1.4.4)

$$0 \to C_{i+1} \underset{s_i}{\overset{d_{i+1}}{\rightleftarrows}} C_i \underset{s_{i-1}}{\overset{d_i}{\rightleftarrows}} C_{i-1} \to$$

$$d_i s_{i-1} d_i = d_i$$

$$0 \to \ker d_i \underset{s_i}{\rightleftarrows} C_i \underset{s_{i-1}}{\overset{d_i}{\rightleftarrows}} \operatorname{im} d_i \to 0$$

$$\to H_{i+1}(C.) \xrightarrow{0} H_i(C.) \xrightarrow{0} H_{i-1}(C.)$$

$$C_i \cong \ker d_i \oplus \operatorname{im} d_i$$

$$H_i(C.) \cong \ker d_i / \operatorname{im} d_{i+1}$$

Let $\pi : C_i \longrightarrow \ker d_i \longrightarrow 0$

$$\begin{array}{ccccc}
\to C_{i+1} & \longrightarrow & C_i & \xrightarrow{d_i} & C_{i-1} \to \\
\downarrow \pi & & \downarrow \pi & & \downarrow \pi \\
\ker d_{i+1} & & \ker d_i & & \ker d_{i-1} \\
\downarrow & & \downarrow & & \downarrow \\
H_{i-2} & \xrightarrow{0} & H_i & \xrightarrow{0} & H_{i-1}
\end{array}$$

Next need a map from $H_i \xrightarrow{?} C_i$

$$\begin{array}{ccc}
H_i & \xrightarrow{?} & C_i \\
\uparrow & \nearrow & \\
\ker d_i & &
\end{array}$$

we need $\ker d_i \cong H_i \oplus \operatorname{im} d_{i+1}$.

Look at $d_{i+1} \cdot s_i(\ker d_i)$: This $\subseteq \operatorname{im} d_{i+1}$ and if $x \in \operatorname{im} d_{i+1}$

identity on $\operatorname{im} d_i$

say $x = d_{i+1} y$

$d_{i+1} s_i d_{i+1} y = d_{i+1} y = x$

So $\ker d_i \xrightarrow{d_{i+1} s_i} \operatorname{im} d_{i+1} \to 0$

So split $\Rightarrow$ $\ker d_i \cong H_i \oplus \operatorname{im} d_{i+1}$

Using $C_i \cong \ker d_i \oplus \operatorname{im} d_i$

$\cong H_i \oplus \operatorname{im} d_{i+1} \oplus \operatorname{im} d_i$

inclusion of H_i in C_i gives the homotopy inverse.

For converse, take $C.$ as

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

$$H.(C.): \quad 0 \to 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0$$

we do have a trivial homotopy equivalence

But $C.$ does not split.

1.a.5) K - category of chain complexes but maps are equivalence classes upto chain homotopy

why is K not abelian?

a) $f \sim g$, $g \sim h$ Then $\exists\, s_1, s_2$ such that

$f - g = s_1 d + d s_1 \qquad g - h = s_2 d + d s_2$

$f - h = (s_1 + s_2) d + d(s_1 + s_2) \Rightarrow f \sim h$

b) $f, g : C \to D \qquad u : B \to C \qquad v : \cancel{C} D \to E$

$f - g = sd + ds$

$vfu - vgu = vsdu - \cancel{vgu}\, vdsu$

$= vsud - dvsu$

$\Rightarrow vfu \sim vgu$

So composition is well defined in K.

c) $f_0, f_1, g_0, g_1 : C \to D$

$f_0 \sim g_0, \quad f_1 \sim g_1$

$f_0 - g_0 = s_0 d - d s_0 \qquad f_1 - g_1 = s_1 d - d s_1$

$f_0 + f_1 - g_0 - g_1 = (s_0 + s_1) d - d(s_0 + s_1)$

So Ab-cat. And 0-object is the 0 chain.

Next ~~Pre~~additivity need products. Take $C \times D$ in Ch and look at quotient.

$\mathrm{Ch} \longrightarrow K$

$f + g \longmapsto [f + g]$

So Abelian functor

~~There are objects $A \in K$ s.t. Hom(A,A) = 0~~

~~$A = 0 \to \mathbb{Z} \xrightarrow{1} \mathbb{Z} \to 0$ and A is split~~

~~exact so identity is null homotopic~~ ~~So Hom(A,A) = 0~~

~~Take $B = 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to 0$~~

~~and map $f : B \to A$.~~

$$\begin{array}{ccccccc} 0 & \to & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \to & 0 \\ & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 2} & & \\ 0 & \to & \mathbb{Z} & \xrightarrow{1} & \mathbb{Z} & \to & 0 \end{array}$$

Mapping cone:

