Jones Polynomial — John

Def^n: Knot – isotopy class of embedding S^1 in $\mathbb{R}^3$

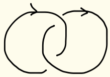
Link – disjoint collection of knots.

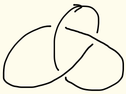
Knot & Link projections:

$P : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ such that

1) $\forall\, \ell \in P(L)$ $\quad P^{-1}(\ell)$ has at most 2 points
2) only points have 2 preimages.

ex:  unknot

 Hopf Link

 Trefoil

 (different) Trefoil

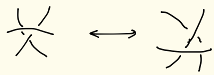
Th^m (Reidemeister)

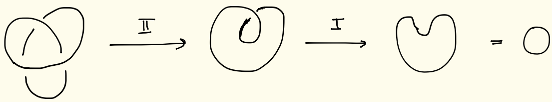
Two equivalent knot projections are related by a series of following moves:

1) 

Reidemester moves

2) 

3) 

eg: 

Def^n: Knot Invariant – Something which is invariant under the Reidemeister moves.

Ex: Crossing Number $cr(L) :=$ minimal number of crossings in a link projection.
This does not distinguish between the two trefoils.

Kaufmann Bracket Polynomial: $\langle\ \rangle$

L - Link projection

X - Set of crossings in L $\qquad |X| = n_+ + n_-$

$n_+ = \#$ +ve crossings

$n_- = \#$ −ve crossings

$\langle\ \rangle$ is defined recursively:

$\langle \phi \rangle = 1$

$\langle \circ \amalg \phi \rangle = (q+q^{-1})\langle L \rangle$ $\qquad$ usually $q = t^{1/2}$

$\langle \times \rangle = \langle \asymp \rangle - q \langle)(\rangle$

$\times \to \asymp$ 0 smoothing

$\times \to)($ 1 smoothing

The Kaufmann bracket is not a knot invariant.

$\langle \Omega \rangle = \langle \Omega \rangle - q\langle \Omega \rangle$

$= (q+q^{-1})L - qL$

$= q^{-1}L \quad = q^{-1}\langle \Omega \rangle$ and so $\langle \Omega \rangle \neq \langle \Omega \rangle$

Def^n: **Jones Polynomial** (normalised)

$\hat{J}(L) = (-1)^{n_-} q^{n_+ - 2n_-} \langle L \rangle$

$J(L) = \hat{J}(L)/(q+q^{-1})$ ← normalised

ex: Hopf Link

Check

$\alpha \in \{0,1\}^X$ is the complete set of smoothings

We add a term $(-1)^r q^r (q+q^{-1})^k$

$r = \#$ 1 smoothing $\qquad k = \#$ circles

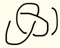
$\langle$  $\rangle = (q+q^{-1})^2 + 2\left(-q(q+q^{-1})\right) + q^2(q+q^{-1})^2$

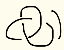
$= q^4 + q^2 + 1 + q^{-2}$

$\hat{J}($  $) = (-1)^0 q^{2-2\cdot 0} \langle \quad \rangle$

$= q^6 + q^4 + q^2 + 1$

$J(\quad) = q^5 + q$

Similarly, $J($  $) = q^2 + q^6 - q^8$

$J($  $) = q^{-2} + q^{-6} - q^{-8}$

Khovanov Homology:

$W = \bigoplus_m W_m$ graded V.S.

$\dim_q W := \sum_m q^m \dim(W_m)$

degree shift by ℓ : $W\{\ell\}_m := W_{m-\ell}$

$\Rightarrow \dim_q W\{\ell\} = q^\ell \cdot \dim_q W$

height shift by s for chain complexes,

$\bar{C} = \longrightarrow C^r \longrightarrow C^{r+1} \longrightarrow \cdots$ graded V.S.

$\bar{C}[s] = \longrightarrow C^{r-s} \longrightarrow C^{r+1-s} \longrightarrow \cdots$

Let $V = \mathbb{R}v_+ \oplus \mathbb{R}v_-$ so $\dim_q V = q + q^{-1}$

grading 1 grading -1

For every $\alpha \in \{0,1\}^X$ assign

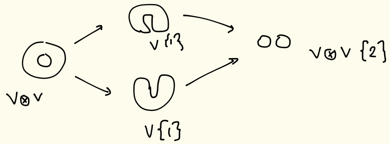
$V_\alpha := V^{\otimes k}\{r\}$ $k = \#$ circles $r = \#$ of 1-smoothings $=: |\alpha|$

Set $[\![L]\!]^r := \bigoplus_{|\alpha|=r} V_\alpha(L)$

Define $C(L) := [\![L]\!][-n_-]\{n_+ - 2n_-\}$

Khovanov Homology := Homology of $c(L)$

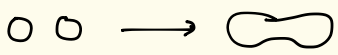
Chain maps



$\vdots$

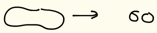
$$0 \longrightarrow V\otimes V \longrightarrow V\{1\}\oplus V\{1\} \longrightarrow V\otimes V\ \{2\} \longrightarrow 0$$

differentials



$\rightsquigarrow \quad V\otimes V \xrightarrow{m} V$

$$m \begin{cases} V_+\otimes V_- \longrightarrow V_- \\ V_-\otimes V_+ \longrightarrow V_- \\ V_+\otimes V_+ \longrightarrow V_+ \\ V_-\otimes V_- \longrightarrow C \end{cases}$$



$\rightsquigarrow \quad V \xrightarrow{\Delta} V\otimes V$

$V_+ \longrightarrow V_+\otimes V_- + V_-\otimes V_+$

$V_- \longrightarrow V_-\otimes V_-$