Problem Set 3 - Riemann–Hurwitz

For today, let X and Y be compact connected Riemann surfaces with atlases (U_i, φ_i) and (V_j, ψ_j), respectively, and let $f : X \to Y$ be a non-constant complex differentiable function between them.

1 Ramified coverings of Riemann surfaces

Definition 1.1. Pick a chart φ_i around a point $z \in X$ and a chart ψ_j around $f(z) \in Y$. Define

$$\mathrm{index}_f(z) := \mathrm{index}_{\psi_j \circ f \circ \varphi_i^{-1}}(\varphi(z))$$

Q. 1. Let $g(z)$ be a non-constant holomorphic function on $U \subseteq \mathbb{C}$ with Taylor expansion

$$g(z) - g(z_0) = a_k(z - z_0)^k + a_{k+1}(z - z_0)^{k+1} + \dots$$

Let ψ be a biholomorphic function with Taylor expansion

$$\psi(w) - \psi(w_0) = a_1(z - w_0) + a_2(z - w_0)^2 + \dots$$

where $w_0 = g(z_0)$. Find the first term in the Taylor expansions of $\psi \circ g$ at z_0 and argue that f and $\psi \circ f$ have the same index at z_0. Similar statement is true if we pre-compose instead of post-compose with a biholomorphic function.

Q. 2. Show that $\mathrm{index}_f(z)$ does not depend on the choice of charts, and hence is well defined.

Q. 3. Using the fact that X and Y are compact, show that the ramification and branch loci of f are finite. [1]

[1] **Hint:** Use the fact that ramification and branch locus are isolated.

Definition 1.2. For a point $w \in Y$, define

$$\deg_f(w) = \sum_{z \in f^{-1}(w)} \operatorname{index}_f(z)$$

This definition makes sense as the right-hand side is finite.

Let w_0 be a point in Y. Suppose $f^{-1}(w_0) = \{z_1, \ldots, z_k\}$ with ramification indices $\{e_1, \ldots, e_k\}$. We can choose sufficiently small neighborhoods $W_i \subseteq X$ around each z_i such that

1. $f : W_i \to f(W_i) \subseteq Y$ is a ramified covering of degree e_i (so that $f(z) \approx z^{e_i}$)
2. $f(W_i) = f(W_j)$ for all $1 \leq i, j \leq k$.

Let $W = f(W_i)$.

Q. 4. Show that

$$\begin{aligned} \deg_f : W &\longrightarrow \mathbb{Z} \\ w &\longmapsto \deg_f w \end{aligned}$$

is a constant function.

Thus, for each point $w \in Y$, we can find an open neighborhood W on which $\deg_f$ is a constant function. Because Y is connected this gives us,

Theorem 1.3. $\deg_f$ *is constant function on* Y.

This constant is called the *degree/order* of the ramified covering.

Q. 5. Find the degree of a non-constant rational function

$$\begin{aligned} f : \mathbb{P}^1 &\longrightarrow \mathbb{P}^1 \\ z &\longmapsto \frac{p(z)}{q(z)} \end{aligned}$$

Q. 6. Show that every non-constant meromorphic function $f : X \to \mathbb{P}^1$ has the same number of zeroes and poles, counting multiplicities.

2 Riemann–Hurwitz formula

General philosophy:[1] Algebra and analysis are used for constructing maps and algebraic topology is used for proving non-existence, by providing obstructions to existence of maps.

[1] Not to be taken seriously.

Theorem 2.1 (Riemann–Hurwitz)**.** *Let X and Y be compact Riemann surfaces and let $f : X \to Y$ be a non-constant complex differentiable map which is a ramified covering of order N with ramification points $z_1, \ldots, z_k$. Then,*

$$\chi(X) = N \cdot \chi(Y) - \sum_{i=1}^{k} (\text{index}_f(z_i) - 1)$$

where χ is the Euler characteristic.

Lemma 2.2. *If we have a covering map $f : X \to Y$ of degree N of compact topological surfaces then $\chi(X) = N \cdot \chi(Y)$.*

Proof. Put a triangulation on Y which is fine enough that its lift is a triangulation on X. If the original triangulation had V, E, F vertices, edges, and faces, respectively, then the lifted triangulation will have NV, NE, NF vertices, edges, and faces, respectively. The result follows. ☐

Proof of Theorem 2.1. Put a triangulation on Y which is fine enough that its lift is a triangulation on X. Assume further that all the branch and ramification points are vertices in this triangulation.

Suppose the triangulation on X has V, E, F vertices, edges, and faces, respectively. If there were no ramification points the triangulation on Y would have NV, NE, NF vertices, edges, and faces, respectively.

