①

— David Chataur

- Algebraic model of a manifold $\longrightarrow$ covers Pontryagin classes.

$$S^1\times S^1\times S^1 \xrightarrow{c} S^3 \xrightarrow{\eta} S^2$$

$\eta\circ c$ trivial on π_*, H_*

if $\eta\circ c \simeq *$ $\exists$ a lift

$$\begin{tikzcd} & S^1 \arrow[d] \\ S^1\times S^1\times S^1 \arrow[r, "c"] \arrow[ur, dashed, "\exists"] & S^3 \end{tikzcd}$$

but this is not impossible.

- Compute $[X,Y]$: Use CW structure of $X \to$ Quillen's ~~Lie~~ model

  Use Postnikov str. of $Y \to$ Sullivan's model

Eckmann–Hilton duality

- $S^p\times S^q$: Cell structure

$$\begin{tikzcd} S^{p+q-1} \arrow[r, hook] \arrow[d] & e^{p+q} \arrow[d] \\ S^p\vee S^q \arrow[r] & S^p\times S^q \end{tikzcd}$$

encodes the Whitehead product

- $[S^3\times S^3, S^3] \longrightarrow [S^3\vee S^3, S^3]$ is surjective

we get a non-trivial extension

$$\pi_6(S^3)\cong \mathbb{Z}/_{12} \xhookrightarrow{\simeq} [S^3\times S^3, S^3] \longrightarrow\!\!\!\!\rightarrow [S^3\vee S^3, S^3] \cong \mathbb{Z}\oplus\mathbb{Z}$$

Stasheff–Halperin,

Harrison cohomology.

Models for geometry.

- <u>Postnikov tower for S^2</u> :

$$S^2 \longrightarrow \underset{\substack{\| \\ \mathbb{CP}^\infty}}{K(\mathbb{Z};2)} \qquad\qquad H^*(K(\mathbb{Z};2);\mathbb{Z}) \cong \mathbb{Z}[x], \quad |x| = 2$$

$$K(\mathbb{Z};2) \longrightarrow K(\mathbb{Z};4) \qquad \text{represents } x^2 \in H^4(K(\mathbb{Z};2);\mathbb{Z})$$

X_3 = homotopy fiber of this map.

$$[S^1 \times S^1 \times S^1, S^2] \cong [S^1 \times S^1 \times S^1, X_3]$$

Because : ~~we have a m~~

$$\begin{tikzcd} F \arrow[r] & S^2 \arrow[r] & X_3 \\ & & S^1 \times S^1 \times S^1 \arrow[u] \arrow[ul, dashed, "\exists!"] \end{tikzcd}$$

the fiber F is 3-connected.

so $\exists$ a lift unique up to homotopy.

- $K(\mathbb{Z};3) \longrightarrow X_3 \longrightarrow K(\mathbb{Z},2)$ fibration gives us

$$H^3[S^1 \times S^1 \times S^1] \hookrightarrow [S^1 \times S^1 \times S^1, S^2] \twoheadrightarrow H^2(S^1 \times S^1 \times S^1)$$

free dga's

- $T(\mu_2) \longrightarrow C^*(S^2;\mathbb{Z})$

  $\mu_2 \longmapsto c_2$

Need to kill $\mu_2 \otimes \mu_2$, define $d\mu_3 = \mu_2 \otimes \mu_2$

- Turns out this is enough

  $$T(\mu_2, \mu_3, \dots)/_{d\mu_3 = \mu_2} \longrightarrow C^*(S^2;\mathbb{Z})$$

  is a quasi-iso.

* $[C^*(S^2;\mathbb{Z}), C^*(S^3;\mathbb{Z})] \cong \cancel{[T(\mu_2}, \mu_3$

cofib. model

$$\begin{tikzcd}
C^*(S^3;\mathbb{Z}) & \\
& T(v_3, v_5, \dots) \arrow[ul, "\sim"'] \arrow[dl, "\sim"] \arrow[d, mapsto] \\
H^*(S^3;\mathbb{Z}) & v_3 \otimes v_5
\end{tikzcd}$$

<u>Def</u>

Def^n: A dga is formal if it is weakly-equivalent to its homology.

