FIO's and Lagrangian Submanifolds

Apurv Nakade

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1. Introduction

These are the notes for a talk given at JHU. The goal of these notes is to understand the connection between Fourier Integral Operators and motivate the construction the Symplectic 'category' $\mathcal{S}ymp$ and to understand the difficulties involved in it's construction.

Lagrangian submanifolds have connections to multiple areas in mathematics, one of them is to the theory of Fourier Integral Operators. FIOs generalize the notion of a Pseudo-differential operator. A Pseudo-differential operator is an FIO associated to the diagonal Lagrangian $\Delta : M \hookrightarrow M \times M$. A Psuedo-differential operator decays rapidly away from the diagonal, similarly an FIO has an associated Lagrangian submanifold and the FIO decays away from it.

When the composition of two FIOs is another FIO we say that the two corresponding Lagrangians compose to produce the third Lagrangian. This is how composition is defined in the category $\mathcal{S}ymp$. The problem arises because of the fact the composition of two FIO's is not always an FIO.

2. Symplectic Geometry Background

A pair (M^{2m}, ω) is called a **symplectic manifold** if M is a $2m$ dimensional manifold without boundary and $\omega \in \Omega^2(M; \mathbb{R})$ is a 2 form such that ω^m is nowhere vanishing.

Given a manifold L^m, an immersion $\iota : L \looparrowright M$ is called a **Lagrangian immersion** if $\iota^*\omega = 0$. Further if $\omega = d\alpha$ for some $\alpha \in \Omega^1(M; \mathbb{R})$ then ι is called **exact** if $\iota^*\alpha = df$ for some $f \in C^\infty(L)$. If ι is an embedding we think of L as a **Lagrangian submanifold** of M.

The only symplectic manifold of concern to us is $M = T^*X$ where X is an m dimensional manifold. Locally if $x_1, \cdots, x_m$ are the coordinates on X and $y_1, \cdots, y_m$ are the *vertical* coordinates for T^*X then one can define

$$\begin{aligned}\alpha &:= \sum_i y_i dx_i \text{ (Liouville form)}\\ \omega &:= d\alpha\\ &= \sum_i dy_i dx_i\end{aligned}$$

It is easy to see that α is invariant under change of coordinates and hence defines a global 1-form and that ω defines a symplectic form on T^*X.

We are concerned with Lagrangian immersions inside T^*X. One important family of Lagrangian immersions comes from graphs of functions. Consider a function $f : X \to \mathbb{R}$. Then f defines a Lagrangian submanifold of M given by

$$\Lambda_f := \{(x, df(x))\} \subseteq T^*X$$

Then $\alpha|_{\Lambda_f} = df$ and hence Λ_f is an exact Lagrangian submanifold of T^*X. We say that f is the generating function of Λ_f.

More generally given an arbitrary Lagrangian immersion of $L \looparrowright T^*X$ one can add auxiliary variable to X locally and realize L as a graph of a function in X and these extra variables. We'll see this in more details in the later sections.

Consider symplectic manifolds $(M_1, \omega_1), (M_2, \omega_2), (M_3, \omega_3)$. Let $\overline{M_1}$ denote the symplectic manifold $(M_1, -\omega_1)$. It is easy to see that $(\overline{M_1} \times M_2, -\omega_1 \times \omega_2)$ is also a symplectic manifold. Consider two Lagrangian submanifolds $L \subseteq \overline{M_1} \times M_2$ and $L' \subseteq \overline{M_2} \times M_3$.

Definition 2.1. We say L and L' are **composable** if $L \times L' \subseteq \overline{M_1} \times M_2 \times \overline{M_2} \times M_3$ intersects the diagonal $M_1 \times \Delta_{M_2} \times M_3 \subseteq \overline{M_1} \times M_2 \times \overline{M_2} \times M_3$ transversally. Denote by $L' \circ L$ the image of $L \times L' \bigcap M_1 \times \Delta_{M_2} \times M_3$ under the natural projection $\overline{M_1} \times M_2 \times \overline{M_2} \times M_3 \to \overline{M_1} \times M_3$.

