DIRAC OPERATORS

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Abstract. This document mostly has solutions to the algebra exercises from the article by Freed, Geometry of Dirac operators leading up to the Atiyah Singer Index theorem.

1. Clifford algebras

We will not be looking at the most general type of Clifford algebras. For us Clifford algebras are defined over an inner product space over $\mathbb{R}$.

Definition 1.1. Over an inner product space $V, (,)$ a Clifford algebra is an image of $T(V)$ given by the relations

$$v.w + w.v = -2(v, w)$$

Denote the Clifford algebras over the standard $\mathbb{R}^n$ by $Cliff_n$.

If $e_1, \cdots, e_n$ is an orthogonal basis for V then $Cliff(V)$ is completely determined by the relations

$$e_i.e_j + e_j.e_i = -2\delta_{i,j}$$

From here on we will denote the standard basis for $\mathbb{R}^n$ by $e_1^n, \cdots, e_n^n$
$Cliff_1 \cong \mathbb{C}$
$Cliff_2 \cong \mathbb{H}$
The next is not the octonians though as $Cliff_3$ is associative.

Because the relations are degree 2 homogeneous $Cliff_n$ gets a $\mathbb{Z}/2$ grading

$$Cliff_n = Cliff_n^+ \otimes Cliff_n^-$$

as vector space.

$Cliff_n^+$ has a basis consisting of $\{e_{i_1}.\cdots.e_{i_2k} | 1 \leq 2k \leq n, i_1 < \cdots < i_{2k}\}$ and hence $\dim(Cliff_n^+) = 2^{n-1}$. Similarly for $Cliff_n^-$.

2.1.6. Define a map $f : Cliff_{n-1} \to Cliff_n^+$ aa

$$e_i^{n-1} \mapsto e_n^n.e_i^n$$

$$\begin{aligned} f(e_i^{n-1}).f(e_j^{n-1}) + f(e_j^{n-1}).f(e_i^{n-1}) &= e_n^n.e_i^n.e_n^n.e_j^n + e_n^n.e_j^n.e_n^n.e_i^n \\ &= -e_n^n.e_n^n.e_i^n.e_j^n - e_n^n.e_n^n.e_j^n.e_i^n \\ &= e_i^n.e_j^n + e_j^n.e_i^n \\ &= -2\delta_{i,j} \end{aligned}$$

Hence the map is morphism of algebras. Isomorphism as sets is trivial.

2.1.7. The morphism which sends

$$e_{i_1}^n.\cdots.e_{i_k}^n \mapsto e_{i_1}^n \wedge \cdots \wedge e_{i_k}^n$$

is an isomorphism of vectors spaces.

2.1.9. By symmetry it is enough to show this for $e = e_1$, w and $v = e_1 \wedge w$ where w does not contain any e_1 . But this is a trivial check.

For $.e_1$ we need to add in grading, that is for a monomial v,

$$v.e = (-1)^{|v|}(\epsilon(e) + \iota(e))$$

2.1.11. The induced filtration on the Clifford algebra consists of monomials with no repeating terms.

2.1.21. $\mathfrak{o}(n)$ sits inside $Cliff_n$ as the set of elements which satisfy

$$g + g^t = 0$$
$$g.v + v.g^t \in \mathbb{R}^n$$

and the action is infinitesimal that is via exponentiation.

2.1.22. $Spin(1) \cong \mathbb{Z}/2$ and the Lie algebra is trivial. $Spin(2) \cong SO(2)$ is the set of elements of the form $\cos\theta + \sin\theta e_1^2 e_2^2$

2.1.23.

$$SU(2) = \{ \begin{bmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{bmatrix} ||\alpha|^2 + |\beta|^2 = 1\}$$

and hence

$$SU(2) \cong S^3$$

as a manifold. $SU(2)$ acts on itself by conjugation with only the center acting trivially, with the center being ± 1 and so we get

$$SU(2) \cong Spin(2)$$

2.1.24. $\wedge^2_+$ has a basis $e_1^4 \wedge e_2^4 + e_3^4 \wedge e_4^4$, $e_1^4 \wedge e_4^4 + e_2^4 \wedge e_3^4$, $e_1^4 \wedge e_3^4 - e_3^4 \wedge e_4^4$ and hence has dimension 3.

For g an element in $SO(4)$ and β in $\wedge^2_+$,

$$\begin{aligned} \alpha \wedge *g\beta &= (\alpha, g\beta) \\ &= (g^t\alpha, \beta) \\ &= (g^t\alpha, *\beta) \\ &= (\alpha, g * \beta) \\ &= \alpha \wedge g * \beta \end{aligned}$$

and hence $SO(4)$ leaves $\wedge^2_+$ invariant, similarly for $\wedge^2_-$

And hence we get a map

$$SO(4) \to SO(3) \times SO(3)$$

Because all the spaces under consideration are compact and the map is symmetric and hence a submersion at ever point and hence is a surjection (in fact a fiber bundle).

It remains to find the kernel. If $g \in SO(4)$ acts trivially on $\wedge^2$ then it will also act trivially on $\wedge^2 \otimes \mathbb{C}$. Here we can look at the action of g on $v \wedge w$ where v, w are its eigenvectors. And hence product of any two eigenvalues should be 1 which means $g = \pm 1$.

2.1.27. For $n = 1$ the isomorphism is given by $e_1^1 \mapsto \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$.

For $n = 2$ the isomorphism is given by $e_1^2 \mapsto \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ and $e_2^2 \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

For the inductive step we are required to show that

$$Cliff_n^{\mathbb{C}} \cong Cliff_{n-2}^{\mathbb{C}} \otimes Cliff_2^{\mathbb{C}}$$

This is given by the maps

$$\begin{aligned} e_i^n \mapsto e_i^{n-2} \otimes \sqrt{-1}.e_1^2 \text{ for } i < n-1 \\ e_{n-1}^n \mapsto 1 \otimes e_2^2 \\ e_n^n \mapsto 1 \otimes e_1^2.e_2^2 \end{aligned}$$

2.1.30. Consider the case first when n is even.

In this case as noted above conjugation by ϵ is identity on $(Cliff_n^{\mathbb{C}})^+$ and -1 on $(Cliff_n^{\mathbb{C}})^-$ and hence if $\omega \in I$ then $\omega^+ \in I$ and $\omega^- \in I$. So assume that ω is even.

