Crash Course on Representation Theory - Day 1

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All the vector spaces will be finite dimensional vector spaces over $\mathbb{C}$. G will denote a finite group. V will denote a vector space of dimension d over $\mathbb{C}$. All representations will be finite dimensional.

1 Introduction

Representation theory is based on the philosophy: When life gives you groups, linearize. Groups in general have very little structure on them (only 1 product) and hence are notoriously difficult to analyze. On the other hand there are a lot ways to exploit matrices: addition, multiplication, diagonalization, eigenvalues and eigenvectors, the various canonical forms, etc. As such it is usually quite fruitful to reduce a problem in group theory to one in linear algebra.

Theorems in basic representation theory can be very broadly broken into two kinds: i) structural theorems about existence and classification of representations coming from linear algebra, ii) constructive theorems which actually construct these representations using other techniques like combinatorics. In this class we'll only look at theorems of the first kind.

1.1 Frobenius determinant

Historically, a generalization of the following question prompted Frobenius to develop representation theory. For variables $x_0, x_1, \ldots, x_{n-1}$ define the Frobenius matrix to be a matrix whose i, j^{th} entry is $x_{(i-j \mod n)}$ i.e. in the j^{th} column x_i is in the row $i + j \mod n$.

$$F_n = \begin{bmatrix} x_0 & x_{n-1} & \cdots & x_1 \\ x_1 & x_0 & \cdots & x_2 \\ \vdots & & \ddots & \vdots \\ x_{n-1} & x_{n-2} & \cdots & x_0 \end{bmatrix} \tag{1.1}$$

Question. What are the irreducible factors of the determinant of F_n?

We'll use representation theory to answer this question. For small dimensions it is a fun exercise to work out the factors by hand.

$$\det \begin{bmatrix} x_0 & x_1 \\ x_1 & x_0 \end{bmatrix} = x_0^2 - x_1^2 = (x_0 - x_1)(x_0 + x_1) \tag{1.2}$$

$$\det \begin{bmatrix} x_0 & x_1 & x_2 \\ x_1 & x_0 & x_2 \\ x_2 & x_1 & x_0 \end{bmatrix} = x_0^3 + x_1^3 + x_2^3 - 3x_0x_1x_2 = ?? \tag{1.3}$$

2 Definitions

A d dimensional **representation** of G is a group homomorphism

$$\rho : G \to GL(V)$$

where V is a d dimensional vector space over $\mathbb{C}$ and $GL(V)$ denotes the group of invertible linear transformations $V \to V$. We say that G acts on V or that V has an action of G on it, denoted $G \circlearrowright V$. It is common to abuse notation and say that V is a representation of G. More explicitly, we assign to each element $g \in G$ a linear transformation $\rho(g)$ satisfying

1. $\rho(e) = I_V$ where e is the identity in G and I_V is the trivial linear transformation on V
2. $\rho(gh) = \rho(g)\rho(h)$

3. $\rho(g^{-1}) = \rho(g)^{-1}$

An **equivariant** map between two representations ρ and τ of G is a linear map $f : V \to W$ such that the induced map $GL(f)$ fits in the following commutative diagram

$$\begin{tikzcd} & GL(V) \arrow[d, "GL(f)"] \\ G \arrow[ur] \arrow[r] & GL(W) \end{tikzcd} \tag{2.1}$$

To be more explicit a linear map $f : V \to W$ is equivariant if for any $v \in V$ we have $f(\rho(g)(v)) = \tau(g)(f(v))$. An **isomorphism** of representations is an invertible equivariant map $f : V \to W$.

A **sub-representation** of V is a subspace $W \subsetneq V$ such that W is itself a representation of G i.e. for every $g \in G$, $w \in W$ we have $\rho(g)w \in W$. We say that W is closed under the action of G. A representation of G which has no sub-representation is called an **irreducible** representation. We say that a representation V is **decomposable** if it there are sub-representations V_1, V_2 such that $V_1 \oplus V_2 \cong V$.

Irreducible representations are the analogues of prime numbers in representation theory. One of the central goals of representation theory is to classify all the possible irreducible representations and to decompose arbitrary representations into irreducible ones.

3 Examples

3.1 Trivial Representation

For any finite group G and any vector space V there is a **trivial** representation $\rho : G \to GL(V)$ which sends every element in G to the identity transformation I_V.