* Replace by $[T(\mu_2, \mu_3, \dots) ; H^*(S^3;\mathbb{Z})]_{dgas}$

$$\cong$$

image of $\mu_3 \cong \mathbb{Z}$

The image of Hopf map $S^3 \to S^2$ in

$$[C^*(S^2; \mathbb{Z}), C^3(S^3; \mathbb{Z})]_{dga} \text{ is non-trivial.}$$

$$[C^*(X; \mathbb{Z}), C^*(S^n, \mathbb{Z})]_{dga} \cong H_{n-1}(\Omega X; \mathbb{Z})$$

up to sign we get Hurewic morphism for ΩX

$$[S^n, X] \cong [S^{n-1}, \Omega X] \longrightarrow [C^*(X; \mathbb{Z}), C^*(S^n; \mathbb{Z})]_{dga} \cong H_{n-1}(\Omega X; \mathbb{Z})$$

- Steenrod squares encode non-commutative of ~~of~~ cup products.

- $E(k) \otimes C^*(-)^{\otimes k} \longrightarrow C^*(-)$

Dold operad: $\mathrm{Dold}(n) = \mathrm{Nat}(C^*(-)^{\otimes n}; C^*(-))$

Natural transformations

. This is a chain complex

. $\mathrm{Dold}(n) \longrightarrow \mathbb{Z}$
evaluation at a point

acyclic model theorem . This map is an ~~iso~~ quasi-isomorphism.

$$\begin{array}{c} \{*\} \\ \| \\ \{\mathrm{comm}(n)\} \\ \| \\ \text{Commutative} \\ \text{operad} \end{array} \qquad \Sigma_n \curvearrowright \mathrm{Comm}(n) \quad \text{trivially}$$

- ~~Dold~~ $\xrightarrow{\sim}$ quasi-iso of operads

  $\mathrm{Dold} \xrightarrow{\sim} \mathrm{Comm}$

- $C^*(-)$ is a natural Dold algebra.

  $$\mathrm{Dold}(n) \otimes (C^*(-))^{\otimes n} \longrightarrow C^*(-)$$

- E_∞ is the cofibrant replacement of Comm !

  (In what model structure ?)

$$\begin{tikzcd} \mathrm{Dold}(n) \arrow[r, "\sim"] & \mathrm{Comm} \\ & E_\infty \arrow[u, "\sim"'] \arrow[ul, dashed, "\exists"] \end{tikzcd}$$

why? · $C^*(-)$ is an E_∞-algebra. (Mcclure-Smith, Berger-Fresse)

- Can we replace $C^*(-)$ by something commutative.

  Problem: cannot transfer a commutative model structure on cdga's Quillen equivalent to E_∞-~~alg~~ dga's unless over a field of char 0.

- $\mathbb{Q}$ is a $\mathbb{Q}[\Sigma_n]$ - projective module. (Check.)

- X, Y 1-connected CW of finite type then $X \simeq Y$ iff their cochain algebras $C^*(X), C^*(Y)$ are homotopy equivalent as E_∞-algebras.

  i.e. $[X, Y] \longrightarrow [C^*(Y; \mathbb{Z}), C^*(X; \mathbb{Z})]_{dga}$

- $\mathbb{F}$ is field of char 0.

  $$\mathrm{Com} \otimes_{E_\infty} - \; : \; E_\infty\text{-dgas} \rightleftarrows \text{cdgas} : \cancel{\text{forget}}\ U$$

  is a Quillen equivalence.

