The Okounkov-Vershik approach to representations of Symmetric Groups

Report by: Apurv Nakade, MSc. Mathematics, Second Year, CMI

Overview:

We studied in this course how the RSK correspondence gives us a description of the irreducible representations of the symmetric groups. However the method used was indirect and though it gave us a complete description of the irreducible reps. it did not give any algebraic relation between Young's tableaus and the irreducible reps.

In this approach on the other hand we will see that the correspondence between Young's tableaus and the irreducible representations of the symmetric groups arise from their inductive natures. We will give an explicit isomorphism using the content vectors of the Young's tableaus and the spectrum vectors of the symmetric groups.

In this report first we study the group S_n and derive some basic properties of its irreducible representations which are derived simply by the relations between its generators. Next we will define the spectrum vectors of S_n and describe the action of S_n on it. Next we will define the content vectors for standard Young's tableaus. Finally we will prove the one to one correspondence between the spectrum vectors and the content vectors which is the main result of the paper. The paper goes on to prove a recursive formula for calculating the characters, we will not state this.

Notations:

$\boldsymbol{S_n} \coloneqq n^{th}$symmetric group $S_0 \coloneqq \{0\}$

$\boldsymbol{S_n^-} \coloneqq$ equivalence classes of the irreducible representations of S_n over $\mathbb{C}$

$\boldsymbol{s_i} \coloneqq (i\ i+1)$

$\boldsymbol{X_i} \in \mathbb{C}[S_n]$ and $X_i \coloneqq (1\ i) + (2\ i) + \cdots + (i-1\ i)$

$\boldsymbol{Z_n} \coloneqq$ center of $\mathbb{C}[S_n]$

$\boldsymbol{A_n} \coloneqq$ subalgebra of $\mathbb{C}[S_n]$ generated by $Z_1, Z_2, \ldots, Z_n$

$\mathbb{C}[\boldsymbol{S_n}]^{\boldsymbol{H}} \coloneqq$ centralizer of H in S_n

$\boldsymbol{Z(l,k)} \coloneqq \mathbb{C}[S_{l+k}]^{S_l}$

The branching graph of S_n

First we will prove that the branching graph is multiplicity free for S_n. This will help us to construct a canonical basis for S_n^- called the GZ-basis.

$\bigcup_{n\geq 0} S_n^-$ is called the **branching graph**. S_n^- is called the n^{th}level of the branching graph. There are edges only between two consecutive branching levels. Between $\mu \in S_n$and $\nu \in S_{n+1}$ there are k edges, where $k = \dim Hom_{S_n}(\mu, \nu)$. This number is well defined thanks to Schur's lemma. If $k \neq 0$, we say $\mu \to \nu$. If $k \leq 1$ for all μ and ν then we say that the graph is **multiplicity free**.

Proposition 1.

$$\mathbb{C}[S_n] = \bigoplus_{\lambda \in S_n^-} End_{\mathbb{C}}(V^\lambda)$$

Proof:

As direct sums of representations is a representation, we get a map from $\Phi: \mathbb{C}[S_n] \to \bigoplus End_{\mathbb{C}}(V^\lambda)$. If $\Phi(x) = \Phi(y)$ for $x, y \in S_n$ then x, y should act the same on every representation of S_n in particular on the regular representation, so $x = y$. Finally $\dim \mathbb{C}[S_n] = \dim \bigoplus End_{\mathbb{C}}(V^\lambda) = n!$..

Proposition 2.

Let $H \subset G$ and let V be an irreducible G-module and U be an irreducible H-module.Then,

1. multiplicity of U in $V = \dim Hom_H(U, V)$
2. $Hom_H(U, V)$ is an irreducible module of $\mathbb{C}[G]^H$

Proof:

The first result is just Schur's lemma.

