- $T_{ij} * \varepsilon = \sum_k T_{ik} \varepsilon_{kj}$
- $(\mathrm{Ad}\, x)\varepsilon = \left(\sum_{k,l} x_{ik} \varepsilon_{kl} x^{-1}_{lj} \right)_{ij}$

$$(\mathrm{Ad}x)\,\varepsilon\, T_{ij} = \left(\rho_x(*\varepsilon)\rho_x^{-1}\right).T_{ij}$$
$$= \rho_x(*\varepsilon) \sum_k T_{ik} x^{-1}_{kj}$$
$$= \rho_x \sum_k (T_{ik} * \varepsilon) x^{-1}_{kj}$$
$$= \ldots$$

finally we get
$$(\mathrm{Ad}\, x)\,\varepsilon = x \varepsilon x^{-1}$$

-- Kannan

Goal: G affine algebraic group / $K = \bar{K}$
H closed subgroup of G.
To view G/H as a quasi-projective variety $\longrightarrow$ open subvariety of a projective variety
i.e. locally closed subset of projective space

$I_H = \{ f \in K[G] \mid f(h) = 0 \ \forall h \in H \}$
$G \circlearrowright K[G]$ via $(g.f)(x) = f(g^{-1}x) =: (\lambda_g f)(x)$

$(\rho_g f)(x) =$

Lemma: $H = \{ g \in G \mid \rho_g(I_H) = I_H \}$
$= \{ g \in G \mid \lambda_g(I_H) = I_H \}$

Proof: $\subseteq$
$\rho_h(I_H)$ $\quad \rho_h f(x) = f(xh)$ for $x \in H$,
$= 0$
$\Rightarrow \rho_h f \in I_H$

$\supseteq$
$h \in \{ \ \} \Rightarrow \rho_h f(x) = 0 \quad \forall x \in H \quad \forall f \in I_H$
$\Rightarrow f(xh) = 0 \quad \forall x \in H \quad \forall f \in I_H$
$\Rightarrow$ in particular $f(h) = 0 \quad \forall f \in I_H$
$\Rightarrow h \in V(I_H)$ ⟵ zero set of I_H
$\quad\quad V(I_H) = H$

Lemma: V finite dim vector space,

- $v_1, \dots, v_r \in V$, $v_1 \wedge v_2 \wedge \dots \wedge v_r = 0$ in $\Lambda^r V$

  $\Leftrightarrow$ $v_1, v_2, \dots, v_r$ are lin. independant

- $\dim \Lambda^r V = \binom{n}{r}$

Proof:

$- \wedge v_r : \Lambda^{r-1} V \longrightarrow \Lambda^r V$

. There is a natural action of $Gl(V)$ on $\Lambda^r V$:

$$g(v_1 \wedge \dots \wedge v_r) = g v_1 \wedge g v_2 \wedge \dots \wedge g v_r$$

(This is because g stabilizes $Sym(V)$ under this action)

$\Lambda^r g$ is this element of $Gl(\Lambda^r V)$.

Lemma: V - vector space$/K$ $M < V$, $\dim M = r$

$x \in GL(V)$, $L = \Lambda^r M$, Then,

1) $L < \Lambda^r V$, $\dim L = 1$

2) $xM = M$ $\Leftrightarrow$ $(\Lambda^r x) L = L$

Proof :

1) easy

2) $\Rightarrow$ easy

$\Leftarrow$ $v_1, \dots v_r \in M$ basis

$\Rightarrow$ $v_1 \wedge \dots \wedge v_r \in \Lambda^r L$ basis for $\Lambda^r L$

$\Rightarrow$ $x v_1 \wedge \dots \wedge x v_r = c \cdot v_1 \wedge \dots \wedge v_r$

~~if $v \in xM$~~ ,

$0 = x v_1 \wedge \dots \wedge x v_r \wedge x v_r = c (v_1 \wedge \dots \wedge v_r) \wedge x v_r$

$\Rightarrow$ $x v_r \in \langle v_1, \dots, v_r \rangle = M$

$\Rightarrow$ $xM \subseteq M$ $\Rightarrow$ $xM = M$.

Th^m (Chevalley)

There is a rational representation $\rho : G \longrightarrow Gl(V)$ and a 1-dim subspace L of V

such that, $H = \{ g \in G \mid \rho(g) L = L \}$

Def^n: Rational representation : $\rho : G \longrightarrow Gl(V)$

Example:

- G_m – Then any representation of G_m is of the form

$$t \mapsto \begin{pmatrix} t^{m_1} & & 0 \\ & \ddots & \\ 0 & & t^{m_n} \end{pmatrix} \qquad m_i \in \mathbb{Z}$$

Any finite subgroup of G_m is of the form $\langle e^{\frac{2\pi i}{r}} \rangle$

Ex: Find the stable subgroup in this case.

- G_a – $\rho : G_a \longrightarrow Gl_2\mathbb{C}$, $z \longmapsto \begin{pmatrix} 1 & z \\ & 1 \end{pmatrix}$. Then $1 \in G_a$ corresponds to the subspace $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Note $\langle 1 \rangle$ is the only subgroup of G_a

- $G = Sl_2\mathbb{C}$
  - H = upper triangular matrices in $Sl_2\mathbb{C}$

$$= \left\{ \begin{pmatrix} t & z \\ 0 & t^{-1} \end{pmatrix} : t \in \mathbb{C}^*,\ z \in \mathbb{C} \right\}$$

$\rho : Sl_2\mathbb{C} \longrightarrow Gl_2\mathbb{C}$ natural inclusion

~~we need~~ ?