$f: B_\bullet \to C_\bullet$ $\quad$ $\text{cone}(f)_n := B_{n-1} \oplus C_n$

$d(b,c) := (-d(b), dc - fb)$

$d^2(b,c) = d(-db, dc-fb) = (0, -dfb + fdb) = (0,0).$

One can think of d as a matrix $\begin{bmatrix} -d_B & \\ f & +d_C \end{bmatrix}$

contra $f: B^\bullet \to C^\bullet$ $\quad$ $\text{cone}(f)_n := B^{n+1} \oplus C^n$

$d(b,c) := (-db, dc - fb)$

$\ldots id\, C_\bullet \xrightarrow{id} C_\bullet$ $\quad$ $\text{cone}(C_\bullet) := \text{cone}(id)$

n^{th} term $= C_{n-1} \oplus C_n$

$d(c_1, c_2) = (-dc_1, dc_2 - c_1)$

$\to C_{n+1} \oplus C_{n+2} \longrightarrow C_n \oplus C_{n+1} \longrightarrow C_{n-1} \oplus C_n \longrightarrow$

$\ker d = \{(c_1, c_2) \mid dc_1 = 0,\ c_1 = dc_2\} = (dc_2, c_2) \cong C_n$

So $C_{n-1} \oplus C_n / \ker d \cong C_{n-1} \Rightarrow$ Split exact?

Reverse map should take C_{n-1} to C_{n-1}

$s(c_1, c_2) = (-c_2, 0)$

$dsd(c_1, c_2) = ds(-dc_1, dc_2 - c_1)$

$= d(c_1 - dc_2, 0)$

$= (-dc_1, dc_2 - c_1) = d(c_1, dc_2)$

- $0 \to C_\bullet \to \text{cone}(f)_\bullet \longrightarrow B[-1] \to 0$

  $c \mapsto (0, c)$

  $(b, c) \longmapsto -b$ $\quad$ why $-$ve?

Long exact seq:

$$\begin{tikzcd}
\to H_n(C_\bullet) \arrow[r] & H_n \text{ cone} \arrow[r] & H_{n-1}(B) \arrow[d, "\partial"] & & \\
& & H_{n-1}(C_\bullet) \arrow[r] & H_{n-1} \text{ cone} \arrow[r] & H_{n-2} B
\end{tikzcd}$$

∂:

$$\begin{tikzcd}
& (-b, 0) \arrow[r] \arrow[d] & b \arrow[d, "d"] \\
fb \arrow[r] & (0, fb) & 0
\end{tikzcd}$$

$\partial[b] = [fb]$

$\partial = f_*$

- f - quasi isomorphis $\Leftrightarrow$ ∂ iso morphism

  $\Leftrightarrow$ H_n cone $= 0$

$Z_n(\text{cone } f) = \{(b,c) \mid db = 0,\ dc = fb\}$, $B_n = \{(db, dc - fb)\}$

$Z_n/B_n = \{(b,c)\}/\sim \qquad b_1c_1 \sim b_2c_2 \quad$ if $\quad b_1 - b_2 = db'$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad c_1 - c_2 = dc' - fb'$

- Topology:

$i : A \to X$ inclusion

cone $(i) = X/A$?

$(B_{n-1} \oplus C_n) \qquad A_{n-1} \oplus X_n$

$$\begin{tikzcd} \to H_n(X) \arrow[r] & H_n(X/A) \arrow[r] & H_{n-1}(A) \arrow[d, "i_*"] & & \\ & & H_{n-1}(X) \arrow[r] & H_{n-1}(X/A) \arrow[r] & H_{n-2}(A) \end{tikzcd}$$

Certainly $(X/A)_* \not\cong$ cone (i).

But we have map between long exact sequences of X/A and cone i. which by five lemma will give that $H_n(X/A) \cong H_n(\text{cone } i)$

$$\begin{tikzcd} \to H_n(A) \arrow[r, "i_*"] & H_n(X) \arrow[r] & H_n(X/A) \arrow[r] & H_{n-1}(A) \arrow[r, "i_*"] & H_{n-1}(X) \\ \to H_n(A) \arrow[r, "i_*"] & H_n(X) \arrow[r] & H_n(\text{cone}) \arrow[u] \arrow[r] & H_{n-1}(A) \arrow[r, "i_*"] & H_{n-1}(X) \end{tikzcd}$$

$(x) \mapsto [x] \qquad [x] \mapsto [\partial x]$

(x, a)
$\uparrow$
(x, a)

$[x] \mapsto [x, 0]$

map in the middle is $(x, a) \mapsto x$, $a = \partial x$

commutativity is because $(x,a) \in Z \Rightarrow a = \partial x$.

