Rotations

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I did not have time to proof-read these notes, these are likely to have more errors than usual :-/

Let's start by analyzing the orthogonal group in 2 dimensions $O(2)$.

$$O(2) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = I_2 \right\} \tag{0.1}$$

$$SO(2) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = I_2, ad - bc = 1 \right\} \tag{0.2}$$

By a direct computation we can show that every element of $O(2)$ is one of the two forms (Exercise 3.1)

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \tag{0.3}$$

These matrices have determinants 1 and -1 respectively and represent rotations and reflections in $\mathbb{R}^2$. The eigenvalues are of the form $e^{i\theta}, e^{-i\theta}$ for the rotation matrices and ± 1 for the reflection ones and so we get

Proposition 0.1. *Every matrix in $O(2)$ is either a rotation, in which case it is similar to a matrix of the form $\begin{bmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{bmatrix}$ or a reflection about a line, in which case it is similar to a matrix of the form $\begin{bmatrix} -1 & \\ & 1 \end{bmatrix}$.*

1 Orthogonal matrices

This method does not generalize to higher dimensions (or does it?) instead we use eigenvalues to analyze the matrices.

Theorem 1.1 (Spectral theorem)**.** *Every matrix in $O(n)$ and $U(n)$ is diagonalizable over the complex numbers.*

Recall that diagonalizable means that the matrix is similar to a diagonal matrix i.e. it becomes diagonal after doing some base change. Even though $O(n)$ has real entries it's eigenvalues and eigenvectors might be complex i.e. the eigenvectors can be vectors in $\mathbb{C}^n$ instead of $\mathbb{R}^n$.

Because $O(n) \subseteq U(n)$ it suffices to analyze the eigenvectors of unitary matrices. Let $M \in U(n)$ be a unitary matrix. By the Spectral theorem there exist n eigenvectors $v_1, \ldots, v_n \in \mathbb{C}^n$ with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ i.e. $Av_i = \lambda_i v_i$. Using the definition of unitary matrices we must have

$$\langle Av_i, Av_i \rangle = \langle v_i, v_i \rangle \tag{1.1}$$

$$\implies \langle \lambda_i v_i, \lambda_i v_i \rangle = \langle v_i, v_i \rangle \tag{1.2}$$

$$\implies \overline{\lambda}_i \lambda_i \langle v_i, v_i \rangle = \langle v_i, v_i \rangle \tag{1.3}$$

$$\implies \overline{\lambda}_i \lambda_i = 1 \tag{1.4}$$

As $O(n) \subseteq U(n)$ the same holds for $O(n)$ so we get the following proposition.

Proposition 1.2. *Every eigenvalue of an unitary or an orthogonal matrix is a complex number of norm 1 and hence is of the form $e^{i\theta}$ for some θ.*

1.1 Orthogonal matrices in 3 dimensions

Consider a matrix $A \in O(3)$, by the previous section A has 3 eigenvalues of the form $\lambda_1 = e^{i\theta_1}$, $\lambda_2 = e^{i\theta_2}$, $\lambda_3 = e^{i\theta_3}$ for some θ_1, θ_2, θ_3. But $O(3)$ has real entries and hence the complex eigenvalues of A should come in conjugate pairs. The only way this can happen is if $\theta_1 = 0$ or π and $\theta_2 = -\theta_3$.

Proposition 1.3. *For any $A \in O(3)$ the eigenvalues of A are of the form $\lambda_1 = \pm 1, \lambda_2 = e^{i\theta}, \lambda_3 = e^{-i\theta}$ for some θ. Further $\lambda_1 = 1$ iff $A \in SO(3)$.*

If $A \in SO(3)$ then A is similar to

$$A \sim \begin{bmatrix} 1 & & \\ & e^{i\theta} & \\ & & e^{-i\theta} \end{bmatrix} \sim \begin{bmatrix} 1 & & \\ & \cos\theta & -\sin\theta \\ & \sin\theta & \cos\theta \end{bmatrix} \tag{1.5}$$

This is saying that any matrix in $SO(3)$ represents rotation around an axis.

If $A \in O(3) \setminus SO(3)$ then A is similar to

$$A \sim \begin{bmatrix} -1 & & \\ & e^{i\theta} & \\ & & e^{-i\theta} \end{bmatrix} \sim \begin{bmatrix} -1 & & \\ & \cos\theta & -\sin\theta \\ & \sin\theta & \cos\theta \end{bmatrix} \tag{1.6}$$

This is saying that any matrix in $O(3) \setminus SO(3)$ represents rotation around an axis followed by a reflection along the perpendicular plane.

Proposition 1.4. *Every linear transformation of $\mathbb{R}^3$ that preserves distances is either a rotation about an axis or a rotation about an axis followed by a rotation about the perpendicular plane.*

2 Quaternions

There is another way to talk about rotations, using quaternions! Recall that **quaternions** form a non-abelian group, denoted $\mathbb{H}$, that is isomorphic as a set to $\mathbb{R}^4$. Elements of $\mathbb{H}$ are of the form $a + bi + cj + dk$ and satisfy the relations

$$i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j \tag{2.1}$$

A quaternion $p \in \mathbb{H}$ defines a linear transformation $\Phi(p) : \mathbb{H} \to \mathbb{H}$ that sends $v \mapsto pvp^{-1}$. These transformation turn out to be rotations when restricted to the unit quaternion group!