$L = \mathbb{C} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the required one dimensional subspace of V

- $H = \left\{ \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \mid z \in \mathbb{C} \right\}$

The above representation will not work

Take $V = (\mathbb{C}^2)^* \oplus \underbrace{Sym^2(\mathbb{C}^2)^*}$ ⇝ dual space of the space of all homogenous polynomials of degree 2 in 2 variables ~~symmetric~~

$$\mathbb{C}^2 \otimes \mathbb{C}^2 \longrightarrow Sym^2(\mathbb{C}^2) \to 0$$

$= \langle x^2, y^2, xy \rangle$

~~$L = \mathbb{C}\begin{pmatrix} 1 \\ a \end{pmatrix} \oplus \ \mathbb{C} \oplus \mathbb{C} x^2$~~

~~$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$~~

$L = \mathbb{C}(x + x^2)$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

$$\lambda(x + x^2) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}(x + x^2) = (ax + by) + (ax + by)^2$$

$$\Rightarrow \quad a^2 = a = \lambda, \quad b = 0$$

Ex] Interesting example. Think.

- $H =$ monomial matrices

$$= \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 0 & -t \\ t^{-1} & 0 \end{pmatrix} \right\} \quad t \in \mathbb{C}^*$$

$$H^0 = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right\}$$

Claim: H is the normalizer of H^0

Take $V =$ space of all homogenous polynomials of degree 2 in 2 variables

$$= Sym^2(\mathbb{C}^{2*})$$

$L = \mathbb{C}xy$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} xy = (ax+by)(cx+dy)$$

$\quad \parallel$

$\lambda xy \qquad \Rightarrow ac = 0, \quad bd = 0$

$\Rightarrow$ monomial matrices!

So L is the subgroup corresponding to H.

What about H^0?

$\mathrm{Hom}_{grp}(Sl_2\mathbb{C} \longrightarrow \mathbb{C}^*)$ is trivial

Because $[Sl_2\mathbb{C}, Sl_2\mathbb{C}] = Sl_2\mathbb{C}$ and $\mathbb{C}^*$ is abelian

$V =$ Space of homogenous ~~forms~~ polynomials of degree 3 in 2 variables

$L = \mathbb{C}.x^2y$

$$\lambda x^2 y = (ax+by)^2(cx+dy)$$

$$= a^2cx^3 + b^2dy^3 + \ldots$$

$a^2c = 0 \qquad a = 0, \; d \neq 0 \qquad b = 0, c = 0$

$b^2d = 0 \qquad b^2 \cancel{cy^2x} \qquad a^2d\,xy$

$\downarrow$

diagonal matrices!

Idea of proof: $W' =$ finite dim subspace generating I_H as an algebra

$W =$ G-span of W', is finite dimensional

$M = W \cap I \qquad$ Claim: $H = \{x \mid \rho_x M = M\}$

Use th^m previously stated to get $L = \Lambda^r M$

Then $(\Lambda^r \rho_x) L = L \iff x \in H \qquad H = \{x \in G \mid (\Lambda^r \rho_x) L = L\}$

- For G_m:

  $H =$ solution set of $x^k - 1 = 0$

  Then look at the representation:

  $$\rho : G_m \longmapsto Gl(2,\mathbb{C})$$
  $$t \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & t^k \end{pmatrix}$$

  $L = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

  Stabilizer of L: $\rho(t)\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, where $\rho(t)\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2t^k \end{pmatrix}$

  $\Rightarrow \lambda = 1, \quad 2t^k = 2 \Rightarrow t^k = 1$

- For $Sl_2(\mathbb{C})\ (=G)$

  $\mathbb{C}[SL_2\mathbb{C}] = \mathbb{C}[T_{11}, T_{12}, T_{21}, T_{22}] \big/ T_{11}T_{22} - T_{12}T_{21} - 1$

  $H = \left\{ \begin{bmatrix} t & z \\ 0 & t^{-1} \end{bmatrix} \right\}$ G-spans

  $I_H = \langle T_{21} \rangle$ $\quad G \cdot I_H = \langle T_{11}, T_{21} \rangle$

  $H = \left\{ \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} \right\}$

  $I_H = \langle T_{21}, T_{11} - 1 \rangle$

  $H = \left\{ \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}, \begin{bmatrix} 0 & t \\ t^{-1} & 0 \end{bmatrix} \right\}$

  $I_H = \langle T_{11}T_{12}, T_{21}T_{22} \rangle$

  $H = \left\{ \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} \right\}$

  $I_H = \langle T_{12}, T_{21} \rangle$

- Action of $Sl_2\mathbb{C}$ on $\mathbb{C}[Sl_2\mathbb{C}]$

  $$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = \begin{bmatrix} aT_{11} + bT_{21} & aT_{12} + bT_{22} \\ cT_{11} + dT_{21} & cT_{12} + dT_{22} \end{bmatrix}$$