FINITE GAUGE GROUP TQFT'S

APURV

Contents

1. An $n+1$ TQFT

We will construct an $n+1$ TQFT using a finite group G. We will not worry about the physics aspects of the theory.

We need a functor

$$Bord_{\langle n,n+1\rangle} \to Vect_{\mathbb{C}}$$

where $Bord_{\langle n,n+1\rangle}$ is the category whose objects are *oriented* closed smooth manifolds and whose morphisms are diffeomorphism classes of bordims relative to the boundary. The smoothness condition is included so that we do not have to worry about composing bordisms. Also I haven't really thought about the requirement of orientation on our manifolds, may be later.

1.1. n **dimensional manifolds.** This is the easy part. For X an n dimensional manifold we want to define $Z(X)$ to be the vector space dual to the vector space with basis elements the isomorphism classes of flat principal G bundles over X . But G being a finite group every bundle is flat and hence the basis is simply the isomorphism classes of all principal G bundles.

It follows from elementary topology that

Proposition 1.1.

$$Z(X) = \mathbb{C}\langle Hom(\pi_1(X), G)\rangle/G$$

where the action of G is by conjugation. In particular this implies that $Z(X)$ is finite dimensional.

Proof. Principal G bundles are classified by unbased homotopy classes of maps $[X, BG]$ which is equal to $[X, BG]_*/G = Hom(\pi(X), G)/G$.

For the finiteness use the nerve lemma to see that $\pi_1(X)$ is always finitely generated. □

1.2. $n+1$ **dimensional manifold.** We want to do something analogous for $n+1$ manifolds. That is we should assign to an $n+1$ manifold Y with boundary X a function which assigns to each bundle P over X the number of bundles over Y which restrict to P . But this naive construction does not satisfy the gluing lemma and requires a little modification.

The way to rectify this is to identify points on the boundaries and use these points as basepoints. This is explained in the lecture notes on TQFT by Dan Freed.

A good way to keep track of these is using the language of groupoids.

2. Groupoids

This section is almost entirely from notes by Qiaochu Yuan.

Recall that a groupoid is a small category in which all the morphisms are invertible. We also assume that each object is endowed with a canonical identity automorphism.

Let *Gpd* be the category of all groupoids with morphisms being functors between groupoids. Of interest to us is the category $FinGpd$ of finite groupoids, that is every point has only finitely many automorphisms and there are finitely many connected components. Note that pullbacks exist in this category.

The reason for looking at groupoids is that the space of principal G bundles over a space X forms a groupoid with morphisms being bundle automorphisms. Further as noted earlier when X is a finite CW complex, in particular a manifold, this groupoid is finite.

As noted earlier it is not enough to keep track of the bundles over a manifold with boundary but also to identify certain points on the boundary so that the gluing axiom holds true. The way to do this categorically is to use look at the category of spans in $FinGpd$,

$$Obj(Span(FinGpd)) = Obj(FinGpd)$$
$$Mor(X,Y) = \{X \leftarrow S \rightarrow Y\}$$

Composition of morphisms is defined by pullbacks,

$$(Y \xleftarrow{g} T \rightarrow Z).(X \leftarrow S \xrightarrow{f} Y) := (X \leftarrow S \times_Y T \rightarrow Z)$$

It is interesting to check associativity.

2.1. **Push-Pull.** Let $\mathbb{C}$ denote the trivial groupoid over $\mathbb{C}$, that is only morphisms are the trivial morphisms. For a groupoid X let $\mathbb{C}^X$ denote the space of functors from X to $\mathbb{C}$. $\mathbb{C}^X$ carries as natural vector space structure and any functor $f \in \mathbb{C}^X$ descends to $f : \pi_0(X) \rightarrow C$. We denote by $Aut[x]$ the size of automorphism group $Aut(x)$ which does not vary across a connected component.

For a functor $F : X \rightarrow Y$ we can define the pullback F^* and the 'integration on fibers' F_* as follows,

$$F^* : \mathbb{C}^Y \rightarrow \mathbb{C}^X, f \mapsto x \mapsto f(F(x))$$

$$F_* : \mathbb{C}^X \rightarrow \mathbb{C}^Y, f \mapsto y \mapsto \sum_{[x]\in\pi_0(X):F([x])=[y]} \frac{f([x])}{Aut[x]} Aut[y]$$

I am not sure how this relates to the integration over fibers for a fiber bundle :(

F^* and F_* are related by the following functional,

$$\int_X : \mathbb{C}^X \rightarrow \mathbb{C}, f \mapsto \sum_{[x]\in\pi_0(X)} \frac{f([x])}{Aut[x]}$$

which is supposed to mean integrating over X. Note that Freed in his notes does this by assigning instead the measure $|Aut[x]|^{-1}$ to $[x]$.

