WCATSS - 2016

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Contents

1. Day 1 : Overview

Formal groups are a bridge between algebraic topology and number theory. Multiplication on a Lie group looks like a formal group (power series) in a small neighborhood of identity.

Theorem 1.1. *The functors $R \to \{$formal group laws $/R\}$ and $R \to \{$automorphisms formal group laws $/R\}$ are representable.*

Theorem 1.2. *For nice cohomology theories E,*

(1) $E^0(\mathbb{CP}^\infty)$ carries a formal group law

(2) We have a theory of chern classes for complex vector bundles

MU = complex bordism is nice and carries the universal formal group law.

Theorem 1.3. *Given R and a fgl F over R, there are algebraic conditions which ensure that $\exists$ a cohomology theory R^* that is nice and whose fgl is F.*

Example 1.4. The simplest Lie groups (which give rise to fgl's) are

(1) Additive group: $F_a(x, y) = x + y$

(2) Multiplicative group: $F_m(x, y) = x + y + xy$

(3) Elliptic curves $(\mathbb{C}/\mathbb{Z}^2)$ which give rise to elliptic cohomology theories

Q. How do families of elliptic curves show up in cohomology theories?

Theorem 1.5. *There is a sheaf of commutative ring spectra on the moduli of elliptic curves.*

Theorem 1.6. *Given a fgl F over a perfect field k of char p then the set of deformations is representable. (We perturb the field k in a nilpotent direction and see what fgl's are deformations of this, this is the Lubin-Tate deformation theory.)*

Theorem 1.7. *There is an essentially unique commutative ring spectrum associated to the universal deformation of F over k. More is true here on a categorical level.*

2. Formal group laws I

2.1. Definitions and examples.

Consider a Lie group, 1 dimensional (hence commutative), (G, e, μ). Pick an analytic coordinate neighborhood $(U, \phi : U \to \mathbb{R})$. Let $U' \subset U$ be an open subset such that $\mu(U' \times U') \subset U$ so that μ defines a multiplication on $\phi(U')$. Consequently μ is a formal power series $F(x, y)$ which should satisfy,

(1) $F(x, 0) = x = F(0, x)$

(2) $F(x, F(y, z)) = F(F(x, y), z)$

(3) $F(x, y) = F(y, x)$

Definition 2.1. A power series $F(x, y) \in R[[x, y]]$ satisfying the above three conditions is called a **formal group law**.

Remark 2.2. When R has no non-zero nilpotents the last condition is redundant!

The same works in the algebraic case if one takes (G, e, μ) to be a smooth 1 dimensional group variety over k and instead of charts one looks at the completion of G at e, $\widehat{G} \cong Spf(k[[x]])$ (non-canonically).

Remark 2.3. A formal group law is a formal group with the choice of a local coordinate.

2.2. Maps between fgl's.

Let $\alpha : G \to H$ be a morphism of algebraic groups. Restricting this map to the completed neighborhoods of identity

$$\begin{gathered} \widehat{\alpha} : \widehat{G} \to \widehat{H} \\ Spf k[[t]] \mapsto Spf k[[t]] \\ f(t) \leftarrow t \end{gathered}$$

then f should satisfy

$$f(F_G(x, y)) = F_H(f(x), f(y))$$

Definition 2.4. A map $f \in R[[x]]$ is a **morphism of fgl's** $f : G \to H$ if $f(0) = 0$ and $f(F_G(x, y)) = F_H(f(x), f(y))$. So that $f(x) = a_1 x +$ higher terms. It is easy to see that f is invertible if a_1 is a unit. If $a_1 = 1$ then f is called a **strict isomorphism**.

A ring map $\phi : R \to S$ and an fgl F over R naturally defines a fgl $\phi^* F$ over S.

2.3. Classification over $\mathbb{Q}$.

Proposition 2.5. *Any fgl's F over a $\mathbb{Q}$ algebra is strictly isomorphic to F_a.*

Proof: Proof is by giving a strict isomorphism $f : F \to F_a$. We want $f(F(x, y)) = f(x) + f(y)$, differentiating with respect to y we get

$$\begin{aligned} f'(x) &= \frac{1}{F_y(x, 0)} \\ \implies f(x) &= \int_0^x \frac{dt}{F_y(t, 0)} \end{aligned}$$

□

Definition 2.6. Such an isomorphism $F \to F_a$ is denoted $\log F$

Example 2.7. $\log F_m = -\log(1-x)$

2.4. The universal fgl.

Definition 2.8. Define the **Lazard ring** $L := \mathbb{Z}[a_{ij}]/\sim$ where the relations on a_{ij} are the conditions than $F_L(x,y) = \sum_{ij} a_{ij}x^i y^j$ to be a fgl. Let $W := L[b_1, b_2, \cdots]$.

Theorem 2.9. $\hom(L,R)$ *is the set of formal group laws over* R, *so that* F_L *is the universal formal group law.* $\hom(W,R)$ *is the set of strict isomorphisms over* R. *For* $a \in \hom(L,R)$ *and* $b \in \hom(W,R)$ *the fgl corresponding to* a *is* a^*F_L *and the isomorphism corresponding to* b *is* $f(x) = x + \sum_{i\geq 1} b(b_i)x^{i+1}$.

We can define a functor

$$\begin{aligned} \text{Rings} &\to \text{Groupoids} \\ R &\mapsto (\text{fgl's(R)}, \text{StrictIso(R)}) \end{aligned}$$

The above theorem says that this functor is represented by the pair (L,W).

This implies that (L,W) is a Hopf algebroid and $(\operatorname{Spec} L, \operatorname{Spec} W)$ is a groupoid scheme with maps

$$\begin{aligned} \eta_L &: L \to W \text{ inclusion} \\ \eta_R &: L \to W \text{ coaction} \\ \epsilon &: W \to L \text{ quotient} \\ \Delta &: W \to W \otimes_L W \text{ composition of strict iso} \end{aligned}$$

We can assign grading to L, W by setting

$$|x| = |y| = -2, |a_{ij}| = 2i + 2j - 2, |b_i| = 2i$$

2.5. $Aut(F_a)$ **over** $\mathbb{Z}/p$. A strict iso $f(x)$ of F_a means that $f(x+y) = f(x)+f(y)$. If $f(x) = \sum_{i\geq 0} b_i t^{i+1}$ then we get the condition $b_i = 0$ unless $i+1 = p^k$. Let $c_k = b_{p^k-1}$ then $|c_k| = 2(p^k-1)$.

$$Aut(F_a) = \operatorname{Spec} \mathbb{Z}/p[c_1, c_2, \cdots]$$

Explicit computation gives us that the coproduct on $Aut(F_a)$ is given by

$$\Delta(c_k) = \sum_{i+j=k} c_i^{p^j} \otimes c_j$$

Remark 2.10. This argument recovers a sub-Hopf algebra of the dual Steenrod algebra $P_* \subset A_*$

$$P_* = \begin{cases} P(\xi_1^2, \xi_2^2, \cdots) & \text{when } p = 2 \\ P(\xi_1, \xi_2, \cdots) & \text{when } p \text{ is odd} \end{cases}$$

2.6. Lazard's theorem.

Theorem 2.11.

$$L \cong \mathbb{Z}[x_1, x_2, \cdots], |x_i| = 2i$$

Let $c_n(x,y) = \gamma_n^{-1}((x+y)^n - x^n - y^n)$ *where* γ_n *is* p *if* n *is a power of* p *and 1 otherwise then the universal formal group law has the property*

$$F_L(x,y) \equiv \sum x_n c_{n+1}(x,y) \mod (x_1, x_2, \cdots)^2$$

3. Introduction to Homotopy theory

Goal: Want to construct and classify maps between topological spaces. Knowing $\pi_n(Y)$ and $H_n(X)$ gives us information about $[X, Y]$.

Homology (ordinary): A functor H_* Spaces $\to$ graded abelian groups.

