Cohomology via Sheaves

Apurva Nakade

Contents

0 Introduction & Motivation

> In mathematics you don't understand things. You just get used to them.
>
> John von Neumann

Cohomology was introduced by Poincare in a series of papers named *Analysis Situs* and now forms the basis of modern Algebraic Topology. To a topological space X we can associate a sequence of vector spaces denoted

$$\check{H}^i(X)$$

for each $i \in \mathbb{Z}_{\geq 0}$ called it's **Cech Cohomology** (pronounced *check cohomology*). In a very loose sense, the dimension of $\check{H}^i(X)$ measures the i^{th} dimensional holes in X.

Why care about the i^{th} dimensional holes? We can use these to rigorously distinguish between spaces. For example, most proofs of the fact that $\mathbb{R}^m$ is not homeomorphic* to $\mathbb{R}^n$ if $m \neq n$ use some cohomology computation. We'll also see that a torus $S^1 \times S^1$ has two *1-dimensional holes* which distinguishes it from a sphere S^2 which has none.

Theorem 0.1. *Two topological spaces X, Y are homeomorphic only if*

$$\check{H}^i(X) \cong \check{H}^i(Y)$$

for all non-negative integers i.

Remark 0.2. The above statement is not an *if and only if* statement. The other direction is easily shown to be false. You'll be able to come up with examples by yourself in a couple of days.

Computing the cohomology requires multiple steps. The goal of this class is to develop the relevant machinery and actually do some cohomology computations.

$$\underset{\text{Topological Space}}{X} \rightsquigarrow \underset{\text{Good cover of } X}{\mathcal{U}} \rightsquigarrow \underset{\text{Cech Complex of } \mathcal{U}}{\mathcal{L}^\bullet(\mathcal{U})} \rightsquigarrow \underset{\text{Cech Cohomology of } X}{\check{H}^*(X)}$$

Note: The number of stars (*) on the problems indicate their difficulty level. The non-starred marked problems are compulsory, the starred problems are optional.

*Homeomorphism is the isomorphism for topological spaces.

1 Topological Preliminaries

> Humpty Dumpty sat on a wall,
> Humpty Dumpty had a great fall.
> All the king's horses and all the king's men
> Could not put Humpty together again.

The first step is to break a space up into *simpler spaces* and try to *glue* the pieces back. Simpler spaces will mean contractible spaces and gluing back will be done using locally constant functions.

$$X \rightsquigarrow \mathcal{U}$$

Topological Space   Good cover of X

Definition 1.1. A topological space[*] X is said to be **contractible** if there exists a point $x_0 \in X$ and a continuous map

$$\Phi : X \times [0,1] \to X$$

such that

$$\Phi(x,0) = x$$
$$\Phi(x,1) = x_0$$

for all $x \in X$ i.e. there are "continuously varying paths" connecting each point in X to x_0.

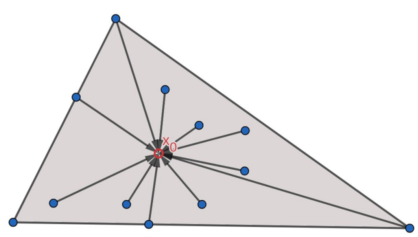


Figure 1: The interior of a triangle is a contractible space.

In some sense, a contractible space is as simple a space as is topologically possible. Most algebro-topological invariants cannot distinguish a contractible space from a point. This makes contractible spaces very useful as one may need infinitely many points to *construct* a space but only finitely many contractible spaces to do so, as we'll see below.

[*] For us a topological space is simply a subset of $\mathbb{R}^2$ or $\mathbb{R}^3$.

Question. 1. A subset X of $\mathbb{R}^n$ is said to be **star-shaped** if there exists a point x_0 such that for any other point $x \in X$ the segment connecting x to x_0 lies entirely in X. Prove that star-shaped subsets of $\mathbb{R}^n$ are contractible.

Remark 1.2. This proves that the sets $\mathbb{R}^n$, $[0,1]$, $(0,1)$, union of x and y axis in $\mathbb{R}^2$, interior of a convex polygon in $\mathbb{R}^2$, interior of a convex polyhedron in $\mathbb{R}^3$ are all contractible.

Not every space is contractible, otherwise topology would have been a very boring subject (which it isn't). The first application of Cech Cohomology will be proving that spaces like the circle S^1, the spheres S^2, S^n, torus, projective space, other higher genus surfaces are not contractible.

Question. 2. Let X, Y be topological spaces.

a) Show that if X, Y are contractible then so is $X \times Y$. In particular, if X is contractible then so is $X \times [0,1]$.

b) Show that if $X \times Y$ is contractible, then so is X.

c) * A subspace $A \subseteq X$ is said to be a **retract** of X if there exists a continuous map

$$r : X \to A$$

such that $r(a) = a$ for all $a \in A$. Show that if X is contractible and A is a retract of X then A is also contractible.

d) Is it true that every subset of a contractible space is contractible?

Definition 1.3. We say that X is **connected**[†] if for any two points $x_0, x_1 \in X$ there exists a continuous map $c : [0,1] \to X$ such that $c(0) = x_0$ and $c(1) = x_1$. Define a relation on the set X as $x_0 \sim_{conn} x_1$ if there exists a path in X connecting x_0 to x_1.

Question. 3. Let X be a topological space.

a) Show that if X is contractible then X is connected.

b) Show that $\sim_{conn}$ is an equivalence relation.

Definition 1.4. Define the **connected components** of X, denoted $\pi_0(X)$, to be the equivalence classes of X under $\sim_{conn}$.

Remark 1.5. All our spaces will have finitely many connected components.

