Let E and B be smooth manifolds and let $\pi : E \to B$ be a fiber bundle with fiber isomorphic to a smooth manifold F . Then the Jet bundle $J(E)$ is a huge fiber bundle over E where all the derivatives of sections of π reside.

1. Jets on Euclidean spaces

Consider the space $\mathbb{R}^n$ with coordinates coordinates be $x_1, \cdots, x_n$.

Definition 1.1. For a local function $s = (s_1, \cdots, s_m) : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ around $x \in U$ and a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$ the α jet of s at x is the tuple

$$\begin{aligned} j_x^\alpha(s) &:= \frac{\partial^\alpha}{\partial x^\alpha} s(x) \\ &= \left(\frac{\partial^\alpha}{\partial x^\alpha} s_i(x) \right)_i \end{aligned}$$

We can think of s as a section of the trivial bundle $\mathbb{R}^{n+m} \to \mathbb{R}^n$.

Definition 1.2. The space J_x^r of r jets at a point $x \in \mathbb{R}^n$ is the space of germs of α jets at x where $|\alpha| \le r$. More explicitly

$$J_x^r := \{(U \ni x, s : U \to \mathbb{R}^{n+m} \text{ a local section})\}/\sim$$

where $(U, s) \sim (V, t)$ if $j_x^\alpha(s) = j_x^\alpha(t)$ for all $|\alpha| \le r$. and the subspace $\bar{J}_x^r \subset J_x^r$ as

$$\bar{J}_x^r := \{(U \ni x, s : U \to \mathbb{R}^{n+m} \text{ a local section and } s(x) = 0)\}/\sim$$

Note that the space $\bar{J}_x^r$ has a natural vector space structure of dimension $\#\{\alpha | |\alpha| \le r\}$, and there is a natural map

$$\begin{aligned} J_x^r &\to \mathbb{R}^m \\ s &\mapsto s(x) \end{aligned}$$

such that the preimage of every value is isomorphic to $\bar{J}_x^r$.

Definition 1.3. The collection of all the spaces J_x^r with the natural compact open topology is the r-jet space J^r. Similarly for $\bar{J}_x^r$.

The topology on J^r can be described more explicitly as follows. Consider the space of functions $C^\infty(\mathbb{R}^n, \mathbb{R}^m)$ endowed with the compact open topology. Then the image of r- jets of a basic open set in $C^\infty(\mathbb{R}^n, \mathbb{R}^m)$ would be a basic open set in J^r. There is a simpler way to describe this topology as shown below.

The space J^0 is simply the space $\mathbb{R}^{n+m}$. An element in J^0 of the form $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^m$ can be represented by a function of the form $j_x^0(\xi)$ where ξ denotes the constant function. So if we thing of our functions as sections of the trivial fibration $\mathbb{R}^{n+m} \to \mathbb{R}^n$ then J^0 is simply the total space of the fibration.

There is a natural map $j_x^1 \to j_x^0$, as two functions which have the same 1 jet also naturally have the same 0 jet, which patches together to give a map $J^1 \to J^0 = \mathbb{R}^{n+m}$. The fiber over each point being the 1 jets with 0 jets given which can be thought of as $\mathbb{R}^m$ valued vector fields which can be thought of as the space $\hom(T_x\mathbb{R}^n, T_{s(x)}\mathbb{R}^m)$.

More generally we have a sequence of bundle maps

$$\cdots J^{r+1} \to J^r \to J^{r-1} \to \cdots \to J^0 = \mathbb{R}^{n+m} \to \mathbb{R}^n$$

The fiber at each points becomes harder to describe, so instead we use the following trick. By construction we have surjective maps

$$C^\infty(\mathbb{R}^n, \mathbb{R}^m) \to J^r$$

sending a function to its r-jet. For each r in fact we can look at polynomials over x_i taking values in $\mathbb{R}^m$ so that we are looking at the functions, so let

$$\begin{aligned} P^r &:= \{ \sum_{|\alpha|\leq r} a_\alpha x^\alpha : a_\alpha \in \mathbb{R}^m \} \\ &\subset \mathbb{R}^m[x_1, \cdots, x_n] \subset C^\infty(\mathbb{R}^n, \mathbb{R}^m) \end{aligned}$$

then it is easy to see that the restriction of the above map

$$P^r \to J^r$$

is a surjection and by Taylor expansion is in fact an isomorphism. We can similarly use this to define a topology on J^r. (Note that P^0 is just constant function and hence is isomorphic to the space $\mathbb{R}^{n+m}$.)

