APURVA

What is homology?

Topological Space $\rightsquigarrow$ Chain Complexes $\rightsquigarrow$ Homology

§ Chain complexes:

A chain complex C_* of $\mathbb{R}$-vector spaces is a collection of vector spaces $\{C_i\}_{i\in\mathbb{Z}}$ along with maps

$$\partial_i : C_i \longrightarrow C_{i-1}$$

∂_i's are called boundary maps

satisfying $\partial_{i-1} \circ \partial_i = 0$.

Index set: $\cdots \leftarrow C_{-1} \leftarrow C_0 \leftarrow C_1 \leftarrow C_2 \leftarrow \cdots$

Examples of chain complexes:

1) $\cdots \leftarrow 0 \leftarrow \mathbb{R} \leftarrow 0 \leftarrow 0 \leftarrow$

2) $\cdots \leftarrow 0 \leftarrow \mathbb{R}^n \xleftarrow{A} \mathbb{R}^m \leftarrow 0 \leftarrow 0 \cdots$ A an $n\times m$ matrix

3) $\cdots \leftarrow 0 \leftarrow \mathbb{R}^{n_1} \xleftarrow{A} \mathbb{R}^{n_2} \xleftarrow{B} \mathbb{R}^{n_2} \leftarrow 0 \cdots$ B an $n_2 \times n_3$ matrix, A an $n_1 \times n_2$ matrix, $AB = 0$

4) $\cdots \leftarrow \mathbb{R} \xleftarrow{0} \mathbb{R} \xleftarrow{0} \mathbb{R} \xleftarrow{0} \mathbb{R} \xleftarrow{0} \mathbb{R} \leftarrow$

Non-examples:

1) $\cdots \leftarrow 0 \leftarrow \mathbb{R}^{n_1} \xleftarrow{A} \mathbb{R}^{n_2} \xleftarrow{B} \mathbb{R}^{n_2} \leftarrow 0 \cdots$ B an $n_2 \times n_3$ matrix, A an $n_1 \times n_2$ matrix, $AB \neq 0$

2) $\cdots \leftarrow 0 \leftarrow \mathbb{R} \xleftarrow{1} \mathbb{R} \xleftarrow{1} \mathbb{R} \leftarrow 0 \leftarrow \cdots$

Because $d_{i-1} \cdot d_i = 0$ we have $\operatorname{im} \partial_i \subseteq \ker \partial_{i-1}$.

We define • $H_i(C_*) = \ker d_i \,/\, \operatorname{im} d_{i+1}$

• These are called the Homology groups (with $\mathbb{R}$ coefficients)

by rank nullity $\dim H_i(C_*) = \dim \ker d_i - \dim \ker d_{i+1}$

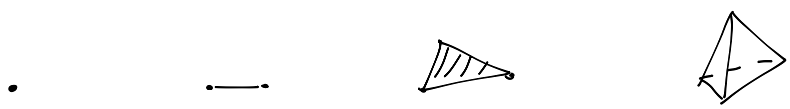
eg: $\cdots \leftarrow \mathbb{R} \xleftarrow{0} \mathbb{R} \xleftarrow{0} \mathbb{R} \xleftarrow{0} \mathbb{R} \xleftarrow{0} \cdots = C_*$

$H_i(C_*) = \mathbb{R}$ for all i.

- If $H_i(C_*) = 0$ then we say that C_* is exact.

Given a topological space X we can define a chain complex associated to it. We'll use the discrete version of a topological space called a cell complex to create a cell complex.

§ Simplices :

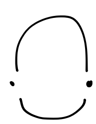


0-simplex 1-simplex 2-simplex 3-simplex

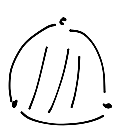
An n-simplex is the set

$$\Delta^n = \{ (x_1, \dots, x_n) \in \mathbb{R}^n : x_1 + \dots + x_n \le 1 , x_i \ge 0 \}$$

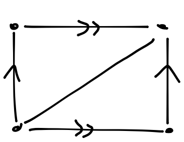
A cell complex is a topological space obtained by gluing cells together cells. The simplices of a cell complexes are called cells.

eg: $S^1 \cong$ 

2 0-cells
2 1-cells

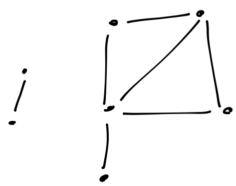
$D^2 \cong$ 

3 0-cells
3 1-cells
1 2-cell

T^2: 

1 0-cell
3 1-cells
2 2-cells

Graphs :



7 0-simplices
7 1-simplices

- To keep things simple we put the following restrictions :
  We only glue simplices via isomorphisms

We can create a chain complex out of a cell complex as follows.

