ALGEBRAIC GEOMETRY - SPRING 2016

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Contents

reference: Görtz and Wedhorn

1. Ch 5: Schemes over fields

1.1. Ch 5.1 - 5.4.

Given a scheme over another scheme the fiber over each point is a scheme over a field so that this is like studying morphisms fiberwise.

X scheme locally finite type, $\operatorname{Spec} k$, k a field,

$$x \in X \text{ is closed } \iff [\kappa(x) : k] < \infty$$

$\kappa(x)$ is the residue field at x. x is a $k-rational$ point if $\kappa(x) = k$. If $k = \overline{k}$ the x is closed $\iff$ x is $k-rational$.

$$X(k) := \operatorname{Hom}_k(\operatorname{Spec} k, X) \cong \{k-rational \text{ points }\}$$

Example 1.1. K/k non trivial finite field extension $X = \operatorname{Spec} K$ then there are no $k-rational$ points.

Definition 1.2. The **dimension** of a scheme X is

$$\dim X := \sup\{l | \exists X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_l \text{ irreducible closed subsets}\}$$

X is **equidimensional** if all irreducible components of X have same dimension.

Recall that if the scheme is (locally) Noetherian then there are (locally) finitely many irreducible components and by Zorn's lemma argument every closed subset is contained in an irreducible component.

Example 1.3. $X = \operatorname{Spec} A$ then the dimension is the Krull dimension of A, which is the maximal chain of prime ideals in A.

- When A is a field or an Artinian ring the dimension is 0.
- When A is Dedekind domain which is not a field then the Krull dimension is 1, because every prime ideal is maximal. This case covers $\mathbb{Z}$, $\operatorname{Spec} k[x] = \mathbb{A}^1_k$.
- $\dim A[t] \geq 1 + \dim A$ and when A is Noetherian then this is an equality.

Lemma 1.4. *X scheme*

(1) $\forall Y \subset X$,

$$\dim Y \leq \dim X$$

If X irreducible and $\dim X < \infty$ an $Y \subsetneq X$ closed, then the inequality is strict.

(2) Let $X = \bigcup_\alpha U_\alpha$ open cover then

$$\dim X = \sup_\alpha \dim U_\alpha$$

(3)

$$\dim X = \sum_{Y \text{ irreducible component}} \dim Y$$

(4)

$$\dim X = \sup_{x \in X} \dim \mathcal{O}_{X,x}$$

Proof: For 1), 2), 3) do the obvious. By 2) we may assume $X = \operatorname{Spec} A$, then $\dim \mathcal{O}_{X,x}$ is the length of chains of prime ideals ending in x and the result follows. □

Corollary 1.5. *$i : Y \to X$ is a closed immersion, that is the image is a closed subscheme of X and the induced map on the structure sheaf is a surjection, (for the affine case this is $\operatorname{Spec}(A/\mathfrak{i}) \to \operatorname{Spec} A$) and X is integral. If $\dim Y = \dim X < \infty$ then i is an isomorphism.*

Proof: By 1) above, Y and X coincide topologically and because X is reduced Y must be equal to X. □

Q. 1.6. *Can we take the last supremum over closed points alone?*

By the above theorem one can reduce understanding $\dim X$ to understanding dim of integral domains.

Proposition 1.7. *Let $f : X \to Y$ be an open (on the level of topological spaces) morphism then $\dim X \geq \dim f(X)$.*

Proof: Without any loss of generality let $Y = f(X)$ and that X and Y are affine. It suffices to show that if $y_0, \cdots, y_n \in Y$ such that $y_{i-1} \in \overline{\{y_i\}}$ for all i then there exist $x_0, \cdots, x_n \in X$ such that $x_{i-1} \in \overline{\{x_i\}}$ for all i and $f(x_i) = y_i$. This is a consequence of the following lemma. □

Lemma 1.8. *Let $f : X \to Y$ open, then $\forall x \in X$ and for all generization y' of $y := f(x)$ (that is $y \in \overline{\{y'\}}$) there exists generization x' of x such that $f(x') = y'$.*

Proof: Suffices to do this in the affine case $f : \operatorname{Spec} B \to \operatorname{Spec} A$. Let Z be the set of all generizations of x that is the set of all prime ideals containing x.

$$\begin{aligned} Z &\cong \operatorname{Spec}(B_x) \\ &\cong \bigcap_{t \in B \setminus x} \operatorname{Spec}(B_t) \\ &=: \bigcap_{t \in B \setminus x} D(t) \end{aligned}$$

$f(D(t))$ open neighborhood of y and hence $f' \in f(D(t)).\forall t$ We have the composite map We look at the composite

$$g : \operatorname{Spec} B_x \to X \to Y$$

□

Remark 1.9. The above theorem is also true if f is closed.

Remark 1.10. Not true without some hypothesis on f. For example take $Y = \operatorname{Spec} \mathbb{Z}_{(2)}$ then $\dim Y = 2$ because $(0) \subseteq (2)$.

1.2. **Ch 5.5: Integral morphisms of affine schemes.**

Definition 1.11. Recall that a ring homomorphism $A \to B$ is **integral** if $\forall b \in B$, $\exists$ a monic $f \in A[t]$ such that $f(b) = 0$ in B.

Proposition 1.12. *Let $f : \operatorname{Spec} B \to \operatorname{Spec} A$ corresponding to integral $\phi : A \to B$. Then $\mathfrak{b} \in B$,*

$$f(V(\mathfrak{b})) = V(\phi^{-1}(\mathfrak{b}))$$

In particular this says that f is a closed map. Moreover

- $\dim f(V(\mathfrak{b})) = \dim V(\mathfrak{b})$

- *If ϕ is injective then f is surjective*

Proof: 1) and 2) are just going up theorem and the incomparibility theorem from commutative algebra.

It is always true that

$$V(\phi^{-1}(\mathfrak{b})) = \overline{f(V(\mathfrak{b}))}$$

that is $f(V(\mathfrak{b}))$ is dense in the left hand side. So we can assume that $\mathfrak{b} = 0$ otherwise replace B by $B/\mathfrak{b}$ and A by $A/\phi^{-1}(\mathfrak{b})$ and then this result follows from 2). $\square$

1.3. Ch 5.6: Dimensions of finite type schemes over fields.

Theorem 1.13 (Noether's Normalization)**.** *Let $A \neq 0$ be a finitely generated k algebra. Then*

- *$\exists t_1, \cdots, t_d \in A$ such that*
$$\begin{aligned} \phi : k[T_1, \cdots, T_d] &\to A \\ T_i &\to t_i \end{aligned}$$
is injective and finite (that is A is a finitely generated module via ϕ. In particular ϕ is an integral extension.)
- *Let $\mathfrak{a}_0 \subsetneq \mathfrak{a}_1 \cdots \subsetneq \mathfrak{a}_r \subsetneq A$ ideals. Then the t_i can be chosen such that*
$$\phi^{-1}(\mathfrak{a}_i) = (T_1, \cdots, T_{h(i)})$$
for $i = 0, \cdots, r$ and suitable $0 \leq h(0) \leq h(1) \cdots \leq h(r) \leq d$.

Remark 1.14. Geometric interpretation: if $X = \operatorname{Spec} A$ then $\exists : f \to \mathbb{A}^d_k$ finite surjective such that if $\phi \subsetneq Z_r \cdots \subsetneq Z_0$ closed subschemes of X, may choose f such that $f(Z_i) = V(T_1, \cdots, T_{h(i)})$.

Remark 1.15. Given a chain $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_r \subsetneq A$ prime ideals then $\phi^{-1}\mathfrak{p}_0 \subsetneq \cdots \subsetneq \phi^{-1}\mathfrak{p}_r \subsetneq \phi^{-1}A = k[T_1, \cdots, T_d]$ which implies $h(0) < h(1) \cdots < h(r) < d$ and $r \leq d$.

Corollary 1.16. *Let $A \neq 0$ finitely generate k algebra. Then* $\dim A = d$ *iff $\exists$ finite injective k algebra homorphism $k[T_1, \cdots, T_d] \hookrightarrow A$.*

Corollary 1.17. $\dim \mathbb{A}^n_k = \dim \mathbb{P}^n_k = n$.

Theorem 1.18. *Let A be finitely generated integral domain over k and $d = \dim k$. Let $\mathfrak{p}_{h(1)} \subsetneq \cdots \subsetneq \mathfrak{p}_{h(r)}$ prime ideals in A such that $\dim \mathfrak{p}_{h(i)} = d - h(i)$ then*

- *$\exists$ finite injective k algebra homorphisms $\phi : k[T_1, \cdots, T_d] \to A$ such that $\phi^{-1}(\mathfrak{p}_{h(i)}) = (T_1, \cdots, T_{h(i)})$ $\forall i$.*
- *$\forall \phi$ as above, the original chain can be complete to chain $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_d$ such that $\phi^{-1}(\mathfrak{p}_j) = (T_1, \cdots, T_j)$ $\forall j$.*

In particular any chain of primes in A can be completed to a maximal chain of length d and all maximal chains have length d.

Proposition 1.19. *Let $X \neq \phi$ be a k scheme of finite type then the following are equivalent:*

1. $\dim X = 0$
2. *X affine , $\Gamma(X, \mathcal{O}_X)$ finite dimensional over k and $\Gamma(X, \mathcal{O}_X) = \prod_{x \in X} \mathcal{O}_{X,x}$*
3. *X discrete as a topological space*
4. *X is finite as a topological space*

Proof: 1) $\implies$ 3) $\implies$ 4) $\implies$ 1) Assume 2) to be true. Then $X = \operatorname{Spec}(ArtinRing) \implies$ 2) , 3) , 4) Conversely, 3) and 4) $\implies$ $X = \operatorname{Spec} A$ and $A = \prod_x \mathcal{O}_{X,x}$. 1) + Noether Normalization $\implies \exists$ finite $k \to A$. $\square$

Corollary 1.20. *X integral finite type k scheme of dimension 0 implies $X = \operatorname{Spec} K$ where K is a finite field extension of k.*

Theorem 1.21 (Main theorem on dimension of finite type schemes over a field)**.** *Let X irreducible k scheme of finite type with generic point η*

(1) $\dim X = \operatorname{tr}\deg_k \kappa(\eta)$ the transcendence degree of fraction field of $\kappa(\eta)$.
(2) Let $x \in X$ be any closed point then $\dim \mathcal{O}_{X,x} = \dim X$.
(3) Let $f : Y \to X$ morphism of k schemes of finite type such that $\eta \in f(Y)$ then $\dim Y \geq \dim X$. In particular $\dim U = \dim X. \forall U \subset X$ non-empty open in X.
(4) Let $f : Y \to X$ morphism of k schemes of finite type with finite fibers then $\dim Y \leq \dim X$.

