From High School Arithmetic to Group Cohomology

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Perhaps I can best describe my experience of doing mathematics in terms of a journey through a dark unexplored mansion. You enter the first room of the mansion and it's completely dark. You stumble around bumping into the furniture, but gradually you learn where each piece of furniture is. Finally after six months or so, you find the light switch, you turn it on, and suddenly it's all illuminated. You can see exactly where you were. Then you move into the next room and spend another six months in the dark. So each of these breakthroughs, while sometimes they're momentary, sometimes over a period of a day or two, they are the culmination of—and couldn't exist without—the many months of stumbling around in the dark that precede them.

Andrew Wiles

Introduction

In this class, our goal is to understand how the simple operation of adding multi-digit numbers using the "carrying process" is really an example of a much general structure, called a group extension, which in turn is related to group cohomology.

We will start with the very concrete world of arithmetic, and gradually increase the level of abstraction and eventually define some form of group cohomology. We will mostly work with abelian groups, and time permitting will switch to non-abelian groups toward the end.

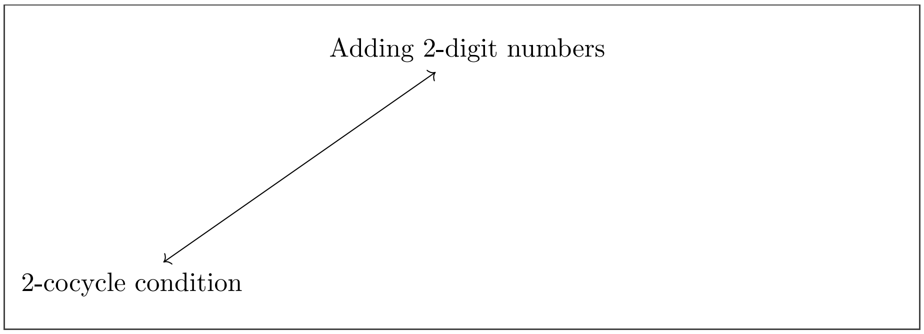
Today, we want to understand how the following are equivalent to each other.

$$\begin{tikzcd} & \text{Adding 2-digit numbers} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ \text{2-cocycle condition} \arrow[rr, leftrightarrow, dashed] & & \text{Group extensions} \end{tikzcd}$$

1 High School Arithmetic

We will consider addition in the group $\mathbb{Z}/100$. Denote the elements of $\mathbb{Z}/100$ by $[\,a\,][\,b\,]$ where $a, b \in \{0, 1, \ldots, 9\}$.

1.1 The carry function



Addition in $\mathbb{Z}/100$ is defined by the formula

$$[\,a_1\,][\,b_1\,] + [\,a_2\,][\,b_2\,] = [\,a_1 + a_2 + c(b_1, b_2)\,][\,b_1 + b_2\,] \tag{1.1}$$

where $c : \mathbb{Z}/10 \times \mathbb{Z}/10 \to \mathbb{Z}/10$ is the “carry” function

Q. 1. Give an explicit definition of the carry function c.

Q. 2. Is c a group homomorphism?

The binary operation on abelian groups satisfies the following three properties:

1. identity: $x + 0 = x = 0 + x$,
2. commutativity: $x + y = y + x$,
3. associativity: $(x + y) + z = x + (y + z)$.

Q. 3. Using Equation (1.1) and the fact that $\mathbb{Z}/100$ is an abelian group, determine the corresponding identities the function c satisfies?

Such a function c has a very fancy name, it is called a

$$\underbrace{\text{normalized}}_{\text{identity}}\ \underbrace{\text{symmetric}}_{\text{commutativity}}\ \underbrace{2-\text{cocycle}}_{\text{associativity}}.$$

Q. 4. Conversely, show that if a function $c : \mathbb{Z}/10 \times \mathbb{Z}/10 \to \mathbb{Z}/10$ is a normalized, symmetric, 2-cocycle then "defining" an addition on $\mathbb{Z}/100$ using Equation 1.1 defines an abelian group structure on it. [1]

Using 2-cocycles, it is possible to create some very exotic "carry functions".

