Vector Fields on Spheres

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This is a summary of the Adams paper titled "Vector fields on Spheres" in which he computes the upper bound on the number of linearly independent non-vanishing vector fields on S^n.

Definition 0.1. For $n = (2a+1)2^b$ define $\rho(n) = \rho'(b) + 1$ where ρ' is defined inductively as

$$\rho'(0) = 0, \rho'(1) = 1, \rho'(2) = 3, \rho'(3) = 7$$

(note: S^0, S^1, S^3, S^7 are trivilizable) and $\rho'(4+b) = 8 + \rho'(b)$.

Theorem 0.1 (Adams)**.** *There do not exist $\rho(n)$ linearly independent vector fields on S^{n-1}. This bound is strict i.e. there exist $\rho(n) - 1$ linearly independent vector fields on S^{n-1}.*

The steps involved in the proof are as follows:

(1) Construct of the $\rho(n) - 1$ vector fields using Clifford algebras

(2) Connect the existence of k vector fields on S^n to coreducibility of *stunted real projective space* i.e. an existence of a map

$$\mathbb{RP}(n+k-1, n-1) \to S^n$$

whose restriction to the n-skeleton is degree 1

(3) Find obstructions to existence of such a map. The restrictions lie in Steenrod Squares when $8 \not| n$ and Adams operations on KO when $8|n$.

(4) The action on Steenrod Squares on $H^*(\mathbb{RP}(n,m); \mathbb{Z}/2)$ is pretty well known. The main crux of Adams paper was computing Adams operations on $\widetilde{KO}(\mathbb{RP}(n,m))$.

Note: Here $K, KO, \widetilde{K}, \widetilde{KO}$ denote the 0^{th} K-groups (and rings), H^* will denote singular cohomology with $\mathbb{Z}$ coefficients.

1. Clifford algebras

The Clifford algebra $\mathcal{C}_n$ over $\mathbb{R}^n$ with standard basis e_i is defined as

$$\mathcal{C}_n := \mathbb{R}[e_i]/(e_i^2 + 1, e_ie_j + e_je_i)$$

$\mathcal{C}_n$ has a vector space basis of monomials in e_i and hence $\dim_{\mathbb{R}} \mathcal{C}_n = 2^n$.

We have the following isomorphisms

$$\begin{aligned}\mathcal{C}_0 &\cong \mathbb{R}\\ \mathcal{C}_1 &\cong \mathbb{C}\\ \mathcal{C}_2 &\cong \mathbb{H}\\ \mathcal{C}_3 &\cong \mathbb{H}\oplus\mathbb{H}\\ \mathcal{C}_{n+8} &\cong \mathcal{C}_n \otimes M_{16}(\mathbb{R}) \text{ (Bott periodicity)}\end{aligned}$$

The first three are just definitions and the last two are proven by giving an explicit isomorphism.

Corollary 1.1. *$\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^4, \mathbb{R}^8$ are modules over $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ respectively. If M is a module over $\mathcal{C}_n$ then $\mathbb{R}^{16} \otimes M$ is a module over $\mathcal{C}_{n+8}$.*

Lemma 1.2. *If $\mathbb{R}^n$ is a $\mathcal{C}_m$ module then there exist m linearly independent vector fields on S^{n-1}.*

Proof: By the module structure we can think of $e_i \in \mathcal{C}_m$ as elements of $GL(n, \mathbb{R})$. So for each $v \in S^{n-1}$ we get $m+1$ linearly independent vectors $v, e_1 v, \cdots, e_m v$. We can then apply the Gram-Schmidt process to get the required m vector fields. □

Corollary 1.3. *There exist $\rho(n) - 1$ linearly independent vector fields on S^{n-1}.*

Proof: It suffices to show that $\mathbb{R}^n$ is a $\mathcal{C}_{\rho(n)-1}$ module. As $\mathbb{R}^n = \mathbb{R}^{2a+1} \otimes \mathbb{R}^{2^b}$ it suffices to show that $\mathbb{R}^{2^b}$ is a $\mathcal{C}_{\rho'(b)}$ module. One can show this by induction. The statement is true for $b = 0, 1, 2, 3$. Assume the statement to be true for an arbitrary b and consider $b+4$. Now $\mathbb{R}^{2^{b+4}} = \mathbb{R}^{16} \otimes \mathbb{R}^{2^b}$ and $\mathcal{C}_{\rho'(b+4)} = \mathcal{C}_{\rho'(b)+8}$ so we are done by the above corollary. □

