Tangent Spaces:

- $p \in X$ affine variety $= V(f) \subseteq \mathbb{A}^2$, $p = (a_1, a_2)$

Tangent to X at p is ~~variety~~ vectorspace defined by

$$\cancel{\left\{ \frac{\partial f}{\partial T_1}\Big|_p (T_1 - a_1) + \frac{\partial f}{\partial T_2}\Big|_p T_2 \right\}} \quad \left\{ \frac{\partial f}{\partial T_1}\Big|_p (T_1 - a_1) + \frac{\partial f}{\partial T_2}\Big|_p (T_2 - a_2) \right\}$$

eg:

$y = x^2$

$T_{(0,0)} = V(y)$

$y^2 = x^3$

$T_{(0,0)} = \mathbb{A}^2$

Def^n: X - variety, $x \in X$

$\mathcal{O}_{X,x}$ - local ring of X at x with max. ideal $\mathfrak{m}_x$

$$T_x X := \left(\mathfrak{m}_x / \mathfrak{m}_x^2\right)^* = \left(\mathrm{Hom}\left(\mathfrak{m}_x/\mathfrak{m}_x^2, K\right)\right)$$

$$K \longrightarrow K[T] \longrightarrow \frac{K[T]}{I(X)} \longrightarrow \underset{\underset{\mathfrak{m}_x}{\cup\!\!\mid}}{\mathcal{O}_{X,x}} \longrightarrow \frac{\mathcal{O}_{X,x}}{\mathfrak{m}_x} \cong K.$$

$$\dim_K (T_x X) = \dim_K \left(\mathfrak{m}_x/\mathfrak{m}_x^2\right) \geq \dim \mathcal{O}_{X,x} = \dim X$$

equality precisely when x is non-singular.

Another description of $T_x X$:

Def^n: Point derivation of $\mathcal{O}_{X,x}$: K linear map $\delta : \mathcal{O}_{X,x} \longrightarrow K$ s.t.

$$\delta(fg) = \delta(f) g(x) + f(x) \delta(g).$$

$\mathcal{D}_x :=$ {point derivations of $\mathcal{O}_{X,x}$}

Then $\mathcal{D}_x \cong \left(\mathfrak{m}_x/\mathfrak{m}_x^2\right)^*$ as a K-vector space

$\longrightarrow$ δ derivation on $\mathcal{O}_{X,x}$ $\Rightarrow$ δ 0 outside $\mathfrak{m}_x$ and on $\mathfrak{m}_x^2$

$\longleftarrow$ $f \in \left(\mathfrak{m}_x/\mathfrak{m}_x^2\right)^*$ $\Rightarrow$ we can extend f to whole of $\mathcal{O}_{X,x}$

X variety, $x \in X$

$(TX)_x = (\mathfrak{m}_x/\mathfrak{m}_x^2)^* = \{\varepsilon : \mathcal{O}_{X,x} \to k \mid \ldots\}$ $\varepsilon(fg) = (\varepsilon f)g + f(\varepsilon g)$

$\varphi : X \to Y$ morphism, $\varphi(x) = y$ $\qquad \varphi^* : k[Y] \to k[X]$

$(d\varphi)_x : (TX)_x \to (TY)_y$ $\qquad \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$

$\mathfrak{m}_y/\mathfrak{m}_y^2 \to \mathfrak{m}_x/\mathfrak{m}_x^2$

$(\mathfrak{m}_x/\mathfrak{m}_x^2)^* \to (\mathfrak{m}_y/\mathfrak{m}_y^2)^*$

Now suppose $\varphi = (\varphi_1, \ldots \varphi_m)$

$X \subseteq A^n$, $Y \subseteq A^m$

$a = (a_1, \ldots a_n) \longrightarrow d\varphi|_x(a) = (b_1, \ldots b_m)$ where $b_k = \cancel{\sum_{i=1}^{n} \frac{\partial \varphi}{\partial T_i}\Big|_x} a_i$

$= \sum_{i=1}^{n} \frac{\partial \varphi_k(x)}{\partial T_i} \cdot a_i$

Ex 1) $\varphi : Gl(n,k) \longrightarrow G_m (\cong K^*)$

$x \longmapsto \det x$

$(d\varphi)_e : M(n,k) \longrightarrow K$

Then, by above formulae

$$(d\varphi)_e (a_{ij}) = \sum_{\substack{i=1\\j=1}}^{n,n} \frac{\partial (\det)}{\partial T_{ij}}(e) \cdot a_{ij}$$

$$= \sum_{i,j} (\text{co-factor of } T_{ij})|_e \cdot a_{ij}$$

$$= Tr(a_{ij})$$

<u>Lie Algebras</u>:

