Stability of moduli spaces of Manifolds - Oscar Randall Manifolds

1. Moduli spaces of Manifolds

M manifold, C^∞, with boundary

$B\,\mathrm{Diff}_\partial(M) =: \mathcal{M}(M)$

fixing boundary

$E(M) = E\mathrm{Diff}_\partial(M) \times_{\mathrm{Diff}_\partial M} M \xrightarrow{\pi} \mathcal{M}(M)$

Fiber bundle with fiber M and structure group $\mathrm{Diff}_\partial(M)$

Because of the ∂, it contains a sub-bundle $E(M)_\partial = E\mathrm{Diff}(M) \times_{\mathrm{Diff}(M)} \partial M = \mathcal{M}(M) \times \partial M$

$\mathcal{M}(M)$ moduli space

$[X, \mathcal{M}(M)] \longrightarrow \left\{ \begin{array}{c} E \\ \downarrow \\ X \end{array} \text{ smooth fiber bundle with fer } M,\ E_\partial \cong X \times \partial M \right\}$

$f \longmapsto \begin{array}{c} f^*E(M) \\ \downarrow \\ X \end{array},\ f^*(E(M))_\partial = X \times \partial M$

Th^m: For good X, this map is an isomorphism.

Th^m: The ring $H^*(\mathcal{M}(M))$ is the characteristic classes of such fibre bundles.

If $K: \partial M \rightsquigarrow P$ is a cobordism, get a manifold $M \cup_{\partial M} K$. Get a homomorphism:

$\mathrm{Diff}_\partial(M) \longrightarrow \mathrm{Diff}_\partial(M \cup_{\partial M} K)$

$\mathcal{M}(M) \longmapsto \mathcal{M}(M \cup_{\partial M} K)$

1. $M = \{1, 2, \dots n\}$

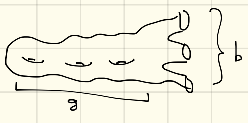
$\mathrm{Diff}_\partial(M) = \Sigma_n$

$\mathcal{M}(\{1,2,\dots,n\}) \simeq B\Sigma_n$

Th^m: $H_i(B\Sigma_n) \longrightarrow H_i(B\Sigma_{n+1})$

isomorphism $i \le \frac{n-2}{2}$ and epi $i \le F_2$

Th^m:

$\Sigma_{g,b} =$ 

 $: \mathcal{M}(\Sigma_{g,b}) \longrightarrow \mathcal{M}(\Sigma_{g+1,b-1})$

epi $3* \le 2g+1$

iso $3* = 2g-2$

 $: \mathcal{M}(\Sigma_{g,b}) \longrightarrow \mathcal{M}(\Sigma_{g,b+1})$

iso $3* \le 2g$

mono all degrees

 $: \mathcal{M}(\Sigma_{g,b}) \longrightarrow \mathcal{M}(\Sigma_{g,b-1})$

epi : $3* \le 2g+3$, all deg if $b-1>0$

iso : $3* \le 2g$,

Th^m: Same sort of thing happens for non-orientable surfaces.

Fix $\theta: B \longrightarrow BO(2)$ a fibration

$$\begin{array}{ccc} \partial\Sigma & \longrightarrow & B \\ \downarrow & \overset{l}{\nearrow} & \downarrow \theta \\ \Sigma & \xrightarrow{\sigma} & BO(2) \end{array}$$

l is a θ-structure

Look at space of all θ-structures on Σ

$\Rightarrow \mathcal{M}^{\theta}(\Sigma_{g,b}) = (\text{space of } \theta \text{ structures on } \Sigma_{g,b}) /\!/ \mathrm{Diff}_{\partial}(\Sigma_{g,b})$

Th^m: $H_i(\mathcal{M}^{\theta}(\Sigma_{g,b}))$ stabilizes with g for all iso $\Longleftrightarrow$ $H_0(\mathcal{M}^{\theta}(\Sigma_{g,b}))$ stabilize with g.

$\theta: BSpin(2) \longrightarrow BO(2)$ is stabilized.

$\theta: BO(2) \times X \longrightarrow BO(2)$ $(\pi_1 X = 0)$

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3. 3-manifold

$\Gamma(M) = \pi_0 \mathrm{Diff}_{\partial}(M^3)$ (Hatcher-Wahl)

$H_*(\Gamma(M \# P^{\# n}))$ stabilize with n. But $B\Gamma(M)$ is not a moduli space.

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4. ≥6-manifolds

Th^m: Let $2n \geq 6$, M be 1-connected $2n$-manifold with non-empty boundary.

$- \cup ([0,1] \times \partial M \# S^n \times S^n) : \mathcal{M}(M) \longrightarrow \mathcal{M}(M \# S^n \times S^n)$

induces an iso for $* \leq \frac{g-3}{2}$ where $g = g(M) = \max\{g \mid M \cong N' \# g S^n \times S^n\}$

Th^m: $V_{g,1} = \#^g S^n \times S^{n+1} \setminus D^{2n+1}$

$- \cup ([0,1] \times S^{n-1} \# S^n \times S^{n+1}) : \mathcal{M}(V_{g,1}) \longrightarrow \mathcal{M}(V_{g+1,1})$

is an isomorphism $* \leq g - 3/2$, for $2n+1 \geq g$

Proof of stability for Σ_n:

S a finite set,

$X_n = \mathrm{Inj}([n], S)$ semi simplicial set

= words in S, where each letter occurs ≤ 1. of length $n+1$

Th^m: $|X_*| \simeq \bigvee S^{|S|-1} \Longleftrightarrow |X_\cdot|$ is $(|S|-2)$ connected.