What is a manifold?

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July 4, 2017

1 Introduction

The goal of this class is to

1. understand what manifolds are
2. how mathematicians manipulate manifolds
3. how to 'visualize' higher dimensional manifolds

Let us start with some motivation. One of the first objects one encounters in geometry is the plane $\mathbb{R}^2$. Two nice mathematical properties that the plane has are that

1. Every point in the plane is determined uniquely by exactly 2 *coordinates*.
2. We can measure distances between any two points on the plane.

In everyday life we hijack these properties of the plane and use them on objects different from the plane. Consider the surface of Earth for example. Within a small area, say a city, every point can be represented by exactly 2 coordinates and we can measure the distance between any points without really worrying about the curvature of the earth.

However things get a little glitchy once we move to larger scales. We have the latitude/longitude coordinate system for all the points on the surface of the Earth but

1. The assignment of longitudes is not continuous, at a few places it jumps abruptly from $-180°$ to $180°$.
2. The poles do not posses a unique longitude.

It also gets increasingly harder to measure distances on larger scales and it becomes necessary to take the curvature of Earth into account. (Along which line should the distance between Seattle and Berlin be measured?) These properties are what characterize a manifold. It is an object which on small scales looks very much a 'Euclidean space', but this is no longer true when you zoom out.

2 Examples of manifolds

Let us try to make the above observations more precise.

Some of the most useful objects studied in mathematics are the Euclidean spaces $\mathbb{R}^n$, the set of n tuples of real numbers, $\mathbb{R}^n = \{(x_1, \cdots, x_n)\}$. We usually think of $\mathbb{R}^1$ as the line, $\mathbb{R}^2$ as the plane, $\mathbb{R}^3$ as the space.

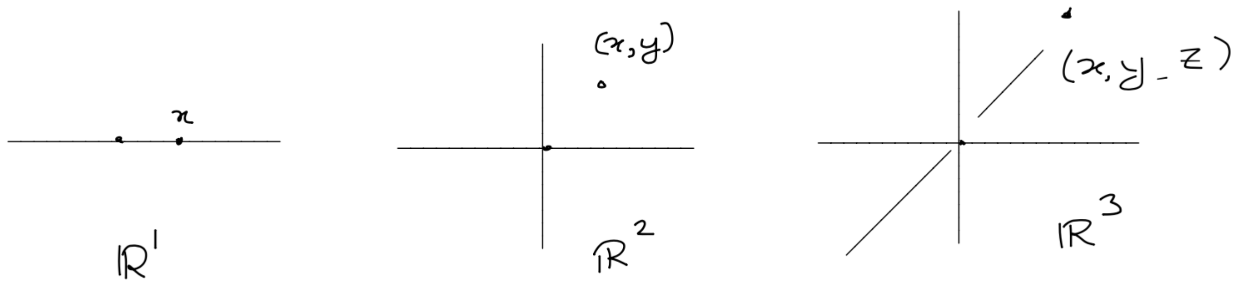

Fig. 1: Euclidean spaces

It is a useful trick in mathematics to take various properties of simple objects and study them one at a time, in the process creating other interesting objects with similar properties. For example, in the case of Euclidean space it is possible to add any two elements in $\mathbb{R}^n$ and get another element in $\mathbb{R}^n$. This generalizes to vector spaces and leads to linear algebra.

Another set of properties that Euclidean spaces have are what might be called *geometric* or *topological* properties.

1. Every point in $\mathbb{R}^n$ is determined uniquely by exactly n coordinates.
2. It is possible to do calculus on $\mathbb{R}^n$.
3. We can measure distances between points in $\mathbb{R}^n$.

The objects which share these properties with Euclidean spaces are called manifolds. A **topological manifold** in an object in which every point can be *locally* described by coordinates. A **smooth manifold** in an object on which it is possible to do calculus. A **Riemannian manifold** is an object on which it is possible to measure lengths of line segments.

