Elliptic Curves / CRS of genus 1

- Λ lattice / $\mathbb{C}$, $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$
- $\mathbb{C} \longrightarrow \mathbb{C}/\Lambda \cong S^1 \times S^1 \cong \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$
  ↳ CRS with complex structure depending on Λ.

eg: $\Lambda' = a\Lambda \quad a \in \mathbb{C}^*$

$$\mathbb{C} \xrightarrow[\text{by } a]{\text{mult}} \mathbb{C} \rightsquigarrow \text{unique map} \quad \mathbb{C}/\Lambda \xrightarrow{\bar{a}} \mathbb{C}/\Lambda'$$

This is an isomorphism

This is the only way two ~~map~~ complex structures on $S^1 \times S^1$ can be biholomorphic.

- Suppose given
  $\varphi : \mathbb{C}/\Lambda \longrightarrow \mathbb{C}/\Lambda'$

  we can lift this, to some

  $\psi : \mathbb{C} \longrightarrow \mathbb{C}$ with $\psi(0) = 0$.

  Claim: $\psi(x+\lambda) - \psi(x) - \psi(\lambda)$ is constant

  Assuming this, we get $\psi'(x+\lambda) = \psi'(x) \quad \forall x, \lambda$

  So we get $\psi(x) = ax + b$

  $\psi(0) = 0$ gives $\psi(0) = b = 0 \Rightarrow \boxed{\psi(x) = ax}$

- Now if we are also given that φ is biholo.
  we will get $\boxed{\Lambda' = a\Lambda}$

- Weierstrass $\wp$ function / $\mathbb{C}$:

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda - \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$$

Doubly periodic: $\wp(z) = \wp(z+\lambda) \quad \lambda \in \Lambda$

$\wp$ - meromorphic function on $\mathbb{C}/\Lambda$

- $\mathcal{M}(\mathbb{C}/\Lambda) = \mathbb{C}(\wp, \wp')$

Can find e_1, e_2 basis for Λ s.t.

$$\left[\wp'^2 = 4(\wp - e_1)(\wp - e_2)\left(\wp - \frac{e_1+e_2}{2}\right) \right]$$

- $\mathbb{C}/\Lambda \xrightarrow{\simeq} E \hookrightarrow \mathbb{P}^2$

  ↳ projective closure of the affine curve
  $$y^2 = 4(x-e_1)(x-e_2)\left(x - \frac{e_1+e_2}{2}\right) \quad \longleftarrow (*)$$

~~In $\mathbb{C}^3$ look at~~

- $$\begin{vmatrix} 1 & 1 & 1 \\ \wp(x) & \wp(y) & \wp(x+y) \\ \wp'(x) & \wp'(y) & \wp'(x+y) \end{vmatrix} = 0$$

  ↳ This gives $[\wp(x):\wp'(x)]$, $[\wp(y):\wp'(y)]$, $[\wp(x+y),\wp'(x+y)]$ are colinear! Fantastic.

  This allows to translate the group structure of $\mathbb{C}/\Lambda$ to the usual group structure traditionally ~~done~~ given on elliplic curves.

- Any ~~sm~~ non-singular cubic curve can be put in the form of $(*)$.

  Having done that we get a way ~~of~~ getting from an elliptic curve to ~~the~~ a complex torus.

  γ be curve on E, Γ corresponding on $\mathbb{C}/\Lambda$

- $$\int_\gamma \frac{dx}{y} = \int_\Gamma \frac{d\,\wp(z)}{\wp'(z)}\,dz = \Gamma(b) - \Gamma(a)$$

  If Γ begins at 0
  $$\int_\gamma \frac{dx}{y} = \text{endpt. of } \Gamma$$
  (indentification with complex plane $\mathbb{C}$)

Def^n: Divisor: $D = \sum a_i\{x_i\} \longleftarrow$ finite sum, $x_i \in X$

$\mathrm{Div}(X) :=$ ~~set~~ grp of divisors

$\mathrm{Div}^0(X) :=$ deg 0

$$\mathrm{Pic}^0(X) := \frac{\mathrm{Div}^0(X)}{\text{Divisors coming from } \mathcal{M}(X)} \quad \text{(zeroes \& poles)}$$

$\parallel$

$J(X)$

↳ Jacobi. Pic

- Abel Jacobi : $\mathbb{C}/\Lambda \longrightarrow \mathrm{Pic}^{\circ}(\mathbb{C}/\Lambda)$ is an isomorphism.

