Linear Groups

Apurva Nakade

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Any mathematical object naturally defines a group $\mathrm{Aut}(X)$, the group of **automorphisms** or symmetries of X which are maps $X \to X$ preserving some structure on X. Our main examples for X will be vector spaces over $\mathbb{R}$ or $\mathbb{C}$ (with additional structures) in which case the automorphisms groups are called **linear groups**. We'll make the structures successively more rigid thereby specializing to smaller and smaller linear groups.

Symmetries of $\mathbb{R}^1$

Symmetries of $\mathbb{R}^1$ are transformations $f : \mathbb{R}^1 \to \mathbb{R}^1$ which preserve some structure. The simplest structure $\mathbb{R}^1$ has is that of a set. The symmetries of $\mathbb{R}^1$ thought of as a set are bijective maps $f : \mathbb{R}^1 \to \mathbb{R}^1$. This group is too big and lacks any interesting structure. $\mathbb{R}^1$ is also a topological space. Symmetries of the topological space $\mathbb{R}^1$ are continuous maps $f : \mathbb{R}^1 \to \mathbb{R}^1$ which are isomorphisms.

From an algebraic point of view $\mathbb{R}^1$ is a vector space over $\mathbb{R}$. Symmetries of the vector space $\mathbb{R}^1$ are linear transformations $f : \mathbb{R}^1 \to \mathbb{R}^1$ which are isomorphisms. A linear transformation f is simply multiplication by a scalar $c \in \mathbb{R}$ and such a transformation is an isomorphism if $c \neq 0$. So the set of symmetries of the vector space $\mathbb{R}^1$ is isomorphic to the group of non-zero reals $\mathbb{R}^\times$. This group is usually denoted by $GL_1(\mathbb{R})$.

Inside $\mathbb{R}^\times$ is the subgroup of positive reals $\mathbb{R}_{>0}$. These are the orientation preserving transformations of $\mathbb{R}^1$ (see Section 2). Another structure that $\mathbb{R}^1$ has is that of distance. Multiplication by a scalar c preserves distances iff $c = \pm 1$, and the set $\{-1, +1\}$ is isomorphic to $\mathbb{Z}/2$. Finally only the identity map preserves both the orientation and the inner product and so the corresponding automorphism group is trivial.

Structures on $\mathbb{R}^1$	Automorphism Groups
Set	Bijections
Topological space	Continuous bijections
Vector space	$\mathbb{R}^\times$
Orientation	$\mathbb{R}_{>0}$
Metric	$\mathbb{Z}/2$
Orientation + Metric	trivial group

1 General linear groups

Denote the set of $n \times n$ matrices with entries in $\mathbb{R}$ (resp. $\mathbb{C}$) by $M_{n\times n}(\mathbb{R})$ (resp. $M_{n\times n}(\mathbb{R})$).

For the vector space $\mathbb{R}^n$ the automorphism group is called the **general linear group** $GL_n(\mathbb{R})$ i.e. $GL_n(\mathbb{R})$ is the group of invertible $n \times n$ matrices with real entries. Consider an $n \times n$ matrix $A \in GL_n(\mathbb{R})$ and let $e_1, e_2, \ldots, e_n$ be the columns of A. Because M is invertible $e_1, e_2, \ldots, e_n$ form a basis for $\mathbb{R}^n$. Conversely any basis $e_1, e_2, \ldots, e_n$ gives an $n \times n$ invertible matrix.

$$\begin{aligned} GL_n(\mathbb{R}) &= \{n \times n \text{ invertible matrices with entries in } \mathbb{R}\} && (1.1)\\ &= \{A \in M_{n\times n}(\mathbb{R}) : \det A \neq 0\} && (1.2)\\ &= \{(e_1, e_2, \cdots, e_n) : (e_i)_{i=1}^n \text{ is a basis for } \mathbb{R}^n\} && (1.3) \end{aligned}$$

2 Orientation

The determinant of an invertible matrix is non-zero. Further the determinant satisfies the property that for two $n \times n$ matrices A, B we have $\det(AB) = \det A \cdot \det B$. Hence we have a group homomorphism

$$\det : GL_n(\mathbb{R}) \to \mathbb{R}^\times \tag{2.1}$$

The preimage of positive reals $\mathbb{R}_{>0}$ under this map is denoted by $GL_n^+(\mathbb{R}) := \det^{-1}(\mathbb{R}_{>0})$. It is easy to see that this is a subgroup of $GL_n(\mathbb{R})$. We can decompose $GL_n(\mathbb{R})$ into cosets

$$GL_n(\mathbb{R}) = \det^{-1}(\mathbb{R}_{>0}) \sqcup \det^{-1}(\mathbb{R}_{<0}) \tag{2.2}$$

This breaks up the set of all bases of $\mathbb{R}^n$ into two. These are called the **orientation classes** of bases of $\mathbb{R}^n$. The bases in the orientation class containing the standard basis are said to have the **standard orientation** and the bases which are not in this orientation class are said to have the **reverse orientation**.