Q. 5. Come up with other examples of normalized, symmetric, 2-cocycles $c : \mathbb{Z}/10 \times \mathbb{Z}/10 \to \mathbb{Z}/10$.

There are exactly 4 isomorphism classes of abelian groups of order 100:

$$\mathbb{Z}/100, \quad \mathbb{Z}/50 \times \mathbb{Z}/2, \quad \mathbb{Z}/20 \times \mathbb{Z}/5, \quad \mathbb{Z}/10 \times \mathbb{Z}/10.$$

Q. 6. Each of the examples of carry functions c you found in Q.5 defines an abelian group structure on the set of 2-digit numbers. Find its isomorphism class.

Q. 7 (Bonus Question)**.** Can you find all the normalized, symmetric, 2-cocycles $c : \mathbb{Z}/10 \times \mathbb{Z}/10 \to \mathbb{Z}/10$? (I do not know any way to answer this question without using group cohomology computations. It would be amazing if you could solve this problem by some elementary methods.)

[1] **Hint:** The only new thing you need to check is that inverses exist.

Summary of Section 1

- Addition in $\mathbb{Z}/100$ is defined by the formula

$$[\,a_1\,]\,[\,b_1\,] + [\,a_2\,]\,[\,b_2\,] = [\,a_1 + a_2 + c(b_1, b_2)\,]\,[\,b_1 + b_2\,]$$

  where $c : \mathbb{Z}/10 \times \mathbb{Z}/10 \to \mathbb{Z}/10$ is the "carry" function. Unravelling the axioms of abelian groups we see that c satisfies the following three identities:

  1. $c(b_1, 0) = 0 = c(0, b_1)$,
  2. $c(b_1, b_2) = c(b_2, b_1)$,
  3. $c(b_1, b_2) + c(b_1 + b_2, b_3) = c(b_1, b_2 + b_3) + c(b_2, b_3)$.

  Such a function c is called a *normalized, symmetric, 2-cocycle.*

- We now flip the tables and "define" an addition on the set of 2-digit numbers by the formula $[\,a_1\,]\,[\,b_1\,] + [\,a_2\,]\,[\,b_2\,] = [\,a_1 + a_2 + c(b_1, b_2)\,]\,[\,b_1 + b_2\,]$ where c is any normalized, symmetric, 2-cocycle.

- Examples:

  1. $c(b_1, b_2) = \left\lfloor \dfrac{b_1 + b_2}{10} \right\rfloor$ defines the standard addition on $\mathbb{Z}/100$.
  2. $c(b_1, b_2) = 0$ defines the addition in which the set of 2-digit numbers becomes $\mathbb{Z}/10 \times \mathbb{Z}/10$.
  3. $c(b_1, b_2) = k \left\lfloor \dfrac{b_1 + b_2}{10} \right\rfloor$ for any integer k defines an addition on the set of 2-digit numbers, the isomorphism class of the resulting abelian group depends on $k \mod 10$.
  4. $c(b_1, b_2) = b_1 b_2$ defines an addition on the set of 2-digit numbers, and resulting group is $\mathbb{Z}/20 \times \mathbb{Z}/5$.

