1)

$\mathrm{Tor}_0^{\bullet}(A,B)$

- $A=\mathbb{Z}$, $\longrightarrow B$
- $A=\mathbb{Q}$, $\longrightarrow B\otimes\mathbb{Q} = B_{free}\otimes\mathbb{Q}$
- $A=\mathbb{Z}/n\mathbb{Z} \rightarrow B/nB$
- $A=\mathbb{Q}/\mathbb{Z} \longrightarrow B\otimes\mathbb{Q}/\mathbb{Z} = B_{free}\otimes\mathbb{Q}/\mathbb{Z}$ $\quad\because\ \mathbb{Q}/\mathbb{Z}$ is divisible

$\mathrm{Tor}_1(A,B)$

$\rightarrow A=\mathbb{Z},\mathbb{Q} \longrightarrow 0$ $\quad$ Torsion free

$\rightarrow A=\mathbb{Z}/n\mathbb{Z} \rightarrow {}_nB$

$\rightarrow A=\mathbb{Q}/\mathbb{Z} \rightarrow B_{torsion}$ $\quad$ by looking at $0\rightarrow\mathbb{Z}\rightarrow\mathbb{Q}\rightarrow\mathbb{Q}/\mathbb{Z}\rightarrow 0$

$\mathrm{Ext}_0(A,B)$

$\rightarrow A=\mathbb{Z}$ $\quad \mathrm{Hom}(\mathbb{Z},B) = B$ $\quad$ ~~$\mathrm{Hom}(B,\mathbb{Z}) = \mathrm{Hom}(B_{free},\mathbb{Z})$~~

$\rightarrow A=\mathbb{Z}/n\mathbb{Z}$ $\quad \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z},B) = {}_nB$

$\rightarrow A=\mathbb{Q}$ $\quad \mathrm{Hom}(\mathbb{Q},B) =$ ~~Hom~~ ?

$\rightarrow A=\mathbb{Q}/\mathbb{Z}$ $\quad \mathrm{Hom}(\mathbb{Q}/\mathbb{Z},B)$

$= \mathrm{Hom}(\bigoplus \mathbb{Z}/p^\infty, B)$

$= \prod_p \mathrm{Hom}(\mathbb{Z}/p^\infty, B)$

$= \prod_p ({}_pB \leftarrow {}_{p^2}B \leftarrow {}_{p^3}B \leftarrow \cdots)$

when $B=\mathbb{Q}/\mathbb{Z}$ this inverse limit is the p-adic integers!

$\rightarrow B=\mathbb{Z}$ $\quad \mathrm{Hom}(A,\mathbb{Z}) = \mathrm{Hom}(A_{free},\mathbb{Z}) \cong$

$\rightarrow B=\mathbb{Q}$ $\quad \mathrm{Hom}(A,\mathbb{Q}) = \mathrm{Hom}(A_{free},\mathbb{Q}) \cong$

$\rightarrow B=\mathbb{Z}/n\mathbb{Z}$ $\quad \mathrm{Hom}(A,\mathbb{Z}/n) =$ ~~$\mathrm{Hom}({}_nA,\mathbb{Z}/n)$~~

$\rightarrow B=\mathbb{Q}/\mathbb{Z}$ $\quad \mathrm{Hom}(A,\mathbb{Q}/\mathbb{Z}) = ?$

- $\mathrm{Ext}^1(A,B)$:

  $\to A = \mathbb{Z}$ $\quad \mathrm{Ext}^1(\mathbb{Z},B) = 0 \quad \because \mathbb{Z}$ is projective

  $\to A = \mathbb{Z}/n\mathbb{Z}$ $\quad \mathrm{Ext}^1(\mathbb{Z}/n\mathbb{Z}, B) = B/nB \qquad 0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$

$$0 \to \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, B) \to B \xrightarrow{n} B \to \mathrm{Ext}^1(\mathbb{Z}/n\mathbb{Z}, B) \to \underset{0}{\underset{\|}{\mathrm{Ext}^1(\mathbb{Z},B)}}$$

  $\to A = \mathbb{Q}$ $\quad \mathrm{Ext}^1(\mathbb{Q},B) = ?$

  $\to A = \mathbb{Q}/\mathbb{Z}$ $\quad \mathrm{Ext}^1(\mathbb{Q}/\mathbb{Z},B) = ?$

we ~~know~~ have iso $\mathbb{Q} \xrightarrow{n} \mathbb{Q}$

So we must have iso $\mathrm{Ext}^1(\mathbb{Q},B) \xleftrightarrow{n} \mathrm{Ext}^1(\mathbb{Q},B)$

Q. For what abelian groups G do we have $G \xleftrightarrow{n} G$ isomorphism?

$\to$ G should be divisible i.e. G ~~also~~ injective $\|$

Same goes for $\mathrm{Ext}^1(\mathbb{Q}/\mathbb{Z}, B)$.