The matrices in $GL_n^+(\mathbb{R})$ are called **orientation preserving** matrices. In the language of automorphisms $GL_n^+(\mathbb{R})$ is the automorphism group of $\mathbb{R}^n$ with a chosen orientation class.

3 Special linear groups

Sitting inside $GL_n(\mathbb{R})$ is the subgroup of $n \times n$ matrices with determinant 1 denoted by $SL_n(\mathbb{R})$, called the **special linear group**.

$$SL_n(\mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) : \det A = 1\} \tag{3.1}$$

We can realize $SL_n(\mathbb{R})$ as volume preserving automorphisms of $\mathbb{R}^n$ i.e. if $A \in SL_n(\mathbb{R})$ and X is a subset of $\mathbb{R}^n$ then the volume of X is the same as the volume of AX. This is because if X is a 'parallelogram' in $\mathbb{R}^n$ then the volume of X is exactly the absolute value of the determinant of X, and any other shape can be approximated by 'parallelograms'.

The converse is almost true. A volume preserving automorphism of $\mathbb{R}^n$ should have determinant ± 1, hence $SL_n(\mathbb{R})$ is the group of automorphism of $\mathbb{R}^n$ which preserve both volume and orientation.

4 Orthogonal groups

On $\mathbb{R}^n$ we have the natural notion of distance and we can look at linear autmorphisms of $\mathbb{R}^n$ which preserve distances. We know from Euclidean geometry that any transformation that preserves distances also preserves angles. Angles and lengths in $\mathbb{R}^n$ can be measured using the dot product, this turns out to be the structure that is easiest to manipulate.

Denote by $(\cdot)$ the standard dot product between vectors in $\mathbb{R}^n$ defined as: for $x = [x_1, x_2, \cdots, x_n]^T$ and $y = [y_1, y_2, \cdots, y_n]^T$ the dot product is

$$x \cdot y := \sum_{i=1}^{n} x_i y_i = x^T y \tag{4.1}$$

Recall that the length of the vectors is given by $\|x\|^2 = x.x$ and we can *define* the angle between the vectors x and y by $\theta := \cos^{-1} \dfrac{x \cdot y}{\|x\|\|y\|}$. The group of linear transformations of $\mathbb{R}^n$ which preserve the standard dot product is called the **orthogonal group** denoted $O(n)$.

$$O(n) := \{A : \mathbb{R}^n \to \mathbb{R}^n : \langle Ax, Ay\rangle = \langle x, y\rangle \text{ for all } x, y \in \mathbb{R}^n\} \tag{4.2}$$

We can simplify the condition $\langle Mx, Ay\rangle = \langle x, y\rangle$ as follows:

$$\begin{aligned} & \langle Mx, My\rangle = \langle x, y\rangle && (4.3)\\ \iff\quad & (Ax)^T(Ay) = x^T y && (4.4)\\ \iff\quad & x^T A^T A y = x^T y && (4.5)\\ \iff\quad & x^T(A^T A - I_n)y = 0 && (4.6)\end{aligned}$$

where I_n denotes the identity matrix of size $n \times n$. For an orthogonal matrix A we want this to be true for all $x, y \in \mathbb{R}^n$. This is equivalent to requiring $A^T A - I_n = 0$ (Exercise 6.1) which gives an alternate definition of $O(n)$.

For an orthogonal matrix $A \in O(n)$ if $e_1, e_2, \cdots, e_n$ are the columns of A then $e_i \cdot e_j = 0$ if $i \neq j$ and $e_i \cdot e_i = 1$, such a basis is called an **orthonormal basis** of $\mathbb{R}^n$.

$$\begin{aligned} O(n) &= \{A \in M_{n\times n}(\mathbb{R}) : \text{ for all } x, y \in \mathbb{R}^n \text{ we have } \langle x, y\rangle = \langle Ax, Ay\rangle\} && (4.7)\\ &= \{A \in M_{n\times n}(\mathbb{R}) : A^T A = I_n\} && (4.8)\\ &= \{(e_1, e_2, \cdots, e_n) : (e_i)_{i=1}^n \text{ is an orthonormal basis for } \mathbb{R}^n\} && (4.9)\end{aligned}$$

4.1 Special orthogonal groups

The intersection $O(n) \cap GL_n^+(\mathbb{R})$ is called the **special orthogonal group** denoted $SO(n)$, these are the matrices which preserve the standard inner product and orientation. As we'll see later for small positive integers n the groups $SO(n)$ have nice descriptions.

