Cohomology via Sheaves

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Contents

0 Introduction & Motivation

> In mathematics you don't understand things. You just get used to them.
>
> John von Neumann

Cohomology was introduced by Poincare in a series of papers named *Analysis Situs* and now forms the basis of modern Algebraic Topology. To a topological space X we can associate a sequence of vector spaces denoted

$$\check{H}^i(X)$$

for each $i \in \mathbb{Z}_{\geq 0}$ called it's **Cech Cohomology** (pronounced *check cohomology*). In a very loose sense, the dimension of $\check{H}^i(X)$ measures the i^{th} dimensional holes in X.

Why care about the i^{th} dimensional holes? We can use these to rigorously distinguish between spaces. For example, most proofs of the fact that $\mathbb{R}^m$ is not homeomorphic[1] to $\mathbb{R}^n$ if $m \neq n$ use some cohomology computation. We'll also see that a torus $S^1 \times S^1$ has two *1-dimensional holes* which distinguishes it from a sphere S^2 which has none.

Theorem 0.1. *Two topological spaces X, Y are homeomorphic only if*

$$\check{H}^i(X) \cong \check{H}^i(Y)$$

for all non-negative integers i.

Remark 0.2. The above statement is not an *if and only if* statement. The other direction is easily shown to be false. You'll be able to come up with examples by yourself in a couple of days.

Computing the cohomology requires multiple steps. The goal of this class is to develop the relevant machinery and actually do some cohomology computations.

$$\underset{\text{Topological Space}}{X} \rightsquigarrow \underset{\text{Good cover of } X}{\mathcal{U}} \rightsquigarrow \underset{\text{Cech Complex of } \mathcal{U}}{\mathcal{L}^\bullet(\mathcal{U})} \rightsquigarrow \underset{\text{Cech Cohomology of } X}{\check{H}^*(X)}$$

Note: The number of stars (*) on the problems indicate their difficulty level. The non-starred marked problems are compulsory, the starred problems are optional.

[1] Homeomorphism is the isomorphism for topological spaces.

1 Topological Preliminaries

Humpty Dumpty sat on a wall,
Humpty Dumpty had a great fall.
All the king's horses and all the king's men
Could not put Humpty together again.

The first step is to break a space up into *simpler spaces* and try to *glue* the pieces back. Simpler spaces will mean contractible spaces and gluing back will be done using locally constant functions.

$$\underset{\text{Topological Space}}{X} \rightsquigarrow \underset{\text{Good cover of } X}{\mathcal{U}}$$

Definition 1.1. A topological space[1] X is said to be **contractible** if there exists a point $x_0 \in X$ and a continuous map

$$\Phi : X \times [0,1] \to X$$

such that

$$\Phi(x,0) = x$$
$$\Phi(x,1) = x_0$$

for all $x \in X$ i.e. there are "continuously varying paths" connecting each point in X to x_0.

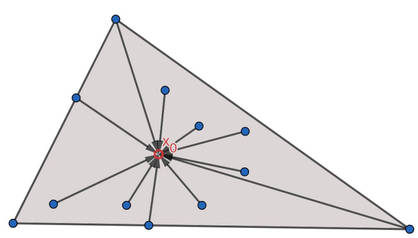


Figure 1: The interior of a triangle is a contractible space.

In some sense, a contractible space is as simple a space as is topologically possible. Most algebro-topological invariants cannot distinguish a contractible space from a point. This makes contractible spaces very useful as one may need infinitely many points to *construct* a space but only finitely many contractible spaces to do so, as we'll see below.

[1]For us a topological space is simply a subset of $\mathbb{R}^2$ or $\mathbb{R}^3$.

Question. 1. A subset X of $\mathbb{R}^n$ is said to be **star-shaped** if there exists a point x_0 such that for any other point $x \in X$ the segment connecting x to x_0 lies entirely in X. Prove that star-shaped subsets of $\mathbb{R}^n$ are contractible.

Remark 1.2. This proves that the sets $\mathbb{R}^n$, $[0,1]$, $(0,1)$, union of x and y axis in $\mathbb{R}^2$, interior of a convex polygon in $\mathbb{R}^2$, interior of a convex polyhedron in $\mathbb{R}^3$ are all contractible.

Not every space is contractible, otherwise topology would have been a very boring subject (which it isn't). The first application of Cech Cohomology will be proving that spaces like the circle S^1, the spheres S^2, S^n, torus, projective space, other higher genus surfaces are not contractible.

