AN APPLICATION OF H-PRINCIPLE TO MANIFOLD CALCULUS

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ABSTRACT. Manifold calculus is a form of functor calculus that analyzes contravariant functors from some categories of manifolds to topological spaces by providing *analytic approximations* to them. In this paper we apply the theory of h-principle to construct several examples of *analytic functors* in this sense. We prove that the analytic approximation of the Lagrangian embeddings functor $\mathrm{emb}_{\mathrm{Lag}}(-, N)$ is the totally real embeddings functor $\mathrm{emb}_{\mathrm{TR}}(-, N)$. Under certain conditions we provide a geometric construction for the homotopy fiber of $\mathrm{emb}(M, N) \to \mathrm{imm}(M, N)$. This construction also provides an example of a functor which is itself empty when evaluated on most manifolds but it's analytic approximation is almost always non-empty.

1. INTRODUCTION

1.1. **Motivation.** This paper is an attempt to apply techniques from homotopy theory to symplectic geometry. The Nearby Lagrangian Conjecture (still open) due to Arnol'd has been a guiding question for several recent advances in symplectic geometry. The current state of art results about the Nearby Lagrangian Conjecture rely on a combination of homotopy theoretic and Floer theoretic techniques; see [Abo12], [Kra11], [AK16].

If N is a smooth compact manifold then T^*N is naturally a symplectic manifold. Let L be another smooth compact manifold of the same dimension as N. Assume that N and L are both simply connected. The following is a weaker homotopy theoretic version of Arnol'd's conjecture:

Nearby Lagrangian Conjecture. The space of Lagrangian embeddings of L in T^*N is contractible if L is diffeomorphic to N, is empty otherwise.

1.2. **Manifold calculus.** In this paper we apply the techniques of manifold calculus to study the space of Lagrangian embeddings. Manifold calculus was introduced by Goodwillie-Weiss in [Wei99] to study the embedding spaces of manifolds. The basic idea of manifold calculus is to try and recover the information about the embedding space by analyzing embeddings of discs inside the source manifold and gluing back the data.

For smooth manifolds M^m, N^n, Weiss creates a sequence of *polynomial approximations* to the embedding space

$$\mathrm{emb}(M, N) \to (T_1 \,\mathrm{emb}(M, N) \leftarrow T_2 \,\mathrm{emb}(M, N) \leftarrow T_3 \,\mathrm{emb}(M, N) \leftarrow \cdots)$$

Thinking of $\mathrm{emb}(-, N)$ as a functor on the space of m dimensional manifolds, the functor $T_k \,\mathrm{emb}(-, N)$ should then be thought of as the k^{th} degree polynomial approximation, in the same sense that the truncated Taylor series provides a polynomial approximation. It turns out that $T_1 \,\mathrm{emb}(M, N)$ is the space of immersions $\mathrm{imm}(M, N)$ and so this construction creates functors interpolating between immersions and embeddings using configuration data. This point of view was further advanced in [BW15] using ∞ categorical language.

The hope then is that we can approximate $\mathrm{emb}(-, N)$ arbitrarily well using the polynomial approximations. That this is indeed true (in the homotopical sense) is a deep theorem due to Goodwillie-Klein-Weiss; see [GW99], [GK08]. They show that when $\dim n - \dim m > 2$ there is a homotopy equivalence

$$\mathrm{emb}(M,N) \xrightarrow{\simeq} \mathrm{holim}_n\, T_n\, \mathrm{emb}(M,N) =: T_\infty\, \mathrm{emb}(M,N)$$

Borrowing the analogy from Taylor series we say that the functor $\mathrm{emb}(-,N)$ is *analytic* in this case. $T_\infty\, \mathrm{emb}(M,N)$, is called the *analytic approximation* of $\mathrm{emb}(M,N)$.

It is natural to ask whether manifold calculus can be used to analyze Lagrangian embeddings. There is an obvious shortcoming in that Weiss' manifold calculus analyzes the embedding spaces by gluing the embeddings data on discs and so the symplectic information is likely to be lost. In Theorem 5.3 we show that not all geometric information is lost when finding the analytic approximation of Lagrangian embeddings.

Theorem 1.1. *Let N be a symplectic manifold with a compatible almost complex structure, and let $n = 2m$. When $m > 2$ there is a natural homotopy equivalence*

$$\mathcal{T}_\infty\, \mathrm{emb}_{\mathrm{Lag}}(M,N) \simeq \mathrm{emb}_{\mathrm{TR}}(M,N)$$

where $\mathrm{emb}_{\mathrm{Lag}}(-,N)$ denotes the space of Lagrangian embeddings and $\mathrm{emb}_{\mathrm{TR}}(-,N)$ denotes the space of totally real embeddings.

This theorem can be used to study embedding spaces of totally real manifolds inside almost complex manifolds.

1.3. **H-principle.** The fundamental reason why Theorem 5.3 holds true is that totally real embeddings satisfy h-principle but Lagrangian embeddings do not. This was proved by Gromov using his technique of convex integration [Gro86]. He used it to show that S^3 can be embedded inside $\mathbb{C}^3$ as a totally real submanifold.

H-principle (homotopy principle) is a collection of techniques for finding approximate solutions to differential equations. When the equality in a differential equation is replaced by an inequality we get a differential relation, H-principles are most useful for finding solutions for these.

Let $\mathrm{Gr}(N)$ by the m-plane Grassmannian of N. Let $A \subseteq \mathrm{Gr}(N)$ be such that $A \to N$ is a fibration and A satisfies the h-principle for directed embeddings. An A-directed embedding is an embedding $e : M \hookrightarrow N$ such that the image of $\mathrm{Gr}(e)$ is in A. In Theorem 4.6 we prove a strong connection between h-principle and manifold calculus.

Theorem 1.2. *For $\dim N - \dim M > 2$ the natural map*

$$\mathrm{emb}_A(M,N) \xrightarrow{\simeq} \mathcal{T}_\infty\, \mathrm{emb}_A(M,N)$$

is a homotopy equivalence, where $\mathrm{emb}_A(-,N)$ denotes the space of A-directed embeddings.

