Euler Characteristic of Surfaces

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Our goal now is to determine the Euler characteristic for more complicated surfaces than a sphere. In the process we'll be able to provide a complete classification of surfaces.

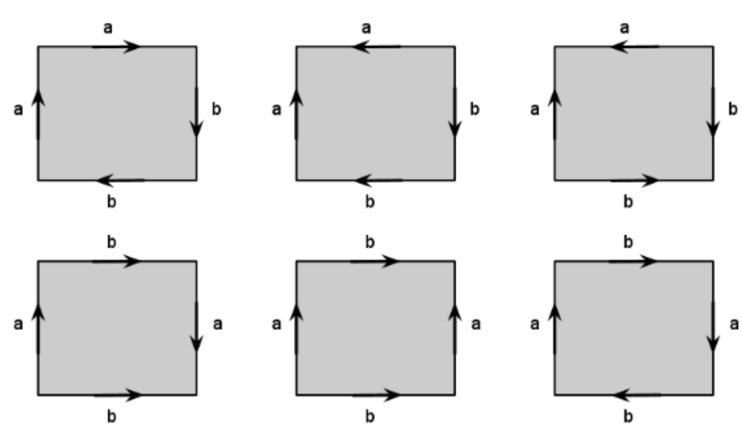
As with a sphere a **polygonal graph** on a surface is a *connected planar* graph $G = (V, E)$ such that V and E are now on the surface and all the faces are polygons.

> **Exercise 1.** Being polygonal is a non-trivial condition for a graph on a surface which is not a sphere. Find a planar connected graph with at least 1 face on a torus with 3 vertices which is not polygonal. Does such a graph exist on a sphere?

This problem does not arise for a sphere because of the Jordan Curve Theorem. Read the statement of the above theorem and try to see why it does not hold true for arbitrary surfaces. This page tries to justify why this simple statement deserves the status of a theorem.

Gluing diagrams naturally give us a polygonal graph on a surface as all the faces are naturally polygons. One can compute the Euler characteristic of these gluing diagrams, care needs to be taken while computing the Euler characteristic as some of the edges and vertices are identified.

> **Exercise 2.** Compute the Euler characteristic for the basic gluing diagrams



Exercise 3. Using the gluing diagrams compute the Euler characteristic of $T\#T$, $T\#T\#T$. Can you generalize this to $T^{\#n}$ (connected sum of T with itself taken n times).

Exercise 4. Compute the Euler characteristic of $\mathbb{RP}^2\#\mathbb{RP}^2$, $\mathbb{RP}^2\#\mathbb{RP}^2\#\mathbb{RP}^2$. Generalize to $\left(\mathbb{RP}^2\right)^{\#n}$.

As a convention let us set $S^2 = T^{\#0}$ to avoid having to deal with S^2 separately.

The above exercises almost prove the following theorem,

Theorem 1.

$$\chi\left(T^{\#n}\right) = 2 - 2n$$
$$\chi\left(\left(\mathbb{RP}^2\right)^{\#n}\right) = 1 - n$$

However we still need to show that the Euler characteristic does not depend on the choice of a polygonal graph. The proof is given in Section 2. We'll assume the fact for now.

1. Excision

The Euler characteristic satisfies a very important property called **excision** which goes as follows,

Theorem 2 (Excision)**.** *For two surfaces M_1, M_2 we have*

$$\chi(M_1\#M_2) = \chi(M_1) + \chi(M_2) - 2$$

Invariants which satisfy excision are the most easily computable ones (relatively speaking). In fact, an invariant is a *homological invariant* only if it satisfies some form of excision. As it turns out Euler characteristic is secretly an invariant coming out of a homology theory called *singular homology* or *simplicial homology.*

Exercise 5. Theorem 1 implies excision directly. Verify that excision holds for the following three cases separately,

(i) $M_1 = T^{\#m}$ and $M_2 = T^{\#n}$

(ii) $M_1 = \left(\mathbb{RP}^2\right)^{\#m}$ and $M_2 = \left(\mathbb{RP}^2\right)^{\#n}$

(iii) $M_1 = T^{\#m}$ and $M_2 = \left(\mathbb{RP}^2\right)^{\#n}$

Exercise 6. It is also possible to first prove excision and then derive Theorem 1 from it. Given a triangulation on M_1 and M_2 construct a triangulation on $M_1\#M_2$. Use this to verify excision.

2. Proof of Euler's theorem

We still need to prove the following statement in order to justify that the Euler characteristic of a surface is well defined.

