

# CLASSICAL LIE GROUPS

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1. Low dimensional Lie groups	2
1.1. Heisenberg group	2
1.2. Affine transformation groups	3
2. $SU_2, SO_3, SL_2(\mathbb{R})$	5
2.1. Geometric description of $SO_{1,3}^+$	6
2.2. Geometric description of $SU_2(\mathbb{R})$	7
3. Representations of $SU(2)$ and $SO(3)$	8
4. Symmetric and Alternating powers of Representations	11
5. Adams operations	14
6. $\mathfrak{sl}(2, \mathbb{C})$ representations	16
6.1. Representations of $SU(2)$	17
6.2. Clebsch - Gordon decomposition	18
7. Spherical Harmonics	20
7.1. Casimir Element	21
8. Weyl Integration Formula	23
8.1. Computations	24
8.2. Proof of the maximal torus theorems	26
9. $U(n)$	27
9.1. Lie algebra	27
9.2. Representation ring	27
10. $SU(n)$	29
10.1. Lie algebra	29
10.2. Representation Ring	29
11. $SO(2n + 1)$	30
11.1. Lie algebra	30
11.2. Representation ring	30
12. $SO(2n)$	32
12.1. Lie algebra	32
13. $Spin(2n + 1)$	33
13.1. Representation ring	33
14. $Spin(2n)$	33
14.1. Representation ring	34
15. $Sp(n)$	34
15.1. Lie algebra	34
15.2. Representation ring	35
Contents	

ABSTRACT. I took these notes while studying the book by Brocker - Tom Dieck. These notes are mainly about computations of various classical Lie groups and contain very little theory. When in doubt  $G$  denotes a compact Lie group,  $T_e G$  denotes the Lie algebra of  $G$ .

## 1. LOW DIMENSIONAL LIE GROUPS

We already encountered  $SU(2), SO(3), SL(2)$  earlier. All of these are dimension 3.

Every dimension 1 manifold is homeomorphic to a circle or a line, and hence there aren't very many interesting dimension 1 Lie groups.

We can form 2 dimensional groups by multiplying two groups of dimension 1. I do not know if there are any other examples.

In dimension 3, there get some very interesting groups.

**1.1. Heisenberg group.** Let  $N \subset Gl(3, \mathbb{R})$  be the subgroup consisting of upper triangular matrices with 1 in the diagonals. The product structure is interesting. As a topological space  $N$  is isomorphic to  $\mathbb{R}^3$ , however the product is given by

$$(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + x_1 y_2)$$

Let  $Z$  be the subgroup of  $N$  consisting of matrices of the form

$$\begin{bmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, n \in \mathbb{Z}$$

**Proposition 1.1.**  *$Z$  is in the center of  $N$ .*

*Proof.*

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a & b+t \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

□

**Definition 1.2.**  $N/Z$  is called the Heisenberg group.

This is an overkill, but because  $N$  is contractible,  $N/Z$  is a classifying space for  $\mathbb{Z}$ .

**Proposition 1.3.**  *$N/Z$  does not have a finite dimensional faithful representation over  $\mathbb{C}$ .*

*Proof.* Let  $T$  be the subgroup isomorphic to  $\mathbb{R}$  sitting in the center of  $N$  consisting of matrices of the form

$$M_t := \begin{bmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Suppose there is a finite dimensional faithful representation  $N/Z \rightarrow Gl(n, \mathbb{C})$ .  $T/Z \cong S^1$  is central,  $\mathbb{C}^n$  decomposes as representations  $V_m$  of  $T/Z$  such that  $M_t$  acts on  $V_m$  via  $e^{imt}$ . Each  $V_m$  is also invariant under the action of  $N/Z$  and this is because  $T/Z$  is central in  $N/Z$ . Now we restrict our attention to  $V_m$  and should that they should all be trivial.

Now here is an interesting trick. It is easy to see that  $N$  is not abelian and the commutator has the property that  $[N, N] \subset T$ . Further by explicitly computing  $aba^{-1}b^{-1}$  for two general matrices  $a, b$  one can easily see that  $T \subset [N, N]$  and

so  $T/Z \subset [N/Z, N/Z]$ . Now a commutator should always map to determinant 1 matrices, and hence  $e^{ikmt} = 1$  for all  $t$  where  $k$  is the dimension of  $V_m$ . But this implies  $m = 0$ .  $\square$

The Lie algebra structure on  $N$  and hence on  $N/Z$  is induced from  $\mathfrak{gl}(3, \mathbb{R})$  is given by

$$[(x_1, y_1, z_1), (x_2, y_2, z_2)] = (0, 0, x_1y_2 - y_1x_2)$$

This gives us yet another Lie algebra structure on  $\mathbb{R}^3$  (we had 2 others coming from  $\mathfrak{su}(2, \mathbb{C})_{\mathbb{R}}, \mathfrak{sl}(2, \mathbb{R})$ ). It is possible to generalize this Heisenberg algebra to abstract vector spaces.

**Definition 1.4.** Let  $(V, \omega)$  be a symplectic vector space, then the corresponding Heisenberg algebra  $\mathfrak{H}(V)$  is a central extension of  $V$  and the Lie algebra structure is given by

$$0 \rightarrow \mathbb{R} \rightarrow \mathfrak{H}(V) \rightarrow V \rightarrow 0$$

$$[(\vec{v}, x), (\vec{w}, y)] = (0, \omega(\vec{v}, \vec{w}))$$

Here is the reason why this algebra is called the Heisenberg algebra. Consider the symplectic space  $(\mathbb{R}^{2n}, \omega)$  and let  $p_i, q_i$  be the Darboux coordinates. Then the universal enveloping algebra of  $\mathfrak{H}$  would have basis consisting of elements of the form

$$p_1^{i_1} \dots p_n^{i_n} q_1^{j_1} \dots q_n^{j_n} z^k$$

and the relationships are  $p_i q_j - q_j p_i = \delta_i^j z$ . If we map  $z$  to 1, then these are precisely the commutativity relations satisfied by the position and momentum operators.

**1.2. Affine transformation groups.** Consider a 1 dimensional subgroup  $H$  of  $Gl(2, \mathbb{R})$ . Because  $H$  is a subgroup of  $Gl(2, \mathbb{R})$ , we can find a matrix  $A \in \mathfrak{gl}(2, \mathbb{R})$  such that

$$H = \{\exp(tA) | t \in \mathbb{R}\}$$

By looking at the eigenvalues of  $H$ , it is easy to see that the only situation when  $H$  is compact is when  $H \cong SO(2)$ .

Define  $E(A)$  to be the affine group related to  $A$ ,

**Definition 1.5.**

$$E(H) := \{x \mapsto \exp(tA)x + b | t \in \mathbb{R}, b \in \mathbb{R}^2\}$$

There is a 3 dimensional faithful representation of  $E(A)$  as

$$(t, [x_1, x_2]) \mapsto \begin{bmatrix} \exp(tA) & b_1 \\ & b_2 \\ & 1 \end{bmatrix}$$

This allows us to compute the Lie algebra structure on  $T_e E(A) \cong \mathbb{R}^3$ ,

$$\begin{aligned} ((t, [x_1, x_2]), (s, [y_1, y_2])) &= (0, t[y_1, y_2]A^T - s[x_1, x_2]A^T) \\ &= (0, [ty_1 - sx_1, ty_2 - sx_2]A^T) \end{aligned}$$

**Proposition 1.6.** The groups  $E(A_1)$  and  $E(A_2)$  are isomorphic iff  $tA_1$  is conjugate to  $sA_2$  for some  $t, s \in \mathbb{R}$ .

*Proof.* We'll prove the theorems for Lie algebras. This theorem would follow from the fact if the Lie algebras are non-isomorphic then so are the groups.

The only trivial such lie algebra occurs when the matrix is 0. So assume that both  $A_1$  and  $A_2$  are non-zero. Notice that we have an exact sequence of Lie algebras,

$$0 \rightarrow \mathbb{R}^2 \rightarrow T_e E(A_i) \rightarrow \mathbb{R} \rightarrow 0$$

The first inclusion is given by  $(x_1, x_2) \mapsto (0, x_1, x_2)$  and Lie algebra structure on  $\mathbb{R}^2$  is the trivial one. Suppose there were an isomorphism  $T_e E(A_1) \rightarrow T_e E(A_2)$ , then because the Lie algebras are non-trivial there cannot be more than one copy of a non-trivial  $\mathbb{R}^2$  sitting in either. Hence the isomorphism should restrict to an isomorphism of  $\mathbb{R}^2$ .

Assume that after a suitable change of basis of  $T_e E(A_2)$ , the isomorphism of  $\mathbb{R}^2$  is just the identity isomorphism. This changes  $A_2$  to one of it's conjugates. We can also easily extend the isomorphism  $\mathbb{R} \rightarrow \mathbb{R}$ . Again by a suitable scaling we can assume that this isomorphism is an identity. If the isomorphism is identity on  $\mathbb{R}^2$  and  $\mathbb{R}^1$  and the two maps are the inclusion and the projection maps are the canonical maps, the isomorphism of the middle part should also be an identity which proves the result.  $\square$

This result produces for us an infinite family of non-isomorphic dimension 3 Lie groups. We can find some representative matrices,

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & t \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} t & 1 \\ -1 & t \end{bmatrix}, t \in \mathbb{R}$$

## 2. $SU_2, SO_3, SL_2(\mathbb{R})$

Segal's book has this remarkable section which describes Lie group isomorphisms in low dimensions. I wish there were more books like this.

$SU_2$  is in fact the space of unit quaternions. The standard generators of quaternions are chosen to be

$$i \leftrightarrow \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, j \leftrightarrow \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, k \leftrightarrow \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

This makes  $SU_2$  homeomorphic to  $S^3$  and hence at least topologically the universal cover of  $SO_3$ . The homomorphism which is a bit opaque is given by

$$\begin{aligned} \cos \theta/2 + \sin \theta/2 i &\mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{bmatrix} \\ \cos \theta/2 + \sin \theta/2 j &\mapsto \begin{bmatrix} \sin \theta & 0 & \cos \theta \\ 0 & 1 & 0 \\ \cos \theta & 0 & -\sin \theta \end{bmatrix} \\ \cos \theta/2 + \sin \theta/2 k &\mapsto \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

It is more easy to see this via Clifford algebras.

If we think of  $\mathbb{R}^4$  as the space of all quaternions, then we have a map

$$\begin{aligned} SU_2 \times SU_2 &\rightarrow SO_4 \\ (g_1, g_2) &\mapsto v \mapsto g_1 v g_2^{-1} \end{aligned}$$

It is easy to show that the kernel should be central and counting dimensions would tell us that this is a 2:1 covering.

