Representation stability - Church

1. Configuration spaces + multiplicity stability
2. Square free polynomials + Combinatorial stability
3. Congruence subgroups + inductive stability
4. FI modules + finite generation
5. FI groups + uniform generating sets
6. Unifying homological stability + representation stability.

Configuration Spaces:

$\mathrm{Conf}_n(M)$ = space of n-element subsets $S \subseteq M$

eg: $\mathrm{Conf}_n(\mathbb{C}) = K(\mathrm{Braid}(n);1)$

Th^m: (Arnold, Cohen)

$$H_*(\mathrm{Conf}_n(\mathbb{C}),\mathbb{Z}) \xrightarrow{\cong} H_*(\mathrm{Conf}_{n+1}(\mathbb{C});\mathbb{Z}) \quad n \geq 2*$$

$$\varinjlim \quad = H_*(\mathrm{Maps}(S^2,S^2);\mathbb{Z})$$

$$H_*(\mathrm{Conf}_n(\mathbb{C}),\mathbb{Q}) = \begin{cases} \mathbb{Q} & *=0 \\ \mathbb{Q} & *=1 \\ 0 & \text{else} \end{cases}$$

Th^m (McDuff, Segal)

for any open M, $H_*(\mathrm{Conf}_n(M);\mathbb{Z}) \xrightarrow{\cong} H_*(\mathrm{Conf}_{n+1}(M);\mathbb{Z}))$ for $n >> *$

Why not M closed?

We do not have a map $\mathrm{Conf}_n(M) \not\longrightarrow \mathrm{Conf}_{n+1}(M)$

$$H_1(\mathrm{Conf}_n(S^2);\mathbb{Z}) = \mathbb{Z}/(2n-2)\mathbb{Z}$$

Th^m (Church)

$$H_*(\mathrm{Conf}_n(M);\mathbb{Q}) \cong H_*(\mathrm{Conf}_{n+1}(M);\mathbb{Q}) \quad \text{for } n > *$$

By transfer: $H^*(\mathrm{Conf}_n(M);\mathbb{Q}) = H^*(\widetilde{\mathrm{Conf}}_n(M);\mathbb{Q})^{S_n}$ space of ordered n-tuples with distinct elements

Not true for $(\,;\mathbb{Z})$ as we need to average

Reduced to understanding S_n-invariants inside S_n-rep $H^*(\widetilde{\mathrm{Conf}}_n(M);\mathbb{Q})$

We do have maps $\widetilde{\mathrm{Conf}}_{n+1} \longrightarrow \widetilde{\mathrm{Conf}}_n$

Th^m: decomposition of $H^i(\widetilde{\mathrm{Conf}}_n(M);\mathbb{Q})$ is stable for $n \geq 4i$.

2.

\# deg-10 polynomials in $\mathbb{F}_3[t] \rightsquigarrow 3^{10}$

\# square-free deg 10 polynomials $\rightsquigarrow 39{,}366 = 2\cdot 3^9$

over all squarefree avg # of linear factors $= 29{,}526 / 39{,}366 \approx 75.0038\%$

Def^n: Conf_n = space of squarefree deg-n polynomials Note: Redefinition of Confn

$\mathrm{Conf}_2 = \{ T^2 + bT + c \mid b^2 - 4c \neq 0 \}$

Grothendieck-Lefschetz:
$$|\mathrm{Conf}_n(\mathbb{F}_q)| = q^n \sum (-1)^i q^{-i} \dim H^i(\mathrm{Conf}_n \mathbb{C}; \mathbb{Q})$$
$$= q^n - q^{n-1}$$

$P(x_1, x_2, \cdots) \rightsquigarrow \chi_P : \mathrm{Conf}_n(\mathbb{F}_q) \longrightarrow \mathbb{Q}$

$\chi_P(f) = P(\#\text{ linear factors, quadratic factors}, \ldots)$

$\chi_P : S_n \longrightarrow \mathbb{Q}$

$\chi_P(\sigma) = P(\#\text{ 1-cycles}, \#\text{2-cycles}, \cdots)$

$$\sum_{f \in \mathrm{Conf}_n(\mathbb{F}_q)} \chi_P(f) = q^n \sum (-1)^i q^{-i} \langle H^i(\mathrm{Conf}_n(\mathbb{C}); \mathbb{Q}), \chi_P \rangle_{S_n}$$ $\rightsquigarrow$ Inner product as S_n representations

Th^m: For any $P(x_1, x_2 \ldots)$

$\langle H^i(\mathrm{Conf}_n(\mathbb{C}), \mathbb{Q}), \chi_P \rangle_{S_n}$ is constant for $n \geq 2i + \deg P$

3.

