§ Reedy categories and Reedy Model Structure

Def: Reedy category $\mathcal{C}$:

- subcategories $\vec{\mathcal{C}}, \overleftarrow{\mathcal{C}}$, every object in $\mathcal{C}$ has a degree, $ob(\mathcal{C}) = ob(\vec{\mathcal{C}}) = ob(\overleftarrow{\mathcal{C}})$
- every non-identity map in $\vec{\mathcal{C}}$ raises degrees
- " " " $\overleftarrow{\mathcal{C}}$ " "

Every morphism $g \in \mathcal{C}$ factors as $g = \vec{g} \circ \overleftarrow{g}$.

eg Δ: $\deg [n] = n$, factors = face $\circ$ degeneracies

eg Δ^{op} also has a dual reedy structure

Def n-filtration of $\mathcal{C}$: $F^n\mathcal{C}$ = ob. of $\deg \leq n$

- This is a Reedy category: $\overrightarrow{F^n\mathcal{C}} = F^n\mathcal{C} \cap \vec{\mathcal{C}}$, ...

We have a filtration: $F^0\mathcal{C} \subseteq F^1\mathcal{C} \subseteq \dots \subseteq \operatorname{colim}_n F^n\mathcal{C} \cong \mathcal{C}$

- Diagrams indexed by Reedy category:

$X \subseteq \mathcal{M}^{\mathcal{C} \leftarrow \text{Reedy}}$ $\leftarrow$ want to construct inductively

- Can start with a functor: $X: F^0\mathcal{C} \longrightarrow \mathcal{M}$
- Suppose $X: F^{n-1}\mathcal{C} \longrightarrow \mathcal{M}$, let $\alpha \in ob\,\mathcal{C}$, $\deg \alpha = n$

to extend X to $F^n\mathcal{C} \longrightarrow \mathcal{M}$ and pick an object $X_\alpha \in ob\,\mathcal{M}$

- then for any cone under α we need a cone under X_α

i.e. Let $I^n: F^{n-1}\mathcal{C} \hookrightarrow F^n\mathcal{C}$ then need a $\operatorname{colim}_{I^n \downarrow \alpha} X \longrightarrow X_\alpha$

- similarly we need $X_\alpha \longrightarrow \lim_{\alpha \downarrow I^n} X$

- these maps will factor $\operatorname{colim}_{I^n\downarrow\alpha} X \longrightarrow X_\alpha \longrightarrow \lim_{\alpha\downarrow I^n} X$

Thm: $\mathcal{C}$ reedy, $\mathcal{M}$ bicomplete (all lim, colim exist), $X: F^{n-1}\mathcal{C} \longrightarrow \mathcal{M}$

If $\alpha \in ob\,\mathcal{C}$, $\deg \alpha = n$ we choose $X_\alpha \in \mathcal{M}$ and factorizations $\operatorname{colim}_{I^n\downarrow\alpha} X \longrightarrow X_\alpha \longrightarrow \lim_{\alpha\downarrow I^n} X$

then this uniquely determines $X: F^n\mathcal{C} \longrightarrow \mathcal{M}$.

- Latching and Matching objects:

Def: $\mathcal{C}$-Reedy

1) Latching category $= \partial(\mathcal{C}\downarrow\alpha)$ maps which raise degrees to α
2) Matching category $= \partial(\alpha\downarrow\mathcal{C})$ " " lower " " α

Def: $X \in \mathcal{M}^{\mathcal{C}}$, we can define objects in $\mathcal{M}^{\partial(I \downarrow i)}$ and $\mathcal{M}^{\partial(i \downarrow I)}$

$X_{\beta \to \alpha} = X_\beta$ & $X_{\alpha \to \beta} = X_\beta$

- Latching object : $\operatorname{colim}_{\partial(\vec{\mathcal{C}} \downarrow \alpha)} X =: L_\alpha X$
- Matching object : $\lim_{\partial(\alpha \downarrow \vec{\mathcal{C}})} X =: M_\alpha X$

Th^m: Can replace the diagram in the previous th^m by

$$L_\alpha X \longrightarrow X \longrightarrow M_\alpha X$$

- $X, Y \in \mathcal{M}^{\mathcal{C}}$ we can define a map $f : X \to Y$ inductively as well:

  a map $f : X \to Y$ is a collection of maps

$$\begin{tikzcd}
L_\alpha X \arrow[r] \arrow[d] & X_\alpha \arrow[r] \arrow[d, dashed] & M_\alpha X \arrow[d] \\
L_\alpha Y \arrow[r] & Y_\alpha \arrow[r] & M_\alpha X
\end{tikzcd}$$

- **homotopy lifting property:**

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & X \arrow[d] \\
B \arrow[r] \arrow[ur, dashed, "?h"] & Y
\end{tikzcd}$$

suppose h on $F^{n-1}\mathcal{C}$ then to lift to $F^n\mathcal{C}$ is same as solving lifting problem:

$$\begin{tikzcd}
A_\alpha \sqcup_{L_\alpha A} L_\alpha B \arrow[r] \arrow[d] & X_\alpha \arrow[d] \\
B_\alpha \arrow[r] \arrow[ur, dashed] & Y_\alpha \times_{M_\alpha Y} M_\alpha X
\end{tikzcd}$$

Th^m: $\exists$ a model structure on $\mathcal{M}^{\mathcal{C}}$:

- $A \longrightarrow B$ is cofib iff $\forall \alpha \in \mathcal{C}$ $\quad A_\alpha \sqcup_{L_\alpha A} L_\alpha B \longrightarrow B_\alpha$ is a cofib
- $X \longrightarrow Y$ is fib iff $\forall \alpha \in \mathcal{C}$ $\quad X_\alpha \longrightarrow Y_\alpha \times_{M_\alpha Y} M_\alpha Y$ is a fib
- w.e. is pointwise

Th^m: 1) If $X \to Y$ is a Reedy (co) fibration then $\forall \alpha$, f_α is a (co) fibration.

2) $f : X \to Y$ is a trivial co-fib $\Leftrightarrow$ $\forall \alpha$. $X_\alpha \sqcup_{L_\alpha X} L_\alpha Y \longrightarrow Y_\alpha$ are trivial cofib

and dually for trivial fibrations.

Prop: If $\mathcal{M}^I$ has a projective and Reedy model structure then $\mathcal{M}^I_{proj} \xrightarrow{\sim} \mathcal{M}^I_{Reedy}$ is Quillen equivalent.

Th^m: $\mathcal{M}^I \underset{c}{\overset{\operatorname{colim}}{\rightleftarrows}} \mathcal{M}$ is a Quillen pair $\Leftrightarrow$ $\partial(i \downarrow \overleftarrow{I})$ are empty or connected.

eg: If $A_1 \leftarrow A_0 \rightarrow A_2$, A_0 is cofibrant, $A_0 \rightarrow A_1$, $A_0 \rightarrow A_2$ cofibrations, $\mathrm{hocolim}\, A \xrightarrow{\simeq} \mathrm{colim}\, A$.