

## PROBLEM SET 08

## PART 1 - FUNDAMENTAL THEOREM OF CALCULUS

**Q.1.** Compute  $F'(x)$  for the following functions\*

$$\begin{array}{ll}
 (1) \int_a^{x^3} \sin^3 t \, dt & (4) \int_a^x \left( \int_b^y \frac{1}{1+t^2+\sin^2 t} dt \right) dy \\
 (2) \int_x^b \frac{1}{1+t^2+\sin^2 t} dt & (5) \int_0^x \frac{1}{1+t^2} dt + \int_0^{1/x} \frac{1}{1+t^2} dt \\
 (3) \int_a^b \frac{x}{1+t^2+\sin^2 t} dt & (6) \int_{-\cos x}^{\sin x} \frac{1}{\sqrt{1-t^2}} dt
 \end{array}$$

**Q.2.** Find  $(f^{-1})'(0)$  if

$$\begin{array}{ll}
 (1) f(x) = \int_0^x 1 + \sin(\sin t) dt & (2) f(x) = \int_1^x \cos(\cos t) dt
 \end{array}$$

**Q.3.** Suppose  $f$  is differentiable with  $f(0) = 0$  and  $0 < f'(x) \leq 1$ . Show that for all  $x \geq 0$

$$\int_0^x f^3 \leq \left( \int_0^x f \right)^2$$

**Q.4.** (1) Find  $F'(x)$  if  $F(x) = \int_0^x x \cdot f(t) dt$  (Be careful: it's not  $x \cdot f(x)$ ).

(2) Prove that<sup>†</sup>

$$\int_0^x f(t)(x-t) dt = \int_0^x \left( \int_0^u f(t) dt \right) du$$

(3) Prove that

$$\int_0^x f(t)(x-t)^2 dt = 2 \int_0^x \left( \int_0^{u_1} \left( \int_0^{u_2} f(t) dt \right) du_2 \right) du_1$$

**Q.5.** (1) Suppose  $G'(x) = g(x)$  and  $F'(x) = f(x)$ . Prove that the function  $y(x)$  satisfies the differential equation (a separable differential equation)

$$g(y) \cdot y' = f(x)$$

for all  $x$  in some interval, if and only if there is a number  $c$  such that

$$G(y) = F(x) + c$$

(2) 'Solve' the following differential equations

$$\begin{array}{ll}
 \text{(a) } y' = \frac{1+x^2}{1+y} & \text{(b) } y' = \frac{-1}{1+5y^4}
 \end{array}$$

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\* Hint: Don't forget the chain rule!

† Hint: Differentiate both sides with respect to  $x$ .

## PART 2 - IMPROPER INTEGRALS

**Q.6.** The limit  $\lim_{x \rightarrow \infty} \int_a^x f$ , if it exists, is denoted by  $\int_a^\infty f$  and called an ‘improper integral’.

Similarly for the improper integral  $\int_{-\infty}^a f$ .

(1) Find  $\int_1^\infty t^r dt$ , when  $r < -1$ .

(2) Using  $\int_1^a t^{-1} dt + \int_1^b t^{-1} dt = \int_1^{ab} t^{-1} dt$ , show that  $\int_1^\infty t^{-1} dt$  does not exist. ‡

**Q.7.** Assume the following the statement:

Suppose that  $f(x) \geq 0$  for  $x \geq 0$  and that  $\int_a^\infty f$  exists. If  $0 \leq g(x) \leq f(x)$  for all  $x \geq 0$  and  $g$  is integrable on the interval  $[0, N]$  for all  $N > 0$  then  $\int_0^\infty g$  exists.

• (Optional) Prove the above statement.

• For which of the following functions does the integral  $\int_0^\infty f$  exist?§

(1)  $\frac{1}{1+x^2}$

(3)  $\frac{x}{1+x^{3/2}}$

(2)  $\frac{1}{\sqrt{1+x^3}}$

(4)  $\frac{1}{\sqrt{1+x}}$

**Q.8.** The improper integral  $\int_{-\infty}^\infty f$  is defined as  $\int_{-\infty}^0 f + \int_0^\infty f$  if both the integrals exist.

(1) Show that  $\int_{-\infty}^\infty \frac{1}{1+x^2} dx$  exists.

(2) Determine the limit  $\lim_{N \rightarrow \infty} \int_{-N}^N x dx$ .

(3) Show that  $\int_{-\infty}^\infty x dx$  does not exist.

**Q.9.** It is possible to have another kind of improper integral, one where the function itself is unbounded but the limits are finite.

For  $-1 < r < 0$  draw the graph of  $x^r$  and determine  $\lim_{\epsilon \rightarrow 0^+} \int_\epsilon^a x^r dx$ .

(This limit is written as  $\int_0^a x^r dx$ .)

‡ Hint: What can you say about  $\int_{-1}^1 t^{-1} dt$ ?

§ Hint: You might have to break  $[0, \infty)$  into multiple intervals and analyze each interval separately.