Honors Single Variable Calculus

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Contents

1 Proofs

We'll start by learning about the various kinds of proofs that we'll encounter in this class.[1]

Proofs are the heart of mathematics. You must come to terms with proofs – you must be able to read, understand and write them. What is the secret? What magic do you need to know? The short answer is: there is no secret, no mystery, no magic. All that is needed is some common sense and a basic understanding of a few trusted and easy to understand techniques.

The Structure of a Proof

The basic structure of a proof is easy: it is just a series of statements, each one being either

- An assumption or
- A conclusion, clearly following from an assumption or previously proved result

And that is all. Occasionally there will be the clarifying remark, but this is just for the reader and has no logical bearing on the structure of the proof.

A well written proof will flow. That is, the reader should feel as though they are being taken on a ride that takes them directly and inevitably to the desired conclusion without any distractions about irrelevant details.

Each step should be clear or at least clearly justified. A good proof is easy to follow. When you are finished with a proof, apply the above simple test to every sentence: is it clearly

1. an assumption
2. a justified conclusion?

If the sentence fails the test, maybe it doesn't belong in the proof.

Example

In order to write proofs, you must be able to read proofs. See if you can follow the proof below. Don't worry about how you would have (or would not have) come up with the idea for the proof. Read the proof with an eye towards the criteria listed above. Is each sentence clearly an assumption or a conclusion? Does the proof flow? Was the theorem in fact proved?

Theorem 1.1. *The square root of 2 is an irrational number.*

Proof. Let's represent the square root of 2 by s. Then, by definition, s satisfies the equation

$$s^2 = 2.$$

[1] This section in taken almost verbatim from http://zimmer.csufresno.edu/~larryc/proofs/proofs.html.

If s were a rational number, then we could write $s = p/q$ where p and q are a pair of integers. In fact, by dividing out the common multiple if necessary, we may even assume p and q have no common multiple (other than 1). If we now substitute this into the first equation we obtain, after a little algebra, the equation

$$p^2 = 2q^2.$$

But now, 2 must appear in the prime factorization of the number p^2 (since it appears in the same number $2q^2$). Since 2 itself is a prime number, 2 must then appear in the prime factorization of the number p. But then, $2 \cdot 2$ would appear in the prime factorization of p^2, and hence in $2q^2$. By dividing out a 2, it then appears that 2 is in the prime factorization of q^2. Like before (with p^2) we can now conclude 2 is a prime factor of q. But now we have p and q sharing a prime factor, namely 2. This violates our assumption above (see if you can find it) that p and q have no common multiple other than 1. □

1.1 Direct Proofs

Most theorems that you want to prove are either explicitly or implicitly in the form

"If P, then Q".

This is the standard form of a theorem (though it can be disguised). A direct proof should be thought of as a flow of implications beginning with P and ending with Q.

$$P \implies \cdots \implies Q$$

Most proofs are (and should be) direct proofs. Always try direct proof first, unless you have a good reason not to. If you find a simple proof, and you are convinced of its correctness, then don't be shy about. Many times proofs are simple and short.

Exercise. 1.2. Prove each of the following.

1. If a divides b and a divides c then a divides $b + c$, where a, b, and c are positive integers.
2. For all real numbers a and b, $a^2 + b^2 \geq 2ab$.
3. If a is a rational number and b is a rational number, then $a + b$ is a rational number.

1.2 Proof by Contradiction

In a proof by contradiction we assume, along with the hypotheses, the logical negation of the result we wish to prove, and then reach some kind of contradiction. That is, if we want to prove "If P, then Q", we assume P and Not Q. The contradiction we arrive at could be some conclusion contradicting one of our assumptions, or something obviously untrue like 1 = 0. The proof of Theorem 1.1 is an example of this.

Exercise. 1.3. Use the method of Proof by Contradiction to prove each of the following.

1. The cube root of 2 is irrational.
2. If a is a rational number and b is an irrational number, then $a + b$ is an irrational number.
3. There are infinitely many prime numbers.[1]

1.3 Proof by Contrapositive

Proof by contrapositive takes advantage of the logical equivalence between "P implies Q" and "Not Q implies Not P". For example, the assertion "If it is my car, then it is red" is equivalent to "If that car is not red, then it is not mine". So, to prove "If P, then Q" by the method of contrapositive means to prove

"If Not Q, then Not P".

