§ Layers of Taylor Tower

$F \leftarrow$ homotopy functor

We have a Taylor tower $\longrightarrow P_k F \longrightarrow P_{k-1} F \longrightarrow \cdots$

- k^{th} layer : $D_k F := \text{hofiber}\left(P_k F \longrightarrow P_{k-1} F \right)$

- k-reduced : $P_{k-1} F \simeq *$
  - 1-reduced + 1-excisive = linear
  - k-reduced + k-excisive = k-homogenous

Prop: $D_k F$ is k-homogenous.

Def^n: $F : \mathcal{C}^k \longrightarrow \mathcal{D}$ is multi-linear if it is $(1, \cdots, 1)$-reduced & excisive.
Note : $F \circ \Delta$ is k-homogenous.

Lemma: $F : Top_* \longrightarrow Top_*$ homotopy $\Rightarrow$ $\forall k$ $\exists$ a fibration
$$P_k F \longrightarrow P_{k-1} F \longrightarrow R_k F$$
where $R_k F$ is k-homogenous.

Th^m: $\Omega^\infty : [Top_*, Sp_*] \longrightarrow [Top_*, Top_*]$ has an inverse B^∞ up to a weak-equivalence when restricting to k-homogenous functors.

Pf: By above sequence
$$F \simeq \Omega R_k F \simeq \cdots \text{ Iterate.}$$
This gives us $\underline{R_k F}$.

Def: $F : \mathcal{C}^k \longrightarrow \mathcal{D}$ is symmetric if $\forall \sigma \in \Sigma_k$ $\exists$ a homeomorphism
$$F(x_1, \cdots, x_k) \xrightarrow{F_\sigma} F(x_{\sigma_1}, \cdots, x_{\sigma_k})$$
and $F_\sigma \circ F_\tau = F_{\sigma \cdot \tau}$

Note : $\exists$ a Σ_k action on a symmetric functor F hence also on $F \circ \Delta_k$. If F is symmetric multilinear then $F \circ \Delta_k$ is k-homogenous.
If F is spectra valued $(F \circ \Delta_k)_{h\Sigma_k}$ k-homogenous (as hocolims, P_k commute)

Def: $L_k(Top_*, Sp_*)$ = symm, multilinear functors $\quad H_k(Top_*, Sp_*)$ = k-homogenous functors

We have just constructed a map

$$- \cdot \Delta_k : L_k \longrightarrow H_k$$

Now we want to construct its inverse.

Def: $F: Top_* \to \mathcal{D}$. The k^{th} cross-effect cube of F is:

$$\left\{ S \longmapsto F\left(\bigvee_{i \notin S} X_i\right) \right\} =: CR_k F(X_1, \dots, X_k)$$

- $cr_k F: Top_*^k \longrightarrow \mathcal{D} = \text{tofiber}\,(CR_k F(X_1, \dots, X_k))$

eg: $cr_2 F(X, Y) = $ tfiber
$$\left\{ \begin{array}{ccc} F(X \vee Y) & \longrightarrow & F(Y) \\ \downarrow & & \downarrow \\ F(X) & \longrightarrow & F(*) \end{array} \right\}$$

Note: $cr_k F$ is symmetric, homotopical, $(1, \dots, 1)$-reduced.

Prop: If F is $(k-1)$ excisive then $cr_k F \simeq *$
If F is k-excisive then $cr_k F$ is multi-linear.

- $cr_k P_k F \simeq cr_k(\text{hofib}\,(P_k F \longrightarrow *))$

$$\simeq \text{hofib}\,(cr_k P_k F \longrightarrow cr_k *)$$

$$\simeq \text{hofib}\,(cr_k P_k F \longrightarrow cr_k P_{k-1} F)$$

$$\simeq cr_k D_k F$$

Prop: If $\theta: F \to G$ is k-homogeneous then $cr_k F \xrightarrow[cr_k \theta]{\simeq} cr_k G$ is w.e.

Th^m: $(- \circ \Delta)_{h\Sigma_k} : L_k \rightleftarrows H_k : cr_k$ are inverses up to weak equivalence.
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- For spectra valued F:

$$D_k F \simeq (cr_k D_k F \cdot \Delta_k)_{h\Sigma_k}$$

$$\left[\begin{array}{l} D^{(k)} F := cr_k(D_k F) \\ \qquad = k\text{-fold differential} \end{array} \right]$$

for top valued F:

$$D_k F \simeq \Omega^\infty (B^\infty cr_k D_k F \circ \Delta_k)_{h\Sigma_k}$$

Lemma: $L \in \mathcal{L}_k$ set $\underline{C}_L := L(S^0, \dots, S^0)$
for all finite complexes $\exists$ a w.e.

$$\underline{C}_L \wedge X_1 \wedge \dots \wedge X_k \longrightarrow L(X_1, \dots, X_k)$$

Apply to $D^{(k)}F$ to get:

$$\partial^{(k)}F(*) = D^{(k)}F(S^0, \dots, S^0)$$
$$= cr_k D_k F(S^0, \dots, S^0)$$

$$\boxed{\Rightarrow \quad D_k F(X) \simeq \left(\partial^{(k)}F(*) \wedge X^{\wedge k}\right)_{h\Sigma_k}}$$