SYMPLECTIC GEOMETRY

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ABSTRACT. These are the notes from the course on Symplectic Geometric taught by Prof. Nitu Kitchloo at JHU. An excellent reference is Notes on Symplectic Geometry, by Ana Cannas Da Silva.

CONTENTS

1. SYMPLECTIC MANIFOLDS

Note 1.1 (Disclaimer)**.** There are so many sign conventions in any text on Symplectic geometry it makes your head spin. For most situations the sign is not very relevant, which might suggest a better way of doing symplectic geometry. I have tried to mention every single convention I've used and tried to stay consistent, but I doubt I've succeeded. Please use your own conventions while solving problems.

I assume familiarity with the theory of symplectic vector spaces from linear algebra.

Definition 1.2 (Symplectic Manifold)**.** (M^m, ω) is a symplectic manifold if M^m is a manifold and $\omega \in H^2(M;\mathbb{R})$ satisfies

- (Closed) $d\omega = 0$
- (Non-degeneracy) Tangent space over each point is a symplectic vector space, that is $\omega : TM \to T^*M$ is an isomorphism.

Proposition 1.3. *Non degeneracy of ω is equivalent to the saying that $\omega^{m/2}$ is nowhere vanishing. Non degeneracy also implies that M^m has even dimension and is orientable.*

Proof. Fix a point $p \in M$. It follows from elementary linear algebra that the fact that ω is non degenerate and antisymmetric gives us a basis $e_1, f_1, e_2, f_2, \cdots$ of T_pM such that

$$\omega(e_i, e_j) = 0, \omega(f_i, f_j) = 0, \omega(e_i, f_j) = \delta_i^j$$

From this we see that M should be even dimensional. It follows from definition of wedge product that

$$\omega_p^{m/2}(e_1, f_1, e_2, f_2, \cdots) = 2^{m/2}$$

and hence $\omega^{m/2}$ is nowhere vanishing. A nowhere vanishing top form gives an orientation to M.

For the other direction, assume that $\omega^{m/2}$ is nowhere vanishing. Now suppose on the contrary that ω is degenerate, that is at some point $p \in M$ there is a vector $v \in T_pM$ such that $\omega(v, -) = 0$. Form a basis with $(v, v_2, \cdots)$ of T_pM, then we have $\omega^{m/2}(v, v_2, \cdots) = 0$ but this is a contradiction as any nonzero top form should give a nonzero value on a tuple of linearly independent elements. □

Remark 1.4. If M is compact, non-degeneracy and closedness give us $[\omega^{m/2}] = [M]$.

Definition 1.5 (Symplectomorphism)**.** A diffeomorphism $f : (M, \omega) \to (N, \tau)$ between two symplectic manifolds is a symplectomorphism if $f^*\tau = \omega$.

Definition 1.6. A vector field X on M is called Hamiltonian if $\omega(X)$ is exact, it is called symplectic if $\omega(X)$ is closed.

Given an arbitrary function $f : M \to \mathbb{R}$ the vector field $\omega^{-1}(df)$ is a Hamiltonian vector field. It is easy to see that every Hamiltonian vector field occurs in this way. If $H^1(M, \mathbb{R}) = 0$ every symplectic vector field is also Hamiltonian.

An easy but very important observation is that if X_f is the Hamiltonian vector field corresponding to f then $\omega(X_f, Y) = df(Y) = Y(f)$. This has a lot of applications, and we will use this later to define a Poisson bracket.

1.1. **Examples.**

Example 1.7. $(\mathbb{R}^{2m} = \mathbb{R} < x_1, y_1, \cdots, x_m, y_m >, \sum_i dx^i dy^i)$ It is easy to see that this is indeed a symplectic manifold. If we identify $\mathbb{R}^{2m}$ with $\mathbb{C}^m$ then the symplectic form becomes $\omega = (\sum \sqrt{-1} d\overline{z_j} dz_j)/2$. It turns out locally every symplectic manifold looks like this. We state without proof the following theorem:

Theorem 1.8 (Darboux's theorem)**.** *If (M, ω) is a symplectic manifold, then around any point $p \in M$ we can pick a local chart $(x_1, y_1, \cdots, x_n, y_n)$ defined on a neighborhood U such that on U the symplectic form ω is equal to $\sum_i dx^i dy^i$.*

Darboux's theorem implies that there are no local invariants of a symplectic manifold like curvature.

Example 1.9 (Cotangent bundle)**.** This is the example which leads to Hamiltonian mechanics in Physics.

For an arbitrary manifold X the symplectic manifold is T^*X . Let $\pi : T^*X \to X$ be the projection map, let $\pi_* : TT^*X \to TX$ be it's differential. Then there exists a one form $\alpha \in H^1(T^*X)$ called the Liouville form defined as,

$$\alpha_{(x,v)}(\psi) = \pi_*(\psi)(v) \text{ for } x \in X, v \in T^*_xX, \psi \in TT^*X$$

Proposition 1.10. *$d\alpha$ is a symplectic form on T^*X .*

Proof. $d\alpha$ being exact is also closed. We need to prove non-degeneracy. Pick a chart $(q_1, p_1, q_2, p_2, \cdots)$ on T^*X such that q_i are the horizontal coordinates and p_i are the corresponding vertical coordinates. Then one can check that the form α is given by

$$\alpha = p_1dq_1 + p_2dq_2 + \cdots$$

One way to show this is by saying that there is a unique form satisfying the Liouville equation and that the form defined above does satisfy it. Then $d\alpha = dp_1dq_1 + dp_2dq_2 + \cdots$ which is non-degenerate. □

Example 1.11 (Space of connections on a Principal Bundle)**.** Given a connected Lie group G with Lie algebra $\mathfrak{g}$ endowed with a G invariant inner product and principal G bundle $\zeta : E \to M$ on a compact symplectic manifold (M^{2m}, ω) , let $A(\zeta)$ be the space of connections on ζ.

Proposition 1.12. *$A(\zeta)$ is an affine space modeled on $\Omega^1(M; E \times_G \mathfrak{g})$, where the action of G on $\mathfrak{g}$ is the adjoint action.*

Proof. The affineness is quite straightforward. It is easy to check that if ∇ and ∇' are two connections then so is $t\nabla + (1-t)\nabla$. The identification follows from the isomorphism $\Omega^*_{G,hor}(E; \mathfrak{g}) \cong \Omega^*(M; E \times_G \mathfrak{g})$. □

Now pick a point in $A(\zeta)$ arbitrarily. The tangent space to $A(\zeta)$ at this point can then be identified with $\Omega^1(M; E \times_G \mathfrak{g})$. The G invariant inner product on $\mathfrak{g}$ pushes forward to a metric on $E \times_G \mathfrak{g}$ via the bundle map $\mathfrak{g} \to E \times_G \mathfrak{g} \to B$. Given $\alpha, \beta \in \Omega^1(M; E \times_G \mathfrak{g})$ define a 2 form $tr(\alpha \wedge \beta) \in \Omega^2(M)$ by

$$tr(\alpha \wedge \beta) \mapsto (u, v) \mapsto (\langle \alpha(u), \beta(v) \rangle - \langle \alpha(v), \beta(u) \rangle)/2$$

Define the symplectic structure on $A(\zeta)$ as

$$\omega_\zeta(\alpha, \beta) = \int_M tr(\alpha \wedge \beta) \wedge \omega^{m-1}$$

