$\dim Gl_n = n^2$

-Krishna 15 / 01 / 13

Note: The topology on $X \times Y$ is not product topology.
For example,
$X = Y = \mathbb{A}^1$. But $\mathbb{A}^2$ has lot more closed sets.

So an Algebraic group need not be a topological group. Because $X \times X \longrightarrow X$ is continuous only in algebraic category.

eg: $\mathbb{A}^1 \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1$ if LHS is given product
$x, y \longmapsto xy$ topology then this map is no more continuous.

- Examples of groups:

1) $\mathbb{A}^1_k$ - $\left.\begin{array}{l}\mu(x,y) := x+y \\ x^{-1} := -x\end{array}\right\}$ denoted G_a

2) $K^* \subseteq K$, $K^* = Gl(1,K) = \mathbb{A}^1 - \{0\}$ $\left.\begin{array}{l}\mu(x,y) := xy \\ x^{-1} := 1/x\end{array}\right\}$ denoted G_m

~~affine~~

$Gl(n,k)$, $Sl(n,k)$. $T(n,k) :=$ Upper Triangular Matrices

$D(n,k) :=$ diagonal $U(n,k) :=$ Upper Triangular Matrices with diagonal entries 1

3) $U(2,k) \cong G_a$

$\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \longmapsto x$ Group + variety isomorphism

→ Let G be an affine algebraic group. Then, there is exactly one irreducible component containing the ~~aff~~ identity - e.

Proof: Suppose $e \in X_1 \cap X_2 \cap \cdots \cap X_n$ $X_i \to$ irreducible, X_i's are all such

So $X_1 \times X_2 \times \cdots \times X_n$ ~~are~~ is also irreducible

look at the map

$$\varphi : X_1 \times X_2 \times \cdots X_n \longrightarrow G$$
$$x_1 \cdots x_n \longmapsto x_1 x_2 \cdots x_n .$$

Image of an irreducible set is irreducible.

But $\cancel{X_1 \cdots} \operatorname{Im}\varphi = X_1 \cdots X_n$ is irreducible

$e \in \operatorname{Im}\varphi$

$\Rightarrow$ $\operatorname{Im}\varphi = X_1$ (say)

But each $X_i \subseteq \operatorname{Im}\varphi$ $\Rightarrow$ There is only $n=1$. ∎

∎

Def^n: G°: identity component of G.

→ X, Y irreducible affine varieties. $X \times Y$ is also irreducible affine.

Proof: Suppose $X \times Y = Z_1 \cup Z_2$ ← closed

$x \in X$, $\underset{\cong \, Y}{\{x\} \times Y}$ closed in $X \times Y$.

Intersect Z_1, Z_2 with $x \times Y$.

$Y \cong x \times Y = [\{x\} \times Y \cap Z_1] \cup [x \times Y \cap Z_2]$

But LHS irreducible

$\Rightarrow \{x\} \times Y \subseteq Z_1$ or Z_2

$X_i = \{x \in X \mid x \times Y \subseteq Z_i\}$

$X = X_1 \cup X_2$

Fix $y \in Y$

$X_{i,y} = \{x \in X \mid (x, y) \in Z_i\}$

Then $X_i = \bigcap_{y \in Y} X_{i,y}$

$X_{i,y} = X \times y \cap Z_i$

From now on always assume algebraic group affine

→ $G^\circ \trianglelefteq G$. $[G : G^\circ] =$ finite $=$ connected components of G

Cosets are ~~connected~~, irreducible.

Proof: $\cdot$ $G^\circ G^\circ$ irreducible, contains e

so $G^\circ G^\circ = G^\circ$.

For $x \in G^\circ$ look at $x^{-1} G^\circ$. $e \in x^{-1} G^\circ$

But $G^\circ \cong x^{-1}G^\circ \Rightarrow$ irreducible

$\Rightarrow \quad x^{-1}G^\circ = G^\circ$.

$\Rightarrow \quad G^\circ$ is a subgroup of G.

Normal because $xG^\circ x^{-1}$ also contains identity.

The coset containing x is xG° which is isomorphic to G°.

So we get each coset is irreducible. $\boxtimes$

Corollary: Algebraic group G irreducible $\Leftrightarrow$ connected.

– Chary

Group Algebra - $k[G] := \{ f : G \to k \mid f \text{ has finite support} \}$

$$\Big(\sum_g a_g g\Big)\Big(\sum_h a_h h\Big) = \sum_{g,h} a_g b_h \, gh$$

$$= \sum_g \Big(\sum_h a_g b_{h^{-1}g} \Big) g$$

$a_g = f(g)$

$k[G]$ modules $\longleftrightarrow$ representations of G.