Let $f, g \in Hom_H(U, V)$ Fix an element $u \in U$. Then there is an H-intertwiner $V \to V$ which takes $f(u)$ to $g(u)$ (we can construct this intertwiner by defining it on each H-invariant subspace of V). Composition with this intertwiner gives a map taking f to g. Because every possible endomorphism of V can be represented by an element of $\mathbb{C}[G]$ we get the irreducibility..

Proposition 3.

$\mathbb{C}[S_{n+1}]^{S_n}$ is abelian. The branching graph is multiplicity free.

Proof:

Notice that for any $\pi \in S_{n+1}$ $\exists \theta \in S_n$ such that $\theta^{-1}\pi\theta = \pi^{-1}$. For this let $(a_0\ a_1 \ldots a_n)$ be a cycle in disjoint cycle decomposition of π with $a_0 > a_i \forall i$ then set $\theta(a_0) = a_0$ and $\theta(a_i) = a_{n+1-i}$ for $i > 0$. Similarly for all other cycles. This permutation does the job, call it θ_π. We also have $\theta_{\pi^{-1}} = \theta_\pi^{-1}$.

By definition, $\mathbb{C}[S_{n+1}]^{S_n} = \{x \in S_{n+1} |\ h^{-1}xh = x\ \forall h \in S_n\}$. So $\mathbb{C}[S_{n+1}]^{S_n}$ is generated as a vector space by elements of the form $\sum_{\theta \in S_n} \theta^{-1}\pi\theta$. So it suffices to prove

$\left[\sum_{\theta \in S_n} \theta^{-1}\pi\theta\right]\left[\sum_{\theta \in S_n} \theta^{-1}\sigma\theta\right] = \left[\sum_{\theta \in S_n} \theta^{-1}\sigma\theta\right]\left[\sum_{\theta \in S_n} \theta^{-1}\pi\theta\right]$

For this we give a one to one bijection between the elements of the two sides. It is easy to check that if $\rho = \theta_1^{-1}\pi\theta_1\theta_2^{-1}\sigma\theta_2$ then $\rho = \left(\theta_2\theta_\rho\right)^{-1}\sigma\left(\theta_2\theta_\rho\right)\left(\theta_1\theta_\rho\right)^{-1}\pi\left(\theta_1\theta_\rho\right)$ and because $\theta_{\pi^{-1}} = \theta_\pi^{-1}$ this is a bijection.

As $Hom_{S_n}(U, V)$ is an irreducible representation over $\mathbb{C}[S_{n+1}]^{S_n}$ for any $U \in S_n^-$, $V \in S_{n+1}^-$ we get that $\dim Hom_{S_n}(U, V) \leq 1$. So the branching graph is multiplicity free..

If $\mu \in S_{n+1}^-$ then,

$$\mu = \bigoplus \nu, \text{direct sum over all } \nu \in S_n^-, \nu \to \mu$$

Repeating this we obtain a basis for μ indexed by all possible chains $T = \mu_0 \to \mu_1 \to \cdots \to \mu_n \to \mu$. This basis normalized according to the S_{n+1} invariant form is called the **GZ-basis**. We will denote these by v_T.

Proposition 4.

> $A_n$is the algebra of all the operators diagonal in the GZ-basis $\{v_T\}$. In particular, it is a maximal commutative subalgebra of $\mathbb{C}[S_n]$. Each v_T is uniquely determined by the eigenvalues of the elements of A_n on it.

Proof:

If $T = \mu_0 \to \mu_1 \to \cdots \to \mu_n$. Let P_{μ_i} denote the projection onto μ_i. Then by Schur's lemma and the fact that the branching graph is multiplicity free we get that $P_{\mu_i} \in Z_i$ so that $P_T = P_{\mu_0}P_{\mu_1} \dots P_{\mu_n} \in A_n$ which is the projection onto v_T. The algebra generated by these $P_T{}'s$ is the algebra of all operators diagonal in GZ-basis. This is a maximal commutative subalgebra so it must be all of A_n..

Proposition 5.