Proposition 1.13. *ω_ζ is a symplectic form on $A(\zeta)$.*

Proof. I do not understand what closedness might mean, so let us try to prove degeneracy. Using partitions of unity we can reduce the problem to showing that given a form $\alpha \in \Omega^1(U, \mathfrak{g})$ there is at least one point in U at which the form $tr(\alpha \wedge \beta) \wedge \omega^{m-1}$ is non zero. Now invoke Darboux's theorem to say that under suitable choice of coordinates on U, ω looks like $\sum dx^i dy^j$. Suppose $\alpha = \alpha_1 \otimes dx^1 + \cdots$ for some $\alpha_1 \in \mathfrak{g}$. Pick an arbitrary non-zero $\beta_1 \in \mathfrak{g}$ such that $tr(\alpha_1, \beta_1) \neq 0$ and let $\beta = \beta_1 \otimes dy_1$. Then $tr(\alpha \wedge \beta) = tr(\alpha_1, \beta_1)dx_1dy_1 + \cdots$ and this is the only term which survives on doing $\wedge\omega^{m-1}$. □

Example 1.14 (Lie-Poisson / Kostant-Kirillov symplectic structure on coadjoint orbits)**.** Let G be a Lie group. Our symplectic manifold is going to be the G orbits in $\mathfrak{g}^*$ where the action is the left coadjoint action.

Let $\zeta \in \mathfrak{g}^*$. Let M_ζ denote the coadjoint orbit of ζ. We wish to define a symplectic form on M.

On $\mathfrak{g}$ define a bilinear form as

$$\omega_\zeta(X, Y) = \zeta([X, Y])$$

Now the G action on $\mathfrak{g}^*$ gives a map $p : G \to \mathfrak{g}^*, g \mapsto Ad(g^{-1})^*\zeta$, such that we have a surjection $p_* : T_eG = \mathfrak{g} \to T_\zeta(\mathfrak{g}^*)$. For $X, Y \in T_\zeta(\mathfrak{g}^*)$ define

$$\omega(X, Y) = \omega_\zeta(p_*^{-1}X, p_*^{-1}Y)$$

Proposition 1.15. *ω is well defined and non-degenerate on $T_\zeta(\mathfrak{g}^*)$.*

Proof. Well defined: Suppose $p_*(X) = 0$. This means that X is in the tangent space of the stabilizer of ζ. We need to show that $\omega_\zeta(X, Y) =$ for all Y.

$$\omega_\zeta(X, Y) = \zeta([X, Y]) = \zeta(ad_X(Y)) = -(coad_X\zeta)(Y) = 0$$

Non-degenerate: The argument is same as above. $T_\zeta(\mathfrak{g}^*)$ can be identified with $\mathfrak{g}/Lie(Stab(\zeta))$ but the kernel of ω_ζ is precisely $Lie(Stab(\zeta))$. □

Now we extend the definition of ω to M_ζ by using a map p_τ for each $\tau \in M_\zeta$. Smoothness should not be an issue and non-degeneracy would follow by applying the previous proposition to each point.

Proposition 1.16. *ω defined as above is closed and hence defines a symplectic form on M_ζ .*

Proof. Let $X, Y, Z \in \mathfrak{g}$. We need to show that $d\omega_\tau(X, Y, Z) = 0$ where now τ is not a constant any more. Expanding out all the terms vanish by the Jacobi identity except for the ones of the form $X(\tau)([Y, Z]) = coad(X)(\tau)([Y, Z]) = -\tau(ad(X)([Y, Z])) = -\tau([X, [Y, Z]])$. So the three terms would again add up to 0. So it's just a consequence of applying the Jacobi again and again. □

2. Compatible Structures

Let (M^{2m}, ω) be a symplectic manifold. Recall that an almost complex structure on M is a bundle map $J : TM \to TM$ such that $J^2 = -1$.

Definition 2.1. We say J is compatible with ω if

(1) $\omega(Jx, Jy) = \omega(x, y)$ for all $x, y \in \Gamma(TM)$
(2) $g = \omega(J-, -)$ is a Riemannian metric on M .

in which case we call (g, ω, J) a compatible triple.

The standard example is $\mathbb{C}^n = \mathbb{C}\langle z_1, \cdots, z_n\rangle$. Let $\langle w, t\rangle = w^*t$ be the standard hermitian structure on it. We can identify tangent space over each point to $\mathbb{C}^n$. Then

$$\omega(w, t) = Im(\langle w, t\rangle), g(w, t) = Real(\langle w, t\rangle), Jw = iw$$

is a compatible triple.

We work with vector spaces for a while before moving on to manifolds.

Look at $(\mathbb{C}^m = \mathbb{R}^{2m}, \omega)$ where $\omega(w, t) = Im(w^*t)$ and w^* is the complex conjugate of w . We are looking at $\mathbb{R}^{2m}$ not as a manifold but simply as a vector space. Then a compatible almost complex structure would be an linear map $\mathbb{R}^{2m} \to \mathbb{R}^{2m}$ with the required compatibility conditions.

2.1. **Polar Decomposition.** Given an arbitrary metric g on $(\mathbb{R}^2 m, \omega)$ we can construct a compatible almost complex structure J via polar decomposition. However we cannot ensure that (ω, g, J) will be a compatible triple.

Let $A : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ be such that $\omega(x, y) = g(x, Ay)$. Let $A^\dagger$ denote the dual of A with respect to g . It is easy to see that $A^\dagger = -A$. Let

$$J = \sqrt{AA^\dagger}^{-1} A$$

$A = \sqrt{AA^\dagger} J$ is called the polar decomposition of A . It is an easy check that the J is indeed a compatible almost complex structure.

Note that if we start out with the inner product $g(-, -) = \omega(-, J-)$ then the polar decomposition gives us back J.

2.2. **Almost complex structures on vector spaces.**

Proposition 2.2. *The space of compatible almost complex structures on $\mathbb{R}^{2m}$ is affine and hence contractible (under the subspace topology induced from $Gl(2m, \mathbb{R})$).*

Proof. Suppose we are given two compatible structures J_0 and J_1 . Look at $g_i(-, -) = \omega(-, J_i-)$. The space of Riemannian metrics is affine. Define $g_t = (1-t)g_0 + tg_1$ and let J_t be the almost complex structure obtained from g_t by Polar decomposition. This then gives a path in the space of almost complex structures joining J_0 and J_1 . □

We can identify the space even further.

Definition 2.3. Define $Sp(2m, \mathbb{R}) \subset Gl(2m, \mathbb{R})$ to be the subgroup of linear transformations which preserve the standard symplectic form on $\mathbb{R}^{2m}$. That is

$$Sp(2m, \mathbb{R}) = \{A : \mathbb{R}^{2m} \to \mathbb{R}^{2m} \mid AIm((Aw)^*(At)) = Im(w^* t)\}$$

Proposition 2.4. *$Sp(2m, \mathbb{R})$ acts transitively on the space of compatible almost complex structures on $\mathbb{C}^m$ via conjugation and the stabilizer of an almost complex structure is $U(m)$.*

Proof. If $A \in Sp(2m, \mathbb{R})$ and J is an almost complex structure then so is AJA^{-1} . Given a J it is easy to construct a symplectic basis of the form $(e_1, Je_1, \cdots, e_m, Je_m)$ which means that there is an $A \in Sp(2m, \mathbb{R})$ such that AJA^{-1} is just multiplication by i. So the action of $Sp(2m, \mathbb{R})$ is transitive.

