Homework set 2 - Solutions

Math 469 – Renato Feres

This is a somewhat long and tedious problem set. You may choose the more interesting items to work out; it is not necessary to write everything down. But make sure you get the main point, which is the connections among the groups $SO(3)$, $SU(2)$, $SL(2,\mathbb{C})$, the Lorentz group, and the Möbius group. You'll find much more information about the topics covered in this assignment in the first volume of *Spinors and space-time* by R. Penrose and W. Rindler, Cambridge Monographs in Mathematical Physics, 1984. Another nice source is by G.L. Nader: *The geometry of Minkowski spacetime*, Dover, 2003.

Here are the main conclusions of this homework set:

- There is a 2-to-1 surjective homomorphism $SU(2) \to SO(3)$. Topologically, this corresponds to the projection $S^3 \to \mathbb{R}P^3 = S^3/\sim$, where the real projective space is defined as the set of equivalence classes of the equivalence relation that identifies pairs of points on the 3-sphere that are each other's antipodes.
- There is a 2-to-1 surjective homomorphism $SL(2,\mathbb{C}) \to SO(1,3)^{\uparrow}$, where the latter is the group of proper, orthochronous Lorentz transformations of Minkowski space. This group is isomorphic with the group of Möbius transformations, or fractional linear transformations, of the extended complex plane (also known as the Riemann sphere).

⋄

1. **The group $SO(3)$ of rotations in $\mathbb{R}^3$.** Recall that $SO(n)$ is the group of matrices $A \in M(n,\mathbb{R})$ satisfying $A^t A = I$ and $\det(A) = 1$. Elements of $SO(n)$ are in one-to-one correspondence with positive orthonormal bases of $\mathbb{R}^n$. In fact, given any such a basis $\{u_1,\dots,u_n\}$ where the u_j are column vectors, the matrix $[u_1,\dots,u_n]$ is in $SO(n)$ and any matrix in $SO(n)$ is of this type.

   (a) If $\langle u,v\rangle := u\cdot v$ is the ordinary dot product in $\mathbb{R}^n$, show that $A \in SO(n)$ iff $\langle Au, Av\rangle = \langle u,v\rangle$ for all $u,v \in \mathbb{R}^n$ and $\det(A) = 1$. [Because lengths of vectors and angles between vectors can be expressed in terms of the dot product, preservation of the dot product means that the transformation preserves shapes; in other words, it is a *rigid* transformation, or and *isometry*, of the Euclidean metric in $\mathbb{R}^n$.]

   (b) Show that if n is odd, then for each $A \in SO(n)$ there exists a non-zero vector $u \in \mathbb{R}^n$ such that $Au = u$. In particular, every $A \in SO(3)$ is a rotation about an axis.

   (c) Show that any rotation in $SO(3)$ can be expressed as a product $R_1 \dots R_m$ where each R_j is either $R_1(\theta)$, for some θ, or $R_3(\phi)$, for some ϕ, where

$$R_1(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \quad R_3(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Suggestion: Let $e = (0,0,1)^t$. For every unit vector u there exist θ and ϕ such that $u = R_3(\phi)R_1(\theta)e$. Thus $(R_3(\phi)R_1(\theta))R_3(\theta')(R_3(\phi)R_1(\theta))^{-1}u = u$ for all θ'.

(d) Convince yourself (you don't need to write down a proof) of the following topological description of $SO(3)$. Consider a solid ball $B \subset \mathbb{R}^3$ centered at the origin and of radius π. Now parametrize elements of $SO(n)$ by points in B as follows: the center corresponds to the identity element and to each nonzero vector $u \in B$ corresponds the counterclockwise rotation by the angle $|u|$ about the axis u. Observe that pairs of antipode points on the boundary of B correspond to the same rotation. This identification of antipode points at the boundary of B (but not in the interior) is formally the quotient space $B/\sim$ under an equivalence relation $\sim$ and equipped with the *quotient topology*. (Look up `https://en.wikipedia.org/wiki/Quotient_space_(topology)` if you don't know what the quotient topology is.) This topological space is known as the *real projective space* in dimension 3, denoted $\mathbb{R}P^3$. Also convince yourself that this same topological space is obtained by the quotient of the 3-dimensional sphere $S^3 := \left\{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \|x\|^2 = 1\right\}$ under the equivalence relation $x \sim -x$. More generally, one defines the *real projective space* in dimension n by the quotient $\mathbb{R}P^n := S^n/\sim$, where pairs of antipode points on the n-dimensional sphere are identified. (Note: only when $n = 3$ do the spaces $SO(n)$ and $\mathbb{R}P^n$ have the same dimension. The projective space is not related to rotation groups in higher dimensions.)

Solution.

(a) This is immediate from the definition of transpose: Given A, A^t may defined by the property $\langle A^t u, v\rangle = \langle u, Av\rangle$. The more familiar definition in terms of the elements of the matrix follow by taking u, v to be elements in the standard basis of $\mathbb{R}^n$.

