

CONNECTIONS ON PRINCIPAL BUNDLES AND CLASSICAL ELECTROMAGNETISM

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ABSTRACT. The goal is to understand how Maxwell's equations can be formulated using the language of connections. There will be a long built up just to define the various notions, the actual mathematics involved is really straightforward. In the end what we get is not new mathematics but a new perspective which is useful for defining field theories in general.

There will be issues about integrating over non-compact manifolds, I do not know their rigorous treatment. For now we can assume that all action integrals are formal.

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1. CONNECTIONS

1.1. Principal G -bundles. We start by defining connections on principal G -bundles. We will assume for convenience that G is a compact linear group, though the statements are true for arbitrary compact groups. Assuming the group to be linear allows us to write the Killing form $\langle a, b \rangle$ as $\text{trace}(ab)$.

Definition 1.1 (Principal G -bundle). A principal G -bundle is a fiber bundle $\pi : E \rightarrow B$ such that G has a right action on E such that

- (1) For each $g \in G$ we have

$$\begin{array}{ccc} E & \xrightarrow{g} & E \\ \pi \downarrow & \swarrow \pi & \\ B & & \end{array}$$

- (2) the action is free and transitive on each fiber.

Note that this forces each fiber to be G itself. We will only deal with the cases when E and B are themselves manifolds.

For each point $p \in E$ let $\phi_p : G \rightarrow E$ be the map which sends $g \in G$ to $p.g$ and let $\psi_p : \pi^{-1}\pi(p) \rightarrow G$ be the left inverse of ϕ_p . Because of these maps, every element λ of the lie algebra \mathfrak{g} gives rise to a vector field λ° on E which at point p is given by $\phi_{p*}\lambda$.

Definition 1.2 (Connection on a principal G -bundle). A connection ∇ on E is a \mathfrak{g} valued 1-form on E which satisfies

- (1) $g^*\nabla = Ad_{g^{-1}}(\nabla)$ where Ad is the adjoint action of G on \mathfrak{g} which is simply conjugation by g^{-1} for linear groups.
- (2) $\nabla(\lambda^\circ) = \lambda$.

There is a g^{-1} in the definition because one action is left and the other is right.

Remark 1.3. There is an equivalent definition which involves prescribing horizontal and vertical splitting of the tangent space of E , we will not pursue this here. In our case the horizontal sub-bundle $T_{hor}E$ is the kernel of ∇ and the vertical sub-bundle $T_{ver}E$ is the kernel of π .

Let us see what a connection looks like in local coordinates. WALOG assume that $B = \mathbb{R}^n$ and $E = B \times G$. Let the coordinates for B be (x^1, x^2, \dots, x^n) and let an orthonormal basis for \mathfrak{g} be $(\alpha^1, \alpha^2, \dots, \alpha^k)$. Let $\partial\alpha^i$ denote the left (L_{g*}) invariant unit vector field corresponding to α_i and let the corresponding dual (L_g^*) invariant one form be $d\alpha_i$.

If $\lambda = \lambda_i \alpha^i$ then λ° is just the left invariant vector field corresponding to λ and hence $\lambda^\circ = \lambda_i \partial\alpha^i$.

If $\nabla = a^i dx_i + b^j d\alpha_j$, where a^i, b^j are \mathfrak{g} valued function, the two conditions for a connection then become,

- (1) $\nabla(\lambda^\circ) = (a^i dx_i + b^j d\alpha_j)(\lambda_k \partial\alpha^k) = b^j \lambda_j$ so the condition $\nabla(\lambda^\circ) = \lambda$ becomes $b^j = \alpha^j$ this determines the coefficients b^j completely.
- (2) the action of g^* does not affect the horizontal coordinates x_i , hence we must have $a_i(g) = g^*a_i(e) = Ad_{g^{-1}}a_i(e)$.

And so we see that the connection ∇ is completely determined by the coordinates $a_1(e), a_2(e), \dots, a_n(e)$ for each fiber, at least for the case of the trivial bundle.

1.2. Associated vector bundle. Given a connection we can define horizontal forms for an associated vector bundle coming from a representation of G .