- $X \longmapsto \mathrm{Comm} \otimes_{E_\infty} F_X$ $\qquad$ F_X = natural cofibrant replacement of $C^*(X; \mathbb{F})$

Summary:

$$\begin{tikzcd}
\mathrm{Top} \arrow[r, shift left] & s\mathrm{Sets} \arrow[l, shift left] \arrow[r] & \text{Simplicial coalgebras}/\mathbb{F} \arrow[r, "(-)^\vee"] & \text{Cosimplicial comm. algebras}/\mathbb{F} \arrow[d] \\
 & & & E_\infty\text{-dga's} \arrow[d, shift left] \\
 & & & \text{cdga's} \arrow[u, shift left]
\end{tikzcd}$$

$\Delta : X \longmapsto X \times X$ induces a cup product on the free vector spaces on the simplicial sets

$\mathbb{F}\langle x_\cdot \rangle \mapsto \mathbb{F}\langle x_\cdot \rangle \otimes \mathbb{F}\langle x_\cdot \rangle$

In Char 0, say $\mathbb{F} = \mathbb{Q}$

$$E_\infty\text{-dga's} \rightleftarrows \text{cdga's} \quad \text{is a Quillen equiv.}$$

$$\begin{tikzcd} \text{cosimplicial commutative algebras over } \mathbb{Q} \arrow[r, bend right, "N_c^\#"'] & \text{cdga's} \arrow[l, bend right, "\Gamma"'] \end{tikzcd}$$

Γ = adjoint of conormalization

- $\text{sSets} \xrightarrow{C^*(-;\mathbb{Q})}$ ~~Algs~~ simplicial co-algebra $\xrightarrow{(-)^\vee}$ cosimplicial comm. algebra $\xrightarrow{N_c^\#}$ cdga's

- The composite is symmetric monoidal.

- This composite is quasi-iso to $\text{comm} \overset{L}{\underset{E_\infty}{\otimes}} C^*(-;\mathbb{Q})$.

PL forms (Sullivan's polynomial forms)

Δ^k = standard k-simplex

$$A^*_{PL}(\Delta^k) = S(t_0, \dots, t_k ; dt_0, \dots, dt_k) / \left(\sum t_i = 1 \ ; \ \sum dt_i = 0 \right)$$

Free cdga over $\mathbb{Q}$

$|t_i| = 0$

$|dt_i| = 1$

compatible with faces and degeneracy maps.

- Using the fact that $X \in \text{sSets}$, $X = \underset{\Delta^k \subseteq X}{\text{colim}}\, \Delta^k$

we can extend A_{PL} to sSets.

$$A_{PL} : \text{sSets} \longrightarrow \text{cdga's}$$

we have $H^*(A^*_{PL}(\mathrm{Sing}(X))) \cong H^*(X;\mathbb{Q})$ as graded algebras

- $A^*_{PL}(\Delta^n) \longrightarrow A^*_{PL}(\partial\Delta^n)$
- $\mathbb{F} \xrightarrow{\cong} \Delta^*_{PL}(\text{point})$
- A_{PL} sends cofibrations to fibrations, preserves w.e.

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§ Cochain theory: Mike Mandell

↳ generalize ~~A~~ Eilenberg-Steenrod axioms to cochain levels.

§ ~~Adjo~~ Quillen adjunction:

$$c^*(-;\mathbb{Z}) : sSets^{op} \rightleftarrows dga : |-|$$

$$[X, |b|] \cong [b, c^*(X;\mathbb{Z})]_{dga's}$$

- $b = T(\alpha_n)$ $\quad |\alpha_n| = n$

$$[T(\alpha_n), c^*(X;\mathbb{Z})]_{dga} \cong H^n(X;\mathbb{Z}) \cong [X, |T(\alpha_n)|]_{dga}$$

$|T(\alpha_n)|$ is the Eilenberg-Mclane space $K(\mathbb{Z}, n)$.

$$\begin{tikzcd} T(\mu_2) \arrow[r, hook] & T(\mu_2, \mu_3)/d\mu_3 = \mu_2 \otimes \mu_2 \arrow[d, "\text{cofiber}"] \\ & T(\mu_3) \end{tikzcd}$$

Applying the 1-1 functor to get a fibration seq

$$K(\mathbb{Z};2) \longleftarrow X_3 \longleftarrow K(\mathbb{Z};3)$$

Applying ~~a~~ adjunction

$$[X, X_3] = [T(M_2, M_3)/_{dM_3 = M_2 \otimes M_2}, C^*(X;\mathbb{Z})]_{dga}$$

$$= \{(a,b) : H^2(X;\mathbb{Z}) \times H^3(X;\mathbb{Z}) : a \cup a = 0\}$$

if $C^*(X;\mathbb{Z})$ is formal.