$\rho : k[G] \longrightarrow \cancel{\mathrm{End}}\ Gl(V) \subseteq \mathrm{End}_k(V)$

Lemma: $\rho : G \longrightarrow Gl(V) \longrightarrow$ rpresentation $\quad V/\mathbb{C}$ finite.
$\hookrightarrow$ finite group abelian

Then $\exists$ basis of V such that image is contained in diagonal matrices.

Proof: Mascke

$V = \oplus W_i \qquad W_i$ - G invariant, 1 dimensional. $\square$

Corollary: G be arbitrary finite group. Then $\rho(g)$ is always diagonalizable.

In arbitrary group G, diagonalizable elements are called semisimple elements.

<u>Characters</u>:

Def^n: χ - character of $\rho : G \longrightarrow Gl(V)$

$\chi : G \longrightarrow \mathbb{C}$

$\chi(g) \longmapsto \mathrm{tr}\, \rho(g)$

~~Character is a class function.~~
~~Also group homomorphism~~

~~$$\chi(h^{-1}gh) = tr(\rho(h^{-1}gh)) = tr(\rho(h^{-1})^{-1}\,\rho(g)\,\rho(h)) = -tr(\rho(h)) + tr(\rho g) + tr\,\rho(h) = 0$$~~

- Character is not a group homomorphism

$$\chi(1) = \dim V$$

$$\chi(s^{-1}) = \overline{\chi(s)}$$

$$\chi(s^{-1}ts) = \chi(t)$$

- Character is called irreducible (faithful, trivial) if ρ is irred (faithful, trivial).

- $$\chi_{\rho_{V_1 \oplus V_2}} = \chi_{V_1} + \chi_{V_2}$$

$$\chi_{V_1 \otimes V_2} = \chi_{V_1}\chi_{V_2}$$

- Character is a class function.

$\mathcal{C}$ – set of all class functions

$\subseteq \mathbb{C}[G]$ $\quad \dim |\mathbb{C}[G]| = |G|$

$\dim(\mathcal{C}) = ?$ no. of conjugacy classes

Characteristic functions of conjugacy classes δ basis for $\mathcal{C}$.

- Inner product on $\mathbb{C}[G]$:

$$\langle f_1, f_2 \rangle = \frac{1}{|G|}\sum_g f_1(g)\overline{f_2(g)}$$

In this inner product the δ's form an orthogonal basis of $\mathcal{C}$.

Th^m: χ_i – irreducible characters then
$$\langle \chi_i, \chi_j \rangle = \delta_{ij}$$

~~Prop~~. $\rho_1 \cong \rho_2 \Rightarrow \chi_1 = \chi_2$.

Prop: $\rho(V)$ be representation of G with character χ_V

$V = W_1 \oplus W_2 \oplus \cdots \oplus W_h$

Let W be an irreducible representation of G.

Then the number of W_i's isomorphic to W

$= \langle \chi_V, \chi_W \rangle$.

Proof:

$$\langle \chi_V, \chi_W \rangle = \langle \chi_{W_1} + \chi_{W_2} + \dots + \chi_{W_h}, \chi_W \rangle$$

Thm: $\rho_1 \cong \rho_2 \Leftarrow \chi_1 = \chi_2$.

Proof: $\cancel{V_1 = W_1 \oplus W_2 \oplus \cdots \oplus W_k}$ W_i's irreducible

$\cancel{V_2 = W_1' \oplus W_2' \oplus \cdots \oplus W_k''}$

$V_1 = W_1^{m_1} \oplus W_2^{m_2} \oplus \cdots \oplus W_k^{m_k}$

$V_2 = W_1^{n_1} \oplus W_2^{n_2} \oplus \cdots \oplus W_k^{n_k}$

$\chi_1 = \chi_2 \Rightarrow \langle \chi_1, \chi_{W_i} \rangle = \langle \chi_2, \chi_{W_i} \rangle$

$\Rightarrow m_i = n_i$

$\Rightarrow V_1 \cong V_2$

→ So we get # of irreducible representations = # of irreducible characters

Prop: ρ irreducible $\Leftrightarrow \langle \chi, \chi \rangle = 1$.

Thm: Every irreducible representation occurs in regular representation.