Question. 2. Let X, Y be topological spaces.

a) Show that if X, Y are contractible then so is $X \times Y$. In particular, if X is contractible then so is $X \times [0,1]$.

b) Show that if $X \times Y$ is contractible, then so is X.

c) * A subspace $A \subseteq X$ is said to be a **retract** of X if there exists a continuous map

$$r : X \to A$$

such that $r(a) = a$ for all $a \in A$. Show that if X is contractible and A is a retract of X then A is also contractible.

d) Is it true that every subset of a contractible space is contractible?

Definition 1.3. We say that X is **connected**[2] if for any two points $x_0, x_1 \in X$ there exists a continuous map $c : [0,1] \to X$ such that $c(0) = x_0$ and $c(1) = x_1$. Define a relation on the set X as $x_0 \sim_{conn} x_1$ if there exists a path in X connecting x_0 to x_1.

Question. 3. Let X be a topological space.

a) Show that if X is contractible then X is connected.

b) Show that $\sim_{conn}$ is an equivalence relation.

Definition 1.4. Define the **connected components** of X, denoted $\pi_0(X)$, to be the equivalence classes of X under $\sim_{conn}$.

Remark 1.5. All our spaces will have finitely many connected components.

[2]Technical point: This should really be called *path-connected* but we will only be dealing with spaces where the two notions coincide.

1.1 Covers

Definition 1.6. A finite collection of open subsets[3] $\mathcal{U} = \{U_1, U_2, \ldots, U_n\}$ of a topological space X is said to be a **cover** of X if $X = U_1 \cup U_2 \cup \cdots \cup U_n$.

Example 1.7.

a) $\mathcal{U} = \{X\}$ is always a cover of X of any space X.

b) If X is a triangle, then $\mathcal{U} = \{U_1, U_2, U_3\}$ with U_i being a side of the triangle is a cover of X.

Not all covers are equal. We need the covers to satisfy the following extra condition.

Definition 1.8. Let $[n] = \{1, 2, \cdots, n\}$. To every non-empty subset $I \subseteq [n]$ we can associate a subset U_I of X defined as

$$U_I := \bigcap_{i \in I} U_i$$

The cover $\mathcal{U} = \{U_1, U_2, \ldots, U_n\}$ is said to be a **good cover** if for every non-empty subset $I \subseteq [n]$ every connected component of the space U_I is contractible.

Remark 1.9. In the above definition, U_I can be empty as every connected component of an empty set is contractible.

Definition 1.10. The dimension of a cover $\mathcal{U}$ is the largest k such that there exists some $I \subseteq [n]$ with $|I| = k + 1$ and U_I non-empty.

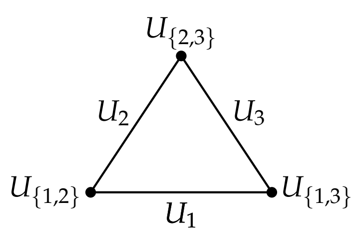


Figure 2: The set of sides $\mathcal{U} = \{U_1, U_2, U_3\}$ is a good cover of the triangle. In this case $U_{\{1,2\}} = U_1 \cap U_2$, $U_{\{2,3\}} = U_2 \cap U_3$, $U_{\{1,3\}} = U_1 \cap U_3$ are the vertices and $U_{\{1,2,3\}} = U_1 \cap U_2 \cap U_3$ is empty.

[3] We'll sometimes be lazy and use simplices instead of open subsets.

Remark 1.11. For the following problems, construct good covers with the least possible dimension. The higher the dimension of the cover, the harder the cohomology computations.

Question. 4. Construct good covers of the following spaces. It's enough (and recommended) to simply draw pictures.

a) $\mathbb{R}^n$

b) S^1 (= the circle)

c) A Tree

d) The bipartite graph $K_{2,3}$

e) $\mathbb{R}^2 \setminus \{(0,0)\}$

f) $\mathbb{R}^2 \setminus \{(0,0),(1,0),\ldots,(k,0)\}$ for some positive integer k

g) S^2 (= the sphere)

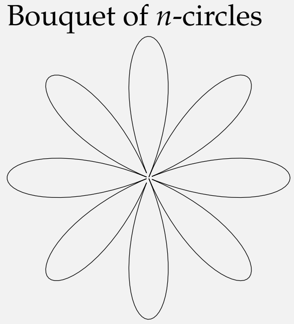
h) S^2 minus a point

i) S^2 minus 2 points

Question. 5. Find good covers of

a) $S^1 \vee S^1$ = two circles glued at a point (pronounced 'S one wedge S one')

b) Bouquet of n-circles



c) $S^1 \vee S^2$ = a circle and a sphere glued at a point (pronounced 'S one wedge S two')

Question. 6. * Let X, X' be topological spaces with good covers $\mathcal{U}, \mathcal{U}'$ respectively. Find a good cover of $X \times X'$. Find a good cover of $S^1 \times S^1$ (= torus). Draw a picture.

