

Model Categories

— alexie

Stronger:

$\{$ axioms & Examples :

\mathcal{C} = category $f: A \rightarrow B, g: X \rightarrow Y$ in $\text{mor } \mathcal{C}$

Def f is a retract of g if \exists a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{i_1} & X & \xrightarrow{r_1} & A \\ f \downarrow & & g \downarrow & & \downarrow f \\ B & \xrightarrow{i_2} & Y & \xrightarrow{r_2} & B \end{array}$$

such that

$$r_1 \circ i_1 = \text{id}_A$$

$$r_2 \circ i_2 = \text{id}_B$$

Def: f has LLP (left lifting property) wrt g iff g has RLP wrt f iff \exists a lift $B \rightarrow X$ st.

$$\begin{array}{ccc} A & \xrightarrow{\quad} & X \\ f \downarrow & \nearrow & \downarrow g \\ B & \xrightarrow{\quad} & Y \end{array}$$

following diagrams all commute.

Def: Model structure: triple (weak equivalences $\xrightarrow{\sim}$,
of cofibrations $\xrightarrow{\circlearrowleft}$,
classes of fibrations $\xrightarrow{\circlearrowright}$)
of morphisms

satisfying 1) (MC1) 2 out of 3

if any 2 morphisms of the following diag. are weak eq. then so are the other 2.

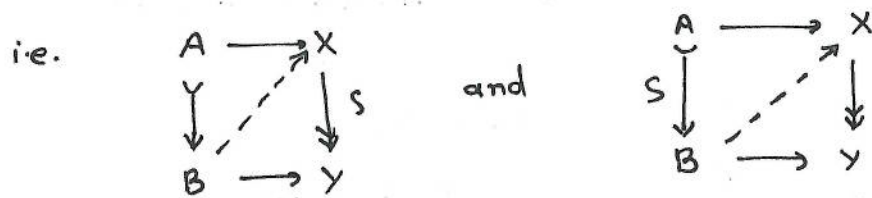
$$\begin{array}{ccc} & \circ & \\ f \nearrow & & \searrow g \\ \circ & \xrightarrow{g \circ f} & \circ \end{array}$$

MC2 : (Retract axiom)

we, fib, cofib are closed under retracts.

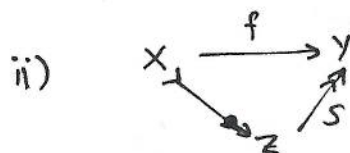
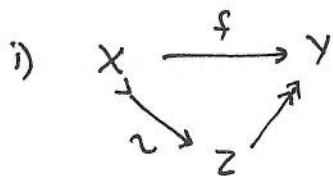
MC3 : (Lifting axiom)

cofib satisfy LLP with respect to trivial fib (= we + fib) (or acyclic)
 fib " RLP " " " cofib (= we + cofib)



MCA : (Factorization)

Any map $f: X \rightarrow Y$ in \mathcal{C} can be factored in the following two ways



Model category : \mathcal{C} contains all finite limits & colimits and has a model structure.

Rmk: $(\mathcal{C}, (w.e., \text{cofib}, \text{fib}))$ is a model category

$\Leftrightarrow (\mathcal{C}^{op}, (w.e., \text{fib}, \text{cofib}))$ is a model category.

it enough to provide either $w.e.$ and cofib or $w.e.$ and fib and the third class is well defined.

Examples:

1) $\mathcal{C} = \text{Top}$ Quillen model structure

$w.e.$ = ~~homotopy~~ weak homotopy equivalence

cofib = retracts of inclusions of relative cell complexes.

$$i: A \hookrightarrow X$$

fib = Serre fibrations

2) $\mathcal{C} = s\text{Sets}$

$w.e.$ = homotopy equivalences of geometric realizations

cof = levelwise injections

fib = Kan fibrations \Leftrightarrow rlp wrt inclusions of horns

$$\Lambda_k^n \rightarrow \Delta^n.$$

3) $\mathcal{C} = \text{Ch}_R^{\geq 0}$

$w.e.$ = quasi-isomorphisms

cofib = levelwise injections with projective kernel

fib = levelwise surjections ~~with~~ for positive ~~surjections~~ degrees.