In general, $\mathrm{Pic}^{\circ}(M^{2g})$ is a torus of genus g. with lot more structure.

- **Lemma** : Λ, Λ' lattices in $\mathbb{C}$ s.t. $\mathbb{C}/\Lambda \cong \mathbb{C}/\Lambda'$ as CRS then $\exists\, a \in \mathbb{C}^*$ such that $\Lambda' = a\Lambda$.

Hence we have a map

$\{\text{Lattices in } \mathbb{C}\} \longleftrightarrow$ Compact R.S. of genus 1

$\cong$

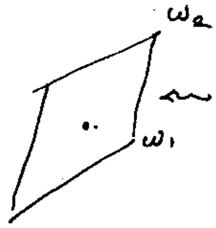
upper half plane $\mathbb{H}/Sl_2(\mathbb{Z}) \rightsquigarrow$ This space is infact a non-compact Riemann Surface

$\exists$ holomorphic $J : \mathbb{H}/Sl(2,\mathbb{Z}) \longrightarrow \mathbb{C}$ bijective

$\mathbb{C}$ becomes moduli space for ~~Rem~~ Complex Torii

- **Weirstrass $\wp$- function** :

Λ lattice in $\mathbb{C}$,



P is the Perimeter of this ||piped.

Lemma : $\displaystyle\sum_{0 \neq \omega \in \Lambda} \frac{1}{|\omega|^3} < \infty$

Proof : (ω_1, ω_2) be basis for Λ

Let $P = \{\alpha\omega_1 \pm \beta\omega_2 \mid \alpha, \beta \geq 0,\ \alpha+\beta = 1\} \cup \{-\alpha\omega_1 \pm \beta\omega_2 \mid \alpha, \beta \geq 0,\ \alpha+\beta = 1\}$

$$\text{LHS} = \sum_{n=1}^{\infty} \sum_{|n_1|+|n_2|=n} \frac{1}{|n_1\omega_1 + n_2\omega_2|^3}$$

$$= \sum_{n=1}^{\infty} \sum_{|n_1|+|n_2|=n} \frac{n^{-3}}{\left|\frac{n_1}{n}\cdot\omega_1 + \frac{n_2}{n}\cdot\omega_2\right|^3}$$ $\rightsquigarrow$ This $\in P$

$$\leq \sum_{n=1}^{\infty} \frac{K}{n^3}$$ for appropriate constant K

$$< \infty$$

Weierstrass $\wp$-function:

$$\wp(\omega) := \frac{1}{z^2} + \sum_{\substack{\omega \neq 0 \\ \omega \in \Lambda}} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right]$$

Claim: $\wp(\omega)$ converges uniformly on compact subsets of $\mathbb{C} - \Lambda$.

Proof: $D_\rho := \{z : |z| < \rho\}$

- if $|\omega| \in 2\rho$ and $z \in \bar{D}_\rho$ then

$$\rightarrow \frac{|z|}{|\omega|} < \frac{1}{2} \quad \text{and} \quad |z-\omega| > \frac{|\omega|}{2}$$

For E compact, suppose $E \subseteq D_\rho$, then

For $z \in E$, $|\omega| > 2\rho$ we get

$$\left| \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right| \leq \frac{10\rho}{|\omega|^3}$$

Since $\sum \frac{1}{|\omega|^3} < \infty$ and there are only finite lattice points inside $D_{2\rho}$ we are done. ▫

$\longrightarrow$ Let $U \subseteq \mathbb{C}$ open, $\{f_n\}$ seq of holomorphic functions on U, converging uniformly on compact subsets of U to f

Note: For a locally compact space, $f_n \to f$ uniformly on compact subsets iff it converges locally uniformly.

