The Adem relations force the Steenrod algebra $\mathcal{S}q$ to have a basis of admissible elements:

deg 0: Sq^0 0

deg 1: Sq^1 1

deg 2: Sq^2 2

deg 3: Sq^3, Sq^2Sq^1 $3, 1$

deg 4: Sq^4, Sq^3Sq^1 $4, 2$

deg 5: Sq^5, Sq^4Sq^1 $5, 3$

deg 6: Sq^6, Sq^5Sq^1, Sq^4Sq^2 $6, 4, 2$

deg 7: Sq^7, Sq^6Sq^1, Sq^5Sq^2, $Sq^4Sq^2Sq^1$ $7, 5, 3, 1$

8: Sq^8, Sq^7Sq^1, Sq^6Sq^2, $Sq^5Sq^2Sq^1$

9: Sq^9, Sq^8Sq^1, Sq^7Sq^2, Sq^6Sq^3, $Sq^6Sq^2Sq^1$

10: Sq^{10}, Sq^9Sq^1, Sq^8Sq^2, Sq^7Sq^3, $Sq^7Sq^2Sq^1$, $Sq^6Sq^3Sq^1$

11: Sq^{11}, $Sq^{10}Sq^1$, Sq^9Sq^2, Sq^8Sq^3, $Sq^8Sq^2Sq^1$, $Sq^7Sq^3Sq^1$

12: Sq^{12}, $Sq^{11}Sq^1$, $Sq^{10}Sq^2$, Sq^9Sq^3, $Sq^9Sq^2Sq^1$, Sq^8Sq^4, $Sq^8Sq^3Sq^1$

Q. How many deg d, excess n elements are there? $\alpha(d,n)$

Every time an excess is "added" it has to precipitate all the way to the top. It is therefore not hard to see that $\alpha(d,n)$ = no. of ways of writing d as sum of n elements of the for 2^k-1.

eg: $7 = 1+1+1+1+1+1+1 \longrightarrow Sq^7$

$= 3+1+1+1+1 \longrightarrow Sq^6Sq^1$

$= 3+3+1 \longrightarrow Sq^5Sq^2$

The ~~exponential~~ generating function for $\alpha(d)$ would be

$$(1+x+x^2+\cdots)(1+x^3+x^6+\cdots)(1+x^7+x^{14}+\cdots) = (1-x)^{-1}(1-x^3)^{-1}(1-x^7)^{-1}\cdots$$

Not sure if this has a closed form.

Because $\mathcal{S}q$ is a Hopf-algebra over $\mathbb{Z}/2$, its dual is also a Hopf algebra $\mathcal{S}q^*$ with a vector space basis given by dual of the admissible elements.

Prop: $\mathcal{S}q^*$ is a polynomial algebra generated by $\{\xi_i\}_{i=0}^{\infty}$ where $\xi_i = [Sq^{2^i}Sq^{2^{i-1}}\cdots Sq^1Sq^0]^*$

What is the product structure? dualize the cop-product.

$$\langle ab, Sq^k\rangle = \sum_i \langle a, Sq^i\rangle \cdot \langle b, Sq^{k-i}\rangle$$

Because the coproduct on $\mathcal{S}q$ is cocommutative, $\mathcal{S}q^*$ is commutative.

The proof is really non-intuitive. For $I = (i_1, i_2, \ldots, i_k)$ admissible we construct a dual element in terms of ξ_i such that it takes Sq^I to 1.

Look at $\xi = \xi_1^{i_1-2i_2} \cdot \xi_2^{i_2-2i_3} \cdot \xi_3^{i_3-2i_4} \cdots \xi_k^{i_k} = \xi^{r(I)}$

$\langle \xi_1^{i_1-2i_2} \xi_2^{i_2-3i_3} \cdots \xi_{k-1}^{i_{k-1}-2i_k} \xi_k^{i_k}, Sq^{i_1}\cdots Sq^{i_k}\rangle$

$= \sum \langle \xi_1^{i_1-2i_2} \xi_2^{i_2-2i_3} \cdots \xi_{k-1}^{i_{k-1}-2i_k} \xi_k^{i_{k-1}}, Sq^{i_1'}\cdots Sq^{i_k''}\rangle$

$\cdot \langle \xi_k, Sq^{i_1''}\cdots Sq^{i_k''}\rangle$

$i_\ell' + i_\ell'' = i_\ell$ but the two sequences are no longer necessarily admissible.

