Goodwille Calculus of Functors:

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Study of approximations

Q. What are polynomial functors?

Q. Approximation?

$\to$ Classification of polynomial functors.

A. Motivation & Examples: (Linear functors)

1. $\tilde{H}_* : Top_* \longrightarrow {}_g Ab$ ← graded abelian groups

$\tilde{H}_*(X \vee Y) \cong \tilde{H}_*(X) \oplus \tilde{H}_*(Y)$

Mayer-Vietoris: $\longrightarrow \tilde{H}_*(U \cap V) \longrightarrow \tilde{H}_*(U) \oplus \tilde{H}_*(V) \longrightarrow \tilde{H}_*(U \cup V) \longrightarrow$

Think of this as linearity.

What is homology an approximation to?

Ans: π_* !!

Hurewicz: $\pi_*(X) \xrightarrow{h} \tilde{H}_*(X)$

h isomorphism for $* \leq k+1$, where X is k-connected.

2. Stable homotopy groups:

$\pi_*^S : Top_* \longrightarrow {}_g Ab$

Generalised homology $\equiv$ Mayer-Vietoris

$\pi_*(X) \longrightarrow \pi_*^S(X)$

Isomorphism in a wider range: $* \leq 2k$, X is k-connected.

3. Manifold / Embedding Calculus (Weiss)

$Im(-, \mathbb{R}^n) : \mathcal{O}(\mathbb{R}^n)^{op} \longrightarrow Top_*$ (open subsets)

$\longrightarrow Im(U \cup V, \mathbb{R}^n) \longrightarrow Imm(U, \mathbb{R}^n) \times_{Im(U \cap V, \mathbb{R}^n)} Imm(V, \mathbb{R}^n)$

Immersions are the linear approximation to embeddings !!

B. Calculus of homotopy functors:

Focus on $F : Top_* \longrightarrow Top_*$ $\left(\begin{array}{l} \text{or } Top_* \longrightarrow Spectra \\ \quad Spectra \longrightarrow Spectra \end{array} \right)$

Preserve weak homotopy equivalences $\longrightarrow$ Homotopy functor

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Def^n: F is 1-excisive "affine linear" if for any homotopy pushout

$$\begin{tikzcd} X_\phi \arrow[r] \arrow[d] & X_2 \arrow[d] \\ X_1 \arrow[r] & X_{12} \end{tikzcd}, \text{ the square } \begin{tikzcd} F(X_\phi) \arrow[r] \arrow[d] & F(X_1) \arrow[d] \\ F(X_2) \arrow[r] & F(X_{12}) \end{tikzcd} \text{ is a homotopy } \underline{\underline{\text{pull-back}}}.$$

Rem: F 1-excisive $\Rightarrow$ $\pi_* F$ has Mayer-Vietoris

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & U \arrow[d] \\ V \arrow[r] & U \cup V \end{tikzcd} \text{ is homotopy pushout}$$

Ex: $\tilde{H}_*(X) \cong \pi_* \Omega^\infty (H\mathbb{Z} \wedge X)$

$\Omega^\infty (H\mathbb{Z} \wedge -) : Top_* \longrightarrow Top_*$ is 1-excisive

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Linear approximation:

Def: homotopy Functor: $F : Top_* \longrightarrow Top_*$, $\quad X \in Top_*$, $\quad \begin{tikzcd} X \arrow[r] \arrow[d] & C(X) \arrow[d] \\ C(X) \arrow[r] & S(X) \end{tikzcd}$ is homotopy pushout

F 1-excisive $\quad F(X) \xrightarrow{\sim} F(CX) \overset{h}{\underset{F(SX)}{\times}} F(CX)$ is a weak equivalence

So in general define: $T_1(F) := F(CX) \overset{h}{\underset{F(SX)}{\times}} F(CX)$

Remark: $T_1 F$ is closer to being linear than F.

Def^n: $P_1 F :=$ hocolim $(F(X) \longrightarrow T_1 F(X) \longrightarrow T_1 T_1 F(X) \longrightarrow \cdots)$

Th^m: (Goodwillie) $\quad P_1 F$ is 1-excisive.

