Error in Weibel:

1) In general it is not true that

$$0 \to A_i \to B_i \Rightarrow 0 \to \oplus A_i \to \oplus B_i$$

eg: we will show that in Sheaves (X) we do not have

$$A_i \to B_i \to 0 \Rightarrow \prod A_i \to \prod B_i \to 0$$

Then in Sheaves$(X)^{op}$ direct sum will not inject.

- Take $X = [0,1]$

  For each $i \in \mathbb{Z}_{>0}$, $\mathcal{U}_i$ be covering of X by finitely many balls of radius $1/i$.

  $\mathcal{F}_U$ be the sheaf: $\mathcal{F}_U(V) := \begin{cases} \mathbb{Z} & \text{if } V \subseteq U \\ 0 & \text{else} \end{cases}$

  for $U \in \mathcal{U}_i$

  $\mathcal{F}_i$ be the sheaf: $\mathcal{F}_i := \prod_{U \in \mathcal{U}_i} \mathcal{F}_U$

- Then we have canonical surjection

$$\mathcal{F}_i \to \mathbb{Z}_{[0,1]} \to 0$$

- But look at

$$\prod_i \mathcal{F}_i \to \prod_i \mathbb{Z}_{[0,1]}$$

Stalk at RHS $= \mathbb{Z}^{\mathbb{Z}}$

But for any $V \subseteq [0,1]$ $\exists\, i$ s.t. V ~~does~~ is not contained in an U_i for $i > n$

So that stalk at any point can have only finite copies of $\mathbb{Z}$.

Hence the map cannot be a surjection.

- Note: In this category we nomore have

$$L_i F(\oplus\ ) = \oplus L_i F$$

2) In general it is not true that

$\varinjlim \cancel{P}\varphi_i$ is F-acyclic if $\forall \varphi_i$ are projective

example?

What we ~~see~~ can show is,

for Tor, in R-mod, this holds:

i.e. $\varinjlim P_i$ is flat

Claim: P_i flat $\Rightarrow \varinjlim P_i$ flat

Proof:

Need to ~~stay~~ say that

$0 \to M \xrightarrow[\alpha]{} N \Rightarrow 0 \to \varinjlim P_i \otimes M \xrightarrow[\alpha]{} \varinjlim P_i \otimes N$

~~But here~~

Suppose

$\cancel{\otimes}\ \alpha \otimes 1 (\Sigma x_i \otimes m_i) = 0$

$\cancel{\Rightarrow \alpha(x_i) \otimes m_i = 0}$

$\Rightarrow x_i \otimes \alpha(m_i) = 0$

Look at a P s.t. $\forall x_i$ ~~can~~ can be represented by an element of P.

So we are reduced to checking

$0 \longrightarrow P \otimes M \longrightarrow P \otimes N$

But this ~~follows~~ is assumed.

- Claim: $\mathrm{Tor}_*(\varinjlim A_i, B) = \varinjlim \mathrm{Tor}_*(A_i, B)$

RHS = ~~$H_*(\varinjlim A_i \otimes \ldots)$~~

$\varinjlim H_* [(B \otimes -) P_i]$ ~~reding~~ $\quad P_{i\cdot} \to A_i$

why directed system?

$$\begin{array}{c} A_i \\ \downarrow \\ A_j \end{array} \quad \text{lifts to} \quad \begin{array}{ccc} P_{i\cdot} & \to & A_i \\ \downarrow & & \downarrow \\ P_{j\cdot} & \to & A_j \end{array}$$

$$\begin{array}{ccc} H_*(B \otimes P_{i\cdot}) & & B \otimes P_{i\cdot} \longrightarrow B \otimes A_i \\ \downarrow & \leftsquigarrow & \downarrow \qquad\qquad \downarrow \\ H_*(B \otimes P_{j\cdot}) & & B \otimes P_{j\cdot} \longrightarrow B \otimes A_j \end{array}$$

LHS $= H_*((B\otimes -)\ \varinjlim P_i)$ Here, we are using $\varinjlim$ is exact and that $\varinjlim P_i$ is fla[t]

$$\varinjlim H_*(B\otimes P_{i\cdot}) \overset{?}{=} H_*(B\otimes \varinjlim P)$$

$$\begin{array}{c} H_*(B\otimes P_{i\cdot}) \\ \downarrow \\ H_*(B\otimes P_{j\cdot}) \end{array}$$

