$\mathbb{R}P^{2n} \hookrightarrow \mathbb{R}P^{2n+1} \longrightarrow \mathbb{C}P^n$

$[x_{2n+1}=0]$ $[x_0; x_1; \dots ; x_{2n+1}] \longrightarrow [x_0+ix_1, x_1+ix_2, \dots, x_{2n}+ix_{2n+1}]$

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Algebraic Topology

Th^m: $\pi_n S^n \longrightarrow \mathbb{Z}$ =: degree is a group isomorphism.

(Brower)

Lemma 1: Given finite sets $U_1, U_2 \in \mathbb{R}^n$, $\exists\ U \subseteq \mathbb{R}^n$ such that $U_1 \subseteq U$, $U_2 \subseteq U^c = \mathbb{R}^n - U$,

$U \cong D^n$ $\partial U \cong \partial D^n$. Do Brute Force.

Read Transversality. Write proof of th^m. very interesting

Write CW approx. proof

Lemma 2: $f : (D^n, \partial D^n) \longrightarrow (D^n, \partial D^n)$ $\partial D^n = S^{n-1}$

then $\deg f = \deg f|_{\partial D^n}$.

~~Proof~~: Proof:

$$\begin{array}{ccc} H_n(D^n, \partial D^n) & \xrightarrow{f_*} & H_n(D^n, \partial D^n) \\ \downarrow & & \downarrow \\ H_{n-1}(\partial D^n) & \xrightarrow{f|_{\partial D^n *}} & H_{n-1}(\partial D^n) \end{array}$$

connecting hom $\longrightarrow$

Enough to show $\deg f = 0 \Rightarrow f \simeq *$.

Assume true for S^{n-1}. Assume f smooth.

~~A~~ Assume $0, \infty$ regular values of f, $f(\infty) = \infty$, [redacted]

Let $S = f^{-1}(\infty)$, $T = f^{-1}(0)$

$f : \mathbb{R}^n - S \longrightarrow \mathbb{R}^n$

By lemma 1, choose ~~*~~ $U \subseteq \mathbb{R}^n$ s.t. $T \subseteq U$, $S \subseteq U^c$.

By local degree property,

$\deg f = \deg f|_U$

~~$\partial U \cong S^{n-1}$~~

~~f~~ O

~~By lemma 2,~~ im

~~$\deg f|_U = \deg f|_{\partial U}$~~

∂U $f : S^{n-1} \to \mathbb{R}^n - 0$

$\downarrow$

S^{n-1}

~~$h : \partial U = S^{n-1} \to S^{n-1}$~~

~~$x \longmapsto f(x) \longmapsto \|f(x)\|^{-1} f(x)$~~

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~~$f: \partial U \to \mathbb{R}^n - \{0\}$~~ $h = \partial U \xrightarrow{f} \mathbb{R}^n - \{0\} \longrightarrow S^{n-1}$
deformation Retract

~~Claim: $\deg f = \deg h$~~

Now we need to modify f so that $f|_{\partial D^n} \subseteq S^{n-1}$

$\bar{U}$ is compact so $f\bar{U}$ is compact.

$\partial\bar{U}$ " $\Rightarrow f\,\partial\bar{U}$ comact!

Suppose $R = \max\{|f(x)| \mid x \in \bar{U}\}$

$R_1 = \min\{|f(x)| \mid x \in \partial\bar{U}\}$

Then f is homotopic to the map

$f': \mathbb{R}^n - S \to \mathbb{R}^n$

$$f'(x) = \begin{cases} f(x) & \text{if } |f(x)| \geq R \\ f(x) \cdot \dfrac{R}{R_1} & \text{if } |f(x)| \leq R_1 \\ \dfrac{f(x)}{|f(x)|} \cdot R & \text{if } R_1 \leq |f(x)| \leq R \end{cases}$$

~~The~~ Call f' f.

Now $f: (\bar{U}, \partial\bar{U}) \longrightarrow (D^n, \partial D^n)$

By lemma 2,

$\deg f = \deg f|_{\partial U} = 0$

~~But~~ By induction

$f|_{\partial U} \simeq *$

$\Rightarrow$ We can extend f to g on U so that

$g(U) \subseteq \partial D^n$

Now g misses the point 0.

So $g \simeq *$ $f \underset{U}{\simeq} g \Rightarrow f \simeq *$.

Q. $f: X \to Y$ weakly homotopic, $\Rightarrow E^n(X) = E^n(Y)\ \forall E$??

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C-W- approximation theorem:

$\forall X$, $\exists Y$ - CW complex such that

$Y \to X$ is a weak ~~equi~~ homotopy equivalence.

If X, Y are m-connected, n-connected resp. then,

$X \vee Y \to X \times Y$ is an $m+n+1$ equivalence.

Using this and the fact that $\pi_n(S^n) = \mathbb{Z}$

[redacted] construct Y.

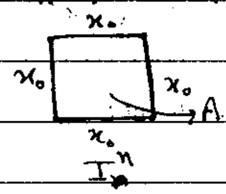
But to show that we have homotopy equivalence we require to invoke homotopy excision formula.

