Higher Categories:

Idea:

1 category: morphisms between objects

2 category: morphisms between 1-morphisms

⋮

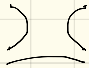
n category: morphisms between n-1 morphisms

⋮

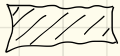
∞ - category

Problem: Not natural. Things hold only upto homotopy in general.

Ex: Category: Objects – 0 morphisms

Morphisms – 1-dim cobordisms 

Won't work. Need to replace things by equivalence classes.

2-morphisms – 2-dim cobordism 

Not well defined

Need to define weaker notions, but there is no consesus on a canonical definition.

Def: An (∞, n)-category is a (weak) ∞-category where k-morphisms are invertible (weakly) for $k > n$.

eg: Cobordism (∞, n)-category

0-manifolds

⋮

n-dim cobordisms $\rightsquigarrow$ n-morphisms

diffeomorphisms $\longrightarrow$ $(n+1)$-morphisms

isotopies $\longrightarrow$ $(n+2)$-morphisms

⋮

Def: An $(\infty, 0)$ category (∞-groupoid) is just a topological space / simplicial set.

X space

points of X $\longrightarrow$ Objects

paths of X $\rightsquigarrow$ 1-morphisms

homotopies between paths $\rightsquigarrow$ 2-morphisms

⋮

Have nice homotopy theory for spaces $\rightsquigarrow$ model category structure. key property: cartesian, internal hom compatible with model structure

Want (∞,n) categories enriched in $(\infty,n-1)$ categories

First approach to $(\infty,1)$ category:

Categories enriched in spaces

$\mathrm{Map}(x,y)$ spaces

Points are now 1-morphisms

Paths are now 2-morphisms

Can do homotopy theory for these $\rightsquigarrow$ model structure

simplicial categories

But this is too strict for examples.

And trying to define 2-categories we lose model structure.

Several other models for $(\infty,1)$ categories:

→ Quasi categories, Segal Categories, Complete Segal spaces, Categories with weak equivalences.

Simplicial spaces: $X: \Delta^{op} \longrightarrow \mathrm{Set}$

Def^n: Segal Space X is a simplicial space such that Segal maps $X_n \longrightarrow X_1 \underset{X_0}{\times} X_1 \underset{X_0}{\times} \cdots \underset{X_0}{\times} X_1$ are weak equivalences $\forall n \geq 2$.

Objects: $X_{0,0}$ mapping spaces, homotopy equivalences.

Two ways to get $(\infty,1)$ categories

1) X_0 discrete $\rightsquigarrow$ Segal Category

2) $X_{hey} \simeq X_0 \rightsquigarrow$ Complete Segal Spaces

Th^m: These model categories are Quillen equivalent to model structure on simplicial categories.

Can enrich in complete Segal Spaces $\rightsquigarrow$ model category of $(\infty,2)$ categories.

But cannot iterate it. So we need to look for higher versions of Complete Segal Spaces.

Θ- construction:

$\mathcal{C}$ category, Define $\Theta\mathcal{C}$: Objects: $[m](c_1 \cdots c_m)$, $[m] \in \Delta$, $c_i \in \mathcal{C}$ $\rightsquigarrow$ $0 \xrightarrow{c_1} 1 \xrightarrow{c_2} 2 \longrightarrow \cdots \xrightarrow{c_m} m$

Morphisms:

$$\begin{tikzcd} 0 \arrow[r, "c_1"] \arrow[d] & 1 \arrow[r, "c_2"] \arrow[d] & 2 \arrow[r, "c_3"] \arrow[d] & 3 \arrow[dl] \\ 0 \arrow[r, "d_1"'] & 1 \arrow[r, "d_2"'] & 3 & \end{tikzcd} \qquad \begin{tikzcd} {[3](c_1, c_2, c_3)} \arrow[d] \\ {[2](d_1, d_2)} \end{tikzcd}$$

Define: $\Theta_0 = *$, $\Theta_n = \Theta\Theta_{n-1}$

Note: $\Theta_1 = \Delta$, Θ_2 ? , $\Theta_2 = \Theta\Delta$

$$[3]([1][0][2])$$

$$0 \xrightarrow{[1]} 1 \xrightarrow{[0]} 2 \xrightarrow{[2]} 3 \rightsquigarrow \begin{tikzcd} 0 \arrow[r, bend left=60] \arrow[r] \arrow[r, phantom, bend left=30, "\Downarrow"] & 1 \arrow[r] & 2 \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, bend left=30, "\Downarrow"] \arrow[r, phantom, bend right=30, "\Downarrow"] & 3 \end{tikzcd}$$

Def^n: A Θ_n space is a functor $X : \Theta_n^{op} \longrightarrow$ Sets satisfying Segal and completeness conditions at all levels.

Th^m: There is a cartesian model structure for Θ_n spaces.

Conjecture: (Θ_{n-1}-Sp) categories are Quillen equivalent to Θ_n-Sp.