Suppose g is in the stabilizer of the standard complex structure. This then means that $g.i.g^{-1} = i$, that is g is complex linear, so that $g \in Gl(m, \mathbb{C})$ is the matrix which preserves the imaginary part of the inner product. But this then means that g also preserves the inner product and hence $g \in U(m)$. □

Corollary 2.5. *U(m) is the largest compact subgroup in $Sp(2m, \mathbb{R})$.*

Proof. Proof follows from the previous two propositions and the fact that every non-compact Lie group has a largest compact subgroup which is such that the quotient is affine. □

2.3. **Almost complex structures on manifolds.** Now we go back to the world of manifolds. By the above results we know that we can put an almost complex structure on the tangent space at each point. Here we prove that we can even do this globally.

Let (M^{2m}, ω) be a symplectic manifold. Define $\mathcal{J} \to M$ be the subbundle of $Gl(TM) \to M$ defined as for each $x \in M$

$$\mathcal{J}_x = \{J \in GL(T_xM) \mid J \text{ is an almost complex structure compatible with } \omega_x \}$$

This then is fiber bundle with each fiber being isomorphic to $Sp(2m, \mathbb{R})/U(m)$. This is contractible and hence the fiber bundle $\mathcal{J}$ a section, in other words

Proposition 2.6. *(M^{2m}, ω) always has a compatible almost complex structure. The space of all the almost complex structures is the space of sections of $\mathcal{J}$.*

Remark 2.7. The converse however is not true. There do exist manifolds which posses almost complex structure but no symplectic structure. For example look at the manifold $M = S^1 \times S^3$. This cannot posses a symplectic structure because of the cohomology ring. But it is quotient of the space $\mathbb{C}^2 \backslash (0,0)$ by the group $\mathbb{Z}$ acting via $(z, w) \to (2z, 2w)$ and hence possesses complex structure.

3. Moment maps

Now we look at group actions on symplectic manifolds.

In this section assume that a Lie group G acts on a symplectic manifold (M^{2m}, ω) via symplectomorphisms. Any $\lambda \in frakg$ induces a vector field on M, denote it by λ^o.

Proposition 3.1. *λ^o is a symplectic vector field.*

Proof.

$$\begin{aligned}
\mathcal{L}_{\lambda^o}\omega &= \lim_{t\to 0}((\exp(t\lambda)\omega) - \omega)/t \\
&= \lim_{t\to 0}(\omega - \omega)/t && \text{because the action is symplectic} \\
&= 0
\end{aligned}$$

$$\mathcal{L}_{\lambda^o}\omega = (d\omega)(\lambda) + d(\omega(\lambda))$$

which implies

$$d(\omega(\lambda)) = 0$$

□

Now if $H^1(M) = 0$ every closed form is exact, and hence ever symplectic vector field is Hamiltonian, that is we can assign a function $M \to \mathbb{R}$ to every element of $\mathfrak{g}$ which is unique up to a constant. We try to make this mapping continuous in the form of a moment map.

Definition 3.2 (Moment map)**.** A moment map is a map $\mu : M \to \mathfrak{g}^*$ such that

1. μ is equivariant, where the action on right is the left coadjoint action
2. The Hamiltonian vector field generated by $\mu(\lambda)$ is λ^o

For example, look at the action of the torus $G = T^m \cong U(1)^{\times m}$ on $\mathbb{C}^m \cong \mathfrak{g}^*$. Then $\mathfrak{g} \cong \mathbb{R}^m$. Define,

$$\mu(z) = -(\|z_1\|^2, \cdots, \|z_m\|^2)/2$$

It is enough to check that this is a moment map for $m = 1$.
The coadjoint action is trivial as the group is abelian and hence the equivariance property is satisfied. An element $\lambda \in \mathfrak{g}^*$ generates the vector field

$$\lambda^o(z) = i\lambda z\partial z - \overline{z}\partial\overline{z})$$

$$\omega(\lambda^o) = idzd\overline{z}/2(i\lambda(z\partial z - \overline{z}\partial\overline{z})) = -\lambda(zdz + \overline{z}d\overline{z})/2$$

The function μ at λ is

$$\mu(\lambda) = -(\lambda z\overline{z})/2$$

$$d\mu(\lambda) = -\lambda(zdz + \overline{z}d\overline{z})/2$$

Not every Lie group action admits a moment map. Moment maps are not unique. We state the following theorems without proof.

Theorem 3.3. *If $H^1(\mathfrak{g};\mathbb{R}) = H^2(\mathfrak{g};\mathbb{R}) = 0$ then moment maps exist for any symplectic G action. If G is a compact Lie group then these conditions are equivalent to saying that G is semisimple, that is $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.*

Theorem 3.4. *For a connected compact Lie group $H^1(\mathfrak{g};\mathbb{R}) = 0$ implies any G moment map is unique.*

Theorem 3.5. *If G is simply connected then a moment map always exists. If the center of G is discrete then the moment map is unique.*

3.1. **Examples.**

Example 3.6 (Linear momentum)**.** Consider $\mathbb{R}^6$ with coordinates $x_1, x_2, x_3, y_1, y_2, y_3$ with symplectic form $\omega = \sum_d x_i dy_i$. Then the action of $\mathbb{R}^3$ on $\mathbb{R}^6$ via

$$a \mapsto t_a, t_a(x, y) = (x + a, y)$$

Define the map

$$\mu : \mathbb{R}^6 \to \mathbb{R}^3, (x, y) \mapsto y$$

This then is the moment map. The equivariance condition is easy. For $a = (a_1, a_2, a_3) \in \mathbb{R}^3$, $\mu_a(x, y) = y.a$ and the Hamiltonian vector field of μ_a is $a_1\frac{\partial}{\partial x_1} + a_2\frac{\partial}{\partial x_2} + a_3\frac{\partial}{\partial x_3}$ but this is precisely the infinitesimal action of a on $\mathbb{R}^6$.
Classically y is the momentum vector and x is the position vector. This moment map is called the linear momentum.

Example 3.7 (Angular Momentum)**.** Now look at the action of $SO(3)$ on R^6 . We can think of $\mathbb{R}^6$ as the cotangent bundle of $\mathbb{R}^3$ and hence the action on $\mathbb{R}^3$ lifts to $\mathbb{R}^6$.

$\mathfrak{so}(3)$ consists of skew symmetric matrices, and we can identify it with the Lie algebra $(\mathbb{R}^3, \times)$ via the map

$$(a_1, a_2, a_3) \mapsto \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

$$a \times b \mapsto AB - BA$$

$$Ma \mapsto MAM^{-1}$$

by $\times$ we mean the cross product. If we use the same identification for $\mathfrak{so}(3)^*$ then the dual pairing simply becomes the dot product.

