Twisted field theories, modular forms and Quillen Connections

- Invariants in top, analysis, QFT
- Modular
- Computations of the easiest twists

- $\hat{A}$-genus Atiyah
- $\hat{W}$-genus Witten

dim 1: $\hat{A}$ genus

$$\begin{array}{ccc} \text{closed oriented genus} & \xrightarrow{\hat{A}} & \mathbb{R} \\ \uparrow & \circlearrowleft & \cup \\ \text{closed spin structure} & \xrightarrow[\text{Dirac op.}]{\text{index of}} & \mathbb{Z} \end{array}$$

Atiyah Singer index theorem

partition function. $= ev(S_t^{1|1})$

quantization of 1-1 dim σ-model with target X

1|1 - Euclidean Field Theory

Modularity comes in because S^1 is replaced by $S^1 \times S^1$

Classical Σ- model: d-dim,

Σ^d - space time - Riemannian manifold

$\mathcal{F}(X)$ - Fields $C^\infty(\Sigma^d, X)$

$A: \mathcal{F}(X) \longrightarrow \mathbb{R}$ action map

$\downarrow$ geometric quantization

$H = \mathcal{L}^2(X)$, time evolution $e^{-t\Delta x}$ after Wick Rotation

$$\int_\varphi e^{-A(\varphi)}$$

This is Bosonic (?).

If Σ has 1-bosonic, 1- fermionic direction

$\Sigma^{1|1} \longrightarrow$ The classical solutions

$$\begin{array}{ccc} P & \longrightarrow & \pi TX \\ \downarrow & & \downarrow \\ TX & \longrightarrow & X \end{array}$$

odd vector space, Fermionic directions

Bosonic part

P has no Lagrangian :(

and hence no polarization

Use the spin structure $\Longrightarrow$ polarization on P

$$\mathcal{H} = \mathcal{L}^2(\$_X)_{Cl_n}$$

$$\begin{tikzcd} P' \arrow[r] \arrow[d] & \pi(TX_g) \oplus (\mathbb{R}^n, -st) \arrow[d] \\ TX \arrow[r] & X \end{tikzcd}$$

$(\mathbb{R}^n, -st)$ ← spin

time evolution $= e^{-t\mathcal{D}_X^2 + \theta \mathcal{D}_X}$

Def^n: A 1-dim Euclidean Field theory is a pair $(V, \alpha_t : V \to V)$ V - topological vector space, α_t - smooth semigroup

$\Leftrightarrow$ symmetric monoidal functor $(1\text{-}E\,Bord, \amalg) \xrightarrow{E} (TVect, \otimes)$ $\otimes$ = projected tensor product

not a topoligacal field theory

ob: compact 0-manifold eg: $pt \longrightarrow V = \mathcal{H}$

mor: $\supset$ $\circ$ $\cdot\!\sim\!\cdot$ with length $\quad \alpha_t(\subset:) : \mathbb{R} \to V \otimes V$

$E(S^1_t) = tr(\alpha_t) =:$ partition function

Now replace $TVect$ to $TVect^{\mathbb{Z}/2}$

Lem: $\{1/1\text{-}EFT\} \underset{\text{equiv. of groupoids}}{\simeq} \left\{ \begin{array}{l} \text{super vector space } V \\ \text{super semigroup of operators} \end{array} \right.$

$\alpha_{t,\theta} : V \to V$

$e^{-tD^2 + \theta D}$, D odd infinitesimal generator

$$\underset{E(S^{1|1}_{t,\theta})}{\overset{\|}{\text{strace}}}(\alpha_{t,\theta}) = \text{sdim}(\ker D) \in \mathbb{Z}$$

"super symmetry cancellation"

Q. index of Dirac operator really lives in $K^{-n}(pt)$??

To get twisted 1|1 EFT:

$$\begin{tikzcd} (1|1\ Ebord, \amalg) \arrow[r, bend left, "T"] \arrow[r, bend right, "\mathbb{1}"'] \arrow[r, phantom, "\Downarrow E"] & (TAlg^{\mathbb{Z}/2}, \otimes) \end{tikzcd}$$

$TAlg^{\mathbb{Z}/2}$ ← 2 category

o-morp: pts, 1mor $=$ $\cap\cdot$, 2mor $=$ isometries

Remark: $TAlg^{\mathbb{Z}/2}$ is a choice of delooping of $TVect$,
$TVect$ is a choice of delooping of $\mathbb{C}$

$T(\mathbb{R}^{0|1}) \in TAlg^{\mathbb{Z}/2}$
$T(\underset{t,\theta}{C^{\cdot}_{\cdot}})$ super semigroup of bimodules

$E(\mathbb{R}^{0|1})$ is a $\mathbb{C}$-$T(\mathbb{R}^{0|1})$ bimodules
eg: $T_n(\mathbb{R}^{0|1}) = Cl_n = Cl_1 \otimes \cdots \otimes Cl_1$ = degree n-twist
$T_n(C^{\cdot}_{t,\theta}) = {}_{Cl_n}{Cl_n}_{Cl_n}$

dim 2:

$$\begin{tikzcd}
\text{closed oriented manifold } X^n \arrow[rr, "\hat{w}"] & & \mathbb{C}[[q]] \\
\text{spin structure} \arrow[u] \arrow[rr] & & \mathbb{Z}[[q]] \arrow[u] \\
\text{if } p_1 X = \text{torsion} \arrow[u] \arrow[rr] & & MF_*^{\mathbb{Z}} \arrow[u, phantom, "\cup|"] \\
\text{first pontryagin class} & & \text{modular form} \\
& \text{twisted 2-1 EFT} \arrow[ul, dashed] \arrow[ur, "\text{partion function}"] &
\end{tikzcd}$$

$\mathbb{C}[[q]]$ ~ Comes from S^1 action on LX

why do we get a modular form?

New input via Factorization Algebras

partion function $= ev(\mathbb{C}/\Gamma)$ $\quad\Gamma$ is a lattice in $\mathbb{R}^2$

Degree comes in since this is a line bundle on the space of lattices.
eg: degree n twist comes from delooping n-th power of Pfaffian line.