There are exactly 4 isomorphism classes of abelian groups of order 100:

$$\mathbb{Z}/100, \quad \mathbb{Z}/50 \times \mathbb{Z}/2, \quad \mathbb{Z}/20 \times \mathbb{Z}/5, \quad \mathbb{Z}/10 \times \mathbb{Z}/10.$$

Q. How do we know which abelian group is being created by using a particular addition?

Answer. The four groups $\mathbb{Z}/100$, $\mathbb{Z}/50 \times \mathbb{Z}/2$, $\mathbb{Z}/20 \times \mathbb{Z}/5$, $\mathbb{Z}/10$ can be differentiated in the following way: $\mathbb{Z}/100$ contains an element of order 100, $\mathbb{Z}/50 \times \mathbb{Z}/2$ contains an element of order 50 but not of order 100, $\mathbb{Z}/20 \times \mathbb{Z}/5$ contains an element of order 20 but not of order 100, the order of every element of $\mathbb{Z}/10 \times \mathbb{Z}/10$ is at most 10. $\square$

2 Group extensions

Today our goal is to understand how addition of 2-digit numbers is equivalent to another abstract construction in group theory called group extensions. Using this interpretation and some facts from group cohomology, we will eventually be able to answer the question,

Q. What are all the normalized, symmetric, 2-cocycles $c : \mathbb{Z}/10 \times \mathbb{Z}/10 \to \mathbb{Z}/10$?

$$\begin{tikzcd} \text{Adding 2-digit numbers} \arrow[dr, leftrightarrow] & \\ & \text{Group extensions} \end{tikzcd}$$

2.1 Review: Group theory

All our groups will be abelian unless otherwise specified.

A map between abelian groups $\varphi : G_1 \longrightarrow G_2$ is a group homomorphism if it satisfies

$$\varphi(a + b) = \varphi(a) + \varphi(b)$$

Q. 8. (Practice problem) Find all the group homomorphisms

1. $\mathbb{Z} \longrightarrow \mathbb{Z}$,
2. $\mathbb{Z} \longrightarrow \mathbb{Z}/n$,
3. $\mathbb{Z}/n \longrightarrow \mathbb{Z}$,
4. $\mathbb{Z}/n \longrightarrow \mathbb{Z}/n^2$,
5. $\mathbb{Z}/n^2 \longrightarrow \mathbb{Z}/n$,

where n is a positive integer.

Q. 9. Let M be an abelian group. Describe all the group homomorphisms $\mathbb{Z}/n \longrightarrow M$, where n is a positive integer. (This set will show up again when we discuss group cohomology.)

Definition 2.1. The *kernel* of a group homomorphism $\varphi : G_1 \longrightarrow G_2$ is the set of elements $g \in G_1$ such that $\varphi(g) = 0$.

$$\ker \varphi = \{g \in G_1 : \varphi(g) = 0\}$$

Definition 2.2. The *image* of a group homomorphism $\varphi : G_1 \longrightarrow G_2$ is the set of elements $\varphi(g) \in G_2$ where $g \in G_1$.

$$\operatorname{im}\varphi = \{\varphi(g) : g \in G_1\}$$

Q. 10. Find the kernel and image of the group homomorphisms you found in Q.8.

Q. 11. Which of the group homomorphisms in Q.8 are

1. injective (=one-to-one)?
2. surjective (=onto)?
3. isomorphisms (=one-to-one and onto)?

Optional practice problems

Q. 12. Show that the image of a group homomorphism $\varphi : G_1 \longrightarrow G_2$ is a subgroup of G_2.

Q. 13. Show that the kernel of a group homomorphism $\varphi : G_1 \longrightarrow G_2$ is a subgroup of G_1.

Q. 14. Show that for a group homomorphism $\varphi : G_1 \longrightarrow G_2$ we have $G_1/\ker\varphi \cong \operatorname{im}\varphi$.

2.2 Group extensions

Q. 15. Consider the two inclusion maps,

$$\begin{aligned} i_u : \mathbb{Z}/10 &\longrightarrow \mathbb{Z}/100 \\ b &\longmapsto [\,0\,][\,b\,] \end{aligned} \qquad\qquad \begin{aligned} i_t : \mathbb{Z}/10 &\longrightarrow \mathbb{Z}/100 \\ a &\longmapsto [\,a\,][\,0\,] \end{aligned}$$

Which of these two maps is a group homomorphism?

