

we do have a trivial homotopy equivalence C. does not split.

1.0.5)

K- category of chain complexes but maps are equivalence classes apto chain homotopy why is K not abelian?

- a) fang, gnh Then $\exists s_1, s_2$ such that $f-g=s_1d+ds$, $g-h=s_2d+ds_2$ $f-h=(s_1+s_2)d+d(s_1+s_2) \Rightarrow fnh$
- b) f,g: C→D u:B→c u:€→D→E

 f-g=sd+ds

 ufu-ugu= usdu-ugud udsu

 = usud-dusu

= ofu~ogu

So composition is well defined in K.

forgo, fing, girc -> D

forgo, fing,

forgo = sod-dso fing, = sid-ds,

forfingo-go = (so+si)d - d(so+si)

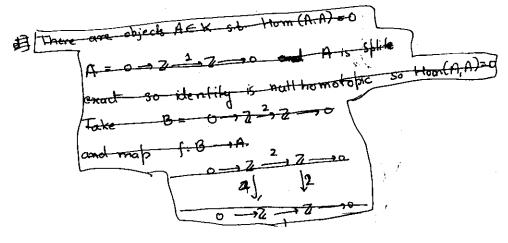
So Ab-cat. And O-object is the ochain.

Next products. Take C.xD. in

Ch and look at quotient.

ch - x
f+g - [f+g]

So Abelian functor



cone
$$(f)_n := B_{n-1} \oplus C_n$$

 $d(b,c) := (-d(b), dc - fb)$

$$d^2(b,c) = d(-db, dc-fb) = (0, -dfb + fdb) = (0,0).$$
One can think of d as a matrix $\begin{bmatrix} -dB \\ f + dc \end{bmatrix}$

contra
$$f: B \rightarrow C$$
 cone $(f)_n := B^{n+1} \oplus C^n$
 $d(b,c) := (-db, dc - fb)$

cone (C.):= cone (Id)

$$n^{m} \text{ term} = c_{n-1} \oplus c_{n}$$

$$d(c_1, c_2) = (-dc_1, dc_2 - c_1)$$

Reverse map should take Cn-1 to Cn-1 in

$$dsd(c_1,c_2) = ds(-dc_1,dc_2-c_1)$$

= $d(c_1-dc_2,0)$

=
$$(-dc_1, dc_2 - c_1) = d(c_1, dc_2)$$

dong exact seq:

Hn-1 (C.) -+ Hn-1 cone -- Hn-2 B

$$fp \longrightarrow (0, tp) \qquad 0 \qquad 9 = t^*$$

$$(p, 0) \longrightarrow p \qquad 9[p] = [tp]$$

$$Z_{n}$$
 (cone f) = $\frac{1}{2}(b,c)$ | $ab=0$, $dc=fb$, $B_{n} = \frac{1}{2}(ab, dc=fb)$ $\frac{1}{2}$ \frac

· Topology:

$$A_{n-1} \oplus X_n$$

(Bn-1 & Cn) An-1 (An-1)

$$\rightarrow H_n(x) \rightarrow H_n(x/A) \rightarrow H_{n-1}(A)$$

$$H_{n-1}(x) \longrightarrow H_{n-1}(x/A) \longrightarrow H_{n-2}(4)$$

teta (XA) & Cone(i).

we have map between long exact sequences of MA and cone in which by five lemma will give that $H_n(x/A) \subseteq M_n(cone i)$

map in the middle is
$$(x,a) \mapsto x$$
commutativity is because $(x,a) \in Z = 0$ as ∂x .