Th^m: Following is exact.

$$\to \pi_2(A) \xrightarrow{i} \pi_2(X) \xrightarrow{\pi} \pi_2(X,A) \xrightarrow{\delta} \pi_1(A) \to \pi_1(X) \to \pi_1(X,A) \to \pi_0(X) \ldots$$

- $\pi_n(A) \xrightarrow{i} \pi_n(X) \xrightarrow{\pi} \pi_n(X,A)$

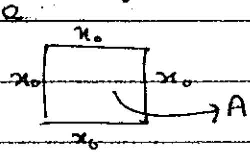
$(\pi \cdot i)$



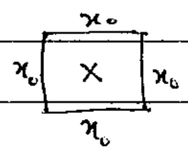
$[I^n, A] = 0$ as I^n is null homotopic

So $\pi \cdot i = 0$

~~ker~~ im i =



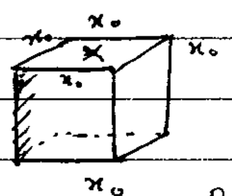
ker π =



$(I^n, \partial I^n) \to (X, x_0)$

$\downarrow$ ~~homotopic~~

$(I^n, \partial_1 I^n, \partial_2 I^n) \to (X, A, x_0)$

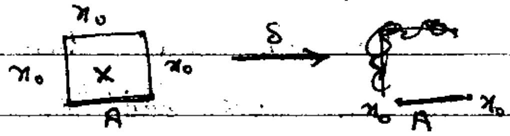


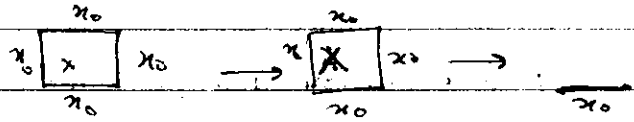
$\downarrow$ homotopic

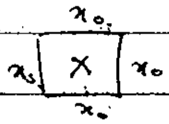
$(I^n, \partial_1 I^n, \partial_2 I^n) \to (x_0, x_0, x_0)$

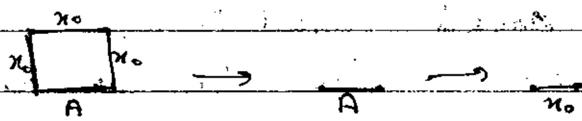
But this homotopy can also be thought as a homotopy between upper face and front face $\Rightarrow$ ker $\pi \subseteq$ im i

• $\pi_n(X) \xrightarrow{\pi} \pi_n(X,A) \xrightarrow{\delta} \pi_{n-1}(A)$



$i \circ \pi$ =  = 0

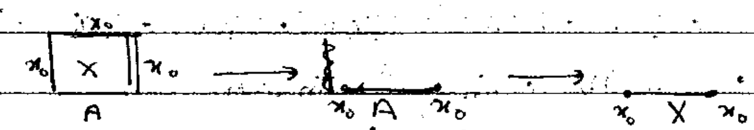
im π = 

ker i = 

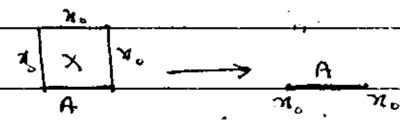
use this homotopy

$(D^n, \partial D^n)$ is a good pair. So extend the homotopy to all of D^n.

• $\pi_n(X,A) \xrightarrow{\delta} \pi_{n-1}(A) \xrightarrow{i} \pi_{n-1}(X)$

$i \circ \delta$ = 

But fig 1 is homotopy of fig 3 to x_0.

im i = 

M_f : Mapping cylinder of f 

Th^m (whitehead) X, Y CW complexes, ~~π~~

$\exists\ f : X \to Y$, s.t. $\pi_n(f) : \pi_n(X) \xrightarrow{\simeq} \pi_n(Y)\ \forall n$

Then f is a homotopy equivalance.

Proof: Make f cellular. This is to make M_f CW complex.

Look at $Z = M_f$; $\pi_n(M_f, X) = 0$

$$\begin{tikzcd} \partial D^n \arrow[r, "g"] \arrow[d] & A \arrow[d] \\ D^n \arrow[r, "g"'] & Y \end{tikzcd}$$

$\to$ induces an element of $\pi_n(Y, A)$

$\pi_n(Y,A) = 0 \Rightarrow g$ is homotopic to a map $D^n \to A$ rel ∂D^n

$f : X \hookrightarrow M_f$ need to show deformation retraction

$M_f = X \cup$ (0 cells) $\cup$ (1 cells) ...

Attach 1 cell at a time

$$\begin{tikzcd} X \arrow[r, leftrightarrow] \arrow[d] & X \arrow[d] \\ X \cup e_k \arrow[r, "f"'] \arrow[ur, dashed] & M_f \end{tikzcd}$$

$\pi_{k+1}(M_f, X) = 0$

homotopic rel X to f

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$\pi_k(\mathbb{C}P^n/\mathbb{C}P^{n-1}) = 0$ for $k \le 2n-1$

$\Rightarrow \pi_k(\mathbb{C}P^{n-1}) \hookrightarrow \pi_k(\mathbb{C}P^n)$ $\begin{cases} \text{isomorphism} & q < 2n-1 \\ \text{surjection} & q = 2n-1 \end{cases}$

$\mathbb{C}P^1 = S^2$

$\Rightarrow \pi_2(\mathbb{C}P^n) = \pi_2(\mathbb{C}P^{n-1}) = \dots = \pi_2(\mathbb{C}P^1) = \mathbb{Z}$

$\pi_2(\mathbb{C}P^\infty) = \mathbb{Z}$ ~~To~~ Argue using compact support image

Ex: $X = \varinjlim X_n$ Then $\pi_k(X) \cong \varinjlim_n \pi_k(X_n)$

$X_n = S^n$ $X = S^\infty$

$\pi_k(S^\infty) = \varinjlim_n \pi_k(S^n) = 0$ $\forall k$

So $* \hookrightarrow S^\infty$ weak homotopy eq.

