Unless otherwise stated a rings are commutative with 1.

Def^n: Valuation ring:

ring A
- A integral domain, field of fractions K
- $x \in K \Rightarrow x \in A$ or $\frac{1}{x} \in A$. · A not a field

Example:

1) $A = \mathbb{C}[x]_{\wp}$ $\quad \wp = x\mathbb{C}[x]$

$K = \mathbb{C}(x)$

Algebraic analogue of La Hospita's rule

2) $A = \mathbb{C}[[x]]_{\wp}$ $\quad \wp = x\,\mathbb{C}[[x]]$

3) A = Convergent power series

Lemma: A is a valuation ring $\Rightarrow$ A local

Proof: Suppose f, g non-units, non-zero. WALOG let $f/g \in A$.

$(1+\frac{f}{g}) \in$ ~~non-unit~~ $A \Rightarrow \frac{f+g}{g} \in A$

if $f+g$ is a unit then $\frac{1}{g} \in A \Rightarrow g$ unit. Contradiction

if fg is a unit then $\frac{1}{fg} \in A \Rightarrow \frac{1}{g} \in A \Rightarrow g$ unit. Contradiction.

This gives non-units form an ideal, hence unique maximal.

Prop: A - $\mathbb{C}$-algebra. Ring of fractions F $\hookrightarrow$ finitely generated, transcendence degree 1.

A valuation ring $\Rightarrow$

$$\mathbb{C} \longrightarrow A \longrightarrow A/m_A$$

is an isomorphism.

Proof: only need to check surjectivity.

Suppose $\exists f \in A$ s.t. $f + m_A$ is not in image of $\mathbb{C} \longrightarrow A/m_A$

But $\mathbb{C}$ alg. closed $\Rightarrow f+m_A$ transcendant over $\mathbb{C}$.

$\Rightarrow H(f+m_A) \neq 0$ for any polynomial H

$\Rightarrow H(f) = $ unit for any polynomial H

Note: $\mathbb{C}(f)$ has transcendence deg 1 over $\mathbb{C}$.

Same statement can be made for $\mathbb{C}(\bar{f})$.

But f unit $\Rightarrow \mathbb{C}(f) \subseteq A \subseteq F$.

Transcendence deg of $\mathbb{C}(f) = F \Rightarrow$ ~~$\mathbb{C}(f)$~~ $[F : \mathbb{C}(f)]$ algebraic

$\Rightarrow$ $\forall g \in F$ satisfies a polynomial over A, say

$$g^n + a_{n-1} g^{n-1} + \dots + a_1 g + a_0 = 0 \qquad a_i \in \mathbb{C}(f) \subseteq A$$

$$\Rightarrow g = -\left(a_{n-1} + a_{n-2} g^{-1} + \dots + a_0 g^{-1-n}\right)$$

$\Rightarrow$ if $\frac{1}{g} \in A$, $g \in A$

$\Rightarrow A = F$

Not possible.

$\mathcal{F} :=$ All transcendence degree 1 fields, finitely generated $/ \mathbb{C}$

Let $F \in \mathcal{F}$

$R(F) = \{A \subseteq F \mid \mathbb{C} \subseteq A$ and A is a valuation ring$\}$

Suppose $A \in R(F)$

Let $t \in \mathfrak{m}_A \setminus \mathfrak{m}_A^2$

$f \in A \Rightarrow f/t$ or t/f is in A

if $f/t \in A$ then $f = tx$

if $t/f \in A$ then $t/f \in \mathfrak{m}_A$

$\Rightarrow t \in \mathfrak{m}_A^2$ Not possible

~~So $\mathfrak{m}_A$ is principal~~

Remains to show $\mathfrak{m}_A \setminus \mathfrak{m}_A^2$ is non-empty.

This will follow from Nakayama if we ~~assum~~ can prove that $\mathfrak{m}_A$ is finitely generated.

Ex: 1. Complete the above proof and conclude that $\mathfrak{m}_A$ is principal.

