Spectra :

- Axioms a category of Spectra should satisfy :
  - Symmetric monoidal w.r.t $\wedge$
  - Σ^∞ left adjoint to Ω^∞

    Σ^∞: suspension spectrum

    Ω^∞: 0^{th} space of the Ω-spectrum
  - Unit of $\wedge$ is $\Sigma^\infty S^0$.
  - Either there is a natural transformation $\Omega^\infty D \wedge \Omega^\infty E \longrightarrow \Omega^\infty(D \wedge E)$

    or $\Sigma^\infty(X \wedge Y) \longrightarrow \Sigma^\infty X \wedge \Sigma^\infty Y$
  - There should be a natural weak equivalence : $\Omega^\infty \Sigma^\infty X \longrightarrow \operatorname{colim}_n \Omega^n \Sigma^n X =: QX$

Th^m (Lewis)

There is no category that satisfies all of these axioms.

- There do exist spectra categories which satisfy MOST of the above axioms: EKMM, Symmetric Spectra, Orthogonal Spectra (all Quillen equivalent)

Coordinate free definition of Spectra :

Let $\mathcal{U}$ be a universe i.e. a real inner product space iso to $\mathbb{R}^\infty$

Def. A spectrum E is a collection of spaces E_V for each finite dim $V \subseteq \mathcal{U}$ such that for $V \subseteq W$ we have homeomorphism

$$\sigma_{V,W} : E_V \xrightarrow{\cong} \Omega^{V-W} E_W = \operatorname{Maps}(S^{V-W}, E_W)$$

S^{V-W}: The one point compactification of $V^\perp \subseteq W$.

- Denote by $\mathcal{S}\mathcal{U}$ the category of spectra over the universe $\mathcal{U}$.

- Given a space X and a spectrum E we can define $E \wedge X$ by $(E \wedge X)_V = E_V \wedge X$ and then spectrify

- External smash product :

  $\bar{\wedge} : \mathcal{S}\mathcal{U} \times \mathcal{S}\mathcal{U}' \longrightarrow \mathcal{S}(\mathcal{U} \oplus \mathcal{U}')$

  $(E \bar{\wedge} E')_{V \oplus W} := E_V \wedge E'_{W'}$

- To make this smash product internal should choose an isometry $\mathcal{U} \oplus \mathcal{U} \longrightarrow \mathcal{U}$

  Does not give us a strictly associative smash product.

Twisted Smash Half product:

Given universes U, U' and an unbased space A with a map $\alpha: A \longrightarrow \mathcal{I}som(U, U')$ and a spectrum indexed on U we get $A \ltimes E$ (depends on α) spectrum indexed on U'.

- when $A = pt$

$$\alpha: * \longrightarrow \mathcal{I}som(U, U') \quad \text{picks out} \quad f: U \longrightarrow U' \in \mathcal{I}som(U, U')$$

$$\text{then} \quad pt \ltimes E = f_* E \quad , \quad (f_* E)_V = E_{f^{-1}(V)}$$

- In general $A \ltimes E$ topologizes union of all $a \ltimes E$ for $a \in A$

Idea: • $E \wedge F$ should be $\mathcal{L}^2(2) \ltimes (E \bar{\wedge} F)$ $\quad (\mathcal{L}(2) = \mathcal{I}som(U \times U, U))$

- More generally, $E_1 \wedge \cdots \wedge E_n := \mathcal{L}(n) \ltimes (E_1 \bar{\wedge} E_2 \cdots \bar{\wedge} E_n)$

- Differentiate between E, $\mathcal{L}(1) \ltimes E$

- Define a monad:

$$\mathbb{L}E := \mathcal{L}(1) \ltimes E$$

$$\mathbb{L}(\mathbb{L}E) = \mathcal{L}(1) \ltimes (\mathcal{L}(1) \ltimes E)$$

$$= (\mathcal{L}(1) \times \mathcal{L}(1)) \ltimes (E) \longrightarrow \mathcal{L}(1) \ltimes E \qquad \text{multiplication}$$

$$E = \{1\} \ltimes E \longrightarrow \mathcal{L}(1) \ltimes E = \mathbb{L}E \qquad \text{identity}$$

Def: An $\mathbb{L}$-spectrum is an algebra over the monad $\mathbb{L}$. i.e. we have a map $\xi: \mathbb{L}M \to M$.

- All suspension spectra are $\mathbb{L}$ spectra

Smash product of $\mathbb{L}$-spectra

$\mathcal{L}(2) \circlearrowleft_r \mathcal{L}(1) \times \mathcal{L}(1)$ right action

$$\begin{array}{ccc} (\mathcal{L}(1) \times \mathcal{L}(1)) \ltimes (M \bar{\wedge} N) & \cong & (\mathcal{L}(1) \ltimes M) \bar{\wedge} (\mathcal{L}(1) \ltimes N) \\ & & \downarrow \xi \bar{\wedge} \xi \\ & & M \bar{\wedge} N \end{array}$$

Coequalize these two actions to get the Smash product:

$$\begin{array}{c} \mathcal{L}(2) \times (\mathcal{L}(1) \times \mathcal{L}(1)) \ltimes (M \bar{\wedge} N) \\ \downarrow \; \downarrow \\ \mathcal{L}(2) \ltimes (M \bar{\wedge} N) \end{array} \qquad \text{coeq} =: M \wedge N$$

- There is a left action $\mathcal{L}(1) \circlearrowright \mathcal{L}(2)$ which make $M \wedge N$ an $\mathcal{L}$-spectrum

Thm: - There are natural isomorphism of $\mathbb{L}$- spectra:

- $M \wedge N \longrightarrow N \wedge M$
- $(M \wedge N) \wedge P \longrightarrow M \wedge (N \wedge P)$
- $(M_1 \wedge \cdots \wedge M_n) \longrightarrow \mathcal{L}(n) \ltimes_{\mathcal{L}(1)^n} (M_1 \bar{\wedge} M_2 \bar{\wedge} \cdots \bar{\wedge} M_n)$

- We get a map $\mathbb{1} : S \wedge N \longrightarrow N$ which is a weak equivalence.

Def: The category of S- modules to be the category s.t. $\mathbb{1}$ is an isomorphism.