§ Goodwillie Functor Calculus

Given a functor $F : \mathcal{C} \to \mathcal{D}$ the goal of functor calculus is to create a sequence of polynomial approximations $P_n F$ which gives us a Taylor tower:

$$\begin{tikzcd} \cdots \arrow[r] & P_{n+1}F \arrow[r] & P_n F \arrow[r] & \cdots \arrow[r] & P_1 F \\ & & & & F \arrow[u] \arrow[ulll] \arrow[ull] \end{tikzcd} \qquad \begin{tikzcd} P_\infty F = \operatorname{holim}_n P_n F \\ F \arrow[u] \end{tikzcd}$$

This is useful if

1) $T_n F$ is "better" than F
2) The maps $T_\infty F \longrightarrow T_n F$ and $F \longrightarrow T_\infty F$ are "well" connected

- A theorem of Goodwillie classifies the "homogenous functors": $\operatorname{hofiber}(P_n F \longrightarrow P_{n-1} F)$ and thereby answer partly the first question.
- The second question is harder to answer and this is where work is needed.

§ Definitions:

Assume all functors are either $Top_* \longrightarrow Top_*$ or $Top_* \longrightarrow Spectra_*$ (and homotopy functors)

Based topological spaces

Based Spectra

- Cube diagrams:

$P(\underline{n})$ = the poset category of the set $\underline{n} = \{1, 2, \ldots, n\}$
$P_0(\underline{n})$ = " " with the initial object removed
$P_1(\underline{n})$ = " " with the terminal object removed

- An n-cube is a diagram $X : P(\underline{n}) \longrightarrow \mathcal{C}$, let
- Total hofiber of X = $\operatorname{hofiber}\left(X(\phi) \longrightarrow \operatorname{holim} X|_{P_0(\underline{n})}\right) = \operatorname{tfiber} X$
- Total hocofiber of X = $\operatorname{hocofiber}\left(\operatorname{hocolim} X|_{P_1(\underline{n})} \longrightarrow X(\underline{n})\right) = \operatorname{tcofiber} X$

- There are equivalent definitions in terms of iterated fibers
  $P(n)$ can be thought of as a map between two of its $n-1$ faces $P(n-1) \longrightarrow P'(n-1)$
  Then

$$\operatorname{tfiber} X \simeq \operatorname{hofiber}\left(\operatorname{tfiber} X|_{P(n-1)} \longrightarrow \operatorname{tfiber} X|_{P'(n-1)}\right)$$

Q, Is there an easy way to see this?

Q. Can we say that $\operatorname{hofiber}\left(\operatorname{holim}_I X_i \longrightarrow \operatorname{holim}_I Y_i\right) \simeq \operatorname{holim}_I \left(\operatorname{hofiber} X_i \longrightarrow Y_i\right)$

Def^n: A diagram $X : P(\underline{n}) \longrightarrow \mathcal{C}$ is

h cartesian if tfiber $X \simeq *$

h cocartesian if $n=1$ or tcofiber $X \simeq *$

strongly h cocartesian $\equiv$ tfiber $X|_{P(i)} \simeq *$ for all $i \geq 2$ dim faces of $P(n)$

- One can show that if $X|_{P(S)}$ is h cocartesian for all 2 dim faces of $P(\underline{n})$ then X is strongly h cocartesian.

Good Functors:

We need the following two properties for $F : \mathcal{C} \to \mathcal{D}$ to have a well defined Taylor tower:

- Homotopy functor:

  F takes weak equivalences to weak equivalences

- Finitary:

  given a filtered diagram $X : \mathcal{I} \to \mathcal{C}$ we get a weak equivalence

  $$\text{hocolim } F(X_i) \longrightarrow F(\text{hocolim } X_i)$$

eg: $X \in Top_*$, $E \in Spectra_*$. The following functors are homotopy:

- $X \longrightarrow X^{\wedge n}$
- $X \longrightarrow \Sigma^\infty X$
- $X \longrightarrow \Omega^\infty \Sigma^\infty X$
- $X \longrightarrow \text{Map}(K, X)$ where K is a CW complex
- $E \longrightarrow \Sigma^\infty \Omega^\infty E$

- A good functor F is reduced if $F(*) \simeq *$

- **Polynomial functors:**

  $F : \mathcal{C} \longrightarrow \mathcal{D}$ is k-excisive if it takes $(k+1)$ strongly h-cocartesian squares to h-cartesian squares.

