Derived Geometry & BV formalism

Rep Theory

LG, affine Kacs-Moody

↑

← Factorization Algebras →

Physics

Chiral conformal FT

Geometry / Topology

Riemann Surfaces

Main ex: Fact algebras arising from Lie algebras

Goal: Explain how to see conformal blocks via factorization methods

Factorization Algebras: like sheaves

Σ - Smooth manifolds

$C^{\otimes}$ - symmetric monoidal category

Def^n: A prefactorization algebra A on Σ with values in $C^{\otimes}$,

- $\forall\, U \subseteq \{\text{open sets in } \Sigma\}$, $A(U) \in C$
- $U \subseteq V \Rightarrow A(U) \longrightarrow A(V)$
- $U_1 \cdots U_n$ disjoint $\Rightarrow A(U_1) \otimes \cdots \otimes A(U_n) \longrightarrow A(V)$
  $U_1 \cup \cdots \cup U_n = V$.
- natural associativity constraints

Another way of saying this, is that there is a coloured operad $\text{Opens}(\Sigma)^{\amalg}$ with colours the opens in Σ. A is an algebra in $C^{\otimes}$ over $\text{Opens}(\Sigma)^{\amalg}$

eg: 1. A - associative algebra, A- factorization algebra

$C^{\otimes} = \text{Vect}^{\otimes}$, $\Sigma = \mathbb{R}$

$A(\cdot\!-\!\cdot) = A$ $\quad A(\cdot\!-\!\cdot) \otimes A(\cdot\!-\!\cdot) \longrightarrow A(\cdot\!-\!\cdot \cup \cdot\!-\!\cdot)$ is the multiplication map

A(countable intervals) = infinite tensor product (?)

2. $V \rightarrow \Sigma$ vector bundle

$\Gamma_c(U)$-compactly supported sections on U

$U \longmapsto \mathrm{Sym}(\mathcal{V}_c(U))$

3. Sheaf of dg Lie algebra $\rightsquigarrow$ prefactorization Lie algebras

Def^n: Local Lie algebra on Σ is a
- $\mathbb{Z}$ graded vector bundle $L \to \Sigma$, $\mathcal{L}$ = sheaf of smooth sections
- $d : L \to L$ degree $+1$, differential, $d^2 = 0$
- $[\,,\,] : L \otimes L \to L$ bidifferential operator such that

such that we have a sheaf of dg-Lie algebras.

eg: · every elliptic complex is an abelian local Lie Algebra
· M oriented 3-manifold. $\mathfrak{g}$ - Lie algebra

$\Omega^*_M \otimes \mathfrak{g}$, $d = d_M \otimes \mathbb{1}_{\mathfrak{g}}$, $[\,,\,] = \wedge \otimes [\,,\,]_{\mathfrak{g}}$

(in Chern Simons)

$$\begin{aligned} CE_* : \text{dg Lie Algebra} &\longrightarrow \text{dg Vect} \\ \mathfrak{g} &\longmapsto (\mathrm{Sym}(\mathfrak{g}[\,,\,]),\ d_{\mathfrak{g}} + d_{[\,,\,]}) \end{aligned}$$

Chevalley Eilenberg Chains

Def^n: Enveloping Factorization Algebra $\mathbb{U}(\mathcal{L})$

$U \longmapsto CE_*(\mathcal{L}_c(U))$

Def^n: The Weiss topology on Σ is generated by a Weiss cover.
Weiss cover $\{U_i\}$ of an open V is a collection such that
$\forall \{x_1 \cdots x_n\} \subseteq V$, $\exists\, U_i$ s.t. $U_i \supseteq \{x_1 \cdots x_n\}$.

Def^n: A factorization algebra is a prefactorization alg $\mathcal{A}$ such that
- $\mathcal{A}$ is a cosheaf for the Weiss topology
- for any pair of disjoint opens U, V

$$\mathcal{A}(U) \otimes \mathcal{A}(V) \xrightarrow{\simeq} \mathcal{A}(U \sqcup V)$$

Main Ex:

Σ = Riemann Surface

$\mathfrak{g}$ = Lie algebra

$\mathcal{G} = \Omega^{0,*}_{\Sigma} \otimes \mathfrak{g}$ - Local Lie algebra

Claim: Enveloping fact. algebra $\mathbb{U}\mathcal{G}$ recovers the vertex algebra for the loop algebra. $L\mathfrak{g} = \mathfrak{g}(t)$

$$V = \mathrm{Ind}_{\mathfrak{g}[[t]]}^{\mathfrak{g}((t))} \mathbb{C}$$

$\mathfrak{g}((t))$ ← Laurent series

$\mathbb{C}$ ← trivial rep

$\mathfrak{g}[[t]]$ ↑ power series

Fact: For any disk D there's a natural dense inclusion

$$V \hookrightarrow H^0(\mathbb{U}\,\mathcal{G}(D))$$

$$H^1(\mathcal{G}_c(D)) \simeq \mathcal{O}(D)^*$$ holomorphic distributions

Modules

Module $\mathcal{M}$ on $\mathcal{A}$

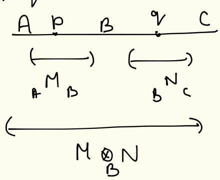
- $\mathcal{M}(U) \in \mathcal{C}$
- $\forall\, U_1, \dots U_n, V \subseteq W$ we have "compatible" maps

$$\mathcal{A}(U_1) \otimes \cdots \otimes \mathcal{A}(U_n) \otimes \mathcal{M}(V) \longrightarrow \mathcal{M}(W)$$

eg: 1. ${}_A M_A$, $\Sigma = \mathbb{R}$

for $p \in \mathbb{R}$, $\mathcal{M}_p(U) = \begin{cases} M & \text{if } p \in U \\ 0 & \text{else} \end{cases}$

2. $p, q \in \mathbb{R}$



- $\mathcal{L} = \Omega^{0,*} \otimes \mathfrak{g}$??

M = sheaf that is also a module for $\mathcal{L}$