> $Z(l, k)$ is generated by the elements
>
> 1. $X_{l+1}, X_{l+2}, \dots, X_{l+k}$
> 2. the group S_k which permutes the elements $\{l + 1, l + 2, \dots, l + k\}$
> 3. Z_l.

Proof:
I have not understood the proof of this result..

Proposition 6.

> $X_1, X_2, \dots, X_n$ generate by A_n.

Proof:

$Z(n) \subset Z(n-1,1) = < Z(n-1), X_n >$ So by induction we get $A_n =< X_1, X_2, \dots, X_n >$..

The spectrum vectors of S_n

In this section we will define spectrum of S_n and give a description of the action of S_n on them. By the end of this section we will have almost proved that the irreducible representations of S_n are in $\mathbb{Q}$.

The GZ-basis $\{v_T\}$ is the common eigenbasis for the X_i's. Define,

$$\boldsymbol{\alpha_T} \coloneqq \alpha(v_T) \coloneqq (a_1, a_2, \dots, a_n) \in \mathbb{C}^n \text{ where } X_i v_T = a_i v_T \text{ and}$$

$$\boldsymbol{spec(n)} \coloneqq \{\alpha_T : v_T \in \text{GZ-basis}\} \subset \mathbb{C}^n$$

For any $\alpha \in spec(n)$ let T_α be the corresponding path and v_α the corresponding eigenvector. Define an equivalence relation ~ on $spec(n)$ as $\alpha \sim \beta \iff v_\alpha$ and v_β are in the same irreducible representation $\iff T_\alpha$ and T_β end at the same vertex. Note that since $X_i' s$ generated A_n a $spec(n)$ vector uniquely determines the corresponding GZ-basis element.

We say s_i is **admissible** with respect to $\alpha \in spec(n)$ if $s_i v_\alpha \in spec(n)$ and $s_i v_\alpha \neq \pm v_\alpha$

Proposition 7.

1. $s_i X_j = X_j s_i, j \neq i, i+1$
2. $s_i X_i + 1 = X_{i+1} s_i$

Proof:

Follows from definition..

Proposition 8.

If V is an irreducible $\mathbb{C}[S_{l+k}]$ module and U an irreducible $\mathbb{C}[S_l]$ module then $\dim Hom_{S_l}(U, V) \leq k!$. In particular if $\mu \to \nu \to \lambda$ then there is at the most one $\nu' \neq \nu$ such that $\mu \to \nu' \to \lambda$.

Proof:

$Hom_{S_l}(U, V)$ is an irreducible module over $Z(l,k) = < Z(l), S_k, X_{l+1}, \dots, X_{l+k} >$. So $Hom_{S_l}(U,V) =< Z(l)v, S_k v, X_{l+1}v, \dots, X_{l+k}v >$ for any $v \in Hom_{S_l}(U,V)$. Let v be the common eigenvector of $Z(l), X_{l+1}, \dots, X_{l+k}$, these commute and hence have a common eigenvector. Now $< Z(l)v, X_{l+1}v, \dots, X_{l+k}v >=< v >$. So $\dim Hom_{S_l}(U,V) = \dim < S_k v > \leq k!$. The second statement follows by putting $k = 2$..

Proposition 9.

Suppose $\alpha = (a_1, a_2, \dots, a_i, a_{i+1}, \dots, a_n) \in spec(n)$ then,

1. $a_i \neq a_{i+1}$
2. if $a_i = a_{i+1} \pm 1$ then $s_i v_\alpha = \pm v_\alpha$

3. if $a_i = a_{i+1} \pm 1$ then $\alpha' = (a_1, a_2, \dots, a_{i+1}, a_i, \dots, a_n) \in spec(n)$ and $\alpha' \sim \alpha$. Further, $v_{\alpha'} = \left(s_i - \frac{1}{a_{i+1}-a_i}\right) v_\alpha$.