We can think of this statement as saying that analytic approximation is compatible with h-principle.

1.4. **Failure of analyticity.** Somewhat dual to the fact that there are non-trivial functions which have trivial Taylor series, we can use h-principle to create functors which are almost always trivial but their analytic approximation is never so.

For a parallelizable manifold M, when $n > 2\dim M$ we consider the space of embeddings of M inside $\mathbb{R}^n$ such that m chosen linearly independent vector fields on M are sent to constant non-varying vector fields on $\mathbb{R}^n$. If M is not a subspace of $\mathbb{R}^m$ no such embedding is possible. However in 6.2 we show that

Theorem 1.3. *When $n-m>2$, the analytic approximation of the* tangentially straightened *embeddings of M inside $\mathbb{R}^n$ is homotopy equivalent to the homotopy fiber of the map* $\operatorname{emb}(M,\mathbb{R}^n)\to\operatorname{imm}(M,\mathbb{R}^n)$

$$\mathcal{T}_\infty \operatorname{emb}_{\mathrm{TS}}(M,\mathbb{R}^n)\simeq \operatorname{hofib}(\operatorname{emb}(M,\mathbb{R}^n)\to\operatorname{imm}(M,\mathbb{R}^n))$$

We borrow the term tangentially straightened from [DH12]. By the weak Whitney embedding theorem $\operatorname{emb}(M,\mathbb{R}^n)$ is non-empty, so unless the space of embeddings retracts onto the space of immersions we'll get a non-trivial homotopy fiber.

This homotopy fiber usually denoted $\overline{\operatorname{emb}}(M,\mathbb{R}^n)$. The above statement is also true when $M=S^1$ as was observed by Sinha in [Sin06], in fact in this case $\overline{\operatorname{emb}}(S^1,\mathbb{R}^n)\simeq\Omega\operatorname{emb}(S^1,\mathbb{R}^n)\times\operatorname{imm}(M,\mathbb{R}^n)$. He then goes on to relate the invariants of the polynomial approximations of this space to finite type knot invariants. Since then this space has been extensively studied, see [MV15, Ch.10].

Outline: In Sections 2 and 3 we provide the necessary background from manifold calculus and h-principle and set up the notations for the rest of the paper. In Section 4 we prove the main theorems of this paper. In Section 5 we describe the main applications of our framework to Lagrangian embeddings. In Section 6 we apply the results of the previous sections to parallelizable manifolds and tangentially straightened embeddings.

Conventions. Throughout this paper we'll assume that M and N are smooth manifolds without boundary of dimensions m and n respectively. For us, every manifold is either closed or the interior of a compact manifold with boundary. Maps between manifolds are all smooth and the mapping spaces are endowed with the weak C^∞ topology. We'll use the terms space and topological space interchangeably. All categories are enriched over spaces and likewise for all the categorical constructions.

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2. Manifold Calculus

In this section we'll recall the basic definitions of manifold calculus defined first in [Wei99]. The theory was reformulated using model category theoretic language in [BW12], we'll describe the definitions and constructions from this. Other easily accessible references for manifold calculus include [MV15, Ch.10], [Wei99], [Mun10], [SV17].

Define the category $\mathcal{M}\mathrm{an}$ as

$$Ob(\mathcal{M}\mathrm{an}):=\{m \text{ dimensional manifolds}\}$$
$$\operatorname{Hom}_{\mathcal{M}\mathrm{an}}(U,V):=\operatorname{emb}(U,V)$$

where $\operatorname{emb}(U,V)$ is the space of embeddings $U\hookrightarrow V$ topologized under the weak C^∞ topology. $\mathcal{M}\mathrm{an}$ has a full subcategory $\mathcal{D}\mathrm{isc}_\infty$ consisting of manifolds diffeomorphic to a disjoint union of finitely many open unit discs.

We wish to apply manifold calculus to manifolds with tangential structures. Further there are variants of h-principle (Section 3.4) which are only applicable to open manifolds. For these reasons we need to restrict to certain subcategories of $\mathcal{M}\mathrm{an}$.

A **manifold category** $\mathcal{M}$ is a full subcategory of $\mathcal{M}\mathrm{an}$ containing the subcategory $\mathcal{D}\mathrm{isc}_\infty$ and is closed under taking open submanifolds (if $M\in\mathcal{M}$ and $M'\subseteq M$ then $M'\in M$).

Example 2.1. The manifold categories we have in mind are the following.

(1) The entire category $\mathcal{M}$an

(2) The category of **open** manifolds. Recall that a manifold M is said to be open if every connected component of M is non-compact.

(3) The category of orientable manifolds.

(4) The category of parallelizable manifolds.

(5) More generally let G be a subgroup $GL_m(\mathbb{R})$. Manifolds whose structure group can be reduced to G form a category of manifolds with G **structure**.

(6) Intersections of any of the above categories.

A morphism $e : M \to M'$ in $\mathcal{M}$ is an **isotopy equivalence** if there exists an embedding $e' : M' \to M$ in $\mathcal{M}$ such that ee' and $e'e$ are isotopic through maps in $\mathcal{M}$ to $id_{M'}$ and id_M, respectively.

For a manifold category $\mathcal{M}$ define the **presheaf category** $\mathrm{PSh}(\mathcal{M})$ to be the functor category whose objects are functors

$$\mathcal{M}^{op} \to \mathcal{T}op$$

which take isotopy equivalences to homotopy equivalences. Manifold calculus studies functors in $\mathrm{PSh}(\mathcal{M})$.

The category $\mathrm{PSh}(\mathcal{M})$ has a natural projective model structure (see [Hir09]) induced by the model structure on $\mathcal{T}op$: the fibrant objects are the presheaves which are object-wise fibrant, the weak equivalences are object-wise weak equivalences, and the cofibrations are presheaves which satisfy the right lifting property with respect to trivial fibrations.

