# PROBLEM SET 01

## Part 1

$\epsilon - \delta$ proofs are at the foundation of all real analysis (and hence of calculus). The hardest step in writing an $\epsilon - \delta$ proof is the first step, once you get the first step correct the rest follows almost mechanically. The goal of this week is to make friends with these $\epsilon$'s and $\delta$'s and hopefully also get good at them.

**Q.1.** Explain in your own words how the provisional definition is equivalent to the more rigorous $\epsilon - \delta$ definition of limit.

**Provisional definition:** The function $f$ approaches a limit $L$ near $a$ i.e. $\lim_{x\to a} f(x) = L$, if we can make $f(x)$ as close to $L$ as we like by requiring that $x$ be sufficiently close to, but unequal to, $a$.

$\epsilon - \delta$ **definition:** The function $f$ approaches a limit $L$ near $a$ i.e. $\lim_{x\to a} f(x) = L$, if for every $\epsilon > 0$ there is some $\delta > 0$ such that, for all $x$,

$$\text{if } 0 < |x - a| < \delta, \text{ then } |f(x) - L| < \epsilon.$$

**Q.2.** For each of the following functions $f$ and real numbers $a$,

- Guess the limit $L = \lim_{x\to a} f(x)$.
- Find a $\delta$ corresponding to $\epsilon = 0.01$ in the $\epsilon - \delta$ definition of limit.
- Find a $\delta$ corresponding to an arbitrary real number $\epsilon$ and use this to **prove** that $L$ is indeed the limit.

(1) $f(x) = 7x + 2, \quad a = 0$

(2) $f(x) = x^2 - 2, \quad a = 0$

(3) $f(x) = |x|, \quad a = 0$

(4) $f(x) = |x|, \quad a = 1$

**Q.3.** (1) Give a rigorous definition of the following statement,

The function $f$ *does not approach* the limit $L$ at $a$

using the $\epsilon - \delta$ notation.

(2) Using the $\epsilon - \delta$ notation prove that the limit $\lim_{x\to 0} x^2 \neq 1$.

(3) For the function

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{otherwise} \end{cases}$$

prove that $\lim_{x\to 0} f(x) \neq L$ for any real number $L$.

## Part 2

**Need to add a problem here.** It is tedious to use $\epsilon - \delta$ proofs in practice, instead we use Theorem 2 and it's analogues.

**Q.4.** (1) Show that for every $\epsilon_1, \epsilon_2 \in \mathbb{R}$ the following holds

$$|\epsilon_1 - \epsilon_2| \leq |\epsilon_1| + |\epsilon_2|$$

(2) For which values of $\epsilon_1, \epsilon_2$ does equality hold?

(3) Using the $\epsilon - \delta$ definition of limit to prove that if $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$ then

$$\lim_{x\to a} (f(x) - g(x)) = l - m$$

**Q.5.** For the function

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{otherwise} \end{cases}$$

determine, with proof, the limits $\lim_{x\to 0^+}, \lim_{x\to 0^-}$. How will you answers change if we replace $x < 0$ by $x \leq 0$ in the definition of $f(x)$?

**Q.6.** (1) Determine, with proof, $\lim_{x\to\infty} 1/x$.

(2) Prove that for no real number $l$ do we have $\lim_{x\to 0^+} 1/x = l$.

**Q.7.** Give examples to show that the following definitions of $\lim_{x\to a} f(x) = L$ are not correct i.e. find functions $f$ which are continuous but do not satisfy the following conditions. (Hint: Think graphically)

- For all $\delta > 0$ there exists an $\epsilon > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.
- For every $\epsilon > 0$ there is some $\delta > 0$ such that, for all $x$, if $|f(x) - L| < \epsilon$ then $0 < |x - a| < \delta$.

## PART 3

To understand continuity it is equally important to understand discontinuity.

**Q.8.** (1) Let $n$ be a positive integer. Use the $\epsilon - \delta$ definition to prove that the function $f(x) = x^n$ is continuous at 0.

(2) Use Theorem 2 to prove that every polynomial $p(x)$ is continuous at 0.

(3) Prove that a function $f(x)$ is continuous at a real number $a$ if and only if the function $g(x) = f(x + a)$ is continuous at 0.

(4) Prove that $x^n$ is continuous at every real number $a$.

(5) Use Theorem 2 to prove that every polynomial $p(x)$ is continuous at every real number $a$.

(6) What are the real numbers at which the ratio of two polynomials $\dfrac{p(x)}{q(x)}$ is continuous?

**Q.9.** Prove that the function

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{otherwise} \end{cases}$$

is discontinuous everywhere. What can you say about the continuity of the function $g(x) = x.f(x)$?

Trigonometric functions, exponential functions and logarithms are continuous wherever they are defined. We will assume this fact without proof for now and come back to it later.