This recovers the topological description of $[X, X_3]$.

$$H^*(K(\mathbb{Z},n);\mathbb{Q}) \cong S(u_n)$$

$\Rightarrow$ $|S(u_n)| \sim K(\mathbb{Q},n)$ is the rationalization.

$$A^*_{PL} : \text{sSets}^{op} \rightleftarrows \text{cdgas} : |-|$$

- in general do this on the Postnikov tower to prove that if X is 1-connected and of $\mathbb{Q}$-finite type then $X \xrightarrow{\sim} |A_{PL}(X)|$.

- Any cdga A has a cofibrant replacement

$$\mathcal{M}_A \twoheadrightarrow A$$

when we forget the differential we can choose $\mathcal{M}_A \cong S(A)$

$$\begin{array}{ccccc} R = S(u_2, & u_4, & u_3 &) \longleftarrow & \underbrace{S(u_2)}_{\text{cofibrant replacement of } R} \\ \downarrow & \downarrow & \downarrow & & \\ 0 & 0 & u_4 - u_2 \otimes u_2 & & \end{array}$$

- odd spheres:

$$S(u_{2n+1}) \longrightarrow \text{model for } S^{2n+1}$$

- even spheres:

$$\begin{array}{ll} S(u_{2n}, u_{4n-1}) & \longrightarrow \text{model for } S^{2n} \\ \quad\quad\quad \downarrow & \\ \quad\quad u_{2n} \otimes u_{2n} & \end{array}$$

- $\mathbb{CP}^n$

$$\begin{array}{ll} S(u_2, u_{2n+1}) & \longrightarrow \text{model for } \mathbb{CP}^n \\ \quad\quad \downarrow & \\ \quad\quad u_2^{n+1} & \end{array}$$

A $\qquad \mathrm{Ind} : A \longmapsto \underbrace{A^+ / A^+ . A^+}_{\text{Indecomposable elements}}$

we can derive this:

$$\mathbb{L}\mathrm{ind} : \mathrm{Ho}(\mathrm{dga}) \longrightarrow \mathrm{Ho}(\mathrm{dg}\text{-}\mathbb{Q}\text{-}\underset{?}{\underline{\mathrm{evs}}})$$

define: $\quad H_*^{\mathbb{Q}}(A) = H_*(\mathbb{L}\mathrm{Ind}(A))$

When A has a minimal model $S(V)$ we get $H_*^{\mathbb{Q}}(S(V)) \simeq S(V)$

cdga's : Postnikov towers

Models of fibrations

cofibrations of cdges.

- Indecomposable elements give you $\pi_*(X) \otimes \mathbb{Q}$.

Quillen :

- Lie $dg_{\mathbb{Q}}$
  - Cellular decompositions
  - Cell attachments.
  - Indecomposable compute $H_*(X; \mathbb{Q})$.

Dichotomy:

Th^m: Let X be 1-connected finite CW complex then either

1) $\sum_{k=1}^{\infty} \dim \pi_k(X) \otimes \mathbb{Q} < \infty$

then we say that X is elliptic.

2) The seq $\dim \pi_k(X) \otimes \mathbb{Q}$ grows exponentially.

$$\sum_{k=1}^{m} \dim \pi_k(X) \otimes \mathbb{Q} > c^m \quad \text{for some } c, \text{ large } m.$$

and X is called hyperbolic.

. An elliptic space satisfies Poincare duality.

Formality :

Example of a non-formal space.

