Let $S$ be a scheme

**Def:** $S$-scheme is a scheme $X$ with a fixed $X \longrightarrow S$.

Morphisms :
$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

**Rem:** For $X$ any scheme and $A$- any commutative ring

$$\mathrm{Hom}_{\text{scheme}}(X, \operatorname{Spec} A) \cong \mathrm{Hom}_{\text{Rings}}(A, \Gamma(X, \mathcal{O}_X))$$

i.e. $\left.\begin{array}{l} A \longrightarrow \operatorname{Spec} A \\ X \longrightarrow \Gamma(X, \mathcal{O}_X) \end{array}\right\}$ adjoint functors

Adjoint maps: $X \longrightarrow \operatorname{Spec} \Gamma(X, \mathcal{O}_X)$ is isomorphism iff $X$ is affine.

**Rem:** $X$ scheme, $x \in X$ have canonical map:

$$\operatorname{Spec} \mathcal{O}_{X,x} \longrightarrow X$$

For this pick $x \in \operatorname{Spec} A$ open affine

$x \longleftrightarrow \mathfrak{p} \subseteq A$ gives $A \longrightarrow A_{\mathfrak{p}} \simeq \mathcal{O}_{X,x}$

gives $\operatorname{Spec} \mathcal{O}_{X,x} \longrightarrow \operatorname{Spec} A \subseteq X$

Why independant of choice of $A$.

Further $\mathcal{O}_{X,x} \longrightarrow K(x)$ gives us :

$$\operatorname{Spec} K(x) \longrightarrow X$$

**Functor of points:**

$\mathcal{C}$ any category (eg: schemes, $S$-schemes)

$X, Z \in \mathrm{Ob}\, \mathcal{C}$

$h_X(Z) := X(Z) := \mathrm{Hom}_{\mathcal{C}}(Z, X) =:$ Set of $Z$ valued points in $X$.

If $\mathcal{C} =$ Schemes or Schemes$/S$ and $Z = \operatorname{Spec} A$ then $X(A) := X(\operatorname{Spec} A)$

$h_X = X(-) = \mathrm{Hom}(-, X)$ is a functor $\mathcal{C}^{op} \longrightarrow$ Sets

Functor of points associated to $X$.

**eg:** $X = \operatorname{Spec} \mathbb{Z}[x_1, \dots, x_n]/(f_1, \dots, f_r)$

$$X(B) = \mathrm{Hom}(\operatorname{Spec} B, X) = \mathrm{Hom}\left(\mathbb{Z}[x_1 \dots x_n]/(f_1 \dots f_r), B\right)$$
$$= \left\{ (y_1 \dots y_n) \in B^{\times n} : f_i(x_1 \dots x_n) = 0 \right\}$$

eg: $\mathbb{A}^1_{\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[x]$

$$\mathbb{A}^1_{\mathbb{Z}}(X) = \mathrm{Hom}(X, \operatorname{Spec}\mathbb{Z}[x]) \cong \mathrm{Hom}(\mathbb{Z}[x], \Gamma(X, \mathcal{O}_X))$$
$$\cong \Gamma(X, \mathcal{O}_X)$$

As it should be for "regular functions" on $X$

More generally: $\mathbb{A}^1_A = \operatorname{Spec} A[x]$ plays same role in the category $A$-Schemes
$\mathbb{A}^1_A(X) \cong \Gamma(X, \mathcal{O}_X)$ for $X \longrightarrow \operatorname{Spec} A$

## §6.6 Representable Functors & Group Schemes

$X \longrightarrow \Gamma(X, \mathcal{O}_X)$

$\mathrm{Schemes}^{op} \longrightarrow \mathrm{Sets}$

Given $X \xrightarrow{f} Y$, we get

$$\begin{tikzcd} \Gamma(X,\mathcal{O}_X) \arrow[d, phantom, "\cong"] & \Gamma(Y,\mathcal{O}_Y) \arrow[l, "f^\#"'] \arrow[d, phantom, "\cong"] \\ \mathbb{A}^1_{\mathbb{Z}}(X) & \mathbb{A}^1_{\mathbb{Z}}(Y) \arrow[l, "f^*"'] \end{tikzcd}$$

$\Rightarrow$ Isomorphism of functors $h_{\mathbb{A}^1_{\mathbb{Z}}}(-) \cong \Gamma(-)$