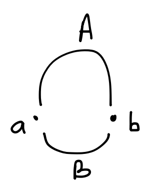
Cellular Chain complex :

Define

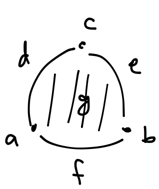
$$C_i(X) = \begin{cases} 0 & \text{if } i < 0 \\ \text{the free } \mathbb{R}\text{-vector space over the set of } i\text{-simplices} & \text{if } i \geq 0 \end{cases}$$

Elements in $C_i(X)$ are called i chains in X.

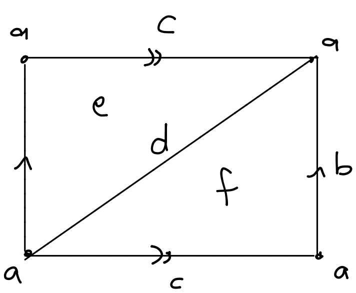
eg :

$S^1 \cong$ 

$C_0(S^1) = \mathbb{R}\langle a, b\rangle$
$C_1(S^1) = \mathbb{R}\langle c, d\rangle$

$\mathbb{D}^2 \cong$ 

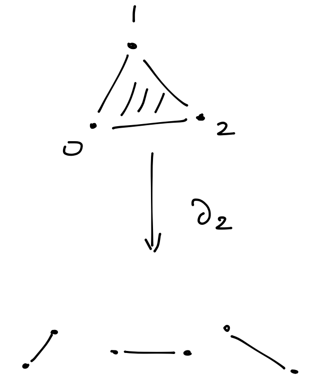
$C_0(\mathbb{D}^2) = \mathbb{R}\langle a, b, c\rangle$
$C_1(\mathbb{D}^2) = \mathbb{R}\langle d, e, f\rangle$
$C_2(\mathbb{D}^2) = \mathbb{R}\langle g\rangle$

T^2 : 

$C_0(T) = \mathbb{R}\langle a\rangle$
$C_1(T) = \mathbb{R}\langle b, c, d\rangle$
$C_2(T) = \mathbb{R}\langle e, f\rangle$

All the other vector spaces are 0.

Defining the boundary map goes as follows:



$[12] - [02] + [01]$

Let $[012\cdots n]$ be a n-chain corresponding to an n-simplex in X

We think of $[012\cdots n]$ as an oriented simplex as follows:

$$
\begin{aligned}
[012\cdots n] &= -[102\cdots n] \\
&= [120\cdots n] \\
&= (-1)^{\mathrm{sign}(\sigma)}[\sigma(1)\,\sigma(2)\cdots\sigma(n)]
\end{aligned}
$$

where σ is a permutation of $\{0,1,2\cdots,n\}$

define:
$$
\begin{aligned}
\partial_n [01\cdots n] &= [\hat{0}12\cdots n] - [0\hat{1}2\cdots n] + \cdots \\
&= \sum_i (-1)^i [01\cdots \hat{i} \cdots n]
\end{aligned}
$$

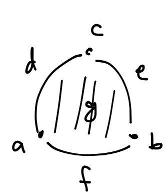
by $[01\cdots\hat{i}\cdots n]$ we mean an $n-1$ simplex obtained by removing the vertex i.

Claim: $\partial_{n-1}\cdot\partial_n [01\cdots n]$

Proof:
$$
\begin{aligned}
\partial_{n-1}\partial_n [01\cdots n] &= \partial_{n-1}\Big[\sum_{i=0}^{n}(-1)^i [01\cdots\hat{i}\cdots n]\Big] \\
&= \sum_{i=0}^{n}(-1)^i\,\partial_{n-1}[01\cdots\hat{i}\cdots n]
\end{aligned}
$$

When we expand out the right hand side we get terms like $[01\cdots\hat{i}\cdots\hat{j}\cdots n]$
This term occurs twice once in $\partial_{n-1}[0\cdots\hat{i}\cdots n]$ and once in $\partial_{n-1}[0\cdots\hat{j}\cdots n]$ but these occur with different signs & hence they cancel.

eg:
$$
\begin{aligned}
\partial_1\partial_2[012] &= \partial_1[12] - \partial_1[02] + \partial_1[01] \\
&= [2]-[1]-([2]-[0]) + [1]-[0] = 0
\end{aligned}
$$

$\mathbb{D}^2 \cong$ 

$\partial_1 d = \partial_1[ac] = a-c$
$\partial_1 e = \partial_1[bc] = c-b$
$\partial_1 f = \partial_1[ab] = b-a$

$$
\begin{aligned}
\partial_2 g &= \partial_2[abc] \\
&= [bc]-[ac]+[ab] \\
&= e-d+f
\end{aligned}
$$