§ Derived Functor Perspective

· Hocolim as derived colim

Top^I has model structure:

Fibrations: Object wise

Weak eq: ″

Cofib : have LLP wrt trivial fibrations

Objectwise cofib are not cofib in Top^I. For any objectwise cofib $D \in Top^I$ we can choose a cofibrant replacement $QD \longrightarrow D$ and $\mathrm{colim}_I QX \cong \mathrm{hocolim}_I X$

Lemma (cofinality) given $\alpha: I \to J$, $X: J \longrightarrow Top$. if $\forall j$, the category $\underset{\substack{\shortparallel\\ (i,\, j \to \alpha(i))}}{i\downarrow\alpha}$ is non-empty and contractible then

$$\mathrm{hocolim}_I\, \alpha^* X \xrightarrow{\simeq} \mathrm{hocolim}_J\, X$$

Cor: If $t \in I$ is terminal and $X: I \longrightarrow Top$ then $X(t) \xrightarrow{\simeq} \mathrm{hocolim}_I X$

Def: Let $X \in Top^I$, define QX to be the diagram

$$i \longmapsto \underset{I\downarrow i}{\mathrm{hocolim}}\ u_i^*(X) \qquad\qquad \begin{aligned} u_i : I\downarrow i &\longrightarrow I \\ (j, j\to i) &\longmapsto j \end{aligned}$$

Lem: There is a natural weak equivalence $Q \longrightarrow id_{Top^I}$

Proof Factor the identity map for each i,

$$X_i = u_i^*(X)\,[i, i=i] \xrightarrow{\simeq} QX_i \longrightarrow \underset{I\downarrow i}{\mathrm{colim}}\ u_i^*(X) \cong X_i$$

2 out of 3 $\Rightarrow$ $QX_i \longrightarrow \underset{I\downarrow i}{\mathrm{colim}}\ u_i^* X \cong X_i$ is a weak equivalence.

· Homotopy coherent maps:

given $X, Y \in Top^I$ a homotopy coherent map consists of

- collection of maps $X_i \longrightarrow Y_i$
- For every $i_0 \to i_1$ we have a homotopy $X_{i_0} \times \Delta^1 \longrightarrow Y_{i_1}$ between

$$\begin{tikzcd} & X_{i_1} \arrow[dr] & \\ X_{i_0} \arrow[ur] \arrow[dr] & & Y_{i_1} \\ & Y_{i_0} \arrow[ur] & \end{tikzcd}$$

- More generally for $i_0 \to \cdots \to i_n$ a map $X_{i_0} \times \Delta^n \longrightarrow Y_{i_n}$

Note: This is a point in $\mathrm{Tot}\Big(\prod_{i_0} Map(X_{i_0}, Y_{i_0}) \rightrightarrows \prod_{i_0 \to i_1} Map(X_{i_0}, Y_{i_1}) \overset{\rightarrow}{\rightrightarrows} \cdots \Big)$

Th^m: (univ property of $\mathrm{hocolim}_I X$)

given $X \in Top^I$, $Z \in Top$ $\qquad hc(X, cZ) \cong Top(\mathrm{hocolim}_I X, Z)$ $\qquad$ here $c: Top \longrightarrow Top^I$ is the constant functor.

Q: Is this true for holim?

Prop: $\mathrm{colim}_I QX \cong \mathrm{hocolim}_I X$

Proof: $\mathrm{Top}(\mathrm{colim}_I QX, Z) \cong \mathrm{Top}^I(QX, cZ) \cong hc(X, cZ) \cong \mathrm{Top}(\mathrm{hocolim}_I X, Z)$

Lem: Let $E \to E'$ be objectwise trivial fibration. Let D be objectwise cofibrant then $hc(D,E) \to hc(D,E')$ is surjective.

Proof: Lift levelwise maps:

$$\begin{tikzcd} \phi \arrow[r, hook] \arrow[d] & E_i \arrow[d] \\ D_i \arrow[r] \arrow[ur, dashed] & E_i' \end{tikzcd}$$

given $i_0 \to i_1$

$$\begin{tikzcd} D_{i_0} \times \partial\Delta^1 \arrow[r] \arrow[d, hook] & E_{i_1} \arrow[d] \\ D_{i_0} \times \Delta^1 \arrow[r] \arrow[ur, dashed, "\exists!"] & E_{i_1}' \end{tikzcd}$$

More generally $i_0 \to \cdots \to i_n$

$$\begin{tikzcd} D_{i_0} \times \partial\Delta^n \arrow[r] \arrow[d, hook] & E_{i_n} \arrow[d] \\ D_{i_0} \times \Delta^n \arrow[r] \arrow[ur, dashed] & E_{i_n} \end{tikzcd}$$

Cor: If $D \in \mathrm{Top}^I$ is objectwise cofibrant then QD is cofibrant.

Proof: Let $Z \to W$ be trivial fibration in Top^I

$$\begin{tikzcd} \mathrm{Top}^I(QD, Z) \arrow[r, two heads] \arrow[d, "\cong"'] & \mathrm{Top}^I(QD, W) \arrow[d, "\cong"] \\ hc(D,Z) \arrow[r, two heads] & hc(D,W) \end{tikzcd}$$

The surjection of the top arrow is implied by the surjection of the bottom arrow.

Alternative construction of the homotopy colim.

Def. $\mathrm{hocolim}'_I X = \mathrm{coeq}\left(\coprod_{i \to j} X_i \times B(j \downarrow I) \rightrightarrows \coprod_i X_i \times B(i \downarrow I) \right)$

Def: $(Q'X)_i := \mathrm{hocolim}'_{(I \downarrow j)} u_i^* X$

One can show these two also give rise to homotopy eq. hocolims.

Tensors of diagrams.

Def: given $F: \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$, the coend $\int^{c \in \mathcal{C}} F(c,c)$ is the coeq $\left(\coprod_{c \to c'} F(c',c) \rightrightarrows \coprod_c F(c,c) \right)$

Def: given $F: \mathcal{C}^{op} \to \mathcal{D}$, $G: \mathcal{C} \to \mathcal{D}$ the tensor product

$$F \otimes_{\mathcal{C}} G := \int^{c \in \mathcal{C}} F_c \times G_c$$

- This satisfies all the usual tensor identities.

eg: 1) Let $\Delta: \Delta \to \mathrm{Top}$ (standard simplex), and $X: \Delta^{op} \to \mathrm{Top}$ then $|X| = X \otimes_{\Delta} \Delta = \int^{[n] \in \Delta} X_n \times \Delta^n$

eg: 3) $X : I \to Top$, then $X \otimes_I \underbrace{B(-\downarrow I)^{op}}_{\cong\, Q(*)} \cong \mathrm{hocolim}_I X$

This shows by properties of tensor products that for any co-fibrant replacement of a pt $\widetilde{Q}(*)$ we have

$$X \otimes \widetilde{Q}(*) \cong \mathrm{hocolim}_I (X)$$