Γ be a closed path in a disc in U,

$$\left| \int_\Gamma (f_n - f)\, dz \right| \leq \left| \int_\Gamma (|f_n| - |f|)\, (dz) \right|$$

$$\leq \sup_{z \in \Gamma} |(f_n - f)(z)| \cdot \text{length}(\Gamma)$$

$$< \varepsilon \qquad \text{for } n > N$$

$\hookrightarrow$ N does not depend on z because of uniform convergence

Hence, $\int_\Gamma f\, dz = \lim_{n} \int_\Gamma f_n\, dz = 0$

Hence, f is holomorphic (because f haves a primitive if it is 0 on every closed loop)

For sufficiently small circles C, we have for $z \in$ interior of C

$$|f_n'(z) - f'(z)| = \left| \frac{1}{2\pi i} \int_C \frac{f_n(w) - f(z)}{(w-z)^2} \, dw \right|$$

$$\leq K \sup_{z \in C} |f_n(w) - f(z)|$$

$\Rightarrow$ $f_n' \longrightarrow f'$ uniformly on interior of C, hence uniformly locally.

Applying this to $\wp(z)$ gives,

<u>Th^m</u>: $\wp(z)$ is holomorphic ~~and~~ on $\mathbb{C} \setminus \Lambda$

$\wp'(z) =$ term by term differentiation of series of $\wp(z)$

$$= -2 \sum_{\omega \in \Lambda} \frac{1}{(z-\omega)^2}$$

$\longrightarrow$ 1) $\wp(z) = \wp(-z)$, $\wp'(z) = -\wp'(-z)$

2) $\wp'(z+\omega) = \wp'(z)$

3) $\wp(z+\omega) = \wp(z)$ look at $(\wp(z+\omega) - \wp(z))'$

$\Rightarrow$ $\wp(z+\omega) - \wp(z) = c(\omega)$

set $z = -\frac{\omega}{2}$ $\Rightarrow$ $\underbrace{\wp\left(\frac{\omega}{2}\right) - \wp\left(-\frac{\omega}{2}\right)}_{= 0} = c(\omega)$ $\because \wp$ even

$\Longrightarrow$ $\wp, \wp'$ meromorphic on $\mathbb{C}/\Lambda$

4) $\wp$ has pole of order 2 at 0. $\because \wp = \frac{1}{z^2} +$ holo near 0.

and this is the only pole in $\mathbb{C}/\Lambda$

$\Rightarrow [\mathcal{M}(\mathbb{C}/\Lambda) : \mathcal{M}(\mathbb{P}^1)] = 2$

$\mathcal{M}(\mathbb{P}^1) \hookrightarrow \mathcal{M}(\mathbb{C}/\Lambda)$

and the image of z is $\wp$

So we identify $\mathcal{M}(\mathbb{P}^1)$ with $\mathbb{C}(\wp)$.

5) $\wp'(z) \neq -\wp'(-z)$

$\Rightarrow$ $\wp'(z) \notin \mathbb{C}(z) \hookrightarrow \mathcal{M}(\mathbb{C}/\Lambda)$

$\Rightarrow$ $\mathcal{M}(\mathbb{C}/\Lambda) = \mathbb{C}(\wp, \wp')$

and $\wp'$ satisfies a quadratic in $\wp$.

6) $\wp' \in \mathcal{M}(\mathbb{C}/\Lambda)$ has pole of order 3 at $0 \in \mathbb{C}/\Lambda$ & no other

for $\omega \in \Lambda$

$$\wp(z+\omega) = \wp(z)$$

$$\Rightarrow \wp'\left(\tfrac{\omega}{2}\right) = -\wp'\left(\tfrac{\omega}{2}\right)$$

$$\Rightarrow \wp'\left(\tfrac{\omega}{2}\right) = 0$$

in particular $\wp'$ has zeroes $\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}$ in $\mathbb{C}/\Lambda$

~~Hence~~, $\wp'$ has exactly there 3 zeroes.
and each 0 is a simple 0.

