Graded Homological Algebra

Recall: Graded ring $A = \bigoplus_i A^i$, need not be commutative (i.e. $ab = (-1)^{|a||b|} ba$)

Module $/_A$: $M = \bigoplus_i M^i$, maps $A^i \otimes M^j \longrightarrow M^{i+j}$, left or right module structure

if R is a commutative ring $\subseteq A^0$ then we can think of A as graded R-algebras.

M, N right, left A-modules then $(M \otimes_A N)^k = \left(\bigoplus_{i+j=k} M^i \otimes N^j \right) \Big/ \langle \alpha a \otimes \beta - \alpha \otimes a\beta \; ; \; a \in A^s, \alpha \in M^{i-s}, \beta \in N^j \rangle$

ideals are also themselves graded (homogeneous)

$\mathrm{Hom}_A(M,N)^k := \{ \varphi \in \prod_{i-j=k} \mathrm{Hom}(M^i, N^j) \mid \varphi(an) = a\varphi(n) \text{ for } \forall a \in A \}$

Note: A projective module is defined as before except that maps are graded. We still have the direct summand property.

A graded object is a comodule $/ \mathbb{Z}[x, x^{-1}]$ ⇜ How? Also what is so special about $\mathbb{Z}$-grading? Any Groupoid will do.

Projective Resolution: $\underbrace{\xrightarrow{\partial_3} P_2(M) \xrightarrow{\partial_2} P_1(M) \xrightarrow{\partial_1} P_0(M)}_{\text{graded right } A\text{-module projective acyclic}} \xrightarrow{\partial_0} M \longrightarrow 0$

Consider $P_{\geq 0}(M) \otimes_A N$,

Define $\mathrm{Tor}^A_{i,j}(M,N) = H_i\left((P_\cdot(M) \otimes_A N)^j \right)$

$\mathrm{Tor}^A_{0,*}(M,N) \cong M \otimes_A N$

i - external grading, j internal grading

Similarly Ext

Ex: Koszul Resolutions

1. $A = R[x_n]$ ⇜ note need conditions on R and n to make it commutative

R has a right A-module structure $R \cong {}^{A}\!/\!_{A_{>0}}$, then we have a projective resolution:

$$0 \longleftarrow R \longleftarrow A \longleftarrow y \otimes A \longleftarrow 0$$

$(y \otimes A)^k = A^{k-n}$

This is bigraded

This map is given by $y \otimes a \longmapsto x_n a$ } is grading preserving and injective and image is $A_{>0}$

Notation: $P_\cdot(R) = \Lambda(y) \otimes A$, $|y| = (1, n)$, $\partial y = x_n$ ⇜ Koszul Resolution

Q. Calculate $\mathrm{Tor}^{R[x_n]}(R, -)$, $\mathrm{Ext}_{R[x_n]}(R, -)$

A. $\mathrm{Tor}^{R[x_n]}(R, R)$:

$$0 \longleftarrow R \longleftarrow R[x_n] \longleftarrow R y \otimes_R R[x_n] \longleftarrow 0$$

Now tensor over $R[x_n]$ with R

$$0 \longleftarrow R \otimes_{R[x_n]} R \longleftarrow R[x_n] \otimes_{R[x_n]} R \longleftarrow R y \otimes_R R[x_n] \otimes_{R[x_n]} R \longleftarrow ??$$

$R \otimes_{R[x_n]} R \cong R \otimes_R R \cong R$, $Ry \otimes_R R[x_n] \otimes_{R[x_n]} R \cong Ry \otimes_R R \otimes_R R \cong Ry$ so that the sequence is

$0 \leftarrow R \leftarrow R \leftarrow Ry$ so we get

$$\mathrm{Tor}^{R[x_n]}_{i,j}(R,R) \cong \begin{cases} R & \text{if } i=1, j=n \quad , i=0, j=0 \\ 0 & \text{else} \end{cases}$$

Ext:

Hom the resolution into R over $R[x_n]$

$$\begin{array}{ccccc} 0 \to \mathrm{Hom}_{R[x_n]}(R,R) & \longrightarrow & \mathrm{Hom}_{R[x_n]}(R[x_n],R) & \longrightarrow & \mathrm{Hom}_{R[x_n]}(Ry \otimes_R R[x_n], R) \\ \wr\wr & & \wr\wr & & \wr\wr \\ \mathrm{Hom}_R(R,R) = R & & R & & \mathrm{Hom}_R(Ry, \mathrm{Hom}_{R[x_n]}(R[x_n],R)) \cong Ry^* \end{array}$$

