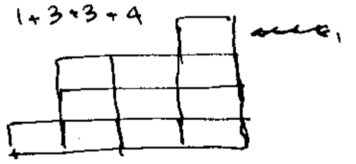
Q.7) A) for G_n - Schubert symbols $(\sigma_1, \cdots, \sigma_n)$ correspond to cells of dim $(\sigma_1 - 1) + (\sigma_2 - 2) + \cdots + (\sigma_n - n)$ where $1 \leq \sigma_1 < \sigma_2 < \cdots < \sigma_n$

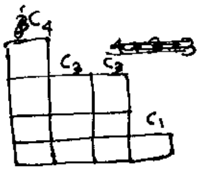
So, if the Chern classes are $c_i \in H^{2i}(G_n)$, (similary $\omega_i \in H^i(G_n(\mathbb{R}$

The cup product $\overset{str.}{\wedge}$ is given by,

$$\sigma_2(\sigma_1, \ldots, \sigma_n) \longmapsto \cancel{c_1^{(\sigma_n - \sigma_{n-1})} \cdot c_2^{(\sigma_{n-1} - \sigma_{n-2})} \cdots c_n^{\sigma_2 - \sigma}}$$

1+3+3+4



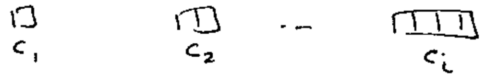


~~1+3+3~~ 4+3+3+1

$n = 4$

$\sigma = (1+1, 2+3, 3+3, 4+4) \longrightarrow c_1^1 \, c_2^0 \, c_3^2 \, c_4^1$

$(\sigma_1, \cdots, \sigma_n) \longmapsto c_1^{\#1\text{'s in dual partion}} \cdots c_i^{\# i\text{'s in dual partition}} \cdots$



Q.7 B), ~~5~~ ??

Q.7) c)

$$\begin{tikzcd} \xi^m \arrow[dr] & & \eta^n \arrow[dl] \\ & B & \end{tikzcd}$$

$\omega_1(\xi^m \otimes \eta^n) = ?$

we will use splitting principal:

1) $m = n = 1$, $\exists f, g : B \longrightarrow \mathbb{R}P^\infty$ s.t. $\xi = f^*\gamma$, $\eta = g^*\gamma$

canonical line bundle

$$\begin{tikzcd} \xi \arrow[r] \arrow[d] & \gamma \arrow[d] & \eta \arrow[l] \arrow[d] \\ B \arrow[r, "f"] & \mathbb{R}P^\infty & B \arrow[l, "g"'] \end{tikzcd}$$

$\omega_1(\xi \otimes \eta) = \omega_1(f^*\gamma \otimes g^*\gamma)$ $\omega_1(\xi) = f^*x$, $\omega_1(\eta) = g^*x$

$\mathbb{R}P^\infty = K(\mathbb{Z}/2, 1) \Rightarrow H^1(B, \mathbb{Z}/2) \cong [B, \mathbb{R}P^\infty]_*$

$\overset{\shortparallel}{G_1}$

$\cancel{f^*(x(\mathbb{R}P^\infty))} \longleftarrow\!\shortmid [f]$

f^*x ↝ generator of $H^1(\mathbb{R}P^\infty, \mathbb{Z}/2)$

$x : S^1 \hookrightarrow \mathbb{R}P$

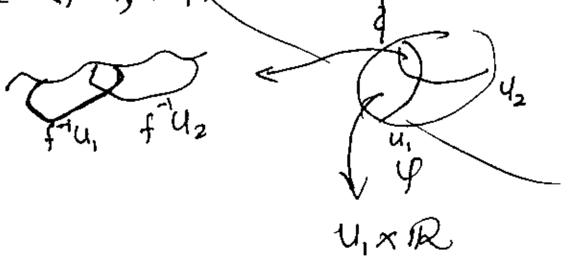
$\mathbb{R}P^\infty \times \mathbb{R}P^\infty \underset{\pi_2}{\overset{\pi_1}{\longrightarrow}} \mathbb{R}P^\infty$

$\tilde{K}(x) = [X, BO]$

~~[illegible]~~, $\text{Vect}^1_{\mathbb{R}}(SX) = [X, Gl_1(\mathbb{R})] = [X, \mathbb{R} - \{0\}]$

~~Transition fns~~ charts of $f^*\gamma = (f^{-1}U, f^*\varphi) = (f^{-1}U, \varphi \circ f)$

I know the transition f^ns for $\xi \otimes \eta$. what are the charts?



How are $E_1 \otimes E_2$, $E_1 \oplus E_2$ related?

$$E_1 \otimes E_2 \downarrow \begin{bmatrix} fg & \\ & g \end{bmatrix} \qquad E_1 \oplus E_2 \downarrow \begin{bmatrix} f & \\ & g \end{bmatrix}$$

$$\begin{bmatrix} -1 & \\ & +1 \end{bmatrix} \qquad \begin{matrix} +1 & \\ & -1 \end{matrix}$$

For $\mathbb{R}P^\infty$, $\mathbb{C}P^\infty$

$$\begin{tikzcd} i^*E \arrow[d] & E \arrow[d] \\ B^{(1)} \arrow[r, hook, "i"] & B \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "f"] & \mathbb{R}P^\infty \\ & \mathbb{R}P^1 \arrow[u] \end{tikzcd}$$

$\omega_1(i^*E) = i^*\omega_1(E)$? ✓

$i^*: H^1(B) \longrightarrow H^1(B^{(1)})$

is ~~surjective~~ injective ? ✓

---

1) $m = n = 1$

Claim: $\omega_1(\xi \otimes \eta) = \omega_1(\xi) \oplus \omega_1(\eta)$

Proof: Let $B^{(1)}$ be 1 skeleton of B. $i: B^{(1)} \hookrightarrow B$

$$\cdots \longleftarrow H^1(B^{(1)}) \xleftarrow{i^*} H^1(B) \longleftarrow H^1(B/B^{(1)}) \longleftarrow \cdots$$

$H^1(B/B^{(1)}) = 0 \quad \because$ No 1-skeleton

$\Rightarrow \quad i^*: H^1(B) \longrightarrow H^1(B^{(1)})$ injective

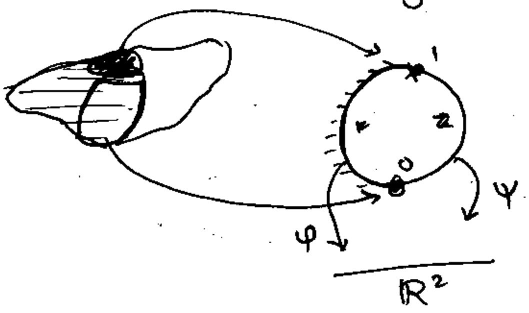
Enough to show

$$\omega_1(i^*\xi \otimes i^*\eta) = \omega_1(i^*\xi) \oplus \omega_1(i^*\eta)$$

So Assume B has only 1 skeleton

Then $\exists\ f, g : B \longrightarrow \mathbb{R}P^1$ s.t. $\xi = f^*\gamma$

$\mathbb{R}P^1 = S^1$, $\quad \eta = g^*\gamma$



Charts on B:

E_1 : ~~$(f^{-1}U_1, \varphi \circ f)$~~ , ~~$(f^{-1}U_2, \psi \circ f)$~~

Transition ~~f^n : ξ φ~~

Next we $\boxed{\text{combine } f, g}$ as

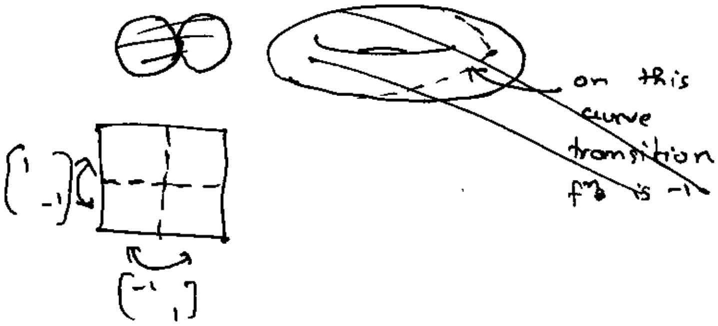
$$\begin{tikzcd} B \arrow[r, "f"] & S^1\times S^1 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ S^1 & & S^1 \end{tikzcd}$$

$\omega_1(f^*\gamma \otimes g^*\gamma)$

$= \omega_1(F^*(\pi_1^*\gamma \otimes \pi_2^*\gamma))$

$= F^*\omega_1(\pi_1^*\gamma \otimes \pi_2^*\gamma)$

γ now is the mobius strip.