Let $e_1 = \wp\left(\frac{\omega_1}{2}\right)$, $e_2 = \wp\left(\frac{\omega_2}{2}\right)$, $e_3 = \wp\left(\frac{\omega_1+\omega_2}{2}\right)$

Then these are distinct

Because $\wp(z) - e_1$ and $\wp'(z)$ are 0 at $\frac{\omega_1}{2}$

So that $\frac{\omega_1}{2}$ is zero of order 2

and hence $\wp(z) - e_1$ has no other 0.

7) Define $f \in \mathbb{C}(\wp, \wp') = \mathcal{M}(\mathbb{C}/\Lambda)$ by

$$f := \frac{\wp'(z)^2}{(\wp(z)-e_1)(\wp(z)-e_2)(\wp(z)-e_3)}$$

Possible poles: $0, \frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}$

By counting orders of vanishiong one can ~~find~~ show that f has no poles!

$\Rightarrow$ f is constant and ~~bad~~ substitution in series will give $f(0) = 4$

$$\Rightarrow \quad \wp'(z)^2 = 4(\wp(z)-e_1)(\wp(z)-e_2)(\wp(z)-e_3)$$

Ex: $$G_k := \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{\omega^{2k}}$$

Show,

$$\wp'(z)^2 = 4\wp(z)^3 - \underbrace{140 \cdot G_2}_{g_2} \cdot \wp(z) - \underbrace{60\, G_3}_{g_3}.$$

- Λ be a lattice $\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ $\qquad \wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right]$

we have shown,

$\rightarrow$ $\wp$ is uniformly convergent on compact subsets of $\mathbb{C}/\Lambda$

$\rightarrow \wp'(z) = -2 \sum_{\omega \in \Lambda} \frac{1}{(z-\omega)^3}$

$\rightarrow \wp'(z)^2 = 4\wp(z)^3 - g_2 \wp(z) - g_3$

$\rightarrow \wp'(z)^2 = 4(\wp(z)-e_1)(\wp(z)-e_2)(\wp(z)-e_3)$

$e_1 = \wp(\frac{\omega_1}{2}) \quad e_2 = \wp(\frac{\omega_2}{2}) \quad e_3 = \wp(\frac{\omega_1+\omega_2}{2})$ e_i's distinct

$\rightarrow \mathfrak{m}(\mathbb{C}/\Lambda) = \mathbb{C}(\wp, \wp')$

Let E be affine curve given by $y^2 = 4(x-e_1)(x-e_2)(x-e_3)$
$\bar{E}$ be its projective closure.

<u>Claim</u>: $\bar{E}$ is a smooth projective curve.

- $\frac{\partial}{\partial y} = 0 \Rightarrow y = 0 \Rightarrow x = e_1$ or e_2 or e_3

at any $\frac{\partial}{\partial x} \neq 0$ as e_i's are distinct.

- At ∞; i.e. $0 = 4x^3$ solution is $[0;1:0]$

Enough to check $z = 4(x-e_1z)(x-e_2z)(x-e_3z)$ has non-zero gradient at $(0,0)$??

- By implicit function th^m;

$\bar{E}$ 1-dim complex manifold, compact

TFAE:

1) $\bar{E}$ is connected

2) $\phi: \mathbb{C}/\Lambda \longrightarrow \bar{E}$ isomorphism
$z \longmapsto (\wp(z), \wp'(z))$ and $0 \longmapsto [0,1,0]$

3) $\bar{E} \cong \mathcal{R}\mathfrak{m}(\mathbb{C}/\Lambda)$

4) If $\wp(a) = \wp(b)$ and $\wp'(a) = \wp'(b)$, then $a = b$

So $\wp, \wp'$ seperate points.

Proof: 3) $\Rightarrow$ 1) $\because \mathcal{R}(K)$ is connected for any function field F

4) $\Rightarrow$ 2) Look at the diagram

$$\begin{tikzcd}
\mathbb{C}/\Lambda \arrow[rr, "\phi"] \arrow[dr, "\wp"'] & & \bar{E} \arrow[dl, "x"] \\
& \mathbb{P}^1 &
\end{tikzcd}$$

fibre cardinality 2 $\rightsquigarrow \wp$ $\qquad$ $x \leftsquigarrow$ fibre cardinality 2

2) ⇒ 3) ?