(b) This is immediate from part (c) of problem 5 of the first homework assignment. You may also show this by noting that the characteristic polynomial of $A \in SO(n)$ has odd degree and so must have a real root (since the complex roots come in pairs.) The real eigenvalues of A can only be 1 or -1 because if $Au = \lambda u$ for a real eigenvector u, then $\|u\|^2 = \langle u, u\rangle = \langle Au, Au\rangle = \lambda^2\|u\|^2$ so $\lambda = \pm 1$. But -1, if it occurs at all, must occur in pairs. Therefore, there must be a real eigenvector u for the eigenvalue 1. So $Au = u$.

(c) The fact that any unit vector u can be written as $u = R_3(\phi)R_1(\theta)e$ is simply the observation that any point on the unit 2-dimensional sphere centered at the origin of $\mathbb{R}^3$ can be expressed in spherical coordinates. Now every rotation A in $SO(3)$ has an axis of rotation $u \in S^2$, that is, $Au = u$. So it corresponds to a two-dimensional rotation on the plane perpendicular to u. But the expression $(R_3(\phi)R_1(\theta))R_3(\theta')(R_3(\phi)R_1(\theta))^{-1}$, for all θ', realizes all such rotations.

$\diamondsuit$

2. **The Möbius group and** $SL(2,\mathbb{C})$**.** Recall that $SL(2,\mathbb{C})$ is the group of matrices in $M(2,\mathbb{C})$ having determinant 1. A *Möbius transformation*, or *linear fractional transformation*, of the complex plane $\mathbb{C}$ is a function of the complex variable z defined by

$$m(z) = \frac{az+b}{cz+d}$$

where a, b, c, d are complex numbers satisfying $ad - bc \neq 0$. (Check that $m(z)$ is constant if $ad = bc$.) It is convenient to *compactify* $\mathbb{C}$ by adding a (single) point at ∞. I'll denote the resulting topological space (with a suitable topology, that you can read about on point-set topology texts) by $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. This space is homeomorphic to the sphere S^2 and is also known as the *Riemann sphere*. We may extend $m(z)$ to the whole Riemann sphere by defining $m(-d/c) = \infty$ and $m(\infty) = a/c$. In this way the Möbius transformations become homeomorphisms of the Riemann sphere. Let $m_A(z)$ denote the Möbius transformation defined by the matrix of coefficients $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Let $\mathscr{M}$ denote the family of all Möbius transformations on $\mathbb{C}_\infty$.

(a) Show that $\mathscr{M}$ is a group and that the correspondence $A \in SL(2,\mathbb{C}) \mapsto m_A \in \mathscr{M}$ is a group homomorphism. Moreover, show that this is a 2-to-1 and surjective homomorphism.

(b) **Stereographic projection.** The stereographic projection is a bijection $\mathscr{S} : S^2 \to \mathbb{C}_\infty$ defined as follows:

$$\mathscr{S}(X) = \begin{cases} \infty & \text{if } X = \mathscr{N} := (0,0,1) \\ \zeta = x + iy & \text{if the line segment from } (x,y,0) \text{ to } \mathscr{N} \text{ intersects } S^2 \text{ at } X. \end{cases}$$

See figure 1.

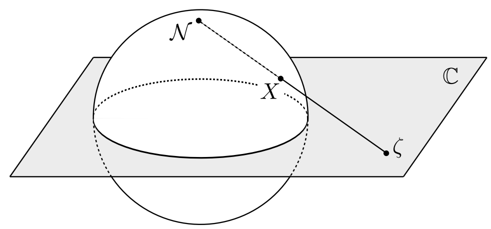

Figure 1: The stereographic projection.

Show that if $X = (X_1, X_2, X_3) \in S^2$, then

$$\mathscr{S}(X) = \frac{X_1 + iX_2}{1 - X_3} \quad \text{and} \quad \mathscr{S}^{-1}(\zeta) = \left(\frac{2\mathrm{Re}(\zeta)}{|\zeta|^2+1}, \frac{2\mathrm{Im}(\zeta)}{|\zeta|^2+1}, \frac{|\zeta|^2-1}{|\zeta|^2+1} \right)$$

(c) We are interested in the transformation that each $m_A \in \mathscr{M}$ induces on S^2 under the stereographic projection. In other words, define $\Lambda_A(X) := \left(\mathscr{S}^{-1} \circ m_A \circ \mathscr{S}\right)(X)$ for each $A \in SL(2,\mathbb{C})$. You will derive below the general form of Λ_A after we introduce the Lorentz group. For the moment, simply show that

$$A = \pm \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \Longrightarrow \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This is the counterclockwise rotation by θ about the X_3-axis. (You may postpone this calculation until problem 5d, where an explicit expression for the general Λ_A is given.

