Tangle Floer Homology and quantum $gl(1|1)$

Categorification: Lift an algebraic structure to a more interesting one

eg: · Homotopy category of Chain complexes $\longrightarrow$ categorifies $\mathbb{Z}$

· X topological space

Singular Homology $\longrightarrow$ categorifies $\chi(X)$ Euler characteristic

· Jones Polynomial $\longrightarrow$ Khovanov homology

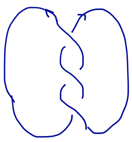
Alexander Polynomial $\longrightarrow$ Knot Floer Cohomology

Reshetikin Turaev way of defining invariants:

TQFT approach to link polynomials

Idea:

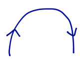
Fix a quantum group $U_q(g)$, a representation V of $U_q(g)$ over $\mathbb{C}$



Start slicing the knot bottoms up

to every cut in the knot we associate a representation

eg:



$\rightsquigarrow V^* \otimes V$

Compose maps between the various slices to get a map

Tangles

$\rightsquigarrow V \otimes V^* \otimes V^* \otimes V$

$\mathbb{C}[q,q^{-1}] \longrightarrow \mathbb{C}[q,q^{-1}]$

$1 \longmapsto \psi_L(1)$

This is an invariant of the knot called the Alexander polynomial

eg: Jones Poly $\rightsquigarrow$ sl_2 U - 2 dim rep

Alexander Poly $\Delta_k \rightsquigarrow$ $gl_{1|1}$ V 2 dim rep

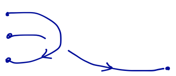
$U_q(gl_{1|1})$ - generated over $\mathbb{C}(q)$ by $E, F, q^{(a,b)}$ $(a,b) \in \mathbb{Z}^2$

Goal: Categorify this construction, for some special cases

· Find a local version of Heegard Floer homology

Tangle Floer Homology:

eg: Tangle (cobordisms)



Left boundary $\partial_L T$

Right boundary $\partial_R T$

Dga $A(\partial_L T)$ $A(\partial_R T)$

$CT(T)$ bimodule over $A(\partial_L T)$ and $A(\partial_R T)$

Th^m

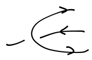
1) $CT(T)$ is an invariant of T.

2) $-T_1 \equiv T_2-$ $\rightsquigarrow$ $CT(T_1 \circ T_2) \simeq CT(T_1) \underset{\partial^R T_1}{\otimes} CT(T_2)$

3) For a link L $CT(L) \simeq HFK(L)$ $\rightsquigarrow$ Knot floer homology

The Algebra $A(P)$: $P \in \{\pm 1\}^n$ $\rightsquigarrow$ sequence of points

· generators over $\mathbb{F}_2$: partial bijections on $\{0,1,\cdots,n\} \supseteq S, T$
$=$ bijections $\varphi : S \longrightarrow T$

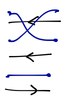
eg:  $= T$

graphically strand diagrams

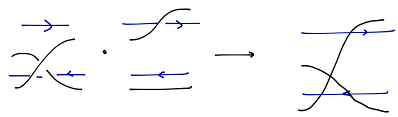
p_i = line $[0,1] \times \{i - 1/2\}$

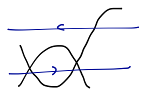
$+ \longrightarrow$ $\quad - \longleftarrow$

$(+,-,-) = P$

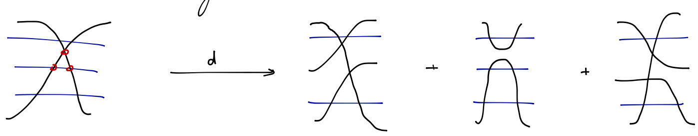
$\varphi = (2,1)(3)$  p_3 p_2 p_1

Strand from $(0,a)$ to $(1, \varphi(a))$

· Multiplication: 

· Relations: Double crossings $= 0$  $= 0$

· Differential: Smooth Crossings



This goes to 0

Relationship to quantum $\mathfrak{gl}_{1|1}$

Th^m: CT categorifies $RT_{\mathfrak{gl}1|1, V}$ $\rightsquigarrow$ Reshethekin Turaev invariants

$K_0(A(P))$ $\rightsquigarrow$ grothendieck group - free module over $\mathbb{Z}[q^{\pm1}]$ module with basis $\{[A(P)e_S \mid S \leq \{0,1,\cdots,n\}\}$ $\rightsquigarrow$ Primitive idempotents

$(P_1, P_2, \cdots, P_n)$

basis for $V_{p_1} \otimes V_{p_2} \otimes \cdots \otimes V_{p_n} \otimes \underbrace{L}_{2 \text{ dim rep}}$
$\shortparallel$
$V_P \otimes L$

·
$$\begin{tikzcd}
A(\partial^L T) \arrow[r, "CT(T)"] \arrow[d] & A(\partial^R T) \arrow[d] \\
K_0(\partial^L T) \arrow[r, "{[CT(T)]}"] \arrow[d, phantom, "\cong" rotate=90] & K_0(\partial^R T) \arrow[d, phantom, "\cong" rotate=90] \\
V_{\partial^L T} \otimes L \arrow[r, "RT_{\mathfrak{gl}1|1, V}"'] & V_{\partial^R T} \otimes L
\end{tikzcd}$$