2) ⇒ 4) $x : \bar{E} \longrightarrow \mathbb{P}^1$ seperate points
$y :$ hence so does $\wp, \wp'$

4) ⇒ 2) ?

It is well known that $\bar{E}$ 1) is true (Shafarevich pg 321)
Here is a different proof:

$$\begin{tikzcd} \mathbb{C}/\Lambda \arrow[rr, "\sim"] \arrow[dr, "\phi"'] & & \mathcal{R}\mathfrak{m}(\mathbb{C}/\Lambda) \arrow[dl, "(\wp, \wp')"] \\ & E & \end{tikzcd}$$

$$\mathbb{C}[\wp, \wp'] = \frac{\mathbb{C}[x,y]}{y^2 - 4(x-e_1)(x-e_2)(x-e_3)}$$

A point $(x_0, y_0) \in E$ corresponds to a maximal ideal of $\mathbb{C}[\wp, \wp']$
By localizing at a dvr, $= \left\{ \frac{f(\wp, \wp')}{g(\wp, \wp')} \;\middle|\; g(x_0, y_0) \neq 0 \right\}$

This dvr is a member of $\mathcal{R}\mathfrak{m}(\mathbb{C}/\Lambda)$ and maps to (x_0, y_0) in E under $(\wp_*, \wp'_*)$, so $(\wp_*, \wp'_*)$ is surjective. Since $\mathcal{R}\mathfrak{m}(\mathbb{C}/\Lambda)$ is connected, E is connected & we have $\bar{E}$ is connected.

• Addition Theorem: Via φ, $\bar{E}$ acquires the structure of an abelian group. ($\bar{E} \subseteq \mathbb{P}^2$, with addition $\oplus$ of Elliptic curve)

<u>Th^m</u>: Suppose $p_1, p_2, p_3 \in \bar{E}$, Then p_1, p_2, p_3 are colinear iff $p_1 \oplus p_2 \oplus p_3 = 0$

- if $p_1 = p_2$, tangent at p_1 meets curve in p_3
- if $p_1 = p_2 = p_3$ we have point of inflection

The only point of inflextion is $[0:1:0]$. The tangent line at $[0:1:0]$ is the line at ∞.

In terms of $\wp$, the addition theorem says for $x, y \in \mathbb{C}/\Lambda \setminus \{0\}$

$$(\wp(x), \wp'(x)),\ (\wp(y), \wp'(y)),\ (\wp(-x-y), \wp'(-x-y))$$

are colinear, i.e.

$$\begin{vmatrix} \wp(x) & \wp'(x) & 1 \\ \wp(y) & \wp'(y) & 1 \\ \underset{\substack{\uparrow \\ \text{even} \\ \text{function}}}{\wp(x+y)} & \underset{\substack{\uparrow \\ \text{odd} \\ \text{function}}}{-\wp'(x+y)} & 1 \end{vmatrix} = 0$$

Addition Theorem

Here x, y can be thought as points in $\mathbb{C}$ or $\mathbb{C}/\Lambda$.

__Proof__:

Fix $u \neq 0$, Define $q(t) := \begin{vmatrix} \wp(t) & \wp'(t) & 1 \\ \wp(u) & \wp'(u) & 1 \\ \wp(t+u) & -\wp'(t+u) & 1 \end{vmatrix}$ for $t \neq -u$

Check that $q(t)$ has poles at most $t=0, t=-u$.
By expanding $\wp(t)$, we can check that there is no pole at $t=0$.
And because $q(t) = -q(-t-u)$ q does not have pole at $t=-u$.

$\Rightarrow$ q is constant, $q(u) = 0 \Rightarrow q \equiv 0$.