Proof: r_G - character of regular representation

$$\langle r_G, \chi_W \rangle = \frac{1}{|G|} \sum_g r_G(g) \cdot \overline{\chi_W(g)}$$

$r_G(g) =$ no. of points fixed by g

$= \begin{cases} |G| & \text{if } g = \text{identity} \\ 0 & \text{else} \end{cases}$

$$= \frac{1}{|G|} \sum \chi_W(1) = \frac{|W|}{\cancel{|G|}} \cdot \cancel{|G|} = |W|.$$

So if $\chi_1 \dots \chi_h$ are irred reps of G, then

$$r_G = |\chi_1|\chi_1 + |\chi_2|\chi_2 + \dots + |\chi_h|\chi_h.$$

Proposition:

1) $|G| = \sum |\chi_i|^2$

2) $s \neq 1$, $\sum_{i=1}^{h} |\chi_i| \chi_i(s) = 0$

Follows by looking at regular representation.

Th^m:

- Irreducible characters form basis for $\mathcal{C}$

$\Rightarrow$ So, no. of irred. representation
= no. of conjugacy classes

17/01/13 — Krishna

Ex1. Every noetherian topological space has finitely many irreducible components.

Prop : Each closed subgroup H of G of finite index contains G°.

Proof: $H < G$, $(G:H) < \infty$

Look at cosets of H: $H = H_1, \dots H_n$

$\Rightarrow$ $H_i = x_i H$ and $x_i : G \longrightarrow G$ is a homeo

$\Rightarrow$ each H_i is closed & open & disjoint

$\Rightarrow$ But G° is connected and contains e

So $G^\circ \subseteq H$.

Remark : • G°, smallest subgroup having finite index

• G irreducible $\Leftrightarrow$ G connected

G° connected algebraic group

Ex2: Open subset of irreducible space is also irreducible.

So • $Gl(n, K)$ open in $\mathbb{A}^{n^2}$

$\Rightarrow$ $Gl(n, K)$ is irreducible.

- $Sl(n,K)$ connected but difficult to prove

Ex3: G algebraic group

U, V dense open in G,

Then $G = U \cdot V$.

~~Enough to show for G°.~~

~~U, V open in G° $\Rightarrow$ $U/V/ \neq \phi^\circ / / / ?$~~

~~V^{-1} open in G°~~

Prop: $H < G$ Then $\bar{H} < G$. $\bar{H}$ - closure of H.

Proof:

Because translation by x is a homeomorphism

$$\overline{xH} = x\bar{H}$$

So $x \in H \Rightarrow \bar{H} = x\bar{H} = \bar{H}x$

$x \in \bar{H} \Rightarrow Hx = H\bar{H} \subseteq \bar{H}$ similarly xH.....

$x\bar{H} = \overline{xH} \subseteq \bar{H}$ ← so closed under composition

- for inverses,

$\overline{H^{-1}} = \bar{H}^{-1}$ $\because$ Inverse is a homeomorphism.

Prop: $X \subseteq$ affine variety, irreducible

$Y \subseteq X$ proper closed subset

Then, $\dim Y < \dim X$.

Proof: Equivalent to saying $\dim K[Y] < \dim K[X]$

(Note: $K[X]$ depends on embedding but one can prove that they are all isomorphic) ~~upto~~

$$K[Y] = K[X]/I \qquad I \neq 0$$

~~So need to say any et~~

So need to say $\dim K[X] > \dim K[X]/I$

Take a chain of prime ideals in $K[X]/I$

$$\wp_1 + I \subseteq \wp_2 + I \subseteq \cdots \subseteq \wp_n + I$$

We get a chain in $K[X]$

$$0 \subseteq \wp_1 \subseteq \wp_2 \subseteq \cdots \subseteq \wp_n$$

So $\dim K[X] > \dim K[Y]/I$

Also true for varieties in general but this proof will not work.

Prop: X irreducible affine, $Y \leqslant X$ irreducible and $\operatorname{codim}_X Y = 1$, Then,

Y is an irreducible component of $V(f)$ for some $f \in K[x]$

Ex1: X - noetherian topological space

$\Rightarrow$ X satisfies dcc for closed subsets

suppose X has infinite irreducible components $\{X_i\}_{i \in I}$

$$X = \bigcup_{i=1}^{\infty} X_i \cup X'$$

Look at $Y_k = \bigcup_{i=k}^{\infty} X_i \cup X'$

So that $Y_1 \supsetneq Y_2 \supsetneq Y_3 \supsetneq \cdots$

is a dc with not stability.

Ex2: $U \subseteq X$ open

if $U = U_1 \cup U_2$, ~~$X = X - U_1 \cup X - U_2$~~

U_1, U_2 closed in U

~~Ex3:~~ By subspace topology $\exists X_1, X_2 \in X$ closed s.t.

$U_1 = X_1 \cap U$ $U_2 = X_2 \cap U$

So $X = X_1 \cup X_2 \cup (X \setminus U)$

Ex3: