RESEARCH STATEMENT

APURVA NAKADE

I work in the area of algebraic topology and homotopy theory. In my research I attempt to apply homotopy theoretical techniques to study problems from geometry, symplectic topology, knot theory, representation theory, and mathematical physics. My research projects are:

1) Studying the space of Lagrangian embeddings using manifold calculus.
2) Finding knot invariants valued in spectra, using spectral lifts of Soergel Bimodules.
3) Future projects, in equivariant homotopy theory:
   a) Extending manifold calculus to manifolds with a group action.
   b) Proving Brown representability theorem for equivariant Goodwillie calculus.

0. Introduction

The main objects of study in homotopy theory are *spectra*. The category of spectra can be thought of as an intermediate category between the category of topological spaces and abelian groups.

$$\text{Topological Spaces} \xrightarrow[\text{Stabilization}]{} \text{Spectra} \xrightarrow[\text{Homotopy Grps}]{} \text{Graded Abelian Groups}$$

It is *nicer* than the category of topological spaces in that the homotopy classes of mapping spaces between spectra are abelian groups. Further in the category of spectra fibrations and cofibrations coincide and so (co)homological computations become substantially easier. Because of these properties, intractable problems about topological spaces become more manageable when translated to spectra. I apply this principle to the study of embedding spaces of Lagrangian manifolds.

Project 1: Lagrangian Submanifolds and Manifold Calculus. (Section 1)
Symplectic manifolds arise naturally in classical mechanics as phase spaces, which are the cotangent bundles of manifolds. The Lagrangian submanifolds of a symplectic submanifold form a very interesting subclass of submanifolds regarding which there are several significant open conjectures. In particular, Arnol'd's nearby Lagrangian conjecture says that the space of Lagrangian submanifolds inside a cotangent bundle is contractible. This conjecture has been a motivation for a lot of advances in Symplectic geometry.

I use a homotopy theoretical technique called manifold calculus, which is a form of functor calculus, to study the homotopy type of the space of Lagrangian submanifolds inside a symplectic manifold.

In the other direction, the category of spectra in more structured than the category of abelian groups. Generally speaking, 1) Spectra have more interesting automorphism groups than abelian groups 2) Spectra are amenable to using simplicial methods, for example, they allow one to define algebraic K-theory 3) They are the natural objects to define highly structured objects like E_∞ rings (the so called *brave new rings*), etc. Thus if we can lift an invariant which is valued in abelian groups to the category of spectra then the resulting new invariant is more powerful and captures more information about the original object. In a *joint work with Vitaly Lorman*, we apply this philosophy to construct knot invariants coming from Soergel Bimodules.

Project 2: Spectral Lift of the Category of Soergel Bimodules and Knot Invariants. (Section 2)
Soergel defined a purely combinatorial category of bimodules associated to any Coxeter group that categorifies its Hecke algebra. This easy to define category contains a lot of information about the BGG category of representations of the complex semisimple Lie algebras associated to the Coxeter group. Soergel bimodules were used to provide an algebraic proof of the Kazhdan-Lusztig conjectures. Khovanov defined a knot invariant, called the triply-graded link homology, using the Hochschild homology of Soergel bimodules of type A.

In an ongoing project (joint with Vitaly Lorman), we aim to construct a category of Soergel bimodules valued in spectra using Bott-Samelson varieties. Our primary goal is to define a spectrum valued knot invariant generalizing Khovanov's triply-graded link homology.

Future projects, in equivariant homotopy theory. Two classical categories that are difficult to study homotopy theoretically are the categories of 1) manifolds, and of 2) spaces with a group action. This is because neither of these two categories behave well under basic categorical constructions like taking (homotopy) limits and (homotopy) colimits. As such a lot of theoretical framework is needed to study these categories using homotopy theory.

Equivariant Manifold Calculus. (Section 3.1)
Goodwillie-Weiss defined manifold calculus to study the embedding spaces of manifolds. This theory encapsulates the homotopical content of classical surgery theory results. We aim to extend this framework to manifolds with a group action.

Brown Representability is the fundamental theorem of cohomology theories. Cohomology theories are also called 1-excisive functors. In [31] Lurie proved a generalized version of Brown representability theorem for n-excisive functors, which can be thought of as higher analogs of cohomology theories. In [12] Dotto defined G-equivariant versions of these n-excisive functors, for a Lie group G.

Equivariant Higher Brown Representability. (Section 3.2)
We aim to prove a generalization of Brown Representability for G-equivariant n-excisive functors.

The long term goal of these projects is to understand the equivariant Goodwillie tower better and use it to analyze equivariant algebraic K-theory.

