①

Orientable vector bundles

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, hook] & \xi \arrow[d] \\ & X \end{tikzcd}$$

ξ - orientable, if $\exists$ orientation

$\exists$ a class $\alpha_x \in H^n(\xi_x, \xi_{x,0})$ + compatibility

$\xi_{x,0} = \xi_x - \{0\}$

Orientation $\Rightarrow$ structure group $SO(n)$

Complex $\Rightarrow$ orientable

$$H^*(\xi, \xi_0^*) \longrightarrow H^*(\xi) \longrightarrow H^*(\xi_0) \longrightarrow H^{*+1}(\xi, \xi_0)$$

$$\begin{tikzcd} \xi \arrow[d] & \xi_0 \arrow[l, hook'] \arrow[d] \\ X & S(\xi) \arrow[l, "p"] \end{tikzcd} \qquad H^*(\xi) \cong H^*(X) \xrightarrow{p^*} H^*(S(\xi)) \cong H^*(\xi_0) \qquad \begin{tikzcd} S(\xi) \arrow[d, "p"] & S^{n-1} \arrow[l, hook'] \\ X \end{tikzcd}$$

One-point compactification of each fibre gives a S^n bundle - S^ξ

$$\begin{tikzcd} S^\xi \arrow[d] \\ X \arrow[u, bend left, "s"' , "0\text{ section}"] \arrow[u, bend right=60, "s'"', "\infty\text{ section}"] \end{tikzcd}$$

$H^*(\xi, \xi_0) \cong H^*(S^\xi, S'(X))$
$S'(X) = \operatorname{im} X$

Comes because
$H^*(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \cong H^*(S^n, \infty)$

$H^*(\xi, \xi_0) \cong H^*(\xi \cup S'(X), \xi_0 \cup S'(X)) \cong H^*(S^\xi, S'(X))$

$\xi_0 \cup S'(X)$ contractible to $S'(X)$

$$\begin{tikzcd} S^n \arrow[r] & S^\xi \arrow[d, "p"] \\ & X \end{tikzcd}$$

$$\pi_k(S^n) \longrightarrow \pi_k(S^\xi) \xrightarrow{p_*} \pi_k(X) \longrightarrow \pi_{k-1}(S^n)$$

$\Rightarrow$ $p_* \cong$ if $k < n$ $\Rightarrow$ (X, S) n-connected
$\twoheadrightarrow$ if $k = n$

$\Rightarrow$ $p_* \cong$ on H_* if $k < n$
$\twoheadrightarrow$ on H_* if $k = n$

$\Rightarrow$ $p^* \cong$ on H^* if $k < n$ Universal co-efficient th^m
$\hookrightarrow$ on H^* if $k = n$

$$\begin{tikzcd}
 & & H^*(s'(X)) \arrow[d, phantom, "\cong"] & \\
\to H^*(\xi,\xi_0) \arrow[r] & H^*(S^\xi) \arrow[ur, "s'^*"] \arrow[r] & H^*(X) \arrow[r] & H^{*+1}(\xi,\xi_0) \to
\end{tikzcd}$$

s' - section $\Rightarrow$ $p\circ s' = id$

$\Rightarrow$ $s'^* \cdot p^* = id$

$p^* \cong \Rightarrow s'^* \cong$

$p^* \hookrightarrow \Rightarrow s'^* \twoheadrightarrow$

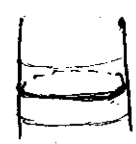
] use in the long exact sequence above

So we get

$$\boxed{\begin{array}{ll} H^k(\xi,\xi_0) = 0 & \text{for } k < n \\ H^n(\xi,\xi_0) = \ker(H^n(S^\xi) \to H^n(X)) & \end{array}}$$

Example:

1. $\xi = S^1 \times \mathbb{R}$



$$H^*(\xi,\xi_0) = \begin{cases} \mathbb{Z} & \text{if } * = 2,1 \\ 0 & \text{else} \end{cases}$$

$\cong$

$H^*(\xi,\xi_0')$

$\cong$

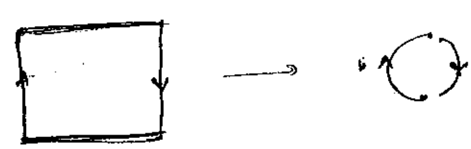
$\tilde{H}^*(\xi/\xi_0') \cong \tilde{H}^*(S^2 \vee S^1)$

2. $\xi =$ Mobius bundle

$H^*(\xi, \xi_0) \cong \tilde{H}^*(\mathbb{RP}^2)$

$\cong \mathbb{Z}/2$ if $* = 2$

0 else



Thom Isomorphism Th^m

- $\mathbb{R}^n \to \xi \downarrow X$ orientable $\iff$ $\exists\, \alpha \in H^n(\xi,\xi_0)$ s.t.

$$\begin{array}{c} H^n(\xi,\xi_0) \\ \downarrow i_x^* \\ H^n(\xi_x, \xi_{x_0}) \end{array} \quad i^*\alpha = \pm 1$$

- for ξ - oriented,

$$H^*(X) \cong H^*(\xi) \xrightarrow{\cup \alpha} H^{*+n}(\xi,\xi_0) \text{ is an isomorphism}$$

For Trivial bundle:

$\xi = B\times\mathbb{R}^n$, $\quad H^n(B\times\mathbb{R}^n, B\times\mathbb{R}^n_0) \cong \tilde{H}^n(B_+\wedge S^n)$

$B\times\mathbb{R}^n \to B$ $\quad \cong \tilde{H}^n(B\times S^n) \cancel{\quad} (B\times\{\infty\})$

$\quad \cong \cancel{H^n(B)\otimes H^n(S^n)} \,/\, \cancel{H^n(B)}$

$$B\times\{\infty\} \longrightarrow B\times S^n \longrightarrow B\times S^n / B\times\{\infty\}$$

$$\longleftarrow H^n(B\times\{\infty\}) \longleftarrow H^n(B\times S^n) \longleftarrow H^n(B\times S^n/B\times\{\infty\}) \longleftarrow$$