These groups are not easy to compute, so we need an intermediate construction C_* which takes values in chain complexes over $\mathbb{Z}$. C_* has nice properties in that it takes disjoint unions to direct sums, pushouts to pushouts (in the derived sense), products to tensor products.

We can mod out spaces by homotopy equivalences and get the Homotopy category, the corresponding construction on the side of chain complexes gives us the derived category $\mathcal{D}(\mathbb{Z})$.

There are several equivalent ways of getting C_*, the most common way being singular chain complexes.

Example 3.1. Associated to a space X we can construct a new space $AG(X)$, the free topological abelian group on X, then a theorem of Dold-Thom says

$$H_n(X) \cong \pi_n(AG(X))$$

From C_* we can construct new homology theories, eg. $H_n(X; A), H^n(X, A)$, which makes $C_*(X)$ universal among these abelian type homology theories. However there are many other geometric theories which are not recovered.

Stable homotopy theory adjusts and captures these other theories.

$$\begin{tikzcd}
\text{Spaces} \arrow[r, "\Sigma^\infty"] \arrow[d] & \text{Spectra} \arrow[d] \\
\text{Homotopy category} \arrow[r] & \text{Stable homotopy category}
\end{tikzcd}$$

and the functor Σ^∞ has nice properties like C_* in that it takes derived colimits to derived colimits, products to products, the stable homotopy category is a triangulated category with mapping cones, long exact sequences, etc.

If E is any object in stable homotopy theory then we get new homology and cohomology theories, $E_*(X), E^*(X)$ which are graded abelian groups,

$$E_n(X) = \pi_*(E \wedge \Sigma^\infty X)$$
$$E^n(X) = [\Sigma^\infty X, E]$$

Ring spectrum: E is a ring spectrum if it has maps $\mu : E \wedge E \to E$, $\eta : \mathbb{S} \to E$. If E is a ring spectrum then $E^*(X)$ naturally has a multiplication.

Every cohomology theory is represented in the stable homotopy category. So we need to understand what makes the stable homotopy category different from say $\mathcal{D}(\mathbb{Z})$ for which we need to compute $[X, Y]$. The tools for doing these constructions are

(1) Obstruction theory:

$$H^*(X, \pi_t(Y)) \implies [X, Y[t-s]]$$

(2) Cellular method: If we know a set of building blocks, for example the spheres $\mathbb{S}$ and all the maps $[\mathbb{S}, \mathbb{S}]$ and $[\mathbb{S}, Y]$ and how to build X out of $\mathbb{S}$ then we can use it to compute $[X, Y]$ which is captured by the spectral sequence

$$Ext^s_{\pi_*\mathbb{S}}(\pi_* X, \pi_* Y) \implies [X, Y[t-s]]$$

(3) Flipped cellular method (Adams): Different building blocks: $H\mathbb{F}_p$ the Eilenberg Maclane spectrum, the corresponding spectral sequence is given by

$$Ext^s_{\mathcal{A}^*}(H^*Y, H^*X[-t]) \implies [X, Y[t-s]]$$

This is saying that the stable homotopy category is similar to $\mathcal{D}(\mathcal{A}^* - mod)$. The dualized steenrod algebra $\mathcal{A}_* \cong \mathbb{F}_p[\xi_i] \otimes \Lambda(\tau_i)$ is easier to handle and the corresponding SS is

$$Ext^s_{\mathcal{A}^*}(H_*X, H_*Y[t]) \implies [X, Y[t-s]]$$

This generalizes. If E is a ring spectrum such that $\pi_*(E \wedge E)$ is a flat π_*E module then the pair $(\pi_*E, \pi_*(E \wedge E))$ form a Hopf algebroid and we get correspondingly the Adams Novikov spectral sequence

$$Ext^s_{(E_*, E_*E)}(E_*X, E_*Y[t]) \implies [X, Y[t-s]]$$

4. Complex bordism theory

Goal: Give a homotopical description of the Lazard hopf algebroid $(\mathcal{L}, W)$.

Definition 4.1. A spectrum is a collection $E = (E_n, \sigma_n)$ where $E_n \in Top$ and $\sigma_n : \Sigma E_n \to E_{n+1}$.

Definition 4.2. A ring spectrum E is **complex orientable** if the induced map $\widetilde{E}^2(\mathbb{CP}^\infty) \to \widetilde{E}^2(\mathbb{CP}^1) = \pi_0(E)$ is surjective.

What this is saying is that there exists an $x^E \in \widetilde{E}^2(\mathbb{CP}^\infty)$ such that $i^*(x^E) = 1 \in \pi_0(E)$ where $i : \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^\infty$.

Example 4.3. (1) $E = H\mathbb{Z}$, $i^* : H^2(\mathbb{CP}^\infty, \mathbb{Z}) \xrightarrow{\cong} H^2(S^2) \cong \mathbb{Z}$ and $x^{H\mathbb{Z}} = c_1$ the first universal chern class.

(2) $E = KU$ the complex K-theory. $\widetilde{KU}^2(\mathbb{CP}^\infty) \cong \widetilde{KU}^0(\mathbb{CP}^\infty)$ by Bott periodicity. In this case $x^{KU} = [l] - 1$ where l is the universal complex line bundle.

Proposition 4.4. *If E is complex orientable with complex orientation x^E then*

$$E^*(\mathbb{CP}^\infty) \cong \pi_*(E)[[x^E]]$$
$$E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \cong \pi_*(E)[[x_1^E, x_2^E]]$$

There exists an H-space structure on $\mathbb{CP}^\infty$ as $\mathbb{CP}^\infty \cong BU(1)$ which classifies the tensor product on line bundles. The induced map

$$m^* : E^*(\mathbb{CP}^\infty) \to E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty)$$
$$(\pi_*E)[[x]] \mapsto (\pi_*E)[[x_1, x_2]]$$
$$x \mapsto \mu^E(x_1, x_2)$$

Proposition 4.5. *$\mu^E(x_1, x_2)$ is a fgl over π_*E.*

Example 4.6. (1) $E = H\mathbb{Z}$, $\mu_E = x_1 + x_2$

(2) $E = KU$, $\mu_E = x_1 + x_2 + x_1x_2$

Definition 4.7. Let $E \to B$ be a real vector bundle of rank r then the Thom space $Th(E) = D(E)/S(E)$, the quotient of the disk bundle by the sphere bundle.

When E is the trivial bundle then $Th(E) = \Sigma^r(B)$.

Definition 4.8. Define

$$MU(n) = Th(\xi^n)$$

where ξ^n is the universal complex vector bundle of rank n over $BU(n)$.

The $2n^{th}$ space the spectrum MU is then defined to be $MU_{2n} := MU(n)$.

Remark 4.9. $\pi_*(MU)$ is the complex bordism group.

One can show that $MU(n) \cong BU(n)/BU(n-1)$ so that $MU_2 = MU(1) \cong \mathbb{CP}^\infty$ and that MU is complex orientable.

Theorem 4.10. *MU is the universal complex orientable theory, that is every complex orientable theory E admits a map $g : MU \to E$ such that $g_*(x^{MU}) = x^E$ and $g^*(\mu^{MU}) = \mu^E$.*

Theorem 4.11 (Quillen)**.** *$\pi_*(MU)$ is isomorphic to the Lazard ring and μ^{MU} is the universal formal group law.*

5. Elliptic curves

5.1. Elliptic curves over $\mathbb{C}$. Given a lattice $\Lambda \subset \mathbb{C}$ we can form the quotient $\mathbb{C}/\Lambda$. Associated to Λ is the Weistrass $\mathfrak{p}$ function

$$\mathfrak{p}(z) = \frac{1}{z^2} + \sum_{0 \neq \lambda \in \Lambda} \left(\frac{1}{(z-\lambda)^2} + \frac{1}{\lambda^2}\right)$$

This determines an embedding of $\mathbb{C}/\Lambda \hookrightarrow \mathbb{CP}^2, z \mapsto (\mathfrak{p}(z) : \mathfrak{p}'(z) : 1)$. The image compactifies to a curve E containing the special point $O = (0:1:0)$.