In general, for I divisible/injective: $\mathrm{Ext}^1(I,B)$ is injective $\quad$ X see below

---

$\to B = \mathbb{Z}$ $\quad \mathrm{Ext}^1(A,\mathbb{Z}) = \dfrac{\mathrm{Hom}(A,\mathbb{Q}/\mathbb{Z})}{(\mathrm{Hom}(A,\mathbb{Q})/\mathrm{Hom}(A,\mathbb{Z}))} \qquad 0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0.$ is an injective resolution

$\to B = \mathbb{Z}/n\mathbb{Z}$ $\quad \mathrm{Ext}^1(A,\mathbb{Z}/n\mathbb{Z}) = ?$ $\quad \mathrm{coker}(\mathrm{Ext}^1(A,\mathbb{Z}) \xrightarrow{n} \mathrm{Ext}^1(A,\mathbb{Z})) \qquad 0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n \to 0$

$\to B = \mathbb{Q}$ $\quad \mathrm{Ext}^1(A,\mathbb{Q}) = 0$

$\to B = \mathbb{Q}/\mathbb{Z}$ $\quad \mathrm{Ext}^1(A,\mathbb{Q}/\mathbb{Z}) = 0$

$\Big\}$ $\mathbb{Q}, \mathbb{Q}/\mathbb{Z}$ injective

G divisible means $G = nG$. It does not mean $G \xrightarrow{n} G$ is an isomorphism.

G divisible + torsionfree $\Rightarrow$ $\mathrm{Ext}^1(G,B)$ is divisible.

1 b) Any finite $\mathbb{Z}/n$ module is also a ~~$\mathbb{Z}$~~ finite $\mathbb{Z}$-module

By structure th^m: $\bigoplus_{m|d} (\mathbb{Z}/m)^{i_m}$ for m prime powers

so enough to calculate $\mathrm{Tor}(\mathbb{Z}/p^k, \mathbb{Z}/q^l)$ for $p^k, q^l | n$

If $n = \alpha\beta$, then as given in Weibel

$$\to \mathbb{Z}/n \xrightarrow{\beta} \mathbb{Z}/n \xrightarrow{\alpha} \mathbb{Z}/n \xrightarrow{1} \mathbb{Z}/\alpha \to 0$$

is a projective resolution of $\mathbb{Z}/\alpha$ so that,

$$\to \mathrm{Tor}_*^{\mathbb{Z}/n}(\mathbb{Z}/\alpha, B) = \cancel{\begin{cases} B/\alpha B & \text{if } * = 0 \\ \alpha B / \beta B \end{cases}}$$

$$= \begin{cases} B/\alpha B & * = 0 \\ {}_\alpha B / \beta B & * \text{ odd} \\ {}_\beta B / \alpha B & * \text{ even} \end{cases}$$

$\to \mathrm{Tor}_*^{\mathbb{Z}/n}(\mathbb{Z}/p^k, \mathbb{Z}/q^l) = 0$ if $p \neq q$ Because q becomes invertible

else if $p = q$

~~$= \mathbb{Z}/p^{\min(k,l)}$ / $* = p$ / Let $p^m || n$.~~

Let $p^m || n$. WALOG assum $k \leq l$

$\mathbb{Z}/p^k \quad * = 0$

for $*$ odd, ${}_{(p^k)}^{\;l}\mathbb{Z}/p^l \big/ p^{m-k}\mathbb{Z}/p^l = \mathbb{Z}/p^k \big/ \mathbb{Z}/p^{l+k-m}$ if $l+k > m$

$= \begin{cases} \mathbb{Z}/p^{m-l} & \text{if } k > m-l \\ \mathbb{Z}/p^k & \text{if } k \leq m-l \end{cases}$ $= \mathbb{Z}/p^k$ if $l+k \leq m$

for $*$ even, $\dfrac{\cancel{\mathbb{Z}/p^k}}{p}$?

