HW QUESTIONS (TOPICS IN TOPOLOGY, FALL 2013)

Characteristic Classes:

(1) Consider the inclusion of the maximal torus:
$$\alpha : \mathrm{BT} = (\mathrm{BSO}(2))^{\times n} \longrightarrow \mathrm{BSO}(2n).$$
Show that α satisfies the equalities in integral cohomology:
$$\alpha^*(p_i) = \sigma_i(x_1^2, x_2^2, \cdots, x_n^2), \quad i \leq n, \quad \alpha^*(e_{2n}) = x_1 \dots x_n,$$
where x_k denotes the Euler class of the tautological line bundle over the k-th factor $\mathrm{BSO}(2)$. Note also that the notation $\sigma_i(y_1, y_2, \cdots, y_n)$ denotes the i-th elementary symmetric polynomial in the variables $y_1, y_2, \dots, y_n$.

(2) Consider the complexification map:
$$\beta : \mathrm{BO}(n) \longrightarrow \mathrm{BU}(n).$$
Show that β satisfies the following equalities in mod 2-cohomology:
$$\beta^*(c_i) = w_i^2 \mod 2, \quad i \leq n.$$

(3) Consider the forgetful map:
$$\gamma : \mathrm{BU}(n) \longrightarrow \mathrm{BSO}(2n).$$
Show that γ satisfies the following equalities in mod 2-cohomology:
$$\gamma^*(w_{2i}) = c_i \mod 2, \quad i \leq n.$$
In addition, let $P_\bullet$ denote the alternating sum of the Pontrjagin classes given by: $P_\bullet = 1 - p_1 + p_2 \dots + (-1)^n p_n$. Show that we have the following integral equalities:
$$\gamma^*(P_\bullet) = (1 - c_1 + c_2 - c_3 + \cdots (-1)^n c_n)(1 + c_1 + c_2 + \cdots + c_n), \qquad \gamma^*(e_{2n}) = c_n.$$

(4) Let δ denote the diagonal inclusion:
$$\delta : \mathrm{BSO}(2n-1) \longrightarrow \mathrm{BSO}(2n).$$
Show that δ satisfies the integral equality: $\delta^*(e_{2n}) = 0$.

(5) Use the splitting principle to show that the Chern classes c_i are the unique family of characteristic classes for complex vector bundles that satisfy the four defining axioms. Show the same fact for the Stiefel-Whitney classes.

(6) Can you construct a complex vector bundle E over a finite dimensional manifold M, so that all its Chern classes are trivial, but the bundle E is non-trivial even after stabilizing (i.e. adding an arbitrary number of trivial bundles to E) ? Can you show that the dimension of M has to be at least 5?
As a hint, consider the map representing the total Chern class:
$$c : \mathbb{Z} \times \mathrm{BU} \longrightarrow \prod_{n \geq 0} \mathrm{K}(\mathbb{Z}, 2n),$$
where $\mathrm{K}(\mathbb{Z}, 2n)$ denotes the Eilenberg-MacLane space representing integral cohomology in degree $2n$. Show that the fiber of c is a CW complex that is 4-connected. Next, show that the 5-skeleton of the fiber admits a non-trivial map to $\mathbb{Z} \times \mathrm{BU}$. Use these observations to answer the question.

Connections, Curvature and the Gauge group:

In the next few questions, let ω be a principal connection on (P, π, B), with structure group G and corresponding Lie algebra $\mathfrak{g}$. Let V be a representation of G. Given $\alpha \in \Omega^p(P, \mathfrak{g})$ and $\beta \in \Omega^q(P, V)$ we define $\alpha \wedge \beta \in \Omega^{p+q}(P, V)$ by the formula:

$$\alpha \wedge \beta\,(x_1, \ldots, x_{p+q}) = \frac{1}{p!\,q!} \sum_{\sigma \in \Sigma_{p+q}} (-1)^\sigma \alpha(x_{\sigma(1)}, \ldots, x_{\sigma(p)}) * \beta(x_{\sigma(p+1)}, \ldots, x_{\sigma(p+q)}).$$

