Complex oriented Cohomology theories

A complex oriented cohomology theory is a theory for which complex vector bundles are orientable.
More precisely,

E_* a multiplicative homology theory. E^* is complex oriented if $\forall$ $\begin{array}{c}\mathbb{C}^n \longrightarrow V \\ \downarrow \\ X\end{array}$ complex vector bundles,
$\exists$ a natural Thom class $u_V \in \tilde{E}^{2n}(X^V)$ X^V = Thom Space
satisfying the usual Thom class axioms:

- Thom isomorphism:

$$E^*X \xrightarrow{\cong} E^*V \xrightarrow{\cup u_V} \tilde{E}^{*+2n}(X^V)$$

- If $\gamma \to \mathbb{CP}^1$ is the canonical line bundle then under the map

$$\tilde{E}^2(\mathbb{CP}^{1^\gamma}) = \tilde{E}^2(\mathbb{CP}^2) \longrightarrow E^2(\gamma) \cong E^2(\mathbb{CP}^1) \cong E^0(S^0)$$

u_V maps to 1.

- H^*, K^*, MU^* are complex oriented
- KO^* is not. To see this consider $\gamma \longrightarrow \mathbb{CP}^1$

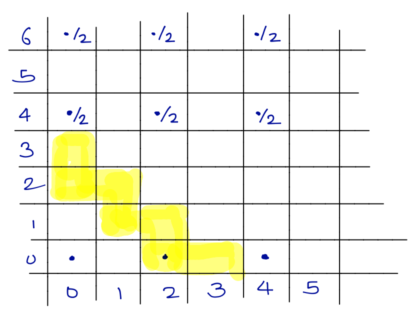
$$\widetilde{KO}^2(\mathbb{CP}^2) \longrightarrow KO^2(\mathbb{CP}^1) \cong KO^2(S^2) \cong KO^0(S^0)$$

n	$\pi_n(KO) = [\tilde{S}^n, KO] = \widetilde{KO}^{-n}(S^0)$
0	$\mathbb{Z}$
1	$\mathbb{Z}/2$
2	$\mathbb{Z}/2$
3	0
4	$\mathbb{Z}/2$
5	0
6	0
7	0
8	$\mathbb{Z}$

We have LES

$$\begin{array}{ccccccccccc}
& & S^4 & & & & & & S^4 & & \\
& & \| & & 0 & & & & \| & & 0 \\
KO^1(\mathbb{CP}^1) & \longrightarrow & \widetilde{KO}^2(\mathbb{CP}^2/\mathbb{CP}^1) & \xrightarrow{0} & KO^2(\mathbb{CP}^2) & \longrightarrow & KO^2(\mathbb{CP}^1) & \longrightarrow & \widetilde{KO}^3(\mathbb{CP}^2/\mathbb{CP}^1) & \xrightarrow{0} & KO^3(\mathbb{CP}^2) \\
\cong & & \cong & & \cong & & \cong & & \cong & & \| \\
KO^{-1}(S^0) & & KO^{-2}(S^0) & & \mathbb{Z} & & \mathbb{Z} & & KO^{-1}(S^0) & & 0 \\
\cong & & \cong & & & & & & \cong & & \\
\pi_1(KO) & & \pi_2(KO) & & & & & & \pi_1(KO) & & \\
\cong & & \| & & & & & & \cong & & \\
\mathbb{Z}/2 & & \mathbb{Z}/2 & & & & & & \mathbb{Z}/2 & &
\end{array}$$

The AHSS for $KO^*(\mathbb{CP}^2)$ gives $KO^2(\mathbb{CP}^2) \cong \mathbb{Z}$
$KO^3(\mathbb{CP}^2) \cong 0$



And so the required map is multiplication by 2 and hence 1 is not in the image.

- Euler Class:

$$\tilde{E}^{2n}(X^V) = E^{2n}(V, V \setminus \{0\}) \longrightarrow E^{2n}(V)$$

The image of the Thom class u_V is the Euler class $e(V)$.

Formal group laws:

- The AHSS gives us

$$E^*(\mathbb{CP}^\infty) \cong E^*[[e(\gamma)]] =: E^*[[x]]$$

$$E^*(\mathbb{CP}^{\infty \times n}) \cong E^*[[e(\gamma_1), e(\gamma_2), \cdots, e(\gamma_n)]] =: E[[x_1 \cdots x_n]]$$

- The tensor product of line bundles gives us the formal group law over E^*:

$$\begin{array}{ccc} \mathbb{CP}^\infty \times \mathbb{CP}^\infty & \longrightarrow & \mathbb{CP}^\infty \\ \| \quad\quad \| & & \| \\ BU(1) \times BU(1) & \longrightarrow & BU(1) \end{array}$$

multiplication of complex numbers

induces

$$\begin{array}{ccc} E^*(\mathbb{CP}^\infty) & \longrightarrow & E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \\ \| & & \| \\ E^*[[x]] & & E^*[[x,y]] \\ x & \longmapsto & f(x,y) =: x +_E y \end{array}$$

- $x +_E y$ is the formal group law over E^*

eg: $x +_H y = x + y$

$x +_K y = x + y + xy$

$x +_{MU} y$ is extremely complicated and classifies all the FGL's.

