Representation & character theory of categorical groups

Def^n: Categorical group / 2 group is a monoidal groupoid with weakly invertible objects.

eg: • Categorical tori :

Two ingredients – 1) lattice $\Lambda^\vee$ 2) integral bilinear form $J : \Lambda^\vee \otimes \Lambda^\vee \longrightarrow \mathbb{Z}$

Step 1: $T := \Lambda^\vee \otimes U(1)$ $\quad \mathfrak{t} = \Lambda^\vee \otimes \mathbb{R}$ = Lie algebra of T

2: construct $\mathcal{T}$, $\quad \mathrm{ob}\,\mathcal{T} = \mathfrak{t}$

$$1_{\mathrm{mor}} = \quad x \xrightarrow{z} x+m \qquad x \in \mathfrak{t},\ m \in \Lambda^\vee,\ z \in U(1)$$

composition:
$$\begin{tikzcd} x \arrow[r, "z"] \arrow[rr, bend right, "zw"'] & x+m \arrow[r, "w"] & x+m+n \end{tikzcd}$$

monoidal structure: ob : $x \cdot y := x+y$

$$1_{\mathrm{mor}} : (x \xrightarrow{z} x+m) \cdot (y \xrightarrow{w} y+n)$$
$$:= x+y \xrightarrow{z \cdot w \cdot e^{-2\pi i J(m,y)}} x+y+m+n$$

+ identity associators, unit isomorphics.

$(\mathcal{T}, \cdot)$ is an example of a (strict) Lie 2 group.

$(*)$ $\mathrm{ob}(\mathcal{T})/\cong \;= T$ and $\mathrm{aut}(1) = U(1)$

Th^m: $(\mathcal{T}, \cdot)$ upto equivalence, only depends on the even symmetric bilinear form $I(m,n) = J(m,n) + J(n,m)$

All Lie 2-groups satisfying $(*)$ is of the above form.

Idea of Proof: $H^4(BT;\mathbb{Z}) \cong H^3_{gp}(T; U(1))$ ⟿ Segal Michison cohomology [Wagemann - Wockel]

set of even symmetric bilinear form on $\Lambda^\vee$

By Schommel Pries

RHS = set of 2 groups satisfying $(*)$

eg: 1) max. torus T of simple simply connected compact Lie group, basic bilinear form I.

2. $\Lambda^{\vee}$ Leech lattice, Nemeyer lattice

Recognition principle:

$G =$ simple, simply connected.

$$\mathbb{Z} = H^3_{gp}(G, U(1)) \cong H^4(BT;\mathbb{Z})^W \longrightarrow H^4(BT;\mathbb{Z}) = H^3_{gp}(T; U(1))$$

Second equivalent description of $\mathcal{T}$:

ob : (t, L) $\quad t \in T$, $L =$ complex hermitian line

mor : $\mathrm{Hom}((t, L_1), (s, L_2)) = \begin{cases} \mathrm{Iso}(L_1, L_2) & \text{if } t = s \\ \phi & \text{else} \end{cases}$

$$(t, L_1)\cdot(s.L_2) = (ts, \mathbb{L}^J_{t,s} \otimes L_1 \otimes L_2) \quad \text{where} \quad \begin{array}{c}\mathbb{L}^J \\ \downarrow \\ T\times T\end{array} = A \times A \times \mathbb{C}/\sim$$

$$(x+m, y+n, z) \sim (x, y, z\cdot e^{2\pi i J(m,y)})$$

(multiplicative bundle gerbe with connection $\overline{\text{on } T}$)

Transgression - Regression [Waldorf]

(L^J, ∇) gives rise to a central extension $\mathcal{L}T$

2-cocycle

$$c(\gamma_1, \gamma_2) = \mathrm{Hol}_{\nabla}(\gamma_1, \gamma_2) = e^{-2\pi i \int_0^1 J(g_1(t), \dot{g}_2(t))\, dt}$$

$$\begin{tikzcd} & \mathfrak{t} \arrow[d] \\ {[0,1]} \arrow[ur, "g_i"] \arrow[r, "\gamma_i"'] & T \end{tikzcd}$$

Conjugacy classes:

Def: 2-group-$\mathcal{G}$. Inertia groupoid of $\mathcal{G}$ is $\Lambda\mathcal{G} = \mathrm{Bicat}(\cdot/\!/\mathbb{Z}, \cdot/\!/\mathcal{G})/2\text{ iso}$

equivalent : $\quad$ ob $\Lambda\mathcal{G} =$ ob $\mathcal{G}$

$\quad$ mor $\Lambda\mathcal{G}$: $g \xrightarrow{s} h \qquad sg \cong hs$

eg: 1) $\mathcal{G} = G$, $\Lambda\mathcal{G} = \quad g \xrightarrow{s} sgs^{-1}$

2) $\mathcal{G} = T$, $\qquad \Lambda T = \left(\begin{array}{c} T\times T \\ \downarrow\downarrow \\ T \end{array}\right)$

3) $\Lambda\mathcal{J} \simeq \left(\begin{array}{c} U(1) \text{ torsor of ?? line bundles on } T \\ \downarrow \\ T \end{array}\right)$