Mapping cylinder:

$f : A \to C$

$cyl(f) := B_n \oplus B_{n-1} \oplus C_n$

$d(b, b', c) := (db + b', -db', dc - fb')$

$d^2(b, b', c) = d(db + b', -db', dc - fb')$
$= (d^2b + db' - db', 0, 0) = (0,0,0)$

- $id : C_\cdot \to C_\cdot \qquad cyl(C_\cdot) := cyl(id)$

n^{th} term $= C_n \oplus C_{n-1} \oplus C_n$

• Suppose $f,g: C_\cdot \to D_\cdot$. $f\sim g$ $\quad f-g = ds+sd$

$$\begin{array}{ccc}
C_n\oplus C_{n-1}\oplus C_n & \xrightarrow{\quad} & C_{n-1}\oplus C_{n-2}\oplus C_{n-1} \\
\downarrow{\scriptstyle (f,s,g)} & & \downarrow \\
D_n & \xrightarrow{\quad} & D_{n-1}
\end{array}$$

??

$(b,b',c) \rightsquigarrow (db+b', -db', dc-fb)$

$(fb, sb', gc) \rightsquigarrow (dfb+sb', -dsb', dgc-fb)$

$$(f,s,g)\,d(c_1,c_2,c_3) = (f,s,g)(dc_1+c_2, -dc_2, dc_3-c_2) = (fdc_1+fc_2,\ -sdc_2,\ gdc_3-gc_2) \quad — *$$

$$d(fsg)(c_1,c_2,c_3) = d(fc_1, sc_2, gc_3) = (dfc_1+sc_2,\ -dsc_2,\ dgc_3-sc_2) \quad — **$$

Put $f = g+sd+ds$ in $*$

$$(gdc_1+\cancel{sddc_1}+\underset{dsd}{\cancel{dsdc_1}}+gc_2+sdc_2+\cancel{dsc_2},\ -sdc_2,\ gdc_3-gc_2)$$

$$\begin{array}{ccc}
C_n\oplus C_{n-1}\oplus C_n & \longrightarrow & C_{n-1}\oplus C_{n-2}\oplus C_{n-1} \\
\downarrow & & \downarrow \\
D_n & \longrightarrow & D_{n-1}
\end{array}$$

$(c_1,c_2,c_3) \longrightarrow (dc_1+c_2, -dc_2, dc_3-c_2) \longrightarrow fdc_1+fc_2-sdc_2+gdc_3-gc_2$

$\downarrow$

$fc_1+sc_2+gc_3 \longrightarrow dfc_1+dsc_2+dgc_3$

Commutativity $\Leftrightarrow$ $dsc_2 = -sdc_2+fc_2-gc_2$

$(ds+sd)c_2 = (f-g)c_2$

$\Leftrightarrow$ $f\sim g$. !!!

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• $0 \to C_\cdot \longrightarrow cyl(f) \longrightarrow cone(-id_B) \longrightarrow 0$

$c \longmapsto (0,0,c)$

$(b,b',c) \longmapsto (b,b')$

cone $(-id)$ is acyclic $\Rightarrow$ $C_\cdot \xrightarrow[\alpha]{} cyl(f)$ quasi isomorphism

1.5.4) $\beta: cyl(f)_n \longrightarrow C_n$

$(b,b',c) \longmapsto f(b)+c$

$\beta\alpha(c) = \beta(0,0,c) = c$

$\alpha\beta(b,b',c) = \alpha(f(b)+c) = (0,0,f(b)+c)$

$s(b,b',c) = (0,b,0)$

$ds+sd\,(b,b',c) = d(0,b,0) + s(db+b', -db', dc-fb')$

$= (b,-db,-fb) + (0, db+b', 0) = (b,b',-fb)$

$\Rightarrow$ $ds + sd = 1 - \alpha\beta$

$\Rightarrow$ C homotopy equivalent to cyl(f). ∎

Scary!

23/01/13

Ex: If $\mathcal{A}$ is an abelian category, what are projective objects in the category of chain complexes of $\mathcal{A}$?

F right exact: $\mathcal{A} \longrightarrow \mathcal{B}$. $L^i F :=$ i^{th} left derived functor.

- $L_0 F = F$
- If P projective $L_i F(P) = 0$ $\forall i > 1$

Def^n: Homological δ-functors: $\mathcal{A} \to \mathcal{B}$ between Abelian categories

- additive functors $T_0, T_1, \ldots$ $(T_n = 0 \; \forall n < 0)$
- for each $0 \to A \to B \to C \to 0$ exact we get a long exact seq.

$$\to T_n A \to T_n B \to T_n C \xrightarrow{\delta} T_{n-1} A \to T_{n-1} B \to \cdots \to T_0 B \to T_0 C \to 0$$

δ are ~~functorial~~ natural maps $T_n(C) \longrightarrow T_{n-1}(A)$ i.e.

given
$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d, "g"'] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$
we get
$$\begin{tikzcd} T_n(C') \arrow[r, "\delta"] \arrow[d, "T_n f"'] & T_{n-1}(A') \arrow[d, "T_{n-1} g"] \\ T_n(C) \arrow[r, "\delta"'] & T_{n-1}(A) \end{tikzcd}$$

δ_n natural transformation between functors $T_n(C), T_{n-1}(A)$.

