# DIRAC OPERATORS

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Abstract. Notes from the stuff I am presenting on Dirac operators. In these notes unless otherwise stated $X$ will always denote a closed Spin manifold of dimension $2n$.



Contents

## 1. Curvature and Chern Weil Theory

I defined connection on a Principal $G$ bundle in my earlier talks.

On a principal $G$ bundle $P \to B$ a connection is a one form

$$\nabla \in \Omega^1_G(P; \mathfrak{g})$$

satisfying certain equivariance and verticalness condition.

*Remark* 1.1. (not relevant here) It follows from the definition of connection that any two connections differ by an element of $\Omega^1_{G,hor}(P;\mathfrak{g})$ and hence the space of connections forms an affine space over $\Omega^1(B; P \times \mathfrak{g})$.

A curvature is a 2 form $\Omega(X,Y) := d\nabla(X,Y) + [\nabla(X), \nabla(Y)]$. The curvature has the nice property that

$$\Omega \in \Omega^2_{G,hor}(P;\mathfrak{g}) \cong \Omega^2(B; P \times_G \mathfrak{g})$$

where the action on $\mathfrak{g}$ is the adjoint action.

### 1.1. Curvature on induced vector bundles.

I'll work with complex vector bundles here, but everything here works equally well for real bundles as well.

**Definition 1.2.** Given a hermitian vector bundle $(E, \langle , \rangle) \to B$, a compatible connection on $E$ is a $\mathbb{C}$ linear map $\nabla : \Gamma(E) \to \Omega^1(B; E)$ such that

- $\nabla_{fX}(s) = f\nabla_X s$ (tensoriality)
- $\nabla_X(fs) = f\nabla_X s + X(f)s$ (derivation)
- $X\langle s,t\rangle = \langle \nabla_X s, t\rangle + \langle s, \nabla_X t\rangle$ (compatibility)

*Remark* 1.3. Note that $\nabla$ is a derivation and not a form.

Locally a connection is of the form

$$\nabla_{\partial x_i}(s) = \partial x_i(s) + \Gamma_i s$$

where $\Gamma_i \in End(E)$. The $\Gamma_i$ are called the Christoffel symbols, and a connection is completely determined by them.

Given a unitary representation $\rho : G \to U(V)$ we can form a vector bundle $\rho(P) := P \times_G V \to B$. The connection $\nabla$ induces a compatible connection on $\rho(P)$ by

$$\Gamma_i|_b = \rho(\nabla(\partial x_i)|_{(b,e)})$$

*Remark* 1.4. If the representation $\rho$ is faithful then we can go in the other direction also.

**Definition 1.5.** The curvature $\Omega$ of the connection $(E, \nabla)$ is defined as

$$\Omega_{X,Y}(s) = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s$$

It turns out that $\Omega \in \Omega^2(X, End(E))$.

The induced curvature has a much better form. Look at the map induced by $\rho$

$$\Omega^2(B; P \times_G \mathfrak{g}) \to \Omega^2(B; P \times_G \mathfrak{u}(V))$$

Note that $P \times_G \mathfrak{u}(V) \cong \mathfrak{u}(\rho(P)) \subset End(E)$ and the induced curvature is nothing but the image of the original curvature under this map.

One usually thinks of curvature at each point as being a matrix in $End(E)$ with entries coming from $\Omega^2(B)$.

Look at the map

$$\Omega^2(B; \mathfrak{u}(\rho(P))) \xrightarrow{invariants} \Omega^2(B)$$

and we can ask what is the image, where by invariants we mean the coefficients of the characteristic polynomial of the matrix over each fiber.

**Theorem 1.6** (Chern - Weil)**.** *The image of the above* invariants *map does not depend on the connection up to homotopy and give us a well defined cohomology class in $H^*(B)$, furthermore*

- *If $E$ is a complex vector bundle,*
$$c(E) = \det\left(1 + \frac{i\Omega}{2\pi}\right)$$
*where $c(E)$ is the total real Chern class of $E$*
- *If $E$ is a real vector bundle,*
$$p(E) = \sqrt{\det\left(1 - \left(\frac{\Omega}{2\pi}\right)^2\right)}$$
*where $p(E)$ is the total real Pontryagin class of $E$*
- *If $E$ is a real vector bundle*
$$e(E) = \sqrt{\det\frac{\Omega}{2\pi}}$$
*where $e(E)$ is the real Euler class of $E$*

**Corollary 1.7.** *If a vector bundle can be given a flat metric then all its characteristic classes are trivial.*

*Remark* 1.8. Note that the matrix corresponding to $\Omega$ takes values in $\mathfrak{u}(E)$ and hence has complex *eigenvalues* and hence $i\Omega$ would have real *eigenvalues*.

The invariants of $\Omega^k$ are a function of the invariants of $\Omega$ and hence these too would be independent of the connection. We can hence generalize the above ideal to arbitrary power series with constant term 1. We use 1 in the constant term so that our computations do not depend on the dimension $E$.

**Definition 1.9** (Multiplicative Sequence)**.** A multiplicative sequence is a power series $f(z) \in \mathbb{R}[[z]]$ with $f(z) = 1$. The genus of a multiplicative sequence is $\det f\left(\frac{i\Omega}{2\pi}\right)$.

*Remark* 1.10. A word of caution. Sometimes the multiplicative sequences are written in terms of the Pontryagin classes instead of the Chern class which leads to a change in sign for the odd terms.

An important invariant which does not come from a multiplicative sequence is the Chern character.

**Definition 1.11.** The Chern character is defined as
$$ch(\Omega) := \operatorname{tr}\exp\left(\frac{i\Omega}{2\pi}\right)$$

Note that the Chern character is not stable, that is if we add a trivial bundle of dimension 1 to a bundle $V$ then
$$ch(V \oplus 1) = ch(V) + 1$$
hence there cannot be a multiplicative sequence whose genus is the Chern character.

1.2. **Chern Roots.** What are the invariants of $\Omega^k$ in terms of the Chern classes? Assume formally that you can write the curvature form as a diagonal matrix
$$\frac{i\Omega}{2\pi} = diag[\lambda_1\lambda_2\cdots]$$
then
$$c_i = sym_i(\lambda_1, \lambda_2, \cdots)$$
Because all the invariants are formal invariants the expression that we get for this particular matrix should give be the general one(!).

For example, suppose we have a multiplicative sequence $1 + z + z^2$, then the corresponding invariants would be given by
$$\begin{aligned}
&(1 + \lambda_1 + \lambda_1^2)(1 + \lambda_2 + \lambda_2^2)\cdots \\
&= 1 + \sum \lambda_i + \sum \lambda_i^2 + \sum \lambda_i\lambda_j + \sum \lambda_i^2\lambda_j + \cdots \\
&= 1 + c_1 + c_1^2 - c_2/2 + c_1c_2 - 3c_3 + \cdots
\end{aligned}$$

If the complex bundle is a complexification of a real vector bundle then the $\lambda_i$ should come in pairs and we would get that the odd Chern classes are all 0.

## 2. Clifford algebra and the Spin Representations

2.1. **Topological index of the Dirac operator.** Going back to the Dirac operator, recall that a Dirac operator acts on Spinor fields which are sections of $\mathbb{S}(X) \otimes V$, where $V$ is a hermitian vector bundle with a compatible connection. Now suppose $TX \otimes \mathbb{C}$ has a curvature $\Omega^X$ and $V$ has a curvature $\Omega^V$.

**Definition 2.1.** The topological index of the Dirac operator is defined as

$$\int_X \hat{A}(\Omega^X) ch(\Omega^V)$$

where $\hat{A}$ is the genus of the multiplicative sequence $\sqrt{\frac{z/2}{\sinh z/2}}$.

2.2. **Clifford algebras.** Our next goal is to define the action of $TX$ on Spinor fields and to explicitly construct the Spin representation.