And so we obtain that $\mathrm{Ext}^{*,*}_{R[x_n]}(R,R) = \begin{cases} R & \text{if } *,*=1,n \text{ or } 0,0 \\ 0 & \text{else} \end{cases}$

Another way of saying the same thing is that $\mathrm{Tor}^{R[x_n]}(R,R) = \Lambda(y)$, $\mathrm{Ext}_{R[x_n]}(R,R) = \Lambda(y^*)$

2. $A = R[x_n]/(x_n^k)$

$R = A^+/A$ then $P_\cdot(\underline{R}) = \Lambda(y) \otimes_R \Gamma(\gamma) \otimes_R R[x_n]/(x_n^k)$ $|y| = (1,n)$ $|\gamma_j| = (2j, jnk)$

$\partial y = x_n$, $\partial \gamma_{j+1} = \gamma_i y x_n^{k-1}$

Show that this is a projective resolution.

3. $A = \Lambda_R(x_n)$

$P_\cdot(R) = \Gamma(\gamma) \otimes_R \Lambda_R(x_n)$, $|\gamma_i| = (i, in)$ $\partial \gamma_{i+1} = \gamma_i x_n$ (Note: if $2=0$ $\Gamma(\gamma) \cong \Lambda(y) \otimes \Gamma(\gamma_2)$)

Assuming A free$/R$

Note: If A is commutative algebra$/R$ then $\mathrm{Tor}^A_{*,*}(R,R)$ is also commutative.

Proof: $A \otimes A \longrightarrow A$ is an algebra map when A commutative.
$a \otimes b \longmapsto ab$

So A modules are also $A \otimes_R A$ modules.

Look at $P_\cdot(R) \otimes_R P_\cdot(R) \longrightarrow R \otimes_R R$ this is a projective $A \otimes A$ resolution of R

Then by acyclicity of $P_\cdot(R)$ and projectivity of $P_\cdot(R) \otimes P_\cdot(R)$ we get a diagram

$$\begin{array}{ccc} P_\cdot(R) \otimes P_\cdot(R) & \longrightarrow & P_\cdot(R) \\ \downarrow & & \downarrow \\ R & \longrightarrow & R \end{array}$$

Tensor this with R over $A \otimes A$

This gives us

$$\begin{array}{ccc} \mathrm{Tor}_A(R,R) \otimes_R \mathrm{Tor}_A(R,R) & \longrightarrow & \mathrm{Tor}_{A \otimes_R A}(R,R) \\ & & \downarrow \\ & & \mathrm{Tor}_A(R,R) \end{array}$$

Now we can flip the two A's in $A \otimes A$

and argue by uniqueness that the Tor is commutative.

Check

Remark: 1) if A is a Hopf algebra (free/R) of finite type then $\mathrm{Ext}_A(R,R)$ is an algebra.

2) if A is cocommutative then $\mathrm{Ext}_A^{*,*}(R,R)$ is commutative. $\longleftarrow$ Prove.

1. $A = R[x_n]$ $\qquad \mathrm{Tor}^A(R,R) = \Lambda(y) \qquad |y| = (1,n)$

Note: $R[x_n]$ can be given a Hopf algebra structure $\quad \Delta : A \longrightarrow A\otimes A$ and extend via algebra map.

$$x_n \longmapsto 1\otimes x_n + x_n \otimes 1$$

Then $\mathrm{Ext}_A^{*,*}(R,R) = \Lambda(y)^* \cong \Lambda(y^*)$

using the coproduct Δ

$$x_n^k \longmapsto (1\otimes x_n + x_n\otimes 1)^k = \sum \binom{k}{i} x_n^i \otimes x_n^{k-i}$$

Q. If $A = \Lambda(x_n)$ find $\mathrm{Tor}_A(R,R)$, $\mathrm{Ext}_A(R,R)$ as algebras.

A. $P(R) = \Gamma(y) \otimes \Lambda(x_n) \qquad |y_i| = (i, in) \qquad \partial y_{i+1} = y_i x_n$

$$\mathrm{Tor}_{**}^A(R,R) = H_*\big(\Gamma(y) \underset{R}{\otimes} \Lambda(x_n) \underset{\Lambda(x_n)}{\otimes} R\big) \overset{?}{=} H_*(\Gamma(y)) = \Gamma(y)$$

$$\mathrm{Ext}_A^{**}(R,R) = H^*\big(\mathrm{Hom}_{\Lambda(x_n)}(\Gamma(y) \underset{R}{\otimes} \Lambda(x_n), R)\big) = H^*\big(\mathrm{Hom}_R(\Gamma(y), \mathrm{Hom}_{\Lambda(x_n)}(\Lambda x_n, R))\big)$$

$$= H^*(\mathrm{Hom}_R(\Gamma(y), R)) = \Gamma(y)^*$$ what is this?