Definition 2.2. For $F : \mathcal{M}^{op} \to \mathcal{T}op$ we define it's **analytic approximation**

$$\mathcal{T}_\infty F : \mathcal{M}^{op} \to \mathcal{T}op$$

to be the right derived Kan extension of $F|_{\mathcal{D}\mathrm{isc}_\infty}$ along the inclusion $\mathcal{D}\mathrm{isc}_\infty \hookrightarrow \mathcal{M}$. More explicitly for $M \in \mathcal{M}$,

$$\mathcal{T}_\infty F(M) := \mathrm{Hom}_{\mathrm{PSh}(\mathcal{D}\mathrm{isc}_\infty)}(Q\,\mathrm{emb}(-, M), F)$$

$$\begin{tikzcd}
\mathcal{D}\mathrm{isc}_\infty \arrow[r, "F|_{\mathcal{D}\mathrm{isc}_\infty}"] \arrow[d] & \mathcal{T}op \\
\mathcal{M} \arrow[ur, dashed, "\mathcal{T}_\infty F"'] &
\end{tikzcd}$$

where $Q\,\mathrm{emb}(-, M)$ denotes the cofibrant replacement of $\mathrm{emb}(-, M)$ in $\mathrm{PSh}(\mathcal{M})$.

Even though we needed the category $\mathcal{M}$ to define the analytic approximation using the model category framework, for evaluating the analytic approximation at a manifold M the ambient category is not necessary. In the original definition of analytic approximation [Wei99] analytic approximation is defined as

$$T_\infty F(M) = \underset{U \in \mathcal{D}\mathrm{isc}_\infty(M)}{\mathrm{holim}} F(U) \tag{2.3}$$

where $\mathcal{D}\mathrm{isc}_\infty(M)$ is the category of submanifolds of M which are diffeomorphic to finitely many open unit discs. The two definitions agree whenever they make sense [BW12, Section 8].

Definition 2.4. We say that a functor $F \in \mathrm{PSh}(\mathcal{M})$ is **analytic** if the natural map

$$F \xrightarrow{\simeq} \mathcal{T}_\infty F$$

is a homotopy equivalence.

Example 2.5. Examples of analytic functors:

(1) By the formal properties of Kan extensions it follows that
$$\mathcal{T}_\infty \mathcal{T}_\infty F = \mathcal{T}_\infty F$$
for any functor $F \in \mathrm{PSh}(\mathcal{M})$ and hence $\mathcal{T}_\infty F$ is always analytic, which justifies the name *analytic approximation*.

(2) [Wei99, Example 2.4] For a smooth manifold N the functor $\mathrm{Maps}(-, N)$ is analytic for all manifold categories $\mathcal{M}$, where $\mathrm{Maps}(M, N)$ is the space of smooth maps.

(3) [Wei99, Example 2.3] Let $\mathrm{imm}(M, N)$ denote the space of immersions of M into N. If $n > m$ then the functor $\mathrm{imm}(-, N)$ is analytic for all manifold categories. If $n = m$ then the functor $\mathrm{imm}(-, N)$ is analytic when restricted to the category of open manifolds.

As mentioned in the introduction manifold calculus was introduced to study the space of embeddings. One of the deepest theorems in manifold calculus states the following [GW99], [GK08].

Theorem 2.6 (Goodwillie-Klein,Goodwillie-Weiss)**.** *For a smooth manifold N, if $\dim N - m > 2$ the functor $\mathrm{emb}(-, N) \in \mathrm{PSh}(\mathcal{M}\mathrm{an})$ is analytic.*

The main theorem in this paper extends this result to directed embeddings.

3. H-principle

In this section we'll recall the basic theory of h-principle as described in [Gro86], [EM01], [Ada93]. We'll restrict to the case of immersions and embeddings instead of arbitrary differential relations. We use some non-standard terminology to keep the exposition lucid.

3.1. Topological h-principle.

Lemma 3.1. *An inclusion $X \subseteq Y$ of topological spaces is a weak homotopy equivalence, if for all good pairs of finite CW complexes $K \subseteq L$ and all maps*
$$\phi : (K, L) \to (X, Y)$$
there exists a homotopy which is constant on K
$$\phi_t : (K, L) \times [0, 1]_t \to (X, Y)$$
*such that $\phi_0 = \phi$ and $\phi_1(L) \subseteq X$. In this case we say that the pair (X, Y) satisfies the **topological h-principle**.*

$$\begin{tikzcd} K \arrow[r, "\phi|_K"] \arrow[d] & X \arrow[d] \\ L \arrow[r, "\phi"] & Y \end{tikzcd} \rightsquigarrow \begin{tikzcd} K \times [0,1] \arrow[r, "\phi_t|_K = \phi|_K"] \arrow[d] & X \arrow[d] \\ L \times [0,1] \arrow[r, dashed, "\phi_t"] & Y \end{tikzcd} + \begin{tikzcd} K \arrow[r, "\phi_1|_K = \phi|_K"] \arrow[d] & X \\ L \arrow[ur, dashed, "\phi_1"] & \end{tikzcd}$$

Proof. To prove the surjectivity of $\pi_k(X) \to \pi_k(Y)$ we let $K = *$ and $L = S^k$, to prove injectivity we let $K = S^k \vee [0, 1] \vee S^k$ and $L = S^k \times [0, 1]$. □

When studying h-principles X is usually the space of genuine solutions and Y is the space of formal solutions. By Whitehead theorem for mapping spaces (see [Pal66] and [Eel66]) a weak homotopy equivalence then implies the usual homotopy equivalence, we'll use the two terms interchangeably.

3.2. **H-principle for immersions.** For $G \subseteq GL_m(\mathbb{R})$ let $\mathcal{M}$ be a manifold category containing manifolds of dimension m with G structure i.e. manifolds whose structure group can be reduced to G.