Proposition 2.2. *If L and L' are composable then $L' \circ L$ is a Lagrangian submanifold of $\overline{M_1} \times M_3$.*

One can also interpret this construction as a fiber product or as a symplectic reduction.

Following is the Symplectic category as defined by Weinstein,

Definition 2.3. Define a **symplectic 'category'** $\mathcal{S}ymp$ as the category whose objects are smooth manifolds and the morphism are

$$\hom(M, N) := \{ \text{ Lagrangian submanifolds inside } \overline{T^*M} \times T^*N \ \}$$

The composition of two morphisms L, L' is defined to be $L' \circ L$ as above when the two Lagrangians are composable.

As is evident the issue with this 'category' is that the morphisms are not always composable. We'll further enhance this category to also take into account the Maslov correction.

3. Generating functions of a Lagrangian

Consider an n dimensional vector bundle $\pi : Z \to X$. Let $S : Z \to \mathbb{R}$ be a function which is Morse when restricted to each fiber, this means that when restricted to each fiber, S has finitely many critical points and the Hessian of S at these critical points is non-degenerate.

Let x be the base coordinates and ξ be the vertical coordinates in some coordinate chart.

Associated to S is the set of critical points

$$C_S := \{(x, \xi) : \partial_\xi S(x, \xi) = 0\}$$

This subset C_S is independent of the choice of coordinates.

By parametrized Morse lemma, C_S would be a submanifold of Z. C_S is the set of points where the *vertical derivative* of S vanishes, and because the Hessian is non-degenerate the total derivative is entirely horizontal which gives us the following proposition,

Proposition 3.1. *If $p \in C_S$ then there exists a unique $\eta_p \in T^*_{\pi(p)}X$ such that $dS_p = \pi^*\eta_p$.*

We'll abuse notation and consider the map

$$\begin{aligned}\partial_x : C_S &\to T^*X \\ p &\mapsto (\pi(p), \eta_p)\end{aligned}$$

Proposition 3.2. *∂_x defines an exact Lagrangian immersion of $C_S \looparrowright T^*X$.*

Definition 3.3. Denote the image by Λ_S. We say that S is a **generating function** for Λ_S.

Note that when $Z = X$, S can be any smooth function and Λ_S is the graph of dS as above.

4. FIO's and associated Lagrangians

Let $\rho : Z \to \mathbb{C}$ be a smooth function in some symbol class, this means that ρ and its derivatives satisfy certain decay conditions.

Suppose to begin with X is an open subset of $\mathbb{R}^m$ and $Z \cong X \times \mathbb{R}^n$.

Definition 4.1. A **Fourier Integral Operator** (FIO) on X with phase function S and symbol ρ is an operator $\Psi_{S,\rho} : C_c^\infty(Z) \to \mathcal{E}'(X)$ defined as

$$(\Psi_{S,\rho}u)(x) := \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}S(x,\xi)}\rho(x,\xi)u(x,\xi)d\xi$$

More generally for an arbitrary manifold X and an arbitrary vector bundle Z an operator Ψ is an **FIO** if it is locally an FIO.

The decay conditions on ρ and the Morse condition on S forces this to be in the appropriate Sobolev space.

Thus we can associate to every FIO $\Psi_{S,\rho}$ an exact Lagrangian immersed manifold Λ_S. It turns out that FIO's are deeply connected to this Lagrangian immersion by the following theorem of Hormander,

Theorem 4.2. *If S and S' are two phase functions on (possibly different dimensional) bundles over X such that $C_S \cong C_{S'}$ and $\Lambda_S = \Lambda_{S'}$ via ∂_x, then for any ρ (in some symbol class) there exists a ρ' (in the same symbol class) such that $\Psi_{S,\rho} = \Psi_{S',\rho'} + \mathcal{O}(\hbar^\infty)$ (i.e. modulo a smoothing operator). In particular the highest order approximations (which are obtained from stationary phase) of $\Psi_{S,\rho}$ and $\Psi_{S',\rho'}$ are the same.*

