Spectra and Stable Homotopy Theory

— Alice Hedenlund

$\mathrm{Top}_* = \begin{cases} \text{ob} & \text{pointed } \cancel{\text{spaces}} \text{ CW complexes} \\ \text{mor} & \text{pointed continuous maps} \end{cases}$

$\mathrm{Ho}(\mathrm{Top}_*)$ = homotopy category

morphisms = $[X, Y]$

Suspension loop adjunction:

$$[\Sigma X, Y] \cong [X, \Omega Y]$$

Stable homotopy theory: study of properties of classical homotopy theory stable under Σ.

§ Spectra:

Def: Spectrum X = seq. of pointed spaces X_n along with structure maps $\Sigma X_n \to X_{n+1}$.
or equivalently $X_n \to \Omega X_{n+1}$.

• X is an Ω spectrum if $X_n \simeq \Omega X_{n+1}$ via .

Def: $\pi_k(X) := \mathrm{colim}_n\, \pi_{k}(\Sigma^{k+n} X_n)$

Sphere spectrum: $\mathbb{S}$

$$\mathbb{S}_n = S^n$$

Suspension spectrum: $K \in Top_*$, $\Sigma^\infty K$ - spectrum

$$(\Sigma^\infty K)_n = \Sigma^n K$$

Eilenberg - Maclane spectrum:

G - abelian group,

$K(G,n)$ char by $\pi_* K(G,n) = \begin{cases} G & \text{if } * = n \\ 0 & \text{else} \end{cases}$

$$K(G,n) \simeq \Omega K(G,n+1)$$

Eilenberg - Maclane spectrum: $HG_n := K(G,n)$

$\mathbb{S}$ Level model structure:

Levelwise w.e., fibrations.

Bousfield localize wrt to $\star$

$f : X \to Y$ s.t. $\pi_*(X) \xrightarrow{\cong} \pi_*(Y)$

is an iso.

<u>Rmk</u>: cofibrant objects: retracts of CW-spectra

fibrant obs : Ω-spectra

w.e. : iso on π_*

<u>Def</u>: This is called the <u>stable homotopy category</u>.