Ex: i) A R-module, $r \in R$ R-commutative

$T_0 A = A/rA$

$T_1 A = {}_r A = \{a \in A \mid rA = 0\}$

δ? $0 \to A \to B \to C \to 0$, $C \cong B/A$

$\delta(T_1 C) \longrightarrow T_0 A = A/rA$

$T_1 C = {}_r C = {}_r B/A = \{b \in B \mid rb \in A\}$

We use snake lemma to

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "r"] & B \arrow[r] \arrow[d, "r"] & C \arrow[r] \arrow[d, "r"] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

Do these form a universal δ-functor?

gives

$$0 \to {}_rA \to {}_rB \to {}_rC \longrightarrow A/rA \longrightarrow B/rB \longrightarrow C/rC \longrightarrow 0$$

$$[b] \longmapsto [rb]$$

Morphism of δ-functors:

$S \to T$ mean natural transforms $S_n \longrightarrow T_n$ compatible with δ.

Universal δ-functor:

- $T = \{T_n\}$ universal if given any δ-functor $S = \{S_n\}$ and a natural transformation $f_0 : S_0 \to T_0$, $\exists!$ morphism of δ functor $f_n : S_n \to T_n$ extending f_0.

Proof that $\{L^iF\}_i$ are δ-functors:

First we need get projective resolutions:

Horse shoe lemma:

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ \arrow[r] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\ \arrow[r] & P_1 \oplus Q_1 \arrow[r] \arrow[d] & P_0 \oplus Q_0 \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\ \arrow[r] & Q_1 \arrow[r] \arrow[d] & Q_0 \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

$$P_i \oplus Q_i \longrightarrow P_{i-1} \oplus Q_{i-1}$$

$$P_i \longrightarrow P_{i-1} \longrightarrow P_{i-1} \oplus Q_{i-1}$$

$$\begin{tikzcd} Q_i \arrow[r] & Q_{i-1} \\ & P_{i-1} \oplus Q_{i-1} \arrow[u] \end{tikzcd} \quad \rightsquigarrow \quad Q_i \longrightarrow P_{i-1} \oplus Q_{i-1}$$

Apply F:

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & & \\ \arrow[r] & FP_1 \arrow[r] \arrow[d] & FP_0 \arrow[r] \arrow[d] & FA \arrow[r] \arrow[d] & 0 \\ \arrow[r] & FP_1 \oplus FQ_1 \arrow[r] \arrow[d] & FP_0 \oplus FQ_0 \arrow[r] \arrow[d] & FB \arrow[r] \arrow[d] & 0 \\ \arrow[r] & FQ_1 \arrow[r] \arrow[d] & FQ_0 \arrow[r] \arrow[d] & FC \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

Salamander

(Jonathan Wise)

Neglect last column and apply H_i :

$\cancel{\to L_i f A \to L_t}$

$$\begin{tikzcd} \arrow[r] & H_i FP \arrow[r] & H_i FP \oplus FQ \arrow[r] & H_i FQ \arrow[d, "\delta"] \\ & & & H_{i-1} FP \arrow[r] & {} \end{tikzcd}$$

δ:

$$\begin{tikzcd} & (0, x) \arrow[r] \arrow[d] & x \in FQ_i \arrow[d] \\ ?? \arrow[r] & ?? & 0 \in FQ_{i-1} \end{tikzcd}$$

~~Naturality~~ of δ:
Well defined :

Horseshoe Lemma:

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \arrow[d] \\ 0 & A' \arrow[l] \arrow[d] & P_0' \arrow[l, two heads] \arrow[r, hook] \arrow[d] & \ker \arrow[d] & P_1' \arrow[l, two heads] \arrow[d] & \arrow[l] \\ 0 & A \arrow[l] \arrow[d] & P_0' \oplus P_0'' \arrow[l, two heads] \arrow[r, hook] \arrow[d] & \ker \arrow[d] & P_1' \oplus P_1'' \arrow[l, two heads] \arrow[d] & \arrow[l] \\ 0 & A'' \arrow[l] \arrow[d] & P_0'' \arrow[l, two heads] \arrow[r, hook] \arrow[d] & \ker \arrow[d] & P_1'' \arrow[l, two heads] \arrow[d] & \arrow[l] \\ & 0 & 0 & 0 & 0 \end{tikzcd}$$

$$P_i' \oplus P_i'' \longrightarrow P_{i-1}' \oplus P_{i-1}'' \qquad d: \begin{bmatrix} d' & * \\ 0 & d'' \end{bmatrix}$$

