Sheaves on a base/basis (for topology) § 2.7

X - top space $\mathcal{B} = \{B_\alpha\}$ basis for topology on X.

$\mathrm{Open}_{\mathcal{B}}(X)$ – category poset

Full subcategory of $\mathrm{Open}(X)$.

$F_{\mathcal{B}} : \mathrm{Open}_{\mathcal{B}} X \hookrightarrow \mathrm{Open}\, X \xrightarrow{F} \mathrm{Sets}$ ⟵ Presheaf on $\mathcal{B}$

similarly Sheaf on $\mathcal{B}$

Prop: Sheaves on $X \xrightarrow{R}$ Sheaves on $\mathcal{B}$ is an equivalence of categories.

⌜ i.e. I) R is fully faithful and essential. or equivalently

II) $\exists\, S$ in the other direction which is a "quasi"-inverse. ⌟

Cor: (Gluing) $X = \bigcup U_\alpha$, F_α a sheaf on U_α satisfy:

1) $\exists\ \phi_{\alpha,\beta} : F_\alpha|_{\alpha\cap\beta} \xrightarrow{\cong} F_\beta|_{\alpha\cap\beta}$ $\forall \alpha, \beta$

2) $\phi_{\beta,\gamma} \circ \phi_{\alpha,\beta} = \phi_{\alpha,\gamma}$ $\phi_{\alpha,\alpha} = 1$ (cocycle)

Then there is a sheaf F over X such that $F|_\alpha \cong F_\alpha$ unique upto a canonical isomorphism.

(This is saying that $U \longrightarrow \left\{ \begin{smallmatrix}\text{sheaves}\\ \text{on}\\ U\end{smallmatrix} \right\}$ is a stack.)

§ 3. Towards affine schemes:

Def^n: A ringed space (X, R): X - topological space
R – sheaf of rings

$\mathrm{Spec}\, A = \left\{ \underset{\mathfrak{p} \subsetneq A}{\text{prime ideals in } A} \right\}$ ⟿ Think of A as "ring of regular functions" on $\mathrm{Spec} A$.
Value of $a \in A$ at the point $\mathfrak{p}$ is
$\mathfrak{p} \longmapsto a + \mathfrak{p} \in A/\mathfrak{p} \subseteq \kappa(\mathfrak{p})$
field of fractions of $A/\mathfrak{p}$

eg: $\cdot\ \mathbb{A}^1_A := \mathrm{Spec}\, A[x]$ $A = k$ algebraically closed field.
$\shortparallel$
$\{ (x-a) \mid a \in k \} \amalg \{(0)\}$
$= (k + \text{a point})$ as a set.
generic point

$\cdot$ $\mathrm{Spec}\, \mathbb{Z} = \{(2), (3), (5), \dots\} \amalg \{(0)\}$

$\cdot$ Convention: $\mathrm{Spec}\, \mathbb{Z} = \phi$

- $\mathbb{A}^1_{\mathbb{R}} = \operatorname{Spec} \mathbb{R}[x] = \{(x-a) \mid a \in \mathbb{R}\} \amalg \{x^2+ax+b \mid a^2 < 4b\} \amalg \{(0)\}$

$$\mathbb{R}[x]/x-a \xrightarrow{\cong} \mathbb{R} \qquad\qquad \mathbb{R}[x]/x^2+ax+b \xrightarrow{\cong} \mathbb{C}$$

$$x \longmapsto a \qquad\qquad x \longmapsto \text{a root of } x^2+ax+b$$

- $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \dots, x_n]$

Weak nullstellensatz: If $k \to K$ is an extension and K is finitely generated as a k-algebra, then K is a finite dim'l$/k$.

$\Rightarrow$ For m maximal in $k[x_1, \dots, x_n]$,

$k[x_1, \dots, x_n]/m$ is a finite extension of k

$\Rightarrow$ If $k = \bar{k}$, $k[x_1, \dots, x_n]/m \xrightarrow{\cong} k$

$\Rightarrow m = (x_1-a_1, x_2-a_2, \dots, x_n-a_n)$ for $a_i \in k$.

Other ideals: (0), $(\underset{\text{irreducible}}{f(x_1, \dots, x_n)})$

- A a ring, I an ideal of A then

$$\left\{ \begin{matrix} \text{prime ideals of} \\ A/I \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{prime ideals of } A \\ \text{containing } I \end{matrix} \right\}$$

$\Longrightarrow \operatorname{Spec} A/I \subseteq \operatorname{Spec} A$

These are going to be the closed subsets of Spec A.

- Let $S \subseteq A$ be a multiplicative subset. We have a canonical map:

$$A \longrightarrow S^{-1}A$$

$$\left\{ \begin{matrix} \text{prime ideals in} \\ S^{-1}A \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{prime ideals } I \text{ such that} \\ I \cap S = \{0\} \end{matrix} \right\}$$

$$\operatorname{Spec} S^{-1}A \subseteq \operatorname{Spec} A$$