Under this identification the infinitesimal action of $\mathbb{R}^3 \cong \mathfrak{so}(3)$ on $\mathbb{R}^6$ will be given by

$$a \to r_a, r_a(x, y) = (a \times x, a \times y)$$

Define the moment map to be

$$\mu : \mathbb{R}^6 \to \mathbb{R}^3 \cong \mathfrak{so}(3)^*, (x, y) \mapsto y \times x$$

The equivariance follows from the identity $Mx \times My = M(x \times y)$ for $M \in SO(3)$. The function $\mu_a = a.(x \times y)$ so that $d\mu_a = a.(dx \times y) + a.(x \times dy)$

$$\begin{aligned}\omega^{-1}(d\mu a) &= -a.(\frac{\partial}{\partial y} \times y) + a.(x \times \frac{\partial}{\partial x}) \\ &= \frac{\partial}{\partial y}.(a \times y) + \frac{\partial}{\partial x}.(a \times x)\end{aligned}$$

which is the infinitesimal action of a.

This moment map is called angular momentum.

4. Symplectic Reduction

Let us recall how symplectic quotients work for vector spaces. From a coisotropic vector space we form another symplectic subspace by quotienting out the extra dimensions.

Given a symplectic vector space (V, ω) , let W be a coisotropic subspace, that is $W^\perp \subset W$. Then

$$W^{red} := W/W^\perp$$

is a symplectic vector space with a symplectic form induced by ω . The important property of this reduction is that if L is a Lagrangian subspace of V then it's image $L^{red} = L \cap W / L \cap W^\perp$ is a Lagrangian subspace of W^{red}.

Now we need to do this in a parametrized manner. For this we need to be able to pick coisotropic sub-bundle of the tangent bundle, reduce each of the tangent spaces and hope than we can 'integrate' this bundle back to get a manifold below.

Back to the world of manifolds.

Theorem 4.1 (Marsden-Weinstein-Meyer)**.** *M, ω, G, μ as above. Suppose G is compact,$a \in \mathfrak{g}^*$ is a regular point of μ and the action of G on $\mu^{-1}(a)$ is free. Then $M//G(a) := \mu^{-1}(a)/G$ is a symplectic manifold with a form ω_{red} compatible with ω on $\mu^{-1}(0)$.*

The conditions we have stated are too rigid. In general we do not require the action to be free. Lacking freeness we get show that we get an orbifold in place of a manifold. When we do the examples we won't check for freeness. But we do always need a to be a regular value of μ.

If $a = 0$ then $M//G(a)$ is usually denoted by $M//G$.

The first step is finding a coisotropic sub-bundle. This bundle is going to be tangent space of $\mu^{-1}(a)$.

Proposition 4.2. *Let C_a be the stabilizer of $a \in \mathfrak{g}$. If a is a regular value then C_a is a finite index subgroup of G and hence has the same Lie algebra as G .*

Proof. Suppose this is not the case, that is G/C_a is a non-zero dimensional Lie group, and so it must contain a one parameter subgroup say H which corresponds to a Lie algebra element $\tau \in G/C_a$. Let λ be a an element in $\mathfrak{g}$ which goes to τ. Then the equivariance condition of the moment map implies that the direction of λ is not in $\mu_*(T(\mu^{-1}(a)))$ which contradicts the regularity of a. $\square$

Proposition 4.3. *$T\mu^{-1}(a)$ is a coisotropic sub-bundle of TM .*

Proof. Let $\mu(p) = a$. Then $v \in T_p\mu^{-1}(a)$ if $\mu_* v = 0$. Regularity implies $\dim \mu^{-1}(a) = \dim M - \dim G$ so that we must have $\dim(T_p\mu^{-1}(a))^\perp = \dim G$.

Now we just need to find these $\dim G$ many linearly independent tangent vectors. Let $\lambda \in \mathfrak{g}$. Look at the induced vector field at λ^o at p . The second property of the moment map tells us that λ^o is the Hamiltonian vector field corresponding to $\mu(\lambda)$. But the function $\mu(\lambda)$ is constant along μ^{-1} and hence λ^o_p is in $(T_p\mu^{-1}(a))^\perp$ and we are done. $\square$

Proof. We assume from differential topology the fact that since G acts freely on $\mu^{-1}(a)$ then $\mu^{-1}(a)/G$ is a manifold. At each tangent space we are precisely getting rid of the extra dimensions by quotienting out by G and the reduction induces a sypmectic form on the image. Closedness is trivial. $\square$

We state two computationally useful propositions.

Proposition 4.4.
1. *If H is a closed subgroup G then the restriction of the moment map extends via the map $\mathfrak{g}^* \to \mathfrak{h}^*$.*
2. *If $G = H \times K$ with moment map μ_G, μ_H, μ_K. Let $a = (h, k)$ be a regular value of μ_G, then $M//H(h)$ admits a K action and $M//G(a) \cong (M//H(h))//K(k)$.*

Proof. The proofs involve a lot of verifications and no non-trivial ideas. I'll skip it here. $\square$

4.1. **Noether's theorem.** The Noether's theorem from physics takes a very simple form in the language of symplectic geometry.

Theorem 4.5 (Noether's theorem)**.** *$f : M \to \mathbb{R}$ is G invariant if and only if μ in constant along the flow lines of X_f .*

Proof.

$$\begin{aligned} &f \text{ is } G \text{ invariant} \\ \Leftrightarrow\ & a^o(f) = 0 \forall a \in \mathfrak{g} \\ \Leftrightarrow\ & X_{\mu_{a^o}}(f) = 0 \\ \Leftrightarrow\ & X_f(\mu_{a^o}) = 0 \end{aligned}$$

$\square$

4.2. **Examples.**

Example 4.6. $(\mathbb{C}^m, (im\langle,\rangle), T^m, \mu)$ This is the same example we did in the last section. Any non-zero value a is a regular value of μ. However T^m acts freely transitively on $\mu^{-1}(a)$ and hence the reduction is trivial.

Example 4.7. $(\mathbb{C}^m, (im\langle , \rangle), U(1), \mu_\Delta)$ Now we look at the subspace $U(1)$ sitting inside T^m as scalar matrices. The moment map extends as

$$\begin{aligned}\mu_\Delta :& \mathbb{C}^m \to \mathbb{R}^m \to \mathbb{R} \\ & (z_1, \cdots, z_m) \mapsto -(\|z_1\|, \cdots, \|z_m\|)/2 \mapsto -(\|z_1\| + \cdots + \|z_m\|)/2\end{aligned}$$

$-1/2$ is a regular value, $\mu^{-1}(-1/2) \cong S^{2m-1}$.

$$\mathbb{C}^m // U(1)(-1/2) = S^{2n-1}/U(1) \cong \mathbb{CP}^{m-1}$$

The symplectic form will be the generator of $H^2(\mathbb{CP}^{m-1}$.

We have the decomposition $T^m \cong U(1) \times T^{m-1}$ where T^{m-1} embeds as diagonal matrices which have the first entry 1. Hence T^{m-1} acts on $\mathbb{CP}^{m-1}$ with a moment map,

$$\begin{aligned}\mathbb{CP}^{m-1} &\to \mathbb{R}^{m-1} \\ [z_1 : \cdots : z_m] &\mapsto \frac{(\|z_2\|^2, \cdots, \|z_m\|^2)}{\|z_1\| + \cdots + \|z_m\|}\end{aligned}$$

Example 4.8 (Space of connections)**.** Let us return to the space of connections $(A(\zeta), \omega_\nabla)$. Let $\mathcal{G}$ denote the gauge group of $zeta$, that is

$$\mathcal{G} = \{\phi : E \to E \mid g.\phi = \phi\}$$

Then the action of $\mathcal{G}$ naturally extends to $A(\zeta)$.