Now suppose $\omega = e_1.f + g$ where f and g do not contain any e_1. $e_1 \omega e_1 = e_1.f - g$ and hence both f and g should also be in I . Applying this to every e_i we finally get that $1 \in I$.

Now consider the case when n is odd. Then notice that $(1 \pm \epsilon)^2 = 2(1 \pm \epsilon)$ and hence the algebra is split as

$$Cliff_n^{\mathbb{C}} = (1+\epsilon)Cliff_n^{\mathbb{C}} \otimes (1-\epsilon)Cliff_n^{\mathbb{C}}$$

Now $(1+\epsilon)Cliff_n^{\mathbb{C}} \cong Cliff_{n-1}^{\mathbb{C}}$ via the map $(1+\epsilon)e_i^n \mapsto 2e_i^{n-1}$ for $i < n$.

2.1.31. We can use the above isomorphism to construct explicit matrix generators for $Cliff^{\mathbb{C}}(4)$ to get

$$\begin{bmatrix} -i & & & \\ & i & & \\ & & i & \\ & & & -i \end{bmatrix}, \begin{bmatrix} & -1 & & \\ 1 & & & \\ & & & 1 \\ & & -1 & \end{bmatrix}, \begin{bmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{bmatrix}, \begin{bmatrix} & & i & \\ & & & i \\ i & & & \\ & i & & \end{bmatrix}$$

2.1.36. For n even $Aut(\mathbb{S}^{\pm})$ is the space $(1 \pm \epsilon)Cliff_n^{\mathbb{C}}$, and hence the action of ϵ on $Aut(\mathbb{S}^{\pm})$ is $\epsilon(1 \pm \epsilon) = (\epsilon \pm 1) = \pm(1 \pm \epsilon)$ which has different eigenvalues ± 1 for the two spaces.

2.1.37. Note that there are two bilinear forms so to avoid confusion let $\langle v, w\rangle$ be the hermitian form on $\mathbb{C}^l$ and let $v^T w$ be the inner product on $\mathbb{R}^{2l}$. We need to check that the map is morphism of algebras, let $x = x_1 \wedge \cdots \wedge x_k$

$$v.w \mapsto x \to (\epsilon v - \iota v)(\epsilon w - \iota w)x$$

$$\begin{aligned} v.w(x) &= v(w \wedge x + \sum_i (-1)^i \langle w, x_i\rangle \hat{x_i} \\ &= v \wedge w \wedge x - \langle v, w\rangle x + \sum_{i,j} (-1)^{f(i,j)} \langle w, x_i\rangle \langle v, x_j\rangle \hat{x_{i,j}} \end{aligned}$$

$$v.w(x) = w \wedge v \wedge x - \langle w, v\rangle x + \sum_{i,j} (-1)^{f(j,i)} \langle w, x_i\rangle \langle v, x_j\rangle \hat{x_{i,j}}$$

adding the two and noting the fact that $f(i,j) = f(j,i) \pm 1$ we get

$$(v.w + w.v)x = -(\langle v, w\rangle + \langle w, v\rangle)x$$

Remains to show that $\langle v, w\rangle + \langle w, v\rangle = 2v^T w$ which is an easy check.

Because *Spin* sits inside the even component of the Clifford algebra, the invariant subspaces are

$$\mathbb{S}^+ = \wedge^{2k}\mathbb{C}^l, \mathbb{S}^- = \wedge^{2k-1}\mathbb{C}^l$$

2.1.39. Essentially we need to provide a $Cliff_n$ map from the i eigenspace of e_1e_2 to the $-i$ eigenspace.

If A is the subalgebra generated by $e_3, e_4, \cdots$ then the i eigenspace is a right A module with generators $(e_1 + ie_2, i + e_1e_2)$ and the $-i$ eigenspace is the right A module with generators $(e_1 - ie_2, i - e_1e_2)$. We define a A map from the i eigenspace to the $-i$ eigenspace which sends

$$e_1 + ie_2 \mapsto e_1 - ie_2$$
$$i + e_1e_2 \mapsto i - e_1e_2$$

It is easy to see that this maps respects the action of e_1, e_2 and hence is a map of right $Cliff_n$ modules as was needed.

Now suppose there are odd number of generators $e_1, e_2, \cdots, e_{2n+1}$ then let B be the subalgebra generated by $e_3, e_4, \cdots, e_{2n}$ then the i eigenspace of e_1e_2 further splits up as two B modules $(-1 + ie_1e_2 - ie_{2n+1} - e_1e_2e_{2n+1})B \oplus (-e_1 - ie_2 - ie_1e_{2n+1} + e_2e_3)B$. It is easy to see that each of these B modules are e_1, e_2, e_{2n+1} invariant. This then is the required splitting.

2.1.40. For the case when $\mathbb{S} = \wedge\mathbb{C}^l$ it follows from direct computation that ϵ and ι are adjoint operators, that is

$$\langle x, \epsilon(v)y\rangle = \langle \iota(v)x, y\rangle$$

and so

$$\langle x, \epsilon(v) - \iota(v)y\rangle = \langle \iota(v) - \epsilon(v)x, y\rangle$$

And so the $Cliff_n^+$ acts via unitary operators, that is it preserves the norm.

For the case $Cliff_n \to Aut(Cliff_n^{\mathbb{C}})$ the natural norm on the right hand side is given by $\langle A, B\rangle := Trace(A^*B)\rangle$

If $g \in Spin(n)$ then $\langle gA, gB\rangle = Trace(A^*g^TgB)\rangle$. We need to prove that ${}^tg = g^T$. Again this is happening because g has even degree. On checking by hand one sees that $e_i^T = -e_i$ and hence $v^T = -v$ for all $v \in \mathbb{R}^n$, for ${}^t(v_1v_2) = v_2v_1$ and $(v_1v_2)^T = v_2^Tv_1^T = v_2v_1$ and similarly for all even degree vectors.

2.1.44. The proof is just following whatever is written in the problem. The only non-trivial step is checking that for n even the half spin representations are either preserved or swapped by conjugation depending on whether 4 divides n or not.