2. Background

2.1. Atiyah Hirzebruch Spectral Sequence.

Theorem 2.1. *For any cohomology theory C^* and a finite CW-complex X, there exists a cohomology spectral sequence, the **AHSS** with*

$$\begin{aligned}E_2^{p,q} &= H^q(X^p; C^*(point))\\ E^{p,q} &\Longrightarrow C^*(X)\end{aligned}$$

where X^p is the p skeleton of X and H^* denotes the singular cohomology.

2.2. Adams Operations.

Theorem 2.2. *Given a finite CW-complex X there exist cohomology operations $\psi^k_{\mathbb{C}} : K(X) \to K(X), k \in \mathbb{Z}$ (called **Adams operations**) which are uniquely determined by the following axioms:*

(1) Naturality: for $f : X \to Y$ we have $f^* \psi^k_{\mathbb{C}} = \psi^k_{\mathbb{C}} f^*$

(2) For a line bundle L we have $\psi^k_{\mathbb{C}}(L) = L^k$, $\psi^0_{\mathbb{C}}(L) = 1$

(3) $\psi^k_{\mathbb{C}} \psi^l_{\mathbb{C}} = \psi^{kl}_{\mathbb{C}}$

(4) If ch^q denotes the $2q^{th}$ component of the Chern character then $ch^q \circ \psi^k_{\mathbb{C}} = k^q . ch^q$. More generally if c_q denotes the q^{th} Chern class then $c_q \circ \psi^k = k^q . c_q$

There exist Adams operations in KO-theory $\psi^k_{\mathbb{R}} : KO(X) \to KO(X)$ satisfying the same axioms and a compatibility condition $\psi^k_{\mathbb{C}}(E \otimes \mathbb{C}) = \psi^k_{\mathbb{R}}(E) \otimes \mathbb{C}$.

Proof: One way to prove this is by using the splitting principle, which is not easy to prove for K theory. Instead Adams uses the exterior algebra representation of $GL(n)$ to define these operations. A $\dim n$ vector bundle can be described by a map $X \to BGL(n)$, the field could be either $\mathbb{R}$ or $\mathbb{C}$. We have exterior power representations

$$E_r : GL(n) \to GL(\binom{n}{r})$$
$$M \mapsto \wedge^r M$$

Now consider the symmetric polynomials $\sigma_1, \cdots, \sigma_n$ in the variables $x_1, \cdots, x_n$. The sum of powers $\sum_i x_i^k$ can be expressed as a polynomial $Q(\sigma_1, \cdots, \sigma_n)$. Define the operation k^{th} Adams operation on $\dim n$ by

$$\psi^k : BGL(n) \to \mathbb{Z} \times BGL$$
$$B(M \mapsto Q(E_1, \cdots, E_n))$$

Note that this could lead to a virtual bundle of $\dim n$. Uniqueness is a bit non-trivial and needs some facts from representation theory over $\mathbb{R}$.

The Chern character can be written in terms of the Chern roots $x_1, \cdots, x_n$ as $ch = (e^{x_1} + \cdots + e^{x_n})$. Because $c_1(L' \otimes L'') = c_1(L') + c_1(L'')$ we see that $ch \circ \psi^k = (e^{kx_1} + \cdots + e^{kx_n})$ which gives us 4). □

2.3. K theory of spheres.

Theorem 2.1 (Bott Periodicity)**.**

$$\widetilde{K}(S^{2n+1}) \cong 0$$
$$\widetilde{K}(S^{2n}) \cong \mathbb{Z}$$
$$\widetilde{KO}(S^{4n}) \cong \mathbb{Z}$$

The complexification map $\otimes\mathbb{C} : \widetilde{KO}(S^{4n}) \to \widetilde{K}(S^{4n})$ is injective and the image is $\mathbb{Z}$ if n is even and $2\mathbb{Z}$ if q is odd. The $2n^{th}$ component of the Chern character $ch^n : \widetilde{K}(S^{2n}) \to \widetilde{H}^{2n}(S^{2n}, \mathbb{Q})$ maps isomorphically onto $\widetilde{H}^{2n}(S^{2n}, \mathbb{Z})$.