Lie algebra associated to $M(n,k) = \mathfrak{gl}(n,k)$

G affine algebraic group, $A = K[G]$

$Der_k A = \{$derivations of $A\}$

Left translation $\qquad \lambda_g : K[G] \longrightarrow K[G]$

$f \longmapsto f \circ g^{-1}$

$\delta_1, \delta_2 \in Der_k A \Rightarrow \delta_1\delta_2 - \delta_2\delta_1 \in Der_k A$

$=: [\delta_1, \delta_2]$

- Lie algebra associated to G:

$\mathcal{L}(G) := \{\delta \in Der\, A \mid \delta\lambda_g = \lambda_g\delta \ \forall g \in G\}$

Eg: 1) $G = G_a \quad \mathcal{L}(G) = ?$

$A = K[G] = K[T]$

$\mathrm{Der}_K A = A \cdot \frac{\partial}{\partial T}$ ← rank 1 free module $/A$

$\delta \in \mathrm{Der}_K A \Rightarrow \delta$ determined by the image of T.

$\delta \in \mathcal{L}(G) \Rightarrow \lambda_{-g}\delta = \delta\lambda_{-g} \quad \forall g \in G.$

$\lambda_{-g}\delta(T) = f(T+g)$ where $\delta(T) = f(T)$

$\delta\lambda_{-g}(T) = \delta(T+g)$

$= f(T)$

$\Rightarrow f(T)$ constant

$\Rightarrow \mathcal{L}(G) = K \cdot \frac{d}{dT}$

2) $G = G_m$

$A = K[G_m] = K[T, \frac{1}{T}]$

By same method as above, ~~$\mathcal{L}(G) = K \cdot \frac{d}{dT}$~~

if $\delta(T) = f(T)$, then $\lambda_{-g}\delta(T) = f(gT)$

rational f^n

$\delta\lambda_g(T) = g f(T)$

So, $\mathcal{L}(G) = \{$rational f^ns $f \mid f(gT) = g(f(T)) \; \forall g \in G\}$

$= \{$linear f^ns$\} \cdot \frac{d}{dT}$

G - algebraic group,

$\mathfrak{g} := (TG)_e$ = point derivations at e

$= \{\delta : K[G] \longrightarrow K \mid \quad \}$

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G alg grp

$\mathcal{L}(G) :=$ left invariant derivations

$\mathfrak{g} := (TG)_e$

$\Theta : \mathcal{L}G \longrightarrow \mathfrak{g}$

$\delta \longmapsto (f \longmapsto \delta f(e)) \qquad f \in K[G]$

- Θ K-linear
- $\Theta\delta \in \mathfrak{g}$ because $\Theta\delta(fg) = \delta(fg)(e) = (\delta f)(e)g(e) + f(e)(\delta g)(e)$

$= (\Theta\delta f \cdot g(e) + f(e)\Theta\delta g)$

$\eta : \mathfrak{g} \longrightarrow \mathcal{L}(G)$

$\epsilon \longmapsto (f \longmapsto f * \epsilon)$ $\qquad f \in K[G]$ $\qquad \lambda_x : K[G] \longrightarrow K[G]$

$\qquad\qquad\qquad \parallel$

$\qquad\qquad x \longmapsto \epsilon(\lambda_{x^{-1}} f)$ $\qquad$ (Recall $\lambda_x f(y) = f(x^{-1}y)$)

- η K-linear
- $\eta(\epsilon) \in \mathcal{L}(G)$

Derivation: $\eta(\epsilon)(fg) = (x \longmapsto \epsilon(\lambda_{x^{-1}}(fg)))$

$$\eta(\epsilon)(fg)(x) = \epsilon\, \lambda_{x^{-1}}(fg)$$
$$\parallel$$
$$\epsilon\, (\lambda_{x^{-1}} f)\, (\lambda_{x^{-1}} g)$$
$$\parallel$$
$$(\lambda_{x^{-1}} f)\, e . \epsilon(\lambda_{x^{-1}} g) + \epsilon(\lambda_{x^{-1}} f) . (\lambda_{x^{-1}} g)(e)$$
$$\parallel$$
$$f(x) . \epsilon(\lambda_{x^{-1}} g) + \epsilon(\lambda_{x^{-1}} f) . g(x)$$

$$\Rightarrow \quad \eta(\epsilon)(fg) = f . (\eta(\epsilon)\, g) + g(\eta(\epsilon) . f) .$$