- Addition th^m implies, that the group law on $\bar{E}$ is given by rational functions. (α, α'), (β, β'), (γ, γ') colinear

$$\Rightarrow \gamma = -\alpha - \beta + \frac{1}{4}\left(\frac{\beta' - \alpha'}{\beta - \alpha}\right)^2$$

$$\Rightarrow \wp(x+y) = -\wp(x) - \wp(y) + \frac{1}{4}\left[\frac{\wp'(x) - \wp'(y)}{\wp(x) - \wp(y)}\right]$$

- Every meromorphic function q on $\mathbb{C}/\Lambda$ satisfies an "addition th^m"
  i.e. $\exists$ a polynomial A s.t. $A(q(x), q(y), q(x+y)) = 0$.

__Abel's theorem__ for $\mathbb{C}/\Lambda$

Let $f \in \mathcal{M}(\mathbb{C}/\Lambda)$, $f \neq 0$. Let $a_1 \dots a_n$ be zeroes of f with repitions allowed.
Let $b_1 \dots b_n$ be poles. Then,

$$a_1 + \dots + a_n = b_1 + \dots + b_n \pmod{\Lambda}$$

Proof of Geometric addition th^m using Abel's theorem:

L line in $\mathbb{P}^2$ $\rightsquigarrow$ $\alpha x + \beta y + \gamma z = 0$ $\quad L \cap \mathbb{A}^2$

$\{P, Q, R\} = L \cap \bar{E}$

Let u, v, w be points inside a fundamental parallelopipedgram corr. to P, Q, R

we need to show $P \oplus Q \oplus R = 0$

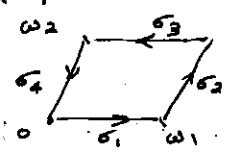
~~By abel's th^m~~ enough to show $u + v + w \equiv 0 \mod \Lambda$

Let f be the doubly periodic function, $f = \alpha \wp + \beta \wp' + \gamma$

zeroes of f are at u, v, w and poles at 0.

By Abel's th^m we are done.

- Pick a fundamental $\parallel$ such that a_i's, b_i's are in the interior



$$a_1 + \dots + a_n - b_1 - \dots - b_n$$
$$= \frac{1}{2\pi i} \int_\sigma \frac{z f'(z)}{f(z)} dz$$

- Using doubly periodicity

$$\int_{\sigma_3} \frac{z f'(z)}{f(z)} . dz = - \int_{\sigma_3} (z + \omega_2) \cdot \frac{f'(z)}{f(z)} dz = \frac{-\omega_2}{\cancel{2}} \int_{\sigma_3} \frac{f'(z)}{f(z)} dz + \int_{\sigma_1} \dots$$

- So the summation becomes

$$-(\omega_2) \int_{\sigma_3} \frac{f'(z)}{f(z)} + \omega_1 \int$$

because winding no this should be an integer

$$\text{ie} = n_1 \omega_2 + n_2 \omega_1$$
$$\equiv 0 \quad (\text{mod } \Lambda)$$

Divisors: $\deg : \operatorname{Div}(X) \longrightarrow \mathbb{Z}$

$$\sum n_i P_i \longmapsto \sum n_i$$

$\operatorname{Div}^0 X = \ker(\deg)$ subgrp of $\operatorname{Div}(X)$

meromorphic f^n divisors sit here

Principal divisors

$$\operatorname{Pic}^0(X) = \frac{\operatorname{Div}^0(X)}{\text{Principal divisors}} \qquad \operatorname{Pic}(X) = \frac{\operatorname{Div}(X)}{\text{Principal divisors}}$$

For $\mathbb{C}/\Lambda$ we have a map

$$j: \cancel{\mathbb{C}/\Lambda}\,\mathrm{Div}^0(\mathbb{C}/\Lambda) \longrightarrow \mathbb{C}/\Lambda$$

$$\sum n_i a_i \longmapsto \sum n_i a_i \quad , \qquad \text{surjective}$$

Abel's $\mathrm{th}^m \Rightarrow$ Principal divisors $\subseteq \ker j$

So we have an Abel-Jacobi map

$$AJ: \mathrm{Pic}^0(\mathbb{C}/\Lambda) \longrightarrow \mathbb{C}/\Lambda$$

~~So we~~

Abel Jacobi Th^m : AJ is an isomorphism.

