PROBLEM SET 12

Part 1 - Infinite series

Q.1. In this problem $a_n \geq 0$.

(1) Show that if there is a number $c < 1$, and an integer N such that $a_n < c^n$ for all $n > N$, then $\sum_{n=0}^{\infty} a_n$ converges.

(2) Show that if there is a number $c > 1$, and an integer N such that $a_n > c^n$ for all $n > N$, then $\sum_{n=0}^{\infty} a_n$ diverges.

(3) Show that if $\lim_{n\to\infty} \sqrt[n]{a_n} = r$ then,

$$\text{if } r < 1 \text{ then } \sum_{n=0}^{\infty} a_n \text{ converges and if } r > 1 \text{ then } \sum_{n=0}^{\infty} a_n \text{ diverges.}$$

This is called the **root test**.

(4) Find examples of positive a_n and b_n with $\lim_{n\to\infty} \sqrt[n]{a_n} = 1 = \lim_{n\to\infty} \sqrt[n]{b_n}$ such that $\sum_{n=0}^{\infty} a_n$ converges and $\sum_{n=0}^{\infty} b_n$ diverges.

(5) This part is surprising difficult to make rigorous, you can give an approximate proof instead.

Show that if $\lim_{n\to\infty} \frac{a_n}{a_{n-1}} = r$ then $\lim_{n\to\infty} \sqrt[n]{a_n} = r$.

(And hence, in theory, the root test is stronger than the ratio test, in practice though the ratio test is easier to use.)

Q.2. Q.1 of Ch.23 on Pages 482-484. Note that there are **20** (small) problems in this question.

PART 2 - ABSOLUTE CONVERGENCE

Q.3. Let a_n be a decreasing sequence with $a_n > 0$. Define

$$s_n = a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n-1} a_n$$

(1) Show that the sequence $b_n = s_{2n}$ is an increasing sequence.

(2) Show that $a_1 > b_n$ for all n, hence conclude that b_n converges.

(3) Show that the sequence $c_n = s_{2n-1}$ is a decreasing sequence.

(4) Show that $0 < c_n$ for all n, hence conclude that c_n converges.

(5) Show that if $\lim_{n\to\infty} a_n = 0$ then

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} c_n$$

(It's ok to give an approximate argument for this last part.)

Q.4. This question is a freebie.

The theorems from this chapter are extremely important, but the proofs are a bit convoluted and it is too late in the semester to spend significant time on them, if you have time you should read their proofs and understand the discussions in the book.

For this problem simply write down the statements of Theorems from 5 to 9.

Part 3 - Integral Computations

Last problem set on Integrals!

Q.5. For this week do Q.7 i)-ix) on Pg.381 and Q.11 problems - i), ii) on Pg.383.