$$\begin{tikzcd} Z \arrow[r, "f'"] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & U \end{tikzcd}$$

$Z$ is pull back.
Then:

- $f$ epic $\Rightarrow$ $f'$ epic, $\ker g \cong \ker g'$.

$X \xrightarrow{f} U \xrightarrow{\alpha} V$ $\qquad \alpha \cdot f = 0 \Rightarrow \alpha = 0$

Suppose $\quad Z \xrightarrow{f'} Y \xrightarrow{\alpha'} V \qquad \alpha' \cdot f' = 0$

Then, $\quad$ ??

→ Weibel 1.3.5

$X_\cdot, Y_\cdot$ chain complexes $\qquad f : X_\cdot \to Y_\cdot$

$\ker f$, $\operatorname{coker} f$ acyclic.

Break $f$ as

$$0 \to \ker f \to X_\cdot \to \operatorname{Im} f \to 0$$

$$0 \to \operatorname{Im} f \to Y_\cdot \to \operatorname{coker} f \to 0$$

$\ker f$, $\operatorname{coker} f$ acyclic $\Rightarrow$ $H_n(X_\cdot) = H_n(\operatorname{Im} f)$
$= H_n(Y_\cdot)$

And the map $H_n(X_\cdot) \to H_n(\operatorname{Im} f) \to H_n(Y_\cdot)$ is just $f_*$.

→ Double complexes:

$$\begin{tikzcd} & \uparrow & \\ \arrow[r] & C_{i,j} \arrow[r] & {} \\ & \uparrow & \end{tikzcd} \qquad d^v\uparrow \qquad \xrightarrow{d^h} \qquad \begin{array}{l} d^v \cdot d^v = 0, \\ d^h \cdot d^h = 0, \end{array} \quad d^v d^h = d^h \cdot d^v$$

Then form a total complex $\operatorname{Tot}(C_\cdot)$ with

$$\operatorname{Tot}(C)_i = \sum_{p+q=i} \oplus\, C_{i,j} \qquad \text{and}$$

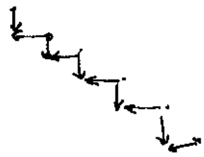
$$d : \operatorname{Tot}(C)_i \to \operatorname{Tot}(C)_{i-1} \qquad d = d^v \oplus (-1)^q d^h$$

$$d^2 = (d^v + (-1)^q d^h)(d^v + (-1)^{q+1} d^h)$$

$$= \underbrace{\cancel{d^v \cdot d^v}}_{0} + (-1)^{q \cdot q+1} \underbrace{\cancel{d^h \cdot d^h}}_{0} + \cancel{d^h \cdot d^v \cdot (-1)^q} + \cancel{d^v \cdot d^h (-1)^{q+1}}$$

$= 0$

$\longrightarrow$ Tot(C) acyclic whenever bounded, exact rows
(Assume $2^{nd}$ Quadrant)



Weibel 1.2.5

$\mathrm{Tot}(C)_n = \bigoplus_{p+q=n} C_{p,q}$

$\cancel{Z_n\big|_{p,q} = \ker(-d^h + d^v(-1)^p)\big|_{p,q}} =$ assume signs included in def$^n$ of $d$

$(x_{0n}, x_{1n-1}, \ldots, x_{n0}) \in Z(C)_n$ then

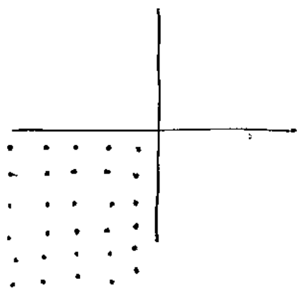
$(d^h x_{0n} + d^v x_{1n-1},\ d^h x_{1n-1} + d^v x_{2n-2}, \ldots,\ d^h x_{n-1\,1} + d^v x_{n0}) = 0$