**Definition 2.2.** Given a vector space $V$ over $\mathbb{R}$ with a quadratic form $Q$ we define the Clifford algebra $\mathrm{Cliff}(V)$ as the quotient of the tensor algebra $T(V)$ with the ideal generated by relations

$$v \otimes v = -Q(v)$$

A quadratic form $Q$ induces a symmetric bilinear form on $V$ as $\langle v, w \rangle := (Q(v + w)^2 - Q(v)^2 - Q(w)^2)/2$, the Clifford condition can then be restated as

$$v.w + w.v = -2\langle v, w \rangle$$

Our main object of interest is $\mathrm{Cliff}(n) := \mathrm{Cliff}(\mathbb{R}^n)$ with the standard quadratic forms $Q(x_1, \cdots, x_n) = x_1^2 + \cdots + x_n^2$.

Some of the properties of $\mathrm{Cliff}(n)$ are

- $\mathrm{Cliff}(n)$ satisfies the universal property that for any unital associative $\mathbb{R}$ algebra $A$ any linear map $f : \mathbb{R}^n \to A$ satisfying $f(v)^2 = -Q(v).1_A$ factors through $\mathrm{Cliff}(n)$
- If $e_1, \cdots, e_n$ is an orthonormal basis then we have the relations
$$e_i e_j + e_j e_i = 0$$
$$e_i^2 = -1$$
and monomials of the form $e_{i_1} \cdots e_{i_m}$ such that $i_1 \leq \cdots \leq i_m$ form a basis for $\mathrm{Cliff}(n)$ and hence the dimension of $\mathrm{Cliff}(n)$ is $2^n$. Note that this identification does depend upon the basis and hence is not canonical.
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$$\mathrm{Cliff}(1) \cong \mathbb{C}$$
$$\mathrm{Cliff}(2) \cong \mathbb{H}$$
- $\mathrm{Cliff}(n)$ is a $\mathbb{Z}/2$ graded algebra with $\mathrm{Cliff}^+(n)$ consisting of elements which can be written as sum of monomials with evenly many terms and $\mathrm{Cliff}^-(n)$ for odd terms. Note that one needs to check that this is well defined.

**Proposition 2.3.** *We have non-canonical isomorphisms*

$$\mathrm{Cliff}(2n) \otimes \mathbb{C} \cong M_{2^n}(\mathbb{C})$$

*Proof.* We prove this by induction.

For $n = 1$, we have the standard quaternion representations

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

For the inductive step it suffices to show that as algebras

$$\mathrm{Cliff}(2n+2) \otimes \mathbb{C} \cong (\mathrm{Cliff}(2n) \otimes \mathbb{C}) \otimes M_2(\mathbb{C})$$

We will simply find $2n+2$ elements in $(\mathrm{Cliff}(2n) \otimes \mathbb{C}) \otimes M_2(\mathbb{C})$ which would satisfy the relations of being a basis for Clifford algebra. Let $e_i \in \mathrm{Cliff}(2n)$ be an orthonormal basis for $\mathbb{R}^{2n}$, then consider the elements

$$e_i \otimes \begin{bmatrix} -1 & \\ & 1 \end{bmatrix}, 1 \otimes \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}, 1 \otimes \begin{bmatrix} & i \\ i & \end{bmatrix}$$

$$(e_i \otimes \begin{bmatrix} -1 & \\ & 1 \end{bmatrix})^2 = (1 \otimes \begin{bmatrix} & 1 \\ -1 & \end{bmatrix})^2 = 1 \otimes \begin{bmatrix} & i \\ i & \end{bmatrix} = -1$$

Hence this gives an embedding of $(\mathrm{Cliff}(2n) \otimes \mathbb{C}) \otimes M_2(\mathbb{C})$ in $\mathrm{Cliff}(2n+2)$, which then gives us an isomorphism by counting dimensions. □

**Corollary 2.4.** *As an* $\mathbb{C}$ *algebra* $\mathrm{Cliff}(2n) \otimes \mathbb{C}$ *is simple, that is it has no two sided ideals.*

**The Spin group.**

**Proposition 2.5.** *The Lie group $G$ (with subspace topology) generated by elements of the form* $\cos\theta + \sin\theta e_i e_j$ *is isomorphic to $Spin(n)$.*

*Proof.* Why is $G$ a Lie group? Because it is compact?

Assume $n > 2$. To do this we need to give a map $G \to SO(n)$ which is a $2:1$. Look at the map

$$g \mapsto (v \mapsto gvg^{-1})$$

This map sends $\cos\theta + \sin\theta e_i e_j \mapsto \cos(2\theta)(E_{ii} + E_{jj}) + \sin(2\theta)(E_{ij} - E_{ji})$ where $E_{kl}$ is the matrix with 1 in the $k$th row and $l$th column.

Clearly $\pm 1$ is in the kernel. We would be done once we show that $G$ is connected. But it is straightforward to construct a path from any element $g$ to the identity and hence we are done. □

**Proposition 2.6.** *This embedding of $Spin(n)$ in* $\mathrm{Cliff}(n)$ *does not depend on the choice of basis.*

*Proof.* We need to show that if we have two orthonormal vectors $f_1, f_2$ then $\cos\theta + f_1 f_2 \sin\theta \in G$. Let $g$ be an orthonormal matrix that takes $e_1, e_2$ to $f_1, f_2$. Then look at a lift $\tilde{g}$ of $g$ in $G$. Then $\tilde{g}(\cos\theta + e_1 e_2 \sin\theta)\tilde{g}^{-1} = \cos\theta + f_1 f_2 \sin\theta$. □

**Corollary 2.7.**

- *$SO(n)$ is generated by elements of the form* $\cos(2\theta)(E_{ii}+E_{jj})+\sin(2\theta)(E_{ij}-E_{ji})$
- $Spin(n) \subset \mathrm{Cliff}^+(n)$

*Remark* 2.8. It is possible to define $Spin(n)$ without using coordinates as the set of invertible elements $g \in \mathrm{Cliff}^+(n)$ such that $gvg^{-1} \in \mathbb{R}^n$ for $v \in \mathbb{R}^n$.

2.3. **The representation $\mathbb{S}$.** Now we restrict our attention to $Spin(2n)$. We assume the following theorem.

**Theorem 2.9.** *There are two non-isomorphic irreducible representations $\mathbb{S}^+$ and $\mathbb{S}^-$ of $Spin(2n)$ which are not representations of $SO(2n)$.*

We give an explicit description of $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$.

Consider $\mathbb{R}^{2n}$ with standard bilinear inner product and an orthonormal basis $e_1, \cdots, e_{2n}$ and $\mathbb{C}^n$ with the standard **hermitian** inner product and unitary basis $v_1, \cdots, v_n$.

**Definition 2.10.** Let $\mathbb{S}^+ := \wedge^{even}\mathbb{C}^n, \mathbb{S}^- := \wedge^{odd}\mathbb{C}^n, \mathbb{S} := \wedge^*\mathbb{C}^n = \mathbb{S}^+ \otimes \mathbb{S}^-$

We need to define an action of $Spin(2n)$ on $\mathbb{S}$. We will do this using the universal property of Clifford algebras, but first the inner product.

We can extend the bilinear form on $\mathbb{C}^n$ to $\wedge^*\mathbb{C}^n$ as follows. Any two forms of different degrees are orthogonal to each other. For two forms of the same degree define the inner product

$$\langle v_1 \wedge v_2 \wedge \cdots, w_1 \wedge w_2 \wedge \cdots \rangle := \det[\langle v_i, w_j \rangle]$$

For each $v \in \wedge^*\mathbb{C}^n$ we have a linear map $v\wedge : \wedge^*\mathbb{C}^n \to \wedge^*\mathbb{C}^n$.