We have $B\otimes P_{i\cdot} \longrightarrow B\otimes \varinjlim P$

~~Chain~~ Chain map? Yes, because $B\otimes -$ is a functor

so descends to $H_*(B\otimes P_{i\cdot}) \longrightarrow H_*(B\otimes \varinjlim P_\cdot)$

~~is this~~

Given
$$\begin{array}{ccc} H_*(B\otimes P_{i\cdot}) & \longrightarrow & M \\ \downarrow & \nearrow & \\ H_*(B\otimes P_{j\cdot}) & & \end{array}$$

- Left adjoint functors commute with colimit.

$\mathbb{Z}/p\mathbb{Z} \xrightarrow{p} \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{p} \cdots$

what is direct limit?

$\mathbb{Z}/p^\infty\mathbb{Z}$

This sits inside $\mathbb{Q}/\mathbb{Z}$.

$Ext^n(A,B)$

- Quotient of injective abelian group is injective.

- $Ext^1(\mathbb{Q}/\mathbb{Z}, \mathbb{Z})$
 $\quad \| $
 $Hom(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$

- $F \longrightarrow G$ natural transf[ormation], left exact contravarian[t]

$F(R) \longrightarrow G(R)$ is iso

Then, $F(A) \longrightarrow G(A)$ iso for finitely presented A.

This proof looks very likely to work. Instead we will look at the other way of calculating Tor:

- Let $P_\cdot \longrightarrow B$ be projective resolution of B.

Look at

$P_\cdot \otimes \varinjlim_I A_k$

Then LHS: $H_i(P_\cdot \otimes \varinjlim A) = Tor_i(\varinjlim A, B)$

RHS:

$Tor_i(A_j, B) = H_i(P_\cdot \otimes A_j)$

How do these form a directed system?

Given $A_j \longrightarrow A_k$

$P_\cdot\otimes$ will give an object in Chain( )

$P_\cdot\otimes A_j \longrightarrow P_\cdot \otimes A_k$

But then H_i is a functor on Chain( )

$H_i(P\otimes A_j) \longrightarrow H_i(P_\cdot \otimes A_k)$

So we can make sense of

$\varinjlim_k H_i(P\otimes A_k)$

The problem is reduced to the following:

$$H_i(P_\cdot \otimes \varinjlim A) = \varinjlim H_i(P_\cdot \otimes A)$$

So we are asking whether the functor

$$H_i(P\otimes -) : R\text{-mod} \longrightarrow \text{Ab Grp}$$

commute with $\varinjlim$?

- $P_\cdot \otimes \varinjlim A = \varinjlim P \otimes A$

  $A_j \longrightarrow A_k$

  $P_\cdot \otimes A_j \longrightarrow P_\cdot \otimes A_k$

  Now because limits exist in R-mod, we can use the same reasoning slot-wise to prove that limits exist in $Ch(R\text{-mod})$ and hence in $Ch(\mathbb{Z})$ and ~~hence~~ that these limits are slotwise limits. i.e. $\varinjlim P_\cdot \otimes A = (\varinjlim P \otimes A)$.

  Next we need to work in each slot and pull $\varinjlim$ out of tensors This follows from the fact that $P\otimes$ is left ~~exact~~ adjoint and $\varinjlim$ commutes with left adjoints.

- So we now need to prove

  $$\varinjlim H_*(P\otimes A) = H_*(\varinjlim P_\cdot \otimes A_k)$$

  i.e. $$\varinjlim H_*(Q_\cdot) = H_*(\varinjlim Q_\cdot)$$

  Left hand side makes sense because maps are chain maps.

  For simplicity we will prove for H_1

  For each index i we have an exact sequence

  $$0 \to (\ker d_1)^i \longrightarrow (\operatorname{im} d_2)^i \longrightarrow H_1^i \longrightarrow 0$$

  for the chain $Q_2^i \xrightarrow{d_2} Q_1^i \xrightarrow{d_1} Q_0^i$

  Since $\varinjlim$ is an exact functor, we get

  $$0 \to \varinjlim(\ker d_1) \to \varinjlim(\operatorname{im} d_1) \longrightarrow \varinjlim H_1^i \to 0$$

  And we have an exact seq

  $$0 \to \ker \varinjlim d_1 \longrightarrow \operatorname{im} \varinjlim d_1 \longrightarrow H_1^i \varinjlim \to 0$$

  So we are left to prove

  $$\varinjlim \ker d_1 = \ker \varinjlim d_1 \quad , \quad \varinjlim \operatorname{im} d_1 = \operatorname{im} \varinjlim d_1$$

  But this precisely the statement of exactness of $\varinjlim$.