Now we can replace the above sequence of bundles by

$$\cdots P^{r+1} \to P^r \to P^{r-1} \to \cdots \to P^0 = \mathbb{R}^{n+m} \to \mathbb{R}^n$$

and the map $P^{r+1} \to P^r$ is map which forgets the highest degree term in the polynomial.

Definition 1.4. Finally define the total jet space J as the inverse limit of the spaces J^r.

In our identification with P^r this is just the space of power series $\mathbb{R}^m[[x_1, \cdots, x_n]]$. The map

$$C^\infty(\mathbb{R}^n, \mathbb{R}^m) \to J \cong \mathbb{R}^m[[x_1, \cdots, x_n]]$$

is the map sending a function to it's (formal) Taylor polynomial. Formal because we are not making any statements about convergence, and as there are many functions which are not analytic there is consider information lost in this map.

2. Jets on Euclidean Open Sets

The first generalization is replacing $\mathbb{R}^n$ and $\mathbb{R}^m$ by open sets $\Omega \subseteq \mathbb{R}^n$ and $\Lambda \subseteq \mathbb{R}^m$. All the construction go through with appropriate restrictions and we get well defined space $J^r(\Omega, \Lambda)$.

The identifications polynomials become a little more subtle. The polynomial space would be

$$\begin{aligned} P^r &:= \{ \sum_{|\alpha|\leq r} a_\alpha x^\alpha : a_\alpha \in \mathbb{R}^m, a_0 \in \Lambda \} \\ &\subset \mathbb{R}^m[x_1, \cdots, x_n] \subset C^\infty(\Omega, \mathbb{R}^m) \end{aligned}$$

And then we have the natural isomorphism

$$P^r \to J^r$$

This is happening because the higher coefficients are really living in the tangent spaces and hence are still allowed to range over $\mathbb{R}^m$.

Finally we have the total jet space J isomorphic to power series in $\mathbb{R}^m[[x_1, \cdots, x_n]]$ with the constant term taking values in Λ the map $C^\infty(\Omega, \Lambda) \to J(\Omega, \Lambda)$ is the (formal) Taylor series map.

3. Jets over Fiber bundles

Now consider a fiber bundle of manifolds $F \to E \to B$. For any small open set $\Omega \subseteq B$ over which the bundle is trivial and some small open subset of $\Lambda \subseteq F$ we can define $J^r(\Omega, \Lambda)$. (Think of (Ω, Λ) as coordinate charts.)

We now interpret $C^\infty(\Omega, \Lambda)$ as local sections of $\Gamma_{loc}(E)$.

Here's the subtle thing. If two local sections s, t have the same r-jets in one coordinate chart (Ω, Λ) then they'll have the same r-jets in any other coordinate chart (Ω', Λ') and consequently these spaces are intrinsically defined and can be glued up together to form nice vector bundles!

I think making this rigorous is more confusing than enlightening hence I'd leave it at this. What we have in the end is

$$\cdots J^{r+1}(E) \to J^r(E) \to J^{r-1}(E) \to \cdots \to J^0(E) = E \to B$$

The polynomials on the other hand do not glue nicely! So they are useful when working locally but useless when trying to describe a global picture.

It is worthwhile to make some of the identifications explicity.

4. The h-principle

We have a natural map

$$\Gamma(B, E) \to \Gamma(B, J(E))$$

sending a section to it's formal Taylor series.

Definition 4.1. The jets which lie in the image of the above map are called **holonomic** section of $J(E)$.

Not all sections are holonomic. A simple example of a section which is not holonomic is a section of E which is not constant but all the higher jets are 0.

Definition 4.2.
(1) A **differential relation** $\mathcal{D}$ of order k is a distribution over B in $J^k(E)$, that is a subbundle of $J^k(E) \to B$.
(2) A **solution** of $\mathcal{D}$ is a section of $\mathcal{D}$ which is holonomic.
(3) Let $hol_{\mathcal{D}}(E)$ be the space of holonomic sections of $\mathcal{D}$.
(4) $\mathcal{D}$ is said to satisfy the h**-principle** (**homotopy principle**) if the inclusion $hol_{\mathcal{D}}(E) \to \Gamma(\mathcal{D})$ is a weak homotopy equivalence.