How Is This Different From Proof by Contradiction? The difference between the Contrapositive method and the Contradiction method is subtle. Let's examine how the two methods work when trying to prove "If P, then Q".

Method of Contradiction: Assume P and Not Q and prove some sort of contradiction.

Method of Contrapositive: Assume Not Q and prove Not P.

The method of Contrapositive has the advantage that your goal is clear: Prove Not P. In the method of Contradiction, your goal is to prove a contradiction, but it is not always clear what the contradiction is going to be at the start.

Exercise. 1.4. Use the method of Proof by Contrapositive to prove each of the following.

1. If the product of two integers is even, then at least one of the two must be even.
2. If the product of two integers is odd, then both must be odd.
3. If the product of two real numbers is an irrational number, then at least one of the two must be an irrational number.

1.4 Converse

The converse of an assertion in the form "If P, then Q" is the assertion

"If Q, then P".

A common logical fallacy is to assume that if an assertion is true then so is its converse.

[1] There are dozens of proofs of this theorem, originally due to Euclid. Feel free to look one up online.

1.4.1 If and Only If

Many theorems are stated in the form "P, if, and only if, Q". Another way to say the same thing is: "Q is necessary, and sufficient for P". This means two things:

$$\text{"If } P\text{, Then } Q\text{"} \qquad \text{and} \qquad \text{"If } Q\text{, Then } P\text{"}.$$

So to prove an "If, and Only If" theorem, you must prove the theorem and also its *converse*.

Exercise. 1.5. Go back to the problems in Exercises 1.2, 1.3, 1.4 and find the ones which are of the form "If P, Then Q". For each of these:

- State the converse.
- Prove or disprove the converse (by providing either a proof or a counterexample).
- For the problems where the converse is also true rewrite the assertion as an "If, and Only If" statement.

1.5 Quantifiers

It is extremely important in mathematics to be able to formulate very precise statements. To prevent ambiguous statements and logical fallacies the vocabulary used is very limited and every term has a well-defined meaning. The two concepts we need to get used to are *logical operators* and *quantifiers*.

Logical Operators: Operators allow us to form complex statements by combining simpler ones. The basic logical operators are *and*, *or*, and *not*. Their usage is the same as in everyday language. The more complicated logical operator that we'll be using a lot is *If - then -*. We've already encountered several examples of its usage in the previous sections.

Quantifiers: Quantifiers allow us to make abstract statements which are "universally true" without having to specify a concrete element. There are two commonly used quantifiers:

For all/every **There exists**

Understanding and formulating complex statements using these requires a lot of practice. You'll get used to these as the course progresses. Here are a few examples,

Example 1.6.

1. *For every* odd integer a, the integer $a + 1$ is even.
2. An integer a is even if and only if *there exists* an integer b such that $a = 2b$.
3. There *do not exist* integers p, q such that $\frac{p}{q} = \sqrt{2}$.
4. *For every* non-zero rational number x *there exists* a rational number y such that $x \cdot y = 1$.

1.5.1 Nesting Quantifiers

When dealing with statements involving multiple quantifiers the order really matters; changing the order changes the meaning of a statement completely. When you're using multiple quantifiers in a single sentence you should always pause and check if you're using the right quantifiers in the right order.

Exercise. 1.7. For each of the following pairs, explain how changing the order of the quantifiers changes the meaning of the statement.[1]

1. (a) *For every* even integer a *there exists* an integer b such that $a = 2b$.
   (b) *There exists* an integer b such that *for every* even integer a, $a = 2b$.
2. (a) *For every* positive integer n *there exists* a positive real number ϵ such that $\epsilon < 1/n$.
   (b) *There exists* a positive real number ϵ such that *for every* positive integer n, $\epsilon < 1/n$.[2]
3. (a) *For every* even integer a, *for every* odd integer b, $a + b$ is odd.
   (b) *For every* odd integer b, *for every* even integer a, $a + b$ is odd.
4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let ϵ be a positive real number.
   (a) *For every* x, *there exists* a $\delta > 0$ such that *for every* y, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.
   (b) *There exists* a $\delta > 0$ such that *for every* x, *for every* y, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.