[†]Technical point: This should really be called *path-connected* but we will only be dealing with spaces where the two notions coincide.

1.1 Covers

Definition 1.6. A finite collection of open subsets[‡] $\mathcal{U} = \{U_1, U_2, \ldots, U_n\}$ of a topological space X is said to be a **cover** of X if $X = U_1 \cup U_2 \cup \cdots \cup U_n$.

Example 1.7.

a) $\mathcal{U} = \{X\}$ is always a cover of X of any space X.

b) If X is a triangle, then $\mathcal{U} = \{U_1, U_2, U_3\}$ with U_i being a "side" of the triangle is a cover of X.

Not all covers are equal. We need the covers to satisfy the following extra condition.

Definition 1.8. Let $[n] = \{1, 2, \cdots, n\}$. To every non-empty subset $I \subseteq [n]$ we can associate a subset U_I of X defined as

$$U_I := \bigcap_{i \in I} U_i$$

The cover $\mathcal{U} = \{U_1, U_2, \ldots, U_n\}$ is said to be a **good cover** if for every non-empty subset $I \subseteq [n]$ every connected component of the space U_I is contractible.

Remark 1.9. In the above definition, U_I can be empty as every connected component of an empty set is contractible.

Definition 1.10. The **dimension** of a cover $\mathcal{U}$ is the largest k such that there exists some $I \subseteq [n]$ with $|I| = k + 1$ and U_I non-empty.

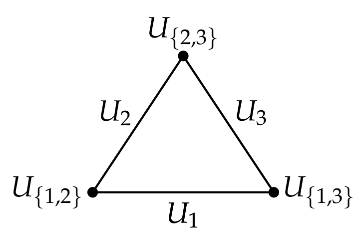


Figure 2: The set of "sides" $\mathcal{U} = \{U_1, U_2, U_3\}$ is a good cover of the triangle. In this case $U_{\{1,2\}} = U_1 \cap U_2$, $U_{\{2,3\}} = U_2 \cap U_3$, $U_{\{1,3\}} = U_1 \cap U_3$ are the vertices and $U_{\{1,2,3\}} = U_1 \cap U_2 \cap U_3$ is empty.

[‡]We'll sometimes be lazy and use simplices instead of open subsets.

Remark 1.11. For the following problems, construct good covers with the least possible dimension; the higher the dimension of the cover, the harder the cohomology computations.

Question. 4. Construct good covers of the following spaces. It's enough (and recommended) to simply draw pictures.

a) $\mathbb{R}^n$

b) S^1 (= the circle)

c) A Tree

d) The bipartite graph $K_{2,3}$

e) $\mathbb{R}^2 \setminus \{(0,0)\}$

f) $\mathbb{R}^2 \setminus \{(0,0),(1,0),\ldots,(k,0)\}$ for some positive integer k

g) S^2 (= the sphere)

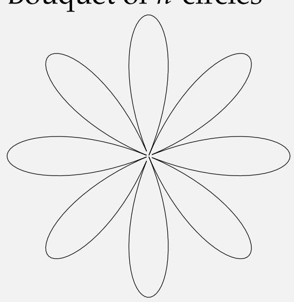
h) S^2 minus a point

i) S^2 minus 2 points

Question. 5. Find good covers of

a) $S^1 \vee S^1$ = two circles glued at a point (pronounced 'S one wedge S one')

b) Bouquet of n-circles



c) $S^1 \vee S^2$ = a circle and a sphere glued at a point (pronounced 'S one wedge S two')

Question. 6. * Let X, X' be topological spaces with good covers $\mathcal{U}, \mathcal{U}'$ respectively. Find a good cover of $X \times X'$. Find a good cover of $S^1 \times S^1$ (= torus). Draw a picture.

Question. 7. * Find a good cover of the solid torus (which is homeomorphic to $[0,1] \times [0,1] \times S^1$).

Question. 8. ** Find a good cover of the g-holed torus.

It is not true that all spaces admit good covers. However, the spaces that do not admit one are very exotic in nature. For example, every smooth manifold admits a good cover (this is called the Nerve Lemma) however this is false if we drop the adjective smooth. For this class we'll assume that all the spaces under consideration admit a good cover.

2 The Cocomplex World of Cochain Complexes

> The introduction of the digit 0 or the group concept was general nonsense too, and mathematics was more or less stagnating for thousands of years because nobody was around to take such childish steps.
>
> Grothendieck

$$\underset{\text{Good cover of } X}{\mathcal{U}} \rightsquigarrow \underset{\text{Cech Complex of } \mathcal{U}}{\mathcal{L}^\bullet(\mathcal{U})}$$

To be able to use algebraic techniques, we need a way to convert topological information into algebraic information. We'll do this using locally constant functions and of course, linear algebra. All our vector spaces will be over the base field $\mathbb{F}_2 = \{0, 1\}$ i.e. all the scalars are either 0 or 1. This is mainly because $-1 = 1$ in $\mathbb{F}_2$ and hence we do not have to worry about signs.

Notation: $V = \mathbb{F}_2\langle v_1, \ldots, v_n\rangle = \mathbb{F}_2\langle\mathcal{B}\rangle$ stands for "V is a vector space over $\mathbb{F}_2$ with basis $\mathcal{B} = \{v_1, \ldots, v_n\}$".

Question. 9. Show that the elements of $\mathbb{F}_2\langle\mathcal{B}\rangle$ can be identified with subsets of $\mathcal{B}$ and hence as a set V has size $2^{|\mathcal{B}|}$, where $|\mathcal{B}|$ denotes the size of $\mathcal{B}$.

