Stable homology of moduli spaces of manifolds:

$\{\mathcal{M}_n\}_{n\in\mathbb{Z}}$ $\quad B\Sigma_n$, $B Br_n$ ← Braid group $\quad BGl_n(\mathbb{Z})$ $\quad BAut(F_n)$ $\quad BDiff(\Sigma_{n,1})$ — exhibit H_* stability

Stable homology:

i) Often given by (iterated) loop spaces

ii) Often a map $\bigvee \mathcal{M}_n \longrightarrow \Omega(?)$ inducing isomorphism in the stable range.
"scanning"

Ex: 1) $BBr_n \longrightarrow \Omega^2 S^2$ is, in H_* onto the path component in the stable range

2) $B\Sigma_n \longrightarrow \Omega^\infty S^0$

3) $BDiff(\Sigma_{g,1}) \longrightarrow \Omega^\infty MTSO(2)$
spectrum whose k^{th} space is $\dfrac{SO(n)}{SO(2) \wedge_{SO(n-2)} S^{n-2}}$

Q. Can we look at $\mathcal{M}_n$ and say that $B\mathcal{M}_n$ will be an infinite loop space?

Scanning: $\quad BBr_n \longrightarrow \Omega^2 S^2$
$\quad\quad \shortparallel$
$\quad C_n(\mathbb{R}^2)\,(Conf_n(\mathbb{R}^2))$

for $A \subseteq \mathbb{R}^2$, $|A| = n$

Define $\alpha(A): \mathbb{R}^2 \cup \{\infty\} \longrightarrow \mathbb{R}^2 \cup \{\infty\} \quad \in \Omega^2_n S^2$

$$p \longmapsto (A-p) \cap B(0,\varepsilon)$$

$$\infty \longmapsto \infty$$

Another way of describing the same map:

Ψ = all discrete $Q \in \mathbb{R}^2$

Q' near Q if it's a small perturbation inside a ball

$$C_n(\mathbb{R}^2) \longrightarrow \Omega^2 \Psi \qquad S^2 \simeq \Psi \text{ homotopy equivalence}$$

$$A \longmapsto \begin{pmatrix} p \in \mathbb{R}^2 \longmapsto A - p \\ \infty \longmapsto \infty \end{pmatrix}$$

Similar technique would work for $B\Sigma_n$

Oriented Surfaces:

Σ — closed + orientable

$$BDiff^+(\Sigma) = \frac{Emb(\Sigma, \mathbb{R}^\infty) \times Or(\Sigma)}{Diff(\Sigma)}$$

$\shortparallel$

$C_\Sigma(\mathbb{R}^\infty)$

$= \{ Q \subseteq \mathbb{R}^\infty \mid Q \simeq \Sigma \}$

Approximated by $C_\Sigma(\mathbb{R}^n)$.

We have a scanning map $C_\Sigma \mathbb{R}^n \xrightarrow{\alpha} \Omega^n$ — 1-point compactification of this

$$Q \subseteq \mathbb{R}^n \longmapsto \begin{pmatrix} p \in \mathbb{R}^n \longmapsto \overbrace{(Q-p) \cap B(0,\varepsilon)}^{\text{empty or affine}} \\ \infty \longmapsto \phi \end{pmatrix}$$

$$\frac{SO(n)}{SO(2)} \underset{SO(n-2)}{\times} \mathbb{R}^{n-2} = \text{space of affine 2-dim oriented subspaces of } \mathbb{R}^n$$

Ways of proving H_* equivalence –

$BB\Gamma_n \longrightarrow \Omega^2_n S^2$ iso in H_*, $* \leq n/2$

$\Updownarrow$ ⟵ Assuming H_* stability

$\mathbb{Z} \times BB\Gamma_\infty \longrightarrow \Omega^2 S^2$ iso in H_*, $\forall *$.

$B\Gamma_\infty = \operatorname{colim}(B\Gamma_n \longrightarrow B\Gamma_{n+1} \longrightarrow \cdots)$

Monoid Structures:

eg: $M = \coprod_n BB\Gamma_n$ $\qquad$ $B\Gamma_n \times B\Gamma_m \longrightarrow B\Gamma_{n+m}$ $\qquad$ similarly for $B\Sigma_n$, $BDiff(\Sigma_{g,1})$

Group completion theorem (Quillen, McDuff-Segal)

Top. Monoid $M = \coprod_{n \geq 0} M_n$ $\quad$ Homotopy Commutative, $\pi_0 M = \mathbb{N}$, Pick $m \in M_1$

$$M \longrightarrow \Omega BM \qquad M_\infty = \operatorname{hocolim}(M_0 \xrightarrow{\cdot m} M_1 \xrightarrow{\cdot m} M_2 \xrightarrow{\cdot m} \cdots)$$

$$M \to \mathbb{Z} \times M_\infty \to \Omega BM$$

Thm: This is an H_* isomorphism

eg: $\mathbb{Z} \times BB\Gamma_\infty \xrightarrow{H_* \text{ iso}} \Omega^2 S^2$

$\Uparrow$

$B(\coprod BB\Gamma_n) \xrightarrow{\simeq} \Omega S^2$ ⟵ Prove Both sides $\simeq \{A \in \psi \mid A \subseteq \mathbb{R} \times (-1,1)\}$

In general moduli spaces of n-dim manifolds:

monoid ⇝ category, but scanning map does not detect all the information

often get too big moduli spaces eg: non-connected 2-manifolds

Rmk: RHS easier to understand

eg: $H^*(BDiff(D^6 \# g S^3 \times S^3); \mathbb{Q}) \cong \underbrace{\mathbb{Q}[\mathcal{K}_c \mid c \in B]}_{\text{Polynomial Ring}}$ $\quad$ for $* \leq \frac{g-3}{2}$

independant of g, for $* \leq \frac{g-3}{2}$

$c \in B = \left\{ p_1^i e^j p_2^k \;\middle|\; \begin{matrix} i,j,k \geq 0 \\ |c| \geq 6 \end{matrix} \right\}$

$|\mathcal{K}_c| = |c| - 6$