2016 West Coast Algebraic Topology Summer School Problem Set

Contents

1 Monday

1.1 Lecture 1: Formal group laws I

1. Prove that for any formal group law $F(x, y)$ over R, x has a formal inverse. That is, there exists an element $i(x) \in R[[x]]$ such that $F(x, i(x)) = 0$.

2. Prove that the additive formal group law and the multiplicative formal group law $F_m(x, y) = x + y + xy$ are not isomorphic over $\mathbb{F}_p$. (Hint: Compare $[p]_{F_a}(x)$ and $[p]_{F_m}(x)$. What can you deduce?)

3. Let F be a formal group law defined over a $\mathbb{Q}$–algebra R. Prove that the formal expression
$$f(x) = \int_0^x \frac{dt}{\frac{\partial F}{\partial y}(t, 0)}$$
satisfies the identity $f(F(x, y)) = f(x) + f(y)$. Conclude that every formal group law defined over a $\mathbb{Q}$–algebra R is isomorphic to the additive formal group law $F_a(x, y) = x + y$. (Remark: The powers series $f(x)$ is called the *logarithm* of F and is often denoted $\log(x) = f(x)$.)

4. Let $g(x) \in R[[x]]$ be of the form $g(x) = x + \sum_{i \geq 1} b_i x^i$.
   (a) Prove that there is a power series $g^{-1}(x)$ such that $g(g^{-1}(x)) = g^{-1}(g(x)) = 1$.
   (b) Prove that $F(x, y) = g(g^{-1}(x) + g^{-1}(y))$ is a formal group law.
   (c) If R is $\mathbb{Z}[\epsilon]/\epsilon^2$. Give a formula for the formal group laws you obtain when $g(x) = 1 + \epsilon x^{n+1}$.

5. Let F be a formal group law over $\mathbb{Z}_p$. Then its logarithm $\log F(X) \in \mathbb{Q}_p[[X]]$ can be computed by $\log F(X) = \lim_{n\to\infty} p^{-n}[p^n](X)$.

6. Let F be a formal group over R. If an integer n is invertible in R, show that the multiplication by n map $[n] : F \to F$ is an isomorphism. Therefore, F is naturally a $\mathbb{Z}_S$-module, where S is the multiplicative subset of $\mathbb{Z}$ consisting of invertible numbers in R.

7. Compute the endomorphism ring of the additive formal group law.

8. Let F defined over a field of characteristic $p > 0$.
   (a) Prove that
$$[p^n](x) \equiv 0 \mod x^{p^n}$$
   (b) Prove that the natural map $\mathbb{Z} \to \mathrm{End}(F)$ factors through extends to $\mathbb{Z}_p$.

9. Let G be a formal group law defined over a ring R which is the ring of integers of a finite extension of $\mathbb{Q}_p$.

(a) Recall that R is a complete local ring with maximal ideal $\mathfrak{m}$. Show that for every $a \in \mathfrak{m}$,

$$\lim_{n\to\infty} [p^n](a) = 0.$$

(b) Prove that for every α in $\mathbb{Z}_p \subseteq R$, there exists an endomorphism $[\alpha](x) \in R[[x]]$ of F such that

$$[\alpha](x) = \alpha x \mod (x^2).$$

10. Let $f : R \to S$ be a surjective homomorphism of commutative rings. Recall that for a formal group law $F(x, y) = \sum a_{i,j} x^i y^j$ over R, $f^*F = \sum f(a_{i,j}) x^i y^j$. Let G be a formal group law over S. Prove that there is a formal group law F over R such that $f^*F = G$. (Hint: use Lazard's theorem.)

11. The augmentation ideal of a graded ring R is $I_R/(I_R)^2$ where

$$I_R = \{x \in R \mid \deg(x) > 0\}$$

Let R and S be graded rings which have no elements in negative degrees. Prove that a homomorphism of graded ring $f : R \to S$ is an isomorphism if and only if the induced maps $R_0 \to S_0$ and $QR \to QS$ are surjective.

12. Let F and H be formal group laws over R. Recall that a morphism $f : F \to H$ is a power series $f(x) \in R[[x]]$ such that $f(F(x, y)) = H(f(x), f(y))$. To motivate this definition, let R be a complete local ring. Let $\mathcal{C}$ be the category of complete local R–algebras. For $S, T \in \mathcal{C}$, $\mathcal{C}(S, T)$ consists of the set of *continuous* R-algebra maps, i.e., R–algebra homomorphism such that $f(\mathfrak{m}_S) \subset f(\mathfrak{m}_T)$. Let $\mathrm{Spf}(R[[x]]) : \mathcal{C} \to \mathit{Sets}$ be the functor $S \mapsto \mathcal{C}(-, R[[x]])$.

   (a) Explain why how a formal group $F \in R[[x, y]]$ can be thought of a the data of a group structure on $\mathrm{Spf}(R[[x]])$ (i.e., a map $\mathrm{Spf}(R[[x]]) \times \mathrm{Spf}(R[[x]]) \to \mathrm{Spf}(R[[x]])$ with a unit should be $\mathrm{Spf}(R) \to \mathrm{Spf}(R[[x]])$ satisfying the usual commutative diagram).

   (b) Now use this to explain why a morphism of formal group laws $f : F \to H$ corresponds to a group homomorphism from $G_F = \mathrm{Spf}(R[[x_F]])$ to $G_H = \mathrm{Spf}(R[[x_H]])$.

1.2 Lecture 2: Introduction to stable homotopy theory

1. Suppose that E is a *homotopy commutative ring spectrum*, meaning that E is a spectrum equipped with maps $\mu : E \wedge E \to E$ (multiplication) and $\eta : S^0 \to E$ (unit) making E a comutative monoid object in the stable homotopy category.

   [(a)]

   (a) Write down explicit diagrams (in the stable homotopy category) expressing what it means for (E, μ, η) to be a homotopy commutative ring spectrum. Prove that $\pi_* E$ has the structure of a commutative ring.

(b) The E-*cooperations* are $E_*E = \pi_*(E \wedge E)$. Applying π_* to the maps $S^0 \wedge E \xrightarrow{\eta \wedge E} E \wedge E$ and $E \wedge S^0 \xrightarrow{E \wedge \eta} E \wedge E$ results in maps $\eta_L : E_* \to E_*E$ and $\eta_R : E_* \to E_*E$ referred to as the *left unit* and *right unit*, respectively. Prove that these maps make E_*E a π_*E-bimodule.

(c) Show that E_*E is a commutative ring.

(d) For the rest of this problem, assume that E_*E is flat as a left π_*E-module. Show that $E \wedge \mu \wedge E : E \wedge E \wedge E \wedge E \to E \wedge E \wedge E$ induces an isomorphism

$$E_*E \otimes_{\pi_*E} E_*E \to \pi_*(E \wedge E \wedge E).$$

(Note that the left-hand side is a tensor product of bimodules where E_*E has right π_*E-module structure via η_R and left π_*E-module structure via η_L.)