(d) **Projective homogeneous coordinates.** The seeming exceptional nature of the point ∞ in the Riemann sphere can be an inconvenience. There is another description of this space that puts all its points on an equal footing based on the identification of the extended complex plane with the *complex projective space* in (complex) dimension 1, denoted $\mathbb{C}P^1$. In order to define this space, let $\sim$ be the equivalence relation on $\mathbb{C}^2 \setminus \{0\}$ given by

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \sim \begin{pmatrix} \xi' \\ \eta' \end{pmatrix} \Longleftrightarrow \text{there exists } \lambda \in \mathbb{C} \setminus \{0\} \text{ such that } \begin{pmatrix} \xi' \\ \eta' \end{pmatrix} = \lambda \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

In other words, an element of $\mathbb{C}P^1$ consists of a complex line in $\mathbb{C}^2$ passing through the origin. (We exclude

0, of course, because it does not specify a line.) To each equivalence class of $\begin{pmatrix} \xi \\ \eta \end{pmatrix}$ for which $\eta \neq 0$ we can associate $\zeta = \frac{\xi}{\eta} \in \mathbb{C}$. Naturally, $\begin{pmatrix} \xi \\ 0 \end{pmatrix}$ corresponds to ∞. In this way each point of $\mathbb{C}_\infty$ is made to correspond bijectively to a point in $\mathbb{C}P^1$ and each point of the latter space represents a one-dimensional (complex) subspace of $\mathbb{C}^2$. (More generally, the projective space $\mathbb{F}P^n$ of dimension n over the field $\mathbb{F}$ is defined as the set of one-dimensional subspaces of $\mathbb{F}^n$.) Convince yourself (not much to do here!) that by identifying the Riemann sphere with $\mathbb{C}P^1$, the Möbius transformations take on the following form:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \left[\begin{pmatrix} \xi \\ \eta \end{pmatrix} \right] = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right]$$

where $[u]$ indicates equivalence class represented by u.

Solution.

(a) That $A \mapsto m_A$ is a homomorphism is a straightforward verification. It is also clear that it is surjective and that $\pm A \mapsto m_A$. To show that it is 2-to-1, suppose $m_A(z) = z$ for all $z \in \mathbb{C}$. That is, assume that

$$\frac{az+b}{cz+d} = z \text{ for all } z \in \mathbb{C}.$$

Multiplying out by the denominator: $az + b = cz^2 + dz$, so $b = c = 0$ and $a = d$. This means that $A = aI$ is a scalar matrix (a times the identity matrix). But $\det(A) = 1$, so $a = \pm 1$ and $A = \pm I$. Therefore, the homomorphism is indeed 2-to-1.

(b) The key point is to note that $\mathscr{S}(X)$ lies in the positive ray $\lambda(X_1.X_2)$, $\lambda \in [0,\infty)$. The value of λ follows from a simple planar geometry argument using similar triangles. The inverse $\mathscr{S}^{-1}$ is obtained just as easily.

(c) See problem 5d.

(d) This is more notation than substance.

$\diamondsuit$

3. **The special linear group $SL(n,\mathbb{C})$ is connected.** Show that $SL(n,\mathbb{C})$ is path-connected. This is Proposition 1.12 in Hall's book, page 18.

Solution. By the Jordan canonical form, any matrix in $SL(n,\mathbb{C})$ can be written as $A = CBC^{-1}$ where B is upper triangular. Define $A(t) = CB(t)C^{-1}$, where

$$B_{ij}(t) = \begin{cases} B_{ij} & \text{if } i = j \\ (1-t)B_{ij} & \text{if } i \neq j \end{cases}.$$

This defines a path in $SL(n,\mathbb{C}$ such that $A(0) = A$ and $A(1)$ is diagonalizable: $A(1) = C\mathrm{diag}(\lambda_1,\dots,\lambda_n)C^{-1}$ with $\lambda_1 \cdots \lambda_n = 1$. It remains to show that the diagonal matrices in $SL(n,\mathbb{C})$ form a connected subgroup. Now, let $D(t) = \mathrm{diag}(\lambda_1(t),\dots,\lambda_n(t))$ where $(\lambda_2(t),\dots,\lambda_n(t))$ is a continuous path in $\mathbb{C}^{n-1}$ connecting $(\lambda_2,\dots,\lambda_n)$ to $(1,\dots,1)$ such that $\lambda_j(t) \neq 0$ for all $j = 2,\dots,n$. (It is easy to describe one such a path explicitly.) Then let $\lambda_1(t)$ be the reciprocal of the product of the $\lambda_j(t)$ for $j \geq 2$. The resulting diagonal matrix will have determinant 1 and will connect D to the identity matrix. Since A was an arbitrary element of $SL(n,\mathbb{C})$, this group must be path-connected.

$\diamondsuit$

4. **The Lorentz group.** The groups $SL(2,\mathbb{C})$ and $\mathscr{M}$ are intimately related to the Lorentz group of relativity theory as will be seen in the next problem. But first we need to become acquainted with the Lorentz group.