This gives us an inner product on $\mathbb{C}^X$ as

$$\langle f, g\rangle = \int_X \bar{f}g$$

Proposition 2.1. *For $f \in \mathbb{C}^X$ and $g \in \mathbb{C}^Y$*

$$\langle F_*f, g\rangle = \langle f, F^*g\rangle$$

Proof.

$$\begin{aligned}
\langle F_*f, g\rangle &= \sum_{[y]\in\pi_0(Y)} \frac{\overline{F_*f([y])}g([y])}{Aut[y]} \\
&= \sum_{[y]\in\pi_0(Y)} \Big(\sum_{[x]\in\pi_0(X):F([x])=[y]} \frac{Aut[y]}{Aut[x]}\overline{f([x])}\Big)\frac{g([y])}{Aut[y]} \\
&= \sum_{[x]\in\pi_0(X)} \frac{\overline{f([x])}g([F(x)])}{Aut[x]} \\
&= \langle f, F^*g\rangle
\end{aligned}$$

□

Proposition 2.2. *For $F : X \rightarrow Y$ and $G : Y \rightarrow Z$, $GF^* = G^*F^*$ and $FG_* = F_*G_*$*

Proposition 2.3. *The first part is trivial. The second part follows from*

$$\begin{aligned}\langle FG_*f, g\rangle &= \langle f, FG^*g\rangle \\ &= \langle f, G^*F^*g\rangle \\ &= \langle F_*G_*f, g\rangle\end{aligned}$$

2.2. **Push-Pull for Spans.** We now define a functor from the category of Spans to that of finite dimensional vector spaces.

$$\mathbb{C}^- : Span(FinGpd) \to FinVect$$
$$(X \xleftarrow{p} S \xrightarrow{q} Y) \mapsto q_*p^* : \mathbb{C}^X \to \mathbb{C}^Y$$

that is

$$\mathbb{C}^{(p,q)} \mapsto f \mapsto y \mapsto \sum_{[s]\in\pi_0(S):q[s]=[y]} \frac{f(p[s])}{Aut[s]} Aut[y]$$

Proposition 2.4. $\mathbb{C}^-$ *is a functor.*

Proof. That the compositions are well defined is reduced to checking the following fact. For a pullback diagram
$$\begin{tikzcd} X \times_U Y \arrow[r, "j"] \arrow[d, "i"'] & Y \arrow[d, "q"] \\ X \arrow[r, "p"] & U \end{tikzcd}$$
the following holds

$$q^*p_* = j_*i^*$$

I am going to do this check by hand. For a functional $f \in \mathbb{C}^X$ and an element $y \in Y$

$$q^*(p_*f)y = \sum_{[x]\in\pi_0(X):p([x])=q([y])} \frac{f([x])}{Aut[x]} Aut[q(y)]$$

and

$$j_*(i^*f)y = \sum_{[x]\in\pi_0(X):p([x])=q([y])} \frac{f([x])}{Aut[(x,y)]} Aut[y]$$

This is false! The problem is that we can add a lot of redundant automorphisms to U thereby only altering the term $Aut[q(y)]$ in the above expressions. To rescue this we need some control over the automorphism groups of U, one way to do this is by looking at G sets instead of arbitrary groupoids.

Proof unfinished. □

2.3. **Finite G sets.** We restrict over attention to the category of finite G sets which is a subcategory of *FinGpd* and which is closed under taking pullbacks. A lot of the identities simplify for this subcategory.

Let $p : X \to Y$ be a functor between finite G sets then

$$\int_X : \mathbb{C}^X \to \mathbb{C}, f \to |G|^{-1} \sum_{x\in X} f(x)$$

$$p_* : \mathbb{C}^X \to \mathbb{C}^Y, f(y) = \frac{Aut[y]}{|G|} \sum_{x \in p^{-1}([y])} f(x)$$

For spans $X \xleftarrow{p} S \xrightarrow{q} Y$ we get

$$\mathbb{C}^{(p,q)} \mapsto f \mapsto y \mapsto \sum_{s \in S:[q(s)]=[y]} \frac{f(p(s))}{|G|} Aut[y]$$

And this is not even the best part. Connected G sets are parametrized by subgroups of G and any map either surjects onto a connected component or misses it completely which simplifies our life a great deal.