Question. 7. * Find a good cover of the solid torus (which is homeomorphic to $[0,1] \times [0,1] \times S^1$).

Question. 8. ** Find a good cover of the g-holed torus.

It is not true that all spaces admit good covers. However, the spaces that do not admit one are very exotic in nature. For example, every smooth manifold admits a good cover (this is called the Nerve Lemma) however this is false if we drop the adjective smooth. For this class we'll assume that all the spaces under consideration admit a good cover.

2 The Cocomplex World of Cochain Complexes

> The introduction of the digit 0 or the group concept was general nonsense too, and mathematics was more or less stagnating for thousands of years because nobody was around to take such childish steps.
>
> Grothendieck

$$\underset{\text{Good cover of } X}{\mathcal{U}} \rightsquigarrow \underset{\text{Cech Complex of } \mathcal{U}}{\mathcal{L}^{\bullet}(\mathcal{U})}$$

We need to convert topological information into algebraic information. We'll do this using locally constant functions and of course, linear algebra. All our vector spaces will be over the base field $\mathbb{F}_2 = \{0, 1\}$ i.e. all the scalars are either 0 or 1. This is mainly because in $\mathbb{F}_2$

$$-1 = 1$$

and hence we do not have to worry about signs. Furthermore, we can also think of $\mathbb{F}_2$ as a topological space with 2 points ($\mathbb{F}_2$ is an example of a topological field).

Notation: We'll shorthand "V is a vector space over $\mathbb{F}_2$ with basis $\mathcal{B} = \{v_1, \ldots, v_n\}$" as

$$V = \mathbb{F}_2\langle v_1, \ldots, v_n\rangle = \mathbb{F}_2\langle \mathcal{B}\rangle$$

Question. 9. Show that the elements of $\mathbb{F}_2\langle \mathcal{B}\rangle$ can be identified with subsets of $\mathcal{B}$ and hence as a set V has size $2^{|\mathcal{B}|}$, where $|\mathcal{B}|$ denotes the size of $\mathcal{B}$.

2.1 Locally Constant Functions

Definition 2.1. For a topological space X, define the vector space of **locally constant functions**, denoted $\mathcal{L}(X)$, to be the space of continuous maps from X to $\mathbb{F}_2$.

$$\mathcal{L}(X) := \{ f : X \to \mathbb{F}_2 \text{ continuous } \}$$

Question. 10. Show that $\mathcal{L}(X)$ is naturally a vector space over $\mathbb{F}_2$.

Question. 11. Show that if X is connected then every continuous function $f : X \to \mathbb{F}_2$ is a constant function and hence $\mathcal{L}(X) \cong \mathbb{F}_2$ as a vector space.

Question. 12. What is $\mathcal{L}(\phi)$ where ϕ is the empty set (which is a legit topological space)?

Let $X_1, X_2, \ldots, X_k$ be the connected components of X. Define k functions $\delta_1, \delta_2, \ldots, \delta_k : X \to \mathbb{F}_2$ as

$$\delta_i(x) \cong \begin{cases} 1 & \text{if } x \in X_i \\ 0 & \text{otherwise} \end{cases}$$

for $1 \leq i \leq k$.

Question. 13. Show that

$$\mathcal{L}(X) = \mathbb{F}_2\langle \delta_1, \delta_2, \ldots, \delta_k \rangle$$

and hence $\dim \mathcal{L}(X) = k = \pi_0(X)$.

Definition 2.2. With the notation as above, we call $\delta_1, \delta_2, \ldots, \delta_k$ the **canonical basis** for $\mathcal{L}(X)$.

This construction is useful as we can use tools from algebra to analyze vector spaces. We'll next analyze the vector spaces $\mathcal{L}(U_I)$ and the various maps induced between them. The space X can be *glued together* from the cover $\mathcal{U}$ using the sets U_I, this gluing information gets translated into the notion of cohomology via $\mathcal{L}(-)$. But first we'll need to talk about cochain complexes.