$\deg \xi = (2^1-1)\cdot(i_1-2i_2) + (2^2-1)(i_2-2i_3)$
$+ \cdots + (2^{k-1}-1)(i_{k-1}-2i_k) + (2^k-1)\, i_k$
$= 2i_1 - \cancel{2^2 i_2} + \cancel{2^2 i_2} - \cancel{2^3 i_3} + \cdots + \cancel{2^{k-1}} i_{k-1} - \cancel{2^k i_k} + \cancel{2^k i_k}$
$- i_1 + 2i_2 - i_2 + 2i_3 + \cdots - i_{k-1} + 2i_k - i_k$
$= i_1 + i_2 + \cdots + i_{k-1} + i_k$

And now one uses induction to say that $\langle \xi_k, Sq^{i_1}\cdots Sq^{i_k}\rangle = 1$ iff $i_1 = 2^{k-1}, i_2 = 2^{k-2}, \cdots, i_k = 1$
and that the corresponding term is non-zero iff $i_1 = j_1, i_2 = j_2, \cdots$

$\boxed{\text{Next we say that } \langle \xi^{r(I)}, Sq^J\rangle = 0 \text{ for all } J < I \text{ again by induction.}}$

And so $Sq^* \cong \mathbb{Z}/2[\xi_0, \xi_1, \xi_2, \cdots]$

The coproduct is given by $\langle \Delta \xi_i, Sq^I \otimes Sq^J\rangle = \langle \xi_i, Sq^I Sq^J\rangle$

Claim: $\Delta \xi_k = \sum_i (\xi_{k-i})^{2^i} \otimes \xi_i$

Again the proof is really convoluted.

But one thing which is easy to notice is $\left[(\xi_{k-i})^{2^i}\right]^* = \left[\xi_1^0 \xi_2^0 \cdots \xi_{k-i}^{2^i}\right]^*$ $\quad 0 = i_1 - 2i_2 = i_2 - 2i_3 = \cdots = i_{k-i-1} - 2i_{k-i}$

$2^i = i_{k-i}$

$= Sq^{2^k} \cdot Sq^{2^{k-1}} \cdots Sq^{2^i}$

$(\xi_i)^* = (\xi_1^0 \cdots \xi_i^1)^* \quad 0 = i_1 - 2i_2 = \cdots = i_{i-1} - 2i_i$

$= Sq^{2^{i-1}} \cdots Sq^2 Sq^1 \quad 1 = i_i$

☆ ⇝ There is a missing argument here. See later.

So that $(\xi_{k-i})^{2i\,*} \cdot (\xi_i)^* = Sq^{2^k} Sq^{2^{k-1}} \cdots Sq^2 Sq^1 = (\xi_k)^*$

hence $\Delta \xi_k = \sum_i (\xi_{k-i})^{2^i} \otimes \xi_i$ + other terms

The hard part is showing that there are no other terms.

We need to look at the action of Sq on $H^*(X; \mathbb{Z}/2)$.

Now here's the trouble — H^* is itself an algebra & Sq has a coproduct. I'm going to try real hard to avoid drawing big diagrams.

What does it mean for $H^*(X)$ to be an algebra over Sq? That to graded.

- $Sq^i(\alpha) \in H^{|\alpha|+i}(X)$
- $Sq^i(\alpha\beta) = \sum Sq^k(\alpha)\, Sq^{i-k}(\beta) = \langle \Delta Sq^i, \alpha \otimes \beta\rangle$

This also makes $H^*(X;\mathbb{Z}/2)$ into a comodule over $\mathcal{S}q^*$ i.e. we want to define a map

$$\lambda^*: H^*(X;\mathbb{Z}/2) \longrightarrow H^*(X;\mathbb{Z}/2)\hat{\otimes}\,\mathcal{S}q^*$$
$$\alpha \longmapsto \sum Sq^I(\alpha)\otimes(Sq^I)^* \qquad I \text{ admissible}$$

eg: $X = K(\mathbb{Z}/2,1)$. $Sq^I(x) = \begin{cases} x^{2^i} & \text{if } I \text{ has excess } 1 \quad \text{i.e. } I = (2^{i-1}, 2^{i-2}, \dots, 2, 1) \\ 0 & \text{else} \end{cases}$

Because $(Sq^{2^{i-1}}\cdots Sq^2 Sq^1)^* = \xi_i$ we get

$$\lambda^*(x) = \sum_i x^{2^i}\otimes \xi_i$$

★ Missing argument:
We really only know that
$$(Sq^{i_1 i_2 \dots i_n})^* = \xi_1^{i_1-2i_2}\,\xi_2^{i_2-2i_3}\cdots\xi_{n-1}^{i_{n-1}-2i_n}\,\xi_n^{i_n} + \xi^{r(J)} + \dots$$
and all we can say is that
$$(\xi_{k-i}^{2^i})^*(\xi_i^*) = \xi_k^* + \text{other terms}$$

Our aim is to understand how λ^*, Δ are connected.