Q. 16. Consider the two projection maps,

$$\begin{aligned} p_u : \mathbb{Z}/100 &\longrightarrow \mathbb{Z}/10 \\ [\,a\,][\,b\,] &\longmapsto b \end{aligned} \qquad\qquad \begin{aligned} p_t : \mathbb{Z}/100 &\longrightarrow \mathbb{Z}/10 \\ [\,a\,][\,b\,] &\longmapsto a \end{aligned}$$

Which of these two maps is a group homomorphism?

Q. 17. For the i and p in Questions 15 and 16 that are group homomorphisms, check that

1. i is injective,
2. p is surjective,
3. $\operatorname{im} i = \ker p$.

Definition 2.3. An *extension* of a group K by H is a group G along with a pair of maps

$$i : H \longrightarrow G \qquad\qquad p : G \longrightarrow K$$

such that

1. i is injective,
2. p is surjective,
3. $\operatorname{im} i = \ker p$.

This is often written as a *short exact sequence*

$$0 \longrightarrow H \longrightarrow G \longrightarrow K \longrightarrow 0$$

Q.17 is saying that the following is a SES

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}/10 \arrow[r, "i"] \arrow[d, equal] & \mathbb{Z}/100 \arrow[r, "p"] \arrow[d, equal] & \mathbb{Z}/10 \arrow[r] \arrow[d, equal] & 0 \\ & \text{tens} & \text{2-digit numbers} & \text{units} & \end{tikzcd}$$

In fact, every group extension arises in this manner. Consider an extension of abelian groups,

$$0 \longrightarrow H \xrightarrow{i} G \xrightarrow{p} K \longrightarrow 0.$$

i.e.

1. i is injective,
2. p is surjective,
3. $\operatorname{im} i = \ker p$.

For an element $a \in H$, denote by $[\,a\,][\,0\,] \in G$ the element $i(a)$. For an element $b \in K$, let $[\,0\,][\,b\,]$ be *some* element in G such that $p([\,0\,][\,b\,]) = b$. Define $[\,a\,][\,b\,]$ to be the element $[\,a\,][\,0\,] + [\,0\,][\,b\,]$ in G.

Q. 18. Show that $p([\,a\,][\,0\,]) = 0$ for any $a \in H$.

Q. 19. Let a_1, a_2 be elements in H and let b_1, b_2 be elements in K, show that [1]

$$[a_1][b_1] = [a_2][b_2] \implies a_1 = a_2 \text{ and } b_1 = b_2.$$

Q. 20. Show that for every element $g \in G$, there exist unique elements $a \in H$ and $b \in K$ such that $g = [\,a\,][\,b\,]$. [2]

[1] **Hint:** Apply p to both sides of $[a_1][b_1] = [a_2][b_2]$.

[2] **Hint:** Look at $p(g)$ and $g - [\,p(g)\,][\,0\,]$.

Summary of Section 2

1. The group $\mathbb{Z}/100$ of 2-digit integers sits in a short exact sequence/group extension

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}/10 \arrow[r, "i"] \arrow[d, equal] & \mathbb{Z}/100 \arrow[r, "p"] \arrow[d, equal] & \mathbb{Z}/10 \arrow[r] \arrow[d, equal] & 0 \\ & \text{tens} & \text{2-digit numbers} & \text{units} & \end{tikzcd}$$

2. Every group extension arises in this manner. For an extension of abelian groups,

$$0 \longrightarrow H \xrightarrow{i} G \xrightarrow{p} K \longrightarrow 0.$$

every element of G can be uniquely written as $[\,a\,][\,b\,]$ for some $a \in H$ and $b \in K$.

Solution to Q.20. We will only prove existence as uniqueness follows by Q.19.

