Stalks:

$$(\mathcal{O}_{\operatorname{Spec} A})_{\mathfrak{p}} \cong A_{\mathfrak{p}} \quad (\mathfrak{p} \in \operatorname{Spec} A)$$

Also, $\tilde{M}_{\mathfrak{p}} \cong M_{\mathfrak{p}} = M \otimes_A A_{\mathfrak{p}}$

Def^n: $(X, \mathcal{O}_X)$ is a locally ringed space if $\mathcal{O}_{X,x}$ is a local ring for all $x \in X$.

$\mathfrak{m}_{X,x} :=$ maximal ideal in $\mathcal{O}_{X,x} =: \mathfrak{m}_x$

$\kappa(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$ residue field at x

For $f \in \Gamma(U, \mathcal{O}_X)$ & $x \in U$,

value of f at x is im of f in $\kappa(x)$.

eg: $X = \operatorname{Spec} A$ $f \in A$ a global section

Value of f at $\mathfrak{p}$ is $\bar{f}$ in $\kappa(\mathfrak{p})$

$$\begin{aligned} A &\longrightarrow A/\mathfrak{p} \longrightarrow \bar{\kappa}(\mathfrak{p}) \\ f &\longmapsto f+\mathfrak{p} \longmapsto \bar{f} \end{aligned}$$

Q. What happens when $A = k[x,y]$ $\mathfrak{p} = f(x,y)$ an irreducible polynomial.

Def: X - locally ringed space, $\mathcal{F}$ sheaf of $\mathcal{O}_X$-modules

Fiber of $\mathcal{F}$ at X is

$$\mathcal{F}|_x := \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$$

Why?

§ 4.4 Examples

1. k-field $\mathbb{A}^2_k = \operatorname{Spec} k[x,y]$

$U = \mathbb{A}^2_k \setminus \{(0,0)\} = \mathbb{A}^2_k \setminus \{(x,y)\}$

U open $\Rightarrow (U, \mathcal{O}_X)$ is a scheme.

$$U = \underset{\{\mathfrak{p} \mid x \notin \mathfrak{p}\}}{D(x)} \cup \underset{\{\mathfrak{p} \mid y \notin \mathfrak{p}\}}{D(y)}$$

$$\mathcal{O}_{\operatorname{Spec} A}(D(x)) = k[x,y,x^{-1}] \qquad \mathcal{O}_{\operatorname{Spec} A}(D(y)) = k[x,y,y^{-1}]$$

$$\mathcal{O}_{\operatorname{Spec} A}(D(x) \cap D(y)) = \mathcal{O}_{\operatorname{Spec} A}(D(xy)) = k[x,y,x^{-1},y^{-1}]$$

Can we show this in higher dimensions?

$\mathcal{O}_{\operatorname{Spec} A}(U) = k[x,y]$ $\rightsquigarrow$ Think about this. Hartog's lemma. See Normality.

In $(U, \mathcal{O}_{\mathrm{Spec}\,\mathbb{A}^2_k})$ we have two functions x, y and $V(x) \cap V(y) = \phi$

But in $k[x,y]$ $(x)+(y) = (x,y)$ is a prime ideal in $k[x,y]$

And hence $k[x,y]$ <u>cannot</u> be an affine scheme.

Q. How to glue schemes? The usual cocycle conditions.

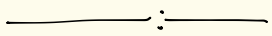
k - field, $X = \mathrm{Spec}\, k[t]$ $\qquad Y = \mathrm{Spec}\, k[u]$

$\quad\cup\!\!\!|\qquad\qquad\qquad\qquad\qquad\cup\!\!\!|$

$U = D[t] = k[t, t^{-1}]$ $\qquad V = D(u) = \mathrm{Spec}\, k[u, u^{-1}]$

Gluing: 1) $\mathbb{A}^1$ with double origin.

$U \xleftrightarrow{\simeq} V$



$u \longleftrightarrow t$

Show not affine.

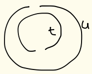
Remark: Similarly consider plane with doubled origin. $\{0\}, \{0'\}$

In this $\underset{\text{without } \{0'\}}{\mathbb{A}^2_k}$ & $\underset{\text{without } \{0\}}{\mathbb{A}^2_k}$ are affine but their intersection is not.

These spaces are "not separated"

Gluing again: $\mathbb{P}^1_k$

$U \longleftrightarrow V$



$t \longleftrightarrow u^{-1}$

Projective coordinates: $[a,b] \sim [ca, cb]$ for $c \notin k^*$

$\frac{b}{a}$ on u-line $= \frac{a}{b}$ on the t-line.

Prop: $\mathbb{P}^1_k$ is not affine.

Proof: $\Gamma(\mathbb{P}^1_k, \mathcal{O}) = \{(p(u), q(t)) \mid$ such that p, q agree on the overlap$\}$

$= \{ p(u) \mid p(u^{-1})$ is a polynomial when $u \neq 0\}$

$\simeq k \rightsquigarrow$ constant polynomials.

Generalization: $\mathbb{P}^n_A$

$R = A[x_0, \dots, x_n]$

$U_i = \operatorname{Spec} A\left[\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\right] \cong \mathbb{A}^n_A$

$$\underbrace{A\left[\frac{x_1}{x_i}, \dots, \frac{x_n}{x_i}, \frac{x_i}{x_j}\right]}_{\overset{\shortparallel}{U_{ij}} \subseteq U_i} = \underbrace{A\left[\frac{x_1}{x_j}, \dots, \frac{x_n}{x_j}, \frac{x_j}{x_i}\right]}_{U_j \supseteq \overset{\shortparallel}{U_{ji}}} \text{ inside } A\left[x_1, \dots, x_n, \frac{1}{x_1 \cdots x_n}\right]$$

These glue to give you $\mathbb{P}^n_A$.