There is also a double covering

$$SL_2(\mathbb{C}) \rightarrow SO_{1,3}^+$$

as follows,

Think of  $\mathbb{R}^4$  as the set of hermitian matrices  $A = \begin{bmatrix} t - x_1 & x_2 + ix_3 \\ x_2 - ix_3 & t + x_1 \end{bmatrix}$ . Define the action of  $g \in SL_2(\mathbb{C})$  on  $\mathbb{R}^4$  as

$$g \mapsto A \mapsto gAg^*$$

Because the determinant is  $t^2 - x_1^2 - x_2^2 - x_3^2$  this gives a map to  $SO_{1,3}$ . Then by using elementary matrices one can check that the kernel is  $\mathbb{Z}/2$ , surjectivity follows by counting dimensions. I could not come up with this myself, this looks very much like the Pauli's 2 component formalism with a twist.

Let us try to see this mapping on the level of Lie algebras.  $\mathfrak{so}_{1,3}$  is a real Lie algebra generated by operators of the form

$$t \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial t}, x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}$$

and as real Lie algebra  $\mathfrak{sl}_2(\mathbb{C})$  has the generators,

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -i & 0 \end{bmatrix}$$

Exponentiating we see that these would be mapping to,

$$2x_1 \frac{\partial}{\partial t} + 2t \frac{\partial}{\partial x_1}, (x_2 \frac{\partial}{\partial t} + t \frac{\partial}{\partial x_2}) + (x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}), (x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}) - (x_2 \frac{\partial}{\partial t} + t \frac{\partial}{\partial x_2})$$

$$2x_2 \frac{\partial}{\partial x_3} - 2x_3 \frac{\partial}{\partial x_2}, (x_3 \frac{\partial}{\partial t} + t \frac{\partial}{\partial x_3}) + (x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_3}), (x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_3}) - (x_3 \frac{\partial}{\partial t} + t \frac{\partial}{\partial x_3})$$

Restricting this action to  $SU_2(\mathbb{R})$  and replacing  $\mathbb{R}^4$  by  $\mathbb{R}^3$  by setting  $z = 0$  we get a double covering

$$SU_2(\mathbb{R}) \rightarrow SO_{1,2}^+$$

**2.1. Geometric description of  $SO_{1,3}^+$ .** We can stereographically project the unit sphere  $S^2$  onto the Riemann Sphere  $\mathbb{CP}^1$  as

$$(x_1, x_2, x_3) \mapsto \frac{x_1 + ix_2}{1 - x_3}$$

$$z_1 + iz_2 \mapsto \left( \frac{2z_1}{1 + z_1^2 + z_2^2}, \frac{2z_2}{1 + z_1^2 + z_2^2}, \frac{z_1^2 + z_2^2 - 1}{1 + z_1^2 + z_2^2} \right)$$

$$re^{i\tau} \mapsto \left( \frac{2r \cos \tau}{r^2 + 1}, \frac{2r \sin \tau}{r^2 + 1}, \frac{r^2 - 1}{r^2 + 1} \right)$$

$$re^{i\tau} \mapsto (\theta = 2 \cot^{-1} r, \phi = \tau)$$

where the last map is the map in spherical coordinates  $(\theta, \phi)$ .

An element of  $\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \in SU(2)$  acts on  $\mathbb{CP}^1$  via mobius transformations

$$(2.1) \quad z \mapsto \frac{az + b}{-\bar{b}z + \bar{a}}$$

It is easy to show that the kernel is  $\pm 1$  and hence the space of Mobius transformations coming from  $SU_2(\mathbb{C})$  is isomorphic to  $SO(3)$ .

Mobius transformations are rational morphisms and hence are conformal. One can see that the stereographic projection is conformal

$$\frac{\partial}{\partial r} \mapsto \frac{-2}{r^2 + 1} \frac{\partial}{\partial \theta}$$

$$r^{-1} \frac{\partial}{\partial \theta} \mapsto r \frac{\partial}{\partial \phi} = \frac{\sin \theta}{r} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} = \frac{1}{r^2 + 1} \frac{2}{\sin \theta} \frac{\partial}{\partial \theta}$$

Thus pushing Mobius transformations back to  $S^2$  we get conformal isomorphisms of  $S^2$ .

It is also known from complex analysis that the only conformal mappings of the Riemann Sphere are the Mobius transformations  $\frac{az + b}{cz + d}$  and we can assume that  $ad - bc = 1$ . So every element of  $SL_2(\mathbb{C})$  gives us a conformal mapping of the

Riemann Sphere, and it is easy to see that the mapping is 2:1 with kernel being the center. But quotienting out the center from  $SL_2(\mathbb{C})$  also gives us  $SO_{1,3}$  which tells us that there is an isomorphism

$$PSL_2(\mathbb{C}) \cong SO_{1,3}^+$$

Similarly the mobius transformations coming from  $SL_2\mathbb{R}$  are the ones preserving the upper half plane and we get

$$PSL_2(\mathbb{R}) \cong SO_{1,2}^+$$

**2.2. Geometric description of  $SU_2(\mathbb{R})$ .** There is a pretty picture in Segal's book, I did not find anything too insightful about the geometric picture, except the following result,

**Proposition 2.1.** *The exponential map  $\mathfrak{sl}_2(\mathbb{R}) \rightarrow SL_2(\mathbb{R})$  is not surjective.*

*Proof.*  $\mathfrak{sl}_2(\mathbb{R})$  is the space of trace 0 matrices and hence all the matrices are either diagonalizable or both the eigenvalues are 0. In either case the exponential of the matrix only has only positive real eigenvalues but  $SL_2(\mathbb{R})$  also contains matrices with complex eigenvalues.  $\square$

In particular it contains entire  $SO(2)$  and also matrices with negative eigenvalues. But once we go to  $SL_2(\mathbb{C})$ , the matrices fall under the image of the exp map. But even here the non-diagonalizable ones  $\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$  are not in the image of the exponential map. To get this element too, one has to increase the domain to  $GL_2(\mathbb{C})$ .

### 3. REPRESENTATIONS OF $SU(2)$ AND $SO(3)$

The irreducible representations of  $SU(2)$  are

$$V_n \subset \mathbb{C}[z_1, z_2] = \{\text{homogenous polynomials of degree } n\}$$

For an element  $e(t) = \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix}$  the corresponding characters are

$$\chi(V_n)(e(t)) = \sum_{k=0}^n e^{i(n-2k)t} = \frac{\sin(n+1)t}{\sin t}$$

**Proposition 3.1.** *If  $f : SU(2) \rightarrow \mathbb{C}$  is a class function then we have*

$$\int_{SU(2)} f = \frac{2}{\pi} \int_0^\pi f(e(t)) \sin^2 t dt$$

*Proof.* To show this let  $f$  be a character of the representation  $V_n$ , then left hand side is  $\langle \chi(V_n), \chi(V_0) \rangle$  and they both being irreducible this should equal  $\delta_n^0$ . Now the right hand side  $\sin(n+1)t \sin t$ . Then we use the fact that these character functions are dense in the space of all class functions to conclude that this result holds for all class functions.  $\square$

While this proof is clever it is not very satisfactory. How would one generalize this to an arbitrary compact Lie group?

**Proposition 3.2.** *On  $V_n$  there is an  $SU(2)$  invariant inner product given by*

$$\langle z_1^k z_2^{n-k}, z_1^l z_2^{n-l} \rangle = \delta_k^l k!(n-k)!$$

*Proof.* We know that up to a scalar there exists a unique  $SU(2)$  invariant inner product on  $V_n$ . Let  $g = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$ , then

$$\begin{aligned} \langle z_1^k z_2^{n-k}, z_1^l z_2^{n-l} \rangle &= \langle g z_1^k g z_2^{n-k}, g z_1^l g z_2^{n-l} \rangle \\ &= \langle e^{i(n-2k)\theta} z_1^k z_2^{n-k}, e^{i(n-2l)\theta} z_1^l z_2^{n-l} \rangle \\ &= e^{i(2l-2k)\theta} \langle z_1^k z_2^{n-k}, z_1^l z_2^{n-l} \rangle \end{aligned}$$

So we get  $\langle z_1^k z_2^{n-k}, z_1^l z_2^{n-l} \rangle = 0$  if  $k \neq l$ . Finally to get the constant, look at the action of  $g = \begin{bmatrix} a & -b \\ \bar{b} & \bar{a} \end{bmatrix}$  on  $z_1^n$  so that

$$\begin{aligned} \langle z_1^n, z_1^n \rangle &= \langle g z_1^n, g z_1^n \rangle \\ &= \langle (a z_1 + b z_2)^n, (a z_1 + b z_2)^n \rangle \\ &= \sum_i \binom{n}{i} (a\bar{a})^i (b\bar{b})^{n-i} \langle z_1^i z_2^{n-i}, z_1^i z_2^{n-i} \rangle \end{aligned}$$

And then plugging in  $\langle z_1^k z_2^{n-k}, z_1^l z_2^{n-l} \rangle = \delta_k^l k!(n-k)!$  verifies the equality.  $\square$

Again this proof is not very insightful as it does not show how to come up with an inner product for a general representation of a Lie group.



**Proposition 3.3.** *The irreducible representations of  $U(n)$  are of the form  $A_m \otimes V_n$  where  $m+n$  is even,  $A_m$  are the irreducible representations of  $U(1)$ . The characters are given by*

$$\begin{aligned}\chi(A_m \otimes V_n)\left(\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix}\right) &= e^{im\theta}(n+1) \\ \chi(A_m \otimes V_n)\left(\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}\right) &= \frac{\sin(n+1)\theta}{\sin\theta}\end{aligned}$$

*Proof.* There is a 2:1 mapping

$$U(1) \times SU(2) \rightarrow U(2), (\theta, A) \rightarrow e^{i\theta} A$$

The rest is a routine check.  $\square$

Because  $SU(2)$  is 2:1 universal cover of  $SO(3)$  the representations of  $SU(2)$  are precisely the projective representations of  $SO(3)$  and the representations  $V_{2n}$  are in fact proper representations of  $SO(3)$  (note that these representations are odd dimensional).

**Proposition 3.4.** *The character for the representation  $V_{2n}$  of  $SO(3)$  is given by*

$$\chi(V_{2n})\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & -\sin 2\theta \\ 0 & \sin 2\theta & \cos 2\theta \end{bmatrix}\right) = \frac{e^{2i(n+1)\theta} - e^{-2i(n+1)\theta}}{e^{2i\theta} - e^{-2i\theta}} = \frac{\sin 2(n+1)\theta}{\sin 2\theta}$$

*Proof.* Preimage of  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & -\sin 2\theta \\ 0 & \sin 2\theta & \cos 2\theta \end{bmatrix}$  in  $SU(2)$  is  $\begin{bmatrix} \pm e^{i\theta} & 0 \\ 0 & \pm e^{-i\theta} \end{bmatrix}$  and since the representation is induced from  $SU(2)$  the character is the same as that of the preimage.  $\square$

There is another description of the space  $V_{2n}$  as a  $SO(3)$  representation in terms of Spherical harmonics. Let  $P_t$  be polynomial functions with complex coefficients in 3 variables  $x_1, x_2, x_3$  thought of as functions over  $\mathbb{R}^3$ . There is a natural  $SO(3)$  action on this. And let  $\Delta$  be the standard Laplacian. Let  $\mathfrak{h}_l$  be the harmonic polynomials of degree  $l$ .