**Definition 2.11.** For each $v \in \wedge^*\mathbb{C}^n$ the interior product $\iota_v$ is defined to be the dual of this map under the above defined inner product, that is

$$\langle v \wedge w, u \rangle = \langle w, \iota_v u \rangle$$

**Proposition 2.12.** *For each $v \in V$*

$$\iota_v(w_0 \wedge w_1 \wedge \cdots) := \sum_i (-1)^i \overline{\langle v, w_i \rangle} \hat{w}_i$$

*Proof.* Follows by taking $w = w_0 \wedge w_1 \wedge \cdots$ and $u = u_1 \wedge u_2 \wedge \cdots$ and computing $\langle v \wedge u, w \rangle$. □

**Definition 2.13.** Define a linear map

$$\sigma : \mathbb{R}^{2n} \to \mathbb{S}, e_{2j-1} \mapsto v_j, e_{2j} \mapsto iv_j$$

and then define an action of $\mathbb{R}^{2n}$ on $\mathbb{S}$ via $\rho : \mathbb{R}^{2n} \to End(\mathbb{S})$ on as

$$e \to (w \to \sigma(e) \wedge w - \iota_{\sigma(e)} w)$$

**Proposition 2.14.** *$\rho(e)^2 = -Q(e)$ for any $e \in \mathbb{R}^{2n}$ and hence $\rho$ extends to* $\mathrm{Cliff}(2n)$ *via the universal property. Restricting the action to $Spin(2n)$ gives us the required representation.*

*Proof.*

$$\begin{aligned}\rho(e)^2 w &= \sigma(e) \wedge (\sigma(e) \wedge w - \iota_{\sigma(e)} w) - \iota_{\sigma(e)}(\sigma(e) \wedge w - \iota_{\sigma(e)} w) \\ &= -\overline{\langle \sigma(e), w \rangle} \sigma(e) - \overline{\langle \sigma(e), \sigma(e) \rangle} w + \overline{\langle \sigma(e), w \rangle} \sigma(e) \\ &= -\overline{\langle \sigma(e), \sigma(e) \rangle} w\end{aligned}$$

The result then follows by noticing that $Q(e) = \overline{\langle \sigma(e), \sigma(e) \rangle}$. □

**Proposition 2.15.** *The complex representations $\mathbb{S}^+$ and $\mathbb{S}^-$ are irreducible and unitary of dimensions $2^{n-1}$.*

*Proof.* For the unitary part notice that $\rho(e_j)$ is a skew hermitian matrix for each $e_j$.

$$\begin{aligned}&(\cos\theta + \rho(e_i)\rho(e_j)\sin\theta)^*(\cos\theta + \rho(e_i)\rho(e_j)\sin\theta) \\ &= (\cos\theta + \rho(e_j)^*\rho(e_i)^*\sin\theta)(\cos\theta + \rho(e_i)\rho(e_j)\sin\theta) \\ &= (\cos\theta + \rho(e_j)\rho(e_i)\sin\theta)(\cos\theta + \rho(e_i)\rho(e_j)\sin\theta) \\ &= 1\end{aligned}$$

Look at the group ring $\mathbb{C}[Spin(2n)]$. If we can show that the image of this ring under $\rho$ is $Gl(\mathbb{S}^+)$ then we would have irreducibility. But notice that $\mathbb{C}[Spin(2n)] = \mathrm{Cliff}^+(2n) \otimes \mathbb{C}$. By an earlier proposition we know that $\mathrm{Cliff}(2n) \otimes \mathbb{C}$ is simple and hence the map $\rho : \mathrm{Cliff}^+(2n) \otimes \mathbb{C} \oplus \mathrm{Cliff}^-(2n) \otimes \mathbb{C} \to Gl(\mathbb{S})$ has to be injective. It is an easy check that the image of $\mathrm{Cliff}^+(2n) \otimes \mathbb{C}$ respects the decomposition $\mathbb{S}^+ \oplus \mathbb{S}^-$ and the image of $\mathrm{Cliff}^-(2n) \otimes \mathbb{C}$ swaps $\mathbb{S}^+$ and $\mathbb{S}^-$. Isomorphism then follows by counting dimensions.

□

## 3. Spin bundles and the Dirac operator

3.1. **Recall.** The following things follow quite directly from what we did so far,

- $\mathbb{S} = \wedge^*\mathbb{C}^n = \wedge^{even}\mathbb{C}^*n \oplus \wedge^{odd}\mathbb{C}^*n = \mathbb{S}^+ \oplus \mathbb{S}^-$ admits a representation of Cliff$(2n)$ and hence of $Spin(2n)$ extending the map which sends $v$ to $\sigma(v) \wedge (-) - i_{\sigma(v)}(-)$ and hence $\mathbb{R}^{2n} \subset$ Cliff$(2n)$ acts via anti hermitian matrices
- The action of $Spin(2n)$ is unitary
- Cliff$^+(2n) \otimes \mathbb{C} \cong End(\mathbb{S}^+) \oplus End(\mathbb{S}^-)$ and Cliff$(2n) \otimes \mathbb{C} \cong End(\mathbb{S})$ and the elements of $\mathbb{R}^{2n} \in$ Cliff$(2n)$ interchange $\mathbb{S}^+$ and $\mathbb{S}^-$

3.2. **More about Spin Groups.** Because $Spin(2n)$ double covers $SO(2n)$ a maximal torus of $Spin(2n)$ would be the pre image of a maximal torus of $SO(2n)$. It is easy to see that if we pick the standard maximal torus for $SO(2n)$ we get the maximal torus of $Spin(2n)$ generated by the elements

$$\cos\theta + \sin\theta e_{2i-1}e_{2i}$$

From here on let us denote the element $e_1e_2\cdots e_{2n-1}e_{2n}$ by $\epsilon$.

**Proposition 3.1.** *The center* $Z(Spin(2n)) = \{\pm 1, \pm\epsilon\}$ *and* $\epsilon^2 = \begin{cases} -1 & \textit{if } 2 \mid n \\ 1 & \textit{otherwise} \end{cases}$

*Proof.* $Z(Spin(2n))$ can at the most be a 2:1 covering of $Z(SO(n)) = \{\pm 1\}$. It is then easy to check that the element $\epsilon$ indeed in the center. In fact $\epsilon$ commutes with Cliff$^+(2n)$ and anti commutes with Cliff$^-(2n)$. $\square$

**Proposition 3.2.** *The representations $\mathbb{S}^+$ and $\mathbb{S}^-$ are non-isomorphic representations of $Spin(2n)$.*

*Proof.* Because $\epsilon$ is in the center it suffices to show that it is sent to different scalars under under the two representations.

$$\begin{aligned} \epsilon.1 &= e_1e_2\cdots e_{2n-1}e_{2n}1 \\ &= (-i)^n.1 && \text{because } e_{2j-1}e_{2j}1 = -i \\ \epsilon.v_1 &= e_1e_2\cdots e_{2n-1}e_{2n}v_1 \\ &= (-i)^{n-1}e_1e_2v_1 \text{ because } e_{2j-1}e_{2j}v_1 = -iv_1 \text{ if } j > 1 \\ &= -(-i)^n v_1 \end{aligned}$$

$\square$

**Proposition 3.3.** $(\mathbb{S}^+)^* \cong \begin{cases} \mathbb{S}^+ & \textit{if } 2|n \\ \mathbb{S}^- & \textit{otherwise} \end{cases}$ *and hence* $\mathbb{S}^* \cong \mathbb{S}$.

