Large Scale Phenomenon in Homotopy Theory

Chromatic Stable Homotopy Theory:

Goal: Calculate $\pi_k(S^0) = \varinjlim \pi_{n+k}(S^k)$

k	0	1	2	3	4	5	6	7	8	9
$\pi_k(S^0)$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/16\cdot 3\cdot 5$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^3$

$\rightsquigarrow$ No pattern

Large Scale Patterns:

Use cohomology theories:

$f, g : X \longrightarrow Y \qquad H^*f \neq H^*g \Rightarrow f \not\simeq g$

Complex K-theory:

$K^0(X) = grp\left(\coprod_n \mathrm{Vect}_n^{\mathbb{C}}(X), \oplus\right)$ ring theory

$K^0(pt) = \mathbb{Z}[\mu^{\pm 1}] \qquad |\mu| = 2$

Bott element

This is 2-periodic. Such things are complex orientable $\Rightarrow$ we have Thom classes, Chern classes

Euler class: $e(V) \in K^{2n}(X) \simeq K^0(X)$ (multiplication by μ)

Gysin seq: $K^0(\mathbb{CP}^\infty) \cong \mathbb{Z}[[x]]$

Formal Group Law $\rightsquigarrow$ $e(L_1 \otimes L_2) =: e(L_1) +_F e(L_2)$

- For K-theory $\quad x +_F y = x + y + xy \qquad e(L) = L - 1$

Different orientations produce different Euler classes $\qquad e'(L) = \varphi(e(L))$

$\varphi(x) \in K^0[[x]]^\times$ gives an isomorphism between the two fgls

- Look at $\quad E^0(X) \longrightarrow E^0(X) \quad$ natural ring morphisms

TR^m (Butlewicz - Turner) $\qquad$ 2 periodic

Such natural ring iso are in 1-1 correspondance with pairs

$(f : E^0 \longrightarrow E^0 \ , \ \varphi : F \longrightarrow f^*F)$

fgl $\qquad$ change of co-efficients of fgl

- $E =$ K-theory

$f : \mathbb{Z} \to \mathbb{Z}$, only two iso $\qquad \varphi(x) = x$

$\varphi(x) = (1+x)^{-1} - 1$

- $\hat{K}$ - p-complete K-theory

$\hat{K}^*(pt) = \mathbb{Z}_p[[\mu^{\pm 1}]] \qquad f : \hat{F} \longrightarrow \hat{F} \qquad x + y + xy$ over $\mathbb{Z}_p[[x]]$

$\mathbb{Z}_p^\times \longrightarrow \mathrm{Aut}(\hat{F})$

$a \longmapsto (1+x)^a - 1$

$p > 2: \quad \mathbb{Z}_p^\times \cong \mathbb{F}_p^\times \times \mathbb{Z}_p \xrightarrow{\simeq} 1 + p\mathbb{Z}_p$

- Back to π_*

$(\tilde{\hat{K}}^0)(S^{2k}) \cong \hat{K}^{-2k}(S^0) \simeq \mathbb{Z}_p \mu^{-k} \simeq \mathbb{Z}_p(-k) \circlearrowleft \psi^a$

write ψ^a operator $(1+x)^a - 1$, $\quad \psi^a(\mu^{-k}) = a^{-k}\mu^{-k}$

$\alpha \in \pi_{2n+2k-1} S^{2n} \quad \rightsquigarrow \quad S^{2n+2k-1} \xrightarrow{\alpha} S^{2n} \longrightarrow C(\alpha) \quad \rightsquigarrow$

Mapping cone of α

$$\begin{array}{ccc} 0 \leftarrow K^0(S^{2n}) & \cong & \mathbb{Z}_p(-n) \\ \uparrow & & \\ K^0(C(\alpha)) & & \\ \uparrow & & \\ K^0(S^{2n+2k}) & \cong & \mathbb{Z}_p(-n-k) \\ \uparrow & & \\ 0 & & \end{array}$$

$$e(\alpha) \in \mathrm{Ext}_{\mathbb{Z}_p^\times}(\mathbb{Z}_p(-n-k), \mathbb{Z}_p(-n))$$

$$\cong \mathrm{Ext}_{\mathbb{Z}_p^\times}(\mathbb{Z}_p, \mathbb{Z}_p(k)) =: H^1(\mathbb{Z}_p^\times, \mathbb{Z}_p(k))$$

$e : \pi_{2k-1} S^0 \longrightarrow H^1(\mathbb{Z}_p^\times, \mathbb{Z}_p(k))$ =: Adams e-invariant

Th^m Adams

for $p>2$ $\quad H^1(\mathbb{Z}_p^\times, \mathbb{Z}_p(k)) \cong \begin{cases} 0 & k \neq p^t s_0(p-1) \quad (s_0, p) = 1 \\ \mathbb{Z}/p^{t+1} & k = p^t s_0 (p-1) \\ \mathbb{Z}_p & k = 0 \end{cases}$

• e is split.