**Proposition 3.5.**  $\mathfrak{h}_l$  is  $SO(3)$  invariant and hence is it's representation, and as a  $SO(3)$  representation

$$\mathfrak{h}_l \cong V_{2l}$$

*Proof.* For an arbitrary metric  $g$  the laplacian is defined as  $g^{ij}\partial_i\partial_j$  and hence is a 0 tensor. The metric  $g^{ij}$  remains unchanged under the action of  $SO(3)$  and hence so does the laplacian.

The dimension of  $\mathfrak{h}_l$  is  $2l+1$  as the only free variables are the coefficients of  $x_2^i x_3^{l-i}$  and  $x_1 x_2^i x_3^{l-i-1}$ .

Look at the element  $(x_2 + ix_3)^l \in \mathfrak{h}_l$ .  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & -\sin 2\theta \\ 0 & \sin 2\theta & \cos 2\theta \end{bmatrix}$  sends it to  $[(\cos 2\theta x_2 + \sin 2\theta x_3) + i(-\sin 2\theta x_2 + \cos 2\theta x_3)]^l = e^{-2il\theta}(x_2 + ix_3)^l$ .  $V_{2n}$  for  $n < l$  does not contain any  $e^{-2il\theta}$  eigenspace for the action of any element of  $SO(3)$  and hence

$\mathfrak{h}_l$  should contain some  $V_{2n}$  with  $n \geq l$ , but the dimension tells us that the only possibility is  $V_{2l}$ .  $\square$

#### 4. SYMMETRIC AND ALTERNATING POWERS OF REPRESENTATIONS

These are really important constructs and it is worthwhile to talk about them carefully. For a non-virtual representation these are easy to define. Suppose we have a non-virtual representation  $G \rightarrow Gl(V)$ , then we can form the tensor algebra  $T^n(V)$  over  $V$  and we have a natural action of  $G$  given by

$$g(v_1 \otimes \cdots \otimes v_n) = (gv_1 \otimes \cdots \otimes gv_n)$$

Then we can identify the  $k$ th exterior algebra  $\wedge^k(V)$  as the subspace of alternating tensors spanned by vectors of the form

$$\sum_{\sigma \in S_k} (-1)^{\text{sign}(\sigma)} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}$$

and  $\wedge(V) = \oplus_i \wedge^i(V)$ .

And the  $k$ th symmetric algebra  $S^k(V)$  as the subspace of symmetric tensors spanned by vectors of the form

$$\sum_{\sigma \in S_k} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}$$

and  $S(V) = \oplus_i S^i(V)$ . Note that both of these subspaces are  $G$  invariant. These definitions are good when one is writing out the elements of  $S(V)$  or  $\wedge(V)$  in a given basis. However these are cumbersome when we wish to think of  $S(V)$  or  $\wedge(V)$  independently without making any reference to tensors. For these situations the following equivalent definition is more useful.

**Proposition 4.1.** *Let  $I_1$  be the ideal of  $T(V)$  generated as an algebra by elements of the form  $v \otimes v$  and let  $I_2$  be the ideal generated by elements of the form  $v \otimes w - w \otimes v$ , then*

$$\begin{aligned} \wedge(V) &\cong T(V)/I_1 \\ S(V) &\cong T(V)/I_2 \end{aligned}$$

and this puts a graded algebra structure on both.

*Proof.* Define the following maps

$$\begin{aligned} T(V) &\rightarrow \wedge(V) \\ v_1 \otimes \cdots \otimes v_k &\mapsto \frac{1}{k!} \left( \sum_{\sigma \in S_k} (-1)^{\text{sign}(\sigma)} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \right) \\ T(V) &\rightarrow S(V) \\ v_1 \otimes \cdots \otimes v_k &\mapsto \frac{1}{k!} \left( \sum_{\sigma \in S_k} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \right) \end{aligned}$$

and extended via linearity. These maps are onto by definition and one can check that the kernels are precisely  $I_1$  and  $I_2$ .  $\square$

**Proposition 4.2.**  $\dim \wedge^k(V) = \binom{n}{k}, \dim S^k(V) = \binom{n+k-1}{k}$  where  $n = \dim V$ .

*Proof.* Suppose  $e_1, \dots, e_n$  is a basis for  $V$  then,  $\{e_{i_1} \wedge \dots \wedge e_{i_k} \mid i_1 < \dots < i_k\}$  is a basis for  $\wedge^k(V)$  and  $\{e_1^{i_1} \dots e_n^{i_n} \mid i_1 + \dots + i_n = k, i_j \geq 0\}$  is a basis for  $S^k(V)$ . Now one counts.  $\square$

**Proposition 4.3.**  $T(V \oplus W) = T(V) \otimes T(W)$ ,  $\wedge(V \oplus W) = \wedge(V) \otimes \wedge(W)$ ,  $S(V \oplus W) = S(V) \otimes S(W)$

*Proof.* Follows by writing out an explicit basis for each of the spaces.  $\square$

And now we see how to define these constructs for virtual representations. We would want  $V \oplus (-V) = 0$ , which means that we should have  $T(V) \otimes T(-V) = \mathbb{C}$ . The  $n$ th graded component on the left hand side is  $\sum_i T^i(V) \otimes T^{n-i}(-V)$  and so we have the following recursive definition,

**Definition 4.4.** For a non-virtual representation  $V$  define

$$T^0(-V) := \mathbb{C}$$

$$T^k(-V) := - \sum_{i=0}^{k-1} T^i(-V) \otimes T^{k-i}(V)$$

We can analogously extend these definitions to  $\wedge(V)$  and  $S(V)$  and formally these will have the exact same form as above. And so we get the following result,

**Proposition 4.5.**  $T^k, \wedge^k, S^k$  are functors from  $R(G)$  to  $R(G)$ . Further if we think of  $R$  as a functor  $R : \text{Grp} \rightarrow \text{Rings}$  then  $T^k, \wedge^k, S^k$  are natural transformations.

Note that while talking about virtual representations we only care about the stable isomorphism classes. Tensors of  $-V$  need not all be purely virtual. For example  $T^2(-V) = T^1(V) \otimes T^1(V) - T^2(V)$  and we have the following beautiful result,

**Proposition 4.6.** For any non-virtual representation  $V$  and any positive integer  $k$ ,  $\sum_{i=0}^k (-1)^i \wedge^i(V) S^{k-i}(V) = 0$  and hence for  $k \leq \dim V$ ,  $\wedge^k(-V) = (-1)^k S^k(V)$

*Proof.* By the splitting principle, it suffices to show this for  $V = V_1 + \dots + V_n$ , where each  $V_i$  is a 1 dimensional representation. In this case

$$(-1)^i \wedge^i(V) = (-1)^i \sum_{j_1 + \dots + j_n = i, j_t \in \{0,1\}} V_1^{j_1} \dots V_n^{j_n}$$

$$S^{k-i}(V) = \sum_{j_1 + \dots + j_n = k-i} V_1^{j_1} \dots V_n^{j_n}$$

When we multiply and add everything out, we get terms of the form  $V_1^{j_1} \dots V_n^{j_n}$  such that  $j_1 + j_2 + \dots + j_n = k$  and the coefficient is  $\sum_S (-1)^{|S|}$  where  $S$  runs over all the subsets of  $\{1, \dots, n\}$  such that  $j_S$  consists of non-zero elements. Because there are as many even sized subsets as there are odd the coefficient turns out to be zero.

The second part follows inductively from the identity  $\wedge^k(-V) = - \sum_{i=0}^{k-1} \wedge^i(-V) \otimes \wedge^{k-i}(V)$   $\square$

A stronger result is true but I have not been able to prove it.

**Proposition 4.7.** *The following sequence is long exact*

$$0 \rightarrow \wedge^k(V) \rightarrow \cdots \wedge^i(V) \otimes S^{k-i}(V) \rightarrow \wedge^{i+1}(V) \otimes S^{k-i-1}(V) \rightarrow \cdots \rightarrow S^k(V) \rightarrow 0$$

where the map  $\wedge^i(V) \otimes S^{k-i}(V) \rightarrow \wedge^{i+1}(V) \otimes S^{k-i-1}(V)$  is given by sending a basis element  $e_1^{j_1} \cdots e_n^{j_n} \otimes e_{l_1} \wedge \cdots \wedge e_{l_n}$  to  $\sum_j (-1)^t e_1^{j_1} \cdots e_n^{j_n} \cdot e_{l_t} \otimes \hat{e}_{l_t}$ ,  $\hat{e}_{l_t}$  denotes  $e_{l_t}$  missing from  $e_{l_1} \wedge \cdots \wedge e_{l_n}$ .

## 5. ADAMS OPERATIONS

Let  $R(G)$  denote the ring of finite dimensional representations of a compact Lie group  $G$  which is the group completion of the semiring of  $G$  finite dimensional representations. The addition law is the addition is given by  $\oplus$  and the product by  $\otimes$ .

**Proposition 5.1** (Splitting principle). *Let  $S$  vary over all closed topologically cyclic subgroups in  $G$  as a closed subgroup. Then we have an injection of rings given by the restriction maps,*

$$(res_S^G) : R(G) \rightarrow \prod_S R(S)$$

*Proof.* This is not a good way of proving this, but here goes. Every element  $g \in G$  lies in some maximal torus. It is easy to see that  $S^1$ 's are dense in a torus (look at the lines with rational slope).  $\square$

Using this then we wish to define Adams operations as the unique natural transformations  $\Psi^k : R \rightarrow R, k \in \mathbb{Z}_{\geq 0}$  where  $R : Grp \rightarrow Rings$  is the representation ring functor, that satisfy the property

$$\psi_G^k(V) = V^k$$

for any one dimensional representation  $V \in R(G)$ . If such a natural transformation were to exist then it would be uniquely determined by the above proposition.

**Proposition 5.2.** *Adams operations exist.*

*Proof.* For any representation  $G \rightarrow Gl(V)$  define the Adams operations  $\psi^k$  as follows. Consider  $n$  formal variables  $x_1, \dots, x_n$  for  $n > k$  and write  $x_1^k + \dots + x_n^k$  as a polynomial in  $\sigma_1, \dots, \sigma_k$ , where  $\sigma_i$  is the  $i$ th elementary symmetric polynomial in  $x_i$ 's

$$x_1^k + \dots + x_n^k = Q(\sigma_1, \dots, \sigma_k)$$

Note that by degree considerations we do not require any elementary symmetric polynomials of degree more than  $k$  and once  $n \geq k$  the form of the polynomial  $Q$  stabilizes and so  $Q$  does not depend on  $n$  or  $x_i$ .