Question. 14. For an inclusion of topological spaces $U \subseteq V \subseteq W$,

a) We can restrict a function $f : V \to \mathbb{F}_2$ to the subspace U and get a function $f|_U : U \to \mathbb{F}_2$. Show that this induces a linear transformation

$$\text{Res}_{V\to U} : \mathcal{L}(V) \to \mathcal{L}(U)$$

b) Express this linear transformation as a matrix in the canonical bases.

c) Show that $\text{Res}_{U\to U}$ is the identity transformation.

d) Show that $\text{Res}_{V\to U} \circ \text{Res}_{W\to V} = \text{Res}_{W\to U}$.

These properties (and some more) are what define a sheaf, hence $\mathcal{L}(X)$ defines the sheaf of locally constant functions on X. More on this later.

2.2 Cochain Complexes

The primary algebraic structure we'll need is a *cochain complex*.

Definition 2.3. A (non-negatively graded, finite) **cochain complex** $\mathcal{V}^\bullet$ of vector spaces consists of the following data:

a) A vector space $\mathcal{V}^i$ for each $i \in Z$, with $\mathcal{V}^i \neq 0$ only if $0 \leq i \leq n$ for some positive integer n.

$$0 \longrightarrow \mathcal{V}^0 \longrightarrow \cdots \longrightarrow \mathcal{V}^{i-1} \xrightarrow{d^{i-1}} \mathcal{V}^i \xrightarrow{d^i} \mathcal{V}^{i+1} \longrightarrow \cdots \longrightarrow \mathcal{V}^n \longrightarrow 0$$

b) For each $i \in \mathbb{Z}$ a linear transformation $d^i : \mathcal{V}^i \to \mathcal{V}^{i+1}$ that satisfies the identity

$$d^i \circ d^{i-1} = 0 \qquad \begin{tikzcd} \mathcal{V}^{i-1} \arrow[r, "d^{i-1}"] \arrow[rr, bend right, "d^i \circ d^{i-1} = 0"'] & \mathcal{V}^i \arrow[r, "d^i"] & \mathcal{V}^{i+1} \end{tikzcd}$$

Question. 15. Show that $\operatorname{im} d^{i-1} \subseteq \ker d^i$.

Definition 2.4. The i^{th} **cohomology** of $\mathcal{V}^\bullet$ is the quotient vector space

$$H^i(\mathcal{V}) := \ker d^i / \operatorname{im} d^{i-1}$$

Remark 2.5. This is well-defined because of the previous exercise. We'll only be interested in the dimensions of the cohomologies

$$\dim H^i(\mathcal{V}) = \dim \ker d^i - \dim \operatorname{im} d^{i-1}$$

Convention: Even though in a cochain complex there is a vector space $\mathcal{V}^i$ for all integers i it is a common convention to explicitly define $\mathcal{V}^i$ only where it is non-zero. It is understood that the rest of the $\mathcal{V}^i$ are all 0. The first non-zero vector space in the cochain complex is understood to be $\mathcal{V}^0$ unless otherwise specified.

Question. 16. Verify that the following are cochain complexes and compute their cohomologies.

a) $0 \longrightarrow \mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2 \longrightarrow 0$

b) $0 \longrightarrow \mathbb{F}_2 \xrightarrow{id} \mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2 \longrightarrow 0$

c) $0 \longrightarrow \mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2 \xrightarrow{id} \mathbb{F}_2 \longrightarrow 0$

d) $0 \longrightarrow \mathbb{F}_2 \xrightarrow{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} \mathbb{F}_2^2 \xrightarrow{\begin{bmatrix} 1 & 1 \end{bmatrix}} \mathbb{F}_2 \longrightarrow 0$

Question. 17. A vector space A can be thought of as a cochain complex as

$$\mathcal{V}^\bullet = 0 \longrightarrow A \longrightarrow 0$$

What are the cohomologies $H^i(\mathcal{V})$, for $i \in \mathbb{Z}$?

Question. 18. A linear transformation $f : A \to B$ can be thought of as a cochain complex as

$$\mathcal{V}^\bullet = 0 \longrightarrow A \xrightarrow{f} B \longrightarrow 0$$

What are the cohomologies $H^i(\mathcal{V})$, for $i \in \mathbb{Z}$?

Question. 19. Given $\mathcal{V}^\bullet = 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$

a) Under what conditions on f, g is $\mathcal{V}^\bullet$ a cochain complex.

b) Under what conditions on f, g is $H^i(\mathcal{V}) = 0$ for all i. In this case we say that $\mathcal{V}^\bullet$ is a **short exact sequence**.

