§3.6 Topological properties of Spec A

- Spec A is not connected $\iff$ $A \cong A_1 \times A_2$ for some A_1, A_2

Proof: $\operatorname{Spec} A = X \sqcup Y$ both X, Y open & closed. $X = V(I)$, $Y = V(J)$ for some radical ideals I, J. Every prime ideal either contains I or J but not both. Look at $A \longrightarrow A/I \times A/J$. This is clearly surjective. Kernel consists precisely of $I \cap J$. Not sure how to get rid of this.

Def^n: X is quasi-compact if every open cover has a finite subcover.

Saw last time, Spec A is quasicompact.

Def^n: X is irreducible if X is not union of finitely many proper closed subsets.

1) irreducible $\Rightarrow$ connected.   Prove all these
2) $\Rightarrow$ Every open subset is dense.
3) $Z \subseteq X$ irred $\Rightarrow$ $\bar{Z} \subseteq X$ irreducible
4) A integral domain $\Rightarrow$ (0) prime   Note: for an $\mathfrak{p} \in \operatorname{Spec} A$
5) $\Rightarrow$ $\overline{(0)}$ irreducible $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$
6) $\Rightarrow$ Spec A irreducible

1) Trivial. 2) Let $U \subseteq X$ be open. If $\bar{U} \cup (X \setminus U) = X$. But X is irreducible $\Rightarrow \bar{U} = X$.

3) $Z \subseteq X$ irreducible. Suppose $\bar{Z} = W \cup T$ both W, T closed inside $\bar{Z}$.

$Z = (W \cap Z) \cup (T \cup Z)$

$\Rightarrow Z = W \cap Z$ or $Z = T \cup Z$ as $Z \subseteq \bar{Z}$ has subspace topology

$\Rightarrow \bar{Z} = W$ or T. $\Rightarrow \bar{Z}$ irreducible.

A better way of thinking of A as functions on Spec A:

$$A \cong \operatorname{Hom}(\mathbb{Z}[x], A) \cong \operatorname{Hom}(\operatorname{Spec} A, \operatorname{Spec} \mathbb{Z}[x])$$

Def: $\mathfrak{p} \in \operatorname{Spec}(A)$ is a closed point if $\{\mathfrak{p}\}$ closed in $\operatorname{Spec} A$.

- $\mathfrak{p}$ closed point $\iff$ $\mathfrak{p}$ is maximal.

Prop: A finitely generated algebra$/k \Rightarrow$ closed points dense in $\operatorname{Spec} A$.

Def: In topological space X, x is <u>specialization</u> of y & y is a <u>generalization</u> of x if $x \in \overline{\{y\}}$.

In $\operatorname{Spec} A$, $\mathfrak{q}$ is specialization of $\mathfrak{p}$

$\iff \mathfrak{q} \supseteq \mathfrak{p}$.

Def^n: $p \in X$ is a generic point for closed $Z \subseteq X$ if $Z = \overline{\{\mathfrak{p}\}}$.

Eg: A integral domain $\Rightarrow$ (0) is generic point for $\operatorname{Spec} A$.

Def^n: An irreducible connected component is a maximal irreducible component

- Every $x \in X$ belongs to some irreducible component. (Zorn's lemma)
- $\mathbb{A}^1_{\mathbb{C}}$ is irreducible in Zariski topology.
- Show: $\mathbb{A}^1_k$ is irreducible for any field k.

Note: $I_1 \subseteq I_2 \subseteq \cdots$ in A

$\Rightarrow V(I_1) \supseteq V(I_2) \supseteq \cdots$ in $\operatorname{Spec} A$

($\Rightarrow \operatorname{Spec} A$ is noetherian.)

Def^n: A topological space X is noetherian if an descending chain of closed subspaces eventually stabilizes.

Converse false:

$k[x_1, x_2, \ldots]/(x_1, x_2, \ldots)^2$ consists entirely of nilpotent elements and so its Spec is noetherian but is not.

Prop: X Noetherian topological space, $Z \subseteq X$ closed. Then $Z = Z_1 \cup Z_2 \cdots \cup Z_n$ for irreducible closed Z_i, unique upto ordering, none Z_i contained inside the other.

Proof: Standard Noetherian argument.

§ 3.7 $I(-)$

Def^n: $I(S) = \bigcap_{\wp \in S} \wp \subseteq A$ "all functions vanishing on S" for $S \subseteq \operatorname{Spec} A$

- $I(S) = I(\bar{S})$
- $I(V(J)) = \sqrt{J}$
- $V(I(S)) = \bar{S}$

$$\Longleftrightarrow \quad \left\{ \begin{array}{c} \text{closed subsets} \\ \text{in Spec } A \end{array} \right\} \underset{V}{\overset{I}{\rightleftarrows}} \left\{ \begin{array}{c} \text{radical ideals} \\ \text{in } A \end{array} \right\}$$

Note: $\{\wp\} \in \operatorname{Spec} A \Rightarrow V(\wp) = \overline{\{\wp\}}$ irreducible

- $S \subseteq \operatorname{Spec} A \Rightarrow I(S)$ is prime.

Proof: WALOG assume $S = V(J)$, J radical

Let $ab \in J \Rightarrow V(J) \subseteq V(ab) = V(a) \cup V(b)$

$\Rightarrow V(J) \subseteq V(a)$ or $V(J) \subseteq V(b)$ as $V(J)$ irreducible

$\Rightarrow a \in J$ or $b \in J$

$\Rightarrow J$ prime.

$$\left\{ \begin{array}{c} \text{prime ideals} \\ \text{in } A \end{array} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{irreducible closed subsets} \\ \text{in Spec } A \end{array} \right\}$$

$$\left\{ \begin{array}{c} \text{minimal prime} \\ \text{ideals} \end{array} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{irreducible components} \\ \text{of Spec } A \end{array} \right\}$$

And so the prime components of the nil radical correspond to the connected components.

Prop: $\mathbb{A}^1_k$ is irreducible.

Proof: $\mathbb{A}^1_k = \operatorname{Spec}(k[x])$. Because $k[x]$ is an integral domain, (0) is prime & it most certainly is minimal. And hence $\mathbb{A}^1_k$ is irreducible. (More generally $\mathbb{A}^n_k$, or Spec of any integral domain would be irreducible.)