   The *Minkowski space*, or space-time in special relativity theory, is $\mathbb{R}^4$ equipped with the indefinite inner product, known as the Lorentz metric, $(X,Y) = X_0Y_0 - X_1Y_1 - X_2Y_2 - X_3Y_3$. In physics $X_0 = ct$ where t is time and c is the speed of light. The Lorentz transformations are the linear transformations of $\mathbb{R}^4$ that leave invariant the Lorentz metric. These transformations constitute the *Lorentz group*

$$O(1,3) := \left\{A \in M(4,\mathbb{R}) : (AX, AY) = (X,Y) \text{ for all } X, Y \in \mathbb{R}^4\right\}.$$

   Equivalently, $A \in O(1,3)$ exactly when $A^t g A = g$ where $g = \mathrm{diag}(1,-1,-1,-1)$.

   (a) Show that the determinant of a Lorentz transformation is either 1 or -1.

   (b) Show that if $A^t g A = g$, then $A_{00}^2 - \left(A_{10}^2 + A_{20}^2 + A_{30}^2\right) = 1$. Thus A_{00} is either greater than 1 or less than -1.

   (c) Show that the map $A \mapsto (\det(A), \mathrm{sign}(A_{00}))$ is a group homomorphism.

   (d) Show that $SO(1,3)^{\uparrow} = \{A \in O(1,3) : \det(A) = 1 \text{ and } A_{00} \geq 1\}$ is a normal subgroup of $O(1,3)$. This is called the *proper orthochronous* Lorentz group.

   (e) Define the diagonal matrices

$$\Lambda_0 = \mathrm{diag}(1,1,1,1),\ \ \Lambda_1 = \mathrm{diag}(1,-1,-1,-1),\ \ \Lambda_2 = \mathrm{diag}(-1,1,1,1),\ \ \Lambda_3 = \mathrm{diag}(-1,-1,-1,-1),$$

   Show that

$$O(1,3) = \Lambda_0 SO(1,3)^{\uparrow} \cup \Lambda_1 SO(1,3)^{\uparrow} \cup \Lambda_2 SO(1,3)^{\uparrow} \cup \Lambda_3 SO(1,3)^{\uparrow}$$

   and that this is the union of disjoint sets.

   (f) Show that the quotient group $O(1,3)/SO(1,3)^{\uparrow}$ is isomorphic to the finite group $\{\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3\}$.

   (g) Show that the subgroup of $SO(1,3)^{\uparrow}$ of all A such that $A_{00} = 1$ is isomorphic to $SO(3)$.

   (h) Show that the subgroup of $SO(1,3)^{\uparrow}$ leaving unchanged the coordinates X_2 and X_3 consists of block-diagonal matrices $\mathrm{diag}(A,1,1)$ such that

$$A = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}, \quad t \in \mathbb{R}.$$

   These transformations are called *boosts.*

   Solution.

   (a) This is because $\det(g) = \det(A^t)\det(g)\det(A)$, hence $\det(A)^2 = 1$.

   (b) The quantity on the left-side of the equation is $(0,0)$-element of the matrix $A^t g A$, whereas the right-hand side of the equation is the $(0,0)$-element of g.

   (c) Note that

$$\begin{aligned}(AB)_{00} &= A_{00}B_{00} + \sum_{i=1}^{3} A_{0i}B_{i0} \\ &= \mathrm{sign}\,(A_{00})\,\mathrm{sign}\,(B_{00})\left(1 + \sum_{i=1}^{3} A_{0i}^2\right)^{1/2}\left(1 + \sum_{i=1}^{3} B_{i0}^2\right)^{1/2} + \sum_{i=1}^{3} A_{0i}B_{i0}\end{aligned}$$

Let us write $a = (A_{01}, A_{02}, A_{0,3})$ and $b = (B_{10}, B_{20}, B_{30})$. Then the above can be written as

$$\left|(AB)_{00} - \operatorname{sign}(A_{00})\operatorname{sign}(B_{00})\left(1+\|a\|^2\right)^{1/2}\left(1+\|b\|^2\right)^{1/2}\right| = |a\cdot b| \le \|a\|\|b\|.$$

Therefore,

$$\operatorname{sign}(A_{00})\operatorname{sign}(B_{00})\left(1+\|a\|^2\right)^{1/2}\left(1+\|b\|^2\right)^{1/2} - \|a\|\|b\| \le (AB)_{00}$$

and

$$(AB)_{00} \le \operatorname{sign}(A_{00})\operatorname{sign}(B_{00})\left(1+\|a\|^2\right)^{1/2}\left(1+\|b\|^2\right)^{1/2} + \|a\|\|b\|.$$

This shows that if A_{00} and B_{00} have the same sign, then $(AB)_{00} > 0$ and if A_{00} and B_{00} have opposite signs, then $(AB)_{00} < 0$. Consequently, the pair $(\det(A), \operatorname{sign}(A_{00}))$ must be an element of the finite (product) group $\{(1,1),(1,-1),(-1,1),(-1,-1)\}$ with multiplication $(a,b)(a',b') = (aa',bb')$. It also follows that the correspondence $A \mapsto (\det(A), \operatorname{sign}(A_{00}))$ is a group homomorphism.