$H^n(B\times\{\infty\}) \cong H^n(B)$, $\quad H^n(B\times S^n) \cong H^n(B)\otimes H^0(S^n) \oplus H^0(B)\otimes H^n(S^n)$

$H^n(S^n) = \mathbb{Z}\alpha$

$$H^n\left(\frac{B\times S^n}{B\times\{\infty\}}\right) = \ker\left(H^0(B)\otimes H^n(S^n) \oplus H^n(B)\otimes H^0(S^n) \longrightarrow H^n(B)\right)$$

$$= H^0(B)\otimes H^n(S^n)$$

- Assume true for $U, V, U\cap V$

$$H^*(\xi U\cup V) \longleftrightarrow H^*(U)\oplus H^*(V) \longleftrightarrow H^*(U\cap V)$$

$$\downarrow \cup\,\alpha_{U\cup V} \qquad \downarrow \cup\,\alpha_U,\alpha_V \qquad \downarrow (\cup\,\alpha_{U\cap V})$$

$$\longrightarrow H^*(\xi_{U\cup V}, \xi_{U\cup V_0}) \longrightarrow H^*(\xi_U, \xi_{U_0})\oplus H^*(\xi_V, \xi_{V_0}) \longrightarrow H^*(\xi_{U\cap V}, \xi_{U\cap V_0}) \longrightarrow$$

$$(\alpha_U, \alpha_V) \longmapsto (\alpha_{U\cap V} - \alpha_{U\cap V}) = 0$$

$$\Rightarrow \exists\, \alpha_{U\cup V} \longmapsto (\alpha_U, \alpha_V) \longrightarrow 0$$

- So result is true for X-compact

- for $X = \varinjlim X_{(n)}$ $\quad X_{(n)}$ compact

$$\Rightarrow \mathrm{Hom}^*(X) \longrightarrow \varprojlim H^*(X_{(n)}) \longrightarrow 0$$

$$\uparrow$$

$$0 \longrightarrow \lim{}^1 (H^{*-1}(X_{(n)}))$$

Corresponding th^m for S^{n-1}-bundles

$$\begin{tikzcd} S^{n-1} \arrow[r] & S \arrow[d] \\ & X \end{tikzcd} \quad \text{orientable} \iff \exists\ \alpha \in \begin{tikzcd} H^{n-1}(S) \arrow[d, "i^*"] \\ H^{n-1}(S_x) \end{tikzcd} \quad i^*(\alpha) = \pm 1$$

$\phi \in H^n(\xi, \xi_0)$ is called the Thom class.

$$\begin{tikzcd} H^n(\xi,\xi_0) \arrow[r] \arrow[dr] & H^n(\xi) \arrow[d, "\cong"] \\ \phi \arrow[r, mapsto] & H^n(X) \end{tikzcd}$$

$\boxed{e(\xi) = \text{Euler class of } \xi.}$ with a chosen orientation.

- Naturality

$$\begin{tikzcd} f^*\xi \arrow[r] \arrow[d] & \xi \arrow[d] \\ Y \arrow[r, "f"] & X \end{tikzcd} \rightsquigarrow \begin{array}{l} f^*\xi \text{ is also oriented} \\ e(f^*\xi) = f^*(e(\xi)) \end{array}$$

$\phi(f^*\xi) = f^*\phi(\xi)$

~~$H^n(f^*\xi, f^*\xi_0)$ $H^n(\xi, \xi_0)$~~

$$\begin{tikzcd} \phi(f^*\xi) \arrow[ddd] & H^n(f^*\xi, f^*\xi_0) \arrow[d] & H^n(\xi,\xi_0) \arrow[l, "f^*"'] \arrow[d] & \phi(\xi) \arrow[dd] \\ & H^n(f^*\xi) \arrow[d] & H^n(\xi) \arrow[l, "f^*"'] \arrow[d] & \\ & H^n(Y) & H^n(X) \arrow[l, "f^*"'] & e(\xi) \\ e(f^*\xi) & & & \end{tikzcd}$$

$\Rightarrow \quad e(f^*\xi) = f^*e(\xi)$

- If ξ - $\mathbb{R}^{2n+1}$ vector bundle,

$$\begin{tikzcd} \mathbb{R}^{2n+1} \arrow[r] & \xi \arrow[d] \\ & X \end{tikzcd} \qquad -\mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n \text{ gives } \begin{tikzcd} \xi \arrow[rr] \arrow[dr] & & \xi \arrow[dl] \\ & X & \end{tikzcd} \quad \text{orientation reversing}$$

- Suppose $s: X \longrightarrow \xi$ is a section s.t. $s(x) \neq 0 \ \forall x$

  Then $e(\xi) = 0$.

- Proof:

$$
\begin{tikzcd}
H^n(\xi, \xi_0) \arrow[r] & H^n(\xi) \arrow[r] \arrow[d, "\cong"] & H^n(\xi_0) \arrow[dl, "s^*"] \\
 & H^n(X) &
\end{tikzcd}
$$

$\phi \longmapsto e$ $\square$

$s: X \longrightarrow \xi_0$

$$
\begin{tikzcd}
\xi_0 \arrow[r, "i"] & \xi \arrow[d, shift left, "p"] \\
 & X \arrow[ul, "s'"] \arrow[u, shift left, "s"]
\end{tikzcd}
$$

this map homotopic to 0-section

- $\underline{\underline{\text{Th}^m}}$ N^n - compact, oriented manifold C^∞

$e(TN) \in H^n(N)$ $\qquad$ $[N] \in H_n(N) \cong \mathbb{Z}$

$\quad \| $

$e(N)$

Then, $\langle e(N), [N] \rangle = \chi(N)$.