By stationary phase (7.1) the operator $\Psi_{S,\rho}$ is *concentrated* at C_S i.e. up to the first order of approximation,

$$\hbar^{n/2}\Psi_{S,\rho} \approx \delta(C_S)A(x,\xi)\exp\left(\frac{iS(x,\xi)}{\hbar}\right)\rho(x,\xi) + \mathcal{O}(\hbar)$$

where $\delta(C_S)$ is the Dirac delta function supported on C_S and $A(x) = (2\pi)^{n/2}|\det H(S)(x)|^{-1/2}e^{i\pi(sgn(H(S)(x))/4}$. The functions A, S, ρ can be thought of as functions on L via the map ∂_x so that as functions on L we have the following clean formula,

$$\hbar^{n/2}\Psi_{S,\rho} \approx \delta(C_S)\exp(iS/\hbar)A\rho + \mathcal{O}(\hbar)$$

5. Maslov bundle

We can ask how does this depend on the generating function S and ρ. That is if we replace S by S' as in theorem (4.2) we should recover the same expression, so really we are asking how does the expression for A depend on S.

For L a Lagrangian submanifold of T^*X, consider the following set

$$L \mapsto \left\{|\det H|^{-1/2}e^{i\pi(sgn(H))/4}.\exp(iS/\hbar).\rho : C_S = L\right\}$$

(1) The components $\exp(iS/\hbar).\rho$ are the sections of a trivial line bundle so we'll neglect them
(2) The component $|\det H|^{-1/2}$ transforms like square root of the top form $|\det L|^{1/2}$ and so can be thought of as a **half-density**

(3) The component $e^{i\pi(sgn(H))/4}$ is the section of a line bundle called **Maslov bundle** $\mathbb{L}$. $\mathbb{L}$ is a flat line bundle with $\mathbb{Z}/4$ holonomy. This line bundle can also be defined geometrically using the Lagrangian Grassmanian.

So that FIO's can be thought of as sections of

$$\begin{tikzcd} \mathbb{L}_L \otimes |\det L|^{1/2} \arrow[d] \\ L \arrow[u, bend right, "FIO"'] \end{tikzcd}$$

6. Composition of FIO's and the Symplectic 'Category'

There is a natural way to think of an FIO on $X \times Y$ as an operator $X \to Y$ via the Schwarz kernel theorem. Given two such FIO's $\Psi : X \to Y$ and $\Psi' : Y \to Z$ one can ask when is the composition $\Psi' \circ \Psi$ an FIO?

Let L and L' be the Lagrangian submanifolds associated to Ψ and Ψ' respectively.

Theorem 6.1. *The composition of Ψ and Ψ' is an FIO iff L' and L are composable. In this case the Lagrangian associated to a $\Psi' \circ \Psi$ is $L' \circ L$.*

Furthermore it turns out that the Maslov bundle $\mathbb{L}_{L' \circ L}$ is the *fiber product* of $\mathbb{L}_L$ and $\mathbb{L}_{L'}$.

Definition 6.2. This suggests that we define a **Symplectic 'category' -** $\mathcal{S}ymp$ whose objects are symplectic manifolds (M, ω) and whose morphisms are given by

$$\hom(M_1, M_2) = \{(L, \sigma)\}$$

where L is a Lagrangian submanifold of $\overline{M_1} \times M_2$ and σ is a section of $\mathbb{L}_L \otimes |\det L|^{1/2}$. Composition is defined as before.

7. Appendix: Analysis Background

FIOs are wave propagators which arise when studying propagation of singularities. One example of wave equation is the Schrodinger equation.

7.1. Schrodinger equation. Consider the space $\mathbb{R}^n \times \mathbb{R}$ with coordinates $(x_1, \cdots, x_n, t)$. We think of t as the time coordinate. Let $\hbar$ be a *formal* variable. Let $\phi(x), V(x) \in C^\infty(\mathbb{R}^n)$ be *well-behaved* functions.