Definition 2.6. Define the **Euler characteristic** of a cochain complex to be

$$\chi(\mathcal{V}) = \sum_{i=0}^{N} (-1)^i \dim H^i(\mathcal{V}) \tag{2.1}$$

$$= \dim H^0(\mathcal{V}) - \dim H^1(\mathcal{V}) + \dim H^2(\mathcal{V}) \pm \cdots + (-1)^n \dim H^n(\mathcal{V}) \tag{2.2}$$

Question. 20.

a) Express the Euler characteristic in terms of the $\dim \ker(d^i)$ and $\dim \operatorname{im}(d^i)$.

b) Show that

$$\chi(V) = \sum_{i=0}^{N} (-1)^i \dim \mathcal{V}^i \tag{2.3}$$

Thus the Euler characteristic can be computed using the dimensions of the vector spaces of the original chain complex, but it is really an invariant of the underlying cohomology!

Definition 2.7. A cochain complex $\mathcal{V}^\bullet$ is said to be **exact** (or **long exact**) if $H^i(\mathcal{V}) = 0$ for all i.

Question. 21.

a) Show that if $\mathcal{V}^\bullet$ is exact then $\chi(\mathcal{V}) = 0$.

b) Find an example of a cochain complex $\mathcal{V}^\bullet$ which is not exact but for which $\chi(\mathcal{V}) = 0$.

Question. 22. A direct sum of chain complexes $\mathcal{V}_1^\bullet \oplus \mathcal{V}_2^\bullet$ is defined as

$$\cdots \longrightarrow \mathcal{V}_1^{i-1} \oplus \mathcal{V}_2^{i-1} \xrightarrow{d_1^{i-1} \oplus d_2^{i-1}} \mathcal{V}_1^i \oplus \mathcal{V}_2^i \xrightarrow{d_1^i \oplus d_2^i} \mathcal{V}_1^{i+1} \oplus \mathcal{V}_2^{i+1} \longrightarrow \cdots$$

Find the cohomology of the direct sum.

Question. 23. * A **morphism of cochain complexes** $\phi : \mathcal{V}_1^\bullet \to \mathcal{V}_2^\bullet$ is a collection of maps $\phi : \mathcal{V}_1^i \to \mathcal{V}_2^i$ for each $i \in \mathbb{Z}$ such that the following diagram commutes

$$\begin{tikzcd} \mathcal{V}_1^i \arrow[r, "d"] \arrow[d, "\phi"'] & \mathcal{V}_1^{i+1} \arrow[d, "\phi"] \\ \mathcal{V}_2^i \arrow[r, "d"] & \mathcal{V}_2^{i+1} \end{tikzcd}$$

a) Show that a morphism $\phi : \mathcal{V}_1^\bullet \to \mathcal{V}_2^\bullet$ between cochain complexes naturally induces a map $\phi^* : H^i(\mathcal{V}_1) \to H^i(\mathcal{V}_2)$ between cohomologies, for all $i \in Z$.

b) Given two morphisms $\phi_1 : \mathcal{V}_1^\bullet \to \mathcal{V}_2^\bullet$ and $\phi_2 : \mathcal{V}_2^\bullet \to \mathcal{V}_3^\bullet$, show that their composition $\phi_2 \circ \phi_1 : \mathcal{V}_1^\bullet \to \mathcal{V}_3^\bullet$ is also a morphism of cochain complexes. Further show that $(\phi_2 \circ \phi_1)^* = \phi_2^* \circ \phi_1^*$.

Question. 24 (Double complexes)**.** * A cochain complex of cochain complexes (i.e. each $\mathcal{V}^i$ is itself a cochain complex and the differential d^i are morphisms of cochain complexes) is called a **double complex**. In this case we require the differential d^i to be a morphism between cochain complexes. Unravel this description of a double complex and describe it more explicitly as a grid of vector spaces.

Question. 25 (Tensor Products). ** The tensor product $(\mathcal{V}_1 \otimes \mathcal{V}_2)^\bullet$ of two chain complexes $\mathcal{V}_1^\bullet, \mathcal{V}_2^\bullet$ is defined as

$$(\mathcal{V}_1 \otimes \mathcal{V}_2)^k := \bigoplus_{i+j=k} \mathcal{V}_1^i \otimes \mathcal{V}_2^j$$

a) Define

Question. 26 (Snake Lemma). ** Show that a morphism ϕ between short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{V}_1^0 \arrow[r] \arrow[d, "\phi^0"] & \mathcal{V}_1^1 \arrow[r] \arrow[d, "\phi^1"] & \mathcal{V}_1^2 \arrow[r] \arrow[d, "\phi^2"] & 0 \\ 0 \arrow[r] & \mathcal{V}_2^0 \arrow[r] & \mathcal{V}_2^1 \arrow[r] & \mathcal{V}_2^2 \arrow[r] & 0 \end{tikzcd} \tag{2.4}$$

induces a natural exact sequence (not a full cochain complex)

$$\ker \phi^0 \longrightarrow \ker \phi^1 \longrightarrow \ker \phi^2 \longrightarrow \operatorname{coker} \phi^0 \longrightarrow \operatorname{coker} \phi^1 \longrightarrow \operatorname{coker} \phi^2 \tag{2.5}$$