Th^m (Charney): $H_*(SL_n \mathbb{Z}; \mathbb{Z}) \longrightarrow H_*(SL_{n+1} \mathbb{Z}; \mathbb{Z})$ $n \geq 3$*

Fails for congruence subgroups:

$\Gamma_n(p) = \ker(SL_n \mathbb{Z} \longrightarrow SL_n(\mathbb{Z}/p))$

ex: $H_1(\Gamma_n(p); \mathbb{Z}) = \mathfrak{sl}_n \mathbb{Z}/p$

$\Gamma_n(p) \quad I + pA \longmapsto A \bmod p$

Def^n: given $T \subseteq \{1, 2, \cdots, n\}$ $SL_T \mathbb{Z} = \{ M \in SL_n(\mathbb{Z}) \mid M_{ij} = \delta_{ij} \text{ if } i \notin T \text{ or } j \notin T \}$, $\Gamma_T(p) = SL_T(\mathbb{Z}) \cap \Gamma_n(p)$

eg: $T = \{1, 2\}$ $SL_T \mathbb{Z} = \begin{pmatrix} * & 0 \\ 0 & \ddots 1 \end{pmatrix}$

Th^m:

For $n \geq 2^{i-1}$, $H_i(\Gamma_n(p); \mathbb{Z}) = \operatorname{colim}_{T \subsetneq [n]} H_i(\Gamma_T(p); \mathbb{Z})$ $\Big\}$ Inductive Stability

4.

FI- category of finite sets and inclusions

FI- module functor $V : FI \longrightarrow R\text{-mod}$

Ex: $T \longmapsto SL_T \mathbb{Z}$ make $SL_* \mathbb{Z}$ an FI group

$T \longmapsto \Gamma_T(p)$ $\Gamma_*(p)$ $\Rightarrow$ $H_i(\Gamma(p)_i; \mathbb{Z})$ is an FI-module

$\widetilde{Conf}_T(M) = Inj(T, M)$

$T \mapsto \widetilde{Conf}_T(M)$ $FI^{op} \longrightarrow$ Spaces, $H^i(\widetilde{Conf}_n(M); \mathbb{Z})$ is an FI-module

Def^n: an FI-module V is finitely generated if $\exists v_1 \cdots v_n$ that "span" V.

Th^m:

V be an FI-module$/\mathbb{Q}$, TFAE

1) V is finitely generated

2) "Multiplicity stability" for S_n-reps V_n

3) Combinatorial stability for $\langle V_n, \chi_p \rangle_{S_n}$

Th^m:

V is an FI-module$/R$, TFAE

1) V is finitely presented

2) inductive stability $V_n = \operatorname{colim}_{T \subsetneq [n]} V_T$

Th^m:

R-noetherian $\Rightarrow$ FI-modules$/R$ are Noetherian

(V finitely generated $\Rightarrow W \subseteq V \Rightarrow W$ finitely generated)

6.

$\underset{\text{free group}}{Aut(F_n)}$, $Aut_n[k] = \ker(Aut(F_n) \longrightarrow Aut(N_n^k))$ k-step nilpotent quotients

Th^m:

$Aut_n[k]$ is generated by elements supported on uniformly small splittings $F_n = A * B$ $\varphi(A) = A$ $\varphi|_B = id$

$1 \longrightarrow Aut_n[1] \longrightarrow Aut^+(F_n) \longrightarrow Sl_n \mathbb{Z} \longrightarrow 1$

normal generators

$SL_n(\mathbb{Z}) = \langle E_{ij} \mid [E_{ij}, E_{jk}] = E_{ik}, \ (E_{12}^{-1} E_{21} E_{12}^{-1})^4 = 1 \rangle$
$\qquad\qquad [E_{ij}, E_{k\ell}] = 1$

all relations lie in some $Sl_T \mathbb{Z}$ with $|T| \leq 4$

$$SL_n \mathbb{Z} = \operatorname*{colim}_{\substack{T \subsetneq [n] \\ |T| \leq 4}} SL_T \mathbb{Z}$$

(for $SL_n R \longleftrightarrow$ stable range for unstable $K_2(R)$)

$G = \Sigma_*, Gl(*)$, (Braids, Surfs)

These have symmetric monoidal product

$\rightsquigarrow$ G-mod: $[G, Ab]$ symmetric monoidal

Algebras in G-mod $\quad X = \{1\} \quad X = R$

- $T = T(X)$ free algebra on $\mathbb{Z}\,\mathrm{Hom}_C(X, -)$
- S free commutative algebra on "

A - constant functor $\mathbb{Z}$

Th^m:

1) $\mathrm{Tor}_i^T(V, \mathbb{N}) \approx 0 \iff$ twisted stability for $H_*(Gl_n \mathbb{Z}; V_n)$
2) $\mathrm{Tor}_i^A(\underline{\mathbb{Z}}, \underline{\mathbb{Z}}) = 0 \iff$ we can prove homological stability

this vanishing $\longleftrightarrow$ $\exists$ highly connected complex by which G_n acts with nice stability?