(e) Applying π_* to the composite

$$E \wedge E \simeq E \wedge S^0 \wedge E \xrightarrow{E \wedge \eta \wedge E} E \wedge E \wedge E$$

results in a map $\Delta : E_*E \to E_*E \otimes_{\pi_*E} E_*E$, the *comultiplication* on E_*E. Show that Δ gives E_*E the structure of a coalgebra.

(f) The multiplication $\mu : E \wedge E \to E$ induces the *counit* $\varepsilon : E_*E \to \pi_*E$, and the twist map $E \wedge E \to E \wedge E$ induces the *antipode* $\chi : E_*E \to E_*E$.

(g) Write down the diagrams defining a groupoid object in affine schemes. Prove that $(\operatorname{Spec} \pi_*E, \operatorname{Spec} E_*E)$ is a groupoid object in affine schemes. A pair of commutative rings (A, Γ) such that $(\operatorname{Spec} A, \operatorname{Spec} \Gamma)$ is a groupoid object in affine schemes is called a *Hopf algebroid*, and we have just seen that (π_*E, E_*E) is a Hopf algebroid.

2. When is a groupoid object in affine schemes actually a group object? Let $H = H\mathbb{F}_2$ denote the mod 2 Eilenberg-MacLane spectrum. Prove that $\operatorname{Spec} H_*H$ is the group scheme of strict automorphisms of the additive formal group law.

3. Let bu denote the 2-complete connective complex K-theory spectrum. Use the following steps to compute π_*bu via the Adams spectral sequence.

   (a) Let $E(1)$ denote the subalgebra of A generated by the Milnor primitives Q_0 and Q_1. Show that $H^*(bu; \mathbb{F}_2) \cong A//E(1)$ as A-modules.

   (b) Prove the change of rings isomorphism $\operatorname{Ext}_A(A//E(1), \mathbb{F}_2) \cong \operatorname{Ext}_{E(1)}(\mathbb{F}_2, \mathbb{F}_2)$.

   (c) Show that

   $$\operatorname{Ext}_{E(1)}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2[h_{10}, h_{22}]$$

   where $|h_{10}| = (1, 1)$ and $|h_{22}| = (2, 4)$.

   (d) Show that the Adams spectral sequence for bu collapses at the E_2 page and

   $$\pi_*bu = \mathbb{Z}_2[\beta]$$

   where $|\beta| = 2$.

4. Let bo denote the 2-complete connective real K-theory spectrum. Use the following steps to compute $\pi_* bo$ via the Adams spectral sequence.
   (a) Let $A(1)$ denote the subalgebra of A generated by Sq^1 and Sq^2. Use the cofiber sequence
   $$\Sigma bo \xrightarrow{\eta} bo \to bu$$
   to show that $H^*(bo;\mathbb{F}_2) \cong A//A(1)$ as A-modules.
   (b) Prove the change of rings isomorphism $\mathrm{Ext}_A(A//A(1), \mathbb{F}_2) \cong \mathrm{Ext}_{A(1)}(\mathbb{F}_2, \mathbb{F}_2)$.
   (c) Show that
   $$\mathrm{Ext}_{A(1)}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2[h_{10}, h_{11}, v, w]/(h_{10}h_{11}, h_{11}^3, vh_{11}, v^2 - h_{10}^2 w)$$
   where $|h_{10}| = (1,1)$, $|h_{11}| = (1,2)$, $|v| = (3,7)$, and $|w| = (4,12)$.
   (d) Show that the Adams spectral sequence for bo collapses at the E_2 page and
   $$\pi_* bo = \mathbb{Z}_2[\eta, \alpha, \beta]/(2\eta, \eta^3, \alpha\eta, \alpha^2 - 4\beta)$$
   where $|\eta| = 1$, $|\alpha| = 4$, and $|\beta| = 8$.

1.3 Lecture 3: Complex bordism theory

1. Give $H\mathbb{Z}$ a complex orientation and compute the corresponding formal group law.

2. Let $\mathrm{MU} \wedge H\mathbb{Z}$ have the complex orientation inherited from MU. Compute the corresponding formal group law.

3. View $U(n-1)$ as the subgroup of $U(n)$ consisting of matrices stabilizing the first basis vector. Show that the Thom space of the Tautological bundle on $BU(n)$ can be identified with the homotopy cofiber $BU(n)/BU(n-1)$. Let $c_n \in H^{2n}(BU(n)/BU(n-1), \mathbb{Z})$ be a Thom class. Show that c_n maps to (a unit times) the nth chern class in $H^{2n}(BU, \mathbb{Z})$.

4. Let $\mathcal{O}(-1)$ denote the tautological complex vector bundle on $\mathbb{CP}^n$. Let ω denote the orthogonal complement, using the standard metric on $\mathbb{C}^n$. Use the fact that the tangent bundle $T\mathbb{CP}^n$ is isomorphic to $\mathrm{Hom}(\mathcal{O}(-1), \omega)$ to show that the ith Chern class c_i of this tangent bundle is $\binom{n+1}{i}(-c_1(\mathcal{O}(-1)))^i$.

1.4 Lecture 4: Elliptic curves

1. *Requires some knowledge about divisors on curves.* An elliptic curve C/S is a smooth proper curve $p : C \to S$ of genus one, equipped with a section $e : S \to C$. For any scheme T over S, define $Pic^{(0)}(C/T)$ to be the group (under tensor product) of isomorphism classes of invertible sheaves on $C_T = C \times_S T$ which are fiberwise of degree zero, modulo linear equivalence. For a T-point P of C, i.e. a map $i_P : T \to C$ over S,

the ideal sheaf $I(P) = \ker(\mathcal{O}_C \to (i_P)_*\mathcal{O}_T)$ is an invertible line bundle over C. If we denote the T-points of E by $E(T)$, we get a map

$$E(T) \to Pic^{(0)}(C/T)$$

given by $P \mapsto I(P)^{-1} \otimes I(e)$. Show that this is a bijection, and conclude that $E(T)$ inherits a (commutative) group structure with e as the neutral element.

2. Given a Weierstrass curve

$$C : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

do the following:

   (a) Describe the points of order 2, i.e. points $P = (x_0, y_0)$ such that $[2]P = 0$. How many are there?

   (b) Suppose 2 is invertible in the base. Transform C to a Weierstrass curve of the form $y^2 = f(x)$, and now describe even more explicitly the points of order 2 on C.

3. Start with an elliptic curve C over $SpecR$ (with structure map $p : C \to SpecR$ and section e), and describe why can C be described, locally on $SpecR$, as the zeros of a Weierstrass equation as follows:

   (a) Use the Riemann-Roch theorem to conclude that the invertible sheaf $I(e)$ has the property that $p_*I(e)^{-n}$ is locally free of rank n over S.

