Aim to construct TFT ?

ob: finite sets of ordered sets :D

mor: bordisms

make things interesting by allowing singularities :D 

bordisms - connected graphs with p-inputs q-outputs

$$F: \text{Bord} \longrightarrow \text{Vect}$$
$$p \longmapsto H_*(M)^{\otimes p}$$

$\{\Gamma$- graph p-inputs, q-ouputs, basept, $B_1(\Gamma) = b\} = \mathcal{C}_{b,p+q}$ cat of such Γ.

Want: $q_\Gamma : H_*^{?}(M)^p \longrightarrow H_*^{?}(M)^q$

Idea: Construct a space of Γ shaped flows in M : $\mathcal{M}_p(M)$

$ev_{in} : \mathcal{M}_p(M) \longrightarrow M^p$

$ev_{out} : \mathcal{M}_q(M) \longrightarrow M^q$ and unkehr maps. !

$$q_\Gamma := (ev_{out})_! \cdot (ev_{in})_*$$

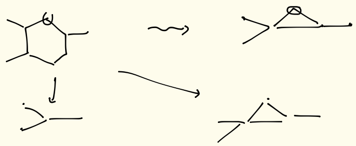
Def: $\mathcal{C}_{b,p+q}$: mor $\Gamma \longrightarrow \Gamma'$

maps of graphs such that

- preimage of vertex is a tree
- preimage of edge is an edge
- basepts preserved

aut $(\Gamma) = \{\Gamma \to \Gamma\}$

ex:



non ex:

Don't understand this.

Def^n: $\mathcal{C}/\Gamma$ is the category ??

Th^m: $|\mathcal{C}/\Gamma|$ is the space of metrics on graphs over Γ.

Pr: point in $|C/\Gamma|$

$$\vec{t}:(t_0 \cdots t_k) \in \Delta^k$$

$$\Gamma_k \xrightarrow{\psi_k} \Gamma_{k-1} \longrightarrow \cdots \longrightarrow \Gamma_1 \xrightarrow{\psi_0} \Gamma_0$$

$E \in \Gamma_r$, $\ell(E) = \sum t_i \lambda_i(E)$

$$\lambda_i(E) = \begin{cases} 1 & \text{if } E \text{ survives to } \Gamma_i \\ 0 & \text{else} \end{cases}$$

$$S_\Gamma := \underset{\mathcal{C}/\Gamma}{\mathrm{hocolim}} \left(\Gamma_k \longmapsto \mathrm{Conf}_{e(\Gamma_k)}(C^\infty(M)) \right)$$

edges

Note this is contractible. WTF.

$\mathrm{Aut}(\Gamma) \circlearrowright \mathcal{C}/\Gamma$ freely and hence also on S_Γ and so

$$S_\Gamma/_{\mathrm{Aut}(\Gamma)} \cong B\,\mathrm{Aut}(\Gamma)$$

$\widetilde{\mathcal{M}}_\Gamma(M)$ = space of flows in M

$$= \{ (\sigma, \gamma) \mid \sigma \in S_\Gamma , \gamma: \Gamma_k \to M \}$$

$$\sigma = (\vec{t}, \vec{\psi}, c)$$

s.t. $\gamma_E : [0, \ell(E)] \longrightarrow \Gamma \xrightarrow{\gamma} M$

satisfies $\frac{d\gamma_E}{dt}(s) + \nabla f_E(\gamma(s)) = 0$.

$$\mathcal{M}_\Gamma(M) = \widetilde{\mathcal{M}}_\Gamma(M)/_{\mathrm{Aut}(\Gamma)}$$

so get evaluation maps $ev_{in/out} : \widetilde{\mathcal{M}}_\Gamma(M) \longrightarrow M^{p/q}$

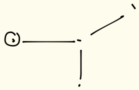
$$ev_{in/out} : \mathcal{M}_\Gamma(M) \xhookrightarrow{\ \ } \mathcal{M}_\Gamma^T(M) \simeq S_\Gamma/_{\mathrm{Aut}(\Gamma)} \times M \xrightarrow{\Delta ?} E\mathrm{Aut}(\Gamma) \times_{\mathrm{Aut}(\Gamma)} M^{p/q}$$

If Γ is a tree

$$\widetilde{\mathcal{M}}_\Gamma(M) \cong S_\Gamma \times M \quad \longleftarrow \quad \text{Topologize } \mathcal{M}_\Gamma(M)$$

$$(\sigma, \gamma) \longmapsto \sigma \times \gamma(v)$$

$$\widetilde{\mathcal{M}}_\Gamma^T(M) = \{ (\sigma, \{\gamma_T\}) \mid T \text{ all maximal trees in } \Gamma ; \gamma_{T'}(v) = \gamma_T(v) \} \cong S_\Gamma \times M.$$

ex:  $= \Gamma$, $\mathrm{Aut}(\Gamma) = \mathbb{Z}/2$

$$\begin{tikzcd}
q^\Gamma : H_*^{\mathbb{Z}/2}(M) \arrow[r, "ev_{in}"] \arrow[d, phantom, "\cong" rotate=90] & H_*(\mathcal{M}_\Gamma(M)) \arrow[r, "(ev_{out})_*"] \arrow[d, phantom, "\ncong" rotate=90] & H_*^{\mathbb{Z}/2}(M^2) \\
H_*(B\mathbb{Z}/2) \otimes H_*(M) & & \text{equivariant diagonal} \arrow[ul, shorten=2em]
\end{tikzcd}$$

$$(q_\Gamma)^* : H^*_{\mathbb{Z}/2}(M^2) \longrightarrow H^*(B\mathbb{Z}/2) \otimes H^*(M)$$

$$(q_\Gamma)^*(\alpha \otimes \alpha) := \sum a^i \otimes Sq^{?}(\alpha) \qquad a \text{ generates } H^i(B\mathbb{Z}/2)$$

$$\alpha \in H^q(M)$$