Now to prove the theorem that we proved above it suffices to look at a pullback diagram of the form

$$\begin{tikzcd} & G/K \arrow[d, "q"] \\ G/H \arrow[r, "p"] & G/L \end{tikzcd}$$

where $K, H \leq L$. To see this note that we can analyze one connected component of U at a time

G/H and G/K are connected groupoids and hence the maps $\mathbb{C}^{G/H} \to \mathbb{C}^{G/K}$ is just a real number, that we need to see where the constant function 1 goes.

Simple computation gives us $q^* p_* 1 = \frac{|L|}{|G|} \frac{|G|}{|H|} = \frac{|L|}{|H|}$.

And $j_* i^* 1 = \sum_{G/H \times_{G/L} G/K} \frac{1}{|G|} |K| = \frac{|G/H \times_{G/L} G/K|.|K|}{|G|}$

But these two are then equal from the fact that

$$|G/H \times_{G/L} G/K| = \frac{|G/H|.|G/K|}{|G/K|} = \frac{|L|.|G|}{|H|.|K|}$$

2.4. G **sets.** Unfortunately this is not enough for us and we need to extend the above result from the category of finite G sets to the category of G sets which need not themselves be finite but are equivalent to some finite G set.

This follows then by saying that if we have G sets X, Y, U such that the horizontal arrows are equivalences,

$$\begin{tikzcd} X \arrow[r] \arrow[d] & G/H \arrow[d] \\ U \arrow[r] & G/L \\ Y \arrow[r] \arrow[u] & G/K \arrow[u] \end{tikzcd}$$

then this extends to an equivalence

$$X \times_U Y \to G/H \times_{G/L} G/K$$

such that all the resulting squares commute and the maps $q^* p_*$ and $j_* i^*$ induce the same maps.

It's too much effort to write all the details for this, hope there is no hole in the argument.

2.5. **Some Categories.** Now we look at the categories that occur naturally and pertinent to the current problem.

2.6. **Fundamental Groupoid.** To every topological space X we can associate the fundamental groupoid $\Pi_1(X)$ whose objects are the points of X and whose morphisms are the homotopy classes of paths between them.

The reason we look at this groupoid instead of the usual fundamental group, even though the two are equivalent as categories, is that the fundamental groupoid is canonical where as the fundamental group involves choosing a basepoint. Where as this is no big issue while making explicit calculations when we try to build a theory this lack of canonical structure makes it really difficult to 'glue' things.

It is in some sense reminiscent of the local global aspect of the tangent space of a manifold. For local computations, it is enough to work with vector fields and flows as locally a manifold imitates the Euclidean space, and this is the point of view a lot of people working with solving differential equations use. But if we are interested in the global structure it is much more convenient to talk of the tangent bundle as a sheaf of germs. Though this only complicates local computations, it helps to study global 'canonical' properties of a manifold.

2.7. **Principal G bundles.** Let BG be the groupoid with just one object and the automorphism group G for a compact group G . Of interest to us is only the case when G is a finite group.

Definition 2.5. $G(X) := BG^{\Pi_1(X)}$ that is the space of functors from $\Pi_1(X)$ to BG.

Note that there is a canonical right action of G on this space, which makes this space into a G set and hence also a groupoid. (Any G set has a canonical groupoid structure coming from the group action.)

Proposition 2.6. *$\pi_0(G(X))$ is isomorphic to the moduli space of principal G bundles over X. For our case as G is a finite group this is the space of all principal G bundles.*

Proof. It is not hard to see that the category $G(X)$ is equivalent to the category $Hom(\pi_1(X), G)$ as 'G groupoids' and hence we need to show simply that the moduli of flat G bundles over X is isomorphic to $\pi_0(Hom(\pi_1(X), G))$.