We have $Sq^I(Sq^J(\alpha)) = (Sq^I Sq^J)(\alpha)$

Dualizing

$$\begin{tikzcd}
\alpha \arrow[r, mapsto, "\lambda^*"] & \sum_I Sq^I(\alpha)\otimes(Sq^I)^* \arrow[r, mapsto] \arrow[d] & \sum_I \lambda^*(Sq^I(\alpha))\otimes(Sq^I)^* \arrow[dl, equal] \\
& \sum_I Sq^I(\alpha)\otimes\Delta(Sq^I)^* &
\end{tikzcd}$$

ie. $\forall\alpha$. $$\sum_I Sq^I(\alpha)\otimes\Delta(Sq^I)^* = \sum_I \lambda^*(Sq^I(\alpha))\otimes(Sq^I)^*$$

Now pick $\alpha = x$ as above

$$LHS = \sum_I Sq^I(\alpha)\otimes\Delta(Sq^{I^*}) = \sum_i x^{2^i}\otimes\Delta\xi_i$$

$$\lambda^*(\alpha)\,\lambda^*(\beta) = \sum_{J'} Sq^{J'}(\alpha)\otimes(Sq^{J'})^* \sum_{J''} Sq^{J''}(\beta)\otimes(Sq^{J''})^* = \sum_{J',J''} Sq^{J'}(\alpha)\cdot Sq^{J''}(\beta)\otimes(Sq^{J'})^*(Sq^{J''})^*$$

$$\lambda^*(\alpha\beta) = \sum_J Sq^J(\alpha\beta)\otimes(Sq^J)^* = \sum_{\substack{J'+J''\\ \text{admissible}}} Sq^{J'}(\alpha)\,Sq^{J''}(\beta)\,[Sq^{J'+J''}]^*$$ Why are these two equal?

$$\begin{tikzcd}
H^*\otimes H^* \arrow[r, "\cdot"] \arrow[d] & H^* \arrow[r, "\lambda^*"] & H^*\hat{\otimes}\,\mathcal{S}q^* \\
H^*\hat{\otimes}\,\mathcal{S}q^*\otimes H^*\hat{\otimes}\,\mathcal{S}q^* \arrow[urr] & &
\end{tikzcd}
\quad \overset{\text{Dualize}}{\rightsquigarrow} \quad
\begin{tikzcd}
H^*\otimes H^* & H^* \arrow[l, "\Delta"'] & H^*\otimes\mathcal{S}q \arrow[l, "\lambda"'] \arrow[dll, "\Delta"] \\
H^*\otimes\mathcal{S}q\otimes H^*\otimes\mathcal{S}q \arrow[u] & &
\end{tikzcd}$$

$$\sum_i \sum_{I'+I''} Sq^{I'}(\alpha_i')\,Sq^{I''}(\alpha_i'') = \sum_i Sq^I(\alpha_i'\alpha_i'') =$$

$$\begin{tikzcd}
& Sq^I(\alpha) \arrow[dl, "\Delta"', "?"] & \alpha\otimes Sq^I \arrow[l, mapsto] \arrow[d] \\
\sum_{i,\,I'+I''=I} Sq^{I'}(\alpha_i')\otimes Sq^{I''}(\alpha_i'') & & (\sum \alpha_i'\otimes\alpha_i'')(\sum Sq^{I'}\otimes Sq^{I''}) \arrow[ll, mapsto]
\end{tikzcd}
\qquad \alpha = \sum_i \alpha_i'\alpha_i''$$

$$\Rightarrow \lambda^*(x^{2^i}) = (\lambda^*(x))^{2^i} = \left(\sum_j x^{2^j} \otimes \xi_j\right)^{2^i} = \sum_j x^{2^{i+j}} \otimes \xi_j^{2^i}$$

$$\text{RHS} = \sum_i \lambda^*(x^{2^i}) \otimes \xi_i$$

$$= \sum_{i,j} x^{2^{i+j}} \otimes \xi_j^{2^i} \otimes \xi_i$$

Comparing LHS, RHS we get $\Delta \xi_i = \sum_j \xi_{k-i}^{2^i} \otimes \xi_i$

Lot of tricks went into proving this. I ought to summarize what we have:

We have 3 interconnected objects: $(\mathcal{S}q, \cdot, \Delta)$, $(\mathcal{S}q^*, \cdot, \Delta)$, $(H^*(X), \cdot)$

We know the Hopf structure on $\mathcal{S}q$ (graded)

- $\mathcal{S}q$ has vector space basis (Sq^I) I admissible
- $\Delta Sq^I = \sum_{I'+I''=I} Sq^{I'} \otimes Sq^{I''}$, $\Delta(Sq^I Sq^J) = (\Delta Sq^I)\cdot(\Delta Sq^J)$
- $Sq^a Sq^b = \sum_c \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c$ if $a < 2b$

We can define a dual basis $\{(Sq^I)^*\}$, I admissible for $\mathcal{S}q^*$.