2.1 Locally Constant Functions

Definition 2.1. For a topological space X, define the vector space of **locally constant functions**, denoted $\mathcal{L}(X)$, to be the space of continuous maps from X to $\mathbb{F}_2$.

$$\mathcal{L}(X) := \{\, f : X \to \mathbb{F}_2 \ \text{ continuous} \,\}$$

Here we are thinking of $\mathbb{F}_2$ as a topological space with 2 points.

Question. 10. Show that $\mathcal{L}(X)$ is naturally a vector space over $\mathbb{F}_2$.

Question. 11. Show that if X is connected then every continuous function $f : X \to \mathbb{F}_2$ is a constant function and hence $\mathcal{L}(X) \cong \mathbb{F}_2$ as a vector space.

Question. 12. What is $\mathcal{L}(\phi)$ where ϕ is the empty set (which is a legit topological space)?

Let $X^1, X^2, \ldots, X^k$ be the connected components of X. Define k functions $\delta^1, \delta^2, \ldots, \delta^k : X \to \mathbb{F}_2$ as

$$\delta^i(x) = \begin{cases} 1 & \text{if } x \in X^i \\ 0 & \text{otherwise} \end{cases}$$

for $1 \leq i \leq k$.

Question. 13. Show that

$$\mathcal{L}(X) = \mathbb{F}_2\langle \delta^1, \delta^2, \ldots, \delta^k \rangle$$

and hence $\dim \mathcal{L}(X) = k = \pi_0(X)$.

Question. 14. * Show that the above statement is false if X has infinitely many connected components, for example, if X is the set of integers.

Definition 2.2. With the notation as above, we call $\delta^1, \delta^2, \ldots, \delta^k$ the **canonical basis** for $\mathcal{L}(X)$.

The following exercise is extremely important, make sure you understand it well.

Question. 15. For an inclusion of topological spaces $X \subseteq Y$,

a) We can restrict a function $f : Y \to \mathbb{F}_2$ to the subspace X and get a function $f|_X : X \to \mathbb{F}_2$. Show that this induces a linear transformation

$$\mathrm{Res}_{Y \to X} : \mathcal{L}(Y) \to \mathcal{L}(X)$$

b) Explicitly compute the matrix for $\mathrm{Res}_{Y \to X}$ in the canonical bases when $Y = \mathbb{R}$ and $X = \{-1, 1\}$.

c) Explicitly compute the matrix for $\mathrm{Res}_{Y \to X}$ in the canonical bases when $Y = \mathbb{R}^2$ minus the y-axis and $X = \{(-1,0), (1,0), (2,0)\}$.

d) * More generally, show that if X^i and Y^j are the connected components of X and Y respectively, for $1 \leq i \leq k$ and $1 \leq j \leq l$, then in the canonical basis $\mathrm{Res}_{Y \to X}$ is a $k \times l$ matrix whose entry in the i^{th} row and j^{th} column is given by

$$[\mathrm{Res}_{Y \to X}]_{i,j} = \begin{cases} 1 & \text{if } X^i \subseteq Y^j \\ 0 & \text{if } X^i \not\subseteq Y^j \end{cases}$$

2.2 Cochain Complexes

The various vector spaces $\mathcal{L}(U_I)$ naturally assemble themselves into a cochain complex, we'll get to that in the next section. First we need to understand what cochain complexes are.

Definition 2.3. A **cochain complex** $\mathcal{V}^\bullet$ consists of the following data:

$$0 \longrightarrow \mathcal{V}^0 \longrightarrow \cdots \longrightarrow \mathcal{V}^{i-1} \xrightarrow{d^{i-1}} \mathcal{V}^i \xrightarrow{d^i} \mathcal{V}^{i+1} \longrightarrow \cdots \longrightarrow \mathcal{V}^n \longrightarrow 0$$

a) A vector space $\mathcal{V}^i$ for each $i \in \mathbb{Z}$, with $\mathcal{V}^i \neq 0$ only if $0 \leq i \leq n$ for some positive integer n.

b) For each $i \in \mathbb{Z}$ a linear transformation $d^i : \mathcal{V}^i \to \mathcal{V}^{i+1}$ that satisfies

$$d^i \circ d^{i-1} = 0 \qquad \begin{tikzcd} \mathcal{V}^{i-1} \arrow[r, "d^{i-1}"] \arrow[rr, bend right, "d^i \circ d^{i-1} = 0"'] & \mathcal{V}^i \arrow[r, "d^i"] & \mathcal{V}^{i+1} \end{tikzcd}$$

> **Question. 16.** Show that $\operatorname{im} d^{i-1} \subseteq \ker d^i$.

Definition 2.4. The i^{th} **cohomology** of $\mathcal{V}^\bullet$ is the quotient vector space

$$H^i(\mathcal{V}) := \ker d^i \,/\, \operatorname{im} d^{i-1}$$

Remark 2.5. This is well-defined because of the previous exercise. We'll only be interested in the dimensions

$$\dim H^i(\mathcal{V}) = \dim \ker d^i - \dim \operatorname{im} d^{i-1}$$

Convention: Even though in a cochain complex there is a vector space $\mathcal{V}^i$ for all integers i it is a common convention to explicitly define $\mathcal{V}^i$ only where it is non-zero, it is understood that the rest of the $\mathcal{V}^i$ are all 0. The first non-zero vector space in the cochain complex is understood to be $\mathcal{V}^0$ unless otherwise specified.

> **Question. 17.** A vector space A can be thought of as a cochain complex as
>
> $$\mathcal{V}^\bullet = 0 \longrightarrow A \longrightarrow 0$$
>
> What are the cohomologies $H^i(\mathcal{V})$, for $i \in \mathbb{Z}$?