Now define  $\psi^k(V) = Q(\wedge^1 V, \dots, \wedge^k V)$ , where  $\wedge^i V$  is the  $i$ th exterior power of  $V$ . That these satisfy the naturality and takes 1-dim representations to  $k$ th powers is easy to check.  $\square$

**Proposition 5.3.** *For  $g \in G$ , and  $V \in R(G)$*

$$\chi(\psi^k(V))(g) = \chi(V)(g^k)$$

*where  $\chi$  denotes the character.*

*Proof.* This is trivially true for line bundles. For arbitrary bundles use the splitting principle (5.1) to reduce to the case of line bundles.  $\square$

**Proposition 5.4.**  $\psi^k(\wedge^i(V)) = \wedge^i(\psi^k(V)).$

*Proof.* Again by proposition 5.1 it is enough to think of  $V$  as a sum of (possibly virtual) line bundles  $V_1 \oplus \cdots \oplus V_n$ .

$$\begin{aligned}
 \wedge^i(\psi^k(V)) &= \wedge^i(V_1^k \oplus \cdots \oplus V_n^k) \\
 &= \oplus_{1 \leq j_1 < \cdots < j_i \leq n} (V_{j_1}^k \otimes \cdots \otimes V_{j_i}^k) \\
 &= \oplus_{1 \leq j_1 < \cdots < j_i \leq n} (V_{j_1} \otimes \cdots \otimes V_{j_i})^k \\
 &= \oplus_{1 \leq j_1 < \cdots < j_i \leq n} \psi^k(V_{j_1} \otimes \cdots \otimes V_{j_i}) \\
 &= \psi^k(\oplus_{1 \leq j_1 < \cdots < j_i \leq n} V_{j_1} \otimes \cdots \otimes V_{j_i}) \\
 &= \psi^k(\wedge^i(V))
 \end{aligned}$$

□

**Proposition 5.5.** *If  $G$  is finite and  $V$  is an irreducible representation of  $G$ . If  $k$  is relatively prime to  $|G|$ , then  $\psi^k(V)$  is also irreducible (and not virtual).*

*Proof.* Let  $\omega$  be a  $|G|$  root of unity. Then every irreducible representation of  $G$  can be realized over the cyclotomic field  $\mathbb{Q}(\omega)$  and so  $V$  can be thought of as a representation over  $\mathbb{Q}(\omega)$ . Because  $k$  is relatively prime to  $|G|$ ,  $\phi : x \mapsto x^k$  is a Galois automorphism of  $\mathbb{Q}(\omega)$ . Look at an element  $g \in G$ . Because  $\chi(\psi^k(V))(g) = \chi(V)(g^k) = \phi(\chi(V)(g))$  and so  $\psi^k(V)$  is nothing but the non-virtual representation obtained by applying  $\phi$  to  $Gl(V)$  and

$$\begin{aligned}
 \sum_{g \in G} \overline{\chi(\psi^k(V))(g)} \chi(\psi^k(V))(g) &= \sum_{g \in G} \overline{\phi(\chi(V)(g))} \phi(\chi(V)(g)) \\
 &= \sum_{g \in G} \phi(\overline{\chi(V)(g)} \chi(V)(g)) \\
 &= \sum_{g \in G} \overline{\chi(V)(g)} \chi(V)(g) \\
 &= 1
 \end{aligned}$$

and hence  $\psi^k(V)$  is irreducible! □

This is false if  $k \nmid |G|$ . For example, take  $G = S_3$  and  $k = 2$  and let  $V$  denote the sign representation of  $G$  and let  $W$  denote the 2 dimensional irreducible representation. Then by looking at characters one can conclude that  $W \otimes W = 1 \oplus V \oplus W$ .

$$\begin{aligned}
 \psi^2(W) &= W^2 - 2 \wedge^2 W \\
 &= (1 + V + W) - 2V \\
 &= 1 - V + W
 \end{aligned}$$

which is virtual.

6.  $\mathfrak{sl}(2, \mathbb{C})$  REPRESENTATIONS

$\mathfrak{sl}(2, \mathbb{C})$  is a 3 dimensional complex Lie subalgebra of  $\mathfrak{gl}(3, \mathbb{C})$  consisting of trace 0 matrices. We can pick three generators

$$\begin{aligned} H &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ X^+ &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ X^- &= \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \end{aligned}$$

satisfying commutativity relations:

$$[H, X^+] = 2X^+, [H, X^-] = -2X^-, [X^+, X^-] = -H$$

Thus  $H$  acts on  $\mathfrak{sl}(2, \mathbb{C})$  via  $[H, -]$  and  $H, X^+, X^-$  is the eigenspace decomposition with eigenvalues 0, 2, -2.

If we exponentiate the Lie algebra we get the Lie group  $SL_2(\mathbb{C})$  with

$$\begin{aligned} \exp_t(H) &= \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \\ \exp_t(X^+) &= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \\ \exp_t(X^-) &= \begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix} \end{aligned}$$

A representation  $E$  of  $\mathfrak{sl}(2, \mathbb{C})$  is just a Lie algebra module. An element  $x \in E$  is called an element of weight  $\lambda$  if

$$Hx = \lambda x$$

In addition,  $x$  is said to be primitive if  $X^+x = 0$ . The set of all weight occurring in  $E$  is called the weight space of  $E$ .

For example we can think of  $\mathfrak{sl}(2, \mathbb{C})$  as a representation over itself. Then  $H, X^+, X^-$  have weights 0, 2, -2 respectively, and  $X^+$  is a primitive element.

Simple finite dimensional  $\mathfrak{sl}(2, \mathbb{C})$  representations can be characterized entirely by it's primitive elements thanks to the following result:

**Proposition 6.1.** *If  $E$  is non-trivial simple finite dimensional representation of  $\mathfrak{sl}(2, \mathbb{C})$  then*

- (1)  *$E$  contains a primitive element. Let  $x$  be a primitive element in  $E$  with weight  $\lambda$ .*
- (2) *Let  $x_j = (X^-)^j x$ . Then  $Hx_j = (x - 2j)x_j$  and  $X^+x_j = (-\lambda j + (j - 1)j)(X^-)^{j-1}x$ .*
- (3)  *$\dim E = \lambda + 1$  and  $x_0, x_1, \dots, x_\lambda$  form a vector space basis for  $E$ .*
- (4) *Any two primitive elements of  $E$  differ by a scalar and hence have the same weight.*

*Proof.* (1)  $H$  would always have an eigenvector in  $E$ , say  $y$  with eigenvalue  $\tau$ . Start acting on  $y$  by  $X^+$ , finite dimensionality implies that there is some relationship between these. But lie algebra module structure tells us that us



that  $HX^+y = X^+Hy + [H, X^+]y = (\tau + 2)X^+y$  and so there cannot be any non-trivial linear dependence between the  $(X^+)^iy$ . And hence  $(X^+)^iy = 0$  for some  $i$ . Then pick  $x = (X^+)^{i-1}y$  and let  $\lambda$  be it's weight.

(2)

$$\begin{aligned} H(X^-)^jx &= X^-H(X^-)^{j-1}x + [H, X^-](X^-)^{j-1}x \\ &= (\lambda - 2j + 2)(X^-)^jx + 2(X^-)^jx \\ &= (\lambda - 2j)(X^-)^jx \end{aligned}$$

$$\begin{aligned} X^+(X^-)^jx &= X^-X^+(X^-)^{j-1}x + [X^+, X^-](X^-)^{j-1}x \\ &= (-\lambda(j-1) - (j-2)(j-1) - (\lambda - 2j + 2))(X^-)^{j-1}x \\ &= (-\lambda j + (j-1)j)(X^-)^{j-1}x \end{aligned}$$

- (3) Because  $E$  is simple  $\mathfrak{sl}(2, \mathbb{C})x = E$ . Because of the lie algebra structure of  $\mathfrak{sl}(2, \mathbb{C})$  it is easy to see that  $\mathfrak{sl}(2, \mathbb{C})x$  has a vector space basis given by  $(X^-)^jx$ . Look at  $x_{\lambda+1}$ .

$$X^+x_{\lambda+1} = (-\lambda(\lambda+1) + (\lambda+1)(\lambda))x_\lambda = 0$$

If  $X^+x_{\lambda+1}$  were non-zero then it would generate a Lie subalgebra completely disjoint from the algebra generated by  $x$  contradicting the simplicity of  $E$  and hence  $x_{\lambda+1} = 0$  and  $x_0, x_1, \dots, x_\lambda$  form a vector space basis for  $E$ .

- (4) Because every primitive element gives a decomposition of  $E$  into weight spaces, with each weight space being one dimensional, all the primitive elements should be in the weight space with the largest weight.

□

So we see that every simple representation is completely determined once we have located a primitive element and found it's weight. From this we get an easy corollary.

**Corollary 6.2.** *For every non-negative integer  $n$  there is exactly one simple representation of  $\mathfrak{sl}(2, \mathbb{C})$  of that dimension.*

**6.1. Representations of  $SU(2)$ .** Look at the real Lie algebra of  $SU(2)$ .  $\mathfrak{su}(2, \mathbb{C})_{\mathbb{R}}$  is a subalgebra of  $\mathfrak{gl}(2, \mathbb{C})$  consists of matrices of the form which satisfy the relations  $\{A | \text{tr}(A) = 0, A + A^* = 0\}$ . If we complexify to get  $\mathfrak{su}(2, \mathbb{C})$ , then we get the following relations,

**Proposition 6.3.**  $\mathfrak{su}(2, \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C})$  as Lie algebras, but  $\mathfrak{su}(2, \mathbb{C})_{\mathbb{R}} \not\cong \mathfrak{sl}(2, \mathbb{R})$

*Proof.*  $\mathfrak{su}(2, \mathbb{C})$  has a vector space basis

$$\begin{aligned} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} &= iH \\ \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} &= i(X^+ - X^-) \\ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} &= X^+ + X^- \end{aligned}$$

And so  $\mathfrak{su}(2, \mathbb{C}) \subset \mathfrak{sl}(2, \mathbb{C})$  but the two have the same dimension.

