

PROBLEM SET 11

PART 1 - SEQUENCES

Q.1. and Q.3. are basically proofs from the book, if you get stuck you should consult them.

- Q.1.** (1) Let a_n be a sequence such that $a_n \neq c$, for any n , and $\lim_{n \rightarrow \infty} a_n = c$. Using the definition, show that if $\lim_{x \rightarrow c} f(x) = l$ then $\lim_{n \rightarrow \infty} f(a_n) = l$.
- (2) Write down the definitions of $\lim_{x \rightarrow c} f(x) \neq l$ and $\lim_{n \rightarrow \infty} a_n \neq k$.
- (3) Show that if $\lim_{x \rightarrow c} f(x) \neq l$ then there is a sequence a_n such that $\lim_{n \rightarrow \infty} a_n = c$ and $\lim_{n \rightarrow \infty} f(a_n) \neq l$.
- (4) Come up with a definition for $\lim_{n \rightarrow \infty} a_n = \infty$. Let a_n be a sequence such that $\lim_{n \rightarrow \infty} a_n = \infty$. Show that if $\lim_{x \rightarrow \infty} f(x) = l$ then $\lim_{n \rightarrow \infty} f(a_n) = l$.

- Q.2.** Do problem 1 (i - ix) and 2 from the book on Pg. 453-454. There are a lot of problems but as before all of them are very short and require a very small argument.

You do not have to be very rigorous for these problems, for most (but not all) problems you can use the part (4) of Q.1.

- Q.3.** (1) Find a sequence a_n such that a_n is bounded from both above and below but $\lim_{n \rightarrow \infty} a_n$ does not exist. No proof needed, draw a picture.
- (2) Let a_n be a non-increasing sequence bounded below. Let $A = \{a_n\}$.
- (a) Argue that $\inf A$ exists.
- (b) Show that $\lim_{n \rightarrow \infty} a_n = \inf A$.

- Q.4.** (1) Prove that if $0 < a < 2$ then $a < \sqrt{2a} < 2$.
- (2) Prove that the sequence

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \dots$$

converges using Theorem 2.

- (3) Find the limit.

- Q.5.** Let $0 < a_1 < b_1$ and define

$$a_{n+1} = \sqrt{a_n b_n} \text{ and } b_{n+1} = \frac{a_n + b_n}{2}$$

- (1) Use Theorem 2 to show that the sequence a_n and b_n converge.
- (2) Show that they converge to the same limit.

PART 2 - SEQUENCES CONTINUED

Q.6. (1) Using integrals show that

$$\frac{1}{n+1} < \log(n+1) - \log(n) < \frac{1}{n}$$

(2) Using Theorem 2 show that the following sequence converges.

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log(n)$$

Q.7. Prove that if $\lim_{n \rightarrow \infty} a_n = l$ then

$$\lim_{n \rightarrow \infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = l$$

(Hint: As a_n converges to l , for every $\epsilon > 0$ there is some constant N such that for all $n > N$ the inequality $|a_n - l| < \epsilon$ holds. For $n > N$ break $\frac{a_1 + a_2 + \cdots + a_n}{n}$ into two fractions and show that one fraction can be made arbitrarily small and the second fraction is close to l .)

PART 3 - INTEGRAL COMPUTATIONS

Q.8. For this week do Q.4 problems - vi) to x) and Q.5) on Pg. 379-380 from Ch.19.