Apply F :

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ 0 & FA' \arrow[l] \arrow[d] & FP_0' \arrow[l] \arrow[d] & FP_1' \arrow[l] \arrow[d] & \arrow[l] \\ 0 & FA \arrow[l] \arrow[d] & FP_0' \oplus FP_0'' \arrow[l] \arrow[d] & FP_1' \oplus FP_1'' \arrow[l] \arrow[d] & \arrow[l] \\ 0 & FA'' \arrow[l] \arrow[d] & FP_0'' \arrow[l] \arrow[d] & FP_1'' \arrow[l] \arrow[d] & \arrow[l] \\ & 0 & 0 & 0 & \end{tikzcd}$$

Neglect first column, apply homology functors to get the desired $L_i F$'s.

• naturality of δ :

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r] \arrow[d, "f'"'] & A \arrow[r] \arrow[d, "f"'] & A'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & P' \arrow[r] \arrow[d, "\text{lift of } f'"'] & P'\oplus P'' \arrow[r] \arrow[d, dashed, "?"] & P'' \arrow[r] \arrow[d, "\text{lift of } f''"] & 0 \\
0 \arrow[r] & Q' \arrow[r] & Q'\oplus Q'' \arrow[r] & Q'' \arrow[r] & 0
\end{tikzcd}$$

Four squares join at $(P'\oplus P'')_i$

$$\begin{tikzcd}
0 \arrow[r] & P_i' \arrow[r] \arrow[d, "f'"'] & P_i'\oplus P_i'' \arrow[r] & P_i'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & Q_i' \arrow[r] & Q_i'\oplus Q_i'' \arrow[r] & Q_i'' \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
P'_{i-1}\oplus P''_{i-1} \arrow[rr] \arrow[d] & & Q'_{i-1}\oplus Q''_{i-1} \\
P_i'\oplus P_i'' & & Q_i'\oplus Q_i'' \arrow[u] \\
P'_{i+1}\oplus P''_{i+1} \arrow[u] & & Q'_{i+1}\oplus Q''_{i+1} \arrow[u]
\end{tikzcd}$$

on P_i' we have no choice has to be f'

on P_i'' we need a lift of

$$\begin{tikzcd}
P_i'' \arrow[r, "id"'] & P_i'' \arrow[d, "f''"] \\
Q_i'\oplus Q_i'' \arrow[r] & Q_i''
\end{tikzcd}
\quad \text{as well as} \quad
\begin{tikzcd}
P_i'' \arrow[r] & P'_{i-1}\oplus P''_{i-1} \arrow[d] \\
Q_i'\oplus Q_i'' \arrow[r] & Q'_{i-1}\oplus Q''_{i-1}
\end{tikzcd}$$

$$\begin{tikzcd}
& P'_{i-1}\oplus P''_{i-1} \arrow[rr] & & P''_{i-1} \arrow[dd] \\
P_i'\oplus P_i'' \arrow[ur] \arrow[rr] \arrow[dd, "?"'] & & P_i'' \arrow[ur] \arrow[dd] & \\
& Q'_{i-1}\oplus Q''_{i-1} & & Q''_{i-1} \\
Q_i'\oplus Q_i'' \arrow[rr] & & Q_i'' \arrow[ur] &
\end{tikzcd}$$

How about lifting

$$\begin{tikzcd}
P_i'' \arrow[r] & P_i'' \arrow[r] & P''_{i-1} \arrow[d, "f''_\bullet"] \\
Q_i'\oplus Q_i' \arrow[r] & Q_i'' \arrow[r] & Q''_{i-1}
\end{tikzcd}$$

same thing.

Injectives:

- There "enough" injectives in $\mathbb{Z}$-mod

$$\mathbb{Z}/n\mathbb{Z} \hookrightarrow \mathbb{Q}/n\mathbb{Z}$$

is $\mathbb{Q}/n\mathbb{Z}$ injective?
look like $x/m + n\mathbb{Z}$ is in $\mathbb{Q}/n\mathbb{Z}$.

