Linear Algebraic Groups

- Krishna

Ref: Humphreys - LAG.

Overview:

~~Prerequisites~~:

K - algebraically closed field

$\mathbb{A}^n_K$ - affine n-space over K $\cong K^n = \{(x_1 \cdots x_n) \mid x_i \in K\}$

$f(T) \in K[T_1 \cdots T_n]$ $\quad T := (T_1 \cdots T_n)$

- $\mathcal{V}(I)$, ~~Z~~ $\quad \mathcal{I}(X)$

zero sets of I closed sets in zariski $\mathbb{A}^n_K$.

Hilbert basis th^m:

R noetherian $\Rightarrow$ R[x] noetherian

Ex: check that this forms a topology.

- Affine varity X : ~~zero set of~~ closed sets of $\mathbb{A}^n$.

- Null stellensatz:

$\cancel{V(\mathcal{I})}$ $\quad \mathcal{I}(V(I)) = \sqrt{I}$

Ex:

Excercise: 1) Prove $\sqrt{I}$ is an ideal

2) Prove $\sqrt{I} = \bigcap_{\wp \supseteq I} \cancel{\sqrt{I}} \wp$

1) ~~Using~~ need to show $f^k \in I$, $g^l \in I \Rightarrow$

$\Rightarrow$ for some $m \in \mathbb{N}$ $(f+g)^m \in I$

Choose $m = k+l$.

2) $\sqrt{I} \subseteq \bigcap_{\wp \in I} \wp$ is clear

$\supseteq$ Enough to show $\bigcap_{\wp} \wp = \sqrt{0}$. ~~Two ways~~

So need to show f not nilpotent $\Rightarrow \exists \wp$ not containing f

1) $R \longrightarrow R_f$ Localize at f

$m^c \longrightarrow m$ look at max ideal, inverse

2) Look at $U = \{I \mid f^n \notin I$ for any $n\}$

This has maximal prime element.

Then

$$\{\text{affine variety}\} \underset{\mathcal{I}}{\overset{V}{\longleftrightarrow}} \{\text{radical ideals}\}$$

$$\{\text{irreducible affine variety}\} \longleftrightarrow \{\text{prime ideal}\}$$

~~Ex~~ Ex:

Ex 3: Prove the above correspondence

4: Any variety will have finite set of maximal irreducible components, which is same as prime ideal decomposition. Also unique.

Def^n 1

- Morphisms

$$\mathrm{Mor}(X, K) = K[T]/\mathcal{I}(X) =: K[X]$$

In $K[X]$ we donot have any nilpotent elements (but we can have ~~&~~ zero divisors). by Nullstellensatz $\mathcal{I}(X)$ is radical

~~So we look at~~

This is called reduced.

Also same as saying $\sqrt{0} = 0 = \bigcap\limits_{\mathfrak{p} \text{ prime}} \mathfrak{p}$

- Now we reverse the process

let R be f.g. reduced K algebra

$\Rightarrow \exists$ an exact seq

$$0 \to I \to K[T] \longrightarrow R \longrightarrow 0$$

This is ~~the~~ a corresponding affine variety.

- Morphism of affine varieties

$X \in \mathbb{A}^n \quad Y \in \mathbb{A}^m$

$\varphi : X \longrightarrow Y$ morphism ~~of~~

$\varphi(x) \longmapsto \varphi_1(x) \ldots \varphi_m(x)$

where $\varphi_1 \cdots \varphi_m$ are polynomial functions on X i.e. in $K[X]$

~~and~~

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, hook] & Y \arrow[d, hook] \\ \mathbb{A}^n \arrow[r] & \mathbb{A}^m \end{tikzcd}$$

True for some representative of $(\varphi_1 \cdots \varphi_m)$

- $\varphi^*: K[Y] \longrightarrow K[X]$
  $f \longmapsto f \circ \varphi$

$$X \xrightarrow{\varphi} Y \xrightarrow{f} K$$

Ex: Ex 5. well defined ? polynomial f^n ?

One can recover φ from φ^* by looking at co-ordinate f^ns.

i.e $\varphi_i(x) = \varphi^*(y_i)$

- Next we need to give an intristic def^n of a variety.

$x \in X$, X irreducible

$\Rightarrow K[X]$ ~~irreducible~~ integral domain

$K(X)$ field of fractions of $K(x)$

– field of rational functions on X.

not really functions but functions on open subsets where denominators do not vanish

$U \in X$, $K(U)$ – ring of rational functions on U

~~such that~~ $f \in K(X)$ s.t. f defined on U.

Ex 3, 4 : Prime decomposition for Noetherian ring implies every ~~prime~~ ideal $I = \wp_1 \cap \wp_2 \ldots \cap \wp_k$

minimal primes

unique upto ordering

$\Rightarrow V(I) = V(\wp_1) \cup V(\wp_2) \ldots \cup V(\wp_k)$

~~Enough to show~~ ~~$V(\wp_i)$ irreducible~~

Ex 5 : $[f_1] = [f_2] \Rightarrow f_1 - f_2 \in \mathcal{I}(Y)$

~~Also~~ Need to show $(f_1 - f_2)\phi \in \mathcal{I}(X)$ i.e. $(f_1 - f_2)\circ\phi(x) = 0 \ \forall x \in X$

But this true $\because$ ~~f_1~~ $\phi(x) \in Y$ and $f_1 - f_2(Y) = 0$.

