MODEL CATEGORIES

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References: Dwyer and Spalinski : Homotopy theories and model categories

1. Motivation

Category of topological space : Top_*
In this we have three kinds of morphisms: weak equivalence eg. homotopy equivalence , fibrations, cofibrations

Weak equivalence is not an isomorphism in Top_* , so we want to create a category in which the weak equivalences would be isomorphisms. So we want to formally invert the weak equivalences and create a Homotopy category.

2. Model categories

Definition 2.1. $X \in ob(\mathcal{C})$ is a retract of $Y \in ob(\mathcal{C})$ if

$$\exists i : X \to Y \text{ and } r : Y \to X$$

such that $r \circ i = 1_X$.

Ex: Consider the category $\mathcal{D} = \{\circ \to \circ\}$. Then the category of functors from $\mathcal{D}$ to $\mathcal{C}$ is the category of morphisms $Mor(\mathcal{C})$, denote this by $\mathcal{C}^{\mathcal{D}}$. We are interested in the retracts in this category,

Definition 2.2. A model category is a category $\mathcal{C}$ with 3 classes of maps

- $\xrightarrow{\sim}$ weak equivalences
- $\twoheadrightarrow$ fibrations
- $\hookrightarrow$ cofibrations

closed under cofibrations and contains 1. And satisfying the axioms

- (bicomplete) finite limits and colimits exist
- (2 out of 3) for morphisms f, g, fg if two of these are weak equivalences then so is the third
- (closed under retract)
- (Lifting property)

$$\begin{tikzcd} \circ \arrow[r] \arrow[d, hook] & \circ \arrow[d, "\text{acyclic fibration}"] \\ \circ \arrow[r] \arrow[ur, dashed, "\exists"] & \circ \end{tikzcd} \qquad \begin{tikzcd} \circ \arrow[r] \arrow[d, "\text{acyclic cofibration}"'] & \circ \arrow[d, two heads] \\ \circ \arrow[r] \arrow[ur, dashed, "\exists"] & \circ \end{tikzcd}$$

- (Factorization) Every morphism can be factored as a composition of a cofibration followed by a fibration. And one can further force either of the maps to be weak equivalences. (Factorization is not functorial.)

Definition 2.3. For ϕ initial object and $*$ the final object, if $\phi \hookrightarrow A$ is a cofibration then A is called a cofibrant object. If $B \twoheadrightarrow *$ is a fibration then B is a fibrant object.

Proposition 2.4. *$\mathcal{C}$ is model category in which all objects are fibrant. Given a cofibrant object A. If there exists*

$$A \xrightarrow{\sim}\xleftarrow{\sim} \cdots \xrightarrow{\sim}\xleftarrow{\sim} B$$

then $A \xrightarrow{\sim} B$.

Proof. It suffices to show this for the case $A \xrightarrow{\sim} \circ \xleftarrow{\sim} B$. In this we have the diagram

$$\begin{tikzcd} \phi \arrow[r] \arrow[d, hook] & B \arrow[d, "\text{acyclic fibration}"] \\ A \arrow[r] & \circ \end{tikzcd}$$

and by the Lifting property axiom the diagonal map exists. □

3. Homotopies

Definition 3.1. A *cylinder object* for A is an object $A \wedge I$ together with a diagram

$$A \coprod A \xrightarrow{i} A \wedge I \xrightarrow{\sim} A$$

which factors the map $1_A + 1_A : A \coprod A \to A$.

Two maps $f, g : A \to X$ are said to be *left homotopic* $f \sim^l g$ if there exists a cylinder object $A \wedge I$ such that the map $f + g : A \coprod A \to X$ extends to a map $H : A \wedge I \to X$.

Definition 3.2. A *path object* for X is an object X^I along with a diagram

$$X \xrightarrow{\sim} X^I \to X \times X$$

which factors the diagonal map $X \to X \times X$.

Two maps $f, g : A \to X$ are said to be *left homotopic* $f \sim^r g$ if there exists a cylinder object $A \wedge I$ such that the map $f \times g : A \to X \times X$ factors through a map $H : A \to X^I$.

Proposition 3.3. *If A is cofibrant then $\sim^l$ is an equivalence relation on* $\hom(A, X)$ *and the equivalence classes are denoted by $\pi^l(A, X)$. If X is fibrant then $\sim^r$ is an equivalence relation on* $\hom(A, X)$ *and the equivalence classes are denoted by $\pi^r(A, X)$. If A is cofibrant and X is fibrant then the two equivalence relations agree and the equivalence classes are simply denoted by $\pi(A, X)$.*

4. Homotopy category

$\mathcal{C}$ model category and S the set of weak equivalences, the localized category $Ho(\mathcal{C}) := \mathcal{C}[S^{-1}]$ is called the homotopy category.
We have a natural functor $\gamma : \mathcal{C} \to Ho(\mathcal{C})$. (In general creating a homotopy category without a model category leads to set theory problems)

Definition 4.1. For any map $\phi \to A$ factor it via $\phi \hookrightarrow QA \twoheadrightarrow A$ such that the fibration is a weak equivalence then QA is called a cofibrant replacement. Similarly a fibrant replacement RA.