1. Manifold Calculus and Lagrangian Embeddings

1.1. **Motivation.** Let N and L be closed manifolds of the same dimension. Recall that T^*N carries a natural symplectic structure. Assume that N and L are both simply connected. The following is a weaker homotopy theoretic version of Arnol'd's Nearby Lagrangian conjecture (still open): *The space* *of Lagrangian embeddings of L in T^*N is contractible if L is diffeomorphic to N, is empty otherwise.* Which leads to the question:

Q. How can we use homotopy theoretical techniques to study the space of Lagrangian submanifolds?

1.2. **Manifold Calculus.** I apply the techniques of manifold calculus to study the space of Lagrangian embeddings. Manifold calculus was introduced by Goodwillie-Weiss in [44], [20]. Denote by $\operatorname{Emb}(-,N)$ the functor which assigns to each m dimensional smooth manifold M the space of embeddings of M inside N. In [44] Weiss defines a tower of *Taylor approximations*

$$\operatorname{Emb}(-,N) \to (\mathcal{T}_1 \operatorname{Emb}(-,N) \leftarrow \mathcal{T}_2 \operatorname{Emb}(-,N) \leftarrow \mathcal{T}_3 \operatorname{Emb}(-,N) \leftarrow \dots)$$

The functor $\mathcal{T}_\infty \operatorname{Emb}(-,N)$ is defined to be the homotopy limit of this tower and is called **analytic approximation** to $\operatorname{Emb}(-,N)$. The analytic approximation $\mathcal{T}_\infty \operatorname{Emb}(-,N)$ is essentially constructed by restricting the functor $\operatorname{Emb}(-,N)$ to the category of manifolds which are diffeomorphic to finitely many $\mathbb{R}^m$.

1.3. **Lagrangian Embeddings.** Let N be a symplectic manifold with the choice of a compatible almost complex structure (which is unique up to homotopy), and let $n = 2m$. Denote by $\operatorname{Emb}_{\mathrm{Lag}}(-,N)$ and $\operatorname{Emb}_{\mathrm{TR}}(-,N)$ the functors that assign to each m dimensional smooth manifold M the space of *Lagrangian* embeddings and *totally real embeddings* of M inside N, respectively. In **[?]** I prove the following homotopy equivalence.

Theorem (Nakade)**.** *With the notation as above, for $m > 2$ there is a homotopy equivalence*

$$\mathcal{T}_\infty \operatorname{Emb}_{\mathrm{Lag}}(M,N) \simeq \operatorname{Emb}_{\mathrm{TR}}(M,N)$$

1.4. **Homotopy Principle.** The main idea behind the proof of the above theorem is connecting the theory of manifold calculus to the theory of h-principle. An h-principle is a method to reduce existence problems in differential geometry to homotopy-theoretic problems.

Let $\mathrm{Gr}_m(N)$ be the m-plane Grassmannian bundle over N, more explicitly, $\mathrm{Gr}_m(N)$ is a fiber bundle over N with fiber over a point $p \in N$ being $\mathrm{Gr}_m(T_pN)$, the space of m planes inside T_pN. Let $\mathcal{A} \subseteq \mathrm{Gr}_m(N)$ be a subfibration over N. An $\mathcal{A}$-directed embedding is an embedding e such that the image of the induced map $\mathrm{Gr}_m(e)$ lies inside $\mathcal{A}$. Denote by $\operatorname{Emb}_{\mathcal{A}}(-,N)$ the functor that assigns to M the space of $\mathcal{A}$-directed embeddings of M inside N.

Theorem (Nakade)**.** *If $n - m > 2$ and $\mathcal{A}$ satisfies the h-principle for directed embeddings then the naturally induced map*

$$\operatorname{Emb}_{\mathcal{A}}(M,N) \xrightarrow{\simeq} \mathcal{T}_\infty \operatorname{Emb}_{\mathcal{A}}(M,N)$$

is a homotopy equivalence for all m dimensional smooth manifolds M.

The result about Lagrangian and totally real embeddings is a direct corollary of this theorem.

1.5. **Tangentially Straightened Embeddings.** Using the above theorem we can also construct several exotic embedding functors. Somewhat analogous to the fact that there are non-trivial functions which have trivial Taylor series, we can use the h-principle to create functors which are almost always trivial but whose analytic approximation is never so.