Let $\pi : E \rightarrow B$ be a vector bundle with fiber V and structure group G (assume G is a subgroup of $Gl(V)$). Pick an arbitrary trivializing chart on B , the transition functions would all lie in G . Construct a principal G -bundle over B with the same

transition functions call this E_G . Note that E_G is unique upto isomorphism, this can be shown either using the classifying space of G or using Čech cohomology.

Definition 1.4 (horizontal G -forms). $\Omega_{G,hor}^k(E_G; \mathfrak{g})$ consists of elements η which satisfy $\eta(v_1, v_2, \dots, v_k) = 0$ if some $v_i = \lambda^\circ$ for $\lambda \in \mathfrak{g}$.

Assume now that the G -bundle E_G is endowed with a connection ∇ . As noted above this allows us to break the bundle E_G into $T_{hor}E_G$ and $T_{ver}E_G$. This gives us a horizontal projection map from $\Omega_G^k(E_G; \mathfrak{g})$ to $\Omega_{G,hor}^k(E_G; \mathfrak{g})$.

Proposition 1.5.

$$\Omega_{G,hor}^k(E_G) \cong \Omega^k(B; E_G \times_G \mathfrak{g})$$

where the action on \mathfrak{g} is the left adjoint action.

Proof. Define the map $\phi : \Omega_{G,hor}^k(E_G) \cong \Omega^k(B; E_G \times_G \mathfrak{g})$ as $\phi_e(\omega)_b(v_1, \dots, v_k) = [e : \omega(w_1, \dots, w_k)]$ for an arbitrary point e in the fiber over b and w_i is the horizontal lift of v_i at the point e .

And the inverse map is defined as $\psi : \Omega^k(B; E_G \times_G \mathfrak{g}) \rightarrow \Omega_{G,hor}^k(E_G)$ as $\psi(\omega)_e(v_1, \dots, v_k)$ is the unique element that satisfies

$$[e, \psi(\omega)_e(v_1, \dots, v_k)] = \omega_{\pi(b)}(\pi_* v_1, \dots, \pi_* v_k)$$

□

If G is abelian then the above proposition says that $\Omega_{G,hor}^k(E_G) \cong \Omega^k(B; \mathfrak{g})$.

1.3. Classical Connection. It is also interesting to see how this definition reconciles with the usual definition of a connection over a vector bundle.

Recall that for a vector bundle (π, E, B) a connection is a map

$$\tilde{\nabla} : \Gamma^0(B; TB) \rightarrow \text{Der}(\Gamma^0(B; E))$$

which is $C^\infty(B)$ linear.

Let us work locally, suppose E is a trivial vector bundle over $B = \mathbb{R}^k$ with fiber \mathbb{R}^n with coordinates y^1, \dots, y^n . Let the coordinates on B be x^1, \dots, x^k . Assume that G is a closed subgroup of $GL(\mathbb{R}^n)$ and hence \mathfrak{g} will be a subgroup of $\mathfrak{gl}(\mathbb{R}^n)$.

Assume that we are given a connection ∇ on the principal bundle E_G . We wish to construct from this a connection $\tilde{\nabla}$ for the vector bundle E .

We noted earlier that a connection is completely determined by its values at a section. For $b \in B$ let $\Gamma_i = \nabla_{(b,e)}(\partial x_i) \in \mathfrak{g}$ and denote its (p, q) individual entries by $\Gamma_{p;i}^q$.

These then are the Christoffel symbols for $\tilde{\nabla}$ that is define the connection $\tilde{\nabla}$ by setting $\tilde{\nabla}_{\partial x_i} y_p = \Gamma_{p;i}^q y_q$. It then is a matter of verifying that this is independent of the choice of coordinates.

2. CURVATURE

We start by defining exterior products on Lie algebra valued forms.

Remark 2.1. From here on we always have a vector bundle E with the structure group G . All the relevant forms would be forms on the associated principal bundle E_G taking values in \mathfrak{g} , there is no loss of information here. It is possible to include the bundle E more explicitly by making our forms take values in V but this adds a level of unnecessary complications.

