^*(U(n),\mathbb{Z}) & H^*(U(n+1);\mathbb{Z})\otimes H^*(U(n+1);\mathbb{Z}) \arrow[l] & & & 1\otimes x_n + x_n\otimes 1 + \sum \text{lower terms}
\end{tikzcd}$$

The only way the lower terms can map to 0 is if they are themselves 0.
The same argument applies to $H^*(SO(n);\mathbb{Z}/2)$.

In fact there is a general theorem of Borel (?) which says that similar result holds for all compact classical Lie Groups.

While we are at it let us use the $H_*(U(n);\mathbb{Z})$ to compute $H^*(BU(n);\mathbb{Z})$.
We have the fibration: $U(n) \to * \to BU(n)$ and the Bar Spectral Sequence then gives:

$$E_2^{r,s} = \mathrm{Ext}^{r,s}_{\underset{\Lambda(x_1, x_2, \dots, x_n)}{\overset{\shortparallel}{H_*(U(n),\mathbb{F}_p)}}}(\mathbb{F}_p, \mathbb{F}_p) \Longrightarrow H^{r+s}(BU(n);\mathbb{F}_p)$$

$|x_n| = 2n-1$

The resolution of $\mathbb{F}_p$ over is $\Gamma(\gamma^1, \gamma^2, \dots, \gamma^n) \otimes_{\mathbb{F}_p} \Lambda(x_1, \dots, x_n)$ where $|\gamma^i_j| = (j, j(2i-1))$
When we do Hom we get $\mathrm{Ext}^{**}_{\Lambda()}(\mathbb{F}_p, \mathbb{F}_p) = \mathbb{F}_p[y_1, y_2, \dots, y_n]$ $\quad |y_i| = (1, 2i-1)$
Notice that the total degree of each y_i is even and so there cannot be any differentials.
Hence we have $E_\infty^{**} = \mathbb{F}_p[y_1, y_2, \dots, y_n]$ $\quad |y_i|$ even

Claim: If $A = I_1 \supseteq I_2 \supseteq \dots$, $Gr(A) = \oplus I_i / I_{i+1} \cong \mathbb{F}_p[y_1 \cdots y_n]$ then $A \cong \mathbb{F}_p[\tilde{y}_1, \dots, \tilde{y}_n]$
To prove this first not that $A \cong Gr(A)$ as $\mathbb{F}_p$ modules simply because $Gr(A)$ is free. Product structures then have to be compatible. This also holds true for arbitrary rings.
So we get:

$$H^*(BU(n);\mathbb{F}_p) \cong \mathbb{F}_p[c_1, \dots, c_n], \quad |c_i| = 2i$$

Borel gave a classification of Hopf algebra over perfect fields:
Th^m (Borel): k-field, char $k = p$, H - hopf algebra / k generated as an algebra by 1 and x, (called mongenic)

a) if $p \neq 2$, $|x|$ odd, then $H \cong \Lambda(x)$
b) $|x|$ even, then $H \cong k[x]/x^{p^i}$
c) if $p = 2$, $H \cong k[x]/x^{2^i}$

If k is perfect then every Hopf algebra is a tensor product of monogenic algebras.

Prop: 1) $H^*(G,\mathbb{Q}) \cong \Lambda(y_1) \otimes \Lambda(y_2) \otimes \dots \otimes \Lambda(y_k)$ where $|y_i| =$ odd
2) $H^*(BG,\mathbb{Q}) \cong \mathbb{Q}[x_1, \dots, x_k]$ where $|x_i| = |y_i| - 1$