Computations in homotopy groups of spheres: — Mark Behrens

— Ravenel's Green Book , ref

$$\begin{tikzcd} H^{4n-1}(S^{2n}) \arrow[r, "HI"'] & \mathbb{Z} \end{tikzcd}$$

$\leftarrow$ Hopf invariant

$$\begin{tikzcd} S^{4n-1} \arrow[r, "\alpha"] & S^{2n} \arrow[r] & C\alpha \end{tikzcd}$$

$x^2 = \pm HI(\alpha) y$

$x \in H^{2n}(C\alpha)$
$y \in H^{4n}(C\alpha)$

$HI(\alpha) = 1$ for $n = 1, 2, 4, 8$ (Adams)

$$\begin{tikzcd} SO(n) \arrow[r] & \Omega^n S^n \end{tikzcd}$$

$A : \mathbb{R}^n \longrightarrow \mathbb{R}^n \longmapsto A^+$

$\Big\downarrow \lim_{n \to \infty}$

$$\begin{tikzcd} SO \arrow[r] & QS^0 = \Omega^\infty \Sigma^\infty S^0 \end{tikzcd}$$

$$\begin{tikzcd} J : \pi_* SO \arrow[r] & \pi_*^S \end{tikzcd}$$

$\pi_* SO = \mathbb{Z}/2 \; 0 \; \mathbb{Z} \; 0 \; 0 \; 0 \; \mathbb{Z} \; \mathbb{Z}/2 \; 0 \; \mathbb{Z} \; 0 \; 0 \; 0 \; \mathbb{Z} \; \ldots$

$\mathbb{Z}/2 \overset{J}{\longmapsto}$ non-trivial

$\# |\mathrm{Im}\, J_{4k-1}| = \mathrm{denom}\left(\frac{B_{2k}}{4k}\right)$

EHP:

$$\begin{tikzcd} & \Omega^{n-1} S^{n-1} \arrow[r, "E"] & \Omega^n S^n \arrow[r, "H"] & \Omega^n S^{2n-1} \\ \Omega^{n+1} S^{2n-1} \arrow[ur, "P"] & & & \end{tikzcd}$$

$$\begin{tikzcd} \pi_{2n-1} S^n \arrow[r, "HI"'] & \pi_{2n-1} S^{2n-1} \cong \mathbb{Z} \end{tikzcd}$$

EHPSS $\quad E_1 = \pi_{n+k} S^{2n-1} \Longrightarrow \pi_k^S$

$$\begin{tikzcd} \arrow[r] & \Omega^n S^n \arrow[r] \arrow[d] & \Omega^{n+1} S^{n+1} \arrow[r] \arrow[d] & \Omega^{n+1} S^{2n+1} \arrow[r] \arrow[d] & {} \arrow[r, squiggly] & {} \\ \arrow[r] & Q\mathbb{RP}^{n-1}_+ \arrow[r] & Q\mathbb{RP}^n_+ \arrow[r] & QS^n \arrow[r] & {} & \end{tikzcd}$$

$$\begin{tikzcd} \text{EHPSS} \arrow[d] & \bigoplus \pi_*(S^{2n+1}) \arrow[r, Rightarrow] \arrow[d] & \pi_*^S \arrow[d] \\ \text{AHSS} & \bigoplus \pi_*^S(S^n) \arrow[r, Rightarrow] & \pi_*^S(\mathbb{RP}^\infty) \end{tikzcd}$$

differentials here are tough

differentials here are easier to compute as the differentials are attaching maps

Mahowald invariant:

Periodicity of differentials in AHSS of $\mathbb{RP}^\infty$ —

$$\begin{tikzcd} \bigoplus_{n\in\mathbb{Z}} \pi_*^s(S^n) \arrow[r, Rightarrow] \arrow[d, phantom, "\cup"] & \pi_* \mathbb{RP}^\infty_{-\infty} \arrow[d, phantom, "\cong"] \\ MI(\alpha) \arrow[r] & \pi_*(S^{-1}) \ni \alpha \end{tikzcd}$$