Using this finitely generated case is taken care of.
what can we do for arbitrary abelian groups?

$$A \longmapsto \prod_{a \in A} (\mathbb{Q}/\mathrm{ann}\, a\mathbb{Z})$$
$$a \longmapsto i_a$$

No need not work. $a \mapsto i_a$, $2a \mapsto i_{2a}$ but $i_{2a} \neq 2 i_a$

Very different strategy:

$$S = \mathrm{Hom}(X, \mathbb{Q}/\mathbb{Z})$$
$$I(A) = \prod_{\alpha \in S} (\mathbb{Q}/\mathbb{Z})_\alpha$$

Then,

$$A \hookrightarrow I(A)$$
$$a \longmapsto (\alpha(a))_\alpha$$

- $I(A)$ injective?
  Product of injectives is injective

- map injective?
  $\forall a \in A \quad \exists \alpha \in S . \quad \alpha(a) \neq 0$

$$\begin{tikzcd} & \mathbb{Q}/\mathbb{Z} \\ \mathbb{Z}a \arrow[ur] \arrow[r, hook] & A \arrow[u, "?"'] \end{tikzcd}$$

send $a \mapsto \frac{1}{\text{~~torsion~~ order } A}$

Now we intend to extend this to arbitrary rings.
For this we ~~requ~~ will use adjoint functors.

• Adjoint functors:

$$A \underset{R}{\overset{L}{\rightleftarrows}} B$$

L, R adjoint is $\forall$ A in $\mathcal{A}$, B in $\mathcal{B}$

$$\mathrm{Hom}(LA, B) \xrightarrow{\simeq} \mathrm{Hom}(A, RB)$$

s.t. $A \xrightarrow{f} A' \rightsquigarrow$

$$\begin{tikzcd}
\mathrm{Hom}(LA,B) \arrow[d, "S"] & \mathrm{Hom}(LA',B) \arrow[l, "Lf"'] \arrow[d, "S"] \\
\mathrm{Hom}(A,RB) & \mathrm{Hom}(A',RB) \arrow[l, "f"]
\end{tikzcd}$$

$B \xrightarrow{g} B' \rightsquigarrow$

$$\begin{tikzcd}
\mathrm{Hom}(LA,B) \arrow[r, "g"] \arrow[d, "S"] & \mathrm{Hom}(LA,B') \arrow[d, "S"] \\
\mathrm{Hom}(A,RB) \arrow[r, "g"] & \mathrm{Hom}(A,RB')
\end{tikzcd}$$

eg: $R\text{-mod} \underset{\text{Free}}{\overset{F}{\rightleftarrows}} \text{Set}$ F - forgetful functor

$$\mathrm{Hom}_{\text{Set}}(S, F(M)) \cong \mathrm{Hom}_{R\text{-mod}}(\text{Free}(S), M)$$

free - left F - right

Additionally we also require natural transformations:

a: $L \cdot R \longrightarrow 1_B$ b: $1_A \longrightarrow RL$

s.t.

following morphisms are identity morphisms

$$L(X) \xrightarrow{L \circ b(X)} LRL(X) \xrightarrow{a \circ L(X)} L(X)$$

$$R(Y) \xrightarrow{b \circ L(Y)} RLR(Y) \xrightarrow{R \cdot a(Y)} R(Y)$$

So for the case (Free, Forget).

a: $\text{Free} \circ \text{Forget}(M) \longrightarrow 1_{R\text{-mod}}(M)$

$e_x \longmapsto x$

~~b: Forget Free (S) → 1_{Set} (S)~~

b: $1_{\text{Set}}(S) \longrightarrow \text{Forget} \cdot \text{Free}(S)$

$x \longmapsto 1 \cdot e_x$.

Dimca

Ex 1.1.7:

~~$a: F\circ G \longrightarrow 1$ isomorphism~~

$a: \mathcal{L}\mathcal{R} \longrightarrow 1$ isomorphism

i.e. $\exists\, c: 1 \longrightarrow \mathcal{L}\mathcal{R}$ inverse of a

Result:

a isomorphism $\Leftrightarrow$ $\mathcal{R}$ fully faithful

b isomorphism $\Leftrightarrow$ $\mathcal{L}$ fully faithful.

Mapping cones return

$$X^\cdot \xrightarrow{f} Y^\cdot \quad \rightsquigarrow \quad \mathrm{Cone}(f)^n = Y^n \oplus X^{n+1} \quad (= Y^\cdot \oplus X^\cdot[1])$$

$$d(y,x) = (dy + fx, -dx)$$

a map between maps, gives rise to a map between complexes

$$\begin{array}{ccc} X^\cdot & \xrightarrow{f} & Y^\cdot \\ \alpha\downarrow & & \downarrow\beta \\ X_1^\cdot & \xrightarrow[g]{} & Y_1^\cdot \end{array} \quad \rightsquigarrow \quad \mathrm{cone}(f) \xrightarrow{(\alpha,\beta)} \mathrm{cone}(g)$$

- A mapping cone gives rise to

$$T_f: \; X^\cdot \xrightarrow{f} Y^\cdot \hookrightarrow \mathrm{cone}\, f \twoheadrightarrow X^\cdot[1]$$

famously written as

$$T_f: \quad \begin{tikzcd} X^\cdot \arrow[r, "f"] & Y^\cdot \arrow[dl] \\ \mathrm{cone}\, f \arrow[u, "{[1]}"] & \end{tikzcd}$$

