Borels Theorem

→ Maps between spectral sequences: $f:(E_r,d_r)_{r>k} \longrightarrow (\tilde{E}_r,\tilde{d}_r)_{r>k'}$ suitable compatibility conditions

eg: 1)

$$\begin{tikzcd} F \arrow[r] \arrow[d] & E \arrow[r, "\pi"] \arrow[d, "f"] & B \arrow[d] \\ \tilde{F} \arrow[r] & \tilde{E} \arrow[r, "\tilde{\pi}"] & \tilde{B} \end{tikzcd}$$
} fibrations

maps between the Serre SS E_2 page onwards and the map $E_\infty(\tilde{\pi}) \longrightarrow E_\infty(\pi)$

$$Gr(H^*(E)) \cong \underset{\parallel}{} \; {A_{p+q}}/{A_{p+q-1}} \xrightarrow{f_\infty^{p,q}} {\tilde{A}_{p+q}}/{\tilde{A}_{p+q-1}} \rightsquigarrow Gr(H^*(\tilde{E}))$$

$$f_\infty^{p,q} = f^*$$

2) If $f_r: E_r \longrightarrow \tilde{E}_r$ is an isomorphism for $r \geq m$, $E, \tilde{E}$ first quadrant map then objects computed by $E_r, \tilde{E}_r$ are isomorphic if f_∞ comes from a global map.

Prove this.

Comparison Theorem:

Let $E_r, \tilde{E}_r$ be cohomologically graded first quadrant SS with $E_2^{p,q} = E_2^{p,0} \otimes E_2^{0,q}$, $\tilde{E}_2^{p,q} = \tilde{E}_2^{p,0} \otimes \tilde{E}_2^{0,q}$.

Assume $f_r: E_r \longrightarrow \tilde{E}_r$ such that

1) $f_2^{p,q} = f_2^{p,0} \otimes f_2^{0,q}$ with $f_2^{0,q}$ isomorphism $\forall q$

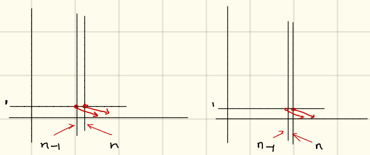
2) $f_\infty^{p,q}$ is an isomorphism $\forall p,q$

Then $f_2^{p,q}$ is an isomorphism $\forall p,q$.

Proof: Assume $f_2^{p,q}$ is an isomorphism for $p \leq n$

$\Rightarrow$ $f_r^{p,q}$ is isomorphism for $p \leq n-r+1$ and

$f_r^{p,q}$ is injective for $n-r+1 < p \leq n$



Now look at the transgression:

$$\begin{tikzcd} E_{n+1}^{0,n} \arrow[r, "d_{n+1}"] \arrow[d] & E_{n+1}^{n+1,0} \arrow[d] \\ \tilde{E}_{n+1}^{0,n} \arrow[r, "\tilde{d}_{n+1}"] & \tilde{E}_{n+1}^{n+1,0} \end{tikzcd}$$

here $f_{n+1}^{0,n}$ is still an isomorphism and $f_{n+2}^{0,n} = f_\infty^{0,n}$, $f_{n+2}^{n+1,0} = f_\infty^{n+1,0}$

Use five lemma here ⇝ to get $f_{n+1}^{n+1,0}$ is an iso.

Next look at $f_n^{1,n-1}$. Use your best argument skills here to conclude $f_n^{n+1,0}$ is an iso. And keep backtracking.