- **Prop:** If F is k-excisive then it is also $k+1$ excisive.

Proof: Consider a $(k+2)$-strongly h-cocartesian square $X : P(k+2) \longrightarrow \mathcal{C}$

Let $P(n+2) = P(n+1) \longrightarrow P'(n+1)$

X strongly h-cocartesian $\Rightarrow$ $X|_{P(n+1)}$ and $X|_{P'(n+1)}$ are also strongly h cocartesian

$\Rightarrow$ $F(X)|_{P(n+1)}$ and $F(X)|_{P'(n+1)}$ are h cartesian

$\Rightarrow$ tfiber $F(X)|_{P(n+1)} \simeq * \simeq$ tfiber $F(X)|_{P'(n+1)}$

$\Rightarrow$ tfiber $F(X) \simeq *$

eg. 0-excisive :

$P(\underline{1}) : X \to Top_*$

$X_0 \to X_1$ $\rightsquigarrow$ every map is 1-cocartesian.

F is 0-excisive if $F(X_0) \to F(X_1)$ is a w.e. $\forall X_0 \to X_1$

$\Rightarrow F(X) \simeq F(*)$ $\forall X$.

eg: 1-excisive

$\Rightarrow$ F takes h-pushouts to h-pullbacks

- $Id : Spectra_* \to Spectra_*$ is 1-excisive
  i.e. pushouts are also pullbacks
  this is because $Spectra_*$ is triangulated.

- But $id : Top_* \to Top_*$ is not 1-excisive
  as not every h-pushout is a h-pullback

$$\begin{tikzcd} S^0 \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & S^1 \end{tikzcd}$$

is a h-pushout but not a h-pullback

eg: For a spectrum E :

$- \wedge E : Top_* \to Spectra_*$ is 1-excisive

- Consider a homotopy pushout :

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd} \longrightarrow \begin{tikzcd} E\wedge X \arrow[r] \arrow[d] & E\wedge Y \arrow[d] \\ E\wedge Z \arrow[r] & E\wedge W \end{tikzcd}$$

need to show h-cartesian

For this we need to show the long exact sequence in π_*-groups

- This is the long exact sequence in the homology $\to E_n(X) \to E_n(Y \times Z) \to E_n(W) \to \cdots$

eg. by the above example $X \mapsto \Sigma^\infty X = \mathbb{S} \wedge X$ is 1-excisive

eg. The functor
$$Top_* \to Top_*$$
$$X \mapsto \Omega^\infty (E \wedge X)$$
is 1-excisive.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd} \rightsquigarrow \begin{tikzcd} \Omega^\infty(E\wedge X) \arrow[r] \arrow[d] & \Omega^\infty(E\wedge Y) \arrow[d] \\ \Omega^\infty(E\wedge Z) \arrow[r] & \Omega^\infty(E\wedge W) \end{tikzcd}$$

Ω^∞ is a right adjoint hence preserves pullbacks

• Constructing n-excisive functors:

Th^m: Suppose $F: \mathcal{C}^k \to \mathcal{D}$ is d_i-excisive in the i^{th} variable. Let $\Delta: C \to C^k$ be the diagonal functor then $\Delta \circ F: \mathcal{C} \to \mathcal{D}$ is $\sum d_i$ excisive.