2. Every ideal of A is of the form $\mathfrak{m}_A^k$.

- Now we try to impose a topology on $R(F)$. !!
  - $\mathbb{P}^F$ = set of maps $F \to \mathbb{P}^1 = \prod_F \mathbb{P}^1$

    Give $\prod_F \mathbb{P}^1$ a product topology. In this we get a compact set, by Tychonoff.

~~$A \in R(F)$~~,

- $A \in \mathcal{R}(F)$. Define

$$\varphi_A : F \longrightarrow \mathbb{P}^1$$
$$f \longmapsto \begin{cases} f + m_A \in A/m_A \cong \mathbb{C} & \text{if } f \in A \\ \infty & \text{else} \end{cases}$$

- Next we define

$$\varphi_\square : \mathcal{R}(F) \longrightarrow \mathbb{P}^F$$
$$A \longmapsto \varphi_A$$

Injectivity: $\varphi_A = \varphi_{A'} \Rightarrow \frac{1}{f} \in A \Leftrightarrow \frac{1}{f} \in A' \Rightarrow A = A'$

So endow $\mathcal{R}(F)$ with the subspace topology using $\varphi_\square$.

- Next we need to show $\mathcal{R}(F) \xrightarrow{\varphi_\square} \mathbb{P}^F$ is a closed embedding. (We will drop $\varphi_\square$ and assume $\mathcal{R}(F) \subseteq \mathbb{P}^F$) i.e. $A = \varphi_A$

Let $\psi : F \longrightarrow \mathbb{P}^1 \in \overline{\mathcal{R}(F)}$

$A_\psi := \{ f \in F \mid \psi(f) \neq \infty \}$

- A_ψ ring

$f, g \in A_\psi \Rightarrow$ Because $\psi \in \overline{\mathcal{R}(F)}$, ~~using~~ ~~the~~ we will

Show $\psi(f+g) = \psi(f) + \psi(g)$

for $\varepsilon > 0$, take discs of radius ε about $\psi(f), \psi(g), \psi(f+g)$. $U_1 U_2 U_3$

Take $U = (\prod \mathbb{P}^1) \cup U_1 \cup U_2 \cup U_3$

Then U open in $\mathbb{P}^F$.

$\Rightarrow \exists A \in \mathcal{R}(F)$ s.t. $\varphi_A \in U$

Consider $|\psi(f) + \psi(g) - \psi(f+g)| < 3\varepsilon$ by triangle in.

Similarly, for fg, $-f$.

- A_ψ valuation ring

Need to show $f \in A_\psi$ or $\frac{1}{f} \in A_\psi$

if $\psi(f) \neq \infty$ we are done

else look at $\psi(\frac{1}{f})$ and approximate this by valuation ring. This will give us that $\psi(\frac{1}{f}) \neq \infty \Rightarrow \frac{1}{f} \in A_\psi$.

What the fuck is going on?

Our intension is to create a C.R.S. whose field of meromorphic functions is exactly F. We are trying to use valuation rings as points.

~~How because~~

Let use see for $\mathbb{P}^1$ itself:

$F = \mathbb{C}(x)$. Q. what are all the valuation rings of $\mathbb{C}(x)$?

$\mathfrak{m}$ a maximal ideal in $\mathbb{C}[x]$, then $\mathbb{C}[x]_{\mathfrak{m}}$ is valuation.

Since $\mathfrak{m}$ correspond to points in $\mathbb{C}$ these give all the ~~ideals~~ points other than ∞.

Q. How to get infinity?

$\mathbb{C}[\frac{1}{t}]_{1/t}$:D

Q. why are these all the valuation rings?

- We had shown that $A/\mathfrak{m}_A$ is $\mathbb{C}$.