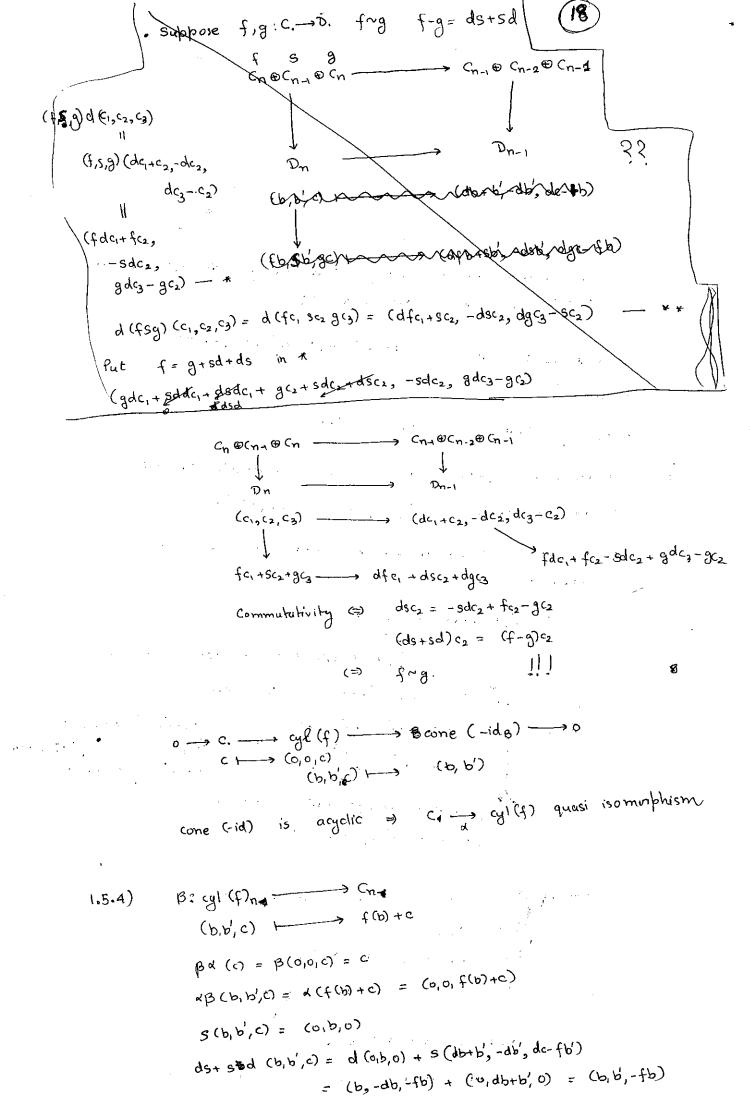
cg linder: Mapping

$$d^{2}(b,b',c) = d(db+b',-db',dc-fb')$$

$$= (d^{2}b+db'-db',o,o) = (0,0,0)$$

. id:
$$C \longrightarrow C_0$$
 grame ((.):= C_1 (id)

with ferm = $C_1 \oplus C_{h-1} \oplus C_h$



=) ds+sd = 1- aB

=> a C homotopy equivalent to cyl (f).

Scary!

23/01/13

If A is an abelian category, what are projective objects in the category of chain complexes of A?

Fright exact: A -B. Lif := ith left derived functor.

- L.F = F

. If P projective Lif(P)=0 4 i>1

Homological S-functors: A -> B between Abelian categories Def:

additive functors To, Ti, -- (Theo 4n <0)

· for each o-A-B-c-o remact we get a long exact seq.

> -> & TnA -> TnB -> TnC -ees $T_{n-1}A \longrightarrow T_{n-1}B \longrightarrow T_0C \longrightarrow 0$

are fundament maps Tn(c) - Tn-1(A) (e.

give n

 $0 \rightarrow A' \rightarrow B' \rightarrow C' \rightarrow 0$ $3 \downarrow \qquad \downarrow f \qquad \text{or jet} \qquad T_{n}(C') \xrightarrow{S} T_{n-1}(A)$ $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ $T_{n}(C') \xrightarrow{S} T_{n-1}(A)$ $T_{n}(C') \xrightarrow{S} T_{n-1}(A)$

8, natural transformation between. functors Tn(c), Tn-1 (A).

Ex: 1 A R-module, rER R-commutative

T. A = A/A TA = A = {aeA|rA=0}

0 -> A --> B --> C --> o 115 B/A

S (aT,C) --- TOA = A/VA rec = rB/A = {beb | rbeA} We use snake lemma to

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

Dollese form a universal S-sandor?

gives

Morphism of S-functors:

5-T mean natural transforms Sn -Tn compatible with S.