Proof: Given a $\sum d_i$ strongly co-cartesian cube, we apply $\Delta \circ F$ and take the tfibers along the first coordinate to get another cube in one less variable and cartesian by induction.

eg: $X \mapsto E \wedge (X_+)^{\wedge k}$, $X \mapsto \Omega^\infty(E \wedge (X_+)^{\wedge k})$

Prop: If F_i, $i \in I$ is an I-diagram of k excisive functors then $\underset{I}{\text{holim}}\, F_i$ is also k-excisive.

Proof:

This is because holims commute.

§ Stably excisiveness

This is a weaker condition than excisiveness.

• Define the following connectivity measures for constants c, τ, k:

• $F: \mathcal{C} \to \mathcal{D}$ is stably k-excisive if it satisfies

$E_k(c,\tau)$: $X: P(k+1) \to \mathcal{C}$ is strongly k cocartesian
if $X_\phi \to X_i$ is d_i-connected and $d_i \geq \tau$
then $F(X)$ is $(-c + \sum d_i)$ connected.

eg: • F is k-excisive if it satisfies $E_k(-\infty, -1)$.

• $\mathrm{Id}: \mathrm{Top}_* \to \mathrm{Top}_*$ satisfies $E_k(k-1, -1)$

$k=2$
$$\begin{array}{ccc} A\cap B & \xrightarrow{d_1} & A \\ {\scriptstyle d_2}\downarrow & & \downarrow \\ B & \longrightarrow & A\cup B \end{array}$$
is $(-1+d_1+d_2)$ connected

this is the homotopy excision theorem.

• For higher k follow from Blakers-Massey theorem

• $F \to G$ agree to order k

$O_k(c,\tau)$: for $d \geq \tau$ and every $(d-1)$ connected X
$F(X) \to G(X)$ is $(-c + (k+1)d)$ connected

For $X \in Top_*$ define a cube

$$X * - : P(\underline{n}) \longrightarrow Top$$
$$S \longmapsto X * S$$ ← join of X and S. another description of Join :

$$X * S = \text{hocolim} \left(\begin{tikzcd} & X \times S \arrow[dl] \arrow[dr] & \\ X & & S \end{tikzcd} \right)$$

Claim: $X * \mathcal{U}$ is strongly homotopy cocartesian.

Proof: Suffices to show this for all 2-faces

A 2 face looks like

$$\begin{tikzcd} S \arrow[r] \arrow[d] & S \cup \{a\} \arrow[d] \\ S \cup \{b\} \arrow[r] & S \cup \{a,b\} \end{tikzcd}$$

so suffices to show

$$\begin{tikzcd} X * S \arrow[r] \arrow[d] & X * (S \cup \{a\}) \arrow[d] \\ X * (S \cup \{b\}) \arrow[r] & X * (S \cup \{a,b\}) \end{tikzcd}$$

This can be checked by hand

So an n-excisive functor would send $X * P(\underline{n+1})$ to a h-cartesian square.
We use this to construct the polynomial approximations.

- **Note** ◦ $X * \phi = X$
  - for $S \neq \phi$, $X * S$ has higher connectivity than S.

    to see this note that we have the excision LES

$$\longrightarrow \tilde{H}_n(X)^{\oplus |S|} \longrightarrow \tilde{H}_n(X) \longrightarrow \tilde{H}_n(X * S) \longrightarrow \tilde{H}_{n-1}(X) \longrightarrow \cdots$$

always surjective $\qquad\qquad\qquad$ $=0 \Rightarrow H_n(X * S) = 0$

Def^n: For $F: \mathcal{C} \longrightarrow \mathcal{D}$ define $T_k F(X) := \text{holim}\, F\left(P_0(\underline{k+1}) * X\right)$

- If F was k-excisive we should have $T_k F(X) \simeq X$ as $X * \phi = X$
- We have a natural map $t_k(X): F(X) \longrightarrow T_k F(X)$
- If F was reduced
  - $T_1 F(X) = \text{holim} \left(\begin{tikzcd} & * \arrow[d] \\ * \arrow[r] & F(\Sigma X) \end{tikzcd} \right) = \Omega F(\Sigma X)$
  - $T_0 F(X) \simeq *$