But flat G bundles are the same as principal G bundles where G is given the discrete topology and hence the moduli space of principal G bundles is precisely what we wanted. □

For a finite group G we extend $G(-)$ to a functor $G : Bord_{\langle n,n+1\rangle} \to Span(FinGpd)$ which sends

$$X^n \mapsto G(X)$$
$$(X_1^n \xrightarrow{p} Y^{n+1} \xleftarrow{q} X_2^n) \mapsto (G(X_1) \xleftarrow{p^*} G(Y) \xrightarrow{q^*} G(X_2))$$

Proposition 2.7. *G is functor .*

Proof. Follows from Seifert Van Kampen theorem for fundamental groupoids. □

3. The TQFT

Finally with the machinery of groupoids we can return to our quest of defining a TQFT.

Definition 3.1.

$$Z_G : Bord_{\langle n,n+1\rangle} \to FinVect$$

is defined to be the composition of functors

$$Bord_{\langle n,n+1\rangle} \xrightarrow{G} Span(FinGpd) \xrightarrow{\mathbb{C}^-} FinVect$$

Proposition 3.2. *Z_G is a TQFT, that is it is a symmetric monoidal functor.*

Proof. $Z_G(X_1 \sqcup X_2) = Z_G(X_1) \otimes Z_G(X_2)$ follows from definition.

G maps the identity morphism $X \xrightarrow{i_0} X \times [0,1] \xleftarrow{i_1}$ to $G(X) \xleftarrow{i_0^*} G(X \times [0,1]) \xrightarrow{i_1^*} G(X)$. Both i_0^* and i_1^* are isomorphisms and hence $\mathbb{C}^-$ maps this to the identity linear transformation. (I should have checked this earlier.)

The gluing axiom follows from the fact that both G and $\mathbb{C}^-$ are functors. □

More explicitly, Z_G sends a bordism $X_1 \to Y \leftarrow X_2$ to a map $Z_G(Y) : Z_G(X_1) \to Z_G(X_2)$. Say $\tilde{P}_i \in Z_G(X_i)$ is dual to the principal G bundle P_i over X_i then the functional $Z_G(Y)\tilde{P}_1$ on P_2 takes the value

$$\sum_{[P]\in\pi_0(Y)} \frac{1}{Aut[P]} Aut[P_2]$$

where P varies over those principal G bundles over Y which restrict to P_1 and P_2 on the boundary manifolds.

3.1. **Closed $n+1$ manifolds.** In particular if Y^{n+1} is a manifold without boundary, the morphism $* \to Y \leftarrow *$ is mapped to the complex number

$$\sum_{[P]\in\pi_0(G(Y))} (Aut[P])^{-1}$$

.

We can further simplify this by recalling that the groupoid $G(Y)$ is isomorphic to the G set $Hom(\pi_1(Y), G)$ where G acts via conjugation.

For the G set $U = G/H$ there is just one orbit and the automorphism group is H. And the sum can be rewritten as

$$|H|^{-1} = \frac{|U|}{|G|}$$

This formula generalizes to the case when there are more than 1 orbits and we get

$$Z_G(Y) = \frac{|Hom(\pi_1(Y), G)|}{|G|}$$

4. Dimension 1 Computations

Here our TQFT is a functor

$$Bord_{\langle 0,1\rangle} \to FinVect$$

As a monoidal category $Bord_{\langle 0,1\rangle}$ is generated by ϕ and $*$ and the morphisms are generated by $0 \to [0,1] \leftarrow 1$,$0 \cup 1 \to [0,1] \leftarrow \phi$ and $\phi \to [0,1] \leftarrow 0 \cup 1$.

$$G(*) = BG$$
$$Z(0 \to [0,1] \leftarrow 1) = \text{ the identity transformation}$$

Next we need to figure out $Z(0 \cup 1 \to [0,1] \leftarrow \phi) : \mathbb{C}^G \otimes \mathbb{C}^G \to \mathbb{C}$.

This is a notational nightmare. This map will send $g^\vee \otimes h^\vee$ to 0 if $g \neq h$ and there is precisely one G bundle over $[0,1]$ which restricts to $g^\vee \otimes g^\vee$ and it has G as the automorphism group. Hence, $g^\vee \otimes g^\vee \mapsto |G|^{-1}$.

The map $Z(\phi \to [0,1] \leftarrow 0 \cup 1) : \mathbb{C} \to \mathbb{C}^G \otimes \mathbb{C}^G$ should be somehow dual to the previous map. Again there is only one bundle on $[0,1]$ which restricts to $g^\vee \otimes g^\vee$ but this time the element $g^\vee \otimes g^\vee$ itself has $|G|^2$ many automoprhisms and the map should send 1 to the element $|G| \sum_{g \in G} g^\vee \otimes g^\vee$

Finally to S^1 we assign the numb. As noted earlier the groupoid over S^1 is the G set $Hom(\pi_1(S^1), G)$ which is isomorphic to the groupoid of G acting on itself. A simple computation then gives us that $Z_G(S^1) = 1$.