By whitehead thm, S^∞ is contractible.

Homotopy Excision: $X = A \cup B$ A, B subcomplexes

$C = A \cap B$

(A, C) – m connected

(B, C) – n connected

Then $(A, C) \longrightarrow (X, B)$ is an $(m+n)$ equivalence

Long Proof: Step 1: Reduce to the case

$A = C \cup e^{m+1}$

$B = C \cup e^{n+1}$

Reduction:

Induction on no. of cells in $A - C$

$X = X' \cup e^M$, $A = A' \cup e^M$ $M \ge m+1$ Use 5-lemma

$$\begin{array}{ccccccccc} \to & \pi_*(A', C) & \longrightarrow & \pi_*(A, C) & \longrightarrow & \pi_*(A, A') & \to \\ & \downarrow & & \downarrow & & \downarrow & \\ \to & \pi_*(X', B) & \longrightarrow & \pi_*(X, B) & \longrightarrow & \pi_*(X, X') & \to \end{array}$$

$B = B' \cup e^N \qquad X = A \cup B'$

$$\begin{array}{ccc} \pi_*(X', B') & \longrightarrow & \pi_*(X, B) \\ \uparrow & & \\ \pi_*(A, C) & & \end{array}$$

For infinite cells, we direct limit & compact image arg.

Step 2: $A = C \cup D^{m+1}$

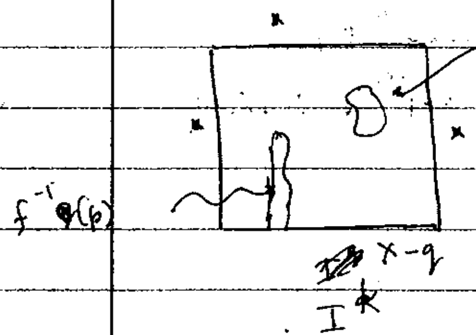
$B = C \cup D^{n+1}$

$X = C \cup D^{m+1} \cup D^{n+1} \qquad p \in D^{m+1} \qquad q \in D^{n+1}$

$$\begin{array}{ccc} (A, C) & \longrightarrow & (X, B) \\ \wr\wr & & \wr\wr \\ (X-p, X-p-q) & \longrightarrow & (X, X-q) \end{array}$$

For $H^k(\ ) = H^k(\ ;\ )$ we need to lift a homotopy $[(I^k, \partial I^k), (X, X-q)]$ to $(X-p, X-p-q)$

i.e. can we omit p from the image of I^k



heuristically $f^{-1}(q)$, $f^{-1}(p)$ are far apart

So we can

Simplicial Approximation

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Simpicial Complex	$K \xrightarrow{f} L$ Δ-complex Δ complex
Simplicial map	f simplicial iff f continuous s.t. $f(\text{int simplex}) \subseteq \text{int}(\text{simplex})$ $f\vert_{\text{int simplex}} = \text{linear}$
	Not all maps ~~bet~~ are simplicial, but $\exists$ a barycentric subdivision such that the map ~~becomes~~ homotopes to a simplicial map
see later	$(K,L) \longrightarrow (X,A)$ $X = A \cup D^{m+1}$ Relative simplicial complex finite $D_1^{m+1} = \{x \mid \lvert x\rvert \le 1/2\} \cap \text{int } D^{m+1}$ $D_0^{m+1} = \{x \mid \lvert x\rvert \le 1/4\}$
Th^m:	$\exists$ a barycentric subdivision of (K,L), $f' : (K,L) \to (X,A)$ $f'\vert_L = f$ f' has the property if $f'(\text{simplex}) \cap D_0^{m+1} \neq \phi$ (Prove this) $\Rightarrow f'(\text{simplex}) \subseteq D_1^{m+1}$, $f'\vert_{\text{simplex}}$ is linear.
Q.1	Prove $[\Sigma^2 X, Y]_*$ is abelian.
2	Γ connected graph. Show $\pi_i(\Gamma) = 0$ for $i > 1$
3	Show $[S^1 \vee S^1, S^2]_* = 0$ Calculate $\pi_2(S^2, S^1 \vee S^1)$.
4.	X - m connected Y n connected $X \vee Y \hookrightarrow X \times Y$ is $m+n-1$ equivalence
5.	Compute $\pi_n(\mathbb{R}P^n, \mathbb{R}P^{n-1})$, $\pi_n(\mathbb{R}P^n / \mathbb{R}P^{n-1})$

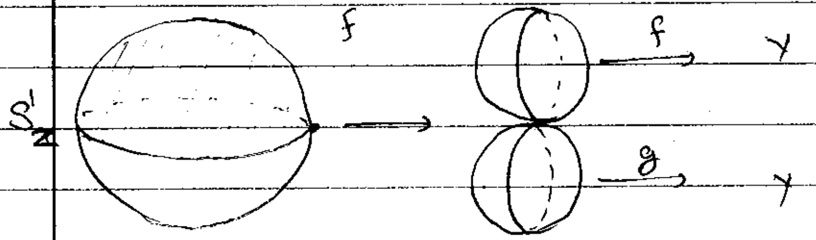
1. $[S^2, Y]_*$ S^1_x, S^1_y, S^1_z denote equators $x=0, y=0, z=0$.

