

Manifold Calculus and the h -principle

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Abstract

in this project we use manifold calculus and h -principle to study a problem in symplectic geometry.

manifold calculus + h -principle \rightsquigarrow symplectic geometry

1 Motivation

a symplectic manifold	is a pair (N, ω) where N is a smooth manifold of dimension $2m$ and $\omega \in \Omega_{DeRham}^2(N)$ satisfying <ol style="list-style-type: none"> 1. $d\omega = 0$ 2. ω^m is nowhere vanishing ω defines an isomorphism $\omega : TM \rightarrow T^*M$
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Example 1.1. $N = \mathbb{R}^{2m}$
 $\omega = \sum_{i=1}^m dp_i \wedge dq_i$

Example 1.2. $N = T^*M$
 ω (locally) $= \sum_{i=1}^m dp_i \wedge dq_i$

where q_i are the base coordinates and p_i are the tangential coordinates. in classical mechanics, T^*M is the phase space, q_i are the position coordinates and p_i are the momentum coordinates.

Darboux's theorem says that *every* symplectic manifold is locally symplectomorphic to \mathbb{R}^{2m} with the standard symplectic form, and so symplectic manifolds have no *local invariants* but they do have very non-trivial *global invariants*; see [Cds01]. many global invariants are constructed out of Lagrangian submanifolds.

a Lagrangian submanifold	is a submanifold $M \subseteq N$ of dimension m such that $\omega _M \equiv 0$ ¹
$\text{Emb}_{\text{Lag}}(M, N)$	is the space of Lagrangian embeddings $M \hookrightarrow N$

The *Nearby Lagrangian Conjecture* (still open) due to Arnol'd has been a guiding question for several recent advances in symplectic geometry. The current state of the art results about the Nearby Lagrangian Conjecture rely on a combination of homotopy theoretic and Floer theoretic techniques.

Conjecture 1.3 (Arnold's Nearby Lagrangian Conjecture). *If L and M are simply connected closed manifolds of dim m , then the space*

$$\text{Emb}_{\text{Lag}}(L, T^*M) \text{ is } \begin{cases} \text{contractible} & \text{if } L \cong M, \\ \text{empty} & \text{otherwise.} \end{cases}$$

the goal of this project is to apply homotopy theoretical methods (manifold calculus) to study $\text{Emb}_{\text{Lag}}(M, N)$.

¹basic example is the position space inside a phase space, in classical mechanics.

2 Statement of the Theorem

Man	the Spaces -enriched category of smooth manifolds of a fixed dimension m , with morphisms being the embedding spaces
Disc_∞	the full subcategory of Man consisting of manifolds which are diffeomorphic to disjoint union of \mathbb{R}^m
Psh(Man)	space valued presheaves on Man , functors $\mathbf{Man}^{op} \rightarrow \mathbf{Spaces}$
$T_\infty F$	right derived Kan extension of $F \in \mathbf{Psh}(\mathbf{Man})$ along the inclusion $\mathbf{Disc}_\infty^{op} \hookrightarrow \mathbf{Man}^{op}$

$$\begin{array}{ccc}
 \mathbf{Disc}_\infty^{op} & \xrightarrow{F|_{\mathbf{Disc}_\infty^{op}}} & \mathbf{Spaces} \\
 \downarrow & \Downarrow & \nearrow T_\infty F \\
 \mathbf{Man}^{op} & &
 \end{array}$$

we say that $T_\infty F$ is the **analytic approximation** of F .

See [Wei99], [BdBW12], [MV15].

Theorem 2.1 (N., [Nak17]). *let (N, ω) be a symplectic manifold with dimensions $n = 2m$. for $m > 2$ there is a homotopy equivalence*

$$T_\infty \text{Emb}_{\text{Lag}}(-, N) \simeq \text{Emb}_{\text{TR}}(-, N)$$

where

$\text{Emb}_{\text{Lag}}(-, N)$ is the space of Lagrangian embeddings and
 $\text{Emb}_{\text{TR}}(-, N)$ is the space of totally real embeddings.

