Formal Group Law:

for a ring R, $fgl(R) = \{ F(x,y) \in R[[x,y]]$ (denoted $x +_F y$) such that $x +_F 0 = x$, $x +_F y = y +_F x$, $(x +_F (y +_F z)) = ((x +_F y) +_F z) \}$

an $\overset{\text{strict}}{\vee}$ isomorphism of fgl's $F \longrightarrow G$ is an element $\varphi \in R[[x]]$ s.t. φ invertible i.e. $\varphi'(0) = 1$

$$\varphi(x +_F y) = \varphi(x) +_G \varphi(y)$$

* See below

So we can think of $fgl(R)$ as a groupoid.

Invertible power series:

We say $\phi \in R[[x]]$ is invertible if $\exists\, \psi \in R[[x]]$ such that $\varphi \circ \psi = \psi \circ \phi = x$.

If $\phi = x + a_2 x^2 + a_3 x^3 + \cdots$ then ϕ is always invertible as one can construct ψ inductively.

Let $\psi = x + b_2 x^2 + b_3 x^3 + \cdots$

for example:

$$\psi \circ \phi = \phi + b_2 \phi^2 + b_3 \phi^3 + \cdots$$

$$= x + x^2 \cdot [a_2 + b_2] + x^3 [a_3 + 2a_2 b_2 + b_3] + \cdots$$

$$\Rightarrow b_2 = -a_2, \quad b_3 = -a_3 - 2a_2 b_2, \ \ldots$$

· Examples:
- $x \underset{u}{+} y = x + y + uxy$, $u \in R$ a unit
- $x \underset{T}{+} y = (x+y)/(1-xy)$
- $x \underset{E}{+} y = (x\sqrt{1-y^4} + y\sqrt{1-x^2})/(1+x^2y^2)$ fgl over $\mathbb{Z}[1/2]$

addition formula of the elliptic integral $\int_0^x \frac{dt}{\sqrt{1-t^4}}$

Logarithm:

An isomorphism $\varphi : +_F \longrightarrow +$ is called a logrithm and denoted $\log_F$.

· Over $\mathbb{Q}$: $+_u \xrightarrow{\cong} +$ via $\log_u(x) = \ln(1 + ux)$

$$\log_u(x +_u y) = \ln(1 + ux + uy + u^2 xy) = \ln(1+ux) + \ln(1+uy)$$

$+_T \xrightarrow{\cong} +$ via $\log_T(x) = \tan^{-1}(x)$

$$\tan^{-1}(x \underset{T}{+} y) = \tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1}(x) + \tan^{-1}(y)$$

Prop: Over $\mathbb{Q}$ any fgl is isomorphic to $+$.

Proof: define

$$\log_F(x) = \int_0^x \frac{1}{(\partial F/\partial y)(t,0)}\, dt$$

$$\frac{\partial}{\partial y}\left[\log_F(x) + \log_F(y)\right] = \frac{1}{F_y(y,0)} \qquad \frac{\partial}{\partial y} \log_F(x +_F y) = \frac{\partial}{\partial y}(x +_F y) \cdot \frac{1}{F_y(x \underset{F}{+} y, 0)}$$

Suffices to show:

$$\frac{1}{F_y(y,0)} = F_y(x,y) \cdot \frac{1}{F_y(x +_F y, 0)} \iff F_y(x +_F y, 0) = F_y(x,y) \cdot F_y(y,0)$$

Sanity check: $F(x,y) = x + y + uxy$

$F_y(x,y) = 1 + ux$

$F(F(x,y), z) = F(x, F(y,z))$ Chain Rule!

