Bousfield - Kuhn functor

— Gils

Rational homotopy groups of a pointed space X arise from

$[S^k, X]$ by inverting action of self-map

$S^k \xrightarrow{p} S^k$ for every prime p.

- What is lost? p-torsion

$$S^k \xrightarrow{p} S^k \longrightarrow S^k/p \quad \text{for mod } p \text{ Moore space}$$

and consider $[S^k/p, X] =: \pi_k(X; \mathbb{Z}/p)$ mod p homotopy grps

- Adams: $\exists$ a self-map

$$v_1 : \Sigma^d S^k/p \longrightarrow S^k/p$$

which induces an iso in K-theory.

for p - odd, $k \geq 3$, $d = 2(p-1)$

$p = 2$, $k \geq 5$, $d = 8$.

- v_1 on K^* acts by some multiplication of the Bott class.

- $$v_1^{-1}\pi_k(X; \mathbb{Z}/p) := \varinjlim \left([S^k/p, X] \xrightarrow{v_1} [\Sigma^d S^k/p, X] \xrightarrow{v_1} [\Sigma^{2d} S^k/p, X] \to \cdots \right)$$

- (mod p) v_1-periodic homotopy groups of X.
- These are (somewhat) computable. (Mahowald).

These are completly known for spheres.

Q. what info is lost? v_1-torsion.

continue: consider maps out of

$$\mathrm{cof}\,(\Sigma^d S^k/p \to S^k/p) =: V(2)$$

Beginning of a heirarchy in which rationalization is the "zeroth" map.

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§ Bousfield classes of finite (p-local) spectra

V finite (p-local) ~~space~~ spectrum, E spectrum

- E is V-null if $F(V,E) \simeq *$
- ~~the~~ finite spectrum V, W have same <u>Bousfield class</u> if

E is V-null $\iff$ E is W-null, $\forall$ spectra E

write $\langle V \rangle = \langle W \rangle$

write $\langle V \rangle \geq \langle W \rangle$ if E is V-null $\Rightarrow E$ is W-null $\forall E$.

$\langle \mathbb{S} \rangle$ is maximal $\qquad$ (E $\mathbb{S}$-null $\Leftrightarrow E \simeq *$)

$\geq$

$\langle \mathbb{S}/p \rangle$ $\qquad$ (E is $\mathbb{S}/p$ local $\Leftrightarrow E$ is rational)

$\geq$

$\langle \Sigma^\infty V(2) \rangle$

$\vdots$

Def: A finite spectrum V is of <u>type n</u> if

$$K(m)_* V = 0 \qquad \text{for } m < n \text{ and}$$

$$K(n)_* V \neq 0$$

where $K(n)$ is the n^{th} Morava K-theory.

$K(0) = H\mathbb{Q}$, $K(1)$ "=" KU/p , $K(n)_* = \mathbb{F}_p[v_n^{\pm 1}]$, $|v_n| = 2(p^n - 1)$

- $\langle \mathbb{S} \rangle$ has type 0.

  $\langle \mathbb{S}/p \rangle$ has type 1.

  $\langle \Sigma^\infty V(2) \rangle$ has type 2.

Th^m (Mitchell) Type n-spectra ~~p~~ exist for all n.

Th^m (Hopkins-Smith) For finite spectra V, W

$$\langle V \rangle \geq \langle W \rangle \Leftrightarrow \text{type of } V \leq \text{type } W$$

i.e. the poset of Bousfield classes of finite p-local spectra is iso to $\mathbb{Z}_{\geq 0}$.

Th^m: (Hopkins-Smith)

A finite type spectrum V admits a v_n self-map

$$v_n : \Sigma^d V \longrightarrow V \quad \text{meaning}$$

$$K(i)_* v_n = \begin{cases} \text{an iso} & \text{if } i = n \\ 0 & \text{if } i \neq n. \end{cases}$$

eg: v_0 self map : $\mathbb{S} \xrightarrow{p} \mathbb{S}$

v_1 Self map : Adams map $\Sigma^d \mathbb{S}/p \xrightarrow{v_1} \mathbb{S}/p$.

Step 0: Start with $[S^k, -]$ and invert p

1: consider $[S^k/p, -]$, invert v_1.

2: consider $[V(2), -]$, invert v_2

$\vdots$

Step n: consider $V(n) = \text{cofib}\left(\Sigma^d V(n-1) \xrightarrow{v_{n-1}} V(n-1)\right)$

consider $[V(n), -]$, invert v_n.