---

Assume C $3^{th}$ quadrant

Then if rows are exact,

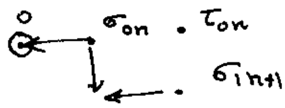
Claim: any element of $H_n(C)$ can be represented by an element of $\mathrm{Tot}_n(C)$ having only 1 non-zero entry in first column $C_{*0}$



Proof:

$[\sigma] \in H_n(C)$

$\Rightarrow \sigma = (\sigma_{0n}, \sigma_{1\,n+1}, \cdots, \sigma_{n0})$




$d\sigma = 0$

$\Rightarrow d^h \sigma_{0n} = 0,\quad d^v \sigma_{0n} + d^h \sigma_{1n+1} = 0$

$\downarrow$

Rows are exact $\Rightarrow \sigma_{0n} = d^h \tau_{0n-1}$ for some $\tau$

Now look at

$\sigma - d\tau$,

By def$^n$ $[\sigma - d\tau] = [\sigma]$

But $(\sigma - d\tau)_{0n} = \sigma_{0n} - \sigma_{0n} = 0$

In this way keep on making top row element 0.

We can continue this even beyond $0^{th}$ column as $H_0$ is also given to be 0. So we will get an element in 1 - column. But these are all 0.

So $H_*(C) = 0$.

Weibel 1.2.6 $\longrightarrow$

a) Example of double complex $C$ with $\mathrm{Tot}^{\Pi}(C)$ not acyclic but $\mathrm{Tot}^{\oplus}C$ acyclic:

$$\begin{array}{ccccc}
\mathbb{Z}\oplus\mathbb{Z} & & \mathbb{Z}\oplus\mathbb{Z} & & \mathbb{Z}\oplus\mathbb{Z} \\
\mathbb{Z}\oplus\mathbb{Z} & & \mathbb{Z}\oplus\mathbb{Z} & & \mathbb{Z}\oplus\mathbb{Z} \\
\vdots & & \vdots & & \vdots \\
\mathbb{Z}\oplus\mathbb{Z} & \leftarrow & \mathbb{Z}\oplus\mathbb{Z} & \leftarrow & \mathbb{Z}\oplus\mathbb{Z} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{Z}\oplus\mathbb{Z} & \leftarrow & \mathbb{Z}\oplus\mathbb{Z} & \leftarrow & \mathbb{Z}\oplus\mathbb{Z}
\end{array}$$

$$d^v = d^h = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Chain complex, double complex is easy to check.

~~$d((x_1,y_1),(x_2y_2),\dots)$~~

~~$= (y_1\ 0)\ (y_2 0)\ \cdots$ except when $(x_1,y_1)$ is in lowest row.~~

~~$Z_i(C.) =$ if $i>0$~~

~~$(*,0)\ (*,0)\cdots,\Sigma$~~

~~if $i\le 0$~~

~~$(*,*),\ (*,0)\cdots$~~

~~For $\mathrm{Tot}^{\oplus}(C.)$~~

for $i^{th}$ cell of total complex,

if $i>0$

$$d(x_1,y_1)\ (x_2y_2)\cdots$$
$$= (y_1\ 0)\ (y_2+y_1\ 0)\ (y_3+y_2\ 0)\cdots$$

if $i\le 0$

$$d(x_1,y_1)\ (x_2\,y_2)\cdots$$
$$= (0,0)\ (y_2+y_1,0)\ (y_3+y_2\ 0)\cdots$$

For $\mathrm{Tot}^{\Pi}(C.)$ :

$$Z_i(C.) = 0 \quad \text{if} \quad i>0$$
$$= (*,y)\ (*,-y)\ (*,y)\ (*,-y) \dots \quad \text{if} \quad i<0$$

$B_i(C.)$ consists only of things with second co-ordinate 0.

So $H_i(\mathrm{Tot}^{\Pi} C_\bullet) = \bigoplus_{\mathbb{Z}} \mathbb{Z}$ for $i<0$
$= 0$ else

For $\mathrm{Tot}^{\oplus} C.$ :

$Z_i(C.) = 0$

$\because$ only finite entries are allowed to be 0.

