Def^n: $f: X \to Y$ be a holo map ~~bet~~, $p \in X$
Ramification index: $e_p(f) = k$ if f is ~~the~~ of the form $z \mapsto z^k$ locally at p.

Prove well-defined.

Remark: Ramification locus: $B = \{p \in X \mid e_p > 1\}$ is always discrete.

Image $D = f(B)$ is called the discriminant locus of f.

Prop: $f: X \longrightarrow Y$ non-constant holomorphic map between compact Riemann Surfaces. B, D as above. Then,

$f|_{X \setminus f^{-1}(D)} : X \setminus f^{-1}(D) \longrightarrow Y \setminus D$ is a covering space.

Proof: $y_0 \in Y \setminus D$, $x \in f^{-1}(y_0) \Rightarrow e_x = 1$.
f local biholomorphism. Result follows.

Remark:
- f above is called ramified / branched cover.
- For X, Y non-compact f needs to be proper for this to work.

Def^n
- $f: X \longrightarrow Y$ non constant
  ~~order~~ degree of $f := \# \{x \mid f(x) = y\}$ for some $y \in Y \setminus D$.
- X has "enough functions" if, for $x_1 \neq x_2 \in X$, $\exists\, \varphi \in \underline{m}(X)$ s.t. $\varphi(x_1) \neq \varphi(x_2)$

Lemma: X have enough functions. $x_1, \dots x_n \in X$ distinct, then $\rightarrow$ $\exists\, \varphi \in \underline{m}(X)$ s.t. $\varphi(x_1) \dots \varphi(x_n)$ distinct.

Proof: $V_{ij} = \{\varphi \in \underline{m}(X) \mid \varphi(x_i) = \varphi(x_j)\}$
Then $V_{ij} \subsetneq \underline{m}(X)$. Since $\mathbb{C}$ is infinite $\underline{m}(X) \neq \bigcup V_{ij}$
choose $\varphi \in \underline{m}(X) \setminus \bigcup V_{ij}$

Prop: $f: X \to Y$ non-constant, n-degree of f.
Let $f^*: \underline{m}(Y) \longrightarrow \underline{m}(X)$ be canonical field extension induced by f.
Then $\forall g \in \underline{m}(X)$ satisfies some algebraic relation:

$$\beta_n g^n + \beta_{n-1} g^{n-1} + \dots + \beta_1 g' + \beta_0 = 0$$

where $\beta_0 \cdots \beta_n \in \underline{m}(\cancel{X}\, Y)$.

<u>Proof</u>: $g \in \underline{m}(X)$. $y \in Y - (D \cup f(\text{poles of } g))$

$\alpha_i(y) =$ i^{th} symmetric polynomial on $\{g(x_1) \cdots g(x_n)\}$

where $\{x_1 \cdots x_n\} = f^{-1}(y)$.

On $X \setminus f^{-1}(\cancel{Y} (D \cup f(\text{poles of } g))) \sim S$

$$\prod_{i=1}^{n} (g(x) - g(x_i)) = 0 \qquad \{x_1 \cdots x_n\} = f^{-1}(f(x))$$

LHS:

$$\sum_{i=0}^{n} \alpha_i(y) g^i(x) \qquad y = f(x).$$

Thus

$$\sum_{i=0}^{n} \alpha_i(f(x)) g^i(x) = 0 \qquad \forall x \in S. \quad \text{— } (*)$$

If we could show $\alpha_i(y)$ meromorphic on Y then by continuity $(*)$ will hold for all of X.

- On S,
  $f|_S : S \longrightarrow f(S)$ is covering.
  Suppose pancakes are U_i around x_i, V around y.
  $f : U_i \longrightarrow V$ biholomorphic, with inverse σ_i
  then
  $\alpha_i =$ i^{th} symmetric poly of $\{g \circ \sigma_1, \dots, g \circ \sigma_n\}$.

- on X,
  enough to show locally
  $(y-y_0)^M \alpha_i(y) \longrightarrow 0$ as $y \longrightarrow y_0$ for $y_0 \in D \cup f(\text{poles of } g)$
  Let $x_0 \in f^{-1}(y_0)$.
  - $\left| \dfrac{f(x) - y_0}{x - x_0} \right|$ bounded close to x_0
  - $\lim_{x \to x_0} |x - x_0|^N g(x) = 0$ for some N.

