Def: $\pi: X \longrightarrow Y$ is separated if $\delta: X \longrightarrow X \times X$ is a closed embedding.

$$\left[\begin{tikzcd} X \arrow[rr, bend left, "id"] \arrow[dr, dashed] \arrow[ddr, bend right, "id"'] & & X \arrow[dd] \\ & X\times X \arrow[ur] \arrow[d] & \\ & X \arrow[r] & Y \end{tikzcd}\right]$$

- Separated morphisms are preserved under base change.

$$\left(\begin{tikzcd} Z \times_Y X \arrow[r] \arrow[d, "sep"'] & X \arrow[d, "sep"] \\ Z \arrow[r] & Y \end{tikzcd}\right)$$

$y \in Y$, $X_y :=$ fiber$/y$

π separated $\Rightarrow$ X_y separated over Spec $K(y)$

$\Rightarrow$ $X_y \hookrightarrow X_y \times_{\mathrm{Spec}\,K(y)} X_y$ is a closed embedding.

- Closed and open embeddings (any monomorphism) are separated.

Prop: $\mathbb{P}^n_A \longrightarrow \mathrm{Spec}\,A$ is separated.

Proof: $U_0, \dots, U_n$ std open affine ... do patching.

Rem: For any $\pi: X \longrightarrow Y$ $\quad U, V \xhookrightarrow{mono} X$

$$\begin{tikzcd} U \times_Y V \arrow[r, hook, "mono"] & X \times_Y X \\ \delta^{-1}(U \times_Y V) \arrow[u] \arrow[r] \arrow[ur, phantom, "\llcorner"] & X \arrow[u, "\delta"'] \end{tikzcd}$$

$$\delta^{-1}(U \times_Y V) = U \cap V = U \times_X V$$

Prop: $\mathrm{Spec}\,A \longrightarrow \mathrm{Spec}\,B$ is always separated. (because $A \otimes_B A \longrightarrow A$, $a_1 \otimes a_2 \longmapsto a_1 a_2$)

Def^n: A variety over a field k is a reduced separated separated finite type scheme / Spec k.
A subvariety is a reduced locally closed reduced subscheme of X.

Prop: $X \to \mathrm{Spec}\,A$ is separated. Let $U, V \subseteq X$ be open affines then $U \cap V$ is affine.

Proof:
$$U \cap V = \underbrace{\underbrace{\left(U \times_{\mathrm{Spec}A} V\right)}_{\text{affine}} \cap\, \delta(X)}$$

$\delta(X)$ is a closed subscheme of $X \times_{\mathrm{Spec}A} X$ so this is a closed subscheme of affine scheme.

Prop: $\pi: X \longrightarrow Y$ quasi-separated if $\delta: X \longrightarrow X \times X$ is quasi-compact.
$\Rightarrow$ π separated $\Rightarrow$ π quasi-separated.

Prop: Separated & Quasi-sep morphisms are stable under base change.

Prop: $\pi : X \longrightarrow Y$ separated $\Leftrightarrow$ for some open cover $Y = \bigcup U_i$ $\pi^{-1}(U_i) \longrightarrow U_i$ is separated.
Similarly q-sep.

Prop: Seperated & q-sep are closed under compositions.

Prop: Every q.proj A-scheme is separated /A.

Prop: $\pi : X \longrightarrow Y$ and $\pi' : X' \longrightarrow Y'$ (q)separated morphisms of S-schemes, then $X \times_S X' \longrightarrow Y \times_S Y'$ is (q) separated.

Rem: Same argument works for any property stable under base change & composition. (Ex. 9.4.F)

Def^n: The graph morphism of $\pi : X \longrightarrow Y$ (over Z) is $X \times_Z Y \xleftarrow{(id, \pi)} X$.

Prop: Γ_π is always locally closed embedding & closed embedding if Y sep $/Z$.

Th^m: Suppose
$$\begin{tikzcd} X \arrow[rr, "\pi"] \arrow[dr, "\tau"'] & & Y \arrow[dl, "\rho"] \\ & Z & \end{tikzcd}$$
commutes. Suppose P is a class of morphisms stable under base change & Composition and $\tau \in P$ and $\delta_\rho : Y \longrightarrow Y \times_Z Y$ is in P then $\pi \in P$.
In particular:

1) Suppose locally closed embedding are in P. Then $\tau \in P \Longrightarrow \pi \in P$.
2) Suppose closed embedding are in P. Then $\tau \in P$, ρ sep $\Rightarrow \pi \in P$