For this note that $c(\epsilon)$ restricted to $\mathcal{S}^+$ is the identity transformation and restricted to $\mathcal{S}^-$ is $-$ identity. $\beta(\epsilon) = \begin{cases} \epsilon & \text{if 4 divides } n \\ -\epsilon & \text{otherwise} \end{cases}$ and hence the result follows.

2.1.46.

$$c : \mathcal{E}^n \otimes \mathcal{S} \to \mathcal{S}$$

That $c(v)$ interchanges $\mathbb{S}^+$ and $\mathbb{S}^-$ follows from the fact that ϵ anticommutes with elements of odd degree. Another way to see this is to explicitly use the model $\mathbb{S} = \wedge\mathbb{C}^l$ and see that $c(v)$ sends an odd degree element to an even degree one and vice versa.

What I do not understand is what is the $Spin(n)$ action on the left.

2.1.47. Not sure if this is the complete answer. But we have the array of maps which give rise to the representation:

$$Spin(n) \subset Cliff_n^+ \cong Cliff_{n-1} \subset Cliff_{n-1}^{\mathbb{C}}$$

So to prove irreducibility it suffices to show that the $\mathbb{C}$ algebra generated by $Spin(n)$ is the entire set $Cliff_n^{\mathbb{C}+}$ but this follows from the fact that $Spin(n)$ contains all the elements $e_i e_j$ and these are precisely the set of generators!

Here we have used Burnside's theorem that for an irreducible representation the group algebra is the entire matrix algebra and Peter Weyl theorem, that $Spin(n)$ being a compact Lie group every finite representation over $\mathbb{C}$ is semisimple.

2.1.48. This problem does not seem entirely correct. We use an explicit model for $\mathbb{S}$.

Also I do not see how the given matrix corresponds to the element $x \in Cliff_n$. Let us instead look directly at $x = \frac{1}{2}(y_1e_1e_2 + y_2e_3e_4 + \cdots) \in Cliff(n = 2l)$. However it easy to see that exponential of the given matrix is indeed equal to e^x.

Then because all the $e_{2i-1}e_{2i}$ commute with each other and all of them formally behave like $\sqrt{-1}$ hence we should get

$$g = e^x = (\cos y_1/2 + e_1e_2 \sin y_1/2)(\cos y_2/2 + e_3e_4 \sin y_2/2) \cdots$$

We can use the explicit model $\mathbb{S} = \wedge\mathbb{C}^l$ and it is an easy computation to see that f_1v is an eigenvector of e_1e_2 with eigenvalue i and v is an eigenvector of e_1e_2 with eigenvalue $-i$ where v does not contain any f_1. And hence the eigenvalue of f_1v for $(\cos y_1/2 + e_1e_2 \sin y_1/2)$ is $e^{iy_1/2}$ and of v is $e^{-iy_1/2}$.

So for g we get an eigenbasis consisting of monomials in $\wedge\mathbb{C}^l$. The eigenvalue for $f_{j_1} \wedge f_{j_2} \wedge \cdots$ is $e^{i(-y/2+y_{j_1}+y_{j_2}+\cdots)}$ where $y = y_1 + y_2 + \cdots$, so that when you sum over all the eigenvalues we get

$$\begin{aligned}
\chi_{\mathbb{S}}(g) &= e^{-iy/2}(1 + e^{iy_1})(1 + e^{iy_2}) \cdots \\
&= (e^{iy_1/2} + e^{-iy_1/2})(e^{iy_2} + e^{-iy_1/2}) \cdots \\
&= (2\cos y_1/2)(2\cos y_2/2) \cdots
\end{aligned}$$

For the other representation we need to sum over the even ones and subtract the odd ones, it is not hard to see that this turns out to be

$$\begin{aligned}
\chi_{\mathbb{S}^+-\mathbb{S}^-}(g) &= (e^{-iy_1/2} - e^{iy_1/2})(e^{-iy_2} + e^{-iy_1/2}) \cdots \\
&= (-2i \sin y_1/2)(-2i \sin y_2/2) \cdots
\end{aligned}$$

2.1.54. $\tilde{U}(l)$ is just the pullback of $Spin(2l)$ along the inclusion $U(l) \to SO(2l)$. The group structure is derived by thinking of $\tilde{U}(l)$ as a subgroup of $Spin(2l)$.

To get the map $\det^{1/2}$ look at the double covering $U(1) \to U(1), x \mapsto x^2$ and lift the map det.

I do not enough about representation theory of $U(n)$ to prove the next fact with certainty :(But my guess is that it is enough to check that the characters agree on the maximal torus to say that the representations are the same.

A maximal torus of $U(l)$ is the set of diagonals in the standard representation, that is $t = \{diag(e^{iy_1}, \cdots, e^{iy_l})\}$. The preimage of this in $Spin(2l)$ is precisely the element we encountered in 2.1.48

$$g = e^x = (\cos y_1/2 + e_1 e_2 \sin y_1/2)(\cos y_2/2 + e_3 e_4 \sin y_2/2) \cdots$$

The 1/2 appear because we are looking a 2 covering.

It is easy to see that the trace of t in the standard representation will be

$$(1 + e^{iy_1}) \cdots (1 + e^{iy_l})$$

and the trace of g in the $det^{-1/2}$ representation will be $e^{-i(y_1/2 + \cdots + y_l/2)}$.

Multiplying out and referring to the computation in 2.1.48 we see that the two traces agree on t.

2.1.58. The isomorphism is given by sending e_i to $\epsilon^{-1} e_i$.

2.1.59. I am going to use $Cliff_\delta$ instead of $Cliff_\epsilon$, then the Clifford multiplication by v_δ is given by

$$c_\delta(v_\delta) = \epsilon(v_\delta) - \delta^2 \iota(v_\delta)$$

And hence as $\delta \to 0$, $c_delta \to v_\delta$.

