# Extended field theories: — Brian Williams

$\mathrm{Bord}_2^{SO}$ : Objs — manifolds oriented

1-mor — oriented manifolds with boundary

2-mor — oriented manifolds with corners

higher morphisms invertible

## Tangential Structures:

$B \to BO(n)$ fibration,

$B$ framing of $m$-manifold $M^m$:

$$\begin{tikzcd} M \arrow[r, "TM"'] \arrow[drr, dashed] & BO(m) \arrow[r, "\times \mathbb{R}^{n-m}"'] & BO(n) \\ & & B \arrow[u] \end{tikzcd}$$

B-framing

ex: $G \to O(n)$

1) $G = SO(n)$, $m=n$ — B-structure $=$ orientation

2) $G = \mathrm{Spin}(n)$, $m=n$ — Spin-structure

3) $G = *$ — Framing of $TM$ i.e. trivialization of $TM$.

4) $m = n-1$, $G = *$, $n=3$

$S^2$ has 3 framing but no 2-framing.

**Def$^n$:** $\mathrm{Bord}_n^G$ Bordism category of $G$ structures.

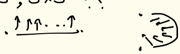
**Note:** $* = G$ Call $\mathrm{Bord}_n^G = \mathrm{Bord}_n^{fr}$

$* \to G \to O(n)$

$\mathrm{Bord}_n^{fr} \to \mathrm{Bord}_n^G \to \mathrm{Bord}_n$

ex: $\mathrm{Bord}_2^{fr}$ obj $= +, -, S^1, S^1 \times S^1$ ??

1-morp =



## Cobordism Hypothesism:

$\{$framed TFT's$\} \longleftrightarrow \{$fully dualizable objects in $\mathcal{C}\}$ — $\mathcal{C}$- symmetric monoidal $(\infty, n)$ category.

- $\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^{fr}, \mathcal{C}) \xrightarrow[\simeq]{ev} (\mathcal{C}^{fd})^{\sim}$

  fully dualizable

ex: $\alpha: F \longrightarrow F'$

Want to show $\alpha_M: F(M) \to F'(M)$ is an equivalence

$$F'(M) \simeq F'(\bar{M})^{\vee} \xrightarrow[\alpha_{\bar{M}}^{\vee}]{} F(\bar{M})^{\vee} \simeq F(M).$$

Cor: $\mathcal{C}^{fd}$ as above, carries an action of $O(n)$, and hence of $G$ via $G \to O(n)$.

Th$^m$: Tangential Cobordism Hypothesis:

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n^{G}, \mathcal{C}) \simeq (\mathcal{C}^{fd})^{G}$$

Fully dualizable:

$\mathcal{C}$ = symmetric monoidal $(\infty,2)$ category

$X \in \mathcal{C}$ fully dualizable $\Leftrightarrow$ $\exists\, X^{\vee}$ $\quad ev: X \otimes X^{\vee} \longrightarrow 1$

$\rightsquigarrow$ s.t. $\quad coev: 1 \longrightarrow X^{\vee} \otimes X$

1) $X \longrightarrow X \qquad (ev \otimes id) \circ (id \otimes coev_X) = id_X$

2) $ev$, $coev$ have left, right adjoints

Prop: $X$ is fully dualizable iff $X^{\vee}$ exists & $ev$ has both adjoints.

Leads to Serre Automorphism.

Extended 2-TFT:

$\mathrm{Bord}_2^{fr}$ : $+, -$ $\qquad (+)^{\vee} = -$

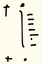
$ev_+$:  $\qquad coev_+$: 

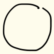
$ev_+^R : \phi \longrightarrow + \sqcup -$ 

$ev_+ \circ ev_+^R \longrightarrow id_{\phi}$ 

$id_{\phi} \longrightarrow ev_+^L \circ ev_+$ 

## Serre Automorphism:

 $\in (+,+)$ Twist:  $= S_+$ $S_+$ generates the group $\mathrm{Aut}(+)$

 $T_\pi S^1 \times \mathbb{R} \xrightarrow{f,g} \mathbb{R}^2$

$f \cdot g^{-1} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \rightsquigarrow$ element $\Omega \overline{Gl_2 \mathbb{R}} \simeq \Omega SO(2) = \mathbb{Z}$

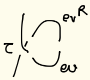
$\rightsquigarrow$ framings of $S^1$ form a $\mathbb{Z}$-torsor

$S^1_n$ – Let $S^1_1$ is the framing coming from an orientation

$S^1_0$ the other one.

$SO(2)$ acts on $\mathcal{C}^{fd}$, $SO(2) \longrightarrow B\mathrm{Aut}((\mathcal{C}^{fd})^\sim)$

Consider $\mathcal{C} = \mathrm{Bord}_2^{fr}$. This action gives an automorphism:

 $=$  $\rightsquigarrow$ $X \to X \otimes 1 \xrightarrow{ev^R} X \otimes X \otimes X^\vee \xrightarrow{\tau} X \otimes X \otimes X^\vee \xrightarrow{ev} X$

$\overrightarrow{S,\ \text{Serre Automorphism}}$

Formulas:
$$ev_X^R = (S \otimes id_{X^\vee}) \circ coev_X$$
$$ev_X^L = (S^{-1} \otimes id_{X^\vee}) \circ coev_X$$

ex: $\mathcal{C} = \mathrm{Alg}_{\mathbb{C}}^{(2)}$ Ob: $\mathbb{C}$-algebras

1-mor: bimodules

2-mor: maps of bimodules

$\forall A \in \mathcal{C}$ is dualizable, $A^\vee = A^{op}$

ev: ${}_{A \otimes A^{op}} A_{\mathbb{C}}$ coev: ${}_{\mathbb{C}} A_{A^{op} \otimes A}$

$F_A : \mathrm{Bord}_2^{fr} \longrightarrow \mathrm{Alg}_{\mathbb{C}}^{(2)}$

$+ \longmapsto A$

$F_A(S^1) = F_A(ev_A \circ coev_A) = A \otimes_{A \otimes A^{op}} A$

Q. When is $A \in \mathrm{Alg}_{\mathbb{C}}^{(2)}$ fully dualizable? $ev_A^L$, $ev_A^R$

Want $\mathrm{Hom}_{A\otimes A^{op}}(ev_A^L \otimes M, N) \cong \mathrm{Hom}(M, \quad A_{A\otimes A^{op}} \otimes N)$

similarly for $ev_A^R$.

This implies that $ev_A^R = \mathrm{Hom}(A, \mathbb{C})$

$$ev_A^L = \mathrm{Hom}_{A\otimes A^{op}}(A, A\otimes A^{op})$$

Suffices for $A$ to be semisimple.

$S_A = {}_A A^*_A$

If we want oriented theory we need trivialization of $S_A$

i.e. ${}_A A^*_A \cong {}_A A_A$

$\Rightarrow$ $A$ is a Frobenius algebra!

$$\left\{\begin{array}{l}\text{extended}\\ \text{oriented}\\ \text{2-TFTs}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\cdot\ \text{semisimple} \\ \cdot\ \text{Symmetric} \quad Tr(ab) = Tr(ba)\end{array} \quad \text{Frobenius algebras}\right\}$$

not necessarily commutative.