$= \mathbb{Z}/p^{\min(k, l, m-k, m-l)}$

first we calculate $\mathrm{Tor}_*^{\mathbb{Z}/n}(\mathbb{Z}/p^k, \mathbb{Z}/p^k)$:

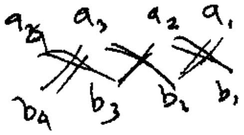
$$\left.\begin{array}{l} \\ \cancel{\mathrm{Tor}^{\mathbb{Z}}} \quad = \begin{cases} \mathbb{Z}/p^k & * = 0 \\ \mathbb{Z}/p^{\min(k,m-k)} & * > 0 \end{cases} \end{array}\right] \text{unnecessary}$$

Now we the short exact sequences:

$$0 \to \mathbb{Z}/p^k \to \mathbb{Z}/p^m \to \mathbb{Z}/p^{m-k} \to 0 \qquad \mathbb{Z}/p^m \text{ is projective}$$

$$0 \to \mathbb{Z}/p^{m-k} \to \mathbb{Z}/p^m \to \mathbb{Z}/p^k \to 0$$

Gives:



$$\mathrm{Tor}_{i+1}(\mathbb{Z}/p^{m-k}, B) = \mathrm{Tor}_i(\mathbb{Z}/p^k, B) \qquad \text{for } i > 0$$

$$\mathrm{Tor}_{i+1}(\mathbb{Z}/p^k, B) = \mathrm{Tor}_i(\mathbb{Z}/p^{m-k}, B) \qquad \text{for } i > 0$$

Combining: $\mathrm{Tor}_i(\mathbb{Z}/p^k, \mathbb{Z}/p^l) = \begin{cases} \mathbb{Z}/p^{\min(k,l)} & i = 0 \\ \mathbb{Z}/p^{\min(k,l,m-k,m-l)} & i > 0 \end{cases}$

Error

Similarly for Ext we are only concerned with $\mathrm{Ext}(\mathbb{Z}/p^k, \mathbb{Z}/q^l)$.
Look at projective resolution of $\mathbb{Z}/p^k$:

$$\cdots \to \mathbb{Z}/p^m \xrightarrow{p^{k}} \mathbb{Z}/p^m \xrightarrow{p^{m-k}} \mathbb{Z}/p^m \xrightarrow{p^{k}} \mathbb{Z}/p^k$$

Hom ~~with~~ into $\mathbb{Z}/q^l$:

$$\leftarrow \mathrm{Hom}(\mathbb{Z}/p^m, \mathbb{Z}/q^l) \xleftarrow{p^{m-k}} \mathrm{Hom}(\mathbb{Z}/p^m, \mathbb{Z}/q^l) \xleftarrow{p^{k}} \mathrm{Hom}(\mathbb{Z}/p^m, \mathbb{Z}/q^l) \leftarrow 0$$

Again if $p \neq q$, all Hom groups are 0. (as $\mathbb{Z}/n$ modules)

$$\mathrm{Hom}(\mathbb{Z}/p^m, \mathbb{Z}/p^l) = \begin{cases} \mathbb{Z}/p^l & \text{if } l \leq m \leftarrow \text{this is always true} \\ \cancel{\text{if } m \leq l} \end{cases}$$

This very much like Tor

$$\mathrm{Ext}^0(\mathbb{Z}/p^k, \mathbb{Z}/p^l) = \mathbb{Z}/p^{\min(k,l)}$$

Ext = Tor !!

2. Let I be a small category.

$(L, R, \mathcal{A}, \mathcal{B})$ -- adjunction.

$\varphi : I \to \mathcal{B}$ be a functor with a $\varprojlim$

Then

$R\circ\varphi$ also has a $\varprojlim$ & $\varprojlim R\circ\varphi = R \varprojlim \varphi$

Unit-counit:

Natural transforms: $\varepsilon : LR \longrightarrow 1_{\mathcal{B}}$

$\eta : 1_{\mathcal{A}} \longrightarrow RL$

(For each $i \in I$, we have maps $\varprojlim \varphi \xrightarrow{\pi_i} \varphi i$)

Next, we are given an object $M \in \mathcal{A}$

and maps $M \xrightarrow{\tau_i} R\varphi i$ compatible with I.