Corollary 2.2. *Combining the above propositions we get that the Adams operations $\psi^k_{\mathbb{C}}$ and $\psi^k_{\mathbb{R}}$ on $\widetilde{K}(S^{2n})$ and $\widetilde{KO}(S^{4n})$ are given by multiplication by k^n and k^{2n} respectively.*

3. Co-reducibility

Definition 3.1. Define the following spaces:

Stunted projective spaces:

$$\mathbb{RP}(n, k) := \mathbb{R}P^n / \mathbb{R}P^k$$
$$\mathbb{CP}(n, k) := \mathbb{C}P^n / \mathbb{C}P^k$$

Canonical line bundles over $\mathbb{RP}^k$:

$$\gamma^k := \{(l, v) | l \in \mathbb{RP}^k, v \in l \subset \mathbb{R}^{k+1}\}$$
$$\gamma^k_\perp := \{(l, v) | l \in \mathbb{RP}^k, v \perp l \subset \mathbb{R}^{k+1}\}$$

First a few facts about the canonical line bundles,

Lemma 3.2. *Normal bundle of $\mathbb{RP}^k$ in $\mathbb{RP}^{n+k}$ is isomorphic to $n\gamma^k$. It's Thom space $T(n\gamma^k) \cong \mathbb{RP}^{n+k}/\mathbb{RP}^{n-1}$.*

Proof: The lift of γ^n to S^n is the trivial bundle $S^n \times \mathbb{R}$, so $\gamma^n \cong S^n \times_{\mathbb{Z}/2} \mathbb{R}$. The normal bundle of S^k inside S^{n+k} is trivial, so the normal bundle of $\mathbb{RP}^k$ in $\mathbb{RP}^{n+k}$ is isomorphic to

$$S^k \times_{\mathbb{Z}/2} \mathbb{R}^n \cong n\gamma^k$$

For the second part it suffices to show that $\mathbb{RP}^{n+k} \setminus \mathbb{RP}^k$ deformation retracts onto $\mathbb{RP}^{n-1}$. Again it is trivial to see this for S^k inside S^{n+k} and then quotient out by $\mathbb{Z}/2$. □

Lemma 3.3. *If there exist $k-1$ linearly independent vector fields on S^{n-1} then the unit sphere bundle $S(n\gamma^{k-1})$ is trivial and hence is homotopy equivalent to $\mathbb{RP}^{k-1} \times S^{n-1}$.*

Proof: Assume that there exist $s_1, s_2, \cdots, s_{k-1}$ orthonormal vector fields on S^{n-1}. Let s_0 be the radial vector field. Then $(s)_p = (s_0(p), \cdots, s_{k-1}(p))$ defines an orthonormal $(n-1) \times k$ matrix for each $p \in S^{n-1}$.

By the above lemma $n\gamma^{k-1} \cong S^{k-1} \times_{\mathbb{Z}/2} \mathbb{R}^n$ so that $S(n\gamma^{k-1}) \cong S^{k-1} \times_{\mathbb{Z}/2} S^{n-1}$.

Define a map

$$\begin{aligned} \mathbb{RP}^{k-1} \times S^{n-1} &\to S^{k-1} \times_{\mathbb{Z}/2} S^{n-1} \\ ([p], q) &\mapsto [p, s(q)p] \end{aligned}$$

Why is this a homotopy equivalence? □

Theorem 3.4 (coreducibility)**.** *Given $k-1$ vector fields on S^{n-1} there exist map*

$$f\colon \mathbb{RP}(n+k-1, n-1) \to S^n$$

such that the composite

$$S^n = \mathbb{RP}(n, n-1) \hookrightarrow \mathbb{RP}(n+k-1, n-1) \xrightarrow{f} S^n$$

has degree 1.