A **manifold** is an object such that every point on it has a *neighborhood* surrounding it which *looks like* a Euclidean space. The exact definition of *looks like* depends upon which geometric/topological property we want the manifold to have. We'll stick to the simplest kind of manifolds - topological manifolds, these are objects in which every point has a neighborhood which can be described uniquely by n coordinates. The number of coordinates **n** is called the **dimension** of the manifold.

Let us see some examples and non-examples of manifolds.

Example 2.1 (S^2)**.** The surface of the Earth that we saw earlier is denoted by S^2, the 2 dimensional sphere

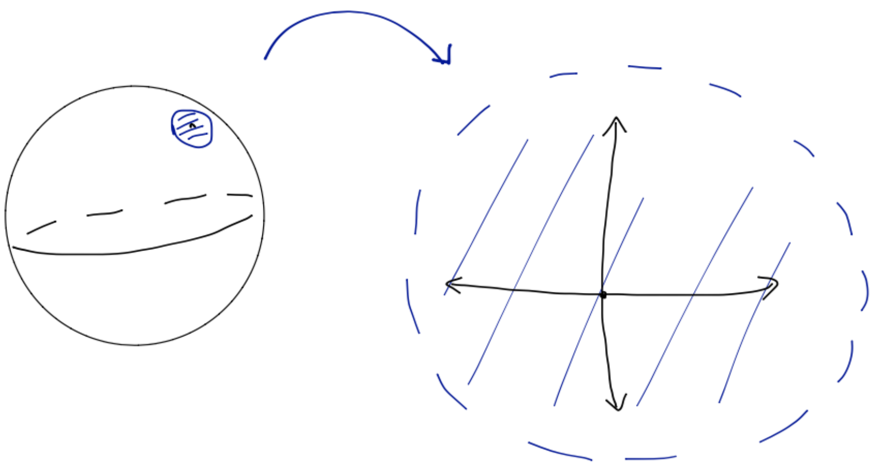

Fig. 2: Every point on S^2 has a neighborhood that *looks like* the plane $\mathbb{R}^2$

Example 2.2 ($\mathbb{R}^n$)**.** Euclidean spaces are themselves manifolds (they better be as manifolds are modeled on them) and $\mathbb{R}^n$ has dimension n.

Example 2.3 (1-dimensional manifolds)**.** 1 dimensional manifolds are objects which locally look like lines.

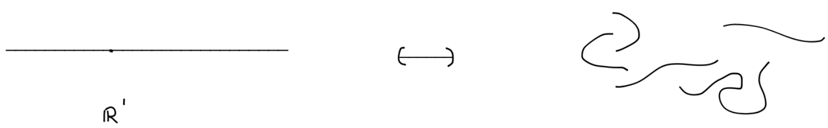

Fig. 3: $\mathbb{R}^1$, Open internval $(a, b) \subset \mathbb{R}^1$ and collection of *open* line segments in $\mathbb{R}^2$ are 1-dim manifolds

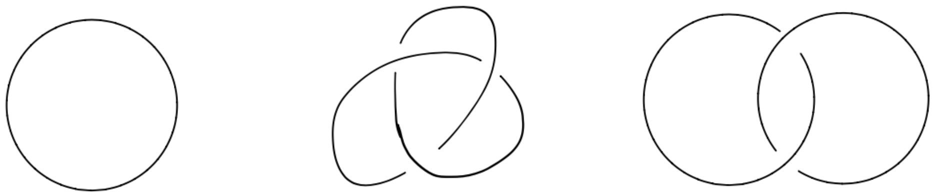

Fig. 4: Circle, trefoil and Hopf link are 1-dim manifolds

The following examples are NOT manifolds. The union of x and y axes is not a manifold as no neighborhood of the origin $(0, 0)$ *looks like* a Euclidean space. The space of non-negative real numbers is not a manifold because the point 0 does not have a neighborhood which looks like a Euclidean space (it only *looks like* half the Euclidean space $\mathbb{R}^1$), same argument holds for non-open intervals $[a, b]$ or $[a, b)$.