The Weistrass function satisfies the equation

$$\mathfrak{p}'(z)^2 = 4\mathfrak{p}^3 - 40G_4\mathfrak{p}(z) - 140G_6$$

Definition 5.1. Let $P, Q \in E$, let L be a line in $\mathbb{CP}^2$ through P, Q and let R be the third point of intersection of $L \cap E$. Let L' be a line through O and R. Then define $P + Q$ to be third point of intersection of $L' \cap E$.

Proposition 5.2. *This defines a group structure on E.*

5.2. Elliptic curves over k.

Definition 5.3. Let R be a ring. A **Weirstrass polynomial** over R is of the form

$$f(x,y) = p^2 + c_1xy + c_3y - x^3 + c_1x + c_6 \in R[x,y]$$

The **universal Weirstrass polynomial** is the same polynomial $f(x,y)$ as above in the ring $\mathbb{Z}[c_1, c_2, c_3, c_4, c_6][x,y]$.

If the characteristic is not 2 or 3 then the Weirstrass equation can be simplified by a change of coordinates to $y^2 = x^3 + c_4x + c_6$.

Definition 5.4. Let k be a field, a **Weirstrass polynomial** will be a projective curve determined by a Weirstrass polynomial over k. An **elliptic curve** over k is a smooth Weirstrass curve.

Remark 5.5. The smooth locus of a Weistrass curve C^{reg} also admits a group structure similar to an elliptic curve.

Proposition 5.6. *Let C be a Weirstrass curve over k and let Δ be the discriminant of C*

(1) *C is smooth, $C^{reg} = C$ and $\Delta \neq 0$ is an elliptic curve*

(2) *C is singular, $\Delta = 0, c_4 \neq 0, c_{\overline{k}}^{reg} \approx \mathbb{G}_{m,\overline{k}}$*

(3) *C is singular, $\Delta = 0, c_4 = 0, c_{\overline{k}}^{reg} \approx \mathbb{G}_{a,\overline{k}}$*

Proposition 5.7. *A curve C is elliptic iff genus is 1, smooth projective, geometrically integral with a k rational point O.*

5.3. Weistrass curves over any ring R. Given f a Weistrass polynomial one can form $\mathrm{Spec}(R[x,y]/f)$. This compactifies to a projective S scheme $C \xrightarrow{\pi} S$ where $S = \mathrm{Spec}\, R$.

We can this of this as a family of weistrass curves over fields, for $s \in S$ the fiber is $C_s = \mathrm{Spec}(k_s[x,y]/f)$ where k_s is the residue field of R at s.

Definition 5.8. An S scheme that is finitely presented, proper flat such that every fiber has arithmetic genus 1 ($= \dim(\mathcal{O}_{C_s})$) is a **relative curve of genus 1**. A **relative elliptic curve** is such a curve which is also smooth over S along with a section $e : S \to C$ (this is the identity for the group structure).

Theorem 5.9. *There exists an S scheme $C^{reg} \to S$ such that $(C^{reg})_s = (C_s)^{reg}$ and $C^{reg} \to S$ is a group.*

5.4. Deriving the fgl associated to $C^{reg} \to S$. The identity section e is a closed embedding

$$0 \to \mathfrak{I}_e \to \mathcal{O} \to e_*\mathcal{O}_S \to 0$$

The formal completion is $\widehat{C^{reg}} = (|S|, \lim_n e^{-1}(\mathcal{O}_{C^{reg}}/\mathfrak{I}_e^n))$, then the fgl defined by C is isomorphic to $\widehat{C^{reg}} \cong Spf(R[[x]])$.

Let E be an elliptic curve over k defined by f and let $O = (0:1:0)$. Suppose $f(x,y) = y^2 - x^3 + c_4x + c_6$. Apply the change of coordinates $x = z/w, y = -1/w$ then the equation of the elliptic curve becomes

$$w = z^3 + c_4z + c_6w^3$$

Now we recursively plug in the w in the right hand side to get a power series in z!! One can then explicitly compute the formal group law.

6. Day 2 : Overview

Lubin Tate deformation theory

(1) height of a fgl
(2) universal deformation of height n fgl over a perfect field k
(3) $End(F)$ acting on the deformations

Local class field theory abelian extensions of a local field F correspond to subgroups $F^\times$
Artin reciprocity: $rec_F : F^\times \to Gal(F^{ab}/F)$

Landweber exact functor theorem F a graded fgl over R. This is classified by $MU_* \to R_*$

Q. When is $X \mapsto MU_*X \otimes_{MU_*} R_*$ a generalized homology theory?

Ans. LEFT gives a weaker condition than flatness.

Modular forms functions on the set of elliptic curves satisfying a modularity condition.

7. Formal group laws II

R commutative ring, $fgl(R)$ is the set of formal group laws.

Alternatively, given $G \in R[[x,y]]$ we get a functor F_G : commutative R algebra $\to$ sets with a binary operation which sends $S \mapsto (\mathcal{N}_S, \oplus)$ where $\mathcal{N}_S$ is the set of nilpotent elements in S. G is a fgl iff F_G factors through Abelian groups. F_G is representable by $(Spf(R[[x]], F)$.

Want to understand the functor Commutative rings $\to$ Sets, $R \mapsto fgl(R)/$strict isomorphisms.

Proposition 7.1. *If R is a $\mathbb{Q}$ algebra then $fgl(R)/$strict isom is a single point.*

Q. What is the analogue of the above theorem in char p?

Let k be a perfect field of char $p > 0$. Let θ denote a nilpotent thickening of k i.e. it is an Artinian local ring, $\mathfrak{m}$ is a maximal ideal, nilpotent. For example, $\theta = \mathbb{Z}/p^2$. We wish to understand $fgl(k)/\sim$, $fgl(\theta)/\sim$.

7.1. Invariants of formal group laws in char p. Fix $G \in fgl(k)$ and $n \in \mathbb{Z}$. We have a natural transformation $F_G \xrightarrow{[n]} F_G$ induced by multiplication by n on groups given by $x \mapsto F(x, F(x, \cdots, x)\cdots)$, n times.

Proposition 7.2. *Either* $[p](t) \equiv 0$ *or* $[p](t) = g(t^{p^n})$ *for some* $n \in \mathbb{Z}, n \geq 1$ *and* g *satisfies* $g \in tR[[t]], g(0) = 0, g'(0) \neq 0$.

Definition 7.3. Height of G is defined to be the n in the above proposition, ∞ otherwise.

It is easy to see that height is invariant under strict isomorphisms.

Example 7.4. G_a has height ∞ and G_m has height 1 so that G_a and G_m are not isomorphic in characteristic p.

Theorem 7.5 (Lazard)**.** $h : fgl(k)/\sim \to \mathbb{Z}_{>0} \cup \{\infty\}$ *is a bijection if* k *is separably closed. For a general* k *the fiber of* h *over* $n \in \mathbb{Z}_{n>0}$ *is* $H^1(Gal(k^{sep}/k), Aut(F)_{k^{sep}})$ *where* F *is any* $fgl(k)$ *of height* n.

Definition 7.6. Given $\Phi \in fgl(k), F, G \in fgl(\theta)$ such that $red(F) = red(G) = \Phi$. A $\star$ isomorphism f between G and F is a map $f \in iso(F, G)$ such that

$$\begin{array}{ccc} F & \longrightarrow & G \\ \downarrow & & \downarrow \\ \Phi & \longrightarrow & \Phi \end{array}$$

commutes.

We wish to study $fgl(\theta)/\star$ iso.

Theorem 7.7. *Fix* $\Phi \in fgl(k)$ *of height* n. *The functor, Nilpotent thickenings* $\to$ *Sets*, $\theta \mapsto Def_\Phi(\theta)$ *is representable by* $Spf(W(k)[[t_1, \cdots, t_{n-1}]] = R, \Gamma)$ *for every* $(\theta, \mathfrak{m})$

$$\hom_{cts}(R, \theta) = \mathfrak{m}^{n-1} \to Def_\Phi(\theta)$$
$$\phi \mapsto \phi^*(\Gamma)$$

is an isomorphism.