Definition 2.2. For $a \in \Omega^p(X; \mathfrak{g})$ and $b \in \Omega^q(X; \mathfrak{g})$ then define $[a, b](v_1, v_2, \dots, v_{p+q})$ as

$$\sum_{\sigma \text{ is a } (p, q)\text{-shuffle}} (-1)^\sigma [a(v_{\sigma_1(1)}, \dots, v_{\sigma_1(p)}), b(v_{\sigma(p+1)}, \dots, v_{\sigma(p+q)})]$$

One needs to be careful with the commutativity relations. In local coordinate the exterior derivative has a much more tangible form, if $a = a^I dx_I$ and $b = b^J dx_J$ where a^I, b^J are \mathfrak{g} valued functions, then $[a, b] = [a^I, b^J] dx_I dx_J$. From these we can derive the relations:

Proposition 2.3. If $a = a^I dx_I, b = b^J dx_J, c = c^K dx_K$ are three \mathfrak{g} valued forms, then

- (1) $[a, b] = (-1)^{1+|a||b|} [b, a]$
- (2) $(-1)^{|a||c|} [[a, b], c] + (-1)^{|b||a|} [[b, c], a] + (-1)^{|c||b|} [[c, a], b] = 0$
- (3) $d[a, b] = [da, b] + (-1)^{|a|} [a, db]$

and hence $\Omega^*(X; \mathfrak{g})$ form as differential graded Lie algebra. The same holds for $\Omega_G^*(X; \mathfrak{g})$.

Definition 2.4 (covariant derivative). The exterior derivative $d_\nabla : \Omega_G^k(E_G; \mathfrak{g}) \rightarrow \Omega_{G,hor}^{k+1}(E_G; \mathfrak{g})$ is defined as composition of maps

$$\Omega_G^k(E_G; \mathfrak{g}) \xrightarrow{d} \Omega_G^{k+1}(E_G; \mathfrak{g}) \longrightarrow \Omega_{G,hor}^{k+1}(E_G; \mathfrak{g})$$

where the second map is the projection onto the horizontal component.

Proposition 2.5. $d_\nabla(\eta) = d\eta + [\nabla, \eta]$ for $\eta \in \Omega_{G,hor}^k(E_G; \mathfrak{g})$

Proof. If $v = (v_1, v_2, \dots, v_{k+1})$ are all horizontal vector fields then, $[\nabla, \eta](v) = 0$ because ∇ is vertical. If more than one of the fields is vertical, then because η is horizontal both sides must be zero. So it remains to prove the statement in the case when $v_1 = \lambda^\circ$ is vertical and the rest all are horizontal. In this situation $d_\nabla \eta(v) = 0$. $d\eta(v) = \lambda^\circ(\eta(v_2, \dots, v_{k+1}))$ because lie bracket of a horizontal and a vertical vector field is trivial, and $[\nabla, \eta](v) = [\lambda, \eta(v_2, \dots, v_{k+1})]$. So we are reduced to proving,

$$\lambda^\circ(f) = [f, \lambda] \text{ where } f \text{ is a } G \text{ equivariant function from } G \text{ to } \mathfrak{g}$$

Let x_t be the flow of λ° then

$$\begin{aligned} [f, \lambda]|_e &= \lim_{t \rightarrow 0} \frac{1}{t} (x_{t*}(f_e^\circ) - f_e^\circ) \\ &= \lim_{t \rightarrow 0} \frac{1}{t} (R_{x_{t*}(e)}(f_e^\circ) - f_e^\circ) \\ &= \lim_{t \rightarrow 0} \frac{1}{t} (R_{x_{t*}(e)} L_{x_{t*}(e)}(f_e^\circ) - f_e^\circ) \\ &= \lim_{t \rightarrow 0} \frac{1}{t} (Ad(x_{t*}(e)^{-1})(f_e^\circ) - f_e^\circ) \\ &= \lim_{t \rightarrow 0} \frac{1}{t} (f(x_t(e)) - f(e)) \\ &= \lambda^\circ(f)|_e \end{aligned}$$

□

Remark 2.6. This definition of a horizontal derivative and the above result can be extended to an arbitrary vector bundle, however one needs to define the lie bracket carefully.

Definition 2.7 (curvature). The curvature Ω for a connection ∇ (which by 1.5 can be thought of as a form in $\Omega_G^1(E_G; \mathfrak{g})$) is defined to be

$$\omega = d\nabla + [\nabla, \nabla]/2$$

A connection is called flat if its curvature is 0.

Proposition 2.8.