5. Dimension 2 computations

2 Dimensional TQFT's form a commutative Frobenius algebra. In our case this corresponds to the Frobenius algebra of class functions on a finite group.

Definition 5.1 (Frobenius algebra)**.** A commutative Frobenius algebra over a field k is a finite dimensional commutative algebra over k which is equipped with a linear trace map $Tr : A \to k$ such that the bilinear form $a, b \mapsto Tr(ab)$ is non degenerate. For us $k = \mathbb{C}$.

Theorem 5.2. *If Z is a 2 TQFT then $Z(S^1)$ has a natural structure of a commutative Frobenius algebra.*

Proof. The pair of pants defines a product on $Z(S^1)$ and the trace map is given by thinking of the disk D^2 as a bordism between S^1 and the null manifold.

Associativity and commutativity are immediate. For non degeneracy notice that $Tr(-.-)$ corresponds to the dual of the identity map corresponding to the bordism $[0,1] \times S^1$ thought of as a bordism between two copies of S^1 and the null manifold.

Note that the unit of the algebra is the element corresponding to $Z(D^2)$ thought of as a bordism from the null manifold to S^1.

For details see the paper of Atiyah or thesis of Lowell Abrams. □

It follows from Artin-Wedderburn theorem that in commutative Frobenius algebras the identity decomposes as sum of orthogonal idempotents, so that once we know what the idempotents are the entire algebra is completely determined.

5.1. **Handle element.** Every commutative Frobenius has a special element called the handle element $e(A)$ which geometrically corresponds to gluing $S^1 \times [0,1]$ to a pair of pants and then thinking of the resulting manifold as a bordism between ϕ and S^1. So we need to find the element corresponding to $\phi \to S^1 \times [0,1] \leftarrow S^1 \cup S^1$.

Now $Z(\phi \to D^2 \leftarrow S^1) = 1 = e_1 + \cdots + e_k$ where e_i are the orthogonal idempotents.

$Z(\phi \to S^1 \times [0,1] \leftarrow S^1 \cup S^1)$ is an element of the form $\sum_i c_i e_i \otimes e_i$ for some constants c_i that we need to figure out. If we cap off one of the S^1 we get $\phi \to$

$D^2 \leftarrow S^1$ which as noted above is the element 1. But capping off is the same thing as taking the trace, and we must have the identity

$$\sum_i c_i Tr(e_i) e_i = 1 = \sum_i e_i$$

which implies that $c_i = Tr(e_i)^{-1}$.

Attaching a pair of pants to this is simply multiplying in our Frobenius algebra, and so we get

$$e(A) = \sum_i Tr(e_i)^{-1} e_i \otimes e_i = \sum_i Tr(e_i)^{-1} e_i$$

5.2. **Riemann Surfaces.** Let us compute the numbers that our TQFT outputs for oriented genus g surfaces.

For S^2 we get the identity

$$Z(S^2) = Tr(1) = \sum_i Tr(e_i)$$

For $S^1 \times S^1$ we should get

$$Z(S^1 \times S^1) = Tr(e(A)) = \sum_i Tr(e_i)^{-1} Tr(e_i) = k$$

Finally for a genus g Riemann surface M^g it is not hard to see that M^g can be obtained by attaching g copies of the handle element to a pants with $k+1$ and capping out the last hole, this in our Frobenius algebra corresponds to

$$Z(M^g) = Tr(e(A)^g) = Tr(\sum_i Tr(e_i)^{-g} e_i) = \sum_i Tr(e_i)^{1-g}$$

which can be rewritten as

$$Z(M) = \sum_i Tr(e_i)^{\chi(M)/2}$$

where χ denotes the Euler characteristic. Note that this covers the previous two cases as well.

6. Representations of Finite groups

As noted in the previous sections $G(S^1)$ is isomorphic to the groupoid $Hom(\pi_1(S^1), G)$ as a G set where the action is via conjugation and hence the corresponding vector space associated to $Z_G(S^1)$ is the space of class functions.

It remains for us to determine the multiplication on $Z(S^1)$ and find the idempotents.