If F is 1-excisive, $F \xrightarrow{\sim} P_1 F$,

In general, $F \longrightarrow P_1 F$ is universal in category $F \longrightarrow$ 1-excisive functors $\left(\begin{tikzcd} F \arrow[r] \arrow[d] & G \arrow[d, "S"] \\ P_1 F \arrow[r] & P_1 G \end{tikzcd} \right)$ Note: $F(\cdot) \xrightarrow{\sim} P_1 F(\cdot)$

Ex: $\mathrm{Id}: \mathrm{Top}_* \longrightarrow \mathrm{Top}_*$ is not 1-excisive

$T_1 \mathrm{Id}\, X = CX \underset{SX}{\times} CX \simeq * \underset{SX}{\times} * \simeq \Omega\Sigma X$

$P_1 \mathrm{Id}\, X = \mathrm{hocolim}\, (X \longrightarrow \Omega\Sigma X \longrightarrow \Omega^\infty \Sigma^\infty X \longrightarrow \cdots)$

$= \Omega^\infty \Sigma^\infty X = QX$ stable homotopy functor

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Classification of Linear Functors:

Th^m: $F: \mathrm{Top}_* \longrightarrow \mathrm{Top}_*$ homotopy, 1-excisive, $F(*) = *$, preserves filtered homotopy colimit (finitary)

$F \simeq \Omega^\infty (E \wedge X)$ for a spectrum E

(Brown Representability)

C. Higher Degree Polynomials

(f quadratic $f(x+y+z) - f(x+y) - \cdots = 0$)

Def^n: $F: \mathrm{Top}_* \longrightarrow \mathrm{Top}_*$ is n-excisive if

for any strongly homotopy co-cartesian $(n+1)$-cube

$$\begin{tikzcd} X_\phi \arrow[r] \arrow[dr] \arrow[d] & X_1 \arrow[r, dash] & X_{1,\dots,n+1} \\ X_{n+1} \arrow[urr, bend right] & X_2 \arrow[ur] & \end{tikzcd} \xrightarrow{\ F\ } \begin{tikzcd} F(X_\phi) \arrow[r] \arrow[d] & F(X_{n+1}) \arrow[d] \\ F(X_n) \arrow[r] & F(X_{1,\dots,n+1}) \end{tikzcd}$$

is a homotopy pullback

Def^n:

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr] & CX \arrow[dr] & \\ CX \arrow[dr] & CX \arrow[r] \arrow[d] & SX \arrow[d] \\ & SX \arrow[r] & \bullet \end{tikzcd}$$

$F(X) \longrightarrow \underset{\phi \neq U \subseteq \{1,\dots,n+1\}}{\mathrm{hocolim}} F(U * X) =: T_n F$

$P_n F = \mathrm{hocolim}\, (F \longrightarrow T_n F \longrightarrow T_n T_n F \longrightarrow \cdots)$

$P_n F$ is n-excisive, and we have a tower — "Taylor series"

$F \longrightarrow \cdots \longrightarrow P_{n+1} F \longrightarrow P_n F \longrightarrow \cdots \longrightarrow P_1 F \longrightarrow F$

Convergence of Taylor Tower:

For a suitable F, X, $F(X) \xrightarrow{\sim} \mathrm{ho}\lim P_n F(X)$

Ex: $F = \mathrm{Id}$, X simply-connected

$X \xrightarrow{\sim} \underset{\longrightarrow n}{P_n F(X)}$, If X is k-connected $X \longrightarrow P_n \mathrm{Id}(X)$ is $(n+1)k$-connected.

$\rightarrow F$ finitary $\Rightarrow P_1 F$ finitary, other examples: $\Omega^\infty A(X)$, $\Omega^\infty \Sigma^\infty \mathrm{Map}(K, X)$ | $X \longrightarrow \Omega^\infty(\Sigma^\infty X \wedge \Sigma^\infty X)$

K theory

ex of quadratic funct