Q. Check that for $A = R[x_n]/x_n^k$ a projective resolution for R is $\Lambda(y) \otimes \Gamma(\gamma) \otimes R[x_n]/x_n^k$ where the degrees are $|y| = (1,n)$, $|\gamma_i| = (2i, 2ink)$

A: external degree ↴

$$\begin{array}{ccccc}
-1 & 0 & 1 & 2 & 3 \\
0 \leftarrow R \leftarrow & 1 & & &
\end{array}$$

$k=4$

$x_n \longleftarrow y$

$x_n^2 \longleftarrow x_n y$

$x_n^2 y \longleftarrow \gamma_1$

$\gamma_1 x_n \longleftarrow \gamma_1 y$

$\gamma_1 x_n^2 \longleftarrow \gamma_1 y x_n \qquad \gamma_1^2/2$

$\gamma_1 y x_n^2 \longleftarrow \gamma_2 \qquad ''$

In general if

$k = 2j$ the k degree term is $R\langle \gamma_j, \gamma_j x_n, \dots, \gamma_j x_n^{k-1} \rangle$

$k = 2j+1$ the k degree term is $R\langle y\gamma_j, y\gamma_j x_n, \dots, y\gamma_j x_n^{k-1} \rangle$

$$\left.\begin{array}{l}
\partial : 2j \longrightarrow 2j-1 \\
\gamma x_n^i \longmapsto \gamma y x_n^{k-1+i} \\
\partial : 2j+1 \longrightarrow 2j \\
\gamma y x_n^i \longmapsto \gamma x_n^{i+1} + \cancel{y^2 x_n}
\end{array}\right\} \Rightarrow \text{exactness}$$

Why $\gamma_i \gamma_j = \binom{i+j}{i} \gamma_{i+j}$?

$$\partial(\gamma_i \gamma_j) = \partial\gamma_i \cdot \gamma_j + \gamma_i \cdot \partial\gamma_j$$

$$= (y \cdot \gamma_{i-1} \cdot x_n^{k-1}) \cdot \gamma_j + \gamma_i \cdot (y \cdot \gamma_{j-1} \cdot x_n^{k-1})$$

$$= y(\gamma_{i-1}\cdot\gamma_j + \gamma_i \cdot \gamma_{j-1}) x_n^{k-1} = y\left(\binom{i+j-1}{i-1} + \binom{i+j-1}{i}\right) \gamma_{i+j-1} x_n^{k-1} = \binom{i+j}{i} \partial(\gamma_{i+j})$$

Why projective? Each grade is simply a copy of R. $\square$

Q. If A is a Hopf algebra which is cocommutative and free over R, and of finite type then $\mathrm{Ext}^{**}(R,R)$ is a commutative graded algebra over R.

Q. Compute product structures for the various Ext groups.

$\longrightarrow A = R[x_n] \qquad \Delta : x_n \longmapsto 1\otimes x_n + x_n \otimes 1$

then $\quad \mathrm{Hom}_R(A,R) = R\langle 1, \gamma_1, \gamma_2, \dots \rangle$

$\gamma_i : x_n^i \longrightarrow 1$ rest all go to 0

and
$$\gamma_i\gamma_j(x_n^k) = (\gamma_i \otimes \gamma_j)\Delta x_n^k$$
$$= (\gamma_i\otimes\gamma_j)\sum_i \binom{k}{i} x_n^i \otimes x_n^{k-i}$$
$$= \begin{cases} \binom{k}{i} & \text{if } i+j=k \\ 0 & \text{else} \end{cases}$$

and hence $\gamma_i\gamma_j = \binom{i+j}{i}\gamma_{i+j}$

Product on A gives diagonal on $\mathrm{Hom}_R(A,A)$, $\delta(\gamma_k) = \sum_i \gamma_i \otimes \gamma_{k-i}$
And so the double dual of A is again A } check