$X = (S^2_a \vee S^2_b \vee S^2_c) \cup e^5$ $\qquad \alpha =$ generator of $\pi_2(S^2_a)$

$[\alpha, [\beta, \gamma]]$

whitehead product.

$$S(a_2, b_2, c_2, \underset{\downarrow}{a_3}, \underset{\downarrow}{b_3}, \underset{\downarrow}{c_3}, \underset{\downarrow}{d_3}, \underset{\downarrow}{e_3}, \underset{\downarrow}{f_3},$$
$$\quad a_2^2 \quad b_2^2 \quad c_2^2 \quad a_2b_2 \quad a_2c_2 \quad b_2c_2$$

$a_3b_2 - a_2b_3$ is a cocycle

$d_3c_2 - b_2e_3$ is a cocycle.

we don't kill $d_3c_2 - b_2e_3$, this cocycle is a Massey product $\langle b_2, a_2, c_2\rangle$. If we had added a class $\xi_4 \mapsto d_3c_2 - b_2e_3$ then under the Lie algebra identification

$$\xi_4 \xmapsto{d} d_3c_2 - b_2e_3$$

$$\xi_4 \rightsquigarrow [[\alpha, \beta], \gamma] - [\beta, [\alpha, \gamma]] = 2\,[[\alpha, \beta], \gamma]$$

- This space is not formal because there exists a non-trivial Massey product.
- One can use Massey products to prove Jacobi identity on Massey product.

<u>Th^m</u>: (Deligne - Griffiths - Morgan - Sullivan)
A compact Kähler manifold is formal.

They prove that the De Rham complex is formal.

- Formality over $\mathbb{R} \Rightarrow$ Formality over $\mathbb{Q}$ (true for finite CW)
- Particular case of Kähler manifold are smooth complex <u>projective</u> varieties.

<u>Open</u> : Does there exist a 1-connected ~~for~~ complex projective variety which is non-formal?

<u>Th^m</u>: An hypersurface ~~is~~ with only isolated singular points is formal.
- Uses : mixed Hodge theory.

→ • $\exists$ symplectic manifolds which are not Kähler.

Formality of Poincaré duality spaces:

<u>Th^m</u>: If X is $(p-1)$ connected, finite CW connected of dim $\leq 3p-2$ then it is formal. Moreover if X satisfies Poincaré duality if it is $(p-1)$-connected of dim $\leq 4p-2$ then it is formal.

eg of non-formal manifolds (1-connected) of dim 7:

$$\begin{tikzcd} S^3 \arrow[r] & S^7 \arrow[d, "\text{Hopf}"'] & X \arrow[l] \arrow[d] \arrow[dl, phantom, "\lrcorner"] & S^3 \arrow[l] \\ & S^4 & S^2 \times S^2 \arrow[l, "\text{degree } 1"'] & \end{tikzcd}$$

<u>Model for X</u> :

- Model for $S^3 \longrightarrow S^7 \longrightarrow S^4$

i.e. want a cofibration of cdga's

$$A^*_{PL}(S^4) \longrightarrow A^*_{PL}(S^7)$$

as S^4 is formal a model is given by

$$\begin{tikzcd}
\begin{matrix} H^*(S^4) \\ \cong \\ \mathbb{Q}[a_4]/a_4^2 \end{matrix} \arrow[r, hook] \arrow[d, leftrightarrow] & H^*(S^4) \otimes_\tau S(\alpha_3) \big/ d\alpha_3 = a_4 \arrow[r] \arrow[d] \arrow[dl, phantom, "\ulcorner"] & S(\alpha_3) \\
\begin{matrix} H^*(S^2 \times S^2) \\ \cong \\ \mathbb{Q}[c_2, d_2]/c_2^2, d_2^2 \end{matrix} \arrow[r] & H^*(S^2) \otimes H^*(S^2) \otimes_\tau S(\alpha_3) \big/ d\alpha_3 = c_2 d_2 &
\end{tikzcd}$$

Using this model : $\exists$ non-trivial Massey products on $H^*(X; \mathbb{Q})$ $\quad \langle c_2, c_2, d_2 \rangle$

As a cohomology algebra (as commutative) $H^*(X; \mathbb{Q}) \cong H^*(S^2 \times S^5 \# S^2 \times S^5 ; \mathbb{Q})$

connected sum of two formal manifolds is formal.