- (1) The curvature Ω is a horizontal form that is, $\Omega \in \Omega_{G,hor}^2(E_G; \mathfrak{g})$
- (2) For two horizontal vector fields η, τ , $\Omega(\eta, \tau) = -\nabla[\eta, \tau]$. In other words, the curvature measures how much the Lie derivative of horizontal vector fields deviates from being horizontal.
- (3) The horizontal sub-bundle $T_{hor}E = \ker(\nabla)$ is integrable if and only if the connection is flat.
- (4) For a flat connection we can choose charts over the base for E_G (and hence also for E) such that the transition functions are locally constant, that is the structure group can be reduced from G to G with discrete topology.

Proof. For vector fields v, w on E_G we have,

$$\begin{aligned} \Omega(v, w) &= d\nabla(v, w) + [\nabla, \nabla](v, w) \\ &= v(\nabla(w)) - w(\nabla(v)) - \nabla([v, w]) + [\nabla(v), \nabla(w)] \\ &= \begin{cases} -\nabla([v, w]) & \text{if } v \text{ horizontal and } w \text{ horizontal} \\ 0 & \text{if } v = \lambda^o \text{ vertical and } w \text{ horizontal} \\ -\nabla([\lambda^o, \tau^o]) + [\lambda, \tau] & \text{if } v = \lambda^o \text{ vertical and } w = \tau^o \text{ vertical} \end{cases} \end{aligned}$$

Here we repeatedly used the fact that ∇ kills horizontal fields and that the lie bracket of a horizontal field and a vertical field should be 0.

This gives us parts 1 and 2. Part 3 is an application of Frobenius theorem.

For the last part we simply choose the charts which come from foliations which come by integrating the $T_{hor}E_G$ over each fiber. Note that here we require the fiber G to be compact. \square

Proposition 2.9.

- (1) $d_{\nabla}d_{\nabla}(\eta) = [\Omega, \eta]$ which is 0 if and only if $\Omega = 0$
- (2) (Bianchi Identity) $d_{\nabla}(\Omega) = 0$

Proof. These follow from the commutativity relations. \square

2.1. Classical Curvature. Let us try to relate this definition of curvature with the usual definition. For a connection $\tilde{\nabla}$ the curvature R is given by the formula

$$R_{X,Y}s = \tilde{\nabla}_X \tilde{\nabla}_Y s - \tilde{\nabla}_Y \tilde{\nabla}_X s - \tilde{\nabla}_{[X,Y]}s$$

Once again let us work locally with the same conventions as in the previous section.

It follows from a standard calculation that the coordinates for R are given by

$$R_{l;i,j}^m = R[\partial x_i, \partial x_j]y_l(y^m)^* = \frac{\partial}{\partial x_i} \Gamma_{l;j}^m + \Gamma_{l;j}^p \Gamma_{p;j}^m - \frac{\partial}{\partial x_j} \Gamma_{l;i}^m + \Gamma_{l;i}^p \Gamma_{p;j}^m$$

For the connection on principal bundle again look at the section (b, e) then here the curvature is determined by

$$\begin{aligned}\Omega(\partial x_i, \partial x_j) &= d\nabla(\partial x_i, \partial x_j) + [\nabla(\partial x_i), \nabla(\partial x_j)] \\ &= \frac{\partial}{\partial x_i}(\nabla(\partial x_j)) - \frac{\partial}{\partial x_j}(\nabla(\partial x_i)) + [\nabla(\partial x_i), \nabla(\partial x_j)] \\ &= \frac{\partial}{\partial x_i}\Gamma_j - \frac{\partial}{\partial x_j}\Gamma_i - \Gamma_i\Gamma_j + \Gamma_j\Gamma_i\end{aligned}$$

which agrees with the classical definition.

What is interesting is the fact that we add the term $\nabla_{[X, Y]}$ to the definition of R to make it a tensor and the term $[\nabla, \nabla]$ to the definition of Ω to make it a horizontal form and both the corrections agree!

3. HODGE DUAL

Note that the symbol for Hodge dual and the usual vector space dual is very similar and hence care needs to be taken.

Definition 3.1.