$H^1(S^1)=\mathbb{Z}/2$ $H^1(S^1\times S^1)=\mathbb{Z}/2\oplus\mathbb{Z}/2$

~~So just need to check~~

$S^1 \xrightarrow{i_1} S^1\times S^1 \xrightarrow{\pi_1} S^1$

$\pi_1\circ i_1 = \mathrm{id}$

$F=i_1$ $B=S^1$ $f=\mathrm{id}$ $g=0$

$\omega_1(f^*\gamma\otimes g^*\gamma) = \omega_1(f^*\gamma) = i_1^*\omega_1(\pi_1^*\gamma\otimes\pi_2^*\gamma)$

$\Rightarrow \omega_1(\pi_1^*\gamma\otimes\pi_2^*\gamma) = \mathbb{Z}/2\oplus\mathbb{Z}/2 = \pi_1^*\omega_1(\gamma)\oplus\pi_2^*\omega_2(\gamma)$

$\Rightarrow \boxed{\omega_1(\xi\otimes\eta) = \omega_1(\xi)\oplus\omega_1(\eta)}$

ii) For general m, n

$$\begin{tikzcd}[column sep=small] \xi \arrow[d] & \eta \arrow[dl] & p^*\xi \arrow[d] & p^*\eta \arrow[dl] & q^*p^*\xi \arrow[d] & q^*p^*\eta \arrow[dl] \\ B & & F(\xi) \arrow[ll, "p"] & & F(p^*\eta) \arrow[ll, "q"] & \end{tikzcd}$$

Both $q^*p^*\xi$, $q^*p^*\eta$ are sum of line bundles

and p^*, q^* injective on cohomologies

So enough to show for sum of line bundles

$\omega(\oplus\xi_i \otimes \oplus\eta_j) = \omega(\bigoplus_{i,j} (\xi_i\otimes\eta_j))$

$= \bigcup_{i,j}\omega(\xi_i\otimes\eta_j)$ $=$

$= \bigcup_{i,j}(1+\omega_1(\xi_i)+\omega_1(\eta_j))$

$=$

Steifel whitney classes :

- Thom isomorphism: $\mathbb{Z}/2$

$\exists u \in H^n(E, E_0)$, $\phi: H^{i}(B) \xrightarrow{\sim} H^i(E) \xrightarrow[\cup u]{\sim} H^{i+n}(E, E_0)$

$u|_{\text{fibre}} =$ generator of $H^n(F, F_0; \mathbb{Z}/2)$.

- $\boxed{w_i^{\xi}(E) = \phi^{-1} \cdot Sq^i \cdot \phi(1)}$

$\Rightarrow \quad w_i(E) \cup u = Sq^i(u)$

0.8. A) Wu's formula :

$$Sq^k(w_m) = \sum_{i=0}^{k} \binom{k-m}{i} w_{k-i} w_{m+i}$$

eg: $\xi = L_1 \oplus L_2 \oplus L_3$

$w_2(\xi) = w_1 L_1 \cup w L_2 + w_1 L_2 \cup w_1 L_3 \oplus w_1 L_1 \cup w_1 L_3$

$Sq^1(w_2) = w_1 L_1 \cup w_1 L_1 \cup w_1 L_2 + \cdots$

RHS $= w_1(\xi)\, w_2(\xi) + w_0 w_3$

$= (w_1 L_1 + w_1 L_2 + w_1 L_3)(w_1 L_1 \cup w_1 L_2 + \cdots + \cdots)$

$+ \; w_1 L_1 \cup w_1 L_2 \cup w_1 L_3$

Proof:

splitting principle

$\xi = L \oplus \eta$ $\qquad$ L - Line bundle

$w_m(\xi) = w_m(\eta) + w_1(L) \cup w_{m-1}(\eta)$

Assume inductively formulae for η

$Sq^k(w_1(L) \cup w_{m-1}(\eta)) = w_1(L) \cup Sq^k w_{m-1}(\eta)$

$+ \; Sq^1 w_1(L) \cup Sq^{k-1} w_{m-1}(\eta)$

$+ \; \underset{0}{Sq^2 (w_1(L))} \cup Sq^{k-2} w_{m-1}(\eta)$

~~By Universal Property, we can assume ξ, L, η bundles over $\mathbb{RP}^\infty$.~~

- ~~if L = trivial, we are done as $w_1(L) = 0$~~
- ~~$L = \gamma_1 \quad w_1(L) = x \in H^1(\mathbb{RP}^\infty, \mathbb{Z}/2)$~~

~~$Sq^1(x) = x^2$~~ $\qquad$ [illegible]

~~$Sq^2(x^2) = x^4 = Sq^0(x^4)\cdot Sq^2(x) + Sq^1(x)\, Sq^1(x) + Sq^2(x)\, Sq^0(x)$~~

Adem's Relations:

$$Sq^i Sq^j = \sum_{K=0}^{[i/2]} Sq^{i+j-K} Sq^K \cdot \binom{j-K-1}{i-2K}$$

$i,j>0$

$i \leq 2j$

$$Sq^1 Sq^1 = Sq^2 \cdot Sq^0 \binom{1-0-1}{1-0} \qquad i=j=1$$

$$= Sq^2$$

$$Sq^1 Sq^1(x) = Sq^1(x^2) = Sq^0(x)\cdot Sq^1(x) + Sq^1(x)\cdot Sq^0(x) = 0$$

$Sq^2(x) = 0$

$$Sq^k(x \cdot \omega_{m-1}(\eta)) = x \cdot Sq^K \omega_{m-1}(\eta) + x^2 \cdot Sq^{K-1} \omega_{m-1}(\eta)$$

$$\text{LHS} = Sq^K(\omega_1 L \cup \omega_{m-1}(\eta)) = \omega_1 L \cup Sq^K \omega_{m-1}(\eta) + \omega_1 L \cup \omega_1 L \cup Sq^{K-1}\omega_{n-1}(\eta)$$

$+ Sq^K \omega_m \eta$

$$= \omega_1 L \cup \left[\omega_K \omega_{m-1} + \binom{K-m+1}{1}\omega_{K-1}\omega_m + \cdots\right]\eta$$

$$Sq^K \omega_m(\eta) + \quad + \omega_1 L \cup \omega_1 L \cup \left[\omega_{K-1}\omega_{m-1} + \binom{K-m}{1}\omega_{K-2}\omega_m + \cdots\right]\eta$$

$$\text{RHS} = \omega_K \omega_m(\xi) + \binom{K-m}{1}\omega_{K-1}\omega_{m+1} + \cdots$$

$$= \binom{K-m}{i}\omega_{K-i}\omega_{m+i}(L \oplus \eta) = \binom{K-m}{i}\omega_{K-i}[\omega_{m+i}(\eta) + \omega_1(L)\,\omega_{m+i-1}$$

$$= Sq^K \omega_m(\eta) + \binom{K-m}{i}\omega_m \omega_{K-i}\;\omega_1(L)\,\omega$$

$$= \binom{K-m}{i}\left\{[\omega_{K-i}(\eta) + \omega_1(L)\cdot\omega_{K-i-1}(\eta)] + [\omega_{m+i}(\eta) + \omega_1(L) + \omega_{m+i-1}(\eta)\right.$$

$$= \binom{K-m}{i}\Big[\omega_{K-i}\cdot\omega_{m+i}\,\eta + \omega_1(L)\,\omega_{K-i-1}(\eta)\cdot\omega_{m+i}(\eta)$$

$$+ \omega_1(L)\,\omega_{K-i}(\eta)\,\omega_{m+i-1}(\eta)$$

$$+ \omega_1(L)\,\omega_1(L)\,\omega_{K-i-1}(\eta)\,\omega_{m+i-1}(\eta)\Big]$$

$$\text{LHS} - \text{RHS} = \omega_1 L \cup \left[\binom{K-m}{i-1}\omega_{K-i}\omega_{m+i-1} - \binom{K-m}{i}\omega_{K-i-1}\cdot\omega_{m+i}\right)\eta$$

$$= 0$$

Base case - Line Bundle.

Q. 8·B)

~~Sq~~ $n =$ smallest no. s.t. $w_n(\cancel{\xi}) \neq 0$

$\Rightarrow k+m=n \;,\; k, m > 0$

$$\underset{\overset{\shortparallel}{0}}{Sq^k(w_m)} = \binom{k-m}{k} \cancel{Sq_0 Sq_n}\, w_0 w_n$$

~~$\Rightarrow \binom{k-m}{k}$ even~~ ~~$\Rightarrow$ always 0~~ ~~take $m=0$~~

$$\Rightarrow \binom{k-m}{k} = \text{even} \quad \forall \quad k, m > 0 \;,\; k+m=n$$

$$\Rightarrow \binom{2k-n}{k} = \text{even}$$

if $n = 2^a b$ $\quad$ b odd, $b > \cancel{0}1$,

Take $k = 2^a$

$$\Rightarrow 0 = \binom{2^{a+1} - 2^a b}{2^a} \quad \cancel{2^a(2-b)\cdot 2}$$

$$= \frac{(2^{a+1} - 2^a b)(2^{a+1} - 2^a b - 1) \cdots (2^{a+1} - 2^a b - 2^a + 1)}{2^a \cdot (2^a - 1) \cdot \;\cdots\; \cdot 1}$$

~~Power of 2 X $2^{a+1} - 2^a b$~~

$$= \frac{(2^a b - 2^{a+1})(2^a b + 1 - 2^{a+1}) \cdots (2^a b + 2^a - 1 - 2^{a+1})}{2^a!}$$

$$= \binom{2^a b - 2^a - 1}{2^a}$$

$$\text{Power of } 2 = \left[\frac{2^a b - 2^a - 1}{2}\right] + \cdots + \left[2^a b - \right.$$

$$- \left[\frac{\cancel{2^a b - 2^a}}{2}\right] + \cdots$$

$$- \left[\frac{2^a b - 2^{a+1} - 1}{2}\right] - \left[\frac{2^a b - 2^{a+1} - 1}{2^2}\right] + \cdots$$

