§ Spectral Sequences for holims and hocolims :

Th^m: There exist a SS's:

$$E^2_{p,q} = H_p(I; K_q(\mathcal{D})) \Rightarrow K_{p+q}(\operatorname{hocolim} \mathcal{D}) \qquad d^r \text{ has deg } (-r, r+1)$$

$$E_2^{p,q} = H^p(I, K^q(\mathcal{D})) \Rightarrow K^{p+q}(\operatorname{hocolim} \mathcal{D}) \qquad d_r \text{ has deg } (r, -r+1)$$

Q. This looks exactly like the Serre SS. Is there a direct connection?

Th^m: Modulo certain technical issues, there exists a SS

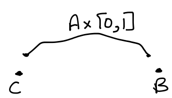
$$E_2^{p,q} = H^p(I; \pi_q \mathcal{D}) \Rightarrow \pi_{q-p}(\operatorname{holim} \mathcal{D}) \quad , \quad \deg d_r = (r, r-1)$$

The issue here is that π_0, π_1 need not be abelian groups.

eg: Pushouts:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \\ C & \end{tikzcd}$$

the hocolim is



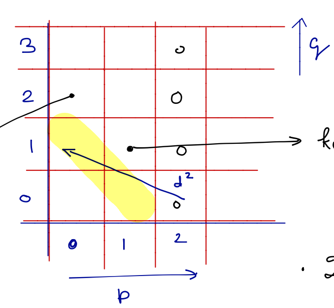
$= U \cup V \qquad U \simeq B,\ V \simeq C,\ U \cap V \simeq A$

$\Rightarrow$ we get an LES in homology / cohomology (for an homology theory K_*)

$$\to K_i(A) \to K_i(B) \oplus K_i(C) \to K_i(\operatorname{hocolim}) \to K_{i-1}(A) \to \cdots$$

- For the diagram:
$$\begin{tikzcd} K_i(A) \arrow[r] \arrow[d] & K_i(B) \\ K_i(C) & \end{tikzcd}$$
we already know the homologies

$E^2_{p,q}$:



$\operatorname{coker}\begin{bmatrix} K_2(A) \\ \downarrow f\oplus g \\ K_2(B) \oplus K_2(C) \end{bmatrix}$

$\ker[K_1(A) \xrightarrow{f \oplus g} K_1(B) \oplus K_1(C)]$

. The SS collapses on the second page $E^2 = E^\infty$

Recall: differentials always go from higher filtration to lower

$$\Rightarrow \quad 0 \to E^2_{0,n} \to K_n(\operatorname{colim}) \to E^2_{1,n-1} \to 0$$

$E^2_{0,n} = \operatorname{coker}(f \oplus g)$, $\quad E^2_{1,n-1} = \ker(f \oplus g)$

Gluing these SES together give us the LES for pushouts.

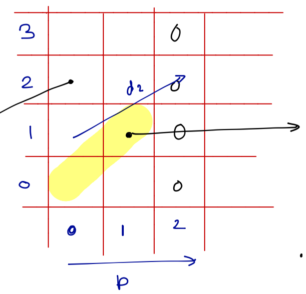
- Pullbacks: $\begin{tikzcd} & B \arrow[d] \\ C \arrow[r] & A \end{tikzcd}$ the holim is the lim of $\begin{tikzcd} & A^{[0,1]} \arrow[d, "\pi"] \\ B\times C \arrow[r] & A\times A \end{tikzcd}$ π is a fibration with fibre ΩA

$\Rightarrow$ we have a fibration $\begin{tikzcd} \Omega A \arrow[r] & \mathrm{holim} \arrow[d] \\ & B\times C \end{tikzcd}$ $\Rightarrow$ we get a LES for homotopy groups

$$\cdots \to \pi_{i+1}(A) \to \pi_i(\mathrm{holim}) \to \pi_i(B)\times\pi_i(C) \to \pi_i(A) \to \cdots$$

The SS:

$E_2^{p,q}$:



ker $\begin{bmatrix} \pi_2(B)\times\pi_2(C) \\ \downarrow f\times g \\ \pi_2(A) \end{bmatrix}$

coker $\pi_1(A) \xrightarrow{f\times g} \pi_1(B)\times\pi_1(C)$

(Assume abelian)

. The SS collapses on the second page $E^2 = E^\infty$

$$\Rightarrow \quad 0 \to \underset{\substack{\| \\ \mathrm{Coker}(f\times g)}}{E^1_{1,n+1}} \to \pi_n(\lim) \to \underset{\substack{\| \\ \ker(f\times g)}}{E^1_{0,n}} \to 0$$

Gluing these SES together give us the LES for pushouts.

- Towers: It is very common to study a space X by building a series of approximations:
$X = \underleftarrow{\mathrm{holim}}\, X_i$ for $\to X_{i+1} \to X_i \to \cdots \to X_1 \to X_0$

The spectral sequence in this case is the same as for pullback with $E^2_{1,n} = \lim^1(\pi_n(X_i))$
$\Rightarrow$ we get a SES: $0 \to \lim^1(\pi_{n+1}(X_i)) \to \pi_n(X) \to \underleftarrow{\lim}(\pi_n(X_i)) \to 0$