*Proof.* Because $\mathbb{S}^+$ and $\mathbb{S}^-$ are the only two representations of $Spin(2n)$ which do not factor through $SO(2n)$ when we take the duals they should get permuted amongst themselves. Then $\rho^*(\epsilon) = \rho(\epsilon)^{-1} = \begin{cases} i^n & \text{for } (\mathbb{S}^+)^* \\ -i^n & \text{for } (\mathbb{S}^-)^* \end{cases}$. The result follows by comparing this with $\rho(\epsilon)$. □

3.3. **Dirac operator.** If we look at the bundle $End(\mathbb{S}(X))$, each fiber of it is a module over $Cliff(TX)$. So we can ask is it possible to define continuous action of $Cliff(TX)$ on $End(\mathbb{S}(X))$. The answer is yes, and the "reason" for this is simply the fact that we have a Spin structure on $X$ which allows us to do this. More rigourosly we have the following facts,

**Proposition 3.4.** *$\rho : \mathbb{R}^{2n} \to End(\mathbb{S}), v \to \sigma(v) \wedge (-) - \iota_{\sigma(v)}(-)$ is $Spin(2n)$ equivariant, where the action of $Spin(2n)$ on the left is via $SO(2n)$ and on the right is via conjugation.*

*Proof.* We have noted above that $End(\mathbb{S}) \cong \mathrm{Cliff}(2n)$ and under this identification this map is nothing but the inclusion $\mathbb{R}^{2n} \hookrightarrow \mathrm{Cliff}(2n)$. It then follows trivially that this map is equivariant. □

Because of this we can extend the map to bundles over $X$ to get

$$Spin(X) \times_{Spin(2n)} \mathbb{R}^{2n} \to SO(X) \times_{SO(2n)} \mathbb{R}^{2n} \to Spin(X) \times_{Spin(2n)} End(\mathbb{S})$$

The last arrow then gives us a map

$$\rho : TX \to End(\mathbb{S}(X))$$

More generally we have an isomorphism

$$\rho : Cliff(TX) \xrightarrow{\simeq} End(\mathbb{S}(X))$$

For a hermitian vector bundle $V$ on $X$ extend the action $\rho$ to $\mathbb{S}(X) \otimes V$ via trivial action on $V$.

Now we can define the Dirac operator as follows:

**Definition 3.5.** The Dirac operator $\not{D}_V$ is a differential operator on the bundle $\mathbb{S}(X) \otimes V$, defined as the following composition

$$\Gamma(\mathbb{S}(X) \otimes V) \xrightarrow{\nabla} \Omega^1(X, \mathbb{S}(X) \otimes V) \xrightarrow{\langle , \rangle} \Gamma(TX \otimes \mathbb{S}(X) \otimes V) \xrightarrow{\rho} \Gamma(\mathbb{S}(X) \otimes V)$$

where $\nabla$ is any compatible connection on $\mathbb{S}(X) \otimes V$ and we have used the inner product on $TX$ to get a tangent vector form a cotangent vector.

The Dirac operator over each point $p \in X$ then becomes

$$\begin{aligned} \not{D}_{Vp} : (\mathbb{S}(X) \otimes V)_p &\to (\mathbb{S}(X) \otimes V)_p \\ \alpha_p &\mapsto \sum_i [\rho(e_i) \nabla_{e_i} \alpha]_p \end{aligned}$$

where $e_i$ form an orthonormal basis of $T_pX$. If the Christoffel symbols are all 0 then the above expression becomes

$$\not{D}_{Vp}\alpha = \sum_i \rho(\partial x_i) \frac{\partial \alpha}{\partial x_i}$$

Note that the Dirac operator interchanges $\mathbb{S}^+(X) \otimes V$ and $\mathbb{S}^-(X) \otimes V$.

On the space of sections $\Gamma(\mathbb{S}(X) \otimes V)$ we can the hermitian inner product with respect to which the completion gives us the $L^2(\mathbb{S}(X) \otimes V)$.

## 4. Multiplicative Sequences

4.1. **Recall.**

- $\mathbb{S}^* \cong \mathbb{S}$
- $\rho : \text{Cliff}(\mathbb{R}^{2n}) \to End(\mathbb{S}), v \mapsto \sigma(v) \wedge (-) - \iota_\sigma(v)(-)$ extends to a bundle map $\rho : \text{Cliff}(TX) \to End(\mathbb{S}) \mapsto End(\mathbb{S} \otimes V)$
- $\Gamma(\mathbb{S}(X) \otimes V) \xrightarrow{\nabla} \Omega^1(X, \mathbb{S}(X) \otimes V) \xrightarrow{\langle,\rangle} \Gamma(TX \otimes \mathbb{S}(X) \otimes V) \xrightarrow{c} \Gamma(\mathbb{S}(X) \otimes V)$
- Locally if $e_i$ is an orthonormal basis of $T_pX$ for $p \in X$, the Dirac operator at $p$ is given by $\not{D}_V|_p s = \sum_i e_i \nabla_{e_i} s_p$
- $\not{D}_V$ interchanges $\mathbb{S}^+(X) \otimes V$ and $\mathbb{S}^-(X) \otimes V$

Assume the following theorem for the moment which I'll prove in a short while.

**Proposition 4.1.** *$\not{D}_V$ can be extended to the $L^2$ (and $L^2_2$ completion of $\Gamma(\mathbb{S}(X) \otimes V)$ and has finite dimensional kernel which consists entirely of smooth sections.*

**Definition 4.2.** Then define the index of the Dirac operator to be

$$\ker \not{D}|_{\mathbb{S}^+(X)\otimes V} - \ker \not{D}|_{\mathbb{S}^-(X)\otimes V}$$

**Theorem 4.3.** *The Atiyah Singer Index for Dirac operators then says that for a closed spin manifold $X$ with a Spin bundle $\mathbb{S}(X)$ and an arbitrary hermitian bundle $V$*

$$\textit{topological index of } \not{D}_V = \textit{analytic index of } \not{D}_V$$

Note that there is a choice of a connection and a spin structure, but the Atiyah-Singer index theorem tells us that the index is independent of any such choice.

Let us see a quick application of the Index theorem.

4.2. **Classical Dirac operators : $\hat{A}$ genus.** Look at the untwisted Dirac operator, that is take $V$ to be the trivial line bundle over $X$. Then the AS theorem tells us,

$$Index(\not{D}) = \int \hat{A}(X)$$

The multiplicative sequence is

$$\begin{aligned} \hat{A}(z) &= \sqrt{\frac{z/2}{\sinh z/2}} \\ &= 1 - \frac{z^2}{48} + \frac{z^4}{2560} + O(z^6) \end{aligned}$$

When $X$ is a 4 manifold, only the first two terms matter. In this case,

$$\int \hat{A}(X) = (1 - \frac{c_1^2 - 2c_2}{48} + \text{ higher terms}) \cap [X]$$

$$= \frac{c_2}{24} \cap [X] \qquad\qquad c_1 = 0 \text{ for real manifolds}$$
$$= -\frac{p_1(X)}{24}$$

and because the index is always an integer this gives us the theorem,

We used the following proposition above,

**Proposition 4.4.** *The real odd Chern classes are 0 for a bundle $E \otimes \mathbb{C}$ where $E$ is a real vector bundle.*

*Proof.* One way to do this is to look at the Chern roots. Because $E$ is a real vector bundle Chern roots will come in pairs and hence all the odd invariants would be 0.

Another way to do this is to look at the bundle $\overline{E \otimes \mathbb{C}}$. Because the transition functions live in $\mathbb{R}$ and hence commute with conjugation both these bundles will be isomorphic. But the Chern classes of these bundles are related as $c_i(E \otimes \mathbb{C}) = (-1)^i c_i(\overline{E \otimes \mathbb{C}})$. Again, this is happening because of the splitting principle. $\square$

**Theorem 4.5.** *If $X$ is a 4 dimensional spin manifold, $p_1(X)$ is divisible by 24.*

*Remark* 4.6. For $\mathbb{C}P^2$ the Pontryagin class up to sign is the first Chern class of $T\mathbb{C}P^2 \otimes_{\mathbb{R}} \mathbb{C} \cong T\mathbb{C}P^2 \oplus T\mathbb{C}P^2$ and so

$$\begin{aligned} p_2(\mathbb{C}P^2) &= -c_1(T\mathbb{C}P^2 \oplus T\mathbb{C}P^2) \\ &= -2c_1(T\mathbb{C}P^2) \\ &= -4x \qquad\qquad \text{where } x \text{ is the generator of } H^2(\mathbb{C}P^2) \end{aligned}$$

which is not divisible by 24 and hence we get a long winded proof of the fact that $\mathbb{C}P^2$ does not carry any Spin structure.

Another way of stating this remarkable theorem is that for an oriented Riemannian manifold $w_1 = w_2 = 0$ implies $24|p_1$.

