1. $l_z, r_z : C^n(R,M) \longrightarrow C^n(R,M)$, $z \in R$ $\qquad M / R / k$

Commutative

$l_z(f)(r_1 \cdots r_n) = z f(r_1 \cdots r_n)$

$r_z(f)(r_1 \cdots r_n) = f(r_1 \cdots r_n) z$

Induces $l_z^*, r_z^*$ which make $H^n(R,Z)$ into a symmetric $R$-bimodule

A: Note that $l_z, r_z$ are left and right $k$-module maps respectively. That they commute with $d$ and hence induce an action of $R$ on $H^n(R,M)$ is also easy.

Symmetry:

Need to show $\exists \beta : C^n(R,M) \longrightarrow C^{n-1}(R,M)$ such that $\pm d\beta \pm \beta d = l_z - r_z$

deg 0: $0 \longrightarrow C^0(R,M) \underset{\beta}{\overset{d}{\rightleftarrows}} C^1(R,M)$

$$\beta d(m) = l_z m - r_z m$$

$$\beta(r \mapsto rm - mr) \quad (zm - mz)$$

$$\boxed{\beta(f) := f(z)}$$

deg 1: $C^0(R,M) \underset{\beta}{\overset{d}{\rightleftarrows}} C^1(R,M) \underset{\beta}{\overset{d}{\rightleftarrows}} C^2(R,M)$

$f$

$$(d\beta \pm \beta d)(f)(r) = zf(r) - f(r)z$$

$$d(\beta(f))(r) \pm \beta d f r = d(f(z))r \pm \beta d f(r) = rf(z) - f(z)r \pm \beta d f(r)$$

$$\Rightarrow \pm \beta d(f)(r) = zf(r) - f(r)z + f(z)r - rf(z)$$

$$\pm \beta((r_1, r_2) \mapsto r_1 f(r_2) - f(r_1 r_2) + f(r_1) r_2)(r)$$

$$\boxed{\begin{array}{l} \beta : C^2(R,M) \longrightarrow C^1(R,M) \\ \quad f \longmapsto (r \longmapsto f(z,r) - f(r,z)) \end{array}}$$

Leap of faith:

$$\beta : C^{n+1}(R,M) \longrightarrow C^n(R,M)$$

$$f \longmapsto ((r_1 \cdots r_n) \longmapsto \underset{\beta_0}{f(z, r_1, \cdots r_n)} - \underset{\beta_1}{f(r_1, z r_2 \cdots r_n)} + \cdots + \underset{\beta_n}{(-1)^n f(r_1 \cdots r_n, z)})$$

$$\beta = \sum_{i=0}^{n} (-1)^i \beta_i$$

Claim: $C^n(R,M) \longrightarrow C^n(R,M)$, $(d\beta + \beta d) f(r_1 \cdots r_n) = zf(r_1 \cdots r_n) - f(r_1 \cdots r_n) z$ How to see this?

What kind of terms can occur in $(d\beta + \beta d) f(r_1 \cdots r_n)$

| Term | Coefficient in $d\beta$ | Co-efficient in $\beta d$ |
|---|---|---|
| $zf(r_1 \cdots r_n)$ | 0 | 1 |
| $f(r_1 \cdots r_n)z$ | 0 | $(-1)^n \cdot (-1)^{n+1} = -1$ |

| Term | $d\beta$ | $\beta d$ |
|---|---|---|
| $f(\cdots \underset{j^{th}\text{ place}}{z} \cdots r_i r_{i+1} \cdots)$ $j<i$ | $(-1)^i \cdot (-1)^{j-1}$ | $(-1)^{i-1} \cdot (-1)^{i+1}$ |
| $f(\cdots r_i r_{i+1} \cdots \underset{j^{th}\text{ place}}{z} \cdots)$ $j>i$ | $(-1)^i \cdot (-1)^j$ | $(-1)^{j+1} (-1)^i$ |
| $f(\cdots r_i z \cdots)$ | $0$ | $(-1)^{i-1} \cdot (-1)^i + (-1)^i \cdot (-1)^i = 0$ |

*Hence proved*

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2. $M \in \text{Mod-}R$, $N \in R\text{-mod}$, $R$ *commutative*/$k$. $H_0(R, M \otimes_k N) = ?$

$M, N \in R\text{-Mod}$, $H^0(R, \text{Hom}_k(M,N)) = ?$

*A.*

$$0 \longleftarrow M \otimes_k N \longleftarrow R \otimes_k M \otimes_k N$$

$$\begin{aligned} &r(m\otimes n) - (m \otimes n) r \longleftarrow r \otimes (m \otimes n) \\ &\quad = m \otimes rn - mr \otimes n \end{aligned}$$

$$H^0 = M \otimes_k N \Big/ (m \otimes rn - mr \otimes n) \cong M \otimes_R N$$

$$0 \longrightarrow \text{Hom}_k(M,N) \longrightarrow \text{Hom}_k(R, \text{Hom}_k(M,N))$$

$$f \longmapsto (r \longmapsto rf - fr)$$

$$H^0 = \{ f \mid rf - fr = 0 \} \cong \text{Hom}_R(M,N)$$

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3. $J = \ker(R \otimes R \longrightarrow R)$. *Show* 1) $H^1(R,J) = 0 \Rightarrow H^1(R,M) = 0 \quad \forall M$

$x \otimes y \longmapsto xy$

2) $H^1(A,J) = 0$ *for $A$ the field of algebraic numbers* $/\mathbb{Q}$.

*A.* 1) $J$ *has the universal property that every derivation factors through it.*

$$\begin{tikzcd} R \arrow[rr, "\phi"] \arrow[dr, "d"'] & & M \\ & J \arrow[ur, "\exists!\,\tilde{\phi}"'] & \end{tikzcd}$$

$H^1(R,J) = 0 \Rightarrow$ *Every derivation on $J$ is inner, in particular $d$*

$\exists\, \alpha \in J$ *such that* $d(r) = r\alpha - \alpha r$

$\Rightarrow \phi(r) = \tilde{\phi} \cdot d(r) = \tilde{\phi}(r\alpha - \alpha r) = r\tilde{\phi}(\alpha) - \tilde{\phi}(\alpha) \cdot r$

$\Rightarrow H^1(R,M) = 0.$

2) *We invoke the theorem that* $H^1(F, M) = 0 \ \forall M \iff F$ *is a finite seperable extension of* $\mathbb{Q}$.

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