Consider the element $g \in G$. Let $b = p(g) \in K$ and let

$$h = g - [\,0\,][\,b\,].$$

Applying p to both sides we get

$$\begin{aligned} p(h) &= p(g - [\,0\,][\,b\,]) \\ &= p(g) - p([\,0\,][\,b\,]) && \text{as } p \text{ is a group homomorphism} \\ &= b - b \\ &= 0 \end{aligned}$$

Hence, $h \in \ker p$, which implies that $h \in \operatorname{im} i$. Hence, $h = i(a) = [\,a\,][\,0\,]$ for some $a \in H$, giving us

$$g = h + [\,0\,][\,b\,] = [\,a\,][\,0\,] + [\,0\,][\,b\,] = [\,a\,][\,b\,].$$

□

3 Group cohomology

Today we will connect the two topics of 2-cocycles and group extensions and finally define the notion of group cohomology.

$$\begin{tikzcd}
& \text{Adding 2-digit numbers} \arrow[dl, <->] \arrow[dr, <->] & \\
\text{2-cocycle condition} \arrow[rr, <->, dashed] & & \text{Group extensions}
\end{tikzcd}$$

In the case of two digit numbers, this correspondence looks as follows.

$$\begin{tikzcd}
& \mathbb{Z}/100 \text{ under standard addition} \arrow[dl, <->] \arrow[dr, <->] & \\
\text{carry: } \mathbb{Z}/10 \times \mathbb{Z}/10 \to \mathbb{Z}/10 \arrow[rr, <->, dashed] & & 0 \to \mathbb{Z}/10 \to \mathbb{Z}/100 \to \mathbb{Z}/10 \to 0
\end{tikzcd}$$

But there is nothing special about $\mathbb{Z}/10$ or $\mathbb{Z}/100$ and all our proofs and correspondences can be generalized to arbitrary group extensions.

$$\begin{tikzcd}
& G = \{[\,a\,][\,b\,] : a \in H, b \in K\} \arrow[dl, <->] \arrow[dr, <->] & \\
c : K \times K \to H \arrow[rr, <->, dashed] & & 0 \to H \to (G, +_c) \to K \to 0
\end{tikzcd}$$

3.1 Maps between extensions

We will fix two abelian groups H and K.

Let G_c and G_d be two group extensions of H and K, given by the 2-cocycles $c : K \times K \to H$ and $d : K \times K \to H$. This means that in G_c and G_d the additions are given by

$$\begin{aligned} [\,a_1\,][\,b_1\,] +_c [\,a_2\,][\,b_2\,] &= [\,a_1 + a_2 + c(b_1, b_2)\,][\,b_1 + b_2\,] \\ [\,a_1\,][\,b_1\,] +_d [\,a_2\,][\,b_2\,] &= [\,a_1 + a_2 + d(b_1, b_2)\,][\,b_1 + b_2\,] \end{aligned} \tag{3.1}$$

where a_1, $a_2 \in H$ and b_1, $b_2 \in K$. And there are short exact sequences

$$0 \longrightarrow H \xrightarrow{i_c} (G_c, +_c) \xrightarrow{p_c} K \longrightarrow 0,$$

$$0 \longrightarrow H \xrightarrow{i_d} (G_d, +_d) \xrightarrow{p_d} K \longrightarrow 0.$$

Definition 3.1. A *morphism between extensions* is a group homomorphism $\varphi : G_c \to G_d$ which satisfies the following properties:

1. φ restricted to H is just the identity map,
2. the map induced by φ on K is the identity map.

In the language of short exact sequences, this is written as

$$\begin{tikzcd} 0 \arrow[r] & H \arrow[r, "i_c"] \arrow[d, "\mathrm{Id}_H"] & G_c \arrow[r, "p_c"] \arrow[d, "\varphi"] & K \arrow[r] \arrow[d, "\mathrm{Id}_K"] & 0 \\ 0 \arrow[r] & H \arrow[r, "i_d"] & G_d \arrow[r, "p_d"] & K \arrow[r] & 0 \end{tikzcd}$$

Q. 21. What are all the group homomorphisms $\mathbb{Z}/100 \to \mathbb{Z}/100$? Of these, which group homomorphisms are also morphisms from the standard extension $0 \to \mathbb{Z}/10 \to \mathbb{Z}/100 \to \mathbb{Z}/10 \to 0$ to itself.