Push τ_i by L: $LM \xrightarrow{L\tau_i} LR\varphi i$

Compose with ε and factor through $\varprojlim$:

$$\begin{tikzcd} LM \arrow[r, "L\tau_i"] \arrow[dr, bend right, "\sigma"'] & LR\varphi i \arrow[r, "\varepsilon(\varphi i)"] & \varphi i \\ & \varprojlim \varphi \arrow[ur, bend right, "\pi_i"'] & \end{tikzcd}$$

$$(\varepsilon(\varphi i))\circ L\tau_i = \pi_i \cdot \sigma$$

Push σ back by R and use η:

$$\begin{tikzcd} RLM \arrow[d, "R\sigma"'] & M \arrow[l, "\eta M"'] \\ R\varprojlim \varphi & \end{tikzcd}$$

Need to check:

$$\begin{tikzcd} M \arrow[rr, "\tau_i"] \arrow[dr, bend right, "R\sigma\circ\eta M"'] & & R\varphi i \\ & R\varprojlim\varphi \arrow[ur, bend right, "R\pi_i"'] & \end{tikzcd}$$

~~But we know~~

$$\begin{tikzcd} M \arrow[r, "\eta"] & RLM \arrow[d, "RL\tau_i"] \\ R(\varphi i) & RLR(\varphi i) \arrow[l, "R\varepsilon(\varphi i)"] \end{tikzcd}$$

$$\begin{aligned} \tau_i &= R\pi_i \circ R\sigma \circ \eta M \\ &= R[\varepsilon(\varphi i)\circ L\tau_i]\circ \eta M \\ &= R\varepsilon(\varphi i) \circ RL\tau_i \circ \eta M \\ &= R\varepsilon(\varphi i) \circ \eta(R(\varphi i)) \circ \tau_i \qquad \because \eta \text{ is natural transform} \\ &= 1_{\varphi i}\circ \tau_i = \tau_i \end{aligned}$$

Deriving Properties of Units & Counits (ε, η):

we have natural iso $\psi: \mathrm{Hom}_B(LX, Y) \xrightarrow{\simeq} \mathrm{Hom}_A(X, RY)$ $\qquad X \in A,\ Y \in B$

Take $Y = LX$, image of $1_{LX} = X \xrightarrow[\eta]{} RLX$ i.e. $\eta X = \psi 1_{LX}$

$X = RY$, image of $1_{RY} = LRY \xrightarrow[\varepsilon]{} Y$ $\qquad \underset{\varepsilon(Y)}{\overset{\shortparallel}{\varepsilon(LRY)}} = \psi^{-1} 1_{RY}$

The most important property:

$$
\begin{tikzcd}
L \arrow[r, "L\eta"] \arrow[rr, bend right, "1_L"'] & LRL \arrow[r, "\varepsilon L"] & L
\end{tikzcd}
\qquad \cancel{1_L = \varepsilon L} \quad \cancel{1_{LX} = L(\eta X)}
$$

$1_L = (\varepsilon L) \cdot (L\eta)$

$1_{LX} = \varepsilon(LX) \cdot L(\eta X)$ ⟵ How to prove this?

$$
\cancel{
\begin{tikzcd}
X \arrow[r, "\eta X"] \arrow[d, squiggly] & RLX \arrow[r, "1_{RLX}"] \arrow[d, squiggly] & RLX \\
LX \arrow[r, "L(\eta X)"] & LRLX &
\end{tikzcd}
}
$$

$$
\begin{tikzcd}
\varepsilon L \cdot L\eta & \mathrm{Hom}_B(LX, LX) \arrow[r, phantom, "\simeq"] & \mathrm{Hom}_A(X, RLX) & \eta \\
\varepsilon L \arrow[u, mapsto] & \mathrm{Hom}_B(LRLX, LX) \arrow[u, "L\eta"] \arrow[r, phantom, "\simeq"] & \mathrm{Hom}_A(RLX, RLX) \arrow[u, "\eta"] & 1 \arrow[u, mapsto]
\end{tikzcd}
$$

Similarly

$$
\begin{tikzcd}
R\varepsilon \cdot \eta(RY) & \mathrm{Hom}_B(RY, RY) \arrow[r, phantom, "\simeq"] & \mathrm{Hom}_A(LRY, Y) \arrow[d, leftrightarrow, "\varepsilon Y"] & \varepsilon \\
R\eta(RY) \arrow[u, mapsto] & \mathrm{Hom}_A(RY, RLRY) \arrow[u, "R\varepsilon"] \arrow[r, phantom, "\simeq"] & \mathrm{Hom}_B(LRY, LRY) & 1 \arrow[u]
\end{tikzcd}
$$

$$
\text{So} \quad
\begin{tikzcd}
R \arrow[r, "\eta R"] \arrow[rr, bend right, "1_R"'] & RLR \arrow[r, "R\varepsilon"] & R
\end{tikzcd}
$$

$1_R = R\varepsilon \cdot \eta R$

- R – Need to ~~prove~~ R is left exact.