(d) The group $SO(1,3)^\uparrow$ is the kernel of the homomorphism obtained in the previous item. Therefore, it must be a normal subgroup of $O(1,3)$.

(e) $O(1,3)$ is the disjoint union $O_{++} \cup O_{+-} \cup O_{-+} \cup O_{--}$ where O_{ab} is the subset of matrices in $O(1,3)$ having determinant a and sign of the top-left element b. By definition, $O_{++} = SO(1,3)^\uparrow$. It is a straightforward matrix multiplication exercise to check that

$$O_{+-} = \Lambda_3 SO(1,3)^\uparrow \quad O_{-+} = \Lambda_1 SO(1,3)^\uparrow, \quad O_{--} = \Lambda_2 SO(1,3)^\uparrow.$$

(f) The multiplicative group generated by the Λ is isomorphic to the group just introduced in the above item 4c. The fact that $O(1,3)/SO(1,3)^\uparrow$ is isomorphic to this group is one of the basic isomorphism properties we learn in beginning group theory. (The quotient $G/\ker(\rho) = \operatorname{im}(\rho)$, where ρ is a group homomorphism, $\ker(\rho)$ is its kernel and $\operatorname{im}(\rho)$ is its image.)

(g) If $A_{00} = 1$, then by the second item of this exercise $A_{10} = A_{20} = A_{30} = 0$. Note that if $A \in SO(1,3)^\uparrow$, then $A^t \in SO(1,3)^\uparrow$. In fact, since A^{-1} is in the group, $(A^{-1})^t g A^{-1} = g$, and by taking the inverse on both sides of this identity (and noting that $g^{-1} = g$) we obtain $AgA^t = g$. Therefore, $A^t \in O(1,3)$. But the transpose does not change the determinant and the sign of the top-left element, so $A^t \in SO(1,3)^\uparrow$. Thus we also conclude that $A_{01} = A_{02} = A_{03} = 0$ and so $A = \operatorname{diag}(1,R)$ for some 3-by-3 matrix R of determinant equal to 1. The equation defining the Lorentz group immediately implies that $R^t R = I$. Therefore, $R \in SO(3)$. This shows that $SO(3)$ is isomorphic to the subgroup of $SO(1,3)^\uparrow$ consisting of matrices with top-left element 1.

(h) If $\operatorname{diag}(A,1,1) \in SO(1,3)^\uparrow$ then $A^t \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Let us write $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then the following equations and inequality are satisfied:

$$a^2 - c^2 = 1, \;\; d^2 - b^2 = 1, \;\; ab - cd = 0, \;\; ad - cb = 1, \;\; a \ge 1.$$

Now if $b = 0$ or $c = 0$, it is easy to obtain that $A = I$. So let us assume that b and c are not zero. The third equation above implies $\lambda = \frac{a}{c} = \frac{d}{b}$. Simple algebraic manipulation of these equations gives $a^2 = d^2 = ad = \lambda^2/(\lambda^2 - 1)$. From this it follows that $a = d$. We also find that $c = b$. Thus $A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ with $a > 0$ and $a^2 - b^2 = 1$. But such a and b can always be written as $a = \cosh t$, $b = \sinh t$ for some $t \in \mathbb{R}$.

$\diamondsuit$

5. **The spin map and the celestial sphere.** We now turn to the relation between $SL(2,\mathbb{C})$ and $SO(1,3)^{\uparrow}$, as well as the relation between $SU(2)$ and $SO(3)$. Define the *Pauli matrices*

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The function

$$\mathscr{P} : \mathbb{R}^4 \mapsto X_0\sigma_0 + X_1\sigma_1 + X_2\sigma_2 + X_3\sigma_3 = \begin{pmatrix} X_0 + X_3 & X_1 - iX_2 \\ X_1 + iX_2 & X_0 - X_3 \end{pmatrix} \in M(2,\mathbb{C}).$$

Note that $\mathscr{P}$ maps elements of Minkowski space into the 4-dimensional subspace $\mathscr{H} \subset M(2,\mathbb{C})$ of Hermitian matrices. That is, $\mathscr{P}(X)^* = \mathscr{P}(X)$. Define the linear map $M_A : \mathscr{H} \to \mathscr{H}$ given by $M_A(B) := ABA^*$ and

$$\Lambda_A := \mathscr{P}^{-1} \circ M_A \circ \mathscr{P}$$

for each $A \in SL(2,\mathbb{C})$.

(a) Show that $\det(\mathscr{P}(X)) = (X,X)$ is the Lorentz norm of X.

(b) Show that $(\Lambda_A X, \Lambda_A X) = (X,X)$. Conclude that $\Lambda_A \in O(1,3)$ for each $A \in SL(2,\mathbb{C})$.

