Recall: (1 lecture missing)

$$\begin{array}{c} \pi : X \to Y \\ \uparrow \\ G \text{ sheaf} \end{array}$$

Then, define:

$$(\pi^{-1}G)^{pre}(U) := \varinjlim_{V \supseteq \pi(U)} G(V)$$

$$\Updownarrow$$

$$\pi^{-1}(V) \subseteq U, \quad V \text{ open in } Y$$

Not a sheaf in general:

$$\pi^{-1}G := (\pi^{-1}G^{pre})^{sh}$$

- π^{-1} is more complicated to define, but π^{-1} has better formal properties than π_*

- π^{-1} is exact !!
  π_* is only left exact, in fact it is right adjoint to π^{-1}.

- eg: $U \hookrightarrow X$ open, $\begin{array}{c} G \\ \downarrow \\ X \end{array}$ sheaf
  then $\pi^{-1}G = G|_U$

Big fact: $\mathrm{Hom}_X(\pi^{-1}G, F) = \mathrm{Hom}_Y(G, \pi_* F)$

What are the derived functors of π_*? Some relative version of Sheaf Cohomology.

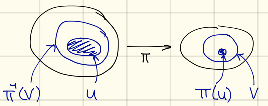
Does π^{-1} have a left adjoint too? Yes! it is called $\pi_!$ Read about this.

Proof:

$$\begin{tikzcd} \pi^{-1}G \arrow[rr] \arrow[dr] & & F \arrow[dl] \\ & X & \end{tikzcd} \quad \overset{\text{what is}}{\underset{\text{the corres?}}{\rightsquigarrow}} \quad \begin{tikzcd} G \arrow[rr] \arrow[dr] & & \pi_* F \arrow[dl] \\ & Y & \end{tikzcd} \qquad \begin{tikzcd} F \arrow[d] & G \arrow[d] \\ X \arrow[r, "\pi"] & Y \end{tikzcd}$$

$$(\pi^{-1}G)(U) = \varinjlim_{\pi^{-1}(V) \subseteq U} G(V) \rightsquigarrow$$

$$\begin{tikzcd} \pi^{-1}G(U) = \varinjlim_{\pi^{-1}(V) \subseteq U} G(V) \arrow[rr, "f_U"] & & F(U) \\ (\pi^{-1}G)(\pi^{-1}V) = G(V) \arrow[u] \arrow[rr, "f_V"] & & F(\pi^{-1}(V)) \arrow[u] \end{tikzcd}$$



$$(\pi_* F)(V) = F(\pi^{-1}(V)) \rightsquigarrow G(V) \xrightarrow{f_V} F(\pi^{-1}(V))$$

Back to morphisms of ringed spaces:

$(X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ $\quad \pi : X \longrightarrow Y$, $\begin{cases} \mathcal{O}_Y \longrightarrow \pi_* \mathcal{O}_X \\ \pi^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_X \end{cases}$

Gives us a category (RgSp)

· Open embedding. $(U, \mathcal{O}_U) \longrightarrow (Y, \mathcal{O}_Y)$ open inclusion, ...
(Also called immersion!)

Rem: $X \subseteq \bigcup_i U_i$ open cover

Can glue compatible morphisms $f_i : (U_i, \mathcal{O}_X|_{U_i}) \longrightarrow (Y, \mathcal{O}_Y)$

Rem: $\pi : X \longrightarrow Y$, $x \longmapsto y$ morphism of ringed spaces, $\pi^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_X$

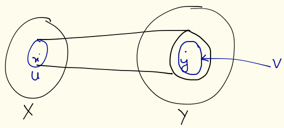
gives: $\mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x}$

as follows:

$$(\pi^{-1}\mathcal{O}_Y)_x = \varinjlim_{U \ni x} (\pi^{-1}\mathcal{O}_Y)(U)$$

$$= \varinjlim_{U \ni x} \left(\varinjlim_{\pi(U) \supseteq V} \mathcal{O}_Y(V) \right)$$

$$= \varinjlim_{\substack{V \ni y \\ \text{open}}} \mathcal{O}_Y(V) = \mathcal{O}_{Y,y}$$