So $\mathrm{Tot}^{\oplus} C$ acyclic.

b) Example of double complex with rows exact, neither $\mathrm{Tot}^{\oplus} C$ ~~not~~ acyclic ~~but~~ nor $\mathrm{Tot}^{\Pi} C$ acyclic

$$\begin{tikzcd}
{} & \mathbb{Z} \arrow[l, "1"'] \arrow[d, "1"] & 0 & & \\
 & \mathbb{Z} & \mathbb{Z} \arrow[l, "1"'] \arrow[d, "1"] & 0 & \\
 & 0 & \mathbb{Z}. & \mathbb{Z} \arrow[l, "1"'] \arrow[d, "1"] & 0 \\
 & & 0 & \mathbb{Z} & \mathbb{Z} \arrow[l, "1"']
\end{tikzcd}$$

$d^h = d^v = 1$

~~Now~~

From $C_0$:

$$d(x_1, x_2 \dots) = (x_1+x_2,\ x_2+x_3, \dots)$$

$C_1$:

$$d(x_1, x_2 \dots) = 0$$

For $\mathrm{Tot}^{\oplus} C$:

$Z_{-1}(C.) = (*, *, *)$

$Z_0(C.) = (x_1, -x_1, x_1, -x_1, \dots) \cap C_1 = 0$

(only finitely many terms are allowed non-zero)

$H_0(C.) = 0$

$H_{-1}(C.) \neq 0$

Look at $(1,0,0\dots)$ in $Z_{-1}(C.)$

if $d(x_1, x_2 \dots) = (1, 0, \dots)$

$\Rightarrow x_1 = 1,\ x_2 = -1,\ x_3 = 1, \dots$

But only finite non-zero terms are allowed.

For $\mathrm{Tot}^{\Pi} C$:

$Z_{-1}(C.) = (*, *, \dots)$

$B_{-1}(C.) = (*, *, \dots)$

$Z_0(C.) = (x_1, -x_1, x_1, -x_1, \dots)$

So $H_0(C.) = \mathbb{Z}$

So we have both rows and columns exact

But neither $\mathrm{Tot}^{\oplus}$ nor $\mathrm{Tot}^{\Pi}$ acyclic.

c) $\mathrm{Tot}^{\oplus}$ not acyclic but $\mathrm{Tot}^{\Pi}$ acyclic ??

Weibel 1.3.2

⟶ 3×3 lemma

$$\begin{tikzcd}
X' & A' \arrow[r] \arrow[d, "i"] & B' & C' \\
X & A & B \arrow[r] \arrow[d] & C \arrow[d] \\
X'' & A'' \arrow[u, phantom, "\pi"] & B'' \arrow[r] & C''
\end{tikzcd}$$

columns are exact. Squares commute.

- Bottom 2 rows exact

$A' = \ker \pi_A$ $\quad B' = \ker \pi_B$ $\quad C' = \ker \pi_C$

$$\begin{tikzcd}
0 & \ker f \arrow[d] & \ker g \arrow[d] \\
& B \arrow[r, "\alpha"] \arrow[d, "f"] & C \arrow[d, "g"] \\
& B'' \arrow[r, "\beta"] & C''
\end{tikzcd}$$

$\ker f \xrightarrow{\hat{\alpha}} \ker g$ which map?

$\alpha|_{\ker f} \subseteq \ker g$ ?

$g\alpha(x) = ?$ $\quad$ if $fx = 0$

$= \beta f(x) = 0$

surjective? $\quad \ker g \xrightarrow{t} T$

$t \cdot \hat{\alpha} = 0 \Rightarrow$

---

We directly say that ~~because~~ the third row is either a kernel or cokernel of the remaining two