Theorem 4.9 (Atiyah-Bott)**.** *The action of $\mathcal{G}$ on $A(\zeta)$ is symplectic and the moment map is given by the curvature map $\nabla \mapsto F_\nabla$.*

The symplectic reduction $A(\zeta)//\mathcal{G}$ is then the moduli space of flat connections, which turns out to be a finite dimensional orbifold.

Suppose $G = S^1$ and M is a Riemann surface. Then $\mathcal{G} \cong C\infty(M, S^1)$, this is because S^1 is abelian and hence there the gauge group can act on the left without disturbing the right action of G. $E \times_G \mathfrak{g} \cong M \times \mathbb{R}$ so that $A(\zeta) \cong \Omega^1(M)$ and $\omega_\zeta(a, b) = \int_M a \wedge b$, that is $\omega_\zeta = [M]$ the fundamental class of M .

The $\mathcal{G}$ action is given by $f \mapsto \nabla \mapsto \nabla - df$ (for the calculation of this see the section on prequantum line bundle) so that it is quite apparent how the action of $\mathcal{G}$ is symplectic.

Now identify $Lie(\mathcal{G})^* \cong (C^\infty(M))^*$ with $\Omega^2(M)$ via the pairing $\beta(f) \mapsto \int_M f.\beta$. Then the map

$$\mu : A(\zeta) \to \Omega^2(M), \nabla \mapsto F_\nabla = d\nabla$$

is a moment map.

The action of (G) on $\Omega^2(M)$ is trivial and it acts on ∇ by adding an exact form to it, so we have G equivariance. Fix $\tau \in C^\infty(M)$ then $\mu_f(\nabla) = \int_M f(d\nabla)$. I am not very clear about what follows. The vector field generated infinitesimal action of $f \in Lie(\mathcal{G})^* \cong C^\infty(M)$ is $X^f = -df$. For any $\nabla \in \Omega^1(M) \cong T(A(\zeta))$, we will have $\omega(X^f, \nabla) = -\int_M df\nabla = \int_M fd\nabla$. Then we say that $\mu_f(\nabla)$ is linear in ∇ and hence its differential is the same, and hence we get $d\mu_f(\nabla) = \omega(X^f, \nabla)$ giving us the moment map!

5. Poisson Brackets and Prequantization

In this section we assume that the manifold M is a closed symplectic manifold and has $H^1(M) = 0$. It is possible to relax this assumption for defining the Poisson brackets and the various Lie algebras, but we need this for prequantization.

Definition 5.1 (Poisson Bracket)**.** On $C^\infty(M)$ define the poisson bracket $\{,\}$ as

$$\{f, g\} = \omega(\omega^{-1}(df), \omega^{-1}(dg))$$

Be careful about the sign of this bracket. There sometimes the opposite sign is used. I am using the convention in the notes by Ana Cannas de Silva.

If X_f is the Hamiltonian vector field corresponding to f then we have

$$\{f, g\} = \omega(X_f, X_g) = df(X_g) = X_g(f)$$

Proposition 5.2. *$(C^\infty(M), \{,\})$ is a Lie algebra.*

Proof. Found this very interesting proof on MO. The trick is to notice that when you expand out the expression $d\omega(X_f, X_g, X_h)$ you end up with twice the Jacobi identity. □

Let $Symp(M, \omega)$ denote the group of symplecto from M to M. This is an infinite dimensional Lie group. We can ask what is the corresponding Lie algebra. $Symp(M) \subset Diff(M)$ the diffeo group from M to M . The Lie algebra of $Diff(M)$ is $\chi(M)$ the vector fields on M . So we are looking at the vector fields whose flow preserve the form ω, that is those X such that $\mathcal{L}_X(\omega) = 0$. But these are precisely the symplectic vector fields. Because we have assumed that $H^1(M) = 0$ these are precisely the Hamiltonian vector fields, and every Hamiltonian vector field is uniquely determined by a function up to a constant and hence we get,

Proposition 5.3. *$Lie_{id}(Symp(M)) = C^\infty(M)/\mathbb{R}$. We identify $C^\infty(M)/\mathbb{R}$ with the space of Hamiltonian vector fields and the Lie algebra structure is given by the Lie bracket of vector fields.*

Remark 5.4. There is a small catch here though. The Lie bracket is not really the Lie bracket which emerges as the Lie bracket of left invariant vector fields. As we will soon see this Lie bracket is more like that of right invariant vector fields.

Proposition 5.5. *$\mathbb{R} \to (C^\infty(M), \{,\}) \to (C^\infty(M)/\mathbb{R}, -[,])$ is a map of Lie algebras.*

Proof. Note that we need to change the sign on the Lie bracket of $C^\infty(M)/\mathbb{R}$ to prevent the map from being an antihomomorphism. We need to show that $X_{\{f,g\}} = -[X_f, X_g]$.

$$\begin{aligned}
\omega([X_f, X_g]) &= \omega(\mathcal{L}_{X_f}\omega^{-1}dg) \\
&= \mathcal{L}_{X_f}dg \\
&= dX_f(g) \\
&= -d\{f, g\}
\end{aligned}$$

□

Note 5.6. From here onwards we will assume that $C^\infty(M)/\mathbb{R}$ has the Lie bracket with the sign flipped. This will turn a lot of antihomomorphisms into homomorphisms.

5.1. **Reinterpretation of the Moment map.** Suppose a compact Lie group G acts on (M, ω). This induces a map

$$G \to Symp(M, \omega)$$

The differential of this map induces a map

$$\mathfrak{g} \to C^\infty/\mathbb{R}, \lambda \mapsto \lambda^o$$

Proposition 5.7. *$\lambda \mapsto \lambda^o$ is a homomorphism of Lie algebras.*

Proof. We need to show that $[\lambda, \tau]^o = -[\lambda^o, \tau^o]$. I am not able to do this in it's full generality, but here is an idea.

Look at G acting on G by left multiplication. Then in local coordinates

$$\begin{aligned} \lambda^o_{gh} &= \lim_{t \to 0} (exp(t\lambda)gh - gh)/t \\ &= R_{h*}(\lambda^o_g) \end{aligned}$$

where R_h denotes right multiplication by h . And so λ^o is in a right invariant vector field, and hence it induces the wrong sign on the Lie algebra. $\square$

Proposition 5.8. *A moment map μ defined earlier is the same as a lift of the above map to $C^\infty(M)$ as a map of Lie algebras.*

Proof. That a lift exists is equivalent to choosing a smoothly varying function whose Hamiltonian action gives the infinitesimal action of G which is the second condition of the moment map.

The G equivariance follows from the fact that the map is a map of Lie algebras.

$$\begin{aligned} \mu_{[\lambda,\tau]} &= \{\mu_\lambda, \mu_\tau\} \\ \Leftrightarrow \mu_{[\lambda,\tau]}(p) &= \{\mu_\lambda, \mu_\tau\}(p) \qquad\qquad \forall p \in M \\ \Leftrightarrow \mu_{ad_\lambda(\tau)}(p) &= X_{\mu_\lambda}\mu_\tau(p) \\ \Leftrightarrow \frac{d}{dt}|_{t=0}\ \mu_{Ad_{\exp t\lambda}(\tau)}(p) &= \frac{d}{dt}|_{t=0}\ \mu_\tau(\exp(t\lambda)p) \end{aligned}$$

The last term is the infinitesimal version of the equivariance condition. Uniqueness of flows will then give us the equivariance condition. $\square$

5.2. **Prequantization.** Now we ask if the short exact sequence of Lie algebras above comes from a sequence of Lie groups that is does there exist a Lie group $\mathfrak{X}(M)$ such that there is a short exact sequence of groups $S^1 \to \mathfrak{X}(M) \to Symp(M)$.

