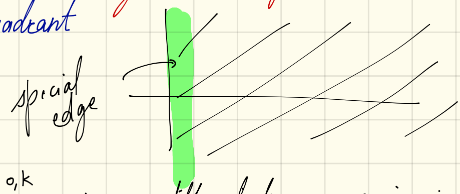
Edge Homomorphisms

1. First & Fourth Quadrant



a) $E_n^{0,k} \subseteq E_{n-1}^{0,k} \subseteq \ldots \subseteq E_2^{0,k}$ as no differentials are coming in

So that $E_\infty^{0,k} \subseteq E_2^{0,k}$

b) Differentials always point towards lower filtration

i.e $A_{p+q} = \lim_{p \to \infty} \bar{A}_p^{p+q}$, $E_\infty^{p,q} = \bar{A}_{p+q,p} / \bar{A}_{p+q,p+1}$

$\Rightarrow E_\infty^{0,k} = A_K / \text{submodule } \bar{A}_{K+1}$

$$\begin{tikzcd} A_K \arrow[r, two heads] \arrow[dr] & A_K / \bar{A}_{K+1} = E_\infty^{K,0} \arrow[d, phantom, "\cap"] \\ & E_2^{K,0} \end{tikzcd}$$

edge homomorphisms

b) First and Second Quadrant Spectral sequence

a) $E_\infty^{K,0} \twoheadleftarrow E_{n+1}^{K,0} \twoheadleftarrow \ldots \twoheadleftarrow E_2^{K,0}$

b) $\bar{A}_{K,K}$ is the lowest filtration of $A_K = \lim_{\substack{\to \\ p}} A_p^k$

$\Rightarrow E_\infty^{k,0} = \bar{A}_{K,K} \subseteq A_K$

the map $E_2^{K,0} \twoheadrightarrow E_\infty^{K,0} = A_K^k \subseteq A_K$ horizontal edge homomorphism

3) Serre SS

a) $H^K(E,R) \longrightarrow E_2^{0,k} = H^k(F;R)$
is simply i^*.

b) $\underset{\substack{\| \\ H^K(B;R)}}{E_2^{K,0}} \longrightarrow H^K(E;R)$
is simply π^*.

4. Transgression $\rightsquigarrow$ def
differential from vertical axis to horizontal axis in a first quadrant cohomological spectral sequence

$$d_r : E_r^{0,r-1} \longrightarrow E_r^{r,0}$$

$$\left(\text{and} \quad E_{r+1}^{0,r-1} = E_\infty^{0,r-1}, \quad E_{r+1}^{r,0} = E_\infty^{r,0}\right)$$

For the Serre SS we have the following geometric interpretation

$$\begin{tikzcd} E_r^{0,r-1} \arrow[r, "d_r"] & E_r^{r,0} \arrow[d] \\ E_2^{0,r-1} \arrow[u, phantom, "\cap"] & E_\infty^{r,0} \\ H^{r-1}(F,R) \arrow[u, phantom, "\|"] & H^r(E,R) \arrow[u, phantom, "\cap"] \end{tikzcd}$$

$\rightsquigarrow$ Consider the diagram

$\pi : (E,F) \longrightarrow (B, b_0)$

which gives :

$$\begin{tikzcd} H^{r-1}(F) \arrow[r] & H^r(E,F) \arrow[r] & H^r(E) \arrow[r] & H^r(F) \\ & H^r(B,b_0) \arrow[u, "\pi^*"] & & \end{tikzcd}$$

Then,

$$E_r^{0,r-1} = \{\alpha \in H^{r-1}(F;R) \mid \delta(\alpha) \in \operatorname{Im} \pi^*\}$$

$$E_r^{r,0} = H^r(B;b_0)/\ker \pi^* \subseteq H^r(E)$$

Then, claim:

$$d_r(\alpha) = [(\pi^*)^{-1}\delta(\alpha)]$$

Examples of Graded Commutative Rings

1. Exterior Algebra

$\Lambda(x_n) = R[x_n]/(x_n^2) \quad |x_n| = n \quad$ eg: $H^*(S^n)$

2. Polynomial Algebra

$R[x_n]$ eg: $H^*(\mathbb{C}P^\infty; \mathbb{Z})$, $H^*(\mathbb{R}P^\infty; \mathbb{Z}/2)$