$LHS = 1 + u(x + y + uxy)$

Lazard Ring $\mathcal{L} \overset{\simeq}{} \mathbb{Z}[x_1, x_2, \dots]$

$\exists!$ a ring $\mathcal{L}$ and a fgl $F(x,y) = \sum a_{ij} x^i y^j \in fgl(\mathcal{L})$ s.t. $\forall$ rings R and $G \in fgl(R)$ $\exists$ a map

$\varphi : \mathcal{L} \longrightarrow R$ such that $G(x,y) = \sum \varphi(a_{ij}) x^i y^j =: \varphi_* F$

Grading: $|x| = |y| = -2$, $|a_{ij}| = 2(i+j-1)$, $|x_i| = 2i$

so that $F(x,y)$ is homogenous of degree -2

Over $\mathbb{Q}$:

- $\mathcal{L} \otimes \mathbb{Q} = \mathbb{Q}[m_1, m_2, \dots]$, $F(x,y) = f^{-1}(f(x) + f(y))$ where $f(x) = x + \sum_{i>0} m_i x^{i+1}$ $\quad |m_i| = 2i$

  Given a $G \in fgl(A)$ we saw that $G(x,y) = \log_G^{-1}(\log_G(x) + \log_G(y))$ if A is a $\mathbb{Q}$-algebra.

  So if $\log_G = x + \sum g_{ij} x^i y^j$ we can define

  $\varphi : \mathbb{Q} \longrightarrow A$, $a_{ij} \longrightarrow g_{ij}$.

- $f(x) \in \mathbb{Z}[m_1, m_2, \dots] =: M$ $\Rightarrow f^{-1}(x) \in M$

  $\Rightarrow$ The map $\mathcal{L} \longrightarrow \mathcal{L} \otimes \mathbb{Q}$ factors through M

Lazard's Thm: $x_i \in \mathcal{L}$ can be chosen so that $\mathcal{L} \longrightarrow \mathcal{L} \otimes \mathbb{Q}$ send x_i to $p m_i$ if $i = p^k$, m_i else. Further $\mathcal{L} \longrightarrow \mathcal{L} \otimes \mathbb{Q}$ is injective.

Remark: There is a subtle point to be made here. Even though all fgl's over $\mathbb{Q}$ has log's, they have non-trivial automorphisms which should be captured by the Lazard ring. Hence it is not correct to say $L \otimes \mathbb{Q} \cong \mathbb{Q}$ and the fgl is $x + y$.

Automorphisms:

Denote by $si(R) = \{ (F, f, G) : F, G \in fgl(R) , f : F \longrightarrow G$ is a strict isomorphism $\}$

- $si(\mathcal{L}) \cong \mathcal{L}\mathcal{B} := \mathcal{L}[b_1, b_2, \dots]$ $\quad |b_i| = 2i$

  as every power series $f(x) = x + \sum b_i x^{i+1}$ gives an isomorphism & vice versa.

Lazard's theorem ...:

$\mathcal{L}\mathcal{B}$ corepresents the functor $si(-)$. Or $(\mathcal{L}, \mathcal{L}\mathcal{B})$ corepresent the monoid valued functor $(fgl(-), si(-))$.

- $(\mathcal{L}, \mathcal{L}\mathcal{B})$ is a Hopf algebroid because $(fgl(-), si(-))$ is a groupoid

obj $\quad \mathcal{L} = \mathbb{Z}[x_1, x_2, \dots]$ $\qquad\qquad \{\cdot\} = \{F\}$

mor: $\quad \mathcal{L}\mathcal{B} = \mathcal{L}[b_1, b_2, \dots]$ $\qquad\qquad \{\cdot \longrightarrow \cdot\} = \{(F, f, G)\}$

$\varepsilon : \mathcal{L}\mathcal{B} \longrightarrow \mathcal{L}$ $\qquad\qquad F = \cdot \longmapsto \cdot\circlearrowright^{id} = (F, id, F)$

$\quad b_i \longmapsto 0$

$\eta_L : \mathcal{L} \longrightarrow \mathcal{L}\mathcal{B}$

inclusion of co-efficients $\qquad\qquad (F, f, G) = \underset{a}{\cdot} \longrightarrow \underset{b}{\cdot} \longmapsto \underset{a}{\cdot} = F$