§ 2 The Homotopy Category :

Moral : \mathcal{C} -model category

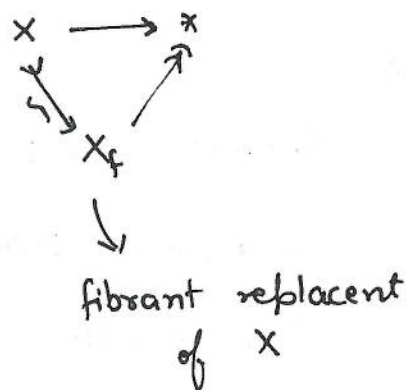
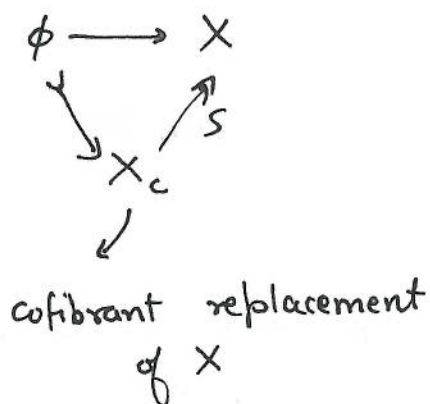
Defⁿ : $Ho(\mathcal{C})$ = homotopy category of \mathcal{C} is the localization wrt weak equivalences.

Def : $\left\{ \begin{array}{l} \phi \rightarrow X \\ \text{initial object} \end{array} \right\} \text{ cofib} \} \Leftrightarrow X \text{ is a cofibrant object}$

$\left\{ \begin{array}{l} X \rightarrow * \\ \text{terminal object} \end{array} \right\} \text{ fib} \} \equiv X \text{ is a fibrant object}$

- All objects are weakly equivalent to cofibrant / fibrant objects.

Follows from the factorizations:



- similarly we get bifibrant (= cofibrant + fibrant) replacement
bifibrant

(6)

Th^m: $\mathcal{C}_b \hookrightarrow \mathcal{C}$ induces an equivalence of categories

$$\mathcal{C}_b / \sim \xrightarrow{\cong} H_0(\mathcal{C})$$

↓

objects: bifibrant objects of \mathcal{C}

$$\text{mor} : \text{Hom}_{\mathcal{C}/\sim}(A, B) = [A, B]$$

Rmk: A map $f: X \rightarrow Y$ in \mathcal{C} is mapped to an iso in $H_*\mathcal{C}$

$$\Leftrightarrow f: X \rightarrow Y \text{ is a w.e.}$$

Def: A Quillen ^{adjunction} ~~functor~~ between model categories \mathcal{C}, \mathcal{D} is an ~~functor~~ adjunction

$$F: \mathcal{C} \rightleftarrows \mathcal{D} : G$$

$\sim F$ preserves cofib and trivial cofib., or equivalently

$\sim G$ preserves fib and trivial fib

Def: This adjunction is Quillen equivalent iff the total derived functors

$$H_0 F: H_0(\mathcal{C}) \rightleftarrows H_0(\mathcal{D}) : H_0(G)$$

is a weak equivalence of the Homotopy categories.

eg: $||: s\text{Sets} \rightleftarrows \text{Top} : \text{Sing}$

Let \mathcal{C}_b be the full subcategory of \mathcal{C} category consisting of all bifibrant objects.

Def: $A \in \text{Ob}(\mathcal{C})$.
Cylinder object $\text{cyl}(A) =$ object ~~together~~ together with a factorization

$$A \sqcup A \rightarrow \text{cyl}(A) \xrightarrow{\sim} A$$

such that the composition is identity on each component.

Def: $f, g: A \rightarrow B$ in \mathcal{C} are left homotopic $f \stackrel{\mathcal{C}}{\sim} g$ if \exists a cylinder object $\text{cyl}(A)$ along with an extension

$$\begin{array}{ccc} A \sqcup A & \xrightarrow{f \sqcup g} & B \\ \downarrow \gamma & \nearrow \exists & \\ \text{cyl}(A) & & \end{array}$$

Prop: a) Left homotopy is an equivalence on $\text{Hom}_{\mathcal{C}}(A, B)$ if A is cofibrant

b) If B is fibrant then the composition in \mathcal{C} descends on classes:

$$\text{Hom}_{\mathcal{C}}(B, A) / \sim \times \text{Hom}_{\mathcal{C}}(A, X) / \sim \longrightarrow \text{Hom}_{\mathcal{C}}(B, X) / \sim$$

$$\text{Hom}_{\mathcal{C}}(B, A) / \sim := [B, A] \quad \text{homotopy classes of maps}$$