1.5.2 Negating Nested Quantifiers

As we saw in Sections 1.2 and 1.3 we often need to negate mathematical statements.

In order to systematically negate complicated logical expressions, we observe that we can negate simple statements in the following manner. (These are called *De Morgan's laws*.)

[1] It's ok to a bit vague in your answers to this question.

[2] Such an ϵ is called an **infinitesimal**. There are number systems, for example the **hypperreal numbers**, which extend the real numbers by incorporating infinitesimals.

Statement	Negation
For every x the statement P is true.	There exists an x such that the statement P is false.
There exists an x such that the statement P is true.	For every x the statement P is false.
P is true and Q is true	P is false or Q is false.
P is true or Q is true	P is false and Q is false.
If P then Q	P but not Q.

If we have multiple quantifiers in a statement then we start with the outermost quantifier and recursively move inwards.

Example 1.8. The negation of *"For every x, there exits a y such that, the statement P is true"* equals *"There exists an x such that, for every y, the statement P is false."*

Exercise. 1.9. Negate each of the following.

1. *For every* even integer a *there exists* an integer b such that $a = 2b$.
2. *There exists* an integer b such that *for every* even integer a, $a = 2b$.
3. *For every* positive integer n *there exists* a positive real number ϵ such that $\epsilon < 1/n$.
4. *There exists* a positive real number ϵ such that *for every* integer n, $\epsilon < 1/n$.
5. *For every* even integer a, *for every* odd integer b, $a + b$ is odd.
6. *For every* $\epsilon > 0$, *for every* x, *there exists* a $\delta > 0$ such that *for every* y, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.
7. *For every* $\epsilon > 0$, *there exists* a $\delta > 0$ such that *for every* x, *for every* y, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.

Word of caution: When the word "and" is used in a list it is not considered a logical operator (this is a deficiency of the English language). For example, negating the following expression

For every even integer a and *for every* odd integer b, $a + b$ is odd,

does not change the *and* to an *or* (what happens if you do change the *and* to an *or*?).

2 Inequalities

In this section we'll learn how to solve inequalities and in the process figure out how to write proofs of statements involving inequalities.

The biggest different between equalities and inequalities is that, unlike equalities, there are usually infinitely many (if any) solutions to inequalities. When we we're trying to solve an inequality we're not looking for the best solution, we're not even looking for a good solution, we're simply looking for **one** solution. For example, both $x = 1.001$ and $x = 10^{100}$ are equally valid solutions to the inequality $x > 1$.

Exercise. 2.1. Find a positive real solution to each of the following inequalities.

1. $x^3 + x < 1$
2. $(1+x)^2 - 1 < 1$
3. $1 - (1-x)^2 < 1$
4. $x^3 + 2x < 1 + x^2$
5. $x^{10} + x > 10$
6. $x^3 - x > 1$

When we try to solve equalities we try to simply the equation until it becomes easy to solve. The same thing is true for inequalities, however, the ways to simplify an inequality are much more subtle.

Example 2.2. If we're trying to solve $P(x) > \epsilon$, where P is some complicated expression that cannot be simplified, then we try to find some simpler $Q(x)$, such that $P(x) > Q(x)$ and try to solve for $Q(x) > \epsilon$ instead. Similarly, if we're trying to solve $P(x) < \epsilon$, then we try to find some simpler $Q(x)$, such that $P(x) < Q(x)$ and try to solve for $Q(x) < \epsilon$ instead.

There is no *canonical way* in which an inequality can be solved. We somehow simplify the inequality and hope that we get a solution.

For solving complicated inequalities, we need a systematic approach towards estimating functions. There are different tricks and identities for estimating different kinds of functions. For now, we'll focus mainly on estimating polynomials.

