

Euler Characteristic for surfaces

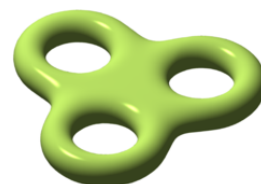
Apurva Nakade

July 13, 2017

Our goal is to determine the Euler characteristic for more complicated surfaces than a sphere. The most general surface that we'll consider is the g holed torus. The Euler characteristic can be defined for much more complicated objects and similar results hold true.

As with the sphere a **surface graph** on a surface is a *connected planar* graph $G = (V, E)$ such that V and E are now on the surface and all the faces are **polygons**.

Exercise 0.1. Being polygonal is a non-trivial condition for a graph on a surface which is not a sphere. Find a planar connected graph with at least 1 face on a torus with 3 vertices which is not polygonal. Does such a graph exist on a sphere?



Example of a surface:
3 holed torus

1 Euler's theorem for surfaces

Let G be a surface graph on the surface S and define $\chi(S)$ to be the Euler characteristic of G .

Theorem 1.1 (Euler characteristic). *The Euler characteristic of a g holed torus is $2 - 2g$.*

The idea behind the proof of this theorem is very similar to what we did for a sphere. However, to prove that the Euler characteristic does not depend on the graph we cannot delete vertices and edges successively as we cannot guarantee that a surface graph would have polygonal faces after deleting edges. Instead we'll *add* edges and vertices. We break up the proof into two steps.

Lemma 1.2. *For any two surface graphs G, G' on a surface S we have $\chi(G) = \chi(G')$. Hence we can define $\chi(S) := \chi(G)$ for any surface graph G .*

Exercise 1.3 (proof of Lemma 1.2). Let S be any surface.

1. Give an example of a surface graph which does not have polygonal faces upon deleting an edge.
2. Show that a surface graphs stays a surface graph and it's Euler characteristic does not change by performing one of the following operations
 - a) adding a new edge joining two existing vertices
 - b) subdividing an edge by declaring it's midpoint as a new vertex
3. Argue that given two different surface graphs G and G' of S it is possible to find a *common refinement* G'' of the two using a sequence of the above two steps.
4. Conclude that $\chi(G) = \chi(G'') = \chi(G')$.

We can now use *any* surface graph to **define** the Euler characteristic of the surface.

Exercise 1.4 (proof of Theorem 1.1).

1. Draw a surface graph on a torus and show that it's Euler characteristic is 0.
2. Glue g many of these surface graphs appropriately to get a surface graph on a g holed torus.
3. Prove that the Euler characteristic of this surface graph is $2 - 2g$.

Exercise 1.5 (suggested by Jan). Show that for any graph G on a surface S , not necessarily with polygonal faces, we have $\chi(G) \geq \chi(S)$. So an alternate way of defining the Euler characteristic is by defining it to be the *minimum* Euler characteristic of any graph on the surface.

1.1 Platonic Solids

A platonic solid is a convex polyhedron all of whose faces are regular polygons which are congruent to each other, i.e. all the edges have the same length and all the faces have the same number of sides e.g. a tetrahedron, a cube, etc. We'll assume that the word convex means that the platonic solid is topologically equivalent to S^2 . We can use the Euler characteristic to prove that there are at most 5 such solids.

Consider a platonic solid S and let V, E, F be its vertices, edges and faces. Suppose all the faces of S are polygons with n number of edges. Further suppose that p edges of S intersect at a single vertex.

Exercise 1.6. Argue that both n and p should be at least 3.

Exercise 1.7. 1. By counting the total number of edges, show that $p|V| = 2|E|$ and $n|F| = 2|E|$.

2. Use $\chi(S^2) = 2$ to show that

$$\frac{1}{n} - \frac{1}{2} + \frac{1}{p} = \frac{1}{|E|} \quad (1.1)$$

and because $|E| > 0$ this implies $\frac{1}{n} + \frac{1}{p} > \frac{1}{2}$.

3. Show that there are at most 5 possible triples $(n, p, |E|)$ that satisfy (1.1). As it turns out all the 5 triples are realizable as regular polyhedra.

You can also do this exercise more geometrically using angle defects.

Exercise 1.8. Does the polyhedron in Fig.2 have i) more 'holes' than you think it should have ii) less 'holes' than you think it should have or iii) the same number of 'holes' as what you think it should have? Verify your answer using the Euler characteristic.

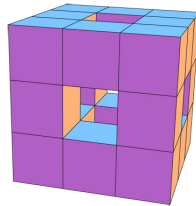


Fig. 2: Rubik's cube with 'holes'

Exercise 1.9. If you know what a Klein bottle and the real projective plane is, find their Euler characteristics.

2 Angle defects

Exercise 2.1. 1. For the cubical torus in Fig.3 find the vertices which have +ve, 0, and -ve curvature without making any computations.

2. Verify your answers by computing the angle defects.

3. Compute the total angle defect and verify Descartes' theorem.

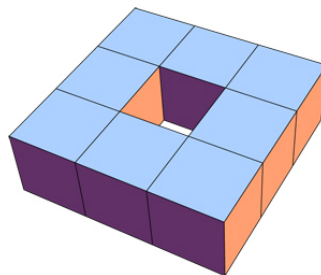


Fig. 3: Cubical torus

Exercise 2.2. For a general polyhedron try to describe what vertices with 0 angle defect can possibly look like.