Proof:

Let v_T be a GZ-basis element. Let $s_i v_T = \sum_{v_R \in GZ-basis} c_R v_R$, then by proposition 7.1 and the linear independence of the GZ-basis and second part of proposition 8, we get that either 1. $s_i v_T = c_T v_T$ or 2. $s_i v_T = c_T v_T + c_{T'} v_{T'}$. Then using proposition 7.2 we get 2 in case 1 and 3 in case 2 and 1 is automatically proved by elimination..

We will prove later using content vectors that all the $a_i's$ are integers and that there exist a sequence of admissible transformations taking v_T to $v_{T'}$ (if v_T and $v_{T'}$ are in the same irreducible representation) so that the action of S_n is by proposition 8.3 which consists of rational numbers so we get the result,

Proposition 10.

All irreducible representations of S_n are over $\mathbb{Q}$.

The Young's graph

In this section we will define the Young's graph and describe some of its basic properties. The correspondence between Young's tableaus and irreducible representations will become apparent.

The n^{th} level of the **Young graph** consists of partitions of n. There are edges only between two vertices on consecutive levels. A partition $\mu = (n-1) \vdash (\mu_1, \mu_2, \dots, \mu_l)$ is connected by an edge to partitions of the form $\mu' = n \vdash (\mu_1, \mu_2, \dots \mu_{i-1}, \mu_i + 1, \mu_{i+1}, \dots, \mu_l)$. We associate to these partitions empty Young tableaus and call these the Young frames.

For any partition $\lambda \vdash n$ we will look at the **Standard Young's Tableau** of shape λ. In these unlike the semistandard young's tableaus, all the entries are distinct. Call this $\boldsymbol{Tab_\lambda}$ and let $\boldsymbol{Tab_n} \coloneqq \cup_{\lambda \vdash n} Tab_\lambda$

For $\lambda = n \vdash (\lambda_1, \dots \lambda_k)$ denote by $\boldsymbol{T^\lambda} \in Tab_\lambda$ be the tableau containing $\{\lambda_1 + \dots + \lambda_{i-1} + 1, \lambda_1 + \dots + \lambda_{i-1} + 2, \dots, \lambda_1 + \dots + \lambda_{i-1} + \lambda_i\}$ in the i^{th} row.

For $\pi \in S_n$ we can define action of π on $T \in Tab_\lambda$ by making π act on each entry of T. We say π is **admissible** with respect to T if $\pi T \in Tab_\lambda$. Define $\boldsymbol{\pi_{T,\lambda}}$ to be the unique permutation such that $\pi_{T,\lambda} T = T^\lambda$.

Proposition 11.

The set Tab_n is in one to one correspondence with the set of all paths till the n^{th} level in the Young graph. In particular, the set Tab_λ is in one to one correspondence with the set of paths ending in λ.

Proof:

If $\mu = \mu_0 \to \mu_1 \to \cdots \to \mu_n$ is a path in the Young graph, one new box is added to the Young frame μ_n at the i^{th} level. Put i in this box. Thus we get a standard young tableau. It is easy to see that this is a bijection..

Proposition 12.

> $T \in Tab_\lambda$ then there is a sequence of admissible transpositions taking T to T_λ. Consequently, if $S, T \in Tab_\lambda$ then S can be obtained from T using an admissible permutation.

Proof:

The proof is by induction on n. Suppose it is true for n. Consider $n + 1$. Suppose $n + 1$ is not in the last box of the last row in T. Then swap $n + 1$ and n, then $n + 1$ $n - 1$ and so on till $n + 1$ reaches the last box of the last row. All these transpositions are admissible. Then by induction we can permute the rest of the tableau *admissibly* so to get T_λ..

The correspondence

Finally we define the content vectors for a Standard Young's tableau and give the one to one correspondence.