- $(Sq^{I'})^* (Sq^{I''})^* = \sum_I (Sq^I)^*$ where I runs over indices $\Delta Sq^I = Sq^{I'} \otimes Sq^{I''}$ + other terms
- $\Delta(Sq^{I*}) = \sum_{\substack{I', I'' \\ \text{admissible}}} Sq^{I'*} \otimes Sq^{I''*}$ where $Sq^{I'} Sq^{I''} = Sq^I$ + other terms
- $\Delta[(Sq^{I*})(Sq^{J*})] = (\Delta Sq^{I*})(\Delta Sq^{J*})$
  this need not equal $\Delta(Sq^I Sq^J)^*$

$\mathcal{S}q$ acts on H^* and $\mathcal{S}q^*$ "coacts" on H^*

- $\lambda: \mathcal{S}q \otimes H^*(X) \longrightarrow H^*(X)$ $\xrightarrow{\text{dualizing}}$ $\lambda^*: H^*(X) \longrightarrow H^*(X) \hat{\otimes} \mathcal{S}q^*$

  $|Sq^I \cdot \alpha| = |\alpha| + |I|$ $\qquad$ $\alpha \longmapsto \sum_{I \text{ admissible}} Sq^I(\alpha) \otimes (Sq^I)^*$

- $Sq^I(\alpha\beta) = \sum_{I'+I''=I} Sq^{I'}(\alpha) Sq^{I''}(\beta)$ $\qquad$ $\lambda^*(\alpha\beta) = \sum_{\substack{I'+I'' \\ \text{admissible}}} Sq^{I'}(\alpha) Sq^{I''}(\beta) \otimes (Sq^{I'+I''})^*$ $= \lambda^*(\alpha)\lambda^*(\beta)$

- $Sq^I(Sq^J(\alpha)) = (Sq^I Sq^J)(\alpha)$ $\qquad$ $(\lambda^* \otimes 1)\lambda^* = (1 \otimes \Delta)\lambda^*$

Miraculously there exists a "simpler" basis for Sq^* : $\{\xi_i\}$ $\xi_i = \left(Sq^{2^i} Sq^{2^{i-1}} \cdots Sq^2 Sq^1\right)^*$

Induction tells us : $\left(Sq^I\right)^* = \xi^{\gamma(I)} + \sum_J c(J)\xi^{\gamma(J)}$ where $\xi^{\gamma(i_1,\dots,i_n)} = \xi_1^{i_1-2i_2} \cdot \xi_2^{i_2-2i_3} \cdot \dots \cdot \xi_{n-1}^{i_{n-1}-2i_n} \xi_n^{i_n}$

and $J < I$ from <u>right</u> lexicographically

and of course $|\xi^{\gamma(J)}| = |\xi^{\gamma(I)}| = |I|$, $c(J) = 0/1$

- $Sq^* = \mathbb{Z}/2\,[\xi_i]$ $\qquad |\xi_i| = 2^{i-1}-1$
- $\Delta \xi_n = \sum_i \xi_{n-i}^{2^i} \otimes \xi_i$

We can derive some immediate computations based on these

1) $(Sq^i)^* = \xi^{\gamma(i,0,\dots 0)} + \sum_{J<(i,0\dots)} c(J)\,\xi^{\gamma(J)}$

But $J < (i,0,\dots) \Rightarrow |J| < i$

$= \xi_1^i$

$= (Sq^1)^{*i}$

2) $\lambda^*(x^{2^i}) = \left(\lambda^*(x)\right)^{2^i} = \left(\sum_j x^{2^j} \otimes \xi_j\right)^{2^i} = \sum_j x^{2^{i+j}} \otimes \xi_j^{2^i}$

$\shortparallel$

$\sum_I Sq^I(x^{2^i}) \otimes (Sq^I)^*$

$= \left[x^{2^i} \otimes 1 + x^{2^{i+1}} \otimes (Sq^{2^i})^* + \dots + x^{2^{i+j}} \otimes \left(Sq^{2^{i+j-1}} Sq^{2^{i+j-2}} \cdots Sq^{2^i}\right)^* + \dots \right]$

Comparing we get $\left(Sq^{2^{i+j-1}} Sq^{2^{i+j-2}} \cdots Sq^{2^i}\right)^* = \xi_j^{2^i}$ This finally proves ☆

There is a flaw in the way I commuted the diagrams involving λ^*. I do not see how to rectify it by hand without just drawing loads of other diagrams.