For $A = \mathbb{C}[t]_t$ $\mathfrak{m}_A = t\,\mathbb{C}[t]_t$

what is the canonical isomorphism $\mathbb{C} \longrightarrow \mathbb{C}[t]_t / t\,\mathbb{C}[t]_t$

Claim: $\dfrac{p(x)}{q(x)} \longmapsto \dfrac{p(0)}{q(0)}$

Proof: if $q(t) = 1$,

$p(t) = t(\quad) + p(0)$ ↳ goes to 0

Need to show $t \mid \dfrac{p(t)}{q(t)} - \dfrac{p(0)}{q(0)}$

This is almost tautology.

- So by associating each ~~point of~~ valuation ring $\mathbb{C}[x]_{\mathfrak{m}}$ to a map $\varphi_{\mathfrak{m}}$, we are just associating to each point evaluation of regular functions at that point!

- So in general we are looking at the co-ordinate ring of a non-singular algebraic curve ~~surface~~. The localizations of these at various points will be my A, the function field will be F.

Lemma: $A \in R(F)$ valuation $\Rightarrow$ A dvr.

Proof: $0 \neq z \in \mathfrak{m}_A \Rightarrow \mathbb{C}[z] \subseteq A$ and $\mathbb{C}[z]$ is a polynomial ring

(Because $z \notin \mathbb{C}$ and $\mathbb{C}$ is algebraically closed, so z must be transcendental $/ \mathbb{C}$).

$S = \{ g \in F \mid g$ satisfies poly on $\mathbb{C}[z]$, integral eq$^n\}$

Then, we know S - is a dedekind domain in F

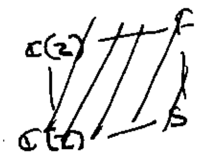
To show this need to show: S is

- dim 1 — using going down & that $\mathbb{C}[z]$ has dim 1
- noetherian domain — !?!
- integrally closed — by def^n

field of fractions of $S = F$?

Yes

$$\begin{tikzcd} S \arrow[r, dash] \arrow[d, dash] & F \arrow[d, dash] \\ \mathbb{C}[z] \arrow[r, dash] & \mathbb{C}(z) \end{tikzcd}$$



~~Define~~ ~~$M = \mathfrak{m}_A \cap S$~~

- A valuation $\Rightarrow$ A integrally closed

  Because suppose $x \in F$ is root of a monic $f(\omega) \in A[\omega]$?

  if $x \notin A \Rightarrow \frac{1}{x} \in A$, but x then can be written as a poly in $\frac{1}{x}$, A and hence is in A contradiction

  $\Rightarrow$ $S \subset A$

- $M_S := \mathfrak{m}_A \cap S$ ideal in S

  $\Rightarrow$ $S_{M_S} \subset A$ ($\because$ we are inverting elements outsides M_S i.e. outside $\mathfrak{m}_A$ which are already units in A)

- S dedekind domain $\Rightarrow$ S_{M_S} is a dvr

- So $A = S_{M_S}$ because $\nexists$ any ring between a dvr and its field of fractions

Note: We have somehow proven here the missing fact that A is noetherian. The hard part still unproven is that "S" is noetherian.

THE local uniformization th^m:

Let $A \subsetneq F$ be a valuation ring with maximal ideal $\mathfrak{m}$.
Then $\exists\ x, y \in \mathfrak{m}$ s.t.

- $xA = \mathfrak{m}$
- $F = \mathbb{C}(x,y)$
- $\exists\ p \in \mathbb{C}[X,Y]$ s.t. $p(x,y) = 0$, $\partial_2 p(x,y) \neq 0$. $\cancel{\partial_2 p(x,y) \neq}$

Proof:

- A dvr $\Rightarrow \mathfrak{m} = xA$ for some $x \in \mathfrak{m}/\mathfrak{m}^2$
- S – integral closure of $\mathbb{C}[x]$ in F, we saw $S \subseteq A$
- Choose $y \in S$ satisfying, $y \notin \mathfrak{m} \cap S$ but y lies in every other maximal ideal containing x in S

$$(x) = (\mathfrak{m} \cap S)\cdot(\mathfrak{P}_1)(\mathfrak{P}_2)\cdots(\mathfrak{P}_k) \quad \text{Dedekind domain}$$

Then we canot have

$$(\mathfrak{m} \cap S) \supseteq (\mathfrak{P}_1)\cdots(\mathfrak{P}_k)$$

- Let $p \in \mathbb{C}[x](Y)$ be minimal poly of $y / \mathbb{C}[x]$.