For showing non-isomorphism note that  $\mathfrak{su}(2, \mathbb{C})_{\mathbb{R}} \cong \mathfrak{so}(2, \mathbb{R}) \cong \mathbb{R}^3$  where the Lie bracket structure on  $\mathbb{R}^3$  is given by the cross product. If there is an isomorphism  $\mathfrak{su}(2, \mathbb{C})_{\mathbb{R}} \cong \mathfrak{sl}(2, \mathbb{R})$  then we should be able to find an element  $h, x^+, x^-$  in  $\mathbb{R}^3$  such that  $h \times x^+ = 2x^+$ . But this is not possible.  $\square$

What this means is that we can differentiate any complex representation of  $SU(2)$  to get a representation of  $\mathfrak{sl}(2, \mathbb{C})$ . By abuse of notation denote the lie algebra representation corresponding to  $V_n$  also by  $V_n$ . We can then try to find the primitive elements in  $V_n$ .  $V_n$  has a basis  $z_1^i z_2^{n-i}$ .

$$\begin{aligned} X^+(z_1^i z_2^{n-i}) &= \frac{d}{dt} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} z_1^i z_2^{n-i} \Big|_{t=0} \\ &= \frac{d}{dt} (z_1 + t z_2)^i z_2^{n-i} \Big|_{t=0} \\ &= \begin{cases} z_1^{i-1} z_2^{n-i+1} & \text{if } i > 0 \\ 0 & \text{if } i = 0 \end{cases} \end{aligned}$$

These basis elements seem to generate the weight spaces, we need to check one more thing to be sure.

$$\begin{aligned} H(z_1^i z_2^{n-i}) &= \frac{d}{dt} \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} z_1^i z_2^{n-i} \Big|_{t=0} \\ &= \frac{d}{dt} e^{2it-nt} z_1^i z_2^{n-i} \Big|_{t=0} \\ &= (2i - n) z_1^i z_2^{n-i} \end{aligned}$$

So these are indeed the weight spaces and the primitive element is  $z_2^n$ .

**6.2. Clebsch - Gordon decomposition.** I had encountered this while doing some quantum mechanics. It is so satisfying to realize that it is just a decomposition of representations. Also, I have finally learnt the correct spelling.

**Proposition 6.4.**

$$V_k \otimes V_l = \bigoplus_{j=0}^q V_{(k-l-2j)}, q = \min\{k, l\}$$

*Proof.* While one can do this using characters of  $SU(2)$ , it is much transparent to treat these as  $\mathfrak{sl}(2, \mathbb{C})$  representations and find the weight spaces. The thing to be careful about is the induced Lie algebra action on  $V_k \otimes V_l$  is induced by a derivation and we will have to invoke the product rule there.

Denote the primitive element of  $V_k$  by  $x^k$ , and let  $x_j^k = (X^+)x_{j+1}^k$  with  $x_0^k = x^k$ . Then the weight spaces of  $V_k \otimes V_l$  are of the form  $x_i^k \otimes x_j^l$ .

$$Hx_i^k \otimes x_j^l = [(k - 2i) + (l - 2j)]x_i^k \otimes x_j^l$$

So the weight spaces get mixed,  $x_i^k \otimes x_{j+1}^l$  and  $x_{i+1}^k \otimes x_j^l$  have the same weight. To find the primitive elements we need to look at elements of the form  $\sum_{i+j=m} c_{i,j} x_i^k \otimes x_j^l$ .

Using a combinatorial argument here one can say that such  $c_{i,j}$  exist iff  $m \leq \min k, l$  and these are unique up to a scalar. The corresponding weight of these is  $(k - 2i) + (l - 2j) = k + l - 2m$ .  $\square$

## 7. SPHERICAL HARMONICS

We had an alternate description for  $V_{2n}$ . This representation could also be identified with space  $\mathfrak{h}_l$ , the degree  $l$  harmonic polynomials in 3 variables and so we should be able to get a corresponding  $\mathfrak{sl}(2, \mathbb{C})$  action on  $\mathfrak{h}_l$ .

First let us identify  $\mathfrak{so}(3)$  with  $\mathfrak{su}(2, \mathbb{C})$ . For this one needs to play around with the generators of  $\mathfrak{su}(2, \mathbb{C})$  to come up with three generators whose product mimics the cross product on  $\mathbb{C}^3$

$$\begin{aligned} Z_1 &= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \leftrightarrow \begin{bmatrix} -i/2 & 0 \\ 0 & i/2 \end{bmatrix} \\ Z_2 &= \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \leftrightarrow \begin{bmatrix} 0 & -i/2 \\ -i/2 & 0 \end{bmatrix} \\ Z_3 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \leftrightarrow \begin{bmatrix} 0 & -1/2 \\ 1/2 & 0 \end{bmatrix} \end{aligned}$$

Translating back to  $\mathfrak{su}(2, \mathbb{C})$  we get the correspondence,

$$\begin{aligned} H &\leftrightarrow \begin{bmatrix} 0 & 2i & 0 \\ -2i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = -2i(x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}) \\ X^+ &\leftrightarrow \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & -1 \\ i & 1 & 0 \end{bmatrix} = i(x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_3}) + (x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3}) \\ X^- &\leftrightarrow \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & -1 \\ -i & 1 & 0 \end{bmatrix} = -i(x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_3}) + (x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3}) \end{aligned}$$

Note that by doing this we have *broken symmetry*. We could have easily permuted 1, 2, 3 cyclically and gotten another identification.

Next we should find the weight spaces in  $\mathfrak{h}_l$ . We had computed the action of  $\exp tH$  on  $(x_1 + ix_2)^l$ . The same element gives us the primitive element,

$$\begin{aligned} H(x_1 - ix_2)^l &= -2i(x_2.l.(x_1 - ix_2)^{l-1} - x_1.l.i.(x_1 - ix_2)^{l-1}) \\ &= 2l(x_1 - ix_2)^l \end{aligned}$$

The other weight spaces then are generated by  $(X^-)^j$ ,

$$\begin{aligned}
 x_j &= (X^-)^j (x_1 - ix_2)^l \\
 &= [-i(x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_3}) + (x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3})]^j (x_1 - ix_2)^l \\
 &= [-ix_3(\frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2}) + i(x_1 + ix_2) \frac{\partial}{\partial x_3}]^j (x_1 - ix_2)^l \\
 &\text{substituting } z = \frac{x_1 + ix_2}{\sqrt{2}}, \bar{z} = \frac{x_1 - ix_2}{\sqrt{2}} \\
 &= (-\sqrt{2}i)^j [x_3 \frac{\partial}{\partial \bar{z}} - z \frac{\partial}{\partial x_3}]^j (\sqrt{2}\bar{z})^l
 \end{aligned}$$

Let  $A = x_3 \frac{\partial}{\partial \bar{z}}$  and  $B = z \frac{\partial}{\partial x_3}$ .  $[A, B] = z \frac{\partial}{\partial \bar{z}} =: C$  and  $[A, C] = [B, C] = 0$ . Further  $B\bar{z}^l = 0$ . Further we are only interested in  $x_j$  up to a scalar. Using these identities we can simplify the above equation to

$$\begin{aligned}
 x_j &= [A - B]^j \bar{z}^l \\
 &= [\sum_{k=0}^{j/2} (-1)^k A^{j-2k} C^k] \bar{z}^l \\
 &= \sum_{k=0}^{j/2} (-1)^k x_3^{j-2k} z^k \frac{\partial^{j-k}}{\partial \bar{z}^{j-k}} \bar{z}^l \\
 &= \sum_{k=0}^{j/2} (-1)^k \frac{l!}{(l-j+k)!} x_3^{j-2k} z^k \bar{z}^{l-j+k}
 \end{aligned}$$

The standard way to write this is to use spherical coordinates and say that  $z = \sin \theta e^{i\phi}/\sqrt{2}$ ,  $x_3 = \cos \theta$

$$\begin{aligned}
 x_j &= \sum_{k=0}^{j/2} (-1)^k \frac{l!}{(l-j+k)!} x_3^{j-2k} z^k \bar{z}^{l-j+k} \\
 &= e^{i(j-l)\phi} \sum_{k=0}^{j/2} (-2)^k \frac{l!}{(l-j+k)!} (\cos^{j-2k} \theta) (\sin^{l-(j-2k)} \theta)
 \end{aligned}$$

These are the spherical harmonics. Usually they are normalized so that their integral over the unit sphere is 1.

**7.1. Casimir Element.** In the last section we identified  $\mathfrak{h}_l$  as an  $\mathfrak{so}(3)$  module and then identified the various weight spaces. While most of it followed quite linearly from the module structure, we have not really understood why the module structure exists in the first place. In particular what has the Laplacian  $\Delta$  anything to do with  $\mathfrak{so}(3)$ .

The connection was that  $\Delta$  was  $SO(3)$  invariant. So more generally if we have an operator which is  $H$  invariant and if  $H$  is a subgroup of  $G$ , then we can look at any eigenspace of an  $H$  equivariant operator on a  $\mathfrak{g}$  module to get a  $\mathfrak{h}$ . There

is a canonical such element called the Casimir element. I do not know the general theory.

Consider the element

$$\begin{aligned}
C &= -Z_1^2 - Z_2^2 - Z_3^2 \\
&= -\left((x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1})^2 + (x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2})^2 + (x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_3})^2\right) \\
&= -(x_1^2 + x_2^2 + x_3^2)\Delta + x_1^2 \frac{\partial^2}{\partial x_1^2} + x_2^2 \frac{\partial^2}{\partial x_2^2} + x_3^2 \frac{\partial^2}{\partial x_3^2} + 2(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}) \\
&= -(x_1^2 + x_2^2 + x_3^2)\Delta + L(L+1) \\
&\text{where } L = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}
\end{aligned}$$

$C$  is the Casimir element of  $\mathfrak{so}(3)$ . Under change of coordinates  $L$  behaves like a 0 tensor, and hence is  $SO(3)$  invariant, and hence so is  $C$ . In particular  $C$  commutes with  $Z_1, Z_2, Z_3$ . In terms of  $H, X^+, H^-$  the  $C$  is

$$C = H^2/4 - (X^+X^- + X^-X^+)/2$$

For  $p \in \mathfrak{h}_l$ ,

$$\begin{aligned}
Lx_1^a x_2^b x_3^c &= (a+b+c)x_1^a x_2^b x_3^c \\
\implies Cp &= L(L+1)p \\
&= l(l+1)p
\end{aligned}$$

Abstractly we could have computed the action of  $C$  by writing it in terms of  $H, X^+, X^-$  and noticing that Schur's lemma tells us that  $C$  should act via scalar multiplication on  $V_m$  and hence, if we act  $C$  on a primitive element  $x$  we get,

$$\begin{aligned}
Cx &= [H^2/4 - (X^+X^- + X^-X^+)/2]x \\
&= m^2/4 - X^+X^-x/2 \\
&= m^2/4 - (X^-X^+ + [X^+, X^-])x/2 \\
&= m^2/4 + m/2
\end{aligned}$$

## 8. WEYL INTEGRATION FORMULA

The aim now is to *justify* the integral formula we had obtained for  $SU(2)$

$$\int_{SU(2)} f = \frac{2}{\pi} \int_0^\pi f(e(t)) \sin^2 t dt$$

In the process of doing this we will end up proving a few facts about maximal tori as well.

Let  $T$  be a maximal torus inside  $G$ . Let  $n = \dim G, k = \dim T$ . Let  $dt$  and  $dg$  be a choice of top forms on  $T$  and  $G$  which are left invariant. We can assume that  $dt$  is in fact a form on  $G$  which has been restricted to  $T$ .