For $T \in Tab_n$ define

1. $\boldsymbol{i_T(i)} \coloneqq number\ of\ the\ row\ in\ which\ i\ appears\ in\ T$
2. $\boldsymbol{j_T(i)} \coloneqq number\ of\ the\ row\ in\ which\ i\ appears\ in\ T$
3. $\boldsymbol{C_T} := (i_T(1) - j_T(1), i_T(2) - j_T(2), \dots, i_T(n) - j_T(n))$

Define the set of content vectors $\boldsymbol{Cont_n}$ as the set of all vectors $\alpha = (a_1, a_2, \dots, a_n) \in \mathbb{Z}^n$ such that

1. $a_1 = 0$
2. $\{a_q + 1, a_q - 1\} \cap \{a_1, a_2, \dots, a_{q-1}\} \neq \{\}$ for $q > 1$
3. If $a_p = a_q$ for some $p < q$ then $\{a_q + 1, a_q - 1\} \subseteq \{a_{p+1}, \dots, a_{q-1}\}$

We define an **equivalence relation** on $Cont_n$ as $\alpha \approx \beta\ if\ \exists \pi \in S_n$ such that $\pi\alpha = \beta$.

Proposition 13.

> If $\alpha = (a_1, a_2, \dots, a_n) \in Cont_n$ then,
>
> 1. if $a_q > 0$ then $a_q - 1 \in \{a_1, a_2, \dots, a_{q-1}\}$ and if $a_q < 0$ then $a_q + 1 \in \{a_1, a_2, \dots, a_{q-1}\}$
> 2. if $a_p = a_q$ and $a_p \notin \{a_{p+1}, \dots, a_{q-1}\}$ then $\exists! s_-, s_+ \in \{p + 1, \dots, q - 1\}$ such that $a_{s_-} = a_p - 1$ and $a_{s_+} = a_p + 1$.

Proposition 14.

For $T \in Tab_n$ we have $C_T \in Cont_n$ and the map $T \to C_T$ is a bijection. If $\alpha, \beta \in Cont_n$ and $\alpha = C_T$ and $\beta = C_S$ then $\alpha \approx \beta$ if and only if T, S have the same shape.

Proposition 15.

$$spec(n) \subseteq Cont_n$$

Proof:

The proofs of the above three theorems are combinatorial, simple but very long. They can be found in the references. Note that proposition 15 completes the proof of proposition 10..

Proposition 16.

If $\alpha \in spec(n)$ and $\alpha \approx \beta, \beta \in Cont_n$ then $\beta \in spec(n)$ and $\alpha \sim \beta$.

Proof:

It suffices to show that if s_i is admissible in Tab_n then the corresponding s_i is admissible in $spec(n)$. In Tab_n, s_i swaps the elements i and $i+1$ so that the corresponding elements a_i and a_{i+1} of C_T are swapped, which is *admissible* in $spec(n)$. So it remains to show that s_i is admissible in Tab_n only if $a_i \neq a_{i+1} \pm 1$. If $a_i = a_{i+1} \pm 1$ then i and $i+1$ are in the adjacent anti-diagonals. In this case one can check that swapping them is not admissible..

Proposition 17.

$spec(n) = Cont_n$ and $\sim\, = \,\approx$.

Proof:

$\#\{spec(n)/\sim\} = \#\{S_n^-\} = \#\{Cont_n/\approx\}$ = number of partitions of n. And each equivalence class of $Cont_n$ is either completely contained inside an equivalence class of $spec(n)$ or does not intersect at all..

Putting in words this final theorem gives a correspondence between the GZ-basis and the standard Young's tableaus of size n.

Final remarks and references

Unlike with the RSK approach it is possible to generalize the above technique to that of groups which have inductive structures. Further, using skew hooks one can give a recursive formula for the characters of S_n^-.

1. Okounkov and Vershik, *A New Approach to Representation Theory of Symmetric Groups*
2. Ceccherini-Silberstein, Scarabotti, Tolli, *Representation theory of Symmetric Groups*