> **Question. 18.** A linear transformation $f : A \to B$ can be thought of as a cochain complex as
>
> $$\mathcal{V}^\bullet = 0 \longrightarrow A \xrightarrow{f} B \longrightarrow 0$$
>
> What are the cohomologies $H^i(\mathcal{V})$, for $i \in \mathbb{Z}$?

For the following exercises it'll be useful to invoke the **Rank-Nullity / Dimension Theorem**.

Question. 19. Verify that the following are cochain complexes and compute their cohomologies.

a) $0 \longrightarrow \mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2 \longrightarrow 0$

b) $0 \longrightarrow \mathbb{F}_2 \xrightarrow{1} \mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2 \longrightarrow 0$

c) $0 \longrightarrow \mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2 \xrightarrow{1} \mathbb{F}_2 \longrightarrow 0$

d) $0 \longrightarrow \mathbb{F}_2 \xrightarrow{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} \mathbb{F}_2^2 \xrightarrow{\begin{bmatrix} 1 & 1 \end{bmatrix}} \mathbb{F}_2 \longrightarrow 0$

Question. 20. Compute the cohomologies of the following cochain complexes.

a) $0 \longrightarrow \mathbb{F}_2^2 \xrightarrow{\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}} \mathbb{F}_2^3 \longrightarrow 0$ (this is computing $\check{H}^*(S^1 \vee S^1)$).

b) $0 \longrightarrow \mathbb{F}_2^3 \xrightarrow{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}} \mathbb{F}_2^3 \xrightarrow{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}} \mathbb{F}_2^2 \longrightarrow 0$ (this is computing $\check{H}^*(S^2)$).

Question. 21. Given $\mathcal{V}^\bullet = 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$

a) Under what conditions on f, g is $\mathcal{V}^\bullet$ a cochain complex.

b) Under what conditions on f, g is $H^i(\mathcal{V}) = 0$ for all i. In this case, we say that $\mathcal{V}^\bullet$ is a **short exact sequence**.

Definition 2.6. More generally, a cochain complex $\mathcal{V}^\bullet$ is said to be **exact** (or **long exact**) if $H^i(\mathcal{V}) = 0$ for all i.

Optional Problems

Definition 2.7. Define the **Euler characteristic** of a cochain complex to be

$$\begin{aligned}\chi(\mathcal{V}) &= \sum_{i\in\mathbb{Z}} (-1)^i \dim H^i(\mathcal{V}) \\ &= \dim H^0(\mathcal{V}) - \dim H^1(\mathcal{V}) + \dim H^2(\mathcal{V}) \pm \cdots + (-1)^n \dim H^n(\mathcal{V})\end{aligned}$$

Question. 22.

a) Express the Euler characteristic in terms of the $\dim\ker(d^i)$ and $\dim\operatorname{im}(d^i)$ for $i \in \mathbb{Z}$.

b) Show that

$$\chi(V) = \sum_{i\in\mathbb{Z}} (-1)^i \dim \mathcal{V}^i$$

Thus the Euler characteristic can be computed using the dimensions of the vector spaces of the original cochain complex, but it is really an invariant of the underlying cohomology!!!

Question. 23. * The direct sum of cochain complexes $(\mathcal{V}_1 \oplus \mathcal{V}_2)^\bullet$ is defined as

$$(\mathcal{V}_1 \oplus \mathcal{V}_2)^i := \mathcal{V}_1^i \oplus \mathcal{V}_2^i$$

and the differential is defined as

$$d^i := d_1^i \oplus d_2^i$$

Find $H^*(\mathcal{V}_1 \oplus \mathcal{V}_2)$ in terms of $H^*(\mathcal{V}_1)$ and $H^*(\mathcal{V}_2)$.

Question. 24. *** The tensor product $(\mathcal{V}_1 \otimes \mathcal{V}_2)^\bullet$ of two cochain complexes $\mathcal{V}_1^\bullet, \mathcal{V}_2^\bullet$ is defined as

$$(\mathcal{V}_1 \otimes \mathcal{V}_2)^k := \bigoplus_{i+j=k} \mathcal{V}_1^i \otimes \mathcal{V}_2^j$$

and the differential is defined as

$$d^k := \bigoplus_{i+j=k} d_1^i \otimes d_2^j$$

Find $H^*(\mathcal{V}_1 \otimes \mathcal{V}_2)$ in terms of $H^*(\mathcal{V}_1)$ and $H^*(\mathcal{V}_2)$.

Question. 25. *** A **morphism of cochain complexes** $\phi : \mathcal{V}_1^\bullet \to \mathcal{V}_2^\bullet$ is a collection of maps $\phi : \mathcal{V}_1^i \to \mathcal{V}_2^i$ for each $i \in \mathbb{Z}$ such that the following diagram commutes

$$\begin{tikzcd} \mathcal{V}_2^i \arrow[r, "d"] & \mathcal{V}_2^{i+1} \\ \mathcal{V}_1^i \arrow[r, "d"] \arrow[u, "\phi"'] & \mathcal{V}_1^{i+1} \arrow[u, "\phi"] \end{tikzcd}$$

a) Show that a morphism $\phi : \mathcal{V}_1^\bullet \to \mathcal{V}_2^\bullet$ between cochain complexes naturally induces a map between cohomologies $\phi^* : H^i(\mathcal{V}_1) \to H^i(\mathcal{V}_2)$, for all $i \in Z$.

b) Given two morphisms $\phi_1 : \mathcal{V}_1^\bullet \to \mathcal{V}_2^\bullet$ and $\phi_2 : \mathcal{V}_2^\bullet \to \mathcal{V}_3^\bullet$, show that their composition $\phi_2 \circ \phi_1 : \mathcal{V}_1^\bullet \to \mathcal{V}_3^\bullet$ is also a morphism of cochain complexes. Further show that $(\phi_2 \circ \phi_1)^* = \phi_2^* \circ \phi_1^*$.