**Proposition 8.1.** *There exists a normalized left  $G$  invariant top form  $dgT$  on  $G/T$  which when pulled back to  $G$  we gives us the relation,*

$$dg = dgT \wedge dt$$

*Proof.* Define the form in the following way. At  $e$ , we have an exact sequence

$$0 \rightarrow T_e T \rightarrow T_e G \rightarrow T_e G/T \rightarrow 0$$

Use a  $G$  bi-invariant inner product on  $G$  to get a splitting  $T_e G \cong T_e G/T \oplus T_e T$ . At  $e$  define  $dgT|_e$  to be the  $n - k$  form corresponding to this decomposition  $\wedge^{n-k}(T_e G/T)^*$ . Let  $l_g$  denote left multiplication and  $r_g$  denote right multiplication by  $g$ . Then

$$dgT|_g := l_{g*} \wedge^{n-k} (T_e G/T)^*$$

This ensures left invariance, however we still need to check that this is well-defined. That is we need to show that if  $g_1 = g_2 t$  for  $t \in T$  then  $l_{g_1*} \wedge^{n-k} (T_e G/T)^* = r_{t*} l_{g_2*} \wedge^{n-k} (T_e G/T)^*$ . Substituting  $l_{g_1*} = l_{g_2*} l_{t*}$  we are reduced to showing that

$$Ad_t(T_e G/T) = T_e G/T \quad \forall t \in T \text{ and } Ad_t \text{ is orientation preserving}$$

However this follows from the fact that the inner product is  $G$  bi-invariant.  $\square$

**Corollary 8.2** (Fubini). *For any real-valued function measurable  $f$  on  $G$ ,*

$$\int_G f dG = \int_{G/H} dgT \left( \int_T dt f(gt) \right)$$

Note that  $\int_H dH f(-h)$  is a well defined function on  $G/H$ .

*Proof.* This follows by looking at the fiber bundle  $T \rightarrow G \rightarrow G/T$ .  $\square$

We wish to do a similar trick but instead of breaking up  $G$  into cosets of  $T$  and applying Fubini we want to break  $G$  into conjugacy classes and then apply Fubini. But conjugacy classes do not come with a canonical form and in fact they are of varying dimensions which restricts our ability to separate variables. However we can restrict our attention to generic conjugacy classes and these do have nice properties.

**Theorem 8.3.** (1) *If  $t \in T$  and  $\langle t \rangle$  denotes the smallest subgroup generated by  $T$ , then for almost all  $t \in T$  (in the sense of measure)  $\langle t \rangle$  is dense in  $T$ .*  
 (2) *If  $N$  is the normalizer of  $T$  in  $G$ , then  $N/T$  is a discrete subgroup called the Weyl group  $W$ .*

- (3) *The conjugacy classes of  $T$  cover  $G$ .*  
 (4) *All the maximal tori are conjugate to each other.*

An element which satisfies the first condition of the above proposition is called a generator of  $T$ . Let  $T'$  denote the set of generators of  $T$ . There is a natural action of  $W$  on  $T$ , the restriction of this action to  $T'$  is free.

We restrict our attention to the subset  $G'$  consisting of elements which are conjugate to some generator. The subset  $G'$  and  $G$  differ in a measure 0 set so they are the same for integration purposes. We can break  $G$  up as conjugacy classes, and the above proposition tells us that these conjugacy class are parametrized by  $T'/W$ . This allows us to use Fubini on  $G$ , if  $f$  is a class function,

$$\begin{aligned} \int_G f(g)dg &= \int_{T'/W} f(t).vol(C_t)dt/W \\ &\quad C_t \text{ is the conjugacy class of } t \in T'/W \\ &= \frac{1}{W} \int_{T'} f(t).vol(C_t)dt \end{aligned}$$

To compute the volume of  $C_t$  one can look at the map  $C_t \rightarrow G/T, g \mapsto gT$ . It is easy to see that this map is a covering space map with the group of Deck transformations being  $W$ .

Consider the differential of this map. Because all the metric are left  $G$  invariant, we can push this to  $e$  and then the map would be a map on the Lie algebras. Let  $g \in C_t$ , pushing this map to  $e$  we get the map

$$\phi : g^{-1}hgh^{-1} \mapsto g^{-1}hgh^{-1}T$$

If  $h \in T$ , then this maps to 0, so the differential of this map in fact restricts to

$$(8.1) \quad d\phi : T_e G/T \mapsto T_e G/T$$

To explicitly determine this map suppose  $h$  is very close to  $e$ ,

$$\begin{aligned} g^{-1}hgh^{-1} &= g^{-1}(e + \epsilon)g(e - \epsilon) \\ &= e + g^{-1}\epsilon g - \epsilon + O(\epsilon^2) \\ d\phi\epsilon &\mapsto g^{-1}\epsilon g - \epsilon \\ d\phi &= Ad_{g^{-1}} - 1 \end{aligned}$$

The local degree then is just the local degree. Further because all the elements in  $C_t$  are conjugates of each other, this determinant would not depend on  $g$  and hence we get,

**Proposition 8.4** (Weyl Integration Formula). *If  $f$  is a class function on  $G$  then*

$$\int_G f(g)dg = \frac{1}{W} \int_T f(t). \det(Ad_{t^{-1}} - 1)dt$$

where  $Ad_{t^{-1}} - 1$  is thought of as a map between  $T_e G/T \mapsto T_e G/T$

**8.1. Computations.** To do the actual computations we do simple trick which simplifies things a lot. We are interested in the determinant of an operator which should be a real number depending on  $t$ . The determinant does not depend on the underlying field so instead of thinking of  $T_e G/T$  we look at the space  $T_e G/T \otimes \mathbb{C}$ . The reason to do this is that the later space has a very neat basis. For matrix



groups we choose the bi-invariant inner product to be  $\langle a, b \rangle = \text{Trace}(a^*b)$ .

Let us see what we get for  $SU(2)$ . A maximal torus  $T$  is one dimensional consisting of matrices of the form  $e(t) = \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix}$ . The space  $T_eSU(2)/T$  has basis  $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ .

$$\begin{aligned} e(-t) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} e(t) - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} 0 & e^{-2it} - 1 \\ 0 & 0 \end{bmatrix} \\ e(-t) \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} e(t) - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ e^{2it} - 1 & 0 \end{bmatrix} \\ \Rightarrow \det(Ad_{t^{-1}} - 1) &= (e^{-2it} - 1)(e^{2it} - 1) \\ &= 4 \sin^2 t \end{aligned}$$

The Weyl group  $W$  is just  $\mathbb{Z}/2$ .

$$\int_{SU(2)} f(g) dg = \frac{1}{2} \int_0^{2\pi} f(e(\theta)) \cdot 4 \sin^2 \theta \frac{d\theta}{2\pi}$$

which agrees with the formula we had obtained earlier.

For  $SO(3)$  the torus  $T$  is one dimensional consisting of matrices of the form  $e(t) = \begin{bmatrix} e^{it} & 0 & 0 \\ 0 & e^{-it} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ . The space  $T_eSO(3)/T$  has basis  $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$ .

$$\begin{aligned} e(-t) \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} e(t) - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} &= \begin{bmatrix} 0 & e^{-2it} - 1 \\ 0 & 0 \end{bmatrix} \\ e(-t) \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} e(t) - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ e^{2it} - 1 & 0 \end{bmatrix} \\ \Rightarrow \det(Ad_{t^{-1}} - 1) &= (e^{-2it} - 1)(e^{2it} - 1) \\ &= 4 \sin^2 t \end{aligned}$$

The Weyl group  $W$  is again just  $\mathbb{Z}/2$ .

We can do the same computation for  $U(n)$  in general. A maximal torus consists of diagonal elements  $t = (t_j)_{j=1}^n, e(t) = \text{diag}(e^{it_j})$ . The Weyl group is  $S_n$ . The orthonormal basis for  $T_eU(n)/T$  consists of off diagonal elementary matrices  $E_j^k, j \neq k$ .

$$\begin{aligned} e(-t) E_j^k e(t) &= e^{i(t_j - t_k)} E_j^k \\ \Rightarrow \det(Ad_{t^{-1}} - 1) &= \prod_{j \neq k} (e^{i(t_j - t_k)} - 1) \\ &= \prod_{j > k} |e^{it_j} - e^{it_k}| \end{aligned}$$

And so we get the formula,

$$\int_{U(n)} f(g) dg = \frac{1}{n!} \int_0^{2\pi} \cdots \int_0^{2\pi} f(e(t)) \prod_{j>k} |e^{it_j} - e^{it_k}| \frac{dt_1}{2\pi} \cdots \frac{dt_n}{2\pi}$$

**8.2. Proof of the maximal torus theorems.** There are two non-trivial results about the maximal torus. The first one says that  $N/T$  is discrete. Look at the lie algebra  $T_e N$ . If  $\dim N > \dim T$  then there is an element  $X \in T_e N, X \perp T_e T$  such that  $Ad(\exp tX)(T_e T) \subset T_e T$ . Differentiating we get  $[X, T_e T] \subset T_e T$ . But the action of  $T_e T$  preserves  $T_e G/T$  and hence  $[X, T_e T] = 0$ , but this contradicts the maximality of  $T$ .

The second part of the theorem takes a lot of effort to prove and is one of the fundamental structure theorems about compact Lie groups. The theorem states that every element of  $G$  belongs to some maximal torus. One way of proving this result is topological! Look at the map  $G/T \rightarrow G/T$  which maps  $hT \mapsto ghT$ . If  $h^{-1}gh \in T$  then  $ghT = hT$  and hence the condition is equivalent to the following proposition:

**Proposition 8.5.** *The map  $g : G/T \rightarrow G/T$  which sends  $hT \mapsto ghT$  has a fixed point.*

*Proof.* We get really lucky here because as it turns out any continuous map  $G/T \rightarrow G/T$  has a fixed point. Lefschetz fixed point theorem tells us that the number of fixed points of any map counting multiplicity homotopic to the identity equals the Euler characteristic of  $G/T$ . So we can use any other simpler map  $G/T \rightarrow G/T$  and count it's fixed points.

Let  $t \in T$  be a generator for  $T$  then we look at the map  $gT \mapsto tgT$ . Then  $gT$  is a fixed point of this map iff the map  $g \in N(T)$ . So we get precisely  $N(T)/T = W$  fixed points. To compute the index note that  $G/T$  is an oriented manifold and these maps are orientation preserving isomorphisms and hence must have index 1.  $\square$

## 9. $U(n)$

In  $U(n)$  a maximal torus  $T$  is the group of diagonal matrices  $(e^{it_j})_{j=1}^n$ . A generic element in  $T$  consists of matrices with distinct eigenvalues and the Weyl group acts via permuting these eigenvalues, so the Weyl group is  $S_n$ .