So

$$|f(x) - y_0|^N g(x) \longrightarrow 0 \quad \text{as} \quad x \longrightarrow x_0$$

$$\Rightarrow \frac{|y - y_0|^{iN} \, |\alpha_i(y)|}{\binom{n}{i}} \longrightarrow 0 \quad \text{as} \quad y \longrightarrow y_0$$

Th^m: $f: X \longrightarrow Y$ non-constant. degree of $f = n$

Suppose X ~~Y~~ have enough functions. Then,

$$[\mathcal{M}(X) : \mathcal{M}(Y)] = n.$$

Proof: Claim: $[\mathcal{M}(X) : \mathcal{M}(Y)] \leq n$

Proof: Suppose not

Choose a subfield of $\mathcal{M}(X)$, F s.t. $[F : \mathcal{M}(Y)] > n$, finite.

This will have a primitive element g

And minimum poly of g will have degree at ~~most~~ least n

contradicting previous proposition.

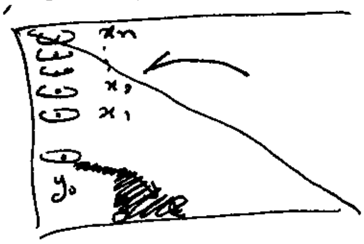
~~Claim~~ Assume $[\mathcal{M}(X) : \mathcal{M}(Y)] = m < n$

g primitive element for this extension i.e. $\mathcal{M}[X] = \mathcal{M}[Y](g)$.

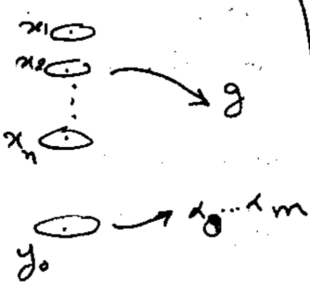
min poly of g of degree m.

Look at a "good" point y_0 in Y. This will have $x_1 \dots x_n$ as inverse images, distinct.

Let minimum poly of g be G, $\deg G < n$, $G \in \mathcal{M}(Y)[t]$



Suppose $G(t) = \sum_{i<n} \alpha_i(y)\, t^i$



Now at y_0 each of $g(x_1) \dots g(x_n)$ ~~has~~ are roots of the poly $G(t)$ which has $\deg < n$.

So atleast for some two x_i, x_j ; $g(x_i) = g(x_j)$.

But $\mathcal{M}(X) = \mathcal{M}(Y)(g)$.

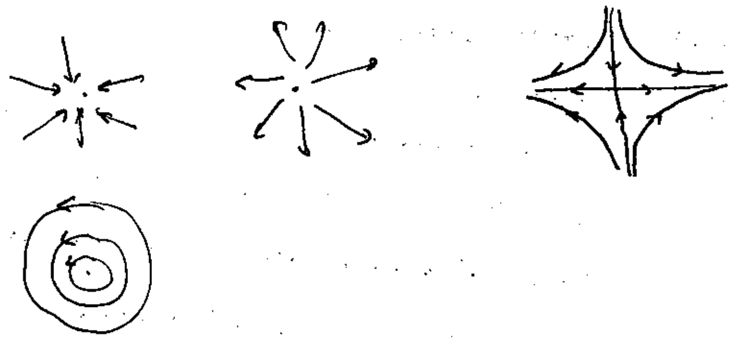
and g_0 does not seperate x_i, x_j. This will give us that $\mathcal{M}[X]$ does not ~~de~~separate $x_i\ x_j$ which is false by our assumption.

Detour:

<u>Fluid Flows:</u>

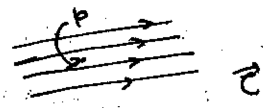
Look at a vector field $\vec{X}$ on a surface, that has ~~an~~ isolated critical pts.

eg:



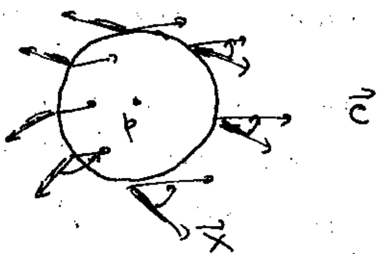
Def^n: Index of $\vec{X}$:

Let p be a point. Near p choose a vector field $\vec{C}$ (called adiabatic vector field) ~~so~~ having no critical point something like $\frac{\partial}{\partial x_i}$



Choose a circle S^1 around p such that $D^1 \setminus \{p\}$ has no critical point.

Look at angles between $\vec{X}, \vec{C}$ at each point on S^1



Then

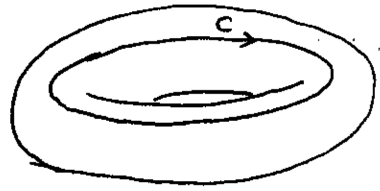
$$\text{ind}_p \vec{X} := \text{change in } \underset{\substack{\uparrow \\ \text{angle bet}^n\ \vec{X}, \vec{C}.}}{\text{angle}} \text{ as we move along } S^1 \Big/ 2\pi$$

How to prove that the index is well-defined?
We have made two choices $S^1, \vec{C}$.