2.1 Totally Real Submanifolds

a compatible almost complex structure on (N, ω)	<p>is a vector space automorphism $J : TN \rightarrow TN$ with $J^2 = -1$ such that</p> <ol style="list-style-type: none"> 1. $\omega(J-, -)$ defines a Riemannian metric on N, 2. $\omega(J-, J-) = \omega(-, -)$. <p>there exists a unique compatible almost complex structure up to homotopy, we'll assume to have chosen one</p>
a totally real manifold	is a submanifold $M \subseteq N$ of dimension m such that $TM \oplus J(TM) \cong TN _M$ ²
$\text{Emb}_{\text{TR}}(M, N)$	<p>is the space of totally real embeddings $M \hookrightarrow N$</p> <p>by compatibility, every Lagrangian submanifold is also a totally real submanifold,</p> $\text{Emb}_{\text{Lag}}(M, N) \subseteq \text{Emb}_{\text{TR}}(M, N)$

3 Proof

following are the steps to prove 2.1:

1. $\text{Imm}_{\text{Lag}}(M, N) \xrightarrow{\simeq} \text{Imm}_{\text{TR}}(M, N)$, see [EM01].
 - immersions are linear, in the sense of manifold calculus
 - so the homotopy type is completely determined by the tangential data
 - on the level of tangent spaces the result follows by the homotopy equivalence $U(m)/O(m) \xrightarrow{\simeq} GL(m, \mathbb{C})/GL(m, \mathbb{R})$, see [Arn67].
2. $T_{\infty} \text{Emb}_{\text{Lag}}(M, N) \xrightarrow{\simeq} T_{\infty} \text{Emb}_{\text{TR}}(M, N)$

²the basic example is $\mathbb{R}^m \subseteq \mathbb{C}^m$.

- this is because of the following homotopy pullback diagram

$$\begin{array}{ccc}
 T_\infty \text{Emb}_{\text{Lag}}(M, N) & \longrightarrow & T_\infty \text{Emb}_{\text{TR}}(M, N) \\
 \downarrow & & \downarrow \\
 \text{Imm}_{\text{Lag}}(M, N) & \longrightarrow & \text{Imm}_{\text{TR}}(M, N)
 \end{array}$$

$$3. \text{Emb}_{\text{TR}}(M, N) \xrightarrow{\simeq} T_\infty \text{Emb}_{\text{TR}}(M, N)$$

- this last step in the proof uses Gromov's *h*-principle for directed embeddings. See [Gro86], [Spr98].

4 Gromov's *h*-principle

$\text{Gr}_m(N) \rightarrow N$	is the m -plane Grassmannian bundle over N
$\text{Lag} \subseteq \text{Gr}_m(N)$	subbundle corresponding to the Lagrangian subspaces of TN
$\text{TR} \subseteq \text{Gr}_m(N)$	subbundle corresponding to the totally real subspaces of TN
	by compatibility, $\text{Lag} \subseteq \text{TR}$.

an embedding $e : M \hookrightarrow N$ induces a map

$$\text{Gr}_m(e) : M \rightarrow \text{Gr}_m(TN)$$

when e is a totally real (resp. Lagrangian) embedding the image of $\text{Gr}_m(e)$ lies in TR (resp. Lag).

Gromov's idea (originally by Smale) was to consider the space of **formal totally real sections** consisting of triples (e, s, γ)

e | an embedding $M \rightarrow N$

s | a section of TR over $e(M)$

γ | a path of sections of $\text{Gr}_m(TN)$ over $e(M)$ from $De(TM)$ to s

a formal section (e, s, γ) is called **holonomic** if $s = De(TM)$ and γ is the constant path.

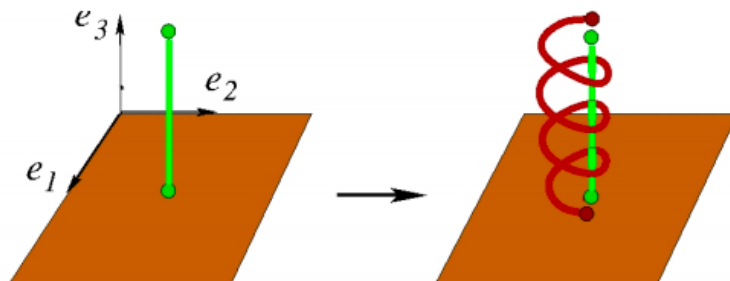


Figure 1: an example of h -principle. in the above image e is a vertical embedding of a segment and E is a horizontal vector field. image taken from <http://math.univ-lyon1.fr/~borrelli/Diablerets/1-diablerets.pdf>.

Gromov showed using his technique of convex integration that the space of formal totally real sections is homotopy equivalent to the space of holonomic totally real sections.

$$\{ \text{formal totally real sections} \} \simeq \{ \text{holonomic totally real sections} \}$$

this is *false* for lagrangian sections. this is because the space TR is *large* but Lag is *small*. more specifically, the complement of TR inside $\text{Gr}_m(N)$ is a variety of codimension ≥ 2 (it is the complement of a thin singularity).

5 Future Work

1. how to incorporate Floer theoretic data in the Goodwillie-Weiss tower?
2. using manifold calculus to study foliations (which are studying using h -principle)
3. extending the above techniques to manifolds with a group action (current work in progress).

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