$= 0$ $\quad$ $\because \nexists$ multiple of 2^{a+1} between $2^a b - 2^a - 1 \,,\, (2^a b - 2^a - 1) - 2^a$

which means $\binom{2k-n}{k}$ odd

So $b = 1$, $n = 2^a$

Thom iso for oriented vector bundles:

$$\phi : H^k(B) \xrightarrow{\sim} H^k(E, \mathbb{Z}) \xrightarrow[\cup u]{\sim} H^{n+k}(E, E_0)$$

$$u|_{\text{fibre}} = \text{generator of } H^n(F, F_0)$$

euler class : $H^{n+k}(E, E_0) \xrightarrow{p^*} H^{n+k}(E) \xrightarrow{\sim} H^{n+k}(B)$ $p : E \longrightarrow E/E_0$

$$u \longmapsto e \in H^{n+k}(B)$$

$$\begin{tikzcd}
H^l(B) \arrow[r, "\sim"] & H^l(E) \arrow[r, "\cup u"] & H^{n+l}(E, E_0) \arrow[d] \\
1 \arrow[r, mapsto] & e \quad H^n(B) \arrow[d] & H^n(E) \arrow[l, "\sim"'] \\
& H^n(E) \arrow[r, "\cup u"'] & H^{n+n}(E, E_0) \\
& & \phi(e)
\end{tikzcd}$$

e is just the Thom class thought of as an element of $H^n(E)$.

$$\Phi(e) = e|_E \cup u = p^* u \cup u$$
$$= u \cup u$$

---

in $\mathbb{Z}/_2$ $e = w_n$

Q.9 - A) we know that $w_i(G_n(\mathbb{R}^\infty))$ generate $H^*(G_n(\mathbb{R}^\infty))$ — (*)

$$\begin{tikzcd}
\gamma^n \oplus \gamma^n \arrow[d] \\
G_n(\mathbb{R}^\infty)
\end{tikzcd}$$

$E \oplus E$ is orientable for any bundle E.

we choose ~~bas~~ orientation as (v, v) for each fibre.

- $w_{2n}(\gamma^n \oplus \gamma^n) = w_n(\gamma^n)\, w_n(\gamma^n) \neq 0$ by (*)

$\Rightarrow$ $e \neq 0$

- Let f denote orientation as above

n-odd f' be opposite orientation, i.e. having basis of the form $(v, -v)$

Then $e(E,f) = -e(E,f')$

Because reversing orientation reverses Euler class

$e(E,f) = e(E,f')$

$$\begin{tikzcd} E' \arrow[r, "F"] \arrow[d] & E \arrow[d] \\ G_n(\mathbb{R}^\infty) \arrow[r, "id"'] & G_n(\mathbb{R}^\infty) \end{tikzcd}$$

$~~F(b,v) =~~$ $F(b,v,\omega) = F(b,v,-\omega)$

F covers identity while sending (E,f) to $(E,-f)$

$\Rightarrow \quad 2e(E,f) = 0.$

Q.9-B) $\mathbb{C}^n$ bundle

$$\begin{tikzcd} \xi^n \arrow[d, "\pi"] \\ G_n(\mathbb{C}^\infty) \end{tikzcd}$$

$i: \mathbb{R}^\infty \hookrightarrow \mathbb{C}^\infty$ inclusion

$i: G_n(\mathbb{R}^\infty) \hookrightarrow G_n(\mathbb{C}^\infty)$

$i^*(\xi^n) = ?$ $\quad i^*(\xi^n) = \{(b,v) \mid b \in \mathbb{R}^\infty,\ v \in \xi^n \ \ i(b) = \pi(v)\}$

b is an n-plane in $\mathbb{R}^\infty$.

i(b) is an n-plane in $\mathbb{C}^\infty$ – $\{v_1 + iv_2 \mid v_i \in b\}$

v is a vector in $\mathbb{C}^\infty$

$\pi(v) = i(b) \Rightarrow \{v = \omega_1 + i\omega_2 \mid \pi(\omega_i) = b$ (i.e. $\omega_i \in b\}$

$i^*\xi^n = \nu^n \oplus \nu^n$

Q.9-c)

$$\begin{tikzcd} TS^n \arrow[d] \\ S^n \end{tikzcd}$$

$A \subseteq S^n \times S^n = \{(x,-x) \mid x \in S^n\}$

- $TS^n \cong S^n \times S^n - A$

Let ρ_x denote stereographic projection from $x \in S^n$.

$\rho_x : S^n - \{x\} \xrightarrow{\cong} TS^n_{-x}$

~~Then~~ $\phi: S^n \times S^n - A \longrightarrow TS^n$

$(x,y) \longmapsto \rho_x(y)$

isomorphism. easy.

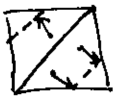
- $E = TS^n \cong S^n\times S^n - A$

$E_0 = S^n \subseteq TS^n \cong \{(x,y) \mid x \neq y\} = S^n\times S^n - A - D\}$

$H^*(E,E_0) \cong H^*(S^n\times S^n - A,$ ~~$S^n\times S^n - A - D$~~ $S^n\times S^n - A - D)$

$\cong H^*(S^n\times S^n, S^n\times S^n - D)$

From the long exact seq^n of triple $(S^n\times S^n, S^n\times S^n - D, A)$ and deformation retract of $S^n\times S^n - D$ onto A



$H: S^n\times S^n - D\times I \longrightarrow A$

$p,t \longmapsto tp + (1-t)q$ where q is the point on the anti-diagonal closest to p.

Note that q is well-defined because D has been removed.

$\cong H^*(S^n\times S^n, A)$

$$\begin{tikzcd} H^*(A) & H^*(S^n\times S^n) \arrow[l] & H^*(S^n\times S^n, A) \arrow[l] \\ H^{*-1}(A) \arrow[urr, bend left] & & \end{tikzcd}$$

- Euler class : $E \cong TS^n$ n-even

$e(\tau) = \phi^{-1}(u \smile u)$

$\phi: H^*(S^n) \xrightarrow{\cong} H^*(TS^n) \xrightarrow{\smile u} H^{*+n}(TS^n, TS^n - S^n)$

$\cong H^{*+n}(S^n\times S^n, A)$

$u \in H^n(TS^n, TS^n - S^n) \cong H^n(S^n\times S^n, A)$

u restricted to each fiber must be generator.

Need to trace each fiber

$F \hookrightarrow E$ $F = T_x S^n$

$$\begin{tikzcd}
(F, F_0) \arrow[r, hook] \arrow[d, "\cong"] & (E, E_0) \arrow[d, "\cong"] \\
((-x)\times S^n, (-x,x)) \arrow[r, hook] & (S^n\times S^n, A)
\end{tikzcd}$$

Now
$$\begin{tikzcd}
0 \arrow[r] & H^n(S^n\times S^n, A) \arrow[r] & H^n(S^n\times S^n) \arrow[r] & H^n(A) \arrow[r] & 0 \\
 & (a) \arrow[r, mapsto] & (a,+a) \quad (a,b) \arrow[r, mapsto] & (a-b) &
\end{tikzcd}$$

This comes by looking a
each projection $S^n\times S^n \longrightarrow S^n$

$$\begin{tikzcd}
(-x\times S^n, (-x,x)) \arrow[r, hook] \arrow[dr, bend right, "id"] \arrow[drr, bend right=60, "trivial"'] & S^n\times S^n, A \arrow[d] \arrow[dr] & \\
 & -x\times S^n, -x,x & (S^n\times x, (-x,x))
\end{tikzcd}$$

$\Rightarrow$ $u =$ generator of $H^n(S^n\times S^n, A)$

$\phi(x) = \pi^*(x) \cup u$ $\qquad$ $\pi: \underset{-D}{S^n\times S^n} \longrightarrow S^n \cong A$

what is $\phi^{-1}(u \cup u)$?

For what x, $\pi^*(x) \cup u = u \cup u$?

$\pi^*(x)$ is in $H^n(S^n\times S^n)$, u is in $H^n(S^n\times S^n, A)$

$u \cup u \in H^{2n}(S^n\times S^n, A) \cong \mathbb{Z}\oplus\mathbb{Z}$

$$H^*(S^n\times S^n, A) \longleftrightarrow \tilde{H}^*(S^n\times S^n)$$

ring homo?

$$\begin{tikzcd}
\mathbb{Z} \quad a \arrow[r, leftrightarrow] & a \quad \mathbb{Z} & * = 2n \\
\mathbb{Z} \quad a \arrow[r, leftrightarrow] & (a,a) \quad \mathbb{Z}\oplus\mathbb{Z} \arrow[d, "\text{cup product}"'] & * = n \\
 & &
\end{tikzcd}$$

$(a,b) \cup (c,d) = ad + bc$ $\quad$ $\because$ n-even

$\Rightarrow$ $a \cup a \longmapsto (a,a) \cup (a,a) = 2a \cup a$

$\times$ So, $\pi^*(x) \cup u = 2 \times$ generator of $H^{2n}(S^n\times S^n, A) \cong \tilde{H}^{2n}(S^n\times S^n)$

we know that ϕ is an isomorphism

So $\phi^{-1}(u \cup u) = 2$ generator of $H^n(S^n)$

* Note: $u \cup u$ will be 0 for n-odd
also the map on cohomolgies will be different.

- Suppose $TS^n = V \oplus W$, $V \to S^n$, $W \to S^n$

$e(TS^n) = e(V) \cdot e(W)$

But cohomology ring of S^n is trivial

$\Rightarrow$ V, or W has dim 0.

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11. Computations in Smooth Manifold – Tough

- $M \hookrightarrow A$ manifolds, M closed embedded

$\nu \to M$ normal bundle

$H^*(\nu, \nu_0) \cong H^*(A, A-M)$

Follows from Excision and Tubular Nbd. Th^m.