$\eta_R : \mathcal{L} \longrightarrow \mathcal{L}\mathcal{B}$ $\qquad\qquad (F, f, G) = \cdot \longrightarrow \cdot \longmapsto \underset{b}{\cdot} = G = f(F)$

product : $\Delta : \mathcal{LB}\ {}_{\eta_R}\!\otimes_{\eta_L} \mathcal{LB} \longrightarrow \mathcal{LB}$

$$(F,f,g) = \overset{a}{\bullet} \longrightarrow \overset{b}{\bullet} \quad \longmapsto \quad \sum_c \begin{tikzcd} a \arrow[rr] \arrow[r, bend left] & \overset{c}{\bullet} \arrow[r, bend left] & b \end{tikzcd} = \sum (F,h,H),(H,g\cdot h^{-1},G)$$

antipode : $c : \mathcal{LB} \longrightarrow \mathcal{LB}$

$$(F,f,G) = \overset{a}{\bullet} \longrightarrow \overset{b}{\bullet} \quad \longmapsto \quad \overset{b}{\bullet} \longrightarrow \overset{a}{\bullet} = (G,f^{-1},F)$$

---

p-typical fgl: – From now on all rings are $\mathbb{Z}_{(p)}$ algebras

$[n](x) := x +_F x +_F + \cdots +_F x$ n-times

$[1/n](x) =$ inverse of $[n](x)$ i.e. $[1/n]([n](x)) = x$

Def^n: $F \in fgl(R)$ is p-typical if $\forall\ (n,p)=1$

$$[1/n](x +_F \theta x + \cdots + \theta^{n-1} x) = 0 \qquad \text{where} \quad \theta = e^{2\pi i/n}$$

We say $F \in fglp(R)$.

Prop: If R is torsion free then F is p-typical iff $\log_F(x) = \sum_i l_i x^{p^i}$.

Proof: If $\log_F(x) = \sum l_i x^{p^i}$ then $\log_F(x +_F y) = \log_F(x) + \log_F(y)$

$$[1/n](x +_F \theta x +_F \cdots + \theta^{n-1} x) = [1/n]\cdot(\log_F^{-1}[\log_F(x) + \log_F(\theta x) + \cdots + \log_F(\theta^{n-1} x)])$$

$$= [1/n] \log_F \Big[\sum_i l_i \big[x^{p^i} + \theta^{p^i} x^{p^i} + \cdots + \theta^{(n-1)p^i} x^{p^i}\big]\Big]$$

$$= [1/n] \log_F \Big[\sum_i l_i x^{p^i} [0] \Big] \qquad \text{as } (n,p)=1$$

$$= 0$$

Conversely suppose $[1/n](x +_F \theta x +_F \cdots +_F \theta^{n-1} x) = 0$. Suppose n is prime.

then suppose $\log_F(x) = \sum m_i x^{i+1}$ where $m_i \in R \otimes \mathbb{Q}$

the above computation gives us

$$\log_F [1/n](x +_F \theta x +_F \cdots + \theta^{n-1} x) = \log_F [1/n]\cdot(\log_F^{-1}[\log_F(x) + \log_F(\theta x) + \cdots + \log_F(\theta^{n-1} x)])$$

$$= 1/n \Big[\sum_i m_{i-1} [x^i + \theta^i x^i + \cdots + \theta^{i(n-1)} x^i] \Big]$$

$$= 1/n \Big(\sum_i m_{i-1} \cdot x^i \cdot [\theta^{i\cdot n} - 1]/[\theta^i - 1] \Big)$$

$$= 1/n \cdot \sum_i m_{i-1} \cdot x^i \cdot \begin{cases} 0 & \text{if } (i,n)=1 \\ n & \text{else} \end{cases}$$

$$= \sum_{i>0} m_{ni-1} x^{ni}$$

So p-typical $\Rightarrow m_{ni-1} = 0$ $\square$.