But now consider a ramified point $z \in X$ with ramification degree e and let $w = f(z) \in Y$ be the corresponding branch point. Suppose there are k triangles with vertex w.

Q. 7. Show that the triangles around w have a total of $1 + k$ vertices, $2k$ edges, and k faces.

Q. 8. Show that in the lifted triangulation, the triangles around z have a total of $1 + k \cdot e$ vertices, $2k \cdot e$ edges, and $k \cdot e$ faces.

If there was no ramification at z then f should have been an $e : 1$ mapping and hence we should have had $(1 + k) \cdot e$ vertices, $2k \cdot e$ edges, and $k \cdot e$ faces. Hence, a ramification of index e at z results in a drop in the Euler characteristic by

$$((1 + k) \cdot e - 2k \cdot e + k \cdot e) - ((1 + k \cdot e) - 2k \cdot e + k \cdot e) = e - 1$$

The result follows. ☐

Q. 9. Explicitly lift the following triangulation for the function $f : \mathbb{P}^1 \to \mathbb{P}^1$, $f(z) = z^2$ and verify the proof of the Riemann–Hurwitz formula.

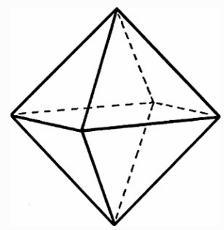

Figure 1: Triangulation of $\mathbb{P}^1$: the north pole is ∞, the south pole is 0, "the square equator" is the unit circle in $\mathbb{C}$.

Using $\chi(X) = 2 - 2g(X)$, we can rewrite Theorem 2.1 as

$$g(X) - 1 = N \cdot (g(Y) - 1) + \sum_{i=1}^{k} (\mathrm{index}_f(z_i) - 1) \cdot 1/2$$

where g is the genus.

Corollary 2.3. *For a compact Riemann surface Y, there are no non-constant differentiable functions $f : \mathbb{P}^1 \to Y$ if $Y \ncong \mathbb{P}^1$.*

Corollary 2.4. *If X and Y are complex tori (genus=1) then any non-constant complex differentiable map $f : X \to Y$ has no ramification points, i.e. the only maps between complex tori are (genuine) covering maps.*

Corollary 2.5. *If X and Y are compact Riemann surfaces and there is a non-constant complex differentiable map $f : X \to Y$ which is not an isomorphism, then $g(X) \geq g(Y)$.*

Q. 10. Prove the above corollaries using Theorem 2.1.

3 Elliptic curves

Analogy: We can construct 1-dimensional real manifolds by looking at solutions to equations $f(x, y) = 0$ inside $\mathbb{R}^2$.

We can do a similar thing for complex manifolds. Consider

$$S_p = \{(z, w) : p(z, w) = 0\} \subseteq \mathbb{C}^2$$

Under certain restrictions on p, this defines a Riemann surface. In particular, this is true when $p(z, w) = z^2 - q(w)$ where $q(w)$ is a degree three polynomial with distinct roots. Further, it is possible to compactify this object by adding a single point at ∞ and the resulting object is called an *elliptic curve.*

$$\mathcal{E}ll_q = \big\{(z, w) : z^2 = q(w)\big\} \cup \{\infty\}$$

There is a natural map

$$\begin{aligned} \mathcal{E}ll_q &\longrightarrow \mathbb{P}^1 \\ (z,w) &\longmapsto w \\ \infty &\longmapsto \infty \end{aligned}$$

Turns out this is a complex differentiable map of degree 2 which has exactly 4 distinct ramification points, the three roots of q and the point at infinity. Plugging in the Riemann–Hurwitz formula we get

$$\begin{aligned} \chi(\mathcal{E}ll_q) &= 2\chi(\mathbb{P}^1) \; + \sum_{\text{4 points}} (2-1) \\ &= 4-4 \\ &= 0 \end{aligned}$$

Hence, $\mathcal{E}ll_q$ is homeomorphic to a torus.

> Almost nothing in this section generalizes arbitrarily. Not all compact Riemann surfaces can be embedded in $\mathbb{P}^2$, not all non-compact Riemann surfaces can be compactified by adding a single point at infinity.
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> But things DO generalize with some effort. All compact Riemann surfaces can be embedded in $\mathbb{P}^3$, many non-compact Riemann surfaces of interest can be compactified by adding multiple points at infinity. It is a very non-trivial theorem in complex analysis that every Riemann surface admits a non-constant meromorphic function.
>
> It is a remarkable accident that things work out to be so nice for elliptic curves.
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> We will make all of this rigorous (as much as possible) in the next two classes.