Remark 1.22. Chevalley's theorem implies that $\eta \in f(Y) \iff f(Y)$ dense in X.

Proof:

(1) We may assume X is affine and reduced then this is just Noether Normalization.
(2) Again we may assume X is affine and reduced then the closed points are the maximal ideals and because all maximal chains have the same length as that of the dimension of X and so the result follows.
(3) $\exists\, \theta \in Y$ such that $f(\theta) = \eta$ and we get an inclusion $\kappa(\theta) \hookleftarrow \kappa(\eta)$.
(4) Assume Y is irreducible with generic point θ and let $x = f(\theta)$ then it suffices to show that $\operatorname{tr}\deg_k \kappa(\theta) \leq \dim X$. We may assume that $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ where B is an A algebra of finite type. Then $f^{-1}(x) = \operatorname{Spec}(B \otimes_A \kappa(x))$ and $\kappa(\theta)$ is as finite extension of $\kappa(x)$ and hence they have the same dimension.

$\square$

Corollary 1.23. *X a k scheme locally finite type, $x \in X$ a closed point. Then*

$$\dim \mathcal{O}_{X,x} = \sup \dim Z$$

where the sup is over Z irreducible components of X containing x.

1.4. Ch 5.7: Local dimension at a point.

Definition 1.24. Let X be a scheme, $x \in X$. The **dimension of X at x** is

$$\dim_x X = \inf_{open U \ni x} \dim U$$

Lemma 1.25. *X a cheme*

(1) Let $U \subset X$ neighborhood of x then $\dim_x U = \dim_x X$.
(2) $\dim X = \sup_{x \in X} \dim_x X$. If X is quasicompact scheme then the supremum can be taken over all closed points only.
(3) $\forall\, n \in \mathbb{Z}$ the set $\{x \in X : \dim_x X \leq n\}$ is open in X.

Proposition 1.26. *Let X be a scheme which is locally of finite type over a field k and let $x \in X$, $I := \{$irreducible components containing $x\}$ Then $\dim_x X = \max_{Z \in I} \dim Z$. If $x \in X$ closed then $\dim_x X = \dim \mathcal{O}_{X,x}$.*

1.5. **Ch 5.8: codimension of closed subschemes.**

Definition 1.27. Let X be a scheme

- Let $Z \in X$ closed irreducible. The **codimension of** Z **in** X is
$$\operatorname{codim}_X Z := \sup\{l : \exists Z_0 \supsetneq Z_1 \supsetneq \cdots \supsetneq Z_l = Z\}$$
closed irreducibles in X.
- Arbitrary closed $Z \subset X$ is equi-codimensional if all irreducible components of Z have the same codimension in X.

Example 1.28. $X = \operatorname{Spec} A$ and $Z = V(\mathfrak{p})$ then
$$\operatorname{codim}_X Z = \dim A_{\mathfrak{p}}$$
called the height of $\mathfrak{p}$.

Example 1.29. More generally for all schemes X for all closed irreducible $Z \subseteq Z$
$$\begin{aligned}\operatorname{codim}_X Z &= \dim \mathcal{O}_{X,\eta} \\ &= \inf_{z \in Z} \dim \mathcal{O}_{X,z}\end{aligned}$$
where η is the generic point of Z. (*generic points correspond to smaller prime ideals*)

Definition 1.30. X any scheme and $Z \subset X$ any subset then **codimension of** Z **in** X is defined as
$$\operatorname{codim}_X Z := \inf_{z \in Z} \dim \mathcal{O}_{X,z}$$

Example 1.31. Consider an infinite collection of closed points inside $\mathbb{A}^1_k$ then by the above definition codimension of this set is 1 but there is no closed irreducible set containing it other than $\mathbb{A}^1$!

Remark 1.32. X scheme

- $Y \subset X$ closed implies $\operatorname{codim}_X Y = \inf_{Z \text{ irred comp of } Y} \operatorname{codim}_X Z$
- Closed $Y \subset X$ is of codim $0 \iff Y$ contains an irreducible component of X.

For $Z \subset X$ closed irreducible then it is obvious that
$$\dim Z + \operatorname{codim}_X Z \leq \dim X$$
But the inequality can be strict!

Example 1.33. Let A be DVR (example being local rings on smooth curves), π a uniformizer that is generator of the maximal ideal. Let $\mathfrak{m} := (\pi t - 1) \subset A[t]$. $A[t]/\mathfrak{m} = Frac(A)$ hence $\mathfrak{m}$ is a maximal ideal in $A[t] \implies \dim V(\mathfrak{m}) = 0$. But since $\mathfrak{m}$ is principal it is easy to see that height of $A[t]_{\mathfrak{m}} = 1$ and hence
$$\dim V(\mathfrak{m}) + \operatorname{codim}_{\mathbb{A}^1_A} V(\mathfrak{m}) = 1 \neq 2 = \dim \mathbb{A}^1_A$$
So there exists chains of prime ideals of length 2, for example $0 \subset (\pi) \subset (\pi, t)$, are in $A[t]$ but none which end in $\mathfrak{m}$!

Proposition 1.34. *X irreducible schemes of finite type over a field k, $d := \dim X$*

1. *All maximal chains of closed irreducible subsets of X have same length d*
2. *For all $Y \subset X$ closed, $\dim Y + \operatorname{codim}_X Y = \dim X$.*

1.6. Ch 5.9: Dimensions of hypersurfaces.

Proposition 1.35. *Let A be a Noetherian UFD, let $X = \operatorname{Spec} A$, $Z \subset X$ be a reduced closed of equi-codimension 1, then $Z = V(f)$ for some $f \in A$. Conversely, for $0 \neq f$ then $V(f)$ is equi-codimensional 1.*

Remark 1.36. Note that this is not an iff statement because there is condition that Z is reduced in the first part which is not required in the second part.

Proof: We first show that an integral scheme $Z = V(\mathfrak{p})$ has codim 1 $\iff$ $Z = V(f)$ some irreducible f.
$\Rightarrow$ Let $\mathfrak{p}$ be of height 1 and let $0 \neq g \in \mathfrak{p}$ then $\mathfrak{p}$ prime implies $\mathfrak{p}$ contains some irreducible divisor f of g is in $\mathfrak{p}$ and hence $(f) = \mathfrak{p}$.
$\Leftarrow$ This follows from Principal ideal theorem.

Let Z be reduced closed of equi-codimension 1 with irreducible components $Z_1, \cdots, Z_r$. We have just shown that $Z_i = V(f_i)$. Because Z is reduced we get $Z = V(f_1, \cdots, f_r)$.

The converse follows trivially from the prime factorization of f. □

Theorem 1.37. *Let X be an integral finite type k scheme and let $0 \neq f \in \Gamma(X, \mathcal{O}_X)$ non-unit then $V(f)$ is equi-codimension 1 in X.*

Proof: Say $Z_1, \cdots, Z_r$ irreducible components of $V(f)$. For all $i, \exists U_i$ open affine containing generic point of Z_i such that $U_i \cap Z_j = \phi. \forall\, j \neq i$. We know that $\dim U_i = \dim X. \forall\, i$ and so we can reduce to $X = \operatorname{Spec} A$ and $V(f)$ irreducible.

Let $\phi : k[T_1, \cdots, T_d] \hookrightarrow A$ be finite injective and let $K := k(T_1, \cdots, T_d)$ and $L := Frac(A)$. Let $g := N_{L/K}(f) = \det_K(L \xrightarrow{\times f} L)$.

One can show that $g \in k[T_1, \cdots, T_d]$ and $X \to \mathbb{A}^d_k, V(f) \mapsto V(g)$ induces finite surjection. Finite surjections preserve dimension so $\dim V(f) = \dim V(g) = d - 1$ by previous proposition. □

Remark 1.38. This is a geometric version of Krull's Prime Ideal theorem. If A is a Noetherian ring, if $\mathfrak{p}$ minimal prime containing some $f \in A$ then $ht(\mathfrak{p}) \leq 1$. Conversely if $ht(\mathfrak{p}) \leq 1$ then $\exists f \in \mathfrak{p}$ such that $\mathfrak{p}$ is minimal amongst primes containing f.

Corollary 1.39. *Let X be a k scheme of finite type, $f_1, \cdots, f_r \in \Gamma(X, \mathcal{O}_K)$ with $V(f_1 \cdots, f_r) \neq \phi$ then $\operatorname{codim} V(f_1, \cdots, f_r) \leq r$.*

Proposition 1.40. *Let $X = \operatorname{Spec} A$ be an integral finite type k scheme, let $Z \subset X$ be an integral closed subscheme of $\operatorname{codim} r > 0$ then $\exists f_1, \cdots, f_r \in A$ such that Z is an irreducible component of $V(f_1, \cdots, f_r)$ reduced.*

Proof: Choose $Z = Z_r \subsetneq Z_{r-1} \subsetneq \cdots \subsetneq Z_0 = X$ be closed irreducible with $\operatorname{codim} Z_i = i$. Prove by induction that $\exists f_1, \cdots, f_r \in A$ such that $\forall i$ Z_i is the irreducible component of the equi-codimensional space $V(f_1, \cdots, f_i)$ reduced by repeatedly applying the previous theorem. □

Proposition 1.41. *Let A Noetherian and $X = \operatorname{Spec} A$, $Z \subset X$ closed irreducible , $r \geq 0$, then the following all equivalent:*

1. $\operatorname{codim}_X Z \leq r$
2. *$\exists f_1, \cdots, f_r \in A$ such that Z irreducible components of $V(f_1, \cdots, f_r)$ reduced.*

1.7. Ch 5.10: Products and extensions of base field.

Proposition 1.42. *Let $X, Y \neq \phi$ be locally finite type k schemes then* $\dim X \times_k Y = \dim X + \dim Y$. *Further if K/k is a field extension then* $\dim X = \dim(X \otimes_k K)$.

Proof: First reduce to the affine case and then use Noether Normalization. □

2. Ch 6: Local properties of Schemes

2.1. Ch 6.1: Zariski tangent space.

Definition 2.1. For any scheme X the **Zariski tangent space** at $x \in X$ is

$$\begin{aligned} T_x X &:= (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee \\ &= \mathrm{Hom}_{\kappa(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2, \kappa(x)) \\ \kappa(x) &= \mathfrak{m}_x/\mathfrak{m}_x^2 \end{aligned}$$

Remark 2.2. $T_x X$ best if X/k and $x \in X(k)$ on the other hand if X integral with generic point η then $T_\eta X = 0$.