(7) Show that d_V (defined as $\rho \circ d$ in class) satisfies the formula:

$$d_V(\eta) = d\eta + \omega \wedge \eta.$$

Show also that d_V^2 satisfies the formula (you will need to use the Jacobi identity):

$$d_V^2(\eta) = (d\omega + \frac{1}{2}\,\omega \wedge \omega) \wedge \eta.$$

(8) Prove the identity $\omega \wedge (\omega \wedge \omega) = 0$. Use this to prove the Bianchi identity:

$$d_{\mathfrak{g}}(\Omega_\omega) = 0.$$

(9) Let $p \in P$ be any point. Show that Ω_ω satisfies:

$$\Omega_\omega(\alpha, \beta)_p = -\omega[\hat{\alpha}, \hat{\beta}]_p,$$

where $\hat{\alpha}$ and $\hat{\beta}$ are any choice of G-invariant horizontal vector fields extending α and β respectively.

(10) Consider the morphism:

$$\begin{tikzcd} P \times V \arrow[r, "\pi_V"] \arrow[d] & P \times_G V \arrow[d] \\ P \arrow[r, "\pi"] & B. \end{tikzcd}$$

Show that this morphism induces an isomorphism of vector spaces:

$$\pi_V^* : \Omega^0(\Lambda^k(T^*B) \otimes \zeta_V) \longrightarrow \Omega^k_{G,hor}(P, V) := \Omega^k(\zeta_V).$$

(11) Given an element in the Gauge group $\varphi \in \mathcal{G}(P)$, let $\underline{\varphi} : P \longrightarrow G$ denote the corresponding Ad-equivariant map. Using the geometric description of a connection (as an invariant horizontal distribution on P), show that $\varphi^*\omega$ is also a principal connection on (P, π, B), and that it satisfies the equality:

$$\Omega_{\varphi^*\omega} = Ad_{\underline{\varphi}^{-1}}(\Omega_\omega).$$

(12) Consider the right action map: $\mu : P \times G \longrightarrow P$. Show that:

$$\mu^*\omega_{(p,g)} = Ad_{g^{-1}}(\omega_p) \times \theta_g,$$

where θ is the tautological left invariant $\mathfrak{g}$-valued one-form on G. Now let the map $\underline{\varphi} : P \longrightarrow G$ represent an element φ of the Gauge group $\mathcal{G}(P)$. Show, using the above observation that:

$$\varphi^*\omega = Ad_{\underline{\varphi}^{-1}}(\omega) + \underline{\varphi}^{-1} d\underline{\varphi}.$$

(13) Let $\psi \in \Omega^0(\zeta_{\mathfrak{g}})$ be a vertical vector field seen as an Ad-equivariant map $\psi : P \longrightarrow \mathfrak{g}$. Let $\underline{\varphi}_t = \exp(t\psi)$ represent a one parameter family φ_t in the Gauge group $\mathcal{G}(P)$. Differentiate the equality in the previous question to give an alternate proof of the eqality:

$$\mathcal{L}_{\psi}\omega := \frac{\partial}{\partial t}\varphi_t^*\omega = d\psi + [\omega, \psi] = d_{\mathfrak{g}}(\psi).$$

(14) Find the error, and correct the proof I presented in class that shows that the expression $\det(I + t\frac{i}{2\pi}\Omega)$ is a formal power series of closed forms on the base manifold. Proceed as follows: First consider the formal power series in Ω:

$$\ln(I + t\frac{i}{2\pi}\Omega) = \sum_{k>0}(-1)^{k-1}(t\frac{i}{2\pi}\Omega)^k,$$

where Ω^k denotes multiplication of matrices with values in forms. Use the Bianchi identity, (and question 11) to show that the trace of this expression $tr\ln(I + t\frac{i}{2\pi}\Omega)$ is a formal power series of closed, real valued forms on the base manifold. Now use the fact that:

$$\det(I + t\frac{i}{2\pi}\Omega) = \exp tr\ln(I + t\frac{i}{2\pi}\Omega).$$

Multiplicative Sequences, Genera and the Index Theorem:

(15) Given a multiplicative sequence $f(x)$, use the naturality of characteristic classes and the Stokes theorem to show that the f-genus of a manifold M is a cobordism invariant of M (i.e. $f(M)$ depends only on the cobordism class of M). Here, by convention, we mean oriented cobordism when talking about real sequences, and complex cobordism for general multiplicative sequences.