Question. 26. ** A cochain complex of cochain complexes (i.e. each $\mathcal{V}^i$ is itself a cochain complex and the differentials d^i are morphisms of cochain complexes) is called a **double complex**. Unravel this description of a double complex and describe it more explicitly as a grid of vector spaces.

3 Gluing it Back Together

> The only way to learn math is to do math.
>
> Paul Halmos

$$\begin{array}{ccccc} \mathcal{U} & \rightsquigarrow & \mathcal{L}^\bullet & \rightsquigarrow & \check{H}^*(X) \\ \text{Good cover of } X & & \text{Cech Complex of } \mathcal{U} & & \text{Cech Cohomology of } X \end{array}$$

We now know enough theory to define the Cech Cohomology of a space. We'll start by defining it in an ad hoc manner and later try and understand it better.

Let $\mathcal{U}$ be a good cover of X consisting of n sets. In order to define the Cech Complex $\mathcal{L}^\bullet$ we need two things:

$$\begin{array}{ll} \text{vector spaces} & \mathcal{L}^k \\ \text{linear maps} & d^k : \mathcal{L}^k \to \mathcal{L}^{k+1} \end{array}$$

Definition 3.1. The vector space $\mathcal{L}^k$ has dimension

$$\dim \mathcal{L}^k = \sum_{|I|=k+1} \text{number of connected components of } U_I$$ [*]

where I varies over the non-empty subsets of $[n]$.

Definition 3.2. The linear map d^k is a matrix whose

rows	correspond to the connected components of all the U_I with $\lvert I\rvert = k+2$
columns	correspond to the connected components of all the U_J with $\lvert J\rvert = k+1$
$(i,j)^{th}$ entry	equals 1 if and only if the i^{th} connected component in U_I is a subset of the j^{th} connected component in U_J.

Definition 3.3. Define the **Cech Cohomology** $\check{H}^k(X)$ of X to be the cohomology of the above cochain complex $\check{H}^k(\mathcal{L})$.

Example 3.4. For the triangle with cover being the three sides $\mathcal{U} = \{U_1, U_2, U_3\}$

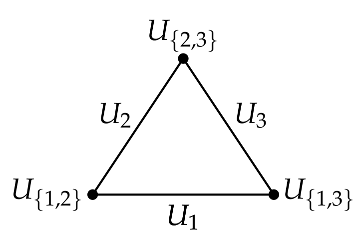


Figure 3: $\mathcal{U} = \{U_1, U_2, U_3\}$ is a good cover of the triangle, $U_{\{1,2\}}, U_{\{2,3\}}, U_{\{1,3\}}$ are the vertices, and $U_{\{1,2,3\}}$ is empty.

$$
\begin{array}{lll}
|I| = 0 & U_1, U_2, U_3 & \mathcal{L}^0 = \mathbb{F}_2^3 \\
|I| = 1 & U_{\{1,2\}}, U_{\{2,3\}}, U_{\{1,3\}} & \mathcal{L}^1 = \mathbb{F}_2^3
\end{array}
$$

We have inclusions $U_{\{1,2\}} \subseteq U_1$, $U_{\{1,2\}} \subseteq U_2$ etc. Hence the differential d^0 looks like

$$
\begin{array}{c|ccc}
 & U_1 & U_2 & U_3 \\
\hline
U_{\{1,2\}} & 1 & 1 & 0 \\
U_{\{2,3\}} & 0 & 1 & 1 \\
U_{\{1,3\}} & 1 & 0 & 1
\end{array} = d^0
$$

So that

$$
\mathcal{L}^\bullet = 0 \to \mathbb{F}_2^3 \xrightarrow{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}} \mathbb{F}_2^3 \to 0
$$

Example 3.5. For the circle S^1 we can find a good cover consisting of semicircles $\mathcal{U} = \{U_1, U_2\}$.

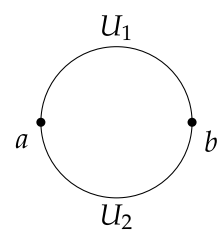


Figure 4: $\{U_1, U_2\}$ is a good cover of the circle with $U_{\{1,2\}} = \{a, b\}$

$$
\begin{array}{lll}
|I| = 0 & U_1, U_2 & \mathcal{L}^0 = \mathbb{F}_2^2 \\
|I| = 1 & U_{\{1,2\}} = \{a, b\} & \mathcal{L}^1 = \mathbb{F}_2^2
\end{array}
$$

We have inclusions $\{a\} \subseteq U_1$, $\{a\} \subseteq U_2$ etc. Hence the differential d^0 looks like

$$
\begin{array}{c|cc}
 & U_1 & U_2 \\
\hline
a & 1 & 1 \\
b & 1 & 1
\end{array} = d^0
$$

So that

$$
\mathcal{L}^\bullet = 0 \to \mathbb{F}_2^2 \xrightarrow{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}} \mathbb{F}_2^2 \to 0
$$

Question. 27. Compute the dimensions of the cohomologies in the above examples.

Compute the dimensions of the cohomologies for the following spaces. Interpret your results topologically.

