

Linear Algebraic Groups

- Krishna

Ref: Humphreys - LAG.

Overview:

Prerequisites:

K - algebraically closed field

A_K^n - affine n -space over $K \cong K^n = \{(x_1, \dots, x_n) \mid x_i \in K\}$

$f(T) \in K[T_1, \dots, T_n]$ $T := (T_1, \dots, T_n)$

• $V(I)$, \mathbb{A}^n $f(x)$

zero sets of I
closed sets in
Zariski A_K^n .

Hilbert basis th^m:

R noetherian

$\Rightarrow R[x]$ noetherian

Ex: check that this forms a topology.

• Affine variety X : zero set of closed sets of A^n .

• Nullstellensatz:

$$\sqrt{V(I)} = V(I)$$

Ex:

Exercise: 1) Prove \sqrt{I} is an ideal

2) Prove $\sqrt{I} = \bigcap_{\mathfrak{p} \supseteq I} \mathfrak{p}$

1) Using need to show $f^k \in I, g^l \in I \Rightarrow$

\Rightarrow for some $m \in \mathbb{N}$ $(f+g)^m \in I$

Choose $m = k+l$.

2) $\sqrt{I} \subseteq \bigcap_{\mathfrak{p} \supseteq I} \mathfrak{p}$ is clear

$\mathbb{R} \geq$ Enough to show $\bigcap \mathfrak{p} = \sqrt{I}$. Two ways

So need to show f not nilpotent $\Rightarrow f \notin \mathfrak{p}$ not containing f

1) $R \rightarrow R_f$ localize at f
 $\mathfrak{m}_f^e \rightarrow \mathfrak{m}_f$ look at max ideal, inverse

2) Look at $U = \{I \mid f^n \notin I\}$
This has maximal for any n prime element.

Then

$$\{\text{affine variety}\} \xleftrightarrow[\mathfrak{g}]{\mathfrak{V}} \{\text{radical ideals}\}$$

$$\{\text{irreducible affine variety}\} \longleftrightarrow \{\text{prime ideal}\}$$

Ex. Ex:

Ex 3: Prove the above correspondence

4: Any variety will have finite set of ^{maximal} irreducible components, which is same as prime ideal decomposition. Also unique.

• Morphisms

Def 1

$$\text{Mor}(X, K) = K[T] / \mathfrak{g}(X) =: K[X]$$

In $K[X]$ we don't have any nilpotent elements (but we can have zero divisors). by Nullstellensatz $\mathfrak{g}(X)$ is radical

So we took it

This is called reduced.

Also same as saying $\sqrt{0} = 0 = \bigcap \mathfrak{p}$ prime

• Now we reverse the process:

Let R be f.g. reduced K algebra

$\Rightarrow \exists$ an exact seq

$$0 \rightarrow I \rightarrow K[T] \rightarrow R \rightarrow 0$$

This is ~~the~~ \mathfrak{g} corresponding affine variety.

• Morphism of affine varieties

$$X \in \mathbb{A}^n \quad Y \in \mathbb{A}^m$$

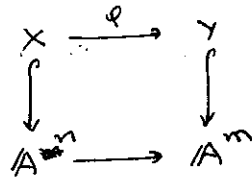
$$\varphi: X \rightarrow Y \quad \text{morphism of}$$

$$\varphi(x) \mapsto \varphi_1(x) \dots \varphi_m(x)$$

②

where ψ_1, \dots, ψ_m are polynomial functions on X i.e. in $K[X]$

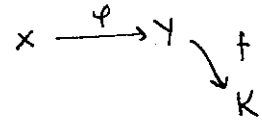
~~and~~



True for some representative of (ψ_1, \dots, ψ_m)

$$\varphi^*: K[Y] \longrightarrow K[X]$$

$$f \longmapsto f \circ \varphi$$



Ex:

Ex 5: well defined?

polynomial f^n ?

One can recover φ from φ^* by looking at ω -coordinate f^n 's.

$$\text{i.e. } \psi_i(x) = \varphi^{*n}(\psi_i)$$

• Next we need to give an intrinsic defⁿ of a variety.

$x \in X$, X irreducible

$\Rightarrow K[x]$ irreducible integral domain

$K(x)$ field of fractions of $K[x]$

- field of rational functions on X .

not really functions but functions on open subsets where denominators do not vanish

$U \in X$, $K(U)$ - ring of rational functions on U

such that $f \in K(x)$ s.t. f defined on U .

Ex 3, 4: Prime decomposition for Noetherian ring implies every prime ideal $I = \mathfrak{p}_1 \cap \mathfrak{p}_2 \dots \cap \mathfrak{p}_k$

\hookrightarrow minimal primes

unique upto ordering

$$\Rightarrow V(I) = V(\mathfrak{p}_1) \cup V(\mathfrak{p}_2) \dots \cup V(\mathfrak{p}_k)$$

Enough to show $\nexists (\mathfrak{p}_i)$ irreducible

Ex 5 :

$$[f_1] = [f_2] \Rightarrow f_1 - f_2 \in \mathcal{O}_x \setminus \mathcal{O}_x^\times$$

Also need to show $(f_1 - f_2)\phi \in \mathcal{O}_x$ i.e. $(f_1 - f_2) \circ \phi(x) = 0 \forall x \in X$

But this true $\because \exists \phi(x) \in \mathcal{O}_x$ and $f_1 - f_2(\gamma) = 0$.