$\sigma_x^\theta, \sigma_y^\theta, \sigma_z^\theta$ denote rotations about x, y, z axis

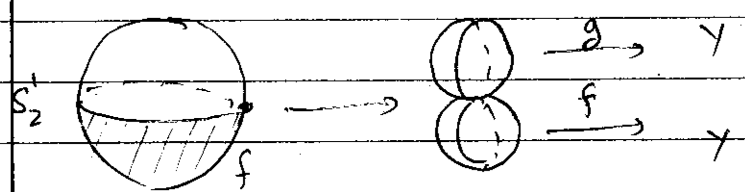
base pt $(1,0,0)$

$$[S^2, Y]_* \cdot [S^2, Y]_* \longrightarrow [S^2, Y]_*$$

$$f \cdot g \longmapsto f \cdot g$$



$$g \cdot f \longmapsto g \cdot f$$



Then ~~~~ Homotopy on S^2

$$F : S^2 \times I \longrightarrow S^2$$

$$F_t(\theta) = \sigma_x^{t \cdot 2\pi}$$

composes to give homotopy between fg & gf

so abelian

~~$[S^2, X]_*$~~ $[S^2_* X]_* \longrightarrow [S^2 X, Y]_*$

~~$[S^2 \times \longrightarrow$~~ $(S^2 \vee S^2)$

$$[\Sigma^2 X, Y]_* \times [\Sigma^2 X, Y]_* \longrightarrow [\Sigma^2 X, Y]_*$$

$$S^2 \wedge X \longrightarrow (S^2 \vee S^2) \wedge X \xrightarrow[g]{f} Y$$

As map is abelian on S^2,

it is abelian for $S^2 \wedge X = \Sigma^2 X$

Q. Universal cover of a CW complex is a CW complex?

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2. Any Universal cover of a graph is a tree.

3. $[S^1 \vee S^1, S^2]_* = [S^1, S^2]_* \oplus [S^1, S^2]_* = 0$

$\underset{\substack{\parallel \\ 0}}{\pi_2(S^1 \vee S^1)} \to \pi_2(S^2) \longrightarrow \pi_2(S^2, S^1 \vee S^1) \longrightarrow \pi_1(S^1 \vee S^1) \longrightarrow \pi_1(S^2)$

$S^1 \vee S^1$ has universal cover cayley graph of F_2

$\Rightarrow \quad \pi_2(S^2, S^1 \vee S^1) = F_2 \oplus \mathbb{Z}$

4. X - m connected . Y - n-connected

Then C-W structure of $X \times Y$ is

$(X \times Y)^{(i)} = \bigcup * \times e^i \cup e'^i \times *$ $\quad i \leq m+n+1$

where e^i is an i cell in Y

e'^i is an i cell in X

So $(X \times Y)^{(i)} = (X \vee Y)^{(i)}$ for $i \leq m+n+1$

$\Rightarrow$ $X \vee Y \hookrightarrow X \times Y$ is an $m+n+1$ equivalence.

5. $\mathbb{RP}^n / \mathbb{RP}^{n-1} = S^n$ $\quad \pi_n(\mathbb{RP}^n / \mathbb{RP}^{n-1}) = \mathbb{Z}$

$$\begin{array}{ccccc} \pi_n(\mathbb{RP}^{n-1}) & \longrightarrow & \pi_n(\mathbb{RP}^n) & \longrightarrow & \pi_n(\mathbb{RP}^n, \mathbb{RP}^{n-1}) \\ & & \parallel & & \downarrow \\ & & \mathbb{Z} & & \pi_{n-1}(\mathbb{RP}^{n-1}) \longrightarrow 0 \\ & & & & \parallel \\ & & & & \mathbb{Z}. \end{array}$$

$\pi_n(\mathbb{RP}^n, \mathbb{RP}^{n-1}) \cong \mathbb{Z} \oplus \mathbb{Z}$

(Need to do more in this case. for $n=2$)

Note: $\mathrm{Hom}(X \times Y, Z) \cong \mathrm{Hom}(X, \mathrm{Hom}(Y, Z))$

in pointed spaces

$[X \wedge Y, Z]_* \cong [X, [Y, Z]_*]_*$

so $[\Sigma^2 X, Z]_* \cong [S^2, [X, Z]_*]_* = \pi_2([X, Z]_*)$

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Simplicial Approx Lemma

$$(K,L) \xrightarrow{f} (X,A) \qquad X = A \cup e^n$$

finite simplicial pair $\qquad e_0^n = \{x \in e^n \mid |x| \le 1/4\}$

$\exists$ a subdivision of (K,L) and a homotopy $f \simeq f' : (K,L) \to (X,A)$ relative to $f^{-1}(A)$, such that if for any simplex σ, $f'(\sigma) \cap e_0^n \neq \phi$ then $f'(\sigma) \subset \text{int } e_1^n$ and $f|_\sigma$ is linear.