Assume a Riemannian metric $\langle , \rangle$. Outside critical pts normalize $\vec{X}$, i.e. assume $\langle \vec{X}, \vec{X} \rangle = 1$. Then angle at a point is just $\langle \vec{X}, \vec{C} \rangle$. (Assuming $\langle \vec{C}, \vec{C} \rangle = 1$). So we have a function $S^1 \longrightarrow S^1$, $p \underset{\cos^{-1}}{\longmapsto} \langle \vec{X}, \vec{C} \rangle_p$

$\text{Ind}_p \vec{X}$ = degree of this?

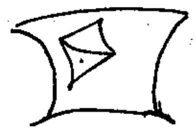
eg: $T = S^1 \times S^1$



$\vec{c}$ - as shown

Let $\vec{X}$ be arbitrary with isolated critical point.

Then, triangulate T so that each isolated critical point is inside a triangle and any triangle has at most 1 critical pt.



By virtue of having a global vector field, if we add all the indices at all critical points we get 0.

$$\sum_{p \text{ critical}} \text{ind}_p X = 0.$$

as along edges the change in angles in two adjacent triangles are in the opposite direction.

- Using similar argument it is possible to ~~defin~~ conclude that for 2 vector field $\vec{X}, \vec{Y}$

$$\sum_{\substack{p \text{ critical} \\ \text{of } \vec{X}}} \text{ind}_p \vec{X} = \sum_{\substack{p \text{ critical} \\ \text{of } \vec{Y}}} \text{ind}_p \vec{Y}$$

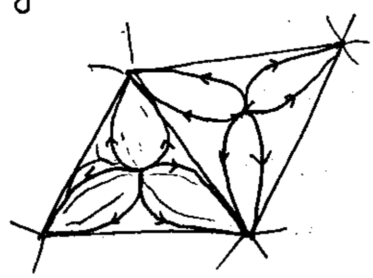
Poincare Hopf Th^m

Call this χ.

Now on genus g surface it is easy to construct a vector field with 1 sink, 1 source, 2g saddle points.

For this we get $\chi = 2 - 2g$.

For a triangulation we can construct a vector field as follows:



a source for each face(f)
a sink for each vertex(v)
a saddle for each edge (e)
each with degree 1.

From this we get $\chi = v - e + f$

Tutorial

— Pierre

Link between Riemannian Geometry and Riemannian Surfaces

Def: M^n - C^k manifold ...

Dual vector field, form: Given an inner product $\langle , \rangle$

- Take vector field X: — ω ?

$v \in T_x M \quad \omega_x(v) = \cancel{g_x(v,} \langle v, X(x) \rangle_x$

- Take 1-form ω: — X ?

unique X satisfying

$\omega_x(\cdot) = \langle \cdot, X(x) \rangle_x$

Th^m: for M-oriented surface, following are ~~equivalent~~ = same

- conformal classes of Riemannian metric on M + + orientation
- complex structures on M.

Def: $M - C^\infty$ $g, \tilde{g}$ metrics

Conformally equivalent $\equiv$ $g = f \cdot \tilde{g}$ for $f: M \to \mathbb{R}_{>0}$ C^∞

~~Exam~~

- $M - C^\omega$ manifold g ~~vari~~ C^ω metric

Then $\exists$ a complex structure which is compatible with this metric.

- $p \in M$ $(U, (x,y))$ - chart, $g = a\,dx^2 + 2b\,dx\,dy + c\,dy^2$

$a>0, c>0, ac - b^2 > 0$

Need another chart $w(z, \bar{z})$ s.t. $= s(dz + \mu d\bar{z})(d\bar{z} + \mu dz)$

$g = s\, dw\, d\bar{w}$

$z = x + iy$

~~So~~ we need to solve for

$dw = dz + \mu d\bar{z}$

$\rightarrow$ Beltrami Eq^n, isothermal co-ordinates

Aim: To ~~comp~~ construct a compatible Riemann Surface given a orientation + Riemmanian metric on analytic surface M.