$T := \mathbb{C}[X,Y] / p(X,Y)$

Look at
$$\mathbb{C}[X,Y] \longrightarrow S$$
$$(X,Y) \longmapsto (x,y)$$
$$p(X,Y) \hookrightarrow \ker$$

Claim: $\ker = p(X,Y)$

Suppose $f(x,y) = 0$

Then because y integral over $\mathbb{C}[x]$ we can use division to get

$$f(x,Y) = p(x,Y)\cdot g(x,Y) \text{ for some } g \in \mathbb{C}[x][Y]$$

$$f(X,Y) - p(X,Y)\, g(X,Y) \quad \forall Y$$

$\Rightarrow x$ satisfies $f(X,Y) - p(X,Y)\,g(X,Y)$ $\forall Y$

$\Rightarrow f(X,Y) - g(X,Y)\,p(X,Y) \equiv 0$

$\Rightarrow p(X,Y) \mid f(X,Y)$

$\Rightarrow \ker = p(X,Y)$.

- ~~So $p(x,Y)$ irred / $\mathbb{C}[x][Y]$ also $p(X,Y)$ irred / $\mathbb{C}[X,Y]$~~

- $T \hookrightarrow S \Rightarrow T$ domain

- $\mathbb{C} \longrightarrow A \longrightarrow A/\mathfrak{m} \cong \mathbb{C}$

$$y \longmapsto c$$

- $x \in \mathfrak{m} \cap T$, $y - c \in \mathfrak{m} \cap T$

$$\frac{T}{(x, y-c)} = \frac{\mathbb{C}[X,Y]}{(p(X,Y), X, Y-c)} = \frac{\mathbb{C}[X,Y]}{(X, Y-c)} \cong \mathbb{C}$$

$\Rightarrow$ $(x, y-c)$ maximal in T

$\Rightarrow$ $(x, y-c) = \mathfrak{m} \cap T$

- $$\begin{tikzcd} \mathbb{C}[x] \arrow[r, hook] & T \arrow[r, hook] \arrow[d, dash] & S \arrow[r, hook] \arrow[d, dash] \arrow[dr, dash] & A \arrow[r, hook] \arrow[d, equal] & F \\ & T_{\mathfrak{m}\cap T} \arrow[r, hook] & S_{\mathfrak{m}\cap T} \arrow[r] & S_{\mathfrak{m}\cap S} & \end{tikzcd}$$

not localizing at $\mathfrak{m}\cap T$ but at multiplicative set $T \setminus \mathfrak{m}\cap T$

maximal ideals in $S_{\mathfrak{m}\cap T}$ $\Longleftrightarrow$ maximal ideals in S containing $\mathfrak{m}\cap T$ $= (x, y-c)$ Why?

Every max ideal containg x contains y except $\mathfrak{m}\cap S$ and hence cannot contain $y-c$

So only max ideal in $S_{\mathfrak{m}\cap T}$ corresponds to $\mathfrak{m}\cap S$

$\Rightarrow$ $S_{\mathfrak{m}\cap T}$ local

- From this somehow conclude that, $T_{\mathfrak{m}\cap T} = S_{\mathfrak{m}\cap T} = S_{\mathfrak{m}\cap S} = A$

- We know that S is a f.g. $\mathbb{C}[x]$ module

hence S is a f.g. T module

hence $S_{\mathfrak{m}\cap T}$ is f.g. $T_{\mathfrak{m}\cap T}$ module

(we are using following ~~etc~~ facts:
$A \subseteq F$ a valuation ring, then ~~a valued~~ A is maximally integrally closed and any such ring is a valuation ring).