Proof:

$e_1^n = \{x \mid |x| \le 1/2\} \qquad e_2^n = \{x \mid |x| \le 3/4\}$

$f^{-1}(e_2^n)$ is compact as K, L is finite

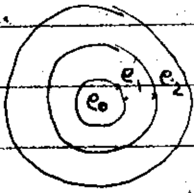
$\Rightarrow f|_{f^{-1}(e_2^n)}$ is uniformly continuous

$\therefore$ Choose $\delta > 0$ such that

$$|x-y| < \delta \Rightarrow |f(x) - f(y)| < 1/4 \qquad x, y \in f^{-1}(e_2^n)$$

Subdivide (K,L) till diameter of simplices becomes less than δ. we get 3 classes of simplices:

- $C_1 = \{\sigma \mid f(\sigma) \in X - e_1^n\}$
- $C_2 = \{\sigma \mid f(\sigma) \in \text{int } e_1^n\}$
- $C_3 = \{\sigma \mid f(\sigma) \cap \partial e_1^n \neq \phi\}$



$$\sigma \in C_3 \Rightarrow \sigma \cap e_0^n = \phi$$

Define f' as follows:

$\sigma = [v_0 \cdots v_k]$

if $\sigma \in C_1$ $\quad f'(\sigma) = f(\sigma)$

if $\sigma \in C_2$ $\quad f'(t_0 v_0 + \cdots + t_k v_k) = t_0 f'(v_0) + \cdots + t_k f'(v_k)$

if $\sigma \in C_3$ $\quad$ define inductively on $\dim \sigma$

$\dim = 0 \quad f'(\sigma) = f(\sigma)$

suppose defined for $\dim \sigma < k$

$\sigma = [v_0 \cdots v_k]$ $b =$ Barycenter of σ

$f'(b) := f(b)$

$x \in \sigma$, $x \neq b$ suppose line joining b to σ ~~...~~ intersects $\partial\sigma$ in $\lambda(x)$

if $x = tb + (1-t)\lambda(x)$ define,

$f'(x) := tf'(b) + (1-t) f'(\lambda(x))$

We do this ~~as~~ there might be simplices half in e_1 ~~in ...~~.

Check Homotopy: $f' \simeq f$ give Linear Homotopy.

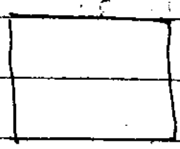
$\square$.

Homotopy Excision: $A = C \cup D^{m+1}$, $B = C \cup D^{n+1}$

$(A, C) \hookrightarrow (A \cup B, B)$ $m+n$ - equivalence.

$q \leq m+n$

x



I^q

$\pi_q(A \cup B, B) = \pi_q(C \cup D^{m+1} \cup D^{n+1}, C \cup D^{n+1})$

$(I^q, I^{q-1} \times \{0\}, \partial I^{q-1} \times [0,1] \cup I^{q-1} \times \{0\}) \longrightarrow (C \cup D^{m+1} \cup D^{n+1}, C \cup D^{n+1}, *)$

We want to show that we can remove D^{n+1} from image so that $(C \cup D^{m+1} \cup D^{n+1}, C \cup D^{n+1}) \to (\underset{A}{C \cup D^{m+1}}, C)$

It is enough to homotope to a map which misses a point in D^{n+1}.

$(I^q, \partial I^q) \longrightarrow (A \cup B, B)$

f

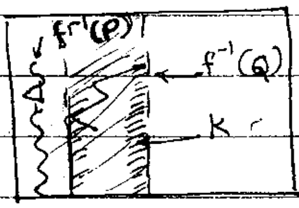
Apply simplicial approximation lemma

if ~~...~~ σ is s.t. $f(\sigma) \cap e_0^n \neq \phi$

f linear on σ so $\dim f(\sigma) \leq m+1$

$\bigcup_{\dim f(\sigma)\le m} f(\sigma) \neq e_0^{n+1}$ by lookig at dim

$\therefore\ \exists\, Q \in e_0^{n+1} - \bigcup_{\dim f(\sigma)\le m} f(\sigma)$



So $\dim f^{-1}(Q) \le q-m-1$

$\pi : I^q \longrightarrow I^{q-1}$

$(a_1, \dots a_q) \longrightarrow (a_1, \dots a_{q-1})$

$K = \pi^{-1}(\pi(f^{-1}Q))$ $\quad \dim K \le q-m \le n$

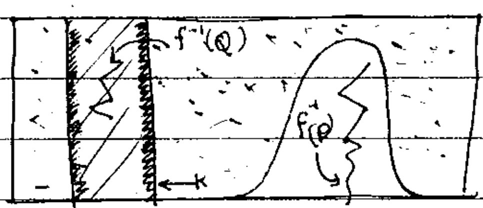
So $f(K) \neq D^{n+1}$

$\Rightarrow \exists\, P \in D^{n+1} - f(K)$

$f^{-1}(P) \cap K = \phi$

Now wee need to miss P

$\underbrace{\pi(K)}_{\text{closed}} \cap \underbrace{\pi(f^{-1}(P))}_{\text{closed}} = \phi$



