PROBLEM SET 01

Day 1

The hardest step in writing an $\epsilon - \delta$ proof is the first step, once you know how to start the proof the rest is mechanical.

Q.1. Explain in your own words how the provisional definition is equivalent to the more rigorous $\epsilon - \delta$ definition of limit.

Provisional definition: The function f approaches a limit l near a i.e. $\lim\limits_{x \to a} f(x) = l$, if we can make $f(x)$ as close to l as we like by requiring that x be sufficiently close to, but unequal to, a.

$\epsilon - \delta$ **definition:** The function f approaches a limit l near a i.e. $\lim\limits_{x \to a} f(x) = l$, if for every $\epsilon > 0$ there is some $\delta > 0$ such that, for all x, if $0 < |x - a| < \delta$, then $|f(x) - l| < \epsilon$.

Explain how this is equivalent to the standard $\epsilon - \delta$ definition.

Q.2. For each of the following functions f and real numbers a,

- Guess the limit $L = \lim\limits_{x \to a} f(x)$.

- Find a δ corresponding to $\epsilon = 0.1$ in the $\epsilon - \delta$ definition of limit.

- Find a δ corresponding to an arbitrary real number ϵ and **prove** that L is indeed the limit.

   (1) $f(x) = x + 2, \quad a = 0$

   (2) $f(x) = 7x + 2, \quad a = 0$

   (3) $f(x) = x^2 - 2, \quad a = 0$

   (4) $f(x) = 1/x, \quad a = 1$

DAY 2

Its tedious to use $\epsilon - \delta$ proofs in practice, instead we use Theorem 2 and it's analogues.

Q.3.
- Let n be a positive integer. Prove that the function $f(x) = x^n$ is continuous at 0.

- Use Theorem 2 to prove that every polynomial $p(x)$ is continuous at 0.

- Prove that a function $f(x)$ is continuous at a if and only if the function $g(x) = f(x + a)$ is continuous at 0.

- Prove that x^n is continuous at every real number a.

- Use Theorem 2 to prove that every polynomial $p(x)$ is continuous at all real numbers.

- What are the real numbers at which the ratio of two polynomials $\dfrac{p(x)}{q(x)}$ is continuous?

Q.4. Using the $\epsilon - \delta$ definition of limit to prove that if $\lim\limits_{x \to a} f(x) = l$ and $\lim\limits_{x \to a} g(x) = m$ then
$$\lim_{x \to a} (f(x) - g(x)) = l - m$$

Trigonometric functions, exponential functions and logarithms are continuous wherever they are defined. We will assume this fact without proof for now and come back to it later.

Day 3

To understand continuity it is equally important to understand discontinuity.

Q.5. • Determine, with proof, $\lim\limits_{x \to \infty} 1/x$.

 • Prove that for no real number l do we have $\lim\limits_{x \to 0^+} 1/x = l$.

Q.6. For the function

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{otherwise} \end{cases}$$

determine, with proof, the limits $\lim\limits_{x \to 0^+}$, $\lim\limits_{x \to 0^-}$. How will you answers change if we replace $x < 0$ by $x \leq 0$ in the definition of $f(x)$?

Q.7. Prove that the function

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{otherwise} \end{cases}$$

is discontinuous everywhere. What can you say about the continuity of the function $g(x) = x.f(x)$?

Q.8. Give examples to show that the following definitions of $\lim\limits_{x \to a} f(x) = L$ are not correct. (Hint: Think graphically)

 • For all $\delta > 0$ there exists an $\epsilon > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

 • For every $\epsilon > 0$ there is some $\delta > 0$ such that, for all x, if $|f(x) - L| < \epsilon$ then $0 < |x - a| < \delta$.