§ 11,12 Two sided bar construction and function spaces:

· Functors as modules:

For a small categories I, J the functors:

- $F: I \longrightarrow \mathcal{M}$ can be thought of as a left I-module $\in I\text{-mod}$
- $G: I^{op} \longrightarrow \mathcal{M}$ can be thought of as a right I-module $\in \text{mod-}I$
- $H: I \times J^{op} \longrightarrow \mathcal{M}$ " " an I-J bi module $\in I\text{-}J$ mod

· If objects of $\mathcal{M}$ are sets, let $f \in F(i)$ for $i \in I$.
For $f = i \to j$ in I, we can define $w.f := F(i \to j) f \in F(j)$
similarly for G, H

· Tensors and coends:

$F: I \times J^{op} \longrightarrow \mathcal{M}$ $\in$ I-J mod

$G: J \times K^{op} \longrightarrow \mathcal{M}$ $\in$ J-K mod

The coend defined as

$$F \underset{J}{\otimes} G = \int^{J} F(-,j) \times G(j,-) = \operatorname{coeq}\Big[\coprod_{i \leftarrow j} F(-,i) \times G(j,-) \rightrightarrows \coprod_{i \in J} F(-,i) \times G(i,-) \Big]$$

is an I-K mod

· Universal property?

For $F \in I \times J^{op}$, $G \in J \times K^{op}$, $H \in I \times K^{op}$ what is the set of maps $\hom_{I\text{-}K}(F \otimes G, H)$?

$$\hom_{I\text{-}K}(F \otimes G, H) = \hom_{I\text{-}K}\Big(\operatorname{coeq}\Big[\coprod_{i \leftarrow j} F(\cdot,i) \times G(j,\cdot) \rightrightarrows \coprod_{i \in J} F(\cdot,i) \times G(i,\cdot) \Big], H(\cdot,\cdot) \Big)$$

$$\hom\Big(\int^{J} F \times G, H\Big) = \operatorname{eq}\Big[\hom_{I\text{-}K}\Big(\coprod_{i \leftarrow j} F(\cdot,i) \times G(j,\cdot), H(\cdot,\cdot) \Big) \leftleftarrows \hom\Big(\coprod_{i} F(\cdot,i) \times G(i,\cdot), H(\cdot,\cdot) \Big) \Big]$$

$$= \operatorname{eq}\Big[\prod_{i \leftarrow j} \hom(F(\cdot,i) \times G(j,\cdot), H(\cdot,\cdot)) \leftleftarrows \prod_{i} \hom(F(\cdot,i) \times G(i,\cdot), H(\cdot,\cdot)) \Big]$$

$$= \int_{J} \hom(F \times G, H)$$ ← This is an end

Q. Can we think of the last thing as bilinear maps $F \times G \to H$?
Yes!!

· $G = \hom_J(i,-) \in J$-mod gives $\hom_{-K}(F \otimes \hom(-,i), H) \cong \hom(F, \hom_J(\hom(-,i), H))$
$\cong \hom(F, \hom(H(i)))$ By Yoneda's lemma

$F = \hom_J(-,i) \in$ mod-J gives $\hom(\hom_J(-,j) \otimes G, H) \cong \hom(\hom_I(-,j), \hom(G,H))$
$\cong \hom_K(G,H)_j \cong \hom_K(G_j, H)$

· $\overset{\text{dende by } I}{I}: I^{op} \times I \longrightarrow$ Sets
$(i,j) \longmapsto \hom_I(j,i)$
$\in I$-I mod

$$\underline{\underline{F \otimes I(k)}} = \operatorname{coeq}\Big(\coprod_{i \leftarrow j} F(i) \times I(k,j) \rightrightarrows \coprod_{i} F(i) \times I(k,i) \Big]$$

$$\underline{\underline{= F(k) = (I \otimes F)(k)}}$$

$$\begin{tikzcd} & F(i) \times I(k,i) \arrow[dr] & \\ F(i) \times I(k,j) \arrow[ur] \arrow[dr] & \arrow[r, phantom, ">"] & F(k) \\ & F(j) \times I(k,j) \arrow[ur] & \end{tikzcd}$$

Pushout diagram

· **Bar construction:**

Bar construction should be thought of as a fattened version of $\otimes$.

· **Def^n:** Given $W \in J\text{-}I\,\mathrm{mod}$, $X \in I\text{-}K\,\mathrm{mod}$, define the two-sided bar construction as

$$B_\bullet(W, I, X)[n] = \coprod_{i_0 \leftarrow i_1 \leftarrow \cdots \leftarrow i_n} W(i_0) \times X(i_n) \quad \in J\text{-}K\text{-}\mathrm{mod}$$

