Morita Theory in Stable homotopy theory – Brooke Shipley

Morita Theory

a) Classical
b) Derived
c) DGA
d) Spectra
e) Results

Ref: Schwede '04, Shipley '07 surveys

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a) Morita theory for Rings:

TFAE

1) Rings R, T are Morita equivalent if $\text{mod-}R \xrightarrow[F]{\simeq} \text{mod-}T$ equivalent categories
2) $\exists$ fin. generated projective generator M such that $\hom_T(M, M) \cong R$
3) $\exists$ R-T bimodule N s.t. $- \otimes_R N : \text{mod-}R \xrightarrow{\simeq} \text{mod-}T$

Prf:

$2 \Rightarrow 1$ $\quad \hom_T(M, -) : \text{mod-}T \rightleftarrows \text{mod-}R : - \otimes_R M$

$M \longmapsto R$
$R \longleftarrow M$

$\hom_T(M, -)$ preserves sums and cokernels

$2 \Rightarrow 3$, $3 \Rightarrow 1$ easy

$1 \Rightarrow 2$ $\quad M := F(R)$ $\quad \hom_T(F(R), F(R)) \overset{\sim}{=} \hom_R(R, R) \cong R$

Ex: R, $M_n(R)$ are Morita equivalent.

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b) Derived

Look at $\mathcal{D}(R) :=$ derived category of mod-R

Th^m: TFAE

1) R, T rings derived equivalent if $\mathcal{D}(R) \underset{\Delta}{\cong} \mathcal{D}(T)$ equivalent as Δ' categories
2) $\exists$ compact generator M in $\mathcal{D}(T)$ s.t. $\mathcal{D}(T)(M, M)_* \cong R$ concentrated in deg 0

   M is called a <u>tilting complex</u>.

Def^n: M is <u>compact</u> if $\text{Hom}(M, -)$ preserves coproducts. In the category $\mathcal{D}(R)$, compact is same as being q-iso to bounded complex of f. gen. projectives.

M is <u>generator</u> if only Δ' subcategory of $\mathcal{C}$ containing M and closed under coproducts is $\mathcal{C}$.

Ex: k-field, $M_3(k)$, $T = \begin{pmatrix} x & x & x \\ o & x & x \\ o & o & x \end{pmatrix}$, $R = \begin{pmatrix} x & x & o \\ o & x & o \\ o & x & x \end{pmatrix}$

R, T are derived equivalent but not Morita equivalent.

Projectives in T: $e_{ii}T = P_i$ $\quad P_3 = (0\,0\,*) \hookrightarrow P_2 - (0 * x) \hookrightarrow P_1 \; (* \, x \, x)$

$M = P_1 \oplus P_2 \oplus P_2/P_3$, Check: $\mathcal{D}_T(M,M) = R$, M is tilting complex.

R, T are not morita equivalent. Check the indecomposables

$T: \; \cdot \longrightarrow \cdot \longrightarrow \cdot \qquad R: \; \cdot \leftarrow \cdot \rightarrow \cdot$

For more examples look at Brane's Conjecture.

Th^m: For R, T $\quad \mathcal{D}(R) \simeq_\Delta \mathcal{D}(T)$ if and only if $Ch_R \overset{\sim}{=} Ch_T$ quillen equivalent.

Special case for rings. Not true for spectra.

Differential Graded Algebras:

Q: For two DGA's A, B are the following equivalent?

1) $\mathcal{D}(A) \simeq_\Delta \mathcal{D}(B)$

2) d.g. mod-A $\simeq_Q$ d.g. mod-B

3) $\exists$ compact generator M in d.g. mod A such that $hom_A(M,M) \simeq B$ q-iso of DGA's.

A: 3) $\Rightarrow$ 2) same proof as before

2) $\Rightarrow$ 1) always

1) $\not\Rightarrow$ 2) counter examples exist

2) $\not\Rightarrow$ 3) instead need equivalence of spectra

FACT: DGA $\simeq_Q$ $H\mathbb{Z}$-algebra

Counter example: $\quad H\mathbb{Z} \xrightarrow{L} H\mathbb{Z} \wedge_S H\mathbb{Z}/2 \xleftarrow{R} H\mathbb{Z}$

Claim: 1) $L \not\simeq R$ as $H\mathbb{Z}$-algebras

2) $L \simeq R$ as S-algebras

Morita theory for Ring Spectra:

Th^m: TFAE

1) Ring spectra R, T are Morita equivalent if mod-R $\simeq_Q$ mod-T are Quillen equivalent

2) $\exists$ compact generator M such that $\underline{hom}_T(M,M) \simeq R$

3) $\exists$ R-T bimodule N such that $- \wedge_R N$: mod-$R \xrightarrow{\simeq_Q}$ mod-T

Also can do for spectral categories (ring spectrum with many objects.)

Related Results :

1) Spaces of Morita equivalences

Th^m: Two DGA's A, B (or DG-categories), $\quad map^h_{DGA's}(A,B) \cong N \quad$ A-B bimodules (free on one generator as a right B-module.)

Th^m: For $(\mathcal{C}, \otimes)$ monoidal model category + some axioms, then

R, T monoids in $\mathcal{C} \qquad map^h(R,T) \cong N \quad$ R-T bimodules

2) Spectral categories upto Morita equivalence.

3) Brauer Group :

Def^n: A, R-algebra is Azumaya algebra if $\exists B$ such that $A \otimes_R B$ and $B \otimes_R A$ are Morita equivalent to R.

Brauer $(R) \cong$ Morita eq. of Azumaya algebras

Derived versions, Brauer spectrum.

Th^m: Brauer $(\mathbb{S}) = 0$