PROBLEM SET 11

Part 1 - Sequences

Q.1. and Q.3. are basically proofs from the book, if you get stuck you should consult them.

Q.1. (1) Let a_n be a sequence such that $a_n \neq c$, for any n, and $\lim_{n\to\infty} a_n = c$. Using the definition, show that if $\lim_{x\to c} f(x) = l$ then $\lim_{n\to\infty} f(a_n) = l$.

(2) Write down the definitions of $\lim_{x\to c} f(x) \neq l$ and $\lim_{n\to\infty} a_n \neq k$.

(3) Show that if $\lim_{x\to c} f(x) \neq l$ then there is a sequence a_n such that $\lim_{n\to\infty} a_n = c$ and $\lim_{n\to\infty} f(a_n) \neq l$.

(4) Come up with a definition for $\lim_{n\to\infty} a_n = \infty$. Let a_n be a sequence such that $\lim_{n\to\infty} a_n = \infty$. Show that if $\lim_{x\to\infty} f(x) = l$ then $\lim_{n\to\infty} f(a_n) = l$.

Q.2. Do problem 1 (i - ix) and 2 from the book on Pg. 453-454. There are a lot of problems but as before all of them are very short and require a very small argument.

You do not have to be very rigorous for these problems, for most (but not all) problems you can use the part (4) of Q.1.

Q.3. (1) Find a sequence a_n such that a_n is bounded from both above and below but $\lim_{n\to\infty} a_n$ does not exist. No proof needed, draw a picture.

(2) Let a_n be a non-increasing sequence bounded below. Let $A = \{a_n\}$.

(a) Argue that $\inf A$ exists.

(b) Show that $\lim_{n\to\infty} a_n = \inf A$.

Q.4. (1) Prove that if $0 < a < 2$ then $a < \sqrt{2a} < 2$.

(2) Prove that the sequence

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \cdots$$

converges using Theorem 2.

(3) Find the limit.

Q.5. Let $0 < a_1 < b_1$ and define

$$a_{n+1} = \sqrt{a_n b_n} \text{ and } b_{n+1} = \frac{a_n + b_n}{2}$$

(1) Use Theorem 2 to show that the sequence a_n and b_n converge.

(2) Show that they converge to the same limit.

Part 2 - Sequences continued

Q.6. Prove directly using the definition of Cauchy sequence, that if a_n is a non-decreasing sequence bounded from above then a_n is a Cauchy sequence.*

Q.7. Q.9 from Ch.22 on Pg.455.

Q.8. (1) Using integrals show that

$$\frac{1}{n+1} < \log(n+1) - \log(n) < \frac{1}{n}$$

(2) Using Theorem 2 show that the following sequence converges.

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log(n)$$

Q.9. Prove that if $\lim\limits_{n\to\infty} a_n = l$ then

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = l$$

(Hint: As a_n converges to l, for every $\epsilon > 0$ there is some constant N such that for all $n > N$ the inequality $|a_n - l| < \epsilon$ holds. For $n > N$ break $\frac{a_1+a_2+\cdots+a_n}{n}$ into two fractions and show that one fraction can be made arbitrarily small and the second fraction is close to l.)

* Hint: Estimate $a_n - a_m$ using $\sup\{a_n\}$.

Part 3 - Integral Computations

Q.10. For this week do Q.4 problems - vi) to x) and Q.5) on Pg. 379-380 from Ch.19.