Homotopy theory seems more and more combinatorial

Cohomology of X_1, X_1 is fiber in $X_1 \longrightarrow K(\mathbb{Z},n) \longrightarrow K(\mathbb{Z}/2, n+2)$

Assume that n is large and the only non-trivial differentials in the relevant range are the transgressions. We have the two maps

$$H^{n+2}(K(\mathbb{Z}/2, n+2)) \longrightarrow H^{n+2}(K(\mathbb{Z},n)) \qquad\qquad H^n(K(\mathbb{Z},n)) \longrightarrow H^n(X)$$

$$i_{n+2} \longmapsto Sq^2 i_n \qquad\qquad\qquad i_n \longmapsto \sigma$$

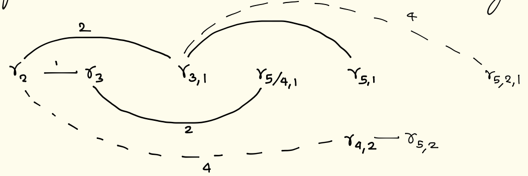
It is now a matter of repeating this till one dies.

$H^{*-1}(X_1)$	$H^*(K(\mathbb{Z}/2, n+2))$	Sq^2	$H^*(K(\mathbb{Z}, n))$	$H^*(X_1)$	$*$
	–	–	i_n	σ_0	n
	i_{n+2}	$Sq^2 i_n$	$Sq^2 i_n$	–	$n+2$
	$Sq^1 i_{n+2}$	$Sq^1 Sq^2 = Sq^3$	$Sq^3 i_n$	–	$n+3$
	–	–	$Sq^4 i_n$	σ_4	
γ_2	$Sq^2 i_{n+2}$	$Sq^2 Sq^2 = Sq^3 Sq^1$			
γ_3	Sq^3	$Sq^3 Sq^2 = 0$	–	-	$n+5$
	$Sq^2 Sq^1$	$Sq^2 Sq^1 Sq^2 = Sq^5 + (4,1)$	$Sq^5 i_n$	–	
	Sq^4	$Sq^4 Sq^2$	$Sq^4 Sq^2 i_n$	-	$+6$
$\gamma_{3,1}$	$Sq^3 Sq^1$	$Sq^3 Sq^1 Sq^2 = \cancel{Sq^5 Sq^1}$	–	.	
			$Sq^6 i_n$	σ_6	
$\gamma_{5/4,1}$	Sq^5	$Sq^5 Sq^2$	$Sq^5 Sq^2 i_n$		$+7$
	$Sq^4 Sq^1$	$(4,1,2) = \binom{2-2}{4-4}(5,2)$			
			$Sq^7 i_n$	σ_7	
	Sq^6	$Sq^6 Sq^2$	$Sq^6 Sq^2 i_n$		$+8$
$\gamma_{5,1}$	$Sq^5 Sq^1$	$(5,1,2) = 0$			
$\gamma_{4,2}$	$Sq^4 Sq^2$	$(4,2,2) = \cancel{(5,1)}$			
			$Sq^8 i_n$	σ_8	
	Sq^7	$Sq^7 Sq^2$	$Sq^7 Sq^2 i_n$		$+9$
	$Sq^6 Sq^1$	$(6,1,2) = (6,3)$	$Sq^6 Sq^3 i_n$		
$\gamma_{5,2}$	$Sq^5 Sq^2$	$(5,2,2) = \cancel{(5,3,1)}$			
	$Sq^4 Sq^2 Sq^1$	$(4,2,1,2) = (9) + \cancel{(\cdot,1)}$	$Sq^9 i_n$		

$H^{*-1}(X_1)$	$H^*(K(\mathbb{Z}/2,n+2))$	Sq^2	$H^*(K(\mathbb{Z},n))$	$H^*(X_1)$	$*$
	8	(8,2)	Sq^8Sq^2		+10
	7,1	(7,3)	Sq^7Sq^3		
$\gamma_{6,2}$	6,2	(6,2,2) = ~~(6,3,1)~~	Sq^{10}	σ_{10}	
$\gamma_{5,2,1}$	5,2,1	(5,2,1,2) = ~~(...,1)~~			
	9	(9,2)	Sq^9Sq^2		+11
	8,1	(8,3)	Sq^8Sq^3		
$\gamma_{7,2}$	7,2	~~(7,3,1)~~			
$\gamma_{6,3}$	6,3	(6,3,2) = 0	Sq^{11}	σ_{11}	
$\gamma_{6,2,1}$	6,2,1	(6,2,3) = (9,2) + (8,3)	~~$Sq^9Sq^2 + Sq^8Sq^3$~~		
	10	(10,2)	$Sq^{10}Sq^2$		+12
	9,1	9,3	Sq^9Sq^3		
$\gamma_{8,2}$	8,2	~~(8,3,1)~~			
$\gamma_{7,3}$	7,3	0			
$\gamma_{7,2,1/9,1}$	7,2,1	(7,2,1) = (9,3)			
$\gamma_{6,3,1}$	6,3,1	(6,3,3) = ~~(6,5,1)~~			

Spending hours trying to figure this out has put me in a philosophical limbo.

A lot of these γ's are related by Steenrod Squares just like the σ's are. We make a choice for the γ's as we are allowed to add any element which transgresses to 0.



Cohomology of X_2:

We look at the SS for the fibration:

$$X_2 \longrightarrow X_1 \xrightarrow[\gamma_2]{} K(\mathbb{Z}, n+3)$$
$$\gamma_2 \longleftarrow i_{n+3}$$

We do the same drill again only this time we do not know how the γ's and σ's are connected.

