

# MODEL CATEGORIES

Talk by **Xiyuan Wang**

References: Dwyer and Spalinski : Homotopy theories and model categories

## 1. MOTIVATION

Category of topological space :  $Top_*$

In this we have three kinds of morphisms: weak equivalence eg. homotopy equivalence, fibrations, cofibrations

Weak equivalence is not an isomorphism in  $Top_*$ , so we want to create a category in which the weak equivalences would be isomorphisms. So we want to formally invert the weak equivalences and create a Homotopy category.

## 2. MODEL CATEGORIES

**Definition 2.1.**  $X \in ob(\mathcal{C})$  is a retract of  $Y \in ob(\mathcal{C})$  if

$$\exists i : X \rightarrow Y \text{ and } r : Y \rightarrow X$$

such that  $r \circ i = 1_X$ .

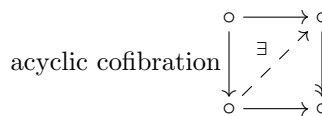
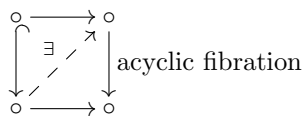
Ex: Consider the category  $\mathcal{D} = \{\circ \rightarrow \circ\}$ . Then the category of functors from  $\mathcal{D}$  to  $\mathcal{C}$  is the category of morphisms  $Mor(\mathcal{C})$ , denote this by  $\mathcal{C}^{\mathcal{D}}$ . We are interested in the retracts in this category,

**Definition 2.2.** A model category is a category  $\mathcal{C}$  with 3 classes of maps

- $\xrightarrow{\sim}$  weak equivalences
- $\rightarrow$  fibrations
- $\hookrightarrow$  cofibrations

closed under cofibrations and contains 1. And satisfying the axioms

- (bicomplete) finite limits and colimits exist
- (2 out of 3) for morphisms  $f, g, fg$  if two of these are weak equivalences then so is the third
- (closed under retract)
- (Lifting property)



- (Factorization) Every morphism can be factored as a composition of a cofibration followed by a fibration. And one can further force either of the maps to be weak equivalences. (Factorization is not functorial.)

**Definition 2.3.** For  $\phi$  initial object and  $*$  the final object, if  $\phi \hookrightarrow A$  is a cofibration then  $A$  is called a cofibrant object. If  $B \rightarrow *$  is a fibration then  $B$  is a fibrant object.

**Proposition 2.4.**  *$\mathcal{C}$  is model category in which all objects are fibrant. Given a cofibrant object  $A$ . If there exists*

$$A \xrightarrow{\sim} \xleftarrow{\sim} \dots \xrightarrow{\sim} \xleftarrow{\sim} B$$

*then  $A \xrightarrow{\sim} B$ .*

*Proof.* It suffices to show this for the case  $A \xrightarrow{\sim} \circ \xleftarrow{\sim} B$ . In this we have the diagram

$$\begin{array}{ccc} \phi & \longrightarrow & B \\ \downarrow & & \downarrow \text{acyclic fibration} \\ A & \longrightarrow & \circ \end{array}$$

and by the Lifting property axiom the diagonal map exists.  $\square$

### 3. HOMOTOPIES

**Definition 3.1.** A *cylinder object* for  $A$  is an object  $A \wedge I$  together with a diagram

$$A \coprod A \xrightarrow{i} A \wedge I \xrightarrow{\sim} A$$

which factors the map  $1_A + 1_A : A \coprod A \rightarrow A$ .

Two maps  $f, g : A \rightarrow X$  are said to be *left homotopic*  $f \sim^l g$  if there exists a cylinder object  $A \wedge I$  such that the map  $f + g : A \coprod A \rightarrow X$  extends to a map  $H : A \wedge I \rightarrow X$ .

**Definition 3.2.** A *path object* for  $X$  is an object  $X^I$  along with a diagram

$$X \xrightarrow{\sim} X^I \rightarrow X \times X$$

which factors the diagonal map  $X \rightarrow X \times X$ .

Two maps  $f, g : A \rightarrow X$  are said to be *left homotopic*  $f \sim^r g$  if there exists a cylinder object  $A \wedge I$  such that the map  $f \times g : A \rightarrow X \times X$  factors through a map  $H : A \rightarrow X^I$ .

**Proposition 3.3.** *If  $A$  is cofibrant then  $\sim^l$  is an equivalence relation on  $\text{hom}(A, X)$  and the equivalence classes are denoted by  $\pi^l(A, X)$ . If  $X$  is fibrant then  $\sim^r$  is an equivalence relation on  $\text{hom}(A, X)$  and the equivalence classes are denoted by  $\pi^r(A, X)$ . If  $A$  is cofibrant and  $X$  is fibrant then the two equivalence relations agree and the equivalence classes are simply denoted by  $\pi(A, X)$ .*

### 4. HOMOTOPY CATEGORY

$\mathcal{C}$  model category and  $S$  the set of weak equivalences, the localized category  $Ho(\mathcal{C}) := \mathcal{C}[S^{-1}]$  is called the homotopy category.