(16) Given a multiplicative sequence $f(x)$, show that the f-genus is multiplicative. In other words, show that $f(M \times N) = f(M)\,f(N)$. Show furthermore that the f-genus is additive with respect to disjoint union: $f(M \coprod N) = f(M) + f(N)$. This question, along with the previous one, shows that genera correspond to ring homomorphisms with domain being the cobordism ring.

(17) Show from the definition of the L-genus that $L(\mathbb{CP}^{2n}) = 1$.

(18) Using the definition of the Todd genus, express $Td(\Sigma_g)$ for a compact complex Riemann surface Σ_g in terms of the genus g. Can you use the Hodge-decomposition of complex deRham cohomology: $H^*(\Sigma_g, \mathbb{C})$ to show that this number is identical to the holomorphic Euler-characteristic of Σ_g? This is a special case of the Riemann-Roch formula, which is a baby case of the Atiyah-Singer Index formula.

(19) Use the splitting principle to show that the Chern character is natural with respect to morphisms of vector bundles, and that it satisfies:

$$ch(E \oplus F) = ch(E) + ch(F), \quad ch(E \otimes F) = ch(E) \cup ch(F).$$

(20) Use the Atiyah-Singer Index theorem for the differential operator:

$$D = d + d^* : \Omega^{ev}(M) \longrightarrow \Omega^{odd}(M)$$

to prove the Gauss-Bonnet theorem:

$$\chi(M) = \int_M e(TM).$$

(21) Let V be a vector space with an inner product, so that V^* gets an induced inner product. Given $\alpha \in V^*$, consider two maps $\wedge(\alpha)$ and $\iota(\alpha)$:

$$\wedge(\alpha), \iota(\alpha) : \Lambda^{ev}(V^*) \longrightarrow \Lambda^{odd}(V^*), \quad \wedge(\alpha)\eta = \alpha \wedge \eta, \quad \iota(\alpha)\eta = \iota_\alpha(\eta),$$

where $\iota_\alpha(x_1 \wedge x_2 \wedge \cdots \wedge x_k)$ is defined as:

$$\iota_\alpha(x_1 \wedge x_2 \wedge \cdots \wedge x_k) = \sum_i (-1)^{i-1} \langle \alpha, x_i \rangle \, x_1 \wedge \cdots \wedge x_{i-1} \wedge x_{i+1} \wedge \cdots \wedge x_k.$$

Show that the map $\sigma(\alpha) = \wedge(\alpha) + \iota(\alpha)$ is an isomorphism for all $\alpha \neq 0$.

(22) Consider the operator:

$$D := d + d^* : \Omega(\Lambda^{ev}(TM^*)) \longrightarrow \Omega(\Lambda^{odd}(TM^*)).$$

What is the order of this differential operator? Show that its symbol $\sigma(D)$ is given by the (fiberwise) polynomial over $T^*(M)$, which restricts to the polynomial given by $\alpha \mapsto \sigma(\alpha)$ on each fiber. In particular, D is elliptic.

Clifford Algebras and the Dirac Operator:

(23) Let $q = \sum x_i^2$ be the standard quadratic form on $\mathbb{R}^k$. Show that there are isomorphisms of real algebras:

$$Cl(\mathbb{R}^k, q) = \mathbb{R},\ \mathbb{C},\ \mathbb{H},\ \mathbb{H} \oplus \mathbb{H},$$

for $k = 0, 1, 2, 3$ respectively.

(24) Let q be as in the above question, show that:

$$Cl(\mathbb{R}^k, -q) = \mathbb{R},\ \mathbb{R} \oplus \mathbb{R},\ M_2(\mathbb{R}),$$

for $k = 0, 1, 2$ respectively.

(25) Let q be the form $\sum x_i^2$ over $\mathbb{C}^k$. Show from first principles, or using the previous question, that there is an isomorphism of complex algebras:

$$Cl(\mathbb{C}^k, q) = \mathbb{C},\ \mathbb{C} \oplus \mathbb{C},\ M_2(\mathbb{C}),$$

for $k = 0, 1, 2$ respectively.