For a manifold N^n let $F_m(N)$ denote the m frame bundle of N. There is a free action of $GL_m(\mathbb{R})$, and hence of G, on $F(N)$. Denote the quotient space by

$$\mathrm{Gr}(N) = F_m(N)/G$$

This is nothing but the **m-plane Grassmannian** bundle with G structure. When $G = GL_m(\mathbb{R})$ this is the standard Grassmannian, when $G = GL_m^+(\mathbb{R})$ this is the oriented Grassmannian, when G is the trivial group we get the entire m frame bundle $F_m(M)$. Note that $\mathrm{Gr}(M) \cong M$ for $M \in \mathcal{M}$.

Let $\mathrm{bs} : \mathrm{Gr}(N) \to N$ (stands for base) denote the natural projection. From this section on we'll let

$$A \subseteq \mathrm{Gr}(N)$$

denote a subset of $\mathrm{Gr}(N)$.

Definition 3.2. For a manifold $M \in \mathcal{M}$ an immersion $i : M \looparrowright N$ naturally lifts to a map $\mathrm{Gr}(i) : M \to \mathrm{Gr}(N)$. We say that i is **A-directed** if the image of $\mathrm{Gr}(i)$ is in A. Denote by $\mathrm{emb}_A(M, N)$ and $\mathrm{imm}_A(M, N)$ the space of A-directed embeddings and immersions respectively.

We have a natural inclusion $\mathrm{imm}_A(M, N) \hookrightarrow \mathrm{Maps}(M, A)$ sending an A-directed immersion i to $\mathrm{Gr}(i)$.

Definition 3.3. We say that A satisfies the **h-principle for directed immersions** for manifolds in $\mathcal{M}$ if the pair $(\mathrm{imm}_A(M, N), \mathrm{Maps}(M, A))$ satisfies the topological h-principle for all $M \in \mathcal{M}$, and hence the map

$$\mathrm{imm}_A(M, N) \to \mathrm{Maps}(M, A)$$

is a weak homotopy equivalence.

3.3. **H-principle for embeddings.** The space of embeddings is not dense in the space of all smooth maps $\mathrm{Maps}(M, \mathrm{Gr}(N))$, it is however still an open subset. Hence in the embeddings version of the h-principle we need to start with a map which is itself an embedding. For this we define the space of tangential homotopies, which is the space of paths in $\mathrm{Maps}(M, \mathrm{Gr}(N))$ lying over an embedding, which connect the embedding to a point in $\mathrm{Maps}(M, A)$.

Definition 3.4. The space of **tangential homotopies** $\mathrm{Tan_A}(M, N)$ is the space of paths

$$\gamma_t : [0, 1]_t \to \mathrm{Maps}(M, \mathrm{Gr}(N))$$

satisfying

$$\begin{aligned} \mathrm{bs}\,\gamma_0 &\in \mathrm{emb}(M, N) \\ \mathrm{bs}\,\gamma_t &= \mathrm{bs}\,\gamma_0 \\ \gamma_1 &\in \mathrm{Maps}(M, A) \end{aligned}$$

topologized as a subspace of $\mathrm{Maps}([0, 1] \times M, \mathrm{Gr}(N))$.

There is a natural inclusion $\mathrm{emb}_A(M, N) \hookrightarrow \mathrm{Tan_A}(M, N)$ sending an A-directed embedding $e : M \to N$ to the constant path at $\mathrm{Gr}(e)$.

Definition 3.5. We say that A satisfies the **h-principle for directed embeddings** for manifolds in $\mathcal{M}$ if the pair $(\mathrm{emb}_A(M, N), \mathrm{Tan_A}(M, A))$ satisfies the topological h-principle for all $M \in \mathcal{M}$, and hence the map

$$\mathrm{emb}_A(M, N) \to \mathrm{Tan_A}(M, N)$$

is a weak homotopy equivalence.

The essential properties that allow us to extend an h-principle for directed immersions to an h-principle for directed embeddings are the following.

(1) (C^0-closeness) The genuine immersion $i : M \looparrowright N$ is C^0-close to the original formal solution $M \to A \xrightarrow{\mathrm{bs}} N$.

(2) (Relative stability) Let M be a submanifold of N with a tubular neighborhood ν. Consider the relation defined as $A_\nu = A \cap R_\nu$ where R_ν is the subset of $\mathrm{Gr}(N)$ defined by the sections of the fiber bundle $\nu \to M$. Then A_ν satisfies the h-principle for directed immersions for all such M, ν.

If A satisfies these conditions then by the h-principle for directed immersions we can perturb an embedding M to get an A-directed immersions which is locally the section of the tubular neighborhood of M. But an immersion which is sufficiently close to an embedding is itself an embedding thereby giving us an A-directed embedding; see [Hir76, Pg.36]. Needless to say that this is an oversimplification and there are several technicalities one needs to deal with to make this work; see [EM01, Section 4.4, 19.4], [Gro86].

3.4. **Examples.** The two references mentioned at the beginning of this section provide an extensive collection of sets for which the h-principle holds. We'll only mention two examples here which are directly relevant to us.

When the manifolds in $\mathcal{M}$ are open we have the following general theorem [EM02, Section 4.4] proven using the technique of Holonomic Approximation.

Theorem 3.6 (Eliashberg-Mishachev)**.** *If the manifolds in $\mathcal{M}$ are open and $n > m$ then every open set $A \subseteq \mathrm{Gr}(N)$ satisfies the h-principle for directed embeddings.*

When the manifolds in $\mathcal{M}$ are not necessarily open, holonomic approximation can no longer be applied. In this case Gromov's technique of Convex Integration allows us to construct a class of sets which satisfy h-principle. One such example is the following.

A set $A \subseteq \mathrm{Gr}(N)$ defines a subset $\mathfrak{R}_A \subseteq J^1(M,N)$ of the 1-jet space of smooth maps from $M \to N$. We say that A is the **complement of a thin singularity** if the complement $J^1(M,N) \setminus \mathfrak{R}_A$ is a subcomplex of $J^1(M,N)$ of codimension $\geq \mathbf{2}$.