## 5. Symbol of the Dirac operator

5.1. **Recall.**

- $\rho : \mathrm{Cliff}(TX) \to End(\mathbb{S}) \to End(\mathbb{S} \otimes V)$
- $\not{D}_V : \Gamma(\mathbb{S}(X) \otimes V) \xrightarrow{\nabla} \Omega^1(X, \mathbb{S}(X) \otimes V) \xrightarrow{\langle , \rangle} \Gamma(TX \otimes \mathbb{S}(X) \otimes V) \xrightarrow{\rho} \Gamma(\mathbb{S}(X) \otimes V)$
- Locally if $e_i$ is an orthonormal basis of $T_pX$ for $p \in X$, the Dirac operator at $p$ is given by $\not{D}_V s = \sum_i e_i \nabla_{e_i} s$

5.2. **Dirac operator is Fredholm.**

**Proposition 5.1.** *$\not{D}_V$ is a hermitian operator on $\Gamma(\mathbb{S}(X) \otimes V)$ under the $L^2$ norm.*

*Proof.* Let $e_1, \cdots, e_{2n}$ be an orthonormal basis of $TX$ and $\eta_1, \cdots, \eta_{2n}$ be the dual basis of $T^*X$ around $p$, then

$$\begin{aligned}
\langle \not{D}_V \alpha, \beta \rangle_p &= \sum_i \langle \rho(e_i) \nabla_{e_i} \alpha, \beta \rangle_p \\
&= \sum_i \langle \nabla_{e_i} \rho(e_i) \alpha, \beta \rangle_p \\
&\quad \text{because } c(e_i) \text{ are locally constant} \\
&= \sum_i e_i \langle \rho(e_i) \alpha, \beta \rangle_p - \langle \rho(e_i) \alpha, \nabla_{e_i \beta} \rangle \\
&= \sum_i e_i \langle \rho(e_i) \alpha, \beta \rangle_p + \langle \alpha, \rho(e_i) \nabla_{e_i} \beta \rangle_p \\
&= \sum_i e_i \langle \rho(e_i) \alpha, \beta \rangle_p + \langle \alpha, \not{D}_V \beta \rangle_p \\
\langle \not{D}_V \alpha, \beta \rangle &= \langle \alpha, \not{D}_V \beta \rangle + \int_X \sum_i e_i \langle \rho(e_i) \alpha, \beta \rangle_p \eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_{2n}
\end{aligned}$$

The term inside the integral sign is the $d$ of the $2n - 1$ form

$$\sum_i (-1)^i \langle \rho(e_i) \alpha, \beta \rangle_p \hat{\eta}_i$$

except that these terms are only defined locally. So we need to show that this form patches up nicely. One way is to come up with a coordinate free description of this.

Now if we look at the vector field $Y$ uniquely defined by the property $\langle Y, Z \rangle_p = \langle c(Z) \alpha, \beta \rangle$, then locally

$$\begin{aligned}
&\eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_{2n}(Y) \qquad \text{the contraction of } \eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_{2n} \text{ with } Y \\
&= \sum_i (-1)^i \eta_i(Y) \hat{\eta}_i \\
&= \sum_i (-1)^i \langle Y, e_i \rangle \hat{\eta}_i
\end{aligned}$$

$$= \sum_i (-1)^i \langle \rho(e_i)\alpha, \beta \rangle \hat{\eta_i}$$

□

### 5.3. **Symbol of the Dirac operator.**

**Definition 5.2.** If $D : \Gamma(E) \to \Gamma(F)$ is a differential operator of degree $k$ such that locally $D = \sum_{|I|=k} c_I \frac{\partial}{\partial x_I} +$ lower terms for $c_I \in Hom(E, F)$ define the symbol of the Dirac operator $\sigma(D)$ to be locally the formal polynomial $(-i)^k \sum_{|I|=k} c_I \zeta^I$.

The product rule then implies that $\sigma(D)$ changes as a symmetric $(k, 0)$ tensor with values in $Hom(E, F)$ and hence is section of $Sym^k(T^*X) \otimes Hom(E, F)$ where by $Sym^k()$ I mean polynomial functions of degree $k$, that is it takes a 1-form and returns a bundle map $E \to F$.

Let us calculate the index of $\not{D}_V$. Locally $\not{D}_V(-) = \sum \rho(e_j)\nabla_{e_j}(-)$ for $e_j$ an orthonormal basis of $TX$. The symbol $\sigma(\not{D}_V) \in Sym^1(T^*X) \otimes End(\mathbb{S} \otimes V) = TX \otimes End(\mathbb{S} \otimes V)$. So we need to formally replace $\nabla_{e_i}$ by $e_i$ so that $\sigma(\not{D}_V) = \sum e_j \otimes \rho(e_j)$.

**Definition 5.3.** The operator $D$ is called elliptic if $\sigma(D)(\eta_1, \cdots, \eta_k)$ is an invertible for all $(\eta_1, \cdots, \eta_k) \neq (0, \cdots, 0)$.

Note that an elliptic operator can exist between two bundles only if they are of the same rank. The following is the theorem that make elliptic operators useful in topology

**Theorem 5.4** (Elliptic regularity theorem)**.** *If $D : \Gamma(E) \to \Gamma(F)$ is an elliptic operator (extended suitably to the $L^2$ completion) between complex vector bundles over a compact manifold, then kernel of $D$ is finite dimensional and consists entirely of smooth sections of $E$. Furthermore, $\Gamma(E) \cong ker(D) \oplus im(D^*)$ where $D^*$ is the adjoint of $D$.*

**Definition 5.5.** A Fredholm operator is a differential operator which is hermitian and has finite dimensional kernel and cokernel.

So that by the Elliptic regularity theorem, a hermitian elliptic operator is Fredholm.

**Proposition 5.6.** $\not{D}_V : \Gamma(\mathbb{S} \otimes V) \to \Gamma(\mathbb{S} \otimes V)$ *is elliptic and hence Fredholm.*

*Proof.* For any non zero $\eta = \sum \alpha_j \eta_j$ where $\eta_j$ is the dual basis of $T^*X$, $\sigma(\not{D}_V)(\eta) = \sum e_j(\eta)\rho(e_j) = \sum \alpha_j \rho(e_j) = \rho(\sum_j \alpha_j e_j)$ which is invertible because it's square is $-\sum_j \alpha_j^2$. □

## 6. Dirac operator on Differential forms

### 6.1. **Recall.**

- Dirac operator $\mathcal{D}_V = \sum_{e_i} \rho(e_i)\nabla_{e_i}(-)$ is Fredholm
- $(\mathbb{S}(X))^* \cong \mathbb{S}(X)$
- $End(\mathbb{S}(X)) \cong \mathrm{Cliff}(TX) \otimes \mathbb{C}$
- On the space of exterior forms over an inner product space there is a natural inner product, and the adjoint of $v\wedge -$ is denoted by $\iota_v$ and $\iota_v(v_1 \wedge v_2 \wedge \cdots) = \langle v_1, v\rangle v_2 \wedge \cdots - \langle v_2, v\rangle v_1 \wedge \cdots + \cdots$

6.2. **Applications of the Index theorem.** Take $V$ to be the vector bundle $\mathbb{S}(X)$ itself. Then we have the following isomorphism:

**Proposition 6.1.** *There is an isomorphism of bundles* $\wedge^*(T^*X) \to \mathrm{Cliff}(TX)$.

*Proof.* Let $e_1, \cdots, e_{2n}$ be an orthonormal basis around $p$ of $TX$ and let $\eta_i$ be the corresponding dual basis for $T^*X$. Then define the bundle map

$$\wedge^*(T^*X) \to \mathrm{Cliff}(TX)$$
$$\eta_{i_1} \wedge \cdots \wedge \eta_{i_k} \mapsto e_{i_1}. \cdots .e_{i_k}$$

and extend this linearly. This map is clearly an isomorphism and patches up properly because the underlying principal $Spin(2n)$ bundles are the same. □

This is a very important isomorphism and we'll from now on identify $\wedge^*(T^*X)$ with $\mathrm{Cliff}(TX)$.