Weibel 2.6.6.

$f : X \to Y$ $\quad f^{-1} : \text{Sheaves}(Y) \longrightarrow \text{Sheaves}(X)$ exact

Defined as:

$$(f^{-1}F)U = \varinjlim_{f(U) \subseteq V} F(V)$$

we have the adjoint relation:

$$\text{Hom}_X(f^{-1}G, F) = \text{Hom}_Y(G, f_* F)$$

So we already know f^{-1} is right exact.

Given a sequence of sheaves on Y.

$$0 \to F \longrightarrow G$$

We have the equivalent condition

$$0 \to F(V) \longrightarrow G(V) \quad \forall V \text{ open in } Y$$

Suffices to show

$$0 \to f^{-1}F(U) \longrightarrow f^{-1}G(U) \quad \forall U \text{ open in } X$$

$$\wr\wr$$

$$0 \to \varinjlim_{f(U) \subseteq V} F(V) \longrightarrow \varinjlim_{f(U) \subseteq V} G(V)$$

But this follows from exactness of $\varinjlim$.

Q. what about f_*? Is this exact?

No.

For example look at F, G on X st.

~~F(X) =~~ $F(U) = 0 = G(U)$ for all $U \subseteq X$ open

except for $G(X) = \mathbb{Z}$

Then the trivial map $F \longrightarrow G$ is an epimorphism

Infact an isomorphism.

Take $Y = *$, $f : X \longrightarrow *$ trivial map

But $f_* F = 0$ $\quad f_* G = \mathbb{Z}$ so that $f_* F \to f_* G$ is

no longer an epimorphism.

Q. what condition do we require on f for f_* to be exact?

Tor and Ext:

- $\mathrm{Tor}_0^{\mathbb{Z}}(\mathbb{Z}/p, B) = \mathrm{Tor}_0^{\mathbb{Z}}(B, \mathbb{Z}/p) = B/pB$

  $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/p, B) = {}_pB = \{b \in B \mid pb = 0\}$

- $\mathrm{Tor}_1^{\mathbb{Z}}(A, B)$ is always torsion.

  $\mathrm{Tor}_1(A,B) = \varinjlim_{A' \leq A} \mathrm{Tor}(A', B)$ A' finitely generated ~~subgrp~~ subgrp of A

  $= \varinjlim_{A' \leq A} \left(\oplus {}_{n}B \right)$

  So remains to show that $\varinjlim$ of torsion abelian group is ~~abelian~~ torsion.
  But this is because each element in $\varinjlim$ can be represented by an element of some group forming the limit.

- $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, B) = \varinjlim_{\mathbb{Z}} \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n, B)$

  where $\mathbb{Z}/n$'s form a directed set as

  $$\begin{array}{ccc} \mathbb{Z}/n & \longrightarrow & \mathbb{Z}/nm \\ 1 & \longmapsto & m \end{array}$$

  And for this directed ~~set~~ system
  $\mathbb{Q}/\mathbb{Z} = \varinjlim_n \mathbb{Z}/n\mathbb{Z}$

  $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, B) = \varinjlim_n {}_nB$ ${}_nB \longrightarrow {}_{mn}B$ is just inclusion

  $= B_{\text{torsion}}$

- $\mathrm{Tor}_1^{\mathbb{Z}}(A, -) = 0 \iff A$ torsion free

  $\mathrm{Tor}_1^{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}) = 0 \Rightarrow A$ torsion free

- $r \in R$ left non-zero divisor, $_rR = 0$

$$\Rightarrow \quad \mathrm{Tor}_*(R/rR, B) = \begin{cases} B/rB & * = 0 \\ {}_rB & * = 1 \\ 0 & \text{else} \end{cases}$$

- $r \in R$ left zero divisor

$$0 \longrightarrow {}_rR \longrightarrow R \xrightarrow{r} R \longrightarrow R/rR \longrightarrow 0$$