Corollary 7.8. $Aut_k(\Phi)$ *acts on* $Def_\Phi(\theta)$.

Proof: Steps:

(1) Construct a specific element $\Gamma \in fgl(k)$
(2) Classify infinitesimal deformation of Φ using $H^2(\Phi)$ which is thought of *the tangent space of* $\mathcal{M}_{fgl}$ *at* Φ.
(3) Show $H^2_k(\Phi) \cong k^{n-1}$
(4) Show Γ is universal

Definition 7.9. F is an r bud if it satisfies the axioms of fgl mod degree $r + 1$.

Given an F_r, r bud, there exists an F_{r+1} an $(r+1)$ bud extending it such that $F'_{r+1} - F''_{r+1} = aC_{r+1}(x, y)$ mod degree $r + 2$. Further if F has height n then $F(x, y) = x + y + aC_p h(x, y) \mod x^{p^{n+1}}$.

Proposition 7.10. $\exists . \Gamma(t)(x, y) \in R[[x, y]]$ *satisfying* $\Gamma^\star(0, \cdots, 0)(x, y) = \phi(x, y)$ *and* $\forall\, i . \Gamma(0, \cdots, 0, t_i, \cdots, t_{n-1})(x, y) = x + y + t.C_p(x, y) \mod x^{p^{n+1}}$

Let $k = \theta/\mathfrak{m}$. We want a way to lift Φ to $k[\epsilon]/\epsilon^2$. Any other lift $F(x, y)$ should look like $\Phi(x, y) +_\Phi G(x, y).\epsilon$. The fgl axioms then imply that $\epsilon(G(y, z) - G(x + y, z) + G(x, y + z) - G(x, y)) = 0$ along with $G(x, 0) = 0 = G(0, x)$ and $G(x, y) = G(y, x)$. This is a symmetric 2-cycle!!!

Further two lifts are isomorphic if $\epsilon(G(x, y) - G'(x, y)) = \epsilon(g(x + y) - g(y) - g(x))$ which is a 2-coboundary condition. □

8. Into to local class field theory

8.1. Definitions and notations.

Definition 8.1. A field k is called a **local field** it is locally compact wrt a non-trivial valuation.

Example 8.2. Finite extensions of $\mathbb{Q}_p$, finite extensions of $\mathbb{F}_p((t))$, $\mathbb{R}$ or $\mathbb{C}$.

Let k be a finite extension of $\mathbb{Q}_p$ and $\mathcal{O}_k$ is the ring integers with units $\mathcal{O}_k^\times$, $\mathfrak{m}_k$ is the maximal ideal, π a prime element, generater of $\mathfrak{m}_k$.

Every element $a \in k^\times$ is of the form $a = u\pi^n$, u a unit, define $ord_k(a) := n$, $|a| := q^{-ord_k(a)}$, k^{al} = separable algebraic closure of k and k^{ab} = union of all abelian extensions of k.

Goal: Classify all finite degree abelian extensions, study the structure of k^{ab}, construct k_π a subfield of k^{ab}.

Let L/k be a finite Galois extension of local fields with Galois group G. Define $G_i = \{\tau_i \in G : ord_L(\tau(x) - x) \geq i + 1. \forall\, x \in \mathcal{O}_L\}$. This gives a filtration $G \supseteq G_0 \supseteq G_1 \supseteq \cdots$.

$G_0 = 1 \iff L/k$ is unramified. Equivalently let π_k and π_L be the uniformizers, then $\pi_k = u\pi_L^e$ where e is the ramification degree.

8.2. Main theorems:

Theorem 8.3. *Let L be a finite unramified extension of k then L/k is Galois and there is an element $\sigma \in Gal(L/k) = G$ such that $\sigma x \equiv x^q. \forall\, x \in \mathcal{O}_L$. Then G is cyclic generate by σ. We call $\sigma = Frob_{L/k}$ the Frobenius element of G.*

Theorem 8.4 (Local reciprocity law)**.** *For any non-archimedean local field, there is a homomorphism $\phi_k : k^\times \to Gal(k^{ab}/k)$ such that*

(1) for any prime $\pi \in \mathfrak{m}_k$ and any finite unramified extension L/K, $\phi_k(\pi)$ acts on L as $Frob_{L/k}$.

(2) For any finite abelian extension L/k, $Norm(L^\times) \subseteq \ker(\phi_k|_L)$

Combining these, ϕ_k induces an isomorphism $\phi_{L/k} : k^\times/Norm_{L/k}(L^\times) \to Gal(L/k)$.

Remark 8.5. The above theorem does not make sense.

Theorem 8.6 (Local existence theorem)**.** *A subgroup $N \subseteq k^\times$ is of the form $Norm_{L/k}(L^\times)$ for some finite abelian extension L iff it is a finite index open subgroup of $k^\times$.*

Taking the inverse limit over L abelian extensions of k we get a map

$$\widehat{\phi}_k : \widehat{k^\times} \to Gal(k^{ab}/k)$$

We have $k^\times \cong \mathcal{O}_k^\times \times \mathbb{Z}$ via the map $a = u\pi^n \mapsto (u, n)$. Similarly $\widehat{k^\times} \cong \mathcal{O}_k^\times \times \widehat{\mathbb{Z}}$.

Let k_π be the subfield of k^{ab} fixed by $\widehat{\phi}_k(\pi)$ and let k^{unr} be the subgroup of k^{ab} fixed by $\widehat{\phi}_k(\mathcal{O}_k^\times)$ so that $k^{ab} = k_\pi.k^{unr}$

Example 8.7. When k has char p, k^{unr} is generated by all roots of unity of order prime to p.

8.3. Lubin-Tate fgls. Let k be a non-archemidean local field and A an $\mathcal{O}_k$ algebra with $i : \mathcal{O}_k \to A$. A formal $\mathcal{O}_k$ module over A is

(1) fgl G over A

(2) a family of power series $[a]_G$ for every $a \in \mathcal{O}_k$

(3) $[a]_G$ together represent a homomorphism

$$\mathcal{O}_k \to End(G)$$
$$a \mapsto [a]_G$$

(4) $[a]_G(x) = i(a)x + O(x^2)$

Example 8.8. G_a is an $\mathcal{O}_k$ module over $\mathcal{O}_k$ algebra A. And when $k = \mathbb{Q}_p$, G_m becomes a formal $\mathcal{O}_K = \mathbb{Z}/p$ module as for $a \in \mathbb{Z}/p$,

$$[a]_{G_m}(x) = (1+x)^a - 1 = \sum_1^\infty \binom{a}{n} x^n \in \mathbb{Z}_p[[x]]$$

Lubin-Tate: Start with a choice $[\pi]_G$ and construct G from this. Let $f \in \mathcal{O}_k[[x]]$ be any power series satisfying a) $f(x) = \pi x + O(x^2)$ and b) $f(x) = x^q \mod \pi$.

Theorem 8.9. *There exists a unique formal $\mathcal{O}_k$ module G_f over $\mathcal{O}_k$ for which $[\pi]_{G_f}(x) = f(x)$. Further, if g is any power series satisfying the two conditions a) and b) as above, then G_f and G_g are isomorphic.*

Theorem 8.10. *$C : Gal(k_\pi/k) \to Aut(T_\pi(G)) \cong \mathcal{O}_k^\times$ is an isomorphism $k_\pi k^{unr} = k^{ab}$ and the Artin reciprocity map $\phi_{k_\pi/k}$ is the unique map which sends π to 1 and $\alpha \in \mathcal{O}_k^\times$ to $C^{-1}(\alpha^{-1})$.*

9. Landweber Exact Functor theorem

Theorem 9.1 (Conner-Floyd)**.** *For $X \in Top$ for K the complex K theory,*

$$K_*(X) \cong MU_*(X) \otimes_{MU_*} K_*$$

Q. Say F is an fgl over R with a map $MU_* \to R$, Is $MU_*(-) \otimes_{MU_*} R$ a homology theory?

If it is then it would have a cup product and a complex orientation.