Question. 28.

a) $\mathbb{R}^n$

b) A tree

c) d points in $\mathbb{R}^2$

d) The bipartite graph $K_{2,3}$

e) $\mathbb{R}^2 \setminus \{(0,0)\}$

f) $\mathbb{R}^2 \setminus \{(0,0),(1,0),\ldots,(d,0)\}$ for some positive integer d

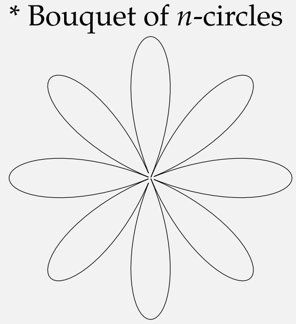
g) S^2 minus a point

h) S^2 minus 2 points

Question. 29.

a) $S^1 \vee S^1$

b) * Bouquet of n-circles



Question. 30.

a) * S^2

b) * $S^1 \vee S^2$

c) ** $S^1 \times S^1$ (= torus)

d) *** g-holed torus

e) Solid torus

The *basis independent* way of defining the Cech Cohomology is as follows. Let $(U_{I_1}, U_{I_2}, \ldots, U_{I_m})$ denote the non-empty sets U_I with $|I| = k+1$. Then L^k is the direct sum

$$\mathcal{L}^k = \mathcal{L}(U_{I_1}) \oplus \mathcal{L}(U_{I_2}) \oplus \cdots \oplus \mathcal{L}(U_{I_m})$$

The differential maps d^k are direct sums of the restriction maps $\mathrm{Res}_{U_I \to U_J}$ for $|I| = k+1, |J| = k+2$.

Question. 31. * Prove this.

Question. 32. *** Following is the proof of the fact that the Cech complex is indeed a cochain complex i.e. it satisfies $d^{k+1} \circ d^k = 0$:

Consider subsets $J \subset I \subseteq [n]$ such that $k+1 = |J| = |I| - 2$ and $I = J \cup \{\alpha, \beta\}$. Let $J_\alpha := J \cup \{\alpha\}$ and $J_\beta := J \cup \{\beta\}$. Show that we have the following induced maps:

$$J \subset J_\alpha \subset I$$
$$J \subset J_\beta \subset I$$

$$U_I \subset U_{J_\alpha} \subset U_J$$
$$U_I \subset U_{J_\beta} \subset U_J$$

$$\phi_\alpha : \mathcal{L}(U_J) \xrightarrow{\mathrm{Res}_{U_J \to U_{J_\alpha}}} \mathcal{L}(U_{J_\alpha}) \xrightarrow{\mathrm{Res}_{U_{J_\alpha} \to U_I}} \mathcal{L}(U_I)$$
$$\phi_\beta : \mathcal{L}(U_J) \xrightarrow{\mathrm{Res}_{U_J \to U_{J_\beta}}} \mathcal{L}(U_{J_\beta}) \xrightarrow{\mathrm{Res}_{U_{J_\beta} \to U_I}} \mathcal{L}(U_I)$$

Show that $d^{k+1} \circ d^k|_{\mathcal{L}(U_J)} = \phi_\alpha + \phi_\beta$. Argue that this implies $d^{k+1} \circ d^k|_{\mathcal{L}(U_J)} = 0$ and hence $d^{k+1} \circ d^k = 0$.

Finally, notice that we're saying that for finding the Cech cohomology of X we needed to choose a good open cover $\mathcal{U}$. It is a deep theorem that this choice does not matter. The only proof I know of this uses heavy homological algebra machinery.

Theorem 3.6. *If $\mathcal{U}, \mathcal{U}'$ are **good covers** of X with associated Cech complexes $\mathcal{L}^\bullet(\mathcal{U})$ and $\mathcal{L}^\bullet(\mathcal{U}')$ then*

$$H^i(\mathcal{L}(\mathcal{U})) \cong H^i(\mathcal{L}(\mathcal{U}')) \tag{3.1}$$

for all $i \in \mathbb{Z}$ i.e. the cohomology of the Cech complex is independent of the good cover and hence is an invariant of the underlying space. Hence, we can define the Cech cohomology of X as

$$\check{H}^i(X; \mathbb{F}_2) := H^i(\mathcal{L}(\mathcal{U})) \tag{3.2}$$

for any good cover $\mathcal{U}$ of X.

4 The Topology behind the Algebra

> It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out.
>
> E. Artin

Most problems in this section are (*) or harder and hence optional.

Question. 33. Show that for a space X if some good cover has dimension n then $\check{H}^i(X) = 0$ for $i > n$.

Definition 4.1. The **Euler characteristic** of a space X is defined as

$$\chi(M) := \sum_{i \in \mathbb{Z}} (-1)^i \dim \check{H}^i(X) = \dim \check{H}^0(X) - \dim \check{H}^1(X) + \dim \check{H}^0(X) \pm \dots$$

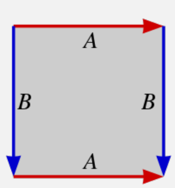
Question. 34. Let M be a g holed torus. Triangulate M and suppose there are V number of vertices, E number of edges, and F number of faces in the triangulation. Find a good cover of M using this triangulation and prove that

$$\chi(M) = V - E + F$$

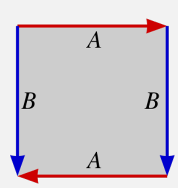
(Recall that we don't need to find the maps d^i to find the Euler characteristic.)

Question. 35. Let G be a connected graph. Let T be a maximal tree in G i.e. T is a subgraph of G which is a tree such that adding any edge of G to it creates a cycle. Find a cover $\mathcal{U}$ of G containing T as an element (i.e. $T \in \mathcal{U}$). Use this cover to find a topological interpretation of $\dim \check{H}^1(G)$.

