Homotopy groups of MO

We have the following standard facts about the Steenrod algebra, the classifying spaces BO and their Thom spaces MO. The multiplicative structures on homologies are given by the canonical maps $BO \times BO \to BO$ and $MO \wedge MO \to MO$.

$$\begin{aligned}
\mathcal{A}^* &\cong \mathbb{Z}/2\langle Sq^I, I = (i_1, \cdots, i_k) \text{ is such that } i_j \geq 2i_{j+1}\rangle && \text{Steenrod algebra} \\
\mathcal{A}_* &\cong \mathbb{Z}/2[\zeta_i] \text{ where } (\zeta_i)^* = Sq^{2^i} Sq^{2^{i-1}} \cdots Sq^1 && \text{Dual Steenrod algebra} \\
H^*(\mathbb{RP}^\infty) &\cong \mathbb{Z}/2[x], |x| = 1 \\
H_*(\mathbb{RP}^\infty) &\cong \mathbb{Z}/2\langle 1, b_1, b_2, \cdots\rangle \text{ where } (b_i)^* = x^i \\
H^*(BO) &\cong \mathbb{Z}/2[w_1, w_2, \cdots], |w_i| = i \\
H_*(BO) &\cong \mathbb{Z}/2[b_1, b_2, \cdots], |b_i| = i \\
H^*(MO) &\cong (\text{as } H^*(BO) \text{ modules}) \ \mathbb{Z}/2\gamma \otimes \mathbb{Z}/2[w_1, w_2, \cdots] && \gamma \text{ is the Thom class} \\
H_*(MO) &\cong \mathbb{Z}/2[b_1, b_2, \cdots]
\end{aligned}$$

Remark 0.1. Note that MO is a based spectrum so when we say $H_*(MO)$ or $H^*(MO)$ we mean the reduced homology or cohomology.

The Steenrod actions are given by:

$$Sq^i(x) = x + x^2 \tag{0.1}$$

$$Sq^i(w_n) = w_i w_n \mod (w_{n+1}, w_{n+2}, \cdots) \tag{0.2}$$

$$Sq^i(\gamma) = w_i \gamma \tag{0.3}$$

Proposition 0.2. *The coaction of $\mathcal{A}_*$ on $H_*(MO)$ has the property that*

$$b_{2^i-1} \mapsto b_{2^i-1} \otimes 1 + 1 \otimes \zeta^i + \sum_{I,J \neq 0} c_{I,J} b_I \otimes \zeta_J$$

Proof: This is equivalent to proving that

$$\begin{aligned}
&(\zeta^i)^* \gamma = (b_{2^i-1})^* + \text{ other linearly independent terms} \\
\iff\ & Sq^{2^i} Sq^{2^{i-1}} \cdots Sq^1 \gamma = \gamma . w_1^{2^i-1} + \text{ lower powers of } w_1 \mod (w_2, w_3, \cdots)
\end{aligned}$$

This we prove by induction. For $i = 1$ this is simply (0.3). Assume the statement to be true for j. For $j + 1$,

$$\begin{aligned}
&Sq^{2^{j+1}} Sq^{2^j} \cdots Sq^1 \gamma \\
&= Sq^{2^{j+1}}(\gamma . w_1^{2^j-1}) + \text{ lower powers of } w_1 && \mod (w_2, w_3, \cdots) \\
&= (Sq^1 \gamma).(Sq^1 w_1)^{2^j-1} + \text{ lower powers of } w_1 && \mod (w_2, w_3, \cdots) \text{ by (0.2)} \\
&= \gamma . w_1^{2^{j+1}-1} + \text{ lower powers of } w_1 && \mod (w_2, w_3, \cdots)
\end{aligned}$$

$\square$

Proposition 0.3. *The coaction map*

$$H_*(MO) \to H_*(MO) \otimes \mathcal{A}_*$$

is a map of algebras.

Proof: This statement is more generally true for any product spectrum. The coaction map is the homotopy functor π_* applied to the 'inclusion' $MO = MO \wedge S^0 \to MO \wedge H\mathbb{Z}/2$. Because the smash product is associative up to homotopy this is a map of algebras. $\square$

Lemma 0.4. *Let $\mathcal{N}_*$ be the quotient of $H_*(MO)$ obtained by quotienting out all the b_{2^i-1}. The map*

$$\Psi : H_*(MO) \to H_*(MO) \otimes \mathcal{A}_* \to \mathcal{N}_* \otimes \mathcal{A}_*$$

is surjective.

Proof: Because all the maps are maps of algebras it suffices to show that all the generators of $\mathcal{N}_* \otimes \mathcal{A}_*$ are in the image.

These generators are of the form $b_i \otimes \zeta_j$ such that $i+1$ is not a power of 2 and $|b_i| = i$ and $|\zeta_j| = 2^j - 1$. Then

$$\begin{aligned}
\Psi(b_i) &= b_i \otimes 1 + \cdots \\
\Psi(b_{2^j-1}) &= b_j \otimes 1 + 1 \otimes \zeta_j + \cdots \\
\implies \Psi(b_i b_{2^j-1}) &= (b_i \otimes 1 + \cdots).(b_{2^j-1} \otimes 1 + 1 \otimes \zeta_j + \cdots) \\
&= b_i b_{2^j-1} \otimes 1 + b_i \otimes \zeta_j + \cdots \\
&= 0 + b_i \otimes \zeta_j + \cdots
\end{aligned}$$

where $\cdots$ contains terms involving (b_i', ζ_j') which are less than (b_i, ζ_j) in the lexicographical ordering and hence by induction over (i, j) we are done. $\square$

Theorem 0.5. *The map ψ as above is an isomorphism and hence so is its dual*

$$\Psi^* : \mathcal{N}^* \otimes \mathcal{A}^* \to H^*(MO)$$

where $\mathcal{N}^* = \mathbb{Z}/2\gamma \otimes \mathbb{Z}/2[w_i, i \neq 2^j - 1]$ is the dual of $\mathcal{N}_*$.

Proof: We have already shown this to be surjective. The injectivity follows by counting dimensions. $\square$

Theorem 0.6. $\pi_*(MO) \cong \mathbb{Z}/2[\beta_i]$ *where $|\beta_i| = i$ and $i \neq 2^j - 1$ for any j.*

Proof: The $\mathbb{Z}/2$ homotopy groups are computed by running the Adams SS. Thom's theorem identifying MO as the classifying space for unoriented bordisms tells us that $\pi_*(MO)$ is entirely 2-torsion. $\square$