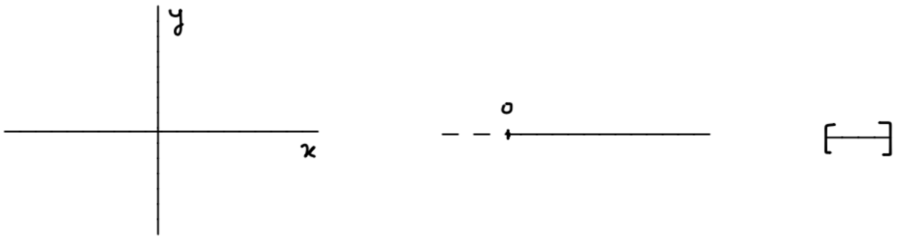

Fig. 5: Union of x and y axes in $\mathbb{R}^2$, the space of non-negative real numbers, closed interval $[a, b]$ are not manifolds

Example 2.4 (Surfaces)**.** The surface of a donut is called a **torus**. Because it has 1 'hole' in it the torus is also called a **genus 1** surface. More generally surfaces with g holes are called genus g surfaces.

Fig. 6: Torus, genus 2 surface, higher genus surfaces

We can remove points from a torus to get a punctured torus. Such manifolds are very interesting mathematically and are extensively studied in hyperbolic geometry.

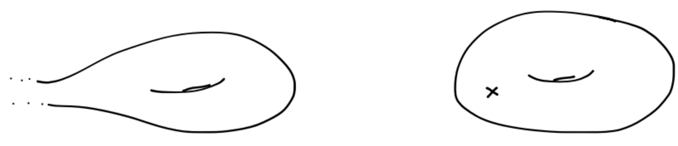

Fig. 7: Torus with an end and a punctured torus

Example 2.5 (Spheres)**.** The set $S^{n-1} \subset \mathbb{R}^n$ is defined as

$$S^{n-1} = \{(x_1, \cdots, x_n) : x_1^2 + x_2^2 + \cdots x_n^2 = 1\}$$

This is the $n - 1$ dimensional **sphere** (and a manifold) sitting inside $\mathbb{R}^n$.

3 Definition of a manifold

To define 'looks like' we need the concept of **continuity** from point set topology. A continuous map is a map which does not break things apart, another way of saying the same thing is that nearby points go to nearby points. Let us make this more precise.

Let A, B be subsets of $\mathbb{R}^n$ where n is a positive integer.

Definition 3.1. A sequence of points $x_1, x_2, \cdots, x_i, \cdots \in A$ is said to converge to a point $x \in A$, denoted $x_i \to x$, if the distance between x_i and x goes to 0 as $i \to \infty$.

$$\lim_{i\to\infty} \|x - x_i\| = 0 \implies x_i \to x \tag{3.1}$$

Definition 3.2. A map $f : A \to B$ is said to be **continuous at a** $\in A$ if for every sequence $x_i \to a$ in A the sequence $f(x_i) \to f(a)$ in B. If $f : A \to B$ is continuous at every point $a \in A$ then f is called a **continuous map**.

We're now ready to define 'looks like':

Definition 3.3. We say A and B are homeomorphic if there exist *continuous* maps $f : A \to B$ and $g : B \to A$ such that for all points $a \in A$ and $b \in B$ we have $f(g(b)) = b$ and $g(f(a)) = a$.

This then is going to be our version of 'looks like'.

Definition 3.4 (Topological manifold)**.** A set $M \subseteq \mathbb{R}^n$ is called a topological manifold if for each point a in A there exists an open subset U containing a, called a **neighborhood** of a, such that U is homeomorphic to $\mathbb{R}^k$ for some k.

To be precise the definition above is that of a **submanifold** of $\mathbb{R}^n$. We can allow M to be a topological space (with some technical restrictions) devoid of any ambient $\mathbb{R}^n$.

For the purposes of this class we'll think of two homeomorphic manifolds as being the same manifold. It is not hard to see that if one manifold can be continuously deformed into another then the two manifolds are homeomorphic. This gives rise to the infamous for-a-topologist-cup-is-the-same-as-a-donut joke.