Let $S\mathbb{H}$ denote the group of unit quaternions i.e. $\{p \in \mathbb{H} : \|p\| = 1\}$. We think of $\mathbb{R}^3$ as the set of *purely imaginary* quaternions i.e. the vector (x, y, z) represents the quaternion $xi + yj + zk$. It turns out to be the case that when $p \in S\mathbb{H}$ the transformation $v \mapsto pvp^{-1}$ preserves the set of purely imaginary quaternions. In fact a much stronger result holds.

Theorem 2.1. *The map sending $p \in S\mathbb{H}$ to $\Phi(p)$ defines a homomorphism*

$$\Phi : S\mathbb{H} \to SO(3) \tag{2.2}$$

This homomorphism is surjective with kernel $\mathbb{Z}/2$.

The proof of this has several steps and is in Exercises in 3.1.

The group $S\mathbb{H}$ shows up in several avatars in various branches of mathematics. It is the spin group in 3 dimensions, denoted $Spin(3)$. Because $SO(3)$ is the group of rotation of $\mathbb{R}^3$ the above theorem is asserting that there are two quaternions over each rotation of $\mathbb{R}^3$. In physics this fact becomes relevant because in quantum mechanics certain systems have $S\mathbb{H}$ as their symmetry groups and for such systems there are is a physical quantity, called **spin** which has two possible values for each value of the angular moment.

3 Exercise

Exercise 3.1. Consider a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in O(2)$.

1. Show that for some θ, ϕ we must have $a = \cos\theta$, $b = \sin\theta$, and $c = \cos\phi$, $d = \sin\phi$.
2. Find the relations between θ and ϕ and prove that every matrix in $O(2)$ is of the form (0.3).
3. Describe the matrices in (0.3) geometrically and compute their eigenvalues.

Exercise 3.2. Show that the matrices

$$\begin{bmatrix} 1 & & \\ & \cos\theta & \sin\theta \\ & \sin\theta & -\cos\theta \end{bmatrix} \qquad \begin{bmatrix} -1 & & \\ & \cos\theta & \sin\theta \\ & \sin\theta & -\cos\theta \end{bmatrix} \tag{3.1}$$

are in $O(3)$. What do these geometrically represent? Find the matrices of type (1.5) or (1.6) to which these are similar.

Exercise 3.3. Describe the matrices in $SO(n)$ geometrically for arbitrary positive integer n. Do these matrices still represent rotations? What is the difference between matrices in $SO(2n)$ and matrices in $SO(2n+1)$.

3.1 Quaternions

The following exercises prove theorem 2.1.

Exercise 3.4. The first step is to figure out how to deal with inner products using quaternions. Let $\Re(a + bi + cj + dk) = a$ denote the real part of quaternions.

1. Show that for two vectors $x, y \in \mathbb{R}^3$ the dot product $\langle x, y\rangle$ is equal to $-\Re(xy)$.
2. Show that for any quaternion $p\overline{p} = \|p\|^2$ and hence if $p \in S\mathbb{H}$ then $p^{-1} = \overline{p}$.
3. Show that for $p \in S\mathbb{H}$ and $v \in \mathbb{H}$ we have $\Re(x) = \Re(pxp^{-1})$. This implies in particular that $\Phi(p)$ takes the purely imaginary quaternions to purely imaginary quaternions.
4. Show that for $p \in S\mathbb{H}$ and $x, y \in \mathbb{R}^3$ we have $\langle x, y\rangle = \langle px\overline{p}, px\overline{p}\rangle$.

Exercise 3.5. Let $p \in S\mathbb{H}$ be a unit quaternion. The above exercise proves that the transformation $\Phi(p)$ preserves the dot product.

1. Show that for $q \in \mathbb{H}$ we have $\Phi(pq) = \Phi(p)\Phi(q)$ and hence we have a group homomorphism $\Phi : S\mathbb{H} \to SO(3)$. (It is $SO(3)$ and not $SO(4)$ because we're looking at the transformations of the space of purely imaginary quaternions.)
2. Argue that because $S\mathbb{H}$ is connected the image of Φ should be a subgroup of $SO(3)$ and hence Φ is a homomorphism $S\mathbb{H} \to SO(3)$.
3. Show that for the unit quaternion $p = \cos(\theta/2) + \sin(\theta/2)(xi + yj + zk)$ the transformation $\Phi(p)$ fixes the vector (x, y, z). Use this to argue that Φ is surjective.
4. Show that the center of $S\mathbb{H}$ is the set of purely real quaternions. Argue that the kernel of Φ is $\mathbb{Z}/2$.

Exercise 3.6.

$$O_8 = \{\pm 1, \pm i, \pm j, \pm k\} \tag{3.2}$$

Let $O_8 \subseteq \mathbb{H}$ be the finite quaternion group. Describe the image of O_8 under the homomorphism Φ (defined in Section **??**).