(c) Show that $A \in SL(2,\mathbb{C}) \mapsto \Lambda_A \in O(1,3)$ is a group homomorphism. That is, $\Lambda_{A_1A_2} = \Lambda_{A_1}\Lambda_{A_2}$.

(d) If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, show that Λ_A has the explicit form

$$\Lambda_A = \begin{pmatrix} \frac{|a|^2+|b|^2+|c|^2+|d|^2}{2} & \mathrm{Re}(a\overline{b}+c\overline{d}) & \mathrm{Im}(a\overline{b}+c\overline{d}) & \frac{|a|^2-|b|^2+|c|^2-|d|^2}{2} \\ \mathrm{Re}(a\overline{c}+b\overline{d}) & \mathrm{Re}(a\overline{d}+b\overline{c}) & \mathrm{Im}(a\overline{d}+c\overline{b}) & \mathrm{Re}(a\overline{c}-b\overline{d}) \\ \mathrm{Im}(c\overline{a}+d\overline{b}) & \mathrm{Im}(d\overline{a}+c\overline{b}) & \mathrm{Re}(a\overline{d}-c\overline{b}) & \mathrm{Im}(c\overline{a}+b\overline{d}) \\ \frac{|a|^2+|b|^2-|c|^2-|d|^2}{2} & \mathrm{Re}(a\overline{b}-c\overline{d}) & \mathrm{Im}(a\overline{b}+d\overline{c}) & \frac{|a|^2-|b|^2-|c|^2+|d|^2}{2} \end{pmatrix}$$

And if $A = \begin{pmatrix} a & -\overline{c} \\ c & \overline{a} \end{pmatrix} \in SU(2)$,

$$\Lambda_A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathrm{Re}(a^2-c^2) & \mathrm{Im}(c^2-a^2) & 2\mathrm{Re}(c\overline{a}) \\ 0 & -\mathrm{Im}(a^2+c^2) & \mathrm{Re}(a^2+c^2) & 2\mathrm{Im}(c\overline{a}) \\ 0 & -2\mathrm{Re}(ac) & -2\mathrm{Im}(ac) & |a|^2-|c|^2 \end{pmatrix}$$

As noted in problem 4g, the matrix R in $\Lambda_A = \mathrm{diag}(1,R)$ is in $SO(3)$.

This is a tedious calculation. I suggest the following approach. Define the matrix H

$$H = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = \begin{pmatrix} X_0 + X_3 & X_1 - iX_2 \\ X_1 + iX_2 & X_0 - X_3 \end{pmatrix}$$

and note that

$$\begin{pmatrix} h_{11} \\ h_{12} \\ h_{21} \\ h_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -i & 0 \\ 0 & 1 & i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{pmatrix}, \quad \begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} h_{11} \\ h_{12} \\ h_{21} \\ h_{22} \end{pmatrix}$$

Let G denote the above 4-by-4 matrix on the left. The matrix on the right is G^{-1}. By expressing the 2-by-2 matrices as 4-dimensional column vectors as above, verify that the equation $H' = AHA^*$ takes the form

$$\begin{pmatrix} h'_{11} \\ h'_{12} \\ h'_{21} \\ h'_{22} \end{pmatrix} = \begin{pmatrix} |a|^2 & a\overline{b} & b\overline{a} & |b|^2 \\ a\overline{c} & a\overline{d} & b\overline{c} & b\overline{d} \\ c\overline{a} & c\overline{b} & d\overline{a} & d\overline{b} \\ |c|^2 & c\overline{d} & d\overline{c} & |d|^2 \end{pmatrix} \begin{pmatrix} h_{11} \\ h_{12} \\ h_{21} \\ h_{22} \end{pmatrix}.$$

Let R_A be the above 4-by-4 matrix. Then $\Lambda_A = G^{-1} \circ R_A \circ G$.

(e) **The spinor map.** The map $A \in SL(2,\mathbb{C}) \to \Lambda_A \in SO(1,3)^\uparrow$ is known as the *spinor map*. Show that the spinor map is a 2-to-1 homomorphism. More specifically, show that $\Lambda_A = \Lambda_B$ if and only if $B = \pm A$.

(f) **Boosts.** Show that if $A = \begin{pmatrix} \cosh\frac{\theta}{2} & \sinh\frac{\theta}{2} \\ \sinh\frac{\theta}{2} & \cosh\frac{\theta}{2} \end{pmatrix}$ then

$$\Lambda_A = \begin{pmatrix} \cosh\theta & \sinh\theta & 0 & 0 \\ \sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(g) **Rotations.** Show that if $A = \begin{pmatrix} \cos(\theta/2)e^{-\frac{i}{2}\phi} & -i\sin(\theta/2)e^{-\frac{i}{2}\phi} \\ -i\sin(\theta/2)e^{\frac{i}{2}\phi} & \cos(\theta/2)e^{\frac{i}{2}\phi} \end{pmatrix} \in SU(2)$ then the following holds: If $\phi = 0$ then $\Lambda_A = \mathrm{diag}(1, R)$ where R is equal to

$$R_1(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

and if $\theta = 0$ then $\Lambda_A = \mathrm{diag}(1, R)$ where R is equal to

$$R_3(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(h) Show that $A \in SU(2) \mapsto \Lambda_A \in \mathrm{diag}(1, SO(3))$ is surjective. (See exercise 1c.)