6.1. **Algebra structure of $Z_G(S^1)$.** Let $[g]$ denote the conjugacy class corresponding to the element $g \in G$ and let $[g]^\vee$ denote the corresponding dual class functions. Then we are interested in finding $[g]^\vee * [h]^\vee$.

Now notice that the pair of pants retracts onto a wedge of two circles and hence the bordism is simply given by the map $\pi_1(S^1) \to \pi_1(S^1 \vee S^1), 1 \mapsto 1 * 1$ where g corresponds to the map $\pi_1(S^1) \to G$ sending 1 to g or geometrically to the G cover of S^1 whose clutching function is g . And so we get

$$[g]^\vee [h]^\vee \mapsto [gh]^\vee$$

Finally the trace map is the Z_G of $S^1 \to D^2 \leftarrow \phi$. The only bundle on D^2 is the trivial bundle and it has G automorphisms and hence we get $Tr : Z(S^1) \to \mathbb{C}$ is the map sending $[e]$ to $|G|^{-1}$ and every other $[g]$ to 0. For an arbitrary class function f this is the same as saying

$$Tr(f) = |G|^{-1} f(e)$$

What we have obtained is the usual convolution on the space of class functions and hence the Frobenius algebra we obtain is the same as that we get in character theory of finite groups.

Let us recall some basic results from representation theory of finite groups.

6.2. **Schur's lemma.** We have the elementary but useful Schur's lemma,

Theorem 6.1. (Schur's Lemma) *Given irreducible representation V, W of G we have,*

$$Hom_{\mathbb{C}[G]}(V, W) = \begin{cases} \mathbb{C} & \text{if } V \cong W \\ 0 & \text{otherwise} \end{cases} \tag{6.1}$$

Proof. Follows by noticing that the kernel and image of a $G - invariant$ map are also $G - invariant$. □

The following corollary is just a reformulation of the above theorem but is computationally more useful,

Corollary 6.2. *For irreducible non-isomorphic G representations V, W we have, the $G-$invariant subset of $V^* \otimes_{\mathbb{C}} W$ is 0 and of $V^* \otimes_{\mathbb{C}} V$ is the one dimensional subspace generated by the identity map, where by V^* we mean the right $\mathbb{C}[G]-$module $Hom_{\mathbb{C}}(V, \mathbb{C})$ and the action of G on $V^* \otimes_C W$ is the diagonal action.*

If V has a basis $e_1, e_2, \cdots, e_n$ and W has a basis $f_1, f_2, \cdots, f_m$ then for a fixed element $h \in G$ we have that the G invariants of $V^* \otimes_{\mathbb{C}} W$ include terms of the form

$$\begin{aligned} \sum_{g \in G} e_i^* g^{-1} \otimes ghf_j &= \sum_{g \in G} (ge_i)^* \otimes ghf_j \\ &= \sum_{g} (\sum_{k} g_{i,k}^V e_k)^* \otimes (\sum_{l} (gh)_{j,l}^W f_l) \\ &= \sum_{k,l} (e_k^* \otimes f_l)(\sum_{g} \overline{g_{i,k}}^V (gh)_{j,l}^W) \end{aligned}$$

Corollary 6.3. *Using the notation as above, for all $0 < i, j \leq n$, $0 < k, l \leq m$ and $h \in G$*

(1) *for non-isomorphic V,W we have*

$$\sum_{g \in G} \overline{g_{i,j}}^V (gh)_{k,l}^W = 0$$

(2) *and if $V = W$ then*

$$\sum_{g \in G} \overline{g_{i,j}}^V g_{k,l}^V = |G|/(dimV).\delta_{i,k}\delta_{j,l}$$

Proof. Proof of the first equation follows straight from 6.1
For the second equation notice that

$$\sum_{g\in G}(gh)^* \otimes (gh)$$

is just a sum of one dimensional projections and hence cannot possibly be zero but it is also a $G-invariant$ sum and hence must be a multiple of unity. Now it is just a matter counting the number of terms to get the required constant $|G|/(dimV)$. □

6.3. **Orthogonality Relations.** These relations are direct consequences of 6.1 and lead to the very non-intuitive result that every finite representation is uniquely determined by its character.