We have a natural functor  $\gamma : \mathcal{C} \rightarrow Ho(\mathcal{C})$ . (In general creating a homotopy category without a model category leads to set theory problems)

**Definition 4.1.** For any map  $\phi \rightarrow A$  factor it via  $\phi \hookrightarrow QA \twoheadrightarrow A$  such that the fibration is a weak equivalence then  $QA$  is called a cofibrant replacement. Similarly a fibrant replacement  $RA$ .

**Proposition 4.2.**

$$\mathrm{hom}_{Ho(\mathcal{C})}(X, Y) = \pi(PQX, PQY)$$

## 5. EXAMPLES

5.1. **Constructing a model category from a model category.** Given that  $\mathcal{C}$  is model category

- $\mathcal{C}^{op}$  has a natural model structure
- For  $a \in ob(\mathcal{C})$  the  $a \rightarrow \mathcal{C}$  has a natural model structure
- If  $\mathcal{D}$  is a “very small” category then  $\mathcal{C}^{\mathcal{D}}$  has a natural model structure.  
This allows to define homotopy limits and colimits.

5.2. **From geometry : Spaces, spectra, simplicial sets,**

- $Top_*$ 
  - $\xrightarrow{\sim}$  weak homotopy equivalence
  - $\hookrightarrow$  is a ‘retract’ of inclusion of a subcomplex inside a CW complex.
  - $\twoheadrightarrow$  Serre fibration
  - Fibrant object: every topological object is fibrant
  - Cofibrant object: hard to describe, but every CW complex is a cofibrant object
  - Homotopy category : classical homotopy category
- $Top_*$ 
  - $\xrightarrow{\sim}$  homotopy equivalence
  - $\hookrightarrow$  closed Hurewicz cofibration
  - $\twoheadrightarrow$  Serre fibration
  - Homotopy category : classical homotopy category
- Simplicial sets
  - $\xrightarrow{\sim} ??$
  - $\hookrightarrow ??$
  - $\twoheadrightarrow ??$

5.3. **From algebra: CDGM, DGA.**

- $(Ch_R)_{\geq 0}$ 
  - $\xrightarrow{\sim}$  quasi isomorphisms
  - $\hookrightarrow$ , for  $f : M \rightarrow N$  and  $k \geq 0$  we want  $f_k : M_k \rightarrow N_k$  to be injective with a projective cokernel.
  - $\twoheadrightarrow$  for  $f : M \rightarrow N$  and  $k > 0$  we want  $f_k : M_k \rightarrow N_k$  is surjective.
  - Fibrant object: every topological object is fibrant
  - Cofibrant object: projective resolutions
  - Cofibrant replacement is a projective resolution
  - Homotopy category: is the Derived category

**Proposition 5.1.**

$$\mathrm{hom}_{Ho(Ch_{\mathbb{R}})}(K(A, m), K(B, n)) \cong Ext_R^{n-m}(A, B)$$

*Proof.*

□

## 6. QUILLEN EQUIVALENCE

Given a model category  $\mathcal{C}$  and a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$ .

**Definition 6.1.** Consider pairs  $(G, s)$  consisting of a functor  $G : Ho(\mathcal{C}) \rightarrow \mathcal{D}$  and a natural transformation  $s : G\gamma \rightarrow F$ . A *left derived functor* is a pair  $(LF, t)$  of this type which is universal from the left, that is for any other pair  $(G, s)$  there exists a unique natural transformation  $s' : G \rightarrow LF$  such that the composite natural transformation

$$G\gamma \xrightarrow{s' \circ \gamma} (LF)\gamma \xrightarrow{t} F$$

is the natural transformation  $s$ .

One can similarly define a right derived functor.

**Definition 6.2.** Given a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  between two model categories a *total left derived functor* is a functor

$$LF : Ho(\mathcal{C}) \rightarrow Ho(\mathcal{D})$$

which is a left derived functor of the composition  $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{\gamma} Ho(\mathcal{D})$ . A right derived functor of the same is called a *total right derived functor*.

**Theorem 6.3.** *If  $F : \mathcal{C} \leftrightarrow \mathcal{D} : G$  are a pair of adjoint functors such that  $F$  preserves cofibrations and  $G$  preserves fibrations then the total derived functors exist and form an adjoint pair,*

$$LF : Ho(\mathcal{C}) \leftrightarrow Ho(\mathcal{D}) : RG$$

*If in addition, for each cofibrant object  $A$  of  $\mathcal{C}$  and fibrant object  $X$  of  $\mathcal{D}$ , a map  $f : A \rightarrow G(X)$  is a weak equivalence in  $\mathcal{C}$  if and only if its adjoint  $f^\flat : F(A) \rightarrow X$  is a weak equivalence in  $\mathcal{D}$ , then  $LF$  and  $RG$  are inverse equivalences of categories.*