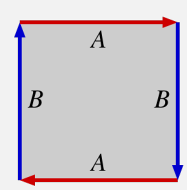
Question. 36. By gluing the sides of a square in funky ways we can create the Klein Bottle and the Real Projective Plane. Find their Cech cohomologies. (We can find open covers for torus, Klein bottle, and projective plane containing exactly 3 sets.)



(a) Torus



(b) Klein Bottle



(c) Projective Plane

We'll now find a topological interpretation for $\check{H}^0$. Let X be a connected space with a good cover $\mathcal{U} = \{U_1, U_2, \dots, U_n\}$. Assume that all the non-empty U_i and $U_{\{i,j\}}$ are connected (this is simplify to our arguments, the proof works without this simplification). We have

$$\mathcal{L}^0 = \mathcal{L}(U_1) \oplus \mathcal{L}(U_2) \oplus \cdots \oplus \mathcal{L}(U_n)$$
$$\mathcal{L}^1 = \mathcal{L}(U_{\{1,2\}}) \oplus \mathcal{L}(U_{\{1,3\}}) \oplus \cdots \oplus \mathcal{L}(U_{\{n-1,n\}})$$

and $\check{H}^0(X)$ is the kernel of the restriction maps $d^0 : \mathcal{L}^0 \to \mathcal{L}^1$.

$$\begin{array}{c|cc} & U_l & \cdots \\ \hline U_{\{i,j\}} & 0 \text{ or } 1 & \\ \vdots & & \end{array} = d^0$$

Question. 37.

a) As $U_{\{i,j\}} = U_i \cap U_j$, argue that every row of the matrix d^0 (in the canonical bases) has exactly 2 non-zero entries.

b) Argue that because X is connected every open set in the cover U_i must intersect some other U_j. What does this imply for the matrix d^0?

c) Show that

$$\ker d^0 = \{[0, 0, \dots, 0]^T, [1, 1, \dots, 1]^T\}$$

and hence $\dim \check{H}^0(X) = 1$.

d) Suppose Y is another topological space. Find the Cech cohomologies of $X \sqcup Y$ (the disjoint union of X and Y) in terms of X and Y. In particular, what is $\check{H}^0(X \sqcup Y)$?

e) Find a topological interpretation of $\dim \check{H}^0(Z)$ for a general topological space Z (not necessarily connected).

Question. 38. Find the Cech cohomologies of $X \vee Y$ in terms of X and Y. (Assume we have appropriate good covers.)

We'll next prove that $\mathbb{R}^n \not\cong \mathbb{R}^m$ (non-homeomorphic) if $n \neq m$. Let

$$S^n = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : x_0^2 + x_1^2 + \cdots + x_n^2 = 1\}$$

Question. 39.

a) Let X, Y be topological spaces such that Y is contractible. If $\mathcal{U}$ is a good cover for X, find a good cover for $X \times Y$. Show that $\check{H}^i(X) \cong \check{H}^i(X \times Y)$ for all $i \in \mathbb{Z}$. In particular, this is true when $Y = (0, 1)$.

b) Show that $\mathbb{R}^n \setminus \{0\} \cong S^{n-1} \times (0, 1)$.

c) Let X, Y be topological spaces. Show that if $X \cong Y$ then $\check{H}^i(X) \cong \check{H}^i(Y)$.

d) Conclude that if $\mathbb{R}^n \cong \mathbb{R}^m$ then $\check{H}^i(S^{n-1}) \cong \check{H}^i(S^{m-1})$ for all $i \in \mathbb{Z}$.

Thus we've reduced the problem to a cohomology computation, which we can do by finding an appropriate good cover of S^n. Let

$$\begin{aligned} S^n_+ &= \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : x_n \geq 0, x_0^2 + x_1^2 + \cdots + x_n^2 = 1\} \\ S^n_- &= \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : x_n \leq 0, x_0^2 + x_1^2 + \cdots + x_n^2 = 1\} \end{aligned}$$

Question. 40.

a) What is S^0?

b) What are the dimensions of the Cech cohomologies of S^0, S^1, and S^2? Based on these make a guess as to what the dimensions of the Cech cohomologies of S^n are.

We'll find the cohomologies of S^n using induction. For a non-empty subset $X \subseteq S^{n-1}$ define the (positive) cone over X, denoted C_+X, to be a subspace of S^n_+ defined as

$$C_+X = \{(tx_0, tx_1, \ldots, tx_{n-1}, x_n) \in S^n : t \in \mathbb{R}_{\geq 0}, (x_0, x_1, \ldots, x_{n-1}) \in X\}$$

Define the cone over the empty set $C_+(\phi)$ to be the single point $(0, \ldots, 0, 1)$.

Question. 41. Let $X, Y \subseteq S^{n-1}$.

a) Let X be some subset of S^1. Draw the cone $C_+X \subseteq S^2$.

b) Prove that C_+X is connected and contractible.

c) Prove that $C_+X \cap C_+Y = C_+(X \cap Y)$ (this is why we need $C_+\phi = (0, \ldots, 0, 1)$).

d) Prove that $C_+X \cap S^{n+1}_- \cong X$.

Question. 42. By induction, suppose S^{n-1} has a good cover $\mathcal{U} = \{U_1, \ldots, U_n\}$ with n elements. (Check the base case.)

a) Prove that $\mathcal{U}' = \{C_+U_1, C_+U_2, \ldots, C_+U_n, S^n_-\}$ is a good cover of S^n (this proves the induction step).

b) What is the dimension of this cover? Conclude that $\check{H}^i(S^n) = 0$ for $i > n$.

