Reflection Groups

— Priyavrat

eg: Symmetry groups of platonic solids

<u>Affine Space</u>: $\mathbb{A}^n$

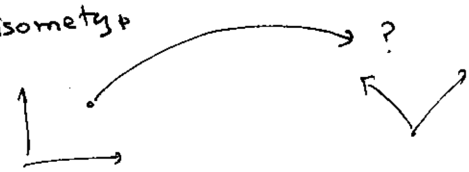
- $\mathbb{R}^n$ ~~vector space structure~~ minus origin info. One can recover vector space structure from affine space structure by fixing an origin
- Affine maps - Translations are allowed.
- Euclidean distance → induces metric
- Isometry

$$\boxed{\mathrm{Iso}(\mathbb{A}^n) \cong \mathbb{R}^n \rtimes O(n)}$$ → orthogonal matrices

Proof: every isometry is of the form $Ax + B$, $A \in O(n)$, $B \in \mathbb{R}^n$] This is what we need to show

This is ~~because~~ by brute force.

~~In general,~~ Need to show that image of an orthonormal basis completely determines the isometry.



~~f~~, Enough to show that if $e_1 \dots e_n$ is an orthonormal basis of $\mathbb{R}^n$, and $f(e_i) = e_i \ \forall i$ then $f \equiv$ identity.

This is because for $v \in \mathbb{R}^n$, we must have

$$f(\langle v, e_i \rangle) = \langle f(v), f(e_i) \rangle = \langle f(v), e_i \rangle \ \forall i$$

$$\Rightarrow \ v = f(v).$$

How does $O(n)$ conjugate?

$\cancel{A^{-1} \circ (Bx+c) \circ A}$

$$(x \mapsto A^{-1}x) \circ (x \mapsto Bx + c) \circ (x \to Ax)$$

$$= x \mapsto A^{-1}BAx + A^{-1}c$$

when $B = Id$, $x \mapsto x + A^{-1}c$

So translation by c becomes translation by $A^{-1}c$.

$$1 \longrightarrow \mathbb{R}^n \longrightarrow iso(\mathbb{A}^n) \longrightarrow O(n) \longrightarrow 1$$

$O(n)$ acts on $\mathbb{R}^n$.

$\Longrightarrow$ What are affine automorphisms of $\mathbb{A}^n$?

$\boxed{Aut(\mathbb{A}^n) \cong \mathbb{R}^n \rtimes Gl(\mathbb{R}^n)}$! How?

Need to show every automorphism is of the form $Ax + b$.

So enough to show, if $f \in Aut(\mathbb{A}^n)$ ~~then~~ such that for some basis $e_1 \ldots e_n$, & $f(e_i) = e_i$

then $f \equiv id$. How?

Now we cannot say inner product is preserved.

what do we mean by $Aut(\mathbb{A}^n)$.

Need a precise definition of $\mathbb{A}^n$?

Th^m: Isometries of S^2 ~~can~~ ~~be~~ is generated by reflections?

3 reflection Th^m Infact by at most 3-reflections.

Also true for $\mathbb{H}$, $\mathbb{R}^2$.

In general, it is true that $O(n)$ is generated by reflections.

Def^n: Given an isometry $\omega = (A, b)$ we say

det $A = 1$ $\Rightarrow$ ω orientation preserving

det $A = -1$ $\Rightarrow$ ω orientation reversing.

Def^n: $\omega \in Iso(\mathbb{R}^n)$ is called reflection, if

1) $\omega^2 = 1$, $\omega \neq$ identity

2) ω fixes a hyperplane.

Def^n: Rayleigh quotient of v wrt A

$$R_A(v) := \frac{\langle Av, v\rangle}{\langle v, v\rangle}$$

- $R_A(v) = R_A(tv)$
- $R_A(v)$ attains maxima, minima because R_A can be restricted to sphere, compact
- R_A differentiable

$$R_A'(v) = \frac{Av + A^t v - 2R_A(v).v}{\langle v, v\rangle}$$

Th^m: If $A \in O(n)$ then A has alleast one invariant subspace of dim 1 or 2.

Proof: Let v be a critical pt. of R_A.

$$0 = R_A'(v) = \frac{Av + A^t v - 2R_A(v)v}{\langle v, v\rangle}$$

$$\Rightarrow \quad Av + A^t v = 2R_A(v)v$$

But $A \in O(n) \Rightarrow A^t = A^{-1}$

$\dim \mathrm{span} \langle Av, A^{-1}v, v \rangle \leq 2.$

Claim: V is invariant under A.

