# PROBLEM SET 06

## PART 1 - MIN-MAX

**Q.1.** If $a_1 < a_2 < \cdots < a_n$ are real numbers,

(1) Find the minimum value of the function $f(x) = \sum_{i=1}^{n} (x - a_i)^2$.

(2) Now find the minimum value of $f(x) = \sum_{i=1}^{n} |x - a_i|$. *

(3) If $a > 0$, find the maximum value of the function

$$f(x) = \frac{1}{1 + |x|} + \frac{1}{1 + |x - a|}$$

**Q.2.** (1) Prove that if $f'(x) \geq M$ for all $x \in [a, b]$ then $f(b) \geq f(a) + M(b - a)$.

(2) Prove that if $f'(x) \leq M$ for all $x \in [a, b]$ then $f(b) \leq f(a) + M(b - a)$.

(3) Formulate a similar theorem when $|f'(x)| \leq M$ for all $x$ in $[a, b]$.

**Q.3.** Suppose that $f'(x) > M > 0$ for all $x$ in $[0, 1]$. Show that there is an interval of length $\frac{1}{4}$ on which $|f| \geq M/4$. †

**Q.4.** Sketch the following functions and find their local maxima and minima:

(1) $\dfrac{x + 1}{x^2 + 1}$

(2) $x + \dfrac{1}{x}$

(3) $\dfrac{x^2}{x^2 - 1}$

(4) $\dfrac{1}{1 + x^2}$

**Q.5.** (Optional) Show that if $f$ is twice differentiable with $f(0) = 0$ and $f(1) = 1$ and $f'(0) = f'(1) = 0$ then $|f''(x)| \geq 4$ for some $x$ in $[0, 1]$.

* Hint: Think graphically.

† Hint: Compare $f(x)$ with the line of slope $M$ passing through $f(0)$.

## Part 2 - Applications

**Q.6.** (1) What is the relationship between the critical points of $f$ and $f^2$?

(2) Consider the straight line described by the equation $Ax + By + C = 0$. Show that the distance from the origin to this line is $\dfrac{C}{\sqrt{A^2 + B^2}}$.

**Q.7.** Show that the sum of a positive number and it's reciprocal is at least 2.

**Q.8.** Prove that if $\dfrac{a_0}{1} + \dfrac{a_1}{2} + \dfrac{a_2}{3} + \cdots + \dfrac{a_n}{n+1} = 0$ then $a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n = 0$ for some $x$ in $[0, 1]$.

**Q.9.** Prove that the function $x^2 = \cos x$ has precisely 2 solutions. (Draw a picture.)

**Q.10.** Prove that if $n > 1$ and $x > 0$ then $(1 + x)^n > 1 + nx$.

**Q.11.** Suppose that $f$ is continuous and differentiable on $[0, 1]$ such that $f(x)$ is in $[0, 1]$ for each $x$, and that $f'(x) \neq 1$ for all $x$ in $[0, 1]$. Show that there is exactly one number $x$ in $[0, 1]$ such that $f(x) = x$.

**Q.12.** Suppose $f$ and $g$ are two differentiable functions which satisfy $f'g - g'f = 0$. Prove that if $a$ and $b$ are adjacent zeroes of $f$, and $g(a) \neq 0$ and $g(b) \neq 0$ then $g(x) = 0$ for some $x$ between $a$ and $b$. ‡

‡ Hint: Proof by contradiction.

## Part 3 - Applications

**Q.13.** Prove that if $f(0) = 0$ and $f'(x)$ is increasing then the function $g(x) = f(x)/x$ is increasing on $(0, \infty)$.

This is a deceptively hard problem, try to work it out yourself, in case you get stuck use the following steps.

(1) Write $g(x)$ as the slope of an appropriate secant line. Then use the Mean Value Theorem to relate $g(x)$ to the slope of some tangent.

(2) Find $g'(x)$ and rewrite it in terms of $f'(x)$ and $g(x)$.

(3) Use parts (1) and (2) to write $g'(x)$ entirely in terms of $f'$ and use this to conclude that $g'(x) > 0$ for all $x > 0$.

**Q.14.** (1) What is wrong with the following application of l'Hospital's rule:

$$\lim_{x\to 1} \frac{x^2 + x - 2}{x^2 - 3x + 2} = \lim_{x\to 1} \frac{3x^2 + 1}{2x - 3} = \lim_{x\to 1} \frac{6x}{2} = 3$$

What is the correct limit?

(2) Find the following limits

(a) $\displaystyle\lim_{x\to 0} \frac{\tan x}{x}$ (b) $\displaystyle\lim_{x\to 0} \frac{\cos^2 x - 1}{x^2}$

**Q.15.** l'Hospital's rule is used in various forms all of which are closely related to each other.

**l'Hospital's rule:** If $\displaystyle\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} g(x)$ and both are equal to either 0 or $\infty$, and if $\displaystyle\lim_{x\to 0^+} \frac{f'(x)}{g'(x)} = l$ then $\displaystyle\lim_{x\to 0^+} \frac{f(x)}{g(x)} = l$.

Using only algebraic manipulations (no complicated proofs) derive the following versions from the above theorem.

(1) If $\displaystyle\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} g(x) = 0$ or $\infty$ and if $\displaystyle\lim_{x\to 0^-} \frac{f'(x)}{g'(x)} = l$ then $\displaystyle\lim_{x\to 0^-} \frac{f(x)}{g(x)} = l$.

(2) If $\displaystyle\lim_{x\to \infty} f(x) = \lim_{x\to \infty} g(x) = 0$ or $\infty$ and if $\displaystyle\lim_{x\to \infty} \frac{f'(x)}{g'(x)} = l$ then $\displaystyle\lim_{x\to \infty} \frac{f(x)}{g(x)} = l$.