Remark 2.3.
- $\mathfrak{m}_x$ finitely generated then $\dim_{\kappa(x)} T_x X$ is the size of the minimal generating set for $\mathfrak{m}_x$. Applies when X is locally Noetherian.
- $U \subset X$ neighborhood of x then $T_x X = T_x U$.
- Functoriality : $X \to Y, x \mapsto y$ induces a map $T_x X \to T_y Y \otimes_{\kappa(y)} \kappa(x)$ if $T_y Y$ finite dimensional or $\kappa(x)/\kappa(y)$ is a finite extension.

2.2. Ch 6.3 $T_x X$ for $x \in X(k)$.

Let $X = \mathbb{A}_k^n$, $x \in \mathbb{A}_k^n(k), x = (x_1, \cdots, x_n)$ and so $\mathfrak{m}_x = (t_1 - x_1, \cdots, t_n - x_n)$ then $\mathfrak{m}_x/\mathfrak{m}_x^2 = k\langle t_1 - x_1, \cdots, t_n - x_n \rangle \cong k^n$. Take a dual basis for $T_x \mathbb{A}_k^n : k^n \to T_x \mathbb{A}_k^n$ as

$$(v_1, \cdots, v_n) \mapsto (\overline{g} \in (\mathfrak{m}_x/\mathfrak{m}_x^2) \mapsto \sum_i v_i \frac{\partial g}{\partial t_i}(x_i))$$

If $f = \{f_1, \cdots, f_r\} \subset k[t_1, \cdots, t_n]$ gives us a map $\mathbb{A}_k^n \xrightarrow{f} \mathbb{A}_k^r$. We want to understand the induced map on the tangent spaces $T_x \mathbb{A}_k^n \to T_{f(x)} \mathbb{A}_x^r$ using the above identification this map should be a matrix $J(f)(x) : k^n \to k^r$ and this matrix is just the **Jacobian** at x

$$J(f)_{i,j}(x) = \left[\frac{\partial f_i}{\partial t_j}(x) \right]$$

Let $X = V(f_1, \cdots, f_r) \in \mathbb{A}_x^n, x \in X(k)$ then

$$\mathcal{O}_{\mathbb{A}_x^r, f(x)=0} \to \mathcal{O}_{\mathbb{A}_k^n, x} \to \mathcal{O}_{\mathbb{A}_x^n}/(f_1, \cdots, f_r) = \mathcal{O}_{X,x}$$

gives us a map

$$\begin{gathered} \mathfrak{m}_{\mathbb{A}_k^r, 0} \to \mathfrak{m}_{\mathbb{A}_k^n, x} \to \mathfrak{m}_{X,x} \to 0 \\ 0 \to T_x X \to T_x \mathbb{A}_x^n \to T_0 \mathbb{A}_k^r \end{gathered}$$

which implies

$$T_x X = \ker J_f(x)$$

Remark 2.4. Further if X is locally finite type over k and $x \in X(k)$ then the above applies to compute $T_x X$.

2.3. **Ch 6.4 $T_x = X(k[\epsilon])$.** $k[\epsilon] = k[t]/t^2$ is called the **ring of dual numbers** over k. We have a natural map $\operatorname{Spec} k[\epsilon] \to \operatorname{Spec} k$ and a section by sending $\epsilon \mapsto 0$.

We have $X(k[\epsilon]) = \operatorname{Hom}(\operatorname{Spec} k[\epsilon], X)$ we can compose with $\epsilon \mapsto 0$, for a fixed element in $x \in X(k)$ the fiber would be denoted by $X(k[\epsilon])_x$ that is all morphisms

$$\operatorname{Spec} k \to \operatorname{Spec} k[\epsilon] \to X$$

whose image is x.

For $f \in X(k[\epsilon])_x$ we get $f^\sharp(\mathfrak{m}_x) \subset k\epsilon = T_\epsilon k[\epsilon]$ which gives us a map $\mathfrak{m}_x/\mathfrak{m}_x^2 \to k$ which is an element of T_xX.

Proposition 2.5. *This defines an isomorphism $X(k[\epsilon])_x \to T_xX$ functorial in (X, x).*

Proof: To construct an inverse map given $f : \mathfrak{m}_x/\mathfrak{m}_x^2 \to k = k\epsilon$ we precompose with the map $\mathfrak{m}_x \to \mathfrak{m}_x^2$ to get a map $\mathfrak{m}_x \to k\epsilon$. We add a copy of k to get a map $k \oplus \mathfrak{m}_x \cong \mathcal{O}_{X,x} \to k \oplus k\epsilon$. □

Remark 2.6. $X(k[\epsilon_1, \epsilon_2]/(\epsilon_1, \epsilon_2)^2)_x \cong T_xX \times T_xX$. There is a natural map $k[\epsilon_1, \epsilon_2] \to k[\epsilon], \epsilon_i \mapsto \epsilon$. The induced map $T_xX \times T_xX \to T_xX$ is the sum! Similarly scalar multiplication by $a \in k$ could be induced by $k[\epsilon] \to k[\epsilon], \epsilon \mapsto a\epsilon$. This explains the natural tangent space structure on $X(k[\epsilon])_x$ and hence on T_xX.

Proposition 2.7. *If X, Y are k schemes and $x \in X(k), y \in Y(k)$ then $T_{(x,y)}X \times_k Y = T_xX \oplus T_yY$.*

Proof: Use the universal property of fiber product and the above notion of dual numbers. □

2.4. **Ch 6.5: Projective schemes.** Similar analysis: If $x = [x_0 : \cdots , x_n] \in \mathbb{P}^n_k(k)$ then $T_x\mathbb{P}^n_k \cong k^{n+1}/(x)$ (vector space quotient) . If $X = V(f_1, \cdots , f_r) \subset \mathbb{P}^n_k$ closed subscheme then $T_xX \cong \ker J_f/(x)$

2.5. **Ch 6.6: Relative tangent space.** Now we work in **the category of S schemes**. Let X be an S scheme. Consider a map $\xi : \operatorname{Spec} k \to X$. Then

Definition 2.8. Define

$$T_\xi(X/S) := X(k[\epsilon])_\xi$$

If $x \in X$ and ξ is the canonical morphism corresponding to the residue field at x then $T_x(X/S)$ is called the **relative tangent space of X/S at x.**

As above $T_\xi(X/S)$ is a k vector space.

Remark 2.9.

(1) $\operatorname{Spec} \kappa(x) \xrightarrow{\overline{x}} X$ then $T_{\overline{x}}(X/S) = T_x(X/S)$, but if $S = \operatorname{Spec} k$ and $\kappa(x) \neq k$ then T_xX and $T_x(X/k)$ are not necessarily isomorphic. For example take $X = \operatorname{Spec} \mathbb{C}[x]$ and $S = \operatorname{Spec} \mathbb{R}$ and $\xi : \operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{C}[x]$ for the point (x),

$$\begin{aligned}
T_\xi(X/S) &= \{f \in \operatorname{Hom}_{\mathbb{R}}(\mathbb{C}[x], \mathbb{C}[\epsilon]) : ((\epsilon \mapsto 0) \circ f)(x) = 0\} \\
&= \{f(i) = a + b\epsilon, f(x) = c\epsilon : a^2 + 2ab\epsilon - b^2 = -1\} \\
&= \{f(i) = \pm i, f(x) = c\epsilon\} \cup \{f(i) = \pm\epsilon, f(x) = c\epsilon\} \\
&\cong \mathbb{C}^4 \\
&\ncong \mathbb{C} \cong T_{(x)}(X) \cong T_{(x)}(X/\operatorname{Spec} \mathbb{C})
\end{aligned}$$

(2) Finding a lift of $\xi : \operatorname{Spec} k \to X$ to $\operatorname{Spec} k[\epsilon]$ is the same as finding a lift of $\operatorname{Spec} k \xrightarrow{\overline{x}} X \times_S \operatorname{Spec} k$ to $\operatorname{Spec} k[\epsilon]$ and hence $T_\xi(X/S) \cong T_{\overline{x}}(X \times_S \operatorname{Spec} k)$ which is the absolute tangent space at $\overline{x}$.
(3) If L/k is a field extension then we can naturally extend $\operatorname{Spec} L \xrightarrow{\rho} \operatorname{Spec} K \xrightarrow{\xi} X$ induces $T_\xi(X/S) \otimes_k L \xrightarrow{\sim} T_{\xi\circ\rho}(X/S)$
(4) Special case X scheme over k , $x \in X(k)$, $k \le k'$ a field extension and $X' := X \times_{\operatorname{Spec} k} \operatorname{Spec} k'$ and let x' be a point in X' which maps to x then

$$T_{x'}X' = T_{x'}(X'/k') \cong T_x X \otimes_k k'$$

Proposition 2.10. *Let X be locally finite type over k then $\forall\, d$ the set $Z := \{x \in X : \dim_{\kappa(x)} T_x(X/k) \ge d\}$ closed in X.*

Proof: Assume $X = \operatorname{Spec} k[t_1, \cdots, t_n]/(f_1, \cdots, f_r)$ then

$$\begin{aligned} T_x(X/k) &= T_x(X \otimes_k \kappa(x)) \\ &= \ker(J_f(x)) \otimes \kappa(x) \end{aligned}$$

And so $d \ge \dim_{\kappa(x)} T_x(X/k) = \dim T_x X$ is the same as saying that the rank of $J(f)(x) \ge r$ which is the same as all the $r-1$ minors vanish which is a closed condition. □

Remark 2.11. $k \mapsto \dim_{\kappa(k)} T_x(X/k)$ is upper semi continuous, i.e. can only go up under specialization.

3. Smooth Morphisms

3.1. Ch 6.8: Definitions.

Definition 3.1. $f : X \to Y$ be a morphisms of (arbitrary) schemes f is **smooth of relative dimension** d **at** $x \in X$ if $\exists$ affine opens $U \ni x, V = \operatorname{Spec} R \ni y$ such that $f(U) \subseteq V$ and $\exists$ an open embedding $U \hookrightarrow \operatorname{Spec} R[t_1, \cdots, t_n]/(f_1, \cdots, f_{n-d})$ such that $J_f(x) \in M_{(n-d)\times n}(\kappa(x))$ has rank $n-d$ (full rank). (that is $T_x(X/Y)$ has $\dim d$.)

f is **smooth** or X is **smooth over** Y if it is smooth at all $x \in X$.

