2 - TQFT's

2D Field theory

$\mathrm{Bord}^{SO}_{\langle 2,1\rangle}$ is free symmetric monoidal containing a commutative Frobenius object

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}^{SO}_{\langle 2,1\rangle}, \mathcal{C}) \simeq \mathrm{cFrob}(\mathcal{C})$$
$$Z \longmapsto Z(S^1)$$

Morse theory, theory & examples of Frobenius algebras.

Morse theory:

locally $f = (x_1)^2 + \cdots + (x_r)^2 - (x_{r+1})^2 - \cdots - (x_n)^2$ then index $f = n-r$

$X : Y_0 \to Y_1$ bordism. $f : X \to \mathbb{R}$ excellent morse function if $(X\text{- compact})$

a) $f(Y_0) = \{a_0\}$, $f(Y_1) = \{a_1\}$

b) c, d critical points $\Rightarrow$ $f(c) \neq f(d)$

$\mathrm{Bord}^{SO}_{\langle 2,1\rangle}$

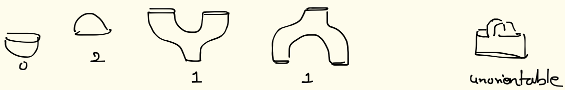
Elementary bordism: $X : Y_0 \to Y_1$ admits an excellent morse function with 1 critical point

This gives us factorization $X = E_n \circ E_{n-1} \circ \cdots \circ E_1$ each E_i elementary.

ex:

Non-unique

All elementary bordisms:



Cerf theory:

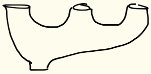
$f : X \to \mathbb{R}$ good if

1) excellent

2) excellent except at 1 pt where it admits birth-death singularity
$[(x^1)^3 + (x^2)^2 + \cdots \quad]$

3) f is excellent except at 2 critical points where it has the same value.

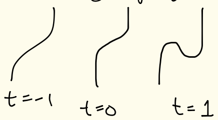
eg:

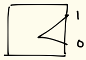


Thm: (Cerf)

f_0, f_1 excellent $\Rightarrow \exists f_t$ good functions connecting f_0 to f_1 excellent at all but finitely many times t.

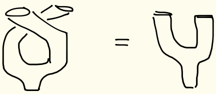
eg: Wall crossing of type t





Relations $\mathrm{Bord}^{(SO)}_{\langle 2,1\rangle}$

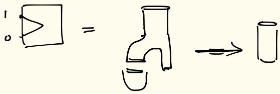
Symmetry:



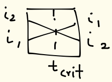
and time reversal

Cerf Theory:

Type α wall crossings



Type β wall crossing

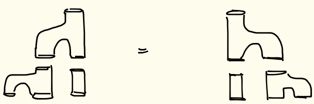


Claim: t_{crit} is connected if our bordism is connected Check

Then $i_1 = i_2 = 1$ Check

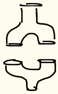
→ Attaching 1-handles changes boundary components by 1.

$3 \to 1$

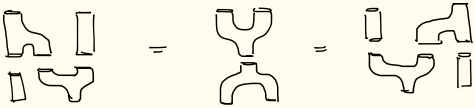


$1 \to 3$ time reversal ↑ time

$1 \to 1$



$2 \to 2$



Frobenius Compatibility

Def: Frobenius Object in $(\mathcal{C}, \otimes, 1)$ Symmetric monoidal category
$A \in \mathcal{C}$ equipped with structure of a monoid (m, η) and a comonoid (Δ, ε) such that Frobenius compatibility holds:

$$(m \otimes 1) \circ 1 \circ \Delta = m \circ \Delta = 1_A \circ m \circ \Delta \circ 1_A$$

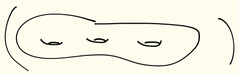
A frobenius object is commutative if 

Frobenius Algebra: Frobenius object in Vect

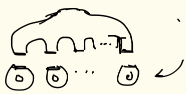
Def^n: Algebra A is frobenius if $\exists$ linear functional $\varepsilon : A \to k$ such that $\langle , \rangle = \varepsilon \circ m$ is non-degenerate.

Prop: This def agrees with previous def^n.

Every Comm. Frobenius algebra contains a cannonical element  $= e(A)$ euler operator

 $= (e(A))^g$

genus g



g of these

Ex: $H^*(\mathbb{CP}^1) = A$

$\varepsilon(\alpha) := \langle \alpha, [\mathbb{CP}^1] \rangle$ then $e(A) = 2x =: e(\mathbb{CP}^1)$ Check

Ex: $A = \mathbb{C}[X]$ $\varepsilon : A \to \mathbb{C}$ determined by $\theta : X \to \mathbb{C}$

finite set

basis: δ_x $x \in X$ orthogonal idempotents

$Z($$) = \sum \theta^{\varepsilon(x)/2}$

dual basis: $\theta_x^{-1} \delta_x$ $e(A) = \sum_{x \in X} \theta_x^{-1} \delta_x$

Ex: G finite group

$\mathbb{C}[G]$ $\quad \varepsilon(x) := \dfrac{tr(x : \mathbb{C}[G] \to \mathbb{C}[G])}{|G|^2}$

$Z(\mathbb{C}[G])$, $\varepsilon|_Z$ is commutative semisimple Frobenius

$$\mathbb{C}[G] \cong \bigoplus_{V \in Irred(G)} End_{\mathbb{C}}(V)$$

$$\Rightarrow \quad Z(\mathbb{C}[G]) \simeq \mathbb{C}[Irred[G])$$

$$\theta_V = \frac{(\dim V)^2}{|G|^2}$$

$$Z_{\mathbb{C}[G]}(\Sigma) = \sum \left(\frac{\dim V}{|G|}\right) \cdot \chi(\Sigma)$$