   (b) Locally, find 7 functions generating $p_*I(e)^{-6}$, and conclude that there is a linear relation between them. Conclude that this gives you a Weierstrass equation.

   (c) Study the choices of your generators to determine how non-unique was the Weierstrass equation thus obtained, and conclude which transformations of the equation preserve the elliptic curve you started with.

4. Take the Weierstrass curve

$$y^2 + ay = x^3,$$

and expand its formal group law as much as you can. (To do this, you can use the procedure described in IV.1 of Silverman's Arithmetic of Elliptic Curves.) Determine as much as you can of the [2]-series of this formal group law.

2 Tuesday

2.1 Lecture 1: Formal group laws II

1. Suppose k is a perfect field of characteristic p containing $\mathbb{F}_{p^n}$ and $\overline{k}$ its algebraic closure.
   (a) Show that the (essentially unique) formal group over $\overline{k}$ is isomorphic to the extension of a formal group Γ over $\mathbb{F}_p \subset k$.
   (b) Show that the set of isomorphism classes of formal groups of height n over k can be identified with the Galois cohomology group $H^1(\mathrm{Gal}(\overline{k}/k), \mathrm{Aut}(\Gamma))$.

2. Suppose that R is a ring with formal group law G of exact height $0 < n < \infty$: in the p-series $[p](x)$, the coefficients u_i of $x, x^p, \ldots, x^{p^{n-1}}$ are nilpotent and the coefficient u of x^{p^n} is a unit. Show that G is strictly isomorphic to a formal group law G' extended from a Lubin–Tate ring of the form $\mathbb{Z}_p[\![u_1, \ldots, u_{n-1}]\!][u^{\pm 1}]$.

3. (a) Prove that the multiplicative formal group law $F_m(x, y) = x + y + xy$ has height 1 over $\mathbb{F}_p$.
   (b) Prove that its endomorphism ring $End(F_m)$ is isomorphic to $\mathbb{Z}_p$, the p–adic integers. Conclude that $\mathbb{G}_1 \cong \mathbb{Z}_p^\times$.
   (c) Prove that $\widetilde{F}_m(x, y) = x + y + xy$ defined over $R = W(\mathbb{F}_p) \cong \mathbb{Z}_p$ is a universal deformation of F_m.
   (d) Conclude that the action of $\mathbb{G}_1$ on R is trivial.
   (e) Compute the action of $\mathbb{G}_1$ on $(E_1)_* = \pi_* E_1$ (Perhaps after hearing Lecture 2.3).

4. Let k be a field of characteristic $p > 0$. Let $F(x, y) = x + y + C_{p^h}(x, y) + \ldots$ be a formal group law over k. Show that the height of F is h. (Here, $C_d(x, y)$ is the additive symmetric cocycle $C_d(x, y) = \frac{1}{\lambda_d}(x^d + y^d - (x + y)^d)$ for $\lambda_d = 1$ if d is not a power of p and p if it is.)

5. (The Honda Formal Group Law) Let $q = p^h$. Let $f(x) = \sum_{n \geq 0} \frac{x^{q^n}}{p^n}$. Define
$$F_h(x, y) = f^{-1}(f(x) + f(y)).$$
   (a) Assume you know that F_h is a formal group law defined over $\mathbb{Z}$. Show that the mod p reduction Γ_h has p–series $[p]_{\Gamma_h}(x) = x^q$.
   (b) Prove that $g(x) = ax$ is an endomorphism of Γ_h over $\overline{\mathbb{F}}_p$ if and only if a is in the subfield $\mathbb{F}_q$.
   (c) Prove $End(\Gamma_h/\mathbb{F}_p)$ is the $\mathbb{Z}_p$–algebra generated by $S(x) = x^p$.
   (d) (Optional) Prove that $End(\Gamma_h/\overline{\mathbb{F}}_p)$ is the $W(\mathbb{F}_q)$–algebra generated by $S(x)$.
   (e) (Optional) Show that F_h is defined over $\mathbb{Z}$ using the Functional Equation-Integrality Lemma (Section 2 of Chapter 1 of Hazewinkel's *Formal Groups and Applications*).

6. Let Γ be a p–typical formal group law of height n over k and let F over $R = W(k)[[u_1, \ldots, u_{n-1}]]$ be a p–typical universal deformation of Γ, with $[p]$–series

$$[p]_F(x) = px +_F u_1 x^p +_F \ldots +_F u_{n-1} x^{p^{n-1}} +_F x^p.$$

Given any automorphism $g : \Gamma \to \Gamma$, there corresponds a ring homomorphism $\varphi_g : R \to R$ and an isomorphism $F \xrightarrow{f_g} \varphi_g^* F$, where

$$f_g = r_0 x +_F r_1 x^p +_F \ldots \in R[[x]].$$

Further, note that

$$f_g([p]_F(x)) = [p]_{\varphi_g^* F}(f_g(x)).$$

Use this to give a formula of $\varphi_g(u_1)$ in terms of the r_i's.

7. (a) Let $C_3 : y^2 = x^3 - x$ be the Weierstrass equation of an elliptic curve defined over $\mathbb{F}_9$. Prove that the formal group law F_C of C has height 2.
   (b) Let $\widetilde{C}_2 : y^2 = 4x^3 + u_1 x^2 + 2x$ be defined over $\mathbb{W}(\mathbb{F}_9)[[u_1]]$. Prove that the formal group law of $\widetilde{C}_3$ is a universal deformation of the formal group law of C. (Hint: Use Proposition 1.1 of Lubin–Tate *Formal moduli for one-parameter formal Lie groups.*)
   (c) Suppose that $(x, y) \mapsto (\lambda^2 x, \lambda^3 y)$ is an automorphism of C_3 for some $\lambda \in \mathbb{F}_9^\times$. Prove that λ has order 4. Conclude that $(\mathbb{F}_9^\times)^2 \subset Aut(C)$, and hence induces an automorphism of its formal group law F_C.
   (d) For $\lambda \in (\mathbb{F}_9^\times)^2 \subset Aut(F_C)$, compute the induced action on $\mathbb{W}(\mathbb{F}_9)[[u_1]]^\times$.
   (e) Do a similar exercise with $C_2 : y^2 + y = x^3$ over $\mathbb{F}_4$ and $\widetilde{C}_2 : y^2 + u_1 xy + y = x^3$ over $\mathbb{W}(\mathbb{F}_4)[[u_1]]$.

2.2 Lecture 2: Introduction to local class field theory

1. Prove Theorem 1 from *The Geometry of Lubin-Tate spaces* by J. Weinstein. To do this, one may proceed following *Cormal Complex Multiplication in Local Fields* by J. Lubin and J. Tate in the manner we now outline. We use the notation from loc. cit. Weinstein.