Consider the **Schrodinger equation**

$$(i\hbar\frac{\partial}{\partial t} + \hbar^2 \Delta_x - V(x))\psi(x,t) = 0 \tag{7.1}$$

with initial condition $\psi(x, 0) = \phi(x)$. (Here $\Delta_x = \sum_i \frac{\partial^2}{\partial x_i^2}$ is the **Lapalcian**.)

We are interested in solutions of the form,

$$\psi(x,t) = \int_{\mathbb{R}^n} U_\hbar(t, \phi, \xi)\widehat{\phi}_\hbar(\xi) d\xi$$

where $\phi_\hbar(\xi)$ is the *semi-classical* Fourier transform of $\phi(x)$.

7.2. WKB approximation. There is a general method for solving these kind of PDEs called the WKB. In the first step we assume that

$$\psi(x,t) = (2\pi\hbar)^n \int e^{\frac{i}{\hbar}S(t,x,\xi)}\widehat{\phi}_\hbar(\xi)d\xi$$

solves the Schrodinger equation (7.1) $\mod \hbar$. Plugging in, this implies that S satisfies the **Eikonal equation**

$$\frac{\partial S}{\partial t} + \sum_j \left(\frac{\partial S}{\partial x_j}\right)^2 + V(x) = 0 \tag{7.2}$$

and the initial condition $S(0,x,\xi) = x \cdot \xi$. (S is called the phase function.)

7.3. General situation. More generally we can replace the Schrodinger equation by any hyperbolic differential equation $P(D)\psi = 0$. Then the general WKB approximation is given by a power series in $\hbar$,

$$\psi(x,t) = \sum_{j=0}^{\infty} \hbar^j \int e^{\frac{i}{\hbar}S(t,x,\xi)} a_j(t,x,\xi)\widehat{\phi}_\hbar(\xi)d\xi$$

We find these functions recursively starting with S. S satisfies the Eikonal equation which is obtained from the principal symbol of $P(D)$ and the a_j's satisfy recursive transport equations.

7.4. Stationary phase approximation. We want to think of $\lim_{\hbar\to 0}$ as the *classical limit*, that is,

$$\lim_{\hbar\to 0}\psi(t,x) = \lim_{\hbar\to 0}\int_{\mathbb{R}^n} e^{\frac{i}{\hbar}S}\widehat{\phi}\, d\xi = ??$$

where S is a solution of the Eikonal equation (7.2).

Theorem 7.1 (Stationary phase)**.** *If $f : \mathbb{R}^n \to \mathbb{R}$ is a Morse function with finitely many critical points $x_1, \cdots, x_k$ and $I_\hbar := \int_{\mathbb{R}^n} e^{\frac{i}{\hbar}f(x)} g(x)dx$ then,*

$$I_\hbar = \hbar^{n/2} \cdot \sum_{j=1}^{k} A(x_j) e^{\frac{if(x_j)}{\hbar}} g(x_j) + \mathcal{O}(\hbar^{n/2+1}) \text{ as } h \to 0$$

$$\text{where } A(x) := (2\pi)^{n/2} \frac{1}{\sqrt{\det H(f)(x)|}} e^{i\pi/4(sgn(H(f)(x))}$$

here $H(f)(x)$ is the Hessian of f at x thought of as a symmetric bilinear form on TX and det *and sgn denote it's determinant and signature respectively.*

This is saying is that the oscillatory integral $I_\hbar$ is *concentrated* at the critical points of the phase function.

Under the assumptions that for each (t,x), $S(t,x,\xi)$ is Morse in ξ with finitely many critical points ξ_j, applying stationary phase (7.1) we get,

$$\psi(t,x)\hbar^{-n/2} \approx \sum_{j=1}^{k} A(\xi_j) \exp\left(\frac{iS(t,x,\xi_j)}{\hbar}\right) \widehat{\phi}(\xi_j)$$