Theorem 3. *For any two polygonal graphs G, G' on a surface S we have $\chi(G) = \chi(G')$. Hence we can define $\chi(S) := \chi(G)$.*

The idea behind the proof of this theorem is basically the same as that for a sphere.

Exercise 7. To prove that the Euler characteristic does not depend on the graph we cannot delete vertices and edges successively as we did for a sphere as one cannot guarantee that a polygonal graph would remain polygonal. So instead we need to *add* edges and vertices. Let S be any surface.

(i) Give an example of a polygonal graph which does not remain polygonal upon deleting an edge.

(ii) Show that the Euler characteristic of a polygonal graph does not change by performing one of the following operations

a) adding a new edge joining two existing vertices

b) subdividing an edge by declaring it's midpoint as a new vertex

(iii) Argue that given two different polygonal graphs G and G' of S it is possible to find a *common refinement* G'' of the two using a sequence of the above two steps. (Try to superimpose G on G' to get G''.)

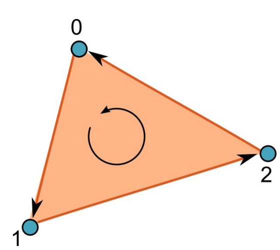
(iv) Conclude that $\chi(G) = \chi(G'') = \chi(G')$.

3. Orientation of surfaces

The final ingredient in the classification of surfaces is the orientation.

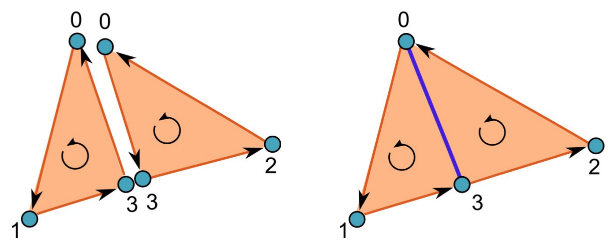
Roughly speaking a surface is called orientable if has two sides. A sphere or a torus are easily seen to be orientable. In fact any surface that can be *embedded* in $\mathbb{R}^3$ is orientable. A Mobius strip on the other hand (which is not a surface as it has a boundary) is not orientable. With a little more mental effort one might be able to see that a Klein bottle or a Projective space are not orientable. We want an equivalent definition which can be used on gluing diagrams.

We begin by defining orientation of a polygon. An **orientation of a polygon** is simply a cyclic ordering of it's vertices. For example this is one of the two possible ways to orient a triangle with vertices $(0, 1, 2)$.



Exercise 8. This is related to the fact that the polygon has two sides via the right hand rule in physics. Do you see the connection?

An orientation of a polygonal graph on a surface is a compatible choice of orientation for each face, where the orientations of two adjacent polygons need to be compatible in the following way,



This forces an edge to be directed in two different directions in adjacent triangles.

A surface is **orientable** if it is possible to find compatible orientations for some triangulation of it. It is not too difficult to show that this property is independent of the choice of the triangulation.

Exercise 9. Add the diagonals to the standard gluing diagrams to obtain triangulations and try to find orientations for them. Conclude that S^2, T are orientable and $\mathbb{RP}^2, K$ are not.

Exercise 10. Use gluing diagrams to show that $T\#T$ is orientable. Generalize this to argue that $T^{\#n}$ are orientable.

It is slightly harder to prove that the connected sum of two non-orientable surfaces is non-orientable and hence $(\mathbb{RP}^2)^{\#n}$ are non-orientable. Assuming this and using Theorem 1 we get the final classification theorem,

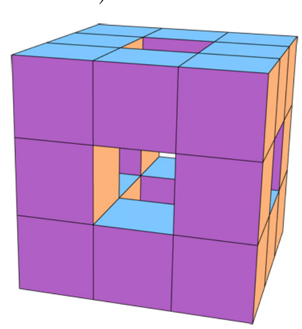
Theorem 4 (Classification of Surfaces)**.** *For a surface M the following two properties completely determine it's topological type*

(i) Orientability

(ii) Euler characteristic

If M is orientable then $M = T^{\#n}$ where $n = \dfrac{2-\chi(M)}{2}$ and if M is non-orientable then $M = (\mathbb{RP}^2)^{\#n}$ where $n = 1 - \chi(M)$.

Exercise 11. Without computing guess the topological type of the following surface (rubik's cube with holes).



Verify your answer by computing the Euler characteristic.