Problem Set 4 - Points at ∞

Corrected Theorem 4.1 from PSet 02:
Theorem 0.1. *Let f be a non-constant holomorphic function defined near $z_0 \in \mathbb{C}$. Suppose the Taylor series of f near z_0 has the form*

$$f(z) - f(z_0) = a_k(z - z_0)^k + a_{k+1}(z - z_0)^{k+1} + a_{k+2}(z - z_0)^{k+2} + \dots$$

with $a_k \neq 0$. Then there exists biholomorphic functions $\psi(z)$, $\phi(z)$ (with appropriate domains) such that

$$\psi \circ f \circ \phi(z) = z^k$$

1 Complex projective space

Projectivization provides a technique for compactifying non-compact surfaces. However, as we will see tomorrow, this technique sometimes creates singularities so should be used with caution.

Definition 1.1. The *complex projective space* of dimension n is the set

$$\mathbb{P}^n := \{(z_0, z_1, \dots, z_n) \mid z_i \in \mathbb{C} \text{ and not all } z_i \text{ are zero}\} / \sim$$

where the equivalence $\sim$ is defined as

$$(z_0, z_1, \dots, z_n) \sim (\lambda z_0, \lambda z_1, \dots, \lambda z_n) \text{ for } \lambda \neq 0 \in \mathbb{C}$$

The elements of $\mathbb{P}^n$ are written as $[z_0 : z_1 : \dots : z_n]$ and the z_i's are called *homogeneous coordinates.*

Q. 1. Convince yourself that $\mathbb{P}^n$ is the space of complex lines passing through the origin in $\mathbb{C}^{n+1}$. (Such spaces are called Grassmannians.)

There is a natural embedding of $\mathbb{C}^n$ in $\mathbb{P}^n$ given by

$$(z_0, z_1, \dots, z_{n-1}) \longmapsto [z_0 : z_1 : \dots : z_{n-1} : 1]$$

It is easy to see that this map is injective. So we can think of $\mathbb{C}^n$ as a subset of $\mathbb{P}^n$. Any element in $\mathbb{P}^n$ of the form $[z_0 : z_1 : \cdots : z_{n-1} : z_n]$ with $z_n \neq 1$ is in the image of the above embedding, hence

$$\begin{aligned}\mathbb{P}^n \setminus \mathbb{C}^n &= \{[z_0 : z_1 : \cdots : z_{n-1} : 0] \mid z_i \in \mathbb{C}\} \\ &\cong \mathbb{P}^{n-1}\end{aligned}$$

We think of the points in $\mathbb{P}^n \setminus \mathbb{C}^n$ as the "points at ∞". So $\mathbb{P}^n$ has "$\mathbb{P}^{n-1}$ many" points at ∞.

We are mainly interested in the space $\mathbb{P}^2$. We will use the notation

$$\mathbb{P}^2 = \{[z : w : t]\}.$$

The points at ∞ are then the points with $t = 0$ i.e. $\{[z : w : 0]\}$.

2 Homogenization

Let $p(z, w)$ be a polynomial in two variables with complex coefficients. And let

$$S_p = \{(z, w) \mid p(z, w) = 0\} \subseteq \mathbb{C}^2$$

Definition 2.1. A polynomial is *homogeneous* if every monomial term in it has the same total degree.

Homogenization turns p into a homogeneous polynomial $\overline{p}(z, w, t)$ in three variables, where we add powers of t as needed to make each term of the same degree.

Example 2.2. If $p(z, w) = z^2 - w^3 - w$ then $\overline{p}(z, w, t) = z^2t - w^3 - wt^2$. If $p(z, w) = z^2 - w^2 - w$ then $\overline{p}(z, w, t) = z^2 - w^2 - wt$.

> **Q. 2.** Let $\overline{p}(z, w, t)$ is a homogeneous polynomial. Prove that (a, b, c) is a root of $\overline{p}$ if and only if $(\lambda a, \lambda b, \lambda c)$ is a root of $\overline{p}$ for $\lambda \neq 0 \in \mathbb{C}$.

Hence, we can ask for solutions of the homogeneous polynomial $p(z, w, t)$ in the projective space $\mathbb{P}^2$.

If S_p is the set $\{(z, w) \mid p(z, w) = 0\} \subseteq \mathbb{C}^2$ then denote

$$\overline{S_p} := \{[z : w : t] \mid p(z, w, t) = 0\} \subseteq \mathbb{P}^2.$$

Definition 2.3. $\overline{S_p}$ is called the *projectivization* of S_p.