But $H^*(X; \mathbb{Q})$ is not formal !!

§ Poincare Duality Algebras:

X - Poincare duality space / $\mathbb{Q}$ of dim n.

$$\varepsilon : H^n(X;\mathbb{Q}) \longrightarrow \mathbb{Q}$$

$$\begin{aligned} \langle -,- \rangle : H^k(X;\mathbb{Q}) \otimes H^{n-k}(X;\mathbb{Q}) &\longrightarrow \mathbb{Q} \\ a\ ,\ b &\longmapsto \varepsilon(a \cup b) \end{aligned}$$

non-degenerate pairing, $\langle a, bc\rangle = \langle ab, c\rangle$

- Want a cdga (A,d) together with a map
$$\begin{tikzcd} \vdots & \vdots \\ A^n \arrow[r, "\varepsilon"] & \mathbb{Q} \\ A^{n-1} \arrow[u] \arrow[r] & 0 \arrow[u] \\ \vdots & \vdots \end{tikzcd}$$
where we think of $\mathbb{Q}$ as a chain complex concentrated in degree n.

- A is quasi-iso to $A^*_{PL}(X)$.

  ε induces a non-degenerate pairing $A^k \otimes A^{n-k} \longrightarrow \mathbb{Q}$.

  and hence Poincare duality.

$\underline{\underline{\text{Th}^m}}$: (Lambretch, Stanley)

Any 1-connected Poincaré duality space has a Poincaré duality algebra model.

- As a consequence you get ~~con~~ a model for configuration of manifolds.

$M \longrightarrow F(M;k)$ = ordered configuration space of k-points in M.

$= \{(x_1, \dots x_k) : \quad x_i \in M, \quad x_i \neq x_j \quad \text{if } i \neq j\}$

Q. Is this a homotopy invariant functor?

$\exists$ M_1, M_2 non-simply connected ~~c~~ compact s.t.

$M_1 \simeq M_2$ but $F(M_1, k) \not\simeq F(M_2, k)$

Thm (N. Idrissi)

M, N 1-connected, closed, C^∞.

If $M \simeq_{\mathbb{R}} N$ then $F(M;k) \simeq_{\mathbb{R}} F(N;k)$

- The proof uses ~~F~~ Fulton-MacPherson compactification and uses the operad action and hence needs the results about $\mathbb{R}$-formality of operads. Question still open in $\mathbb{Q}$.

Algebraic model for $F(M,k)$:

$$G_A(k) = \left(\frac{A^{\otimes k} \otimes \Lambda(g_{i,j})}{I}, d \right) \qquad |g_{i,j}| = \dim M - 1$$

$$1 \leq i \neq j \leq k$$

model for $F(M;k)$

$(A; d_A)$ - Poincare model for M.

- $\omega_A \in \bigoplus_j A^j \otimes A^{\dim M - j}$ diagonal class.
obtained by dualizing A.
coproduct of the unit

- $\pi_i^* : A \longrightarrow A^{\otimes k}$
$a \longmapsto 1 \otimes \cdots \otimes a \otimes \cdots \otimes 1$

$\pi_{i,j}^* : A \otimes A \longrightarrow A^{\otimes k}$
$a \otimes b \longmapsto \underset{i^{th}}{1 \otimes \cdots \otimes a \otimes \cdots \otimes 1} - \underset{j^{th}}{1 \otimes \cdots \otimes b \otimes \cdots \otimes 1}$

$d(g_{i,j}) = \pi_{i,j}^* (\omega_A)$

I is generated by $g_{i,j} \cdot g_{j,l} + g_{j,l} \cdot g_{l,i} + g_{l,i}\, g_{i,j}$

and the symmetry relations: $(\pi_i^* a - \pi_j^* a) \cdot g_{i,j}$

These are braid relations due to Arnol'd.

