Jordan - Chevalley decomposition:

Def^n: $x \in \text{End } V$

Nilpotent $\Rightarrow$ $x^n = 0$

Semisimple $\Rightarrow$ minimal poly of x has distinct roots m_x

Diagonalizable $\Rightarrow$ $\exists$ basis consisting of eigen vectors

Lemma:

Nilpotent $\Leftrightarrow$ Diagonalizable

Proof:
$$K[x] \cong K[T]/\langle m_x \rangle = \bigoplus_i K[T]/_{T-\alpha_i} \quad \text{where } m_x T = (T-\alpha_1)\dots(T-\alpha_n)$$
$$\cong K^r, \quad \text{isomorphism is as rings}$$

Jordan Decomposition:

$x \in \text{End}(V)$, $\exists!$ x_s semisimple & x_n nilpotent, such that

1) $x = x_s + x_n$

2) $\exists$ polynomials $p(T)$, $q(T)$ with constant co-efficients 0 such that $p(x) = x_s$, $q(x) = x_n$

3) $\forall y \in \text{End } V$, $xy = yx \Leftrightarrow x_s y = y x_s$, $x_n y = y x_n$.

4) $A \subseteq B \subseteq V$ subspaces, if $xB \subseteq A$ then $x_s B \subseteq A$, $x_n B \subseteq A$.

5) If $xy = yx$, then $(x+y)_s = x_s + y_s$, $(x+y)_n = x_n + y_n$.

Def^n: Unipotent $\Rightarrow$ $x - 1_V$ nilpotent

Lemma: Chevalley Decomposition $x \in Gl(V)$, Then $\exists!$ semisimple $x_s \in Gl(V)$, a unipotent $x_u \in Gl(V)$ such that,

i) $x = x_s x_u = x_u x_s$

ii) x_s, x_u commute with every element $y \in \text{End } V$ which commutes with x

iii) $B \subseteq V$ then if $xB = B$ then $x_s B = B$, $x_u B = B$.

iv) If $xy = yx$, $(xy)_s = x_s y_s$, $(xy)_u = x_u y_u$

Proof: Write $x = x_s + x_n$

$\because x_s$ is semisimple x_s is invertible

$$x = x_s(1 + x_s^{-1} x_n)$$

Take $x_u = 1 + x_s^{-1} x_n$, ($\because x_s^{-1} x_n = x_n x_s^{-1}$ & x_n nilpotent, x_u unipotent)

Excercise: 1) Set of all unipotent elements of $Gl(V)$ is a closed subvariety

2) Set of all semisimple elements is dense in $Gl(V)$.

1) If $\dim V = n$, unipotent matrices $= \bigcup_{i=1}^{n} \mathcal{U}_i$

$$\mathcal{U}_i = \{x \in Gl(V) \mid x^i - 1 = 0\}$$
$$= \ker [x^i - 1]$$

2) Set of all semisimple elements – means every open set contains a semisimple element – must be true for semisimple elements with distinct eigenvalues

. Unipotent elements

In char $= p > 0$, x unipotent $\Leftrightarrow$ $x^{p^i} = 1_V$ for some i.

In char $= 0$, x unipotent $\Leftrightarrow$ x has infinite order
$\neq 1$

Th^m: $u \in Gl(V) \neq 1$ unipotent, $K = \mathbb{C}$
G = closure of $H = \langle u \rangle$. Then $G \cong (G_a, +)$ as an algebraic grp.

Proof: Recall, formally $\exp x = \sum_i \frac{x^i}{i!}$ $\log(y+1) = \sum_i \frac{y^i}{i}(-1)^{i+1}$

n nilpotent $\Rightarrow$ $\exp n$ is a polynomial in n.

u unipotent $\Rightarrow$ $\log u$ is a polynomial in u

Let $u = \cancel{1+n}$, $n = \log u$ Define, by, $\varphi: G_a \longrightarrow Gl(V)$, $t \longmapsto \exp(tn)$

φ is a homomorphism of algebraic groups.

Using $\log(\exp tn) = tn$
$$= \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} (\varphi(tn) - 1)^i$$, we get

$$\psi : \varphi(G_a) \longrightarrow G_a$$

So, φ is an isomorphism of algebraic groups.