$(\mathcal{A}, \mathcal{B}, \mathcal{L}, \mathcal{R})$ $\quad R : \mathcal{B} \longrightarrow \mathcal{A}$ $\quad \mathcal{L} : \mathcal{A} \longrightarrow \mathcal{B}$

Given $\quad 0 \to X \to Y \to Z \to 0 \quad$ in $\mathcal{B}$, we need

$$0 \to RX \to RY \to RZ$$

we know for $T \in \mathcal{A}$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(\mathcal{L}T, X) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}(\mathcal{L}T, Y) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}(\mathcal{L}T, Z) \arrow[d, "\cong"] \\
0 \arrow[r] & \mathrm{Hom}(T, RX) \arrow[r] & \mathrm{Hom}(T, RY) \arrow[r] & \mathrm{Hom}(T, RZ)
\end{tikzcd}$$

Thinking of 0 as $\mathrm{Hom}(T, 0)$, we are reduced to showing

$$\mathrm{Hom}(T, X) \xrightarrow{f_*} \mathrm{Hom}(T, Y) \xrightarrow{g_*} \mathrm{Hom}(T, Z) \quad \text{exact}$$

$$\Rightarrow \quad X \xrightarrow{f} Y \xrightarrow{g} Z$$

- Take $T = X$, $g_* f_* (1_X) = 0$

$$\Rightarrow \quad g \circ f = 0$$

$$\Rightarrow \quad \ker g \supseteq \operatorname{im} f$$

- Take $T = \ker g$, look at inclusion $i : T \longrightarrow Y$

Then $\quad g_* i = 0 \quad \Rightarrow \quad i = f_* j$ for some $j : T \to X$

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \\
 & \ker g \arrow[ul, dashed, "j"] \arrow[u, "i"] &
\end{tikzcd}$$

i.e. $\ker g \subseteq \operatorname{im}(j \circ f)$
$\subseteq \operatorname{im}(f)$

∎

For arbitrary categories (not necessarily set)

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \\
 & \ker g \arrow[ul, "j"] \arrow[u] \\
 & 0 \arrow[u]
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
0 \arrow[r] & \operatorname{im} f \arrow[r, hook, "\phi"] & Y \\
 & & \ker g \arrow[ul, "f \cdot j"] \arrow[u, "i"] \\
 & & 0 \arrow[u]
\end{tikzcd}$$

3. ~~For~~ $S^{-1}R$ is a flat R-module $\leftarrow$ Th^m: 3.2.2 Weibel

S – central, multiplicatively closed

Look at the small category I

$$ob(I) = S$$

if $s_1 s_2 = s_3$ there is an arrow $s_1 \longrightarrow s_3$ indexed by s_2.

category? id is there

$$\begin{tikzcd} s_1 \arrow[r, "s_2"] \arrow[rr, bend right, "s_2 s_4"'] & s_3 \arrow[r, "s_4"] & s_5 \end{tikzcd}$$

composition is also there
need commutativity here

$\varphi : I \longrightarrow R\text{-mod}$

$\varphi(s_i) = R \qquad \varphi(s_i \xrightarrow{s_j} s_k) = R \xrightarrow{s_j} R$ multiplication by s_j

Claim: $\varinjlim \varphi = S^{-1}R$ $\qquad$ I – directed system?

$$\begin{tikzcd} s_1 \arrow[dr, "s_2"] & \\ & s_1 s_2 \\ s_2 \arrow[ur, "s_1"'] & \end{tikzcd}$$

$$\begin{tikzcd} s_1 \arrow[r, shift left, "s_2"] \arrow[r, shift right, "s_3"'] & s_4 \arrow[r, "s_1"] & s_1 s_4 \end{tikzcd}$$

$s_1(s_2 - s_3) = 0$

• Maps

$$\begin{array}{ccc} \varphi(s) & \longrightarrow & S^{-1}R \\ \wr\wr & & \\ R & & \\ 1 & \longmapsto & \frac{1}{s} \end{array}$$

• Natural map?

$$\begin{tikzcd} R \cong \varphi(s_1) \arrow[d, "s_2"'] \arrow[r] & S^{-1}R \\ R \cong \varphi(s_1 s_2) \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd} 1 \arrow[r, mapsto] \arrow[d, mapsto] & 1/s_1 \\ s_2 \arrow[ur, mapsto] & \end{tikzcd}$$