Prop: Dimca 1.1.20

i) composition of any two consecutive terms is null homotopic

- $$X^\cdot \xrightarrow{f} Y^\cdot \hookrightarrow \mathrm{cone}\, f^\cdot$$

$$x \longmapsto fx \longmapsto (fx, 0)$$

Give the map

$$s: X^n \longrightarrow \mathrm{cone}\, f^{n-1}$$

which is inclusion in the $X^{\cdot}$ component

~~Then~~ $\cancel{(sd+ds)}$ $\cancel{sd(x) = s\,dx = (0, dx)}$

$\cancel{ds(x) = d(0,x) = (fx, -dx)}$

$$\begin{tikzcd}
X^{n-1} \arrow[r] \arrow[d, "{f,0}"] & X^{n} \arrow[r] \arrow[dl, "s"'] \arrow[d, "{f,0}"] & X^{n+1} \arrow[dl, "s"'] \arrow[d, "{f,0}"] \\
Y^{n-1}\oplus X^{n} \arrow[r] & Y^{n}\oplus X^{n+1} \arrow[r] & Y^{n+1}\oplus X^{n+2}
\end{tikzcd}$$

Tracing:

$$\begin{tikzcd}
& x \arrow[rr, "d"] \arrow[dl, "s"'] & & dx \arrow[dl, "s"] \\
(0,x) \arrow[r, "d"'] & (fx,-dx) & (0,dx) &
\end{tikzcd}$$

$$sd+ds\,(x) = (0,dx) + (fx,-dx) = (fx,0)$$

- $$\begin{array}{ccccc} Y^{\cdot} & \longrightarrow & \operatorname{cone} f^{\cdot} & \longrightarrow & X^{\cdot}[1] \\ y & \longmapsto & (y,0) & \longmapsto & 0 \end{array}$$

- $$\begin{array}{ccccc} \operatorname{cone} f^{\cdot} & \longrightarrow & X^{\cdot}[1] & \longrightarrow & Y^{\cdot}[1] \\ (y,x) & \longmapsto & x & \longmapsto & fx \end{array}$$

$$\begin{tikzcd}
Y^{n-1}\oplus X^{n} \arrow[r] & Y^{n}\oplus X^{n+1} \arrow[r] \arrow[dl, "{(-1)^n,0}"'] \arrow[d, "{0,f}"] & Y^{n+1}\oplus X^{n+2} \arrow[dl, "{(-1)^{n+1},0}"] \\
Y^{n} \arrow[r] & Y^{n+1} \arrow[r] & Y^{n+2}
\end{tikzcd}$$

Tracing:

$$\begin{tikzcd}
& y,x \arrow[rr] \arrow[dl] \arrow[d, squiggly] & & (dy+fx,-dx) \arrow[dl] \\
(-1)^n y \arrow[r] & (-1)^n dy & (-1)^{n+1} dy + (-1)^{n+1} fx &
\end{tikzcd}$$

$$(sd+ds)\,(y,x) = (-1)^{n+1} fx$$

There is an issue of sign.

iii) $u, v : X^\cdot \longrightarrow Y^\cdot$ $\quad u \sim v \quad$ i.e. $\exists s : X^\cdot \longrightarrow Y^\cdot[-1]$

s.t. $ds + sd = u - v$

$\cancel{\phi =}$

need $\phi : \text{cone } u^\cdot \longrightarrow \text{cone } v^\cdot$ ~~e~~ isomorphism s.t.

$$\begin{tikzcd}
 & \text{cone } u^\cdot \arrow[dd, "\phi"] \arrow[dr] & \\
Y^\cdot \arrow[ur] \arrow[dr] & & X^\cdot[1] \\
 & \text{cone } v^\cdot \arrow[ur] &
\end{tikzcd}$$

on $Y^\cdot$ component of cone u we do not have a choice on $X^\cdot$? Trouble is with ϕ being chain map

$\phi(0, x) = (\phi'(x), x)$.

what is ϕ'? need $d\phi(\underset{y,x}{\cancel{\ \ }}) = \phi d(y,x)$

$$\begin{aligned}
LHS &= d(y + \phi'(x), x) \\
&= (dy + d\phi'(x) + vx, -dx) \\
RHS &= \phi(dy + ux, -dx) \\
&= (dy + ux - \phi'(dx), -dx).
\end{aligned}$$

So we require,

$$d\phi' x + vx = ux - \phi' dx$$

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Take $\phi' = s$.