Rem: $\pi : \operatorname{Spec} A \longrightarrow \operatorname{Spec} B$ coming from $\pi^{\#} : A \longrightarrow B$

Need: $\mathcal{O}_{\operatorname{Spec} B} \longrightarrow \pi_* \mathcal{O}_{\operatorname{Spec} A}$

Suffices to define on $D(g) \subseteq \operatorname{Spec} B$ for $g \in B$

$$\begin{array}{ccc} \mathcal{O}_{\operatorname{Spec} B}(D(g)) & \dashrightarrow & \pi_*(\mathcal{O}_{\operatorname{Spec} B}(D(g))) = \mathcal{O}_{\operatorname{Spec} A}(D(\pi^{\#}(g))) \\ \| & & \| \\ B_g & \longrightarrow & A_{\pi^{\#}(g)} \end{array}$$

This gives us a functor:

$\operatorname{Spec} : \text{Rings} \longrightarrow \text{RgSp}$

§ Morphisms of locally ringed spaces:

Def: A morphism of locally ringed spaces

$$\pi : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$

is morphism of ringed spaces s.t. $\forall x \in X$,

$$\pi^\# : \mathcal{O}_{Y, \pi(x)} \longrightarrow \mathcal{O}_{X,x}$$

is a local homomorphism of local rings.

Rem: π as above, $f \in \Gamma(V, \mathcal{O}_Y)$ then

pullback of vanishing locus of f in V = vanishing locus in X of pullback of f

Prop: given $\pi : \mathcal{O}_{\mathrm{Spec}\,A} \longrightarrow \mathcal{O}_{\mathrm{Spec}\,B}$ a morphism of locally ringed spaces, then π is the map induced by $\pi^\# : B \to A$. i.e.

$\mathrm{Spec} : \mathrm{Rings}^{op} \longrightarrow \mathrm{LocRgSp}$ is fully faithful. i.e.

$$\mathrm{Hom}(B, A) \cong \mathrm{Hom}_{l.r.s}(\mathrm{Spec}\,A, \mathrm{Spec}\,B).$$

Proof: Given $\pi : \mathcal{O}_{\mathrm{Spec}\,A} \longrightarrow \mathcal{O}_{\mathrm{Spec}\,B}$

Look at $\pi^\# : \underset{\parallel\atop B}{\Gamma(\mathrm{Spec}\,B)} \longrightarrow \underset{\parallel\atop A}{\Gamma(\mathrm{Spec}\,A)}$

$$\begin{tikzcd}
\mathcal{O}_{\mathrm{Spec}\,A} \arrow[d] & \mathcal{O}_{\mathrm{Spec}\,B} \arrow[l, "\pi"'] \arrow[d] \\
\mathrm{Spec}\,A \arrow[r, "\pi"] & \mathrm{Spec}\,B \\
\mathfrak{p} \arrow[r, mapsto] & \pi(\mathfrak{p})
\end{tikzcd}$$

Enough to describe the map on principal opens.

Map on topological spaces: *Claim:* $\pi(\mathfrak{p}) = (\pi^\#)^{-1}(\mathfrak{p})$ for $\mathfrak{p} \in \mathrm{Spec}\,A$

Proof: Look at map on stalks: $A_\mathfrak{p} \longleftarrow B_{\pi(\mathfrak{p})} : \pi^\#$

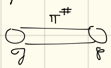
$(\pi^\#)^{-1}\mathfrak{p} = \pi(\mathfrak{p}) \quad \Leftarrow \quad \mathfrak{p} \longleftarrow \pi(\mathfrak{p}) \rightsquigarrow$ By locality

Map on sections: $\pi(D_f) = \pi\{\mathfrak{p} \subseteq A \mid f \notin \mathfrak{p}\}$

$$= \{(\pi^\#)^{-1}\mathfrak{p} \mid f \notin \mathfrak{p}\}$$

$$= \{\mathfrak{q} \mid (\pi^\#) f \notin$$

$\mathfrak{q} = (\pi^\#)^{-1}\mathfrak{p}$



Def: A morphism of schemes $(X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ is a morphism in category of l.r.s.