## Chevalley's Theorem:

$G$ - algebraic group, $H < G$ closed, then
$\exists$ rational representation $\varphi : G \longrightarrow Gl(V)$ and a 1-dim subspace $L$ of $V$
such that $H = \{x \in G \mid \varphi(x) L = L\}$.

we have a homomorphism $\varphi|_H : H \longrightarrow Gl(L)$

$$Gl(L) \cong K^* = Gl(1,K) = G_m$$

So we are interested in homomorphisms $H \longrightarrow G_m$

### Characters:

. Def$^n$: Homomorphism $\chi : G \longrightarrow G_m$

Example:

1. $G_a$ - ~~$\chi \in G_a$~~ only trivial character

Let $X(G) :=$ Set of all characters of $G$

<u>Lemma</u>: $X(G)$ is an abelian group ~~gr~~ with multiplication given by
$\chi_1, \chi_2 \in X(G) \quad (\chi_1 \cdot \chi_2)(g) = \chi_1(g)\, \chi_2(g)$

$$\left( \begin{array}{ccccc} G & \longrightarrow & G_m \times G_m & \xrightarrow{\mu} & G_m \\ g & \longmapsto & (\chi_1(g), \chi_2(g)) & & \\ & & (g,h) & \longmapsto & gh \end{array} \right)$$

2. $G_m$ - $X(G_m) \cong \mathbb{Z}$
any character is of type $t \longmapsto t^m$

3. $G = D(n,k)$ invertible diagonal matrices
- $X(G) \cong \mathbb{Z}^n$ (Why?)

4. $G = Gl(n,k)$ - $X(G) \cong \mathbb{Z}$
$\det^m$

5. $G = Sl(n,k)$ - $X(G) = \{1\}$
because $[Sl(n,k), Sl(n,k)] = Sl(n,k)$

6. $X(G_1 \times G_2) \cong X(G_1) \oplus X(G_2)$

Ex: Prove all these

$\longrightarrow$ $N \trianglelefteq G$, closed, then $G$ acts on $X(N)$ by automorphisms

$g \in G, \quad \chi : N \longrightarrow G_m$

$g.\chi(n) = \chi(g^{-1}ng)$ (why character?)

~~Def~~$^n$: $V$ rational representation of $G$

$\chi \in X(N) \leftarrow$ weight of $V$ wrt $N$ if

$\exists v \in V$ s.t. $n.v = \chi(n).v \quad \forall n \in N$

$V_\chi := \{v \in V \mid n.v = \chi(n).v \quad \forall n \in N\}$

1 Lemma: $\bigoplus_{\chi \in X(N)} V_\chi$ is a $G$-submodule of $V$.

Proof:

- $\forall g \in G, \ \forall \chi \in X(N)$ we have
  $g.V_\chi = V_{g.\chi}$ (Prove this)

  $\subseteq \ g.v \in V_{g.\chi}? \Leftarrow n.g.v = g\chi(n).g.v = \chi(g^{-1}ng)\,gv = g^{-1}ng.gv$

2 Lemma: $\exists$ only finitely many $\chi$'s in $X(N)$ for which $V_\chi \neq 0$.

3 Lemma: Let $\chi_1, \dots \chi_n$ be finitely many ~~represen~~ characters of $N$ ~~Then~~ $V$ be a rational representation of $N$, then $V_{\chi_j}$'s are linearly independant.

4 Lemma: By Chevalley th$^m$,

$\exists \, \rho : G \longrightarrow Gl(V)$ such that

$V = \oplus V_\chi$ is non-zero $G$-submodule of $V$.

Proof of Chevalley's $th^m$:

→ $I_H \subset K[G]$. Look at the finite dimensional subspace ~~span~~ which generates $I_H$. Then look at the G-span of this. By earlier $th^m$ this is finite dimensional. Call this $W$.