- The Thom class of ν, in $(A, A-M)$ is called fundamental class of M in A — denoted by u.

$$\begin{tikzcd}
u \arrow[rr, mapsto] & & e \\
H^n(\nu,\nu_0) \arrow[r] \arrow[d, leftrightarrow] \arrow[rd, phantom, "\circlearrowright"] & H^n(\nu) \arrow[r] \arrow[d, leftrightarrow] \arrow[rd, phantom, "\circlearrowright"] & H^n(M) \arrow[d, leftrightarrow, "\|"] \\
H^n(A,A-M) \arrow[r] & H^n(A) \arrow[r] & H^n(M) \\
u \arrow[rr, mapsto] & & e
\end{tikzcd}$$

By def^n of euler class

Similarly for $\mathbb{Z}/2$ coeff.

image of u in $H^n(A) =: u'$ dual class of M in A

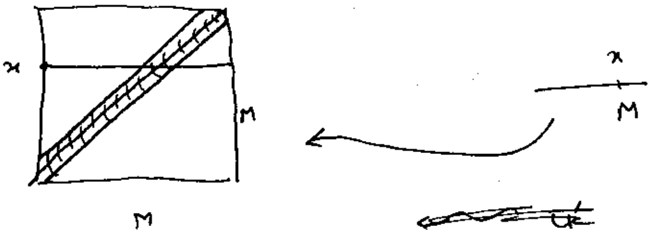
Cor: if $\bar{w}_k(TM) \neq 0$ Then M cannot be embedded in $\mathbb{R}^{n+k}$

$\bar{w} = \frac{1}{w}$

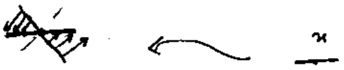
- $\Delta : M \hookrightarrow M\times M$ ν of M in $M\times M \cong TM$, u'- fundamental class of M in $M\times M$

$$j_x : (M, M-x) \longrightarrow (M\times M, M\times M - \Delta)$$
$$y \longmapsto (x,y)$$

$j_x^*(u') = u_x$ u_x - orientation of M at x



$$\begin{tikzcd} M \arrow[r, "\Delta"] & M\times M \\ & M \arrow[u, "j_x"'] \end{tikzcd}$$



$$\begin{tikzcd} e & u' \arrow[l] \arrow[d] \\ & ? \end{tikzcd}$$

Reason: locally j_x is homotopic to diagonal

Now diagonal map maps thom class to u_x

By homotopy, j_x maps u' to u_x $\because$ normal bundle $\cong TM$.

- $u' \in H^n(M\times M, M\times M - \Delta) \longrightarrow H^n(M\times M)$
  $u' \longmapsto u''$ diagonal cohomology class of M

- $a \in H^*(M)$

$$(1\times a) \cup u'' = (a\times 1) \cup u''$$

$$\begin{tikzcd} M\times M \arrow[r, "p_1"] \arrow[d, "p_2"] & M \\ M & \end{tikzcd}$$

$1\times a = p_1^*(a)$
$a\times 1 = p_2^*(a)$

$(M\times M, M\times M - \Delta) \cong (N_\varepsilon, N_\varepsilon - M)$ excision , N_ε - tube

Inside N_ε - $p_1 \simeq p_2$
rotation by 90°

so $p_1^*(a) = p_2^*(a)$ in $H^*(N_\varepsilon, N_\varepsilon - M)$

- $/ : H^{p+q}(X\times Y) \otimes H_q(Y) \longrightarrow H^p(X)$ "field" co-efficients

$$\begin{tikzcd} H^{p+q}(X\times Y)\otimes H_q(Y) \arrow[d, phantom, "\cong" rotate=90] \arrow[r] & H^p(X) \\ H^*(X)\otimes H^*(Y)\otimes H_*(Y) \arrow[r] & H^*(X) \\ \alpha,\ \beta,\ \mu \arrow[r, mapsto] & \alpha\langle\beta,\mu\rangle \end{tikzcd}$$

$(\alpha\times\beta)/_{\mu} = \alpha\langle\beta,\mu\rangle$

$[(\alpha\times 1)\cup\beta]/_{\mu} = \alpha\cup(\beta/\mu)$

- M compact, ~~β~~ $[M]\in H_n M$ $\qquad u''\in H^n(M\times M, M\times M-\Delta)$

$u''/_{[M]} = 1$

$$\begin{tikzcd} x \arrow[r, hook] & M & \\ H_n^*(M, M-x) & H_n^*(M) \arrow[l, leftrightarrow] \\ {[M_x]} & {[M]} \arrow[l] \end{tikzcd}$$

- In field co-efficients

1) $u'' = \sum_{\sigma} (-1)^{|\sigma|}\,\sigma\times\tilde{\sigma}$ $\qquad \{\sigma\}$ generate $H^*(M)$ as a vector space

$\langle\tilde{\sigma}, \sigma\cap[M]\rangle = 1$

dual basis

2) $\langle e(TM), [M]\rangle = \chi(M)$

$e(TM) = \Delta^*\mu''$

- Wu's formula

$w_i = \phi^{-1}Sq^i\phi^*(1)$

$= \phi^{-1}Sq^i(u)$

$\Rightarrow \pi^* w_i \cup u = Sq^i(u)$ $\qquad \pi^*: E\longrightarrow M$

$\phi: H^*(E,E_0) \xleftrightarrow{\cup u} H^{*-n}(M)$

Thom iso.

$E = TM$

$$\begin{tikzcd} (E,E_0) \arrow[r, phantom, "\cong"] \arrow[dr, "\pi"'] & (M\times M, M\times M-\Delta) \arrow[d, "\pi_1\cong\pi_2"] \\ & M \end{tikzcd}$$

~~Apply~~ By naturality,

$\pi_1^* w_i \cup u' = Sq^i(u')$

(or π_2^*)

$\pi_1^* \alpha = \alpha \times 1$ So, $(\omega_i \times 1) \cup u' = Sq^i u'$

Again by naturality, $H^*(M \times M, M \times M - \Delta) \longrightarrow H^*(M \times M)$

$$(\omega_i \times 1) \cup u'' = Sq^i(u'')$$

Applying $/[M]$,

$$(\omega_i \times 1) \cup u''/_{[M]} = Sq^i(u'')/_{[M]}$$

$$\begin{matrix} \parallel \\ \omega_i \cup u''/_{[M]} \\ \parallel \\ \omega_i \end{matrix}$$

$$\boxed{\omega_i = Sq^i(u'')/_{[M]}}$$

- $x \longmapsto \langle Sq^i(x), [M] \rangle$ $x \in H^{n-i}(M)$

By Poincare duality: (as we are in $\mathbb{Z}/2$)

$\exists\, v_i$ s.t. $v_i \in H^i(M)$

$x \cup v_i = Sq^i(x)$

Then

$$\omega_K = \sum_{i+j=k} Sq^i(v_j)$$

$$\omega = Sq(v)$$

Q.11 - A)

$$\mathbb{P}^n \hookrightarrow \mathbb{P}^{n+1} \longrightarrow S^{n+1}$$

$\Rightarrow H^*(\mathbb{P}^{n+1})/_{H^{n+1}(\mathbb{P}^{n+1})} \cong H^*(\mathbb{P}^n)$ in $\mathbb{Z}/2$

$\cong \mathbb{Z}/_2[x]/x^{n+1}$ by induction

By duality,

if $H^{n+1}(\mathbb{P}^{n+1}) = \mathbb{Z}/2y$

$\langle y, [\mathbb{P}^{n+1}] \rangle = 1$

$\parallel$

$\langle x^1 \cup x^{n}, [\mathbb{P}^{n+1}] \rangle$

$\because$ dual of $H^1 \in H^{n}$, $\Rightarrow x^1$ dual to x^n

$\Rightarrow x^1 \cup x^n = y$

$\Rightarrow y = x^{n+1}$

Base case. $P^1 = S^1$.

Q.9-B]

$$\cap [M] : H^k(M) \xrightarrow{\cong} H_{n-k}(M)$$

$$[\alpha] \longmapsto [\alpha \cap [M]]$$

$$u'' \in H^{n}(M \times M)$$

$$u''/ \ : \ H_{n-k}(M) \longrightarrow H^k(M)$$

1) $$(u''/) \circ (\cap [M]) \circ (\alpha) = u''/\alpha \cap [M] \quad \overset{?}{=} \alpha \cdot (-1)^{|\alpha||M|}$$

$$u'' = \sum_{\alpha} (-1)^{|\alpha|} \cdot \alpha \times \tilde{\alpha}$$

$$u''/\alpha \cap [M] = \sum_{\beta} (-1)^{|\beta|} \cdot \beta \times \tilde{\beta}/\alpha \cap [M]$$

$$= \sum_{\beta} (-1)^{|\beta|} \cdot \beta < \tilde{\beta}, \alpha \cap [M] >$$

$$= \sum_{\beta} (-1)^{|\beta|} \cdot \beta < \tilde{\beta} \cup \alpha, [M] >$$

Take α to be a basis element

$$= \cancel{\sum_{\beta}} (-1)^{|\alpha|} \alpha < \cancel{\beta} \tilde{\alpha} \cup \alpha, [M] >$$

$$= (-1)^{|\alpha| + (|\alpha| + |M|)(|\alpha|)} \cdot \alpha$$

$$= (-1)^{|\alpha| \cdot |M|} \cdot \alpha$$

2) $$(\cap [M]) \cdot (u''/ \ ) \circ \mu = u''/\mu \cap [M] \overset{?}{=} \mu \cdot (-1)^{|\mu| \cdot |M|}$$

$$= \sum_{\beta} (-1)^{\beta} \ \beta \times \tilde{\beta}/\mu \ \cap [M]$$

$$= \sum_{\beta} (-1)^{\beta} \beta < \tilde{\beta}, \mu > \ \cap [M]$$

Take μ to be a basis dual to α i.e. $<\alpha, \mu> = 1$

$$= (-1)^{|\tilde{\alpha}|} \tilde{\alpha} \cap [M] \qquad <\alpha, \tilde{\alpha} \cap [M]> = <\alpha \cup \tilde{\alpha}, [M]>$$

$$= 1$$

$$= (-1)^{|\tilde{\alpha}|} \mu$$

Sign problem

Q.~~8~~ 11 c) ~~$A^p \hookrightarrow M^m \qquad k = m - p$~~

$M^m \hookrightarrow A^p \qquad k = p - m$

$\cap$ ~~M~~ $[A] : H^k(A) \longrightarrow H_m(A)$

$u' \in$ ~~$H^k(A)$~~

$u' \in H^k(A, A-M)$

Q.11 D)

