Th^n (Chevalley)

$\pi: X \longrightarrow Y$ finite type, X, Y noetherian then U constructible $\subseteq X \Rightarrow \pi(U)$ constructible.

Proof:

Enough to show: $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$ irreducible

Lemma - Generic freeness:

$\exists$ open dense $U \subseteq Y$ such that

$$U \subseteq \pi(X) \quad \text{or} \quad U \cap \pi(X) = \phi$$

We induct on $\dim Y$.

If $\dim Y = 0$ or $\pi(X) = Y$ then we are done

Suppose $\pi(X) \neq Y$, then $\exists U$ open s.t. $U \subseteq \pi(X)$ or $U \cap \pi(X) = \phi$

Let $Y' = Y \setminus U = \operatorname{Spec} B/J$

Th^m: Fundamental th^m of elimination theory

$\pi : \mathbb{P}^n_A \longrightarrow \operatorname{Spec} A$ is closed

Proof: Let $Z \subseteq \mathbb{P}^n_A$ closed, $Z = V_{\mathbb{P}^n_A}(f_\alpha)$ $\quad f_\alpha$ homogenous of +ve degree

$\wp \in \operatorname{Spec} A$ $\qquad \wp \in \pi(Z) \iff \bar{f}_\alpha$ have a common 0 in $\operatorname{Proj} K(\wp)[x_0, \dots, x_n]$

$\uparrow$ reduction mod $\wp$

$\iff \bar{f}_\alpha$ have a common 0 in $\operatorname{Spec} K(\wp)[x_0 \dots x_n]$ different from the origin

$\iff V_{\mathbb{A}^{n+1}}(\bar{f}_\alpha) \nsubseteq V(x_0, \dots, x_n) \subseteq \operatorname{Spec} \mathbb{A}^{n+1}$

$\iff (x_0, \dots, x_n)^N \nsubseteq (\bar{f}_\alpha)$ for any N

$\iff S_N \nsubseteq (\bar{f}_\alpha) \qquad S_N = \{\text{homogenous poly of deg } N\}$

$\iff S_N \nsubseteq \sum_\alpha \bar{f}_\alpha S_{N - \deg f_\alpha}$

$\iff \bigoplus_\alpha S_{N-\deg f_\alpha} \longrightarrow S_N$ is not surjective

$\sum_\alpha x_\alpha \longmapsto \sum \bar{f}_\alpha x_\alpha$

$\iff$ for any $d \times d$ minor of the above linear transformation vanishes !

$\iff \wp \in V_A(\text{all } d \times d \text{ minors})$

which is a closed condition.

Ch 8: Closed embeddings

Def: $\pi : X \longrightarrow Y$ is a closed immersion / embedding & $\forall$ open $\operatorname{Spec} B \subseteq Y$ say with $\pi^{-1}(\operatorname{Spec} B) = \operatorname{Spec} A$ we have $B \twoheadrightarrow A$.

eg: $Y = \operatorname{Spec} B \Rightarrow$ all closed embeddings of form $\operatorname{Spec} B/I \hookrightarrow \operatorname{Spec} B$.

If $\pi : X \hookrightarrow Y$ closed subset natural inclusion, then X closed subscheme of Y.

Rem: Every closed subset of scheme admits some scheme structure, but usually many.

Easy: • π closed $\Rightarrow$ π finite

• closed stable under composition

• affine local on the target

$\pi : X \to Y$ closed embedding gives rise to an exact seq of sheaves

$$0 \to \mathcal{I}_{X/Y} \to \mathcal{O}_Y \to \pi_* \mathcal{O}_X \to 0$$

ideal sheaf (i.e. subsheaf of $\mathcal{O}_Y$ of ideals)

Can recover X from $\mathcal{I}_{X/Y}$: $\forall$ open $\operatorname{Spec} B \subseteq Y$

$$X \cap \operatorname{Spec} B = \operatorname{Spec}(B/\mathcal{I}(B))$$

Q. Does any ideal Sheaf $\mathcal{I} \subseteq \mathcal{O}_Y$ give closed subscheme? No

Prop: An ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_Y$ "comes from" closed embedding

$\Leftrightarrow$ $\forall$ open $\operatorname{Spec} B \subseteq Y$, $\mathcal{I}(B)_f \xrightarrow{\simeq} \mathcal{I}(B_f)$ $\forall f \in B$.

Proof: $\Rightarrow$ $0 \to \mathcal{I} \to \mathcal{O}_Y \to \pi_* \mathcal{O}_X \to 0$

Restricting to $\operatorname{Spec} A$: $0 \to \mathcal{I} \to B \to A \to 0$

Further rest. to B_f $0 \to \mathcal{I}(B_f) \to B_f \to A_f \to 0$

$$\Rightarrow \mathcal{I}(B_f) \cong \mathcal{I}(B)_f$$

$\Leftarrow$ $\forall$ open affine $\operatorname{Spec} B \subseteq Y$ define $X_{\operatorname{Spec} B} := \operatorname{Spec} B/\mathcal{I}(B) \xrightarrow{\text{closed}} \operatorname{Spec} B$

Must show $X_{\operatorname{Spec} B}$'s glue

$$X_{\operatorname{Spec} B} \cap X_{\operatorname{Spec} B'} = \bigcup_i \operatorname{Spec} A_i$$

Simultaneously distinguished in $\operatorname{Spec} B$, $\operatorname{Spec} B'$.

Say $A_i = B_f = B'_{f'}$

$$0 \to \mathcal{I}(B) \to B \to B/\mathcal{I}(B) \to 0$$

$$0 \to \mathcal{I}(A_i) \to A_i \to A_i/\mathcal{I}(A_i) \to 0$$ because of our assumption

$$0 \to \mathcal{I}(B') \to B' \to B'/\mathcal{I}(B') \to 0$$ And hence we have patching

Rem: • $\mathcal{I}$ as in the proposition quasi-coherent.

• $\left\{ \begin{matrix} \text{q. coherent ideal} \\ \text{sheaves } \mathcal{I} \subseteq \mathcal{O}_Y \end{matrix} \right\} \xleftrightarrow{1-1} \left\{ \begin{matrix} \text{closed subscheme} \\ \text{of } Y \end{matrix} \right\}$