Def^n: Gallery: Γ- sequence $C_0, \dots C_\ell \in \mathcal{C}(\Sigma)$

C_i & C_{i+1} are adjacent.

ℓ - length of gallery

Minimal Gallery: Shortest length gallery joining ~~&~~ endpoints.
Geodesic

distance between two chambers is length of minimal gallery joining the two.

Propos: Any two chambers can be connected by a gallery. Hence by a minimal gallery. $d(C,D) = \#$ hyperplanes seperating C, D.

Prop. A gallery is minimal $\Leftrightarrow$ it crosses each wall exactly once.

~~Polyhedra~~:

Def^n: Intersections of closed half spaces. - Polyhedra - Δ

Bounded Polyhedra - Polytope

- If $F \in \mathcal{F}(\Sigma)$, $F \cap \Delta \neq \{\}$ then $F \subseteq \Delta$.
  i.e. Δ is union of faces.

Cones, Polyhedral Cones

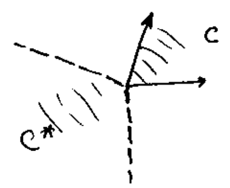
- A vector $u \in C$ cone is simple iff it cannot be written as +ve linear combination of vectors not colinear with u.

Proposition: Let Π finite, +ve set (some hyperplane has Π on one side)
no 3 vectors colinear.
C - cone generated by Π, Then,
Π - contains a unique simple set of generators of C.

Def^n: Dual Cone:

Cone C, Dual - $C^* = \{u \in \mathbb{R}^n \mid \langle u, v \rangle \leq 0 \ \forall v \in C\}$

eg:



Th^m: If C is a polyhedral cone, so is C^* and $C^{**} = C$.

Def^n: fin. gen. C is called simplicial if it is spanned by n - linearly independant vectors.

. Dual of a simplicial ~~comple~~ cone is simplicial.

- $\Pi = \{u_1 \ldots u_n\} \subseteq \mathbb{R}^n$.

  C_Π simplicial cone

  C_Π^* - generated by $\{u_1^* \ldots u_n^*\} \subseteq \mathbb{R}^n$ where u_i^* satisfy

  $$\langle u_i^*, u_j \rangle = \begin{cases} -1 & \text{if } i=j \\ 0 & \text{else} \end{cases}$$

  $H_i = \{x \in \mathbb{R}^n \mid \langle x, u_i^* \rangle = 0\}$

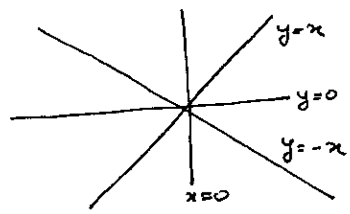
  for $1 \leq k \leq n$ $\quad C_k := C \cap H_k$

  Then, C_k is a simplicial cone in H_k generated by $\Pi \setminus \{u_i\}$. - co dim 1 face

Mirror Systems:

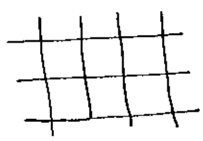
An arragement of hyperplanes Σ is mirror system if $H, H' \in \Sigma \Rightarrow s_H(H') \in \Sigma$

eg:

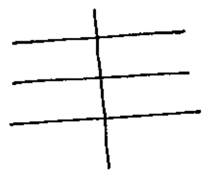


If Σ is infinite we want Σ to be locally finite ie any neighborhood of a point intersects only finite planes in Σ.

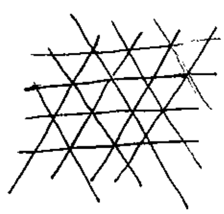
eg:



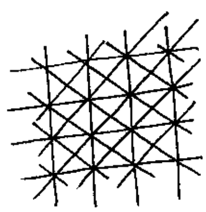
$\tilde{A}_1 \oplus \tilde{A}_1$



$A_1 \oplus \tilde{A}_1$



$\tilde{A}_2$



BC_2

Lemma. Σ - finite system of mirrors
Then Σ generates a finite subgroup of isometries.

Def^n: Spherical reflection / coxeter groups / finite reflection groups
G, $|G| < \infty$ ~~origin fixing isometries preserving origin~~
$G \leq O(n)$.

Th^m: 1) Finite reflection group has a fixed point.
2) All mirrors in a finite mirror system have a point in common.

Proof. 1) $|W| < \infty$ $\Rightarrow$ all orbits in $\mathbb{R}^n$ are finite

$W \curvearrowright \mathbb{R}^n$ Prove that this implies we have a fixed point.

2) fixed points of W are fixed by every reflection in a mirror. These fixed pts are common to all mirrors.

[Now on pick one of these fixed pts as origin.]

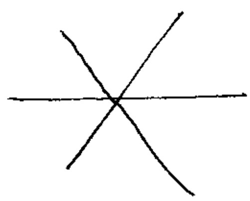
ex: 1) Dihedral groups:

$D_n = \langle a, b \mid a^2 = b^2 = (ab)^n = 1 \rangle$ Symmetries of an n-gon

$|D_n| = 2n$

$[D_n : \langle ab \rangle] = 2$

$n = 3$ $D_3 = S_3 = \{1, a, b, ab, ba, \underset{\substack{\parallel \\ bab}}{aba}\}$



Root Systems

s_H - reflection in H,

$v \in \mathbb{R}^n$ $v \perp H$. $\mathbb{R}^n = H \oplus \mathbb{R}v$

$s_H(u) = u - \dfrac{\langle u, v \rangle v}{\langle v, v \rangle}$

$\parallel$

$s_v(u)$

$\langle s_v(u), s_v(u') \rangle = \langle u, u' \rangle$, $s_v = s_{cv}$ $c \neq 0$

Def^n: Root System:

Φ finite set of vectors in $\mathbb{R}^n$

i) $\Phi \cap \mathbb{R}v = \{v, -v\}$ $\forall v \in \Phi$

ii) $s_v(\Phi) = \Phi$ $\forall v \in \Phi$