Surfaces

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1 Real projective space $\mathbb{RP}^2$

One way to define the projective plane is by gluing the antipodal points on the disc. In this section we'll see how this corresponds to lines in $\mathbb{R}^3$.

Consider the upper hemisphere

$$S^2_+ = \{(x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$$

Every line in $\mathbb{R}^3$ intersects S^2_+ in exactly one point, unless the line is along the $x - y$ plane in which case it intersects S^2_+ boundary in antipodal points. Thus the set of lines in $\mathbb{R}^3$ is in 1-1 correspondence with the points on S^2_+ but with the antipodal points on the boundary circle glued, but this is precisely $\mathbb{RP}^2$!

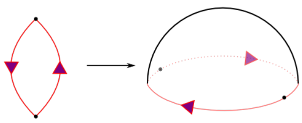

Fig. 1: Real projective plane

Exercise 1.1. Describe the space $\mathbb{RP}^1$, the space of lines in $\mathbb{R}^2$.

2 Poincaré conjecture

Poincaré conjecture is perhaps the most celebrated conjecture in the theory of manifolds. It's solution(s) led to creation of a lot of amazing mathematics. For 3 dimensional manifolds the conjecture is as follows.

Theorem 2.1 (Poincaré conjecture)**.** *Let M be a compact connected 3 dimensional manifold (without boundary). Suppose every loop on M can be shrunk to a point then M is homeomorphic to the 3 dimensional sphere S^3.*

There is a generalization of this conjecture to higher dimensions which requires a little more language to state. Surprisingly the conjecture was first proven to be true for dimensions ≥ 5 by Smale, next for dimension 4 by Freedman and finally for dimension 3 by Perelman. The conjecture is **false** in the 'category' of smooth manifolds i.e. if we replace homeomorphism by something stronger like diffeomorphism. The fact that the conjecture is true in the topological 'category' but false in the smooth 'category' implies the existence of **exotic manifolds**. Exotic spheres were first discovered by Milnor and only exist in dimensions 7 and higher. Freedman proved that $\mathbb{R}^4$ is the only Euclidean space which supports an exotic structure.

The Poincaré conjecture in dimension 2 follows from the classification theorem for surfaces.

Theorem 2.2 (Classification of surfaces)**.** *Every compact connected 2-dimensional manifold (without boundary) is homeomorphic to one of the following:*

1. S^2
2. $T^{\#k}$ *for some* k
3. $(\mathbb{RP}^2)^{\#k}$ *for some* k

Exercise 2.3. Find a loop on each of the following spaces T, $\mathbb{RP}^2$ and K which cannot be shrunk to a point.

Exercise 2.4. Show that for any compact connected manifold M there is a loop in each of the spaces $M\#T$ and $M\#\mathbb{RP}^2$ which cannot be shrunk to a point.

Exercise 2.5. For a compact connected 2 dimensional manifold M (without boundary) show that if every loop on M can be shrunk to a point then M is homeomorphic to S^2.