Th^m Cartier: Every fglp over a $\mathbb{Z}_{(p)}$ algebra is isomorphic to a p-typical one.

Universal p-typical fgl:

$\exists$ a ring $V = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$, $|v_n| = 2(p^n - 1)$ and a universal fgl G/V which co-represents $fglp(-)$

The map $L \otimes \mathbb{Z}_{(p)} \longrightarrow V$ is split i.e. V is a direct summand of $L \otimes \mathbb{Z}_{(p)}$

The endomorphisms $sip(-)$ is corepresented by $V[t_1, t_2, \dots]$. $\left[sip(R) = \{(F, f, G) : F, G \in fglp(R), \ f \text{ strict iso}\} \right]$

$|t_i| = 2(p^i - 1)$

$\wr$

every endomorphism is of the form

$\sum t_i x^{p^i}$

Height filtration: - All rings are $\mathbb{Z}/p$ algebras and all fgl are p-typical

Def: A fgl has height n if $[p](x) = a_n x^{p^n} +$ higher order terms

- $x + y$: $[p](x) = 0 \Rightarrow ht = \infty$
- $x +_u y = x + y + uxy$ $\qquad \log_u(x) = \log(1 + ux)$

$\log_u([p](x)) = p \cdot \log_u(x) = p \cdot \log(1 + ux) = \log\left[(1+ux)^p\right]$ formally

$\Rightarrow 1 + u[p]x = (1+ux)^p$

$\Rightarrow [p]x = \frac{1}{u} \cdot [(1+ux)^p - 1] = u^{p-1} x^p$

$\Rightarrow$ height $= 1$

- $x +_G y = \dfrac{x+y}{1-xy}$ $\qquad$ over prime $p = 2$ $\qquad [2](x) = 0 \Rightarrow ht = \infty$

over odd prime p $\qquad [p](x) = \tan(p \cdot \tan^{-1}(x))$

- $x +_H y = (x\sqrt{1-y^4} + y\sqrt{1-x^4})/(1 + x^2y^2)$ over $\mathbb{Z}[1/2]$ $\qquad \log_H(x) = \int_0^x \frac{dt}{\sqrt{1-t^4}}$

Q: How to find the height of G and H?

Every fgl defined using an elliptic curve has height 1 or 2.

Def^n: For n define a p-typical formal group law F_n of height n induced by

$\theta : V \longrightarrow R$, $\quad v_n \longmapsto 1$

$v_i \longmapsto 0$ for $i \neq n$.

Prop: Any fgl of height n is isomorphic to F_n.

Endomorphisms of F_n:

E = ring of endomorphisms of F_n

- E is a ring under composition and formal sum $+_{F_n}$
- E is a domain, $ht(f +_{F_n} g) = ht(f) + ht(g)$

Witt rings :

$\mathbb{F}_q$ field of q elements, $q = p^n$

$h_n(x) \in \mathbb{Z}/(p)[x]$ be lifting of an irreducible poly of degree n over $\mathbb{Z}/p$

Def: Witt rings $W(\mathbb{F}_{p^n}) \cong \mathbb{Z}_p[x]/h_n(x)$

$E_n := W(\mathbb{F}_{p^n})[S]/I$, $I = (S^n - p,\ S\omega - \omega^{\sigma} S \ .\ \forall\ \omega \in W(\mathbb{F}_{p^n}))$

Thm: Endomorphism ring of $F_n \cong E_n$, $\omega \leftrightarrow x \mapsto \bar{\omega} x$, $S \leftrightarrow x \mapsto x^p$

For $\mathbb{F}_\infty(x,y) = x+y$ $\mathrm{End}(F_\infty) \cong$ the non-commutative power series $R\langle\langle S\rangle\rangle / (Sa = a^p S\ ,\ a \in R)$

$a \leftrightarrow x \mapsto ax$, $S \leftrightarrow x \mapsto x^p$