   (a) Choose f and g satisfying the conditions. Choose a linear polynomial

   $$L(x_1, \ldots, x_n) = a_1 x_1 + a_2 x_2 + \ldots + a_n x_n$$

   with coefficients a_i in $\mathcal{O}_F$. Show that there is a unique power series $F(x_1, \ldots, x_n)$ with coefficients in $\mathcal{O}_F$ such that

   $$F(x_1, \ldots, x_n) = L(x_1, \ldots, x_n) + \text{ degree } \geq 2 \text{ terms}$$

and

$$f(F(x_1,\ldots,x_n)) = F(g(x_1),\ldots,g(x_n)).$$

Indeed for each r, the congruences

$$F_r(x_1,\ldots,x_n) = L(x_1,\ldots,x_n) \mod \langle x_1,\ldots,x_n\rangle^2$$

and

$$f(F_r(x_1,\ldots,x_n)) = F_r(g(x_1),\ldots,g(x_n)) \mod \langle x_1,\ldots,x_n\rangle^{r+1}$$

have a unique solution mod $\langle x_1,\ldots,x_n\rangle^{r+1}$.

Let F_f be the unique solution to

$$F_f(x_1,x_2) = x_1 + x_2 \mod \langle x_1,x_2\rangle^2$$

and

$$f(F_f(x_1,x_2)) = F_f(f(x_1),f(x_2)).$$

Choose f and g satisfying the conditions. For each a in $\mathcal{O}_F$, let $[a]_{f,g}(x)$ be the unique solution of

$$[a]_{f,g}(x) = ax \mod \langle x\rangle^2$$

and

$$f([a]_{f,g}(x)) = [a]_{f,g}(g(x)).$$

Show the following

(b) $F_f(x,y) = F_f(y,x)$

(c) $F_f(F_f(x,y),z) = F_f(x,F_f(y,z))$

(d) $F_f([a]_{f,g}x,[a]_{f,g}y) = [a]_{f,g}F_g(x,y)$

(e) $[a]_{f,g}([b]_{g,h}(x)) = [ab]_{f,h}(x)$

(f) $[a+b]_{f,g}(x) = F_f([a]_{f,g}(x),[b]_{f,g}(x))$

(g) $[\pi]_{f,f} = f$. $[1]_{f,f} = 1$.

(h) Complete the proof of Theorem 1

2. Let $K = \mathbb{Q}_p(\zeta_p)$, where ζ_p is a primitive pth root of unity.

   (a) Show that $\pi = 1 - \zeta_p$ is a uniformizer.

   (b) Give an explicit description of the totally ramified abelian extension L of K whose norm group is the subgroup of K^* generated by π and the unites congruent to 1 mod π^2. Write down the image in $\mathrm{Gal}(L/K)$ of some elements of K^* under the reciprocity map.

The following exercises on Kummer theory are from J. Rabinoff's course notes on class field theory.

Kummer Theory

Let K be a field, let m be a positive integer prime to $\mathrm{char}(K)$, and assume that $\mu_m \subset K$. Let $B \subset K^\times/K^{\times m}$ be a subgroup and let $K_B = K(\sqrt[m]{b} \,:\, b \in B)$. Then K_B/K is an abelian extension of exponent dividing m, and the bilinear form

$$\langle \cdot, \cdot \rangle \;:\; \mathrm{Gal}(K_B/K) \times B \to \mu_m \text{ defined by } \langle \sigma, b \rangle = \frac{\sigma(\sqrt[m]{b})}{\sqrt[m]{b}} \tag{2.2.1}$$

is a perfect pairing of a profinite group with a discrete group. Moreover, every abelian extension L/K of exponent dividing m is of the form $L = K_B$ for a unique subgroup $B \subset K^\times/K^{\times m}$. We have $[K_B : K] < \infty$ if and only if B is finite, in which case $[K_B : K] = \#B$.

Taking $B = K^\times/K^{\times m}$, we have a perfect pairing

$$(G_K^{\mathrm{ab}}/mG_K^{\mathrm{ab}}) \times (K^\times/K^{\times m}) \to \mu_m. \tag{2.2.2}$$

Kummer theory and local class field theory

Let K be a local field, let m be a positive integer prime to $\mathrm{char}(K)$, and assume that $\mu_m \subset K^\times$. Let

$$\Psi_K : K^\times \to G_K^{\mathrm{ab}}$$

denote the reciprocity map. It induces an isomorphism $K^\times/K^{\times m} \cong G_K^{\mathrm{ab}}/mG_K^{\mathrm{ab}}$. Composing with the pairing (2.2.2) of Kummer theory gives a bilinear form

$$(\cdot, \cdot)_K \;:\; (K^\times/K^{\times m}) \times (K^\times/K^{\times m}) \to \mu_m \text{ defined by } (a, b)_K = \frac{\Psi_K(b)(\sqrt[m]{a})}{\sqrt[m]{a}}. \tag{2.2.3}$$

This perfect pairing of finite abelian groups is called the *norm residue symbol*. Since $(a, b)_K$ only depends on the image of b under the relative reciprocity map $\Psi_{K(\sqrt[m]{a})/K}$, we have $(a, b)_K = 1$ if and only if b is a norm from $K(\sqrt[m]{a})$.

We have the following facts:

1. $(a, b)_K = 1$ if $a + b \in K^m$.

2. For $a, b \in K^\times$ we have
$$(a, b)_K \, (b, a)_K = 1.$$

Example

Let $m = 2$ and let $x, z \in K$ and $a, b \in K^\times$. We have

$$N_{K(\sqrt{a})/K}(z + x\sqrt{a}) = (z + x\sqrt{a})(z - x\sqrt{a}) = z^2 - ax^2.$$

Therefore $(a, b)_K = 1$ if and only if there exist $z, x \in K$ such that $b = z^2 - ax^2$, i.e. such that $z^2 = ax^2 + b$.

Suppose that there exist $x, y, z \in K$, not all zero, such that $z^2 = ax^2 + by^2$. If $y \neq 0$ then $by^2 = z^2 - ax^2$ is a norm from $K(\sqrt{a})$, so

$$1 = (a, by^2)_K = (a, b)_K (a, y)_K^2 = (a, b)_K.$$

Similarly, if $x \neq 0$ then $ax^2 = z^2 - by^2$ is a norm from $K(\sqrt{a})$, so

$$1 = (ax^2, b)_K = (a, b)_K (x, b)_K^2 = (a, b)_K.$$

Therefore

$$(a, b)_K = \begin{cases} 1 & \text{if } z^2 = ax^2 + by^2 \text{ has a nonzero solution with } x, y, z \in K \\ -1 & \text{otherwise.} \end{cases}$$

This says that for $m = 2$, the norm residue symbol coincides with the *Hilbert symbol*.

1. Calculate $(\cdot, \cdot)_{\mathbb{Q}_2}$ for $m = 2$. Prove in particular that

$$(p, q)_2 = (-1)^{\frac{(p-1)(q-1)}{4}} \qquad (p, 2)_2 = (-1)^{\frac{p^2-1}{8}} \qquad (p, -1)_2 = (-1)^{\frac{p-1}{2}}$$

   for rational prime numbers $p, q > 2$.