Wu's classes : v_k satisfy $\quad v_k \in H^k(M)$

$v_k \cup x = Sq^k(x) \quad$ for $x \in H^{n-k}(M)$

For 3-manifold:

$1 = v_0, v_1, v_2, v_3$

$v_3 \cup x = Sq^3(x) = 0$ ~~$x \in H^0$~~

$\Rightarrow v_3 = 0$

$v_2 \cup x = Sq^2(x) \qquad x \in$ ~~Sq^1~~ H^1

$\Rightarrow v_2 = 0$

So $\quad 1, v_1$

v_1 satisfies $\boxed{v_1 \cup x = Sq^1(x)} \quad \forall x \in H^2(M)$

$w_1 = v_1$

if M is orientable, $w_1 = 0 \Rightarrow v_1 = 0$.

(The statement cannot be true for M non-orientable).

Q.11 E)

$$Sq : H^*(M) \longrightarrow H^*(M)$$
$$x \longmapsto x + Sq^1(x) + \dots + Sq^i(x) + \dots$$

why automorphism?

- Injectivity, Ring homo-morphism is clear
- Surjectivity:

Can we invert $(1 + Sq^1 + \dots)$

~~$(1 - (Sq^1 + ..) + (Sq^1 + ..)^2 - (\;)^3 + \dots)$~~

Yes. Because they are ring operators

Do it by hand.

~~$\langle v\cdot x, [M]\rangle = \langle Sq\, x, [M]\rangle$~~

~~$Sq\, v = w \Rightarrow v = \overline{Sq}\, w$~~

~~$\Rightarrow \langle \overline{Sq}\, w \cdot x, [M]\rangle = \langle Sq\, x, [M]\rangle$~~

~~$y = Sq\, x \Rightarrow x = \overline{Sq}\, y$~~

~~$\Rightarrow \langle \overline{Sq}\,(w\cdot y), [M]\rangle = \langle y, [M]\rangle$~~

~~$\Rightarrow \langle w\cdot y, [M]\rangle = \langle Sq\, y, [M]\rangle$!! WTF.~~

Problem: $\langle x, [M]\rangle = \langle (z), [M]\rangle$
$\langle \not\Rightarrow \rangle$ $\langle Sq\, x, [M]\rangle = \langle Sq\,(z), [M]\rangle$

How does $\overline{Sq}$ look?

$Sq\,(x) = x + Sq^1(x) + \cdots$

$Sq\,(Sq^1(x)) = Sq^1(x) + Sq^1 Sq^1(x) + \cdots$

So $\overline{Sq}(x) = x - Sq^1(x) - Sq^2(x) + Sq^1 Sq^1(x) - \cdots$

So Call $\overline{Sq}^i(x) =$ component of $Sq(x)$ in $H^{n+i}(M)$.

---

we need to show, for $x \in H^{n-k}(M)$

$\bar{w}_k \cup x = \overline{Sq}^k(x)$.

i.e. in degree n,

$\bar{w}\cdot x \overset{?}{=} \overline{Sq}(x)$

Now, $\exists z$ s.t. $x = Sq(v\cdot z)$ $\quad (z = \bar{v}\cdot \overline{Sq}(x))$

$\Rightarrow$ To show in deg n, ..

$\bar{w}\cdot Sq\,(v\cdot z) \overset{?}{=} \overline{Sq}\,(\cdot\, Sq\,(v\cdot x))$

But $\bar{w} = Sq\cdot\bar{v}$

$\Rightarrow Sq\,(\bar{v})\cdot Sq\,(v\cdot z) \overset{?}{=} \overline{Sq}\,(Sq\,(v\cdot))$ in deg n

$\Rightarrow$ ~~$v \cdot z$~~ $Sq(z) \overset{?}{=} v \cdot z$ in deg n

But this is simply original Wu's formula.

So,

$$\langle \bar{Sq}^i(x), [M] \rangle = \langle \bar{\omega}^i \cup x, [M] \rangle.$$

for $i = n$, $x = 1$

$$\Rightarrow \langle \bar{Sq}^n(1), [M] \rangle = \langle \bar{\omega}^n, [M] \rangle$$

$$\overset{\|}{0}$$

$$\Rightarrow \bar{\omega}^n = 0 \qquad \because \bar{Sq}(1) = 1 \Rightarrow \bar{Sq}^n(1) = 0$$

for $i = n-1$,

$$\langle \bar{Sq}^{n-1}(x), [M] \rangle = \langle \bar{\omega}^{n-1} x, [M] \rangle$$

- or simply $\bar{Sq}^{n-1}(x) = \bar{\omega}^{n-1} x$ for $x \in H^1(M)$

Need to show $\bar{Sq}^{n-1} = 0$

$$Sq(x) = x + Sq^1(x) = x + x^2 \quad \text{for } x \in H^1$$

~~[illegible]~~

~~$Sq(Sq^1(x)) = Sq^1(x) + Sq^1 Sq^1(x) + Sq^2(Sq^1(x))$~~

$$Sq(x + x^2) =$$

$$Sq(x) = x + x^2 \Rightarrow Sq(x^i) = (x + x^2)^i$$
$$= x^i (1 + x)^i$$

So Sq of what $= x$?

$$Sq(x + a_2 x^2 + \cdots + a_i x^i + \cdots) = x$$

$$\Rightarrow (x + x^2) + a_2 (x + x^2)^2 + \cdots + a_i (x + x^2)^i + \cdots = x$$

By trial and error one gets

$$a_{2^i} = 1 \quad , \quad a_i = 0 \quad \text{if } i \text{ not a power of } 2$$

This is because
$$(x+x^2)^2 = x^2 + x^4$$
$$(x+x^2)^4 = x^4 + x^8$$
$$(x+x^2)^{2^i} = x^{2^i} + x^{2^{i+1}}$$

☆ So $\bar{Sq}(x) = x + x^2 + \dots + x^{2^i} + \dots$ ☆

So if n is not a power of 2

$\bar{Sq}^{n-1}(x) = 0$ as there is no n degree term in $\bar{Sq}(x)$

$$\Rightarrow \bar{\omega}^{n-1}(x) = 0 .$$

11. F)

$$Sq^i : H_k(X) \longrightarrow H_{k-i}(X)$$
$$\langle x, Sq^i(\beta) \rangle = \langle \bar{Sq}^i(x), \beta \rangle \qquad |x| + i = |\beta|$$

- $Sq^i(a \cap \beta) \overset{?}{=} \sum Sq^k(a) \cap Sq^l(\beta) \longrightarrow -|a| + |\beta| + k - l$

$\parallel$

$\deg = -i + \deg a + \deg \beta$ $\longrightarrow$ $\boxed{k - l = -i}$

$= -i + |a| + |\beta|$

$$\langle x, Sq^i(a \cap \beta) \rangle = \langle \bar{Sq}^i(x), a \cap \beta \rangle$$
$$= \langle \bar{Sq}^i(x) \cup a, \beta \rangle$$

$$\sum_{k-l=i} \langle x, Sq^k(a) \cap Sq^l(\beta) \rangle = \sum_{k-l=-i} \langle x \cup Sq^k(a), Sq^l(\beta) \rangle$$
$$= \sum_{k-l=-i} \langle \bar{Sq}^l(x) \bar{Sq}^l(Sq^k(a)), \beta \rangle$$

$$\bar{Sq}^i(x) \cup a \overset{?}{=} \sum_{k-l=-i} \bar{Sq}^l(x) . \bar{Sq}^l(Sq^k(a))$$
$$= \sum_{k=l-i} \bar{Sq}^l(x . Sq^k(a)) = \sum_{\substack{m+n=l \\ k=l-i}} \bar{Sq}^m(x) . \bar{Sq}^n Sq^k($$

~~If true for all i, we will get~~

$$\bar{Sq}(x) a = \sum_{k \le l} \bar{Sq}^l(x . Sq^k(a))$$

Applying $Sq \longrightarrow$

$$x \cdot Sq(a) = \sum_{k \le \ell} Sq(\bar{Sq}^{\ell}(x \cdot Sq^{k}(a)))$$

- $\langle x, Sq(\beta)\rangle = \langle \bar{Sq}(x), \beta\rangle$

$\langle x, Sq(a \cap \beta)\rangle = \langle \bar{Sq}(x) \cup a, \beta\rangle$

$\langle x, Sq(a) \cap Sq(\beta)\rangle = \langle x \cup Sq(a), Sq(\beta)\rangle$

$= \langle \bar{Sq}(x \cup Sq(a)), \beta\rangle$

$= \langle \bar{Sq}(x) \cup a, \beta\rangle$

$\Rightarrow \quad Sq(a \cap \beta) = Sq(a) \cap Sq(\beta)$

- $Sq(u''/\beta)$

Slant product:

co-officients in a field

$$/ : H^{p+q}(X \times Y) \otimes H_q(Y) \longrightarrow H^p(X)$$

$\cong$

$H^{p+i}(X) \otimes H^{q-i}(Y)$

$$(a,b)/\mu \longmapsto a\langle b, \mu\rangle.$$

- $(a \times b)/\mu = a\langle b, \mu\rangle$

$(a \times 1) \cup p/\mu = a \cup (p/\beta)$

No idea how to do this problem.