- (U, x, y) be a chart on M ~~such~~

  g is given locally by

$$g = a\,dx^2 + 2b\,dx\,dy + c\,dy^2$$

Set $z = x+iy$ $\quad dz = dx + i\,dy \quad d\bar{z} = dx - i\,dy$

$$g = A\,dz^2 + 2B\,dz\,d\bar{z} + \bar{A}\,d\bar{z}^2$$

$$(= (A+2B+\bar{A})\,dx^2 + 2i[A - \bar{A}]\,dx\,dy + (2B - A - \bar{A})\,dy^2)$$

$$= s(dz + \mu\,d\bar{z})(d\bar{z} + \bar{\mu}\,dz) \quad s > 0$$

$$s = \frac{2B}{1+|\mu|^2} \qquad \mu = \frac{\bar{A}}{2B}(1+|\mu|^2)$$

Note: μ is unique only upto modulus: $\mu' = e^{i\theta}\mu$ will also do

we need to make this look like

$$s\,dw\,d\bar{w}$$

So we need functions w on $(z, \bar{z})$ s.t.

$$dw = dz + \mu\,d\bar{z}$$
$$d\bar{w} = d\bar{z} + \bar{\mu}\,dz$$

or the other way & w orientation preserving

$$dw\left(\frac{\partial}{\partial z}\right) = 1 = \frac{\partial w}{\partial z} \qquad dw\left(\frac{\partial}{\partial \bar{z}}\right) = \mu = \frac{\partial w}{\partial \bar{z}}$$

$$\Rightarrow \boxed{\frac{\partial w}{\partial \bar{z}} = \mu \frac{\partial w}{\partial z}} \quad \leftarrow \text{Beltrami Eq}^n \quad \left(\frac{\partial \bar{w}}{\partial z} = \bar{\mu}\frac{\partial \bar{w}}{\partial \bar{z}}\right)$$

This eq^n is enough as any ~~scalar multiple of w~~ w can be rescaled to get $\frac{\partial w'}{\partial z} = 1$ or simply absorb the extra factor in s.

- Assuming solutions of Beltrami Eq^n, we need to show that the w that we have obtained above is,

  1) complex structure
  2) compatible with g.

1) w_1, w_2 satisfy suppose

$$\frac{\partial w_1}{\partial \bar{z}} = \mu \frac{\partial w_1}{\partial z}, \qquad \frac{\partial w_2}{\partial \bar{z}} = \mu' \frac{\partial w_2}{\partial z}$$

$$\mu' = e^{i\theta}\mu$$

$$S_1\, d\omega_1 d\bar{\omega}_1 = S_2\, d\omega_2 d\bar{\omega}_2$$

$$d\omega_1 = \frac{\partial\omega_1}{\partial\omega_2} d\omega_2 + \frac{\partial\omega_1}{\partial\bar{\omega}_2} \cdot d\bar{\omega}_2$$

$$d\bar{\omega}_1 = \frac{\partial\bar{\omega}_1}{\partial\omega_2} d\omega_2 + \frac{\partial\bar{\omega}_1}{\partial\bar{\omega}_2} d\bar{\omega}_2$$

$$d\omega_1 d\bar{\omega}_1 = d\omega_2^2 \underbrace{\left[\frac{\partial\bar{\omega}_1}{\partial\omega_2} \cdot \frac{\partial\bar{\omega}_1}{\partial\omega_2}\right]}_{=0} + d\omega_2 d\bar{\omega}_2 \left[\ \dots\ \right] + d\bar{\omega}_2^2 \underbrace{\left[\frac{\partial\omega_1}{\partial\bar{\omega}_2} \cdot \frac{\partial\bar{\omega}_1}{\partial\bar{\omega}_2}\right]}_{=0}$$

$$\overline{\frac{\partial\bar{\omega}_1}{\partial\omega_2}} = \frac{\partial\omega_1}{\partial\bar{\omega}_2}$$

So it suffices to say $\frac{\partial\omega_1}{\partial\omega_2} = 0$ or $\frac{\partial\bar{\omega}_1}{\partial\omega_2} = 0$
& implies

But ω_1, ω_2 have same orientation so $\frac{\partial\bar{\omega}_1}{\partial\omega_2} = 0$

$\Rightarrow$ ω_1, ω_2 transition function holomorphic.

$\Leftarrow$ is straightforward because by choice of ω

$$g = s\, d\omega d\bar{\omega}$$

Solution of Beltrami uses an interesting ~~lem~~ result from complex analysis:

$\bar{\partial}$-lemma: $V \subseteq \mathbb{C}$ open, $g \in C^k(V)$, define

$$f(z) := \frac{1}{2\pi i} \int_V \frac{g(\omega)}{\omega - z}\, d\omega \wedge d\bar{\omega}$$

then $f \in C^{k+1}(V)$ and $\frac{\partial f}{\partial \bar{z}} = g$.