**Proposition 6.2.** *We can pull back the algebra structure of* $\mathrm{Cliff}(TX)$ *on to* $\wedge^*(T^*X)$. *The left Clifford multiplication by an element* $\eta \in T^*X$ *would then be given by*

$$\eta.\omega = \eta \wedge \omega - \iota_\eta \omega$$

*where as before* $\iota_\eta$ *is the adjoint to* $\eta \wedge -$.

*Proof.* It suffices to check that this holds when $\eta = \eta_{i_0}$ and $\omega = \eta_{i_1} \wedge \cdots \wedge \eta_{i_k}$ which is an easy check. □

This then extends to a series of bundle isomorphisms,

$$\wedge^*(T^*X) \otimes \mathbb{C} \cong \mathrm{Cliff}(TX) \otimes \mathbb{C} \cong End(\mathbb{S}(X)) \cong \mathbb{S}(X)^* \otimes \mathbb{S}(X) \cong \mathbb{S}(X) \otimes \mathbb{S}(X)$$

On this last bundle we have a Dirac operator, so we can ask what is the corresponding Dirac operator on the space of forms.

**Theorem 6.3.** *Around any point* $p$ *of a Riemannian manifold* $M$ *there exists orthonormal vector fields* $e_i$ *such that the Levi Civita connection is given by* $\nabla_{e_i}(s)|_p = e_i(s)_p$, *that is the Christoffel symbols are all 0 at* $p$. *These coordinates are called normal/geodesic coordinates.*

*Proof.* To Do. □

This also applies to the bundles associated to $TX$ like $T^*X, \mathrm{Cliff}(TX), \mathbb{S}(X)$.

So that on $\wedge^*(T^*X)$ the Dirac operator at the point $p$ in geodesic coordinates becomes

$$\begin{aligned}
\omega &= f\eta_{i_1} \wedge \cdots \wedge \eta_{i_k} \\
\not{D}(\omega)|_p &= \sum_{e_i} e_i(f)\rho(e_i)\eta_{i_1} \wedge \cdots \wedge \eta_{i_k}|_p \\
&= \sum_{e_i} e_i(f)(\eta_i \wedge \eta_{i_1} \wedge \cdots \wedge \eta_{i_k} - \iota_{\eta_i}\eta_{i_1} \wedge \cdots \wedge \eta_{i_k})|_p \\
&= d\omega - \sum_{e_i} e_i(f)\iota_{\eta_i}\eta_{i_1} \wedge \cdots \wedge \eta_{i_k}|_p
\end{aligned}$$

The second term here is called the Hodge dual $d^*$ of $d$, so that the Dirac operator then becomes

$$\not{D}(\omega) = (d + d^*)(\omega)$$

This brings us to some Hodge theory.

6.3. **Hodge star.** Given a hermitian vector space $(V, \langle,\rangle)$ of dimension $2n$ we can extend the inner product to $\wedge^*V$ as we did earlier. We also have a natural non degenerate pairing

$$\wedge : \wedge^k V \otimes \wedge^{2n-k} V \to \wedge^{2n} V \cong \mathbb{C}$$

where we have made a choice of a top form $\epsilon_V$ in identifying $\wedge^{2n}V$ with $\mathbb{C}$.

We can invert the above pairing using the inner product to get a dual map

$$* : \wedge^{2n-k} V \xrightarrow{\simeq} \wedge^k V^* \cong \wedge^k V$$

Another way of stating the same thing is,

**Definition 6.4.** The Hodge dual of an element $\alpha \in \wedge^k V$ is the unique element $*\alpha \in \wedge^{2n-k}V$ satisfying

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \epsilon_V$$

for all $\beta \in \wedge^k V$ and a fixed top for $\epsilon_V \in \wedge^{2n} V$.

If $e_1, \cdots, e_{2n}$ is an orthonormal basis for $V$ such that $\epsilon_V = e_1 \wedge \cdots \wedge e_{2n}$ then the Hodge star is given by

$$*e_1 \wedge \cdots \wedge e_k = e_{k+1} \wedge \cdots \wedge e_{2n}$$

And we have the identity

$$*^2 = (-1)^k$$

Coming back to our manifold $X$ we now want to do the Hodge dual construction on it's differential forms. We can do it over each point, the only thing to be careful about is the fact that we are required to choose a top form. But because $X$ is oriented it comes with canonical top form and hence we get a well defined Hodge dual

$$* : \Omega^k(X, \mathbb{C}) \to \Omega^{2n-k}(X, \mathbb{C})$$

**Definition 6.5.** Define the codifferential $d^*$ to be the dual of the differential $d$ under the $L^2$ norm. That is $d^* : \Omega^{k+1}(X,\mathbb{C}) \to \Omega^k(X,\mathbb{C})$ such that

$$\int_X \langle d\alpha, \beta \rangle = \int_X \langle \alpha, d^*\beta \rangle$$

**Proposition 6.6.**

$$d^* = (-1)^{k+1} * d *$$

*Proof.* If $\alpha, \beta \in \Omega^k(X,\mathbb{C})$

$$\begin{aligned}
\langle \alpha, *d*\beta \rangle &= \int_X \langle \alpha_p, (*d*\beta)_p \rangle \\
&= \int_X \langle \alpha_p, *(d*\beta)_p \rangle \\
&= \int_X (-1)^k (\alpha \wedge d*\beta)_p && \text{because } *^2 = (-1)^k \\
&= \int_X (-1)^{k+1} (d\alpha \wedge *\beta)_p && \text{by Stoke's theorem} \\
&= \int_X (-1)^{k+1} \langle d\alpha, \beta \rangle_p \\
&= (-1)^{k+1} \langle \alpha, d^*\beta \rangle
\end{aligned}$$

□

Relating back to the problem at hand,

**Proposition 6.7.**

$$d^*(f\eta_{i_1} \wedge \cdots \wedge \eta_{i_k}) = -\sum_{e_i} e_i(f)\iota_{\eta_i}\eta_{i_1} \wedge \cdots \wedge \eta_{i_k}$$

*Proof.* Suffices to check for the exterior forms of form $f\eta_1 \wedge \cdots \wedge \eta_k$. On this we have

$$\begin{aligned}
d^* f\eta_1 \wedge \cdots \wedge \eta_k &= (-1)^k * d * f\eta_1 \wedge \cdots \wedge \eta_k \\
&= (-1)^k * df\eta_{k+1} \wedge \cdots \wedge \eta_{2n} \\
&= (-1)^k * \sum_{i \le k} e_i(f)\eta_i \wedge \eta_{k+1} \wedge \cdots \wedge \eta_{2n} \\
&= \sum_{i \le k} (-1)^i e_i(f)\eta_1 \wedge \cdots \wedge \eta_{i-1} \wedge \eta_{i+1} \wedge \cdots \wedge \eta_k \\
&= -\sum_{e_i} e_i(f)\iota_{\eta_i}\eta_1 \wedge \cdots \wedge \eta_k
\end{aligned}$$

□

And so we get the proposition,

**Corollary 6.8.** *On $\wedge^*(T^*X) \otimes \mathbb{C}$ the Dirac operator is given by $d + d^*$.*

## 7. Classical Dirac operators

7.1. **Recall.**

- We have a canonical decomposition of bundles $\wedge^*(T^*X) \otimes \mathbb{C} \cong \text{Cliff}(TX)$
- This allows us to define a Dirac operator on $\wedge^*(T^*X) \otimes \mathbb{C}$ which is given by $\not{D} = d + d^*$

7.2. **Hodge decomposition of the cohomology groups.** Because $d^*d^* = *d**d* = 0$, on the space of differential forms we have two differentials $d$ and $d^*$. We can ask how are their cohomologies related.

