Rationalization

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Space = sSet

<u>Def^n</u>: $X \to Y$ rational equivalence if the induced map $H_*(X;\mathbb{Q}) \to H_*(Y;\mathbb{Q})$ is an isomorphism (or equivalently for H^*).

- Z is rational if $X \to Y$ rational equivalence

  $\Rightarrow$ $[X, Z] \to [Y, Z]$ is an equivalence.

  ($\Leftrightarrow$ $R\mathrm{Map}(Y, Z) \to R\mathrm{Map}(X, Z)$ equivalence)

- $Z \to Z_{\mathbb{Q}}$ is called rationalization if it is an equivalence and $Z_{\mathbb{Q}}$ is rational.

<u>Ex</u>: • $K(\mathbb{Q};n)$ is rational. for $X \to Y$ rational equivalence

$$\begin{tikzcd}
{[Y, K(\mathbb{Q};n)]} \arrow[r, "\simeq"] \arrow[d] & H^*(Y;\mathbb{Q}) \\
{[X, K(\mathbb{Q};n)]} \arrow[r, "\simeq"] & H^*(X;\mathbb{Q}) \arrow[u, "\text{is}"']
\end{tikzcd}$$

- $V = \mathbb{Q}$ vector space $\quad K(V;n)$ is rational.

- Homotopy limits of rational spaces are rational.

$$Z = \operatorname{holim} Z_i$$

$$\begin{tikzcd}
\mathrm{Rmap}(Y, Z) \arrow[d] \arrow[r, phantom, "="] & \operatorname{holim} \mathrm{Rmap}(Y, Z_i) \arrow[d, "\wr"] \\
\mathrm{Rmap}(X, Z) \arrow[r, phantom, "="] & \operatorname{holim} \mathrm{Rmap}(X, Z_i)
\end{tikzcd}$$

- Z simply-connected space s.t. $\pi_*(Z)$ are rational.

See this by induction on Postnikov tower:

$$Z = \operatorname{holim} (\tau_{\leq n} Z)$$

$$\pi_*(\tau_{\leq n} Z) \cong \begin{cases} 0 & * > n \\ \pi_* Z & * \leq n \end{cases}$$

$$\begin{tikzcd}
\tau_{\leq n} Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\
\tau_{\leq n-1} Z \arrow[r] & K(\pi_* Z, n-1)
\end{tikzcd}$$

is a homotopy pullback.

- n-odd integer

$$S^n \to K(\cancel{\mathbb{Z}}\,\mathbb{Q}, n)$$

$$H^*(K(\mathbb{Q},n)) \cong H^*(S^n) \quad \text{because } n\text{-odd}$$

so we get a rational eq.

n-even

$$\text{hofib}\ \cancel{\text{s}}\left[K(\mathbb{Q},n) \xrightarrow{Sq} K(\mathbb{Q},2n)\right]$$

$$\uparrow$$

$$S^n$$

is a ~~ho~~ rational ~~homotopy~~ equivalence.

- $Ho(sSets)_{\mathbb{Q}} \hookrightarrow Ho(sSets)_{*}$ This has a left adjoint $(-)_{\mathbb{Q}}$ which is the rationalization functor.

Full subcategory of rational spaces

Thm: ① 1-connected space X is rational iff $\pi_* X$ is rational.

② $X \to Y$ ~~is~~ between 1-connected spaces is a rational equivalence $\Leftrightarrow$ $\pi_* X \otimes \mathbb{Q} \to \pi_* Y \otimes \mathbb{Q}$ is an isomorphism

3 X 1-connected, $\exists\ X \to X_{\mathbb{Q}}$ with X- 1-connected.

Proof: 2 Assume $X \to Y$ induces iso on $\pi_*(-) \otimes \mathbb{Q}$.

Want to show rational eq.

homotopy fiber: F

then the Serre SS gives

$$H^*(Y;\, H^*(F;\mathbb{Q})) \Rightarrow H^*(X;\mathbb{Q})$$

enough to show $H_*(F;\mathbb{Q}) = 0$, $* \geq 1$.