There is a natural embedding $S_p \to \overline{S_p}$ which sends a solution (z, w) of p to the solution $[z : w : 1]$ of $\overline{p}$. The points in $\overline{S_p} \setminus S_p$ are called the points at ∞.

For any homogeneous polynomial $\overline{p}$ the space $\overline{S_p}$ is compact. The proof of this is essentially the fact that closed subsets of compact sets are compact and zero sets of polynomials are closed. But because of the points at ∞ the argument is a bit more intricate, we won't go over the details.

Q. 3. Let $q(w)$ be a complex polynomial and let $p(z, w) = z^2 - q(w)$. Find the number of points at ∞ for $\overline{S_p}$ for the following polynomials

1. $q(w) = w + b$, where $b \in \mathbb{C}$.
2. $q(w) = w^2 + bw + c$, where $b, c \in \mathbb{C}$.
3. $q(w) = w^n + a_{n-1}w^{n-1} + \ldots a_1 w + a_0$, where $a_i \in \mathbb{C}$ and $n \geq 3$.

Q. 4. Let $p(z, w)$ be an arbitrary polynomial with homogenization $\overline{p}(z, w, t)$. What can you say about the number of points at ∞ for $\overline{S_p}$? Can you interpret this result in terms of limits $\lim_{z\to\infty} z/w$?

Fact: when $q(w)$ is a non-constant polynomial of degree ≤ 3 with distinct roots and $p(z, w) = z^2 - q(w)$ the space $\overline{S_p}$ is a Riemann surface and the projection map

$$\begin{aligned} \pi : \overline{S_p} &\longrightarrow \mathbb{P}^1 \\ [z : w : 1] &\longmapsto w \\ [z : w : 0] &\longmapsto \infty \end{aligned}$$

is a complex differentiable map.

Q. 5. Using the Riemann–Hurwitz formula for the projection π, find the genus of the curves $\overline{S_p}$ when $p(z, w) = z^2 - q(w)$ and q is a non-constant polynomial of degree ≤ 3 with distinct roots.

3 Fermat's conjecture for function fields

Theorem 3.1. *The are no non-constant complex coefficient polynomial solutions to the equation*

$$(x(t))^d + (y(t))^d = (z(t))^d$$

if $d > 2$, *with* $\gcd(x(t), y(t), z(t)) = 1$.

Proof. Consider the polynomial $p(z, w) = z^d + w^d - 1$ with homogenization $p(z, w, t) = z^d + w^d - t^d$. $\overline{S_p}$ has exactly d points at ∞ given by $z^d + w^d = 0$, namely

$$[\zeta_1 : 1 : 0]\,,\ [\zeta_2 : 1 : 0]\,, \ldots,\ [\zeta_d : 1 : 0] \text{ where } \zeta_i^d = -1$$

One can show that the projection map

$$\begin{aligned} \pi : \overline{S_p} &\longrightarrow \mathbb{P}^1 \\ [z : w : 1] &\longmapsto w \\ [z : w : 0] &\longmapsto \infty \end{aligned}$$

is a complex differentiable map, and hence a ramified covering.

Consider a point $w \in \mathbb{C}$. The points in $\pi^{-1}(w)$ are the elements $[z : w : 1]$ such that $z^d = 1 - w^d$. Hence,

1. $\pi^{-1}(w)$ has size d if $1 - w^d \neq 0$,
2. $\pi^{-1}(w)$ has size 1 if $w^d = 0$,
3. $\pi^{-1}(\infty)$ has size d.

This tells us that there are d branch points given by the d^{th} roots of unity, call them $\tau_1, \ldots, \tau_d$, and the fiber over each branch point is single element $[0 : \tau_i : 1]$.

Plugging this in the Riemann–Hurwitz formula we get

$$\begin{aligned} \chi(\overline{S_p}) &= d \cdot \chi(\mathbb{P}^1) - \sum_d (d-1) \\ &= 3d - d^2 \end{aligned}$$

Hence if $d > 2$, $\chi(\overline{S_p}) < 2$ and hence genus of $\overline{S_p} > 0$. Hence if $d > 2$, there are no non-constant complex differentiable maps

$$\mathbb{P}^1 \longrightarrow \overline{S_p}$$

But a solution $(x(s), y(s), z(s))$ to the equation $x^d + y^d = z^d$ defines a complex differentiable map

$$\begin{aligned} \mathbb{P}^1 &\longrightarrow \overline{S_p} \\ s &\longmapsto [x(s) : y(s) : z(s)] \end{aligned}$$

extended to ∞ by taking the limit. This map would be non-constant if $\gcd(x, y, z) = 1$. But no such map exists. $\square$