(26) Consider the map $Cl(\mathbb{C}^{2n-1}, q) \longrightarrow Cl(\mathbb{C}^{2n}, q)^{ev}$ given by:

$$A^{ev} + B^{odd} \mapsto A + B\, e_{2n}.$$

Show that this map is an isomorphism of rings.

(27) Recall the representation Δ of $Cl(\mathbb{C}^{2n}, q)$ given by the induced representation:

$$\Delta = Cl(\mathbb{C}^{2n}, q) \otimes_{\Lambda^*(\mathbb{C}^n_-)} \mathbb{C} = \Lambda^*(\mathbb{C}^n_+) \otimes 1.$$

Show that Δ restricts to a sum of two irreducibles $\Delta = \Delta_+ \oplus \Delta_-$ under the restriction to $Cl(\mathbb{C}^{2n-1}, q)$ described in the previous question.

(28) Show that $\Delta_\pm$ are distinct representations of $Cl(\mathbb{C}^{2n}, q)^{ev}$. As a hint, consider the action of the element $e_1 \cdots e_{2n}$ on $\Delta_\pm$.

(29) Show that the odd complex projective spaces $\mathbb{CP}^{2n+1}$ admit unique spin structures. Calculate $\hat{A}(\mathbb{CP}^{2n+1})$. What happens if you apply the Index formula to calculate $\hat{A}(\mathbb{CP}^{2n})$?

(30) Let M be a $2n$-dimensional spin manifold. Let $\not{D} \otimes \Delta_\pm$ denote the the Dirac operator on M twisted by the representations $\Delta_\pm$ respectively. Show that:

$$\text{Index}(\not{D} \otimes \Delta_\pm) = \frac{L(M) \pm \chi(M)}{2},$$

where $L(M)$ denotes the signature of M, and $\chi(M)$ denotes the Euler charecteristic of M. Hint: Twist $\not{D}$ with bundles $E = \Delta_+ + \Delta_-$ and $E = \Delta_+ - \Delta_-$ respectively. Note in particular, you have shown that $L(M) + \chi(M)$ is an even number.

Symplectic Manifolds:

(31) Let X^m be a smooth m-dimensional manifold, with cotangent bundle T^*X. Consider the projection map:

$$\pi : T^*X \longrightarrow X, \quad \pi(x,v) = x, \quad v \in T^*_x X.$$

Let $\alpha \in \Omega^1(T^*X)$ be the one form defined by:

$$\alpha_{(x,v)}(\zeta) = \langle v, d\pi(\zeta)\rangle,$$

where $d\pi$ is the derivative of π, and the above pairing denotes the canonical pairing between the cotangent vectors and tangent vectors. Show that $\omega := d\alpha$ is a symplectic form on T^*X.

(32) Find an example of an almost complex manifold that admits no symplectic structure. Can you find an example of a complex manifold that does not admit any symplectic structure?

(33) Let G be a compact connected Lie group with lie algebra $\mathfrak{g}$ endowed with an invariant inner product. Let ζ be a principal G bundle over a compact $2m$-dimensional symplectic manifold (M, ω). Let $\mathcal{A}(\zeta)$ denote the space of connections on ζ. Recall that $\mathcal{A}(\zeta)$ is an affine manifold modeled on the vector space $\Omega^1(M, E \times_G \mathfrak{g})$.

Given $\alpha, \beta \in \Omega^1(M, E \times_G \mathfrak{g})$, define a two form $tr(\alpha \wedge \beta) \in \Omega^2(M)$ via:

$$tr(\alpha \wedge \beta)(u,v) := \frac{\langle \alpha(u), \beta(v)\rangle - \langle \beta(u), \alpha(v)\rangle}{2}.$$

Show that the following two form on $\mathcal{A}(\zeta)$ is a symplectic form:

$$\omega_{\mathcal{A}}(\alpha, \beta) := \int_M tr(\alpha \wedge \beta) \wedge \omega^{(m-1)}.$$

Symplectic reduction:

(34) Let $H \subseteq G$ be a closed subgroup of a Lie group G. Let $\iota : \mathfrak{h} \subseteq \mathfrak{g}$ denote the corresponding inclusion of Lie algebras. Assume that μ_G is a moment map for the action of G on a symplectic manifold (M, ω). Show that H also acts via symplectomorphisms and admits a moment map μ_H given by:

$$\mu_H = \iota^* \circ \mu_G : M \longrightarrow \mathfrak{g}^* \longrightarrow \mathfrak{h}^*.$$

(35) Assume that one has a product of Lie groups $H \times Z = G$, with corresponding decomposition of Lie algebras: $\mathfrak{h} \times \mathfrak{z} = \mathfrak{g}$. Let μ_G denote the moment map for a G-action on a symplectic manifold (M, ω).

Assume that $a = (h, z) \in \mathfrak{h}^* \times \mathfrak{z}^*$ is a regular value of μ_G. Then show that $M//H\,(h)$ admits an induced Z-action which supports a moment map μ_Z. Furthermore, show that there is a canonical isomorphism of *symplectic manifolds*:

$$M//G\,(a) = (M//H)\,(h)//Z\,(z).$$

(36) Describe a canonical action of the n-torus $T = (S^1)^{\times n}$ on $\mathbb{CP}^n$ that admits a moment map so that the corresponding reduction is a point.

Symplectomorphism groups and pre-quantization:

(37) Consider the following pairing on $C^\infty(M)$ for a symplectic manifold (M, ω) known as the Poisson structure:

$$\{f, g\} := \omega(\omega^{-1}(dg), \omega^{-1}(df)).$$

Show that $\{f, g\}$ is a Lie bracket that satisfies the derivation property:

$$\{f, g\, h\} = g\{f, h\} + h\{f, g\}.$$

Furthermore, show that the projection map $C^\infty(M) \longrightarrow C^\infty(M)/\mathbb{R}$ is a map of Lie algebras, where $C^\infty(M)/\mathbb{R}$ is identified with the Lie algebra of Symplectic vector fields on M.

(38) Let (M, ω) be a compact symplectic manifold. Assume $\zeta \in \mathbb{C}^\infty(M)/\mathbb{R}$ is a symplectic vector field exponentiating to a flow that gives rise to an action of the real line by symplectomorphisms:

$$\exp(t\,\zeta) := \varphi_\zeta : \mathbb{R} \longrightarrow \mathrm{Symp}(M, \omega).$$

Show that this action supports a moment map $H_\zeta : M \longrightarrow \mathbb{R}$. Also show that H_ζ is invariant along φ_ζ. In other words, show that φ_ζ preserves the level-sets of H_ζ.

(39) Assume that (M, ω) is a symplectic manifold with pre quantum line bundle $(\mathcal{L}, \nabla)$. Given a Lie group G, assume that one has a group homomorphism:

$$\varphi : G \longrightarrow \mathrm{Aut}(\mathcal{L}, \nabla).$$

Show that the symplectic reduction $M//G\,(a)$ supports a canonical pre quantum line bundle denoted by $\mathcal{L}//G\,(a)$, with connection $\nabla//G, (a)$. Show as a consequence that $\mathbb{CP}^n$ is canonically pre-quantized.

(40) Consider a symplectic vector space (V, ω) seen as a symplectic manifold. Consider diffeomorphisms of V given by translation:

$$\varphi : V \longrightarrow \mathrm{Diff}(V), \quad \varphi(x)\, v = x + v.$$

Show that φ belongs to the symplectomorphism group. Fix a pre-quantum line bundle $(\mathcal{L}, \nabla)$ on (V, ω). In particular, one obtains a central extension called the Heisenberg group by restricting the central extension of $\mathrm{Symp}(V, \omega)$:

$$1 \to S^1 \longrightarrow \tilde{V} \longrightarrow V \to 1.$$

Describe the structure of the (Heisenberg) Lie algebra of $\tilde{V}$ in terms of (V, ω).

(41) Correct a wrong claim in made in class about the action of the Lie algebra of $\mathrm{Aut}(\mathcal{L}, \nabla)$ on the space of sections: $\Omega(\mathcal{L})$ in the case $M = \mathbb{C}$.