Need to check the Steenrod axioms:

(1) homotopy functor - automatically true
(2) excision - automatically true
(3) additivity - automatically true
(4) LES of a pair - true if R flat but this is too strict a condition

Need a weaker notion of flatness.

Theorem 9.2 (LEFT)**.** *Say M is an MU_* module, if $(p, v_1, v_2, \cdots) \in L$ is a regular sequence for $M, \forall p$ then $MU_*(-) \otimes_{MU_*} M$ is a homology theory. Here if $MU_* \cong \mathbb{Z}[x_1, x_2, \cdots]$ then $v_n = x_{p^n-1}$. Moreover if M is a commutative algebra over MU_* then we get a complex orientable homology theory.*

Definition 9.3. If M satisfies the conditions above then it is called **Landweber exact**.

Recall that a sequence $(x_1, x_2, \cdots) \in R$ is regular for an R mod M if $\forall\, n$ the action of x_n on $M/(x_1, x_2, \cdots, x_{n-1})$ has no kernel.

Remark 9.4. If F in a fgl, v_n is the coefficient of x^{p^n}. v_n is not invariant, but is invariant $\mod (p, v_1, v_2, \cdots, v_{n-1})$.

Example 9.5.
(1) $G_m = x + y$ over $R = \mathbb{Z}$. $[p]x = px$ so that $v_i = 0$ for all i. Is $(p, v_1, v_2, \cdots)$ regular? No as 0 is not invertible in $\mathbb{Z}/p$.
(2) $G_m = x + y + \beta^{-1}xy$ over $R = \mathbb{Z}[\beta^\pm]$. In this case $[p]x = \beta^{1-p}x^p \mod p$ so that $v_1 = \beta^{1-p}$ and $v_i = 0$. In this case we do get Landweber flatness.
(3) $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \cdots]$ (Brown-Peterson) a quotient of MU_{*p} and in this case again we get a regular sequence which corresponds to a homology theory BP.

Remark 9.6. BP classifies p typical fgl's.

Proof: We work 1 prime at a time. We have a BP_* module M.

Proposition 9.7 (Landweber invariant prime ideal theorem)**.** *All prime ideals of BP_* that are fixed by the coaction of BP_*BP are $(p, v_1, \cdots, v_n)$ for all n.*

□

Wish to apply this theorem to construct three homology theories:

(1) Johnson-Wilson theory $E(n)$
(2) Morava K theory $K(n)$
(3) Lubin - Tate theories (Morava E theory)

9.1. Johnson-Wilson theory. Fix a prime p. Let $R = \mathbb{Z}_{(p)}[v_1, \cdots, v_{n-1}, v_n^{\pm}]$ and the module structure is given by the natural map $BP_* \to R$. Applying LEFT to this produces $E(n)$, the Johnson-Wilson theory of height n. One thinks of $E(n)$ as encoding information of height $\leq n$.

9.2. Morava K theory. Now take $R = \mathbb{Z}/p[v_n^{\pm}]$ with the natural BP_* algebra structure. This unfortunately does not satisfy LEFT as p itself is not invertible. But one can construct $K(n)$ out of $E(n)$. We need to kill the lower v_i in $E(n)$. Consider the map

$$v_i : S^{2(p^i-1)} \to E(n)$$

To kill the v_i then we form the following cofiber

$$S^{2(p^2-1)} \wedge E(n) \to E(n) \wedge E(n) \to E(n) \xrightarrow{cofiber} E(n)/v_i$$

Repeatedly doing this kills produces the $K(n)$.

(1) These theories form the building blocks of chromatic homotopy theory.
(2) When $n = 1$, $K(1)_* = \mathbb{Z}/p[v_1^{\pm}], |v_1| = 2(p-1)$. It turns out that $K(1)$ is one of the $(p-1)$ summands of mod p complex K theory.
(3) These theories also satisfy the **Kunneth isomorphism**

$$K(n)_*(X \times Y) \cong K(n)_*(X) \otimes_{K(n)_*} K(n)_*(Y)$$

(4) $K(n)$ are homotopy commutative for $p \geq 3$ but for $p = 2$ these are homotopy associative. In fact, $K(n)$ are A_∞ ring spectrum.

9.3. Lubin-Tate homology theory. $E(k, \Gamma) = W(k)[[u_1, \cdots, u_{n-1}]][u^{\pm}]$ with $|u_i| = 0, |u| = -2$ and Γ is the universal formal group law, with the natural BP_* structure. This turns out to be Landweber exact and hence gives us Lubin-Tate homology theories.

When $K = \mathbb{F}_{p^n}$ and Γ is the Honda formal group law we get $[p]_p = x^{p^n}$ which gives us E_n, the Morava E theories. These are highly structured E_∞ ring spectra.

10. Modular Forms

Generalized curve $p : C \to S$ with a section e which is locally a Weirstrass curve. Think of this as a family of Elliptic curves parametrized by S.

Now look at WCs up to Weirstrass equivalence and strict isomorphisms (of the corresponding elliptic curves).

Assume that these WCs can be described by an equation involving $a_1, \cdots, a_6$ which are functions of s. Then the space of elliptic curves along with their automorphisms is given by a Hopf algebroid given by (A, Γ) where $A = \mathbb{Z}[a_1, \cdots, a_6], \Gamma = A[r, s, t], |a_i| = 2i, |r| = 4, |s| = 2, |t| = 6$.

Given $C \to S$ define $\omega_{C/S}$ to be the cotangent space of C along S and let π be a section of $\omega_{C/S}$.

11. Day 3 : Overview

Deformations of fgl's Honda fgl $\to \phi$ of height n over $k = \mathbb{F}_{p^n}$ and $[p]_\phi x = x^{p^n}$.

There exists a (Γ, R) with $R \cong W(\mathbb{F}_{p^n})[[u_1, \cdots, u_{n-1}]]$ where $W(\mathbb{F}_{p^n} = \mathcal{O}_L$ representing deformations of (ϕ, x). The automorphisms $Aut(\phi)$ acts on R.

Denote $\mathbb{S}_n = Aut(\phi) \cong \mathcal{O}_D^\times$ the **Morava stabilizer group**. Let $\mathbb{G}_n \cong \mathbb{S}_n \rtimes Gal(\mathbb{F}_{p^n}/\mathbb{F}_p)$.

Γ is the fgl for E_n the Morava E theory, Landweber exact with $(E_n)_* \cong W(\mathbb{F}_{p^n}[[u_1, \cdots, u_{n-1}]][u^\pm]]$. ϕ is the fgl for $K(n)$ the Morava K theory with $K(n)_* \cong \mathbb{F}_{p^n}[u^\pm]$. Γ is the universal deformation of ϕ.

$Aut(\Phi)$ acts on E_n.

A spectrum X is built from other simpler spectra $L_{K(n)}X$, which interpolates between $X \otimes \mathbb{Q}$ and $X_p^\wedge$.

Theorem 11.1 (Hopkins-Devinatz)**.**

$$L_{K(n)}S \cong E_n^{h\mathbb{G}_n}$$

Elliptic curves and elliptic cohomology theories E be a complex oriented ring spectrum coming from an elliptic curve with fgl F_E.

There are spectral sequences

$$\begin{aligned} H^*(\mathbb{G}_n, (E_2)_*) &\implies \pi_* L_{K(2)}S \\ H^*(A, \Gamma) &\implies \pi_* L_{K(2)} tmf \end{aligned}$$

12. Local chromatic homotopy theory

We wish to compute π_*X. The Adams Novikov spectral sequence starts with $Ext_{BP_*BP}(BP_*, BP_*X)$ and computes the p part of π_*X.

The chromatic spectral sequence computes the $Ext_{BP_*BP}(BP_*, BP_*X)$ via breaking up X into chromatic layers determined by families of periodic self maps.

12.1. Bousfield localization.

Definition 12.1. Let X be a spectrum, E_* be a homology theory. We say X is E_* local if for any W such that $E_*W = 0$ we have $[W, X] = 0$.