What does slant product do?

- $\langle x, \bar{\omega} \cap \mu\rangle = \langle x \cup \bar{\omega}, \mu\rangle$

$= \langle \bar{Sq}(x), \mu\rangle$ with sign issues

$= \langle x, Sq(\mu)\rangle$

but we are using $\mathbb{Z}/2$ co-eff So no problem

$\langle x, v \cap \mu\rangle = \langle x \cup v, \mu\rangle$

$= \langle Sq(x), \mu\rangle = \langle x, \bar{Sq}(\mu)\rangle.$

~~12. obstr~~ (39)

12. obstructions

Need to read Local co-efficients

- Skifel Manifold Bundle

$$\begin{tikzcd} E \arrow[d, "\pi"] \\ B \end{tikzcd}$$ $\mathbb{R}^n$- vector bundle

$$\begin{tikzcd} & V_k(E) \arrow[d] \\ V_k(\mathbb{R}^n) \arrow[ur, hook] & B \end{tikzcd}$$

$V(E)$ - k-frames in each fibre. orthonormal

$= \{(x, (v_1, \cdots, v_k)) \mid \pi(v_i) = x, \ v_i\text{'s linearly ind}\}$

$V_k(\mathbb{R}^n)$ - $n-k-1$ connected ??

(problem is finding cross-section over $n-k+1$ skeleton of B)

Requires bcal coefficients.

~~Alternately we can look at only~~ normal ~~orthogonal~~ frames.

~~This is the same manifold bundle because GL_k deformation retracts onto O_k.~~

- $V_k(\mathbb{R}^n)$ - $(n-k-1)$ connected

  $V_k(\mathbb{C}^n)$ - $2(n-k)$ connected

Proof: $\quad l < k \leq n$

$$\exists \quad \begin{tikzcd} V_{k-l}(\mathbb{R}^{n-l}) \arrow[r] & V_k(\mathbb{R}^n) \arrow[d, "p"] \\ & V_l(\mathbb{R}^n) \end{tikzcd}$$

$p : ((v_1, v_2, \cdots, v_k), x) = ((v_1, \cdots, v_l), x)$

Restricting to $l=1$

$V_{k-1}(\mathbb{R}^{n-1}) \longrightarrow V_k(\mathbb{R}^n) \longrightarrow S^{n-1}$

$\pi_i(V_k \mathbb{R}^n) \longrightarrow \pi_i(S^{n-1}) \longrightarrow$

$\rightarrow \pi_i(V_{k-1} \mathbb{R}^{n-1}) \longrightarrow \pi_i(V_k \mathbb{R}^n) \longrightarrow \pi_i(S^{n-1}) \longrightarrow$

$i < n-1 \qquad \pi_i(V_{k-1} \mathbb{R}^{n-1}) = \pi_i(V_k \mathbb{R}^n)$

for $i < n-k \qquad \pi_i(V_k \mathbb{R}^n) = \pi_i(V_{k-1} \mathbb{R}^{n-1})$

$\vdots$

$= \pi_i(V_1(\mathbb{R}^{n-k+1}))$

$= \pi_i(S^{n-k}) = 0$

For $\mathbb{C}^n$,

$V_{k-1}(\mathbb{C}^{n-1}) \longrightarrow V_k(\mathbb{C}^n) \longrightarrow S^{2n-1}$

- Gysin Sequence :

$$\begin{tikzcd}
{} \arrow[r] & H^i(E,E_0) \arrow[r] \arrow[d, dash, "\cup u"', "\cong"] & H^i(E) \arrow[r] \arrow[dd, dash, "\cong"] & H^i(E_0) \arrow[r] & H^{i+1}(E,E_0) \arrow[r] & {} \\
& H^{i-n}(E) \arrow[d, dash, "\cong"] \arrow[r, phantom, "\circlearrowleft"] & {} & & & \\
& H^{i-n}(B) \arrow[r] & H^i(B) & & &
\end{tikzcd}$$

~~u: Thom class~~

$$\begin{tikzcd}
E \arrow[d, "\pi"] \\
B
\end{tikzcd}$$ oriented

u : Thom class $\in H^n(E,E_0)$

Follow left + top → $x \longmapsto \pi^*x \longmapsto \pi^*x \cup u$

$$\downarrow$$

$$\pi^*x \cup u|_E$$

Follow ~~right~~ bottom + right → $x \longmapsto ? \longmapsto \pi^*?$

$$\pi^*? = \pi^*x \cup u|_E$$

$$= \pi^*(x \cup e)$$

e- euler class

$$\Rightarrow \quad ? = x \cup e$$

$$\longrightarrow H^{i-n}(B) \xrightarrow{\cup e} H^i(B) \longrightarrow H^i(E_0) \longrightarrow H^{i+1-n}(B) \longrightarrow$$

for un-oriented, $e \longmapsto w_n$

For $\tilde{B} \longrightarrow B$ .. two cover

$$\begin{tikzcd}
\tilde{B} \times \mathbb{R}/\mathbb{Z}/2 \arrow[d] \\
B
\end{tikzcd}$$

Line bundle ←

$\mathbb{Z}/2 \circlearrowleft \mathbb{R}$ by antipode

$\mathbb{Z}/2 \circlearrowleft \tilde{B}$ via Deck

~~is a~~ [illegible]

Here $E = \tilde{B} \times \mathbb{R}/\mathbb{Z}/2$ $\quad E_0 \cong \tilde{B}$

↓ This is because

$$E_0 \cong \tilde{B} \times \mathbb{R}-\{0\}/\mathbb{Z}/2$$

$$\cong \tilde{B} \times S^0/\mathbb{Z}/2$$

$$\cong \tilde{B}$$

- Oriented Grasmanian

$\tilde{G}_n(\mathbb{R}^{n+k})$ two covering of $G_n(\mathbb{R}^{n+k})$

being a two cover use above lemma

$$\begin{tikzcd}
\tilde{\gamma}^n \arrow[r] \arrow[d] & \gamma^n \arrow[d] \\
\tilde{G}_n(\mathbb{R}^{n+k}) \arrow[r] & G_n(\mathbb{R}^{n+k})
\end{tikzcd}$$

↗ universal property wrt oriented bundles

Two cover $\Rightarrow$ $\longrightarrow H^i(B) \xrightarrow[\cdot \cup \omega]{} H^{i+1}(B) \longrightarrow H^{i+1}(\tilde{B}) \longrightarrow H^{i+1}(B) \xrightarrow[\cup \omega_1]{}$

$\mathbb{Z}/2$ co-eff

$B = G_n(\mathbb{R}^{n+k})$

$\tilde{B} = \tilde{G}_n(\mathbb{R}^{n+k})$

$\omega_1 \in H^1(G_n(\mathbb{R}^{n+k})) = \mathbb{Z}/_2 \omega_1(\gamma^n)$

$\omega_1 = 0 \Rightarrow$

$0 \longrightarrow H^{i+1}(B) \longrightarrow H^{i+1}(\tilde{B}) \longrightarrow H^{i+1}(B) \longrightarrow 0$

at $i = -1$ we get

$H^{i+1}(\tilde{B}) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \Rightarrow \tilde{G}_n(\mathbb{R}^{n+k})$ not connected

Why is $\tilde{G}$ connected?

~~Since G is connected, enough to show ∃ path taking $(p, [p]) \longrightarrow (p, -[p])$~~

$\hookrightarrow$ orientation of p.

$$\begin{tikzcd} Gl^+(\mathbb{R}^k) \arrow[r] & V_n(\mathbb{R}^{n+k}) \arrow[d] \\ & \tilde{Gl}_n(\mathbb{R}^{n+k}) \end{tikzcd}$$

$\pi_0(Gl^+ \mathbb{R}^k) \longrightarrow \pi_0(V_n) \longrightarrow \pi_0(\tilde{Gl}_n) \longrightarrow 0$

$\pi_1(\tilde{Gl}_n) \longleftarrow \pi_1(V_n) \longleftarrow \pi_1(Gl^+(\mathbb{R}^k)) \longleftarrow$

So $\pi_0(V_n) = 0 \Rightarrow \pi_0(\tilde{G}l_n) = 0$

~~In particular~~

or Simply, V_n-connected $\Rightarrow \tilde{G}l_n$ connected

$\Rightarrow \omega_1(\tilde{B}) \neq 0 \Rightarrow \omega_1(\tilde{B}) = \omega_1(\tilde{\gamma}^n)$

But $H^*(B)$ generated by $\omega_1(\gamma^n), \dots, \omega_n(\gamma^n)$

$\Rightarrow 0 \longrightarrow H^i(B) \xrightarrow[\cup \omega_1]{} H^{i+1}(B) \longrightarrow H^{i+1}(\tilde{B}) \longrightarrow 0$

$H^{i+1}(\tilde{B})$ generated by images of $\omega_2, \dots, \omega_n(\gamma^n)$

$$\begin{tikzcd} H^i(E) \arrow[r] \arrow[d, "\wr"] & H^i(E_0) \arrow[d, "\wr"] \\ H^i(B) \arrow[r, "?"] & H^{i+1}(\tilde{B}) \end{tikzcd}$$

$\leadsto$ But image of $\omega_2(\gamma^n) = \omega_2(\tilde{\gamma}^n)$

$$\begin{tikzcd} E_0 \arrow[r, hook] \arrow[d] & E \arrow[d] \\ \tilde{B} \arrow[r, "?"] & B \end{tikzcd}$$

? is the projection map

So $H^*(\tilde{G}(\mathbb{R}^{n+k}))$ generated by

$\omega_2(\tilde{\gamma}^n), \dots, \omega_n(\tilde{\gamma}^n)$ and

$\omega_1(\tilde{\gamma}^n) = 0$.