First fix the basis $\mathbb{C} = \mathbb{R}\langle x \rangle \oplus \mathbb{R}\langle y \rangle$, where $y = i\,x$. Under this identification, the symplectic form is given by: $\omega = dy \wedge dx$, and the Louville form is given by: $\alpha = y\, dx$. Now proceed as follows:

Recall that for general pre-quantized symplectic manifolds M, we established an identification in lecture of $C^\infty(M)$ with the Lie algebra of $\mathrm{Aut}(\mathcal{L}, \nabla)$, in terms of vertical and horizontal lifts of vector fields as:

$$f \mapsto -f\mathbf{1} \oplus \omega^{-1}(df).$$

Motivated by this (see remark below), define an operator on the space of sections $\Omega(\mathcal{L} \times_{S^1} \overline{\mathbb{C}})$ given by:

$$\underline{f} = \sqrt{-1}\, m(f) + \nabla_{\omega^{-1}(df)},$$

where $m(f)$ denotes the multiplication operator by f, and $\nabla_{\omega^{-1}(df)}$ is the operator given by covariant derivative along $\omega^{-1}(df)$. Show that this extends to an action of the Lie algebra $C^\infty(M)$ on $\Omega(\mathcal{L} \times_{S^1} \overline{\mathbb{C}})$.

Now in the special case of $M = \mathbb{C}$, show that $\omega^{-1}(dx) = \frac{\partial}{\partial y}$ and $\omega^{-1}(dy) = -\frac{\partial}{\partial x}$. In particular, show that the operators x and y in the Lie algebra of $\mathrm{Aut}(\mathcal{L}, \nabla)$ acts on functions in the variables x and y as the operators $\underline{x}$ and $\underline{y}$ given by:

$$\underline{x} = \sqrt{-1}\, m(x) + \frac{\partial}{\partial y}, \quad \underline{y} = -\frac{\partial}{\partial x}.$$

Notice in particular that functions in x are preserved under the action of $\mathrm{Aut}(\mathcal{L}, \nabla)$.

Remark 0.1. *The operator $\underline{f}$ can be identified with the induced action of the vector field $-f\mathbf{1} \oplus \omega^{-1}(df)$ on the space: $\Omega(\mathcal{L} \times_{S^1} \overline{\mathbb{C}}) = \mathrm{Map}_{S^1}(\mathcal{L}, \overline{\mathbb{C}})$. Notice that the complex conjugation map identifies $\Omega(\mathcal{L} \times_{S^1} \overline{\mathbb{C}})$ canonically with $\Omega(\mathcal{L} \times_{S^1} \mathbb{C})$.*

Lagrangians and Polarizations:

(42) Consider the cotangent bundle (T^*X, ω). Show that the fibers of the projection map $\pi : T^*X \longrightarrow X$ are lagrangian. In addition, given a smooth map $f : X \longrightarrow \mathbb{R}$, consider the graph of df as a section of π:

$$L := \text{graph}(df) \subset T^*X.$$

Show that L is a lagrangian. Hint for question: restrict the Louville form α to L.

(43) Show that the space of lagrangian subspaces of $\mathbb{C}^n$ (with its standard symplectic form) can be identified with the lagrangian Grassmannian:

$$\mathrm{U(n)} \,/\, \mathrm{O(n)}\,.$$

Compact Toric Symplectic Manifolds:

(44) Given a symplectic toric manifold $M(\Delta)$ corresponding to a delzant polytope Δ, let $\mathbb{T}$ denote the corresponding torus that acts on $M(\Delta)$. Let $\mathbb{T}_J \subseteq \mathbb{T}$ denote a subtorus corresponding to a face of Δ indexed by J. Show that the fixed point space $M(\Delta)^{\mathbb{T}_J}$ is a symplectic toric manifold in its own right. What is the corresponding delzant polytope?

(45) Show that scaling a Delzant polytope corresponds to scaling the symplectic form for the corresponding toric symplectic manifold.

(46) Show that as a $\mathbb{T}$-space, $M(\Delta)$ is homeomorphic to a homotopy colimit of a functor $\mathcal{F}$ taking values in $\mathbb{T}$-spaces and defined on the poset category (under reverse inclusion) of faces of Δ.