Example 12.2. If X is any spectrum, E is a ring spectrum, then $E \wedge X$ is E local.

Definition 12.3. An E localization of a spectrum X is an E local spectrum L_EX and a map $\lambda : X \to L_EX$ such that $E_*(\lambda)$ is an isomorphism.

Theorem 12.4 (Bousfield localization)**.** *Such localizations exist and are unique.*

Example 12.5. $E = K$ be the ordinary complex K theory and X be the sphere spectrum then $\pi_{-2}(L_kS^0) = \mathbb{Q}/\mathbb{Z}$ and $\pi_{-i}(L_kS^0) \neq 0$ for infinitely many i.

Definition 12.6. Spectrum E and F are Bousfield equivalent if for any X, $E \wedge X \cong * \iff F \wedge X \cong *$. Denote the equivalence class of E by $[E]$.

Theorem 12.7. *$L_E = L_F$ iff they are Bousfield equivalent.*

The Bousfield classes satisfy the following properties:

(1) $[E] \wedge [F] = [E \wedge F]$
(2) $[E] \vee [F] = [E \vee F]$
(3) Partial ordering: write $[E] \geq [F]$ if $E \wedge X \cong * \implies F \wedge X \cong *$.

Theorem 12.8. *If $[E] \geq [F]$ then $L_FL_E = L_F$ and we have a map $L_E \to L_F$.*

12.2. Classical chromatic homotopy theory.

Fix a prime p.

Theorem 12.9. *There exist homology theories $K(n)_*$ such that*

1. *$K(0)_*(X)$ is the ordinary cohomology with rational coefficients*
2. *$K(1)$ is one of the summands of* $\mod p$ *complex K theory*
3. *$K(0)_* = \mathbb{Q}$ and $K(n)_* = \mathbb{F}_p[v_n^{\pm}], |v_n| = 2(p^n - 1)$*
4. *$K(n)_*(X \times Y) \cong K(n)_*(X) \otimes_{K(n)_*} K(n)_*(Y)$*
5. *Any graded module over $K(n)_*$ is free*

Remark 12.10. $K(n)$ is a complex oriented cohomology theory with $[p](x) = x^{p^n}$.

Definition 12.11. A finite spectrum X is called p local if $H(X;\mathbb{Z}) = H(X;\mathbb{Z}) \otimes \mathbb{Z}_{(p)}$. Equivalently, $X = L_{H\mathbb{Z}_{(p)}}X$ or that X is built out of p local spheres.

Theorem 12.12. *$K(n)$ has a detection property. Let X be a finite p local spectrum, if $K(n)_*(X) = 0$ then $K(n-1)_*(X) = 0$.*

Definition 12.13. The minimum n in the above theorem for which $K(n)_*(X) \neq 0$ is called type of X.

Theorem 12.14 (Class invariance theorem)**.** *Let X, Y be finite p local of type m and n then $[X] = [Y]$ iff $m = n$ and $[X] < [Y]$ iff $m > n$.*

Recall that $E(n) = \mathbb{Z}_{(p)}[v_1, \cdots, v_n, v_n^{-1}]$. Denote $L_n X := L_{K(n)}X$. Want maps $L_n \to L_{n-1}$.

It turns out $[E(n)] = \bigvee_{i=0}^{n} [K(n)]$ so we have a natural map $L_n \to L_{n-1}$, this is the **chromatic tower**.

Theorem 12.15 (Chromatic convergence theorem)**.**

$$X = \lim_n L_n X$$

Definition 12.16. A **chromatic filtration** is given by $\ker(\pi_* X \to \pi_* L_n X)$.

Chromatic fracture square:

$$\begin{tikzcd} L_n X \arrow[r] \arrow[d] & L_{K(n)}X \arrow[d] \\ L_{n-1}X \arrow[r] & L_{n-1}L_{K(n)}X \end{tikzcd}$$

is a pullback square.

What can we say about $L_{K(n)}X$?

If X is of type n then $Ext_{BP_*BP}(BP_*, BP_* L_{K(n)}X) \cong H^*(\mathbb{G}_n, (E_n)_* X)$ which converges to $\pi_* L_{K(n)}X$.

13. Lubin Tate cohomology

13.1. Operads.

$C = Top$.

Definition 13.1. An operad C in Top is a collection of spaces $\{C(n)\}_{n \geq 0}$ equipped with

1. a right action of Σ_n on $C(n)$
2. a distinguished $1 \in C(1)$
3. maps $C(k) \times C(i_1) \times \cdots \times C(i_k) \to C(i_1 + \cdots + i_k)$

Example 13.2.

1. Endomorphism operad: $End(X)$ of $X \in Top$ with $End(X)_n = \hom(X^n, X)$
2. Associative operad: $As(n) = \Sigma_n$
3. Commutative operad: $Com(n) = *$
4. E is an E_∞ operad if each $E(n)$ is contractible and the action of Σ_n on $E(n)$ is free.

(5) A is an A_∞ operad if each $A(n)$ is homotopy equivalent to Σ_n.

To every operad C one can associate an endofunctor $C : Top \to Top$ sending $C(n) : \coprod_{n\geq 0} C(n) \times_{\Sigma_n} X^n$. C is in fact a monad.

Definition 13.3. X is a C algebra if it is equipped with a map $\lambda : C(X) \to X$ + appropriate diagrams, equivalently, X is equipped with a collection of maps $C(n) \times_{\Sigma_n} X^n \to X$ with appropriate relations.

Example 13.4. (1) $C(X)$ is a free C algebra

(2) An algebra over Com is a commutative ring.

We want to define $E_n^{h\mathbb{G}_n}$, and hopefully it'll be an E_∞ ring spectrum. First we need to show that E_n itself is E_∞/A_∞ and $\mathbb{G}_n$ acts on E_n by E_∞/A_∞ maps.

Assuming E_n is C - algebra, for an A_∞ algebra C, we'll show that

$$\hom_C(E_n, E_n) \cong \hom_{fgl}((\mathbb{F}_{p^n}, \Gamma), (\mathbb{F}_{p^n}, \Gamma))$$

and then check that E_n is a C algebra.

Let C be an A_∞ operad with E, F algebras over it.

14. Day 4 : Overview

Finitely generated abelian group $A \cong A_{free} \oplus A_p$ the p torsion. Think of this as A lives over $\operatorname{Spec}\mathbb{Z}$, A_{free} lives over $\operatorname{Spec}\mathbb{Q}$ and A_p over $\mathbb{F}_p$.

(Finite) Spectra: The corresponding primes are the $K(n)$ and so we study where a spectrum E is $K(n)$ local or acyclic.

The formal neighborhood of $K(n)$ is the Lubin Tate space, these are governed by E_n and the Morava stabilizer group $\mathbb{G}_n$.

The period map theorem says that this neighborhood can produce an etale cover of $\mathbb{P}^1$.

15. Topology and the moduli of formal groups

Definition 15.1. A **prestack** is a functor $Coverings \cong Aff^{op} \to Grpds$ satisfying a sheaf condition. A **stack** is a prestack with descent data.

Example 15.2. A Hopf algebroid $(\operatorname{Spec} A, \operatorname{Spec}\Gamma)$ is an example, it gives a groupoid: $(\hom(A, -), \hom(\Gamma, -))$, the ring R maps the equalizer of

$$\hom(A, R) \leftarrow \hom(\Gamma, R) \leftarrow \hom(\Gamma, R) \times_{\hom(R),A)} \hom(\Gamma, R)$$

Definition 15.3. An **algebraic stack** is a stack M with affine diagonal and $\exists$ a cover $f : U \to M$ with an affine cover such that f is affine, surjective, flat.

A stack is **rigidified** is algebraic and we have a choice of presentation $\operatorname{Spec} A \to M$.

