Problem Set 1 - The Riemann Sphere

Corrected open mapping theorem statement:

Theorem 0.1 (Open mapping theorem). If $f: U \to \mathbb{C}$ is a non-constant complex differentiable then for any open set $V \subseteq U$, the set f(V) is an open subset of \mathbb{C} .

Holomorphic functions on \mathbb{P}^1

A complex differentiable function $f: X \to \mathbb{C}$ is called a *holomorphic* function on X.

The Riemann sphere \mathbb{P}^1 is the set $\mathbb{C} \cup \{\infty\}$. We write \mathbb{P}^1 as the union of two sets

$$U_1 = \mathbb{C}$$
 $U_2 = \mathbb{C} \setminus \{0\} \cup \{\infty\}$

The set U_1 is the standard complex plane, but the set U_2 is not. We can turn U_2 into the complex plane by using the following function

$$\varphi_2: U_2 \longrightarrow \mathbb{C}$$

$$z \mapsto z^{-1} \text{ if } z \neq \infty$$

$$\infty \mapsto 0$$

Thus we can think of \mathbb{P}^1 as two copies of the complex plane $(U_1$ and $\varphi_2(U_2))$ glued together.

A function $f: \mathbb{P}^1 \to \mathbb{C}$ is defined as a pair of functions

$$f_1: U_1 \to \mathbb{C}$$
 $f_2: U_2 \to \mathbb{C}$

such that f_1 and f_2 agree on $U_1 \cap U_2$.

We can only make sense of the complex differentiable functions when both the source and target are open subsets of \mathbb{C} . For this reason, we define holomorphic functions on \mathbb{P}^1 as follows.

A function $f: \mathbb{P}^1 \to \mathbb{C}$ is holomorphic if

- 1. $f|_{U_1}$ is holomorphic,
- 2. $f \circ \varphi_2^{-1}|_{\varphi(U_2)}$ is holomorphic.

$$U_1 \xrightarrow{f|_{U_1}} \mathbb{C}$$

$$\begin{array}{c} U_2 \stackrel{f}{\longrightarrow} \mathbb{C} \\ \varphi_2^{-1} \bigwedge^{\uparrow} \varphi_2 & \\ \mathbb{C} \end{array}$$

Q. 1. Check that defining a function $f: \mathbb{P}^1 \to \mathbb{C}$ is equivalent to defining a pair of functions

$$f_1: U_1 \longrightarrow \mathbb{C}$$

 $f_2 \circ \varphi_2^{-1}: \varphi(U_2) \longrightarrow \mathbb{C}$

such that

$$f_1(z) = f_2 \circ \varphi_2^{-1}(z^{-1})$$
 whenever $z \neq 0$.

Note that the source of both f_1 and $f_2 \circ \varphi_2^{-1}$ is \mathbb{C} .

It is kinda hard to come up with examples because of the following theorem.

Theorem 0.2. The only holomorphic functions on \mathbb{P}^1 are the constant functions.

The proof is in the following exercises.

A subset V of a topological space X is compact if it has the following property.

Every infinite sequence has a convergent subsequence i.e. for every infinite sequence of points a_1, a_2, \ldots in V there exists a subsequence $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$ which converges to a point in V.

- **Q. 2.** Prove that compact subsets V of a (nice) topological space X are closed i.e. if a sequence of points a_1, a_2, \ldots in V converges to point $a \in X$ then a is in V.
- **Q. 3.** Prove that every compact subset V of \mathbb{C} is bounded i.e. there exists a real number M such that z < M for all $z \in V$.

Q. 4. Use the fact that every infinite sequence has a convergent subsequence to argue that for any continuous function $g:X\to Y$ the image of a compact set is compact.

Assume the following fact:

The Riemann sphere is a compact topological space.

One way to see this is that the Riemann sphere is topologically isomorphic to the sphere S^2 in \mathbb{R}^3 (hence the name) which is a closed and bounded subset of \mathbb{R}^3 . It is not hard to show that such subsets are compact.

Q. 5. Argue that the image of any continuous function $f: \mathbb{P}^1 \to \mathbb{C}$ is bounded.

Q. 6. Using Liouville's theorem, argue that if f is a holomorphic function on \mathbb{P}^1 then $f|_{U_1}$ is a constant function.

Q. 7. Using continuity, argue that if f is a holomorphic function on \mathbb{P}^1 then f is a constant function.

Meromorphic functions on \mathbb{C}

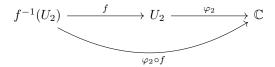
A complex differentiable function $f:X\to\mathbb{P}^1$ is called a meromorphic function on X

We can only make sense of the complex differentiable functions when both the source and target are open subsets of \mathbb{C} . For this reason, we define meromorphic functions on X as follows.

A function $f: X \to \mathbb{P}^1$ is meromorphic if

- 1. f is holomorphic when restricted to $f^{-1}(U_1)$,
- 2. $\varphi_2 \circ f$ is holomorphic when restricted to $f^{-1}(U_2)$.