2. Spinors on Manifolds

Just to expand out the definition of a Dirac operator on $\mathbb{R}^n$, suppose $\mathbb{S} = \mathbb{C}\langle s^1, s^2, \cdots\rangle$ then a spinor field is a bunch of functions $f_j : \mathbb{R}^n \to \mathbb{R}$ such that $f : \mathbb{R}^n \to \mathbb{S}$ is given by $f(x) = f_1(x)s^1 + f_2(x)s^2 + \cdots$. Let $e_j = c(dx_j)$, then Dirac operator is

$$\begin{aligned} D : C^\infty(\mathbb{R}^n, \mathbb{S}) &\to C^\infty(\mathbb{R}^n, \mathbb{S}) \\ f &\mapsto \frac{\partial f_j}{\partial x^k} e_k(s^j) \end{aligned}$$

Then

$$\begin{aligned} D^2 f &= \frac{\partial}{\partial x^l}(\frac{\partial f_j}{\partial x^k}) e_l e_k(s^j) \\ &= \frac{\partial^2 f_j}{\partial x_l \partial x^k} e_l e_k(s^j) \\ &= \sum_j -\frac{\partial^2 f_j}{\partial x^{l\ 2}} s^j \end{aligned}$$

A spin manifold is an oriented manifold X with a Spin structure, that is where the structure group of the tangent bundle can be reduced to $Spin(n)$. The best way of saying this is that the there exists a lift of the map $TX \to BSO(n)$ to $BSpin(n) \to BSO(n)$.

2.2.17. w_2 is nothing but the unique generator of $H^2(BSO(n); \mathbb{Z}/2)$. There is a fibration sequence

$$BSpin \to BSO \to K(\mathbb{Z}/2; 2)$$

and because we know of manifolds which do not have Spin structures the last map cannot be trivial. But the only other option for it is w_2 and hence a lift exists if and only if $w_2 = 0$.

For the action part look at the fibration sequence

$$B\mathbb{Z}/2 \to BSpin \to BSO$$

which on applying the functor $[X, -]$ gives rise the sequence

$$H^1(X; \mathbb{Z}/2) \to [X, BSpin] \to [X, BSO]$$

and we are looking at the preimage of $TM \in [X, BSO]$ which is going to be exactly $H^1(X)$.

2.2.17? X, Y spin manifold implies their $w_1 = w_2 = 0$. It is easy that $w_1(X \times Y) = w_2(X \times Y) = 0$ and hence $X \times Y$ is also a Spin manifold. But the question is naturality. We need to answer if there is any canonical map

$$Spin(n) \times Spin(m) \to Spin(n)$$

There are two ways of seeing this. A unique continuous map exists simply because of unique lifting property of covering spaces. The question is why is it a group homomorphism. One way to answer this is by looking at the induced map on the Lie algebras which is a Lie algebra map and hence corresponding map on the Lie groups should be a group homomorphism(?).

2.2.18. As above the Spin structures are in one-one correspondence with $H^1(S^1;\mathbb{Z}/2) \cong \mathbb{Z}/2$. Spin(1) is just the group $\mathbb{Z}/2$ and hence we are looking at complex line bundles over S^1. Topologically there is only one, but an $\mathbb{Z}/2$ principal bundles we get 2.

Since all the Riemann Surfaces can be given a complex structure, w_2 is really the Euler class $mod 2$ which is 0 and hence a Spin structure always exists. On a Riemann Surface of genus g there should be 2^{2g} spin structures. These come by doubling the holonomy around each of the generating S^1's.

2.2.19. $\mathbb{CP}^2$ does not admit a Spin structure. To see this, the cohomology ring $H^*(\mathbb{CP}^2) = \mathbb{Z}/2[x]/x^2$ tells us that the Wu class $v_2 = x$ and hence $w_2 = v_2 \neq 0$ and hence no Spin structure.

w_2 of 2 or 3 dimensional manifolds are always 0 and hence they always admit Spin structures.

2.2.20. We use a different notation here. Let X be given a spin structure $Spin(X) : X \to BSpin(n)$ which will be a $Spin(n)$ *left* principal bundle. We tensor this with the Spin representation $\mathbb{S}$ to get a vector bundle over X, this is called the Spinor bundle $\mathbb{S}(X) = \mathbb{S} \otimes Spin(X)$. A spinor field on X is a section ψ of the bundle $\mathbb{S} \otimes Spin(X)$.

Let ∇ denote the Levi Civita connection on $SO(X)$ we abuse notation and use the same symbol for all connections of interest, because the Christoffel symbols for all of them are really the same.

We can lift ∇ to $Spin(X)$ via the double covering map $Spin(X) \to SO(X)$ and it would still be a connection because $Spin(n)$ and $SO(n)$ share their Lie algebras. And finally use the construction of transferring a connection from a principal bundle to a vector bundle to get a connection on $\mathbb{S}(X)$. Now the Dirac operator $\mathcal{D}$ is the following composition:

$$\Gamma(\mathbb{S}(X)) \xrightarrow{\nabla} T^*X \otimes Spin(X) \otimes \mathbb{S} \xrightarrow{g} TX \otimes Spin(X) \otimes \mathbb{S} \xrightarrow{\cdot} \Gamma(\mathbb{S}(X))$$

The first arrow is the connection on $\mathbb{S}(X)$, the second arrow is the Riemannian metric on X and the third arrow is the Clifford multiplication.

A lot of subtle things need to be emphasized about the above definitions,

(1) $Spin(X)$ is a left $Spin(n)$ bundle instead of the usual right bundle. This I guess is to have the Dirac operator act on left as differential operators normally do
(2) $\mathbb{S}(X)$ is not canonical and there are $H^1(X;\mathbb{Z}/2)$ many of them up to isomorphism
(3) The standard Spin representation comes with a Hermitian inner product which induces a fiber wise inner product on each fiber and the representation being unitary the inner product is compatible with the original metric and the induced connection
(4) If an orthonormal coordinate system were to exist at each point locally we could get rid of the metric and the cotangent bundle from our system. Another way of saying this is that $\mathcal{D}^2$ would be $-$ Laplacian for flat metrics.

Let e_i form a local orthonormal system of vector fields on X and η_i be the corresponding dual basis. Let s be a local section of $\mathbb{S}(X)$. Then,

$$\mathcal{D}s = e_i.\nabla_{e_i} s$$

We can use the above definition for an arbitrary $Spin(n)$ representation E endowed with a compatible connection.