Exercise. 2.3. Prove the following extremely useful set of inequalities

$$\begin{array}{ll} \text{if } 1 \geq x > 0 & \text{then } 1 \geq x \geq x^2 \geq x^3 \geq \dots \\ \text{if } 1 \leq x & \text{then } 1 \leq x \leq x^2 \leq x^3 \leq \dots . \end{array}$$

Word of Caution: You should be very careful when negative numbers are involved. Multiplying an inequality by a negative number changes it's sign, for example, $2 < 3$ but $-2 > -3$. So,

$$\begin{array}{ll} \text{if } 1 \geq x > 0 & \text{then } -1 \leq -x \leq -x^2 \leq -x^3 \leq \dots \\ \text{if } 1 \leq x & \text{then } -1 \geq -x \geq -x^2 \geq -x^3 \geq \dots . \end{array}$$

2.1 Proofs involving Inequalities

We'll encounter several statements in this class like: *For every $\epsilon > 0$ there exists a positive real number x such that $x^3 + 3x < \epsilon$.* When we want to prove such a statement, we are essentially trying to solve $x^3 + 3x < \epsilon$. (Why?) For example,

Q. 2.4. Prove that for every $\epsilon > 0$ there exists a positive real number x such that $x^3 + 3x < \epsilon$.

We're asking for a solution to the equation $x^3 + 3x < \epsilon$ for an arbitrary positive real number ϵ.

1. The terms x^3 and x have different degrees. We can try to use the inequalities in Exercise 2.3 to simplify their sum, for this we need to know if $x \geq 1$ or $x \leq 1$.
2. We're probably looking for a small x so we could assume $x \leq 1$. If this does not work we'll come back try something else. (Remember that we're trying to find **one** solution.)
3. As $x \leq 1$, $x^3 \leq x$ so that $x^3 + 3x \leq x + 3x = 4x$.
4. As $x^3 + 3x \leq 4x$ and we want to find a solution to $x^3 + 3x < \epsilon$, it suffices to solve $4x < \epsilon$. As ϵ is positive, **one** solution for this is $x = \epsilon/5$.
5. We still need to check if this solution actually works. We can do this by trying to write down a direct proof and making sure that all the implications are logically sound.
6. We required the condition $x \leq 1$ for this solution to work. To ensure this we'll set $x = \min\{1, \epsilon/5\}$.

Now we know what the solution is, but we still need to write down a proof for the original statement. A proof should always start with some assumptions and logically derive the required conclusion. For us the assumption will be $\epsilon > 0$ and $x = \min\{1, \epsilon/5\}$ and the expected conclusion is $x^3 + 3x < \epsilon$.

Proof of Q. 2.4. Let $\epsilon > 0$ and let $x = \min\{1, \epsilon/5\}$. Note that $x \leq 1$ and hence $x^3 \leq x$. Then,

$$\begin{aligned} x^3 + 3x &\leq x + 3x \\ &= 4x \\ &= 4\min\{1, \epsilon/5\} \\ &\leq 4\epsilon/5 \\ &< \epsilon && \text{as } \epsilon \text{ is positive} \end{aligned}$$

Hence, $x = \min\{1, \epsilon/5\}$ is a solution of $x^3 + 3x < \epsilon$, which proves the proposition. □

First we had to solve the inequality then reverse the steps: start with the solution and write a *direct proof* for the proposition. This is how proofs are usually discovered. You make an "educated guess" and hope that it works. Sometimes it doesn't, so you go back to finding another "educated guess".

Remark 2.5 (A note on quantifiers)**.** For proving the statement *"prove that for every $\epsilon > 0$ there exists a positive real number x such that $x^3 + 3x < \epsilon$"* we had to start with the statement *"let $\epsilon > 0$ and $x = \min\{1, \epsilon/5\}$"*. The variable x depends on the variable ϵ. This is fine because of the order of quantifiers: *"...for every $\epsilon > 0$ there exists a positive real number x ..."*; the quantifier for ϵ comes before the quantifier for x and hence the variable x can depend on the variable ϵ.

