Homotopy theory of Kac-Moody groups — Nitu

G – Compact Lie group, simple, simply connected (eg: $SU(n)$)

$T \leq G$ the maximal torus of rank n (eg: $T = \overset{n-1}{S^1 \times \cdots \times S^1}$, rank $n-1$)

Weyl group: $W := N_G(T)/T$, $W \curvearrowright T$ and so $W \curvearrowright \pi_1(T)$

W is a finite group, reflection group generated by n-reflections

G can be recovered from this action.

Rank 2-example

$a, b \in \mathbb{Z}_{>0}$, $W(a,b) = \left\langle r_1 = \begin{pmatrix} -1 & b \\ 0 & 1 \end{pmatrix};\ r_2 = \begin{pmatrix} 1 & 0 \\ a & -1 \end{pmatrix} \right\rangle$ ⟵ is a dihedral group

$$r_1 r_2 = \begin{pmatrix} -1+ab & -b \\ a & -1 \end{pmatrix}, \quad (r_1 r_2)^2 = \Big(1+ab+a^2b^2 \quad 2b-ab^2$$

$$(r_1 r_2)^n = \begin{pmatrix} d_{2k+1} & -d_{2k} \\ c_{2k} & -c_{2k-1} \end{pmatrix} \qquad \begin{matrix} c_0 = d_0 = 0 \\ c_1 = d_1 = 1 \end{matrix} \qquad \begin{matrix} c_{i+1} = a d_i - c_{i-1} \\ d_{i+1} = b c_i - d_{i-1} \end{matrix}$$

eg:

c_i	$(a,b) = (1,1)$	$(a,b) = (2,2)$	$(3,3)$
	0	0	0
	1	1	1
	1	2	3
	0	3	8
	-1	4	21
	-1	5	55
	0	6	144
	1		
	1		

even fibonacci nos.

Q. When is $W(a,b)$ finite?

$(a,b) = (1,1)$	$(r_1r_2)^3 = 1$	$G = SU(3)$
$(2,1)$	$(r_1r_2)^4 = 1$	$G = Spin(5)$
$(3,1)$	$(r_1r_2)^6 = 1$	$G = G_2$

For all else $|W(a,b)|$ is infinite.

Q. $D_\infty = \mathbb{Z} \ltimes \mathbb{Z}/2$ infinite dihedral

What are the quotients?

Q. Is there a simply connected topological group whose Weyl group is $W(a,b)$? Yes!

$K(a,b)$ is pushout (amalgam colimit)

$$\begin{tikzcd} T \arrow[r, "\begin{pmatrix} 0 & 1 \\ 1 & -b/2 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & 0 \\ -a/2 & 1 \end{pmatrix}"'] & S^1 \times SU(2) \arrow[d] \\ S^1 \times SU(2) \arrow[r] & K(a,b) \end{tikzcd}$$

in the category of topological groups

The Weyl group of $K(a,b)$ is $W(a,b)$!

What do know about $K(a,b)$?

$$H^{2k}(K(a,b);\mathbb{Z}) \cong H^{2k+3}(K(a,b);\mathbb{Z}) = \mathbb{Z}/(c_k,d_k)\ \text{gcd}$$

p prime,

$H_*(K(a,b);\mathbb{F}_p) = E(x_3; x_{2m-1}) \otimes \mathbb{F}_p[X_{2m}]$ m = smallest integer s.t. that $p \mid (c_k, d_k)$

$H^{2k}(K(a,b)/T;\mathbb{Z}) = \mathbb{Z}\langle S_k\rangle \oplus \mathbb{Z}\langle \tau_k\rangle \quad k>0 \rightsquigarrow$ Schubert basis

$$S_i \cup S_j = c(i,j)\, S_{i+j}, \quad c(i,j) = \frac{c_{i+j}\, c_{i+j-1}\cdots c_1}{c_i\, c_{i-1}\cdots c_1\; c_j\, c_{j-1}\cdots c_1}$$

$\longleftarrow$ This should be an integer always.