So we have a map between complexes ~~X''~~

$0 \to X'' \to X \to X' \to 0$ $\quad$ exact

So we get long exact sequence

~~$0 \to \ker A' \to \ker B'/\ker A \to \ker\, \mathrm{im}\, B'$~~

~~$\downarrow$~~

~~$\ker A \to \ker B/\ker A$~~

$$\begin{tikzcd}
0 \arrow[r] & \ker A' \arrow[r] & \ker A \arrow[r] & \ker A'' \arrow[d] \\
& & & \frac{\ker B'}{\mathrm{im} A'} \arrow[r] & \frac{\ker B}{\mathrm{im} A} \arrow[r] & \frac{\ker B''}{\mathrm{im} A} \arrow[d] \\
& & & & & C'/\mathrm{im} B' \arrow[r] & C/\mathrm{im} B \arrow[r] & C''/\mathrm{im} B'' \arrow[r] & 0
\end{tikzcd}$$

Any two rows acyclic $\Rightarrow$ 3rd acyclic.

$\longrightarrow$ Weibel 1.4.1

a) $R_3 \xrightarrow{d} R_2 \xrightarrow{d} R_1 \longrightarrow 0$

Splits because of freeness

Remains to show

$\ker R_2 \xrightarrow{d} R_1$ is also free

Incomplete

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16 / 01 / 13

Category ch(R-mod)

Comparison Th$^m$:

Given a complex

$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$ $\qquad P_i$ Projective

Given $f: M \longrightarrow N$. Then for any resolution $Q_\cdot \to N$

$\exists$ a map $P_\cdot \to Q_\cdot$ lifting $f$, unique upto chain homotopy.

Def$^n$: Resolution: of $M$

~~Exact~~ Acyclic chain ending in $M \to 0$.

$\cdots \to Q_2 \to Q_1 \to Q_0 \to N \to 0$

Projective Module: $P$

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d] & \\ M \arrow[r, "\pi"] & N \arrow[r] & 0 \end{tikzcd}$$

Given $f: P \to N$, $\pi: M \to N$

Then $f$ lifts to $M$.

eg: 1) Every free is projective

2) Direct summand of projective is projective

3) Direct summand of free

Hom$(P,-)$ exact.

→ Free (set $S$) adjoint to forgetful functor

$$\mathrm{Hom}_{Set}(S, \mathrm{Set}(M)) \xrightarrow{\simeq} \mathrm{Hom}_{R\text{-mod}}(\mathrm{Free}(S), M)$$

So ~~Forget~~ $\{ \mathrm{Hom}_{Set}(-, \mathrm{Set}\,-) = \mathrm{Hom}_{R\text{-mod}}(\mathrm{Free}\,-, -)$

1) Free ⇒ Projective

$$\begin{tikzcd} \mathrm{Free}(S) \arrow[dr] & & \\ N \arrow[r, two heads] & M \arrow[r] & 0 \end{tikzcd}$$

gives

$$\begin{tikzcd} S \arrow[dr] & & \\ N \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

this lifts so result follows

2) ~~submod~~ summand of projective

$$\begin{tikzcd} A \arrow[dr] & & \\ N \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} A\oplus B \arrow[dr, shift left] \arrow[dd, dashed] & & \\ & A \arrow[ul, shift left] \arrow[dr] & \\ N \arrow[rr] & & M \arrow[r] & 0 \end{tikzcd}$$

This exists

so $A \to M$ lifts to $A \to N$.

3) ~~summand~~ projective ⇔ sumand of free ?

4) $\mathrm{Hom}(P,-)$ exact

~~Break~~ Error is only in surjectivity

In general $N \to M \to 0$

does not imply $\mathrm{Hom}(\cancel{N,P}\ P,N) \to \mathrm{Hom}(P,M) \to 0$

But for projective this holds.

In particular this is saying that

$\mathrm{Ext}^1_R(P,A) = 0$ ~~(or $\mathrm{Ext}^1_R(A,P)=0$)~~ $\forall A$

which one?

3)

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d, "\text{identity}"] & \\ \mathrm{Free}(P) \arrow[r] & P \arrow[r] & 0 \end{tikzcd}$$

This makes $P$ into a direct summand of free $(P)$.