Since, order of $u = \infty$ H is infinite, so $\overline{H}$ has +ve dimension.

and also $\underbrace{\varphi(G_a) \supseteq \overline{H}}_{?}$ $\Rightarrow$ $\varphi(G_a) = H$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix}$$

~~want~~ $\cancel{1 + tn + \cdots + \frac{t^k n^k}{k!}} = \cancel{n+1}$

only need to say that log of a unipotent element is nilpotent.

=

Let $G = Gl(V)$, $E = End(V)$, $d = \det$

$K[Gl(V)] = K[E]_{(d)}$

• $x \in Gl(V)$, $\rho_x dy = d(yx) = d(y) \cdot d(x)$

$\Rightarrow \quad \rho_x(dy) = (\det x)\, d(y)$

Let $s \in Gl(V)$ semisimple, then $\exists$ basis $v_1 \cdots v_n$ of V such that

$$s v_i = s_i v_i, \quad s_i \in K^*$$

Look at linear functions X_{ij} on $Gl(V)$ such that $X_{ij}(A) = A_{ij}$.

$$\rho_s X_{ij} = s_j X_{ij} \qquad \text{because } s \text{ is diagonal.}$$

• Hence, the matrix of ρ_x wrt basis $\{X_{ij}\}$ of E is diagonal.

• Can do this with tensor algebra, symmetric algebra ...

• So we get matrix of ρ_s is diagonal for any finite dim ρ stable subspace of $K[E]$.

$K[Gl(V)] = \bigoplus_m K[E]\, d^{-m} \Rightarrow \rho_s$ diagonal for $K[Gl(W)] \subseteq K[Gl(V)]$ for any W such that $\rho_s W = W$.

If u is unipotent, ρ_u is unipotent in $Gl(W)$ for finite dim $W \subseteq V$ which is stable under ρ_s.

Summarising, we have

• Theorem: We have the Jordan-Chevalley decomposition for ρ_x in $K[Gl(V)]$ such that $(\rho_x = (\rho_x)_s (\rho_x)_u)$ such that $(\rho_x)_s = \rho_{x_s}$, $(\rho_x)_u = \rho_{x_u}$

· Jordan-Chevalley Decomposition for arbitrary groups:

G algebraic group $\Rightarrow$ 1) $\forall x \in G$, $\exists! s, u \in G$ s.t. $su = us = x$

and $(\rho_x)_s = \rho_s$, $(\rho_x)_u = \rho_u$

s- semisimple part
u- unipotent part

2) $\varphi : G_1 \longrightarrow G_2$ homomorphism, then

$\varphi(x_s) = \varphi(x)_s$, $\varphi(x_u) = \varphi(x)_u$.

· Proof 1) Embed G in $Gl(n,K)$. Let $x = su$ be Jordan-Chevalley decomposition in $Gl(n,K)$.

By previous th^m, $(\rho_x)_s = \rho_s$, $(\rho_x)_u = \rho_u$

Let I be ideal of regular functions vanishing on G in $Gl(n,K)$.

So we have, $(\rho_x) I = I$

But by J-C decomposition, $(\rho_x)_s I = I$, $(\rho_x)_u I = I$.

So we have, $x_s = s \in G$, $u \in G$.

2) $G_1 \twoheadrightarrow \varphi(G_1) \hookrightarrow G_2$

Inclusion part follows from part 1).

For the onto case, $\varphi^* : K[G_1] \longleftarrow K[G_2]$ is 1-1.

$\varphi(x) = \varphi(x_s)\varphi(x_u)$.

Look at $\rho_{x_s}|_{K[G_2]} = \rho_{\varphi(x_s)}$ by definition of φ^*.

This gives $\rho_{\varphi(x_s)} \rho_{\varphi(x_u)}$ is Jordan-C decomposition for $\rho_{\varphi(x)}$

Corollary : $\varphi : G_1 \longrightarrow G_2$, $\varphi(\text{semisimple}) = \text{semisimple}$
$\varphi(\text{unipotent}) = \text{unipotent}$

· $G_s :=$ set of all semisimple elements of G

$G_u :=$ " " " unipotent " " "

· Lemma: M be collection of commuting elements in End(V).

a) Then there is a basis $v_1 \dots v_n$ of V such that M is simultaneously upper triangularisable.

b) If every element of M is semisimple, then there is a basis of V such that matrix of every element of M is diagonal.

Proof · Induction on $\dim V$.

· $\dim V = 1$, clear

· Assume $\dim V \geq 2$.

If every element of M is scalar then we are done.