1.1.23)

$$0 \to X^\cdot \xrightarrow{u} Y^\cdot \xrightarrow{v} Z^\cdot \to 0$$

i) need $m : Y^\cdot \xrightarrow{\sim} Z^\cdot$ s.t.

$$\begin{tikzcd}
C_u \arrow[rr, "m", "\sim"'] & & Z^\cdot \\
 & Y^\cdot \arrow[ul, leftrightarrow] \arrow[ur, "v"'] &
\end{tikzcd}$$

we are going to use 5-lemma for the two short exacts:

$$0 \to X^\cdot \to Y^\cdot \to Z^\cdot \to 0 \qquad \text{and}$$

$$0 \to Y^\cdot \to \text{cone } u^\cdot \longrightarrow X^\cdot[1] \longrightarrow 0$$

we don't really have an option with m: Need m

$$Y^n \oplus X^{n+1} \longrightarrow Z^n$$

There does not exist a canonical map $X^{n+1} \to Z^n$. So define

$$m(y,x) = vy$$

So we need to show quasi-isomorphism : look at long exact seq

$$\begin{tikzcd} H^n(Y) \arrow[r] & H^n(Z) \arrow[r] & H^{n+1}(X) \\ H^n(Y) \arrow[r] \arrow[u, "1"] & H^n(\text{cone}) \arrow[r] \arrow[u, "m^*"] & H^{n+1}(X) \arrow[u, "-1"] \end{tikzcd}$$

Enough to show commutativity, 5-lemma will do the rest

$$\begin{tikzcd} {[y]} \arrow[r] & {[vy]} \\ {[y]} \arrow[r] \arrow[u] & {[(y,0)]} \arrow[u] \end{tikzcd} \checkmark \qquad \begin{tikzcd} vy \arrow[r, "\delta"] & ? \\ {[(y,x)]} \arrow[r] \arrow[u] & x \arrow[u, "-1"] \end{tikzcd}$$

$d(y,x) = 0 \Rightarrow (dy + ux, -dx) = 0 \Rightarrow dy = -ux$

δ:

$$\begin{tikzcd} y \arrow[r] \arrow[d] & vy \\ -x \arrow[r] & dy = -ux \end{tikzcd}$$

$\delta(vy) = -x$

So commutative. ∎

ii) $s : Z \to Y$ s.t. $vs = 1_Z$

Define : $\hat{s} : Z^n \longrightarrow \text{cone } u^n = Y^n \oplus X^{n+1}$

$z \longmapsto (sz, 0)$

- $m \cdot \hat{s}(z) = m(sz, 0) = vsz = z$
- $\hat{s}m\underset{(y,x)}{\cancel{()}} = \hat{s}(vy) = (svy, 0)$

$vsvy = vy \Rightarrow v(sv - 1)y = 0$

$\Rightarrow (sv - 1)y = ux'$ for some $x' \in X^n$

$$\begin{tikzcd} & Y^n \oplus X^{n+1} \arrow[r, "d"] \arrow[dl, "\alpha"'] \arrow[d, "sv-1"] & \cancel{Y}^{n+1} \oplus X^{n+2} \arrow[dl, "\alpha"] \\ Y^{n-1} \oplus X^{n} \arrow[r] & Y^{n} \oplus X^{n+1} & \end{tikzcd}$$

define:

$\alpha : Y^n \longrightarrow X^n$

$y \longmapsto u^{-1}(sv-1)y$

Tracing: (y, x)

$$(0, u^{-1}(sv-1)y) \longrightarrow ((sv-1)y,\ u^{-1}d(sv-1)y)$$

$$(y,x) \longrightarrow (dy + ux, -dx)$$

$$\swarrow$$

$$(0, u^{-1}(sv-1)(dy+ux)) = (0, u^{-1}(sv-1)dy - x)$$

$$-\kappa d + d\kappa = ((sv-1)y, x + u^{-1}[(sv-1)dy - d(sv-1)]y)$$

$$\qquad\qquad\qquad\qquad\qquad\qquad svd - dsv$$

So ~~we need~~ ~~sv to be a chain map.~~

we want $u^{-1}[svd - dsv]y = 0$

i.e. $svdy = dsvy$

But sv differs from 1 by element of x

~~dy~~ ??

Result is true if sv is a chain map. ▯

=

Homotopical Category: $K^{\alpha}(A)$

Morphisms are equivalence classes upto chain homotopy.

Tr 3

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f_1"'] & Y \arrow[r, "v"] \arrow[d, "f_2"] & Z \arrow[r, "w"] \arrow[d, "f_3"] & X[1] \arrow[d, "f_1[1]"] \\ A \arrow[r, "\alpha"'] & B \arrow[r] & \mathrm{Cone}\,\alpha \arrow[r] & A[1] \end{tikzcd}$$

distinguished

Need to show 1) $u[1]w = 0$. follows from above iso.

2) $f_2[1] \cdot u[1] = \alpha[1] f_1[1]$

But taking care of gradation this is just

$f_2 \cdot u = \alpha \cdot f_1$