Left invariant: $\cancel{\lambda_y \eta \epsilon}$

$$[\lambda_y (\eta \epsilon)(f)](x) = (\eta \epsilon) f(y^{-1}x)$$
$$= \epsilon(\lambda_{x^{-1}y} f)$$
$$= \cancel{\epsilon(\lambda_{(\lambda_y(x))^{-1}} f)}\ \epsilon(\lambda_{x^{-1}}(\lambda_y f))$$
$$= \cancel{(\eta \epsilon)\, f(\lambda_y x)}\ (\eta \epsilon)\ (\lambda_y f)(x)$$

$$\Rightarrow \lambda_y(\eta \epsilon) = (\eta \epsilon) \lambda_y$$

Claim: $\theta \eta = 1_{\mathfrak{g}}$ $\qquad \eta \theta = 1_{\mathcal{L}(G)}$

Proof:

- $$\eta \theta(\delta)(f)(x) = (f * \theta \delta)(x)$$
$$= \theta \delta(\lambda_{x^{-1}} f)$$
$$= [\delta(\lambda_{x^{-1}} f)](e)$$
$$= \lambda_{x^{-1}}(\delta f)(e)$$
$$= (\delta f)(x) \quad \cancel{= \delta f}$$

- $(\theta\eta)(\varepsilon)(f)$ $\qquad\qquad\qquad$ $\varepsilon \in \mathfrak{g}$ $\quad f \in K[G]$

$$= \eta(\varepsilon) f|_e$$

$$= \varepsilon(\lambda_e f)$$

$$= \varepsilon(f) \qquad \blacksquare$$

Th^m: 1) $\theta : \mathcal{L}(G) \longrightarrow \mathfrak{g}'$ isomorphism of vector spaces

2) $\varphi : G \longrightarrow G'$ morphism of algebras, then

$d\varphi_e : \mathfrak{g} \longrightarrow \mathfrak{g}'$ is a morphism of Lie algebras.

=

$\varphi : X \longrightarrow Y$ map of varieties

$d\varphi_x : TX_x \longrightarrow TY_{\varphi(x)}$ $\qquad$ $\varphi^* : K[Y] \longrightarrow K[X]$

$d\varphi_x$ is just $(\varphi^*)^*$ $\qquad$ $\because$ $TX_x = \{K[X] \to K | ..\}$ $\quad$ $TY_y = \{K[Y] \to K | ...\}$

1) $\mu : G \times G \longrightarrow G$

$(x, y) \longmapsto xy$

$T(G \times G)_{(e,e)} = ?$

$= \mathfrak{g} \oplus \mathfrak{g}$

$(x, y) \in X \times Y$

$K[X \times Y] = K[X] \otimes K[Y]$

$\mathrm{Der}(A \otimes B, K) \simeq \mathrm{Der}(A, K) \oplus \mathrm{Der}(B, K)$

$$\begin{tikzcd} K[X] \arrow[dr] & & \\ & K[X] \otimes K[Y] \arrow[r] & K \\ K[Y] \arrow[ur] & & \end{tikzcd}$$

$$\begin{tikzcd} K[G] \arrow[r] & K[G] \otimes K[G] \arrow[d, "(\varepsilon', \varepsilon'')"] \\ & K \end{tikzcd}$$

$f \longmapsto (\varepsilon' \oplus \varepsilon'') f$

So we will get

$$d\mu_{(e,e)}(x, y) = x + y$$

2) $i : G \longrightarrow G$ $\qquad$ $di_e = \mathfrak{g} \longrightarrow \mathfrak{g}$

$x \longmapsto x^{-1}$

$$G \xrightarrow{1 \times i} G \times G \xrightarrow{\mu} G$$

$$x \longmapsto (x, x^{-1}) \longrightarrow 1$$

$d\,(\text{composite}) = 0$ ← Ex: Prove this

$\Rightarrow d\mu_{(e,e)} \cdot d(1\times i)_e = 0$

$\Rightarrow \quad d(1)_e + d(i)_e = 0$

$\Rightarrow \quad d(i)_e = -d(1)_e = -1_{\mathfrak{g}}$

- $G = Gl(n,k)$

Claim: Lie algebra of $Gl(n,k) = \mathfrak{gl}(n,k)$.