Proof: Look at the following diagram,

$$\begin{tikzcd}
S^n \cong \mathbb{RP}(n, n-1) \arrow[r, hook] & \mathbb{RP}(n+k-1, n-1) \arrow[d, "\cong"] \\
& Th(n\gamma^{k-1}) \arrow[d, "\cong"] \\
& Th(\mathbb{RP}^{k-1} \times S^{n-1}) \arrow[d, "\cong"] \\
& \mathbb{RP}^{k-1}_+ \wedge S^n \arrow[d] \\
& S^n
\end{tikzcd}$$

Define f to be the composition of the vertical arrows. □

4. $\widetilde{KO}(\mathbb{RP}(n, m))$

The computation of these groups are quite complicated as $\mathbb{RP}^n$ has cells in all dimensions. The results of this section will be proven at the end.

Theorem 4.1. *If $b \geq 3$ where $n = (2a+1)2^b$ then for any positive integer k*

$$\widetilde{KO}(\mathbb{RP}(n+k, n-1)) = \mathbb{Z}\mu \oplus \mathbb{Z}/2^{b+1}\lambda$$

The Adams operations on the generators are given by

$$\begin{aligned} \psi^3_{\mathbb{R}}(\lambda) &= \lambda \\ \psi^3_{\mathbb{R}}(\mu) &= 3^{n/2}\mu + \frac{3^{n/2}-1}{2}\lambda \end{aligned}$$

5. Proof of the main theorem

Proof: Proof by **contradiction**. Assume that there exists an n such that there are $\rho(n)$ linearly independent vector fields on S^{n-1}. By the coreducibility theorem (3.4) we get an $f\colon \mathbb{RP}(n+\rho(n), n-1) \to S^n$.

Claim 5.1. *n is divisible by 8.*

Proof: By looking at the Euler characteristic we should have $2|n$. Recall that $H^*(\mathbb{RP}^n;\mathbb{Z}/2) = \mathbb{Z}/2[x]/x^{n+1}$ and $Sq^i(x^j) = \binom{j}{i}x^{i+j}$ for $j \geq i$.

Tracing the generator of $H^n(S^n;\mathbb{Z}/2)$ in the following diagram

$$\begin{tikzcd}
H^n(S^n) \arrow[r, "f^*"] \arrow[d, "Sq^{\rho(n)}"'] & H^n(\mathbb{RP}(n+\rho(n), n-1)) \arrow[r] \arrow[d, "Sq^{\rho(n)}"'] & H^n(\mathbb{RP}^{n+\rho(n)}) \arrow[d, "Sq^{\rho(n)}"'] \\
H^{n+\rho(n)}(S^n) \arrow[r, "f^*"] & H^{n+\rho(n)}(\mathbb{RP}(n+\rho(n), n-1))) \arrow[r] & H^{n+\rho(n)}(\mathbb{RP}^{n+\rho(n)})
\end{tikzcd}
\qquad
\begin{tikzcd}
\alpha \arrow[r, mapsto] \arrow[d, mapsto] & x^n \arrow[d, mapsto] \\
0 \arrow[r, mapsto] & \binom{n}{\rho(n)}x^{n+\rho(n)}
\end{tikzcd}$$

we get $2|\binom{n+\rho(n)}{n}$. An easy check shows that this forces $8|n$. □

The map f should map a generator α of $\widetilde{KO}(S^n)$ to a non-torsion generator of $\mathbb{RP}(n+\rho(n), n-1)$. As $8|n$ we can invoke theorem (4.1) so that for some integer N

$$f^*\alpha = \mu + N\lambda$$

Applying $\psi^3_{\mathbb{R}}$ and using (2.2) and (4.1) we get

$$\begin{aligned}
& & f^*\psi^3_{\mathbb{R}}\alpha &= \psi^3_{\mathbb{R}}\mu + N\psi^3_{\mathbb{R}}\lambda \\
&\implies & f^*(3^{n/2}\alpha) &= (3^{n/2}\mu + \frac{3^{n/2}-1}{2}\lambda) + N\lambda \\
&\implies & 3^{n/2}\mu + 3^{n/2}.N\lambda &= 3^{n/2}\mu + ((3^{n/2}-1)/2 + N)\lambda \\
&\implies & 3^{n/2}.N &\equiv (3^{n/2}-1)/2 + N \mod 2^{b+1} \\
&\implies & (3^{n/2}-1)(N-1/2) &\equiv 0 \mod 2^{b+1} \\
&\implies & 3^{n/2} &\equiv 1 \mod 2^{b+2}
\end{aligned}$$

