Spectral Sequences:

Examples:

1. Filtered Complexes

$C_* \in Ch$ R-mod, assume we have a filtration $F_pC_* \subseteq F_{p+1}C_* \subseteq F_{p+2}C_* \subseteq \cdots$ such that $C_* = \bigcup_p F_pC_*$ eg: X - CW complex $C_* = S_*(X)$, $F_pC_* = S_*(X_p)$

Show: $H_*(C_\cdot) = \varinjlim_p H_*(F_pC_\cdot)$

We get LES:

$$\begin{array}{ccccc} \to H_n(F_pC_*) & & & & \\ \downarrow i & & & & \\ H_n(F_{p+1}C_*) & \xrightarrow[j]{} & H_n(F_{p+1}C_*/F_pC_*) & \xrightarrow[k]{} & H_{n-1}(F_pC_*) \\ & & & & \downarrow \\ & & & & \vdots \end{array}$$

Homologically graded exact couple $\rightsquigarrow$ Computes the homology of $H_*(C_\cdot)$

$$E^1_{p,q} = H_{p+q}(F_pC_\cdot/F_{p-1}C_\cdot) \Rightarrow \varinjlim_p H_{p+q}(F_pC_\cdot) = H_{p+q}(C_\cdot)$$

Q. How explicitly can we write the differentials. Convergence??

2. Double Complex:

$\{(C_{s,t}, s,t \in \mathbb{Z}), \partial', \partial''\}$ satisfying: $\partial': C_{s,t} \longrightarrow C_{s-1,t}$, $\partial'': C_{s,t} \longrightarrow C_{s,t-1}$, $\partial'^2 = \partial''^2 = \partial'\partial'' + \partial''\partial' = 0$

eg: for $(C_\cdot, \partial^C), (D_\cdot, \partial^D) \in Ch$ R-mod. then we have double complex $\{(C_p \otimes_R D_q), \partial^C, \partial^D\}$

$\mathrm{Tot}(C_{\cdot\cdot}) \in Ch$-$R$-Mod is given by $(\mathrm{Tot}\, C_{\cdot\cdot})_n = \bigoplus_{p+q=n} C_{p,q}$, $\partial = \partial' + \partial''$

Two natural filtrations: $${}'F_p \mathrm{Tot}(C)_n = \bigoplus_{\substack{s+t=n \\ s \le p}} C_{s,t}$$

$${}''F_p \mathrm{Tot}(C)_n = \bigoplus_{\substack{s+t=n \\ t \le p}} C_{s,t}$$

For 1, $E^1_{p,q} = H_{p+q}({}'F_p/{}'F_{p-1}, \partial)$

$$= H_q(C_{p,*}\,;\partial'')$$

$$\begin{array}{ccc} d_1 : E^1_{p,q} & \longrightarrow & E^1_{p-1,q} \\ \| & & \| \\ H_q(C_{p,*},\partial'') & & H_q(C_{p,*},\partial'') \end{array}$$ is the differential induced by ∂'

$$\Rightarrow \quad E^2_{p,q} = H_{p+q}(H_q(C_{*,*},\partial''),\partial')$$

For II we get,

$$E^1_{p,q} = H_p(C_{*,q},\partial')$$

$$E^2_{p,q} = H_{p+q}(H_p(C_{*,*}),\partial'),\partial'')$$

Both converge to $\mathrm{Tot}(C_{**})$.

Th^m: $\mathrm{Tor}^R_i(M,N) = \mathrm{Tor}^R_i(N,M)$

Proof: Let $P^M_* \longrightarrow M$, $P^N_* \longrightarrow N$ be projective/free/flat resolutions

Consider bicomplex $P^M_\cdot \otimes P^N_\cdot$ ⟿ each component is projective/free/flat ?

$$E^1_{p,q} = H_q(P^M_p \otimes P^N_\cdot,\ \partial^N) = P^M_p \otimes H_q(P^N_\cdot,\partial^N) = \begin{cases} 0 & \text{if } q>0 \\ P^M_p \otimes N & \text{if } q=0 \end{cases}$$

$$E^2_{p,q} = \begin{cases} \mathrm{Tor}^R_p(M,N) & \text{if } q=0 \\ 0 & \text{else} \end{cases}$$

$$\|$$

$$E^\infty_{p,q} = \mathrm{Tot}(P^M_\cdot \otimes P^N_\cdot)$$

Now look at the second filtration

$$''E^1_{p,q} = \begin{cases} M \underset{R}{\otimes} P^N_q & \text{if } p=0 \\ 0 & \text{else} \end{cases}$$

$$''E^2_{p,q} = \begin{cases} \mathrm{Tor}_q(N,M) & \text{if } p=0 \\ 0 & \end{cases} = {''E}^\infty_{p,q} = H_{p+q}(\mathrm{Tot}\ P^M_\cdot \otimes P^N_\cdot)$$

Together this implies $\mathrm{Tor}^R_p(M,N) = \mathrm{Tor}^R_p(N,M)$