Th$^m$: Every projective module over $K[x_1, \ldots x_n]$ is free.

(Read)

Example of a projective module which is not free:

- $R = \mathbb{Z}/6\mathbb{Z}$

  $M = \mathbb{Z}/6\mathbb{Z}$ $\quad P = 3\mathbb{Z}/6\mathbb{Z}$ (or $2\mathbb{Z}/6\mathbb{Z}$)

  Then $\underset{\substack{\| \\ R}}{M} = 3\mathbb{Z}/6\mathbb{Z} \oplus 2\mathbb{Z}/6\mathbb{Z}$ but $3\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$ which is not free.

Q. Prove that $P$ is projective iff $\mathrm{Ext}^1_R(P, A) = 0 \ \forall A$. using earlier defined $\mathrm{Ext}^1_R(\ ,\ )$.

$\longrightarrow$ $\quad 0 \to A \to * \to P \to 0$

But Projective $\Rightarrow$
$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d, "id"] & \\ * \arrow[r] & P \arrow[r] & 0 \end{tikzcd}$$
lifts.

So this splits $\Rightarrow * = A \oplus P$

$\Rightarrow \mathrm{Ext}^1(P, A) = 0$.

---

We usually assume that ~~an~~ give category, for every object $M$ there is a projective ~~module~~ $P$ s.t. $P \to M \to 0$.

Then we can construct a projective ~~modu~~ resolution of $M$.

$$\begin{tikzcd}[column sep=small] \arrow[r] & P_2 \arrow[rr] \arrow[dr] & & P_1 \arrow[rr, "f_1"] \arrow[dr] & & P \arrow[r, "f_0"] & M \arrow[r] & 0 \\ & & \ker f_1 \arrow[ur] \arrow[dr] & & \ker f_0 \arrow[ur] \arrow[dr] & & & \\ & 0 \arrow[ur] & & 0 \arrow[ur] & & 0 & & \end{tikzcd}$$

Def$^n$: $C_\cdot$ - chain in R-mod $\boxed{\text{split}}$ if $\exists s: C_n \to C_{n+1}$ s.t. $dsd = d$.

Note: $\quad f := ds + sd \qquad\qquad s: C_n \to D_{n+1}$

$df = dsd = d$ in this case

$fd = dsd = d$ $\quad$ "

So $f$ is a chain map $C_\cdot \to D_\cdot$ !

what is $f_*$ on $H_n(C_\cdot)$?

~~z s.t.~~ $[z] \in Z_n(C_\cdot) \qquad dz = 0$

$f_*(z) = [dsz + sdz]$

$= [dsz] = 0$

Ans: 0 map

Such a map is called null-homotopic. $(ds + sd)$

Def$^n$: • $f$ null-homotopic if $f = ds + sd$ for some $s$

$f \sim g$ if $f - g = ds + sd$

• $f : C_\bullet \to D_\bullet$ is homotopy equivalence if

$\exists g : D_\bullet \to C_\bullet$ s.t. $fg \sim 1_D$ $gf \sim 1_C$

• $1$ null homotopic $\Leftrightarrow$ $C_\bullet$ exact

Does the converse hold? i.e. $C_\bullet$ exact $\Rightarrow$ $1$ null-homotopic

Ex: Define a new category of chain complexes: $K$

$$\mathrm{Hom}_K(C_\bullet, D_\bullet) := \mathrm{Hom}(C_\bullet, D_\bullet)/\sim$$

Prove: ~~Ab~~ Additive category but not Abelian.

