# Exact Couples & Spectral Sequences

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

eg: $$\begin{tikzcd} H^\bullet(X_*) \arrow[rr] & & H^\bullet(X_*) \arrow[dl] \\ & H^\bullet(X_*, X_{*-1}) \arrow[ul] & \end{tikzcd}$$

where $X_*$ is a filtration of a topological space $X$.

use this example to keep track of indices

Cohomological $\rightsquigarrow$

$i: H^n(X_p) \longrightarrow H^n(X_{p-1})$

bidegree : $(n,p) \longmapsto (0,-1)$

$(p,q) \longmapsto (-1,1)$

$j: H^n(X_p) \longrightarrow H^{n+1}(X_{p+1}, X_p)$

bidegree: $(n,p) \longmapsto (1,1)$

$(p,q) \longmapsto (1,0)$

$k: H^n(X_p, X_{p-1}) \longrightarrow H^n(X_p)$

bidegree : $(n,p) \longmapsto (0,0)$

$(p,q) \longmapsto (0,0)$

$d = j \cdot k : H^n(X_p, X_{p-1}) \longrightarrow H^{n+1}(X_{p+1}, X_p)$

bidegree : $(n,p) \longmapsto (1,1)$

$(p,q) \longmapsto (1,0)$

For homology, j and k are interchanged

This is just the grading for the first couple.

In the $r^{th}$ stage :

$$\begin{tikzcd} i^r(A) \arrow[rr] & & i^r(A) \arrow[dl] \\ & E_r \arrow[ul] & \end{tikzcd}$$

$i^r(A)^n_p = i^r(A^n_{p+r})$

$Z_2 = \ker j \cdot k = \{x \in E^n_p \mid jkx = 0\}$

$= \{x \in E^n_p \mid kx \in iA\}$

$= k^{-1}(iA^n_{p+1})$

$B_2 = \operatorname{im} j \cdot k$

$= j(\ker i)$

Remember : bidegrees of $i, k$ do not change and $d$ always increases

$d^r: (p,q) \longmapsto (p+r, q-r+1)$

$(n,p) \longmapsto (n+1, p+r)$

In general,

$$E_r = Z_r / B_r = \frac{k^{-1}(i^{r-1} A^n_{p+1-r})}{j(\ker i^{r-1})}$$

Q. How to use this shit in practice?

## Serre SS – Always good to remember:

$$\begin{tikzcd} F \arrow[r, "i"] & X \arrow[d, "\pi"] \\ & Y \end{tikzcd}$$

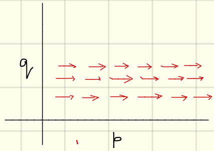
For appropriate filtration of X,

$A^n_p = ??$

$E_1^{p,q} = C^p(Y, H^q(F))$

The first $d$ is simply the differential on $C^*$

$E_*^{p,q} \Rightarrow H^{p+q}(X)$




The maps $i, \pi$ induce natural maps

## SS for a filtered complex:

This is the most boring maths I have ever seen.

# Multiplicative Cohomological Exact Couples:

Assume we have products:

$$C^{n_1}_{p_1} \otimes C^{n_2}_{p_2} \longrightarrow C^{n_1+n_2}_{p_1+p_2}$$

and that each $A^*_p$ is a ring (Not $A^n_p$ but whole collection $\bigoplus_n A^n_p$)
such that

1. $i$'s are ring maps
2. $j$'s are algebra maps
3. $d_s$ are graded derivations

i.e. $d_s(cc') = d_s c . c' + (-1)^{|c|} c . d_s c'$ $\rightsquigarrow$ comes down to checking it is a derivation.

($\Rightarrow$ each $E_r$ is a graded ring)

This is called multiplicative structure on SS.

$Th^m$: Under convergence conditions, given multiplicative structure,

$$\bigoplus_{p+q=n} E^{p,q}_\infty \cong \bigoplus_p \bar{A}^n_p / \bar{A}^n_{p+1} \quad \text{as algebras.}$$

Ex: $A^n_p = H^n(X_p)$, $C^n_p = H^n(X_p, X_{p-1})$

Does $d_1$ derivation $\Downarrow$ $d_2$ derivation.