Manifold Calculus and H-principle

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Manifold categories

$$
\begin{aligned}
M^m,\ N^n & \quad \text{smooth manifolds without boundary} \\
\mathrm{emb}(M,N) = & \quad \text{space of embeddings of } M \hookrightarrow N \\
\mathcal{M}\mathrm{an} : & \quad \text{category of smooth manifolds of a } \textit{fixed dimension } m \\
& \quad \text{with weak (Whitney) topology} \\
\mathcal{M}\mathrm{an}(M_1, M_2) = & \quad \mathrm{emb}(M_1, M_2)
\end{aligned}
$$

Main objects of interest

Top valued homotopy presheaves on $\mathcal{M}\mathrm{an}$ i.e. functors of the form,

$$F : \mathcal{M}\mathrm{an}^{op} \to \mathrm{Top}$$

which take isotopy equivalences to weak equivalences.

Gluing Discs

$$\begin{aligned} \mathcal{D}\mathrm{isc}_\infty \subseteq \mathcal{M}\mathrm{an} : & \quad \text{full subcategory of } \mathcal{M}\mathrm{an} \\ Ob(\mathcal{D}\mathrm{isc}_\infty) : & \quad \text{manifolds diffeomorphic to} \\ & \quad \text{disjoint union of finitely many open discs} \end{aligned}$$

Manifold Calculus (Goodwillie-Weiss, Boavida-Weiss)

Try to recover F from it's restriction to $\mathcal{D}\mathrm{isc}_\infty$.

Analytic approximation

Definition (analytic approximation)

$T_\infty F :=$ right derived Kan extension of F along $\mathcal{D}\mathrm{isc}_\infty{}^{op} \hookrightarrow \mathcal{M}\mathrm{an}^{op}$.

$$\begin{tikzcd}
\mathcal{D}\mathrm{isc}_\infty{}^{op} \arrow[rr, "F|_{\mathcal{D}\mathrm{isc}_\infty{}^{op}}"] \arrow[d] & \arrow[d, phantom, "\Downarrow"] & \mathrm{Top} \\
\mathcal{M}\mathrm{an}^{op} \arrow[urr, "T_\infty F"'] & &
\end{tikzcd}$$

We have a natural map

$$F \longrightarrow T_\infty F$$

Analytic Functor

Consider the projective model structure on the category of functors

$$\mathcal{C} = \{\mathcal{D}\mathrm{isc}_{\infty}{}^{op} \to \mathrm{Top}\}$$

then

$$T_{\infty}F(N) = \hom_{\mathcal{C}}(Q\,\mathrm{emb}(-, N), F)$$

where $Q\,\mathrm{emb}(-, N)$ is the cofibrant replacement of $\mathrm{emb}(-, N)$ in the projective model structure.

Definition (analytic functor)

F is **analytic** if $F(M) \longrightarrow T_{\infty}F(M)$ is a weak equivalence for all M.

Analyticity of the Embeddings functor

Theorem (Goodwillie-Weiss, Goodwillie-Klein)

When $n - m > 2$, the functor $\mathrm{emb}(-, N)$ is analytic.

Question

What can we say about directed embeddings $\mathrm{emb}_A(-, N)$?

$$T_\infty \, \mathrm{emb}_A(-, N) = ?$$

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Lagrangian Embeddings

(N, ω) : a symplectic manifold, $\dim N = 2 \dim M$
N has a compatible almost complex structure
i.e. structure group of N can be reduced to $U(m)$

$\mathrm{emb}_{Lag}(M, N) =$ space of embeddings of $M \hookrightarrow N$
as a Lagrangian submanifold
$(V^m \subseteq \mathbb{R}^{2m}$ is Lagrangian if $\omega|_V \cong 0)$

$\mathrm{emb}_{TR}(M, N) =$ space of embeddings of $M \hookrightarrow N$
as a totally real submanifold
$(V^m \subseteq \mathbb{C}^m$ is totally real if $\mathbb{C}^m \cong V \oplus iV)$

Lagrangian Embeddings

Theorem (-)

When $n - m > 2$ there is a natural homotopy equivalence,

$$T_\infty \operatorname{emb}_{Lag}(-, N) \simeq \operatorname{emb}_{TR}(-, N)$$

We can think of this as saying that

manifold calculus sees only the underlying almost complex structure on N.