Proposition 3.2. *$f : X \to Y$ is smooth of relative dimension d at $x \in X$*
- *if and only if $\exists U \ni x$ open such that $f|_U$ is smooth of relative dimension d.*
- *$Y' \to Y$ any morphism then $X \times_Y Y' \to Y'$ smooth at all points projecting onto x.*
- *Composite of smooth maps are smooth and the dimensions add.*
- *Open embeddings have smooth relative dimension 0.*

Definition 3.3. The **smooth locus** of X is $X_{sm} := \{x \in X : f \text{ smooth at } x\}$ (which is an open set of X).

Definition 3.4. A smooth map f of relative dimension 0 is called **etale**.

Remark 3.5. (1) A map between fields is etale if it is a finite and seperable field extension.
(2) f smooth of relative dimension d then the fiber $X_{f(x)}$ is smooth over $\kappa(f(x))$ at x and $\dim_x X_{f(x)} = d$.
(3) f is smooth if f is locally finitely presented.

Example 3.6. (1) $\mathbb{A}^n_S$ and $\mathbb{P}^n_S$ are smooth of relative dimension n over S.

(2) k field, X locally finite type over k and $k \in X(k)$ then X is smooth at x iff $\dim T_x X = \dim_x X = \dim \mathcal{O}_{X,x}$ this ismplies that the smoothness condition at x is independent of $U \ni x$ used in the definition of smoothness.

(3) If $k = \overline{k}$ of characteristic not 2, $f(x) \in k[x]$ and let $Z = V(y^2 - f(x)) \in \mathbb{A}^2_k$. $J_{y^2 - f(x)} = [-f' \ 2y]$ so (x, y) is non smooth iff $y = 0$ and $f'(x) = 0$ iff $f(x) = f'(x) = 0$. So non smoothness if and only if no repeated roots.

3.2. **Ch 6.9 Existence of smooth points - "Sard's theorem".**

Lemma 3.7. *Let X, Y be integral locally finite type schemes over k, $x \in X, y \in Y$ and $\phi : \mathcal{O}_{Y,y} \xrightarrow{\sim} \mathcal{O}_{X,x}$ isomorphisms of k algebras then $\exists U \ni x, V \ni y$ open and an isomorphism of schemes $h : U \xrightarrow{\sim} V$, $h(x) = y$ and $h|_x = \phi$.*

Proof: This is saying that if we have an isomorphism on direct limits then there should be an isomorphism at some finite stage. □

Remark 3.8. This lemma holds without the integrality condition.

Proposition 3.9. *Let X be integral finite type k scheme of* $\dim d$. *Suppose $K(X)/k$ is as separable extension ($K(X) \times_k k'$ is reduced for all field extension k'/k). Then $\exists$ dense open $U \subset X$ such that U is isomorphic to a dense open $V \subset$* $\operatorname{Spec} k[t_1, \cdots, t_d, t]/(g)$ *where g is separable irreducible as a polynomial in $(k[t_1, \cdots, t_d])[t]$. (Recall : $K(X)$ is the function field of X which is the same as the fraction field of any dense open subset of X which is also the stalk at the generic point)*

Proof: Noether normalization gives us that $K(X)$ has a transcendental basis $t_1, \cdots, t_d$ such that $K(X)/k(t_1, \cdots, t_d)$ is a finite separable extension and hence is generated by some $\alpha \in K(X)$. Replace α by $f\alpha$ for some $f \in k[t_1, \cdots, t_d]$ so that we may assume that g of $f\alpha$ has coefficientns in $k[t_1, \cdots, t_d]$.

Now define $B := \operatorname{Spec} k[t_1, \cdots, t_d][t]/(g)$, $Y = \operatorname{Spec} B$ then $K(X) \cong Frac(B) = K(Y)$. Apply the previous lemma to the stalk of the generic point. □

Theorem 3.10 (Sard's theorem)**.** *Let k be a perfect field, $X \neq \phi$ non empty reduced locally finite type scheme $/k$ then the X_{sm} is open dense in X.*

Proof: We have already show the openness. To show denseness we restrict to each irreducible component, and hence integral. By the previous proposition we may assume that $X = \operatorname{Spec} k[t_1, \cdots, t_d][t]/(g)$. The Jacobian $J_g = [\frac{\partial g}{\partial t_1}, \cdots, \frac{\partial g}{\partial t_n}]$. For smoothness we need the partials to not vanish identically on X that is $g \nmid \frac{\partial g}{\partial t_1} \cdots \frac{\partial g}{\partial t_n}$ the perfectness of k then gives us that this is not possible because $K(X) = k(t_1, \cdots, t_d)[t]/(g)$. □

Remark 3.11. Instead of k perfect it suffices to assume that $\forall$ irreducible components $Z \subset X$, $K(Z)/k$ is separable, that is X is geometrically reduced i.e. $X \otimes_k k'$ is reduced for all field extension k'/k.

Proposition 3.12. *If X is locally finite type over k and X_{red} is geometrically reduced then $\{x \in X : x \text{ closed }, \kappa(x)/x \text{ separable }\}$ is a dense in X.*

3.3. **Ch 6.10 Complete local rings.**

Theorem 3.13 (Implicit function theorem)**.** *Let $f : \mathbb{R}^{n+d} = \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^d$ be a C^∞ at $x = (x_0, x_1) \in f^{-1}(0) \subset \mathbb{R}^n \times \mathbb{R}^d$ such that the rightmost $d \times d$ minor of $J_f(x)$ is non-singular then $\exists U \ni x_0, V \ni x_1$ and $g : U \to V$, C^∞ such that $f^{-1}(0) \cap (U \times V)$ is the graph of $g = \{(x, g(x))\}$,*

" the vanishing locus looks like $\mathbb{R}^n$ "

Let R be a ring, $\mathcal{R}_n = R[[x_1, \cdots, x_n]]$ be the formal power series ring.

Lemma 3.14. *Let $\phi : \mathcal{R}_n \to \mathcal{R}_n$ is an R algebra homomorphism with $\phi(x_i) \in (x_1, \cdots, x_n) \forall i$, then ϕ is an isomorphism iff $J_\phi(0) = \left[\frac{\partial \phi(x_i)}{\partial x_j}(0)\right]$ is invertible.*

Proof: ϕ is an isomorphism iff the associated map on associated graded $gr(\phi) : gr(\mathcal{R}_n) \to gr(\mathcal{R}_n)$ is an isomorphism ($gr(\mathcal{R}_n)$ is the polynomial ring $R[x_1, \cdots, x_n]$) iff the degree 1 part $gr^1(\phi) : (x_1, \cdots, x_n)/(x_1, \cdots, x_n)^2 \to (x_1, \cdots, x_n)/(x_1, \cdots, x_n)^2$ is an isomorphism, but this is just the differential of the map ϕ so that $gr^1(\phi) = J_\phi^T(0)$. □

Proposition 3.15. *If X is a k scheme , $x \in X(k)$ smooth point of* $\dim d$ *then the completed ring $\widehat{\mathcal{O}_{X,x}} := \varprojlim \mathcal{O}_{X,x}/\mathfrak{m}_x^n$ is isomorphic to $k[[x_1, \cdots, x_d]]$.*

Proof: Assume that $X = V((f)) = V(f_1, \cdots, f_{n-d}) \subset \mathbb{A}_k^n$ and $J_f(x)$ is rank $n-d$. Assume $x = 0, \iff f_i \in (x_1, \cdots, x_{n-d})$. Further assume that the rightmost $n-d \times n-d$ minor is invertible. By the previous lemma there is an isomorphism

$$k[[y_1, \cdots, y_n]] \xrightarrow{\sim} k[[x_1, \cdots, x_n]] = \widehat{\mathcal{O}_{\mathbb{A}^n,x}}, y_i \mapsto \begin{cases} x_i & \text{if } 1 \le i \le d \\ f_i & \text{otherwise} \end{cases}$$

This implies that $\widehat{\mathcal{O}_{X,x}} \cong \widehat{\mathcal{O}_{\mathbb{A}^n,x}} \cong k[[x_1, \cdots, x_{n-d}]]$ □ Should think of this theorem as a formal version of the implicit function theorem.

3.4. **Ch 6.11 Regular Schemes.** In commutative algebra a local Noetherian ring $(A, \mathfrak{m}, \kappa)$ is regular iff

(1) $\mathfrak{m}$ can be generate by $\dim A$ elements
(2) $\dim_\kappa \mathfrak{m}/\mathfrak{m}^2 = \dim A$
(3) $gr^{\mathfrak{m}}(A) \cong \kappa[x_1, \cdots, x_{\dim A}]$

Noetherian ring A is **regular** if $\forall \mathfrak{m} \in A$, $A_\mathfrak{m}$ is regular.

(1) For any Noetherian local ring $(A, \mathfrak{m}, \kappa)$, $\dim A \le \dim_\kappa \mathfrak{m}/\mathfrak{m}^2$
(2) If $a_1, \cdots, a_{\dim A}$ generate $\mathfrak{m}$ then these are automatically a regular sequence and $\kappa[x_1, \cdots, x_d] \cong gr^{\mathfrak{m}}(A)$ sending $t_i \mapsto a_i$
(3) A regular implies $A_\mathfrak{p}$ regular for all primes $\mathfrak{p}$
(4) If $(A, \mathfrak{m}, \kappa)$ is Noetherian local regular ring then
   (a) A is a UFD
   (b) For $a_1, \cdots, a_r \in \mathfrak{m}$ then the images $\mod \mathfrak{m}^2$ linearly independent over κ iff $A/(a_1, \cdots, a_r)$ is regular
   (c) $A[x]$ and $A[[x]]$ are regular
(5) Locally noetherian A is regular iff $\widehat{A}$ is regular.

Definition 3.16. A locally Noetherian scheme X, $x \in X$ is **regular** if $\mathcal{O}_{X,x}$ is regular. X is regular if every point is regular.

Remark 3.17. (1) $\operatorname{Spec} A$ is regular iff A is regular
(2) $x \in X$ regular iff $\dim_{\kappa(x)} T_x X = \dim \mathcal{O}_{X,x}$
(3) A Noetherian scheme X is regular iff all closed points are regular.
(4) $X_{reg} := \{x \in X : \mathcal{O}_{X,x} \text{ regular}\}$ openness is not always true! If X is finite type over field k then this is true. Over a perfect field regularity and smoothness coincide.

