Chapter 4

Resolving Trees

A natural way to compare knots comes from changing the crossings. This section seeks to formalize this relationship between knot diagrams differing by a single crossing. Relations between knots differing by a crossing are called *skein relations.* Here we discuss resolving trees, a method by which one changes the crossings in a knot one by one until a collection of trivial knots is obtained. This method can be used with various skein relations to define several knot invariants, one of which is our old favorite, the Alexander polynomial.

4.1   Resolving Trees and the Conway Polynomial

Given a crossing in an oriented knot diagram, there are two ways that you can change the crossing while maintaining the orientation in the rest of the knot. You can switch the overcrossing and undercrossing, or you can split the strands of the knot and rejoin them to an adjacent strand with the same orientation. When doing this it is important to keep track of the signs of the crossings given previously in Definition 14.

We may use this method of deconstructing knots to define knot invariants, specifically knot polynomials. One example is the Conway polynomial, denoted $\nabla(K)$.

Definition 20. The Conway polynomial is constructed by three rules as follows.

   1. $\nabla(0_1) = 1$
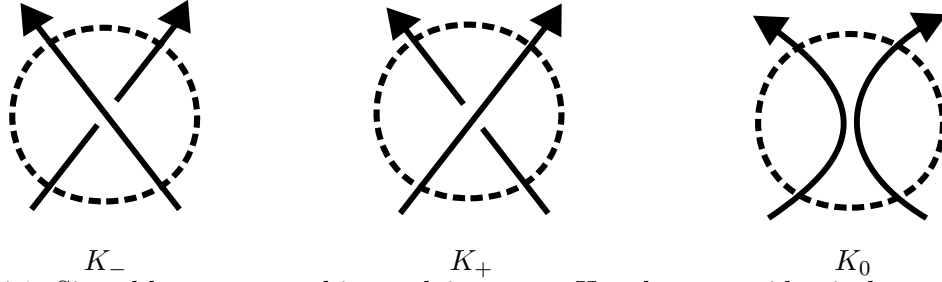
$$K_- \qquad\qquad K_+ \qquad\qquad K_0$$

Figure 4.1: Signed knots are used in resolving trees. Here knots are identical except in the neighborhood of a single crossing. We consider three types of 'crossings', negative or 'left handed crossings' (left), positive or 'right handed' crossings (middle), and zero crossings (right) for which strands are not actually crossed.

2. If K is our knot and K_+, K_-, and K_0 denote K with a positive, negative, or zero crossing respectively, as shown in Figure 4.1 then $\nabla(K_+) - \nabla(K_-) = z\nabla(K_0)$

3. The Conway polynomials of two ambient isotopic knots are equal.

Our goal is to reduce the knot's complexity by changing crossings in this way until we 'unknot' the knot, to obtain trivial knots or links. By keeping track of the signs of the crossings as we go and using the relationships given in Definition 20 this yields the Conway polynomial. This process gives a binary tree of knot diagrams. We call these *resolving trees*. The polynomial found is independent of the resolving tree used [10]. The resolving tree for one knot is given in Figure 4.2 below.

Before doing an example of computing the Conway polynomial of a knot, it is useful to have the following theorem.

Theorem 8. *If L is a split link, including trivial links then $\nabla(L) = 0$*

Proof. Let D_0 be the diagram of a split link with link components L_1 and L_2 and let D_+, D_- be the link with components connected by a positive or negative crossing respectively. D_+ can be obtained from D_- by rotating one link component, say L_1, by 180 degrees. Thus D_+ and D_- are equivalent links and hence $\nabla(D_+) = \nabla(D_-)$. Applying our skein relation we have that $\nabla(D_+) - \nabla(D_-) = z\nabla(D_0)$ and so $\nabla(D_0) = 0$. $\square$

Figure 4.2: Here is a resolving tree for a more complicated knot. The dotted circle indicates which crossing is resolved at each stage. Ambient isotopic knots are denoted with ∼, while +, − and 0 denote positive, negative and zero crossings respectively.

