Brown representability gives us

$$H^n(-,\mathbb{Z}/2) \xrightarrow{\cong} [-, K(\mathbb{Z}/2,n)]$$

Yoneda lemma then tells us that every natural transformation $H^n(-;\mathbb{Z}/2) \longrightarrow H^m(-,\mathbb{Z}/2)$ should come from a map $K(\mathbb{Z}/2,n) \longrightarrow K(\mathbb{Z}/2,m)$

Stable operations are those that make the following diagram commute:

$$\begin{tikzcd} H^n(-,\mathbb{Z}/2) \arrow[r, "\varphi"] \arrow[d, phantom, "\cong" sloped] & H^m(-,\mathbb{Z}/2) \arrow[d, phantom, "\cong" sloped] \\ H^{n+1}(\Sigma -,\mathbb{Z}/2) \arrow[r, "\Sigma\varphi"] & H^{m+1}(\Sigma -,\mathbb{Z}/2) \end{tikzcd}$$

Or equivalently:

$$\begin{tikzcd} K(\mathbb{Z}/2,n) \arrow[r, "\varphi"] \arrow[d, phantom, "\cong" sloped] & K(\mathbb{Z}/2,m) \arrow[d, phantom, "\cong" sloped] \\ \Omega K(\mathbb{Z}/2,n+1) \arrow[r, "\Omega\varphi"] & \Omega K(\mathbb{Z}/2,m+1) \end{tikzcd}$$

This is where the Steenrod squares live.

Steenrod Squares:

1. $Sq^i : \Sigma^* H^n(-,\mathbb{Z}/2) \to \Sigma^* H^{n+i}(-,\mathbb{Z}/2)$
2. $Sq^i(x) = 0$ if $|x| < i$
3. $Sq^0(x) = x$
4. $Sq^{|x|}(x) = x^2$
5. $Sq^1 =$ Bockstein for $\mathbb{Z}/2$
6. $Sq^i(xy) = \sum Sq^j(x)\, Sq^{i-j}(y) \qquad (Sq(xy) = Sq(x)\,Sq(y))$
7. For $a < 2b$, $Sq^a Sq^b(x) = \sum_c \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c$

I do not know how to define this thing yet.

Bockstein:

Sq^1 is the Bockstein. For $x \in H^1(X;\mathbb{Z}/2)$ we should have $\beta(x) = x^2$. Let us verify this.

$$\begin{tikzcd} & C^1(X;\mathbb{Z}/4) \arrow[r] \arrow[d, "d"] & C^1(X;\mathbb{Z}/2) \arrow[r] & 0 \\ 0 \arrow[r] & C^1(X;\mathbb{Z}/2) \arrow[r] & C^2(X;\mathbb{Z}/4) & \end{tikzcd}$$

$$\begin{tikzcd} & x' \arrow[r] \arrow[d, "d"] & x \\ \beta x \arrow[r] & dx' & \end{tikzcd}$$

The bockstein satisfies

$2\beta x \equiv dx' \pmod 4$

where $x' \equiv x \pmod 2$

Claim: $\beta x = x^2 \pmod 2$

Suffices to show $(2x^2) \equiv dx' \pmod 4$. Let us act on a simplex $[0,1,2]$

upto an exact form

Let me remove some abstraction & <u>choose</u> an x'.

$x'(01) = x(01)$ for all <u>basis</u> elements $01 \in C_1(X)$

thought of as an element in $\mathbb{Z}/4$.

$dx'(012) = x(01) - x(02) + x(12) \pmod 4$

$2x^2(012) = 2 \cdot x(01)\, x(12)$

If either of $x(01)$ or $x(12)$ is 0, $x(02) = x(01) + x(12)$ even $\pmod 4$

and we are good.

If both $x(01) = x(12) = 1$, $x(02) = 0 \Rightarrow dx'(012) = 2 = 2x^2(012)$.