The coproduct on $\mathrm{Hom}(A,R)$ is induced by multiplication in $R[x_n]$

$\delta : \mathrm{Hom}(A,R) \longrightarrow \mathrm{Hom}(A,R)\otimes \mathrm{Hom}(A,R) \qquad \delta(f)(x\otimes y) = f(xy)$

Claim: $\delta(\gamma_k) = \sum_i \gamma_i \otimes \gamma_{k-i}$

$\delta(\gamma_k)(x_n^i \otimes x_n^j) = \gamma_k(x_n^{i+j}) = \delta_{k,i+j}$
$\left(\sum_t x_n^t \otimes x_n^{k-t}\right)(x_n^i\otimes x_n^j) = \delta_{k,i+j}$
} as x_n^k is a basis for $R[x_n]$ result follows.

Claim: $\mathrm{Hom}(\mathrm{Hom}(A,R),R) \cong A$

We have a natural map $\quad A \longrightarrow \mathrm{Hom}(\mathrm{Hom}(A,R),R)$
$\qquad a \longmapsto (f \longmapsto f(a))$

This map clearly respects product and coproduct structure.
Isomorphism as sets follows because A is of finite type.

Need to verify the multiplicative structures on Tor and Ext. Lets do it for $A = \Lambda_R(y_n)$

$P.(R) = \Gamma(\bar{\gamma}) \otimes \Lambda(y_n) \qquad |\gamma_i| = (i, in)$

Suffices to say that tensor of Hopf algebras is a Hopf algebra:

$$(a\otimes b)\cdot(c\otimes d) := (-1)^{|b||c|}(ac\otimes bd)$$

$$\Delta : A\otimes B \longrightarrow (A\otimes A)\otimes(B\otimes B) \longrightarrow (A\otimes B)\otimes(A\otimes B)$$

$$(a\otimes b) \longmapsto \Delta a\otimes\Delta b \longmapsto \tau(\Delta a\otimes\Delta b)$$

This should be an algebra map

$$\Delta[(a\otimes b)\cdot(c\otimes d)] = \Delta((-1)^{|b||c|} ac\otimes bd) = (-1)^{|b||c|}\cdot\tau\cdot \Delta(ac)\otimes\Delta bd = (-1)^{|b||c|}\cdot\tau\cdot(\Delta a)(\Delta c)\otimes(\Delta b)(\Delta d)$$

$$\Delta(a\otimes b)\cdot\Delta(c\otimes d) = \tau(\Delta a\otimes\Delta b)\cdot(\Delta c\otimes\Delta d) = \tau\cdot(\Delta a\cdot\Delta c\otimes\Delta b\cdot\Delta d)\cdot(-1)^{|b||c|} \quad \text{as } |\Delta a| = |a|.$$

$\mathrm{Tor}_{**}^{\Lambda(y_n)}(R,R) \cong \Gamma(\bar{\gamma})$ as R-modules.

Multiplication:

$$\mathrm{Tor}^A_{p,q}(R,R)\otimes\mathrm{Tor}_{r,s}(R,R) = H_p((P.R\otimes_A R)_q)\otimes H_r((P.R\otimes_A R)_s) \longrightarrow H_{p+r}\big((P.R\otimes_A R)_q\otimes_R(P.R\otimes_A R)_s\big)$$

$$\downarrow$$

$$H_{p+r}((P.R\otimes R)_{q+s}) = \mathrm{Tor}^{\Lambda(y)}_{p+r,q+s}(R,R)$$

$\mathrm{Ext}^{**}_{\Lambda(y_n)}(R,R) \cong R[x_n]$ as R-modules $\quad x_{ni}(\gamma_j) = \delta_{ij}$

Multiplication:

$$\mathrm{Ext}^{p,q}_A(R,R)\otimes\mathrm{Ext}^{r,s}_A(R,R) = H^p(\mathrm{Hom}^q_A(P.R,R))\otimes H^r(\mathrm{Hom}^s_A(P.R,R)) \longrightarrow H^{p+r}(\mathrm{Hom}^q(P.R,R)\otimes\mathrm{Hom}^s(P.R,R))$$

$$\downarrow \Delta^*$$

$$\mathrm{Ext}^{p+s,q+r}_{\Lambda(y)}(R,R) = H^{p+r}(\mathrm{Hom}^{q+s}(P.R,R))$$

Looks like we only have to understand

$\cdot$ and Δ^* on $P.R\otimes P.R$ and $\mathrm{Hom}(P.R,R)$.