12 - A) $\begin{array}{c} E \\ \downarrow \\ B \end{array}$ orientable $\iff$

$$\begin{tikzcd} E \arrow[r] \arrow[d] & \tilde{\gamma} \arrow[d] \\ B \arrow[r, "\exists f"] & \tilde{G} \end{tikzcd}$$

Since $\omega_1(\tilde{\gamma}) = 0$, Result follows.

(This implies

$\phi : H^k(B) \longrightarrow H^{k+n}(E, E_0)$

$x \longmapsto \pi^* x \cup u$

- $\phi^{-1} Sq^1 \phi(1) = 0$

$\Rightarrow \phi^{-1} Sq^1 u = 0$

ϕ being an isomorphism, we get

$\boxed{Sq^1 u = 0}$

Sq^1 is the Bockstein, $0 \to \mathbb{Z}/2 \xrightarrow[2]{} \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$

what does this mean?

- Also $\omega_1(E \oplus F) = \omega_1(E) \oplus \omega_1(F)$

$\Rightarrow$ $E \oplus F$ orientable iff both or none of E, F orientable

- For a manifold, by Wu's formula

$\omega_1 = \upsilon_1 = 0$

$\Rightarrow$ $Sq^1(x) = 0$ $\forall x \in H^{n-1}(M)$)

12-B) By 11-) $\omega_i(M) =$ [illegible]

$M^{(i)}$ - i^{th} CW skeleton of M.

$M^{(0)}$ - Assume single point.

- $TM|_{M^{(1)}}$ is trivial

Look at a

$$\begin{tikzcd} S^1 \arrow[r, hook, "i"] & M^{(1)} \arrow[r] & M \\ i^* TM \arrow[u] \arrow[r] & TM \arrow[u] \arrow[r] & TM \arrow[u] \end{tikzcd}$$

Then, $i^*(\underset{0}{\underset{\shortparallel}{\omega_1 TM}}) = \omega_1(i_* TM)$

on S^1 a 3- bundle can ~~have~~ be

Mobius $\oplus$ Trivial or Trivial.

In first case $\omega_1 \neq 0$.

- $TM|_{M^{(2)}}$ is trivial

Let (D^2, ϕ) be a two cell in M, $\phi : \partial D^2 \longrightarrow M^{(1)}$ attaching map.

~~Becaus~~e, $TM|_{M^{(1)}}$ is trivial,

So on boundary of D^2 we have a trivial bundle i.e. this is bunelle on S^2

we need to check whe~~te~~her bundle is trivial on D^2 rel ∂D^2 given that w_i's are all 0.

~~ES^2~~ $\mathrm{Vect}^3_{\mathbb{R}}(S^2) = [S^1, SO(3)] = \pi_1(SO(3)) = \pi_1(\mathbb{R}P^3)$

$= \mathbb{Z}/2$

So, $\exists$ only 2 non-eq. $\mathbb{R}^3$ bundles on S^2.

Enough to show that $\exists$ bundle on S^2 with $w_2 \neq 0$.

Look at $\tilde{G}_3(\mathbb{R}^\infty)$ – cohomology generated by $\tilde{w}_2, \tilde{w}_3$

So $\pi_2(\tilde{G}_3) \otimes \mathbb{Z}/2 = \mathbb{Z}/2$, $(\because \pi_1 = 0)$

$\Rightarrow$ $S^3 \longrightarrow \tilde{G}_3$ iso. on $\pi_2 \otimes \mathbb{Z}/2$

Pull back $\tilde{\gamma}_3$. This will have $w_2 \neq 0$. □

So we get that $TM|_{M^2} = 0$

- TM is trivial

$(D^3, \phi) \hookrightarrow M$

Bundle trivial on $\partial D^3 \Rightarrow$ bundle on S^3

~~ES^3~~, $\mathrm{Vect}^3_{\mathbb{R}}(S^3) = \pi_2(\mathbb{R}P^3) = \pi_2(S^3) = 0$

$\Rightarrow$ Bundle trivial on D^3

$\boxtimes$

12-c) $\longrightarrow H^{i-1}(B) \xrightarrow{\cup w_1} H^i(B) \longrightarrow H^i(\tilde{B}) \longrightarrow H^{i}(B)$

$\tilde{B} = S^n \Rightarrow$ $H^{i}(B) \cong H^{i-1}(B) \cup w_1$.

~~$DM \to M$~~

12.D)

$$\begin{tikzcd}\tilde{G}_n(\mathbb{R}^{n+k}) \arrow[d] \\ G_n(\mathbb{R}^{n+k})\end{tikzcd}$$

two covering $\Rightarrow$ $\tilde{G}_n$ is C^∞,

$\pi_1(G_n \mathbb{R}^{n+k}) = \mathbb{Z}/2$

$\Rightarrow \pi_1(\tilde{G}_n ~~\mathbb{R}~~) = 0$

$\Rightarrow \tilde{G}_n$ orientable

$\tilde{G}_n$ quotient of $(G_n \times \mathbb{Z}/2)^\alpha$. ↖ compact by Tychonoff

$\Rightarrow \tilde{G}_n$ compact.

$$\phi : [b_1 \cdots b_n] \longmapsto \frac{b_1 \wedge b_2 \wedge \cdots \wedge b_n}{|b_1 \wedge \cdots \wedge b_n|}$$

Well defined:

$[c_1 \cdots c_n] = [b_1 \cdots b_n]$

Let $c_i = \sum \alpha_{ij} b_j$,

$\Rightarrow c_1 \wedge \cdots c_n = \{\det \alpha\} \cdot b_1 \wedge \cdots \wedge b_n$.

$$\Rightarrow \frac{c_1 \wedge \cdots \wedge c_n}{|c_1 \cdots \wedge c_n|} = \frac{b_1 \wedge \cdots \wedge b_n}{|b_1 \wedge \cdots \wedge b_n|} \cdot \operatorname{sign} \det \alpha$$

But orientation $\Rightarrow$ sign det $\alpha = \pm 1$.

Injectivity:

easy

Smooth:

Near $x_0 \in \tilde{G}_n(\mathbb{R}^{n+k})$, with basis $e_1 \cdots e_n$,

choose $f_1 \cdots f_k$ ~~in~~ basis for $x_0^\perp$.

Then basis for x near x_0, chart is given by:

Choose a basis of x of the form

$v_i = e_i + f_i'$ where $f_i' \in x^{\perp}$.

then

$$\{f_i'\} \left[\quad T \quad \right] \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix} [e_i] \qquad T \in \mathrm{Hom}(x_0, x_0^{\perp})$$

i.e. $f_i' = Te_i$

Then, co-ordinates of x are ~~values~~ entries of T

So,

$$\Phi[v_1, \dots, v_n] = \frac{(e_1+f_1') \wedge (\dots) \wedge (e_n+f_n')}{| \qquad |}$$

in local co-ordinates

~~$= \wedge\,(e_i + T_{i1}e_1 + T_{i2}e_2 + \dots + T_{in}e_n)$~~

~~$\wedge(e_i+$~~

$$= \frac{(e_1+Te_1) \wedge \dots \wedge (e_n+Te_n)}{| \qquad |}.$$

$$= \frac{(e_1 + T_1^1 f_1 + \dots + T_1^n f_n) \wedge \dots \wedge (e_n + T_n^1 f_1 + \dots + T_n^n f_n)}{| \qquad |}$$

which is smooth in T_i^j ?.

Q. 13 - A)

$J : E(\xi) \longrightarrow E(\xi)$ $\qquad \xi$ - $2n$ dim $\mathbb{R}$ vector bundle

~~satisfies~~

$$\begin{tikzcd} \xi \arrow[d] \\ B \end{tikzcd} \qquad u \subseteq B \qquad \pi^{-1}(u) \cong u \times \mathbb{R}^{2n}$$

Then, ~~$\pi^{-1}(u) \cong u \times \mathbb{C}^n$ via~~

Choose a basis for $\mathbb{R}^{2n}$:

$e_1, Je_1, \dots, e_n, Je_n$

$$\begin{tikzcd}
\mathbb{R}^{2n} \arrow[r, "d"] \arrow[d, "T"'] & \mathbb{C}^n \arrow[d, "\tilde{T}"] \\
\mathbb{R}^{2n} \arrow[r, "\phi"] & \mathbb{C}^n
\end{tikzcd}$$

$a_1e_1 + \cdots + a_ne_n + b_1Je_1 + \cdots + b_nJe_n \longmapsto (a_1 + ib_1, \cdots, a_n + ib_n)$

e_i'

is $\tilde{T}$ $\mathbb{C}$-linear ?