$\exists\, \phi : I^{q-1} \longrightarrow [0,1]$ by ~~Tietze extension th~~ Uryzon Lemma

$\varphi|_{\pi(K)} = 0$

$\varphi|_{\pi f^{-1}(P)} = 1$

$H : I^q \times I \longrightarrow I$

$H(a,t,s) \longmapsto (a, t(1-s\varphi(a)))$

Check : $f \circ H$ is a chomotopy ~~which~~ between

$f : (I^q, \partial I^q) \longrightarrow (A\cup B, A\cup B - Q)$ to

$g : (I^q, \partial I^q) \longrightarrow (A\cup B - P, A\cup B - P - Q)$

$(A\cup B - P, A\cup B - P - Q)$ deformation retracts onto (A, C)

$\Rightarrow \pi_q(A,C) \longrightarrow \pi_q(A\cup B, B)$ is surjective.

Exercise : Prove injectivity.

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Freudenthal Suspension Th^m

X $(n-1)$ connected.

$$\pi_k(X) \xrightarrow{\Sigma} \pi_{k+1}(\Sigma X)$$

$$f: S^k \to X \longmapsto \Sigma f: S^{k+1} \to \Sigma X$$

Σ isomorphism if $k \le 2n-2$

Σ surjection if $k = 2n-1$

Th^m: (X,A) X- n connected A- s connected

$\Rightarrow$ $\pi_q(X,A) \longrightarrow \pi_q(X/A, *)$ isomorphism $q \le n+s$

surjection $q = n+s+1$

Th^m: (Y,X) n-connected via $f: X \to Y$

$\Rightarrow$ $[Z,X]_* \xrightarrow{f_*} [Z,Y]_*$ isomorphism $\dim Z < n$

(Z CW-complex) surjection $\dim Z = n$

Cor: X n-connected

$X \longrightarrow *$ $n+1$ connected

$\Rightarrow$ $[Z,X] = *$ $\forall$ CW Z, $\dim Z \le n$

Cor $X \xrightarrow{f} Y$ ∞-connected i.e. weak equivalence

$\Rightarrow$ $H_n(X) \underset{H_n(f)}{\xrightarrow{\cong}} H_n(Y)$ $\forall n$

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Hurewicz $h : \pi_n(X) \longrightarrow H_n(X)$

$[f] \longmapsto f_*[\alpha]$ where α generator of $H_n(S^n)$.

Thm: X $k-1$ connected, $k \geq 2$. Then h is an isomorphism.

Relative: $h : \pi_n(X, A, x_0) = [I^n, I^{n-1}\times\{0\}, \partial I^{n-1}\times I \cup I^{n-1}\times\{1\}]$

$$\downarrow f$$

$$[X, A, x_0]$$

$\pi_n(I^n, \partial I^n) \xmapsto{f_*} H_n(X, A)$

Thm: (X, A) $k-1$ connected $k \geq 2$

A simply connected.

$h : \pi_k(X, A) \longrightarrow H_n(X, A)$ is isomorphism.

Proof:

For S^n true

(X, A) - CW pair (X, A) $k-1$ connected

$\downarrow$ A 1 connected

$(X/A, *)$ $k+1$ equivalence

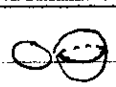
$$\begin{array}{ccc} \pi_n(X,A) & \xrightarrow{h} & H_k(X,A) \\ \approx \downarrow & & \downarrow \approx \\ \pi_n(X/A) & \xrightarrow[h]{} & H_k(X/A) \end{array}$$

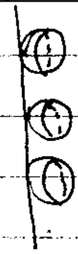
$\approx$ because X/A can be reduced to wedge of k spheres.

Then for infinite complex use direct limits.

For non CW complexes, use CW approximation.

Note: $\pi_n(\varinjlim_k A_k) = \varinjlim_k (\pi_n(A_k))$ only holds when A_k is CW & Topology is direct limit CW topology

Ex: $S^1 \vee S^2$ 

Universal cover $\longrightarrow$ 

$\pi_1(S^1 \vee S^2) = \mathbb{Z}$

$\pi_2(S^1 \vee S^2) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots$

$\pi_n(S^1 \vee S^n) = \pi_n(\vee S^n)$

Thm (Serre): $\pi_{n+k}(S^n) =$ finite for $k > 0$ and except following

$\pi_{2n-1}(S^{2n}) = \mathbb{Z} \oplus$ finite

Thm: $A \xrightarrow{f} X$ $A; X$ simply connected

$H_n(A) \xrightarrow{f_*} H_n(X)$ is isomorphism for all n

Then, $\pi_n(A) \xrightarrow{f_*} \pi_n(X)$ is an isomorphism $\forall n$.