4. Ch 6.12 Regular and smooth schemes / k

Lemma 4.1. *X locally finite type over k , $x \in X$ smooth of relative $\dim d/k$ then $\mathcal{O}_{X,x}$ is regular of $\dim \leq d$ with equality if x is closed.*

Proof: Let open smooth $U \ni x$. $\exists$ closed $x' \in U \cap \overline{\{x\}}$. Suffices to show that $\mathcal{O}_{X,x'}$ is regular of $\dim d$ since $\mathcal{O}_{X,x}$ is localization of $\mathcal{O}_{X,x'}$. So assume x is closed.

By smoothness assume $x \in V(f_1, \cdots, f_{n-d}) \subset \mathbb{A}^n_k$ and $\mathcal{O}_{X,x} \cong \mathcal{O}_{\mathbb{A}^n,x}/(f_1, \cdots, f_{n-d})$. Want $\overline{f}_1, \cdots, \overline{f}_{n-d}$ are linearly independent in $T\mathbb{A}^n_{k,x}$. Use J_f somehow to conclude this. $\square$

Lemma 4.2. *$X = V(g_1, \cdots, g_s) \subset \mathbb{A}^n_k$, let $x \in X$ closed such that $rank_{\kappa(x)} J_g(x) = n - \dim \mathcal{O}_{X,x}$ the x is a smooth point of X.*

Proof: $d := \dim \mathcal{O}_{X,x}$ assume that the first $n-d$ minor of $J_g(x)$ is non-singular. Let $Y := V(g_1, \cdots, g_{n-d}), x \in X \subset Y \subset \mathbb{A}^n_k$. The rank of $J_g(x)$ tells us that x is smooth in Y. By lemma $\mathcal{O}_{Y,x}$ is regular of $\dim d$ and hence $\mathcal{O}_{Y,x}$ surjects onto $\mathcal{O}_{X,x}$ and the two have the same dimension. This is in fact an isomorphism and hence x is smooth in X. $\square$

Theorem 4.3. *Let X be locally finite type over k and let $x \in X$ be a closed point, $d \geq 0, K/k$ with $\overline{K} = K$ then TFAE*

(1) X is smooth over k of dimension d at x
(2) $\forall \overline{x} \in X_K = X \otimes_k K/x$, X_K is smooth over K of relative dimension d at $\overline{x}$
(3) $\forall \overline{k} \in X_k/x$, $\widehat{\mathcal{O}_{X_K,\overline{x}}} \cong K[[x_1, \cdots, x_d]]$
(4) $\forall \overline{x} \in X_K/x$, $\mathcal{O}_{X,x}$ is regular of $\dim d$
(5) $\dim_{\kappa(x)} T_x(X/k) = \dim \mathcal{O}_{X,x} = d$

These imply $\mathcal{O}_{X,x}$ is regular of $\dim d$. If $\kappa(x) = k$ then this statement implies the others.

Proof: We need the fact that $\dim \mathcal{O}_{X,x} = \dim \mathcal{O}_{X_K,\overline{x}}$. The previous lemma says that smoothness can be expressed as a condition on the rank of the Jacobian, but the rank of a matrix does not depend on the base field and hence we can pass to K. $\square$

Corollary 4.4. *For X irreducible finite type over k , $x \in X(k)$, x is smooth $\iff$ x is regular $\iff$ $\dim_k T_x = \dim_X$.*

Remark 4.5. (1) In 2), 3), 4) one can replace $\forall \overline{x}$ by $\exists \overline{x}$.
(2) The theorem says that X is smooth over k iff $X \otimes_k L$ smooth over any field extension L/k.
(3) This is happening because $\operatorname{Spec} L \to \operatorname{Spec} k$ is faithfully flat (flat and surjective on the level of topological spaces). There are many **local** properties which behave like this under **faithfully flat base change**.
(4) Regularity is NOT preserved under base change.
(5) If $\mathcal{O}_{X,x}$ is regular and $\kappa(x)/k$ is separable then X is smooth at x.

Corollary 4.6. *Let $X = V(g_1, \cdots, g_s) \subset \mathbb{A}^n_k, x \in X$ be a closed point then $d = \dim \mathcal{O}_{X,x}$ then x is a smooth point of X iff $\operatorname{rank}_{\kappa(x)} J_g(x) = n - d$. Further when the conditions of the theorem hold*

(1) $s \geq n - d$
(2) Assume the first $n-d$ minor of $J_g(x)$ is non-singular then $\exists$ an open neighborhood $x \in U \subset V(g_1, \cdots, g_{n-d})$ such that U open immerses in X.

Corollary 4.7. *If X is locally finite type, TFAE*

1. *X smooth over k*
2. *X is geometrically regular, that is X_L is regular for all extensions L/k.*
3. *$X_{\overline{k}}$ is regular.*

Remark 4.8.
1. More generally X locally finite type over a **perfect** field k is regular iff it is smooth.
2. X be locally finite type over k, K/k is a field extension then X_K is regular implies X_k is regular.
3. Regular geometrically integral finite type over k does not imply smoothness.

Example 4.9. $f : k = \mathbb{F}_p(t) \hookrightarrow \mathbb{F}_p(t^{1/p}) = L$, then $J_f \equiv 0 \neq 1 - \dim \operatorname{Spec} L$. Steps missing

Also $L \otimes_k L = L[x]/(x^p - t) = L[x]/(x - t^{1/p})^p$ which is not separable.

4.1. Ch 6.13 Normal schemes.

Definition 4.10. A ring A is normal if $A_\mathfrak{p}$ is integral closed for all primes $\mathfrak{p} \in A$.

Note that we do not require A to be a domain itself.

Remark 4.11.
1. If A is normal then $S^{-1}A$ is normal for all multiplicative $S \subset A$
2. If A is an integral domain then A is normal iff $A_\mathfrak{m}$ is integrally closed for all maximal $\mathfrak{m}$ iff A is integrally closed.
3. If A Noetherian integral domain then A is normal iff
   - (a) $\forall$ height 1 primes $\mathfrak{p} \subset A$, $A_\mathfrak{p}$ is a DVR (and hence regular)
   - (b) $A = \bigcap A_\mathfrak{p}$, $\mathfrak{p}$ height 1 primes.
4. A normal implies $A[x]$ is normal.
5. A is regular
   $\implies$ $A_\mathfrak{p}$ is regular $\forall\, \mathfrak{p}$
   $\implies$ $A_\mathfrak{p}$ is a UFD
   $\implies$ $A_\mathfrak{p}$ is integrally closed
   $\implies$ A is normal

Definition 4.12. A scheme X is normal at $x \in X$ if $\mathcal{O}_{X,x}$ is a normal domain. X is **normal** if it is normal at $x \in X$.

Remark 4.13. $\mathcal{O}_{X,x}$ is a domain $\forall\, x$ then all irreducible components in X are pairwise disjoint. If the set of irreducible components is locally finite (e.g. in the locally Noetherian case) then this implies $X = \coprod$ irreducible components topologically.

Proposition 4.14. *Let X be a locally Noetherian scheme.*

1. *$x \in X$ regular $\implies$ x is a normal point.*
2. *$x \in X$ normal point such that $\dim \mathcal{O}_{X,x} \leq 1 \implies x$ regular.*

Lemma 4.15.
1. *If X is locally Noetherian normal scheme, $U \subset X$ connected open, then $\Gamma(U, \mathcal{O}_X)$ is a normal domain.*
2. *Let X be a quasicompact scheme then $\mathcal{O}_{X,x}$ is normal $\forall\, x \in X \implies X$ is normal.*
3. *If $X = \bigcup_i \operatorname{Spec} A_i$ open cover such that A_i is normal for all i then X is normal.*

Proof: Reduce the case to $U = X$ an integral domain. Then $\Gamma(X, \mathcal{O}_X) = \bigcap_x \mathcal{O}_{X,x}$. Normality follows from the normality of stalks.

We use the quasicompactness to say that any closed subset contains a closed point. $\square$

Definition 4.16. A locally Noetherian X is **regular in** codim 1 if $\forall x \in X$ with $\mathcal{O}_{X,x} = 1$, $\mathcal{O}_{X,x}$ is regular.

Proposition 4.17. *$X = \operatorname{Spec} A$ be a regular Noetherian scheme, $Z = V(f) \subset X$ be a closed integral subscheme, then Z is normal iff Z is regular in* codim 1.

Definition 4.18.

$$X_{norm} := \{x \in X : \mathcal{O}_{X,x} \text{ is normal}\}$$

This is open under mild conditions e.g. locally finite type over a field.

Remark 4.19. "Normality" is not stable under base change e.g. L/K an inseparable field extension

Recall

Theorem 4.20 (Hartog)**.** *$n \geq 2$, $U \subset \mathbb{C}^n$ open, $x \in U$. Then any holomorphic function $f : U \setminus \{x\} \to \mathbb{C}$ extends to a holomorphic function on U.*

analogously for normal schemes,

Theorem 4.21. *Let X be a locally Noetherian normal scheme, $U \subset X$ open such that* $\operatorname{codim}_X(X \setminus U) \geq 2$ *then* $\Gamma(X, \mathcal{O}_X) \xrightarrow{\simeq} \Gamma(U, \mathcal{O}_X)$.

Proof: Assume $X = \operatorname{Spec} A$ where A is a normal integral domain. $\forall$ open $V \subset X$, $\Gamma(V, \mathcal{O}_X) \subset K(X) = Frac(A)$. Let $Z \subset X$ be irreducible codim 1. We can argue that $Z \cap U \neq \phi$, hence the generic point $\eta_Z \in U \implies \Gamma(U, \mathcal{O}_X) \subset \mathcal{O}_{X,\eta_Z}$. So that $\Gamma(U, \mathcal{O}_X) \subset \bigcap_Z \mathcal{O}_{X,\eta_Z}$. The righthand side is simply A as A is normal. Trivially $A \subset \Gamma(U, \mathcal{O}_X)$ so that $A = \Gamma(U, \mathcal{O}_X)$. $\square$

4.2. **Ch 12.10 Normalization.** Let A be a ring, B an A algebra. $A = \{b \in B : b$ be integral over $A\}$ subring of B called integral closure of $A \in B$.

Let X be a scheme, $\mathcal{B}$ is quasicoherent (i.e. the restriction map of sections are obtained by tensoring) $\mathcal{O}_X$ algebra. Define $\mathcal{A}'$ by $\Gamma(U, \mathcal{A}') := \{b \in \Gamma(U, \mathcal{B}) : b$ integral over $\Gamma(U, \mathcal{O}_X)\}$. It is easy to check that $\mathcal{A}'$ is a quasicoherent subsheaf of $\mathcal{B}$.

Definition 4.22. Define $\operatorname{Spec} \mathcal{A}'$ to be **integral closure** of X in $\mathcal{B}$.