The following theorem is due to Gromov [Gro86]; see also [EM02, Theorems 18.4.2 and 19.4.1].

Theorem 3.7 (Gromov)**.** *If A is the complement of a thin singularity and $n > m$ then A satisfies the h-principle for directed embeddings for all manifolds.*

4. Main theorems

In this section we connect the two theories of manifold calculus and h-principle. The basic idea is that h-principle gives us various homotopy equivalences which fit into a homotopy pullback diagram. Since the analytic approximation is defined as a homotopy limit, the two commute.

From now on we'll restrict to the case when

$$\mathrm{bs} : A \to N$$

is a fibration. This allows us to deduce a stronger homotopy theoretic statement from the h-principle for directed embeddings. All interesting examples satisfy this condition.

We'll also assume that the space $\mathrm{emb}(M,N)$ is non-empty for all $M \in \mathcal{M}$. Note that the condition of $\mathrm{emb}(M,N)$ being non-empty is closed under taking submanifolds and hence can be enforced upon the entire manifold class $\mathcal{M}$.

We'll also assume that all h-principles are h-principles for directed embeddings for manifolds in $\mathcal{M}$, unless otherwise stated.

Theorem 4.1. *If $A \subseteq \mathrm{Gr}(N)$ satisfies h-principle and $A \to N$ is a fibration, then for every $M \in \mathcal{M}$ the following square is a homotopy pullback square.*

$$\begin{tikzcd} \mathrm{emb}_A(M,N) \arrow[r] \arrow[d] & \mathrm{Maps}(M,A) \arrow[d] \\ \mathrm{emb}(M,N) \arrow[r] & \mathrm{Maps}(M,\mathrm{Gr}(N)) \end{tikzcd} \tag{4.2}$$

where the horizontal maps are defined as $f \mapsto \mathrm{Gr}(f)$.

Proof. Let Ξ denote the homotopy pullback of the diagram

$$\begin{tikzcd} & \mathrm{Maps}(M,A) \arrow[d] \\ \mathrm{emb}(M,N) \arrow[r] & \mathrm{Maps}(M,\mathrm{Gr}(N)) \end{tikzcd}$$

We'll identify Ξ with the space of paths

$$\Xi = \{\gamma : [0,1] \to \mathrm{Maps}(M,\mathrm{Gr}(N)) : \gamma(0) \in \mathrm{Gr}(\mathrm{emb}(M,N)), \gamma(1) \in \mathrm{Maps}(M,A)\}$$

The map $\mathrm{emb}_A(M,N) \to \Xi$ sends e to the path which is constant at $\mathrm{Gr}(e)$. This map factors through the natural inclusion

$$\begin{tikzcd} & \mathrm{emb}_A(M,N) \arrow[dl, dashed] \arrow[d] \\ \mathrm{Tan}_{\mathrm{A}}(M,N) \arrow[r, hook] & \Xi \end{tikzcd}$$

It suffices to show that the pair $(\mathrm{Tan}_{\mathrm{A}}(M,N), \Xi)$ satisfies the topological h-principle. This is a direct consequence of the fact that $A \to N$ and hence also $\mathrm{Maps}(M,A) \to \mathrm{Maps}(M,N)$ is a fibration and we can pull the tangential data along the maps in the base $\mathrm{Maps}(M,N)$ to lie over a fixed embedding. The rest of the proof provides the relevant technical details to make this precise and can be skipped by the reader. It is a standard homotopy-lifting-property argument, however, as we're dealing with path spaces there is an extra dimension that we need to keep track of.

Suppose we are given a good pair of finite CW complexes (K,L) with a map

$$\phi : (K,L) \to (\mathrm{Tan}_{\mathrm{A}}(M,N), \Xi)$$

This is equivalent to a map

$$\phi_s : [0,1]_s \times L \to \mathrm{Maps}(M,\mathrm{Gr}(N))$$

with $\mathrm{bs}\,\phi_0 \in \mathrm{emb}(M,N)$, $\phi_1 \in \mathrm{Maps}(M,A)$ and $\mathrm{bs}\,\phi|_K$ is the constant path at an embedding. We need to construct a homotopy connecting this to a path lying entirely in $\mathrm{Tan}_{\mathrm{A}}(M,N)$ which is constant over K.

Define a map $\psi_{s,t} : [0,1]_s \times [0,1]_t \times L \to \mathrm{Maps}(M,\mathrm{Gr}(N))$ as

$$\psi_{s,t}|_K = \mathrm{bs}\,\phi_s|_K$$
$$\psi_{s,t} = \begin{cases} \mathrm{bs}\,\phi_s & \text{if } s \le 1-t \\ \mathrm{bs}\,\phi_{1-t} & \text{otherwise} \end{cases}$$

This is well defined as $\phi(K) \in \mathrm{Tan}_{\mathrm{A}}(M,N)$ and hence $\mathrm{bs}\,\phi_s|_K = \mathrm{bs}\,\phi_0|_K$.

Let

$$\begin{aligned} S_1 &= (\{1\}_s \times [0,1]_t \times K) \cup (\{1\}_s \times \{0\}_t \times L) \\ S_2 &= ([0,1]_s \times [0,1]_t \times K) \cup ([0,1]_s \times \{0\}_t \times L) \cup (\{0\}_s \times [0,1]_t \times L) \end{aligned}$$

Because (K, L) is a good pair, the inclusions

$$\begin{aligned} S_1 &\hookrightarrow \{1\}_s \times [0,1]_t \times L \\ S_2 &\hookrightarrow [0,1]_s \times [0,1]_t \times L \end{aligned}$$

are trivial cofibrations. Define $\Phi_{s,t}$ on S_2 as

$$\begin{aligned} \Phi_{s,t}|_K &= \phi_s|_K \\ \Phi_{s,0} &= \phi_s \\ \Phi_{0,t} &= \phi_0 \end{aligned}$$