Existence of map is important

$S^2 \times S^2$ $\qquad$ $S^2 \vee S^2 \vee S^4$

$H_* = \mathbb{Z}, 0, \mathbb{Z}^2, 0, \mathbb{Z}$ $\qquad$ $\mathbb{Z}, 0, \mathbb{Z}^2, 0, \mathbb{Z}$

But cohomology ring of first is non-trivial but that of the second is not.

Fibre Bundle

$$\begin{array}{c} F \hookrightarrow E \\ \downarrow p \\ X \end{array}$$

Q. Some $p: E \to X$ s.t. $p^{-1}(x) = F$ but p not a fibre bundle.

Ans: $X \xrightarrow{\cdot} X$ two different topologies on X.

Homotopy lifting

Homotopy lifting for fibre bundles is true.

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Fibration: $\begin{array}{c}E \\ \downarrow p \\ X\end{array}$ fibration if homotopy lifting holds.

$$\begin{tikzcd} A\times\{0\} \arrow[r] \arrow[d, hook] & E \arrow[d] \\ A\times I \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

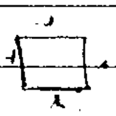
1. Hurewicz fibration $\forall A$
2. Serre fibration $A = D^n$

$$\begin{tikzcd} F \arrow[r, hook] & E \arrow[d, "p"] \\ & X \end{tikzcd}$$

$p^{-1}(*) = F$

Claim: $p : (E, F) \longrightarrow (X, *)$ is an isomorphism

Surjective:



$$\begin{tikzcd} I^{n-1}\times\{0\} \arrow[r, "\text{trivial}"] \arrow[d] & E \arrow[d] \\ I^n, \partial I^n \arrow[r] \arrow[ur, dashed] & X, * \end{tikzcd}$$

lift will be an element of $\pi_n(E, F)$.

Injective:

$f \in \pi_n(E, F)$ $\quad H : p(f) \simeq *$

$$\begin{tikzcd} I^n, \partial I^n \arrow[r, "f"] \arrow[d] & E, F \arrow[d, "p"] \\ I\times I^n, \partial I^n \arrow[r, "H"] \arrow[ur, dashed, "\tilde{H}"] & X \end{tikzcd}$$

lift gives $\tilde{H}_t : (I^n, \partial I^n) \to (E, F)$

$\tilde{H}_0 = f$

$\tilde{H}_1 \in F$

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X - based space $\Omega X = \mathrm{Maps}_*(S^1, X) \leftarrow$ based loop space

$\pi_n(\Omega X) = \pi_{n+1}(X)$

Path-Loop Fibration

$PX = \{\gamma \in X^I \mid \gamma(0) = *\}$. $\Omega X \hookrightarrow PX$

$$\begin{tikzcd} PX \arrow[d, "\pi"] \\ X \end{tikzcd} \quad \pi(\gamma) = \gamma(1) \qquad \begin{tikzcd} \Omega X \arrow[r, hook] & PX \arrow[d] \\ & X \end{tikzcd}$$

PX - contractible.

Proposition X - connected CW-complex. $\begin{matrix} E \\ \downarrow \\ X \end{matrix}$ fibration $\Rightarrow$ all fibres are weakly equivalent.

Pullback
$$\begin{tikzcd} f^*(E) \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$
$f^*(E) = \{(e, y) \mid p(e) = f(y)\} \subseteq E \times Y = E \times_X Y$

Universal Property: $Z \longrightarrow f^*(E)$ Product is a special case of pullback

$\Leftrightarrow$
$$\begin{tikzcd} Z \arrow[r] \arrow[d] & E \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

f^*E is a fibration:
$$\begin{tikzcd} S^n \times I \arrow[r] \arrow[d] & E \times_X Y \arrow[r] \arrow[d] & E \arrow[d] \\ S^n \arrow[r] \arrow[ur, dashed] \arrow[urr, dashed] & Y \arrow[r] & X \end{tikzcd}$$

$$\begin{tikzcd} F \arrow[r, hook] & E \arrow[d] & f^*E \arrow[l] \arrow[d] & F \arrow[l, hook] \\ & X & Y \arrow[l, "f"] & \end{tikzcd}$$

Long Exact Sequence:
$$\begin{tikzcd} \arrow[r] & \pi_{k-1}(F) \arrow[r] \arrow[d, "id"] & \pi_k(f^*E) \arrow[r] \arrow[d] & \pi_k(Y) \arrow[r] \arrow[d, "f"] & {} \\ \arrow[r] & \pi_k(F) \arrow[r] & \pi_k(E) \arrow[r] & \pi_k(X) \arrow[r] & {} \end{tikzcd}$$

if "f" is weak equivalence

$\Rightarrow f^*E \longrightarrow E$ is also a weak equivalence

$$\gamma : [0,1] \longrightarrow X \qquad \gamma(0)=x \qquad \gamma(1)=y$$

$$\begin{array}{ccccc} F_x & \longrightarrow & \gamma^*E & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow \\ \{0\} & \longrightarrow & [0,1] & \xrightarrow{\gamma} & X \end{array}$$

. $\{0\} \hookrightarrow [0,1]$ is a weak equivalence

$\Rightarrow F_x \simeq F_{[0,1]} \simeq F_y$

Eilenberg Maclane

$K(A,n)$ – CW complex, A – finitely generated abelian group $n\geq 1$

$$\pi_i(K(A,n)) = \begin{cases} A & \text{if } i=n \\ 0 & \text{else} \end{cases}$$

$$\underset{\varepsilon_n}{\tilde H^n(K(\mathbb{Z},n))} \cong \underset{1}{\mathbb{Z}}$$

$$\psi \quad [X; K(\mathbb{Z},n)]_* \longrightarrow \cancel{H^n(K(\mathbb{Z},n))}\ H^n(X)$$

$$f : X \to K(\mathbb{Z},n) \longmapsto f^*\varepsilon_n$$

Theorem: $\tilde H^n(X) \cong [X, K(\mathbb{Z},n)]_*$ via above map.