This gives us infinite elements in π_n of higher & higher torsion.

• Elements of order p:

Define $\quad \Sigma^n V(0) =$ cofiber of $S^n \xrightarrow{p} S^n$

Construct a map $\quad f_n : \Sigma^{k+n} V(0) \longrightarrow \Sigma^n V(0)$ with $f_{n+1} = \Sigma f_n$ (map of spectra)

Adams: $\quad K^*(V(0)) \cong \mathbb{F}_p[u^{\pm 1}]$

$v_1 : \Sigma^{2k(p-1)} V(0) \longrightarrow V(0)$

$K^*(v_1) = - \cdot \mu^{p-1}$ ⇜ isomorphism as μ is a unit

$\Rightarrow K^*(v_1^t) \neq 0$

$\Rightarrow$ None of the v_1^t are null-homotopic

$\alpha_t : S^{2t(p-1)} \longrightarrow \Sigma^{2t(p-1)} V(0) \xrightarrow{v_1^t} V(0) \longrightarrow S^1$

This is the element of order p

$e(\alpha_t) \in H^1(\mathbb{Z}_p^\times, \mathbb{Z}_p(t))$

• Rephrasing: $\quad H^s(\mathbb{Z}_p^\times, \mathbb{Z}_p(t)) \Rightarrow \pi_{2t-s} L_{K(1)} S^0 \qquad \begin{matrix} X \\ \downarrow \\ L_{K(1)} X \end{matrix} =$ terminal K^* iso out of X

$H^s(\mathbb{Z}_p^\times, K^t(X)) \Rightarrow \pi_{t-s} L_{K(1)} S^0$

We have a map $\quad \pi_* V(0) \longrightarrow v_1^{-1} \pi_* V(0) \longrightarrow \pi_* L_{K(1)} V(0)$

Telescope conjecture: This is an isomorphism
Proved for $K(1)$.

• Q. How to generalize?

Generalised K-theory : Morava E-theory

$$(E_n)^* \cong W(\mathbb{F}_{p^n})[[\mu_1, \dots, \mu_{n-1}]][\mu^{\pm 1}]$$

FGL $= F_n$ complicated, $\quad [p]_F(x) = x +_F \cdots +_F x = x^{p^n} + \cdots \quad$ height n

$$\mathbb{Z}_p^\times = \mathrm{Aut}(\mathbb{F}_p) \overset{\text{replace}}{\rightsquigarrow} \mathbb{G}_n = \mathrm{Aut}(\mathbb{F}_p/\mathbb{F}_{p^n}) \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$$

$$H^s(\mathbb{G}_n, (E_n)_t X) \Rightarrow \pi_{t-s} L_{K(n)} X$$

• *Ex:* $V(1) := \mathrm{cone}\{ v_1 : \Sigma^{2(p-1)} V(0) \longrightarrow V(0) \}$

$E_2^*(V(1)) \cong \mathbb{F}_{p^2}[\mu^{\pm 1}] \qquad p \geq 5$

$\exists\, v_2 : \Sigma^{2(p^2-1)} V(1) \longrightarrow V(1)$

$E_2^* v_2 = \mu^{p^2-1}$ not nilpotent

$$\underset{\substack{\| \\ K^*(V(1))}}{H^*(\mathbb{Z}_p, \mathbb{F}_p[\mu^{\pm 1}])} \cong \mathbb{F}_p[\underset{\substack{\| \\ \mu^{p-1}}}{v_1^{\pm 1}}] \otimes \Lambda(\rho)$$

$H^*(\mathbb{G}_2, E_*(V(1))) = \mathbb{F}_p[v_2^{\pm 1}] \otimes \Lambda(\rho) \otimes A \qquad \cong \pi_* L_{K(2)} V(1)$

$\hookrightarrow A$ poincare algebra

$v_2 \in H^0(\mathbb{G}_2, E_{2(p^2-1)} V(1)) \qquad |v_2| = (0, 2(p^2-1))$

Telescope conjecture: $v_2^{-1} \pi_* V(1) \longrightarrow \pi_* L_{K(2)} V(1)$

onto? iso?