- From this somehow conclude that $T_{M\cap T} = S_{M\cap S} = A$

Clearly $T_{M\cap T}$ has field of fractions $\mathbb{C}(x,y)$

Therefore, $F = \mathbb{C}(x,y)$.

- Claim: $\partial_2 p(0,c) \neq 0$

Suppose not. Then $(Y-c)^2 \mid p(0,Y)$, say $p(0,Y) = (Y-c)^2\, g(Y)$

Now, $\left(T/xT\right)_{\mathfrak{m}\cap T} = \dfrac{T_{\mathfrak{m}\cap T}}{xT_{\mathfrak{m}\cap T}} = A/xA = A/\mathfrak{m} = \mathbb{C}$

$\frac{T}{xT} = \frac{\mathbb{C}[Y]}{(Y-c)^2 g(Y)}$, which would give that $\left(\frac{T}{xT}\right)_{m \cap T}$ is not a domain

but it a field, which is a contradiction.

Converse: (Jacobson criteria for plane curves)

Let $F = \mathbb{C}(x,y)$ be a function field $/\mathbb{C}$, $p(X,Y) \in \mathbb{C}[X,Y]$ such that $p(x,y)=0$. Let $(a,b) \in \mathbb{C}^2$ s.t. $p(a,b)=0$, $\partial_2 p(a,b) \neq 0$.

Then, $\exists!\ A \in R(F)$ s.t. $(x-a, y-b) \in m_A$ and $m_A = (x-a)A$.

Proof: ~~There exists a proof.~~

~~Hence Proved.~~

Any such A contains $\mathbb{C}[x,y]$

Note, if $g(X,Y) \in \mathbb{C}[X,Y]$ then

$$g(x,y) = 0 \iff \varphi_A(g(x,y)) = 0$$

for such an A, $g(x,y) - g(a,b) \in m_A$

Therefore such an A contains all fractions of the form

$\frac{f(x,y)}{g(x,y)}$ with $f,g \in \mathbb{C}[T_1,T_2]$ & $g(a,b) \neq 0$

$\because g(a,b)$ is a unit, $g(x,y)$ not a unit in m_A

~~So, A contains all fractions of the form $\frac{f(x,y)}{g(x,y)}$~~

i.e. A contains ring of fractions of $\mathbb{C}[x,y]_m$

m = maximal ideal of $\mathbb{C}[x,y]$ generated by $(x-a, y-b)$

We wish to see: $\mathbb{C}[x,y]_m$ is a DVR

If we show this then as $A \supseteq \mathbb{C}[x,y]_m$

we will have $A = \mathbb{C}[x,y]_m$ by maximality of DVR

This will prove uniqueness and existance

By Taylor's theorem,

$$0 = p(x,y) = \partial_1 p(a,b)(x-a) + \partial_2 p(a,b)(y-b) \mod m^2$$

$$\Rightarrow \quad y-b = -\frac{\partial_1 p(a,b)}{\partial_1 p(a,b)}(x-a)$$

Thus, $(x-a)$ generated m/m^2 and hence by Nakayama

$(x-a)$ generates $m\,\mathbb{C}[x,y]_m$ which means DVR. $\blacksquare$

$p(x,y)=0$ plane curve with (a,b) smooth point i.e. $\partial_2 p|_{(a,b)} \neq 0$
Then, by implicit function theorem, y can be regarded as a holomorphic function of x.

Def^n: Regular Local Ring: $(A,\mathfrak{m})$ Noetherian local, $K = A/\mathfrak{m}$ residue field
if $\dim A = \dim_K(\mathfrak{m}/\mathfrak{m}^2)$ then A regular local.

Implicit function th^m:
$p \in \mathbb{C}[T_1,T_2]$, $(a,b)\in\mathbb{C}^2$, $p(a,b)=0$, $\partial_2 p(a,b)\neq 0$,
Then, $\exists\, \epsilon>0, \eta>0$ and a holomorphic function
$h: D_1 \ni a \longrightarrow D_2 \ni b$ s.t.
$\forall\, (x,y)\in D_1\times D_2 \quad p(x,y)=0 \iff y=h(x)$.