(Recall the cokernel of a map $f : A \to B$ is $B/(\operatorname{im} f)$.) This sequence does not naturally extend to a full chain complex.

Remark 2.8. Note that (2.4) is a double complex which gives rise to the exact sequence (2.5). Larger double complexes give rise to more complicated structures called **spectral sequences**.

3 Gluing it Back Together

Associated to any cover $\mathcal{U}$ is a cochain complex called the **Cech complex**. When the cover is a *good cover* this cohomology does not depend on the cover and equals the singular cohomology of the underlying space. To define a complex we need two things: the spaces $\mathcal{L}^i(\mathcal{U})$, and the differential maps d^i.

Recall that to every non-empty subset $I \subseteq [n]$ we can associate an open set U_I defined as

$$U_I := \bigcap_{i \in I} U_i$$

Definition 3.1. For $0 \leq k < n$, let $(U_{I_1}, U_{I_2}, \ldots, U_{I_m})$ denote the non-empty sets U_I with $|I| = k + 1$. Define

$$\mathcal{L}^k(U) := \mathbb{F}_2^m$$

Let $(U_{J_1}, U_{J_2}, \ldots, U_{J_n})$ denote the non-empty sets U_I with $|I| = k + 2$, then $d^k : \mathcal{L}^k(U) \to \mathcal{L}^{k+1}(U)$ is an $n \times m$ matrix defined as

$$\text{the } (a, b) \text{ entry of } d^k = \begin{cases} 1 & \text{if } U_{J_a} \subseteq U_{I_b} \\ 0 & \text{otherwise} \end{cases}$$

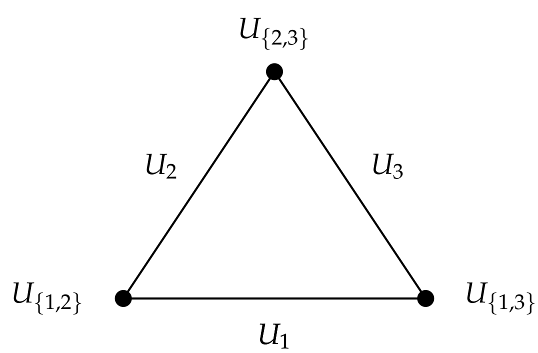


Figure 3: $\mathcal{U} = \{U_1, U_2, U_3\}$ is a good cover of the triangle, $U_{\{1,2\}}, U_{\{2,3\}}, U_{\{1,3\}}$ are the vertices, and $U_{\{1,2,3\}}$ is empty.

Example 3.2. For a triangle in Figure 3 we have the following non-empty sets

$$\begin{array}{ll} |I| = 0 & U_1, U_2, U_3 \\ |I| = 1 & U_{\{1,2\}}, U_{\{2,3\}}, U_{\{1,3\}} \end{array}$$

Hence the spaces in the chain complex are

$$0 \to \mathbb{F}_2^3 \to \mathbb{F}_2^3 \to 0$$

Because of the non-empty inclusions $U_{\{1,2\}} \subseteq U_1$ etc. the differential looks like

	U_1	U_2	U_3
$U_{\{1,2\}}$	1	1	0
$U_{\{2,3\}}$	0	1	1
$U_{\{1,3\}}$	1	0	1

So that

$$\mathcal{L}^\bullet(\mathcal{U}) = 0 \to \mathbb{F}_2^3 \xrightarrow{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}} \mathbb{F}_2^3 \to 0$$

Question. 27. Compute the cohomology of the Cech Complex found in Example 3.2.

Question. 28. Find $\check{H}(-)$ for the following spaces.

a) $\mathbb{R}^m$

b) S^1 = the circle

c) A Tree

d) $\mathbb{R}^2 \setminus \{(0,0)\}$

e) S^2 minus a point

f) S^2 minus 2 points

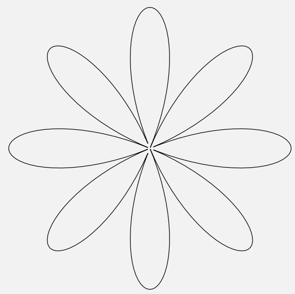
Question. 29. Find the cohomology of a finite planar graph (thought of as a subset of $\mathbb{R}^2$).