Theorem 6.4. (First Orthogonality Relations) *For irreducible irreducible representations V, W we have*

(1) *If V and W are non-isomorphic*

$$\chi_V * \chi_W = 0$$

(2)

$$\chi_V * \chi_V = |G|/(dimV)\chi_V$$

where χ_V denotes the character of V and $*$ denotes convolution defined as, given two class functions f, g, $f * g$ is defined as

$$(f * g)(a) = \sum_{bc=a} f(b)\overline{g(c)}$$

Proof. The first equation after choosing coordinates becomes,

$$= \sum_{g\in G}(\sum_i (g_{ii}^V)^{-1})(\sum_j (gh)_{jj}^W) \tag{6.2}$$

$$= \sum_{g\in G}(\sum_i \overline{g_{ii}}^V)(\sum_j (gh)_{jj}^W) \tag{6.3}$$

$$= \sum_{i,j}(\sum_{g\in G} \overline{g_{ii}}^V (gh)_{jj}^W) \tag{6.4}$$

which is 0 by 6.3

For the second equation it suffices to prove that $\chi_V * \chi_V$ is a multiple of χ_V. To do this use the fact that the regular representation is a unit under the convolution operation and that $\mathbb{C}[G]$ is semisimple. □

6.4. **Idempotents.** Note that under the convolution operation defined above $\rho/|G|$ is the identity element, where ρ is the character of the regular representation.

In a semisimple ring identity breaks up as a sum of orthogonal idempotents which corresponds to the splitting:

$$\rho = \sum_V \dim V \chi_V$$

where the sum is over irreducible representations V of G so that the idempotents are $\frac{\dim V \chi_V}{G}$ and the traces are $Tr(\dim V\chi_V/G) = |G|^{-2}\dim V\chi_V(e) = (\dim V/|G|)^2$.

Finally combining all our previous computations we obtain the beautiful theorem,

Theorem 6.5.
$$\sum_V \left(\frac{dimV}{|G|}\right)^{2-2g} = \frac{|\{u_1,\cdots,u_{2g}\in G|[u_1,u_2].\cdots.[u_{2g-1},u_{2g}]\}|}{|G|}$$
where the sum is over irreducible representations V of G .

Proof. We noted earlier that for a genus g surface M^g, $Z_G(M) = |Hom(\pi_1(M))|/|G|$. Elementary topology tells us that this is the right hand side of the above equation.

But $Z_G(M)$ was also equal to $\sum_i Tr(e_i)^{1-g}$ which is the left hand side. □

Corollary 6.6.
(1) $|\{$*irreducible representations of* $G\}| = |\{$*number of conjugacy classes in* $G\}$
(2) $\sum_V (\dim V)^2 = |G|$

Proof. Follows by substituting $g = 1, 0$ respectively in the theorem above. □

This concludes the systematic study of finite gauge theories. From here on we can try to tweak our TQFT and try to obtain do more involved computations.

7. Further reading

This section serves as a reminder for me and does not contain any maths.

7.1. **Twistings.** There was one step in our whole earlier process which was non canonical. For a finite groupoid X we defined an inner product on the space $\mathbb{C}^X$ as
$$\int_X : \mathbb{C}^X \to \mathbb{C}, f \mapsto \sum_{[x]\in\pi_0(X)} \frac{f([x])}{Aut[x]}$$
For an arbitrary groupoid there might not be any other natural inner product, but we are only interested in the groupoid of principal G bundles over an n and $n+1$ dimensional manifold and we use this inner product to define F_* to be the adjoint of F^* .

It is standard trick to modify the inner product by picking a cohomohology sitting in $H^d(BG;\mathbb{R}/\mathbb{Z})$ or $H^{d+1}(BG;\mathbb{R}/\mathbb{Z})$.

There are hints of how to do this in Teleman's notes.

7.2. **Non finite groups.** Instead of looking at finite groups we can look at compact Lie groups. There instead of looking at the moduli space of all G bundles, one looks at the moduli space of flat G bundles.

There are two very obvious problems here. The moduli space is not a finite set anymore, nor does it possess any kind of a smooth structure. Instead I believe it as a Stack of same kind(?) and one needs to use the Chern Simons functional to define an inner product on the space.

7.3. **Orientation.** We can ask when a TQFT can be extended to unoriented manifolds, at least for the Z_G that we have defined this can be done and it gives us another functional on the space of irreducible representations.

7.4. **Extended TQFT.** Finally we can ask if this TQFT can be extended to one more dimension and turns out it can be done, but I do not understand it yet.