Definition 5.9 (Prequantum Line Bundle)**.** (M, ω) is called prequantizable if $[\omega]/2\pi$ is in the image of $H^2(M, \mathbb{Z})$. If M is prequantizable, then a prequantum bundle on M is an S^1 bundle $\mathcal{L}$ over M along with a connection ∇ such that the the curvature F_∇ is equal to $i\omega$, where we identify $Lie(S^1)$ with $i\mathbb{R}$.

Equivalently one can look at complex line bundles.

Proposition 5.10. *On every prequantizable manifold M there exists a prequantum line bundle.*

Proof. Because complex line bundles are classified by there Euler class which lies in $H^2(M, \mathbb{Z})$, we can always construct a line bundle and connection $(\mathcal{L}, \nabla)$ such that $F_\nabla = i[\omega]$. We need to get rid of the cohomology classes.

Suppose $F_\nabla - i\omega = d\tau$ for some $\tau \in \Omega^1(M, i\mathbb{R})$, then $\nabla - \tau$ is also a connection on $\mathcal{L}$ and the curvature is $F_{\nabla-\tau} = F_\nabla - d\tau = i\omega$. $\square$

Proposition 5.11. *Any two prequantum line bundles over M are isomorphic.*

Proof. Suppose we are given two prequantum line bundles $(\mathcal{L}_1, \nabla_1)$ and $(\mathcal{L}_2, \nabla_2)$. Because complex line bundles are classified by their curvature we have an isomorphism of bundles $\phi : \mathcal{L}_1 \to \mathcal{L}_2$.

$d(\phi^*\nabla_2 - \nabla_1) = \phi^* F_{\nabla_2} - F_{\nabla_1} = 0$. Now we invoke the fact that $H^1(M) = 0$ to get a function f such that $\phi^*\nabla_2 - \nabla_1 = idf$. Replace ϕ by $\phi.e^{-if}$ (this is called gauge fixing). $(e^{-if})^*$ adds the term $-idf$ to $\pi^*\nabla_2$ and corrects the error. The best way to see this is perhaps in local coordinates. Given a connection form $\nabla = a_j dx^j + idi$ multiplication by e^{if} is simply translation by f in the local coordinates, and hence it induces the transformation $di \mapsto di + df$. □

Remark 5.12. The above proof also tells us that any two isomorphisms between $(\mathcal{L}_1, \nabla_1)$ and $(\mathcal{L}_2, \nabla_2)$ can only differ by a function of the form e^{ic} where c is constant.

Definition 5.13.

$$\mathfrak{X} := Aut(\mathcal{L}, \nabla) = \left\{ \begin{tikzcd} \mathcal{L} \arrow[r, "g"] \arrow[d] & \mathcal{L} \arrow[d] \\ M \arrow[r, "\overline{g}"] & M \end{tikzcd} \mid g^*\nabla = \nabla \right\}$$

Proposition 5.14. *There exists a central extension*

$$S^1 \to \mathfrak{X} \to Symp(M)$$

Proof. Define the map

$$\mathfrak{X} \to Symp(M), (g, \overline{g}) \mapsto \overline{g}$$

That the kernel is precisely S^1 follows from the remark made above.

Well-defined: Because $g^*\nabla = \nabla$, $\overline{g}^* F_\nabla = F_\nabla$ and because $i\omega = F_\nabla$, $\overline{g}$ is a symplecto. That it is a group homomorphism is trivial.

Surjective Given a symplecto f, $f^*\mathcal{L}$ will be a prequantum line bundle. We can then use gauge fixing to construct a g such that $\overline{g} = f$. □

Proposition 5.15. *There is an isomorphism of Lie algebras*

$$C^\infty(M) \to Lie(\mathfrak{X})$$

compatible with projection onto $Lie(Symp(M))$.

Proof. Let $X \in Lie(\mathfrak{X}) \subset Vect(\mathcal{L})$ be a vector field which leaves ∇ invariant, that is $\mathcal{L}_X(\nabla) = 0$. This is the same as saying that $d(\nabla(X)) + (d\nabla)(X) = 0$ but because the Lie algebra on S^1 is trivial $d\nabla = F_\nabla = i\omega$ and so we get

$$d(\nabla(X)) + F_\nabla(X) = 0$$

Now break $X = X_{hor} + X_{ver}$ using the connection, so the above equation becomes

$$d(\nabla(X_{ver})) + F_\nabla(X_{hor}) = 0$$

We further need that we should be able to push the flow of X down to M and get a Hamiltonian flow there. But this can be done only if X is G invariant. This then means that we can further simplify, $X_{ver} = h\partial i$ where $h \in C^\infty(M)$ and $X_{hor} = \overline{Y}$ is a horizontal lift of a vector field $Y \in Vect(M)$, the above equation becomes

$$dh + \omega(Y) = 0$$

which forces Y to be the Hamiltonian vector field corresponding to $-h$, and hence we get the isomorphism

$$C^\infty(M) \cong Lie(\mathfrak{X}), -h \mapsto h\partial i + \overline{\omega^{-1}(dh)}$$

We need to compute $-[-h_1\partial i + \overline{X_{h_1}}, -h_2\partial i + \overline{X_{h_2}}]$, $X_h = \omega^{-1}(dh)$ (we again are using the - sign to make things commute rather than anticommute)

$$\begin{aligned}
&[-h_1\partial i + \overline{X_{h_1}}, -h_2\partial i + \overline{X_{h_2}}] \\
&= -[h_1\partial i, \overline{X_{h_2}}] - [\overline{X_{h_1}}, h_2\partial i] + [\overline{X_{h_1}}, \overline{X_{h_2}}] \\
&= \overline{X_{h_2}}(h_1)\partial i - \overline{X_{h_1}}(h_2)\partial i + [\overline{X_{h_1}}, \overline{X_{h_2}}]_{hor} + [\overline{X_{h_1}}, \overline{X_{h_2}}]_{ver} \\
&= \overline{X_{h_2}}(h_1)\partial i - \overline{X_{h_1}}(h_2)\partial i + [\overline{X_{h_1}}, \overline{X_{h_2}}]_{hor} - \overline{\omega(X_{h_1}, X_{h_2})}\partial i \\
&= (X_{h_2}(h_1) - X_{h_1}(h_2) - \{h_1, h_2\})\partial i + [\overline{X_{h_1}}, \overline{X_{h_2}}]_{hor} \\
&= (-\{h_2, h_1\} + \{h_1, h_2\} - \{h_1, h_2\})\partial i + \overline{[X_{h_1}, X_{h_2}]} \\
&= \{h_1, h_2\}\partial i - \overline{X_{\{h_1,h_2\}}}
\end{aligned}$$

And so the Lie algebra structure in precisely given by $\{,\}$. □

$\mathfrak{X}$ acts very naturally on the sections of $\mathcal{L}$. But the space of sections of a principal bundles are no fun. So instead we look at the complex line bundle E corresponding to $\mathcal{L}$ via standard representation of $U(1)$. The element $(g, \overline{g})$ acts on $s : M \to E$ by sending it to $g \circ s \circ \overline{g}^{-1}$. The group action preserves the hermitian metric on E and also the connection. We use the same notation for the induced connection and curvature. Now the corresponding Lie algebra action would be,

$$(-f\partial i + X_f) \mapsto (s \mapsto -ifs + \nabla_{X_f} s)$$

But here is another mess. In physics the bundle that emerges is not this one but it's conjugate, and so we need to flip the sign on it.