4 Cech Cohomology continued

In all the problems in this section assume that all the spaces are connected and have good covers.

Question. 30. Find the Cech cohomology of S^2.

Question. 31. Find the relationship between Cech Cohomology of X, Y and $X \vee Y$. Hence, find the Cech cohomology of a bouquet of n circles.



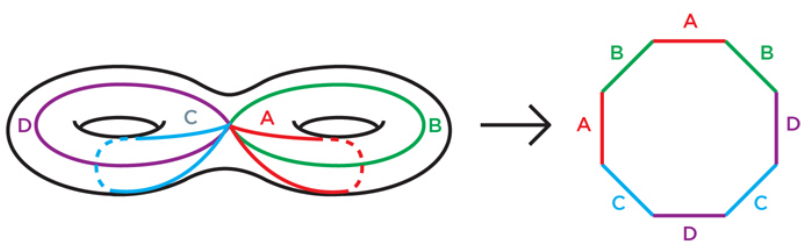


Figure 4: Gluing diagram for constructing a 2 holed torus M^2 using an octagon. (Googled image.)

Question. 32. We can construct a 2 holed torus M^2 by gluing the sides of an octagon as in Figure 5. Use the gluing diagram to construct a good cover of M^2. Find the cohomology using this good cover.

Question. 33. * More generally, it is possible to construct a g holed torus by gluing polygons of $4g$ sides. Repeat Question 32 for this surface.

Question. 34. * By gluing the sides of a square in more funky ways we can create the Klein Bottle and the Real Projective Plane. Repeat Question 32 for these spaces.

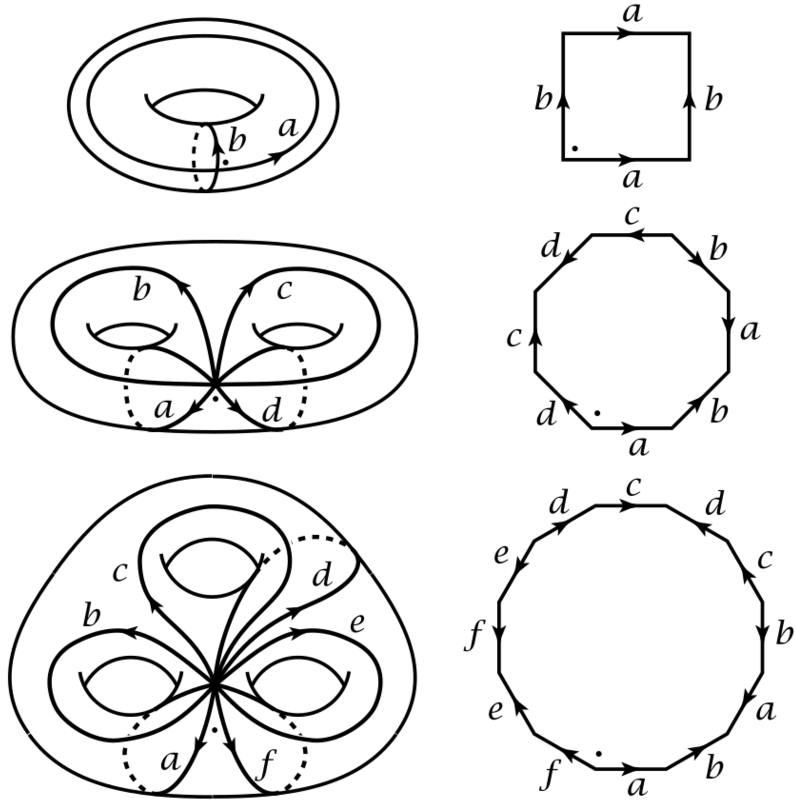
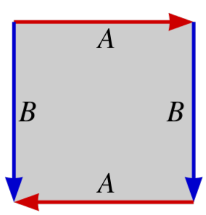
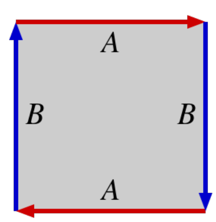


Figure 5: Constructing a g holed torus M^g using a $4g$-gon. (Image from Hatcher.)



(a) Klein Bottle



(b) Projective Plane

5 Appendix: Sheaves

> It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out.
>
> E. Artin

The assignment $U \mapsto \mathcal{L}(U)$ is an example of a sheaf on X. It is perhaps the simplest non-trivial sheaf. Let's see the definition of a sheaf.