Use induction on Postnikov tower of F:

$$H_n(F;\mathbb{Q}) \cong H_n(\tau_{\leq n} F;\mathbb{Q})$$

Serre SS for

$$K(A;n) \longrightarrow \tau_{\leq n} F \longrightarrow \tau_{\leq n-1} F$$

A - torsion

$\Rightarrow$ Reduces the claim to show that

$$H_*(K(A;n);\mathbb{Q}) = 0 \qquad * \geq n.$$

By the fibration $K(A, n-1) \to * \to K(A,n)$

Reduces to $K(A,1)$ and this can be shown using group homology of A.

<u>lem</u>: We construct for every 1-connected space X a map

$X \longrightarrow X_{\mathbb{Q}}$ such that (functorial in X)

1) $\pi_*(X_{\mathbb{Q}})$ are rational

2) iso on $\pi_*(-) \otimes \mathbb{Q}$.

This proves ①, ② and ③.

Construct $X_{\mathbb{Q}}$ by induction over Postnikov tower.

if $X = K(A, n)$

$X_{\mathbb{Q}} := K(A \otimes \mathbb{Q}, n)$

. $$\begin{tikzcd} F \arrow[r] & X \arrow[r] \arrow[d] & Y \arrow[d] \\ & X_{\mathbb{Q}} \arrow[r] & Y_{\mathbb{Q}} \end{tikzcd}$$ ~~for~~

← suppose such a diagram exists (can be shown by using some cylinder object argument)

$F_{\mathbb{Q}}$ = hofiber of $X_{\mathbb{Q}} \to Y_{\mathbb{Q}}$

Then $F \longrightarrow F_{\mathbb{Q}}$ ~~is~~ ~~a~~ satisfies the ~~b~~ conditions in the lemma.

. Apply this to
$$\begin{array}{c} \tau_{\leq n} X \\ \downarrow \\ \tau_{\leq n-1} X \end{array}$$
and use $X = \underleftarrow{\mathrm{holim}}\ \tau_{\leq n} X$

$X_{\mathbb{Q}} := \underleftarrow{\mathrm{holim}}\ (\tau_{\leq n} X)_{\mathbb{Q}}$

Corollary: $\pi_*(S^n) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & * = n \\ \mathbb{Q} & * = 2n-1 \quad \text{if } n\text{-even} \\ 0 & \text{else} \end{cases}$

<u>Th^m</u> (Bousfield)

Every space admits a rationalization.

More precisely there is a Quillen model structure on sSet

- Cofib = mono
- w.e. = rational equivalences
- fib objects = rational Kan complexes.

- If M is a model category then ~~th~~ a left Bousfield localization is a model category M' with same underlying category and same Cofib but more weak equivalences. $(W_M \subseteq W_{M'})$

- $Ho(M') \xrightarrow[Rid]{} Ho(M)$

This functor is fully faithful. Rid has a right adjoint Lid this is like the ~~rationa~~ fibrant replacement.

- $W_{M'} \supseteq W_M$, $Fib(M') \subseteq Fib(M)$

  (This determines what the cofib objects are.

  But it is hard to prove that this is indeed a model structure.

"Sketch of proof":

- fibrations have RLP w.r.t acyclic cofib.
- Can use Kan-Quillen model structure to derive most of factorization th^{m}s.

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, hook, "\sim"'] & & Y \\ & F \arrow[ur] & \end{tikzcd}$$

need to construct this factorization

Idea:
- Use small object argument for $X \overset{\sim}{\rightarrowtail} Y$ rational equivalence and cofib.
- restrict to mono $S = \{ X \overset{\sim}{\rightarrowtail} Y \text{ with } Y \text{ countable} \}$
- This gives us $X \overset{\sim}{\hookrightarrow} F \to Y$ RLP with respect to S

Key lemma: Every ~~rational~~ trivial cofib $X \hookrightarrow Y$ is a transfinite compositions of pushouts of maps in S.

- Given any map $A \to B$ want a lift

$$\begin{tikzcd} X \arrow[r] \arrow[d] & A \arrow[d] \\ Y \arrow[r] & B \end{tikzcd}$$

Replace B by pushout P

$\swarrow$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & A \arrow[d] & \\ Y \arrow[r] & P \arrow[r] & B \end{tikzcd}$$

- Use colimit of all such P's say P_∞

and repeat. ~~This is term~~

- Then repeat using transfinite induction.
- By small object argument this process "terminates".