$$Sq^1Sq^1(\sigma_n) = Sq^1(\sigma_1\sigma_n) = (x_1^2+\cdots+x_n^2)\sigma_n + \sigma_1^2\cdot\sigma_n$$
$$= 0$$

So clearly the Sq's are not algebraically independant. The following is a weaker version of Adem relations.

Defn: A seq $I=(i_1,i_2,\cdots)$ is admissible if $i_1 \geq 2i_2,\ i_2 \geq 2i_3, \cdots$

Time for some ugly computations:

Let $[2,1,0,0,\cdots]$ denote $\sum x_i^2x_j$ summed over all i,j

$$Sq^1Sq^i(\sigma_{i+1}) = Sq^1(\sigma_i\sigma_{i+1})$$
$$= \sigma_1\sigma_i\sigma_{i+1} + \sigma_{i+1}Sq^1(\sigma_i)$$
$$= \Big([1,0,\cdots].[\underbrace{1,1,\cdots,1}_{i},0].[\underbrace{1,1,\cdots,1}_{i+1}]\Big) + \Big([\underbrace{1,1,\cdots,1}_{i+1}].\ [2,\underbrace{1,\cdots,1}_{i},0]\Big)$$
$$= [2,1,\cdots,0)[1,\cdots,1] + (i+1)[1,\cdots,1]\,[1,\cdots,1] + [1,1,\cdots,1].[2,1\cdots,1,0]$$
$$= (i+1)[2,2,\cdots,2]$$

$$Sq^iSq^1(\sigma_{i+1}) = Sq^i(\sigma_1\sigma_{i+1})$$
$$= \sigma_1\sigma_i\sigma_{i+1} + Sq^1(\sigma_1).\ \sigma_{i-1}\cdot\sigma_{i+1}$$
$$= [1,0,\cdots,0]\cdot[1,1,\cdots1,0][1,\cdots,1] + [2,0,\cdots,0]\,[\overbrace{1,\cdots,1}^{i-1},0,0]\,[1\cdots1]$$
$$= (i+1)[2,2,\cdots,2] + 2\cdot[\cancel{3,2,\cdots},2,1] + [4,2,\cdots,2,1,1]$$

$$Sq^2Sq^i(\sigma_{i+2}) = Sq^2(\sigma_i\sigma_{i+2})$$
$$= \sigma_i\sigma_2\sigma_{i+2} + Sq^2(\sigma_i)\sigma_{i+2} + Sq^1\sigma_i\ \sigma_1\sigma_{i+2}$$
$$= [1,1,0\cdots][\underbrace{1,1,\cdots,1}_{i},0,0][1,1\cdots,1] + [2,2,\underbrace{1,\cdots,1}_{i-2},0,0](1,\cdots,1] + [2,\underbrace{1,\cdots,1}_{i-1},0,0][1,0,\cdots,0][1,\cdots,1)$$
$$= \Big\{[2,2,1,\cdots,1,0,0] + \binom{i+2}{2}(1,\cdots,1] + i\cdot[2,\underbrace{1,\cdots,1}_{i},0] + [2,2,1,\cdots1,0,0] + i\,[2,\underbrace{1,\cdots,1}_{i},0] + 2.[2,2,1,\cdots,1,0] + [3,\underbrace{1,\cdots,1}_{i-1},0,0]\Big\}[1,\cdots,1]$$
$$= \binom{i+2}{2}[2,\cdots,2] + [4,2,\cdots,2,1,1]$$

So $Sq^1Sq^i = (i+1)\,Sq^{i+1}$

$Sq^2Sq^i + Sq^{i+1}Sq^1 = \left[(i+1)+\binom{i+2}{2}\right]Sq^{i+2}$

Are there any patterns?

Let $a \geqslant 2b$, $n \gg a$

$$Sq^a Sq^b(\sigma_n) = Sq^a(\sigma_b \sigma_n)$$

$$= \sigma_b\sigma_a\sigma_n + Sq^1(\sigma_b)\cdot\sigma_{a-1}\sigma_n + Sq^2(\sigma_b)\,\sigma_{a-2}\,\sigma_n + \cdots + Sq^{b-1}(\sigma_b)\sigma_1\sigma_n + Sq^b(\sigma_b)\sigma_n$$

$$= \Big([\underbrace{1,\dots,1}_{b},0\dots]\cdot[\underbrace{1,\dots,1}_{a},0] + [2,\underbrace{1,\dots,1}_{b-1},0][\underbrace{1,\dots,1}_{a-1},0] + [2,2,\underbrace{1,\dots,1}_{b-2},0][\underbrace{1,\dots,1}_{a-2},0] + \cdots + [\underbrace{2,2,\dots,2}_{b-1},1,0][\underset{\overset{\uparrow}{a-b+1}}{1},0,\dots] + [\underbrace{2,2,\dots,2}_{b},0][\underbrace{1,\dots,1}_{a-b}0]\Big)\cdot\sigma_n$$

$$= \text{madness}$$

$$Sq^{2i-1}Sq^i(\sigma_n) = Sq^{2i-1}[\underbrace{2,2,\dots,2}_{i},\underbrace{1,\dots,1}_{n-i}]$$

I do not see any way of working these out by hand.