$\rightsquigarrow$ Generalised binomial coefficients

$$\begin{tikzcd} H^*(BK(a,b);\mathbb{F}_p) = \mathbb{F}_p[X_4;X_{2m}]\otimes E(x_{2m}) \arrow[d] \arrow[dr, "p>2"] & \\ H^*(BT;\mathbb{F}_p) \arrow[r] & H^*(BT;\mathbb{F}_p)^{W(a,b)} \end{tikzcd}$$

eg: $a=b=2$, $c_k=d_k=k$, $H^*(BK(2,2);\mathbb{F}_p) = \mathbb{F}_p[x_4, x_{2p}] \oplus E(x_{2p+1})$

$BK(2,2) = LBSU(2)\langle 3\rangle$

$x_4 \longleftrightarrow ax^2 + by^2 - abxy \in H^4(BT;\mathbb{F}_p)$

$H^*(BK(a,b);\mathbb{Q}) = \mathbb{Q}[X_4]$

Generalisation to higher rank:

Let $\mathfrak{h} = \bigoplus_{i=1}^{n} \mathbb{Z}h_i$

n-reflections $\quad r_i(h_j) = h_j - a_{ji}h_i \quad ; a_{ii}=2$

$A = (a_{ij})$

Assumptions: $a_{ij}=0 \iff a_{ji}=0$

$a_{ij}\le 0 \quad (i\neq j)$

Fact: if $J \subseteq \{1,2,\dots,n\}$ such that $W(J) = \langle r_j;\, j\in J\rangle$ is finite, then it is the Weyl group of a compact Lie group $K(J) \longleftarrow$ functorial

if $J=\phi \quad K(\phi) = \mathfrak{h}\otimes\mathbb{R}/\mathbb{Z}$

$K(\phi) \subseteq K(J)$ is a maximal torus $\forall J$

Def^n: Let $\mathcal{C}$ = poset of $J\subseteq \{1,2,\dots,n\}$ s.t. $|W(J)|<\infty$

Define $K(A) = \operatorname{colim}_{\mathcal{C}} K(J) \rightsquigarrow$ category of topological groups

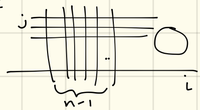
Properties of $K(A)$:

- $W(A) = \langle r_i \,;\, i \in \{1,2,\dots,n\} \rangle$ is the Weyl group of $K(A)$
- $H^{2k}(K(A)/T; \mathbb{Z})$ = free indexed on elements of $W(A)$ of length k
  - Schubert basis
- $H_*(K(A); \mathbb{F}_p)$ - finitely generated ring
- $\exists\ E(x_{2i_1-1}, x_{2i_2-1}, \dots, x_{2i_n-1}) \subseteq H_*(F(A); \mathbb{F}_p)$
  abelian Hopf algebra
- $K(A)$ is always simply connected
- $H_*(K(A); \mathbb{F}_p) \underset{\mathcal{H}}{\otimes} \mathbb{F}_p$ is evenly graded. This is the image of $H_*(K(A)) \subseteq H_*(K(A)/T)$

Facts about $BK(A)$

$$\operatorname{hocolim}_{\mathcal{L}} BK(J) \xrightarrow{\sim} BK(A)$$
$$\operatorname{hocolim}_{\mathcal{L}} BW(J) \xrightarrow{\sim} BW(A)$$

This gives a Mayer Vietoris SS, $\varprojlim_{\mathcal{L}} H^j(BK(J)) \Rightarrow H^{i+j}(BK(A))$



$\Rightarrow$ $H^*(BK(A); \mathbb{F}_p)$ is finitely generated as an algebra

if $p > n+1$ then SS collapses. ($Gl_n(\mathbb{Z})$ cannot have torsion $> n+1$)

if $p > n+1$ $H^*(BK(X)); \mathbb{F}_p) \twoheadrightarrow H^*(BT; \mathbb{F}_p)^W$

$\cdot\ BN_K(T) \longrightarrow BK(A)$
admits a stable splitting

$K(A)/N(T)$ is not a finite complex acyclic if $p > n+1$.

Q. Which lattices give rise to interesting Kac-Moody groups.