→ $M = W \cap I_H$. Finite dimensional v.s. of dim $d$

→ Claim: $H = \{h \in G \mid \lambda_h M = M\}$

$h \in H \Rightarrow \lambda_h f(x) = f(h^{-1}x)$

For $f \in I_H$ $\quad = 0$ for $x \in H$

$\Rightarrow \lambda_h f \in I_H$

for $f \in W$, $\lambda_h f \in W$ $\because W$ is G-stable

$\lambda_g M = M \Rightarrow f \in W \cap I_H \Rightarrow \lambda_h f \in W \cap I_H$

Because all algebra generators of $I_H$ survive in the intersection $(f_1 \cdots f_n)$

~~Apply $\lambda_h f$ to identity, we get $f(h^{-1}) = 0$ for all algebra generators and hence all of $I_H$.~~

~~we have $f(h) = 0$ for all~~

$\therefore \lambda_h f_i(e) = 0 \Rightarrow f_i(h) = 0 \quad \forall i$

$\Rightarrow I_H(h) = 0$

$\Rightarrow h \in H$.

→ ~~dim~~ $M \subseteq W$ Look at $\Lambda^d M \subseteq \Lambda^d W$

By previous lemma, ~~hence~~

$\lambda_h M = M \iff \lambda_h \Lambda^d M = \Lambda^d M$

$\Updownarrow$

$h \in H$ by claim above.

So, the rational map is $G \longrightarrow Gl(\Lambda^d W)$ and $L = \Lambda^d M$.

Characters:

- $G = G_1 \times G_2$

$$\begin{tikzcd} 1 \arrow[r] & G_1 \arrow[r] & G_1\times G_2 \arrow[r] & G_2 \arrow[l, dashed, bend left] \arrow[r] & 1 \end{tikzcd}$$

$$\begin{tikzcd} 1 \arrow[r] & G_2 \arrow[r] & G_1\times G_2 \arrow[r] & G_1 \arrow[l, dashed, bend left] \arrow[r] & 1 \end{tikzcd}$$

Gives $\quad X(G_1\times G_2) = X(G_1) \oplus X(G_2)$ ~~≅~~ because of splitting and functoriality of $X$.

- So $X(D(n,k)) = \mathbb{Z}^n$

- $Gl(n,k)$ –

  $Sl(n,k)$ ~~is~~ ~~in the~~ is in the commutator of $Gl(n,k)$

  So, $Gl(n,k) \longrightarrow G_m$ splits via $Sl(n,k)$

$$\begin{tikzcd} Gl(n,k) \arrow[rr] \arrow[dr, "\det"'] & & G_m \\ & Gl(n,k)/Sl(n,k) \arrow[ur] & \end{tikzcd}$$

$\quad Gl(n,k)/Sl(n,k) \cong G_m$

$\rightsquigarrow$ only maps here are $t \longmapsto t^m$

So, $Gl(n,k) \longrightarrow G_m$

$\quad t \longmapsto (\det t)^m$

Remains to show

$$[Gl(n,k), Sl(n,k)] = Gl(n,k)$$

Proof of lemma 1:

First we prove only summation.

$\forall g \in G$ we have

$$g\cdot V_\chi = V_{g\cdot\chi}$$

$(\subseteq)$ $n\cdot g\cdot v = g\cdot n' v = g\chi(n')\,\cancel{g}\,v = \chi(n')\,g v \quad \because n' = g^{-1}ng$

$$= (g\cdot\chi)(n)\, g v$$

So $g\cdot V_\chi \subseteq V_{g\cdot\chi}$

$\quad\; \cup\!\!\!\mid$

$\quad gv$

$(\supseteq)$ $\quad g^{-1}(g\cdot V_\chi) = g^{-1}V_{g\cdot\chi} \subseteq V_\chi$

So $V_{g\cdot\chi} \subseteq gV_\chi$

Proof of lemma 3 :

Suppose $v_i \in V_{\chi_i}$ such that

$$\sum_{i=1}^{n} v_i = 0 \qquad - \quad *$$

WALOG assume $n \geq 2$, no $v_i = 0$.