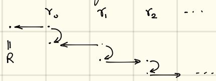
Useful table:

A	A^*	$P_\bullet(R)$	$\mathrm{Tor}^A_{**}(R,R)$	$\mathrm{Ext}^{**}_A(R,R)$
$\Lambda(y)$ $\lvert y\rvert=(0,n)$	$\Lambda(y^*)$ $\lvert y^*\rvert=(0,n)$ $y^*(y)=1$	$\Gamma(\bar{\gamma})\otimes\Lambda(y)$ $\lvert\gamma_i\rvert=(i,in)$	$\Gamma(\bar{\gamma})$	$R[x]$ $\lvert x\rvert=(1,n)$
$R[x]$ $\lvert x\rvert=(0,n)$	$\Gamma(\bar{\gamma})$ $\lvert\gamma_i\rvert=(i,in)$	$\Lambda(y)\otimes R[x]$ $\lvert y\rvert=(1,n)$	$\Lambda(y)$	$\Lambda(y^*)$
$\Gamma(\bar{\gamma})$ $\lvert\gamma_i\rvert=(0,in)$	$R[x]$ $\lvert x\rvert=(0,n)$	?? ??	??	?? ??
$R[x]/x^k$ $\lvert x\rvert=n$	??	$\Lambda(y)\otimes\Gamma(\gamma)\otimes R[x]/x^k$ $\lvert y\rvert=(1,n)$ $\lvert\gamma_i\rvert=(2i,2ink)$	$\Lambda(y)\otimes\Gamma(\gamma)$	$\Lambda(y^*)\otimes R[z]$ $\lvert y^*\rvert=(1,n)$ $\lvert z\rvert=(2,nk)$

Product structures on resolutions:

• $A = \Lambda(x_n) = R[x_n]/x_n^2$. Product structure on $P_\cdot(R) = ?$

Resolution of R:



$P_\ell(R) = \Lambda(x_n) \otimes \gamma_\ell$. $\quad \partial \gamma_\ell = \gamma_{\ell-1}$, $\quad |\gamma_\ell| = \ell n$

Q. What is $P_\ell(R) \otimes_A P_k(R) \longrightarrow P_{\ell+k}(R)$?

or $\gamma_\ell \gamma_k = c_{\ell k} \gamma_{\ell+k}$, what are $c_{\ell k}$?

This product comes by looking at the map of resolutions:

$$P_\cdot(R \otimes R) \longrightarrow P_\cdot(R)$$

On the m^{th} level we get a commutative diagram:

$$\begin{tikzcd} \bigoplus_{i+j=m} \gamma_i \otimes \gamma_j \arrow[r] \arrow[d, "\partial"] & \gamma_{i+j} \arrow[d, "\partial"] \\ \bigoplus_{i+j=m-1} \gamma_i \otimes \gamma_j \arrow[r] & \gamma_{i+j-1} \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \gamma_i \otimes \gamma_j \arrow[r] \arrow[dd] & c_{ij} \gamma_{i+j} \arrow[d] \\ & c_{ij} \gamma_{i+j-1} \arrow[d, phantom, "\parallel"] \\ \gamma_{i-1} \otimes \gamma_j + \gamma_i \otimes \gamma_{j-1} \arrow[r] & (c_{ij-1} + c_{i-1\,j}) \gamma_{i+j-1} \end{tikzcd}$$

So we get commutative relations:

$$c_{ij} = c_{i-1\,j} + c_{ij-1}$$

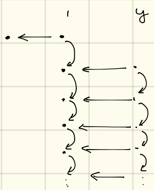
Induction then tells us:

$$c_{ij} = \binom{i+j}{i} \qquad \text{i.e.} \qquad \gamma_i \gamma_j = \binom{i+j}{i} \gamma_{i+j}$$

$$\Rightarrow \quad P_\cdot R = \Gamma(\gamma) \otimes \Lambda(x_n)$$

• $A = R[x_n]$ $\qquad |x_n| = n$

$P_\cdot(R) =$



$P_0(R) = R[x_n] \qquad P_1(R) = y R[x_n]$, $|y| = n$

There are no higher terms, so we must have

$$P_\cdot(R) = R[x] \otimes \Lambda(y)$$