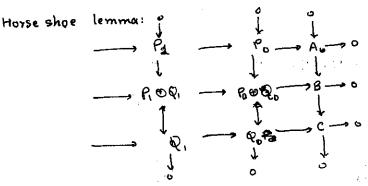
Universal 8-functor:

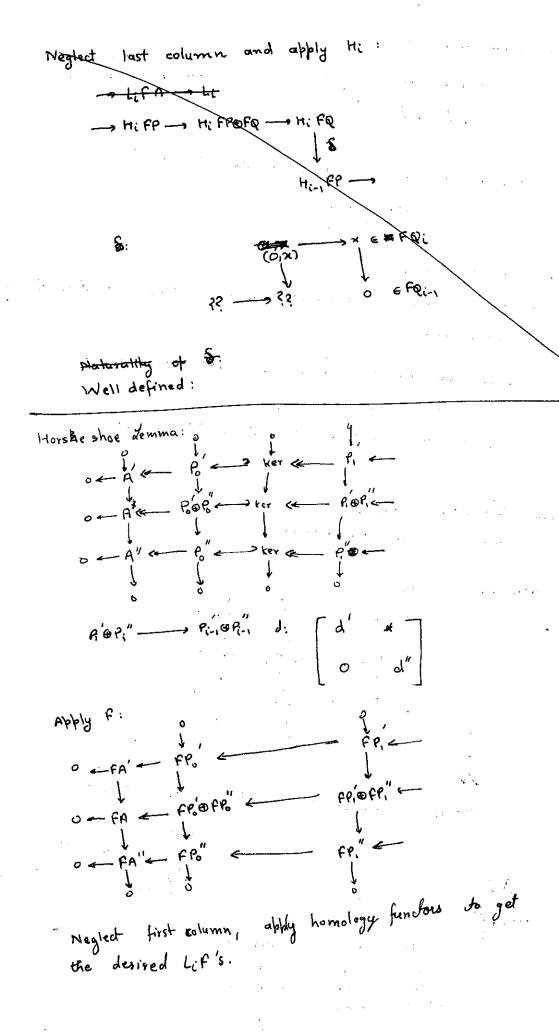
. T = fTn? universal if given any 8-functor S = 15n3 and a nutural transformation fo: So→To,

∃! morphism of 8 functor fn: Sn→Tn extending fo.

Roof that {Lif} are 8-functors:

First we need get projective resolutions:





· # naturality of 8:

Four squares join at (P'&P")i

on Pi' we have no choice has to be f'
on Pi'' we need a lift of

$$P_i'' \xrightarrow{id} P_i''$$
 $\downarrow f''$

as well as

 $Q_i' O Q_i'' \longrightarrow Q_i''$

How about lifting

Pi" Pi-1

for

Same thing.

Injectives:

. There "enough" injectives in Z-mod

Znz - Qnz

is R/nz injective? look like * + nZ is in QnZ.

Using this finitely generated case is taken care of. what can we do for asbiteary abelian groups?

No need not work. a is but iza # Zia

Very different strategy:

S= Hom (x, \$Q/Z)

I(A) = II (Q/Z)X

Then,

 $A \longrightarrow \mathcal{I}(A)$ al (K(a)) x

- . I(A) injective? Product of injectives is injective
- map injective?

YaeA JuES. a(a) +0

RIZ send a > 1 tersion A order

Now we intend to extend this to arbitrary ritys. for this we required use adjoint functors.

Adjoint functors:

Z, R adjoint is * AinA, BinB

Hom (ZA, B) = Hom (BA, RB)

A f, A' mos Hom (ZA,B) = Zf Hom (ZA',B)

Hom (A, RB)

B = B' Hom (ZAZ, B) = Hom (ZA, B')

Hum (A, RB) 3 Hom (A, RB')

eg: R-mod _____ Set F- forgetful functor

Hown (SETS, F(M)) = Hown (Free (S), M)
Set

free-left F-right

Additionally we also require natural transformations:

a. LR --- 18

following merphisms are identity morphisms

 $Z(x) \xrightarrow{Z_0 b(x)} ZRZ(x) \xrightarrow{a.Z(x)} Z(x)$

R(Y) bod(BY) RRLR(Y) ROa(Y), REL(Y)

for the case (Free, Forget). 50

a: Free. Forget (RM) - 1 R-mod (M)

 $e_n \longrightarrow x$

by Forget Free (15) 3 1 Set (5)

Ex 1.1-7:

a: Z.R - 1 isomorphism
i.e. Jc: 1 - XR sinverse of a

Result:

a icomorphism (=) R fully faithful b isomorphism (=) & fully faithful.