- (1) Given an inner product space V, \langle, \rangle . Let X^1, \dots, X^n be an orthonormal basis, the Hodge star is a linear map $*$: $\wedge^k(V) \rightarrow \wedge^{n-k}(V)$ defined as $*(X^1 \wedge \dots \wedge X^k) = X^{k+1} \wedge \dots \wedge X^n$. $*X$ is also called the Hodge dual of X . For differential forms we find the Hodge dual fiberwise.
- (2) The codifferential is a map $d^* : \Omega^k(B) \rightarrow \Omega^{k-1}(B)$ defined as $d^* = (-1)^{nk+k+1} * d *$.
- (3) The Laplacian is a map $\Delta : \Omega^k(B) \rightarrow \Omega^k(B)$ defined as $\Delta = d^*d + dd^*$.

3.1. Extending the metric. Given an a vector space V with a symmetric non-degenerate bilinear form \langle, \rangle , we get an isomorphism $*$: $V \rightarrow V^*$ given by $X \mapsto \langle X, - \rangle$. Now we can push \langle, \rangle from V to V^* using this map, we will abuse some notation and also denote this form by \langle, \rangle so that $\langle \langle X, - \rangle, \langle Y, - \rangle \rangle := \langle X, Y \rangle$.

This is nothing but the raising and lowering of indices using the metric. We can also stretch this metric to $\wedge^k(V)$ by the formula: $\langle X_1 \wedge \dots \wedge X_k \wedge Y_1 \wedge \dots \wedge Y_k \rangle := |\det(\langle (X_i, Y_j) \rangle)|$.

Applying this to the tangent space of a B we get,

Proposition 3.2. *g descends to a point-wise metric on the space $\Omega^k(B)$.*

We can do better and make it a metric on the entire $\Omega^k(B)$, but that we need the L^2 norm and it's completion which would make us leave the domain of smooth functions. I would avoid that for now.

Proposition 3.3. *For the space $\Omega^*(B)$,*

- (1) *The Hodge dual is independent of the choice of basis.*
- (2) *The Hodge dual $*\eta$ is the unique element that satisfies $\omega \wedge *\eta = g(\omega, \eta)$*
- (3) *Locally we have*

$$(*\omega)_{i_1, \dots, i_{n-k}} := \frac{1}{k!} \omega^{j_1, \dots, j_k} \epsilon_{j_1, \dots, j_k, i_1, \dots, i_{n-k}}$$

- (4) $** = (-1)^{k(n-k)}$

Proof. The existence is explicitly given by the formula stated here. And uniqueness follows from the non-degeneracy of the metric. \square

Remark 3.4. The Hodge dual is in fact dual in the actual sense if we put an L^2 norm on the space of forms. But the proof requires more than just linear algebra and I cannot make it rigorous.

Proposition 3.5.

- (1) $g(d\eta, \omega) = g(\eta, d^*\omega)$
- (2) $g(\Delta\eta, \omega) = g(\eta, \Delta\omega)$
- (3) $g(\Delta\eta, \eta) = g(d\eta, d\eta) + g(d^*\eta, d^*\eta)$

Proposition 3.6. *Over \mathbb{R}^3 with the Euclidean metric, if v, w are 1-forms we have,*

- (1) $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$
- (2) $*(v \wedge w) = v \times w$
- (3) $*(dv) = \text{curl}(v)$
- (4) $d^*v = \text{div}(v)$

Remark 3.7. We can extend the Hodge star, codifferential and the Laplacian operator to arbitrary coefficients naturally. We would be interested in the case when the coefficients lie in \mathfrak{g} .

3.2. Hodge Decomposition. Assume now that (B, g) is a closed Riemannian manifold, then we have the following theorem due to Hodge,

Theorem 3.8 (Hodge). *For each cohomology class $[\omega] \in H_{DR}^*(B)$ there a unique harmonic form τ that is $\nabla\tau = 0$ such that $\tau \in [\omega]$.*

Proposition 3.9. *There is a canonical decomposition*

$$\Omega^k(B) = \Omega_{exact}^k(B) \oplus \Omega_{harmonic}^k(B) \oplus \Omega_{coexact}^k(B)$$

where $\Omega_{exact}^k(B) = \{d\omega\}$, $\Omega_{coexact}^k(B) = \{d^*\omega\}$ and $\Omega_{harmonic}^k(B) = \{\omega | d\omega = d^*\omega = 0\}$

Proof. $g(\omega, d^*\tau) = g(d\omega, \tau) = 0$, gives us

$$\Omega_{exact}^k(B) \perp \Omega_{harmonic}^k(B) \perp \Omega_{coexact}^k(B)$$

Next we claim that $\Omega_{exact}^k(B)$ and $\Omega_{coexact}^k(B)$ are Banach spaces. (This will not be true if B is not compact.) It suffices to show this for $\Omega_{exact}^k(B)$. A k form is a function from B to $\wedge^k(T^*B)$ and the metric g induces the topology of uniform convergence, hence $\Omega^k(B)$ is Banach.