5 Complex matrix groups

Everything that we did above can be generalized for vector spaces over complex numbers. We have natural generalizations of the vector spaces $GL_n(\mathbb{C})$ and $SL_n(\mathbb{C})$.

The generalization of $O(n)$ on the other hand needs a little tweaking. On complex vector spaces instead of a regular inner product (.) we have a **hermitian inner product** defined by

$$\langle x, y \rangle = \overline{x_1} y_1 + \overline{x_2} y_2 + \cdots + \overline{x_n} y_n \tag{5.1}$$

Matrices which preserve this structure are called **unitary matrices** and the columns of these form an orthonormal basis under the hermitian inner product.

$$\begin{aligned} U(n) &= \{A \in M_{n\times n}(\mathbb{C}) : \text{ for all } x, y \in \mathbb{C}^n \text{ we have } \langle x, y\rangle = \langle Ax, Ay\rangle\} && (5.2)\\ &= \{A \in M_{n\times n}(\mathbb{C}) : A^* A = I_n\} && (5.3)\\ &= \{(e_1, e_2, \cdots, e_n) : (e_i)_{i=1}^n \text{ is an orthonormal basis for } \mathbb{C}^n\} && (5.4)\end{aligned}$$

where A^* is the complex conjugate of transpose of A. There is also the **special unitary group** $SU(n)$ defined to be the group of unitary matrices with determinant 1 (see 6.2).

We have a natural inclusion $\mathbb{R}^n \to \mathbb{C}^n$ induced by $\mathbb{R} \subseteq \mathbb{C}$. This induces a group homomorphism $\Phi : GL_n(\mathbb{R}) \hookrightarrow GL_n(\mathbb{C})$. Furthermore the restriction of the standard hermitian bilinear on $\mathbb{C}^n$ to $\mathbb{R}^n$ is the standard inner product which implies that $\Phi^{-1}(U(n)) \cong O(n)$ and $\Phi^{-1}(SU(n)) \cong SO(n)$. There are also maps going in the other direction, see Exercise

To summarize we have the following hierarchy of matrices

$$\begin{tikzcd} SO(n) \arrow[r, hook] \arrow[dr, hook] & SL(n) \arrow[r, hook] & GL_n^+(\mathbb{R}) \arrow[r, hook] & GL_n(\mathbb{R}) \\ & O(n) \arrow[urr, hook] & & \end{tikzcd} \qquad \begin{tikzcd} SU(n) \arrow[r, hook] & U(n) \arrow[r, hook] & GL_n(\mathbb{C}) \end{tikzcd}$$

6 Exercises

Exercise 6.1. For $A \in M_{n\times n}(\mathbb{R})$ show that if $x^T A y = 0$ for all vectors $x, y \in \mathbb{R}^n$ then $A = 0$.

Exercise 6.2. Find the images of $O(n)$, $SO(n)$, $GL_n(\mathbb{C})$, $U(n)$ under the determinant map.

Exercise 6.3. Does it make sense to define $GL_n^+(\mathbb{C})$?

Exercise 6.4. Identify the following quotients:

1. $GL_n(\mathbb{R})/GL_n^+(\mathbb{R})$
2. $GL_n(\mathbb{R})/SL_n(\mathbb{R})$
3. $GL_n^+(\mathbb{R})/SL_n(\mathbb{R})$
4. $O(n)/SO(n)$
5. $U(n)/SU(n)$

Exercise 6.5. Find the possible eigenvalues for matrices in $O(n)$ and $U(n)$.

Exercise 6.6.

1. Because $\mathbb{C}^n$ can be thought of as a vector space over $\mathbb{R}$ of dimension $2n$ and every $\mathbb{C}$ linear transformation is also $\mathbb{R}$ linear, we can define a map $\Psi : GL_n(\mathbb{C}) \to GL_{2n}(\mathbb{R})$. Explicitly describe the map Ψ.
2. Is Ψ a group homomorphism?
3. Describe the map $\Psi \circ \Phi$.

Exercise 6.7. Is $O(n)$ a normal subgroup of $GL_n(\mathbb{R})$? Read about the **QR-decomposition** of matrices and identify the set $GL_n(\mathbb{R})/O(n)$. What structure(s) does this set have?

Exercise 6.8. $M_{n\times n}(\mathbb{R})$ is naturally isomorphic to $\mathbb{R}^{(n^2)}$, similarly for $\mathbb{C}$, so we can talk about connectedness of its subsets.

1. Show that $GL_n(\mathbb{R})$ and $O(n)$ are not connected.
2. Can you think of a way of showing that $GL_n^+(\mathbb{R})$, $SO(n)$ and $GL_n(\mathbb{C})$ are connected?