$\longrightarrow$ Proof of comparison th$^m$:

Existense of extension:

$$\begin{tikzcd} \arrow[r] & P_2 \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] \arrow[d, dashed, "f_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\ \arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

$f_0$:

$$\begin{tikzcd} P_0 \arrow[dr, "d"] \arrow[dd, dashed] & & \\ & M \arrow[dr, "f"] & \\ Q \arrow[rr] & & N \arrow[r] & 0 \end{tikzcd}$$

lift $fd$ by projectivity

$f_i$:

$$\begin{tikzcd} P_i \arrow[dr, "d"] & & \\ & P_{i-1} \arrow[dr, "f_{i-1}"] & \\ Q_i \arrow[rr, "d"'] & & Q_{i-1} \end{tikzcd}$$

Need to lift ~~d~~ $f_{i-1} \cdot d$

for this we need

$\underset{\text{Ker}\,d}{\overset{\|}{\mathrm{im}\, d}} \supseteq \mathrm{im}\, f_{i-1} \cdot d$

Need $d \cdot f_{i-1} \cdot d = 0$

But $d \cdot f_{i-1} = \cancel{d} f_{i-2} \cdot d$ So done

Homotopy equivalence of lifts:

Need to find $s$ such that $f_i - g_i = sd + ds$

$s_{-1}$:

$$\begin{tikzcd} & M \arrow[r] \arrow[d, "0"] \arrow[dl, "s_{-1}"'] & 0 \\ Q \arrow[r, "d"'] & N \arrow[r] & 0 \end{tikzcd}$$

$s_{-1} = 0$

$S_0$:

$$\begin{tikzcd} P_1 \arrow[r] & P_0 \arrow[r, "d"] \arrow[d, "f_0-g_0"] \arrow[dl, dashed, "?"'] & M \arrow[dl, "0"] \\ Q_1 \arrow[r, "d"'] & Q_0 \arrow[r] & N \end{tikzcd}$$

To lift $f_0-g_0$ by $d$
we need
$\operatorname{im} d \subseteq \operatorname{im}(f_0-g_0)$
$\overset{\shortparallel}{\ker d}$
need $d(f_0-g_0)=0$
$\overset{\shortparallel}{fd-fd}$. So done.

$S_i$:

$$\begin{tikzcd} P_{i+1} \arrow[r] & P_i \arrow[r] \arrow[d, "f_i-g_i"] \arrow[dl, dashed] & P_{i-1} \arrow[d, "f_{i-1}-g_{i-1}"] \arrow[dl, "s_{i-1}"'] \\ Q_{i+1} \arrow[r] & Q_i \arrow[r] & Q_{i-1} \end{tikzcd}$$

want $f_i-g_i = ds_i + s_{i-1}d$

Need to lift $f_i-g_i-s_{i-1}d$ along $d$

For this need

$$d[f_i-g_i-s_{i-1}d]=0$$

$$d[f_i-g_i] = [f_{i-1}-g_{i-1}]d$$
$$= [ds_{i-1}+s_{i-2}d]d = ds_{i-1}d$$

So $d[f_i-g_i-s_{i-1}d]=0$

Note: did not require exactness of projective resolution. $\boxtimes$

Proof of Excercise:

Additive: $C_\bullet, D_\bullet \in Ob(\mathscr{C}K)$.

$(C_\bullet \oplus D_\bullet)_n \overset{?}{=} C_n \oplus D_n$

Abelian ?

Define: Injective objects $I$

1) $\operatorname{Hom}(-, I)$ is exact

2) ~~Lifting~~ Factoring / Extension property

$$\begin{tikzcd} I & & \\ A \arrow[u, dashed] & B \arrow[l] \arrow[ul] & 0 \arrow[l] \end{tikzcd}$$

example: 1) if $R=\mathbb{Z}$, $I=\mathbb{Z}$, $A=\mathbb{Z}$, $B=\mathbb{Q}$

$$\begin{tikzcd} & \mathbb{Z} & \\ 0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[u, "id"] & \mathbb{Q} \arrow[ul, dashed] \end{tikzcd}$$

no extension of $\mathbb{Q}$ exists

$\rightarrow$ Abelian group $I$ injective $\Leftrightarrow$ $I$ divisible

→ Prop: If extension property is true for $0 \to \mathcal{I} \to R$ then it is true for all $0 \to A \to B$.
ideal, ring; R-modules

Proof:

- $0 \to \wp \to R \quad \forall\ \wp < R$

  $\Rightarrow \quad 0 \to \wp \to \mathfrak{m}$ how?