Now let ω_i be a sequence of exact forms converging to ω . For any immersed compact k submanifold M we have $\int_M \omega_i = 0$, and hence because of compactness $\int_M \omega = 0$ and hence ω is also exact.

For $\omega \in \Omega^k(B)$ let $d^*\tau$ be the closest point to ω in $\Omega_{coexact}^k(B)$, such a τ exists as we just showed Banach. Hence we must have $g(\omega - d^*\tau, d^*\sigma) = 0$ for all σ which is possible if and only if $d(\omega - d^*\tau) = 0$ and so we obtain

$$\Omega^k(B) = \Omega_{closed}^k(B) \oplus \Omega_{coexact}^k(B)$$

Finally $\Omega_{closed}^k(B) = \Omega_{exact}^k(B) \oplus \Omega_{harmonic}^k(B)$ follows by invoking Hodge's theorem and noting that $g(d^*\omega, d^*\omega) = g(\omega, dd^*\omega)$. \square

4. GAUGE GROUP

Definition 4.1. The group of gauge transformations \mathcal{G} of E_G (or of E) is the space of G -equivariant maps preserving the fiber, that is

$$\mathcal{G} := \{\phi : E_G \rightarrow E_G \mid \pi \circ \phi = \pi \text{ and } g \cdot \phi = \phi \cdot g \text{ for all } g \in G\}$$

Denote the space of connection on a bundle E_G by $\mathcal{A}(E)$, the gauge group acts very naturally on this space.

Proposition 4.2. *If ∇ is a connection on E and ϕ is a gauge transformation then so is $\phi_* \nabla := (\phi^{-1})^* \nabla$, however this action need not be free in general.*

Proof.

- (1) $g^* \phi_* \nabla = \phi_* g^* \nabla = \phi_* \text{Ad}_{g^{-1}} \nabla = \text{Ad}_{g^{-1}} (\phi_* \nabla)$
- (2) Suffices to show that $\phi_* \lambda^o = \lambda^o$. $\lambda^o|_p$ can be thought of as a path in G given by $t \mapsto p \cdot \exp(t\lambda)$ then the push forward of this ϕ will be the path at $\phi(p)$ given by $t \mapsto \phi(p \cdot \exp(t\lambda)) = t \mapsto \phi(p) \cdot \exp(t\lambda)$ which is the same as $\lambda^o|_{\phi(p)}$.

□

Proposition 4.3. *For the trivial bundle $E_G = B \times G$ the gauge group \mathcal{G} is isomorphic to $\text{Hom}(B, G)$ and the induced action of a function $\phi : B \rightarrow G$ on a connection $\nabla = a_i dx^i + \alpha_j d\alpha^j$ is given by $\phi_*(\nabla) = (\phi^{-1} a_i \phi + \partial_i \log \phi) dx^i + \alpha_j d\alpha^j$.*

Proof. The isomorphism $\text{Hom}(M, G) \rightarrow \mathcal{G}$ is given by $\phi \mapsto ((b, g) \mapsto (b, \phi(b)g))$.

$$\begin{aligned} \phi_* a_i(b, e) &= (\phi_* \nabla)(\partial x_i)(b, \phi(b)) \\ &= (\phi^{-1*} \nabla)(\partial x_i)(b, \phi(b)) \\ &= (\nabla(b, \phi(b)))(\phi_* \partial x_i)(b, \phi(b)) \end{aligned}$$

Now write ϕ locally as $\phi = \exp(\phi^j \alpha_j)$ then

$$\phi_* \partial x_i = \partial x_i + \frac{\partial \phi_j}{\partial x_i} \partial \alpha^j$$

and

$$\nabla(b, \phi(b)) = \phi^{-1}(b) a_i(b) \phi(b) dx^i + \alpha_j d\alpha^j$$

Combining the two we get the desired result.

