Local structures in groups and classifying spaces – Bob Oliver

p-local structure on a finite group G
- a Sylow subgroup of G
- conjugacy classes

G, H finite groups, $S \in Syl_p(G)$, $T \in Syl_p(H)$

Say $G \sim_p H$ same p-local structure if

$\exists S \xrightarrow[\cong]{\varphi} T$ such that

$\forall\ P \xhookrightarrow[\alpha]{} Q \qquad \varphi(P) \xhookrightarrow[\varphi_\alpha]{} \varphi(Q)$

then, $\alpha \in Iso_G(P, Q) \leftarrow$ conjugaty in G

$\Longleftrightarrow \varphi_\alpha \varphi^{-1} \in Iso(\varphi(P), \varphi(Q))$

Can show that the def^n is independant of choice of S, T.

X, Y have same p-local structure if "$X_p \cong Y_p$" (Bousfield Kan p-completion)

Note: X p-good if . . . ?

Martino Priddy Conjecture:

$\forall\, G, H$ finite groups $\quad BG^\wedge_p \cong BH^\wedge_p \Longleftrightarrow G \sim_p H$

(Not true for infinite groups, take G, H without p-torsion)

Application:

Th^m: Assume $\mathbb{G}$ is a connected reductive group scheme $/\mathbb{Z}$ (eg: Gl_n, Sl_n, Sp_{2n}, E_6, ...), p prime, q, q' prime powers, such that $p \nmid qq'$ then

$$\mathbb{G}(\mathbb{F}_q) \sim_p \mathbb{G}(\mathbb{F}_{q'}) \text{ if } \overline{\langle q \rangle} = \overline{\langle q' \rangle} \leq \mathbb{Z}_p^*$$

Completely group theoretic th^m but the only known proof uses classifying spaces.

Remark:

when $p=2$, $\overline{\langle q \rangle} = \overline{\langle q' \rangle} \Longleftrightarrow q \equiv q' \pmod 8$ and $v_2(q^2-1) = v_2(q'^2-1)$

Fusion Systems:

Encode p-local information in a group

$\rightarrow$ G finite group, $S \in Syl_p(G)$

$\mathcal{F}_S(G)$ category: $Ob = \{P \in S\}$

$$Mor(P, Q) = Hom_G(P, Q) = \{\varphi : P \longrightarrow Q \mid \varphi = \text{conjugation}\}$$

$S \leq G$, $T \leq H$, $G \sim_p H \iff \exists\, \varphi : S \xrightarrow{\cong} T$ which induces isomorphisms $\mathcal{F}_S(G) \xrightarrow{\cong} \mathcal{F}_T(H)$

Def^n: For a finite p-group S, a fusion system over S is a category where the objects of $\mathcal{F}$ are subgroups of S and $\forall P, Q$: $Mor_{\mathcal{F}}(P,Q) = Inj(P,Q)$ + axioms.

Rmk: For a finite group G, $BG \simeq \underset{G/P}{hocolim}\, BP$

Want to define a <u>classifying space</u> for an abstract fusion system.
$\underset{\overrightarrow{\mathcal{F}}}{hocolim}\, \mathcal{F}(-)$ does not work.

Given a fusion system $\mathcal{F}/S$, $P \leq S$ is $\mathcal{F}$ centric if $\forall P' \underset{\mathcal{F}}{\cong} P$, $\overset{\text{Centralizer}}{C_S(P')} \leq P'$

(eg: If all the sylow subgroups are D_8 then the $\mathcal{F}$ centric subgroups will be of order 4)

$\underset{\Theta}{\overset{\shortparallel}{\Theta(\mathcal{F}^c)}}$: $Ob(\Theta) = \left\{ \begin{array}{l} P \leq S \\ \mathcal{F}\text{-centric} \end{array} \right\}$

$$Mor_{\Theta}(P,Q) = Mor_{\mathcal{F}}(P,Q) / Inn(Q)$$

$$\begin{aligned} B : \Theta(\mathcal{F}) &\longrightarrow hoTop \\ P &\longmapsto BP \end{aligned}$$

Def^n: A classification space for $\mathcal{F}$ is of the form

$$B\mathcal{F} := \underset{\Theta(\mathcal{F}^c)}{hocolim}\, (\hat{B}) \quad \text{for any } \hat{B} : \Theta(\mathcal{F}^c) \longrightarrow \text{Topological rigidification of } B$$

Th^m: $\forall \mathcal{F}$, $\exists$ a classifying sp $B\mathcal{F}$, unique up to homotopy type.

Th^m: If $\mathcal{F} = \mathcal{F}_S(G)$ then $B\mathcal{F}^{\wedge}_p \simeq BG^{\wedge}_p$.

Homotopy properties of Classifying Spaces:

BG: $H^*(BG; \mathbb{F}_p) \cong \varprojlim H(-; \mathbb{F}_p)$

$\forall p$, group Q, $[BQ, BG^{\wedge}_p] \cong Hom(Q/G)/_{Inn(G)}$

$Out(BG^{\wedge}_p) \subseteq [BG^{\wedge}_p, BG^{\wedge}_p]$ described by automorphisms of its fusion

BF: $H^*(BF, \mathbb{F}_p) \simeq \underset{\overleftarrow{\mathcal{F}}}{\lim}\, H^*(B-, \mathbb{F}_p)$

$[BQ, B\mathcal{F}^{\wedge}_p) \cong Hom(Q, S)/\sim$

$Out(\mathcal{F}^{\wedge}_p) \cong$ approximately $Out(\mathcal{F})$

Def^n: A discrete p-toral group is an extension

$$1 \longrightarrow \underset{\leq T^r}{(\mathbb{Z}/p^{\infty})^r} \longrightarrow S \longrightarrow \begin{array}{l}\text{finite}\\ p\text{-group}\end{array} \longrightarrow 1$$

$$\left(\begin{array}{l} \forall\, 1 \to \underset{T_{\infty}}{\overset{\vee\!\shortparallel}{T}} \to \bar{S} \to \bar{S}/T \to 1 \\ 1 \to T/T_{\infty} \to \bar{S}/T_{\infty} \to \bar{S}/T \to 1 \end{array} \right)$$

eg: in $O(2)$: $p = 2$
$S = \langle 2\text{-power torsion in torus}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rangle$

$\forall$ compact Lie group G, $\exists$ max $S \leq G$, discrete p-toral, unique upto conjugation.

Can extend the theory to compact Lie groups. Q. $H^*(B\mathcal{F}, \mathbb{F}_p) = ??$