Definition 5.16 (Prequantization)**.** Prequantization is the map

$$Q : C^\infty(M) \to Hom(\Gamma(\overline{E}), \Gamma(\overline{E})), f \mapsto if + \nabla_{X_f}$$

Because $\mathfrak{X}$ action preserves the metric, it's lie algebra would act via skew Hermitian matrices. The skew hermitian property would extend to space $L^2(\Gamma(E))$.

5.3. **Examples.** Whenever we have the situation that $\omega = d\alpha$ (note that this forces M to be non-compact) we can construct a canonical prequantum line bundle over M as follows,

Let $\mathcal{L} = M \times S^1$ and let $\nabla = idi + i\alpha$ where idi is the trivial connection. Then $F_\nabla = d\nabla = id\alpha = i\omega$.

This allows to construct prequantum line bundles over T^*M and $\mathbb{C}^m$.

Another example is $\mathbb{CP}^m$, follows by looking at the cohomology ring.

Remark 5.17. There is a partly resolved conjecture by Guillemin and Sternberg which says that under suitable conditions symplectic reduction 'commutes' with prequantization. The above example of $\mathbb{CP}^m$ is an example of it.

6. Polarization

This section contains a lot of definitions and barely any results. The point of this section is to link prequantization with physics.

Proposition 6.1. *The space of Lagrangians in* $(\mathbb{C}^m, im\langle,\rangle)$ *is homeomorphic to* $U(n)/O(n)$ *.*

Proof. Given a lagrangian L pick a basis $(e_1, \cdots, e_m)$. We can normalize this so that e_i forms a unitary basis and hence $U(m)$ acts transitively on the space of Lagrangians. The stabilizer of the standard basis is $Gl(m,\mathbb{R}) \cap U(m) \cong O(2m)$. □

Definition 6.2. On (M, ω) a Lagrangian distribution is a subbundle L of TM such that L_p is a Lagrangian subspace of T_pM . If $N \subset M$ is a submanifold such that TN is a Lagrangian distribution of TM then N is called a Lagrangian submanifold.

Definition 6.3. A real polarization on M is a Lagrangian distribution which is also integrable or equivalently involutive.

We can put a compatible almost complex structure on M and reduce the structure group to $U(m)$ so that TM is a map $TM : M \to BU(m)$. Then by the above proposition L is simply the splitting of TM as $M \xrightarrow{L} BO(m) \to BU(m)$.

Whenever we have a Lagrangian submanifold the restriction of the symplectic form is 0, hence if there were a prequantum line bundle the restriction of it to the submanifold will be flat.

We are interested in finding the flat sections of these restricted bundles, by flat sections we mean sections whose covariant derivative is 0.

Example 6.4. Given $\pi : T^*X \to X$, T^*X has a family of Lagrangian submanifolds given by $L_p = ker\pi_{*p}$ for each $p \in X$, follows directly from the construction of the symplectic form. Restricted the the fiber bundle the Liouville form α vanishes.

Example 6.5. $f : X \to \mathbb{R}$ then graph of df seen as a submanifold of T^*X is a Lagrangian submanifold. To see this, notice that locally the tangent space of $L_f = \{(x, df_x)\}$ is spanned by $e_i = \frac{\partial}{\partial q_i} + \sum_j \frac{\partial^2 f}{\partial q^i \partial q^j}\frac{\partial}{\partial p_j}$ where as before q_i are the base coordinates and p_i are the vertical coordinates. Then

$$\begin{aligned}\omega(e_k, e_l) &= \sum_i dp_i dq_i\Big(\frac{\partial}{\partial q_k} + \sum_j \frac{\partial^2 f}{\partial q^k \partial q^j}\frac{\partial}{\partial p_j}, \frac{\partial}{\partial q_l} + \sum_j \frac{\partial^2 f}{\partial q^l \partial q^j}\frac{\partial}{\partial p_j}\Big)\\ &= (dp_l dq_l + dp_k dq_k)\Big(\frac{\partial}{\partial q_k} + \frac{\partial^2 f}{\partial q^k \partial q^l}\frac{\partial}{\partial p_l}, \frac{\partial}{\partial q_l} + \frac{\partial^2 f}{\partial q^l \partial q^k}\frac{\partial}{\partial p_k}\Big)\\ &= 0\end{aligned}$$

Notice that $df : X \to L_f$ is an isomorphism, and hence we can pull back the form $\alpha|_{L_f}$ to X. Then $df^*(\alpha)_x(\frac{\partial}{\partial q_i}) = (\alpha)_{(x,df_x)}(e_i) = \frac{\partial f}{\partial x_i}$ and hence $df^*(\alpha) = df$. This is another way of seeing that the manifold is Lagrangian.

Assume now that we have the prequantum line bundle E,

Definition 6.6. Given a real polarization $D \subset TM$ a quantum state space is defined as

$$\Omega^0_D(E) = \{s \in \Gamma(E) \mid \nabla_X s = 0 \forall X \in \Gamma(D)\}$$

Again be cautious if we are dealing with E or $\overline{E}$ when doing concrete examples.

Example 6.7. In the example where $M = T^*X$ and $D = ker\pi_*$ and $\nabla = idi + i\alpha$. Because α dies when restricted to the vertical fiber, the covariantly flat sections are the ones which are constant along the vertical fiber

$$\Omega^0_D \cong C^\infty(M)$$

Note however that polarizations need not exist and even if they do the quantum state space might turn out to be trivial.

Example 6.8. Look at $\mathbb{R}^2$ and this time consider the foliation of $\mathbb{R}^2$ by circles $r = constant$. This produces a Lagrangian distribution which has a trivial Quantum state space.

Definition 6.9. Given a polarization D , a function f is called polarization preserving if $[\omega^{-1}(X_f), X] \in Gamma(D)$ for all $X \in \Gamma(D)$. Denote the space of polarization preserving functions by $C^\infty_D(M)$.

Proposition 6.10. *$C^\infty_D(M)$ is closed under Poisson brackets.*

Proof. This is simply because as noted earlier $Lie((X)) \cong C^infty(M)$ and the lie bracket is precisely the Poisson bracket.

So if $f, g \in C^\infty_D(M)$ and $X \in D$ then ,

$$\begin{aligned}
[X, X_{f,g}] &= [X, [X_f, X_g]] \\
&= [[X, X_f], X_g] + [X_f, [X, X_g]] \\
&= 0
\end{aligned}$$

□

Proposition 6.11. *If $f \in C^\infty_D(M)$ then Q_f preserves quantum state space Ω^0_D .*

Proof. Let $s \in \Gamma(D)$ and $X \in \Gamma(D)$ then we have the following conditions,

$$\nabla_X s = 0, [X_f, X] = 0$$

Then we get,

$$\begin{aligned}
\nabla_X(Q_f s) &= i\nabla_X(fs) + \nabla_X \nabla_{X_f} s \\
&= iX(f)s + if\nabla_X s + \nabla_X \nabla_{X_f} s \\
&= -i\omega(X, X_f)s + F_\nabla(X, X_f) \\
&= 0
\end{aligned}$$

□

Example 6.12. On $\mathbb{R}^2 = \mathbb{R}\langle q, p\rangle$ the symplectic form is $\omega = d\alpha = d(pdq)$. And the prequantum bundle is trivial, so that sections are simply $C^\infty(\mathbb{R}^2, \mathbb{C})$ and the connection is

$$\nabla_X f = X(f) + i\alpha(X)f$$

Then the function q is mapped to the operator $\frac{\partial}{\partial p} + iq$ and the function p is mapped to $-\frac{\partial}{\partial q}$.