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Aim: To define ~~vari~~ affine variety intrinsically

Ex:

$$\mathbb{A}^n_{\mathbb{A}^2_k}$$

Def: $D(f) := \mathbb{A}^n_k - V(f)$

$$\left. \begin{aligned} X_1 &= \mathbb{A}^2_k - \{(0,0)\} \\ X_2 &= D(T_1) \end{aligned} \right\} \text{ open}$$

Are these varieties in higher embeddings?
 X_1 No, X_2 Yes

X - irreducible affine variety

Defⁿ $K[X], K(x), \mathcal{O}_x := \left\{ \frac{f}{g} \in K(X) \mid g(x) \neq 0 \right\} \subseteq K(x)$
rational functions at $x \in X$

Ex 1. check \mathcal{O}_x is well defined.

$$\rightarrow K[X] \subseteq \mathcal{O}_x \subseteq K(x) \text{ as rings}$$

$$\rightarrow \mathcal{O}_x \text{ local ring. } \uparrow \\ \text{maximal ideal} - \left\{ \frac{f}{g} \mid f(x) = 0 \right\} \subseteq \mathcal{O}_x$$

Defⁿ $\mathcal{O}_x(U) := \bigcap_{x \in U} \mathcal{O}_x \leftarrow \begin{array}{l} U \text{ open in } X \\ \text{ring of rational f's on } U \end{array}$

$$\rightarrow K[X] \subseteq \mathcal{O}_x(U) \subseteq K(x) \text{ as rings}$$

\mathcal{O}_x - sheaf on rings

Defⁿ Variety:

X - irreducible topological space

\mathcal{O}_x - sheaf of rings on X

\exists finite open covering $\{U_i\}_{i \in I}$ of X s.t.

$$(U_i, \mathcal{O}_x|_{U_i}) \cong (V_i, \mathcal{O}_{V_i}) \quad \forall i,$$

\uparrow
affine variety

+ a topological condition.

Defⁿ:

$P^n_k \equiv$ Projective n -space / k
 $:= A^{n+1}_k / \sim$ Scalar
 - for
 Projective variety etc.

Defⁿ:

(Linear) Algebraic Group

 X - X variety, \ast groupbinary ~~operation~~ multiplication, inverse are morphisms of varieties- Linear if $X \subseteq GL_n(k)$ closed subgroupeg: $GL_n(k) \subseteq M_n(k) \cong A^{n^2}$ ~~char~~ $D(\det) \leftarrow$ Principal open $\Rightarrow GL_n(k)$ open in affine in A^{n^2+1} SL_n, SO_n, \dots, Sp_n

- Narsimhan Chary

Representation Theory
of finite groups $G \curvearrowright X$ From X what can be said about G ? V/k - vector space over k dim n Representation: $\rho: G \rightarrow GL(V)$ group homo
 \cong
 $GL_n(k)$ via a basis

ex:

1) $G \rightarrow GL(V)$

$g \mapsto \text{identity}$

2) $G \rightarrow GL(V) \cong k^*$ if $n=1$
character

3) $V = k[G]$

$G \rightarrow GL(V)$

$g \mapsto (x \mapsto gx)$

regular representation - left

4) X - finite $G \curvearrowright X$,
similar $V = k\langle X \rangle, G \rightarrow GL(V)$

Invariant Subspace, faithful representation, irreducible rep.
Maschke's Th^m.

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Product of Affine varieties

$$A^n \ni x - I_x \subseteq K[T_1]$$

$$y \in A^m - I_y \subseteq K[T']$$

$$x \times y \in A^{n+m} - (I_x + I_y) \subseteq K[T, T']$$

$$K[x \times y] = K[x] \otimes_K K[y].$$

X, Y irreducible $\Rightarrow X \times Y$ irreducible

i.e. $K[x], K[y]$ domains $\Rightarrow K[x] \otimes_K K[y]$ domain

(Not true in general, tensor product can make tensor-product domains into non-domains, and here $K[x]$ is f.g. algebra over K , hence it works).

$X \subseteq A^3 = \{(t, t^2, t^3) \mid t \in K\}$ show X is closed in A^3 .

$p: A^1 \rightarrow A^3$ call the image Y .

$$t \mapsto (t, t^2, t^3)$$

$\phi: A^1 \rightarrow Y$ is a homeomorphism.

but is not an isomorphism of varieties.

$$K[A^1] = K[T] \quad K[Y] = K[T_1, T_2] / (T_1^3 - T_2^2)$$

Dimension: $X \subseteq A^n$ for X -irreducible

1) $\dim X$ as topological space
i.e. height of prime ideals

2) Krull dim $K[X]$

3) transcendence degree of $K(x)$ over K .

Similarly for arbitrary varieties.

$$\text{What is } K(x)? \quad K(x) := \lim_{u \rightarrow} Q_x(u)$$

$X_f \subseteq X \subseteq A^n$ - affine X -irreducible.

Ex. $\dim X_f = \dim X$, X_f irreducible.

In general, \dim open subset of $X = \dim X$.