3.2 Cyclic Groups

For the abelian group $\mathbb{Z}/n$ with generator a every representation is completely determined by where a is mapped. Any $n \times n$ matrix A satisfying $A^n = 1$ gives a representation $\mathbb{Z}/n \to GL(V), a \mapsto A$. In particular for each $0 \leq k < n$ the n^{th} root of unity $e^{2\pi ik/n}$ gives us a 1 dimensional representation of $\mathbb{Z}/n$. It is easy to see that no two of these representations are isomorphic.

3.3 Symmetric Group

Every symmetric group S_n has an n dimensional representation called the **standard representation**. S_n acts on $\mathbb{C}^n$ as follows: If $e_1, \cdots, e_n$ is the standard basis for $\mathbb{C}^n$ then the permutation $g \in S_n$ acts on $\mathbb{C}^n$ via

$$\rho(g)e_i = e_{\sigma(i)} \tag{3.1}$$

The matrices of $\rho(g)$ in the standard basis are called the **permutation matrices**.

The standard representation of S_n is not reducible. Consider the 1-dimensional subspace generated by the element $e_1 + e_2 + \cdots + e_n$. This subspace is invariant under the action of S_n and hence is a sub-representation of $\mathbb{C}^n$. Let $W^\perp$ be the vector space of $\mathbb{C}^n$ consisting of vectors which are perpendicular to W i.e. $W^\perp = \{c_1.e_1 + \cdots + c_n.e_n : c_1 + \cdots + c_n = 0\}$. It is easy to see that $W^\perp$ is also a sub-representation of $\mathbb{C}^n$ and hence $V \cong W_1 \oplus W_2$ as representations

3.4 Sign Representation

Every symmetric group S_n has a 1 dimensional representation sign : $S_n \to GL_1(\mathbb{C})$ called **sign representation** defined as follows: Every transposition (i, j) maps to -1. Every permutation can be written (non-uniquely) as a product of transpositions and hence we can extend this map to the entire S_n. We can then show that this extension is well defined.

3.5 Dihedral group

Let D_{2n} denote the **dihedral group** which is the group of symmetries of a regular n sided polygon in $\mathbb{R}^2$. D_{2n} has a presentation

$$D_{2n} = \langle x, y \mid x^2 = 1, y^n = 1, xyxy = 1\rangle \tag{3.2}$$

x denotes reflection about a line that passes through the center of the polygon and y denotes a reflection about the center of the polygon by an angle of $2\pi/n$. We can show that D_{2n} contains exactly $2n$ elements and every element is of the form y^d or xy^d for some $0 \leq d \leq n-1$, hence the subscript $2n$. D_{2n} has a natural 2 dimensional representation

$$\rho : D_{2n} \to GL_2(\mathbb{C}) \tag{3.3}$$

$$x \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{3.4}$$

$$y \mapsto \begin{bmatrix} \cos(2\pi/n) & -\sin(2\pi/n) \\ sin(2\pi/n) & \cos(2\pi/n) \end{bmatrix} \tag{3.5}$$

One can show that the standard 2 dimensional representation of D_{2n} is irreducible.

3.6 Quaternions

The **quaternion group** Q_8 defined by the presentation

$$Q_8 = \langle i, j, k, -1 \mid i^2 = j^2 = k^2 = -1 = ijk, (-1)^2 = 1 \rangle \tag{3.6}$$

has a 1 dimensional **sign representation** given by mapping i to 1 and j, k to -1, and similarly two other sign representations. Q_8 also has a 2-dimensional representation $Q_8 \to GL(\mathbb{C}^2)$ given by

$$i \mapsto \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \qquad j \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad k \mapsto \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \tag{3.7}$$

3.7 Regular Representation

Every group G has a $|G|$ dimensional representation called a **regular representation**. Consider the free vector space V over G i.e. V is a $|G|$ dimensional vector space with a basis given by $\{e_g : g \in G\}$. G acts on V via

$$\rho(g)(e_h) = e_{gh} \tag{3.8}$$

The regular representation is not irreducible as it contains a 1 dimensional vector space spanned by $\sum_{g \in G} e_g$ which is invariant under the action of G.

4 'Prime' representations

Representation theory of finite groups over $\mathbb{C}$ asserts the existence of finitely many irreducible representations up to isomorphism. Further these representations can be detected by their characters.

Recall that two elements $g_1, g_2 \in G$ are called **conjugates** of each other if there exists an $h \in G$ such that $h^{-1}g_1h = g_2$. Being a conjugate is an equivalence relation and the equivalence classes are called **conjugacy classes**.

