Compact Riemann Surfaces

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Def^n: Ringed space: $(X, \mathcal{O}_X)$ over field k

X - topological space
$\mathcal{O}_X$ - subsheaf of k-algebras of sheaf of $\mathcal{F}_{X,K}$ - functions on X to k

category of ringed spaces,

Def^n: Riemann surface: Ringed space over $\mathbb{C}$ $(X, \mathcal{O}_X)$ ring of holomorphic functions on X.

X - Hausdorff
$\forall x \in X$, $\exists\, U \ni x$, open s.t. $(U, \mathcal{O}_X|_U) \cong (V, \mathcal{O}_V)$
for some $V \subseteq \mathbb{C}$ open,
$\mathcal{O}_V$ - sheaf of holomorphic functions on V

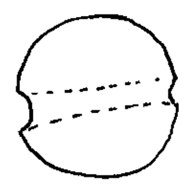
Riemann Surface $\Rightarrow$ Orientable C^∞-manifold

Notation:
$$\frac{\partial}{\partial z_\alpha} := \frac{1}{2}\left(\frac{\partial}{\partial x_\alpha} - i\frac{\partial}{\partial y_\alpha}\right) \quad \text{if } z_\alpha = x_\alpha + i y_\alpha$$
$$\frac{\partial}{\partial \bar{z}_\alpha} := \frac{1}{2}\left(\frac{\partial}{\partial x_\alpha} + i\frac{\partial}{\partial y_\alpha}\right)$$

Th^m: Any compact C^∞ 2-surface is ~~either~~ homeomorphic to ~~a sphere~~,
1) sphere
2) connected sum of torii
3) connected sum of $\mathbb{RP}^2$

Lemma: Klein Bottle homeomorphic to $\mathbb{RP}^2 \# \mathbb{RP}^2$ $\to$ connected sums.

Looks like
Two spheres attached at 2 holes

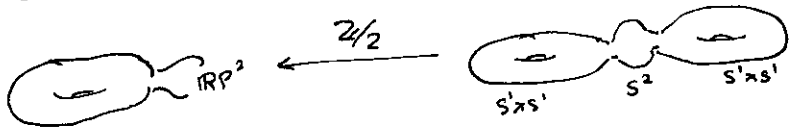


quotiented by $\mathbb{Z}/2$

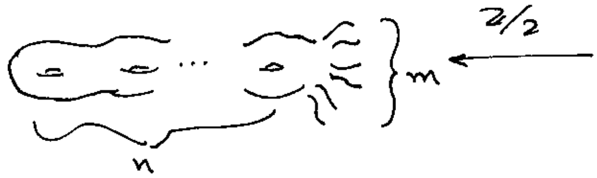
has a torus as a double cover.
So is a Klein bottle.

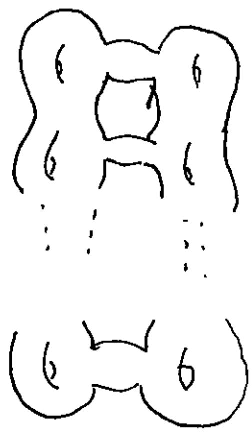
Q. what about Sum of Tori and $\mathbb{R}P^2$?

$S^1 \times S^1 \# \mathbb{R}P^2$ has 2-cover



$$(\underbrace{T^2 \# T^2 \# \cdots \# T^2}_{n}) \# (\underbrace{\mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2}_{m})$$





has a 2-covers of genus $2n+m-1$.

09/01/13

Classification of 2-manifolds using Morse-Theory:

$\boxed{M - C^\infty \text{ compact}}$

Def^n: critical point : $f: M \longrightarrow \mathbb{R}$ C^∞

p critical pt if

$f_*|_p = 0$

critical value: image of a critical point.

regular point, regular value.

[See - Milnor Morse theory + notes:]

eg, 1) S^2 — $f(x,y,z) = z$

$\{(x,y,z) \mid x^2+y^2+z^2=1\}$

critical pts: $(0,0,1)$
$(0,0,-1)$

See Conway's Proof for classification of 2-manifolds. Zip Proof ?

2) $\mathbb{P}^2 = \{[x:y:z]\} = S^2/\sim$

$f[x:y:z] = x^2+2y^2+3z^2$

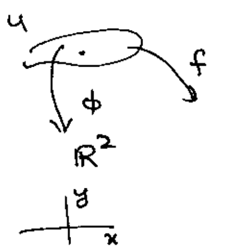
Then $[1:0:0]$, $[0:1:0]$, $[0:0:1]$ critical points

3) M will have at least 2 critical points (for M compact.).

Th^m: (Morse)

M - 2 dim C^∞, $f: M \longrightarrow \mathbb{R}$. $p \in M$ be non-degenerate critical pt. of f. Then $\exists$ chart (ϕ, U) around p & co-ordinates (x,y): s.t.

$$f \circ \phi^{-1}(x,y) = f(p) + g_i(x,y) \qquad i \in \{0,1,2\} \quad \cancel{0 \le i \le 2}$$



$$g_i(x,y) = \begin{cases} x^2+y^2 & i=0 \\ x^2-y^2 & i=1 \\ -x^2-y^2 & i=2 \end{cases}$$

i is called index of f at p.