□

Proposition 4.4. *For an arbitrary G bundle E there is an isomorphism*

$$\begin{aligned} F : \{f \in \text{Hom}(E, G) \mid f(pg) = g^{-1} p g \forall p \in E\} &\rightarrow \mathcal{G} \\ f &\mapsto (p \mapsto p f(p)) \end{aligned}$$

Proof. To see that this is a well defined function,

$$\begin{aligned} F(f)(pg) &= p g f(pg) \\ &= p g g^{-1} f(p) g \\ &= p f(p) g \\ &= F(f)(p) g \end{aligned}$$

To prove isomorphism we simply give an inverse function. Define H in the other direction as $H(\phi) = p^{-1}\phi(p)$, again one checks that it is well defined and inverse of F by computation.

Note that $\phi(p)p^{-1}$ is constant along each fiber but $p^{-1}\phi(p)$ is not. \square

Proposition 4.5. \mathcal{G} forms an infinite dimensional Lie group with Lie algebra $\Omega_G^0(E; \mathfrak{g})$

Proof. We work with $\mathcal{G} = \{f \in \text{Hom}(E, G) | f(pg) = g^{-1}pg \forall p \in E\}$. The group structure is given by lifting the group structure on G .

To see the Lie algebra structure notice an element of $\Omega_G^0(E; \mathfrak{g})$ is of the form $\alpha : E \rightarrow \mathfrak{g}$ such that $\alpha(pg) = g^{-1}\alpha(p)g$. We can exponentiate this to get a map $\exp(\alpha) : E \rightarrow G$ which will satisfy $\exp(\alpha)(pg) = g^{-1}\exp(\alpha)g$. \square

The action of the gauge group can also be extended very naturally to the curvature simply by the naturality of d and the Lie bracket.

Proposition 4.6. If $\phi \in \mathcal{G}$ then $\Omega_{\phi^*\nabla} = \phi^*\Omega_\nabla$.

Denote the space of flat connections on E by \mathcal{A}^{flat} .

Corollary 4.7. \mathcal{G} acts on \mathcal{A}^{flat} that is the gauge group preserves flatness.

5. YANG MILLS THEORY

Now we have the language to describe the Yang Mills theory. What we describe is not the most general theory, but will suffice for now.

Start with an \mathbb{R}^k vector bundle (π, E, B) with structure group G and let the corresponding principal G bundle be E_G . Assume B closed oriented manifold with a non-degenerate metric $\langle \cdot, \cdot \rangle$ and let the volume form be ϵ .

Our space of fields is \mathcal{A} , the space of flat connections on E . Using the killing form we can extend the metric $\langle \cdot, \cdot \rangle$ on $\Omega^*(B)$ to $\Omega^*(B; \mathfrak{g})$.

Definition 5.1 (Yang-Mills action). The Yang-Mills functional or action is defined as

$$\mathcal{Y} : \mathcal{A} \rightarrow \mathbb{R}$$

$$\nabla \mapsto \frac{1}{2} \int_B \langle \Omega_\nabla, \Omega_\nabla \rangle \epsilon$$

Proposition 5.2. \mathcal{Y} is invariant under gauge transformations and hence factors through \mathcal{A}/\mathcal{G} .

Proof. This is because the Killing form is G -invariant. \square

Proposition 5.3. The critical points of \mathcal{Y} are given by $d_\nabla^* \Omega = 0$.

Proof. Suppose we vary ∇ in the direction of $\eta \in \Omega_G^1(E; \mathfrak{g})$, we want

$$\left. \frac{d}{dt} \right|_{t=0} \int_B \langle \Omega_{\nabla+t\eta}, \Omega_{\nabla+t\eta} \rangle \epsilon = 0 \text{ for all } \eta$$

Expanding out $\Omega_{\nabla+t\eta} = \Omega_\nabla + t(d\eta + [\nabla, \eta]) + O(t^2)$ we get the equation

$$\langle \Omega_\nabla, d_\nabla \eta \rangle = 0 \text{ for all } \eta$$

By Hodge duality this is equivalent to requiring that $d_\nabla^* \Omega_\nabla = 0$. \square

Combining this with the Bianchi Identity, $d_{\nabla}\Omega_{\nabla} = 0$ we get the Yang-Mills conditions in vacuum,

Definition 5.4. The Yang-Mills connections in vacuum on E are defined as the critical points of \mathcal{Y} that is those $\nabla \in \mathcal{A}$ which satisfy the equations

$$d_{\nabla}^*\Omega_{\nabla} = d_{\nabla}\Omega_{\nabla} = 0$$

6. CLASSICAL ELECTROMAGNETISM

In this section we see how the language we have developed so far can be used to describe classical theory of electromagnetism. The Yang-Mills equations will give us the classical Maxwell's equations.