(warning: in graded setting $x_n^2 = (-1)^{n^2} x_n^2$, if n is odd $\Rightarrow 2x_n^2 = 0$)

3. Truncated Polynomial Algebra

4. Divided Polynomial Algebra

Sometimes we need to assume n - even

$$\Gamma[x_n] := R\langle 1, \gamma_1, \gamma_2, \dots \mid \gamma_i \gamma_j = \binom{i+j}{i} \gamma_{i+j}, |\gamma_i| = i \cdot n \rangle$$

eg: $H^*(\Omega S^{2n+1}; R)$

Tensoring of G.C. Rings gives graded commutative Rings

A, B graded commutative / R

$A \otimes_R B$ is g.c.

$$(A \otimes_R B)_n = \bigoplus_{i+j=n} A_i \otimes B_j$$

$$(a \otimes b) \cdot (c \otimes d) = (-1)^{|c||b|} (ac \otimes bd)$$

Ex: Show $\Gamma_{\mathbb{F}_2}[x_n] = \bigotimes_{i=1}^{\infty} \mathbb{F}_2[\gamma_{2^i}]/(\gamma_{2^i}^2)$ ⇝ related to

$$\binom{a_1 a_2 \cdots a_k}{b_1 b_2 \cdots b_k} = \prod_{i=1}^{k} \binom{a_i}{b_i} \bmod 2$$

binary expansions

Ex: Let C_R be category of G.C. Rings /R, which of the 4 examples are free objects in C_R.

Note: A is free on a set S if given a map $f: S \to B$ ⇝ G.C. Ring/R if $\exists!\, \tilde{f}$ extending f.

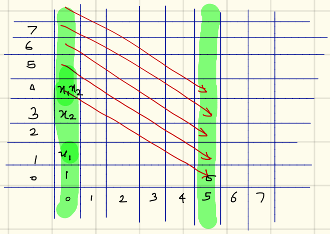
$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, phantom, "\cap"] & B \\ A \arrow[ur, "\exists!\ \tilde{f}"'] & \end{tikzcd}$$

Computations:

1. $U(n)$ $\qquad U(n-1) \longrightarrow U(n) \to S^{2n-1}$

Claim: $H^*(U(n); R) \cong \Lambda_R(x_1, x_2, \dots, x_{2n-1})$
$\cong \Lambda_R(x_1) \otimes_R \Lambda_R(x_2) \otimes_R \cdots \otimes \Lambda_R(x_{2n-1})$

Look at Serre SS:



$n = 3$ say
only possible differentials are d_5

⇜ Induction: $H^*(U(2)) = \Lambda_R(x_1) \otimes \Lambda(x_2)$

Because we have $d_5 x_1 \in E_2^{5,-3} = 0$, similarly for x_2, $d_5 x_i = 0$

By multiplicativity $d_5(x_1 x_2) = 0$.

I feel like there is an argument missing here, but this then implies that all differentials are 0. So remains to figure out the multiplicative structure.

We have the following relation coming from the filtration:

$$0 \longrightarrow E_\infty^{p,0} \longrightarrow H^p(U(3)) \longrightarrow E_\infty^{0,p} \longrightarrow 0$$

So look at some preimages $\bar{x}_1, \bar{x}_2$ of x_1, x_2. These are uniquely determined as the kernel $E_\infty^{p,0}$ is trivial in these degrees.

As $x_1 x_2 \neq 0 \Rightarrow \overline{x_1 x_2} \neq 0$, $x_1\sigma$ is the generator of $E_\infty^{1,5}$ and hence $\bar{x}_1\sigma \neq 0$.

And $\sigma^2 = 0$ so that $H^*(U(3)) = \Lambda_R(\bar{x}_1) \otimes \Lambda_R(\bar{x}_2) \otimes \Lambda_R(\sigma)$.

For arbitrary n, we still have

$$0 \longrightarrow E_\infty^{p,0} \longrightarrow H^p(U(n+1)) \longrightarrow E_\infty^{0,p} \longrightarrow 0$$

$$H^*(U(n)) = \Lambda(x_1) \otimes \Lambda(x_2) \otimes \cdots \otimes \Lambda(x_n), \qquad E_\infty^{0,2n+1} = R\sigma$$

Again let the preimages be $\bar{x}_i$, by the same argument as above all the monomials survive. So we get the desired result.