$$Sq^a(x_1\cdots x_n) = \sum_{\substack{\Sigma i_k = a\\ i_k\in[0,1]}} Sq^{i_1}x_1\cdots Sq^{i_n}x_n$$

Not all x_i's distinct

$$= \sum_{\substack{\Sigma i_n = a\\ i_n\in[0,1]}} x_1^{i_1}\cdots x_n^{i_n}\cdot(x_1\cdots x_n)$$

$$Sq^a(x_1^2\cdots x_b^2\cdots x_n) = \sum_{\substack{\Sigma i_n = a\\ i_k\in[0,1\}}} x_1^{i_1+i_2}\cdots x_b^{i_{2b-1}+i_{2b}}\cdot x_{b+1}^{i_{2b+1}}\cdots x_n^{i_n}\cdot(x_1^2\cdots x_b^2\cdots x_n)$$

Th

Th^m (proved inductively) Adem's Relations:

if $a<2b$, $$Sq^aSq^b = \sum\binom{b-c-1}{a-2c}Sq^{a+b-c}Sq^c$$

$$Sq^{2n-1}Sq^n = \cancelto{0}{\binom{n-0-1}{2n-1}}Sq^{3n-1}Sq^0 + \cancelto{0}{\binom{n-1-1}{2n-3}}\cdots+\cdots+\cancelto{0}{\binom{n-(n-2)-1}{2n-1-2(n-2)}} + \cancelto{0}{\binom{0}{1}}. \quad = 0$$

Now define the Steenrod algebra $\mathcal{A}$ as the free tensor algebra over Sq^i's modulo the Adem relations. This is terrible notation, I am going to denote the Steenrod algebra by $\mathbb{S}q$

Note that this is a highly non-commutative algebra. However there is a coproduct on this which makes it cocommutative.

Prop: $\amalg : \mathbb{S}q \to \mathbb{S}q\otimes\mathbb{S}q$ $\quad Sq^i \mapsto \sum_j Sq^j\otimes Sq^{i-j}$ is a coproduct.

Proof: Lets do this carefully,

- Counit: $\varepsilon : \mathbb{S}q \longrightarrow \mathbb{Z}/2$
  $Sq^i \longmapsto 0$
  $Sq^0 \longmapsto 1$

$$(1\otimes\varepsilon)\cdot\amalg(Sq^n) = (1\otimes\varepsilon)\Big(\sum_i Sq^i\otimes Sq^{n-i}\Big) = Sq^n = (\varepsilon\otimes 1)\cdot\amalg\cdot Sq^n$$

- Coassociativity:

$$\begin{tikzcd} Sq^n \arrow[r, mapsto, "\amalg"] & \sum\limits_{i+j=n} Sq^i\otimes Sq^j \arrow[r, shift left, "1\otimes\amalg"] \arrow[r, shift right, "\amalg\otimes 1"'] & \sum\limits_{i+j+k=n} Sq^i\otimes Sq^j\otimes Sq^k \end{tikzcd}$$

- Compatibility with multiplication :

We need to show that if $a < 2b$

$$\amalg(Sq^a \cdot Sq^b) = \sum \binom{b-c-1}{a-2c} \amalg(Sq^{a+b-c} Sq^c)$$

$$\begin{aligned}\amalg(Sq^a \cdot Sq^b) &= (\amalg Sq^a) \cdot (\amalg Sq^b)\\ &= \left(\textstyle\sum_i Sq^i \otimes Sq^{a-i}\right)\left(\textstyle\sum_j Sq^j \otimes Sq^{b-j}\right)\\ &= \sum Sq^i Sq^j \otimes Sq^{a-i} Sq^{b-j}\end{aligned}$$

I again do not see how to show that there are equal.

So we need to go back to products of projective spaces & $K(\mathbb{Z}/2, n)$.

$$\sum \binom{b-c-1}{a-2c} \amalg(Sq^{a+b-c} Sq^c) = \sum \binom{b+c-1}{a-2c} \left(\sum_i Sq^i \otimes Sq^{a+b-c-i}\right) \cdot \left(\sum_j Sq^j \otimes Sq^{c-j}\right)$$

$$= \sum_c \binom{b-c-1}{a-2c} \sum_{i,j} Sq^i Sq^j \otimes Sq^{a+b-c-i} Sq^{c-j}$$

Failed idea

By Serre's th^m we know that cohomology of $K(\mathbb{Z}/2, n)$ is generated by $Sq^I \sigma_n$ where I is admissible with excess $< n$. Excess of $Sq^{a+b-c} Sq^c = (a+b-c-c) + c \leq a+b$

So for n large the right hand side is detected by $H^*(K(\mathbb{Z}/2, n); \mathbb{Z}/2)$.

$K(\mathbb{Z}/2, n) \cong \Omega K(\mathbb{Z}/2, n+1)$ and hence has a natural Hopf algebra structure. If we show that this coproduct agrees with our coproduct then compatibility would follow.

Even this is false $\ddot{\frown}$ $\Delta(Sq^n(i_n)) = \Delta(i_n^2) = i_n^2 \otimes 1 + 1 \otimes i_n^2 \neq \sum_i Sq^i(i_n) \otimes Sq^{n-i}(i_n)$

The Adem relations are simply algebraic relations between the various symmetric polynomials and so I think the compatibility with multiplication is a tautology, not sure