If instead, suppose we were trying to prove *"there exists a positive real number x such that for every $\epsilon > 0$, $x^3 + 3x < \epsilon$"*. We're still trying to solve the equation $x^3 + 3x < \epsilon$ but now the quantifier for x comes before the quantifier for ϵ and hence x cannot depend on the variable ϵ but instead should be a constant that universally solves the equation $x^3 + 3x < \epsilon$ for every $\epsilon > 0$. No such x exists (why?) and hence the statement *"there exists a positive real number x such that for every $\epsilon > 0$, $x^3 + 3x < \epsilon$"* is false.

Exercise. 2.6. Prove that for every $\epsilon > 0$ there exists a positive real number x such that ...

1. $x^3 + x < \epsilon$.
2. $(1 + x)^2 - 1 < \epsilon$.
3. $1 - (1 - x)^2 < \epsilon$.
4. $x^3 + 2x < \epsilon + x^2$.
5. $x^{10} + x > \epsilon$.
6. $x^3 - x > \epsilon$. [1]

Finally, we are sometimes required to find not one solution but a range of solutions. We try to come up an "educated guess" by the same method, and many a times this gives us the required range for free with a few changes in the final proof.

Q. 2.7. Prove that for every $\epsilon > 0$, there exists a positive real number δ such that, for all x, if $0 < x < \delta$ then $x^3 + 3x < \epsilon$.

Proof. Let $\epsilon > 0$, let $\delta = \min\{1, \epsilon/5\}$ and let $0 < x < \delta$. Because $\delta \leq 1$, we have $x < 1$ and hence $x^3 < x$. As before,

$$\begin{aligned} x^3 + 3x &\leq x + 3x \\ &= 4x \\ &< 4\delta \\ &= 4\min\{1, \epsilon/5\} \\ &\leq 4\epsilon/5 \\ &< \epsilon \qquad \text{as } \epsilon \text{ is positive} \end{aligned}$$

Hence, every $0 < x < \min\{1, \epsilon/5\}$ is a solution of $x^3 + 3x < \epsilon$, which proves the proposition. □

[1] Hint: Break x^3 as $\frac{x^3}{2} + \frac{x^3}{2}$ and find the conditions on x for which $\left(x - \frac{x^3}{2}\right) > 0$.

Exercise. 2.8. Prove that for every $\epsilon > 0$, there exists a positive real number δ such that, for all x, if $0 < x < \delta$ then ...

1. $x^3 + x < \epsilon$.
2. $(1+x)^2 - 1 < \epsilon$.
3. $1 - (1-x)^2 < \epsilon$.
4. $x^3 + 2x < \epsilon + x^2$.

Prove that for every $\epsilon > 0$, there exists a positive real number δ such that, for all x, if $x > \delta$ then

1. $x^{10} + x > \epsilon$.
2. $x^3 - x > \epsilon$.

Optional Problems

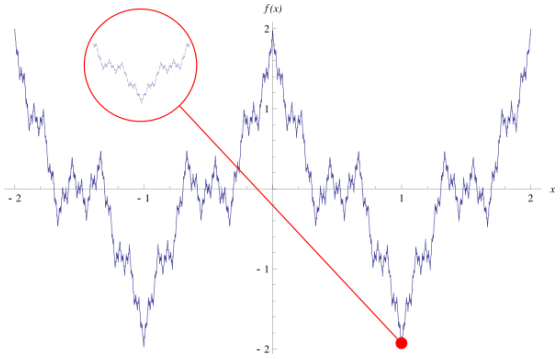
Exercise. 2.9.
1. Prove that for every $\epsilon > 0$, for every real number $x > 0$, there exists a $\delta > 0$ such that, for all real numbers $y > 0$, if $0 < y - x < \delta$ then $y^2 - x^2 < \epsilon$.
2. Prove that the following statement is false: For every $\epsilon > 0$, there exists a $\delta > 0$ such that, for every real number $x > 0$, for all real numbers $y > 0$, if $0 < y - x < \delta$ then $y^2 - x^2 < \epsilon$.

3 Limits & Continuity

We'll now use the proof techniques we've learned so far to study functions.

We'll start with rigorously defining limits. At first this might seem unnecessarily complicated, but having precise definitions will allow us to make more and more sophisticated constructions and later on in the course enable us to prove statements about derivatives and integrals.