Let $M \subseteq \mathbb{R}^n$ be a parallelizable manifold, with a choice of m linearly independent non-vanishing vector fields $X_1, \ldots, X_m$. Denote by $\mathrm{Emb}_{\mathrm{TS}}(-, \mathbb{R}^n)$ the functor on the category of parallelizable manifolds which sends M to the space of embeddings $e : M \hookrightarrow \mathbb{R}^n$ such that $\mathrm{D}e(X_1), \ldots, \mathrm{D}e(X_m)$ are constant non-varying vector fields. Borrowing the terminology from [13] we call $\mathrm{Emb}_{\mathrm{TS}}(M, \mathbb{R}^n)$ the space of *tangentially straightened embeddings*. We show that

Theorem (Nakade). *When $n - m > 2$, there is a homotopy equivalence*
$$\mathcal{T}_\infty \mathrm{Emb}_{\mathrm{TS}}(M, \mathbb{R}^n) \simeq \mathrm{hofib}(\mathrm{Emb}(M, \mathbb{R}^n) \to \mathrm{Imm}(M, \mathbb{R}^n))$$

If M is not diffeomorphic to a submanifold of $\mathbb{R}^m$, it is easy to see that no tangentially straightened embedding is possible and hence $\mathrm{Emb}_{\mathrm{TS}}(M, N)$ is empty. However, as $M \subseteq \mathbb{R}^n$ the homotopy fiber $\mathrm{hofib}(\mathrm{Emb}(M, \mathbb{R}^n) \to \mathrm{Imm}(M, \mathbb{R}^n))$ is non-empty. Thus the functor $\mathrm{Emb}_{\mathrm{TS}}(-, \mathbb{R}^n)$ provides an example of a highly *non-analytic* functor.

1.6. **Future Directions.** Manifold calculus does not see symplectic geometry because it is constructed using $\mathcal{D}\mathrm{isc}_\infty$ and symplectic geometry is locally trivial. One possible remedy might be to replace discs by more structured objects which retain some geometric, particularly Floer theoretic information, however what these structured objects should be remains unclear.

2. Spectral Soergel Bimodules and Knot Invariants

This is a joint work with Vitaly Lorman.

Soergel bimodules were introduced by Soergel in [42]. In [43]. further established a deep link between these bimodules and representations of algebraic groups in positive characteristic. In [36] Rouquier constructed explicit braid invariants using these Soergel bimodules. In [26] Khovanov constructed a knot invariant using Hochschild homology of Rouquier complexes. Topological analogues of these bimodules were studied by Kitchloo in [27].

Goal. Find lifts of these bimodules in the category of spectra.

2.1. **Classical Version.** Let $R = \mathbb{Q}[x_1, x_2, \ldots, x_n]$. The n^{th} symmetric group S_n, and hence also the Braid group $\mathcal{B}_n$, acts on R by permuting the variables x_i. Let σ_i denote the generator of $\mathcal{B}_n$ that swaps x_i and x_{i+1} for $1 \le i \le n-1$, so that $R^{\sigma_i} = \mathbb{Q}[x_1, x_2, \ldots, x_i x_{i+1}, x_i + x_{i+1}, \ldots, x_n]$. Define the **Soergel bimodules** and **Rouquier complexes** as

Soergel bimodules:	B_i	$R \otimes_{R^{\sigma_i}} R$
Rouquier comlpexes:	$F(\sigma_i)$	$0 \to R \longrightarrow B_i \to 0$ $a \mapsto a((x_i - x_{i+1}) \otimes 1 + 1 \otimes (x_i - x_{i+1}))$
	$F(\sigma_i^{-1})$	$0 \to B_i \longrightarrow R \to 0$ $a \otimes b \mapsto ab$

To a braid word $\widetilde{w} = \sigma_{i_1} \sigma_{i_2} \ldots \sigma_{i_k}$ we associate the complex:
$$F(\widetilde{w}) = F(\sigma_{i_1}) \otimes_R F(\sigma_{i_2}) \otimes_R \cdots \otimes_R F(\sigma_{i_k})$$

Theorem 2.1 (Soergel-Rouquier)**.** *There are isomorphisms of R-bimodules*

$$(B_i \otimes_R B_{i+1} \otimes_R B_i) \oplus B_{i+1} \cong (B_{i+1} \otimes_R B_i \otimes_R B_{i+1}) \oplus B_i$$
$$B_i \otimes_R B_j \cong B_j \otimes_R B_i \qquad \text{if } |i-j| > 1$$

As a consequence, if $\widetilde{w_1} = \widetilde{w_2}$ inside the Braid group $\mathcal{B}_n$ then the complexes $F(\widetilde{w_1})$ and $F(\widetilde{w_2})$ are quasi-isomorphic. So we can associate to an element $w \in \mathcal{B}_n$ the complex $F(w) := F(\widetilde{w})$ where $\widetilde{w}$ is any word representing w.