Proposition 4.2.

$$\hom_{Ho(\mathcal{C})}(X,Y) = \pi(PQX, PQY)$$

5. Examples

5.1. **Constructing a model category from a model category.** Given that $\mathcal{C}$ is model category

- $\mathcal{C}^{op}$ has a natural model structure
- For $a \in ob(\mathcal{C})$ the $a \to \mathcal{C}$ has a natural model structure structure.
- If $\mathcal{D}$ is a "very small" category then $\mathcal{C}^{\mathcal{D}}$ has as natural model structure. This allows to define homotopy limits and colimits.

5.2. **From geometry : Spaces, spectra, simplicial sets,**

- Top_*
  - $\xrightarrow{\sim}$ weak homotopy equivalence
  - $\hookrightarrow$ is a 'retract' of inclusion of a subcomplex inside a CW complex.
  - $\twoheadrightarrow$ Serre fibration
  - Fibrant object: every topological object is fibrant
  - Cofibrant object: hard to describe, but every CW complex is a cofibrant object
  - Homotopy category : classical homotopy category
- Top_*
  - $\xrightarrow{\sim}$ homotopy equivalence
  - $\hookrightarrow$ closed Hurewicz cofibration
  - $\twoheadrightarrow$ Serre fibration
  - Homotopy category : classical homotopy category
- Simplicial sets
  - $\xrightarrow{\sim}$??
  - $\hookrightarrow$??
  - $\twoheadrightarrow$??

5.3. **From algebra: CDGM, DGA.**

- $(Ch_R)_{\geq 0}$
  - $\xrightarrow{\sim}$ quasi isomorphisms
  - $\hookrightarrow$, for $f : M \to N$ and $k \geq 0$ we want $f_k : M_k \to N_k$ to be injective with a projective cokernel.
  - $\twoheadrightarrow$ for $f : M \to N$ and $k > 0$ we want $f_k : M_k \to N_k$ is surjective.
  - Fibrant object: every topological object is fibrant
  - Cofibrant object: projective resolutions
  - Cofibrant replacement is a projective resolution
  - Homotopy category: is the Derived category

Proposition 5.1.

$$\hom_{Ho(Ch_{\mathbb{R}})}(K(A,m), K(B,n)) \cong Ext_R^{n-m}(A,B)$$

Proof. □

6. Quillen equivalence

Given a model category $\mathcal{C}$ and a functor $F : \mathcal{C} \to \mathcal{D}$.

Definition 6.1. Consider pairs (G, s) consisting of a functor $G : Ho(\mathcal{C}) \to \mathcal{D}$ and a natural transformation $s : G\gamma \to F$. A *left derived functor* is a pair (LF, t) of this type which is universal from the left, that is for any other pair (G, s) there exists a unique natural transformation $s' : G \to LF$ such that the composite natural transformation

$$G\gamma \xrightarrow{s' \circ \gamma} (LF)\gamma \xrightarrow{t} F$$

is the natural transformation s.

One can similarly define a right derived functor.

Definition 6.2. Given a functor $F : \mathcal{C} \to \mathcal{D}$ between two model categories a *total left derived functor* is a functor

$$\boldsymbol{L}F : Ho(\mathcal{C}) \to Ho(\mathcal{D})$$

which is a left derived functor of the composition $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{\gamma} Ho(\mathcal{D})$. A right derived functor of the same is called a *total right derived functor*.

Theorem 6.3. *If $F : \mathcal{C} \leftrightarrow \mathcal{D} : G$ are a pair of adjoint functors such that F preserves cofibrations and G preserves fibrations then the total derived functors exist and form an adjoint pair,*

$$\boldsymbol{L}F : Ho(\mathcal{C}) \leftrightarrow Ho(\mathcal{D}) : \boldsymbol{R}G$$

If in addition, for each cofibrant object A of $\mathcal{C}$ and fibrant object X of $\mathcal{D}$, a map $f : A \to G(X)$ is a weak equivalence in C if and only if its adjoint $f^\flat : F(A) \to X$ is a weak equivalence in $\mathcal{D}$, then $\boldsymbol{L}F$ and $\boldsymbol{R}G$ are inverse equivalences of categories.