  Restrict the extension to $R$ to $\mathfrak{m}$

  ~~$0 \to A \to B$~~

  ~~For an element~~ $x$ ~~, Let $I_x$ denote annihilator of~~ $x$

- $0 \to M \to M + Ra$

  Then $M \cap Ra = \mathcal{I}a$ for some $\mathcal{I} < R$

  we know how to extend from $\mathcal{I}$ to $R$. Use this to extend from $\mathcal{I}a$ to $Ra$

- Now form a poset of all extensions

  $0 \to A \to C \qquad C \leq B$.

  with $0 \to A \to C_1 < 0 \to A \to C_2$ if

  $0 \to A \to C_1 \to C_2$

  Then given a chain

  $0 \to A \to C_1 \to C_2 \to \cdots$

  we have an upper bound $C_1 \cup C_2 \cup \cdots$

  This gives us by Zorn's lemma a maximal element $M$.

  If $M \neq B$ take $x \in M \setminus B$, and we know that

  ~~$0 \to B \to B$~~ $0 \to M \to M + Rx$

  contradicting maximality of $M$.

  So $M = B$.

  (Note that there is subtelty as to why we cannot apply Zorn's lemma to "all" extensions of $A$. Because all extensions of $A$ do not form a set while all subextensions of $B$ do)

---

- For $R = \mathbb{Z}$, injective module $I \Rightarrow 0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z}$ is an extension

  ie. if $x \in I$ so does "$\frac{x}{n}$".

  Such abelian groups are called divisible.

- How to embed an arbitrary R-module into an injective module?

Def$^n$: Left derived functor of a right exact functor: $\mathcal{F} : \mathcal{A} \longrightarrow \mathcal{B}$
eg: $- \otimes_R M$

$L_i\mathcal{F}$ — Given object $A$
$i \geq 0$ Take Projective resolution of $A$ - $P$.
Apply ~~P~~ $\mathcal{F}$ to $P$. Take $i^{th}$ homology of ~~P~~ $\mathcal{F}P$.

$$L_i\mathcal{F}(A) := H_i(\mathcal{F}P.)$$

- $L_0\mathcal{F} = \mathcal{F}$ $\rightsquigarrow$ because $P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$
  gives $\mathcal{F}(P_1) \longrightarrow \mathcal{F}(P_0) \longrightarrow \mathcal{F}(A) \longrightarrow 0$
  So $\mathcal{F}(A) = \mathcal{F}(P_0) / \text{im}\, \mathcal{F}(P_1)$
  $= H_0(\mathcal{F}P.)$

[Aim: To show $L_n\mathcal{F}$ is a universal $\delta$-functor.]

- well defined by ~~looking~~ invoking comparison th$^m$ between extension of identity to two projective resolutions.

$$\begin{tikzcd} P. \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "id"] & 0 \\ Q. \arrow[r] & A \arrow[r] & 0 \end{tikzcd} \qquad \begin{tikzcd} FP. \arrow[r] \arrow[d] & FA \arrow[r] \arrow[d] & 0 \\ FQ. \arrow[r] & FA \arrow[r] & 0 \end{tikzcd}$$

- $L_i\mathcal{F}$ is an additive functor ~~of on~~ $\mathcal{A} \longrightarrow \mathcal{B}$

  i.e. $A \xrightarrow{f} B \xrightarrow{g} C$

  gives $L_i\mathcal{F}(g \circ f) = L_i\mathcal{F}(g) \cdot L_i\mathcal{F}(f)$

$$\begin{tikzcd} P. \arrow[r] \arrow[d] & A \arrow[d, "f"] \\ Q. \arrow[r] \arrow[d] & B \arrow[d, "g"] \\ R. \arrow[r] & C \end{tikzcd} \longrightarrow \begin{tikzcd} \mathcal{F}P. \arrow[r] \arrow[d] & \mathcal{F}A \arrow[d, "\mathcal{F}f"] \arrow[dd, bend left=60, "\mathcal{F}(f \circ g)"] \\ \mathcal{F}Q. \arrow[r] \arrow[d] & \mathcal{F}B \arrow[d, "\mathcal{F}g"] \\ \mathcal{F}R. \arrow[r] & \mathcal{F}C \end{tikzcd}$$