Q. 22. What are all the group homomorphisms $\mathbb{Z}/10 \times \mathbb{Z}/10 \to \mathbb{Z}/10 \times \mathbb{Z}/10$? Of these, which group homomorphisms are also morphisms from the extension $0 \to \mathbb{Z}/10 \to \mathbb{Z}/10 \times \mathbb{Z}/10 \to \mathbb{Z}/10 \to 0$ to itself.

Q. 23. For $a \in H$ and $b \in K$, show that $\varphi([\,a\,][\,0\,]) = [\,a\,][\,0\,]$ and $\varphi([\,0\,][\,b\,]) = [\,a'\,][\,b\,]$ for some $a' \in H$.

For each $b \in K$, let $\alpha(b)$ be the element in H such that $p([\,0\,][\,b\,]) = [\,\alpha(b)\,][\,b\,]$, so that α is a function (not a group homomorphism) $K \to H$.

Q. 24. For $a \in H$ and $b \in K$, show that $\varphi([\,a\,][\,b\,]) = [\,a + \alpha(b)\,][\,b\,]$.

Q. 25. Show that every morphism between extensions G_c and G_d is bijective.

As we did with group axioms, we want to rewrite what a group homomorphism means in terms of the 2-cocycles c and d. The group homomorphism $\varphi : (G_1, +_c) \to (G_2, +_d)$ satisfies the identity

$$\varphi([\,a_1\,][\,b_1\,] +_c [\,a_2\,][\,b_2\,]) = \varphi([\,a_1\,][\,b_1\,]) +_d \varphi([\,a_2\,][\,b_2\,]) \tag{3.2}$$

Q. 26. Expand the identity (3.2) using the equation (3.1) and find a new identity involving the functions c, d, and h.

Definition 3.2. A *normalized 2-coboundary* is a map $e(b_1, b_2) : K \times K \to H$ such that

$$e(b_1, b_2) = \alpha(b_1 + b_2) - \alpha(b_1) - \alpha(b_2)$$

for some function $h : K \to H$.

Q. 27. Check that the identity in Q.26 is saying that $c - d$ is a normalized 2-coboundary.

Q. 28. Show that a normalized 2-coboundary is also a normalized, symmetric, 2-cocycle.

3.2 Group cohomology

Q. 29. Show that the set of normalized, symmetric, 2-cocycles $c : K \times K \to H$ forms a group under addition. This group is denoted $\mathcal{Z}^2(K; H)$.

Q. 30. Show that the set of normalized 2-coboundaries $c : K \times K \to H$ forms a group under addition. This group is denoted $\mathcal{B}^2(K; H)$.

Q. 31. Show that $\mathcal{B}^2(K; H)$ is a subgroup of $\mathcal{Z}^2(K; H)$.

Definition 3.3. The second cohomology group of K with coefficients H is defined as

$$H^2(K; H) := \mathcal{Z}^2(K; H)/\mathcal{B}^2(K; H)$$

We say that two extensions are equivalent if there is a morphism between them.

Q. 32. Show that this defines an equivalence relation on the set of group extensions.

Denote by $\mathrm{Ext}^1(K; H)$ the equivalence classes of extensions under this equivalence relation.

Q. 33. Prove that there is a 1-1 correspondence between $H^2(K; H)$ and $\mathrm{Ext}^1(K; H)$.

4 What's going on?

We get a lot of mileage by looking at the following correspondence.

$$\{\text{2-cocycles}\} \longleftrightarrow \{\text{Group extensions}\}$$

There is no direct way to go from a group extension to a 2-cocycle, we need to first write the group G as 2-digit numbers $[\,a\,][\,b\,]$ where $a \in H$ and $b \in K$. The 2-cocycle then arises as the carry function.

This correspondence is not a one-to-one correspondence. There are many more cocycles than there are group extensions. To get a 1-1 correspondence, we need to take the quotient of the left-hand side by the set of 2-coboundaries and the right-hand side by isomorphisms.

$$\begin{tikzcd}
\dfrac{\{\text{2-cocycles}\}}{\{\text{2-coboundaries}\}} \arrow[rr, leftrightarrow, dashed, "1-1"] \arrow[d, equal] & & \dfrac{\{\text{Group extensions}\}}{\{\text{isomorphisms}\}} \arrow[dd, equal] \\
\dfrac{\mathcal{Z}^2(K;H)}{\mathcal{B}^2(K;H)} \arrow[d, equal] & & \\
\mathcal{H}^2(K;H) \arrow[rr, leftrightarrow, "1-1"] & & \mathrm{Ext}^1(K;H)
\end{tikzcd}$$

Let us try to understand this correspondence by an example.

4.1 Example

Consider the case that we started with $H = K = \mathbb{Z}/10$ so that the group extensions are groups consisting of 2-digit numbers and hence have sizes 100. Let $c_k = k\left\lfloor \dfrac{b_1 + b_2}{10} \right\rfloor$ be the standard carry function multiplied by k. By repeatedly adding $[\,0\,][\,1\,]$ to itself, one can easily find the isomorphism types of the resulting group extensions.

carry function	isomorphism class
c_1	$\mathbb{Z}/100$
c_2	$\mathbb{Z}/50 \times \mathbb{Z}/2$
c_3	$\mathbb{Z}/100$
c_4	$\mathbb{Z}/50 \times \mathbb{Z}/2$
c_5	$\mathbb{Z}/20 \times \mathbb{Z}/5$
c_6	$\mathbb{Z}/50 \times \mathbb{Z}/2$
c_7	$\mathbb{Z}/100$
c_8	$\mathbb{Z}/50 \times \mathbb{Z}/2$
c_9	$\mathbb{Z}/100$
c_0	$\mathbb{Z}/10 \times \mathbb{Z}/10$

Turns out, these are all the group extensions upto isomorphism. The following is a theorem from algebraic topology,

Theorem 4.1. *The second group cohomology of $\mathbb{Z}/10$ with coefficients in $\mathbb{Z}/10$ is given by*

$$H^2(\mathbb{Z}/10; \mathbb{Z}/10) \cong \mathbb{Z}/10.$$

A generator for this cohomology group is given by c_1.

We can use this theorem to make rapid computions of isomorphism classes.

Proposition 4.2. *The group extension corresponding to the carry function $d(b_1, b_2) = 2b_1b_2$ is given by $\mathbb{Z}/10 \times \mathbb{Z}/10$.*

Proof. We know that c_0 gives rise to the group extension $\mathbb{Z}/10 \times \mathbb{Z}/10$. It suffices to show that there exists a 2-coboundary $\alpha : \mathbb{Z}/10 \to \mathbb{Z}/10$ such that

$$\begin{aligned} d(b_1) - c_0(b_1) &= \alpha(b_1) + \alpha(b_2) - \alpha(b_1 + b_2) \\ \iff 2b_1b_2 &= \alpha(b_1) + \alpha(b_2) - \alpha(b_1 + b_2) \end{aligned}$$

$\alpha(x) = -x^2$ works. ☐

Proposition 4.3. *The group extension corresponding to the carry function $e(b_1, b_2) = b_1b_2$ is either $\mathbb{Z}/20 \times \mathbb{Z}/5$ or $\mathbb{Z}/10 \times \mathbb{Z}/10$.*

Proof. This is because $d = 2e$ where d is as in the previous proposition. And $c_0 = 2c_5$ and $c_0 = 2c_0$. Hence, e must correspond to either $\mathbb{Z}/20 \times \mathbb{Z}/5$ or $\mathbb{Z}/10 \times \mathbb{Z}/10$. ☐

Thus cohomology allows us to reduce questions about reduce questions about isomorphism classes of group extensions to solving identities among functions. However, many a times it only provides partial information and we still need to put in more effort to find the final answer.