X is a scheme, $\mathcal{B}$ is a quasi-coherence $\mathcal{O}_X$ algebra. $\operatorname{Spec} \mathcal{B} \xrightarrow{f} X$ is scheme such that $\forall$ open affine $\operatorname{Spec} A \subset X$, $f^{-1}(U) = \operatorname{Spec} \mathcal{B}(U)$ which is an A algebra. These $\mathcal{B}(U)$ glue as we vary $U = \operatorname{Spec} A \subset X$

4.3. **Ch 12.11 Normalization.** X integral scheme, $L/K(X)$ a field extension. $L_X :=$ constant sheaf on X. On an irreducible space X the constant pre-sheaf is a sheaf. $L_X(U) = L$ for all $U \subset X$ and hence is a quasi coherent $\mathcal{O}_X$ algebra.

Definition 4.23. Normalization of X **in** L is the integral closure of X in L_X. **Normalization of** X is normalization of X in $K(X)$.

Proposition 4.24. *X an integral domain, $L/K(X)$ algebraic field extension, $\pi : X' \to X$ be the normalization of X in L. Then,*

(1) X' is integral and normal, $K(X') = L$
(2) π is integral and surjective and $\dim X' = \dim X$
(3) $U \subset X$ non empty open then $\pi^{-1}(U) \to U$ is the normalization of U in L
(4) If $X = \operatorname{Spec} A$ and $S := A \setminus 0$ and A' the integral closure of A in L then $X' = \operatorname{Spec} A'$ and $S^{-1}A' = L$.

Proof: $X' = \operatorname{Spec} A'$ by definition of normalization. $S^{-1}A'$ is the integral closure of $S^{-1}A = K(X)$ inside L which gives us 4). $\square$

Proposition 4.25. *$\pi : X' \to X$ be a morphism of integral schemes and X' normal then TFAE*

(1) π is a the normalization of X
(2) π is integral, dominant and induces an isomorphism $K(X) \xrightarrow{\sim} K(X')$
(3) $\forall$ integral normal Y, any dominant map $Y \to X$ factors uniquely through X'.

Proof: We use the fact that for any dominant map $Y \to \operatorname{Spec} A$ the map $A \to \Gamma(Y, \mathcal{O}_Y)$ is injective and $\Gamma(Y, \mathcal{O}_Y)$ is integrally closed inside $Frac(\Gamma(Y, \mathcal{O}_Y))$. $\square$

Corollary 4.26. *Y an integral scheme. Y is normal iff $\forall$ dominant $Y \xrightarrow{f} X$ with X integral and normalization $X' \to X$, $\exists$ a lift $Y \to X'$ of f.*

Example 4.27. Cuspidal cubic: $X = \operatorname{Spec} k[x,y]/(y^3 - x^2)$ where k is a field.

$$k[x,y]/(y^3 - x^2) \longrightarrow k[t^2, t^3] \subset k[t]$$
$$(x, y) \mapsto (t^3, t^2)$$

Hence X an integral scheme which is not normal. $K(X) \cong k(t)$ so that $\operatorname{Spec} k[t] = \mathbb{A}^1_k$ is the normalization of X.

The map $\pi : \mathbb{A}^1_k \to X$ is injective on points and isomorphism on all residue fields. This implies that this map is universally injective i.e. it is injective under base change. π integral implies it is universally closed.

This gives us π is a finite universal homeomorphism, isomorphism on residue fields, but it is still not an isomorphism.

4.4. **Ch 12.12 Finiteness of normalization.** For an integral scheme X the normalization map $\pi : X' \to X$ is not necessarily a finite morphism. But π is finite if X is locally finite type over a field.

Definition 4.28. A locally noetherian scheme X is **quasi-excellent** if

(1) $\forall\, x \in X$ every fiber of $\operatorname{Spec} \widehat{\mathcal{O}_{X,x}} \to \operatorname{Spec} \mathcal{O}_{X,x}$ is geometrically regular
(2) $\forall$ finite type $Y \to X$, Y_{reg} is open in Y

X is **excellent** if in addition $\forall$ finite type $Y \to X$ if $Z \subset Z' \subset Y$ irreducible closed subsets, every maximal chain $Z = Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_r = Z'$ of closed irreducibles has same length r. In this case we say X is **universally catenary**.

Remark 4.29. (1) If X quasi-excellent implies X_{norm} is open.
(2) If X is quasi-excellent integral, $L/K(X)$ is a finite extension then the normalization of X in L is finite over X.
(3) If X is quasi-excellent, $x \in X$ then $\mathcal{O}_{X,x}$ is integrally closed domain iff $\widehat{\mathcal{O}_{X,x}}$ is. ($\exists$ a locally closed integral domain X such that $\widehat{A}$ is not even a domain.)

(4) If X is excellent and $f : X' \to X$ is locally finite type then X' is excellent.
(5) R is complete locally Noetherian ring or Dedekind domain of characteristic 0 then $\operatorname{Spec} R$ is excellent.

Corollary 4.30. *Let X be locally finite type over k then*

(1) If X is integral then $\forall$ finite type $L/K(X)$ normalization of X in L is finite over X.
(2) X_{reg} and X_{norm} are open in X.

For this we need the following fact from commutative algebra

Proposition 4.31. *If X is a normal Noetherian integral, $L/K(X)$ is a finite seperable extension then normalization of X in L is finite over X.*

5. Divisors

5.1. **Ch 11.9 Divisors on integral schemes.** Let X be an integral scheme, let $\mathcal{K}_X$ be the constant sheaf on X from $K(X)$.

Definition 5.1. A **Cartier divisor** on X is a global section of $\mathcal{K}_X^\times/\mathcal{O}_X^\times$. Concretely a divisor is given by (U_i, f_i) where U_i is an open cover of X, $f_i \in K(X)^\times$ such that $\dfrac{f_i}{f_j} \in \Gamma(U_i \cap U_j, \mathcal{O}_X^\times)$.

$Div(X)$ denotes the set of Cartier divisors on X. If $D = (U_i, f_i)$ and $E = (U_i, g_i)$ then $D + E := (U_i, f_i g_i)$.

A cartier divisor is **principal** if it is (X, f) for some $f \in K(X)$.

Cartier divisors D and E are equivalent if $D - E$ is principal.

The divisor class group $DivCl(X)$ is $Div(X)/$(principal divisors).

$$1 \to \Gamma(X, \mathcal{O}_X)^\times \to K(X)^\times \to Div(X) \to DivCl(X) \to 1$$

A cartier divisor $D = (U_i, f_i)$ gives rise to line bundle $\mathcal{O}_X(D)$ as follows

$$\Gamma(V, \mathcal{O}_X(D)) = \{f \in K(X) : f_i f \in \Gamma(U_i \cap V, \mathcal{O}_X)\, \forall\, i\}$$

If $V \subset U_i$ then

$$\Gamma(V, \mathcal{O}_X(D)) = \mathcal{O}_X(V).f_i^{-1} \subset K(X)$$

And so $\mathcal{O}_X(D)$ is free of rank 1 and hence a line bundle.

Definition 5.2. The **support of a divisor** D is

$$Supp(D) := \{x \in X : (f_i)_x \notin \mathcal{O}_{X,x}^\times \text{ for some } i \text{ with } x \in U_i\}$$

that is where f_i has a zero or a pole.

5.2. **Ch 11.10 Sheaves of fractions and rational functions.**

Definition 5.3. In a ring A, $a \in A$ is called **regular** if it is not a zero divisor. Total fraction ring $Frac(A) := R^{-1}A$ where R is the subset of all regular elements in A.

For a scheme X, the **sheaf of total fractions** $\mathcal{K}_X$ is the sheaf associated to the presheaf $\mathcal{K}_X(U) := Frac(\Gamma(U, \mathcal{O}_X))$.

For $f \in \Gamma(X, \mathcal{K}_X)$ define $domain(f) = \{x \in X : f_x \in \mathcal{O}_{X,x} \subset \mathcal{K}_{X,x}\}$.

Remark 5.4. $\mathcal{K}_X$ is not always quasicoherent, even if X is Noetherian, but it is if X is Noetherian and reduced or X is integral.

Remark 5.5. If $U \subset X$ is a dense open subscheme and $f, g \in \Gamma(X, \mathcal{O}_X)$ such that $f|_U = g|_U$ then it is not necessary true that $f = g$. Example, take $X = \operatorname{Spec} k[x,y]/(xy, y^2)$ and $U = D(x)$ then $y|_U = 0$ but $y \neq 0$. The problem is that between U and X there can be a Z closed.

Definition 5.6. Open subscheme $U \subset X$ is **schematically dense** if $\forall$ open $V \subset X$ the only closed subscheme of V containing $U \cap V$ is V itself.

An open embedding $j : Y \to X$ is **schematically dominant** if $j(Y)$ is schematically dense.

Proposition 5.7. *Let X be a scheme over S and let $j : U \to X$ open subscheme then TFAE*

(1) *U is schematically dense in X*
(2) *$j^\flat : \mathcal{O}_X \to j_*\mathcal{O}_U$ is injective*
(3) *$\forall$ open $V \subset X$, $\forall$ seperated S schemes Y, $\forall f, g : V \to Y$ such that $f = g$ on $U \cap V$ then $f = g$ everywhere.*

Remark 5.8. Schematically dense implies dense. If X is reduced then the converse is true.

Definition 5.9. Let A be a ring and a prime ideal $\mathfrak{p}$ is **associated** if $\mathfrak{p} = Ann(a)$ for some $a \in A$. Let $Ass(A)$ denote the set of associated primes.

Example 5.10. When A is integral domain then $Ass(A) = \{(0)\}$.

$Ass(k[x,y]/(xy, y^2)) = \{(x,y), (y)\}$.

Definition 5.11. If X is a locally Noetherian scheme then $x \in X$ is an **associated point** if $\mathfrak{p}_x$ is associated in $\mathcal{O}_{X,x}$. Let $Ass(X)$ be the associated points in X.

Remark 5.12. $Ass(S^{-1}A) = Ass(A) \cap \operatorname{Spec}(S^{-1}A)$ so $Ass(A) = Ass(\operatorname{Spec}(A))$.

Minimal primes of A are always associated, one proves this by localizing at the minimal prime ideal and arguing that every ring must have at least one associated prime ideal. This implies that the generic points of X are associated.

Definition 5.13. If $x \in X$ is an associated point then call $\overline{\{x\}}$ an **associated component** and an **embedded component** if x is not a generic point.

For example, in $Ass(k[x,y]/(xy, y^2))$ the origin $(0,0)$ is an embedded component.