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Directed embeddings

$\mathrm{Gr}_m(N)$	the m plane Grassmannian bundle of N
$A \subseteq \mathrm{Gr}_m(N)$	a subset of $\mathrm{Gr}_m(N)$.

A-directed embedding

An embedding $f : M \hookrightarrow N$ is called **A-directed** if the image of the induced lift $\mathrm{Gr}_m(f) : M \to \mathrm{Gr}_m(N)$ is in A.

(Note that $\mathrm{Gr}_m(M) = M$.)

H-principle for directed embeddings

We say that A satisfies the **h-principle for directed embeddings** if for all manifolds M of dimension m the following property holds:

$$\vdots$$

H-principle for directed embeddings

For $s \in [0,1]^k$ rel $\partial[0,1]^k$, given a parametrized family of embeddings,

$f_0^s : M \hookrightarrow N$	parametrized embedding
$\mathrm{Gr}_m(f_0^s) : M \to \mathrm{Gr}_m(N)$	lift to Gr_m
$\exists\ \mathrm{Gr}_m(f_0^s) \sim G_1 : M \to A$	homotopy of the lift over f_0^s
$\implies \ \exists\ f_0^s \sim f_1^s : M \to N$	homotopy of the base map

such that f_1^s is A-directed.

H-principle for directed embeddings

For a smooth family of embeddings

$$f_0^s : M \to N$$

parametrized over $s \in I^k$*, such that for* $s \in \partial I^k$ *the embeddings* f_0^s *are A-directed, if the induced lifts* $\mathrm{Gr}_m(f_0^s) : M \to \mathrm{Gr}_m(N)$ *can be homotoped (rel* ∂I^k*) over* f_0^s *to maps*

$$G_1^s : M \to A$$

then the maps f_0^s *can be homotoped (rel* ∂I^k*) to A-directed embeddings*

$$f_1^s : M \to N$$

Homotopical content of the h-principle

This h-principle is a *relative* h-principle.

Lemma

If $A \to N$ is a fibration and A satisfies h-principle (for directed embeddings) then the following square is a homotopy pullback square.

$$
\begin{tikzcd}
\mathrm{emb}_A(M, N) \arrow[r] \arrow[d] & \mathrm{Maps}(M, A) \arrow[d] & \simeq \mathrm{imm}_A(M, N) \\
\mathrm{emb}(M, N) \arrow[r] & \mathrm{Maps}(M, \mathrm{Gr}_m(N)) & \simeq \mathrm{imm}(M, N)
\end{tikzcd}
$$

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H-principle for directed embeddings

Using the formal properties of Kan extensions we can prove the following.

Theorem (-)

If $A \to N$ is a fibration that satisfies the h-principle and $n - m > 2$ then

$$T_\infty \operatorname{emb}_A(M, N) \simeq \operatorname{emb}_A(M, N)$$

Theorem (Gromov, Eliashberg-Mishachev)

If M is open with $n > m$ then any open subset $A \subseteq \operatorname{Gr}_m(N)$ satisfies the h-principle for directed embeddings.

Note: We need to restrict to the class of open manifolds.

Connection between Lagrangian and Totally Real

- Space of Lagrangian subspaces of $\mathbb{R}^{2m} = Sp(2m)/O(m)$.
- Space of Totally real subspaces of $\mathbb{C}^m = U(m)/O(m)$.
- There is a natural homotopy equivalence,

$$Sp(2m)/O(m) \simeq U(m)/O(m)$$

Connection between Lagrangian and Totally Real

By a local object argument, this induces a homotopy equivalence

$$T_\infty \operatorname{emb}_{Lag}(M, N) \simeq T_\infty \operatorname{emb}_{TR}(M, N)$$

Finally we have the following theorem due to Gromov:

Theorem (Gromov)

TR satisfies the h-principle.

Connection between Lagrangian and Totally Real

Putting it all together we get the desired equivalence.

Theorem (-)

When $n - m > 2$ there is a natural homotopy equivalence,

$$T_\infty \operatorname{emb}_{Lag}(-, N) \simeq \operatorname{emb}_{TR}(-, N)$$

Thank you!