Applying $h \in N$

$$\sum_{i=1}^{n} \chi_i(h) v_i = 0 \qquad - \quad *'$$

$*$, $*'$ will allow us to reduce $n$ by 1. (Need to choose $h$ such that $\chi_1(h) \neq \chi_2(h)$)

Continue & we get the result.

Proof of lemma 2 :

Follows from 3 $\because V_{\chi_i} \subseteq V$, $V$ finite dimensional

Proof of lemma 4 :

Use the representation we get in Chevalley's th$^m$

Then $L$ will be the common eigenvector of $N$.

**Theorem:** $N \trianglelefteq G$ closed. Then, there is a rational rep. $\psi : G \longrightarrow Gl(W)$ such that $\ker \psi = N$.

Proof: By Chevalley's th$^m$, $\exists \varphi : G \rightarrow Gl(V)$, $L \subseteq V$ such that $N = \{ g \in G \mid \varphi(g) L = L \}$

Using this we get a character $\chi_0$ of $N$ such that $\varphi(x) v = \chi_0(x) v \quad \forall x \in N, \forall v \in L$.

Now, we may assume that $V = \bigoplus_{\chi \in X(N)} V_\chi$

Let $W$ be the subspace of $\mathrm{End}(V)$ of all $T$ in $\mathrm{End}(V)$ such that $T V_\chi \subseteq V_\chi$ for all $\chi \in X(N)$.

i.e. $\quad W = \bigoplus_{\chi} \mathrm{End}(V_\chi)$

Consider the conjugation action of $\varphi(G)$ on $\mathrm{End}(V)$.

Claim: $\varphi(x) W \varphi(x)^{-1} = W$

~~To~~ for $T \in W$

$$\varphi(x) . T . \varphi(x)^{-1} V_\chi = (\varphi(x).T).(\varphi(x^{-1})^{-1} V_\chi)$$

$$= \varphi(x).T. V_{x^{-1}\chi}$$

$$= \varphi(x) V_{x^{-1}\chi}$$

$$= V_\chi \, .$$

Define: $\psi : G \longrightarrow Gl(W)$

$$x \longmapsto (T \longmapsto \varphi(x) T \varphi(x)^{-1})$$

Claim: $\ker \psi = N$

$(\supseteq)$ $\quad \psi(n) T v = \varphi(n) T \varphi(n^{-1}) v \qquad v \in V_\chi$

$$= \varphi(n) \; T \; \chi(n^{-1}) v$$

$$= \chi(n^{-1}) \varphi(n) \; Tv \qquad Tv \in V_\chi$$

$$= Tv$$

$(\subseteq)$ $\quad g \in \ker \psi \Rightarrow \varphi(g) T \varphi(g^{-1}) = T$

$\Rightarrow \varphi(g)$ acts by scalar multiplication on each $V_\chi$ in particular on $V_{\chi_0}$

$\Rightarrow \varphi(g) L = L$

$\Rightarrow g \in N.$ (By Chevalley's theorem).

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By Chevalley's th$^m$, for $H < G$, we have a $\varphi : G \longrightarrow Gl(V)$ and an $L \subseteq V$ such that $H =$ ~~Stab(L)~~ $\{g \mid gL = L\}$

Now $L$ is a point in $\mathbb{P}(V)$. So we get an ~~orbit~~ map

$$\tilde{\varphi} : G \longrightarrow \mathbb{P}(V)$$
$$g \longmapsto \varphi(g) L$$

$$\begin{tikzcd} G \arrow[r] \arrow[d, "\tilde{\varphi}"'] & G/H \arrow[dl, dashed] \\ G.L & \end{tikzcd}$$

$G.L \subseteq \mathbb{P}(V)$

Aside: look at $\mathbb{P}^1 \longrightarrow \mathbb{P}^2$

$$x:y \longmapsto x^2 : xy : y^2$$

Though this is a ~~is~~ isomorphism of varieties it is not isomorphism of all sheaves? line bundles in particular are different.