Time permitting, towards the end of the semester, we'll construct the Weierstrass function, a function defined on real numbers which is continuous everywhere but differentiable nowhere! But it all starts with the definition of a limit.



Weierstrass function (Image from Wikipedia)

3.1 Limits

At the heart of all analysis (and hence calculus) lies the notion of a limit. Almost every single concept that we'll define will be a limit of some kind.

Definition 3.1 (Provisional definition of limit)**.** For a function f and a real number a, we say that the function f **approaches a limit** L **at** a, for some real number L, if we can make $f(x)$ as close to L as we like by requiring that x be sufficiently close to, but unequal to, a. This is denoted

$$\lim_{x \to a} f(x) = L$$

There are 3 problems with this definition:

1. The terms *"as close to L as we like"* and *"sufficiently close"* are not precise. (How close is sufficiently close?)
2. The order of implication is unclear.
3. The definition does not give us a way to come up with the limit L.

First, we'll make the terms precise. We can measure the distance between two real numbers x, y by taking their difference $x - y$, however, the difference can be negative so we use the absolute value $|x - y|$ instead. So that

x sufficiently close to, but unequal to, a	translates to	$0 < \lvert x - a \rvert < \delta$
$f(x)$ *as close to L as we like*	translates to	*for every* $\epsilon > 0$, $\lvert f(x) - L \rvert < \epsilon$

and the definition of limit becomes

> ... f **approaches a limit** L **at** a, for some real number L, if we can make $|f(x) - L| < \epsilon$ for every $\epsilon > 0$ by requiring that $0 < |x - a| < \delta$ for some $\delta > 0$.

Next, to clarify the order of implication we rephrase the statement in the more standard *"If ... then"* form and put all the quantifiers in the front,

> ... f **approaches a limit** L **at** a, for some real number L, if for every $\epsilon > 0$, there exists a $\delta > 0$ such that, for all x, if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

The third problem cannot be fixed! There is no systematic way of finding the limit of a general function, the only thing we can do is *guess* the limit and then use the definition to *prove* that it is indeed the limit. For special functions like polynomials, trig functions, exponential, and logarithms, we can find limits explicitly, which is what makes these functions useful for approximating and estimating more complicated functions.

Definition 3.2 (Formal definition of limit)**.** We say that the function f **approaches a limit** L **at** a, for some real number L, if for every $\epsilon > 0$, there exists a $\delta > 0$ such that, for all x, if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

Exercise. 3.3. Come up with formal definitions for the following:

1. The function f approaches a limit L at a **from the right**, for some real number L, if we can make $f(x)$ as close to L as we like by requiring that x be sufficiently close to, but strictly greater than a. This is denoted
$$\lim_{x \to a^+} f(x) = L.$$
2. The function f approaches a limit L at a **from the left**, for some real number L, if we can make $f(x)$ as close to L as we like by requiring that x be sufficiently close to, but strictly smaller than a. This is denoted
$$\lim_{x \to a^-} f(x) = L.$$
3. The function f **approaches the limit** ∞ **at** a, if $f(x)$ can be made as large as we like by requiring that x be sufficiently close to, but unequal to, a. This is denoted [1]
$$\lim_{x \to a} f(x) = \infty.$$
4. The function f **approaches a limit** L **at** ∞, for some real number L if we can make $f(x)$ as close to L as we like by requiring that x be sufficiently large. This is denoted
$$\lim_{x \to \infty} f(x) = L.$$

[1] In mathematics ∞ means several things, in fact, there are infinitely many infinities. For us, ∞ is a *placeholder* for a limit of a function that grows very large. ∞ is not a real number and hence cannot be used in equations.

Exercise. 3.4. For each of the following, first guess the limit, then use the formal definition of limit to prove that it is indeed the limit.

1. $\lim_{x \to 1^+} 2x + 1$
2. $\lim_{x \to 1^+} x^2$
3. $\lim_{x \to \infty} x^{10} + x$
4. $\lim_{x \to 0^+} 1/x$
5. $\lim_{x \to 0} f(x)$ where $f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

Optional Problems

Exercise. 3.5. Give examples to show that the following definitions of $\lim_{x \to a} f(x) = L$ are not correct.