2.2.23. Using partitions of unity assume that s, t are spinor fields with compact support and that an orthonormal system of vector fields (e_i) exist on the base,

$$\begin{aligned}
\langle \mathcal{D}s, t\rangle &= \int \langle \mathcal{D}s, t\rangle dX \\
&= \int \langle e_i.\nabla^i s, t\rangle dX \\
&= \int \langle -\nabla_i s, e_i.t\rangle dX && \text{see 2.1.40} \\
&= \int \langle s, \nabla_i e_i.t\rangle - e_i\langle s, e_i.t\rangle dX && \text{compatibility of metric and connection} \\
&= \langle s, \mathcal{D}t\rangle - \int_X e_i\langle s, e_i.t\rangle dX \\
&= \langle s, \mathcal{D}t\rangle && \text{Stoke's theorem}
\end{aligned}$$

2.2.25. On S^1 the two $\mathbb{S}(S^1)$ bundles can be identified as $[0,1]\times\mathbb{C}/(0,t)$ $(1,\lambda t)$ where $\lambda = \pm 1$. The tangent bundle is trivial with an orthonormal vector field $\frac{\partial}{\partial\theta}$ and the dual form $d\theta$ and theta should act by Clifford multiplication, hence for any section f for either of the two bundles,

$$d\theta.f = if$$

On $\mathbb{C}$ there is only the trivial $Spin(2) \cong U(1)$ bundle $\mathbb{C}\times\mathbb{C}^2$ which would split up as two representations $\mathbb{S}^+(\mathbb{C})$ and $\mathbb{S}^-(\mathbb{C})$ and under the standard representation

$$\partial x_1 \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \partial x_2 \mapsto \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$$

And so for a spinor field (f, g) the Dirac operator will be given by

$$\mathcal{D} = \begin{bmatrix} i\partial x_2 & \partial x_1 \\ -\partial x_1 & i\partial x_2 \end{bmatrix}$$
$$\mathcal{D}(f,g) = (\frac{\partial g}{\partial x_1} + i\frac{\partial f}{\partial x_2}, -\frac{\partial f}{\partial x_1} + i\frac{\partial g}{\partial x_2})$$

2.2.27.

$$\begin{aligned}
\mathcal{D}(fs) &= e_i.\nabla_i fs \\
&= e_i.((e_i f)s + f\nabla_i s) \\
&= e_i.((e_i f)s) + f\mathcal{D}s \\
&= \eta^i.((e_i f)s) + f\mathcal{D}s && \text{because we have an orthonormal system} \\
&= (e_i f)\eta^i s + f\mathcal{D}s && \text{because the Clifford action is fiber wise} \\
&= df.s + f\mathcal{D}s
\end{aligned}$$

2.2.33. Let us denote the connection on the bundle by $\nabla^{\mathbb{S}}$ from here on to avoid confusion.

$$\begin{aligned}
\mathcal{D}^2 s &= (e_i.\nabla_{e_i})(e_i.\nabla_{e_i} s) \\
&= e_i.e_j(\nabla_i \nabla_j s) \\
&= -\nabla_i \nabla_i s - \sum_{i<j} (e_i.e_j.(\nabla_i \nabla_j - \nabla_j \nabla_i)(s)/2 \\
&= -\sum_i \nabla_i \nabla_i s - \sum_{i<j} (e_i.e_j.R^{\mathbb{S}}_{i,j} s)/2 \\
&= (\overline{\nabla}\nabla - \sum_{i<j} (e_i.e_j.R^{\mathbb{S}}_{i,j})/2)s
\end{aligned}$$

I do not understand how the next step works but applying converting $R^{\mathbb{S}}$ into the curvature for X gives us that the second term is just the scalar curvature of X $R_{Sc}/4$ giving us

$$\mathcal{D}^2 = \overline{\nabla}\nabla + R_{Sc}/4$$

2.2.35. Suppose f is a harmonic spinor,

$$\begin{aligned}
0 &= \langle \mathcal{D} f, \mathcal{D} f \rangle \\
&= \langle \mathcal{D}^2 f, f \rangle \\
&= \langle \nabla^* \nabla f, f \rangle + R/4.\langle f, f \rangle \\
&= \langle \nabla f, \nabla f \rangle + R/4.\langle f, f \rangle \\
&> 0
\end{aligned}$$

2.2.36. On $\mathbb{C}$ the spinor $(ix_1 - x_2, ix_1 + x_2)$ is harmonic.

3. Generalized Dirac operators

This is quite straight forward. Suppose $V \to X$ is a Hermitian bundle with a connection. Now we have two bundles $\mathbb{S}(X)$ and V, let their inner products be $\langle,\rangle^{\mathbb{S}}$ and $\langle,\rangle^{V}$ and their connections be $\nabla^{\mathbb{S}}$ and ∇^{V}.

Then on $\mathbb{S}(X) \otimes V$ it is easy to check that $\nabla^{\mathbb{S}} \otimes 1_V + 1_{\mathbb{S}} \otimes \nabla^V$ is a connection compatible with the hermitian metric $\langle,\rangle^{\mathbb{S}}.\langle,\rangle^{V}$ and so we can define the coupled Dirac operator on $\mathbb{S}(X) \otimes V$ as

2.3.4?

$$\mathcal{D} = e_i.\nabla_i^{\mathbb{S}\otimes V}$$

2.3.4. As before one has to use

$$\mathcal{D}^2 = (\overline{\nabla}\nabla - \sum_{i<j}(e_i.e_j.R_{i,j}^{\mathbb{S}\otimes V})/2)$$

Again I do not know how to do the calculations.

2.3.6. Using the fact that $d = \epsilon(\eta^k)e_k$ we claim that $d^* = -\iota(\eta^k)e_k$. Assume first that the Christoffel symbols are all 0.

$$\begin{aligned}
\langle d\omega, \tau\rangle &= \langle \epsilon(\eta^k)e_k\omega, \tau\rangle \\
&= \langle e_k\omega, \iota(\eta^k)\tau\rangle \\
&= -\langle \omega, \iota(\eta^k)e_k\tau\rangle + e_k\langle \omega, \iota(\eta^k)\tau\rangle \quad && \text{here we used the fact that the Christoffel symbols are all 0} \\
&= -\langle \omega, \iota(\eta^k)e_k\tau\rangle && \text{by Stoke's theorem}
\end{aligned}$$

Now I am not sure if this is the valid argument, the codifferential is an operator which does not depend on the connection and hence this same formula should be valid even for non trivial connections.

