‘PROOF’ OF THOM’S THEOREM

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Theorem 0.1 (Thom)**.** *A manifold M is null-bordant if and only if all of its Steifel-Whitney numbers are 0.*

Definition 0.2. Ω_n is the group of cobordism classes of n dimensional manifolds. $BO(n)$ is classifying space of n dimensional real vector bundles and $MO(n)$ is the corresponding Thom space. BO and MO are the respective spectra.

Lemma 0.3. *We have the following series of isomorphisms*

(1) *$H^n(BO;\mathbb{Z}/2)$ is the free polynomial algebra $\mathbb{Z}/2[w_i]$ where w_i are the Steifel-Whitney classes. And we have an isomorphism* $\mathrm{Hom}(H^n(BO;\mathbb{Z}/2),\mathbb{Z}/2)) \xrightarrow{\simeq} H_n(BO;\mathbb{Z}/2)$

(2) *The Pontryagin-Thom collapse gives us an isomorphism $\Omega_n \xrightarrow{\simeq} \pi_n(MO)$.*

(3) *There is (Thom) an isomorphism of rings*
$$H_*(MO;\mathbb{Z}/2) \xrightarrow{\simeq} H_*(BO;\mathbb{Z}/2)$$

(4) *The Hurewicz map*
$$\pi_n(MO) \to H_n(MO;\mathbb{Z}/2)$$
is an inclusion.

Assuming these isomorphisms and appropriate commutativity of diagrams(non-trivial) here is how the proof goes:

A SW number of a closed manifold M^n corresponding to the cohomology class $w \in H^n(BO;\mathbb{Z}/2)$ is the cap product
$$\langle TM^*(w), [M]\rangle \in \mathbb{Z}/2$$
where we think of $TM : M \to BO$ as the map classifying the tangent bundle of BO and $[M]$ is the fundamental class of M in $H_n(M;\mathbb{Z}/2)$. So we can think of the SW numbers of a manifold M as a map
$$H^n(BO;\mathbb{Z}/2) \to \mathbb{Z}/2$$

It is easy to see that SW numbers are cobordism invariants and hence SW numbers can be thought of as a map
$$SW : \Omega_n \to \mathrm{Hom}(H^n(BO;\mathbb{Z}/2),\mathbb{Z}/2)$$
$$M \mapsto (w \mapsto \langle TM^*(w), [M]\rangle)$$

Using 1) and 2) SW numbers can be thought of as a map
$$SW : \pi_n(MO) \to H_n(BO;\mathbb{Z}/2)$$

Finally using 3) SW numbers can be interpreted as a map
$$SW : \pi_n(MO) \to H_n(MO;\mathbb{Z}/2)$$
this map turns out to be the Hurewicz map which is injective by 4) which proves Thom’s theorem.

For oriented cobordisms one needs to replace BO by BSO whose $\mathbb{Q}$ cohomology is generated by the Pontryagin classes instead of the SW classes and a similar proof works, but I do not know all the details.