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§ v_n periodic homotopy groups of spaces:

- Type-n space $V(n)$ with a v_n self map

$$v : \Sigma^d V(n) \to V(n)$$

form

$$v_1^{-1}\pi_*(X; V(n)) := \operatorname{colim}\left(\begin{array}{c}[V(n), X] \\ \downarrow v^* \\ [\Sigma^d V(n), X] \\ \downarrow v^* \\ [\Sigma^{2d} V(n), X] \\ \downarrow \cdots \end{array}\right)$$

- These are the v_n-periodic homotopy groups of coeffs in $V(n)$.
- These homotopy groups are periodic with period d.

- Slightly better construction:

spectrum : $(\Phi_v X)_0 = \mathrm{Maps}_*(V(n), X)$

$$(\Phi_v X)_{k} = (\Phi_v X)_{d+k} \qquad \text{for any } k.$$

Structure maps : $(\Phi_v X)_{kd} \longrightarrow \Omega^d (\Phi_v X)_{(k+1)d}$

adjoint

$$v : \Sigma^d V(n) \longrightarrow V(n) \qquad \circ \ \mathrm{Maps}(-, X)$$

- Φ_v = telescopic functor associated to $V(n)$ (and v).

§ The Bousfield-Kuhn functor : at height n.

$\Phi : Top_* \longrightarrow Spectra$, packages all the Φ_V's into 1.

for different choices of type n spectrum $V(n)$.

Properties :

1) $F(\Sigma^\infty V(n), \Phi(X)) \simeq \Phi_V(X)$

2) $\Phi \Omega^\infty \simeq L_{T(n)}$

$T(n)$ = telescope of a v_n self-map

$\text{hocolim}\left(\Sigma^\infty V(n) \xrightarrow{v} \Sigma^{-d}\Sigma^\infty V(n) \xrightarrow{v} \cdots \right)$

$= \text{"} v^{-1} \Sigma^\infty V(n) \text{"}$

- $L_{T(1)} = L_{K(1)}$

3) $\Phi(X)$ is a $T(n)$ local spectrum.

<u>Kuhn</u> : Φ is characterized by w.e. by 1) & 3).

§ Analogue on rational homotopy :

Slogan : Φ is like the forgetful functor

$L_k(Ch_{\mathbb{Q}}) \longrightarrow Ch_{\mathbb{Q}}$

$$\mathrm{Ho}(\cancel{\mathrm{Sp}}(\mathrm{Top}_*^{\mathbb{Q}})^{\geq 2}) \underset{\Omega^\infty}{\overset{L_{\mathbb{Q}}\Sigma^\infty}{\rightleftarrows}} \mathrm{Ho}(\mathrm{Sp}_{\mathbb{Q}}) \simeq \mathrm{Ho}(\mathrm{Ch}_{\mathbb{Q}})$$

$\wr\!\!\mid$ Quillen

$$\mathrm{Ho}(\mathrm{Lie}(\mathrm{Ch}_{\mathbb{Q}})^{\geq 1})$$

$\downarrow\uparrow$ free - forget

$$\mathrm{Ho}(\mathrm{Ch}_{\mathbb{Q}}^{\geq 1})$$

- $\mathrm{Ho}(M_n^f)$ = category obtained from $\mathrm{Ho}(\mathrm{Top}_*)$ by inverting maps which induce isomorphisms on v_n-periodic homotopy groups.

(Bousfield showed this is possible)

analogue of $\mathrm{Top}_*^{\mathbb{Q}}$ for $n \geq 0$.

$$\mathrm{Ho}(L_{T(n)}\mathrm{Top}_*) \underset{\Phi}{\overset{\Theta}{\rightleftarrows}} \mathrm{Ho}(M_n^f) \underset{\Omega^\infty_{T(n)}}{\overset{L_{T(n)}\Sigma^\infty}{\rightleftarrows}} \mathrm{Ho}(L_{T(n)}\mathrm{Sp})$$

- f = finitary or something

- X - 1-conn, $\mathbb{Q}$ finite

$$[S^n, X] \otimes \mathbb{Q} \cong [S^n, |A^*_{PL}(X)|]$$

$$\cong [A^*_{PL}(X), A^*_{PL}(S^n)]$$

$$\cong [A^*_{PL}(X), H^*(S^n;\mathbb{Q})]$$

$$\cong [S(V), H^*(S^n;\mathbb{Q})] \qquad \text{minimal model}$$

$$\cong \mathrm{Hom}(V_n, \mathbb{Q})$$

$$\cong \mathrm{Hom}(H_n^{\mathbb{Q}}(A^*_{PL}(X)); \mathbb{Q})$$

~~[illegible]~~

$$H^{\mathbb{Q}}_{k-1}(A) \longrightarrow \bigoplus_{l+m=k} H^{\mathbb{Q}}_{l}(A) \otimes H^{\mathbb{Q}}_{m}(A)$$

need to shift degrees when we dualize.

[Baus- ~~[illegible]~~ Lemaire Conjecture]