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Aim: To define ~~var~~ affine variety intrinsically

Ex:

In $\mathbb{A}^2_k$,

Def: $\boxed{D(f) := \mathbb{A}^n_k - V(f)}$

$$\left.\begin{array}{l} X_1 = \{\mathbb{A}^2_k - \{(0,0)\}\} \\ X_2 = D(T_1) \end{array}\right\} \text{ open}$$

Are these varieties in higher embeddings?
X_1 No, X_2 Yes

Def^n

X - irreducible affine variety

$K[X]$, $K(X)$, $\mathcal{O}_x := \{\frac{f}{g} \in K(X) \mid g(x) \neq 0\} \leq K(X)$

rational functions at $x \in X$

Ex 1. Check $\mathcal{O}_x$ is well defined.

$\longrightarrow$ $K[X] \leq \mathcal{O}_x \leq K(X)$ as rings

$\longrightarrow$ $\mathcal{O}_x$ local ring.
maximal ideal - $\{\frac{f}{g} \mid f(x) = 0\} \leq \mathcal{O}_x$

Def^n:

$\mathcal{O}_X(U) := \bigcap_{x \in U} \mathcal{O}_x$ ← U open in X
ring of rational f^{n}s on U

$\longrightarrow$ $K[X] \leq \mathcal{O}_X(U) \leq K(X)$ as rings

$\mathcal{O}_X$ - sheaf on rings

Def^n:

Variety:

X - irreducible topological space

$\mathcal{O}_X$ - sheaf of rings on X

$\exists$ finite open covering $\{U_i\}_{i \in I}$ of X s.t.

$$(U_i, \mathcal{O}_X|_{U_i}) \cong (Y_i, \mathcal{O}_{Y_i}) \quad \forall i,$$

↑ affine variety

+ a topological condition.

Def^n: $\mathbb{P}^n_k \equiv$ Projective n-space / k
$:= \mathbb{A}^{n+1}_k - \{0\} / \sim$ Scalar

Projective variety etc.

Def^n: (Linear) Algebraic Group

X - X variety, ~~a~~ group
~~binary oper~~ multiplication, inverse are morphisms of varieties

- Linear if $X \subseteq Gl_n(K)$ closed subgroup

eg: $Gl_n(K) \subseteq M_n(K) \cong \mathbb{A}^{n^2}$
~~clos~~ $\parallel$ $D(\mathrm{Det}) \leftarrow$ Principal open
$\Rightarrow Gl_n(K)$ ~~open~~ ~~in~~ affine in $\mathbb{A}^{n^2+1}$
Sl_n, So_n, ~~so~~ Sp_n

-Narsimhan Chary

Representation Theory of finite groups

$G \curvearrowright X$ From X what can be said about G?

V/k - vector space over k dim n

Representation: $\rho: G \longrightarrow Gl(V)$ group homo
$\cong$ $Gl_n(K)$ via a basis

~~eg~~:

1) $G \longrightarrow Gl(V)$
$g \longmapsto$ identity

2) $G \longrightarrow Gl(V) \cong k^*$ if $n=1$
character

3) $V = K[G]$
$G \longmapsto Gl(V)$
$g \longmapsto (x \longmapsto gx)$
regular representation - left

4) X - finite $G \curvearrowright X$,
Similar $V = K\langle X \rangle$, $G \longrightarrow Gl(V)$

Invariant Subspace, faithful representation, irreducible rep.
Maschke's Th^m.

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Def^n: Product of Affine varieties

$\mathbb{A}^n \supseteq X$ — $I_X \subseteq K[T_0]$ $Y \subseteq \mathbb{A}^m$ — $I_Y \subseteq K[T']$

$X \times Y \subseteq \mathbb{A}^{n+m}$ — $(I_X + I_Y) \subseteq K[T, T']$

$K[X \times Y] = K[X] \otimes_K K[Y]$.

X, Y irreducible $\Rightarrow$ $X \times Y$ irreducible

i.e. $K[X], K[Y]$ domains $\Rightarrow K[X] \otimes_K K[Y]$ domain

(Not true in general, tensor product can make ~~tensor~~ product domains into non-domains, ~~and~~ here $K[x]$ is f.g. algebra over K, hence it works.)

Ex: $X \subseteq \mathbb{A}^3 = \{(t, t^2, t^3) \mid t \in K\}$ Show X is closed in $\mathbb{A}^3$.

Ex: $\phi: \mathbb{A}^1 \longrightarrow \mathbb{A}^2$ call the image Y.

$t \longmapsto (t^2, t^3)$

$\phi: \mathbb{A}^1 \longrightarrow Y$ is a homeomorphism.
but is not an isomorphism of varieties.

$K[\mathbb{A}^1] = K[T]$. $K[Y] = \dfrac{K[T_1, T_2]}{(T_1^3 - T_2^2)}$

Def^n: Dimension: $X \subseteq \mathbb{A}^n$ for X-irreducible

1) $\dim X$ as topological space
i.e. height of prime ideals
2) Krull dim $K[X]$
3) transcendence degree of $K(x)$ over K.

Similarly for arbitrary varieties.
What is $K(X)$? $K(X) := \varinjlim_{U} \mathcal{O}_X(U)$

Def: $X_f \subseteq X \subseteq \mathbb{A}^n$ — affine X-irreducible.

Ex. $\dim X_f = \dim X$, X_f irreducible.

In general, dim open subset of $X = \dim X$.