### 9.1. Lie algebra.

- Lie algebra: The lie algebra  $\mathfrak{u}(n)$  consists of skew-hermitian matrices but upon tensoring with  $\mathbb{C}$  Lie algebra reduces to  $\mathfrak{gl}(n, \mathbb{C})$ , we'll only consider the  $\mathbb{C}$  case. The inner product on  $\mathfrak{u}(n)$  can be chosen to be  $\langle a, b \rangle = \text{trace}(a^*b)$  which then restricts to  $\mathfrak{h}$ . In everything that follows to avoid confusing notations,  $\pi_l$  would denote projection onto the  $l$ th coordinate appropriately interpreted.
- Cartan subalgebra: The cartan subalgebra corresponding to  $T$  is  $\mathfrak{h} = \{\text{diagonal matrices}\}$  which has a basis  $E_{jj}$ .
- Roots: The eigenbasis of  $\mathfrak{u}(n)/\mathfrak{h}$  for the adjoint representation of  $\mathfrak{h}$  is given by  $E_{jk}, j \neq k$  and the corresponding root is

$$[E_{jj}, E_{kl}] = (\delta_{kj} - \delta_{lj})E_{kl}$$

or more generally for a diagonal matrix  $\sum c_j E_{jj}$  the root is given by

$$[\sum c_j E_{jj}, E_{kl}] = (c_k - c_l)E_{kl}$$

The roots are  $\pi_j - \pi_k$  and all the lengths are  $\sqrt{2}$ . Note that the roots do not span  $\mathfrak{h}^*$  and this is because the center of  $U(n)$  is non-trivial.

- Simple roots: We can pick a system of simple roots to be  $\pi_j - \pi_{j+1}$ . The positive roots are  $\pi_j - \pi_k$  with  $j < k$ . The fundamental Weyl chamber is given by those elements which make an acute angle with  $\pi_j - \pi_{j+1}$ , these are

$$\{\sum c_j \pi_j | c_j > c_{j+1}\}$$

The angle between two consecutive simple roots is

$$\begin{aligned} \cos^{-1} \langle \pi_j - \pi_{j+1}, \pi_{j+1} - \pi_{j+2} \rangle / 2 &= \cos^{-1}(-1)/2 \\ &= 2\pi/3 \end{aligned}$$

And hence the root system is  $A_n$ .

**9.2. Representation ring.** The representation ring of  $U(n)$  is given by

$$\begin{aligned} RU(n) &\cong RT^W \\ &= Z[x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}]^{S_n} \\ &= Z[\sigma_1, \sigma_2, \dots, \sigma_n, \sigma_n^{-1}] \end{aligned}$$

where  $\sigma_i$  denotes the  $i$ th elementary symmetric polynomial

It is possible to identify these representations explicitly. The representation  $\sigma_i$  corresponds to the representation  $\wedge^i(\rho)$  where  $\rho$  is the standard representation  $\rho : U(n) \hookrightarrow GL(n, \mathbb{C})$ . The representation  $\sigma_n$  is then just the determinant representation, and the fact that it is invertible is simply saying that  $\det^{-1}$  is also a

representation. So if  $\lambda^i$  denotes  $\wedge^i(\rho)$  we have another identification

$$\begin{aligned} RU(n) &\cong \mathbb{Z}[\lambda^1, \lambda^2, \dots, \lambda^n, (\lambda^n)^{-1}] \\ &\cong \mathbb{Z}[\lambda^1, \lambda^2, \dots, \lambda^{n-1}] \otimes \mathbb{Z}[\lambda^n, (\lambda^n)^{-1}] \end{aligned}$$

which is a split isomorphism coming from

$$(9.1) \quad 1 \rightarrow SU(n) \rightarrow U(n) \xrightarrow{\det} S^1 \rightarrow 1$$

## 10. $SU(n)$

In  $SU(n)$  a maximal torus  $T$  is the group of diagonal matrices  $(e^{it_j})_{j=1}^n$  with  $\sum t_j = 0$ . A generic element in  $T$  again consists of matrices with distinct eigenvalues and the Weyl group acts via permuting these eigenvalues, so the Weyl group is  $S_n$ .

**10.1. Lie algebra.** The complexified Lie algebra  $\mathfrak{su}(n)$  consists of tracesless matrices in  $\mathfrak{gl}(n, \mathbb{C})$ . This is a subalgebra of  $\mathfrak{u}(n)$  and the coroots we found for  $\mathfrak{u}(n)$  in fact lie in  $\mathfrak{su}(n)$  and hence the root systems are exactly the same.

**10.2. Representation Ring.** The representations of  $U(n)$  restrict to representations of  $SU(n)$  and so we get a map

$$0 \rightarrow (\sigma_n - 1) \rightarrow RU(n) \rightarrow RSU(n)$$

One can work out the ring  $RT^W$  explicitly and show that the last map is in fact a surjection and we get the identification

$$RSU(n) \cong \mathbb{Z}[\lambda^1, \lambda^2, \dots, \lambda^{n-1}]$$

11.  $SO(2n+1)$ 

In  $SO(2n+1)$  a maximal torus  $T$  is the group of block diagonal matrices  $(R(\theta_j)_{j=1}^n, 1)$ , where  $R(\theta_j)$ , is a  $2 \times 2$  matrix which represents rotation about the angle  $\theta_j$  and there is an extra 1 towards the end. A generic element in  $T$  consists of matrices with distinct blocks and the Weyl group acts via permuting these blocks, but now each of these blocks can be flipped  $\theta_j \mapsto -\theta_j$  and so the Weyl group

$$W \cong (\mathbb{Z}/2)^n \rtimes S_n$$

## 11.1. Lie algebra.

- Lie algebra: After tensoring with  $\mathbb{C}$  the lie algebra  $\mathfrak{so}(2n+1)$  consists of skew-symmetric matrices. Denote by  $X_{i,j}$  the matrix  $E_{ij} - E_{ji}$ .
- Cartan subalgebra: The cartan subalgebra has basis  $X_{2j-1,2j}$ .
- Roots: The roots are slightly harder to compute but with a little effort one sees an the eigenbasis consists of the elements  $v_{k,l} := (X_{2k-1,2l-1} + iX_{2k,2l-1}) \pm i(X_{2k-1,2l} + iX_{2k,2l})$ ,  $\bar{v}_{k,l} := (X_{2k-1,2l-1} - iX_{2k,2l-1}) \pm i(X_{2k-1,2l} - iX_{2k,2l})$ ,  $w_{\pm k} := X_{2k-1,2n+1} \pm iX_{2k,2n+1}$  and the corresponding roots are

$$\begin{aligned} [X_{2j-1,2j}, v_{k,l}] &= i(\delta_{k,j} \pm \delta_{l,j})v_{k,l} \\ [X_{2j-1,2j}, \bar{v}_{k,l}] &= i(-\delta_{k,j} \pm \delta_{l,j})\bar{v}_{k,l} \\ [X_{2j-1,2j}, w_{\pm k}] &= \pm i\delta_{k,j}w_{\pm k} \end{aligned}$$

The corresponding roots up to the scalar  $i$  are  $\pm\pi_k, \pm\pi_k \pm \pi_l$ .

- Simple roots: A basis of simple roots would be given by  $\pi_n, \pi_j - \pi_{j+1}$  one of which has length 1 and the rest are length  $\sqrt{2}$  and Weyl fundamental chamber would consist of weights of the form

$$\{\sum c_j \pi_j | c_j > c_{j+1} > 0\}$$

The angle between two consecutive simple roots is

$$\begin{aligned} \cos^{-1} \langle \pi_j - \pi_{j+1}, \pi_{j+1} - \pi_{j+2} \rangle / 2 &= \cos^{-1}(-1)/2 \\ &= 2\pi/3 \\ \cos^{-1} \langle \pi_{n-1} - \pi_n, \pi_n \rangle / \sqrt{2} &= \cos^{-1}(-1)/2 \\ &= 3\pi/4 \end{aligned}$$

And hence the root system is  $B_n$ .

**11.2. Representation ring.** In this case the  $\mathbb{Z}/2$  part of  $W$  sends  $x_i \mapsto x_i^{-1}$  and so we get the isomorphism

$$RSO(2n+1) \cong Z[\sigma'_1, \sigma'_2, \dots, \sigma'_n]$$

where  $\sigma'_i$  denotes the  $i$ th elementary symmetric polynomial in  $x_j + x_j^{-1}$ . The representation  $x_j + x_j^{-1}$  is nothing but the representation  $\theta_j \mapsto R(\theta_j)$  and as before we can identify this with the alternating powers of the standard representation.

Let  $V$  be the *complexified* standard representation  $\rho : SO(2n+1) \hookrightarrow Gl(2n+1, \mathbb{C})$ . Then we can apply a change of basis transformation to  $Gl(2n+1, \mathbb{C})$  to change the map  $\rho$  on the maximal torus to

$$\rho|_T(\cdots, R(\theta_j), \cdots, 1) = (\cdots, e^{i\theta_j}, e^{-i\theta_j}, \cdots, 1)$$

But this is the representation  $\sum_j x_j + x_j^{-1}$  and we deduce the fact that  $\sigma_1 = V$  in  $R(G)$ . The higher symmetric polynomials can always be written in terms of exterior powers and so setting  $\lambda^j = \wedge^j V$  we get the identity,

$$RSO(2n+1) \cong \mathbb{Z}[\lambda^1, \lambda^2, \cdots, \lambda^n]$$

12.  $SO(2n)$ 

In  $SO(2n)$  a maximal torus  $T$  is the group of block diagonal matrices  $R(\theta_j)_{j=1}^n$ , where  $R(\theta_j)$  is a  $2 \times 2$  matrix which represents rotation about the angle  $\theta_j$ . A generic element in  $T$  consists of matrices with distinct blocks and the Weyl group  $W$  is almost the same as before, except  $\theta_j \mapsto -\theta_j$  needs conjugation by an element outside of  $SO(2n)$  but  $(\theta_j, \theta_k) \mapsto (-\theta_j, -\theta_k)$  is still feasible and so

$$W \cong (\mathbb{Z}/2)^{n-1} \rtimes S_n$$

I do not see how to write the representation of  $SO(2n)$  abstractly. But in terms of  $\lambda$ 's it is quite doable. Let  $\lambda^1$  be the representation corresponding to  $\sum_j x_j + x_j^{-1}$  and let  $\lambda^j = \wedge^j \lambda^1$ . If the Weyl group of  $SO(2n)$  had been the same as  $SO(2n+1)$  we would get the same representation ring, but because the Weyl group is only half as big. Now we need a combinatorial observation. Suppose we decompose  $\lambda^j$  into monomials in  $x_j$ 's and suppose  $j < n$ . Then one can see that the action of the Weyl group is still transitive on the set of these monomials and hence these  $\lambda^j$  remain irreducible. But for  $\lambda^n$  the monomials break into two orbits, those which contain even number of inverses and those monomials which contain odd number of inverses.

$$RSO(2n) \cong \mathbb{Z}[\lambda^1, \lambda^2, \dots, \lambda^{n-1}, \lambda_+^n, \lambda_-^n]$$

As an example of this splitting in the case when  $n = 1$  the standard representation  $SO(2) \hookrightarrow Gl(2, \mathbb{C})$  is not irreducible and breaks up into the two standard representations of  $S^1 \hookrightarrow \mathbb{C}^\times$ .