Proof:

$$\Delta: N \longrightarrow N \times N$$
$$x \longmapsto (x, x)$$

$$T(N \times N) \cong TN \oplus TN$$

$\nu_\Delta \cong$ Normal bundle of ΔN in $N \times N \cong TN$

$H^*(\nu_\Delta, \nu_{\Delta 0})$

In general if ν is normal bundle of $i: N \longrightarrow W$, Then $H^*(\nu, \nu_0) \cong H^*(W, W - N)$

$\Rightarrow \phi_\nu \cap [W] = [N]$

Read Milnor, Stasheff

Phodu Proof

(Simpler proof in Bott, Tu)

Milnor-Stasheff notes

- Homology, Cohomology

  $(C^n)\, C_n X$ - (co) - ~~cycles~~ chains

  $(Z^n)\, Z_n X$ - (co) - cycles

  $(B^n)\, B_n X$ - (co) - boundaries

- If $H_{n-1}(X)$ is free, $H^{n} \cong \mathrm{Hom}(H_n, G)$.

  - $\alpha \in \mathrm{Hom}(H_n(X), G)$ $\qquad H_n(X) = Z_n(X)/B_n(X)$

$$\begin{array}{c} \alpha : Z_n(X)/B_n(X) \longrightarrow G \\ \uparrow \\ Z_n(X) \end{array}$$

$$C_n(X)/Z_n(X) = \underbrace{B_{n-1}(X)}_{\text{free}} \hookrightarrow Z_{n-1}(X) \hookrightarrow C_{n-1}(X)$$

$$\begin{array}{ccccccccc} \Rightarrow & 0 \to B_{n-1}(X) & \to & C_n(X) & \to & Z_n(X) & \to & 0 & \\ & & & & & \downarrow & & & \\ G & \xleftarrow{\alpha} & Z_n/B_n(X) & \longleftarrow & & Z_n(X) & \leftarrow & B_n(X) & \leftarrow 0 \end{array}$$

Following the arrow we get an element of $H^n(X, G)$.

- $\alpha \in H^n(X, G)$ ~~defin~~

  we get $\tilde{\alpha} \in \mathrm{Hom}(H_n, G)$

  $\tilde{\alpha}([\sigma]) = \alpha(\sigma)$.

- If $\tilde{\alpha} \to 0 \Rightarrow \alpha(\sigma) = 0$ on all cycles

  Need to show $\alpha = \delta\beta \qquad \beta \in H^{n-1}(X, G)$.

  $\alpha : C_n(X) \longrightarrow G \qquad \alpha|\, Z_n(X) \longrightarrow 0$

  $\alpha = \delta\beta \equiv \alpha$ depends only on boundary

  $\beta : B_{n-1}(X) \longrightarrow G$

  $\beta(\tau) = \alpha(\sigma)$ for some σ, $\partial\sigma = \tau$

  $= \alpha \cdot \partial^{-1} \tau$

  $H_{n-1} = Z_{n-1}/B_{n-1}$ = free $\Rightarrow B_{n-1} \overset{\text{splits}}{\hookrightarrow} Z_{n-1} \hookrightarrow C_{n-1}$

  Extend β to C_{n-1}. ⊠

In general

$$0 \to \mathrm{Ext}(H_{n-1}(X), G) \longrightarrow H^n(X, G) \longrightarrow \mathrm{Hom}(H_n(X), G) \to 0.$$

• 9.3-f

$\xi \;\; \eta$ vector bundles over B $\dashrightarrow$ Tychonoff i.e. $\forall\, p \in B,\ V \subseteq B,\ V$ closed, $p \notin V$
$\exists f : B \to [0,1]$
$f^{-1}(0) = p, \quad f^{-1}(1) = V.$

$S(\xi) = \{$continuous $s : B \to \xi$, section$\}$

$C^0(B) = \{ f : B \to \mathbb{R},$ continuous$\}$

$S(\xi)$ is $C^0(B)$ module

1. $S(\xi \oplus \eta)$

$\Psi : S(\xi) \oplus S(\eta) \xrightarrow{\cong} S(\xi \oplus \eta)$

$\Psi(f\alpha + g\beta)(p) = f(p)\,\alpha(p) + g(p)\,\beta(p)$

Injectivity is clear

Surjectivity

$$\begin{tikzcd} \xi \oplus \eta \arrow[r, "\Delta^*"] \arrow[d] & \xi \times \eta \arrow[d] \\ B \arrow[r, "\Delta"] & B \times B \end{tikzcd} \qquad \begin{tikzcd} & \xi \oplus \eta \arrow[dl] \arrow[d] \arrow[dr] & \\ \eta \arrow[r] & B & \xi \arrow[l] \end{tikzcd}$$

$(\xi \oplus \eta)_p = (\xi_p, \eta_p)$

$\xi = B \times \mathbb{R}^n$

$\xi = \oplus^n \mathbb{1} \qquad \mathbb{1} = B \times \mathbb{R}$

Enough to show $\mathbb{1} = C^0(B)$ which is by def^n

So $\xi = C^0(B)^n$ free

2. If $\xi \oplus \eta = B \times \mathbb{R}^n$

$S(\xi) \oplus S(\eta) = (C^0 B)^n$

So $S(\xi), S(\eta)$ projective

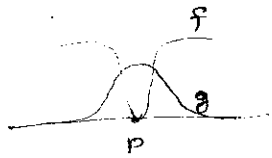
3. $\xi \xrightarrow{\tau} \eta$ τ isomorphism

$S(\xi) \xrightarrow{\tau_*} S(\eta)$ $\quad \tau_*$ isomorphism

$f \longmapsto \tau \circ f$

Given $\tau: S(\xi) \longrightarrow S(\eta)$ isomorphism

$\mathfrak{m}_p = \{ f : B \to \mathbb{R} \mid f(p) = 0 \} \subseteq C^0(B)$



maximal : $g \notin \mathfrak{m}_p \Rightarrow g(p) \neq 0$

$\exists\, U \ni p,\ U \subseteq B$, open s.t.

