Def: $\pi: X \to Y$ locally closed embedding (or simply embedding / immersion) if

$$\exists \quad \begin{tikzcd} X \arrow[r, "\text{closed}", "\text{emb}"'] \arrow[rr, bend right, "\pi"'] & U \arrow[r, "\text{open}", "\text{emb}"'] & Y \end{tikzcd}$$

Locally closed subscheme = closed subscheme of an open subscheme.

Rem: $X \xhookrightarrow{\text{open}} Z \xrightarrow{\text{closed}} Y$, say $X = Z \cap U$ some open $U \subseteq Y$

$\Rightarrow \underset{Z\cap U}{\overset{\|}{X}} \xhookrightarrow{\text{closed}} U \longrightarrow Y \quad \Rightarrow \quad X$ locally closed subscheme.

But converse is not true.

Rem: · locally closed embeddings are locally finite type, stable under composition.

§ 8.2 Projective setting

$\mathbb{P}^n_A = \operatorname{Proj} A[x_0, \dots, x_n]$, covered by $\left(\operatorname{Spec} A[x_0, \dots, x_n]_{x_i}\right)_0 \cong \operatorname{Spec}\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right]$

$f \in A[x_0 \dots x_n]$ homogenous then

$U_i \supseteq V\left(f(x_0 \dots x_n)/x_i^{\deg f}\right)$ closed subscheme

$U_i \cap U_j = \operatorname{Spec}\left(A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}, \frac{x_i}{x_j}\right]\right)$

$$\frac{f(x_0 \cdots x_n)}{x_i^{\deg f}} = \Big(\underbrace{\frac{x_j}{x_i}}_{\text{unit}}\Big)^{\deg f} \frac{f(x_0 \cdots x_n)}{x_j^{\deg f}}$$

So these patch to give a global subscheme in $\mathbb{P}^n_A \supseteq V(f)$

Similarly get V(any homogenous ideal)

Facts: · $S_\cdot \twoheadrightarrow R_\cdot$ gives a closed embedding $\operatorname{Proj} R_\cdot \hookrightarrow \operatorname{Proj} S_\cdot$

· $X \hookrightarrow \operatorname{Proj} S_\cdot$, closed subscheme $\Rightarrow X \cong \operatorname{Proj}(S_\cdot/I)$

$S_\cdot$ finitely gen A-algebra $\qquad I$ homogenuous ideal in $S_\cdot$

· Closed subschemes in $\mathbb{P}^n_k$, k-field,

cut out by a single polynomial f then $\deg f =:$ deg. of hypersurface

· Special hypersurfaces: deg 1 – hyperplane
deg 2 – quadric
3 – cubic, quartic, quintic, sextic

$n=2$ hyper surfaces are curves in $\mathbb{P}^2_k$
deg 1 – line
deg 2 – conic

$n=3$ hyper surfaces are surfaces.

§ 8.3 Interesting closed subschemes:

Images:

$\pi : X \to Y$, $\pi(X)$ needn't in general carry a reasonable scheme structure.

Def. $i : Z \hookrightarrow Y$ closed subscheme $\quad 0 \to \mathcal{I} \to \mathcal{O}_Y \to \pi_* \mathcal{O}_Z \to 0$ (corresponding ideal sheaf)

Say im of $\pi : X \to Y$ contained in Z if $\mathcal{I} \to \mathcal{O}_Y \to \pi_* \mathcal{O}_Z$ is 0.

i.e. induces

$$\begin{tikzcd} \mathcal{O}_Y \arrow[r] \arrow[d] & \pi_* \mathcal{O}_X \\ \pi_* \mathcal{O}_Z \arrow[ur, dashed] & \end{tikzcd}$$

Def: Scheme theoretic image : $\bigcap$ closed subscheme containing im of π, itself a closed subscheme

Need fiber product to make this rigorous.

· $\pi : X \to \operatorname{Spec} B \rightsquigarrow$ Say $I = \ker\left(B \to \Gamma(X, \mathcal{O}_X)\right)$

Scheme theoretic image of $\pi = \operatorname{Spec} B/I$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \operatorname{Spec} B \\ \operatorname{Spec} B/I \arrow[ur, dashed] & \end{tikzcd}$$

Heuristic: In good cases the scheme-theoretic image is the closure of set theoretic image with minimal closed subscheme structure.

eg: · $X = \coprod_{n \geq 1} \operatorname{Spec} k[\varepsilon_n]/(\varepsilon_n^n) \longrightarrow \operatorname{Spec} k[x] = \mathbb{A}^1_k$

$\varepsilon_n \longleftarrow x$

Set theoretic image $= V((x))$

scheme th image $= \operatorname{Spec} k[x]/f(x)$, $(f(x)) = \ker\left[k[x] \to \Gamma(X, \mathcal{O}_X)\right] = 0 \Rightarrow \mathrm{im} = \operatorname{Spec} k[x]/(0) = \mathbb{A}^1_k$

Thm: If $\pi : X \to Y$ qsct or X reduced then scheme theoretic image of π is affine local on X.

Proof: $\overline{\pi(X)}$:= Scheme th image

Must show that for all open $\operatorname{Spec} B \subseteq Y$, $\overline{\pi(X)} \cap \operatorname{Spec} B$ cut out by functions in B pulling back to 0.

$\Leftrightarrow$ Must show $0 \to \mathcal{I} \to \mathcal{O}_Y \to \pi_* \mathcal{O}_X$

$\mathcal{I}$ is quasi-coherent i.e. $\forall \operatorname{Spec} B \overset{\text{open}}{\subseteq} Y$

$$\mathcal{I}(B_f) \xrightarrow{\sim} \mathcal{I}(B)_f \quad \forall f \in B$$

To do

Cor: Under the above hypothesis the scheme theoretic image is the closure of the set theoretic image.

Def: Scheme theoretic closure of locally closed embedding $\pi : X \to Y$ is scheme-theoretic image of π.

Def: $X \subset Y$ ~ scheme

closed subset.

Reduced scheme structure on X is defined by q.coh ideal sheaf $\mathcal{I}$ s.t. $\forall$ open $\operatorname{Spec} B \subseteq Y$ $\mathcal{I}(B) = \sqrt{I}$ for any ideal I s.t. $V(I) = X \cap \operatorname{Spec} B$ as topological space

Easy to check $\mathcal{I}$ is quasi-coherent and hence this makes I a closed subscheme of Y

eg: · X scheme, $X_{red} \hookrightarrow X$ closed subscheme with same top space

$\forall$ open $U \subseteq \operatorname{Spec} B \subseteq X$,

$$\Gamma(U, \mathcal{O}_{X_{red}}) = B/\sqrt{0}$$

$\mathcal{I}$ = nilpotents