Acyclic model theorem:

About functors $F: K \longrightarrow ACh_R$

K: Arbitrary Category; ACh_R: Augemented Chain Complexes on R

Model : $\mathfrak{M} = \coprod_{n \geq 0} \mathfrak{M}_{(n)}$ each $\mathfrak{M}_{(n)}$ = collection of objects in K

Free : $F: K \longrightarrow ACh_R$ free over $\mathfrak{M}$ if $\exists\ \sigma_n(M) \subseteq F(M)_{(n)}$, $\forall M \in \mathfrak{M}_{(n)}$
such that $\forall X$, $F(X)_{(n)} = \bigoplus_{\substack{f: M \longrightarrow X \\ M \in \mathfrak{M}_n}} R f_*(\sigma_n(M))$

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eg: 1) $\mathfrak{M}_n = \{\Delta_n\}$ $\qquad F(X) = S(X,R)$

singular chains with co-efficients in R

$$F(\Delta_n)$$
$$\| $$
$$\cdots \to R(\Delta_m \to \Delta_n) \longrightarrow R(\Delta_{m-1} \longrightarrow \Delta_n) \longrightarrow \cdots \longrightarrow R(\Delta_0 \longrightarrow \Delta_m) \longrightarrow R \longrightarrow 0 \cdots$$

$\sigma_n(\Delta_n) = id: \Delta_n \longrightarrow \Delta_n$

$$F(X)_n = \bigoplus_{f: \Delta_n \longrightarrow X} Rf = \bigoplus_{f: \Delta_n \longrightarrow X} R f_* \sigma_n(\Delta_n)$$

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2) $\mathfrak{M}_n = \{\Delta_n \times \Delta_n\}$, $\quad F: Top \times Top \longrightarrow ACh_R$

$\qquad X, Y \longmapsto S_*(X \times Y, R)$

$\sigma_n(\Delta_n \times \Delta_n) \in F(\Delta_n \times \Delta_n)_n = \bigoplus R(f: \Delta_n \longrightarrow \Delta_n \times \Delta_n)$

Pick σ_n to be the diagonal map

$$F(X,Y)_n = S_n(X \times Y; R) = \bigoplus_f R(f: \Delta_n \longrightarrow X \times Y)$$
$$= \bigoplus_f R(\Delta_n \xrightarrow{\sigma_n} \Delta_n \times \Delta_n \xrightarrow{f} X \times Y)$$
$$= \bigoplus_f R f_*(\sigma_n)$$

$\Rightarrow$ free

3. $\mathcal{M}_n = \{(\Delta_i, \Delta_{n-i})\}_{i\geq 0}$, $F : \mathrm{Top}\times\mathrm{Top} \longrightarrow \mathcal{A}Ch_R$

$$X, Y \longmapsto S_*(X,R) \otimes_R S_*(Y,R)$$

$\sigma_n(\Delta_i, \Delta_{n-i}) \in F(\Delta_i, \Delta_{n-i}) = \bigoplus_j R(f : \Delta_j \longrightarrow \Delta_i) \otimes (g : \Delta_{n-j} \longrightarrow \Delta_{n-i})$

Again pick σ_n to be $id \otimes id$

Acyclic: $F : K \longrightarrow \mathcal{A}Ch_R$ acyclic wrt $\mathcal{M}$ if $\forall\, M \in \mathcal{M}$, $F(M)$ is acyclic.

eg: 1) $\mathcal{M}_n = \{\Delta_n\}$ $\quad F = S_*(-;R)$

acyclic as Δ_n contractible

2) $\mathcal{M}_n = \{\Delta_n \times \Delta_n\}$ $\quad F(X,Y) = S_*(X\times Y; R)$ $\rightsquigarrow$ easy

$G(X,Y) = S_*(X;R) \otimes S_*(Y,R)$ $\rightsquigarrow$ Tensor of two acyclic chain complexes acyclic?

$\rightsquigarrow$ Not necessarily but we have free and acyclic

Look at the double complex:

$$S_p(X;R) \otimes_R S_q(Y;R)$$

$\rightsquigarrow$ This will collapse at E_2 as rows and columns are exact (we required freeness or flatness here.)

3) Similarly for $\mathcal{M}_n = \{\Delta_i \times \Delta_{n-i}\}_{i \geq 0}$

Acyclic model theorem:

$F, F' : K \longrightarrow \mathcal{A}Ch_R$, F free, F' acylic wrt $\mathcal{M}$ then,

$\exists\; T : F \longrightarrow F'$ unique upto chain homotopy.