Theorem 15.4. *There is a 1-1 correspondence between rigidified stacks and flat Hopf algebroids.*

Proposition 15.5. *In the Adams Novikov spectral sequence*

$$E_2^{p,*} \cong Ext^{p,*}_{(MU_*, MU_*MU)}(MU_*, MU_*X) \cong H^p(\mathcal{M}^s_{FG}, \mathcal{F}_X)$$

$\mathcal{M}^s_{fg} = \operatorname{Spec}(L)//G$ where $G = L[b_1, b_2, \cdots]$ and the elements of G act via strict isomorphisms.

$\mathcal{M}_{fg} = \operatorname{Spec}(L)//G \rtimes \mathbb{G}_m$ where now we allow non-strict isomorphisms.

Proposition 15.6. $\operatorname{Spec} L \to \mathcal{M}^s_{fg}$ *is a presentable stack for* $\mathcal{M}^s_{fg}$ *a rigidified stack*

$$\mathcal{M}^s_{fg} \cong \mathcal{M}_{(L,W)} \cong \mathcal{M}_{(MU_*, MU_*MU)}$$

and $\mathcal{M}_{fg} \cong \mathcal{M}_{(MU_*, MU_*MU[b_0^{\pm}])}$

Theorem 15.7. *There is a 1-1 correspondence between quasi-coherent* $\mathcal{O}_{\mathcal{M}(A,\Gamma)}$ *modules and* (A,Γ) *comodules.*

Definition 15.8. A quasi-coherent module assigns to a map $f : \operatorname{Spec} R \to \mathcal{M}_{(A,\Gamma)}$ an R module $\mathcal{M}(f,g)$ such tat for any map $R \to R'$ there is an iso $\alpha : \mathcal{M}(R',g) \cong \mathcal{M}(R,f) \otimes_R R'$.

Definition 15.9. A (A,Γ) comodule is an A module M and an A module map $\phi : M \to M \otimes_A \Gamma$ which is counital and associative.

$\mathcal{F}_X$ is the quasi-coherent $\mathcal{O}_{(MU_*, MU_*MU)}$ module associated to MU_*X.

Proof: Let $\mathcal{M} = \mathcal{M}_{(MU_*, MU_*MU)}$. We have the Cech spectral sequence $H^s(\operatorname{Spec} L^{\times_{\mathcal{M}} t}; \mathcal{F}_X) \implies H^{s+t}(\mathcal{M}, \mathcal{F}_X)$. When $\mathcal{F}_X$ is quasi-coherent and collapses to the $s = 0$ line. Somehow this gives the resolution of MU_*X. $\square$

One should think of $\mathcal{M}_{fg}$ = the stackification of $(\hom(MU_*, MU_*MU))$ as giving a 1 dimensional simplicial object. When we plug in $\mathbb{Q}$ we get an object with 1 connected component, when we plug in $\overline{\mathbb{F}}_p$ we get $\mathbb{Z}_{n\geq 0}$ many connected components one for each height.

Height filtration of $\mathcal{M}^s_{fg}$:

We have a 1-1 correspondence similar to algebraic geometry

closed sub-stack $\leftrightarrow$ quasi coherent ideal sheaves over$\mathcal{M}_{fg}$ $\leftrightarrow$ Ideals of A that are (A,Γ) comodules

Definition 15.10. An invariant prime ideal is an ideal I such that $\eta_L I \subset \Gamma I$ or equivalently $\eta_R I \subset I\Gamma$

Theorem 15.11 (Landweber filtration theorem)**.** *All invariant prime ideals of* MU_* *are* $(p, v_1, \cdots, v_n)$.

This gives us a maximal chain of substacks:

$$\mathcal{M}_{fg} \supseteq \mathcal{M}^{\geq 1}_{fg} \supseteq \mathcal{M}^{\geq 2}_{fg} \supseteq \cdots$$

So we have correspondences

Stack		Classifies	
$\mathcal{M}_{fg}$		formal groups	$\mathcal{M}^s_{(L,W)}$
$\mathcal{M}^{\geq n}_{fg}$	closed	formal groups of height $\geq n$	$\operatorname{Spec} L/(p, \cdots, v_{n-1})//\text{strict iso}$
$\mathcal{M}^{\leq n}_{fg}$	open	formal groups of height $\leq n$	$\operatorname{Spec} \mathbb{Z}_{(p)}/(p, \cdots, v_{n-1}, v_n^{\pm})//\text{strict iso}$
$\mathcal{M}^{n}_{fg}$	neither	formal groups of height $= n$	$\operatorname{Spec} \mathbb{F}_p[v_n^{\pm}]//\text{strict iso}$

Given a spectrum E such that E_*E is flat over E_* then (E_*, E_*E) is a Hopf algebroid and hence gives a stack $\mathcal{M}_{(E_*, E_*E)}$

Proposition 15.12. *If* E *is Landweber exact then* $(E_*, E_*E) \cong (E_*, E_* \otimes_{MU_*} MU_*MU \otimes_{MU_*} E_*)$.

Example 15.13. Consider the homology theory $E(n)_* = \mathbb{Z}_{(p)}[v_1, \cdots, v_{n-1}, v_n^{\pm}]$ which gives rise to the stack $\mathcal{M}^{\leq n}_{fg}$ which gives rise to a commutative diagram

$$\begin{tikzcd}
H^*(\mathcal{M}^s_{fg}; \mathcal{F}_X) \arrow[r, Rightarrow] \arrow[d] & \pi_* X_{(p)} \arrow[d] \\
H^*(\mathcal{M}^{\leq n}_{fg}; \mathcal{F}_X) \arrow[r, Rightarrow] & \pi_* L_{E(n)} X
\end{tikzcd}$$

Landweber exactness is saying that when $\mathcal{M}$ is an MU_* module with a map $u : \operatorname{Spec} L \to \mathcal{M}_{fg}$

(1) $u_*\mathcal{M}$ is flat as a quasi-coherent sheaf over $\mathcal{M}_{fg}$

(2) $(p, v_1, v_2, \cdots)$ is as regular sequence on M.

Theorem 15.14 (Landweber Hopkins)**.** *Both the conditions above are equivalent and in this case $\mathcal{M}$ gives rise to a cohomology theory.*

16. Moduli of Elliptic curves

Example 16.1 (Landweber-Ravenel-Stong)**.** Consider the elliptic curve $C : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$ over $\widetilde{A} := \mathbb{Z}[a_1, \cdots, a_6, \Delta^{-1}]$. This gives rise to a formal group $\widehat{C}$ and a map from $L \to \widetilde{A}$.

Proposition 16.2. *$\widehat{C}$ is Landweber exact.*

Proof: Steps:

(1) $v_0 = p \implies \widetilde{A}/p = \mathbb{F}_p[a_1, \cdots, a_6, \Delta^{-1}]$

(2) $v_1 \in \widetilde{A}/p$ and v_1 invertible $\implies$ there does not exist any supersingular curves

(3) $v_2 \in \widetilde{A}/(p, v_1)$ is in fact invertible

□

So this gives us a cohomology theory E_{Weir} with

$$E_{Weir}(X) = \widetilde{A} \otimes_L MU_*(X)$$

Definition 16.3. An **elliptic cohomology theory** over R consists of the following data:

(1) Elliptic curve C/R

(2) E a spectrum (weakly even periodic)

(3) Isomorphisms $\pi_0 E \cong R$ and $Spf(E^*(\mathbb{CP}^\infty)) \cong \widehat{C}$

Moduli of elliptic curves:

(1) Over $\mathbb{C}$: $\mathcal{M}_{ell}$ is $\mathcal{H}//SL_2(\mathbb{Z}) = \mathcal{M}_{ell} \times \operatorname{Spec}\mathbb{C}$ with a map j to $\mathbb{C}$, the J invariant

(2) Over char $\neq 0$: Consider the map $[p] : C \to C$ If p is invertible, $C[p] = (\mathbb{Z}/p)^2$. If $p = 0$ then $C[p]$ can be $\mathbb{Z}/p$ with formal height 1 (ordinary) or $\{0\}$ with formal height 2 (super singular)

Construct $\mathcal{M}_{ell}$ $\widetilde{A} = \mathbb{Z}[a_1, \cdots, a_6, \Delta^{-1}]$ and $\operatorname{Spec}\widetilde{A}$ is the moduli of smooth Weirstrass equation isomorphisms. $\widetilde{\Gamma} = \widetilde{A}[u^{\pm}, r, s, t]$.