1) For S^2: at $(0,0,1)$

$$f(x,y,z) = z$$
$$= \sqrt{1-x^2-y^2}$$
$$? \quad \sim 1 - x_1^2 - y_1^2$$

2) for $\mathbb{P}^2$:

$$f([x:y:z]) = x^2+2y^2+3z^2$$

$$U_1 = \{[x:y:z] \mid x \neq 0\}$$

$$\phi_1 : \mathbb{R}^2 \longrightarrow U$$
$$(u,v) \longmapsto \frac{[1:u:v]}{\sqrt{1+u^2+v^2}}$$

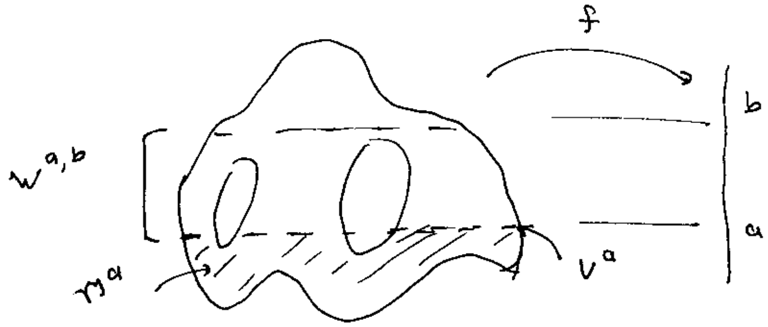
$$\theta: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(u,v) \longmapsto \left(\frac{u}{\sqrt{1+u^2+v^2}}, \frac{v\sqrt{2}}{\sqrt{1+u^2+v^2}}\right)$$

$$f \cdot \phi_1 \cdot \theta\,(u,v) \underset{?}{=} 1+u^2+v^2$$

Notation:

$f: M \to \mathbb{R}$ Morse i.e. all critical pt non-degen

$$M^a = \{x \in M \mid f(x) \le a\} = f^{-1}(-\infty, a]$$
$$(M^a)^c = f^{-1}[a, \infty)$$
$$V^a = M^a \cap (M^a)^c = f^{-1}(a)$$
$$M^{a,b} = M^b \cap (M^a)^c$$



Th^m:

$f : M \longrightarrow \mathbb{R}$ $a < b$ regular values.

$M^{a,b}$ does not contain critical points.

Then,

- $M^a \cong_{\text{diffeo}} M^b$
- $V^a \cong_{\text{diffeo}} V^b$
- $M^{a,b} \cong_{\text{diffeo}} V^a \times [0,1]$

Def^n: Ordered Morse function on C^∞ surface

$\{x_i\}$ — index 0 ordered $\Rightarrow$ $\forall\, i, j, k$

$\{y_j\}$ - index 1 $f(x_i) \leq f(y_j) \leq f(z_k)$

$\{z_i\}$ - index 2

Th^m: Ordered ~~critical points~~ Morse functions also exist.

Remark:

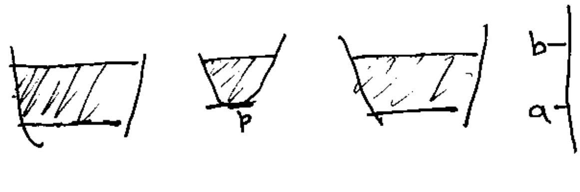
- Non degen critical points are isolated
- Model nbds of g_i's are ~~$\mathbb{H}$~~, look like

$i = 0$ $i = 2$ $i = 1$

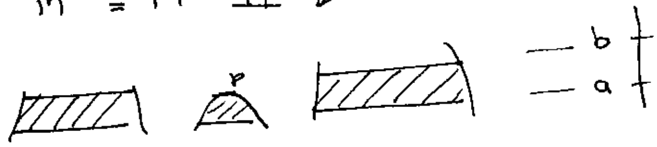
saddle

Th^m: $f : M \longrightarrow \mathbb{R}$ p critical, non-degen, $a \leq f(p) \leq b$ $f^{-1}[a,b]$ has only p as critical point

p index 0 : $M^b \cong M^a \amalg \mathbb{D}^2 \to$ canonical nbd of p



index 2: $M^a \cong M^b \sqcup \mathbb{D}^2$



Prop: $f: M \to \mathbb{R}$ ordered Morse
↳ surface, connected

If f has no index 1 critical point then M is homeomorphic to S^2.

In this case there can be only 2 critical points.

Rm: index 1:

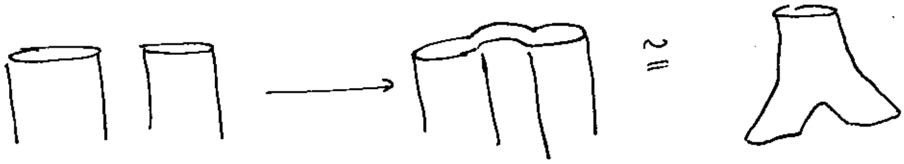
$$M^a \cong M^b \bigcup_f \mathbb{D}^2 \qquad f \text{ attaching map}$$

f can be of two types

1)

orientable

2)



Non-orien~~table~~ed

1)

2)