So we have a map $\varphi \longrightarrow \Delta S^{-1}R$

• Suppose we have maps

$$\begin{tikzcd} R \cong \varphi(s_1) \arrow[d, "s_2"'] \arrow[r, "f_{s_1}"] & M \\ R \cong \varphi(s_1 s_2) \arrow[ur, "f_{s_1 s_2}"'] & \end{tikzcd}$$

~~3. Th^m 3.2.2 :~~

$$\cancel{0 \to M \to N \to P \to 0 \quad \text{as } R\text{-modules}}$$

$$\cancel{\Rightarrow \quad 0 \to S^{-1}M \to S^{-1}N \to S^{-1}P \to 0}$$

~~We wish to construct a directed system I~~
~~so that $S^{-1}M = \varinjlim \varphi_M$ for appropriate functors~~

$$\cancel{0 \to \varphi_M \to \varphi_N \to \varphi_P \to 0}$$

Then we need to define a map: $S^{-1}R \to M$

Define: $f\left(\frac{r}{s}\right) = f_s(r)$

Well defined: $\frac{r_1}{s_1} = \frac{r_2}{s_2} \iff \exists s_3 .\ (s_2 r_1 - s_1 r_2) s_3 = 0$

$$\iff \quad s_2 s_3 r_1 = s_1 s_3 r_2$$

By the diagram:

$$\begin{tikzcd}
\varphi(s_1) \arrow[dr, "s_2 s_3"] \arrow[drr, bend left] & & \\
& \varphi(s_1 s_2 s_3) \arrow[r] & M \\
\varphi(s_2) \arrow[ur, "s_1 s_3"] \arrow[urr, bend right] & &
\end{tikzcd}$$

we get
$$f_{s_1}(r_1) = f_{s_1 s_2 s_3}(r_1 s_2 s_3)$$
$$= f_{s_1 s_2 s_3}(r_2 s_1 s_3)$$
$$= f_{s_2}(r_2) \qquad \square$$

R-module map: $f\left(r_1 \cdot \frac{r}{s}\right) = f_s(r_1 \cdot r) = r_1 f_s(r) = r_1 f\left(\frac{r}{s}\right)$

So, $\varinjlim \varphi \cong S^{-1}R$

Uniqueness: $S^{-1}R$ is generated by $\{1/s\}$ as R-module

Limit of flats is flat

R

- P_i flat $\Rightarrow$ $\varinjlim P$ is flat where $P : I \to R\text{-mod}$

~~to~~ Need to show

$$0 \to A_R \to B_R \to C_R \to 0$$

$$\Rightarrow 0 \to A \otimes \varinjlim P \xrightarrow{f} B \otimes \varinjlim P$$

Suppose $f\left(\sum a_i \otimes \bar{l}_i\right) = 0$

$\Rightarrow \sum_i f(a_i) \otimes \bar{l}_i = 0$ for $l_i \in \varinjlim P$

$\Rightarrow$ ~~for each l_i~~
We can pick a module P' so that $\bar{l}_i$ is the image of some element in P' l_i

Now we have diagram

$$\begin{tikzcd}
0 \arrow[r] & A \otimes P' \arrow[r, hook] \arrow[d] & B \otimes P' \arrow[d] \\
& A \otimes \varinjlim P \arrow[r] & B \otimes \varinjlim P
\end{tikzcd}$$

as P' is given flat

~~$\Rightarrow \sum f(a_i) \otimes l_i = 0$ for some $l_i \in P'$~~

Now since $\varinjlim B \otimes P = B \otimes \varinjlim P$ $\longrightarrow$??

we have $\overline{\sum f(a_i) \otimes l_i} = 0$

$\Rightarrow \exists P''$ s.t. $\sum f(a_i) \otimes l_i = 0$ for $l_i \in P''$

$\Rightarrow$ But P'' is flat $\Rightarrow$ $\sum a_i \otimes l_i = 0$

$\Rightarrow \sum a_i \otimes \bar{l}_i = 0$

$\Rightarrow 0 \to A \otimes \varinjlim P \to B \otimes \varinjlim P$

$\varinjlim (B \otimes P) =$ ~~lim~~ $B \otimes \varinjlim P$

This is because $B \otimes$ is a left adjoint and hence commutes with $\varinjlim$.

$$\begin{tikzcd}
B \otimes P_i \arrow[r] \arrow[d] & M \\
B \otimes P_j \arrow[ur] &
\end{tikzcd}$$