Break this up as $\qquad X = \{rs \mid s \in R\} \leq R$

$$0 \longrightarrow {}_rR \longrightarrow R \longrightarrow X \longrightarrow 0$$
$$0 \longrightarrow X \xrightarrow{i} R \longrightarrow R/rR \longrightarrow 0$$

we get

$$0 \longrightarrow \mathrm{Tor}_1(X, B) \longrightarrow {}_rR \otimes B \longrightarrow B \longrightarrow X \otimes B \longrightarrow 0$$
$$0 \longrightarrow \mathrm{Tor}_{i+1}(X, B) \longrightarrow \mathrm{Tor}_i({}_rR, B) \longrightarrow 0 \qquad i > 0$$

$$0 \longrightarrow \mathrm{Tor}_1(R/rB, B) \longrightarrow X \otimes B \longrightarrow B \longrightarrow R/rB \otimes B \longrightarrow 0$$
$$0 \longrightarrow \mathrm{Tor}_{i+1}(R/rR, B) \longrightarrow \mathrm{Tor}_i(X, B) \longrightarrow 0 \qquad i > 0$$

- $\mathrm{Tor}_1(F/R, B) = B_{\text{torsion}}$ $\qquad$ R- commutative domain, F- field of fractions of R

- $\mathrm{Tor}_1^R(R/I, R/J) = \dfrac{I \cap J}{IJ}$ $\qquad$ I- right ideal of R, J- left ideal of R

From the exact seq

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

we get

$$0 \longrightarrow \mathrm{Tor}_1(R/I, R/J) \longrightarrow I \otimes R/J \longrightarrow R/J \longrightarrow R/I \otimes R/J \longrightarrow 0$$

Enough to show $\dfrac{I \cap J}{IJ}$ is kernel of $I \otimes R/J \longrightarrow R/J$

$\sum i_\ell \otimes (r_\ell + J) \in \ker \Rightarrow \sum i_\ell r_\ell \in J$ $\quad$ But each $i_\ell \in I$

$$\Rightarrow \sum i_\ell r_\ell \in I \cap J$$

So we have an exact seq $\quad I \cap J \longrightarrow I \otimes R/J \longrightarrow R/J$

what is the kernel of $I \cap J \longrightarrow I \otimes R/J$

$$r \longmapsto r \otimes T$$

This is same as kernel of $I \cap J \longrightarrow (I \cap J) \otimes R/J$

for this we look at $0 \to J \to R \to R/J \to 0$

Tensoring we get $(I \cap J) \otimes J \longrightarrow I \cap J \longrightarrow I \cap J \otimes R/J \longrightarrow 0$

So $\ker = \operatorname{im}((I \cap J) \otimes J \longrightarrow I \cap J)$

$= IJ$

$$\Rightarrow \quad \operatorname{Tor}_1(R/I, R/J) = \frac{I \cap J}{IJ}.$$

- S centrally multiplicative closed set in R
  $\Rightarrow S^{-1}R$ flat.

- $0 \to A \to B \to C \to 0$

  Apply $\otimes_R M$

$$\begin{array}{ccccccc}
 & & A \otimes M & \longrightarrow & B \otimes M & \longrightarrow & C \otimes M \longrightarrow 0 \\
 & & \uparrow & & & & \\
\operatorname{Tor}_1(A,M) \longrightarrow \operatorname{Tor}_1(B,M) & \longrightarrow & \operatorname{Tor}_1(C,M) & & & & \\
\uparrow & & & & & & \\
\operatorname{Tor}_2(C,M) & & & & & &
\end{array}$$

A, C flat $\Rightarrow B$ flat

B, C flat $\Rightarrow A$ flat

- Ex 32.3 Weibel

$R = k[x,y] \qquad I = (x,y)R \qquad k = R/I$

$$0 \longrightarrow R \xrightarrow{\begin{bmatrix} -y \\ x \end{bmatrix}} R^2 \xrightarrow{(x \ y)} R \longrightarrow k \longrightarrow 0$$

$$\alpha \longmapsto (-\alpha y, \alpha x)$$

$$(\alpha, \beta) \longmapsto (\alpha x + \beta y)$$

Exactness: $\alpha x + \beta y = 0$

$\Rightarrow y \mid \alpha, \; x \mid \beta$