2.2. **Homotopical Version.** Let $G = PU(n)$, the projectivized unitary group with Weyl group W being the symmetric group Σ_n. Let T be the maximal torus consisting of diagonal matrices. Let G_w be the Levi component of the Parabolic subgroup corresponding to an element $w \in W$.

For a word $\widetilde{w} = \sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_k}$ the Borel construction of the **Bott-Samelson variety** is defined to be

$$P_{\widetilde{w}} = EG \times_T (G_{\sigma_{i_1}} \times_T G_{\sigma_{i_1}} \times_T G_{\sigma_{i_2}} \times_T \cdots \times_T G_{\sigma_{i_1}}/T)$$

Bott-Samelson varities are smooth and provide a resolution of Schubert varieties BG_w where $\widetilde{w}$ is any word representing w. An explicit computation of singular cohomologies gives us

$$H^*(P(\widetilde{w})) \cong F(\widetilde{w})$$

The various spectra $P(\widetilde{w})$ come with a natural BT-bicomodule structure. Which leads us to the question:

Q. *Are there isomorphisms*

$$(P(\sigma_i) \boxtimes_{BT} P(\sigma_{i+1}) \boxtimes_{BT} P(\sigma_i)) \vee P(\sigma_{i+1}) \cong (P(\sigma_{i+1}) \boxtimes_{BT} P(\sigma_i) \boxtimes_{BT} P(\sigma_{i+1})) \vee P(\sigma_i)$$
$$P(\sigma_i) \boxtimes_{BT} P(\sigma_j) \cong P(\sigma_j) \boxtimes_{BT} P(\sigma_i) \qquad \text{if } |i-j| > 1$$

Our calculation suggests that this is not correct. Instead we conjecture that

Conjecture. *There are isomorphisms of BT-bicomodules*

$$\mathrm{Th}\,(P(\sigma_i) \boxtimes_{BT} P(\sigma_{i+1}) \boxtimes_{BT} P(\sigma_i)) \vee \mathrm{Th}(P(\sigma_{i+1})) \cong \mathrm{Th}\,(P(\sigma_{i+1}) \boxtimes_{BT} P(\sigma_i) \boxtimes_{BT} P(\sigma_{i+1})) \vee \mathrm{Th}(P(\sigma_i))$$
$$P(\sigma_i) \boxtimes_{BT} P(\sigma_j) \cong P(\sigma_j) \boxtimes_{BT} P(\sigma_i) \qquad \text{if } |i-j| > 1$$

where $\mathrm{Th}(X)$ *denotes the Thom space of a suitably chosen vector bundle over X.*

This will then allow us to define *spectral valued* Rouquier complexes. Furthermore, we expect the algebraic K-theory of the category of spectral Rouquier complexes to be a knot invariant which lifts Khovanov's triply-graded link homology to spectra.

3. Future Projects, in Equivariant Homotopy Theory

3.1. **Equivariant Manifold Calculus.** Let G be a Lie group. In [12] Dotto generalized Goodwillie's functor calculus to functors with a G-action. The difficulty lies in generalizing the notions of cartersian and strongly co-cartesian squares to G-cartesian and G-cocartesian squares.

Goal. Generalize manifold calculus to manifolds with G-action.

We do not expect the methods similar to those in [12] to extend to the category of manifolds, as the homotopy fixed points under a group action on a manifold are not necessarily homotopy equivalent to a manifold. However, because the fixed points of a manifold under a group action are submanifolds we expect the following conjecture to hold.

Conjecture. *Let M, N be manifolds with G-action. Let $\mathrm{Emb}^G(M, N)$ denote the space of equivariant embeddings of M inside N. The functor $\mathrm{Emb}^G(-, N)$ is analytic on M if $\dim(N^H) - \dim(M^H) > 2$ for all subgroups $H \leq G$.*

We further expect to extend the above conjecture to $\mathrm{Emb}^G_{\mathcal{A}}(M, N)$, equivariant embeddings with tangential structure $\mathcal{A} \subseteq TN$.

3.2. **Equivariant Higher Brown Representability.** The Goodwillie tower defined in [17], [18], [19] extends the notion of cohomology theories to higher cohomology theories via n-excisive functors. In [31] Lurie showed that these n-excisive functors satisfy Brown representability. In [12] Dotto generalized Goodwillie's tower to functors with a G-action.

Conjecture. *G-equivariant n-excisive functors on the category of spectra are equivalent to the functors on the category of finite G-sets of size $\leq n$.*

The long term goal of these projects is to understand the equivariant Goodwillie tower better and use it to analyze equivariant algebraic K-theory.

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