And so we get

$$d + d^* = (\epsilon(\eta^k) + \iota(\eta^k))e_k$$

which looks very much like the Dirac operator for even dimensions.

Now $\mathbb{S}(X) \otimes \mathbb{S}(X)$ is just the bundle formed by using the representation $\mathbb{S} \otimes \mathbb{S}$ on $Spin(X)$, note that on $\mathbb{S} \otimes \mathbb{S}$ the action of $Spin(n)$ on the second $\mathbb{S}$ is trivial. Next we use the isomorphism $\mathbb{S}^* \cong \mathbb{S}$ and 2.1.40 to get the required isomorphism.

The kernel consists of harmonic forms by Hodge decomposition, that is exact and coexact forms are orthogonal to each other. I do not see how this part would be any different for odd dimensions.

2.3.7. This is again like the Hodge decomposition. The kernel will consist of harmonic forms in the even dimensions and the kernel of the adjoint would consist harmonic forms in the odd dimensions.

First note that,

$$\langle da, d^*b\rangle = \langle d^2a, b\rangle = 0$$
$$\langle d_i d_{i+1}^* a, a\rangle = \langle d_{i+1}^* a, d_{i+1}^* a\rangle$$

using which we get

$$ker(d_i + d_i^*) = kerd_i \cap kerd_i^*$$
$$kerd_i \cap imd_{i+1}^* = 0$$

I'll assume the Hodge decomposition, I do not know it's proof.

Proposition 3.1.

$$C^\infty(E_i) = imd_{i-1} \oplus imd^*_{i+1} \oplus (kerd_i \cap kerd^*_i)$$

From this and the previous identities we get,

$$H^i \cong kerd_i \cap kerd^*_i \cong ker(d+d^*)$$

The fact about Euler characteristic is then straightforward.

2.3.12. We are using the isomorphisms $\mathbb{S} \otimes \mathbb{S} \cong \mathbb{S}^* \otimes \mathbb{S} \cong Cliff^{\mathbb{C}}_n$. If $4 \mid n$ then $(\mathbb{S}^+)* \cong \mathbb{S}^+$ and hence the isomorphisms split up as

$$Cliff^{\mathbb{C}}_n \cong \mathbb{S}^+ \otimes \mathbb{S}^+ \oplus \mathbb{S}^- \otimes \mathbb{S}^-$$

If $4 \nmid n$ then $(\mathbb{S}^+)* \cong \mathbb{S}^-$ and hence the splitting is

$$Cliff^{\mathbb{C}}_n \cong \mathbb{S}^+ \otimes \mathbb{S}^- \oplus \mathbb{S}^- \otimes \mathbb{S}^+$$

This takes care of the first part.

Finally these are the even and odd parts because $Cliff^{\mathbb{C}+}_n$ is the invariant subspace of the transformation $x \mapsto \epsilon x \epsilon$ which under the above identification corresponds to the map $c(\epsilon) \otimes c(\epsilon)$.

The coupled Dirac operator operates on a single bundle and should have the same domain and target. One way to extend this is by introducing grading. So by $\mathbb{S}^+ - \mathbb{S}^-$ we mean that $\mathbb{S}^+$ has grading 0 and S^- has grading 1 . Say $4 \mid n$, then we have a series of isomorphisms,

$$\Omega^+_p(X) \cong \wedge^+(\mathbb{R}^n) \otimes \mathbb{C} \cong Cliff^{\mathbb{C}+}_n \cong \mathbb{S}^+ \otimes \mathbb{S}^+ \oplus \mathbb{S}^- \otimes \mathbb{S}^-$$

which is the 0 graded component of $\mathbb{S} \otimes (\mathbb{S}^+ - \mathbb{S}^-)$.

But we should avoid the grading altogether, though this also works, but the Atiyah Singer is for non-graded vector spaces. So another way of doing this is by saying that $-\mathbb{S}$ is isomorphic to $\mathbb{S}$ as a vector space, however the $Spin(n)$ action is reversed in sign, that it is $-\mathbb{S}$ as a representation. It is easy to see that this also gives the same results.

Then by the previous problem the index of the chiral $d+d^*$ is $\chi(X)$. We require compactness for the Hodge decomposition to hold.

2.3.15. One thing to be careful about is that the isomorphism $Cliff^{\mathbb{C}}_n \cong \wedge(\mathbb{R}^n) \otimes \mathbb{C}$ is not the Spin representation.

Assume that $e_1, \cdots, e_n$ is an orthonormal basis for $\mathbb{R}^n$. It is enough to check 2.3.17 for the element $e_1 \cdots e_p$. This formula only holds for n even.

$$\begin{aligned}
\epsilon . e_1 \cdots e_p &= i^{n(n+1)/2} e_1 \cdots e_n e_1 \cdots e_p \\
&= i^{n(n+1)/2} (-1)^{(n-1)+\cdots+(n-p)} (-1)^p e_{p+1} \cdots e_n \\
&= i^{n(n+1)/2} (-1)^{1+\cdots+p-1} e_{p+1} \cdots e_n && \text{as } n \text{ even} \\
&= i^{n(n+1)/2+p(p-1)} * e_1 \cdots e_p
\end{aligned}$$

Similar calculation gives that multiplication on the right is $i^{n(n+1)/2+p(p+1)}*$.

Now we again invoke the isomorphism $Cliff^{\mathbb{C}}_n \cong \mathbb{S} \otimes \mathbb{S}$ where the action is only the first component to get $\wedge_+(\mathbb{R}^n) \cong \mathbb{S}^+ \otimes \mathbb{S}$.

2.3.20. I do not think stuff can get any more confusing. The thing to note here is that $\Omega_+(X)$ is different from the even forms $\Omega^+(X)$.

The result is straightforward as we saw earlier that the operator $d + d^*$ is the Dirac operator for $V = S$ and hence takes $\mathbb{S}^+$ to $\mathbb{S}^-$. The only question left to answer is the identification of the kernel of $d + d^*$.

