I am honestly clueless about what we are doing here.

Everything here is $\mathbb{Z}/2$. Even the Hurewicz th^m's. Assume n is sufficiently large.

$H^*(K(\mathbb{Z},n);\mathbb{Z}/2) \cong \mathbb{Z}/2\,[Sq^I(i_n)]$ with excess $(I) < n$, I admissible and does not contain Sq^1.

$\Rightarrow H^{n+2}(K(\mathbb{Z},n);\mathbb{Z}/2) \cong \mathbb{Z}/2\,\langle Sq^2(i_n)\rangle$

$\Rightarrow Sq^2(i_n) : K(\mathbb{Z},n) \longrightarrow K(\mathbb{Z}/2, n+2)$

$$Sq^2(i_n) \longmapsfrom i_{n+2}$$

Extend this to a fibration:

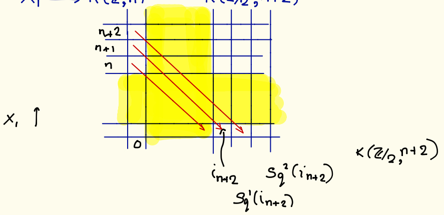
$$\begin{tikzcd}
K(\mathbb{Z},n-1) \arrow[d, equal] & K(\mathbb{Z}/2,n+1) \arrow[d, equal] & K(\mathbb{Z},n) \arrow[r] & K(\mathbb{Z}/2,n+2) \\
\Omega K(\mathbb{Z},n) \arrow[r] & \Omega K(\mathbb{Z}/2,n+2) \arrow[r] & X_1 \arrow[u] &
\end{tikzcd}$$

$$0 \longrightarrow \pi_n(X) \longrightarrow \mathbb{Z} \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \pi_{n+1}(X) \longrightarrow 0 \qquad \text{rest all } 0$$

$X_1 \longrightarrow K(\mathbb{Z},n) \longrightarrow K(\mathbb{Z}/2, n+2)$

$H^{n+2}(X_1;\mathbb{Z}/2) = ?$



$$\begin{tikzcd}
H^{n+2}(K(\mathbb{Z},n);\mathbb{Z}/2) \arrow[d, "Sq^1"] & H^{n+2}(K(\mathbb{Z}/2,n);\mathbb{Z}/2) \arrow[l] \arrow[d, "Sq^1"] \\
H^{n+3}(K(\mathbb{Z},n);\mathbb{Z}/2) & H^{n+3}(K(\mathbb{Z}/2,n);\mathbb{Z}/2) \arrow[l]
\end{tikzcd}$$

$Sq^2(i_n)$ goes to i_{n+2} so it cannot be hit by a differential. $H^{n+1}(K(\mathbb{Z},n)) = 0 \Rightarrow H^{n+1}(X_1;\mathbb{Z}/2) = 0$.

$$\begin{tikzcd}
i_{n+2} \arrow[r, leftrightarrow] \arrow[d, mapsto] & Sq^2(i_n) \arrow[d, mapsto] \\
Sq^1(i_{n+2}) & Sq^1Sq^2(i_n) = Sq^3(i_n) \arrow[l, mapsto]
\end{tikzcd}$$

$Sq^3(i_n)$ goes to $Sq^1(i_{n+2})$ so again it cannot be hit by a differential. $H^{n+2}(K(\mathbb{Z},n)) = \mathbb{Z}/2 \Rightarrow \boxed{H^{n+2}(X_1) = 0.}$

Look at $S^n \xrightarrow{i_n} K(\mathbb{Z},n) \xrightarrow{Sq^2} K(\mathbb{Z}/2,n+2)$. The composition is trivial as $H^{n+2}(S^n) = 0 \Rightarrow$

$$\begin{tikzcd}
S^n \arrow[r, "i_n"] \arrow[dr, dashed, "\exists f_1"'] & K(\mathbb{Z},n) \\
& X_1 \arrow[u]
\end{tikzcd}$$

Why did we do all this? $H_*(S^n;\mathbb{Z}/2) \longrightarrow H_*(K(\mathbb{Z},n);\mathbb{Z}/2)$ is $\begin{cases} \text{iso for } * \le n+1 \\ \underline{\text{not}} \text{ epic for } * = n+2 \end{cases}$

$\pi_n(S^n) = \mathbb{Z} \Leftarrow \pi_*(K(\mathbb{Z},n), S^n) = 0 \Leftarrow \overset{??}{\overset{\shortparallel}{H_{n+2}}}(K(\mathbb{Z},n), S^n) \longrightarrow H_{n+1}(K(\mathbb{Z},n)) \xrightarrow{\cong} H_{n+1}(S^n) \longrightarrow H_n(K(\mathbb{Z},n), S^n) = 0$

$\pi_{n+1}(S^n) = ??$ for $* \le n+1$

Next because $H_*(X_1) \cong H_*(S^n)$ for $* \le n+2$ by the same argument we get

$\boxed{\pi_{n+1}(X_1) = \pi_{n+1}(S^n)}$ and $\boxed{\pi_{n+1}(X_1) = \mathbb{Z}/2}$

To find $\pi_{n+2}(S^n)$ then we should find a space with $H_*(X_2) \cong H_*(S^n)$ for $* \le n+3$.
So we need to look at X_1 and kill its H_{n+3}

Back to the SS: $X_1 \longrightarrow K(\mathbb{Z},n) \longrightarrow K(\mathbb{Z}/2, n+2)$

$$H^{n+3}(K(\mathbb{Z},n)) = \mathbb{Z}/2 \langle Sq^3(i_n)\rangle = \mathbb{Z}/2\langle Sq^1 Sq^2(i_n)\rangle$$

$$H^{n+3}(K(\mathbb{Z}/2, n+2)) = \mathbb{Z}/2 \langle Sq^1(i_{n+2})\rangle$$

So two things are possible

$H^{n+3}(X_1) = 0$ or there is a non-trivial differential on it.

$$H^{n+4}(K(\mathbb{Z},n)) = \mathbb{Z}/2 \langle Sq^4(i_n)\rangle$$

$$H^{n+4}(K(\mathbb{Z}/2, n+2)) = \mathbb{Z}/2 \langle Sq^2 i_{n+2}\rangle$$

$$\begin{tikzcd} Sq^2 i_n \arrow[d, mapsto] & i_{n+2} \arrow[l, mapsto] \arrow[d, mapsto] \\ Sq^2 Sq^2 i_n \arrow[d, equal] & Sq^2 i_{n+2} \arrow[l, mapsto] \\ Sq^3 Sq^1 i_n = 0 & \end{tikzcd}$$

So some term in $H^{n+3}(X_1)$ should hit $Sq^2 i_{n+2}$

$\Rightarrow$ $H^{n+3}(X_1) \cong \mathbb{Z}/2$ Further this is $Sq^2(\alpha)$, $\alpha \in H^n(X_1)$

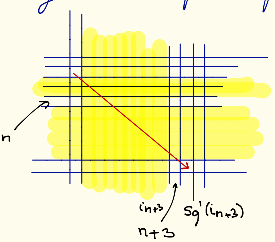
Look at $X_2 \longrightarrow X_1 \longrightarrow K(\mathbb{Z}/2, n+3)$

We now need $\pi_{n+2}(X_2)$. $\pi_*(X) = \begin{cases} n \longmapsto \mathbb{Z}/2 \\ n+1 \longmapsto \mathbb{Z}/2 \\ n+2 \longmapsto \mathbb{Z}/2 \end{cases}$

$\Rightarrow$ $\boxed{\pi_{n+2}(S^n) = \mathbb{Z}/2}$

Now we need $H^{n+4}(X_2)$ and need to verify that $H^{n+3}(X_2) = 0$.

To do this I'll be need to know $H^{n+4}(X_1)$ and the Sq module structure of it ⌢̈
Let me try it in a separate file.



$H^{n+3}(X_1) = \mathbb{Z}/2\langle\gamma_2\rangle \longleftarrow i_{n+3}$

$H^{n+4}(X_1) = \mathbb{Z}/2\, \gamma_3$ We can pick γ_3 such that $Sq^1(\gamma_2) = \gamma_3$

$\mathbb{Z}/2\, \sigma_4$ So the differential on $n+3 \longrightarrow n+4$ is 0.

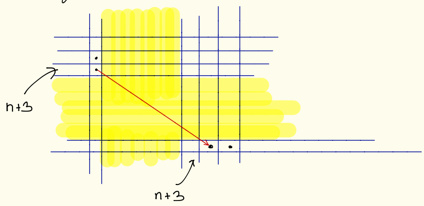
Life is cruel. This method fails in the next stage. Look at the fibration

$$Y \to X_2 \xrightarrow[\tau_4]{} K(\mathbb{Z}/2, n+4)$$

What is $H^{n+4}(Y)$? There is a $\delta_3 \in H^{n+5}(X_2)$. Serre SS for $K(\mathbb{Z}/2, n+3) \to Y \to X_2$

The transgression $n+4 \to n+5$ would be

$$Sq^1 i_{n+3} \longrightarrow Sq^1 \tau_4$$



$\Rightarrow H^{n+4}(X_2) = 0$ iff $Sq^1\tau_4 = \delta_3$ Is this true? No.

In fact we have $d_3\tau_4 = \delta_3$!!

So the correct mapping is $X_2 \longrightarrow K(\mathbb{Z}/8, n+4)$

Look at $X_3 \longrightarrow X_2 \longrightarrow K(\mathbb{Z}/2, n+4)$. The Serre SS for the fibration $K(\mathbb{Z}/8, n+3) \longrightarrow X_3 \longrightarrow X_2$ would have the first two cohomologies related via d_3 and the second differential can kill $H^{n+4}(X_1)$. [It has to kill it for some $\mathbb{Z}/2^k$] $\Rightarrow H^{n+4}(X_3) = 0$

$$\pi_*(X_2) = \begin{cases} \mathbb{Z} & * = n \\ \mathbb{Z}/2 & * = n+1 \\ \mathbb{Z}/2 & * = n+2 \\ \mathbb{Z}/8 & * = n+3 \end{cases}$$

$$\Rightarrow \boxed{\pi_{n+3}(S^n) \cong \mathbb{Z}/8}$$

Next time we get very lucky. $H^*(X_3; \mathbb{Z}/2) = \begin{cases} 0 & \text{for } n < * \le n+6 \\ \mathbb{Z}/2 & \text{for } * = n+7 \end{cases}$

Further this $n+7$, $\mathbb{Z}/2$ has trivial Bockstein

$$\Rightarrow \boxed{\pi_{n+4}(S^n) \cong 0}$$

$$\boxed{\pi_{n+5}(S^n) \cong 0}$$

$$\boxed{\pi_{n+6}(S^n) \cong \mathbb{Z}/8}$$