6.1. The bundle. We start with a \mathbb{C} bundle E over $B = \mathbb{R}^{1+3}$ with structure group $G = U(1)$ and let the corresponding principal bundle be E_G . B is written in this form to distinguish the first time coordinate from the space coordinates. The lie algebra $\mathfrak{u}(1)$ is isomorphic to \mathbb{R} , we will denote the generator of this space by ι this is because sometimes we think of $U(1)$ as unit circle in \mathbb{C} and $\mathfrak{u}(1)$ as the imaginary axis. Let $g = (-1, 1, 1, 1)$ be a constant Lorentz metric on B .

Let $\nabla = \iota A_i dx^i + \iota dt$ be a connection on E_G , then as we noted earlier ∇ is completely determined by $A_i(1)$. Classically, $A_0(1)$ is called the Electric potential and $A_i(1)$ is called the Magnetic potential.

If we apply a gauge transform ϕ , which the bundle being trivial can be identified with a map $Hom(B, G)$ then we know that ιA_i changes to $\iota \phi^{-1} A_i \phi + \partial_i \phi$ and so A_i 's are not physically measurable as anything 'real' should be gauge invariant.

Because we are dealing with a one dimensional lie algebra the lie bracket is trivial and so we get, $\Omega = d\nabla$ and in local coordinates

$$\Omega_{i,j} = \Omega(\partial x_i, \partial x_j) = d\nabla(\partial x_i, \partial x_j) = \iota(\partial_i A_j - \partial_j A_i)$$

We usually neglect the ι .

Now a connection is not a tensor but curvature is a 2-tensor and hence represents a physical quantity. The quantities $\Omega_{0,1}, \Omega_{0,2}, \Omega_{0,3}$ comprise the electric field, and $\Omega_{1,2}, \Omega_{2,3}, \Omega_{3,1}$ comprise the magnetic field, up to a scaling factor, however as we just saw these are not independent of each other and occur together as a 2-tensor.

Just to clear up the notational fog, let us rewrite the curvature in usual physics symbols. Assuming $c = 1$, suppose $\vec{E} = (E_x, E_y, E_z)$ is the Electric field and $\vec{B} = (B_x, B_y, B_z)$ is the magnetic field, then the curvature tensor which in physics is called the Faraday tensor is given by,

$$\Omega_{-,-} = F_{-,-} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$\Omega = E_x dx_0 dx_1 + E_y dx_0 dx_2 + E_z dx_0 dx_3 - (B_x dx_2 dx_3 + B_y dx_3 dx_1 + B_z dx_1 dx_2)$$

6.2. Maxwell's equations in a vacuum. Now we apply Yang-Mill's conditions to the above setup.

The first equation is just the Bianchi identity which as the Lie algebra is trivial simply says that $d\Omega = 0$ expanding out we get

$$\partial_k \Omega_{i,j} + \partial_i \Omega_{j,k} + \partial_j \Omega_{k,i} = 0$$

$$\partial_x B_x + \partial_y B_y + \partial_z B_z = 0$$

$$\partial_t B_x - \partial_z E_y + \partial_y E_z = 0$$

the last two are more succinctly written as

$$\nabla \cdot B = 0 \text{ and } \frac{\partial B}{\partial t} + \nabla \times E = 0$$

Hodge dual of the curvature is given by

$$\Omega = -(E_x dx_2 dx_3 + E_y dx_3 dx_1 + E_z dx_1 dx_2) - (B_x dx_0 dx_1 + B_y dx_0 dx_2 + B_z dx_0 dx_3)$$

which gives the other pair of equations

$$\nabla \cdot E = 0 \text{ and } \frac{\partial E}{\partial t} - \nabla \times B = 0$$