$0 \notin g(U)$.

let f be f^n seperating U^c and p, $f \in \mathfrak{m}_p$

$f^2 + g^2 > 0 \Rightarrow$ unit

$S(\xi) / \mathfrak{m}_p S(\xi)$ — module over $C^0(B)/\mathfrak{m}_p = \mathbb{R}$

Then, $\xi_p \cong S(\xi)/\mathfrak{m}_p S(\xi)$

$s(p) \longleftarrow [s]$

Define: $\tau^* : \xi \longrightarrow \eta$

$$\begin{array}{ccc} \xi_p & \longrightarrow & \eta_p \\ \cong & & \cong \\ S(\xi)/\mathfrak{m}_p S(\xi) & & S(\eta)/\mathfrak{m}_p S(\eta) \end{array}$$

Remains to show τ^* is continuous.

this you can do because,
τ^* takes sections to sections.

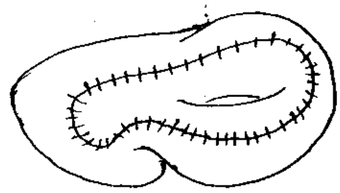
- Tubular Neighborhood Th^m:

$M \hookrightarrow A$ submanifold

$$\begin{array}{ccccc} TM & \hookrightarrow & i^*TA & \longrightarrow & \nu_i \cong i^*TA/TM \\ & \searrow & \swarrow & & \\ & & M & & \end{array}$$

ν_i ← normal bundle of $i: M \hookrightarrow A$

$\exists$ open set $U \supseteq M$ in A, such that $U \cong \nu_i$ with M as zero section.



Q. $f: U \longrightarrow V$ smooth

f bijective $\Rightarrow f$ diffeomorphism?

Claim: $\begin{array}{c} TM \\ \downarrow \\ M \end{array}$ $\langle e(TM), [M] \rangle = \chi(M)$

Proof: • By Tubular Nbd, $H^*(A, A-M) \cong H^*(\nu_i, \nu_i - M) \cong H^*(\nu_i, \nu_{i0})$

$H_{\dim(A)-*}(M) \cong$ Poincare Duality

$\dim \nu_i = \dim A - \dim M$

(Poincare dual of $i_*[M]$) = orientation of ν_i

$e(\nu_i) = i^*(\text{Poincare Dual}^{-1}(i_*[M]))$

$M \overset{\Delta}{\hookrightarrow} M \times M \qquad \nu_\Delta = TM$

- If co-efficient group is $\mathbb{Q}$,

$$H^*(M, \mathbb{Q}) \otimes H^{\dim M - *}(M, \mathbb{Q}) \longrightarrow \mathbb{Q}$$
$$(\alpha, \beta) \longmapsto (\alpha \cup \beta)[M]$$

Poincare duality $\Rightarrow$ non-singular bilinear ~~map~~ graded symmetric form

basis $H^*(M, \mathbb{Q}) \quad \alpha_1 \dots \alpha_r \qquad (\alpha_i \cup \alpha_j^*)[M] = \delta_{ij}$

$H^{\dim M - *}(M, \mathbb{Q}) \quad \alpha_1^* \dots \alpha_r^*$

• $H^*(M\times M) \cong H^*(M)\otimes H^*(M)$ in $\mathbb{Q}$ coefficients

$\{\alpha_i\}$ $\qquad$ $\{\alpha_i^*\}$ dual basis

elements of form $A_{ij}(\alpha_i\otimes\alpha_j^*)$

• τ - orientation class $\nu \in H^*(M\times M, M\times M - \Delta(M)) \longrightarrow H^*(M\times M)$

$\tau \cap [M\times M] = \Delta_*[M]$

$\tau = \sum_{i,j} A_{ij}\,\alpha_i^*\otimes\alpha_j$ & $A_{ij}=0$ if $|\alpha_i|\neq|\alpha_j|$

• $\langle(\alpha_i\otimes\alpha_j^*)\cup\tau,[M\times M]\rangle$

$=\langle\alpha_i\otimes\alpha_j^*,\tau\cap[M\times M]\rangle$

$=\langle\alpha_i\otimes\alpha_j^*,\Delta_*[M]\rangle$

$=\langle\Delta^*(\alpha_i\otimes\alpha_j^*),[M]\rangle$

$=\langle\alpha_i\cup\alpha_j^*,[M]\rangle$

$=\delta_{ij}$

$=\langle(\alpha_i\otimes\alpha_j^*)\cup\sum A_{kl}(\alpha_k^*\otimes\alpha_l),[M\times M]\rangle$

$=(-1)^{(\,)}A_{ij}\langle(\alpha_i\cup\alpha_i^*)\otimes(\alpha_j^*\cup\alpha_j),[M]\otimes[M]\rangle$

$=(-1)^{|\alpha_i|^2}A_{ij}\langle\alpha_i\cup\alpha_i^*,[M]\rangle\langle\alpha_j^*\cup\alpha_j,[M]\rangle$

$=A_{ij}(-1)^{n|\alpha_i|}$

$$A_{ij}=(-1)^{n|\alpha_i|}\delta_{ij}$$

$$\Rightarrow \tau=\sum_i(-1)^{n|\alpha_i|}\alpha_i^*\otimes\alpha_i$$

$$\Rightarrow e(TM)=\Delta^*\tau$$

$$=\sum_i(-1)^{n|\alpha_i|}(\alpha_i^*\cup\alpha_i)$$

$$\langle e(TM),[M]\rangle=\sum_i(-1)^{n|\alpha_i|}\langle\alpha_i^*\cup\alpha_i,[M]\rangle$$

$$=\sum_i(-1)^{|\alpha_i|}$$

$$=\chi(M)$$

⊠

Corollary: Hairy ball theorem.