On $X \times Y$, Kunneth formula and naturality tells us

$$Sq^i(u \times v) = \sum_j Sq^j(u) \times Sq^{i-j}(v)$$

i.e. $Sq(u \times v) = Sq(u) \times Sq(v)$

Claim: For $u \in H^1(X;\mathbb{Z}/2)$ $\quad Sq^i(u^j) = \binom{j}{i} u^{i+j}$

Proof: $Sq(u^j) = (Sq(u))^j = (u+u^2)^j = u^j(1+u)^j = \sum_i \binom{j}{i} u^{i+j}$

Q. What about $Sq(u^j)$ for $u \in H^2(X;\mathbb{Z}/2)$?

Now we have $Sq^0(u) = u$, $Sq^1(u) = \beta u$, $Sq^2(u) = u^2$

So, $$Sq(u^j) = (\underset{2}{u} + \underset{3}{\beta(u)} + \underset{4}{u^2})^j = \sum_n \sum_{2a+3b+4c=n} \frac{n!}{a!\,b!\,c!}\, u^{a+2c} \cdot \beta(u)^b$$

Already this is getting abstract. How tangible is Sq^i for higher i?

One trick is to look at $\prod_{i=1}^{n} K(\mathbb{Z}_2, 1)$ instead of $K(\mathbb{Z}/2, n)$.

By Kunneth,

$$H^*\left(\prod_{i=1}^{n} K(\mathbb{Z}/2, 1); \mathbb{Z}/2\right) \cong \bigotimes_n \underbrace{H^*(K(\mathbb{Z}/2, 1), \mathbb{Z}/2)}_{\cong \mathbb{Z}/2[x_1 \cdots x_n]}$$

So if $x_1, x_2, \dots, x_n$ are the generators of $H^1(K(\mathbb{Z}/2, 1), \mathbb{Z}/2)$ then let $\sigma_n = x_1 \times x_2 \times \dots \times x_n$

Let σ_i denote the i^{th} elementary symmetric polynomial, so

$\sigma_i \in H^i(K_n, \mathbb{Z}/2)$

for eg: $\sigma_1 = x_1 + x_2 + \dots + x_n$ by that we mean $x_1 \times 1 \times \dots + 1 \times x_2 \times \dots + \dots + 1 \times 1 \times \dots \times x_n$

$\sigma_2 = x_1 x_2 + x_1 x_3 + \dots$

$\sigma_n = x_1 x_2 \dots x_n$

☆ $Sq^i(\sigma_n) = Sq^i(x_1 \times \dots \times x_n)$

$= \sigma_n \cdot \sigma_i$

Now we can write $\sigma_n : K_n \longrightarrow K(\mathbb{Z}/2, n)$ with $\sigma_n^*(i_n) = \sigma_n$ where $\mathbb{Z}_2 i_n = H^n(K(\mathbb{Z}/2, n), \mathbb{Z}/2)$

Because of the commutative diagram

$$\begin{tikzcd}
H^n(K_n, \mathbb{Z}/2) \arrow[d, "Sq^i"'] & H^n(K_n, \mathbb{Z}/2) \arrow[l, "\sigma_n^*"'] \arrow[d, "Sq^i"] \\
H^{n+i}(K_n, \mathbb{Z}/2) & H^{n+i}(K_n, \mathbb{Z}/2) \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}
\sigma_n \arrow[d] & i_n \arrow[l] \arrow[d] \\
\sigma_n \cdot \sigma_i & Sq^i i_n \arrow[l]
\end{tikzcd}$$

we get a few elements in $H^{n+i}(K(\mathbb{Z}/2, n), \mathbb{Z}/2)$.

If $H^*(K(\mathbb{Z}/2, n); \mathbb{Z}/2) \xrightarrow{\sigma_n} H^*(K_n; \mathbb{Z}/2)$ were injective we could use this as a definition of Sq^i. Turn out it is,

Th^m Serre: $H^*(K(\mathbb{Z}/2, n); \mathbb{Z}/2)$ is generated by $Sq^I(i_n)$ for appropriate I.

The proof of this needs some prep work.

Assuming this th^m, <u>define</u> $Sq^i(i_n)$ to be the unique element that maps to $\sigma_n \sigma_i$.

Claim: Sq^1 is the Bockstein β.

Proof: β is multiplicative, and hence $\beta(\sigma_n) = \sigma_n \sigma_1 \; \sigma_n^* Sq^1(i_n)$.