PROBLEM SET 08

PART 1 - FUNDAMENTAL THEOREM OF CALCULUS

Q.1. Compute $F'(x)$ for the following functions

(1) $\int\limits_a^{x^3} \sin^3 t\, dt$

(2) $\int\limits_x^b \dfrac{1}{1+t^2+\sin^2 t}\, dt$

(3) $\int\limits_a^b \dfrac{x}{1+t^2+\sin^2 t}\, dt$

(4) $\int\limits_a^x \left(\int\limits_b^y \dfrac{1}{1+t^2+\sin^2 t}\, dt \right) dy$ $*$

(5) $\int\limits_0^x \dfrac{1}{1+t^2}\, dt + \int\limits_0^{1/x} \dfrac{1}{1+t^2}\, dt$

(6) $\int\limits_{-\cos x}^{\sin x} \dfrac{1}{\sqrt{1-t^2}}\, dt$

Q.2. Find $(f^{-1})'(0)$ if

(1) $f(x) = \int\limits_0^x 1 + \sin(\sin t)\, dt$

(2) $f(x) = \int\limits_1^x \cos(\cos t)\, dt$

Q.3. Suppose f is differentiable with $f(0) = 0$ and $0 < f'(x) \le 1$. Show that for all $x \ge 0$

$$\int_0^x f^3 \le \left(\int_0^x f \right)^2$$

Q.4. (1) Find $F'(x)$ if $F(x) = \int\limits_0^x x.f(t)\, dt$ (Be careful: it's not $x.f(x)$).

(2) Prove that†

$$\int_0^x f(t)(x - t)\, dt = \int_0^x \left(\int_0^u f(t)\, dt \right) du$$

(3) Prove that

$$\int_0^x f(t)(x - t)^2\, dt = 2 \int_0^x \left(\int_0^{u_1} \left(\int_0^{u_2} f(t)\, dt \right) du_2 \right) du_1$$

Q.5. (1) Suppose $G'(x) = g(x)$ and $F'(x) = f(x)$. Prove that the function $y(x)$ satisfies the differential equation (*a separable differential equation*)

$$g(y).y' = f(x)$$

for all x in some interval, if and only if there is a number c such that

$$G(y) = F(x) + c$$

(2) 'Solve' the following differential equations

(a) $y' = \dfrac{1 + x^2}{1 + y}$

(b) $y' = \dfrac{-1}{1 + 5y^4}$

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*Hint: Don't forget the chain rule!

†Hint: Differentiate both sides with respect to x.

Part 2 - Improper Integrals

Q.6. The limit $\lim\limits_{x \to \infty} \int\limits_a^x f$, if it exists, is denoted by $\int\limits_a^\infty f$ and called an 'improper integral'.

Similarly for the improper integral $\int\limits_{-\infty}^a f$.

(1) Find $\int\limits_1^\infty t^r \, dt$, when $r < -1$.

(2) Using $\int\limits_1^a t^{-1} \, dt + \int\limits_1^b t^{-1} \, dt = \int\limits_1^{ab} t^{-1} \, dt$, show that $\int\limits_1^\infty t^{-1} \, dt$ does not exist. [‡]

Q.7. Assume the following the statement:

Suppose that $f(x) \geq 0$ for $x \geq 0$ and that $\int\limits_a^\infty f$ exists. If $0 \leq g(x) \leq f(x)$ for all $x \geq 0$ and g is integrable on the interval $[0, N]$ for all $N > 0$ then $\int\limits_0^\infty g$ exists.

- (Optional) Prove the above statement.

- For which of the following functions does the integral $\int\limits_0^\infty f$ exist?[§]

$$\text{(1)} \quad \frac{1}{1 + x^2} \qquad\qquad \text{(3)} \quad \frac{x}{1 + x^{3/2}}$$

$$\text{(2)} \quad \frac{1}{\sqrt{1 + x^3}} \qquad\qquad \text{(4)} \quad \frac{1}{x\sqrt{1 + x}}$$

Q.8. The improper integral $\int\limits_{-\infty}^\infty f$ is defined as $\int\limits_{-\infty}^0 f + \int\limits_0^\infty f$ if both the integrals exist.

(1) Show that $\int\limits_{-\infty}^\infty \frac{1}{1 + x^2} \, dx$ exists.

(2) Determine the limit $\lim\limits_{N \to \infty} \int\limits_{-N}^N x \, dx$.

(3) Show that $\int\limits_{-\infty}^\infty x \, dx$ does not exist.

Q.9. It is possible to have another kind of improper integral, one where the function itself is unbounded but the limits are finite.

For $-1 < r < 0$ draw the graph of x^r and determine $\lim\limits_{\epsilon \to 0^+} \int\limits_\epsilon^a x^r \, dx$.

(This limit is written as $\int\limits_0^a x^r \, dx$.)

[‡] Hint: What can you say about $\int\limits_n^{2n} t^{-1} \, dt$?

[§] Hint: You might have to break $[0, \infty)$ into multiple intervals and analyze each interval separately.