There are two types of intersections $\mathcal{U}'_I$:
(type I) ones obtained by intersecting the sets $\{C_+U_1, C_+U_2, \ldots, C_+U_n\}$ and
(type II) ones obtained by intersecting S^n_- with the some of the sets from $\{C_+U_1, C_+U_2, \ldots, C_+U_n\}$.

Question. 43.

a) Show that the intersections of type I are all contractible and the intersections of type II are the ones that show up in the Cech complex for S^{n-1} and hence the Cech complex for S^n contains a copy of the Cech complex of S^{n-1} *right* shifted by one.

b) Use this to inductively prove that for $n \geq 1$

$$\check{H}^i(S^n) \cong \begin{cases} \mathbb{Z} & \text{for } i = 0, n \\ 0 & \text{otherwise} \end{cases}$$

c) Conclude that $S^n \ncong S^m$ and $\mathbb{R}^n \ncong \mathbb{R}^m$ if $n \neq m$.

5 Sheaves

Look on my works, ye Mighty, and despair!

Percy Shelley

The assignment $U \mapsto \mathcal{L}(U)$ is an example of a sheaf on X. It is perhaps the simplest non-trivial sheaf. Let's see the definition of a sheaf.

Notation: We'll let $\mathcal{C}$ denote the collection of one of the following: sets, groups, abelian groups, topological spaces, modules over a ring R. A **morphism** between two objects in $\mathcal{C}$ will denote a map which preserves the appropriate structure: set maps, group homomorphisms, abelian group homomorphisms, continuous maps, R-module morphisms respectively.*

Definition 5.1. On a topological spaces X, a **presheaf** $\mathcal{P}$ valued in $\mathcal{C}$, consists of the following data:

a) For each open† subset $U \subseteq X$ an assignment of a vector space

$$U \mapsto \mathcal{P}(U)$$

b) For each inclusion of open subsets $U \subseteq V$ a morphism

$$\mathrm{Res}_{V \to U} : \mathcal{P}(V) \to \mathcal{P}(U)$$

satisfying the following conditions

- For each open subset U, $\mathrm{Res}_{U \to U}$ it the identity map.
- For each open subset $U \subseteq V \subseteq W$

$$\mathrm{Res}_{V \to U} \circ \mathrm{Res}_{W \to V} = \mathrm{Res}_{W \to U} \qquad \begin{tikzcd} \mathcal{P}(W) \arrow[r, "\mathrm{Res}_{W \to V}"] \arrow[rr, bend right, "\mathrm{Res}_{W \to U}"'] & \mathcal{P}(V) \arrow[r, "\mathrm{Res}_{V \to U}"] & \mathcal{P}(U) \end{tikzcd}$$

> **Question. 44.** Define natural maps Res on $\mathcal{L}$ which turn it into a presheaf valued in vector spaces.

Definition 5.2. A **sheaf** on X is a presheaf $\mathcal{P}$ that further satisfies the following conditions. Let $\mathcal{U} = \{U_i\}_{i \in I}$ is a cover of $V \subseteq X$.

a) *(Identity axiom)* If a section $s \in \mathcal{P}(V)$ is such that $\mathrm{Res}_{U \to U_i}\, s = 0$ for all U_i then $s = 0$.

*This data is what makes $\mathcal{C}$ a **category**.

†If you do not know what an *open* set is you can neglect the adjective *open* for now, it's won't be the end of the world.

b) *(Gluing axiom)* If there exist a collection of sections $s_i \in \mathcal{P}(U_i)$ such that for all $i, j \in I$ the intersections are compatible

$$\mathrm{Res}_{U_i \to U_i \cap U_j} s_i = \mathrm{Res}_{U_j \to U_i \cap U_j} s_j$$

then there exists a section $s \in \mathcal{P}(V)$ such that

$$s_i = \mathrm{Res}_{V \to U_i} s$$

The identity axiom is a uniqueness axiom. It ensures that there if two sections look the same when restricted to small sets, then the two sets should have been the same to begin with.

The gluing axiom is a constructive axiom. It is saying that you can glue sections on small open subsets to get a section on a large open subset as long as the sections on the smaller sets agree on intersections.

Question. 45. Verify that $\mathcal{L}$ is a sheaf.

Question. 46. Show that for a topological space X and a set Y the assignment

$$U \to \{\text{set maps from } U \to Y\}$$

with the Res maps being restrictions of functions, is a sheaf on X valued in sets.

Definition 5.3. Given a map $\phi : X \to Y$ a **section** of ϕ on a subset $U \subseteq Y$ is a map $s : U \to X$ satisfying $\phi \circ s(u) = u$ for all $u \in U$.

$$\begin{tikzcd} & X \arrow[d, "\phi"] \\ U \arrow[ur, "s"] \arrow[r, hook] & Y \end{tikzcd}$$

Question. 47. *a*) For a map $\phi : X \to Y$ define

$$\mathcal{P}^{\phi}(U) := \text{ The set of sections of } \phi \text{ over } U.$$

Define the Res maps as restrictions of functions. Show that this defines a sheaf on Y valued in sets.

b) Show that the sheaf defined in Question 46 is a special case of this.

c) Show that the sheaf $\mathcal{L}$ is a special case of this if we only allow our sections to be continuous functions.

Remark 5.4. Every sheaf is morally of the above type with the generalization that we can require the map ϕ and the sections s to have certain properties like being continuous, smooth, holomorphic, meromorphic etc. Because of this, for an arbitrary sheaf the set $\mathcal{P}(U)$ is called the sections of $\mathcal{P}$ over U.