$\phi(\xi \oplus \xi') = \phi(\xi) \cup \phi(\xi')$

$e(\xi \oplus \xi') = e(\xi) \cup e(\xi')$

- $$\begin{tikzcd} \arrow[r] & H^*(\xi, \xi_0) \arrow[r] & H^*(\xi) \arrow[r] \arrow[d, "\cong"] & H^*(\xi_0) \arrow[r] & {} \\ & H^{*-n}(X) \arrow[u, "\cong", "\text{Thom}"'] \arrow[r, bend left, "\cup e"] & H^*(X) & & \end{tikzcd}$$

$$\longrightarrow H^{*-n}(X) \xrightarrow{\cup e} H^*(X) \longrightarrow H^*(\xi_0) \longrightarrow$$ Gysin Sequence

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Stiefel Whitney Classes:

$\mathbb{R}^n \to \xi \to B$

$w_i(\xi) \in H^i(B; \mathbb{Z}/2)$

1) $w_0(\xi) = 1$, $w_i(\xi) = 0$ if $\dim \xi < i$
2) $w_i(\xi \oplus \eta) = \sum_{k+l=i} w_k(\xi) \cup w_l(\eta)$
3) $f^*(w_i(\xi)) = w_i(f^*\xi)$
4) $w_1(\mathcal{L}) \neq 0$ where $\mathcal{L} \to \mathbb{R}P^1$ canonical line bundle

Chern Classes:

$\mathbb{C}^n \to \xi \to B$

$c_i(\xi) \in H^{2i}(B, \mathbb{Z})$

1) $c_0(\xi) = 1$, $c_i(\xi) = 0$ if $2i > \dim \xi$
2) $c_i(\xi \oplus \eta) = \sum_{k+l=i} c_k(\eta) \cup c_l(\eta)$
3) $f^*(c_i(\xi)) = c_i(f^*\xi)$
4) $c_1(\mathcal{L}) \doteq$ where $\mathcal{L} \to \mathbb{C}P^1$ canonical line bundle

Then $w = 1 + w_1 + w_2 + \dots$ Total S-W class

$c = 1 + c_1 + c_2 + \dots$ Total Chern class

- If $\eta \oplus \xi =$ trivial

  $w(\eta) = \bar{w}(\xi)$ $\quad (\bar{w}(\xi) \cdot w(\xi) = 1)$

- $w(TS^n) = 1$ $\quad \because \; TS^n \oplus 1 = n+1$

- $$\begin{array}{ccc} \mathcal{L} & \dashrightarrow & \mathcal{L}_n \\ \downarrow & & \downarrow \\ \mathbb{R}P^1 & \dashrightarrow & \mathbb{R}P^n \end{array}$$ So $i^*\omega_1(\mathcal{L}_n) = \omega_1(\mathcal{L}) \neq 0$

  $\Rightarrow \omega_1(\mathcal{L}_n) \neq 0$

- $T\mathbb{R}P^n \cong \mathrm{Hom}(\mathcal{L}, \mathcal{L}^\perp)$

  $\mathrm{Hom}(\mathcal{L}, \mathcal{L}^\perp) \oplus \mathrm{Hom}(\mathcal{L}, \mathcal{L}) = \mathrm{Hom}(\mathcal{L}, n+1)$

  $\Rightarrow \mathrm{Hom}(\mathcal{L}, \mathcal{L}^\perp) \oplus 1$ $\qquad$ * ~~Hom~~ $\oplus^{n+1} \mathrm{Hom}(\mathcal{L}, 1) \cong \mathcal{L}$

  $$\begin{aligned} \Rightarrow \omega_*(T\mathbb{R}P^n) &= \omega_*(\mathrm{Hom}(\mathcal{L}, \mathcal{L}^\perp)) \\ &= (\omega_*(\mathrm{Hom}(\mathcal{L}, 1)))^n \\ &= (1+\alpha)^n \qquad \mathbb{Z}\alpha = H^1(\mathbb{R}P^n) \end{aligned}$$

  $\omega_i(T\mathbb{R}P^n) = \binom{n+1}{i}\alpha^i$ for $i \leq n$

- $T\mathbb{R}P^n \cong n \Rightarrow \omega(T\mathbb{R}P^n) = 1$

  $\Rightarrow \binom{n+1}{i} = 0 \pmod 2$ for $k < i \leq n$

  $\Rightarrow n = 2^k - 1$ for some k.

- $\mathbb{R}P^n \overset{i}{\hookrightarrow} \mathbb{R}^{n+k}$ immersion

  Let ν be ~~the~~ normal bundle of i

  $\Rightarrow T\mathbb{R}P^n \oplus \nu = n+k$

  $\Rightarrow \omega(T\mathbb{R}P^n) \cdot \omega(\nu) = 1$

  $\Rightarrow \bar{\omega}_i(T\mathbb{R}P^n) = 0$ for $i > k$

  for $n = 2^r$ $\quad \binom{n+1}{i} = \binom{2^r+1}{i} = \binom{2^r}{i-1} + \binom{2^r}{i} = 0$ for $i \neq n, 1$

  $$\begin{aligned} \omega(T\mathbb{R}P^{2^r}) &= 1 + \alpha + \alpha^{2^r} \\ \bar{\omega}(T\mathbb{R}P^{2^r}) &= 1 + (\alpha + \alpha^{2^r}) + (\alpha + \alpha^{2^r})^2 + (\alpha + \alpha^{2^r})^3 + \dots \\ &= 1 + \alpha + \alpha^2 + \dots + \alpha^{2^r} \end{aligned}$$

  $\Rightarrow \mathbb{R}P^{2^r}$ can only be immersed in $\mathbb{R}P^{2^{r+1}-1}$

Cohomology, Poincare Duality

$$(X,A)\times(Y\times B) = (X\times Y,\ X\times B \cup Y\times A)$$

- $H^m(X,A) \xrightarrow{\cong} H^{m+n}((X,A)\times(\mathbb{R}^n, \mathbb{R}^n - \{0\}))$

$$a \longmapsto a\times e \qquad e \in H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\})$$

