- Lee Cohn

Quantum Chern Simons:

$$\mathrm{Bord}^{\omega, p_1}_{\langle 3,2,1\rangle} \longrightarrow \mathrm{Cat}_{\mathbb{C}}$$

$Z_{\mathcal{C}}(S^1) = \mathcal{C}$ modular tensor category
in particular, $\mathcal{C}$ is braided, has duals, is semisimple.

A p_1-structure is null homotopy data of

$$M \longrightarrow BO \longrightarrow K(\mathbb{Z}, 4)$$

Modular tensor categories categorify commutative Frobenius algebras.

$\mathcal{C}$ braids $\rightsquigarrow$ $K^0(\mathcal{C})$ is commutative

$$\mathrm{Tr} : K^0(\mathcal{C}) \longrightarrow \mathbb{C}$$
$$V \longmapsto \mathrm{Tr}(\mathrm{id}_V)$$
$$\mathbb{C} \xrightarrow{\mathrm{coev}} V \otimes V^* \xrightarrow{\delta_V \otimes \mathrm{id}} V^{**} \otimes V^* \xrightarrow{\mathrm{ev}} \mathbb{C}$$

$K^0(\mathcal{C})$ - Verlinde Ring $\qquad$ $K^0(\mathcal{C}) \otimes \mathbb{C}$ - Verlinde Algebra

2-dim reduction:

$$Z_{\mathcal{C}} : \mathrm{Bord}^{\omega, p_1}_{\langle 3,2,1\rangle} \longrightarrow \mathrm{Cat}_{\mathbb{C}} \rightsquigarrow Z_{\mathcal{C}} : \mathrm{Bord}_{\langle 2,1,0\rangle} \longrightarrow \mathrm{Cat}_{\mathbb{C}}$$

$$Z^1_{\mathcal{C}}(M) := Z_{\mathcal{C}}(S^1 \times M)$$

$$Z'_{\mathcal{C}}(pt) = \mathcal{C} \qquad Z'_{\mathcal{C}}(S^1) \cong HH_0(\mathcal{C})$$

Claim: $K^0(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{C} \cong HH_0(\mathcal{C})$

$K^0(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{C}$ is the frobenius algebra controlling the (2,1)-dimensional reduction of $Z_{\mathcal{C}}$.

Ex of MTC:

- G finite group. MTC $\mathrm{Vect}_G(G)$

  obj: Vector spaces $\{V_g\}_{g\in G}$ with $V_g \cong V_{hgh^{-1}}$ + some equivariance conditions

Monoidal Structure: Convolution $(V \overset{c}{\otimes} W)_x := \bigoplus_{x_1 x_2 = x} V_{x_1} \otimes W_{x_2}$

$$V \overset{c}{\otimes} W := m_*(\pi_1^* V \otimes \pi_2^* W)$$

$K_G(G)$ the Grothendieck group of $\mathrm{Vect}_G(G)$, is a commutative Frobenius ring.

Twisted version:

$$\alpha \in H^4(BG, \mathbb{Z}) \longrightarrow H^3_G(G;\mathbb{Z}) \cong H^2_G(G, U(1)) \cong H^1_G(G, \{\text{line bundle}\})$$

$\rightsquigarrow$ Lines $L_{x,y}$ with isomorphisms $L_{yxy^{-1},z} \otimes L_{x,y} \cong L_{x,zy}$ + ...

MTC: $\mathrm{Vect}^\alpha_G(G)$ obj: Vector spaces $\{V_x\}_{x\in G}$ $L_{xy} \otimes V_x \cong V_{yxy^{-1}}$

similar monoidal structure as above.

$K^\alpha_G(G)$ commutative Frobenius algebra.

Loop Groups:

G- compact, simply connected, simple Lie group $\Rightarrow H^4(BG;\mathbb{Z}) \cong \mathbb{Z}$

$LG := \mathrm{Maps}(S^1, G)$

Universal central extension:

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \widetilde{LG} \longrightarrow LG \longrightarrow 1$$

Def^n: Positive Energy Representation at level α, V is

1) Rep. of $\widetilde{LG}$ with $\mathbb{C}^*$ acting as scalars.
2) Action extends to $\widetilde{LG} \rtimes \mathrm{Rot}(S^1) \circlearrowright V$ inducing eigen decomposition

$$V = \bigoplus_{n \geq 0} V(n) \qquad V(n) = \{v \in V \mid R_\theta v = e^{in\theta} v\}$$

Rmk: V irreducible $\Rightarrow$ determined by level α & its lowest energy eigenspace as a rep of G.

Th^m: $\mathrm{Rep}^\alpha(LG)$ is a modular tensor category in particular it is semisimple.

Ex: $G = SU(2)$, $\alpha = k$ $\quad \mathrm{Rep}(SU(2)) = \mathbb{C}[t, t^{-1}]^{\mathbb{Z}_2}$

Irreps of $SU(2)$ are $\longleftrightarrow$ poly $t^n + t^{n-2} + \dots + t^{-n}$

$V(n)$ with $\dim(V(n)) = n+1$

$$\mathrm{Ver}_k(SU(2)) \underset{\mathbb{Z}}{\otimes} \mathbb{C} \cong \mathrm{Rep}(SU(2)) \Big/ \begin{matrix} v_{k+1} = 0 \\ V_n \oplus V_{2k+2-n} = 0 \end{matrix}$$

Dan Freed:

Heisenberg group