By an inductive argument one can show that this can never happen which completes the proof. □

6. Computations of the K groups

6.1. $K(\mathbb{CP}^n)$**.** Because the only non-zero cohomology groups of $\mathbb{CP}^n$ are even, the AHSS collapses on page 2. $H^*(\mathbb{CP}^n) = \mathbb{Z}[x]/x^{n+1}$ gives us

$$K(\mathbb{CP}^n) = K^0(\mathbb{CP}^n) = \mathbb{Z}[y]/y^{n+1}$$

.

The generator y of $\mathbb{CP}^n$ is precisely the **canonical line bundle** $\epsilon - 1$ over $\mathbb{CP}^n$. The Adams operations are given by

$$\begin{aligned}
\psi^k(y) &= \psi^k(\epsilon - 1) \\
&= \epsilon^k - 1 \\
&= (y+1)^k - 1
\end{aligned}$$

6.2. $K(\mathbb{RP}^{2n})$**.** $H^*(\mathbb{RP}^{2n})$ is non-zero only in the even dimensions so again the AHSS collapses at the E_2 page. The AHSS gives us

$$E^0_\infty = gr(K(\mathbb{RP}^{2n})) = \mathbb{Z} \oplus (\mathbb{Z}/2)^{\oplus n}$$

Because of torsion we cannot compute $K(\mathbb{RP}^{2n})$ from the SS alone.

Claim 6.1.

$$K(\mathbb{RP}^{2n}) \cong \mathbb{Z} \oplus \mathbb{Z}/2^n$$

Proof: In the AHSS for $K(\mathbb{CP}^n)$, $E^0_\infty = gr(K(\mathbb{CP}^{2n})) = \mathbb{Z} \oplus \mathbb{Z}y \oplus \cdots \mathbb{Z}y^n$ so that for $\mathbb{RP}^{2n}$, $E^0_\infty = gr(K(\mathbb{RP}^{2n})) = \mathbb{Z} \oplus (\mathbb{Z}/2)\pi^*y \oplus \cdots (\mathbb{Z}/2)\pi^*y^n$. So we see that $(\pi^*y)^n \neq 0$. The bounds on the AHSS imply $(\pi^*y)^{n+1} = 0$.

By comparing the first Chern class we see that $\pi^*y = \pi^*(\epsilon - 1) = \mathbb{C} \otimes (\gamma^{2n} - 1)$ so $(\pi^*y+1)^2 = \mathbb{C} \otimes (\gamma^{2n} \otimes \gamma^{2n}) = 1$, so that $(\pi^*y)^2 = -2(\pi^*y) \implies (\pi^*y)^{k+1} = (-2)(\pi^*y)^k = (-2)^k(\pi^*y)$. Combining the results we get

$$2^{n-1}(\pi^*y) \neq 0, 2^n(\pi^*y) = 0$$

This and the size of $gr(K(\mathbb{RP}^{2n}))$ give us $K(\mathbb{RP}^{2n}) \cong \mathbb{Z} \oplus \mathbb{Z}/2^n(\pi^*y)$.

The action of the Adams operations would be

$$\begin{aligned}
\psi^k(\pi^*y) &= \pi^*(\psi^k y) \\
&= ((\pi^*y+1)^k - 1) \\
&= \sum_{i=0}^{k-1} \binom{k}{i+1} (\pi^*y)^{i+1} \\
&= \left(\sum_{i=0}^{k-1} \binom{k}{i+1} (-2)^i \right) \pi^*y \\
&= ((1-2)^k - 1)\pi^*y/(-2) = \begin{cases} \pi^*y & \text{if } k \text{ odd} \\ 0 & \text{if } k \text{ even} \end{cases}
\end{aligned}$$

□

6.3. $K(\mathbb{RP}^{2n+1})$**.** The AHSS for $K(\mathbb{RP}^{2n+1})$ does have non-zero elements in odd dimensions but all the differentials on the *reduced part* are of the form $\mathbb{Z}/2 \to \mathbb{Z}$ and hence 0. So the AHSS again collapses at the E_2 page, so that

$$K(\mathbb{RP}^{2n+1}) \cong K(\mathbb{RP}^{2n})$$

Note that because of the collapse of the spectral sequence we can conclude that $K^1(\mathbb{RP}^n)) = 0$.

