§ Homology & Cohomology of diagrams in Abelian categories

· Projective objects in an abelian category $\mathcal{C}$:

This is a straightforward generalisation of projective modules

Def: Projective class $(P, \mathcal{E}) \subseteq (Ob\, \mathcal{C}, Mor\, \mathcal{C})$:

$$\begin{tikzcd} & X \arrow[d, "\in \mathcal{E}"] \\ U \arrow[ur, dashed, "\exists"] \arrow[r] & Y \end{tikzcd}$$

1) $U \in P \iff \forall\, X \to Y \in \mathcal{E}$ and for any $U \to Y$ $\exists$ a lift

2) $X \to Y \in \mathcal{E} \iff \forall\, U \in P$ and for any map $U \to Y$ $\exists$ "

3) $\forall\, X \in \mathcal{C}$, $\exists\, U \in P$ and $U \to X \in \mathcal{E}$

$(P, \mathcal{E})$ is called (P-projectives, P-epimorphisms).

· A complex $A \to B \to C$ is called P-exact if $\forall\, U \in P$ the complex $hom(U,A) \to hom(U,B) \to hom(U,C)$ is exact.

Def: We can now define P left-derived functors:

Given a right P-exact additive functor $F: \mathcal{C} \to Ab$, consider a P-resolution $U_* \to X = U_{-1}$

define the left derived functors to be $L^P_k F = H_k(F(U_*))$

One needs to check these are independent of resolution

eg) For R-mod, $(P, \mathcal{E}) =$ (projective modules, surjective morphisms)

for any $U \in P$ $hom(U,-)$ is exact hence $A \to B \to C$ is P-exact iff it is exact in R-mod

$L^k F$ are then the usual left derived functors.

· Projective classes for Ab^I:

Ab^I is an abelian category with kernels, cokernels taken pointwise.

Def: The free objects in Ab^I are defined as:

$\forall A \in Ab$, $i \in I$, define $F_i(A)(j) := \coprod_{hom(i,j)} A \in Ab^I$

· We have the free-forget adjunction: $hom_{Ab^I}(F_i(A), \mathcal{D}) \cong hom_{Ab}(A, \mathcal{D}_i)$

Def: The projectives in Ab^I are $P = \{$retracts of coproducts of $F_i(A)\}$

The projective covers in Ab^I are $\mathcal{E} = \{\mathcal{D} \to \mathcal{D}' : \mathcal{D}(i) \to \mathcal{D}'(i)$ is a split-epi for all $i\}$

· P makes sense as projecties in R-mod are also direct summand. Why does $\mathcal{E}$ only have SPLIT-epi?

· By the adjunction $hom(colim^I \mathcal{D}, X) \cong hom(\mathcal{D}, cX)$ $colim^I(-)$ is left exact.

Def: Define $H_p(I,-)$ to be the left derived functors of $colim^I(-)$ (also denoted $colim^I_p(-)$).

· One can show that the chain complex associated to $srep\, \mathcal{D} \to colim\, \mathcal{D}$ comes from a projective resolution of $\mathcal{D}$ and hence can be used as an alternate definition of $H_k(I, \mathcal{D})$.

- Injective objects:
  Dualise everything to get cofree objects or $\mathcal{I}$-injectives in $Ab^{\mathcal{I}}$
  these would be

$$\mathcal{I}_i(A) = \prod_{hom(j,i)} A \quad \in Ab^{\mathcal{I}} \qquad \left[hom_{Ab}(A, \mathcal{D}_i) \cong hom_{Ab^{\mathcal{I}}}(\mathcal{I}_i(A), \mathcal{D}) \right]$$

Q. How are these related to standard projective and injective modules?

Q. It seems like lim, colim are special in that they have homotopical versions.
What about other functors? $\otimes$ and hom ?

Q. While finding higher derived functors of a functor F (colim, lim in our case) it suffices to use F-acyclic objects instead of projectives/injectives. Is this what is happening?
Are $F_i(A)$ or $\coprod_i \mathcal{D}_i$ colim-acyclic ??

