

PROBLEM SET 08

PART 1 - FUNDAMENTAL THEOREM OF CALCULUS

Q.1. Compute $F'(x)$ for the following functions

$$\begin{array}{ll}
 (1) \int_a^{x^3} \sin^3 t \, dt & (4) \int_a^x \left(\int_b^y \frac{1}{1+t^2+\sin^2 t} dt \right) dy \quad * \\
 (2) \int_x^b \frac{1}{1+t^2+\sin^2 t} dt & (5) \int_0^x \frac{1}{1+t^2} dt + \int_0^{1/x} \frac{1}{1+t^2} dt \\
 (3) \int_a^b \frac{x}{1+t^2+\sin^2 t} dt & (6) \int_{-\cos x}^{\sin x} \frac{1}{\sqrt{1-t^2}} dt
 \end{array}$$

Q.2. Find $(f^{-1})'(0)$ if

$$\begin{array}{ll}
 (1) f(x) = \int_0^x 1 + \sin(\sin t) dt & (2) f(x) = \int_1^x \cos(\cos t) dt
 \end{array}$$

Q.3. Suppose f is differentiable with $f(0) = 0$ and $0 < f'(x) \leq 1$. Show that for all $x \geq 0$

$$\int_0^x f^3 \leq \left(\int_0^x f \right)^2$$

Q.4. (1) Find $F'(x)$ if $F(x) = \int_0^x x \cdot f(t) dt$ (Be careful: it's not $x \cdot f(x)$).

(2) Prove that[†]

$$\int_0^x f(t)(x-t) dt = \int_0^x \left(\int_0^u f(t) dt \right) du$$

(3) Prove that

$$\int_0^x f(t)(x-t)^2 dt = 2 \int_0^x \left(\int_0^{u_1} \left(\int_0^{u_2} f(t) dt \right) du_2 \right) du_1$$

Q.5. (1) Suppose $G'(x) = g(x)$ and $F'(x) = f(x)$. Prove that the function $y(x)$ satisfies the differential equation (a separable differential equation)

$$g(y) \cdot y' = f(x)$$

for all x in some interval, if and only if there is a number c such that

$$G(y) = F(x) + c$$

(2) 'Solve' the following differential equations

$$\begin{array}{ll}
 \text{(a) } y' = \frac{1+x^2}{1+y} & \text{(b) } y' = \frac{-1}{1+5y^4}
 \end{array}$$

* Hint: Don't forget the chain rule!

† Hint: Differentiate both sides with respect to x .

PART 2 - IMPROPER INTEGRALS

Q.6. The limit $\lim_{x \rightarrow \infty} \int_a^x f$, if it exists, is denoted by $\int_a^\infty f$ and called an ‘improper integral’.

Similarly for the improper integral $\int_{-\infty}^a f$.

(1) Find $\int_1^\infty t^r dt$, when $r < -1$.

(2) Using $\int_1^a t^{-1} dt + \int_1^b t^{-1} dt = \int_1^{ab} t^{-1} dt$, show that $\int_1^\infty t^{-1} dt$ does not exist. ‡

Q.7. Assume the following the statement:

Suppose that $f(x) \geq 0$ for $x \geq 0$ and that $\int_a^\infty f$ exists. If $0 \leq g(x) \leq f(x)$ for all $x \geq 0$ and g is integrable on the interval $[0, N]$ for all $N > 0$ then $\int_0^\infty g$ exists.

• (Optional) Prove the above statement.

• For which of the following functions does the integral $\int_0^\infty f$ exist?§

(1) $\frac{1}{1+x^2}$

(3) $\frac{x}{1+x^{3/2}}$

(2) $\frac{1}{\sqrt{1+x^3}}$

(4) $\frac{1}{\sqrt{1+x}}$

Q.8. The improper integral $\int_{-\infty}^\infty f$ is defined as $\int_{-\infty}^0 f + \int_0^\infty f$ if both the integrals exist.

(1) Show that $\int_{-\infty}^\infty \frac{1}{1+x^2} dx$ exists.

(2) Determine the limit $\lim_{N \rightarrow \infty} \int_{-N}^N x dx$.

(3) Show that $\int_{-\infty}^\infty x dx$ does not exist.

Q.9. It is possible to have another kind of improper integral, one where the function itself is unbounded but the limits are finite.

For $-1 < r < 0$ draw the graph of x^r and determine $\lim_{\epsilon \rightarrow 0^+} \int_\epsilon^a x^r dx$.

(This limit is written as $\int_0^a x^r dx$.)

‡ Hint: What can you say about $\int_{-1}^1 t^{-1} dt$?

§ Hint: You might have to break $[0, \infty)$ into multiple intervals and analyze each interval separately.