4.2 Group cohomology

For each positive integer n, let $\mathcal{C}^n(K;H)$ be the set of functions from $K^{\times n}$ to H.

$$\mathcal{C}^n(K;H) = \{\underbrace{K \times \cdots \times K}_{n-\text{times}} \to H\}$$

There is a differential function $d^n : \mathcal{C}^n(K;H) \to \mathcal{C}^{n+1}(K;H)$ which takes a function that has n inputs and produces a function that has $n+1$ inputs, defined as follows

$$\begin{aligned}(d^n\varphi)(k_1, k_2, k_3, \ldots, k_n, k_{n+1}) := {} & \varphi(k_2, k_3, \ldots, k_n, k_{n+1}) \\ & - \varphi(k_1 + k_2, k_3, \ldots, k_n, k_{n+1}) \\ & + \varphi(k_1, k_2 + k_3, \ldots, k_n, k_{n+1}) \\ & \mp \cdots \pm \\ & (-1)^n \varphi(k_1, k_2 + k_3, \ldots, k_n + k_{n+1}) \\ & (-1)^{n+1} \varphi(k_1, k_2 + k_3, \ldots, k_n)\end{aligned}$$

where φ is a function $K^{\times n} \to H$.

Example 4.4. For $c : K \times K \to H$

$$(d^2\varphi)(k_1, k_2, k_3) = \varphi(k_2, k_3) - \varphi(k_1 + k_2, k_3) + \varphi(k_1, k_2 + k_3) - \varphi(k_1, k_2).$$

Example 4.5. For $\alpha : K \to H$

$$(d^1\alpha)(k_1, k_2) = \alpha(k_1) - \alpha(k_1 + k_2) + \alpha(k_2)$$

From the above examples, one can see that c is a 2-cocycle precisely when $d^2c = 0$ and c is a 2-coboundary when $c = d^1\alpha$ for some α.

Definition 4.6. The set of n-cocycles is the kernel of d^n and the set of n-coboundaries is the image of d^{n-1}.

$$\begin{aligned}\mathcal{Z}^n(K;H) &:= \{n - \text{cocycles}\} = \ker d^n \\ \mathcal{B}^n(K;H) &:= \{(n-1) - \text{cocycles}\} = \operatorname{im} d^{n-1}\end{aligned}$$

The n^{th} group cohomology group of the group K with coefficients in H is the quotient

$$\mathcal{H}^n(K;H) := \mathcal{Z}^n(K;H)/\mathcal{B}^n(K;H) = \ker d^n / \operatorname{im} d^{n-1}$$

The other group cohomologies also have different meanings.

Example 4.7. A 1-coboundary is a map $\alpha : K \to H$ such that $d^1\alpha = 0$. Expanding this out, we get

$$\begin{aligned} \alpha(k_1) - \alpha(k_1 + k_2) + \alpha(k_2) &= 0 \\ \alpha(k_1) + \alpha(k_2) &= \alpha(k_1 + k_2) \end{aligned}$$

But this means that α is a group homomorphism! Thus, 1-cocycles are exactly group homomorphisms. Further, there are no 1-coboundaries and hence

$$\mathcal{H}^1(K; H) := \{\text{group homomorphisms } K \to H\}$$

Thus we have

$$\begin{aligned} \text{first group cohomology} &\longleftrightarrow \text{ group homomorphisms} \\ \text{second group cohomology} &\longleftrightarrow \text{ group extensions} \\ \text{higher group cohomology} &\longleftrightarrow \text{ ??} \end{aligned}$$

What the higher group cohomologies mean is a very subtle question and computing and studying this forms a branch of mathematics called *homological algebra*.

References

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