$\tilde{T}(a_1 + ib_1, \ldots, a_n + ib_n) =$ ~~$\phi(Ta_1, TJb_1, \cdots)$~~

~~$= \phi(Ta_1, J\cdot Tb.$~~

$= \phi(Ta_1e_1 + T\cdot Jb_1e_1 + \cdots + a_nTe_n + b_nTJe_n)$

$=$

$J\cdot Te_1 = T\cdot Je_1$

$$\tilde{T}(a_i + ib_i) = \phi\cdot T\cdot\phi^{-1}(a_i + ib_i)$$

$$= \phi T(a_ie_i + b_iJe_i)$$

$$= a_i\phi T(e_i) + b_i\,\phi\cdot T\cdot J\cdot e_i$$

$$= a_i\cdot(\phi\cdot T)e_i + b_iJ(\phi\cdot T)e_i$$

a_1, rest 0

$\tilde{T}(a_1) = a_1(\phi\cdot T)e_1$

$\tilde{T}(ia_1) = a_1 J\cdot(\phi\cdot T)e_1$

$= i\,a_1(\phi\cdot T)e_1 = i\,\tilde{T}(a_1)$

b_1, rest 0

$\tilde{T}(ib_1) = ib_1(\phi\cdot T)e_1$

$\tilde{T}(ib_1) = -b_1\cdot(\phi\cdot T)e_1$

$= i\,\tilde{T}(b_1)$

13 - B)

M - complex manifold

U, V charts
ϕ ψ

$$\psi\circ\phi^{-1} : \underset{\mathbb{C}^n}{\underset{\cap}{\phi(U\cap V)}} \longrightarrow \underset{\mathbb{C}^n}{\underset{\cap}{\psi(U\cap V)}}$$

holomorphic.

on TM - charts

$\pi^{-1}U$, $\pi^{-1}V$
$\tilde{\phi}$, $\tilde{\psi}$

$$\tilde{\psi}\cdot\tilde{\phi}^{-1} : \tilde{\phi}(U\cap V) \longrightarrow \tilde{\psi}(U\cap V)$$

$$z_i \longmapsto \psi\cdot\phi^{-1}_i(z)$$

$$\frac{\partial}{\partial z_i} \longmapsto \left[D\psi\circ\phi^{-1}_{e}\left(\frac{\partial}{\partial z}\right)\right]_i$$

$f: M \to N$ holomorphic

$Df: TM \longrightarrow TN$
locally is (f, Df), hence holomorphic.

13.c)

$f: M \longrightarrow \mathbb{C}$
holo, compact

f atains maxima say at p

By maximum modulus around p, f constant.

13.D)

$P^n(\mathbb{C})$ - $\mathbb{C}P^n$

Chart: $(z_0, \cdots, z_n) \longmapsto \left(\frac{z_1}{z_0}, \cdots, \frac{z_n}{z_0}\right)$ at $z_0 \neq 0$.

$G_n(\mathbb{C}^{n+k})$

Chart: for $x_0 \in G_n(\mathbb{C}^{n+k})$, $\mathbb{C}\langle e_1 \cdots e_n\rangle = x_0$, $\mathbb{C}\langle f_1 \cdots f_k\rangle = x_0^\perp$

$x \longmapsto$ if x = spanned by $e_1 + f_1', \cdots, e_n + f_n'$
and $f_i' = Te_i$

T.

Holomorphic?

$x_0, x_1 \in G_n(\mathbb{C}^{n+k})$

bases for x:

$e_{01} + f'_{01}, \ldots, e_{0n} + f'_{0n}$

$e_{11} + f'_{11}, \ldots, e_{1n} + f'_{1n}$

~~e~~ $f_{0i} = T_0 e_{0i}$

$f_{1i} = T_1 e_{1i}$

Then, $T_0 \longrightarrow T_1$ holo?

$B[e_{01} \cdots e_{1n}\ f_{01} \cdots f_{0k}] = [e_{\varphi 1} \cdots e_{\varphi n}\ f_{\varphi 1} \cdots f_{\varphi k}]$

$e_{1i} + f'_{1i} = e_{1i} + T_1 e_{1i}$

~~$e_{1i} + T_1$~~ $= Be_{0i} + T_1 Be_{0i}$

How to find T_1 in terms of T_0?

T_0, T_1 can be thought as in Hom(x, V)

Then we have the identities

$e_i = B e_{0i}$

f_{1i} ~~f_{ei}~~ $= B$ ~~[illegible]~~ f_{0i}

$f'_{0i} = T e_{0i}$, $f'_{1i} = T e_{1i}$

B change of basis $e_0, f_0 \downarrow e_1 f_1$ in $V(n+k)$.

$$\begin{array}{ccc} \langle e_{0i} + f'_{0i} \rangle & = & \langle e_{1i} + f'_{1i} \rangle \\ \| & & \| \\ \langle (1+T_0)\, e_{0i} \rangle & & \langle (1+T_1)\, e_{1i} \rangle \\ \| & & \| \\ (1+T_0) \langle e_{0i} \rangle & & (1+T_1) B \langle e_{0i} \rangle \end{array}$$

$\Rightarrow \quad T_1 = (1+T_0) \cdot B^{-1} - 1$

which is holomorphic?

13-E)

$$\begin{tikzcd} \gamma_n^1 \arrow[d] \\ \mathbb{C}P^n \end{tikzcd}$$

- Need to show γ_n^1 ~~ho~~ does not have any ~~ex~~ holo. cross section. Need not be non-vanishing.

$$\begin{tikzcd} c: \mathbb{C}P^n \arrow[r] & \gamma_n^1 \subseteq \mathbb{C}P^n \times \mathbb{C}^{n+1} \arrow[d] \\ & \mathbb{C}^{n+1} \end{tikzcd}$$

cross section

compose →

c holomorphic , $\mathbb{C}P^n$ - compact ⇒ constant

But the only point common to all lines is 0.

- ~~$\mathbb{C}P^n$~~ $\mathrm{Hom}_{\mathbb{C}}(\gamma_n^1, \mathbb{C})$

section : $p_i : \mathbb{C}P^n \longrightarrow \mathrm{Hom}_{\mathbb{C}}(\gamma_n^1, \mathbb{C})$

projection onto the i^{th} co-ordinate

$c_i([z_0 : \cdots z_n], (z_0, \cdots, z_n)) = z_i$

$\mathbb{C}$- linearly independant ? — $\sum \alpha_i c_i = 0$

$\Rightarrow \sum_i \alpha_i z_i = 0$. ∎

13.F)

M - TM — $\mathrm{Hom}_{\mathbb{R}} TM =: T^*M$

$T^*M \otimes \mathbb{C} \cong T^*_{\mathbb{C}}M \oplus \overline{T^*_{\mathbb{C}}M} = \overline{\mathrm{Hom}_{\mathbb{C}}(TM, \mathbb{C})}$

$T^*_{\mathbb{C}}M = \mathrm{Hom}_{\mathbb{C}}(TM, \mathbb{C})$

- $\mathrm{Hom}_{\mathbb{R}}(\xi, \mathbb{C}) = \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}^n, \mathbb{C})$ fibrewise

  - Holo part of $f = \frac{f + i f(-i)}{2}$

    Anti holo of $f = \frac{f - i(f \circ -i)}{2}$.

$e_n : x \longmapsto \frac{x+iy}{2} + \frac{x-iy}{2}$

$\frac{x+iy}{2} = z/2$, $\quad z/2 + \bar{z}/2$

- dz_i - holo? $\quad dz_i = dx_i + i\, dy_i$

  $dz_i\left(\frac{\partial}{\partial x_j}\right) = \delta_{ij}$ $\qquad dz_i\left(\frac{\partial}{\partial y_j}\right) = i\, \delta_{ij}$

  Change of variable:

  $z_i \longrightarrow w_i$

  $dz_i \longrightarrow dw_i$

  Jacobian : $\left[\frac{\partial w_i}{\partial z_i}\right]$ holomorphic

Chern - Classes

$$\begin{tikzcd} \xi \arrow[d, "\pi"'] \\ B \end{tikzcd}$$

complex n- vector bundle

$c_n(\xi) = e(\xi)$ ← euler class.

Construct a vector bundle on "$E(\xi)_0$". fibre at a point $v \in \pi^{-1}(x)$ is $v^\perp$ where wrt some hermitian inner product is given

Call this ξ_0.

$$\begin{tikzcd} \xi_0 \arrow[d] \\ E(\xi)_0 \end{tikzcd}$$

$n-1$ dim complex vector bundle

Then $c_{n-1}(\xi) := c_{n-1}(\xi_0) = e(\xi_0)$ pushed forward to $H^{2n-2}(B)$. via

i.e. $H^i(E(\xi)_0) \longleftrightarrow H^i(B)$

This can be done as in the Gysin seq

$H^i(E, E_0) \cong H^{i-2n}(B) = 0$ for $i < 2n$.

- Grassmannian

$$H^*(G_n(\mathbb{C}^\infty)) = \mathbb{Z}[c_1(\gamma^n), \ldots, c_n(\gamma^n)]$$

- $c(\omega \oplus \phi) = c(\omega) \cdot c(\phi)$

- $c(\bar{\omega}) = 1 - c_1(\omega) + c_2(\omega) - \cdots$

- $\omega_1(\gamma^1_n) = -$ generator of $H^2(\mathbb{C}P^n) = -x$

- $c_n(T\mathbb{C}P^n) = e(T\mathbb{C}P^n) = (n+1)(-x)^n$.