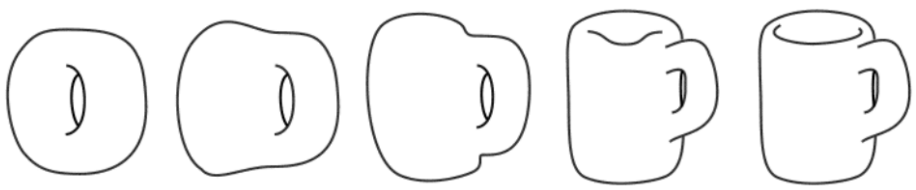

Fig. 8: A cup is diffeomorphic to a donut

4 Exercises

Exercise 4.1. Use the above deformation diagram to explicitly 'construct' a homeomorphism between a torus and teacup.

Exercise 4.2. Show that homeomorphism is an equivalence relation.

Exercise 4.3. Show that any two open intervals (a, b) and (c, d) are homeomorphic.

Exercise 4.4. Prove that $[0, 1]$ is not a manifold.

Exercise 4.5. Use the following exercise to explicitly prove that S^{n-1} is a manifold of $\dim n - 1$.

1. Consider the set $S^1 = \{(x_1, x_2) : x_1^2 + x_2^2 = 1\}$. Let $N = (0, 1) \in S^1$ be the 'north pole'. Explicitly describe the *stereographic projection* map from the north pole

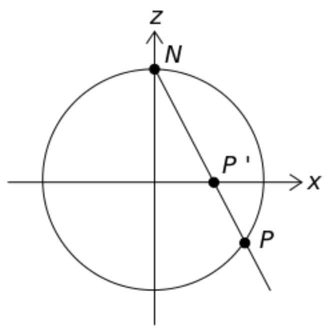

Fig. 9: $f : S^1 \setminus \{n\} \to \mathbb{R}^2$, $P \mapsto P'$

2. Repeat the same for S^2 and generalize to S^{n-1}.

4.1 Obstructions

To show that two sets are homeomorphic or to show that a space is a manifold we need to construct appropriate maps. But showing two spaces are NOT homeomorphic or that a space is NOT a manifold requires a completely different proof.

In topology such proof techniques fall under the umbrella of **obstruction theory**. The simplest obstruction is the size of a set, a finite set cannot be homeomorphic to an infinite one and two finite sets of different sizes cannot be homeomorphic to each other.

The next obstruction is connectedness.

Exercise 4.6. A set $A \subset \mathbb{R}^n$ is called (path)connected if any two points in A can be connected by a path lying entirely in A.

1. Show that if A is connected and A is homeomorphic to B then B is also connected.
2. Show that no two of the spaces $(0,1)$, $[1,0]$ and S^1 are homeomorphic to each other.
3. Show that $\mathbb{R}^1$ is not homeomorphic to $\mathbb{R}^2$.
4. Show that a 1 dimensional manifold is not homeomorphic to a 2 dimensional manifold.

The set of connected components of A is denoted by $\pi_0(A)$. By the above exercise A is homeomorphic to B only if $\pi_0(A) \cong \pi_0(B)$. Hence we say that π_0 is a topological invariant. π_0 is a very useful invariant when trying to distinguish 1 dimensional manifolds but is quite weak for higher dimensional ones.

To be able to distinguish higher dimensional manifolds topologists have created several invariants like higher homotopy groups $\pi_1, \pi_2, \cdots$, homology groups $H_0, H_1, \cdots$, cohomology groups $H^0, H^1, \cdots$, generalized (co)homology theories E_*, E^* which provide stronger obstructions to existence of homeomorphisms.

Try the following exercise which is beyond the scope of this class and needs the next obstruction π_1 or H_1.

Exercise 4.7. Is the torus homeomorphic to the sphere S^2? Is $\mathbb{R}^2$ homeomorphic to $\mathbb{R}^3$?