§ 8.4 Effective Cartier Divisor, reg. sequences & reg. embeddings

Def: $\pi: X \longrightarrow Y$ closed embedding
π locally principal if $\exists$ open cover $Y = \bigcup_i \operatorname{Spec} A_i$ s.t. $\forall i$ ideal of X in A_i is principal.

π (or X) is an effective Cartier divisor if furthermore ideal shf loc generated by non-zero divisor.

Rem: Notions not independent of the affine cover.

Def: $M \in A$-mod. An M-regular sequence in A is $x_1, \cdots, x_r \in A$ such that
1) $\forall i$, x_i is not zero divisor on $M/(x_1, \cdots, x_{i-1})$
2) $(x_1, \cdots, x_r) M \neq M$.

Df: An A-regular sequence is a regular seq.

Th^m: $(A, \mathfrak{m})$ Noetherian local ring, M f.g. A-mod. Then M-regular sequence $x_1, \cdots, x_r$ in $\mathfrak{m}$ then $x_1, \cdots, x_r$ remains regular after reordering.

Def^n: Say that a locally closed embedding $\pi: X \hookrightarrow Y$ is a regular embedding of codim r at a point $p \in X$ if in $\mathcal{O}_{X,p}$ ideal of X generated by regular sequence of length r.
Say π is a regular embedding (of codim r) if reg embedding (of codim r) at every pt $p \in X$.

Def^n: A codim r complete intersection in a scheme Y is a closed subscheme X expressible as scheme theoretic intersection of r effective Cartier divisors $D_1, \dots, D_r$ i.e. eqns corresponding to $D_1, \cdots, D_r$ form a regular seq in $\mathcal{O}_{Y,p}$ $\forall p \in X$.

§ 9 Fiber products and Base change

§1 Existence of Fibered product:

Def^n: In any category $\mathcal{C}$, fibered product of
$$\begin{tikzcd} & Y \arrow[d, "\beta"] \\ X \arrow[r, "\alpha"'] & Z \end{tikzcd}$$
is $X \times_Z Y$ the equalizer of α, β. i.e. $\operatorname{Hom}(W, X \times_Z Y) \cong \operatorname{Hom}(W, X) \times_Z \operatorname{Hom}(W, Y)$

Th^m: Fibered products exist in Sch.

"Proof": Say given

$$\begin{tikzcd} & Y \arrow[d, "\beta"] \\ X \arrow[r, "\alpha"] & Z \end{tikzcd}$$

Step 1: $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, $Z = \operatorname{Spec} C$

then $X \times_Z Y = \operatorname{Spec} A \otimes_C B$

Step 2: If α, β factor through open $U \subseteq Z$ then

$$X \times_Z Y \cong X \times_U Y$$

Step 3: Cover Z by open affines W_i, cover $\alpha^{-1}(W_i)$ with open affines U_{ij}, $\beta^{-1}(W_i)$ by V_{ik}.

Step 4: Glue

Warning: Topological space underlying $X \times_Z Y$ is not same as $\operatorname{top} X \times_{\operatorname{top} Z} \operatorname{top} Y$

eg: · eg: $\mathbb{A}^1_k \times \mathbb{A}^1_k \cong \mathbb{A}^2_k$

· If X, Y finite type $/ \operatorname{Spec} \bar{k}$

$$\left\{ \begin{matrix} \text{closed pts in} \\ X \times_{\bar{k}} Y \end{matrix} \right\} \cong \left\{ \begin{matrix} \text{closed pts in} \\ X \end{matrix} \right\} \times \left\{ \begin{matrix} \text{closed pts in} \\ Y \end{matrix} \right\}$$

· Categorically speaking this is saying that $\operatorname{Hom}(-, X) \times_{\operatorname{Hom}(-, Z)} \operatorname{Hom}(-, Y)$ is representable.

§ 9.2 Fibered products in practice.

1)
$$\begin{tikzcd} & Y \arrow[d, "\beta"] \\ U \arrow[r, phantom, "\subseteq_{\text{open}}"] & X \end{tikzcd} \qquad U \times_Z Y = \beta^{-1}(U)$$

2) $A \otimes_B B[t] = A[t] \implies$

$$\begin{tikzcd} \mathbb{A}^1_A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathbb{A}^1_B \arrow[d] \\ \operatorname{Spec} A \arrow[r] & \operatorname{Spec} B \end{tikzcd}$$

Def: $\mathbb{A}^1_S := \mathbb{A}^1_{\mathbb{Z}} \times_{\mathbb{Z}} S$

Also if $S \in k\text{-Sch}$

$$\begin{tikzcd} S \arrow[r] & \operatorname{Spec} k \arrow[r] & \operatorname{Spec} \mathbb{Z} \\ \mathbb{A}^1_S \arrow[r] \arrow[u] \arrow[ur, phantom, "\ulcorner"] & \mathbb{A}^1_k \arrow[r] \arrow[u] \arrow[ur, phantom, "\ulcorner"] & \mathbb{A}^1_{\mathbb{Z}} \arrow[u] \end{tikzcd}$$

- $\begin{tikzcd} & A \\ B/I & B \arrow[l] \arrow[u, "\phi"'] \end{tikzcd}$ $\rightsquigarrow$ $A \otimes_B B/I \cong A/\phi(I)$ so if

  $X \to Z$ is closed embedding

  so is $X \times_Z Y \hookrightarrow Y$

- $\begin{tikzcd} & A \\ S^{-1}B & B \arrow[l] \arrow[u, "\phi"'] \end{tikzcd}$ $\Rightarrow$ $A \otimes_B S^{-1}B \cong \phi(S)^{-1}A$

  $\Rightarrow$ "fibered products commute with localization."

- $\begin{tikzcd} & \operatorname{Spec} A \arrow[d] \\ D(f) \arrow[r] & \operatorname{Spec} B \end{tikzcd}$ $\operatorname{Spec} A \times_{\operatorname{Spec} B} D(f) \cong \pi^{-1}(D(f)) \cong \operatorname{Spec} A_f$

- $\mathbb{P}^n_S := S \times_{\operatorname{Spec} \mathbb{Z}} \mathbb{P}^n_{\mathbb{Z}}$ for any Scheme S.