(i) Show that $A \in SL(2,\mathbb{C}) \mapsto \Lambda_A \in SO(1,3)^\uparrow$ is surjective. You may use the proposition stated at the end of this assignment. Also see exercise 4h.

Solution.

(a) This is a straightforward computation of determinant:

$$\det(\mathscr{P}(X)) = (X_0 + X_3)(X_0 - X_3) - (X_1 + iX_2)(X_1 - iX_2) = X_0^2 - X_3^2 - X_1^2 - X_2^2 = (X, X).$$

(b) From $\mathscr{P}(\Lambda_A X) = M_A \mathscr{P}(X) = A\mathscr{P}(X)A^*$ we obtain, by taking determinants:

$$(\Lambda_A X, \Lambda_A X) = \det((P)(\Lambda_A X)) = \det(A\mathscr{P}(X)A^*) = \det(A)\det(\mathscr{P}(X))\det(A^*) = \det(\mathscr{P}(X)) = (X, X).$$

Note: For all $X, Y \in \mathbb{R}^4$ we have

$$(X,Y) = \frac{1}{2}\{(X+Y, X+Y) - (X,X) - (Y,Y)\}.$$

Thus if $(\Lambda_A X, \Lambda_A X) = (X,X)$ for all X, it follows that $(\Lambda_A X, \Lambda_A Y) = (X,Y)$ for all X, Y. But this means that $\Lambda_A \in O(1,3)$ for all $A \in SL(2,\mathbb{C})$.

(c) To prove the group homomorphism property first note:

$$M_{A_1A_2}(H) = A_1A_2H(A_1A_2)^* = A_1\left[A_2HA_2^*\right]A_1^* = M_{A_1}\circ M_{A_2}(H).$$

Therefore,

$$\Lambda_{A_1A_2} = \mathscr{P}^{-1}\circ M_{A_1A_2}\circ\mathscr{P} = \left(\mathscr{P}^{-1}\circ M_{A_1}\circ\mathscr{P}\right)\circ\left(\mathscr{P}^{-1}\circ M_{A_2}\circ\mathscr{P}\right) = \Lambda_{A_1}\circ\Lambda_{A_2}.$$

(d) As already said above, this is a tedious calculation, but the suggestion given will make it orderly and help avoid mistakes. (No assurances that I haven't made a few myself in the stated form of the matrix.)

(e) It is clear that $\pm A$ map to the same Λ_A. We need to show that $\Lambda_A = \Lambda_B$ if and only if $B = \pm A$. Note that $\Lambda_A = \Lambda_B$ if and only if $\Lambda_{AB^{-1}}$ is the identity. Let $AB^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. By equating the general form of Λ_A with the identity matrix and solving the system of equations for a,b,c,d, together with the determinant equation $ad - bc = 1$, it is not hard to obtain that $b = c$, $a = d = \pm 1$. Therefore, $AB^{-1} = \pm I$ and $A = \pm B$ as claimed.

(f) This is also a straightforward verification using the general form of Λ_A given above and standard identities involving cosh and sinh.

(g) The same remark just made for boosts applies to rotations.

(h) Substitute the elements of this A into the general form of Λ_A given above.

(i) This follows from exercise 1c.

(j) We already know that all the rotations $\text{diag}(1,R)$, $R \in SO(3)$ already are in $SO(1,3)^{\uparrow}$. Given exercise 4h and the fact that general boosts are obtained from the $A \in SL(2\mathbb{C})$ just given above, surjectivity follows from the proposition given at the end of this assignment.

♢

6. **Möbius transformations on the celestial sphere.** Let $\mathscr{C} \subset \mathbb{R}^4$ denote the *light-cone*. This is the set of vectors $X = (X_0, X_1, X_2, X_3)$ such that $(X,X) = X_0^2 - (X_1^2 + X_2^2 + X_3^2) = 0$. The positive (respectively, negative) light cone is the set of $X \in \mathscr{C}$ such that $X_0 > 0$ (respectively, $X_0 < 0$.) In $\mathscr{C}^{\pm}$ define the equivalence relation $X \sim Y$ if and only if $Y = \lambda X$ for some $\lambda > 0$. The equivalence classes, denoted $[X]$, constitute the sets $P\mathscr{C}^{\pm}$ (P for "projective.") It makes sense to call $P\mathscr{C}^{\pm}$ the (*past* for $-$ and *future* for $+$) *celestial sphere*. See Figure 2. The following are all equivalent descriptions (i.e., these sets are in bijective correspondence)

$$P\mathscr{C}^{\pm} \cong S^2_{\pm} = \{X \in \mathscr{C} : X_0 = \pm 1\} \cong S^2 = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\} \cong \mathbb{C}P^1.$$

(Recall the definition of $\mathbb{C}P^1$ in exercise 2d.)