- $K \subset L \subset M \qquad M - L \subseteq M - K$

$$H_i(M, M-L) \longrightarrow H_i(M, M-K)$$
$$\alpha \longmapsto \rho_K(\alpha)$$
$$C_i(M-L) \hookrightarrow C_i(M-K)$$
$$C_i(M)/C_i(M-L) \twoheadrightarrow C_i(M)/C_i(M-K)$$

- Prop: $\quad H_i(M, M-K) = 0 \quad$ for $\quad i > \dim M$

$\alpha \in H_i(M, M-K) \qquad \rho_x : H_i(M, M-K) \longrightarrow H_i(M, M-x)$

Then,

$$\alpha = 0 \iff \rho_x(\alpha) = 0 \quad \forall\, x \in K$$

- Local Orientation:

$\mu_x \in H_n(M, M-x)$ s.t. $\forall x\ \exists\, B \ni x$ satisfying $\exists\, \alpha$ s.t.

$$H_n(M, M-x) \longleftarrow H_n(M, M-B) \longrightarrow H_n(M, M-y)$$
$$\mu_x \longleftarrow\!\shortmid\ \alpha \longmapsto \mu_y \qquad \forall y \in B.$$

- Global Orientation:

Given a local orientation, $\forall\, K \subseteq M$ compact,

$\exists\, \mu_K \in H_n(M, M-K)$ s.t. $\rho_x(\mu_K) = \mu_x \quad \forall\, x \in K$.

- $H^i_{comp}(M) = \varinjlim_{K \text{ compact}} H^i(M, M-K) \qquad H^i(M, M-K) \longrightarrow H^i(M, M-L)$

- M oriented

$$H^n_{comp}(M) \longrightarrow \mathbb{Z}$$
$$a \longmapsto \langle a', \mu_K\rangle \qquad \text{for } a' \in H^n(M, M-K)$$

$$H^i(M, M-K) \longrightarrow H^i_{comp}(M)$$
$$a' \longmapsto a$$

Integration /

Kronecker product with the fundamental class

- $$\cap: C^i X \otimes C_n X \longrightarrow C_{n-i}(X)$$
  $$\langle a, b \cap \xi \rangle = \langle a \cup b, \xi \rangle$$

- Poincare Duality:
  M compact, oriented
  $$H^i M \xrightarrow{\cong} H_{n-i} M$$
  $$a \longmapsto a \cap \mu_M$$

- $$\cap: C^i(X,A) \otimes C_n(X, A \cup B) \longrightarrow C_{n-i}(X,B)$$
  $$D: H^i_{comp.} M \xrightarrow{\cong} H_{n-i} M$$
  $$a \longmapsto a' \cap \mu_K$$

- $$H^i_{comp}(M) \xrightarrow{\cong} H_{n-i}(M, \partial M)$$ M with boundary
  $$H^i_{comp}(M, \partial M) \xrightarrow{\cong} H_{n-i}(\mathring{M})$$

- Alexandar duality:
  $K \subseteq_{comp} S^n$, good
  $$H^i K \cong \varinjlim_{U \supseteq K} H^i(U)$$
  $$H^i(S^n, K) \cong \varinjlim H^i(S^n, U) \cong H^i_{comp}(S^n - K)$$
  $$\cong H_{n-i}(S^n - K)$$
  $$\Rightarrow \tilde{H}^{i-1}(K) \cong \tilde{H}_{n-i}(S^n - K)$$

Ch. 4)

A)

$$\begin{tikzcd} \xi \arrow[d] & \eta \arrow[d] \\ X & Y \end{tikzcd} \qquad \begin{tikzcd} \hat{\xi} \arrow[r, dashed] \arrow[d] & \xi \arrow[d] \\ X \times Y \arrow[r, "\pi_1"] & X \end{tikzcd} \qquad \begin{tikzcd} \hat{\eta} \arrow[r, dashed] \arrow[d] & \eta \arrow[d] \\ X \times Y \arrow[r, "\pi_2"] & Y \end{tikzcd}$$

Then,

$$\begin{tikzcd} \xi \times \eta \arrow[r] \arrow[d] & \hat{\xi} \times \hat{\eta} \arrow[d] \\ X \times Y \arrow[r, "\Delta"] & (X \times Y) \times (X \times Y) \end{tikzcd}$$

i.e. $\xi \times \eta = \hat{\xi} \oplus \hat{\eta}$

So $\omega(\xi \times \eta) = \omega(\hat{\xi} \oplus \hat{\eta}) = \omega(\hat{\xi}) \cdot \omega(\hat{\eta})$

$= \omega(\pi_1^* \xi)\, \omega(\pi_2^* \eta)$

$= \pi_1^* \omega(\xi)\ \pi_2^* \omega(\eta)$

$= \omega(\xi) \times \omega(\eta)$.

B) $n+1 = 2^r m$

$\binom{n+1}{2^r} = \dfrac{2^r m \cdot (2^r m - 1) \cdots}{2^r \cdot (2^r - 1) \cdots} = \text{odd}$

$\omega_{2^r}(\mathbb{R}P^n) = \omega_{n+1-2^r}(\mathbb{R}P^n) \neq 0.$

if $T\mathbb{R}P^n = 2^r + \eta$ $\qquad \dim \eta = n - 2^r$

then $\omega_{n+1-2^r}(T\mathbb{R}P^n) = 0$

C) $\mathbb{R}P^n$ — $\omega(\mathbb{R}P^n) = \cancel{n+1}\,(1+a)^{n+1}$ $\qquad a \in H^1(\mathbb{R}P^n, \mathbb{Z}_2) \neq 0$

For n-odd we have a non-vanishing vector field.