Th^m Quillen ⟵ The Theorem

fgls are co-represented by the complex oriented theory $(MU_*, MU_* MU)$

p-typical fgls are corepresented by a complex oriented theory $(BP_*, BP_* BP)$

$\exists$ a natural functor e which assigns to each fgl F a p-typical fgl G and a natural is $e_F : F \longrightarrow G$, such that if F is p-typical then $e_F = id$. ($e =:$ Cartier idempotent)

Milnor?

$$MU_* \cong \mathbb{Z}[x_1, x_2, \cdots] \qquad |x_i| = 2i$$

$$MU_* MU \cong MU_*[b_1, b_2, \ldots] \qquad |b_i| = 2i$$

$$BP_* \cong \mathbb{Z}_{(p)}[v_1, v_2, \ldots] \qquad |v_i| = 2(p^i - 1)$$

$$BP_* BP \cong BP_*[t_1, t_2, \cdots] \qquad |t_i| = 2(p^i - 1)$$

Landweber exactness:

Let F be a fgl over A given by a map $F : MU_* \longrightarrow A$ we can ask when is F the fgl of a complex oriented homology theory?

The candidate for this is

$$X \longmapsto A \underset{L}{\otimes} MU_* X$$

Q: When is this a homology theory? does $\exists\, \mathbb{E}$. $\mathbb{E}_* X = [S, \mathbb{E} \wedge X] \cong A \underset{L}{\otimes} MU_* X$?

Th^m: $A \otimes_L MU_* X$ represents a homology theory if $A \underset{L}{\otimes} -$ is an exact functor in the category of (L, W) modules.

Examples :

- *Johnson Wilson theories:*

$$E(n)_* \cong \mathbb{Z}_{(p)}[v_1, v_2, \cdots, v_n^{\pm 1}]$$

$$[p](x) = px +_F v_1 x^p + \cdots +_F v_n x^{p^n}$$

- *K-theory:*

$$KU_* \cong \mathbb{Z}[v_1^{\pm 1}]$$ v_1 = Bott element

$$x +_G y = x + y + v_1 xy$$ Euler class : $e(\gamma) = v_1^{-1}(\gamma - 1)$ γ = Canonical line bundle over $\mathbb{CP}^\infty$

$$e(\gamma_1 \otimes \gamma_2) = v_1^{-1}(\gamma_1 \otimes \gamma_2 - 1) = v_1(v_1^{-1}\gamma_1 - 1) \otimes (v_2^{-1}\gamma_2 - 1) + (v_1^{-1}\gamma_1 - 1) \otimes 1 + 1 \otimes (v_2^{-1}\gamma_2 - 1)$$

- *Lubin Tate theory:*

$$E_{n*} := \underset{\underset{\textstyle W(\mathbb{F}_{p^n})[[u_1, \ldots, u_{n-1}]]}{\parallel}}{R(\mathbb{F}_{p^n}, \Gamma_n)}[u^{\pm 1}]$$ degree 0

$\exists$ a faithfully flat extension

$$E(n)_* \longrightarrow E_{n*}, \qquad v_i \longmapsto \begin{cases} u_i u^{p^i - 1} & , i < n \\ u^{p^n - 1} & , n = 1 \end{cases}$$

$$E_{n*}E_n \cong \text{maps}(G_n, E_{n*}) \qquad G_n = Gal(\mathbb{F}_{p^n} : \mathbb{F}_p) \rtimes S_n$$

Automorphisms of $F_n \to$ fglp $(\mathbb{Z}/p)$ of ht. n

- *Morava K-theory:* (not Landweber exact)

$$K(n)_* = \mathbb{F}_p[v_n^{\pm 1}]$$

F_n = Honda fgl = unique fglp with $[p](x) = v_n x^{p^n}$

$$K(n)_* K(n) \cong \mathbb{F}_p[v_n^{-1}][t_1, t_2, \cdots] / (t_i^{p^n} - v_n^{p^i - 1} t_i)$$

If we consider an ungraded fglp Γ_n instead of F_n with $[p]_{\Gamma_n}(x) = x^{p^n}$ then the endomorphisms of Γ_n is $S(n)$ = Morava Stabilizer group

$$= \mathbb{F}_p[x_0^{\pm 1}, x_1, \cdots] / (x_i^{p^n} - x_i)$$