Proposition 5.14. *X locally Noetherian, an open $U \subset X$ is schematically dense iff it contains $Ass(X)$.*

Proof: $Ass(V) = Ass(X) \cap V \, \forall V$ and being schematically dense is also a local property. So we can assume $X = \operatorname{Spec} A$.

Lemma 5.15. *Let A be a ring and let $U \subset X$ be an open subset, let $\mathfrak{a} \in A$ be such that $V(\mathfrak{a}) = X \setminus U$ then for A Noetherian TFAE:*

(1) *$\exists$ a regular element $t \in \mathfrak{a}$*
(2) *U contains a principal open $D(t)$ for some t regular*
(3) *U schematically dense*
(4) *$Ann(U) = 0$*
(5) *U contains $Ass(A)$*

□

Lemma 5.16. *$domain(f)$ is a schematically dense open subset of X.*

Definition 5.17. A **rational function** on X is an equivalence class of (U, f), $U \subset X$ is a schematically dense subset and $f \in \Gamma(U, \mathcal{O}_X)$, where $(U, f) = (V, g)$ if $f|_{U \cap V} = g|_{U \cap V}$. $R(X)$ denotes the set of all rational functions and $U \mapsto R(U)$ is a sheaf $\mathcal{R}_X$ on X.

There exists a natural map $\alpha : \mathcal{K}_X \to R(X)$ sending $f \mapsto (domain(f), f)$. The natural map $\Gamma(V, \mathcal{O}_X) \to R(U)$ is injective for all schematically dense $V \subset U$.

Lecture missing

5.3. **Weil Divisors.** Let X be a Noetherian scheme, X^1 be the set of closed integral codim 1 subschemes in X.

Definition 5.18. $Z^1(X) := \mathbb{Z}^{X^1}$ is the set of **Weil divisors**. The set of generators X^1 is called the set **prime divisors**.

The goal is to define a cycle map $Div(X) \to Z^1(X)$.

If X is normal then for any codim 1 subscheme Z the local ring $\mathcal{O}_{X,Z}$ is a DVR. Then this cycle map would be the valuation map.

Lemma 5.19. *Let A be a Noetherian local ring of* dim *1, $f = a/b \in Frac(A)^\times$ for regular elements $a, b \in A$. Define $ord_A(f) := length_A(A/(a)) - length_A(A/(b))$ (here length is the length of longest chain of submodules). Then this is a well defined group homomorphism $ord_A : Frac(A)^\times \to \mathbb{Z}$ and $A^\times \subset \ker(ord_A)$.*

Proof: Let $a \in A$ be a regular element then $\dim A/(a) = 0$ which implies $A/(a)$ is Artin local and hence of finite length. b regular implies we have a short exact sequence

$$0 \to A/(a) \cong bA/(ab) \to A/(ab) \to A/(b) \to 0$$

which implies $length(A/(ab)) = length(A/(a)) + length(A/(b))$. □

Note that a regular element of A would always have positive order as we can choose b to be 1. But it is not true that an element of positive order is an element of A. This true though if A is a DVR.

Now let $D = (U_i, f_i)$ be a Cartier divisor in $Div(X)$ and let C be a prime Weil divisor. Say $\eta_C \in U_i$.

Definition 5.20. Let $f := (f_i)_{\eta_C} \in \mathcal{K}^\times_{X,\eta_C} = Frac(\mathcal{O}_{X,C})$. f is well defined up to $\mathcal{O}^\times_{X,C}$. Define

$$ord_C(D) := ord_{\mathcal{O}_{X,C}}(f)$$

We know that $supp(D)$ has codim ≥ 1 which implies that for every $C \in supp(D)$ would be an irreducible component of the support. X Noetherian implies that $supp(D)$ is Noetherian which implies that $supp(D)$ has finitely many irreducible components and hence finitely many prime Weil divisors $C \in X^1$ such that $ord_C(D) \neq 0$.

Definition 5.21. Define

$$\begin{aligned} cyc : Div(X) &\to Z^1(X) \\ D &\mapsto \sum_{C \in X^1} ord_C(D)[C] \end{aligned}$$

For an $f \in \Gamma(X, \mathcal{K}_X^\times)$ we can define $ord_C(f) = ord_C(div(f))$. Then

$$cyc(f) := cyc(div(f))$$

A Weil divisor D is principal if $D = cyc(f)$ for some $f \in \Gamma(X, \mathcal{K}_X^\times)$. $Z^1_{princ}(X)$ is the set of such principal Weil divisors. Define the **Class group** to be

$$ClG(X) := Z^1(X)/Z^1_{princ}(X)$$

and we get a well defined map

$$cyc : DivCl(X) \to ClG(X)$$

Definition 5.22. X is locally factorial if $\forall\, x \in X$, $\mathcal{O}_{X,x}$ is a UFD.

Proposition 5.23. *X is locally Noetherian. Consider the following properties*

1. *X is regular*
2. *X is locally factorial*
3. *$\forall\, x \in X$, $\mathcal{O}_{X,x}$ is a domain and every closed integral $C \subset X$ of $\operatorname{codim} 1$ is regularly immersed (i.e. $\forall\, x \in C$, $C \cap \operatorname{Spec} \mathcal{O}_{X,x}$ is $V(f)$ for some regular f in $\mathcal{O}_{X,x}$).*
4. *X is normal*
5. *$\forall$ closed integral $C \subset X$ of $\operatorname{codim} 1$, $\mathcal{O}_{X,C}$ is a DVR*

Then

$$1) \implies 2) \iff 3) \implies 4) \implies 5)$$

Theorem 5.24. *Let X be a Noetherian scheme*

1. *If X is normal then cyc is injective (all of them).*
2. *If X is locally factorial then cyc is a bijection.*

Remark 5.25. X integral locally factorizable implies $Z^1(X) \cong Div(X) \cong$ the set of invertible fractional ideals in $\mathcal{O}_X$.

Divisors naturally form a presheaf. For $U \subset X$ open we have natural maps

$$\begin{aligned} Z^1(X) &\to Z^1(U) \\ \sum_{C \in X'} n_C[C] &\mapsto \sum_{C \in X', C \cap U \neq \phi} n_C[C \cap U] \end{aligned}$$

The restriction of principal divisors is again principal and so we get induced maps

$$ClG(X) \to ClG(U)$$

Similar results hold for Cartier divisors.

Proposition 5.26. *X Noetherian scheme, $Z \subset X$ a closed subcheme of $\operatorname{codim} \geq 1$ and $U := X \setminus Z$ (so that U is dense topologically). Let $Z_1, \cdots, Z_r$ be the irreducible components of Z of $\operatorname{codim} 1$ in X. Then*

$$0 \to \oplus_i \mathbb{Z}[Z_i] \to Z^1(X) \to Z^1(U) \to 0$$

is an exact sequence. If further U is schematically dense then we get an exact sequence

$$\oplus_i \mathbb{Z}[Z_i] \to ClG(X) \to ClG(U) \to 0$$

We get a diagram

$$\begin{tikzcd} Pic(X) \arrow[d] & DivCl(X) \arrow[l] \arrow[r] \arrow[d] & ClG(X) \arrow[d] \\ Pic(U) & DivCl(U) \arrow[l] \arrow[r] & ClG(U) \end{tikzcd}$$

and the vertical arrows are isomorphisms when U is schematically dense.

Corollary 5.27. *X Noetherian locally factorial, $Z \subset X$ is closed of* $\operatorname{codim} \geq 1$ *and $U := X \setminus Z$ then $Pic(X) \to Pic(U)$ surjective and an isomorphism if* $\operatorname{codim} Z \geq 2$.

6. Ch 11.14: Examples

Example 6.1. A be a Noetherian normal integral domain. A is a UFD iff every height 1 prime ideal $\mathfrak{p}$ is principal, so every prime Weil divisor on $X = \operatorname{Spec} A$ is principal and hence $ClG(\operatorname{Spec} A) = 0$ iff A is a UFD. In this case we also have $DivCl(X) = Pic(X) = 0$.

In particular this holds for $X = \mathbb{A}^n_k$.

Example 6.2. k be a field, $n \geq 1$, $S = k[t_0, \cdots, t_n]$. Let $f \in S$ be irreducible homogenous then $V(f) \subset \mathbb{A}^{n+1}_k$ is a prime Weil divisor and $V_+(f) \subset \mathbb{P}^n_k$ is a prime Weil divisor. These are exactly the codim 1 subvarieties of $\mathbb{P}^n_k$.

Define $\mathcal{R}$ be the set of rational polynomials. Let $f = f_1^{d_1} \cdots f_r^{d_r}$, f_i irreducible. We have a map

$$\begin{aligned} Z : \mathcal{R} &\to Z^1(\mathbb{P}^n_k) \\ f &\mapsto \sum d_i[V_+(f_i)] \end{aligned}$$

which is surjective. The principal Weil divisors come from f which has degree 0 and hence the class group $ClG(\mathbb{P}^k_n) \cong \mathbb{Z} \cong Pic(\mathbb{P}^k_n)$ via the degree map.

For $f \in S$ of degree d we have the prime Weil divisor $[V_+(f)]$ and the Cartier divisor $(D_+(t_i), ft_i^{-\deg f})$. For $d \in \mathbb{Z}$ define $\mathcal{O}_{\mathbb{P}^n_k}(d)$ to be the isomorphism class of $\mathcal{O}_{\mathbb{P}^n_k}(Z(f))$.

$$\mathcal{O}_{\mathbb{P}^n_k}(d)|_{D_+(t_i)} = \mathcal{O}_{D_+(t_i)}.t_i^{\deg f} f^{-1}$$

Proposition 6.3. $\Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d)) \cong k[t_1, \cdots, t_n]_d$ *homogenous polynomials of degree d and hence*

$$\dim_k \Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k(d)}) = \begin{cases} 0 & \text{if } d < 0 \\ \binom{n+d}{d} & \text{otherwise} \end{cases}$$

Example 6.4. Similarly when $X = \mathbb{P}^{n_1}_k \times \mathbb{P}^{n_r}_k$ then

$$Pic(X) = DivCl(X) = ClG(X) = \mathbb{Z}^r$$

We can use this to conclude that X is not isomorphic to $\mathbb{P}^m_k$ for any m if $r > 1$.

A ring A is a **Dedekind domain** if A is Noetherian integral domain such that for all maximal ideals $\mathfrak{m} \subset A$, $A_{\mathfrak{m}}$ is a DVR. This is same as saying that A is a regular integral domain and $\dim A \leq 1$.