~~Define:~~

$f \in K[G]$, $\mu : G\times G \longrightarrow G$

$$\mu^*(f) = \sum f_i \otimes g_i$$

Define:

$$\varepsilon\otimes\varepsilon' : K[G]\otimes K[G] \longrightarrow K$$
$$f\otimes g \longmapsto \varepsilon f \cdot \varepsilon' g$$

$$\varepsilon\cdot\varepsilon' : K[G] \longrightarrow K$$
$$f \longmapsto (\varepsilon\otimes\varepsilon')\,\mu^* f$$

Ex: Prove that $\varepsilon\cdot\varepsilon' \in \mathcal{L}(G)$.

Co-oridnates on $K[G]$: T_{ij}, $\frac{1}{\det(T_{ij})}$

$\mathfrak{g}$ has canonical basis $\frac{\partial}{\partial T_{ij}}$. This can be seen from dimension considerations.

$$\mathfrak{g} \longrightarrow M(n,k)$$
$$\varepsilon \longmapsto (\varepsilon(T_{ij}))_{ij}$$

where does $\varepsilon\cdot\varepsilon'$ goes under this map?

$$\varepsilon\cdot\varepsilon'(T_{ij}) = (\varepsilon\otimes\varepsilon')\,\mu^*(T_{ij})$$

where $\mu^*(T_{ij}) = \sum_k f_{ij}^k \otimes g_{ij}^k$

$$= (\varepsilon\otimes\varepsilon')\left(\sum_k f_{ij}^k \otimes g_{ij}^k\right)$$

$$= \sum_k \varepsilon(f_{ij}^k)\,\varepsilon'(g_{ij}^k)$$

$$(\varepsilon(T_{ij})\cdot\varepsilon'(T_{ij}))_{ij} = \sum_k \varepsilon(T_{ik})\cdot\varepsilon'(T_{kj})$$

What is $\mu^*(T_{ij})$?

$$\mu^*: K[G] \longrightarrow K[G] \otimes K[G]$$

$$\mu: G \times G \longrightarrow G \xrightarrow[T_{ij}]{} K$$

$$x, y \longmapsto (xy) \longmapsto (xy)_{ij}$$

$$\Rightarrow \mu^*(T_{ij}) = \sum_{k=1}^{n} T_{ik} \otimes T_{kj}$$

Putting this in $(\varepsilon \cdot \varepsilon') T_{ij}$

$$= \sum_{k} \varepsilon(T_{ik})\, \varepsilon'(T_{kj})$$

So the map $\mathfrak{g} \to M(n, K)$ respects multiplication.

Next we need to say that ~~*~~ product over $\mathcal{L}(G) \subseteq \mathrm{Act}\,(K[G])$ is the same as $\mathfrak{g}$ under the map θ.

$\delta, \delta' \in \mathcal{L}(G)$

Claim: $(\theta\, \delta) \cdot (\theta\, \delta') = \theta(\delta \circ \delta')$

$$\theta(\delta \circ \delta')(f) = (\delta \circ \delta') f(e)$$

$$(\theta\delta) \cdot (\theta\delta')(f) = (\theta\delta \otimes \theta\delta')\, \mu^*(f)$$

Again choose $f = T_{ij}$

$$= (\theta\delta \otimes \theta\delta')\left(\sum_k T_{ik} \otimes T_{kj}\right)$$

$$= \sum_k (\delta\, T_{ik})(e) \cdot (\delta' T_{kj})(e)$$

why are the two equal?

Ex : Complete proof of the above statements.

$H \leq G$ closed subgroup

$\varphi: H \hookrightarrow G$

$d\varphi_e: \mathfrak{h} \longrightarrow \mathfrak{g}$

} φ is an isomorphism of varieties onto its image

$$\varphi^*: K[G] \longrightarrow K[H] \cong \frac{K[G]}{I(H)}$$

$G \longrightarrow G'$ iso of algebraic groups

$x \longmapsto y$

$TG_x \longrightarrow TG'_y$ is an isomorphism of Lie algebras

$TG_x = \mathfrak{g}$, $TG'_y = \mathfrak{g}'$

(Ex: Prove this)

- So $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$

$$Sl(n,k) \hookrightarrow Gl(n,k) \xrightarrow{\det} k^*$$

$$d(\det): \mathfrak{gl}(n,k) \longrightarrow \mathfrak{gl}(1,k)$$

$$\varepsilon \longmapsto \operatorname{trace} \varepsilon$$

$\mathfrak{sl}(n,k) = \ker(\text{trace})$

$= \text{trace } 0 \text{ matrices}$

Q. Given a Lie subalgebra, of $\mathfrak{gl}(n,k)$ do we have a corresponding group?

Yes. How?