And if one finds the commutator $[Q_q, Q_p] = [\frac{\partial}{\partial p} + iq, \frac{\partial}{\partial q}] = i$. So these operators do indeed satisfy the commutator relations that quantized position and momentum operators satisfy.

Identify $\mathbb{R}^2$ with $T^*\mathbb{R}$ and let $D = ker\pi*$. Then the set $C^\infty_D(\mathbb{R})$ is precisely the set $C^\infty(\mathbb{R})$ and the state space is again $C^\infty(\mathbb{R})$ both functions of q alone.

6.1. **Some philosophy.** Let us try to summarize what we have. We start with a symplectic manifold which is supposed to be the classical state space. The operators of the classical world are the world are the Hamiltonian vector fields which act on the space on $C^\infty M$ and the have a poisson bracket on them.

Now from these classical operators we want an algorithm to construct a quantum state space and the corresponding operators. The first approach is to consider the state space to be the space of functions C^∞ and try to find operators on this. But being topologists we twist this space to instead look at a prequantum line bundle $(M, \omega, \mathcal{L}, \nabla)$ and the state space is then the space of it's sections.

But there is an issue. It does not agree with the classical situation even in the simple case of $M = \mathbb{R}^2$. And the reason it does not is that we think of position x and momentum p as basis vectors for the same state space and hence there is a relationship between these two set of operators. To remedy this we try to get rid of 'half' of the state space by looking at polarization D and considering covariantly flat sections on it, this then is our state space Ω^0_D and the space of operators is $C^\infty_D(M)$. The Poisson bracket lifts appropriately and we get the commutator relations.

There is a catch tough. It only works in the most trivial of the cases. So we need a novel way of finding polarizations such that the corresponding state space and operator space has enough information.

7. Bits and Pieces

7.1. **Symplectic category.** It is tempting to define a morphism two symplectic manifolds to be a smooth map that preserves the symplectic form. For some reason this view is not very useful. My guess is because this does not take Lagrangians to Lagrangians. Also this is not how things arise in Physics as there our Symplectic manifolds are closely related to the cobordism category.

So instead we consider the following category, whose objects are symplectic manifolds and whose morphisms instead are

$$Mor(M, N) = \{L \subset \overline{M} \times N \mid L \text{ is a Lagrangian submanifold }\}$$

where by $\overline{M}$ we mean the Lagrangian submanifold with the sign of the symplectic form flipped.

Proposition 7.1. *If $f : M \to N$ is a symplecto then graph of f, L_f is a Lagrangian submanifold of $\overline{M} \times N$.*

Proof. If ω is the symplectic form on N then $f^*\omega$ is the one on M. $T(L_f)$ comprises of tangent vectors of the form (v, f_*v). Then $(-f^*\omega, \omega)((v, f_*v), (w, f_*w)) = -f^*\omega(v, w) + \omega(f_*v, f_*w) = 0$. □

We need to define composition of two morphisms. Suppose L_1, L_2 are a Lagrangian submanifolds of $\overline{M_1} \times M_2$ and $\overline{M_2} \times M_3$ respectively. Let $\pi_1(L_1)$ and $\pi_2(L_2)$ be the projections to M_2.

Proposition 7.2. *If $\pi_1(L_1)$ and $\pi_2(L_2)$ are submanifolds of M_2 and if they intersect transversally then $L_1 \times_N L_2$ embeds inside $\overline{M_1} \times M_2$ as a Lagrangian submanifold.*

The proof is quite straightforward. This allows us to define $L_1 \circ L_2 := L_1 \times_N L_2$. But this works only in a few special cases.

So we do not get a category, and there does not seem to be a way of extending this to a full category without using some stable or derived structure. I am not sure if this question has been resolved completely.

7.2. **Heisenberg group.** The S^1 extension of $S^1 \to \mathfrak{X} \to Symp(M, \omega)$ allows us to construct S^1 extensions of various subgroups of $Sypm(M)$.

Consider a symplectic vector space (V, ω) as a symplectic manifold. Every element of V defines a symplecto

$$V \to Symp(V), v \mapsto (x \mapsto x + v)$$

and we can look at the S^1 extension of V , $\overline{V} \subset \mathfrak{X}$. The universal cover of $\mathfrak{X}$ is called the Heisenberg group, $H(V)$. The corresponding Lie algebras would fit in a short exact sequence,

$$\mathbb{R}^1 \to \mathfrak{h}(V) \to V$$

Now the function whose Hamiltonian vector field is v at each point is precisely $x \mapsto \omega(v, x) + c$. Then it is easy to see that the poisson bracket is $\{\omega(v, x) + c, \omega(w, x) + d\} = \omega(v, w)$ (one can use symplectic coordinates over each tangent space to see this).

And so the Lie algebra structure on $\mathfrak{h}(V)$ is $[(v, c), (w, d)] = \omega(v, w)$. Choose a symplectic basis for V say $(e_1, \cdots, e_m, f_1, \cdots, f_m)$ and let $v = (v_e, v_f)$ in this basis, where v_e and v_f are both m vectors. Then it is easy to see that the map

$$\mathfrak{h}(V) \to \mathfrak{gl}(m+2), (v, c) \to \begin{bmatrix} 0 & v_e & c \\ 0 & 0_m & v_f \\ 0 & 0 & 0 \end{bmatrix}$$

is a map of Lie algebras.

This allows us to write $H(V)$ as a subgroup of $GL(m+2)$ consisting of elements of the form $\begin{bmatrix} 1 & \alpha & c \\ 0 & 1_m & \beta \\ 0 & 0 & 1 \end{bmatrix}$.

$\mathfrak{h}(V)$ has generators $(0, 1)$, $(e_i, 0)$ and $(f_i, 0)$ .The universal enveloping algebra would then be the free algebra over generators $1, x_i, y_i$ satisfying the relations $x_i y_j - x_j y_i = \delta^i_j$. But this is precisely the Weyl algebra.

Look at the canonical prequantum bundle $(\mathcal{L}, \nabla) \to \mathbb{C}^m$ discussed earlier. We have the inclusion of groups $U(m) \subset Sp(2m, \mathbb{R}) \subset Symp(\mathbb{C}^m)$. We can restrict the S^1 extensions to each of these subgroups to get a sequence of groups

$$\tilde{U}(m) \subset Mp(2m, \mathbb{R}) \subset \mathfrak{X}$$

Definition 7.3. $Mp(2m, \mathbb{R})$ is called the metaplectic extension of $Sp(2n, \mathbb{R})$.

7.3. **Things to work on.** I need to figure out how geometric quantization allows us to quantize field theories like Yang Mills and Chern Simons. Also how the symplectic category is related to these field theories. Another important topic is the symplectic action on Toric varieties.