We can define the following spaces

$$\begin{aligned}
\Omega^k_{closed}(X) &= ker(d) = \{\omega | d\omega = 0\} \\
\Omega^k_{exact}(X) &= im(d) = \{\omega | \exists \tau . d\tau = \omega\} \subset \Omega^k_{closed}(X) \\
\Omega^k_{coclosed}(X) &= ker(d^*) = \{\omega | d^*\omega = 0\} \\
\Omega^k_{coexact}(X) &= im(d^*) = \{\omega | \exists \tau . d\tau = \omega\} \subset \Omega^k_{coclosed}(X) \\
\Omega^k_{harmonic}(X) &= ker(d) \cap ker(d^*) = \{\omega | d\omega = d^*\omega = 0\}
\end{aligned}$$

**Proposition 7.1.** *$\Omega^k_{closed}(X) \perp \Omega^k_{coexact}(X)$ and $\Omega^k_{coclosed}(X) \perp \Omega^k_{exact}(X)$. And as a consequence*

$$\Omega^k_{coexact}(X) \perp \Omega^k_{harmonic} \perp \Omega^k_{exact}$$

*And*

$$\Omega^k_{harmonic} = \ker(dd^* + d^*d) = \ker(d + d^*)$$

*Proof.* This is a simple consequence of the adjointness of $d$ and $d^*$.

$$\langle \omega, d^*\tau \rangle = \langle d\omega, \tau \rangle = 0$$

same argument applies for the other pair.

For the second part, suppose $(dd^* + d^*d)\omega = 0$. Then because $\Omega^k_{exact}(X) \perp \Omega^k_{coexact}(X)$ we must have $dd^*\omega = d^*d\omega = 0$. But these being adjoint we then have $d\omega = d^*\omega = 0$. □

The operator $dd^* + d^*d$ is called the Hodge laplacian and is equal to the usual Laplacian if the metric is flat. The theorem of Hodge then says that this decomposition is spans the entire space, that is

**Theorem 7.2** (Hodge decomposition)**.**

$$\Omega^k(X) = \Omega^k_{exact}(X) \oplus \Omega^k_{coexact}(X) \oplus \Omega^k_{harmonic}(X)$$

*And hence*

$$H^k(X, \mathbb{C}) \cong \Omega^k_{harmonic}(X)$$

*that is each cohomology class can be represented uniquely by a harmonic form.*

*Proof.* The proof of the first part requires the theory of elliptic operators and I'll sketch a proof of it later.

Assuming the first part we have, $\Omega^k_{closed}(X) \perp \Omega^k_{coexact}(X)$ which implies $\Omega^k_{closed} \subset \Omega^k_{exact} \otimes \Omega^k_{harmonic}$ which will give us $\Omega^k_{closed} = \Omega^k_{exact} \otimes \Omega^k_{harmonic}$ from which the second part follows. □

7.3. **Classical Dirac operators: Euler characteristic.** In this section we will derive the generalized Gauss Bonnet theorem as a special case of the Index theorem.

We can restrict $d + d^*$ to get $d + d^* : \wedge^{even}(T^*X) \otimes \mathbb{C} \to \wedge^{odd}(T^X) \otimes \mathbb{C}$. In this case the analytic index would be $\sum H^{even}(X) - \sum H^{odd}(X) = \chi(X)$, the euler characteristic of $X$. We need to compute the topological index of $d + d^*$. For this purpose we need to break $\wedge^*(T^*X) \otimes \mathbb{C}$ as $\mathbb{S}^{\pm} \otimes -$. We have the following isomorphisms

$$\wedge^*(T^*X) \otimes \mathbb{C} \cong \text{Cliff}(TX) \otimes \mathbb{C} \cong End(\mathbb{S}(X)) \cong \mathbb{S}^*(X) \otimes \mathbb{S}(X) \cong \mathbb{S}(X) \otimes \mathbb{S}(X)$$

$\wedge^{even}(T^*X) \cong \text{Cliff}^+(TX) \otimes \mathbb{C}$ which then consists of matrices of the form $\begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}$. Similarly $\wedge^{odd}(T^*X)$ consists of block matrices of the form $\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$.

And so the isomorphism goes

$\wedge^{even}(T^*X) \cong End(\mathbb{S}^+(X)) \otimes End(\mathbb{S}^-(X)) \cong (\mathbb{S}^+(X))^* \otimes \mathbb{S}^+(X) \oplus (\mathbb{S}^-(X))^* \otimes \mathbb{S}^-(X)$

The Dirac operator on $\text{Cliff}(TX) \otimes \mathbb{C}$ is via left Clifford multiplication. Left multiplication when translated over to $\mathbb{S}^*(X) \otimes \mathbb{S}(X)$ only acts on the second component, and hence we get that the topological index will then be the index of the Dirac operator $\not{D}_{(\mathbb{S}^+)^*} - \not{D}_{(\mathbb{S}^-)^*}$.

**Proposition 7.3.** *If $X$ is a closed spin manifold of* $\dim 2n$ *then*

$$\chi(X) = index \not{D}_{(\mathbb{S}^+)^*} - index \not{D}_{(\mathbb{S}^-)^*}$$

*where $\chi(X)$ is the Euler characteristic of $X$.*

7.4. **Classical Dirac operators: Signature.** Now assume that $X$ is a closed $4k$ spin manifold. Then $X$ has a very important topological invariant called the signature.

Look at the middle cohomology of $X$, by Poincare duality we have a non degenerate bilinear symmetric pairing

$$\wedge : H^{2k}(X) \otimes H^{2k}(X) \to H^{4k}(X) \cong \mathbb{R}$$

**Definition 7.4.** The signature of the above bilinear pairing is called the signature $Sign(X)$ of the manifold $X$.

One way of finding the signature is by noticing that if $\alpha \wedge \alpha$ is $\pm$ the top form then $*\alpha = \pm\alpha$. Turns out we can express this in terms of Clifford actions.

**Proposition 7.5.** *Under the isomorphism* $\wedge^*(T^*X) \otimes \mathrm{Cliff}^*(TX) \otimes \mathbb{C}$ *the isomorphism* $(-1)^{p(p-1)/2+k} * : \wedge^p(T^*X) \to \wedge^{4k-p}(T^*X)$ *translates to left multiplication by* $(-1)^k \epsilon$.

*Proof.* Follows by checking this on the basis element $e_1.\cdots.e_p$.

$$\begin{aligned}\epsilon.e_1.\cdots.e_p &= e_1.\cdots.e_{4k}.(e_1.\cdots.e_p) \\ &= (-1)^{(4k-1)+(4k-2)+\cdots+(4k-p)} e_1.e_1.\cdots.e_p.e_p.e_{p+1}.\cdots.e_{4k} \\ &= (-1)^{p(p-1)/2} e_{p+1}.\cdots.e_{4k} \\ &= (-1)^{p(p-1)/2} * (e_1.\cdots.e_p)\end{aligned}$$

□

Extend this operator $(-1)^{p(p-1)/2+k}*$ via linearity to the entire space of exterior forms and call it $\tau$. $\tau^2 = 1$ and hence it breaks the space of exterior forms into $\pm 1$ eigenspaces. Let $\wedge_+(X)$ be the $+1$ eigenspace of the operator $\tau : \wedge^*(T^*X) \otimes \mathbb{C} \to \wedge^*(T^*X) \otimes \mathbb{C}$ and let $\wedge_-(X)$ be the $-1$ eigenspace. $\epsilon$ anti commutes with the Clifford action of $TX$ and hence $d + d^*$ sends $\wedge_+(X)$ to $\wedge_-(X)$.