$H^{*-1}(X_2)$	$H^*(K(\mathbb{Z}/2,n+3))$	Sq^2	$H^*(X_1)$	$H^*(X_2)$	$*$
			σ_0	τ_0	n
	ι_{n+3}	$\longrightarrow$	γ_2		$n+3$
	$Sq^1\iota_{n+3}$	$\longrightarrow$	γ_3		$+4$
			$\sigma_4 = Sq^4\sigma_0$	τ_4	
	$Sq^2\iota_{n+3}$	$\longrightarrow$	$\gamma_{3,1}$		$+5$
			σ_6	τ_6	$+6$
	Sq^2Sq^1	$\longrightarrow$	$\gamma_{5/4,1}$		
S_3	$Sq^3\iota_{n+3}$				
			σ_7	τ_7	$+7$
	$Sq^3Sq^1\iota_{n+3}$	$\longrightarrow$	$\gamma_{5,1}$		
	Sq^4	$\longrightarrow$	$\gamma_{4,2}$		
			σ_8	τ_8	$+8$
S_5	Sq^4Sq^1	$\longrightarrow$	$\gamma_{5,2}$		
	Sq^5	$\longrightarrow$			
	Sq^4Sq^2	$\longrightarrow$	$\gamma_{5,2,1}$		

Next step is computing the Bocksteins:

- On $K(\mathbb{Z}/2,n)$ $Sq^1Sq^n = (n-1)Sq^{n+1}$. *So* $d_1 Sq^{2n} = Sq^{2n+1}$. *These then would give us Bockstein relations on* $H^*(X_1)$.

$K(\mathbb{Z}/2,n+1) \longrightarrow X_1 \longrightarrow K(\mathbb{Z}/2,n)$

- $Sq^1\iota_{n+1} \longmapsto Sq^3\iota_n$

  $Sq^2\iota_n \longmapsto 0$ no information

- $Sq^2Sq^1\iota_{n+1} \longmapsto Sq^5\iota_n$

  $d_1 Sq^2Sq^1 = Sq^3Sq^1 \longleftarrow \gamma_{3,1}$ $\qquad$ $\sigma_4 \downarrow$ $\quad Sq^4\iota_n \xrightarrow{d_1} Sq^5\iota_n$

  $\Rightarrow$ $\boxed{d_2\sigma_4 = \gamma_{3,1}}$, $\boxed{d_1\gamma_{3,1} = 0}$

- $Sq^5\iota_{n+1} \longmapsto Sq^5Sq^2\iota_n$

  $d_1 Sq^5 = Sq^6 \longmapsto 0$ $\qquad$ nothing

- $Sq^4Sq^2Sq^1\iota_{n+1} \longmapsto Sq^9\iota_n$

  $d_1 Sq^4Sq^2Sq^1 \longleftarrow \gamma_{5,2,1}$ $\qquad$ $\sigma_8 \downarrow$ $\quad Sq^8\iota_n \xrightarrow{d_1} Sq^9\iota_n$

  $\Rightarrow$ $\boxed{d_2\sigma_8 = \gamma_{5,2,1}}$, $\boxed{d_1\gamma_{5,2,1} = 0}$

- $Sq^6Sq^3\iota_{n+1} \longmapsto Sq^{11}$

  $\Rightarrow$ $\boxed{d_2\sigma_{10} = \gamma_{7,3}}$, $\boxed{d_1\gamma_{7,3} = 0}$

Let us do the same thing for $H^*(X_2)$:

$$Sq^2 Sq^3 = \binom{2}{2} Sq^5 + \binom{1}{0} Sq^4 Sq^1 \quad \Rightarrow \quad \delta_{5/4,1} = Sq^2 \delta_3 + \text{possibly some } \tau\text{'s}$$

Look at the fibration: $\quad K(\mathbb{Z}/2, n+2) \longrightarrow X_2 \longrightarrow X_1$

- $Sq^2\, i_{n+2} \longmapsto \gamma_{3,1}$

  $d_1 Sq^2 i_{n+2} \longleftarrow \delta_3$

  $$\begin{array}{c} \tau_4 \\ \downarrow \\ \sigma_4 \xrightarrow{d_2} \gamma_{3,1} \end{array}$$

  $\Rightarrow \quad \boxed{d_3 \tau_4 = \delta_3}$

- $Sq^4 Sq^2 i_{n+2} \longmapsto \gamma_{5,2,1}$

  $d_1 Sq^4 Sq^2 \longleftarrow\!\!\mapsto \delta_{5,2}$

  $$\begin{array}{c} \tau_8 \\ \downarrow \\ \sigma_8 \xmapsto{d_2} \gamma_{5,2,1} \end{array}$$

  $\Rightarrow \quad \boxed{d_3 \tau_8 = \delta_{5,2}}$

Let us look at the Serre SS for

$$K(\mathbb{Z}/8, n+3) \longrightarrow X_3 \longrightarrow X_2$$

$H^*(X_3)$	$H^*(K(\mathbb{Z}/8, n+3))$	Sq^2	$H^*(X_2)$	$H^*(X_3)$	$*$
			τ_0	ϵ_0	n
	i_{n+3}				$n+3$
	$d_3 i_{n+3}$		τ_4		$n+4$
	$Sq^2 i_{n+3}$		$\delta_3 = d_3 \tau_4$		$n+5$
	$Sq^3 i_{n+3}$		τ_6		$n+6$
	$Sq^2 d_3 i_{n+3}$				
η_4	$Sq^4 i_{n+3}$		τ_7		
	$Sq^3 d_3 i_{n+3}$	computation missing	$\delta_{5/4,1}$		
			$\delta_{5,1}$		