As $A \to N$ and $\mathrm{Gr}(N) \to N$ are fibrations, the maps $\mathrm{bs} : \mathrm{Maps}(M, A) \to \mathrm{Maps}(M, N)$ and $\mathrm{bs} : \mathrm{Maps}(M, \mathrm{Gr}(N)) \to \mathrm{Maps}(M, N)$ are also fibrations. We can use the homotopy lifting property of these fibrations to extend $\Phi_{s,t}$ to the entire space $[0,1]_s \times [0,1]_t \times L$.

$$\begin{tikzcd} (S_1, S_2) \arrow[r, "\Phi_{s,t}"] \arrow[d] & (\mathrm{Maps}(M,A), \mathrm{Maps}(M, \mathrm{Gr}(N))) \arrow[d, "\mathrm{bs}"] \\ (\{1\}_s \times [0,1]_t \times L, [0,1]_s \times [0,1]_t \times L) \arrow[r, "\psi_{s,t}"] \arrow[ur, dashed] & \mathrm{Maps}(M, N) \end{tikzcd}$$

This extension of $\Phi_{s,t}$ is then the homotopy that witnesses the topological h-principle. ☐

If A satisfies the h-principle for directed embeddings then, more or less by definition, it also satisfies the h-principle for directed immersions. The following is an immediate corollary.

Corollary 4.3. *If $A \to N$ is a fibration that satisfies the h-principle, then the following square is a homotopy pullback square for all $M \in \mathcal{M}$.*

$$\begin{tikzcd} \mathrm{emb}_A(M,N) \arrow[r] \arrow[d] & \mathrm{imm}_A(M,N) \arrow[d] \\ \mathrm{emb}(M,N) \arrow[r] & \mathrm{imm}(M,N) \end{tikzcd} \tag{4.4}$$

With the above setup the analyticity of emb_A follows from formal properties of Kan extensions.

Lemma 4.5. *Given a small I shaped diagram of analytic functors $F : I \to \mathrm{PSh}(\mathcal{M})$, the homotopy limit $\mathrm{holim}_{i \in I} F_i$ is also analytic.*

Proof. For a diagram of analytic functors $F : I \to \mathrm{PSh}(\mathcal{M})$ we have,

$$\begin{aligned} (\mathcal{T}_\infty \operatorname*{holim}_I F_i)(M) &= \mathrm{Hom}_{\mathcal{D}\mathrm{isc}_\infty}(Q\,\mathrm{emb}(-, M), \operatorname*{holim}_I F_i) \\ &\simeq \operatorname*{holim}_I \mathrm{Hom}_{\mathcal{D}\mathrm{isc}_\infty}(Q\,\mathrm{emb}(-, M), F_i) \\ &= \operatorname*{holim}_I (\mathcal{T}_\infty F_i)(M) \\ &\simeq \operatorname*{holim}_I F_i(M) \end{aligned}$$

where the equalities are by the definition of $\mathcal{T}_\infty$ and the homotopy equivalences follow from the universal property of enriched holim and analyticity of F. ☐

Theorem 4.6. *Let $n - m > 2$.*

(1) *If $A \to N$ is a fibration that satisfies h-principle, then the functor* $\mathrm{emb}_A(-, N)$ *is analytic i.e. the natural map*

$$\mathrm{emb}_A(M, N) \xrightarrow{\simeq} \mathcal{T}_\infty \, \mathrm{emb}_A(M, N)$$

is a homotopy equivalence.

(2) *Suppose $A \subseteq B \subseteq \mathrm{Gr}(N)$ are both fibrations over N. If A is homotopy equivalent to B then the induced map on the analytic approximations are homotopy equivalences.*

$$\mathcal{T}_\infty \, \mathrm{emb}_A(M, N) \xrightarrow{\simeq} \mathcal{T}_\infty \, \mathrm{emb}_B(M, N)$$

Proof. As mentioned in Example 2.5 and Theorem 2.6 the three functors $\mathrm{Maps}(-, A)$, $\mathrm{Maps}(-, \mathrm{Gr}(N))$, and $\mathrm{emb}(-, N)$ are analytic. Further if $A \hookrightarrow B$ is a homotopy equivalence then so is the inclusion $\mathrm{Maps}(M, A) \hookrightarrow \mathrm{Maps}(M, B)$. The theorem then follows by applying Lemma 4.5 to the homotopy pullback diagram in Theorem 4.1. □

Remark 4.7. All the above statements and proofs remain true for finite approximations $\mathcal{T}_k$ instead of $\mathcal{T}_\infty$.

5. Lagrangian embeddings

In this section we apply the above framework to study the Lagrangian embeddings functor using manifold calculus. Let us start by recalling some definitions from symplectic geometry. A good reference for the basic notions of symplectic geometry is [Sil01]. For this section $\mathcal{M} = \mathcal{M}\mathrm{an}$, we need the target manifold to have a symplectic structure but we do not need any additional structure on the source manifold.

A manifold N along with a closed, non-degenerate 2 form $\omega \in \Omega^2_{DR}(N)$ is called a **symplectic manifold**. Existence of a symplectic form on N forces it to be even dimensional. An **almost complex structure** on N is a bundle map $J : TN \to TN$ which satisfies the property that $J^2 = -1$. This is equivalent to requiring that N has even dimensions and it's structure group can be reduced to $GL_{n/2}(\mathbb{C})$. On every symplectic manifold N there exists a compatible almost complex structure which is unique up to homotopy where compatibility simply means that J commutes with the symplectic form along with a certain positivity condition. Compatibility implies that every Lagrangian submanifold is also a totally real submanifold.

For the rest of this section we'll assume that (N, ω) is a symplectic manifold with a compatible almost complex structure J. There are several classes of sumbanifolds of symplectic manifolds which are of interest to us. Let M be a submanifold of N, M is called ...

(1) ... **isotropic** if $\omega|_M \equiv 0$.

(2) ... is called **Lagrangian** if it is isotropic and $\dim M = n/2$.