Because $\mathcal{F}$ is an additive functor.

addition: $L_i\mathcal{F}(f+g) = \cancel{H_i(\mathcal{F}(f+g))}\; L_i\mathcal{F}f + L_i\mathcal{F}g$

Result follows because $(f+g)_* = f_* + g_*$

Ex 1.4.1.

a) $\xrightarrow{d} R_i \xrightarrow{d} \cdots \xrightarrow{d} R_2 \xrightarrow{d} R_1 \xrightarrow{d} R_0 \to 0$

$R_i$ free $R$-module

- $\to R_1 \xrightarrow{d_1} R_0 \to 0$ $\qquad R_0$-free

$\Rightarrow \exists\, s_0 : R_0 \to R_1$ s.t. $d_1 s_0 = 1$

so that $R_1 \cong R_0 \oplus \ker d_1 \cong \cancel{R_0 \oplus \operatorname{im} d_1} \cong \operatorname{im} d_1 \oplus \ker d_1$

$\to R_2 \xrightarrow{d_2} \underset{R_1}{R_1} \to \operatorname{im} d_1 \to 0$

~~im d₁ submodule of R₁ ⇒ projective~~

~~So R₁ → im d₁ lifts to R₂ → im d~~

$\to R_2 \xrightarrow{d_2} \operatorname{im} d_2 \to 0 \qquad 0 \to \operatorname{im} d_2 \cong \ker d_1 \to R_1 \underset{s_0}{\rightleftarrows} \operatorname{im} d_1 \to 0$, $\operatorname{im} d_1 \cong R_0$

$\operatorname{im} d_2$ projective

$$\begin{tikzcd} & \operatorname{im} d_2 \arrow[dl, dashed] \arrow[d, "id"] & \\ R_2 \arrow[r] & \operatorname{im} d_2 \arrow[r] & 0 \end{tikzcd}$$

lifting identity gives $s_1 : \operatorname{im} d_2 \to R_2$

Further

$$\begin{tikzcd} & R_1 \arrow[dl, dashed] \arrow[d, "s_1"] \\ R_2 \arrow[r, "d_2"'] & \operatorname{im} d_2 \end{tikzcd}$$

lifts $\qquad s_1 : R_1 \to R_2$

- $\to R_2 \xrightarrow{d_2} R_1 \to \operatorname{im} d_1 \to 0$

$$\begin{tikzcd} & R_1 \arrow[dl, dashed] \arrow[d, "\pi"] & \\ R_2 \arrow[r, "d_2"'] & \operatorname{im} d_2 \arrow[r] & 0 \end{tikzcd}$$

$R_1 \cong \operatorname{im} d_1 \oplus R_0$

lift $\pi$ along $d_2$ to get $s_1$ s.t.

$d_2 \cdot s_1 = \pi$

$\Rightarrow d_2 \cdot s_1 \cdot d_2 = \pi \cdot d_2 = d_2$

$\Rightarrow R_2 \cong \operatorname{im} d_2 \oplus \ker d_2$

- Induction: $R_i \cong \operatorname{Im} d_i \oplus \ker d_i$

$$\begin{tikzcd} & R_i \arrow[dl, dashed, "s_i"'] \arrow[d, "\pi"] & \\ R_{i+1} \arrow[r, "d_{i+1}"'] & \operatorname{im} d_{i+1} \arrow[r] & 0 \end{tikzcd}$$

lift $\pi$ along $d_{i+1}$ to get $s_i$

b) we needed $\operatorname{im} d_i$ to be projective.

for $\mathbb{Z}$-module we have submodule of free is free.

So result follows. $\boxtimes$