Prop: If F satisfies $E_k(c, \tau)$ then
1) $T_k F$ satisfies $E_k(c-1, \tau-1)$
2) $t_k : F \longrightarrow T_k F$ satisfies $O_k(c, \tau)$

Proof: 1) Take a cube $X: P(\underline{k+1}) \longrightarrow \mathcal{C}$ s.t. $X_\phi \to X_i$ are d_i-connected for $d_i \geq \tau - 1$

What is the connectivity of the map

$$\begin{tikzcd}
Y \arrow[d] & \rightsquigarrow \text{fiber} \\
T_k F(X_\phi) \arrow[r] & \underset{S \in P_0(k+1)}{\text{holim}}\, T_k F(X_S)
\end{tikzcd}$$

$$T_kF(X_\phi) = \underset{U \in P_0(\underline{k+1})}{\text{holim}} F(X_\phi * U)$$

$$\underset{S \in P_0(k+1)}{\text{holim}}\, T_k F(X_S) = \underset{S \in P_0(k+1)}{\text{holim}} \ \underset{U \in P_0(k+1)}{\text{holim}} F(X_S * U)$$

$$= \underset{U \in P_0(k+1)}{\text{holim}} \ \underset{S \in P_0(k+1)}{\text{holim}} F(X_S * U)$$

For each $U \in P_0(\underline{k+1})$

$$Y_u \longrightarrow F(X_\phi * U) \longrightarrow \underset{S \in P_0(k+1)}{\text{holim}} F(X_S * U) \quad \text{is} \quad -c + \sum (d_i + 1) = -c + (k+1) + \sum d_i \quad \text{connected}$$

Doing iterated fibers (Why?)

$$Y = \underset{U \in P_0(\underline{k+1})}{\text{holim}} \ Y_u \qquad \underbrace{}_{-c + k + \sum d_i} \ \text{connected}$$

Use spectral sequence (?) to conclude $-c + \sum d_i$ How?

So that $T_k F$ satisfies $E_k(c-1, \tau - 1)$.

2) $T_k F(X) = \underset{S \in P_0(\underline{k+1})}{\text{holim}} F(X * S)$

and so $F(X) \longrightarrow T_k F(X)$ is precisely the map in the condition $O_k(c, \tau)$.

Def: $$P_k F(X) = \underset{n}{\text{hocolim}}\, T_k^n (X)$$

eg. $$P_1(\text{Id}(X)) = \lim_n \Omega^n \Sigma^n X = \Omega^\infty \Sigma^\infty X$$

- We have natural map $F \to P_k F$

- $P_0(k) \hookrightarrow P_0(\underline{k+1}) \Rightarrow$ we have a natural map $T_k F \longrightarrow T_{k-1} F$ and hence a natural map $P_k F \longrightarrow P_{k-1} F$

Combining these maps we get the Taylor tower :

$$\begin{tikzcd}
 & P_n F \arrow[d] \\
F \arrow[ur] \arrow[r] \arrow[ddr] & P_{n-1} F \arrow[d] \\
 & \vdots \\
 & P_1 F
\end{tikzcd}$$

Prop : If F satisfies $E_k(c, \tau)$ then

1) $P_k F$ satisfies $E_k(-\infty, -1)$
2) $t_k : F \longrightarrow P_k F$ satisfies $O_k(c, \tau)$

However a much stronger statement is true :

Thm : 1) $P_k F$ is k-excisive.

2) If $F \longrightarrow G$ satisfy $O_k(c, \tau)$ then $P_k F \longrightarrow P_k G$ is a weak equivalence.

Q What is involved in the proof of this theorem?

Q Would it be possible to have a simple possibly non-rigorous proof ?