(3) ... is called **totally real** if $T_pN \cong T_pM \oplus JT_pM$ for every $p \in M$.

We define subsets of $\mathrm{Gr}(N)$ corresponding to these classes of submanifolds, Iso, Lag, and TR respectively, so that $\mathrm{emb}_{\mathrm{Iso}}(M, N)$, $\mathrm{emb}_{\mathrm{Lag}}(M, N)$, and $\mathrm{emb}_{\mathrm{TR}}(M, N)$ are the spaces of embeddings of M into N as isotropic, Lagrangian, and totally real submanifolds respectively.

Theorem 12.4.1 in [EM02] says that Iso satisfies the h-principle if $n > 2m$ and hence in this case a direct application of Theorem 4.6 gives us

Theorem 5.1. *If N is a symplectic manifold with $n - m > 2$ and $n > 2m$, then the functor* $\mathrm{emb}_{\mathrm{Iso}}(-, N)$ *is analytic.*

This is false when $n = 2m$ which is the case of Lagrangian embeddings. A theorem of [Lee76] and [Gro86] says that the Lagrangian *immersions* functor does in fact satisfy the h-principle for immersions, which then gives us

$$\mathcal{T}_1 \operatorname{emb}_{\mathrm{Lag}}(-, N) \simeq \operatorname{imm}_{\mathrm{Lag}}(-, N)$$

However this does not extend to embeddings.

One consequence of the compatibility of the almost complex structure and the symplectic structure is that every Lagrangian submanifold is also a totally real submanifold. Hence there is a natural inclusion $\mathrm{Lag} \subseteq \mathrm{TR}$.

Proposition 5.2. *When $n = 2m$ the inclusion $\mathrm{Lag} \hookrightarrow \mathrm{TR}$ is a homotopy equivalence.*

Proof. $GL_m(\mathbb{C})$ acts transitively on the space of all totally real m dimensional subspaces of $\mathbb{C}^m$. The stabilizer of each subspace is $GL_m(\mathbb{R})$ and hence the fiber of the fiber bundle $\mathrm{TR} \to N$ is diffeomorphic to $GL_m(\mathbb{C})/GL_m(\mathbb{R})$. There is a natural homotopy equivalence

$$GL_m(\mathbb{C})/GL_m(\mathbb{R}) \simeq U(m)/O(m)$$

The fiber of the bundle $\mathrm{Lag} \to N$, which is also called the Lagrangian Grassmannian, is known to be diffeomorphic to $U(m)/O(m)$; see for example [Arn67]. □

By Theorem 19.3.1 in [EM02] totally real embeddings satisfy the h-principle. This together with Theorems 5.2 and 4.6 give us the following result.

Theorem 5.3. *Let $n - m > 2$, $n = 2m$ and let N be a symplectic manifold with a compatible almost complex structure. Then the functor $\operatorname{emb}_{\mathrm{Lag}}(-, N)$ is not analytic. The analytic approximation of $\operatorname{emb}_{\mathrm{Lag}}(-, N)$ is $\operatorname{emb}_{\mathrm{TR}}(-, N)$*

$$\operatorname{emb}_{\mathrm{TR}}(M, N) \xrightarrow{\simeq} \mathcal{T}_\infty \operatorname{emb}_{\mathrm{TR}}(M, N) \longleftarrow \mathcal{T}_\infty \operatorname{emb}_{\mathrm{Lag}}(M, N)$$

Restricting to the case of almost complex manifolds we see that the following square is a homotopy pullback square.

$$\begin{tikzcd}
\operatorname{emb}_{\mathrm{TR}}(M, N) \arrow[r] \arrow[d] & \operatorname{Maps}(M, \mathrm{TR}) \arrow[d] \\
\operatorname{emb}(M, N) \arrow[r] & \operatorname{Maps}(M, \mathrm{Gr}(N))
\end{tikzcd}$$

$\pi_0(\operatorname{emb}_{\mathrm{TR}}(M, \mathbb{C}^m))$ was computed by Audin in [Aud88]. This pullback square was used in [Bor02] (without actually using manifold calculus) to compute $\pi_1(\operatorname{emb}_{\mathrm{TR}}(M, \mathbb{C}^m))$.

6. Tangentially straightened embeddings

Suppose now $\mathcal{M}$ is the category of parallelizable manifolds and $N = \mathbb{R}^n$. In this case $\mathrm{Gr}(N) \cong \mathbb{R}^n \times F_m(N)$. Let $\overline{\operatorname{emb}}(M, N)$ be the homotopy fiber of $\operatorname{emb}(M, N) \to \operatorname{imm}(M, N)$.

The space $\operatorname{imm}(M, N)$ is not necessarily connected and hence the homotopy fiber is not well-defined. The first step is to find conditions to rectify this.

$$\begin{aligned}
\operatorname{imm}(M, N) &= \operatorname{imm}(M, F_m(\mathbb{R}^n)) \\
&\simeq \operatorname{imm}(M, V_m(\mathbb{R}^n))
\end{aligned}$$

where $V_m(\mathbb{R}^n)$ is the Stiefel manifold consisting of orthogonal m frames in $\mathbb{R}^n$. The Stiefel manifolds fit into a fibration

$$V_{m-1}(\mathbb{R}^n) \to V_m(\mathbb{R}^n) \to S^{m-1} \tag{6.1}$$

and so the smallest non-trivial homotopy group of $V_m(\mathbb{R}^n)$ is in the dimension $n-m$. If $n-m>m$ the space $\mathrm{imm}(M,N)$ is connected, so we'll assume that

$$n > 2m$$

By the weak Whitney embedding theorem this also ensures that $\mathrm{emb}(M,N)$ is non-empty. Let $\overline{\mathrm{emb}}(M,N)$ be the homotopy fiber over any embedding $e \in \mathrm{emb}(M,N)$.