The kernel would consist of harmonic forms of the form $\alpha + \tau(\alpha)$ and the kernel of the adjoint would consist of elements of the form $\alpha - \tau(\alpha)$. It is an easy that if α is harmonic then so is $\tau(\alpha)$. A thing to note here is that τ does not commute with d or d^* alone and hence does not descend to $H^*(X)$ without the identification using Hodge's theorem.

2.3.22. The best trick for calculating the trace is noting that if the pairing matrix is orthonormal then the original

For $\mathbb{CP}^2 n$ the cohomology is $\mathbb{C}[x]/x^{2n+1}$ and hence the signature is just 1 .

For $(S^1)^{4n}$ the standard basis has the pairing matrix as a permutation matrix with no fixed points and hence the signature which will just be the trace in this case will be 0.

By looking at the ± 1 eigenvalues it is easy to see that $Sign(X \times Y) = Sign(X).Sign(Y)$.

2.3.24. This is really interesting.

First note that Poincare duality tells that the pairing is non-degenerate.

Next from the previous question we have,

$$index(d + d^*) = \dim\{\alpha + \tau(\alpha) \mid (d + d^*)\alpha = 0\} - \dim\{\alpha - \tau(\alpha) \mid (d + d^*)\alpha = 0\}$$

If a harmonic form α is not in the middle dimension then both $\alpha + \tau(\alpha)$ and $\alpha - \tau(\alpha)$ are distinct and non-zero and Hodge's decomposition tells us that they are all linearly independent, so these forms do not contribute anything to the index.

Finally as noted above τ descends to $H^*_\nabla(X)$ and hence we can redefine the index as

$$index(d + d^*) = \dim\{\alpha \in H^{n/2}_\nabla(X) \mid *\alpha = \alpha\} - \dim\{\alpha \in H^{n/2}_\nabla(X) \mid *\alpha = -\alpha\}$$

It is a triviality to see that this is indeed the signature!

2.3.25. If as above $\tau'(\alpha) = i^{n(n+1)/2+p(p+1)}\alpha$ is now right multiplication by ϵ then the Dirac operator will correspond to $\mathbb{S} \otimes \mathbb{S}^+$ and $\mathbb{S} \otimes \mathbb{S}^-$ and the bundle in terms of forms will be the eigenspaces of $\tau(\alpha)$. As before the map will be defined over reals if the $n(n+1)/2$ is even, in particular if $4 \mid n$.

The Dirac operator would corresponding to the map $\mathbb{S}^+ \otimes \mathbb{S}^+ \to \mathbb{S}^+ \otimes \mathbb{S}^-$. $\mathbb{S}^+ \otimes \mathbb{S}^+$ is the space invariant under the action of both τ and τ', this is also the same thing as saying the space invariant under τ and $\tau \otimes \tau' = c(\epsilon) \otimes c(\epsilon)$ Let us write this as

$$d + d^* : \Omega^+ \cap \Omega_+ \to \Omega^+ \cap \Omega_-$$

I am not sure what the complex 2.3.26 means, but we can identify the index as,

$$\begin{aligned} index(d + d^*) = &\dim\{\alpha + \tau(\alpha) \mid (d + d^*)\alpha = 0,\ |\alpha| \text{ even }\} \\ &- \dim\{\alpha - \tau(\alpha) \mid (d + d^*)\alpha = 0\ |\alpha| \text{ odd }\} \end{aligned}$$

This then equates to $\sum_{2i \neq n/2} \dim H^{2i}/2 - \sum \dim H^{2i+1}/2 + \beta$ where β is the dimension of the subspace of $H^{n/2}(X)$ on which the signature pairing is positive, so that $\beta = (\dim H^{n/2}(X) + Sign(X))/2$ and so we get the index of the operator is

$$(\chi(X) + Sign(X))/2$$

and the index of the anti dual operator will be $(\chi(X) - Sign(X))/2$

2.3.27. I am not sure what he is trying to say, but a Kahler manifold is simply an almost complex symplectic manifold and because $O(2m) \cap Sp(2m) = U(m)$ the structure group can be reduced from $O(2m)$ to $U(m)$ where $U(m)$ sits inside $O(2m)$ by the canonical identification $\mathbb{R}^2 \cong \mathbb{C}$.

2.3.29. On a complex manifold the almost complex structure is borrowed from the one on $\mathbb{C}^m$. In a local complex chart $(z^1 \cdots z^m) = (x^1, x^2, \cdots, x^{2m-1}, x^{2m})$, $\Omega^{p,q}(X)$ has a basis consisting of $dz^{i_1} \cdots dz^{i_p} dz^{i_{p+1}} \cdots dz^{i_{p+q}}$.

The $\overline{\partial}$ operator is locally given by $\overline{\partial} f_I dz^I = \frac{\partial f_I}{\partial \overline{z}_k} d\overline{z^k} dz^I$ where $\frac{\partial f}{\partial \overline{z}_k} = 1/2(\frac{\partial f}{\partial x_{2k-1}} + \sqrt{-1}\frac{\partial f}{\partial x_{2k}})$, $d\overline{z^k} = dx^{2k-1} + \sqrt{-1}dx^{2k}$. We can write this more succinctly as

$$\overline{\partial} = d\overline{z}^k \overline{\partial_k}$$

The formalism for d and $\overline{\partial}$ is exactly the same and hence by the same reasoning as for d we get that

$$\overline{\partial}^* = \iota(d\overline{z}^k)\partial_k$$

2.3.35. $\pi_1(U(n)) = \mathbb{Z}$ and hence a unique double cover $\tilde{U}(n)$ exists. We want to show that a map $X \to BSpin(2n)$ always lifts to $B\tilde{U}(n)$. The cofiber Y of $B\tilde{U}(n) \to BSpin(2n)$ is the same as the cofiber for $BU(n) \to BSO(2n)$ and so the map $X \to BSpin(2n) \to Y$ factors as $X \to BU(n) \to BSO(2n) \to Y$ which is 0 and hence a lift exists.

The correspondence follows from the fact that the fiber of both $BSpin(2n) \to BSO(2n)$ and $B\tilde{U}(n) \to BU(n)$ is $B\mathbb{Z}/2$.