1. For every $\delta > 0$, there exists an $\epsilon > 0$ such that, for all x, if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.
2. For every $\epsilon > 0$, there exists a $\delta > 0$ such that, for all x, if $|f(x) - L| < \epsilon$ then $0 < |x - a| < \delta$.

Exercise. 3.6. Prove that if $\lim_{x \to a} f(x) = L$ and $L \neq 0$ then $\lim_{x \to a} 1/f(x) = 1/L$.

3.1.1 Triangle Inequality

To prove abstract theorems involving absolute values we will need the following very important inequality called the **triangle inequality**.

$$|x - y| \leq |x| + |y|$$

If you think of the points 0, x, y (on the real axis) as being the three vertices of a (degenerate) triangle, then $|x|$, $|y|$, and $|x - y|$ are the lengths of the three sides and the triangle inequality is saying that: *the sum of the lengths of two sides of a triangle is greater than or equal to the length of the third side.*

There are other forms in which the triangle inequality is commonly used, e.g.

$$\begin{aligned} |x + y| &\leq |x| + |y| \\ |x + y| - |y| &\leq |x| \\ |x| &\leq |x - y| + |y| \end{aligned}$$

These inequalities are central to a lot of analysis proofs.

Exercise. 3.7.

1. Prove that $|x| + |x - 1| \geq 1$ for any real number x.
2. Prove that for any real number x, at least one of $|x|$ and $|x - 1|$ is $\geq 1/2$.[1]

We'll use these to prove inequalities about the non-existence of limits.

Definition 3.8. If f does not approach the limit L, for any real number or ∞ or $-\infty$, then we say that the limit of f at a **does not exist**.

Exercise. 3.9. Negate the formal definitions of limits and come up with an ϵ, δ definition for the following.

1. f **does not** approach the limit L, where L is a real number, at a.
2. f **does not** approach the limit ∞ at a. (Similarly for $-\infty$.)

Exercise. 3.10. Let $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$

1. Prove that for every real number L, $\lim_{x \to 0} f(x) \neq L$.
2. Prove that $\lim_{x \to 0} f(x) \neq \infty$. (Similarly for $-\infty$.)

Hence the limit of f at 0 does not exist.

Exercise. 3.11.

1. Prove that $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^+} f(x) = L$ and $\lim_{x \to a^-} f(x) = L$.[2] (Similarly for ∞, $-\infty$.)
2. Prove that $\lim_{x \to 0} \frac{1}{x}$ does not exist.

[1] Hint: Proof by Contradiction.

[2] Hint: This is very easy, don't overthink! Simply write down the formal definitions.

3.2 Non-existence of limits

Let's move backwards and try and understand what it means *intuitively* for a function to not approach a limit L near a. The solution to Exercise 3.9 is the following:

> *(Formal definition)* The function f **does not approach thelimit** L **at** a if there exists an $\epsilon > 0$, such that for all $\delta > 0$, there exists an x such that, $0 < |x - a| < \delta$ and $|f(x) - L| \geq \epsilon$.

Going back to how to we came up with the Formal Definition of Limit 3.2 from the Provisional Definition of Limit 3.1 and replacing the absolute values by the distance between points, we can work backwards and get the following provisional definition:

> *(Provisional definition)* The function f **does not approach the limit** L **at** a if no matter how close we are to a, there is some x for which $f(x)$ is not close to L.

Optional Problems

Exercise. 3.12.

1. Explain how the provisional definition captures the notion of the failure of the function to approach a limit L at a.
2. Starting from the provisional definition try to systematically come up with the formal definition using arguments similar to the ones we made to go from Definition 3.1 to Definition 3.2.

Exercise. 3.13. It is easy to show that for the function

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

the limit $\lim_{x \to 0} f(x)$ does not exist by computing the limits from the left and the right.

Try to prove that, $\lim_{x \to 0} f(x) \neq L$ for some real numbers, say $L = 0, 1, 1/2$, and more generally for any real number, using the formal definition and try to explain what the variables ϵ, δ, x in your proof mean *geometrically*.