Boundary maps: $B_\bullet(W,I,X)[n] \xrightarrow{d_i} \coprod B_\bullet(W,I,W)[n-1]$ The differentials correspond to BI^{op}

$$\begin{array}{ccl} & & i_1 \leftarrow i_2 \leftarrow \cdots \leftarrow i_n \\ i_0 \leftarrow i_1 \leftarrow \cdots \leftarrow i_n & \xrightarrow{d_0} & W(i_1) \times X(i_n) \\ W(i_0) \times X(i_n) & \xrightarrow{d_j} & i_0 \leftarrow \cdots \leftarrow (i_{j-1} \leftarrow i_j \leftarrow i_{j+1}) \leftarrow \cdots \leftarrow i_n \\ & & W(i_0) \times X(i_n) \\ & \xrightarrow{d_n} & i_0 \leftarrow i_1 \leftarrow \cdots \leftarrow i_{n-1} \\ & & W(i_0) \times X(i_{n-1}) \end{array}$$

define $|B_\bullet(W,I,X)| =: B(W,I,X)$

· $B(*, I, *) = B(I^{op})$

· $\mathrm{coeq}\left(B_1(W,I,X) \rightrightarrows B_0(W,I,X)\right) = \mathrm{coeq}\left(\coprod_{i_0 \leftarrow i_1} W(i_0) \times X(i_1) \rightrightarrows \coprod_i W(i) \times X(i)\right)$

$$= W \underset{I}{\otimes} X$$

We have a natural map $B_\bullet(W,I,X) \longrightarrow W \otimes X$

(this is true for Top)

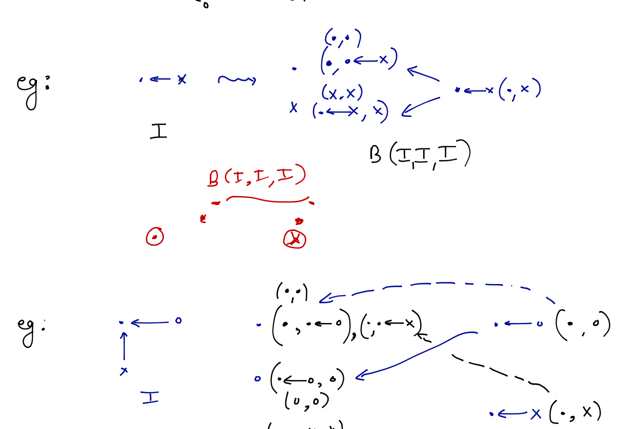
Th^m: If $\forall Z \in \mathcal{M}$ cofibrant, $Z \times (-)$ preserves weak equivalences between cofibrant objects then any objectwise w.e. $W \to W'$, $X \to X'$ induces a w.e. $B(W', I, X) \longrightarrow B(W, I, X')$.

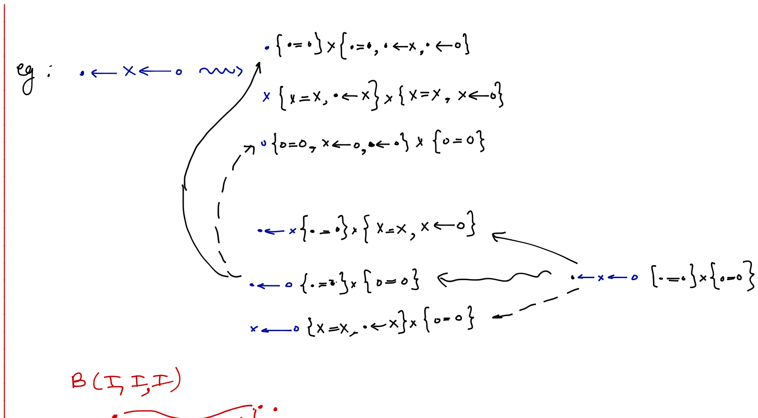
Th^m: $B(W,I,X) \otimes Y \cong B(W,I,X \otimes Y)$, $\quad Z \otimes B(W,I,X) \cong B(Z \otimes W, I, X)$

· $B(W, I, X) \cong W \otimes B_\bullet(I,I,I) \otimes X$ Q. What is $B(I,I,I)$?

This also justifies why $B_\bullet$ is like a fattened version of $\otimes$.

· $B_n(I,I,I) = \coprod_{i_0 \leftarrow \cdots \leftarrow i_n} \hom_I(i_0, -) \times \hom(-, i_n)$ This is an I-I mod in sSets





- $B_\bullet(*, I, X)[n] = \coprod_{i_0 \leftarrow \cdots \leftarrow i_n} X(i_n) = \text{Srep}\, X$

$\Rightarrow \quad B(*, I, X) = |\text{Srep}(X)| = \text{hocolim}_I X$

- $B(I, I, X)[n] = \coprod_{i_0 \leftarrow \cdots \leftarrow i_n} \hom(i_0, -) \times X(i_n)$

$B(I, I, X)[n](k) = \coprod_{i_0 \leftarrow \cdots \leftarrow i_n} \hom(i_0, k) \times X(i_n)$ Note: $B(I, I, X) \in I\text{-mod}$

$= \coprod_{k \leftarrow i_0 \leftarrow \cdots \leftarrow i_n} X(i_n)$

$= B(*, I \downarrow k, X)[n]$

$\Rightarrow B(*, I, X)(-) = \text{hocolim}_{I \downarrow -} X$

$\Rightarrow \quad B(*, I, X) = QX$

- $B(W, I, X) \simeq B(X, I^{op}, W)$ Q. Is this true?

$\Rightarrow$ all the above statements can be rewritten for I^{op}.

- Function Spaces:

We have a categorical version of hom as well:

- Def For $X, Y \in I\text{-mod}$ define:

$$F_I(X, Y) := \text{eq}\left(\prod_i \hom(X(i), Y(i)) \rightrightarrows \prod_{i \to j} \text{Map}(X(i), Y(j)) \right)$$

This gives us the much awaited adjunctions: for $Z \in K\text{-mod}$, $X \in I\text{-}K\text{ mod}$, $Y \in I\text{-mod}$

- $F_K(Z, F_I(X, Y)) = \hom_I(Z \otimes_K X, Y)$