- $A \cdot v \in V$
- $A \cdot (A^{-1}v) \in V$
- $A^2 v = -A(A^t v - 2R_A(v) \cdot v)$

$= -v + 2R_A(v) \cdot v \quad \in V$

So V is invariant under A and $\dim V \leq 2$.

$\longrightarrow \quad A \in O(n) \Rightarrow R_A(v) \leq 1$

equality iff $Av = \pm v$

Th^m: O_n generated by reflections. Each element by atmost n reflections.

Proof: Let $A \in O_n$

- if $\exists v.\ R_A(v) = 1$

$\Rightarrow \quad Av = \pm v$

Restrict A to $v^\perp$ and induct.

- if $\forall v,\ R_A(v) < 1$

$\Rightarrow \quad Av \neq \pm v$ for any v

$\Rightarrow \quad A$ has an invariant 2 dim subspace. — say V.

Def^n: $P \leq \mathbb{R}^n$, $\dim P = 2$, $A \in O(n)$

P - reflection plane of A if $AP = P$, $A|_P$ = reflection

- rotation plane of A if $AP = P$, $A|_P$ = rotation.

then V is a rotation plane of A.

Then by 2 reflections we can make A identity on V.

Induct on $V^\perp$.

Def^n: Affine hyperplane

H: zero set of $a_1x_1 + a_2x_2 + \cdots + a_nx_n - c = f_H(x)$

H^+ $\quad f_H(x) > 0$
H^- $\quad f_H(x) < 0$ } open half spaces

$\bar{H}^+$ $\quad f_H(x) \geq 0$
$\bar{H}^-$ $\quad f_H(x) \leq 0$. } closed half spaces

Hyperplane arrangement: Finite collection of hyperplanes

$$\Sigma = \{H_1, \ldots H_k\}$$

H_i's are called walls.

$a, b \in \mathbb{R}^n$ - similarly positioned: $f_{H_i}(a)$, $f_{H_i}(b)$ have same sign for $\forall \{H_i\}$ $\quad \forall H_i$.

Equivalence relation, Eq. classes are called faces of Σ.

$\mathcal{F}(\Sigma) :=$ set of all faces.

Support of a face F:

$$\text{supp } F := \begin{cases} \mathbb{A}^n \cap \{H \in \Sigma' \mid F \subseteq H\} \\ \mathbb{A}^n ~~\text{if}~~ \text{ no } H ~~\text{intersects } F~~ \end{cases}$$

$$\dim F := \dim \text{Supp} F$$

Lemma: For every $F \in \mathcal{F}(\Sigma)$

1) F is open, convex of supp F
2) $\partial F = (\bar{F} \setminus F^\circ)$ is union of faces of strictly lower dimension
3) $\bar{F} = \bar{F}' \Rightarrow F = F'$.

Chamber: Top dimensional faces.

$\mathcal{C}(\Sigma)$ - set of chambers.

Proposition: $C, C' \in \mathcal{C}(\Sigma)$. Following are equivalent

i) C, C' are seperated by one hyperplane } C, C' adjacent
ii) C, C' have a panel in common
iii) C, C' have a unique panel in common.

Lemma: Let $C, C' \in \mathcal{C}(\Sigma)$ be distinct with a common panel

i) Wall H that contains P is the only wall with a non-trivial intersection with $C \cup P \cup C'$.
ii) $C \cup P \cup C'$ is a convex set

Def^n: Gallery: Γ - sequence $C_0, \dots C_\ell \in \mathcal{C}(\Sigma)$

C_i & C_{i+1} are adjacent.

ℓ - length of gallery

Minimal Gallery: Shortest length gallery joining endpoints.

Geodesic

distance between two chambers is length of minimal gallery joining the two.

Prop: Any two chambers can be connected by a gallery. Hence by a minimal gallery. $d(C,D) = \#$ hyperplanes seperating C, D.

Prop. A gallery is minimal $\Leftrightarrow$ it crosses each wall exactly once.

Def^n: Intersections of closed half spaces. - Polyhedra - Δ

Bounded Polyhedra - Polytope

- If $F \in \mathcal{F}(\Sigma)$, $F \cap \Delta \neq \{\}$ then $F \subseteq \Delta$.

  i.e. Δ is union of faces.