§ 10.2

Lem:
$$\begin{tikzcd} X \arrow[r, shift left, "\pi"] \arrow[r, shift right, "\pi'"'] \arrow[dr] & Y \arrow[d] \\ & Z \end{tikzcd}$$
then locus where π and π' agree is naturally a locally closed subscheme of X; and is closed if Y is separated $/Z$.

Proof: Define L to have the universal property

$$L(T) = \{ f \in X(T) \mid \pi f = \pi' f \} = X \times_Y X$$

$$\begin{tikzcd} L \arrow[r, hook] \arrow[d] & X \arrow[d] \\ Y \arrow[r, "\Delta"] & Y \times_Z Y \end{tikzcd}$$

Prop: $X \underset{\pi'}{\overset{\pi}{\rightrightarrows}} Y$ morphisms of $\bar{k}$-varieties such that $\pi(x) = \pi'(x)$ $\forall$ closed pts. $x \in X$ then $\pi = \pi'$.

Proof: Y separated $\Rightarrow$ locus L where $\pi = \pi'$ is closed in X.
By hypothesis L contains all closed points. $\xRightarrow[\text{closed pts are dense}]{k = \bar{k}}$ $L = X$ as sets.
Since X is a variety, its reduces $\Rightarrow L = X$ as schemes.

Th^m ("reduced to separated th")
$$\begin{tikzcd} U \arrow[r, shift left, "\pi"] \arrow[r, shift right, "\pi'"'] \arrow[dr] & Z \arrow[d] \\ & S \end{tikzcd}$$
agree on a dense open subset of U. If U is reduced & Z is separated then the two morphisms are equal.

Proof: Same as above.

§ 10.3 Proper morphism:

Def^n: Morphism $\pi : X \longrightarrow Y$ is proper if π is separed, finite type & universally closed:
$Z \times_Y X \longrightarrow Z$ closed map $\forall Z \to Y$.

When $Y = \operatorname{Spec} k$ we say X is complete$/k$.

eg: • Closed embeddings

Th^m: Finite morphisms are proper.

Prop: • Proper morphisms are stable under base change, composition, products.
• $$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[dr, "\tau"'] \arrow[dr, phantom, "\Omega", shift left=3] & Y \arrow[d, "\varsigma"] \\ & Z \end{tikzcd}$$
τ proper & ς separated $\Rightarrow$ π proper.

Th^m: Proj A schemes are proper / Spec A

$$\begin{tikzcd} X \text{ proj}/\operatorname{Spec} A \iff X \arrow[r, hook, dashed] \arrow[dr] & \mathbb{P}^n_A \arrow[d] \\ & \operatorname{Spec} A \end{tikzcd}$$

Proof: Suffices to show $\mathbb{P}^n_A \to \operatorname{Spec} A$ is proper.

We know $\mathbb{P}^n_A \to \operatorname{Spec} A$ is closed.

$$\Rightarrow \underbrace{\mathbb{P}^n_A \times_{\operatorname{Spec} A} \operatorname{Spec} B}_{= \mathbb{P}^n_B} \longrightarrow \operatorname{Spec} B \text{ is closed.}$$

$$\Rightarrow \mathbb{P}^n_A \times_{\operatorname{Spec} A} X \longrightarrow X \text{ is closed.}$$

Rem: Hard to find morphisms which are proper but not projective.

Prop: Suppose X connected, reduced, proper / Spec $\bar{k}$, then $\Gamma(X, \mathcal{O}_X) = \bar{k}$.

Proof: $\Gamma(X, \mathcal{O}_X) \cong \operatorname{Hom}_{k\text{-sch}}(X, \mathbb{A}^1_k)$

$f \longmapsto \pi_f$

$$\begin{tikzcd} X \arrow[r, dashed, "\pi_f"] \arrow[rr, dashed, bend left, "\pi_f'"] \arrow[dr] & \mathbb{A}^1_k \arrow[r, hook] \arrow[d] & \mathbb{P}^1_k \\ & \operatorname{Spec} k \end{tikzcd}$$

$\Rightarrow \pi_f'$ proper

$\Rightarrow \pi_f'$ has closed image $=$ finite collection of points

$\Rightarrow X$ connected $\Rightarrow$ image is a single closed point.

§ 12.7 Valuative Criteria:

Recall: A DVR - A is a Noetherian local domain in which maximal ideal is generated by 1 elem.

Spec A = {closed pt, generic point} because every ideal is 0 or power of maximal ideal.

Th^m: Y locally Noetherian, $f: X \to Y$ finite type. TFAE:

1) π seperated (resp. proper)

2) $\forall$ diagrams

$$\begin{tikzcd} \operatorname{Spec} \operatorname{Frac} A \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ \operatorname{Spec} A \arrow[r] & Y \end{tikzcd}$$

$\exists$ at most 1 $f: \operatorname{Spec} A \to X$

(for proper we want exactly one f)

Geometric Intuition:

Spec (DVR) is "germ of a smooth curve"

Spec (Frac DVR) is "punctured germ"

The above theorem is talking about sequential compactness.