- Examples of homology of categories:

eg) $\mathcal{I} : i \rightrightarrows j \qquad \mathcal{D} : X \overset{f}{\underset{g}{\rightrightarrows}} Y \qquad F_i(A) = A \rightrightarrows A \oplus A \qquad F_j(A) = 0 \rightrightarrows A$

The resolution: $\begin{matrix} X \\ f \downarrow\downarrow g \\ Y \end{matrix} \longleftarrow \begin{matrix} X \\ \downarrow\downarrow \\ X \oplus X \oplus Y \end{matrix} \longleftarrow \begin{matrix} 0 \\ \downarrow\downarrow \\ \{(x_1, x_2) : f(x_1) = g(x_2)\} \end{matrix} \longleftarrow 0$

Colim : $\quad coker(f-g) \longleftarrow X \oplus Y \longleftarrow X \times_Y X \longleftarrow 0$

$$H_k(\mathcal{I}, \mathcal{D}) = \begin{cases} coker(f-g) & k = 0 \\ ker f \cap ker g & k = 1 \\ 0 & else \end{cases}$$

eg) $\mathcal{I} = \begin{tikzcd} i \arrow[r] \arrow[d] & j \\ k \end{tikzcd} \qquad F_i(A) : \begin{tikzcd} A \arrow[r] \arrow[d] & A \\ A \end{tikzcd} \qquad F_j(A) : \begin{tikzcd} 0 \arrow[r] \arrow[d] & A \\ 0 \end{tikzcd} \qquad F_k(A) = \begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \\ A \end{tikzcd}$

For $\mathcal{D} = \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C \end{tikzcd}$ we get

$$\boxed{colim_p(\mathcal{D}) = \begin{cases} coker(f \oplus g) & p = 0 \\ ker(f \oplus g) & p = 1 \\ 0 & else \end{cases}}$$

eg) $\mathcal{I} = \begin{tikzcd} i & j \arrow[l] \\ k \arrow[u] \end{tikzcd} \qquad \mathcal{I}_i(A) = \begin{tikzcd} A & A \arrow[l] \\ A \arrow[u] \end{tikzcd} \qquad \mathcal{I}_j(A) = \begin{tikzcd} 0 & A \arrow[l] \\ 0 \arrow[u] \end{tikzcd} \qquad \mathcal{I}_k(A) = \begin{tikzcd} 0 & 0 \arrow[l] \\ A \arrow[u] \end{tikzcd}$

For $\mathcal{D} = \begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & A \end{tikzcd}$ we have

$$\boxed{lim^p(\mathcal{D}) = \begin{cases} ker(f \times g) & p = 0 \\ coker(f \times g) & p = 1 \end{cases}}$$

chain complexes

eg) $I = 0 \to 1 \to 2 \to 3 \to \cdots$ $\quad F_n(A): 0 \to 0 \to \cdots \to 0 \to \overset{[n]}{A} \to A \to A \to \cdots$

For $\mathcal{D} = A_0 \xrightarrow{f} A_1 \xrightarrow{f} \cdots$ Resolution becomes: $0 \to \bigoplus_i A_i \xrightarrow{id \oplus f} \bigoplus_i A_i \to 0$

when $\mathcal{Ab} = R\text{-mod}$ we get $\boxed{\operatorname{colim}_* \mathcal{D} = \begin{cases} \operatorname{colim} \mathcal{D} & \text{if } * = 0 \\ 0 & \text{else} \end{cases}}$

eg) $I = 0 \leftarrow 1 \leftarrow 2 \leftarrow \cdots$ $\quad G_n(A) = 0 \leftarrow \cdots \leftarrow 0 \leftarrow \overset{[n]}{A} \leftarrow A \leftarrow A \cdots$

For $\mathcal{D} = A_0 \xleftarrow{f} A_1 \xleftarrow{f} A_2 \leftarrow \cdots$ the resolution becomes $0 \to \prod_i A_i \xrightarrow[id - f]{} \prod_i A_i \to 0$

For R-mod we get $\boxed{\lim^*(\mathcal{D}) = \begin{cases} \lim \mathcal{D} = \{(a_i)_{i \in I} : f a_i = a_{i+1}\} \\ \lim^1 \mathcal{D} \\ 0 & * \geq 1 \end{cases}}$

For $\mathcal{D}: \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{2} \mathbb{Z} \xleftarrow{2} \cdots$ $\quad \lim(\mathcal{D}) = 0$ and $\lim^1(\mathcal{D}) = \hat{\mathbb{Z}}_2 / \mathbb{Z}$