We perform compactifications:

$$\mathcal{M}_{(\widetilde{A},\widetilde{\Gamma})} \to \overline{\mathcal{M}_{ell}}\text{add nodes} \to \overline{\mathcal{M}_{ell}}^{+}\text{add nodes + cusps} = \mathcal{M}_{(A,\Gamma)}$$

We get a map of stacks

$$\phi : \mathcal{M}_{ell} \to \mathcal{M}_{FG}$$
$$C \mapsto \widehat{C}$$

Theorem 16.4 (Stacky Landweber exact functor theorem)**.** *ϕ is flat.*

Proof: We form the pullback

$$\begin{tikzcd} PB \arrow[r] \arrow[d] & \mathcal{M}_{ell} \arrow[d] \\ \operatorname{Spec} R \arrow[r] & \mathcal{M}_{FG} \end{tikzcd}$$

First show that PB is affine i.e. ϕ is representable, then show that PB is flat over R. □

Corollary 16.5. *Given a flat map* $\operatorname{Spec} R \xrightarrow{f} \mathcal{M}_{ell}$ *we get an elliptic cohomology theory* E_f. *In other words, there is a presheaf* $\mathcal{O}^{hom}$ *of homology theories on* $\mathcal{M}_{ell}$.

Look at the geometric fibers of ϕ. Fix $k = \overline{k}$. We want to study the pullback

$$\begin{tikzcd} PB \arrow[r] \arrow[d] & \mathcal{M}_{ell} \arrow[d, "\phi"] \\ \operatorname{Spec} k \arrow[r] & \mathcal{M}_{FG} \end{tikzcd}$$

In char p: Consider the height 2. In this case, $\mathcal{M}_{FG}$ is single point with the automorphism group given by the $\mathbb{G}_n$, so that this stack is $B\mathbb{G}_n$. $\mathcal{M}_{ell} = \mathcal{M}_{ell}^{ss}$ the supersingular curves, so that $PB = \mathcal{M}_{ell}^{ss} \times \mathbb{G}_n$ but the problem is that the map (Geometric point) $\to$ (Stack / scheme) is not flat. One can replace the geometric point by its formal neighborhood which then is flat.

Example 16.6.

$$\begin{array}{ccc} \operatorname{Spec} R/\mathfrak{m} & Spf(R^{\wedge}_{\mathfrak{m}}) & \operatorname{Spec} R \\ K(2)_* = \mathbb{F}_p[v_2^{\pm}] & E_2 & \mathcal{M}_{fg} \\ \text{Elliptic curve}/k & \text{Elliptic curve}/R & \mathcal{M}_{ell} \end{array}$$

Theorem 16.7. *For a field* K *with char* p *and* R *a nilpotent thickening we have a pullback diagram*

$$\begin{tikzcd} \text{SS elliptic curves } /R \arrow[r] \arrow[d] & \{FG/R\} \arrow[d] \\ \text{SS elliptic curves } /K \arrow[r] & \{FG/k\} \end{tikzcd}$$

A modular form of weight n is a global section of $\omega^{\otimes n} \in H^0(\mathcal{M}_{ell}, \omega^{\otimes n})$.

The Adams Novikov spectral sequence is

$$\begin{aligned} Ext(MU_*, MU_*X) &= Rhom(\mathcal{O}_{\mathcal{M}^s_{FG}}, \mathcal{F}_X) \\ &= H^*(\mathcal{M}^s_{FG}, \mathcal{F}_X) \\ H^*(\mathcal{M}_{ell}, \omega^{\otimes a}) &= H^*(\mathcal{M}_{FG}, \mathcal{F} \otimes \omega^{\otimes a}) \implies \pi_* TMF \end{aligned}$$

Q. Is there a universal elliptic cohomology theory? Can we lift the sheaf $\mathcal{O}^{hom}$ to a presheaf taking values in spectrum?

Theorem 16.8 (Goerss, Hopkins, Miller)**.** $\mathcal{O}^{hom}$ *lifts to a sheaf of* E_∞ *spectrum* $\mathcal{O}^{top}$.

Construction:

(1) Assemble $\mathcal{O}^{top}$ from an arithmetic square

$$\begin{tikzcd} \mathcal{O}^{top} \arrow[r] \arrow[d] & \prod_p i_{p*}\mathcal{O}^{top}_p \arrow[d] \\ i_{\mathbb{Q}*}\mathcal{O}^{top}_{\mathbb{Q}} \arrow[r] & \left(\prod_p i_{p*}\mathcal{O}^{top}_p\right) \end{tikzcd}$$

(2) $\mathcal{O}^{top}_{\mathbb{Q}}$ lives over $\mathcal{M}_{ell,\mathbb{Q}}$, is easy to construct and $\mathcal{O}^{top}_p$ lives over $\mathcal{M}_{ell,p}$ are hard to assemble using Hasse squares $\mathcal{M}^{ord}_{ell} \xrightarrow{i_{ord}} (\mathcal{M}_{ell})_p \xleftarrow{i_{ss}} \mathcal{M}^{ss}_{ell}$.

Definition 16.9. Mother of all elliptic cohomology theories: The spectrum **topological modular forms** TMF is the space of sections $\Gamma(\mathcal{O}^{top}, -)$. TMF is even periodic with period 256.

$Tmf = \Gamma(\mathcal{O}^{top}, \overline{\mathcal{M}}_{ell})$ is non-periodic. tmf is the connective cover of Tmf.

Think of $\mathcal{O}^{top}$ as a topological analogue of $\omega^{\otimes *}$ and $MF_* = \Gamma(\omega^{\otimes *}, \mathcal{M}_{ell})$.

Q. What is $\pi_*(TMF)$?

Ans. There exists a spectral sequence with E_2 term given by

$$E_2^{pq} = H^q(\mathcal{M}_{ell}, \pi_p^+(\mathcal{O}^{top}) \implies \pi_{p-q}(TMF))$$

to get

$$\pi_*(TMF) \otimes \mathbb{Q} \cong MF_* = \mathbb{Z}[c_4, c_6, \Delta^{-1}]/(c_4^3 - c_6^2 - 1728\Delta) \text{ and } \pi_*(tmf) = mf_*$$

.

Construction of the spectral sequence works in the following greater generality:

$$\begin{aligned} \mathcal{M} &: \text{ A site with coproducts} \\ \mathcal{O} : \mathcal{M}^{op} \to S_p &: \text{ a sheaf} \\ C = \{U_i \to \mathcal{U}\} &: \text{ a cover} \end{aligned}$$

17. Goerss-Hopkins period map

$k = \mathbb{F}_{p^n}$ and $W = W(k)$ be the Witt vectors. $K = Frac(W)$. G_0 the Honda formal group of height n with $[p] = x^{p^n}$. Lubin-Tate space $LT/Spf(W)$ is non-canonically a unit $n-1$ dimensional ball. $LT(R) = ((G, i), G$ a formal group law over $R, i : G_k \to G_0)$. $\mathbb{G}_n = Aut(G_0) \rtimes Gal(\mathbb{F}_{p^n}/\mathbb{F}_p)$.

Theorem 17.1. *There is an etale surjective map*

$$\pi_{GH} : LT_K \to \mathbb{P}_K^{n-1}$$

(LT_K rigid analytic fiber ??, functions on LT_K are elements of $K[[t_1, \cdots, t_{n-1}]]$ that converge for all $|t_i| < 1$, $t_i \in \overline{K}$) equivariant for the $\mathbb{G}_n$ action.

This theorem gives us a tool for studying equivariant sheaves on LT.

(1) Why should $\mathbb{P}^1$ have connected etale covers?
(2) Why pass to the generic fiber?
(3) Why is this a period map?
(4) What does $\pi_{GH}(X)$ remember about X?