For n-even $\quad n = 2m$

If $T\mathbb{R}P^{2m} = \eta + \varepsilon$ $\qquad \dim \eta = 1$, $\dim \varepsilon = 2m-1$

$\Rightarrow \omega(\eta) = (1+a)$ or 1

$\Rightarrow \varepsilon = (1+a)^{2m}$ or $(1+a)^{2m+1}$

Both not possible as $\dim \varepsilon < 2m$

For $\mathbb{R}P^4$, $\quad \omega = (1+a)^5 = 1+a+a^4$

If $\quad T\mathbb{R}P^4 = \eta + \varepsilon \qquad \dim\eta = \dim\varepsilon = 2$

$\Rightarrow \omega(\eta) = 1+a+a^2 \quad$ or $\quad 1+a \quad$ or $\quad (1+a)^2 \quad$ or $\quad 1$

$\Rightarrow \omega(\varepsilon) = 1+a^2+a^3+a^4 \quad$ or $\quad 1+a^4 \quad$ or $\quad 1+a+a^2+a^3 \quad$ or $\quad 1+a+a^4$

Not possible as $\dim\varepsilon < 3$

For $\mathbb{R}P^6$, $\quad \omega = (1+a)^7 = 1+a+a^2+a^3+a^4+a^5+a^6$.

If $T\mathbb{R}P^6 = \eta + \varepsilon \qquad \dim\eta = 2 \qquad \dim\varepsilon = 4$

$\Rightarrow \omega(\eta) = 1+a+a^2 \quad$ or $\quad (1+a)^2 \quad$ or $\quad 1+a \quad$ or $\quad 1$

$\Rightarrow \omega(\varepsilon) = 1+a^3+a^6 \quad$ or $\quad 1+a+a^4+a^5 \quad$ or $\quad 1+a^2+a^4+a^6 \quad$ or $\quad (1+a)^6$

$\dim\varepsilon < 5$.

D) $\quad M^m \longrightarrow \mathbb{R}^{m+1} \quad$ immersion

$\Rightarrow \quad TM \oplus \eta = m+1 \qquad \dim\eta = 1 \qquad \cancel{\omega_1(\eta) = \alpha}$

$\omega(\eta) = 1 \quad$ or $\quad 1+$

$\Rightarrow \quad \omega(M) = \dfrac{1}{\omega(\eta)} = \dfrac{1}{1+\omega_1(\eta)} = 1+\omega_1(\eta)+\omega_1(\eta)^2+\cdots$

$\Rightarrow \quad \omega_i(M) = \omega_1(M)^i$

For $\mathbb{R}P^n$, this implies

$\omega(\mathbb{R}P^n) = 1 \quad$ or $\quad \omega(\mathbb{R}P^n) = \dfrac{1}{1+a}$

$\Rightarrow \quad (1+a)^{n+1} = 1 \quad$ or $\quad (1+a)^{n+2} = 1$

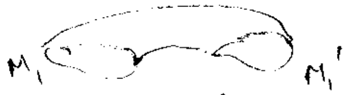
$\Rightarrow \quad \binom{n+1}{i} = 0 \quad$ or $\quad \binom{n+2}{i} = 0 \quad \forall\, i$ non-trivial

$\Rightarrow \quad n = 2^{r-1} \quad$ or $\quad 2^{r-2}$

E) $\mathcal{N}_n$ = Co-bordism classes of n-manifolds

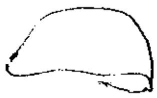
Additive structure:

- $[M_1] + [M_2] = [M_1 \sqcup M_2]$



$[M_1] = [M_1'] \Rightarrow [M_1 \sqcup M_2] = [M_1' \sqcup M_2]$

- $[M] + [M] = 0$ ← $\mathbb{Z}/2$ module
- [Boundary] = 0



Characterized by Stiefel Whitney nos.

$P^2 \times P^2$ $\quad w(P^2 \times P^2) = w(P^2) \times w(P^2)$

$$= (1+a+a^2) \times (1+a+a^2)$$

$$= 1 + 1 \times a + 1 \times a^2 + a \times 1 + a \times a + a \times a^2 + a^2 \times 1 + a^2 \times a + a^2 \times a^2$$

$w_1^4 = 0$

$$= 1 + (1 \times a + a \times 1) + (1 \times a^2 + a \times a + a^2 \times 1) + (a \times a^2 + a^2 \times a) + a^2 \times a^2$$

P^4 $\quad w(P^4) = (1+a)^5$

$$= 1 + a + a^4$$

$w_1^4 = w_4$

So $[P^2 \times P^2] \neq [P^4]$

$$\begin{array}{c} \mathbb{C} \longrightarrow \mathcal{L} \\ \downarrow \\ \mathbb{C}P^n \end{array}$$ canonical line bundle

Gysin sequence:

$$H^k(\mathbb{C}P^n) \xrightarrow[-\cup e(L)]{} H^{k+2}(\mathbb{C}P^n) \longrightarrow H^{k+2}(\mathcal{L}_0) \longrightarrow H^{k+1}(\mathbb{C}P^n)$$

?

$$\begin{array}{c} \mathcal{L}_0 \\ \downarrow \\ \mathbb{C}P^n \end{array} \cong \mathbb{C}^{n+1} - \{0\} \simeq S^{2n+1} \Rightarrow H^*(\mathcal{L}_0) = \begin{cases} 0 & * < 2n+1 \end{cases}$$

$$H^{k}(\mathbb{C}P^n) \xrightarrow[-e(L)]{} H^{k+2}(\mathbb{C}P^n)$$ is an isomorphism for $k \leq 2n+1$

This can only happen if

$$\boxed{e(L) = \pm x.}$$

- $\begin{array}{c} \mathcal{L} \\ \downarrow \\ \mathbb{C}P^n \end{array}$ $e(L) \Rightarrow \{1, -e(L), (-e(L))^2, \dots, (-e(L))^n\}$ is a basis for $H^*(\mathbb{C}P^n)$.