Notation: $A\in R(F)$, $f\in F$
$f_*A := f_*(\varphi_A) := \varphi_A(f) = \begin{cases} \infty & \text{if } f\notin A \\ f+\mathfrak{m}_A & \text{if } f\in A\end{cases}$

- Complex Structure on $R(F)$:

Prop: Let $A_0\in R(F)$ (compact Hausdroff) and let t generates $\mathfrak{m}_0 := \mathfrak{m}_{A_0}$. ~~There~~
Then, t_* maps some open nbd N of A_0 homeomorphically onto an open subset of $\mathbb{C}$. and
Moreover, for any such N, if $f\in A_0$ and $N_f := \{A\in N \mid f\in A\} = N\cap\{f \mid \varphi_A(f)\neq\infty\}$ is also an open nbd. of A_0 and $f_*\circ t_*^{-1}: N_f \longrightarrow \mathbb{C}$ is holomorphic.

Proof: By proof of local uniformization,
$\exists\, y\in\mathfrak{m}_0$ s.t. $F=\mathbb{C}(t,y)$,
$\exists\, p$, poly in 2 variables, $p(t,y)=0$, $\partial_2 p(0,0)\neq 0$

Pick ε, η, h as in implicit f^n th^m taking care that
$\partial_2 p(a,b)\neq 0 \quad \forall\,(a,b)\in D_1\times D_2$

Map the open set
$\{A\in R(F) \mid t_*A\neq\infty,\ y_*A\neq\infty\}$
into $\{(a,b)\in\mathbb{C}^2 \mid p(a,b)\neq 0\}$ using

$$(t_*,y_*): R(F) \longrightarrow \mathbb{P}^1\times\mathbb{P}^1$$
$$A \longmapsto (t_*A,\ y_*A)$$

$D_\varepsilon :=$ ε-neighborhood of 0 in $\mathbb{C}$

$N^* := (t_*, y_*)^{-1}\left(\overline{D_{\varepsilon/2} \times D_{\varepsilon/2}} \cap \{(a,b) \mid p(a,b)=0\}\right)$

By converse of local uniformization:

$$N^* \longrightarrow \overline{D_{\varepsilon/2} \times D_{\varepsilon/2}} \cap \{ \quad \} \text{ is bijective}$$

$\Rightarrow$ Prove that t_*, f_* are continuous.

This will then give that the above map is a homeo.

Remains to show $f_* \circ t_*^{-1}$ holo.

By converse of local uniformization th^m,

$$f \in A_0 \Rightarrow f = \frac{g_1(t,y)}{g_2(t,y)} \qquad g_2(0,0) \neq 0$$

...

- Prop: $\mathcal{R}(F)$ connected. Every meromorphic function on $\mathcal{R}(F)$ is of the form f_* for a unique $f \in F$.

Proof: $\underline{\mathcal{M}}$ - ring of meromorphic functions on $\mathcal{R}(F)$

Have
$$F \longrightarrow \underline{\mathcal{M}}$$
$$f \longmapsto f_* \quad \text{injective}$$

Let $f \in F \setminus \mathbb{C}$, we will show

$$[F : \mathbb{C}(f)] = [\underline{\mathcal{M}} : \mathbb{C}(f)]$$

→ Have to show $\mathcal{R} : \mathcal{F} \longrightarrow \mathcal{X}$ is a functor.
($\mathcal{F}$: category of fields; $\mathcal{X}$: category of CRS ~~funct~~)

→ for $X \in \mathcal{X}$ have to define a natural transformation

$$X \longrightarrow \mathcal{R}\,\underline{\mathcal{M}}(X)$$

which should be an isomorphism.

$$x \longmapsto A_x = \{ f \in \underline{\mathcal{M}}(X) \mid f(x) \neq \infty \}$$