- Geometric addition says

$$\{a_1\}+\{a_2\}+\{-a_1-a_2\} - 3.0 \text{ is principal}$$

- Take $a_2=0$ above to get

$$\{a_1\}+\{-a_1\} - 2.\{0\} \text{ is principal}$$

Lemma: $\{a_1\}+\{a_2\}+\cdots+\{a_n\}+\{-a_1-a_2\cdots-a_n\} - (n+1)\{0\} = 0$ in $\mathrm{Pic}^0(\mathbb{C}/\Lambda)$ ~~Div/Principal~~

Proof:

$n = 1, 2$ as above

$n > 3$

$$(a_1+\cdots+a_{n-1} + (-a_1-\cdots-a_{n-1}) - n.0)$$

$$+ (a_1+\cdots+a_{n-1}) + a_n + (-a_1-\cdots-a_{n-1}-a_n) - 3.0$$

$$- (a_1+\cdots+a_{n-1}) - (-a_1-\cdots-a_n) + 2.0 \quad = 0$$

$$\text{'' LHS}$$

Proposition: Let $a_1 \dots a_n,\ b_1 \cdots b_n \in \mathbb{C}$. Let $c = \sum_{i=1}^n a_i - \sum_{i=1}^n b_i$

Then,

1) $\sum a_i + \sum b_i = -\sum -a_i + \sum -b_i$ in $\mathrm{Pic}^0(\mathbb{C}/\Lambda)$

2) $\sum a_i - \sum b_i = \{c\} - \{0\}$ in $\mathrm{Pic}^0(\mathbb{C}/\Lambda)$

1) $\sum a_i + \sum -a_i = -\{\sum a_i - \sum a_i\} + (n+1)\{0\}$

$= -\{\sum b_i - \sum b_i\} + (n+1)\{0\} = \sum b_i + \sum -b_i$

2) $(-\sum b_i + \{0\}) - (\{-\sum a_i\} + \{c\}) = +\{a_i - c\} - \{\sum b_i\} = 0$ $\square$

Using 1)

$j: \mathrm{Div}^0(\mathbb{C}/\Lambda) \longrightarrow \mathbb{C}/\Lambda$ induces

$AJ: \mathrm{Pic}^0(\mathbb{C}/\Lambda) \longrightarrow \mathbb{C}/\Lambda$

Abel Jacobi: AJ is an isomorphism.

Proof: Enough to show ~~ke~~ $x \in \ker j \Rightarrow x$ principal

$x \in \ker j \Rightarrow x = \sum a_i - \sum b_i$

$= \{c\} - \{0\}$ + principal

$=$ ~~principal~~ 0 + principal ~~mod?~~ $\because x \in \ker, \Rightarrow c = 0$

$\Rightarrow x \in$ principal.

Example: $D = \{p\} - \{0\}$. Suppose D were principal then ramification degree would be 1. Making Riemann surface isomorphic to $\mathbb{C}\mathbb{P}^1$.

$L(D) := \{f \in \mathcal{M}(X) \mid f \neq 0,\ (f) + D \geq 0\} \cup \{0\}$

- Effective divisor $:= D = \sum n_i p_i$ s.t. $n_i \geq 0\ \forall i$
- Equivalence $-$ $D \sim D'$ if $D - D' =$ principal divisor

Riemann:

$$\dim L(D) \geq \deg D + 1 - g$$

Riemann-Roch:

$$\dim L(D) - \dim L(K-D) = \deg D + 1 - g$$

$K =$ "canonical divisor"

For $X = \mathbb{C}/\Lambda$, ~~$\dim L(K-D) = 0$ always because degree of a canonical divisor is $2g-2$.~~

~~$\dim L(D) = \begin{cases} 0 & \text{if } \deg D \leq 0 \\ 1 & \text{if } \deg D = 0 \end{cases}$~~

- $\deg D < 0 \Rightarrow \dim L(D) = 0$
- $\deg D = 0$ $\quad f \in \mathcal{O}_D \Rightarrow (f) + D \geq 0$

$\deg((f) + D) = 0$

$\Rightarrow (f) + D = 0$

$\Rightarrow D$ principal

$L(D) = \mathbb{C}^1$ if D principal

$= 0$ else