Euler Characteristic of the Sphere

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A topological invariant assigns to a topological space, in our case a surface, an algebraic object such as a number, a polynomial or a vector space. If two surfaces have different topological invariants then they must be topologically inequivalent. (However the converse is not always true in that two inequivalent surfaces can have the same topological invariants.) The simplest non-trivial topological invariant is the Euler characteristic. The Euler characteristic assigns to each surface an integer and can be thought of as a way of quantifying shapes.

Euler characteristic for surfaces is computed using graphs. We begin with the simplest surface, the sphere S^2.

Example 0.1. For a polyhedron P let v, e, f denote the number of vertices, edges and faces respectively. When P is a tetrahedron $v = 4, e = 6, f = 4$ and hence $v - e + f = 2$. When P is a cube $v = 8, e = 12, f = 6$ and hence $v - e + f = 2$.

What does the cube and the tetrahedron have in common? They can all be continuously deformed to the 2 dimensional sphere S^2.

1 Planar graphs

We'll start by computing the Euler characteristic of a planar graph. For us a graph G is a pair of finite sets (V, E) where the vertices, V, are distinct points in the standard plane $\mathbb{R}^2$ and the edges, E, are segments connecting two vertices. We say that a graph is **connected** if each vertex is connected to every other vertex via a sequence of edges. We say that a graph is **planar** if no two edges intersect in a point outside the set of vertices.

A planar graph divides the plane $\mathbb{R}^2$ into **faces**. Denote the set of faces by F.* (It is possible for the set F to be empty.) The **Euler characteristic** of a planar graph G is defined to be

$$\chi(G) := |V| - |E| + |F|$$

where $|S|$ denotes the size of the set S.

Theorem 1.1. *The Euler characteristic of a connected planar graph is 1.*

Exercise 1.3 describes one proof of this theorem using induction on the number of faces. First we prove the theorem in the case when the graph has no faces.

A *connected* graph G without any face, i.e. when F is an empty set or equivalently $|F| = 0$, is called a **tree**. (A graph with no faces but which is not necessarily connected is called a forest!) A vertex in a tree with only one edge attached to it is called a **leaf**.

Exercise 1.2. For a tree G what is the relationship between $|V|$ and $|E|$? What is $\chi(G)$?

Exercise 1.3. The following is the proof of (1.1)

1. For a connected planar graph G with at least 1 face, show that it is possible to delete an edge and obtain a graph G' such that G' has exactly one less face than G.
2. What is the relationship between the Euler characteristic of G and G'?
3. Induct on $|F|$ to complete the proof of Theorem 1.1. (What is the base case for induction here?)

Exercise 1.4. What is the Euler characteristic of a planar graph which is not necessarily connected?

*We won't include the unbounded face in F, for us all the faces are (possibly non-convex) polygons.

2 Euler characteristic of S^2

A **surface graph** on a sphere S^2 is a *connected planar* graph $G = (V, E)$ such that V and E are now on S^2 and all the faces are **polygons**. The tetrahedron, the cube, and the octahedron, provide examples of such surface graphs. A surface graph is called a **triangulation** if all the faces are triangles.

Theorem 2.1. *The Euler characteristic of any surface graph G on S^2 is 2. Hence we can define the Euler characteristic of S^2 as $\chi(S^2) := \chi(G)$ and we have,*

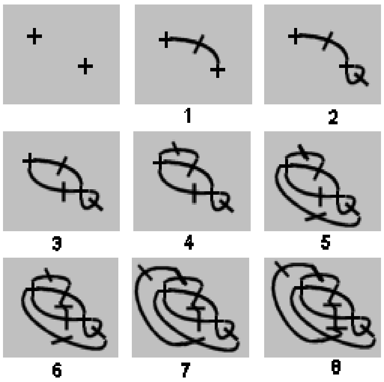
$$\chi(S^2) = 2$$

Exercise 2.2. Theorem 2.1 follows directly from Theorem 1.1 for planar graphs,

1. Explain how a surface graph on S^2 gives rise to a planar graph on $\mathbb{R}^2$.
2. Draw the planar graphs for the cube, the tetrahedron and the octahedron.
3. Show that the Euler characteristic of a graph on S^2 is 2.

3 Brussel Sprouts

The game of **Brussel Sprouts** starts with 2 crosses. Each move involves joining two free ends with a curve not crossing any existing line and then putting a short stroke across the line to create two new free ends. The game ends when no such move is possible.



A game of Brussel Sprouts

Exercise 3.1. Play a few games of Brussel Sprouts!

It turns out that every game of Brussel Sprouts always ends in the same number of steps! The following exercises describe a proof of it,

Exercise 3.2. Let G be the connected planar graph (vertices are the crosses) at the end of the game.

1. What happens to the number of free ends after each move? How many free ends are there in the end?
2. Argue that at each stage of the game every face should have at least one open end on it's boundary. Further, argue that there cannot be two or more open ends on the boundary of a face at the end of the game. Hence every face of G should have exactly 1 open end on it's boundary. The same is true for the *unbounded face*.
3. Conclude that G has 7 faces.

Exercise 3.3. Assume that the game ends in **n** steps.

1. How many vertices and edges are added after each move? Argue that $|E| = 2n$ and $|V| = 2 + n$.
2. Use Theorem 1.1 to find n.

Can you generalize the above proof to k crosses in the beginning? How about playing Brussel Sprouts on a Torus?