Let us define the map $\mathscr{R} : \mathbb{C}P^1 \to P\mathscr{C}_+$ that associates to each equivalence class in $\mathbb{C}P^1$ a future light ray in $P\mathscr{C}^+$

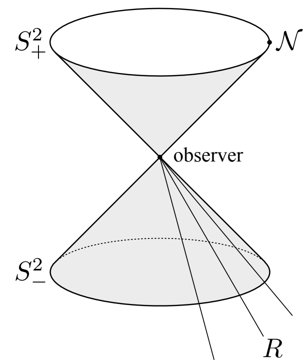

Figure 2: The light cone in Minkowski space is the set of $X \in \mathbb{R}^4$ such that $(X,X) = 0$. Define the spheres $S^2_\pm = \{X : X_0 = \pm 1\}$. The semi-infinite lines issuing from the origin (the observer) and passing through a point of S^2_- are thought of as light rays from the celestial sphere reaching the eye of the observer. We identify S^2_+ with the Riemann sphere. The north pole is $\mathcal{N} = (1,0,0,1)$. A proper orthochronous Lorentz transformation is completely determine by its action on the celestial sphere.

such that, on class representatives,

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \mapsto \begin{pmatrix} |\xi|^2 + |\eta|^2 \\ 2\mathrm{Re}(\xi\overline{\eta}) \\ 2\mathrm{Im}(\xi\overline{\eta}) \\ |\xi|^2 - |\eta|^2 \end{pmatrix}.$$

The group $SL(2,\mathbb{C})$ acts as a group of transformations on $\mathbb{C}P^1$ as seen in exercise 2 and $SO(1,3)^\uparrow$ naturally acts as a group of transformations on $P\mathscr{C}_+$.

(a) Show that $\mathscr{R}(Au) = \Lambda_A \mathscr{R}(u)$ for all $A \in SL(2,\mathbb{C})$ and all $u \in \mathbb{C}P^1$. More explicitly, consider the correspondence

$$\begin{aligned} X_0 &= |\xi|^2 + |\eta|^2 \\ X_1 &= 2\mathrm{Re}(\xi\overline{\eta}) \\ X_2 &= 2\mathrm{Im}(\xi\overline{\eta}) \\ X_3 &= |\xi|^2 - |\eta|^2 \end{aligned}$$

and observe that

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \begin{pmatrix} \overline{\xi} & \overline{\eta} \end{pmatrix} = \begin{pmatrix} |\xi|^2 & \xi\overline{\eta} \\ \eta\overline{\xi} & |\eta|^2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} X_0 + X_3 & X_1 + iX_2 \\ X_1 - iX_2 & X_0 - X_3 \end{pmatrix} = \frac{1}{2}\mathscr{P}(X_0, X_1, -X_2, X_3).$$

From this conclude that

$$\mathscr{R} : A\begin{pmatrix} \xi \\ \eta \end{pmatrix} \mapsto \Lambda_A \begin{pmatrix} X_0 \\ X_1 \\ -X_2 \\ X_3 \end{pmatrix}$$

where we suppressed the factor $1/2$ because it does not change the equivalence class. The slightly unpleasant $-X_2$ could be fixed if we had defined the second Pauli matrix as $\sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$.

(b) The following properties of Möbius transformations of the Riemann sphere $\mathbb{C}_\infty$ hold: any such transformation that is not the identity fixes at least one and at most two points. Moreover, a Möbius transformation on the Riemann sphere is completely determined by by their values on any three distinct points (that may include ∞.) Use these facts to conclude: (1) A proper, orthochronous Lorentz transformation, if not the identity, leaves invariant at least one and at most two past null directions. And (2) a proper, orthochronous Lorentz transformation is completely determined by its action on any three past null directions.

Solution.

(a) Given the remarks in the statement of the exercise, the key point is that

$$A\begin{pmatrix}\xi\\ \eta\end{pmatrix}\left(A\begin{pmatrix}\xi\\ \eta\end{pmatrix}\right)^* = A\begin{pmatrix}\xi\\ \eta\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix}^* A^* = \frac{1}{2}A\mathscr{P}(X_0, X_1, -X_2, X_3)A^* = \Lambda_A(X_0, X_1, -X_2, X_3)^t.$$

(b) By extending past null directions into the future cone, we may substitute future null directions for the past null directions in this statement. Because the action on the Riemann sphere of $SL(2,\mathbb{C})$ exactly corresponds to the action of $SO(1,3)^\uparrow$ on the future null directions as we have shown, the facts about Möbius transformations directly translated into the similar facts about Lorentz transformations.

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