Computations in Motivic Homotopy Theory: — Dan ? , Wayne State

1. Motivic homotopy theory:

Start with category of smooth varieties over k.

Cannot do homotopy theory here — Not enough gluing constructions

1) Add in formal hocolimits to get simplicial pre-sheaves
2) Restore some desired relations:

$$\begin{array}{ccc} U\cap V & \longrightarrow & U \\ \downarrow & & \downarrow \\ V & \longrightarrow & U\cup V \end{array}$$

should be declared a homotopy pushout

3) Declare that $X\times\mathbb{A}^1 \longrightarrow X$ is an equivalence

This gives rise to unstable motivic homotopy theory

Ex: $\mathbb{A}^1/\mathbb{A}^1-0 \cong \Sigma^1(\mathbb{A}^1-0)$

Stable motivic homotopy theory:

$S^{1,0}$ = simplicial circle

$S^{1,1} = \mathbb{A}^1 - 0$

stabilize with respect to both the spheres

Warning: Not every spectrum is built out of spheres.

Can focus on "cellular" motivic homotopy theory. But lose a lot interesting varieties this way.

Are elliptic curves, field extensions cellular.

Theory of motives:

Motive is supposed to retain only the cohomological information

$H\mathbb{Z}\wedge\Sigma^\infty X$ is the motive of X. Why not simply $\Sigma^\infty X$?

Realisation of functors:

mot. homotopy theory$/\mathbb{C}$ $\longrightarrow$ classical homotopy

mot. homotopy theory$/\mathbb{R}$ $\longrightarrow$ $\mathbb{Z}_2$ - equivariant homotopy theory

$S^{1,0} \longrightarrow S^{1,0}$ trivial action

$S^{1,1} \longrightarrow S^{1,1}$ non-trivial action

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2. Survey of computations of $\pi_{**}(S^{0,0})$

$\pi_{p,q} = 0$ if $p<q$ (Morell)

$\pi_{p,p}$ - Milnor Witt K-theory (Morell)

$\pi_{p+1,p}$ - " , $\eta \in \pi_{1,1}$, $\nu \in \pi_{3,2}$, $\sigma \in \pi_{7,4}$, $\eta\nu = 0$, $\nu\sigma = 0$

$\pi_{p,0}$ over $\mathbb{C}$ is the same as the classical π_p.

Adams Motivic SS (2-complete)

Algebraic input (Voevodsky): $H^{**}(pt) = M_2 = \mathbb{F}_2[\tau]$ $|\tau| = (0,1)$

(dual) Steenrod Algebra: $\mathbb{M}_2[\tau, \xi_i]/\tau_i^2 = \tau\xi_{i+1}$

inverting τ reverts back to the classical situation.

$E_2 = \mathrm{Ext}_A(M_2, M_2)$

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4. Classical homotopy groups:

5. η-local sphere

$S^\infty[\eta^{-1}] = \mathrm{colim}\,(S \xrightarrow{\eta} S \xrightarrow{\eta} S \xrightarrow{\eta} S \to \dots)$

Adams SS $\quad \mathrm{Ext}[h_1^{-1}] \Longrightarrow \pi_{**} S^\infty[\eta^{-1}]$

^{m}

Th^m: $\quad \mathrm{Ext}[h_1^{-1}] = \mathbb{F}_2[v_1^{4}, v_2, \dots v_n, \dots]$

Adams: $d_2 v_3 = v_2^2$, $d_2 v_4 = v_3^2$

Conjecture: $d_2 v_n = v_{n-1}^2$, $\pi_{**} S^\infty[\eta^{-1}] = \mathbb{F}_2[\beta, \xi]/\xi^2$

Also the nilpotence theorem fails here.

Rigid connection between

motivic Adams over $\mathbb{C}$ and classical Adams Novikov

Q. Machine computation of E_2 ANSS.

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6. Preliminary calculations over $\mathbb{R}$:

ImJ is different. $h_0^4 h_3 \neq 0$, $h_0^8 h_4 \neq 0$, ... Guess: Extra 2 in dim 7 mod 8.