If not, choose $m_0 \in M$, $\lambda_0 \in K$ s.t. the eigenspace of λ_0 is a non-zero proper subspace of V, say V_{λ_0}.

Since every element of M commutes with m_0, we have $M.V_{\lambda_0} \subseteq V_{\lambda_0}$

a) We are done by induction, if we can find a M-stable complement of V_{λ_0} (basis $u_1 \ldots u_r$)

Look at V/V_{λ_0} for this also we are done by induction.

Chose basis $\bar{u}_{r+1}, \ldots, \bar{u}_n$.

Then $u_1 \ldots u_r\, u_{r+1} \cdots u_n$ will work for V.

b) For semisimple, we can break V as direct sum of Eigenspaces. ~~and~~

· Th^m on Commutative algebraic Groups:

G a commutative ~~sub~~group. then

1) G_s, G_u are closed subgroups of G.

2) $\mu : G_s \times G_u \longrightarrow G$ is an isomorphism of groups.

3) Both G_s, G_u connected if G is connected.

· Proof 2) $\Rightarrow$ 3)

$$U(n,k) \subseteq B(n,k) \subseteq Gl(n,k)$$

unipotent, upper triangular

$$B(n,k) \supseteq D(n,k)$$

diagonal

$$B(n,k) = D(n,k) \ltimes U(n,k)$$

normal subgroup of $B(n,k)$.

Embed G in $Gl(V)$.

G commutative, hence choose a basis $u_1 \ldots u_n$ such that $G \subseteq B(n,k)$

1) One can modify proof of above lemma to get $G_s \subseteq D(n,k)$ and $D(n,k) \cap G_s = G_s$

This gives G_s closed subgroup.

Similarly $G_u = G \cap U(n,k)$.

This gives G_u closed subgroup of G.

2) $\mu : G_s \times G_u \longrightarrow G$

· homomorphism follows from G commutative

· iso by Jordan-Chevalley decomposition.

· morphism - inverse? will follow from following lemma

claim: $\varphi_s : G \longrightarrow G_s$ is a morphism

$x \longmapsto x_s$

For $\varphi : \mathcal{B}(n,k) \longrightarrow D(n,k)$ the map is

$$\begin{pmatrix} x_{11} & * & * \\ & x_{22} & * \\ 0 & & \ddots \; x_{nn} \end{pmatrix} \longmapsto \begin{pmatrix} x_{11} & & 0 \\ & x_{22} & \\ 0 & & \ddots \; x_{nn} \end{pmatrix}$$

which is a morphism. ← above splitting for $B(n,k)$

Now for $x \in G$, $x = su$ ← ...
$= x_s x_u$ ← Jor-Ch for

Then $s^{-1} x_s = u (x_u)^{-1} \quad \Rightarrow \quad s^{-1} x_s = 1 = u \cdot x_u^{-1}$

$s^{-1}x_s \in D(n,k)$, $u(x_u)^{-1} \in U(n,k) \quad \Rightarrow \quad s = x_s, \; u = x_u$.

So φ_s is simply φ restricted to G.

=

· Recall $X(G)$ = group of characters of G
$\subseteq K[G]$ because $G_m \subseteq \mathbb{A}^1$

Given $\varphi : G_1 \longrightarrow G_2$, we have

$$\begin{array}{ccc} G_1 & \xrightarrow{\varphi} & G_2 \\ & \searrow^{\chi\cdot\varphi} & \downarrow \chi \\ & & G_m \hookrightarrow \mathbb{A}^1 \end{array}$$

$\varphi^* : X(G_2) \longrightarrow X(G_1)$

· Def^n: d-group G: $X(G)$ generates $K[G]$ as a K-vector space.

· <u>Lemma</u>: $X(G)$ is a linearly independant subset of $K[G]$.

Proof: Assuming contrary, choose minimal $\psi_1, \cdots \psi_n$ such that

$$a_1\psi_1 + a_2\psi_2 + \cdots + a_n\psi_n = 0 \qquad a_i \in K^* \quad \text{--- (1)}$$

Choose $y \in G$ s.t. $\psi_1(y) \neq \psi_n(y)$

Apply to xy,

$$a_1\psi_1(x)\psi_1(y) + a_2\psi_2(x)\psi_2(y) + \cdots + a_n\psi_n(x)\psi_n(y) = 0 \quad \text{- (2)}$$

Then $(1) \times \psi_n(y) - (2)$ gives a contradiction on minimality of ψ_i's.