Def: A-graded ring of finite type has simple system of generators $\alpha_1 \dots \alpha_k \dots$ $|\alpha_i| \le |\alpha_{i+1}|$
A has a R-module basis of $\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_k}$ $i_1 < i_2 < \dots < i_k$ $\forall (i_1, \dots i_k)$

eg: $\Lambda(x_1, \dots x_n)$, $\mathbb{F}_2[x^k]$

Th^m (Borel): Let E be a cohomological first quadrant multiplicative SS such that

a) $E_2^{p,q} = E_2^{p,0} \otimes E_2^{0,q}$ as rings

b) $E_\infty^{p,q} = \begin{cases} 0 & (p,q) \neq (0,0) \\ R & (p,q) = (0,0) \end{cases}$

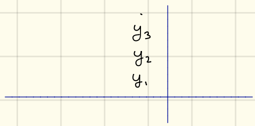
c) $E_2^{0,*}$ has a simple system of transgressive generators α_i

Then $E_2^{0,*}$ is a polynomial algebra on any lift of $d(\alpha_i)$.

eg: 1) $H^*(SO(n); \mathbb{F}_2)$ has simple system of generators. Show these are transgressives.
$H^*(U(n); \mathbb{Z}) = \Lambda(x_1 x_2 \dots x_n)$
$\Rightarrow$ $H^*(BSO(n); \mathbb{F}_2) = \mathbb{F}_2[w_2 \dots w_n]$ $|w_i| = i$
$H^*(BU(n); \mathbb{Z}) = \mathbb{Z}[c_1 \dots c_n]$ $|c_i| = 2i$

Converse: $F \to * \to B$ fibration with $\pi_1(B) = 0$ and $H^*(B;R) = R[x_1 \dots x_k]$ then $H^*(F;R)$ supports a simple system of transgressive generators.

Proof: Invoke the Eilenberg Moore SS:
$$H^*(F;R) \Leftarrow \mathrm{Tor}_{H^*(B;R)}(R,R) = \Lambda(y_1, y_2, \dots, y_k) \qquad |y_i| = (1, |x_i|)$$
y_i gives an element
in $H^{|x_i|-1}(F;R)$
which will be a basis as a R-module.
Why transgressive?



Caution: Does not apply to $\Omega S^3 \to * \to S^3$ because the generators of $H^*(\Omega S^3)$ are not transgressive :(

Proof: Let $[\beta_i] = d_{|\alpha_i|+1}(\alpha_i)$ $\beta_i \in E_2^{0,|\alpha_i|+1}$
Consider $(\tilde{E}_r, \tilde{d}_r)$ with $\tilde{E}_2^{**} = E_2^{0,*} \otimes R[\beta_1, \beta_2, \dots]$
Differentials defined as follows:
$\tilde{d}_{|\alpha_i|+1}(\alpha_i) = \beta_i$ extended by multiplication

$S_k = \{\alpha_i \mid |\alpha_i| = k\}$, $\tilde{S}_k = \{R\text{-module of } E_2^{0,*} \text{ generated by } \alpha_{i_1} \cdots \alpha_{i_j} \quad i_j \in \Lambda_k \quad i_1 < i_2 < \dots < i_j\} \oplus R$

$E_2^{0,*} = \tilde{\Lambda}_1 \otimes M_2$, $M_2 = \tilde{\Lambda}_2 \otimes \tilde{\Lambda}_3 \otimes \dots$

$E_2 = (\tilde{\Lambda}_1 \otimes M_2) \otimes R[\beta_1 \cdots]$

$d_2|_{M_2} = 0 \qquad d_2 \alpha_i = \beta_i \qquad |\alpha_i| = 1$

$= \bar{\Lambda}_1 \otimes M_2 \otimes R[\beta_{s+1} \cdots] \qquad \bar{\Lambda}_1 = \tilde{\Lambda}_1 \otimes R[\beta_1 \cdots \beta_s]$ $|\beta_i| = 2$ is simply the Koszul Resolution

$E_3 = M_2 \otimes R[\beta_{s+1} \cdots] \quad \Leftarrow$

So $(\tilde{E}_r, \tilde{d}_r)$ is a well defined SS.

Now construct $f_r : \tilde{E}_r \longrightarrow E_r \qquad \alpha_i \longrightarrow \alpha_i \ , \quad \beta_i \longrightarrow$ lift of $d_{|\alpha_i|+1}$

Isomorphism on $\tilde{E}_2^{0,*} \longrightarrow E_2^{0,*}$ and $\tilde{E}_\infty^{*,*} \longrightarrow E_\infty^{**}$ so by the comparison theorem we have iso. ∵