- $\begin{array}{c} \xi \hookleftarrow \mathbb{C}^n \\ \downarrow \\ X \end{array}$ $x \in X$ $\xi_x \cong \mathbb{C}^n$

  lines in $\xi_x \cong \mathbb{C}P^{n-1}$

$$\begin{array}{c} \mathbb{C}^n \hookrightarrow \xi \\ \downarrow p \\ X \end{array} \longrightarrow \begin{array}{c} \mathbb{C}P^{n-1} \hookrightarrow P(\xi) \\ \downarrow p \\ X \end{array}$$ Projectivisation of ξ^n.

Form a canonical line bundle over $P(\xi)$

$$\begin{array}{c} \mathbb{C} \longrightarrow \mathcal{L}_\xi \\ \downarrow \\ P(\xi) \end{array} \quad (\mathcal{L}_\xi)_{(x,\ell)} = \ell$$

- $$\begin{array}{ccc} \mathcal{L} & \xhookrightarrow{i^*} & \mathcal{L}_\xi \\ \downarrow & & \downarrow \\ \mathbb{C}P^n & \xhookrightarrow{i} & P(\xi) \end{array}$$ Now let $y = e(\mathcal{L}_\xi) \in H^2(P(\xi))$

~~So~~ Pulling back y via i^*, we get

$$e(\mathcal{L}) = i^* e(\mathcal{L}_{\xi}) = i^* y$$

$\Rightarrow \quad (1, i^*y, (i^*y)^2, \dots, (i^*y)^{n-1}) \leftarrow$ basis for $\mathbb{C}P^{n-1}$

By Leray Hirsch Th^m:

$$\boxed{H^*(P(\xi)) = (p^*H^*(X))\{1, y, y^2, \cdots, y^{n-1}\}}$$

$$\begin{tikzcd} P(\xi) \arrow[d, "p"] \\ X \end{tikzcd}$$

Then, $\exists!$ eq^n

$$y^n - c_1 y^{n-1} + c_2 y^{n-2} - + \dots + (-1)^n c_n = 0 \qquad c_i \in H^{2i}(X).$$

Then:

$$c_i = i^{th} \text{ Chern class of } \begin{matrix}\xi \\ \downarrow \\ X\end{matrix}$$

- Need to check axioms:

1. Naturality

$$\begin{tikzcd} f^*\xi \arrow[r] \arrow[d] & \xi \arrow[d] \\ Y \arrow[r, "f"] & X \end{tikzcd} \longrightarrow \begin{tikzcd} L_{f^*\xi} \arrow[r, "f^*"] \arrow[d] & \mathcal{L}_{\xi} \arrow[d] \\ f^*P(\xi) \arrow[r, "f^*"] \arrow[d] & P(\xi) \arrow[d] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

2. $c_i(\xi) = 0$ for $i > n$, $c_0(\xi) = 1$

3. $c(\xi \oplus \eta) = c(\xi)\, c(\eta)$

Trick:

$$\mathbb{C}P^n - \mathbb{C}P^i \underset{\substack{\text{def} \\ \text{retract}}}{\simeq} \mathbb{C}P^{n-i-1}$$

LHS $= [x_0 : \dots : x_i : x_{i+1} : \dots : x_n]$ s.t. atleast 1 of $x_{i+1} \cdots x_n \neq 0$

$\downarrow$

$[tx_0 \cdots tx_i : x_{i+1} \cdots : x_{n+1}] \leftarrow$ def retract

at $t=0$ RHS

at $t=1$ LHS

Now do same thing for $P(V \oplus W)$

$$P(V \oplus W) - P(V) \xrightarrow[\text{def retracts}]{} P(W)$$

Do this fibre wise

$$U = P(\xi \oplus \eta) - P(\xi) \xrightarrow[\text{def}]{\sim} P(\eta)$$

$$V = P(\xi \oplus \eta) - P(\eta) \xrightarrow[\text{def.}]{\sim} P(\xi)$$

$$\begin{tikzcd}
 & H^*(P(\xi\oplus\eta), V) \arrow[d] & H^*(P(\xi\oplus\eta), U) \arrow[r] \arrow[d] & H^*(P(\xi\oplus\eta), U \cup V) = 0 \arrow[d] \\
\text{Exact} \quad \omega_1 \arrow[d] & H^*(P(\xi\oplus\eta)) \arrow[r, phantom, "\otimes \text{ Cup}"] \arrow[d] & \omega_2 \; H^*(P(\xi\oplus\eta)) \arrow[r] \arrow[d] & H^*(P(\xi\oplus\eta)) \\
0 & H^*(P(\xi)) & 0 \quad H^*(P(\eta)) &
\end{tikzcd}$$

Cup product

$$y_{\xi\oplus\eta} = e(\mathcal{L}_{P(\xi\oplus\eta)})$$

$$\omega_1 = y_{\xi\oplus\eta}^{n} - c_1(\xi)\, y_{\xi\oplus\eta}^{n-1} + \cdots + (-1)^n c_n(\xi)$$

$$\omega_2 = y_{\xi\oplus\eta}^{m} - c_1(\eta)\, y_{\xi\oplus\eta}^{n-1} + \cdots + (-1)^m c_m(\xi).$$

As, $\omega_1 \xrightarrow{i^*} 0$, $\omega_2 \xrightarrow{i^*} 0$

$\omega_1 \in H^*(P(\xi\oplus\eta), V)$ $\quad \omega_2 \in H^*(P(\xi\oplus\eta), U)$

$\Rightarrow \quad \omega_1 \cdot \omega_2 = 0$

$\Rightarrow$ whitney product formula.

4) $\begin{matrix} \mathcal{L} \\ \downarrow \\ \mathbb{C}P^n \end{matrix} \Rightarrow c_1(\mathcal{L}) = -x$ $\qquad \begin{matrix} \mathcal{L} \\ \downarrow \\ \mathbb{C}P^n \end{matrix} \longrightarrow \begin{matrix} P(\mathcal{L}) \cong \mathbb{C}P^n \\ \downarrow \\ \mathbb{C}P^n \end{matrix}$