Example 7. Consider the knot with its resolving tree as shown in Figure 4.2. We use this resolving tree to derive the Conway polynomial for this knot. The rule $\nabla(K_+) - \nabla(K_-) = z\nabla(K_0)$ actually gives us two formulas to use, depending on whether we are starting with a positive, or negative crossing. Solving for each of these we have:

$$\nabla(K_+) = \nabla(K_-) + z\nabla(K_0)$$
$$\nabla(K_-) = \nabla(K_+) - z\nabla(K_0)$$.

We see that K has a negative crossing in the region shown, and splits in the resolving tree to K_0 on the right and K_+ on the left. Thus, using our relationships between the signed crossings, we have that $\nabla(K) = \nabla(K_+) - z\nabla(K_0)$. As K_+ is equivalent to the unknot, moving to the second row we get $\nabla(K) = 1 - z\left(\nabla(K_{00}) - z\nabla(K_{0+})\right)$. Continuing in this manner shows that the Conway polynomial for this knot is given by,

$$
\begin{aligned}
\nabla(K) &= 1 - z(1(1(-z) - z) - z) \\
&= 1 + 3z^2.
\end{aligned}
$$

Resolving trees can can also be used to obtain our favorite knot invariant, the Alexander polynomial.

4.2   The Alexander Polynomial

To obtain the Alexander polynomial of a knot from its resolving tree, we need only to change z to $t^{-1/2} - t^{1/2}$. This relationship is given by the following theorem.

Theorem 9. *If K_+, K_- and K_0 are defined as above then*

$$\Delta(K_+) - \Delta(K_-) = (t^{-1/2} - t^{1/2})\Delta(K_0).$$

This result can be proved using our knowledge of Seifert surfaces and Alexander Matrices from Chapter 3. However, we omit the proof that these methods produce the same polynomial invariant.

This gives the following two formulas to employ depending on whether we are changing a positive or negative crossing:

$$\Delta(K_+) = \Delta(K_-) + (t^{-1/2} - t^{1/2})\Delta(K_0)$$
$$\Delta(K_-) = \Delta(K_+) - (t^{-1/2} - t^{1/2})\Delta(K_0).$$

To see this in action, we outline the process of obtaining the Alexander polynomial from a resolving tree for the same knot discussed above.

Example 8. For this we will use the resolving tree in Figure 4.2. For the Alexander polynomial, as with the Conway polynomial, we can use the fact that $\Delta(0_1) = 1$. Additionally, we need to know the Alexander Polynomial of a split link.

Corollary 10. *The Alexander polynomial of a split link is zero.* □

Given this we need only to change the way we express the relationship between the original knot and the knot differing by a single crossing.

Returning to our resolving tree, we find at the top of the tree that K has a negative crossing in the region shown. This knot branches to K_0 (right) and K_+ (left). Thus, using our relationships between the signed crossings, we have that $\Delta(K) = \Delta(K_+) - (t^{-1/2} - t^{1/2})\Delta(K_0)$. As K_+ is equivalent to the unknot, moving to the second row of our resolving tree we get $\Delta(K) = 1 - (t^{-1/2} - t^{1/2})(\Delta(K_{0+}) - (t^{-1/2} - t^{1/2})\Delta(K_{00})$. Continuing in this manner we finally obtain that the Alexander polynomial for this knot is given by,

$$
\begin{aligned}
\nabla(K) &= 1 - (t^{1/2} - t^{-1/2})(1(1(-(t^{1/2} - t^{-1/2})) - (t^{1/2} - t^{-1/2})) - (t^{1/2} - t^{-1/2})) \\
&= 3t - 5 + 3t^{-1}.
\end{aligned}
$$

Note that the Alexander polynomial for this knot could also have been obtained directly from the Conway polynomial by substituting $z = t^{1/2} - t^{-1/2}$.

Tables of Alexander and Conway polynomial for knots up to nine crossings are included in most knot books, for example [4].