## 12.1. Lie algebra.

- Lie algebra: After tensoring with  $\mathbb{C}$  the lie algebra  $\mathfrak{so}(2n)$  again consists of skew-symmetric matrices. Denote by  $X_{i,j}$  the matrix  $E_{ij} - E_{ji}$ .
- Cartan subalgebra: The cartan subalgebra has basis  $X_{2j-1,2j}$ .
- Roots: The roots are slightly harder to compute but with a little effort one sees an eigenbasis consists of the elements  $v_{k,l} := (X_{2k-1,2l-1} + iX_{2k,2l}) \pm i(X_{2k-1,2l} - iX_{2k,2l-1})$ ,  $\bar{v}_{k,l} := (X_{2k-1,2l-1} - iX_{2k,2l-1}) \pm i(X_{2k-1,2l} + iX_{2k,2l-1})$  and the corresponding roots are

$$\begin{aligned} [X_{2j-1,2j}, v_{k,l}] &= i(\delta_{k,j} \pm \delta_{l,j})v_{k,l} \\ [X_{2j-1,2j}, \bar{v}_{k,l}] &= i(-\delta_{k,j} \pm \delta_{l,j})\bar{v}_{k,l} \end{aligned}$$

- The corresponding roots up to the scalar  $i$  are  $\pm\pi_k \pm \pi_l$  have length  $\sqrt{2}$ .
- Simple roots: A basis of simple roots would be given by  $\pi_j - \pi_{j+1}$ ,  $\pi_{n-1} + \pi_n$  and Weyl fundamental chamber would consist of weights of the form

$$\left\{ \sum c_j \pi_j \mid c_j > c_{j+1}, c_{n-1} + c_n > 0 \right\}$$

The angle between two consecutive simple roots is

$$\begin{aligned} \cos^{-1} \langle \pi_j - \pi_{j+1}, \pi_{j+1} - \pi_{j+2} \rangle / 2 &= \cos^{-1}(-1)/2 \\ &= 2\pi/3 \\ \cos^{-1} \langle \pi_{n-2} - \pi_{n-1}, \pi_{n-1} + \pi_n \rangle / 2 &= \cos^{-1}(-1)/2 \\ &= 2\pi/3 \end{aligned}$$

And hence the root system is  $D_n$ .



13.  $Spin(2n+1)$ 

The Lie algebra of  $Spin(2n+1)$  is the same as that for  $SO(2n+1)$ . Because  $Spin(2n+1)$  is a double cover of  $SO(2n+1)$  it has the same Weyl group and a maximal torus  $\tilde{T}$  is just a double cover of a maximal torus  $T$  of  $SO(2n+1)$ . This is slightly confusing. Assuming some knowledge of Clifford algebras we can say that

$$\begin{aligned} \tilde{T}(n) &= \left\{ \prod_{j=1}^n (\cos \theta_j + e_{2j-1} e_{2j} \sin \theta_j) \right\} \\ \cos \theta_j + e_{2j-1} e_{2j} \sin \theta_j &\mapsto R(\theta_j) \end{aligned}$$

It is slightly tricky and non-intuitive how to write the minimal set of generators for  $\tilde{T}$ . Abstractly however there is not much problem. Suppose  $\hat{T}$  is an  $n$  torus and we have a map

$$\begin{aligned} \hat{T} &\rightarrow \tilde{T} \\ (\theta_1, \theta_2, \dots, \theta_n) &\mapsto \prod_{j=1}^n (\cos \theta_j + e_{2j-1} e_{2j} \sin \theta_j) \end{aligned}$$

This is a covering map and the Deck transformations is  $(\mathbb{Z}/2)^{n-1}$  and consist of transformations generated by picking a pair  $(i, j)$  and sending it to  $(i + \pi, j + \pi)$ .

**13.1. Representation ring.** On the level of representations this sends  $x_i \mapsto -x_i, x_j \mapsto -x_j$ . The representation ring is obtained by further quotienting out the action of  $W$  and so we get

$$\begin{aligned} RSpin(2n+1) &\cong \mathbb{Z}[x_1, x_2, \dots, x_n]^{(\mathbb{Z}/2)^{n-1} \rtimes W} \\ &= \mathbb{Z}[x_1^2, x_2^2, \dots, x_n^2, x_1 x_2 \dots x_n]^W \\ &\cong \mathbb{Z}[\sigma_1'^2, \sigma_2'^2, \dots, \sigma_{n-1}'^2, \sigma_n'] \end{aligned}$$

where as before  $\sigma_j'$  is the  $j$ th elementary symmetric function in  $x_i + x_i^{-1}$ . The mapping  $Spin(2n+1) \rightarrow SO(2n+1)$  would then induce a map on the representation rings given by

$$\begin{aligned} \mathbb{Z}[\sigma_1', \sigma_2', \dots, \sigma_n'] &\rightarrow \mathbb{Z}[\sigma_1'^2, \sigma_2'^2, \dots, \sigma_n'^2, \sigma_n'] \\ x_1 &\mapsto x_1^2 \text{ sorry for the clumsy notation} \end{aligned}$$

We see that the image does not capture exactly one representation of  $Spin(2n+1)$  and that is the Spin representation.

 14.  $Spin(2n)$ 

$Spin(2n)$  is a double cover of  $SO(2n)$  and hence they share the Lie algebra. The first part of the computation is exactly as for  $Spin(2n+1)$ .

$$\begin{aligned} \tilde{T}(n) &= \left\{ \prod_{j=1}^n (\cos \theta_j + e_{2j-1} e_{2j} \sin \theta_j) \right\} \\ \cos \theta_j + e_{2j-1} e_{2j} \sin \theta_j &\mapsto R(\theta_j) \end{aligned}$$

$\hat{T}$  is an  $n$  torus and we have a map

$$\begin{aligned} \hat{T} &\rightarrow \tilde{T} \\ (\theta_1, \theta_2, \dots, \theta_n) &\mapsto \prod_{j=1}^n (\cos \theta_j + e_{2j-1} e_{2j} \sin \theta_j) \end{aligned}$$

As before this is a covering map and the Deck transformations is  $(\mathbb{Z}/2)^{n-1}$  and consist of transformations generated by picking a pair  $(i, j)$  and sending it to  $(i + \pi, j + \pi)$ .

**14.1. Representation ring.** We get

$$\begin{aligned} RSpin(2n) &\cong \mathbb{Z}[x_1, x_2, \dots, x_n]^{(\mathbb{Z}/2)^{n-1} \rtimes W} \\ &= \mathbb{Z}[x_1^2, x_2^2, \dots, x_n^2, x_1 x_2 \dots x_n]^W \\ &\cong \mathbb{Z}[\sigma_1'^2, \sigma_2'^2, \dots, \sigma_{n-1}'^2, \sigma_{n+}'^2, \sigma_{n-}'^2] \end{aligned}$$

The mapping  $Spin(2n) \rightarrow SO(2n)$  again gives an inclusion and the image does not capture exactly two representation of  $Spin(2n)$  and these are the Spin representation.

## 15. $Sp(n)$

This is a very confusing group and I'll use the following ad hoc description of it.

$$Sp(n) = \left\{ \begin{bmatrix} A & B \\ -\bar{B} & \bar{A} \end{bmatrix} \mid AA^* + BB^* = 1, AB^T = BA^T \right\} \subset U(2n)$$

It takes some work to see that the maximal torus  $T$  consists of diagonal matrices in  $Sp(n)$ . The permutation group  $S_n$  acts on  $T$  but it is also possible now to swap two of the corresponding diagonal entries in  $A$  and  $A^*$  and again

$$W \cong (\mathbb{Z}/2)^n \rtimes S_n$$

**15.1. Lie algebra.**

- Lie algebra: The lie algebra  $\mathfrak{sp}(n)$  upon tensoring with  $\mathbb{C}$  reduces to the Lie algebra  $\mathfrak{sp}(n, \mathbb{C}) = \left\{ \begin{bmatrix} A & B \\ B & -A \end{bmatrix} \right\}$  where  $A, B$  are  $n \times n$  matrices, I haven't checked this.
- Cartan subalgebra: The cartan subalgebra corresponding to  $T$  is  $\mathfrak{h} = \left\{ \begin{bmatrix} D & 0 \\ 0 & -D \end{bmatrix} \mid D \text{ is a diagonal matrix} \right\}$  which has a basis  $E_{jj} - E_{n+j, n+j}$ .
- Roots: The eigenbasis of  $\mathfrak{sp}(n, \mathbb{C})/\mathfrak{h}$  for the adjoint representation of  $\mathfrak{h}$  is given by  $E_{n+l, k} - E_{k, n+l}$  and the corresponding root is

$$[E_{jj} - E_{n+j, n+j}, E_{n+l, k} - E_{k, n+l}] = \delta_{jl}(E_{n+l, k} - E_{k, n+l})$$

or more generally for a diagonal matrix  $\sum c_j E_{jj}$  the root is given by

$$[\sum c_j E_{jj} - E_{n+j, n+j}, E_{n+l, k} - E_{k, n+l}] = c_l E_{n+l, k} - E_{k, n+l}$$

The roots are  $\pi_j - \pi_k$  and all the lengths are  $\sqrt{2}$ . Note that the roots do not span  $\mathfrak{h}^*$  and this is because the center of  $U(n)$  is non-trivial.

- Simple roots: We can pick a system of simple roots to be  $\pi_j - \pi_{j+1}$ . The positive roots are  $\pi_j - \pi_k$  with  $j < k$ . The fundamental Weyl chamber is given by those elements which make an acute angle with  $\pi_j - \pi_{j+1}$ , these are

$$\{\sum c_j \pi_j | c_j > c_{j+1}\}$$

The angle between two consecutive simple roots is

$$\begin{aligned} \cos^{-1} \langle \pi_j - \pi_{j+1}, \pi_{j+1} - \pi_{j+2} \rangle / 2 &= \cos^{-1}(-1)/2 \\ &= 2\pi/3 \end{aligned}$$

And hence the root system is  $A_n$ .

**15.2. Representation ring.** This is abstractly isomorphic to that of  $SO(2n+1)$  and so we would have

$$RSp(n) \cong \mathbb{Z}[\lambda^1, \lambda^2, \dots, \lambda^n]$$

where  $\lambda^i$  is the  $i$ th exterior power of the standard representation.