2. If K is Archimedean, prove that $(a, b)_K = 1$ unless $K \cong \mathbb{R}$, both a and b are negative, and $m = 2$, in which case $(a, b)_K = -1$. In particular, $(\cdot, \cdot)_K$ is not strongly antisymmetric in the sense that $(x, x)_K = 1$ for all $x \in K^\times$.

3. Suppose that K is non-Archimedean with residue field k. Assume that $\mathrm{char}(k) \nmid m$. For $a \in \mathcal{O}_K^\times$ and $b \in K^\times$ prove that $(a, b)_K$ is the unique element of μ_m such that

$$(a, b)_K \equiv a^{\mathrm{ord}_K(b)(\#k-1)/m} \mod \mathfrak{m}_K.$$

   Deduce that $(a, b)_K = 1$ if $a, b \in \mathcal{O}_K^\times$.

4. Suppose that K is non-Archimedean with residue field k and uniformizer ϖ. Assume that $\mathrm{char}(k) \nmid m$. For $a \in \mathcal{O}_K^\times$ prove that the following are equivalent:

   (a) $(a, \varpi)_K = 1$.

   (b) a is an mth power in K.

   (c) $a \mod \mathfrak{m}_K$ is an mth power in k.

2.3 Lecture 3: Lifting formal group laws to topology

1. If two formal group laws on a graded ring R_* are strictly isomorphic, show that the resulting homology functors are naturally isomorphic.

2. Show that K-theory is Landweber exact and that integral cohomology is not by calculating the p-series of their formal group laws.

3. If $R_* \to S_*$ is a faithfully flat map of graded rings and R_* has a formal group law such that the composite $MU_* \to R_* \to S_*$ is Landweber exact, show that $MU_* \to R_*$ is also Landweber exact.

4. Suppose we have a graded ring E_* with a formal group law that gives rise to a Landweber exact cohomology theory E. Show that there is an isomorphism

$$E_*E = E_* \otimes_{MU_*} MU_*MU \otimes_{MU_*} E_*$$

and describe the universal property of this ring.

5. If R is a ring with elements b and c, there is a formal group law

$$x +_F y = \frac{x + y + bxy}{1 - cxy}$$

Give conditions on R, b, and c that describe when the resulting formal group will be Landweber exact. (You might assume for simplicity that R is torsion-free over $\mathbb{Z}$.)

2.4 Lecture 4: Modular forms

1. [Silverman, *Arithmetic of elliptic curves*, Exercise 4.5] Let E be the elliptic curve $y^2 = x^3 + Ax$.

   (a) Let
   $$w(z) = z^3(1 + A_1 z + A_2 z^2 + \cdots) \in \mathbb{Z}[a_1, \ldots, a_6][[z]]$$
   denote the expansion of $w = -1/y$ in terms of $z = -x/y$. Prove that $A_n = 0$ when $n \not\equiv 3 \pmod 4$.

   (b) Let $F(X, Y)$ denote the formal group law for E and let $F_n(X, Y)$ denote the homogeneous degree n summand of $F(X, Y)$. Prove that $F_n = 0$ when $n \not\equiv 1 \pmod 4$.

   (c) What are the analogues of (a) and (b) for the curve $y^2 = x^3 + B$?

2. Compute the homotopy groups of $tmf[1/6]$ in the following fashion.

   (a) Assuming 2 is invertible, find a Weierstrass transformation eliminating a_1 and a_3.

   (b) Assuming 3 is invertible, further eliminate a_2 via a Weierstrass transformation.

   (c) Show that a Weierstrass curve of the form $y^2 = x^3 + a_4 x + a_6$ admits only the identity Weierstrass transformation to other Weierstrass curves of the same form.

(d) Conclude that the discrete Hopf algebroid $(A[1/6], \Gamma[1/6])$ is equivalent to the "discrete" Hopf algebroid

$$(\mathbb{Z}[1/6][a_4, a_6], \mathbb{Z}[1/6][a_4, a_6]).$$

(e) Show that $H^{*,*}(A[1/6], \Gamma[1/6]) = H^{0,*}(A[1/6], \Gamma[1/6]) = \mathbb{Z}[1/6][a_4, a_6]$.

(f) Given that there is a spectral sequence of the form

$$E_2 = H^{s,t}(A[1/6], \Gamma[1/6]) \implies \pi_{2t-s} tmf[1/6]$$

with differentials of Adams-Novikov type, conclude that

$$\pi_* tmf[1/6] \cong \mathbb{Z}[1/6][a_4, a_6]$$

with $|a_4| = 8$ and $|a_6| = 12$.

(g) Similarly,

$$\pi_* TMF[1/6] \cong \mathbb{Z}[1/6][a_4, a_6, \Delta^{-1}].$$

Compute the discriminant Δ of $y^2 = x^3 + a_4 x + a_6$ in order to make this formula completely explicit.

3. In this problem, we will explore level $\Gamma_1(N)$ and level $\Gamma_0(N)$ modular forms.

(a) Fix a positive integer N. A $\Gamma_1(N)$-*level structure* is a pair (E, P) consisting of an elliptic curve E and point $P \in E$ of exact order N. A $\Gamma_0(N)$-*level structure* is a pair (E, H) consisting of an elliptic curve E and subgroup $H \leq E$ which is cyclic of order N. For $i \in \{0, 1\}$, define level $\Gamma_i(N)$ modular forms in a manner consistent with our definition of modular forms. Note that you will replicate the definition of modular forms when $N = 1$.

(b) Suppose (E, P) is a $\Gamma_1(N)$ structure where E is in Weierstrass form over $\mathbb{Z}[1/N]$ and P has affine coordinates (α, β). Find a Wierstrass transformation taking (E, P) to $(E', (0, 0))$. What can you say about the a_6 coefficient of the Weierstrass equation for E'?

(c) Find a Weierstrass transformation which eliminates the a_4 coefficient from E' and leaves $(0, 0)$ fixed. The result will be a Weierstrass curve of the form

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2$$

called the *Tate normal form* of (E, P).

(d) Suppose $N = 5$. Use the relation $[3](0, 0) = [-2](0, 0)$ to show that

$$a_2^3 + a_3^2 = a_1 a_2 a_3.$$

(e) Let $M_*(\Gamma_1(5))$ denote the graded ring of level $\Gamma_1(5)$ modular forms over $\mathbb{Z}[1/5]$. Prove that

$$M_*(\Gamma_1(5)) = \mathbb{Z}[1/5][a_1, a_2, a_3, \Delta^{-1}]/(a_2^3 + a_3^2 - a_1a_2a_3)$$

where $|a_i| = i$.