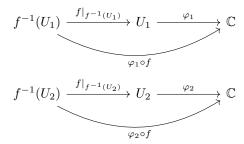
$$f^{-1}(U_1) \xrightarrow{f} U_1 = \mathbb{C}$$



It gets tedious to keep track of all the inverses and the sources and targets. We use the following shorthand notation to simplify the clutter. Let $\varphi_1: U_1 \to \mathbb{C}$ be the identity function, $\varphi_1(z) = z$. Then a function $f: X \to \mathbb{P}^1$ is meromorphic if the two functions

- 1. $\varphi_1 \circ f$,
- $2. \varphi_2 \circ f$

are holomorphic wherever they make sense.



Q. 8. Which of the following functions are meromorphic functions on \mathbb{C} ?

- 1. f(z) = z

$$f(z) = \begin{cases} z^{-1} & \text{if } z \neq 0\\ \infty & \text{if } z = 0 \end{cases}$$

3.

$$f(z) = \begin{cases} e^{1/z} & \text{if } z \neq 0\\ \infty & \text{if } z = 0 \end{cases}$$

Meromorphic functions on \mathbb{P}^1

Q. 9. Show that a function $f: \mathbb{P}^1 \to \mathbb{P}^1$ is meromorphic if the four functions

- 1. $\varphi_1 \circ f \circ \varphi_1^{-1}$, 2. $\varphi_1 \circ f \circ \varphi_2^{-1}$, 3. $\varphi_2 \circ f \circ \varphi_1^{-1}$, 4. $\varphi_2 \circ f \circ \varphi_2^{-1}$,

are holomorphic wherever they make sense.

Q. 10. Let p(z) and q(z) be polynomials with no common roots. Assume that q(z) is not the 0 polynomial.

Show that the following function is a meromorphic function on \mathbb{P}^1 .

$$f(z) = \begin{cases} \frac{p(z)}{q(z)} & \text{if } z \neq \infty, q(z) \neq 0, \\ \infty & \text{if } z \neq \infty, q(z) = 0, \\ \lim_{z \to \infty} \frac{p(z)}{q(z)} & \text{if } z = \infty. \end{cases}$$

Such a function is called a rational function. It is common to simply write $f(z) = \frac{p(z)}{q(z)}$.

Turns out these are all the meromorphic functions on \mathbb{P}^1 .

Theorem 0.3. Every meromorphic function on \mathbb{P}^1 is a rational function.

The following exercises provide the proof of this theorem.

Let $f: \mathbb{P}^1 \to \mathbb{P}^1$ be a meromorphic function. Let $\mathcal{Z} = f^{-1}(0) \cap \mathbb{C}$ and $\mathcal{P} = f^{-1}(\infty) \cap \mathbb{C}$. \mathcal{Z} is called the set of zeroes and \mathcal{P} is called the set of poles.

Q. 11. Using the isolated zeroes property of complex differentiable functions argue that both the sets \mathcal{Z} and \mathcal{P} are isolated i.e. for every point $x \in \mathcal{Z}$ there exists a neighborhood U of x such that $U \cap \mathcal{Z} = \{x\}$. Similarly, for \mathcal{P} .

Q. 12. Using the fact that every infinite sequence in a compact set has a convergent subsequence, and that \mathbb{P}^1 is compact, argue that both \mathcal{Z} and \mathcal{P} are finite sets.

Let $\mathcal{Z} = \{z_1, \ldots, z_m\}$ and $\mathcal{P} = \{p_1, \ldots, p_n\}$. Assume the following fact for now. We'll prove it in class tomorrow.

The function

$$g(z) = f(z) \cdot \frac{(z - p_1)^{k_1} \dots (z - p_n)^{k_n}}{(z - z_1)^{\ell_1} \dots (z - z_m)^{\ell_m}}$$

is meromorphic and has no zeroes or poles, for some positive integers k_1, \ldots, k_n and ℓ_1, \ldots, ℓ_m .

Q. 13. Check that the open mapping theorem 0.1 extends verbatim to meromorphic functions on \mathbb{P}^1 .

Q. 14. Using the open mapping theorem and Q.4 argue that g is either a constant function or the image of g is all of \mathbb{P}^1 .

Because g has no zeroes or poles the image of g cannot be all of \mathbb{P}^1 . Hence, g is a constant function i.e. g(z)=c for some $c\in\mathbb{C}$. Hence,

$$f(z) \cdot \frac{(z - p_1)^{k_1} \dots (z - p_n)^{k_n}}{(z - z_1)^{\ell_1} \dots (z - z_m)^{\ell_m}} = c$$

$$\implies f(z) = c \frac{(z - z_1)^{\ell_1} \dots (z - z_m)^{\ell_m}}{(z - p_1)^{k_1} \dots (z - p_n)^{k_n}}.$$

Hint: You will need to use the fact that the only open and closed subsets of \mathbb{P}^1 are the empty set and \mathbb{P}^1 itself.