2.3.36. This follows directly from 2.1.54 no work needed. We had already seen how this works out for the real case, the complex case is entirely analogous.

2.3.39. By Dolbeault's theorem this has the same cohomology as the operator $(d + d^*)/2$ and hence we get that the index is $\chi(X)/2$, so that for a genus g surface it will be $1 - g$.

4. Atiyah Singer Index theorem

There are a lot of theorems about Dirac operators, which sadly also include the final Atiyah Singer Index theorem, which I do not comprehend. I'd have loved to understand these theorems in detail but they involve too much analysis.

Proposition 4.1.
- *$\mathcal{D}^2$ is a self adjoint operator and only has countably many non-negative eigenvalues, say λ_n with $\lambda_n < \lambda_{n+1}$.*
- *For any $a > 0$ there are finitely many $\lambda_n < a$.*
- $\lim_{n\to\infty} \lambda_n = \infty$.
- *The* eigenspinors *of $\mathcal{D}^2$, ψ_n satisfying $\mathcal{D}^2\psi_n = \lambda_n\psi_n$ are smooth.*
- *$\mathcal{D}$ and $\mathcal{D}^2$ are Fredholm operators.*

5.1.8. There is a slight error in the identifications. We have the isomorphisms,

$$Spin(n) \subset Cliff_n^{\mathbb{C}+} \cong Cliff_{n-1} \cong Aut(\mathbb{S}^+ \otimes \mathbb{S}^-)$$

And so $Cliff_n^{\mathbb{C}+} \cong Aut(\mathbb{S})^+$ and $Cliff_n^{\mathbb{C}} \cong Aut(\mathbb{S})$.

5.1.10. The orientation is a top form with norm 1. Because top forms form a 1 dimensional vector space there is only one such candidate.

5.2.2. This is the crucial lemma connecting the Dirac operator to the trace of the heat kernel.

5.5.1. Since the 2x2 blocks do not interact with each other it is enough to find $\hat{A}$ for a single block. Let $\Omega_0 = \begin{bmatrix} 0 & 2\pi x \\ -2\pi x & 0 \end{bmatrix}$. We use the identity $\exp(\begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix}) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and that $\sinh(x) = (e^x - e^{-x})/2$ to get $\sinh(\Omega_0/4\pi i) = \begin{bmatrix} 0 & \sin x/2i \\ -\sin x/2i & 0 \end{bmatrix}$. So that,

$$\hat{A}(\Omega_0) = \frac{x}{\sinh x}$$

Now for the fact that the curvature can be written in this form, we only need to do this over a point. This is because the $\hat{A}$ only depends on the values of the curvature at a point and not at it's rate of change. Now over a point this follows from the simple fact the curvature is an antisymmetric bilinear form on TX taking values in $T^*X \otimes T^*X$ and hence we can pick a good basis. The x_i 's should be 2 tensors, that these themselves are antisymmetric follows from the symmetries of the curvature.

5.5.4. The induced connection is *defined* as $\nabla^{V_\rho} = \dot{\rho}(\nabla^X)$ and hence the curvature also has the same form.

Because $\dot{\rho}$ is the differential of ρ, $Tr(e^{i/2\pi\dot{\rho}(\Omega_0)}) = Tr(\rho(e^{i\Omega_0/2\pi}))$. Same computation as before shows that $e^{i\Omega_0/2\pi}$ is matrix with 2x2 blocks where each block is of the form $\begin{bmatrix} \cos ix_j & \sin ix_j \\ -\sin ix_j & \cos ix_j \end{bmatrix}$.

Let Ω_0^1 be the matrix obtained by interchanging x_1 and x_2 . It is easy to see that in this case $e^{i\Omega_0/2\pi}$ and $e^{i\Omega_0^1/2\pi}$ are conjugates of each other in $Spin(n)$ and hence so are their images under ρ. The result follows from the fact that conjugates have the same trace.

By definition this is nothing but the Chern character.

5.5.8. The first part is just substituting in the computation already done in 2.1.48. Finding invariant expressions:

$$(-1)^l \prod_j x_j = (-1)^l \sqrt{\det(\Omega/2\pi)}$$

$$\prod_j \frac{x_j}{\tanh x_j/2} = \sqrt{\det \frac{\Omega/2\pi i}{\tanh \Omega/4\pi i}}$$

$$\prod_j \frac{x_j}{\tanh x_j} = \sqrt{\det \frac{\Omega/2\pi i}{\tanh \Omega/2\pi i}}$$

The amazing thing is that the two expressions in tanh have the same degree l. Because each x_i is a degree 2 form this is saying that the top form of degree $2l$ sitting in each of the expressions is the same for both and hence we could use either of the two for integrating.

To show that the degree l term is the same in both, notice that at 0 the degree l term in the Taylor series expansion of $\prod \frac{x_i/2}{\tanh x_i/2}$ is the same as the degree l term in $\prod \frac{x_i}{\tanh x_i}$ but watered down by the factor of 2^l.

$\tanh x$ only has odd derivatives and hence the Taylor series expansion of $x/\ tanhx$ does not have any odd terms. When l is odd the degree l term hence is 0.

5.5.12. It is quite straightforward. Note that because there is an i involved this formula only makes sense when n is even. It might be an interesting question to ask if there any generalization of this to non spin manifolds or odd manifolds.

5.5.15. The Atiyah Singer index theorem in this case gives us the form 5.5.10, but then we use the equivalence of the degree l terms in 5.5.10 and 5.5.11 and get the desired result.

5.5.18. The coupling vector space for the self dual complex is $\mathbb{S}^+$, which gives us

$$P_{\mathbb{S}^+} = (P_{\mathbb{S}} + P_{\mathbb{S}-\mathbb{S}^-})/2$$

5.5.19. For the representation $\rho = \det^{1/2}$ the lie algebra map is $\dot{\rho} = trace/2$ and so we get the expression for $\chi_{\det^{1/2}}$ and we get the expression for $P_{\det^{1/2}}$ by expanding out $\sinh x$.

Finally Hirzebruch Riemann Roch follows from Atiyah Singer applied to $det^{1/2}$.