(f) Use a computer to determine relations on

$$M_*(\Gamma_1(5))[r, s, t]$$

which will guarantee that a Weierstrass transformation $\varphi_{r,s,t,1}$ takes a curve in Tate normal form to another Tate normal curve with order 5 torsion at $(0, 0)$.

(g) Let $\Lambda_0(5)$ denote the quotient of $M_*(\Gamma_1(5))[r, s, t]$ by the relations from part (f). Determine the Hopf algebroid structure on

$$(M_*(\Gamma_1(5)), \Lambda_0(5))$$

which stackifies to the moduli space of $\Gamma_0(5)$-level structures. Use this to determine

$$M_*(\Gamma_0(5)),$$

the ring of level $\Gamma_0(5)$ modular forms over $\mathbb{Z}[1/5]$.

3 Wednesday

3.1 Lecture 1: Local chromatic homotopy theory

1. In this exercise, we will look at the spectral sequence

$$H^s(\mathbb{G}_1, (E_1)_t) \Longrightarrow \pi_{t-s}L_{K(1)}S$$

for odd primes p, where the cohomology $H^*(\mathbb{G}_1, (E_1)_*)$ is the *continuous* cohomology of $\mathbb{G}_1$. Recall that $\mathbb{G}_1 \cong \mathbb{Z}_p^\times$ and $(E_1)_* \cong (KU_p)_* \cong \mathbb{Z}_p[u^{\pm 1}]$, where u is a generator in degree -2. Further, $\mathbb{Z}_p \cong C_{p-1} \times U_1$ where

$$U_i = \{a \in \mathbb{Z}_p \mid a \equiv 1 \mod p^i\}.$$

So one can use the Lyndon-Hochschild-Serre spectral sequence

$$H^r(\mathbb{Z}_p, H^q(C_{p-1}, (E_1)_*)) \Longrightarrow H^{r+q}(\mathbb{Z}_p, H^q(C_{p-1}, (E_1)_*)).$$

to compute the E_2–term.

(a) Prove that the action of $\lambda \in \mathbb{G}_1$ on u is given by

$$\lambda_*(u) = \lambda u$$

where λ on the right is considered as a coefficient in $\mathbb{Z}_p[u^{\pm}]$.

(b) Compute the fixed points $(E_1)_*^{C_p}$.

(c) Let $\mathbb{Z}_p[[U]] = \varprojlim_n \mathbb{Z}_p[U/U_n]$. Let $\lambda = 1+p \in U_1$. Let $\varepsilon : \mathbb{Z}_p[[U]] \to \mathbb{Z}_p$ be function determined by $\varepsilon(g) = 1$ for $g \in U$. Prove that

$$0 \to \mathbb{Z}_p[[U_1]] \xrightarrow{1-\lambda} \mathbb{Z}_p[[U_1]] \xrightarrow{\varepsilon} \mathbb{Z}_p \to 0$$

is a projective resolution of $\mathbb{Z}_p$ as a continuous $\mathbb{Z}_p[[U]]$–module.

(d) Use this resolution to compute

$$H^*(U_1, (E_1)_*^{C_p}).$$

(e) Use this information to compute $\pi_* L_{K(1)} S$.

(f) Now let $p = 2$, noting that $\mathbb{Z}_2^\times = C_2 \times U_2$. Use similar methods to compute

$$H^s(\mathbb{G}_1, (E_1)_t/2) \Longrightarrow \pi_{t-s} L_{K(1)} V(0)$$

where $V(0)$ is the cofiber of $S \xrightarrow{2} S$. (Hint: It may help to know that $\eta^3 = 0$ in $\pi_* V(0)$.) Now try to do the computation for S when $p = 2$.

2. Let E_* be any homology theory.

   (a) An inverse limit of E_*–local spectra is E_*–local.

   (b) If $W \to X \to Y$ is a cofiber sequence of spectra and two of W, X, Y are E_*–local, then so is the third.

   (c) If $X \vee Y$ is E_*–local, then so are X and Y.

3. The localization functors L_E and L_F are the same if and only if $\langle E \rangle = \langle F \rangle$. If $\langle E \rangle \leq \langle F \rangle$ then $L_E L_F = L_E$ and there is a natural transformation $L_F \to L_E$. Conclude that there are maps $L_n X \to L_{K(n)} X$ for every n.

4. Prove that $L_{E \vee F} L_E = L_E = L_E L_{E \vee F}$.

5. (a) Let X be a finite p–local spectrum and let $\mathcal{C}_X$ be the smallest thick subcategory of finite p–local spectra that contains X. Prove that if $Y \in \mathcal{C}_X$, then $\langle Y \rangle \leq \langle X \rangle$.

   (b) Use the thick subcategory theorem to prove that if X is a type m spectrum and Y is a type n spectrum, then $\langle X \rangle = \langle Y \rangle$ if and only if $m = n$.

   (c) For X and Y as above, prove that and $\langle X \rangle < \langle Y \rangle$ if and only if $m > n$.

3.2 Lecture 2: Lubin–Tate cohomology

4 Thursday

4.1 Lecture 1: Basic moduli theory

1. Show that there is a natural map $\mathcal{M}_{fg} \to B\mathbb{G}_m$ from the moduli of formal groups to the classifying stack of the multiplicative group. Interpret this in terms of line bundles. Show that the fiber is the moduli of formal groups and *strict* isomorphisms.

2. Suppose $A \to B$ is a ring map and (A, Γ) is a Hopf algebroid.

   (a) Give a Hopf algebroid structure on $(B, B \otimes_A \Gamma \otimes_A B)$.

   (b) Suppose R is a ring, and take R-points. Show that one gets a fully faithful functor of groupoids of R-points, and that the essential image consists of those maps $A \to R$ which can be factored through B.

   (c) Now introduce a Grothendieck topology (such as the flat topology) and consider the associated map of stacks $\mathcal{M}_{(B,B\otimes_A\Gamma\otimes_A B)} \to \mathcal{M}_{(A,\Gamma)}$. If $A \to B$ is flat, show that this gives a fully faithful functor and determine the essential image.

   (d) Show that if there exists a ring R and a map $B \otimes_A \Gamma \to R$ such that the composite $A \to B \otimes_A \Gamma \to R$ is a cover in a Grothendieck topology, then the associated map of stacks is an equivalence.

3. Give an étale cover of the moduli of elliptic curves by affine open subsets.

4. Suppose that G is a group. Describe what it means to give an action of G on a stack. (Take care: if N is a normal subgroup of G, then G/N should act on the classifying stack BN.)

5. Show that the moduli of formal groups does not have a presentation by a Hopf algebroid (A, Γ) where Γ is étale over A.

4.2 Lecture 2: Topology and the moduli of formal group laws

1. Describe the bundles $\mathcal{O}(n)$ on $B\mathbb{G}_m$ as quasicoherent sheaves on the associated Hopf algebroid $(\mathbb{Z}, \mathbb{Z}[t^{\pm 1}])$.

