Properties of Schemes

§ 5.1 Topological properties:

- k field $\Rightarrow$ $\mathbb{P}^n_k$ irreducible (b/c $\mathbb{A}^n_k$ is)
- $\{\text{points in scheme } X\} \xleftrightarrow{1-1} \left\{\begin{matrix}\text{irreducible closed} \\ \text{subsets in } X\end{matrix}\right\}$

- $X = \bigcup_{i=1}^{n} \operatorname{Spec} A_i$, each A_i Noetherian
  $\Rightarrow X$ noetherian
- scheme X quasi compact $\Longleftrightarrow$ finite unions of open affines
- X quasi compact scheme, $x \in A$
  $\Longrightarrow \exists$ a closed pt $y \in \overline{\{x\}}$ (Check)
  Can fail at non-quasi compact scheme.

Cor: X quasi-compact and some open property P holds at all closed points
$\Rightarrow P$ holds at all points on X.

Def^n: A topological X is quasi-separated if intersection of any 2 q-compact open is q-compact.

Prop: 1) Scheme X is q. separated
$\Leftrightarrow$ $\cap$ of any 2 open affines is a finite union of open affines.
(eg: Plane with a doubled origin.)

2) $X = \operatorname{Spec} A$, then X is quasi-separated. (Use $D(f)$).

3) X quasi-compact and quasi-separated
$\Longleftrightarrow$ X covered by finitely many open affines
intersection of any 2 of which coverable by finitely many open affines.

4) All projective A-schemes are quasi-compact & quasi-separated.

§ 5.2 Reducedness and Integrality:

Def: Ring A is reduced if $\mathfrak{N}(A) = 0$ ← nilpotents
Scheme X is reduced if $\Gamma(U, \mathcal{O}_X)$ is a reduced ring for all $U \subseteq X$ open.

Prop: 1) X reduced $\Leftrightarrow$ Stalk $\mathcal{O}_{X,x}$ reduced $\forall x \in X$.
2) ring A reduced $\Leftrightarrow$ $\operatorname{Spec} A$ reduced scheme
3) X quasi compact scheme, X reduced $\Leftrightarrow$ $\mathcal{O}_{X,x}$ reduced $\forall$ closed $x \in X$.

Remark: 1) Reducedness is not an open condition in general.
2) On a non-affine scheme X, can have $\Gamma(X, \mathcal{O}_X)$ reduced but X-non-reduced.

Def^n: Scheme X is integral if $\neq \phi$ & $\Gamma(U, \mathcal{O}_X)$ integral domain $\forall\ \phi \neq U \subseteq X$ open.

Prop: • X integral $\Leftrightarrow$ X reduced & irreducible.
• $\operatorname{Spec} A$ integral $\Leftrightarrow$ A integral domain.
• X integral scheme, $\eta \in X$ a generic point

$$\mathcal{O}_{X,\eta} = \operatorname{Frac} A \quad \text{for any open affine } \operatorname{Spec} A \subseteq X$$
$$\| $$
$K(X)$ function on X
if X is connected.

§ 5.3 Affine local properties:

Prop: X scheme, $\operatorname{Spec} A, \operatorname{Spec} B \overset{\text{open}}{\subseteq} X$ Then $\operatorname{Spec} A \cap \operatorname{Spec} B$ is union of open affines which are simultaneously distinguished in $\operatorname{Spec} A$ & $\operatorname{Spec} B$.

Proof: $\mathfrak{p} \in \operatorname{Spec} A \cap \operatorname{Spec} B$

Say $\mathfrak{p} \in \operatorname{Spec} A_f \subseteq \operatorname{Spec} A \cap \operatorname{Spec} B$

Further suppose $\mathfrak{p} \in \operatorname{Spec} B_g \subseteq \operatorname{Spec} A_f$

We get a map $\mathcal{O}_X(\operatorname{Spec} B) \longrightarrow \mathcal{O}_X(\operatorname{Spec} A_f)$

$$g \longmapsto g'$$

So $\operatorname{Spec} B_g = \operatorname{Spec} A_{fg'}$

Affine Connection Lemma:

Let P be a property such that

i) open affine $\operatorname{Spec} A \hookrightarrow X$ has P

$\Rightarrow$ So does $\operatorname{Spec} A_f \hookrightarrow X$ $\forall f \in A$.

ii) $(f_1, \dots, f_n) = A$ & $\operatorname{Spec} A_{f_i} \hookrightarrow X$ has P for all i

$\Rightarrow \operatorname{Spec} A \hookrightarrow X$ has P

Suppose $X = \bigcup_{i \in I} \operatorname{Spec} A_i$ where $\operatorname{Spec} A_i$ has P. Then every open affine in X has P.

Def: A property satisfying (i), (ii) is called affine local.

- Open subschemes inherit affine-local properties.
- Stalk-local properties are automatically affine local.

Prop: A ring, $(f_1, \dots, f_n) = A$

a) A reduced $\Rightarrow A_{f_i}$ reduced & A_{f_i} reduced $\forall i \Rightarrow A$ reduced

b) A Noetherian $\Leftrightarrow A_{f_i}$ Noetherian $\forall i$

c) B ring, A is a B-algebra, then

A finitely generated$/B \Leftrightarrow A_{f_i}$ finitely generated$/B$ $\forall i$.

Def: Say scheme X locally Noetherian if we can cover X by open affines $\operatorname{Spec} A$ with A noetherian. If furthermore X quasi compact then X noetherian.