Notation: We'll let $\mathcal{C}$ denote the collection of one of the following: sets, groups, abelian groups, topological spaces, modules over a ring R. A **morphism** between two objects in $\mathcal{C}$ will denote a map which preserves the appropriate structure: set maps, group homomorphisms, abelian group homomorphisms, continuous maps, R-module morphisms respectively.[1]

Definition 5.1. On a topological spaces X, a **presheaf** $\mathcal{P}$ valued in $\mathcal{C}$, consists of the following data:

a) For each open[2] subset $U \subseteq X$ an assignment of a vector space

$$U \mapsto \mathcal{P}(U)$$

b) For each inclusion of open subsets $U \subseteq V$ a morphism

$$\mathrm{Res}_{V \to U} : \mathcal{P}(V) \to \mathcal{P}(U)$$

satisfying the following conditions

- For each open subset U, $\mathrm{Res}_{U \to U}$ it the identity map.
- For each open subset $U \subseteq V \subseteq W$

$$\mathrm{Res}_{V \to U} \circ \mathrm{Res}_{W \to V} = \mathrm{Res}_{W \to U} \qquad \begin{tikzcd} \mathcal{P}(W) \arrow[r, "\mathrm{Res}_{W\to V}"] \arrow[rr, bend right, "\mathrm{Res}_{W\to U}"'] & \mathcal{P}(V) \arrow[r, "\mathrm{Res}_{V\to U}"] & \mathcal{P}(U) \end{tikzcd}$$

> **Question. 35.** Define natural maps Res on $\mathcal{L}$ which turn it into a presheaf valued in vector spaces.

Definition 5.2. A **sheaf** on X is a presheaf $\mathcal{P}$ that further satisfies the following conditions. Let $\mathcal{U} = \{U_i\}_{i \in I}$ is a cover of $V \subseteq X$.

[1] This data is what makes $\mathcal{C}$ a **category**.

[2] If you do not know what an *open* set is you can neglect the adjective *open* for now, it's won't be the end of the world.

a) *(Identity axiom)* If a section $s \in \mathcal{P}(V)$ is such that $\mathrm{Res}_{U \to U_i} s = 0$ for all U_i then $s = 0$.

b) *(Gluing axiom)* If there exist a collection of sections $s_i \in \mathcal{P}(U_i)$ such that for all $i, j \in I$ the intersections are compatible

$$\mathrm{Res}_{U_i \to U_i \cap U_j} s_i = \mathrm{Res}_{U_j \to U_i \cap U_j} s_j$$

then there exists a section $s \in \mathcal{P}(V)$ such that

$$s_i = \mathrm{Res}_{V \to U_i} s$$

The identity axiom is a uniqueness axiom. It ensures that there if two sections look the same when restricted to small sets, then the two sets should have been the same to begin with.

The gluing axiom is a constructive axiom. It is saying that you can glue sections on small open subsets to get a section on a large open subset as long as the sections on the smaller sets agree on intersections.

Question. 36. Verify that $\mathcal{L}$ is a sheaf.

Question. 37. Show that for a topological space X and a set Y the assignment

$$U \to \{\text{set maps from } U \to Y\}$$

with the Res maps being restrictions of functions, is a sheaf on X valued in sets.

Definition 5.3. Given a map $\phi : X \to Y$ a **section** of ϕ on a subset $U \subseteq Y$ is a map $s : U \to X$ satisfying $\phi \circ s(u) = u$ for all $u \in U$.

$$\begin{tikzcd}
& X \arrow[d, "\phi"] \\
U \arrow[ur, "s"] \arrow[r, hook] & Y
\end{tikzcd}$$

Question. 38. *a*) For a map $\phi : X \to Y$ define

$$\mathcal{P}^{\phi}(U) := \text{ The set of sections of } \phi \text{ over } U.$$

Define the Res maps as restrictions of functions. Show that this defines a sheaf on Y valued in sets.

b) Show that the sheaf defined in Question 37 is a special case of this.

c) Show that the sheaf $\mathcal{L}$ is a special case of this if we only allow our sections to be continuous functions.

Remark 5.4. Every sheaf is morally of the above type with the generalization that we can require the map ϕ and the sections s to have certain properties like being continuous, smooth, holomorphic, meromorphic etc. Because of this, for an arbitrary sheaf the set $\mathcal{P}(U)$ is called the sections of $\mathcal{P}$ over U.