FINAL

Q.1. (15 points) Determine, with proof, if the following series converge:

(1) $\sum\limits_{n=2}^{\infty} \dfrac{1}{n.(\log n)^{\pi}}$ (2) $\sum\limits_{n=1}^{\infty} \dfrac{\pi^n n!}{n^n}$

Q.2. (10 points) Compute the value of $\sqrt{e}$ correct up to 5 decimals. (Do not simplify your final answer, just prove that it is correct up to 5 decimal places.)

Q.3. (20 points) Compute the following integrals:

(1) $\int\limits_0^1 x(1-x)^{2017} dx$

(2) $\int\limits_0^{2\pi} (x-\pi)^{2017}(1+\sin^{2018} x) dx$

(3) $\int\limits_0^{\infty} x^n e^{-x} dx$, where n is a positive integer

(4) $\int \dfrac{1}{\sin^4 x + \cos^4 x} dx$ (I do not know how to solve this one.)

Q.4. (15 points) Prove the following by explicit computation.

$$\int\limits_0^1 \frac{x^4(1-x)^4}{1+x^2} dx = \frac{22}{7} - \pi$$

and hence conclude that $22/7 > \pi$.

Q.5. (20 points) Suppose $y(x)$ has the following Taylor series expansion at $x = 0$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$

(1) If y satisfies the differential equation

$$y - y' = 1$$

find the relations that a_i satisfy. If further $y(0) = 1$, determine y. Rewrite your final solution in terms of standard functions.

(2) If y satisfies the differential equation

$$y + y'' = 0$$

find the relations that a_i satisfy. If further $y(0) = 0, y'(0) = 1$, determine y. Rewrite your final solution in terms of standard functions.

Q.6. (20 points) Suppose f is a continuous function such that

$$\lim_{x\to\infty} \frac{f(x)}{x^{2017}} = 0 = \lim_{x\to-\infty} \frac{f(x)}{x^{2017}}$$

Prove that there exists a number c such that

$$f(c) = c^{2017}$$

Q.7. (20 points) Prove that there does not exist a continuous function f on $\mathbb{R}$ which takes every value exactly twice (i.e. for every $c \in \mathbb{R}$ there exist exactly two real numbers x, y such that $f(x) = c = f(y)$).

Q.8. (25 points) Let a_n be a sequence bounded from above and below i.e. there are constants l, k such that $l < a_n < k$ for all n.

Define the following sequences

$$x_n = \sup\{a_m : m \geq n\} = \sup\{a_n, a_{n+1}, a_{n+2}, \dots\}$$
$$y_n = \inf\{a_m : m \geq n\} = \inf\{a_n, a_{n+1}, a_{n+2}, \dots\}$$

(1) Determine x_n, y_n when $a_n = \dfrac{(-1)^n}{n}$.

(2) Prove that $\lim\limits_{n\to\infty} x_n$ and $\lim\limits_{n\to\infty} y_n$ always exist.

(3) Prove that if $\lim\limits_{n\to\infty} a_n = L$ then $\lim\limits_{n\to\infty} x_n = L = \lim\limits_{n\to\infty} y_n$.