Let $\{\vec{e}_i\}_{i=1}^n$ be the standard basis for $\mathbb{R}^n$. And let $\mathrm{TS} = \mathbb{R}^n \times \{(\vec{e}_1, \cdots, \vec{e}_m)\} \subset F_m(\mathbb{R}^n)$. The space $\mathrm{emb}_{\mathrm{TS}}(M,N)$ can be thought of as the space of **tangentially straightened** embeddings of M in N. We borrow the terminology from [DH12] where it is used in a slightly different context. TS is too small a set to satisfy any kind of h-principle.

It is easy to see that there is a natural diffeomorphism

$$\mathrm{emb}_{\mathrm{TS}}(M,\mathbb{R}^n) \cong \mathrm{emb}(M,\mathbb{R}^m) \times \mathbb{R}^n$$

So unless M is an open subset of $\mathbb{R}^m$ the space $\mathrm{emb}_{\mathrm{TS}}(M,N)$ is empty. However it's analytic approximation is almost always non trivial.

Theorem 6.2. *With M, N,* TS *as above and $n-m>2$ then there is a natural homotopy equivalence*

$$\mathcal{T}_\infty \,\mathrm{emb}_{\mathrm{TS}}(M,\mathbb{R}^n) \simeq \overline{\mathrm{emb}}(M,\mathbb{R}^n)$$

Note that there is no map from $\mathcal{T}_\infty \,\mathrm{emb}_{\mathrm{TS}}(M,\mathbb{R}^n) \to \mathrm{emb}(M,\mathbb{R}^n)$, instead the homotopy equivalence is induced by maps

$$\mathcal{T}_\infty \,\mathrm{emb}_{\mathrm{TS}}(M,\mathbb{R}^n) \longrightarrow \mathcal{T}_\infty \,\mathrm{emb}(M,\mathbb{R}^n) \xleftarrow{\simeq} \mathrm{emb}(M,\mathbb{R}^n)$$

Proof. As TS is contractible, by Theorems 4.1 and 4.6 it suffices to find a space $A \subseteq \mathrm{Gr}(N)$ along with a map $\mathrm{TS} \to A$ which is a homotopy equivalence such that A satisfies the h-principle for parallelizable manifolds and $A \to N$ is a fibration.

This is easy to do when M is open. Let TS_ϵ be a small open ϵ ball around TS inside $F_m(\mathbb{R}^n)$. The space TS_ϵ is open by definition and hence satisfies the h-principle for directed embeddings by the Holonomic Approximation Theorem 3.6. For sufficiently small ϵ the inclusion $\mathrm{TS} \hookrightarrow \mathrm{TS}_\epsilon$ is a homotopy equivalence.

If M is not open we need to construct a subspace $A' \subset F_m(\mathbb{R}^m)$ such that the analogous differential relation $\mathfrak{R}_{A'} \subset J^1(M,N)$ is the complement of a thin singularity, we'll then be done by Theorem 3.7. We construct the space A' as follows.

As $F_m(\mathbb{R}^n) \simeq V_m(\mathbb{R}^n)$ we can think of TS as a subspace of $V_m(\mathbb{R}^n)$ instead, this will simply the arguments significantly. Define $A_m^n \subset V_m(\mathbb{R}^n)$ as the set of orthogonal of m-frames $(\vec{v}_1, \ldots, \vec{v}_m)$ satisfying the condition

$$v_i \neq -e_i$$

The largest cell of the complement of A_m^n in $V_m(\mathbb{R}^n)$ corresponds to the space of frames with exactly one $v_i = -e_i$, this has codimension $\geq n-(m-1)$ which is greater than 1. The same is true for the corresponding differential relation in $J^1(M,\mathbb{R}^n)$. To finish the proof we need the following lemma. ∎

Lemma 6.3. *The space A_m^n defined above is contractible. Hence the natural inclusion of $TS \hookrightarrow A_m^n$ is a homotopy equivalence.*

Proof. When $m = 1$, the diffeomorphism $A_1^n \cong S^{n-1} \setminus \{-\vec{e}_1\}$ implies A_1^n is contractible.

For $m > 1$, the fibration of Stiefel manifolds (6.1) restricts to these spaces to give a fibration sequence

$$A_{m-1}^{n-1} \to A_m^n \to (S^{n-1} \setminus \{-\vec{e}_1\})$$

The base is contractible and hence $A_{m-1}^{n-1} \simeq A_m^n$. We're then done by induction on m.

□

7. Final Remarks

7.1. **Manifolds with boundary.** There are variants of manifold calculus [BW12, Section 9] for manifolds with boundary. In this case we replace the category $\mathcal{M}\mathrm{an}$ with the category $\mathcal{M}\mathrm{an}_Z$ of manifolds with a **fixed** boundary manifold Z, we make similar modifications to the category $\mathcal{D}\mathrm{isc}_\infty$. Similarly there are variants of h-principle which are applicable to manifolds with boundary. The h-principles which are proven by convex integration stay true for manifolds with boundary while the ones proven by holonomic approximation do not (manifolds with boundary behave like compact manifolds).

As such the results in this paper about analyticity of totally real embeddings and the tangential straightened embeddings remain true, with appropriate modifications, for manifolds with boundary.

7.2. **Future directions.**

(1) In this paper we used h-principle to prove analyticity of directed embedding functors. There are results in manifold calculus about the existence of loop space structures on embedding spaces. It would be interesting to see whether these higher structures can be used to extract geometric information.

(2) Manifold calculus does not see symplectic geometry because it is constructed using $\mathcal{D}\mathrm{isc}_\infty$ and symplectic geometry is locally trivial. One possible remedy might be to replace discs by more structured objects which retain some geometric information, however what these structured objects should be is unclear.

(3) The dimension restriction $n - m > 0$ in h-principles is a simple requirement as it allows us to perturb the submanifold M inside N. The dimension restriction $n - m > 2$ in manifold calculus on the other hand comes from surgery theory and is much more intricate. It would be interesting to try to use h-principles to simply some of the proofs in [GK08] and possibly improve upon the dimension restriction.

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