$MI(\alpha)$ detects α.

deg $MI(\alpha) <$ deg α

deg $MI(\alpha) >$ deg α

$$2 \xrightarrow{MI} \eta \xrightarrow{MI} \nu \xrightarrow{MI} \sigma \xrightarrow{MI} \sigma^2$$

Adams SS:

E – Ring spectrum, nice

(E_*, E_*E) – Hopf Algebroid, (Algebra + coAlgebra)

$E_*E \longrightarrow E_*E \otimes_{E_*} E_*E$ (co alg)

$E_*X \longrightarrow E_*E \otimes_{E_*} E_*X$ (co-mod)

$$E_2 = \mathrm{Ext}^{s,t}_{E_*E}(E_*, E_*X) \Longrightarrow \pi_{t-s} X^{\wedge}_E$$

$E = H\mathbb{F}_p$, $E_*E = A_*$ ⇝ Steenrod ⟵ Classic Case

In ASS, $HI(x)=1 \iff x$ detected by h_j

$d_2(h_j) = h_{j-1}^2 h_0$ $j \geq 4$ (Hopf invariant - 1 thm)

Kervaire invariant:

π_*^S = framed cobordism groups Ω_*^{fr} Θ_n = group of exotic n-spheres

$\Theta_n \longrightarrow \pi_n^s/\mathrm{Im}J$ obtained by giving a framing to the exotic spheres

$n \equiv 2 \pmod 4$ $\Theta_n \longrightarrow \pi_n^s/\mathrm{Im}J \xrightarrow{\mathrm{Im}J} \mathbb{Z}/2$ Ker inv 1 = element cannot be represented by an exotic sphere

Thm: Browder

$KI(x)=1 \iff x$ detected by h_j^2 in ASS

h_j^2 permanent cycles $j \leq 5$

h_j^2 not perm. cycles $j \geq 7$

h_6^2 ?? $d_r(h_j^2) = ??$

Thm (Mahowald)

$h_1 h_j$ (called η_j) are all permanent cycles for $j \geq 2$.

This Th^m was counter example to doomsday conjecture.

Th^m (Mahowald - Ravenel) $\quad HI(h_j x) = MI(x)$

$$HJ(\eta_j) = HI(h_1 h_j) = MI(\eta) = \nu$$

Q. $h_0 h_j$ are these permanent cycles for $p>2$.

Adams Novikow spectral seq:

$E = BP \qquad \pi_* BP = \mathbb{Z}_{(p)}[v_1, v_2, \cdots] \qquad |v_i| = 2(p^i - 1)$ direct summand of MU

Very few differentials for odd primes

Th^m (Miller) for p odd.
$$\begin{tikzcd} E_2^{ASS} \arrow[r, Rightarrow] \arrow[rr, bend right] & E_2^{ANSS} \arrow[r, Rightarrow] & \pi_*^S \end{tikzcd}$$

Toda differential: $d_r(\beta_{p/p}) = \beta_1^p \alpha_1$, β_{p^i/p^j} odd prime KI

Th^m (Ravenel) : β_{p^i/p^i} is not permanent cycle for $p \geq 5$, $i \geq 1$.

Chromatic homotopy theory:

$$\mathrm{Ext}_{BP_*BP}\left(BP_*, \frac{BP_*}{(p^\infty, v_1^\infty, \cdots, v_{n-1}^\infty)}[v_n^{-1}]\right) \Longrightarrow \mathrm{Ext}_{BP_*BP}(BP_*, BP_*) = E_2^{ANSS}$$

v_n periodicity

$|v_n| = 2(p^n - 1)$

$HI(\alpha_{i/j}) = \alpha_{i-j}$

$MI(p^i) = \alpha_i$

$$\left.\begin{array}{l} HI(\beta_{i/jk}) = \beta_{i-j/k} \\ MI(\alpha_{i/j}) = \beta_{i/j} \end{array}\right\} \text{open questions}$$

Telescope Conjecture??