1. Complex analysis theorems

Let U be an open subset of $\mathbb{C}$. Let $f : U \to \mathbb{C}$ be a continuous function.

f is called (once) *complex differentiable* at $a \in U$ if the limit

$$f'(z) := \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists for all z in some neighborhood of a.

Theorem 1.1 (Taylor series). *If f is once differentiable at a, then f is infinitely differentiable at a and f has a Taylor series converging to it in a neighborhood of a.*

$$f(z) = f(a) + f'(a)(z-a) + f''(a)\frac{(z-a)^2}{2!} + \cdots + f^{(n)}(a)\frac{(z-a)^n}{n!} + \ldots$$

Theorem 1.2 (Open mapping theorem). *If f is complex differentiable then for any open set $V \subseteq U$, the set $f(V)$ is an open subset of $\mathbb{C}$.*

Theorem 1.3 (Isolated zeroes). *If f is complex differentiable then the set of zeroes of f are isolated i.e. if $f(a) = 0$ for some $a \in U$ then there exists a neighborhood V of a such that a is the only zero of f in V.*

A complex differentiable function $g : \mathbb{C} \to \mathbb{C}$ is called an *entire* function.

Theorem 1.4 (Liouville's theorem). *Let g be an entire function. If there exists a real number M such that $|g(z)| < M$ for all $z \in \mathbb{C}$ then $g(z)$ is a constant function.*

Theorem 1.5 (Little Picard's theorem). *Let g be an entire function. If g is not a constant function then the image of g is either the whole complex plane or the plane minus a single point.*