**Proposition 7.6.** *$Sign(X)$ is the index of the operator* $d + d^* : \wedge_+(X) \to \wedge_-(X)$.

*Proof.* If $\omega$ is a $p$ harmonic form with $p \neq 2k$ then $\omega + \tau(\omega)$ is in the kernel of $d + d^*$ and $\omega - \tau(\omega)$ is in the cokernel. If $p = 2k$ then $\tau(\omega) = \omega$ that is it is in the kernel if and only if $*\omega = \omega$ and $\tau(\omega) = -\omega$ that is it is in the cokernel if and only if $*\omega = -\omega$. And hence the index is precisely the signature of $X$. □

In $End(\mathbb{S}(X))$ the $+1$ eigenspace of $(-1)^k \rho(\epsilon)$ is matrices of the form $\begin{bmatrix} * & * \\ 0 & 0 \end{bmatrix}$ and the $-1$ eigenspace is $\begin{bmatrix} 0 & 0 \\ * & * \end{bmatrix}$. This is precisely the decomposition $\mathbb{S}^*(X) \otimes \mathbb{S}^+(X)$ and $\mathbb{S}^*(X) \otimes \mathbb{S}^-(X)$ which then inside $\mathbb{S}(X) \otimes \mathbb{S}(X)$ translates into a decomposition of the form $\mathbb{S}(X) \otimes \mathbb{S}^+(X)$ and $\mathbb{S}(X) \otimes \mathbb{S}^-(X)$. Because of the assymetry it is important to note that the Clifford action of the Dirac operator is only on the second component, and we get

**Proposition 7.7.** *If $X$ is a closed spin manifold of* $\dim 4k$ *then*

$$Sign(X) = index \not{D}_{\mathbb{S}}$$

*where $Sign(X)$ is the signature of $X$.*

## 8. Hirzerbruch Signature theorem

### 8.1. Recall.

- $\chi(X) = index\not{D}_{(\mathbb{S}^+)^*} - index\not{D}_{(\mathbb{S}^-)^*}$
- $Sign(X) = index\not{D}_{\mathbb{S}}$

**Index computations.** Recall that the index is given by

$$index(\not{D}_V) = \int_X \hat{A}(X)ch(V)$$

where $\hat{A}(X)$ is the genus of the multiplicative sequence $\sqrt{\dfrac{z/2}{\sinh z/2}}$

We will use Chern roots to simplify this. Formally suppose the curvature $\Omega^X \in \mathfrak{so}(2n)$ on the bundle $TX \otimes \mathbb{C}$ is of the form $\Omega^X = \begin{bmatrix} 0 & 2\pi x_1 & & & \\ -2\pi x_1 & 0 & & & \\ & & 0 & 2\pi x_2 & \\ & & -2\pi x_2 & 0 & \\ & & & & \ddots \end{bmatrix}$.

Then $\hat{A}(X) = \sqrt{\dfrac{i\Omega_X/4\pi i}{\sinh i\Omega_X/4\pi i}} = \prod \dfrac{x_i/2}{\sinh x_i/2}$

To find the Chern character of $\mathbb{S}^+$ and $\mathbb{S}^-$, we need to push $\Omega^X$ forward by the representation $\rho$. The Chern character is $Tr(\exp \dfrac{i\Omega_{\mathbb{S}_\pm}}{2\pi})$ which would be $Tr(\exp \dfrac{i\rho(\Omega_X)}{2\pi}) = Tr(\rho(\exp \dfrac{i\Omega_X}{2\pi}))$.

Now the trick is to look at $\rho(\exp \dfrac{\Omega_X}{2\pi})$ instead of $\rho(\exp \dfrac{i\Omega_X}{2\pi})$.

If we think of $\exp \dfrac{\Omega_X}{2\pi}$ as an element of $SO(2n)$ the element then is

$$\begin{bmatrix} \cos x_1 & \sin x_1 & & & \\ -\sin x_1 & \cos x_1 & & & \\ & & \cos x_2 & \sin 2x_2 & \\ & & -\sin x_2 & \cos 2x_2 & \\ & & & & \ddots \end{bmatrix}$$

in $Spin(2n)$ which then lifts to the element

$$\exp \frac{\Omega_X}{2\pi} = (\cos x_1/2 + e_1 e_2 sin x_1/2)(\cos x_2/2 + e_3 e_4 \sin x_2/2) \cdots$$

Recall that for $v_1, v_2, \cdots, v_n$ a the basis for $\mathbb{C}^n$, $\mathbb{S}$ is the space of exterior forms $\wedge^* \mathbb{C}^n$ and $v_{i_1} \wedge \cdots v_{i_k}$ is an eigenvector of $e_1 e_2$ with eigenvalue $i$ if one of the $i_k's$ is 1 and it will have eigenvalue $-i$ else. Similarly for the other elements.

So that in general we get that $v_{i_1}\wedge\cdots v_{i_k}$ is an eigenvector of $(\cos x_1/2+e_1e_2 sin x_1/2)(\cos x_2/2+e_3e_4\sin x_2/2)\cdots$ with eigenvalue $(\cos x_1/2\pm i\sin x_1/2)(\cos x_2/2\pm i\sin x_2/2)\cdots=\prod\exp(\pm ix_j/2)$.

Hence the eigenvalues of $\exp\dfrac{i\Omega_X}{2\pi}$ would just be $\prod\exp(\mp x_j/2)$ corresponding to the eigenvector $v_{i_1}\wedge\cdots v_{i_k}$ depending on whether $j$ is one of the $i_k$'s.

The basis for $\mathbb{S}$ consists of all the forms $\wedge^*\mathbb{C}^n$ and these every possible combination of $\pm$ would occur when adding the terms $\sum\prod\exp(\mp x_j/2)$ we would get the trace to be

$$(\exp(x_1/2)+\exp(-x_1/2)).(\exp(x_2/2)+\exp(-x_2/2)).\cdots=2^n\prod\cosh x_j/2$$

For $\mathbb{S}^+-\mathbb{S}^-$ the basis again consists of all the forms $\wedge^*\mathbb{C}^n$ but this time the odd ones have the signs flipped and hence the trace $\sum_{even}\prod\exp(\mp x_j/2)-\sum_{odd}\prod\exp(\mp x_j/2)$ would be

$$(\exp(-x_1/2)-\exp(x_1/2)).(\exp(-x_2/2)-\exp(x_2/2)).\cdots=(-2)^n\prod\sinh x_j/2$$

We now have the identity $(\mathbb{S}^+)^*-(\mathbb{S}^-)^*=(-1)^n(\mathbb{S}^+-\mathbb{S}^-)$ which then gives us

So that the index of $\mathcal{D}_{(\mathbb{S}^+)^*}-\mathcal{D}_{(\mathbb{S}^-)^*}$ is

$$\begin{aligned}\prod\frac{x_i/2}{\sinh x_i/2}.2^n\prod\sinh x_i/2&=\prod x_i\\&=\int_X\det\sqrt{\Omega/2\pi}\end{aligned}$$

And the index of $\mathcal{D}_{\mathbb{S}}$ is

$$\prod\frac{x_i/2}{\sinh x_i/2}.2^n\prod\cosh x_i/2=\prod\frac{x_i}{\tanh x_i/2}$$

This though is not the form is which the signature theorem is classically stated. One sees that the degree $n$ term in $\prod_{i=1}^n\dfrac{x_i}{\tanh x_i/2}$ is the same as the degree $n$ term in $\prod\dfrac{x_i}{\tanh x_i}$ so that

$$\begin{aligned}\prod\frac{x_i/2}{\sinh x_i/2}.2^n\prod\cosh x_i/2&=\prod\frac{x_i}{\tanh x_i/2}\\&=\int_X\det\sqrt{\frac{i\Omega/2\pi}{\tanh i\Omega/4\pi}}\\&=\int_X\det\sqrt{\frac{i\Omega/2\pi}{\tanh i\Omega/2\pi}}\\&=\text{genus of }\sqrt{\frac{z}{\tanh z}}\end{aligned}$$

**Theorem 8.1** (Gauss Bonnet Theorem)**.** *If $X$ is a closed spin manifold of* $\dim 2n$ *then*

$$\chi(X) = \int_X \det \sqrt{\Omega/2\pi}$$

*where $\chi(X)$ is the Euler characteristic of $X$.*

**Theorem 8.2** (Hirzerbruch Signature theorem)**.** *If $X$ is a closed spin manifold of* $\dim 4k$ *then*

$$Sign(X) = \int_X \det \sqrt{\frac{i\Omega/2\pi}{\tanh i\Omega/2\pi}}$$

*where $Sign(X)$ is the signature of $X$.*

## References

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