2. Show that the cofiber sequences $S^2 \to \mathbb{CP}^2 \to S^4$ and $S^4 \to \mathbb{HP}^2 \to S^8$ become short exact sequences of sheaves on the moduli of formal groups, and calculate Ext to determine how many possibilities there are for each of them. Which actually occur?

3. Cohomology satisfies base change. In the following, suppose $R \to S$ is a map of rings and (A, Γ) is a Hopf algebroid over R (so that $(S \otimes_R A, S \otimes_R \Gamma)$ is a Hopf algebroid over S).

(a) Show flat base change for cohomology: if $R \to S$ is flat, then the cohomology of the Hopf algebroid $(S \otimes_R A, S \otimes_R \Gamma)$ over S is the obtained by tensoring with S.

(b) Non-flat base change for cohomology: if S has a finite resolution by flat R-modules, then there is a spectral sequence

$$Tor_*^R(S, H^*(A,\Gamma)) \Rightarrow H^*(S \otimes_R A, S \otimes_R \Gamma).$$

(Hint: Tensor with a resolution of S.)

(c) If (A,Γ) is a flat Hopf algebroid and $x \in A$ is a non-zero-divisor satisfying $\eta_R(x) = \eta_L(x)$, then there is a long exact sequence

$$0 \to H^0(A,\Gamma) \xrightarrow{x} H^0(A,\Gamma) \to H^0(A/(x), \Gamma/(x)) \to H^1(A,\Gamma) \to \dots$$

4.3 Lecture 3: Moduli of elliptic curves

1. Suppose C is an elliptic curve over an algebraically closed field K. Depending on the j-invariant of C, classify all possibilities for the automorphism group of C/K. What does this indicate regarding representability of the moduli problem of elliptic curves?

2. (a) Suppose C is an elliptic curve over a base S. Show that multiplication by n on C is a finite locally free map of degree n^2. Show that if n is invertible on S, its kernel $C[n]$ is finite étale over the base S and locally isomorphic to $(\mathbb{Z}/n)^2$.

   (b) On the other hand, suppose C is an elliptic curve over an field of characteristic p. What does $C[p^k]$ look like then?

   (c) A full level n structure, aka $\Gamma(n)$-structure, on an elliptic curve C is defined to be a homomorphism $\varphi : (Z/n)^2 \to C$ of group schemes which restricts to an isomorphism $(Z/n)^2 \to C[n]$. Show that the moduli problem $\mathcal{M}(n)$ assigning to a ring R with $1/n \in R$ the groupoid of elliptic curves over R equipped with a level n structure is representable if $n > 2$.

   (In this groupoid, an isomorphism $f : (C, \varphi : (\mathbb{Z}/n)^2 \to C) \to (C', \varphi' : (\mathbb{Z}/n)^2 \to C')$ is a commutative diagram

$$\begin{tikzcd} (\mathbb{Z}/n)^2 \arrow[r, "\varphi"] \arrow[d, equal] & C \arrow[d, "f"] \\ (\mathbb{Z}/n)^2 \arrow[r, "\varphi'"] & C' \end{tikzcd}$$

   in which f is an isomorphism of elliptic curves.)

   What are the possible automorphism groups for an elliptic curve C equipped with a level 2 structure?

   (d) Forgetting the level structure gives a map $\mathcal{M}(n) \to \mathcal{M}_{ell}[1/n]$. Show that this map is an étale torsor for the group $GL_2(\mathbb{Z}/n)$.

4.4 Lecture 4: The Gross-Hopkins period map

5 Friday

5.1 Lecture 1: p-divisible groups I

1. Consider the elliptic curve
$$y^2 + y = x^3$$
over $\mathbb{F}_2$. Show that it has no two-torsion points but that the two-torsion subscheme has rank four. Relate the square of the Frobenius endomorphism $(x, y) \mapsto (x^2, y^2)$ to the multiplication-by-two map on this curve.

2. Suppose A is a commutative algebraic group over a field k of characteristic $p > 0$, such that $p : A \to A$ is surjective. Show that $Ker(p)$ is finite. Let $A(p) = \cup_j Ker(p^j)$; show that $A(p)$ is a p-divisible group with connected formal part equal to $\cup_j Ker(F^j)$. Now look at some examples:

   - What is $A(p)$ for $A = \mu_\infty$?
   - If A is an abelian variety of dimension g, the rank of $Ker(p)$ is $2g$. What is the height of $A(p)$?

5.2 Lecture 2: p-divisible groups II

1. Let k be a a perfect field of characteristic $p > 0$, and let $h \geq 1$ be an integer.

   (a) Show that there exists a Dieudonné module M over $W(k)$ with the following properties:

      i. $\dim_k M/FM = 1$, and

      ii. F is topologically nilpotent on M.

   (b) Now assume that k is algebraically closed. Show that the above M is unique up to isomorphism.

   (c) Conclude that up to isomorphism there exists a unique connected p-divisible group G of dimension 1 over k.

   (d) Describe the ring of endomorphisms of G.

2. Keep the assumption that k is an algebraically closed field of characteristic p. We now generalize the previous exercise. Let $h \geq 1$ be an integer, and let $0 \leq d \leq h$ be relatively prime with h. Let $M_{d/h}$ be the Dieudonné module with basis $v, Fv, \ldots, F^{h-1}v$, where the action of F is determined by $F^h v = p^d v$. Show that $M_{d/h}$ really is a Dieudonné module. Also show that $M_{d/h}[1/p]$ is irreducible (that is, it does not contain any proper $W(k)[1/p]$-submodules which are F-invariant). What does this say about the corresponding p-divisible group? Compute the ring of endomorphisms of $M_{d/h}$.

5.3 Lecture 3: Building cohomology theories from p-divisible groups

1. Describe how Lurie's theorem is a strict strengthening of the Hopkins–Miller theorem constructing Lubin–Tate spectra.

2. We can cheat in our definitions: instead of picking a quadratic imaginary extension of $\mathbb{Q}$, we can pick $\mathbb{Q} \times \mathbb{Q}$ with ring of integers $\mathcal{O} = \mathbb{Z} \times \mathbb{Z}$. Show that the moduli of 2-dimensional abelian varieties with an action of $\mathcal{O}$ is isomorphic to the moduli of elliptic curves.

3. Over the complex numbers, a polarized abelian variety A is equivalent to a complex vector space V (the tangent space of A) containing a lattice $L \cong H_1(A)$, together with a positive definite symmetric Hermitian inner product $(-,-)$ whose imaginary part takes integer values on L. We can identify A, as a complex manifold, with V/L. The following exercises refer to this case.

   (a) Show that, if $V \cong \mathbb{C}$, every such Hermitian inner product is a scalar multiple of a canonical one by a natural number. Conclude that every elliptic curve E over $\mathbb{C}$ has a canonical polarization and a canonical "anti-polarization" (its negative).