6.4. $K(\mathbb{RP}(n,2m))$**.** The sequence $\mathbb{RP}^{2m} \to \mathbb{RP}^n \to \mathbb{RP}(n,2m)$ gives rise to a short exact sequence $0 \leftarrow \widetilde{K}(\mathbb{RP}^{2m}) \leftarrow \widetilde{K}(\mathbb{RP}^n) \leftarrow \widetilde{K}(\mathbb{RP}(n,2m)) \leftarrow \widetilde{K}^1(\mathbb{RP}^{2m}) = 0$ which gives us

$$\widetilde{K}(\mathbb{RP}(n,2m)) = \mathbb{Z}/2^{[n/2]-m}$$

And the action of the Adams operations is same as before.

6.5. $K(\mathbb{RP}(n, 2m-1))$**.** The sequence $\mathbb{RP}(2m, 2m-1) \to \mathbb{RP}(n, 2m-1) \to \mathbb{RP}(n, 2m)$ gives rise to a short exact sequence $0 \leftarrow \widetilde{K}(\mathbb{RP}(2m, 2m-1)) = \mathbb{Z} \leftarrow \widetilde{K}(\mathbb{RP}(n, 2m-1)) \leftarrow \widetilde{K}(\mathbb{RP}(n, 2m)) \leftarrow \widetilde{K}^1(\mathbb{RP}^{2m}) = 0$ which gives us

$$\begin{aligned}\widetilde{K}(\mathbb{RP}(n, 2m-1)) &\cong \widetilde{K}(\mathbb{RP}(n, 2m)) \oplus \mathbb{Z}\mu \\ &= \mathbb{Z}/2^{[n/2]-m}\lambda \oplus \mathbb{Z}\mu\end{aligned}$$

μ maps to the generator of $\widetilde{K}(\mathbb{RP}(2m, 2m-1)) = \widetilde{K}(S^{2m})$ so that

$$\psi^k\mu = k^m\mu + c_k.\lambda$$

for some constant c_k. To compute c_k consider the projection $\mathbb{RP}(n, 2m-1) \to \mathbb{RP}(n, 2m-2)$, μ maps to the generator of $\widetilde{KO}(\mathbb{RP}(n, 2m-2))$. Tracing action of ψ^k we get,

$$c_k = \begin{cases} k^m/2 & \text{if } k \text{ even} \\ (k^m-1)/2 & \text{if } k \text{ odd} \end{cases}$$

7. Computation of the KO groups

Definition 7.1. For integers $m > n$ define $\phi(m, n)$ to be the number of integers $k \in (n, m]$ such that $KO^k(*) \neq 0$.

The $KO(\mathbb{RP}^n)$ groups are of the same form as the $K(\mathbb{RP}^n)$ groups. Because there is more torsion in KO^* one needs to make more complicated combinatorial arguments but the ideas are exactly the same as above. I'll only state the results here,

$$\begin{aligned}\widetilde{KO}(\mathbb{RP}^n) &= \mathbb{Z}/2^{\phi(n,0)} \\ \widetilde{KO}(\mathbb{RP}(n,m)) &= \begin{cases} \mathbb{Z}/2^{\phi(n,m)}\lambda & \text{if } m \not\equiv -1 \mod 4 \\ \mathbb{Z}/2^{\phi(n,m)}\lambda \oplus \mathbb{Z}\mu & \text{if } m \equiv -1 \mod 4 \end{cases} \\ \psi^k\lambda &= \begin{cases} \lambda & \text{if } k \text{ odd} \\ 0 & \text{if } k \text{ even} \end{cases} \\ \psi^k\mu &= k^m\mu + \lambda.\begin{cases} k^m/2 & \text{if } k \text{ even} \\ (k^m-1)/2 & \text{if } k \text{ odd} \end{cases}\end{aligned}$$