Theorem 4.1. *Up to isomorphism there are finitely many irreducible representations of G. Suppose $\rho_1, \rho_2, \ldots, \rho_r$ are the distinct irreducible representations of G with dimensions d_1, d_2, $\ldots$, d_r respectively then,*

1. *n equals the number of conjugacy classes in G.*
2. *$d_i \mid |G|$ where $|G|$ denotes the size of G.*
3. *$|G| = d_1^2 + d_2^2 + \cdots + d_r^2$*

Theorem 4.2 (Maschke's theorem)**.** *Every reducible representation is decomposable.*

Theorem 4.3 (Schur's lemma)**.** *Using the same notation as in the previous theorem, every finite dimensional representation τ of G has a unique decomposition*

$$\tau \cong \rho_1^{\oplus k_1} \oplus \rho_2^{\oplus k_2} \oplus \cdots \oplus \rho_r^{\oplus k_r} \tag{4.1}$$

for some positive integers $k_1, k_2, \cdots, k_r$ where by $\rho_i \oplus \rho_j$ we mean the representation $G \xrightarrow{\rho_i \oplus \rho_j} GL(V_i \oplus V_j)$.

In terms of matrices $(\rho_i \oplus \rho_j)(g)$ is the block matrix with two blocks given by $\rho_i(g)$ and $\rho_j(g)$ respectively. This theorem can be interpreted as saying that for every representation $\tau : G \to GL(V)$ it is possible to choose a basis for V such that all the matrices $\tau(g)$ become block diagonal in this basis, further each of the blocks are obtained from the irreducible representations.

These two theorems should be thought of as saying that in the world of G representations there are finitely many 'primes'. Every other representation can be uniquely written as a 'product'(=direct sum) of these primes.

4.1 Examples

Dihedral group D_6: D_6 which is the same as the symmetric group S_3 has 3 conjugacy classes $\{\{1\}, \{(1,2); (1,3); (2,3)\}, \{(1,2,3); (1,3,2)\}\}$ and hence has 3 irreducible representations of dimensions say d_1, d_2, d_3 which satisfy i) $d_i|6$ and ii) $d_1^2+d_2^2+d_3^2 = 6$. The only such numbers are $1, 1, 2$. Further we know what they are: the trivial representation, the sign representation and the standard representation of the dihedral group D_6.

Quaternion group Q_8: The quaternion group Q_8 has 5 conjugacy classes $\{1\}$, $\{-1\}$, $\{i, -i\}$, $\{j, -j\}$, and $\{k, -k\}$ and hence has 5 distinct irreducible representations. But we know 5 irreducible representations of Q_8: the trivial representation, the 3 sign representations, and the two dimensional representation.

Cyclic group: Let G be the cyclic group $\mathbb{Z}/n$. As G is abelian $ghg^{-1} = h$ for all h i.e. each conjugacy class contains exactly 1 element and hence the number of conjugacy classes of G is n. So G has exactly n distinct irreducible representations. Suppose their dimensions are $d_1, \cdots, d_n$ then we must have $n = d_1^2 + \cdots + d_n^2$. The only possibility is $d_i = 1$ for all i i.e. all the irreducible representations of $\mathbb{Z}/n$ are 1 dimensional. But we already know n one dimensional representations of $\mathbb{Z}/n$ (given by the roots of unity). More generally we get

Proposition 4.4. *Every irreducible representation of an a abelian group is 1 dimensional.*

p^2 groups: Let G be a group of size p^2 where p is a prime. Suppose G has r irreducible representations of dimensions $d_1, d_2, \cdots, d_r$ then we must have $d_i|p^2$ so each d_i is in the set $\{1, p, p^2\}$. We also have $p^2 = d_1^2 + d_2^2 + \cdots + d_r^2$. Every group has a trivial representation so that one of the $d_i's$ is 1. The only possibility is $d_i = 1$ for all i and hence $r = p^2$. But this forces G to have p^2 conjugacy classes and hence every conjugacy class contains exactly 1 element i.e. G is abelian.

Proposition 4.5. *Every group of size p^2 is abelian.*

Symmetric groups The symmetric groups S_n have size $n!$. The conjugacy classes of S_n are given by cycle types. One can show that the number of cycle types is equal to the number of ways of partitioning n and hence for every partition of the number n we get an irreducible representation of S_n.