   (b) Show that if we are given such an E with its canonical polarization, every $n \times n$ integer matrix which is symmetric and positive definite gives a polarization on the n-fold product E^n.

   (c) The ring of endomorphisms $\mathrm{End}(A)$ is the set of linear maps $V \to V$ which preserve L. Show how the Hermitian pairing determines an involution $(-)^\dagger$ of $\mathrm{End}(A) \otimes \mathbb{Q}$ (the Rosati involution).

   (d) Suppose that F is a quadratic imaginary extension of $\mathbb{Q}$ and $\mathcal{O}_F$ is its ring of integers. Suppose that a lattice L with $\mathcal{O}_F$-action has an $\mathcal{O}_F$-Hermitian pairing $\langle -,- \rangle$. Then the vector space $V = L \otimes R$ has an action of $\mathcal{O}_F \otimes \mathbb{R} \cong \mathbb{C}$, giving it one complex structure. Show that the set of complex structures on V that will make this into a polarized abelian variety with $\mathcal{O}_F$-action such that $\alpha^\dagger = \overline{\alpha}$ is the same as the set of direct sum decompositions $V = V^+ + V^-$, where V^+ consists of vectors of positive length under $\langle -,- \rangle$ and V^- consists of vectors of negative length.

   (e) In the previous case, describe how $(-,-)$ and $\langle -,- \rangle$ are related.

4. Show that the product of a polarized $\mathcal{O}_F$-linear abelian variety of type (n,m) with one of type (n',m') is one of type $(n+n', m+m')$. If E is an elliptic curve with $\mathcal{O}_F$-multiplication, use this to construct embeddings from Shimura varieties of type $(1, n-1)$ to type $(1,n)$.

5. Show that there exists a Weierstrass curve E over a discrete valuation ring $\mathcal{O}_K$, with the base change E_K to the extension field K an elliptic curve, such that the p-divisible group of E_K does not extend to a p-divisible group over $\mathrm{Spec}(\mathcal{O}_K)$.

6. Given an example of a Landweber exact cohomology theory that does not arise from

5.4 Lecture 4: Advanced topics in the moduli of p-divisible groups

Let $h \geq 1$, let $k = \overline{\mathbf{F}}_p$, and let G/k be a connected p-divisible group of height h and dimension 1. We consider the moduli space M_0 of deformations of G_0. That is: given a complete noetherian $W(k)$-algebra R, $M(R)$ is the set of pairs (G, ι), where G/R is a p-divisible group and $\iota\colon G_0 \otimes_k R/p \to G \otimes_R R/p$ is a quasi-isogeny. Then (Lubin-Tate) M_0 is isomorphic to $\mathbb{Z}$ copies of $\operatorname{Spf} W(k)[\![u_1, \dots, u_{h-1}]\!]$; that is, it is $\mathbb{Z}$ copies of a formal open unit ball of dimension $h-1$. Let $\mathscr{M}_0$ be the rigid generic fiber of M_0. By adding p^n-level structures we obtain a tower of rigid spaces $\{\mathscr{M}_n\}$ for $n \geq 1$. Our objective is to understand the limit $\mathscr{M}_\infty = \varprojlim \mathscr{M}_n$, which is a $\mathrm{GL}_n(\mathbb{Q}_p)$-torsor over $\mathbf{P}^{h-1}$ via the Gross-Hopkins map.

1. First examine the case $h = 1$, so that G_0 is isomorphic to the multiplicative formal group. Then $M_0 = \sqcup_{\mathbb{Z}} \operatorname{Spf} W(k)$ and $\mathscr{M}_0 = \sqcup_{\mathbb{Z}} \operatorname{Spm} W(k)[1/p] = \operatorname{Spm} \breve{\mathbb{Q}}_p$, where $\breve{\mathbb{Q}}_p$ is the completion of the maximal unramified extension of $\mathbb{Q}_p$. Also $\mathscr{M}_n = \sqcup_{\mathbb{Z}} \operatorname{Spm} \breve{\mathbb{Q}}_p(\mu_{p^n})$. Let K be the completion of $\cup_{n\geq 1} \breve{\mathbb{Q}}_p(\mu_{p^n})$, a complete valued field.

   (a) Let $\mathcal{O}_K$ be the ring of integers of K. Show that the Frobenius map Frob is surjective on $\mathcal{O}_K/p$ (this shows that K is a *perfectoid field*).

   (b) Show that $\varprojlim_{\mathrm{Frob}} \mathcal{O}_K/p$ is a domain, and let $K^\flat$ be its fraction field. Show that $K^\flat$ is a perfect valued field of characteristic p. In fact, $K^\flat \cong k(\!(t^{1/p^\infty})\!)$, the completion of the perfect closure of the Laurent series field $k(\!(t)\!)$.

2. Now let $h \geq 1$ be general. Let $G_h/W(k)$ be the Honda formal group: the underlying formal scheme is $G_h = \operatorname{Spf} W(k)[\![x]\!]$, and the formal logarithm is the series

$$L(x) = x + \frac{x^{p^h}}{p} + \frac{x^{p^{2h}}}{p^2} + \dots.$$

   More precisely, L is a homomorphism of rigid spaces $\mathscr{G}_h \to \mathbf{G}_a$, where $\mathscr{G}_h$ is the rigid generic fiber of G_h.

   Let $\tilde{G}_h = \varprojlim_p G_h$. Show that $\tilde{G}_h \cong \operatorname{Spf} W(k)[\![x^{1/p^\infty}]\!]$, and the composite of the projection $\tilde{G}_h \to G_h$ with the logarithm is the series

$$L_\infty(x) = \sum_{n\in\mathbb{Z}} \frac{x^{p^{hn}}}{p^n}.$$

3. Assume that $G_0 = G_h \otimes_{W(k)} k$. The space M_0 comes equipped with a universal deformation G. The quasi-isogeny $\iota\colon G_h \to G$ defined on M_0 modulo p induces an isomorphism (!) $\tilde{G}_h \to \tilde{G}$ of formal schemes over M_0.

4. Over the rigid space $\mathscr{M}_0$, we have the generic fiber $\mathscr{G}$, and the pullback of $\mathscr{G}$ over $\mathscr{M}_\infty = \varprojlim \mathscr{M}_n$ admits n sections $X_1, \ldots, X_n \in \tilde{\mathscr{G}}$. Therefore the sections X_i of $\tilde{\mathscr{G}}$ give us sections x_i of $\tilde{\mathscr{G}_h}$. The Gross-Hopkins map $\mathscr{M}_\infty \to \mathscr{M} \to \mathbf{P}^{h-1}$ has the following description in terms of the coordinates x_i (and their pth power roots): it is the $(h-1)$-plane in $\mathbf{A}^h$ spanned by the (linearly dependent) vectors

$$[L_\infty(x_i), L_\infty(x_i^p), \ldots, L_\infty(x_i^{p^{h-1}})].$$