Mapping cones return $x' = f \cdot y$ A cone $(f)_e^n = y' \cdot \oplus x' \cdot \Box$ d(y,x) = (dy + fx - dx)

a majo between majos, gives rise to a majo between complexes

· A mapping cone gives note to

T₅: x + y - cone of -> x [1]

femously written as

x + y

T₅: x - y

Cone f

Kop: Dimca 1.1.20

 $5d+ds \in x) = (0, dx) + (fx, -dx) = (fx, 0)$

• cone.f $\rightarrow \times [1]$ $\longrightarrow Y[1]$ $(y,x) \longmapsto x \longmapsto fx$

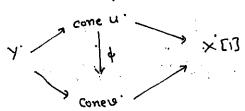
Tracing: $y, n \rightarrow (dy + fx, -dx)$ $(-1)^{n}y, -(-1)^{n+1}dy + (-1)^{n+1}dy + (-1)^{n+1}dy + (-1)^{n+1}fx$ $(sd+ds)(y, n) = (-1)^{n+1}fx$

There is an issue of sign.

u,u: X ---> Y unu ce 3s: X ---> YE-1]

s.l. ds+sd=u-v

φ: cone u' cone u' comorphism s.f.



on y component of cone i we do not have a choice on x? Trouble is with & being chain map $\phi(o(x) = (\phi(x), x).$

what is \$? need do () = bd(y,x)

LHS = d (y+ p'(x), 2)

= (dy + do(x) + un, -dn)

RHS = 4 (dy+ 4x, -dx)

= (dy + 4x - \$ (dx) , - \$ (dx) .

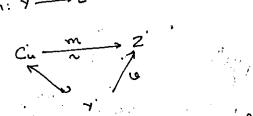
So we require,

do'x + wx = ux - b'dx

Tate \$ = S.

1.1.23)

i) need m: y ~ z st.



we are going to use 5-lemma for the two short exacts:

o->x'->z'->o and o -> y' -- cone u' -- x(1) --> 0 B

we don't really have an option with m: Need m yn xn+1 ---> 2n There toes not exist a canonical, map X" > 2" So define pm (y,x) = uy To we need to show quasi-isomorphism. look at lung exact seap Enough to show commutativity, 5-lemma will do the rest $\begin{array}{c} \Gamma y \uparrow \longrightarrow \Gamma y \downarrow \\ \uparrow & \uparrow \\ \Gamma y \uparrow \longrightarrow \Gamma (y, o) \uparrow \end{array}$ $\begin{array}{c} \Gamma y \uparrow \longrightarrow \Gamma (y, o) \uparrow \longrightarrow \Gamma (y, x) \uparrow \longrightarrow \Gamma ($ $d(y,x)=0 \Rightarrow (dy+ux,-dx)=0 \Rightarrow dy=-ux$ y = vy S(y) = -x

So commutative. 8 Define: S: Z cone u = y & x x x m3 (2) = m (52,0) = 482 = Z 3m = 3(uy) = (suy, 0) 434y = 4y => 6(54-1)y = 0 >) (su-i)y = ux' for some x'e X" y -- 4-1 (SU-1) y Traceing:

(0, 4 (60-1)y) --- (60-1)y, 4 d(50-1)y)

(y, x) (dy+ux, dx)

(o, u' (su-1) (dy+ux)) = (o, u' (su-1)dy-ux)

- xd+dx = ((su-1)y, x + [u'] [(su-1)dy - d(su-1)]y)

so whel next for to be a sud-dsiu

chain map.

we want u' [sud-dsiu]y = o

i.e. sudy = dsuy

But su differs from # 1 by element of x

Result is true if so is a chain map.

Homotopical Category: K(A)

Morphisms are equivalence classes up to chain
homotopy.

Tr3 $\times \frac{u}{1} \times \frac{v}{1} \times \frac{v}{2} \times \frac{v}{2}$

A PART OF THE PROPERTY.

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