<u>Th^m</u> (Sullivan)

Let X be 1-connected, satisfies $\mathbb{Q}$- Poincare duality. of dim 4k, $k \neq 1$ then $\exists$ a smooth manifold M with at map $\phi : M \longrightarrow X$ which induces a rational equivalence iff :

there is a "nice set" of Pontryagin classes and if the intersection form can be lifted over $\mathbb{Z}$. $\{p_i \in H^{4i}(X;\mathbb{Q})\}$

- $\mathbb{L}^{\bullet}_*(-)$: generalized homology theory (Ranicki)

$$\mathbb{L}^{\bullet}_j(-) \otimes \mathbb{Q} \cong \bigoplus_{n \geq 0} H_{j-4n}(X;\mathbb{Q})$$

when M^m is a manifold $\exists$ a fundamental class

$$[M]^+_{\mathbb{L}} \in \mathbb{L}^{\bullet}_m(M) \otimes \mathbb{Q} \cong \bigoplus_n H_{m-n}(M;\mathbb{Q})$$

These classes in H_* are dual to "$\mathbb{L}$-classes" of M which then give rise to Pontryagin classes.

- $\exists$ $\mathbb{L}^{\bullet}_*(X) \otimes \mathbb{Q} \cong \Omega^{Witt}_*(X) \otimes \mathbb{Q}$ (Witt-bordism)

$$\Omega^{Witt}_*(\text{point}) = \text{Witt ring of } \mathbb{Q}.$$

Witt spaces: Singular spaces (pseudo manifolds) that satisfy Poincaré duality for $\{IH^*_{\bar{p}}\}$ intersection cohomology ($\bar{p} \to$ perversity).

§ Intersection cohomology:

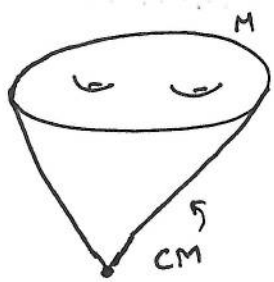
$$\mathbb{IH}^*_{\bar{p}}(- ; \mathbb{Q}) : \text{Stratified Spaces} \longrightarrow \text{Graded Vector Spaces}$$

$\bar{p}$ = perversity = sequence of natural numbers

Cup product : $-\cup-$: $\mathbb{IH}^k_{\bar{p}}(X; \mathbb{Q}) \otimes \mathbb{IH}^l_{\bar{q}}(X; \mathbb{Q}) \longrightarrow \mathbb{IH}^{k+l}_{\bar{p}+\bar{q}}(X; \mathbb{Q})$

- $\mathbb{IH}^*_{\bar{p}}$ is not a homotopy invariant.

eg:



- Intersection cohomology of CM is non-trivial
- dim M = m

  CM has 2 strata: the singular point and $M \times [0,1)$.

$$\mathbb{IH}^*_{\bar{p}}(CM) = \begin{cases} H^*(M) & \text{when } * \leq \bar{p}(m+1) \\ 0 & \end{cases}$$

Not sure what this is.

- $\mathbb{IH}^*_{\bar{p}}$ satisfies stratified homotopy invariants and Mayer-Vietoris.

- Want to lift the cup product to the level of chain complexes.

<u>Th^m</u>: (M. Saralegi, D. Tanre, D. Chataur)

$$\exists \text{ a functor: } \begin{Bmatrix} \text{Stratisfied} \\ \text{spaces} \end{Bmatrix}^{op} \longrightarrow \text{Perverse cdgas}$$

$$X \longmapsto IA^*_{PL,\bar{p}}(X;\mathbb{Q})$$

Theory of minimal model.

<u>Problem</u>: Formality of complex projective varieties.

- This functor factors through

$$\begin{Bmatrix} \text{sSets} \\ \downarrow \\ \text{Posets} \end{Bmatrix}^{op}$$