A scheme X is a **Dedekind scheme** if it is Noetherian integral and $\forall$ open affine $U \subset X$, $\Gamma(U, \mathcal{O}_X)$ is a Dedekind ring. This is the same as requiring that X is a Noetherian integral regular of dim ≤ 1.

Examples of such schemes are

(1) $X = \operatorname{Spec} A$ for A a Dedekind ring, e.g. A is the integral closure of $\mathbb{Z}$ in a number field
(2) $\mathbb{A}^1_k$
(3) $\mathbb{P}^1_k$
(4) any regular integral curve over k

Example 6.5. Let X be a Dedekind scheme. Let X_0 be the set of closed points in X and let $x \in X_0$. So $\mathcal{O}_{X,x}$ is a DVR and $Frac(\mathcal{O}_{X,x}) = K(X) = K$.

So we get a valuation for each x, $v_x : K^\times \to \mathbb{Z}$ with $v_x(0) = \infty$. If $D = \sum_{x \in X_0} n_x[x]$ is a Divisor then,

$$\Gamma(U, \mathcal{O}_{X,x}) = \bigcap_{x \in U \cap X_0} \mathcal{O}_{X,x} = \{a \in K : v_x(a) \geq 0 \,\forall\, x \in U \cap X_0\}$$
$$\Gamma(U, \mathcal{O}_X(D)) = \{a \in K : v_x(a) \geq n_x \,\forall\, x \in U \cap x_0\}$$

If $X = \operatorname{Spec} \mathcal{O}_K$ and K is a number field then both $ClG(\mathcal{O}_K)$ and $Pic(\mathcal{O}_K)$ are equal to the usual divisor class group in algebraic number theory which is a finite group.

7. Ch 13.19: Blowing up

Definition 7.1. X scheme and Z a closed subscheme. The **blow-up** of X along Z is a morphism $Bl_Z(X) \xrightarrow{\pi} X$ such that $\pi^{-1}(Z)$ is an effective Cartier divisor which is universal in the sense that, any map $Y \xrightarrow{\psi} X$ such that $\psi^{-1}(Z)$ is an effective Cartier divisor in Y, factors through π uniquely.

$\pi^{-1}(Z)$ is called the **exceptional divisor** and Z is called the **center of the blowup**.

Remark 7.2. If Z is an effective Cartier divisor then $Bl_Z(X) = X$. If $Z = X$ then $Bl_Z(X)$ is the empty scheme.

Proposition 7.3. *X a scheme, $i : Z \hookrightarrow X$ a closed subscheme with ideal sheaf $\mathcal{I}$. Let $f : X' \to X$ be a morphism and let $Z' := f^{-1}(Z)$ then*

(1) Bl is functorial, that is

$$\begin{tikzcd} Bl_{Z'}(X) \arrow[r] \arrow[d] & Bl_Z(X) \arrow[d, "\pi"] \\ X' \arrow[r] & X \end{tikzcd}$$

(2) if f is flat then $Bl_{Z'}(X) \times_x X'$
(3) $\pi^{-1}(X \setminus Z)$ is isomorphic to $X \setminus Z$
(4) If $\Gamma(V, \mathcal{I})$ contains a regular element for all open affine V then π is birational, that is π is an isomorphism between schematically open dense subsets.

If $p : X \to Y$ is a flat map and E is an effective Cartier divisor in Y then $p^{-1}(E)$ is an effective Cartier divisor.

Construction: Z , X , $\mathcal{I}$ as before. Consider $\mathcal{B} := \oplus_{d \geq 0} \mathcal{I}^d$, with $\mathcal{I}^0 = \mathcal{O}_X$, a graded quasi-coherent $\mathcal{O}_X$ algebra generated in degree 1. Define $\widetilde{X} := Proj(B)$.

Theorem 7.4. *$\widetilde{X}$ is the blow-up of X along Z. The exceptional divisor E is $Proj \oplus_{d\geq 0} \mathcal{I}^d/\mathcal{I}^{d+1}$.*

Example 7.5. Let $X = \mathbb{A}^n_R$ and $Z = \{0\}$. Then $I = (t_1, \cdots, t_n)$ and $B = \oplus_n I^n$,

$$Bl_Z(X) = Proj(B) = V_+(T_iX_j = T_jX_i) \subset \mathbb{P}^{n-1}_R \times_{\operatorname{Spec} R} \mathbb{A}^n_R = \mathbb{P}^{n-1}_A$$

Example 7.6. If X is a locally Noetherian scheme $x \in X$ a closed point and $\mathcal{O}_{X,x}$ is regular of $\dim d$ then $E = \pi^{-1}(x) = Proj \oplus_{d\geq 0} \mathfrak{m}_x^d/\mathfrak{m}_x^{d+1} \cong \mathbb{P}^{d-1}_{\kappa(x)}$.

Definition 7.7. $Bl_Z(X)$ is **projective** over X means that there exists closed embedding $Bl_Z(X) \hookrightarrow \mathbb{P}(\mathcal{E}) = Proj(Sym_{\mathcal{O}_X}\mathcal{E})$ as schemes over X where $\mathcal{E}$ is some quasi-coherent $\mathcal{O}_X$ module of finite type.

$\mathcal{I}\mathcal{O}_{Bl_Z(X)}$ is **very ample** means that $\mathcal{I}\mathcal{O}_{Bl_Z(X)}$ is $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|_{Bl_Z(X)}$.

Proposition 7.8. *$X, Z, \mathcal{I}$ as above.*

1. *If $\mathcal{I}$ is finite type (= locally on X there exists a surjection $\mathcal{O}_X^n \to \mathcal{I}$) (eg if X is locally Noetherian then $Bl_Z(X)$ is projective over X and $\mathcal{I}\mathcal{O}_{X,Z}$ is very ample for $Bl_Z(X) \xrightarrow{\pi} X$*
2. *If $i : Y \hookrightarrow X$ is a closed subscheme then $Bl(i) : Bl_{Z\cap Y}(Y) \to Bl_Z(X)$ is a closed embedding.*

Definition 7.9. In the situation of 2) the blow up $Bl_{Y\cap Z}(Y)$ is called the **strict transform** of Y.

Lemma 7.10.

$$Bl_{Y\cap Z}(Y) = \overline{\pi^{-1}(Y \setminus Z)} \subset Bl_Z(X)$$

Corollary 7.11. *Let X be an integral scheme and let $Z \subsetneq X$ be a closed subscheme such that $\mathcal{I}_Z$ is of finite type (eg. if X is locally Noetherian). Then $Bl_Z(X)$ is integral and $Bl_Z(X) \to X$ is birational, projective and surjective.*

Example 7.12. $\mathbb{A}^2_k = \operatorname{Spec} k[x, y] \supset X = V(x^2 - y^3)$. The blowup at the origin $Bl_0(\mathbb{A}^2_k)$ is covered by two affine charts $U^x = \operatorname{Spec} k[x, y/x]$ and $U^y = \operatorname{Spec} k[x/y, x]$. The exceptional divisor is E defined by $x = 0$ in U^x and $y = 0$ in U^y.

$Bl_0(X) = \overline{\pi^{-1}(X \setminus \{0\})} \subset Bl_0(\mathbb{A}^2)$ and $Bl_0(X) \cap U^x = \operatorname{Spec} k[x, y/x]/(x.(y/x)^3 - 1) \cong \operatorname{Spec} k[y/x, x/y]$ and so this chart does not intersect the exceptional divisor. $Bl_0(X) \cap U^y = \operatorname{Spec} k[x/y, y]/(y-(x/y)^2) \cong \operatorname{Spec} k[x/y]$ and the exceptional divisor is $x/y = 0$.

Example 7.13. $\mathcal{O}_F$ be a DVR, t be a uniformizer, k be the residue field. Consider $\mathbb{A}^2_{\mathcal{O}_F} = \operatorname{Spec} \mathcal{O}_F[x, y]$. Consider the subscheme $X = V(xy - t^2) \subset \mathbb{A}^2_{\mathcal{O}_F}$. X is not regular at $(0, 0) =: O$.

Blowing up $\mathbb{A}^2_{\mathcal{O}_F}$ at O whose ideal is (x, y, t) we have 3 affine charts on Bl_O :
$U^t := \operatorname{Spec} \mathcal{O}_F[x/t, y/t]$ and $Bl_O \cap U^t \cong \operatorname{Spec} \mathcal{O}_F[x/t, t/x]$
$U^x := \operatorname{Spec} \mathcal{O}_F[t/x, y/x]$ and $Bl_O \cap U^x \cong \operatorname{Spec} \mathcal{O}_F[x, t/x]$ this is regular
$U^y := \operatorname{Spec} \mathcal{O}_F[t/y, x/y]$ and $Bl_O \cap U^y \cong \operatorname{Spec} \mathcal{O}_F[y, t/y]$ which is again regular.

8. Cech cohomology

Let X be a scheme and $\mathcal{F}$ be a quasicoherent $\mathcal{O}_X$ module then the cohomology of $\mathcal{F}$ means the derived functors H^i of the global sections functor of $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$.

Desired properties:

1. $H^i : QCoh_X \to Mod_{\Gamma(X,\mathcal{O}_X)}$ is a covariant additive functor

(2) $H^0 = \Gamma$
(3) A short exact sequence in $QCoh_X$ gives rise to a long exact sequence in cohomology.
(4) if $\pi : X \to Y$ is a quasi compact quasi separated morphism of schemes then there is a natural map $H^i(Y, \pi_* \mathcal{F}) \to H^i(X, \mathcal{F})$
(5) π affine $\implies H^i(Y, \pi_* \mathcal{F}) \to H^i(X, \mathcal{F})$ is an isomorphism.
(6) if X is covered by n open affines then $H^i(X, \mathcal{F}) = 0 \, \forall \, i \geq n$ in particular if X itself is affine then all the higher cohomologies vanish.
(7) cohomology commutes with filtered colimits: $H^i(X, \varinjlim_j \mathcal{F}_j) = \varinjlim_j H^i(X, \mathcal{F}_j)$.

Fact: The derived functor cohomology is computed by Cech cohomology and all of these are achieved, when X is separated and quasi compact.

We will assume X is separated and quasi compact from now on.

Let $\mathcal{F} \in QCoh_X$ and $\mathcal{U} = \{U_i\}_{i=1}^n$ be an affine covering of X. For $I \subset \{1, 2, \cdots, n\}$ let $U_I = \bigcap_{i \in I} U_i$.

The check complex:

$$C^{p-1}(\mathcal{U}) := \prod_{\#I = p} \mathcal{F}(U_I)$$
$$d : C^{p-1}(\mathcal{U}) \to C^p(\mathcal{U})$$

Notes missing after this point :(