(Also true $\tilde H^n(X, A) \cong [X, K(A,n)]_*$

Proof:

1. $X = S^n$

$$[S^n, K(\mathbb{Z},n)] \xrightarrow{\cong} \mathbb{Z}$$

2. Group Structure on $[X, K(\mathbb{Z},n)]_*$

$$\begin{aligned}[X, K(\mathbb{Z},n)]_* &\cong [X, \Omega^2 K(\mathbb{Z},n+2)]_* \\ &\cong [\Sigma^2 X, K(\mathbb{Z},n+2)]_*\end{aligned}$$

3. Group Homomorphism

$$[X, K(\mathbb{Z},n)]_* = [\Sigma^2 X, K(\mathbb{Z},n+2)]_* \ni f, g$$

$f+g$: $\cancel{[\Sigma^2 X, K(\mathbb{Z},n+2)] \longrightarrow [\Sigma^2 X \vee \Sigma^2 X, K(\mathbb{Z},n+2)]}$

$$\Sigma^2 X \longrightarrow \Sigma^2 X \vee \Sigma^2 X \xrightarrow{f \vee g} K(\mathbb{Z},n+2)$$

$$\tilde H^n(X,\mathbb{Z}) \cong \tilde H^{n+2}(\Sigma^2 X, \mathbb{Z}) \ni a, b$$

$$\Sigma^2 X \longrightarrow \Sigma^2 X \vee \Sigma^2 X \xrightarrow{a \vee b} K(\mathbb{Z},n+2)$$

$$\begin{array}{ccccc} \tilde H^{n+2}(\Sigma^2 X, \mathbb{Z}) & \longleftarrow & \tilde H^{2+n}(\Sigma^2 X, \mathbb{Z}) \oplus \tilde H^{n+2}(\Sigma^2 X, \mathbb{Z}) & \longleftarrow & \tilde H^{n+2}(K(\mathbb{Z},n+2), \mathbb{Z}) \\ & & & & \uparrow \psi \\ & & (a & \longleftarrow & \psi^*(\varepsilon_{n+2}) \\ & & & & [K(\mathbb{Z},n+2), K(\mathbb{Z},n+2)] \end{array}$$

4. Cofibration Sequence

$$\begin{array}{ccccccc}
[A, K(\mathbb{Z},n)]_* & \longleftarrow & [X, K(\mathbb{Z},n)] & \longleftarrow & [X \cup CA, K(\mathbb{Z},n)]_* & \longleftarrow \\
\downarrow & & \downarrow & & \downarrow & \\
H^n(A,\mathbb{Z}) & \longleftarrow & H^n(X,\mathbb{Z}) & \longleftarrow & H^n(X \cup CA, \mathbb{Z}) & \longleftarrow
\end{array}$$

5. X - CW Complex

$X^{(k)}$ - k^{th} skeleton of X

$X^{(k+1)} = X^{(k)} \cup CS^k$

attaching maps

Use cofibration sequence and 5-lemma

6. Use CW approximation for arbitrary spaces.

□

3. Group Homomorphism

$$H^n(X,A) \longleftarrow [X, K(\mathbb{Z},n)]_* \quad : \Psi$$

Mayer → $\cong \updownarrow$ $\qquad \cong \uparrow$

Vietoris $\quad H^{n+2}(\Sigma^2 X, A) \longleftarrow [\Sigma^2 X, K(\mathbb{Z},n+2)] : \Psi$

Claim: $\Psi(f+g) = \Psi(f) + \Psi(g)$.

$g, f : \Sigma^2 X \longrightarrow K(\mathbb{Z}, n+2)$

$\widetilde{f+g}: \quad \Sigma^2 X \longrightarrow \Sigma^2 X \vee \Sigma^2 X \xrightarrow{f \vee g} K(\mathbb{Z}, n+2)$

$\Psi(f+g) = \cancel{k_*}\,(\widetilde{f+g})_* \, \varepsilon_{n+2}$

$$\Sigma^2 X \longrightarrow \Sigma^2 X \vee \Sigma^2 X \longrightarrow K(\mathbb{Z}, n+2)$$

$$H^{n+2}(\Sigma^2 X) \longleftarrow H^{n+2}(\Sigma^2 X) \oplus H^{n+2}(\Sigma^2 X) \longleftarrow H^{n+2} K(\mathbb{Z}, n+2)$$

$$f_* \varepsilon_{n+2} + g_* \varepsilon_{n+2} \longleftarrow f_* \varepsilon_{n+2} + g_* \varepsilon_{n+2} \longleftarrow \varepsilon_{n+2}$$