**Def: Representable functors:**

a functor $F: \mathcal{C}^{op} \longrightarrow \mathrm{Sets}$ is represented by $Y \in \mathrm{Ob}\,\mathcal{C}$
$F \cong h_Y$

**Yoneda's lemma:**

$$\mathrm{Hom}_{\text{functors}}(h_Y, F) \cong F(Y)$$

**Proof:**

$\phi_-: h_Y \longrightarrow F \rightsquigarrow \phi_Y(\mathrm{id})$

$\alpha_Z: h_Y(Z) \longrightarrow F(Z) \leftsquigarrow \alpha \in F(Y)$
$\quad\;\; \| $
$\mathrm{Hom}(Z, Y)$

$f \longmapsto \alpha_{Z*}(f)$

$$\begin{tikzcd} f = f_*(\mathrm{id}_Y) \in \mathrm{Hom}(Y,Z) \arrow[r, "\phi"] & F(Z) \ni \phi(f) = \phi(f_* \mathrm{id}_Y) = f_*(\phi(\mathrm{id}_Y)) \\ \mathrm{id}_Y \in \mathrm{Hom}(Y,Y) \arrow[u] \arrow[r, "\phi"'] & F(Y) \ni \phi(\mathrm{id}_Y) \arrow[u] \end{tikzcd}$$

**Corrolary:**

$$\begin{aligned} \mathcal{C} &\longrightarrow \mathrm{Hom}(\mathcal{C}^{op}, \mathrm{Sets}) \\ Y &\longmapsto h_Y \end{aligned}$$

is fully faithful because

$$\mathrm{Hom}(h_Y, h_Z) \cong h_Z(Y) \overset{\text{is}}{=} \mathrm{Hom}_{\mathcal{C}}(Y, Z)$$

**Cor:** $F$ representable then the representing object is unique upto iso.

**Eg:**
- In any $\mathcal{C}$, $X \times Y$ means any $Z$ which represents $h_X \times h_Y$
- More generally,

$D : \mathcal{D} \to \mathcal{C}$ any covariant functor.

$$\begin{aligned} F : \mathcal{C} &\longrightarrow \mathrm{Sets} \\ T &\longmapsto \varprojlim \mathrm{Hom}(T, D) \end{aligned}$$

$\varprojlim D$ is just ob $X$ in $\mathcal{C}$ representing $F$. i.e.

$$\mathrm{Hom}(T, \varprojlim D) \cong \varprojlim \mathrm{Hom}(T, D)$$

**eg:** In Schemes:

$$X \longrightarrow \Gamma(X, \mathcal{O}_X^*)$$

is represented by $\mathrm{Spec}\, \mathbb{Z}[x, x^{-1}]$

**Rem:**

$$\begin{tikzcd} \Gamma(X, \mathcal{O}_X)^* \arrow[r, hook] \arrow[d, squiggly, "\text{Yoneda}"] & \Gamma(X, \mathcal{O}_X) \\ \mathrm{Spec}\, \mathbb{Z}[x, x^{-1}] \arrow[r] \arrow[d, squiggly] & \mathrm{Spec}\, \mathbb{Z}[x] \\ \mathbb{Z}[x, x^{-1}] & \mathbb{Z}[x] \arrow[l] \end{tikzcd}$$

**Def:** $\mathbb{G}_m := \mathrm{Spec}\, \mathbb{Z}[x, x^{-1}]$ multiplicative group

**Def:** A group object in a cat $\mathcal{C}$ is $Y \in \mathrm{Ob}\, \mathcal{C}$ equipped with

$$\begin{tikzcd} & \mathrm{Groups} \arrow[d] \\ \mathcal{C}^{op} \arrow[ur, dashed] \arrow[r, "h_Y"'] & \mathrm{Sets} \end{tikzcd}$$

i.e. $h_Y(T)$ is a group $\forall T$ & $\mathrm{Hom}(T, Y) \to \mathrm{Hom}(S, Y)$ is a group Hom.

**eg:** In Schemes,
- $\mathbb{G}_m$ is a group scheme
- $\mathbb{A}^1_{\mathbb{Z}}$ is a group scheme ( $\mathbb{A}^1_{\mathbb{Z}} =: \mathbb{G}_a$ , additive group.)