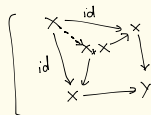


Def: $\pi: X \rightarrow Y$ is separated if $\delta: X \rightarrow X \times X$ is a closed embedding.



• Separated morphisms are preserved under base change.

$$\left(\begin{array}{ccc} \bigcup_{i \in I} X_i & \longrightarrow & X \\ \text{sep} \downarrow & & \downarrow \text{sep} \\ Z & \longrightarrow & Y \end{array} \right)$$

$y \in Y, X_y := \text{fiber}/y$

π separated $\Rightarrow X_y$ separated over $\text{Spec } K(y)$

$\Rightarrow X_y \hookrightarrow X_y \times_{\text{Spec } K(y)} X_y$ is a closed embedding.

• Closed and open embeddings (any monomorphism) are separated.

Prop: $\mathbb{P}_A^n \rightarrow \text{Spec } A$ is separated.

Proof: U_0, \dots, U_n std open affine ... do patching.

Rem: For any $\pi: X \rightarrow Y, u, v \hookrightarrow X$

$$\begin{array}{ccc} U_{X,Y} \vee & \xhookrightarrow{\text{mono}} & X \times_Y X \\ \uparrow \wr & & \uparrow \delta \\ \delta^{-1}(u_X \vee) & \longrightarrow & X \end{array}$$

$$\delta^{-1}(u_X \vee) = u \cap v = u_X \vee$$

Prop: $\text{Spec } A \rightarrow \text{Spec } B$ is always separated (because $A \otimes_B A \rightarrow A$
 $a, b \otimes a_2 \mapsto aa_2$)

Def: A variety over a field k is a reduced separated finite type scheme / $\text{Spec } k$.
 A subvariety is a reduced locally closed reduced subscheme of X .

Prop: $X \rightarrow \text{Spec } A$ is separated. Let $u, v \subseteq X$ be open affines then $u \cap v$ is affine.

Proof: $u \cap v = \underbrace{(u_X \times_{\text{Spec } A} v)}_{\text{affine}} \cap \delta(X)$

$\delta(X)$ is a closed subscheme of $X \times_{\text{Spec } A} X$ so

this is a closed subscheme of affine scheme.

Prop: $\pi: X \rightarrow Y$ quasi-separated if $\delta: X \rightarrow X \times X$ is quasi-compact.

$\Rightarrow \pi$ separated $\Rightarrow \pi$ quasi-separated.

Prop: Separated & Quasi-sep morphisms are stable under base change.

Prop: $\pi: X \rightarrow Y$ separated \Leftrightarrow for some open cover $Y = \bigcup U_i$; $\pi^{-1}(U_i) \rightarrow U_i$ is separated.
Similarly q-sep.

Prop: Separated & q-sep are closed under compositions.

Prop: Every q-proj A -scheme is separated $/A$.

Prop: $\pi: X \rightarrow Y$ and $\pi': X' \rightarrow Y'$ (q)separated morphisms of S -schemes, then $X_S X' \rightarrow Y_S Y'$ is (q) separated.

Rem: Same argument works for any property stable under base change & composition. (Ex. 9.4.F)

Def: The graph morphism of $\pi: X \rightarrow Y$ is $X \times_Y Y \xleftarrow{(\text{id}, \pi)} X$.

Prop: Γ_π is always locally closed embedding & closed embedding if $Y \text{ sep}/Z$.

Th^m: Suppose $\begin{array}{ccc} X & \xrightarrow{\pi} & Y \\ \tau \searrow & & \swarrow s \\ & Z & \end{array}$ commutes. Suppose P is a class of morphisms stable under base change & composition and $\tau \in P$ and $s_\pi: Y \rightarrow Y \times_Z Y$ is in P then $\pi \in P$.

In particular:

1) Suppose locally closed embedding are in P . Then $\tau \in P \Rightarrow \pi \in P$.

2) closed $\tau \in P, \text{ } s \text{ sep} \Rightarrow \pi \in P$