Classification of Symmetric Bilinear Forms

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We'll assume that all of our vector spaces are over $\mathbb{R}$. Let V denote the vector space $\mathbb{R}^n$.

Definition 0.1. A **non-degenerate symmetric bilinear form** on a vector space V is a function

$$\langle -, - \rangle : V \times V \to \mathbb{R} \tag{0.1}$$

satisfying the following conditions

1. (bilinearity) For all $x, y, z \in \mathbb{R}^n$ and $a \in \mathbb{R}$ we have $\langle ax + y, z\rangle = a\langle x, z\rangle + \langle y, z\rangle$.
2. (symmetry) For all $x, y \in \mathbb{R}^n$ we have $\langle x, y\rangle = \langle y, x\rangle$.
3. (non-degeneracy) For all $x \in \mathbb{R}^n$ there exists a vector $y \in \mathbb{R}^n$ such that $\langle x, y\rangle \neq 0$.

We're going to be lazy and call a non-degenerate symmetric bilinear form an **inner product**. We say that an inner product is **positive definite** if for all non-zero vectors $v \neq 0 \in V$ we have $\langle v, v\rangle > 0$. A vector space V with an inner product is called an **inner product space**. An isomorphism of two inner product spaces is a linear transformation between them which preserves the inner product structure.

Inner products can be defined by matrices as well. Choose a basis $e_1, e_2, \cdots, e_n$ for V and let A be the matrix whose i, j^{th} entry is $\langle e_i, e_j\rangle$. For vectors $x = \sum_{i=1}^n x_i e_i$ and $y = \sum_{i=1}^n y_i e_i$ we have

$$\begin{aligned}
\langle x, y\rangle &= \langle \sum_{i=1}^n x_i e_i, \sum_{i=1}^n y_i e_i \rangle && (0.2)\\
&= \sum_{i,j=1}^n x_i \langle e_i, e_j\rangle y_j && \text{by bilinearity} \quad (0.3)\\
&= x^T A y && (0.4)
\end{aligned}$$

Symmetry of the inner product implies that the matrix A is symmetric. Conversely any symmetric matrix A defines a symmetric bilinear form by $\langle x, y\rangle_A = x^T A y$ where x, y are the vectors written in the standard basis.

Proposition 0.2. *The inner product $\langle -, -\rangle_A$ is non-degenerate iff the matrix A is invertible.*

Proof: A is not invertible iff there is a vector $x \in V$ such that $Ax = 0$ in which case for any other vector $y \in V$ we have $\langle y, x\rangle = y^T A x = 0$. □

We can now ask what inner product looks like in a different basis. Changing a basis corresponds to multiplying with an invertible matrix S. We have

$$\begin{aligned}
\langle Sx, Sy\rangle_A &= (Sx)^T A (Sy) && (0.5)\\
&= x^T S^T A S y && (0.6)\\
&= \langle x, y\rangle_{S^T A S} && (0.7)
\end{aligned}$$

which suggests the following definition.

Definition 0.3. We say two $n \times n$ matrices A, B are **congruent**, $A \equiv B$, if there exists an invertible matrix S such that $A = S^T B S$.

If A and B are congruent then the inner product spaces $\mathbb{R}^n, \langle -, -\rangle_A$ and $\mathbb{R}^n, \langle -, -\rangle_B$ are isomorphic, in fact the isomorphism can be though of as a change of basis and hence in a sense two congruent matrices represent the same inner product.

Theorem 0.4 (Sylvester's theorem)**.** *Every symmetric invertible $n \times n$ matrix is congruent to a diagonal matrix with entries ± 1. The number of positive (resp. negative) entries is equal to the number of positive (resp. negative) eigenvalues.*

$$\begin{bmatrix} \pm 1 & & & \\ & \pm 1 & & \\ & & \ddots & \\ & & & \pm 1 \end{bmatrix} \tag{0.8}$$

Equivalently, for any inner product $\langle -, - \rangle$ on V there exists a basis $e_1, e_2, \cdots, e_n$ such that $\langle e_i, e_j \rangle = 0$ if $i \neq j$ and $\langle e_i, e_i \rangle = \pm 1$.

It is possible to prove this theorem directly by induction on the dimension of V. Instead we'll cheat and invoke the spectral theorem and provide a really short proof.

Proof: By the Spectral theorem for symmetric matrices [1] there exits an orthogonal matrix S and a diagonal matrix D with real entries such that $A = S^{-1}DS$. S being orthogonal implies $S^{-1} = S^T$. As A is invertible A does not have 0 as an eigenvalue and hence the diagonal entries of D are non-zero. If the entries of D are $(d_1, \cdots, d_n)$ then we let U be the diagonal matrix with entries $(\sqrt{|d_1|}, \cdots, \sqrt{|d_n|})$. The above equation simplifies to $A = S^T U^T D' U S$ where the entries of D' are $(\operatorname{sign} d_1, \cdots, \operatorname{sign} d_n)$ which completes the proof. □

Corollary 0.6. *If a symmetric matrix A is positive definite i.e. for all vectors $v \neq 0$ we have $v^T A v > 0$ then $A = P^T P$ for some matrix P.*

Proof: We can write $A = P^T D P$ for some invertible matrix P and diagonal matrix D with entries $d_i = \pm 1$. Let $v = P^{-1} e_i$ where e_i is the i^{th} entry in the basis. Because of positive definiteness

$$0 < v^T A v = (Pv)^{-1} D (Pv) = e_i^T D e_i = d_i$$

and so D is the identity matrix. □

Definition 0.7. For a symmetric invertible matrix A let n_+ denote the number of positive eigenvalues and let n_- denote the number of negative eigenvalues of A. The pair (n_+, n_-) is called the **signature** of the induced inner product $\langle -, - \rangle_A$.

By Sylvester's theorem any two congruent matrices have the same signature and hence signature of an inner product is well-defined and does not depend upon the choice of a basis. Signature is as fundamental to a symmetric bilinear form as eigenvalues are to linear transformations. For example the Minkowski space $\mathbb{R}^{1+3}$ can now be defined to be any 4 dimensional vector space with an inner product which has signature $(1, 3)$. This definition is independent of the choice of a basis as is in some ways more fundamental.

The diagonal matrix $D_{(n,m)}$ having n many $1's$ and m many $-1's$. This induces an inner product $\langle -, - \rangle_{D_{(n,m)}}$ on $\mathbb{R}^{n+m}$. The group of automorphisms of this inner product space is denoted by $O(n, m)$. As every inner product space is isomorphic $\mathbb{R}^{n+m}$ for some (n, m) and the automorphism group of every inner product space is isomorphic to $O(n, m)$. This simplifies greatly the study of inner product spaces and their automorphism groups.

[1]

Theorem 0.5 (Spectral theorem for symmetric matrices)**.** *For any symmetric matrix A there exists a diagonal matrix D with real entries and an orthogonal matrix S such that*

$$A = S^{-1} D S \tag{0.9}$$

The entries of D are the eigenvalues of A.