- $x \in G$,

$$\operatorname{Int} x : G \longrightarrow G$$
$$y \longmapsto xyx^{-1}$$

$\operatorname{Ad} x := d(\operatorname{Int} x) : \mathfrak{g} \to \mathfrak{g}$ — Adjoint of x

$\in \operatorname{Aut}(\mathfrak{g}) \subseteq Gl(\mathfrak{g})$

So, we have a map

$$\operatorname{Ad} : G \longrightarrow Gl(\mathfrak{g})$$
$$x \longmapsto \operatorname{Ad} x$$

— Adjoint representation of G

$\longrightarrow$ Ad homomorphism of groups

- For $f \in K[G]$ denote by xfx^{-1} the element of $K[G]$ satisfying $xfx^{-1}(y) = f(xyx^{-1})$.

Then $d(\operatorname{Int} x)(f) = xfx^{-1}$

- $\operatorname{Ad} x : \mathfrak{g} \longrightarrow \mathfrak{g}$
$$\varepsilon \longmapsto x\varepsilon x^{-1}$$

- From this one can get
$$(\operatorname{Ad} x)(\operatorname{Ad} y) = (\operatorname{Ad} xy)$$

- $\operatorname{Ad} x : \mathfrak{g} \longrightarrow \mathfrak{g}$

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & \mathfrak{g} \\ \wr\!| & & \wr\!| \\ \mathcal{L}(G) & & \mathcal{L}(G) \end{array}$$

$\delta \in \mathcal{L}(G)$

$$\operatorname{Ad} x(\delta) = \rho_x \delta \rho_x^{-1}$$

Ex: Prove this.

- For $G = Gl(n,k)$, $\mathfrak{g} = \mathfrak{gl}(n,k)$

$x \in G$, $\varepsilon \in \mathfrak{g}$, $K[G] = K[T_{ij}, \frac{1}{\det T_{ij}}]$

$$\rho_x T_{ij} = \sum_k T_{ik} x_{kj}$$

- $T_{ij} * \varepsilon = \sum_{k} T_{ik} \varepsilon_{kj}$
- $(\mathrm{Ad}\, x)\varepsilon = \left(\sum_{k,l} x_{ik} \varepsilon_{kl} x^{-1}_{lj} \right)_{ij}$

$$(\mathrm{Ad} x)\varepsilon\, T_{ij} = \left(\rho_x (*\varepsilon) \rho_x^{-1}\right). T_{ij}$$
$$= \rho_x (*\varepsilon) \sum_k T_{ik} x^{-1}_{kj}$$
$$= \rho_x \sum_k (T_{ik} * \varepsilon) x^{-1}_{kj}$$
$$= \cdots$$

finally we get

$$(\mathrm{Ad}\, x)\varepsilon = x \varepsilon x^{-1}$$

— Kannan

Goal: G affine algebraic group / $K = \bar{K}$

H closed subgroup of G.

To view G/H as a quasi-projective variety $\longrightarrow$ open subvariety of a projective variety

i.e. locally closed subset of projective space

$I_H = \{ f \in K[G] \mid f(h) = 0 \ \forall h \in H \}$

$G \circlearrowright K[G]$ via $(g.f)(x) = f(g^{-1}x) =: (\lambda_g f)(x)$

$(\rho_g f)(x) =$

Lemma: $H = \{ g \in G \mid \rho_g (I_H) = I_H \}$

$= \{ g \in G \mid \lambda_g (I_H) = I_H \}$

Proof: $\subseteq$

$\rho_h(I_H)$ $\quad \rho_h f(x) = f(xh)$ for $x \in H$,

$= 0$

$\Rightarrow \rho_h f \in I_H$

$\supseteq$

$h \in \{ \ \} \Rightarrow \rho_h f(x) = 0 \quad \forall x \in H \quad \forall f \in I_H$

$\Rightarrow f(xh) = 0 \quad \forall x \in H \quad \forall f \in I_H$

$\Rightarrow$ in particular $f(h) = 0 \quad \forall f \in I_H$

$\Rightarrow h \in V(I_H)$ — zero set of I_H

$= H$