Friedrich-Alexander-Universität
Erlangen-Nürnberg

Decision Theory

Lecture 5

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Decision Theory

Recap: what did we do?

- Decision under certainty
- Implicit description of solutions
  - Linear programming
  - Knapsack problem
- Multi-criteria decision making
  - Pareto efficiency
  - Weighted sum method
  - ϵ-constraint method
  - Goal programming
- Data Envelopment Analysis (DEA)

Decision Theory

Single-Criterion Decision Making

- It’s simple to choose the best from a list
- If solutions are given trough an implicit description, can be tricky!
- Linear programming:

$$\begin{aligned} \max\ & c^t x \\ \text{s.t. } & Ax \leq b \\ & x \geq 0 \end{aligned}$$

- Greedy for knapsack

Decision Theory

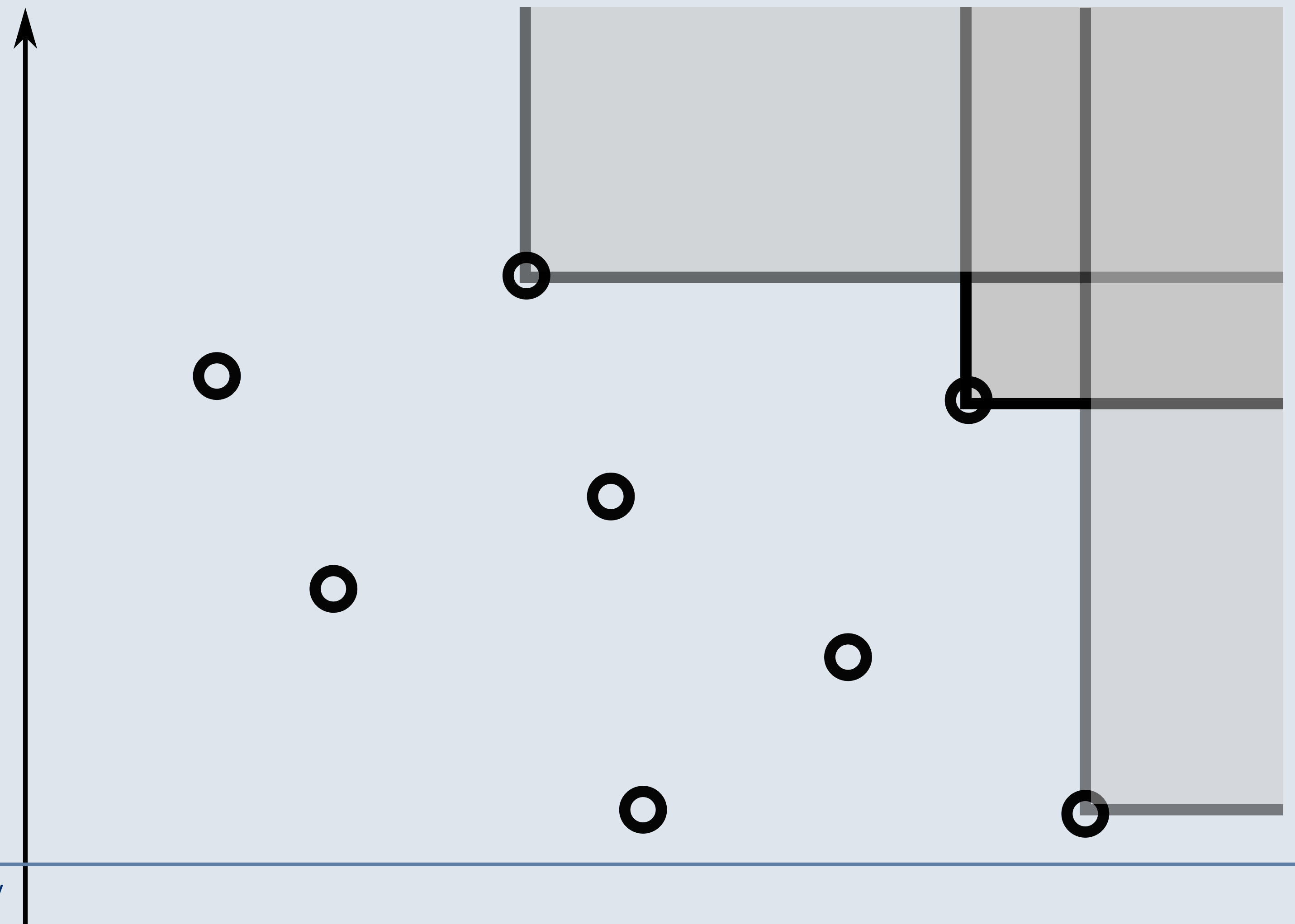
Pareto Efficiency

- Want to maximize
- For solution x, let $(e_x^1, \dots, e_x^K)$ be the K criteria values
- x is Pareto efficient if there is no y such that

$$e_y^i \geq e_x^i \text{ for all } i \in [K]$$
$$e_y^i > e_x^i \text{ for at least one } i \in [K]$$

Decision Theory

Pareto Efficiency



Decision Theory

Fantastic Solutions and How to Find Them

Multi-criteria problem:

$$
\begin{aligned}
\max\ & c^t x \\
\max\ & d^t x \\
\text{s.t. } & Ax \leq b \\
& x \geq 0
\end{aligned}
$$

Weighted sum:

$$
\begin{aligned}
\max\ & \lambda \cdot c^t x + (1-\lambda) \cdot d^t x \\
\text{s.t. } & Ax \leq b \\
& x \geq 0
\end{aligned}
$$

Decision Theory

Fantastic Solutions and How to Find Them

Multi-criteria problem:

$$\begin{aligned} \max\ & c^t x \\ \max\ & d^t x \\ \text{s.t. } & Ax \leq b \\ & x \geq 0 \end{aligned}$$

ϵ-constraint:

$$\begin{aligned} \max\ & c^t x \\ \text{s.t. } & Ax \leq b \\ & d^t x \geq \epsilon \\ & x \geq 0 \end{aligned}$$

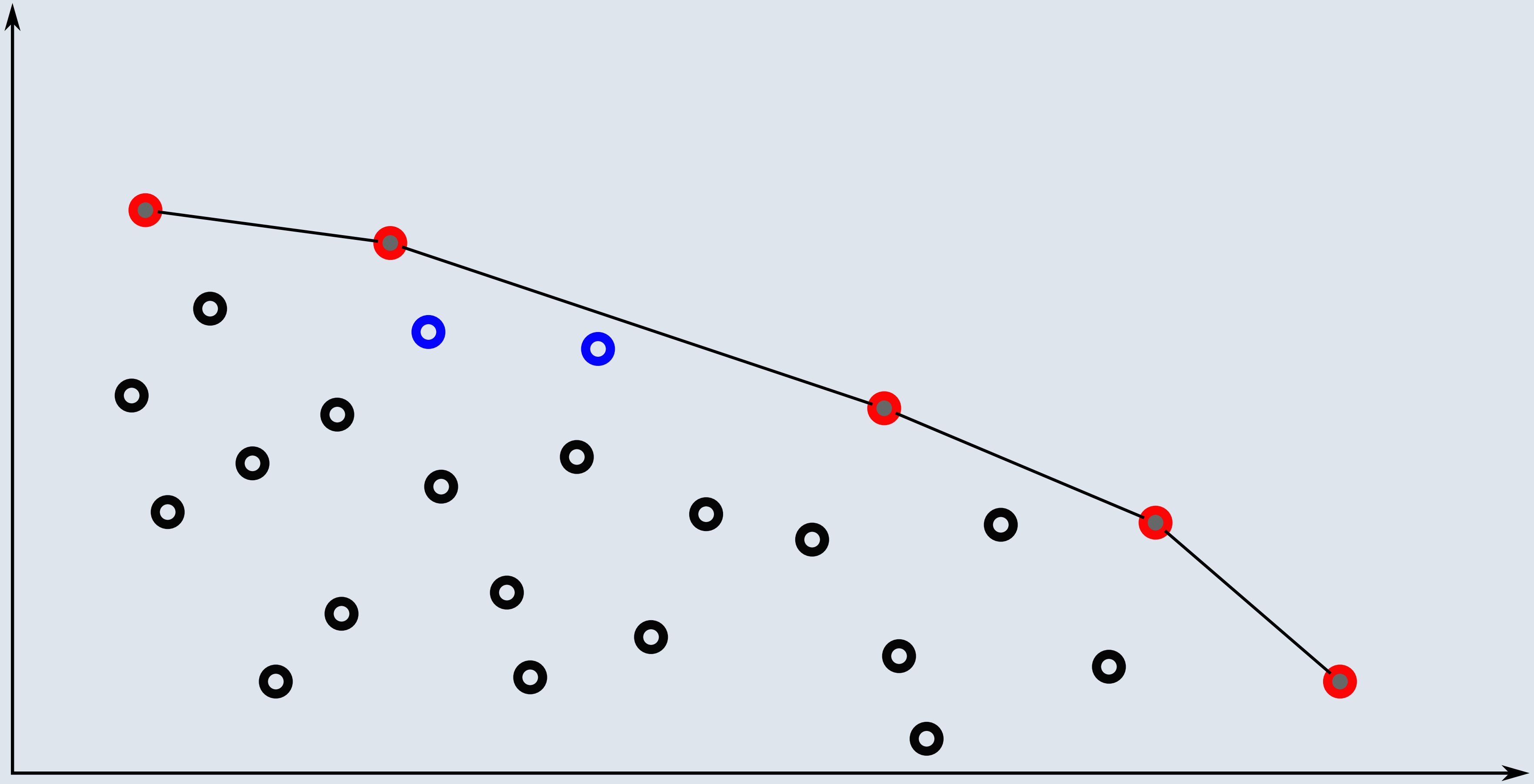
Decision Theory

Supported/Unsupported

- If a solution is Pareto efficient, you can further check If it is supported or unsupported
- Supported: on the boundary of the convex hull of points in the objective space
- Unsupported: inside the convex hull
- Weighted sum can only find supported Pareto solutions

Decision Theory

Supported/Unsupported



Data Envelopment Analysis

- Given multiple decision making units (DMUs)
- Input values
- Output values
- For example, DMU = dairy farm
  - Input 1: energy expense
  - Input 2: vet expense
  - Input 3: number of cows
  - Output 1: milk volume

Decision Theory

Setting

$$\text{efficiency} = \frac{\text{output}}{\text{input}}$$

- Evaluate a DMU relative to the other DMUs
- Not efficient, if we can combine the other DMUs such that
  - At least the same amount of output
  - Is produced with less input
- Notation:
  - $X_i = (x_{i1}, \dots, x_{iN})$ vector of inputs for DMU i
  - $Y_i = (y_{i1}, \dots, y_{iM})$ vector of outputs for DMU i
  - Efficiency of DMU i is θ_i

Decision Theory

Setting

Formulation as linear program:

$$
\begin{aligned}
\min\ & \theta_j & \\
\text{s.t.}\ & \sum_{i \in I} \lambda_i x_{ik} \le \theta_j x_{jk} & \forall k \in [N] \\
& \sum_{i \in I} \lambda_i y_{ik} \ge y_{jk} & \forall k \in [M] \\
& \theta_j \in \mathbb{R} & \\
& \lambda_i \in \mathbb{R}_+ & \forall i \in I
\end{aligned}
$$

Decision Theory

Setting

Formulation as linear program:

$$
\begin{aligned}
\min\ & \theta_j \\
\text{s.t.}\ & \sum_{i \in I} \lambda_i x_{ik} \leq \theta_j x_{jk} && \forall k \in [N] \\
& \sum_{i \in I} \lambda_i y_{ik} \geq y_{jk} && \forall k \in [M] \\
& \theta_j \in \mathbb{R} \\
& \lambda_i \in \mathbb{R}_+ && \forall i \in I
\end{aligned}
$$

for a fixed DMU j, find the worst combination of DMUs

Decision Theory

Setting

Formulation as linear program:

$$\begin{aligned}
&\min\ \theta_j \\
&\text{s.t.}\ \sum_{i \in I} \lambda_i x_{ik} \leq \theta_j x_{jk} && \forall k \in [N] \\
&\quad\ \sum_{i \in I} \lambda_i y_{ik} \geq y_{jk} && \forall k \in [M] \\
&\quad\ \theta_j \in \mathbb{R} \\
&\quad\ \lambda_i \in \mathbb{R}_+ && \forall i \in I
\end{aligned}$$

such that each output is at least as much

Decision Theory

Setting

Formulation as linear program:

$$
\begin{aligned}
\min\ & \theta_j \\
\text{s.t. } & \sum_{i \in I} \lambda_i x_{ik} \leq \theta_j x_{jk} && \forall k \in [N] \\
& \sum_{i \in I} \lambda_i y_{ik} \geq y_{jk} && \forall k \in [M] \\
& \theta_j \in \mathbb{R} \\
& \lambda_i \in \mathbb{R}_+ && \forall i \in I
\end{aligned}
$$

and each input is less

Decision Theory

Example

farm no	Input 1 energy	Input 2 vet	Input 3 cows	Output 1 milk
1	117.9	21.2	121	86.3
2	72.0	43.9	80	60.6
3	158.5	54.6	95	86.6
4	66.8	45.5	87	66.2
5	101.7	81.6	125	100.3

Decision Theory

Example

How efficient is DMU 1?

- Determine θ_1: min θ_1
- Find combination of all DMUs: $\lambda_1, \ldots, \lambda_5$
- Such that the combination is less with respect to energy

$$117.9\lambda_1 + 72.0\lambda_2 + 158.5\lambda_3 + 66.8\lambda_4 + 101.7\lambda_5 \leq \theta_1 117.9$$

- In the same way better with respect to vet and cows
- And better (more) with respect to output milk

$$86.3\lambda_1 + 60.6\lambda_2 + 86.6\lambda_3 + 66.2\lambda_4 + 100.3\lambda_5 \geq 86.3$$

Decision Theory

Efficiency Score

- Solving the linear program for one DMU, we get a value θ_j
- Value cannot be larger than 1
- Value = 1: DMU is efficient
- Value $<$ 1: DMU is not efficient, and we get a set of comparison DMUs (where $\lambda > 0$)

Alternative Model

- Choose (imaginary) price u_k for each input $k \in [N]$, and v_k for each output $k \in [M]$
- Efficiency of DMU i is

$$\text{efficiency} = \frac{\text{output}}{\text{input}} = \frac{v^t Y_j}{u^t X_j}$$

- Is there any set of prices such that DMU i looks good?
- Under the constraint that all efficiencies are ≤ 1

Alternative Model

$$
\begin{aligned}
\max \quad & \sum_{k\in[M]} v_k y_{jk} \\
\text{s.t.} \quad & \sum_{k\in[N]} u_k x_{jk} = 1 \\
& \sum_{k\in[M]} v_k y_{ik} - \sum_{k\in[N]} u_k x_{ik} \leq 0 && \forall i \in I \\
& u \in \mathbb{R}_+^N, v \in \mathbb{R}_+^M
\end{aligned}
$$

Decision Theory

Alternative Model

$$
\begin{aligned}
\max \ & \sum_{k\in[M]} v_k y_{jk} \\
\text{s.t.} \ & \sum_{k\in[N]} u_k x_{jk} = 1 \\
& \sum_{k\in[M]} v_k y_{ik} - \sum_{k\in[N]} u_k x_{ik} \leq 0 && \forall i \in I \\
& u \in \mathbb{R}^N_+, v \in \mathbb{R}^M_+
\end{aligned}
$$

chose prices u_k, v_k

Decision Theory

Alternative Model

$$
\begin{aligned}
\max \quad & \sum_{k\in[M]} v_k y_{jk} \\
\text{s.t.} \quad & \sum_{k\in[N]} u_k x_{jk} = 1 \\
& \sum_{k\in[M]} v_k y_{ik} - \sum_{k\in[N]} u_k x_{ik} \leq 0 && \forall i \in I \\
& u \in \mathbb{R}_+^N, v \in \mathbb{R}_+^M
\end{aligned}
$$

that maximize the efficiency of DMU j

Decision Theory

Alternative Model

$$
\begin{aligned}
\max \quad & \sum_{k\in[M]} v_k y_{jk} \\
\text{s.t.} \quad & \sum_{k\in[N]} u_k x_{jk} = 1 \\
& \sum_{k\in[M]} v_k y_{ik} - \sum_{k\in[N]} u_k x_{ik} \leq 0 && \forall i \in I \\
& u \in \mathbb{R}_+^N,\ v \in \mathbb{R}_+^M
\end{aligned}
$$

while all efficiencies are ≤ 1

Decision Theory

Alternative Model

- Objective value of first and second model are the same
- Can be seen through duality
- We are free to choose which model we prefer

Decision Theory

DEA

- Each DMU is a black box, need to know nothing but input and output
- Very flexible, general, can be applied to a wide range of problems
- Popular in the literature
- Need to determine what value is input, what is output

Decision Theory

Today

- Analytic Hierarchy Process (AHP)
- Multi-Attribute Utility Theory (MAUT)

Decision Theory

AHP

- Analytic Hierarchy Process (AHP)
- Alternatives are given again
  - E.g., where to go on vacation?
  - a_1 =Rome
  - a_2 =Barcelona
  - a_3 =Reykjavik

Decision Theory

AHP

- Goal: achieve a normalized vector indicating the quality of alternatives
  - e.g., $w = (6/9, 2/9, 1/9)$
- What we do: pairwise comparisons in a matrix

$$R = \begin{pmatrix} r_{11} & \dots & r_{1n} \\ \vdots & \vdots & \vdots \\ r_{n1} & \dots & r_{nn} \end{pmatrix}$$

- $r_{ij} > 0$ indicates the preference of a_i over a_j

Decision Theory

AHP – Example

- Rome is "3 times better" than Barcelona
- Rome is "6 times better" than Reykjavik
- Barcelona is "2 times better" than Reykjavik
- Results in:

$$R = \begin{pmatrix} 1 & 3 & 6 \\ 1/3 & 1 & 2 \\ 1/6 & 1/2 & 1 \end{pmatrix}$$

Decision Theory

AHP

- Typically:

$$R = \begin{pmatrix} 1 & r_{12} & \dots & r_{1n} \\ 1/r_{12} & 1 & \dots & r_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 1/r_{1n} & 1/r_{2n} & \dots & 1 \end{pmatrix}$$

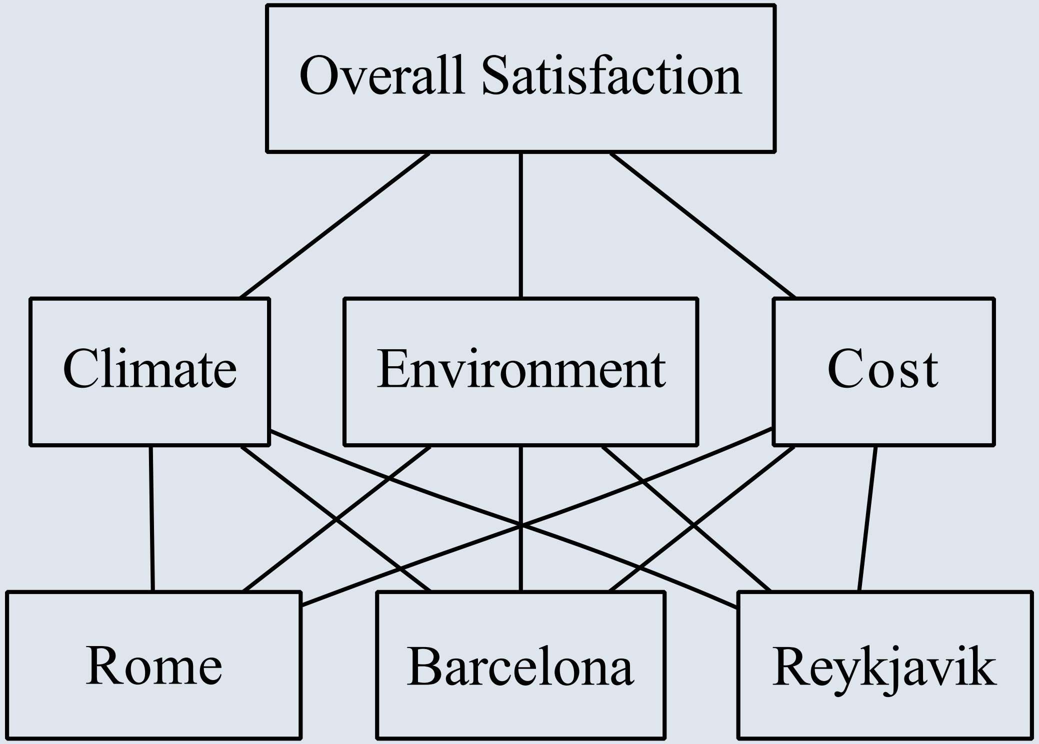
- If the matrix is derived from a preference vector, then

$$r_{ij} \approx \frac{w_i}{w_j}$$

Decision Theory

Multiple Criteria

- What if there are multiple criteria?
- Example: Climate, Environmental Friendliness, Costs
- ”Hierarchy” in AHP through relations between
  - Overall goal
  - Criteria
  - Alternatives



Decision Theory

Multiple Criteria

- Build matrix R^k for each criterion k
- Example: three matrices

$$R^c = \begin{pmatrix} 1 & 1 & 4 \\ 1 & 1 & 4 \\ 1/4 & 1/4 & 1 \end{pmatrix} \text{ (climate)}$$

$$R^e = \begin{pmatrix} 1 & 1/2 & 1/8 \\ 2 & 1 & 1/4 \\ 8 & 4 & 1 \end{pmatrix} \text{ (environment)}$$

$$R^s = \begin{pmatrix} 1 & 2 & 6 \\ 1/2 & 1 & 3 \\ 1/6 & 1/3 & 1 \end{pmatrix} \text{ (cost)}$$

Decision Theory

Weight Vectors

- Assuming, based on matrices, we can estimate three weight vectors
- Normalized: $\sum_{i=1}^{n} w_i = 1$
- Recall, we want: $r_{ij} \approx w_i / w_j$

$$w^c = \begin{pmatrix} 4/9 \\ 4/9 \\ 1/9 \end{pmatrix}, w^e = \begin{pmatrix} 1/11 \\ 2/11 \\ 8/11 \end{pmatrix}, w^s = \begin{pmatrix} 6/10 \\ 3/10 \\ 1/10 \end{pmatrix}$$

Decision Theory

Weighting

- Now, we want to find a compromise solution
- Average of criteria not meaningful
- Approach: again a comparison matrix, this time between the criteria
- ”How much more important is climate than the environment?”
- Example:

$$\hat{R} = \begin{pmatrix} 1 & 1/2 & 1/4 \\ 2 & 1 & 1/2 \\ 4 & 2 & 1 \end{pmatrix}$$

Decision Theory

Weighting

- Assuming, we can build a weight vector again from $\hat{R}$:

$$\hat{w} = \begin{pmatrix} 1/7 \\ 2/7 \\ 4/7 \end{pmatrix}$$

- Use this to weight the three criteria vectors:

$$\begin{aligned} w &= \hat{w}_1 w^c + \hat{w}_3 w^e + \hat{w}_2 w^s \\ &= \frac{1}{7}\begin{pmatrix} 4/9 \\ 4/9 \\ 1/9 \end{pmatrix} + \frac{4}{7}\begin{pmatrix} 1/11 \\ 2/11 \\ 8/11 \end{pmatrix} + \frac{2}{7}\begin{pmatrix} 6/10 \\ 3/10 \\ 1/10 \end{pmatrix} \approx \begin{pmatrix} 0.287 \\ 0.253 \\ 0.460 \end{pmatrix} \end{aligned}$$

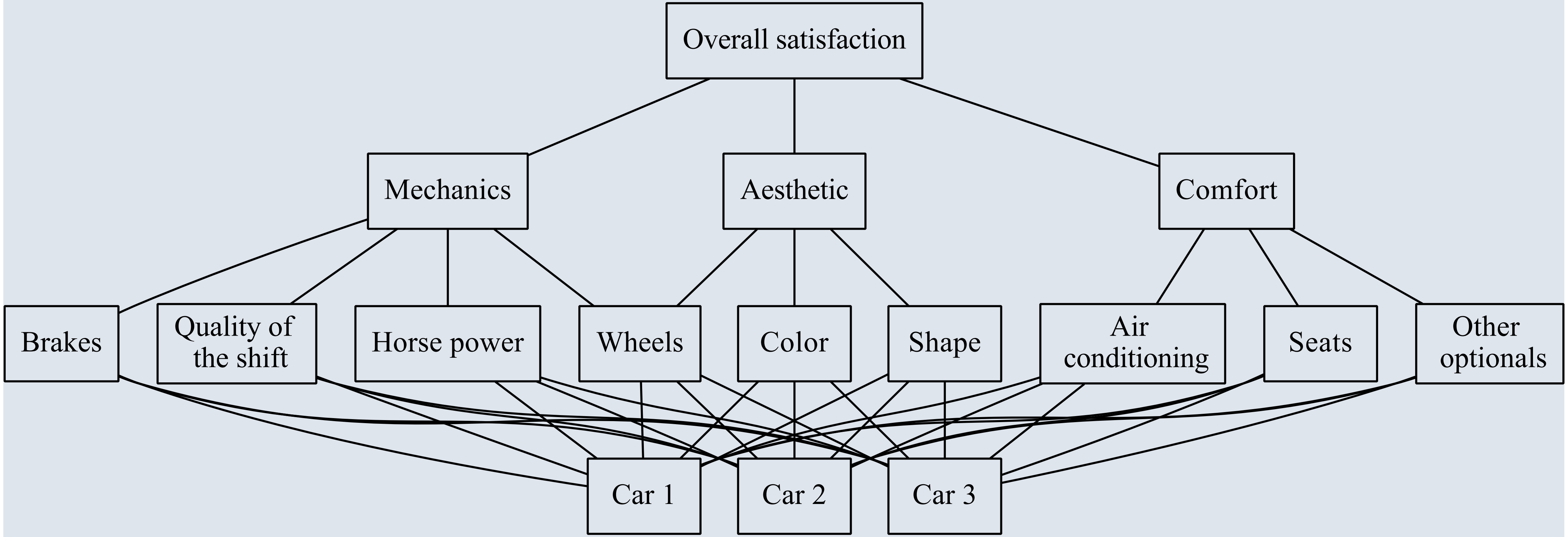
- Answer: Reykjavik!

Decision Theory

Summary AHP

1. Structure the problem, define hierarchy
2. Pairwise comparisons
3. Calculate weight vectors
4. Choose alternative with the best weighting

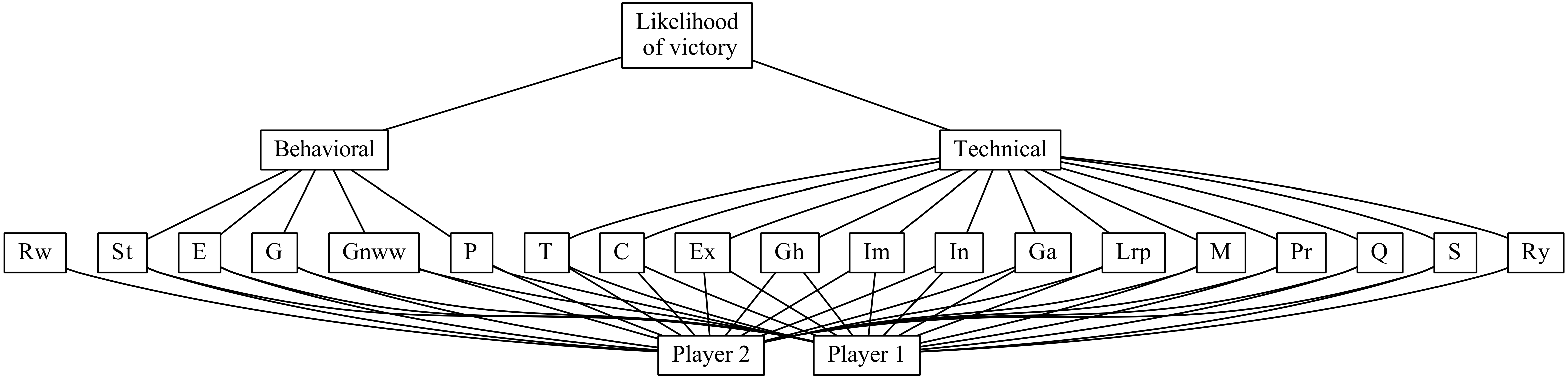
Application: Car Purchase



Decision Theory

Applications

- ”Livability index” of cities
- As a precursor to the Human Development Index (instead of just GDP)
- Allocation of organ donations
- Prediction of wins in a chess tournament



Decision Theory

Drawbacks

- "Rank reversal": adding a new alternative changes the evaluations of other alternatives
- How to say ”Barcelona is 4 times better than Reykjavik”?
  - E.g., set a scale

Verbal description	Saaty’s scale	Balanced scale
Indifference	1	1
—	2	1.22
Moderate preference	3	1.5
—	4	1.86
Strong preference	5	2.33
—	6	3
Very strong or demonstrated preference	7	4
—	8	5.67
Extreme preference	9	9

Decision Theory

Weight Vectors

- Still missing: Given comparison matrix, how do I determine appropriate weight vectors?
- Want to find w so that $r_{ij} \approx w_i / w_j$
  - Find eigenvector
  - Calculate geometric mean
  - Method of least squares

Decision Theory

Eigenvector

$$Rw = \begin{pmatrix} w_1/w_1 & w_1/w_2 & \dots & w_1/w_n \\ w_2/w_1 & w_2/w_2 & \dots & w_2/w_n \\ \vdots & \vdots & \vdots & \vdots \\ w_n/w_1 & w_n/w_2 & \dots & w_n/w_n \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} nw_1 \\ \vdots \\ nw_n \end{pmatrix} = nw$$

- A vector w such that $Rw = \lambda w$ is called an eigenvector
- λ is called an eigenvalue
- Easily determined using mathematical software
- Choose eigenvector corresponding to the largest eigenvalue

Decision Theory

Geometric Mean

$$w_i = \frac{\left(\prod_{j=1}^{n} r_{ij}\right)^{1/n}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} r_{ij}\right)^{1/n}}$$

- Advantage: closed formula
- If there exists w with $r_{ij} = w_i/w_j$, it will be found

Decision Theory

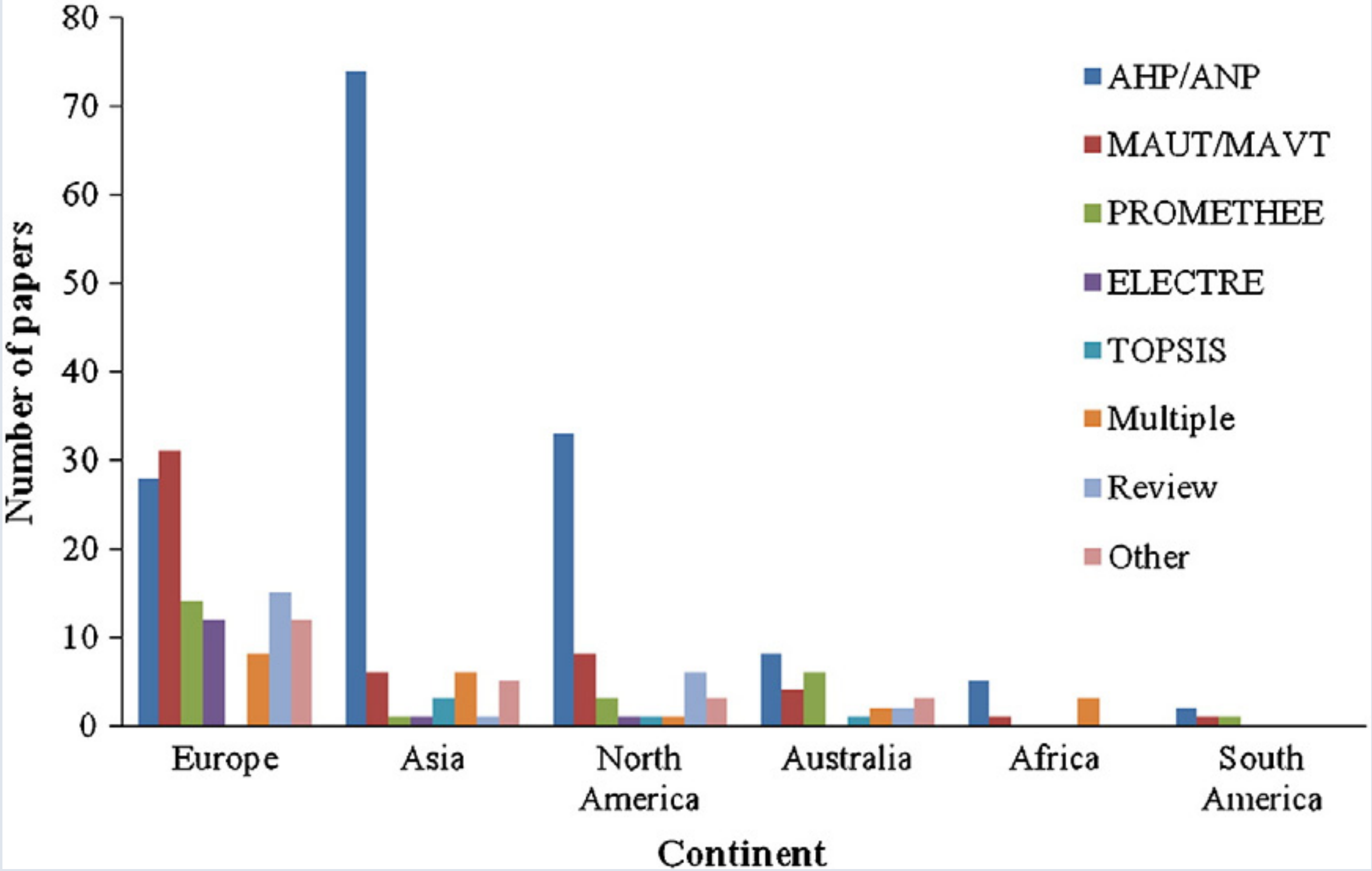
Method of Least Squares

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} \left(r_{ij} - \frac{w_i}{w_j} \right)^2$$

$$\text{s.t.} \quad \sum_{i=1}^{n} w_i = 1$$

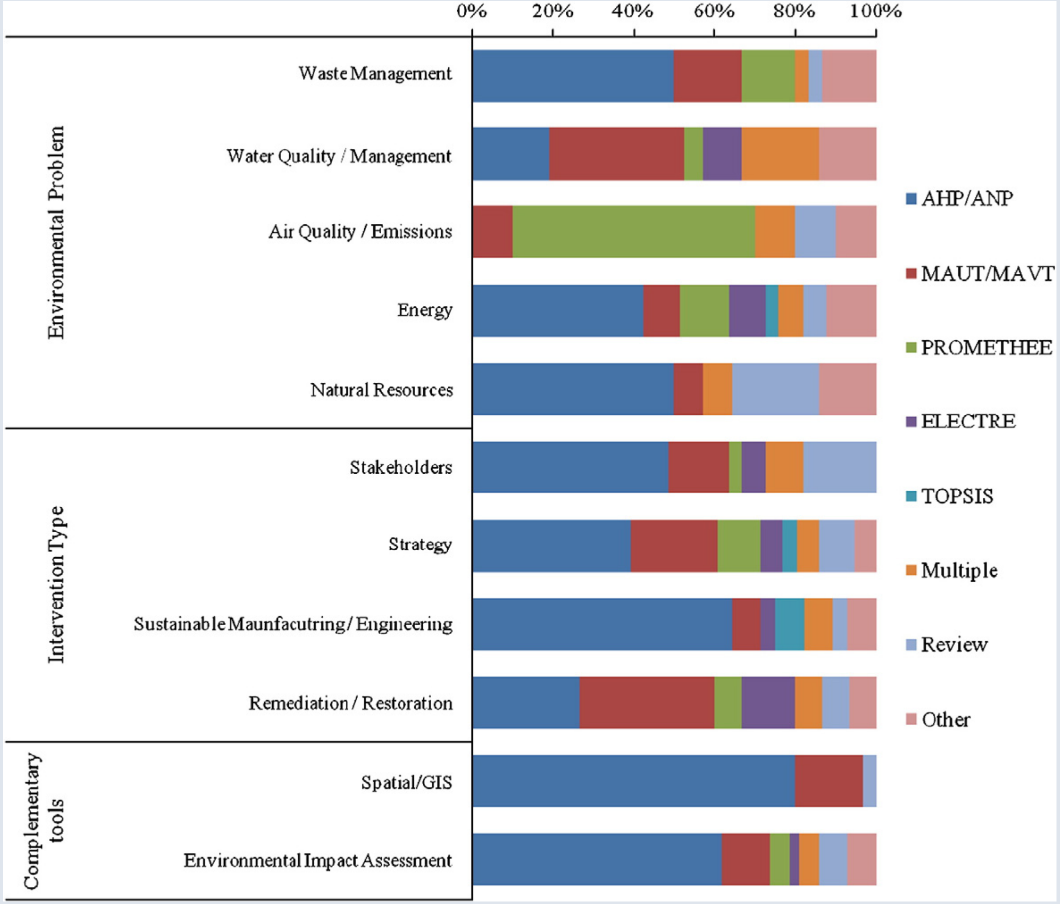
Example: Choice of Transportation

Using AHP


80
70
60
50
40
30
20
10
0
Number of papers
AHP/ANP
MAUT/MAVT
PROMETHEE
ELECTRE
TOPSIS
Multiple
Review
Other
Europe
Asia
North America
Australia
Africa
South America
Continent

Decision Theory

Using AHP



Decision Theory

MAUT

- Now: MAUT (multiattributive utility theory)

MAUT Setting

- Similar to AHP: multiple alternatives and criteria, how to weigh them?
- In AHP: pairwise comparisons
- Determine weights λ_k and functions u_k for

$$U(a_i) = \sum_{k \in [K]} \lambda_k u_k(e_i^k)$$

- u is a utility function that should be consistent with preference relations
- AHP is a special case

Decision Theory

Example

- Mr. Müller wants to go on vacation
- Vacation in Calabria, Mallorca, or Sylt?
- He asks for our help

Decision Theory

Example

- Mr. Müller wants to go on vacation
- Vacation in Calabria, Mallorca, or Sylt?
- He asks for our help

Calabria				
Mallorca				
Sylt				

Decision Theory

Example

- Mr. Müller wants to go on vacation
- Vacation in Calabria, Mallorca, or Sylt?
- He asks for our help

	Air Temp.	Water Temp.	Cost	
Calabria				
Mallorca				
Sylt				

Decision Theory

Example

- Mr. Müller wants to go on vacation
- Vacation in Calabria, Mallorca, or Sylt?
- He asks for our help

	Air Temp.	Water Temp.	Cost	
Calabria	27	25	1800	
Mallorca	25	23	2000	
Sylt	15	13	1500	

Decision Theory

Example

- Mr. Müller wants to go on vacation
- Vacation in Calabria, Mallorca, or Sylt?
- He asks for our help

	Air Temp.	Utility	Water Temp.	Utility	Cost	Utility	
Calabria	27		25		1800		
Mallorca	25		23		2000		
Sylt	15		13		1500		

Decision Theory

Example

- Mr. Müller wants to go on vacation
- Vacation in Calabria, Mallorca, or Sylt?
- He asks for our help

	Air Temp.	Utility	Water Temp.	Utility	Cost	Utility	
Calabria	27	1.0	25	1.0	1800	0.7	
Mallorca	25	0.8	23	0.9	2000	0.3	
Sylt	15	0.5	13	0.5	1500	1.0	

Decision Theory

Example

- Mr. Müller wants to go on vacation
- Vacation in Calabria, Mallorca, or Sylt?
- He asks for our help

	Air Temp.	Utility	Water Temp.	Utility	Cost	Utility	
Calabria	27	1.0	25	1.0	1800	0.7	
Mallorca	25	0.8	23	0.9	2000	0.3	
Sylt	15	0.5	13	0.5	1500	1.0	
Importance							

Decision Theory

Example

- Mr. Müller wants to go on vacation
- Vacation in Calabria, Mallorca, or Sylt?
- He asks for our help

	Air Temp.	Utility	Water Temp.	Utility	Cost	Utility
Calabria	27	1.0	25	1.0	1800	0.7
Mallorca	25	0.8	23	0.9	2000	0.3
Sylt	15	0.5	13	0.5	1500	1.0
Importance		0.3		0.5		0.2

Decision Theory

Example

- Mr. Müller wants to go on vacation
- Vacation in Calabria, Mallorca, or Sylt?
- He asks for our help

	Air Temp.	Utility	Water Temp.	Utility	Cost	Utility	Overall
Calabria	27	1.0	25	1.0	1800	0.7	
Mallorca	25	0.8	23	0.9	2000	0.3	
Sylt	15	0.5	13	0.5	1500	1.0	
Importance		0.3		0.5		0.2	

Decision Theory

Example

- Mr. Müller wants to go on vacation
- Vacation in Calabria, Mallorca, or Sylt?
- He asks for our help

	Air Temp.	Utility	Water Temp.	Utility	Cost	Utility	Overall
Calabria	27	1.0	25	1.0	1800	0.7	0.94
Mallorca	25	0.8	23	0.9	2000	0.3	0.75
Sylt	15	0.5	13	0.5	1500	1.0	0.60
Importance		0.3		0.5		0.2	

Decision Theory

MAUT Workflow

- Decision-maker defines goals (goal hierarchy)
- Decision-maker determines alternatives
- Analyst queries decision-maker for value preferences u_k
  - Introduce methods
  - AHP: pairwise comparisons
- Analyst queries decision-maker for goal weights λ_k
  - Introduce methods
  - AHP: pairwise comparisons
- Determination of overall utility
- Sensitivity analysis

Value Preferences u_k: Direct Rating

- Set the result of the worst alternative to 0
- Set the result of the best alternative to 100
- Obtain scores from the decision-maker
- Normalize all values to $[0, 1]$
- Normalization allows us to determine goal weights sensibly

Decision Theory

Example

Water Temp.	13	16.5	20	23.5	27
Points	0				100

Decision Theory

Example

Water Temp.	13	16.5	20	23.5	27
Points	0	60			100

Decision Theory

Example

Water Temp.	13	16.5	20	23.5	27
Points	0	60	90		100

Decision Theory

Example

Water Temp.	13	16.5	20	23.5	27
Points	0	60	90	95	100

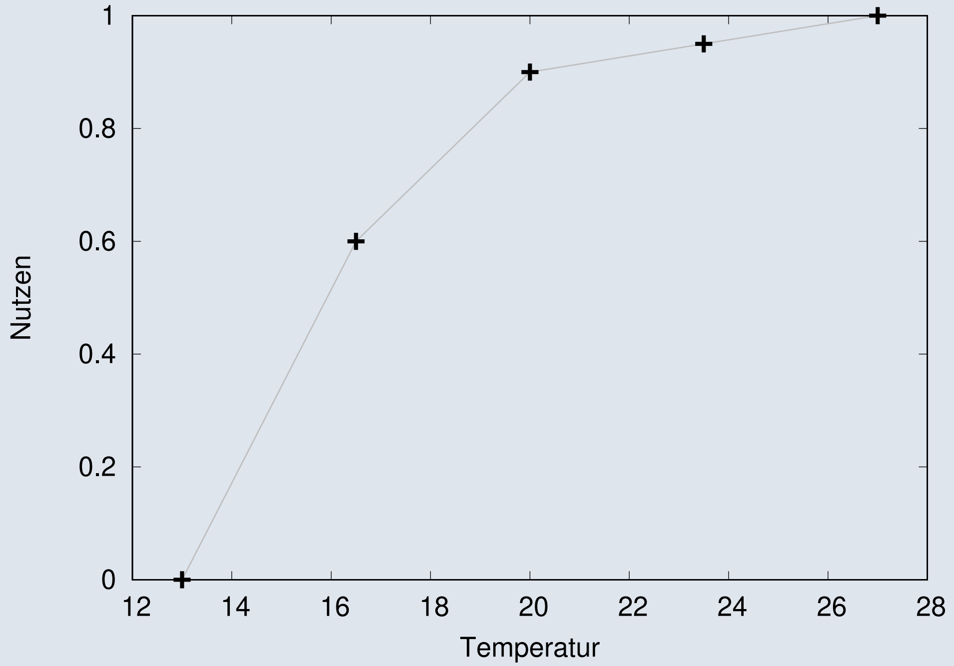
Decision Theory

Example

Water Temp.	13	16.5	20	23.5	27
Points	0	60	90	95	100
Normalized	0.00	0.60	0.90	0.95	1.00

Example

Water Temp.	13	16.5	20	23.5	27
Points	0	60	90	95	100
Normalized	0.00	0.60	0.90	0.95	1.00



Decision Theory

Value Preferences u_k: Direct Rating

- We practically provide no support
- Solution "for emergencies"
- Helpful when consequences are not measurable
- Consistency check:
  - Are preferences correctly reflected?
  - If a_1 is better than a_2, then a better score for a_1.
  - Are differences correctly reflected?
  - For $u(4) = 0$, $u(5) = 0.6$, and $u(6) = 1$, is it true that the utility difference between 4 and 5 is greater than between 5 and 6?

Value Preferences u_k: Bisection Method

- Write x_i^v for the manifestation of goal i with $u_i(x_i^v) = v$
- Set utility of the worst result to 0: $u_i(x_i^0) = 0$
- Set utility of the best result to 1: $u_i(x_i^1) = 1$
- Query the utility median $x_i^{0.5}$, i.e., the value z such that

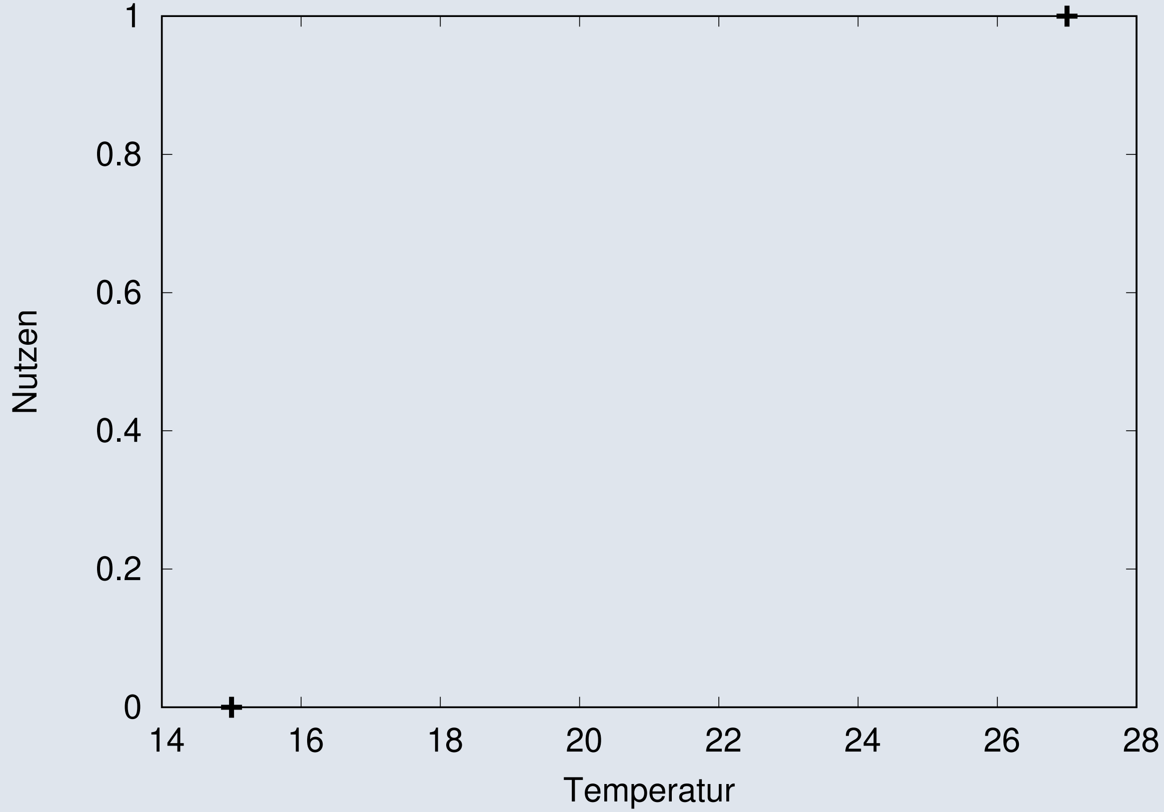
$$u_i(z) - u_i(x_i^0) = u_i(x_i^1) - u_i(z)$$

- Successively halve all subintervals until the desired accuracy is reached

Decision Theory

Example

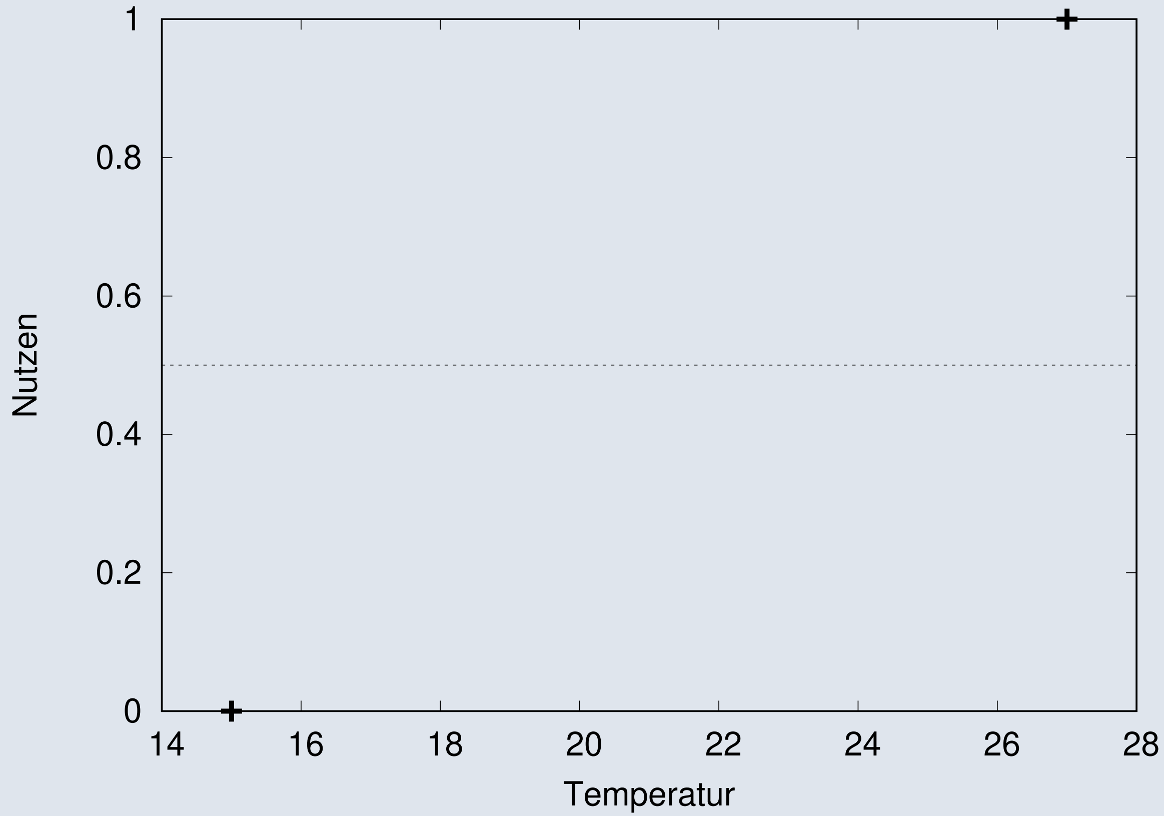
	x_i^0				x_i^1
Utility	0	-	-	-	1
Air Temp.	15	-	-	-	27



Decision Theory

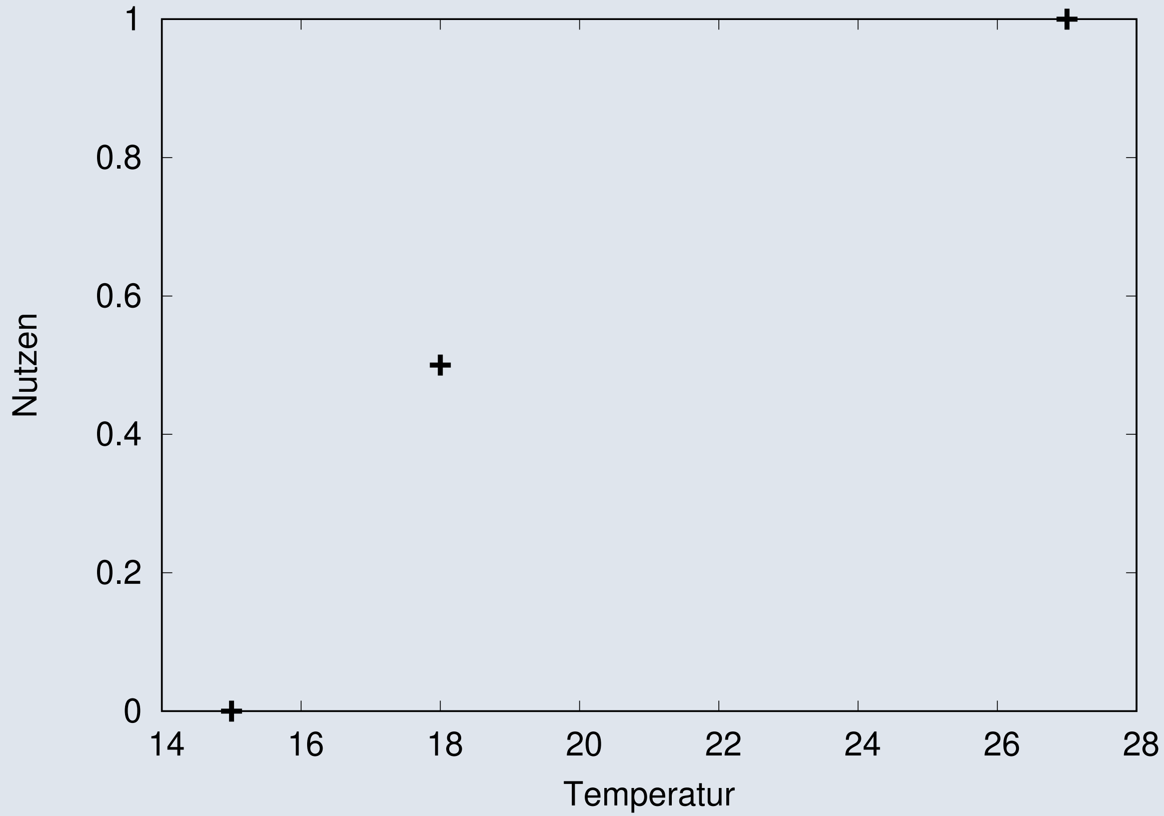
Example

	x_i^0		$x_i^{0.5}$		x_i^1
Utility	0	-	0.5	-	1
Air Temp.	15	-	?	-	27



Example

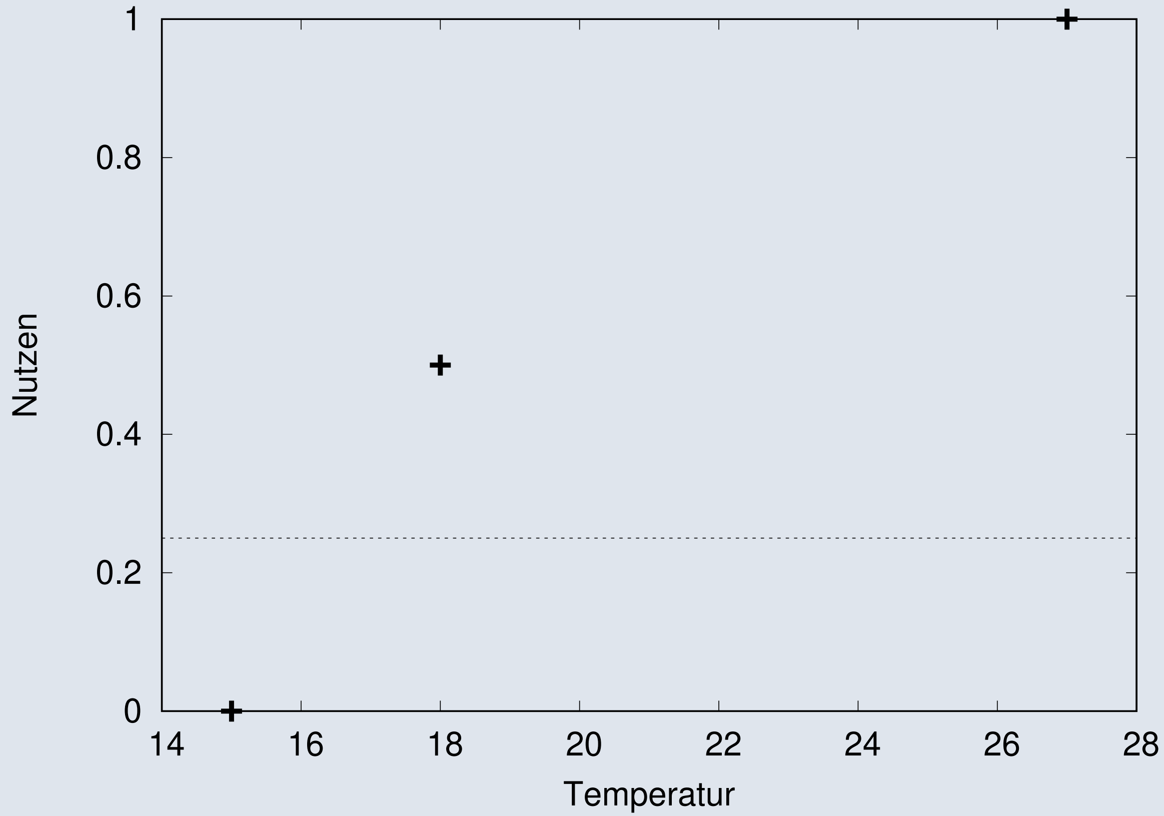
	x_i^0		$x_i^{0.5}$		x_i^1
Utility	0	-	0.5	-	1
Air Temp.	15	-	18	-	27



Decision Theory

Example

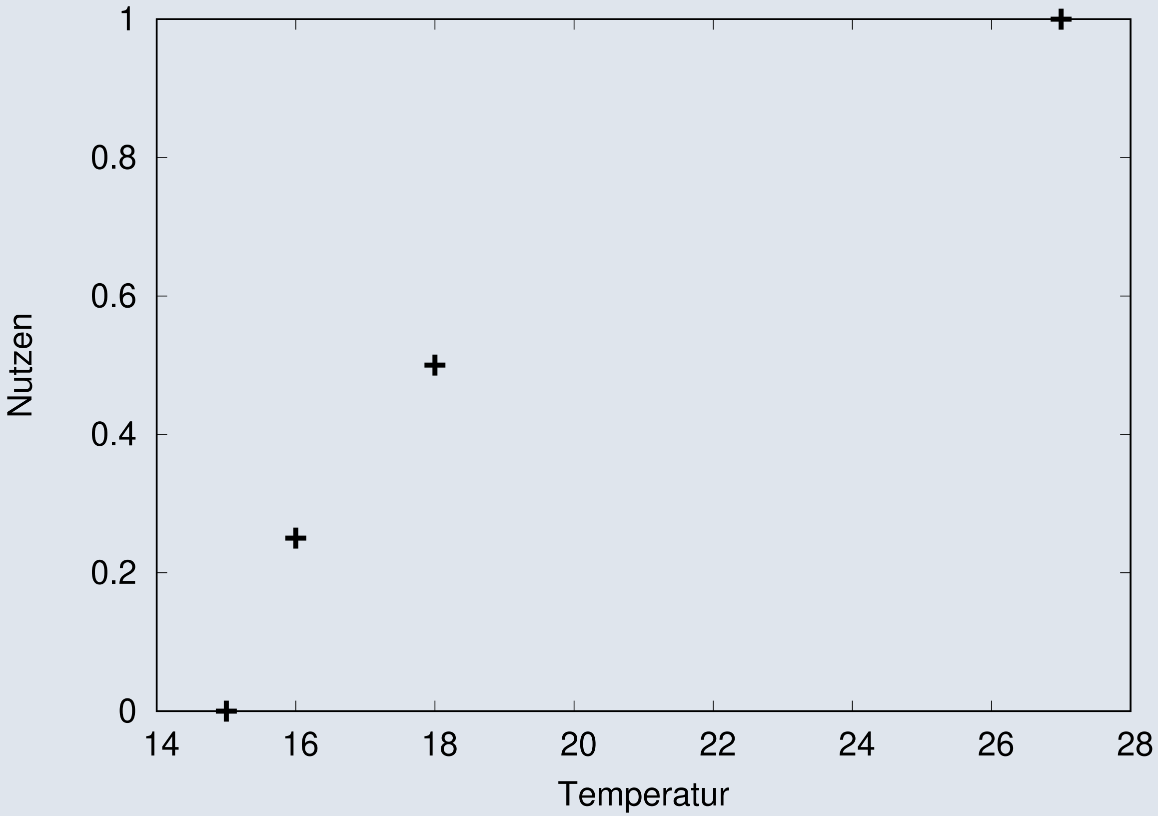
	x_i^0	$x_i^{0.25}$	$x_i^{0.5}$		x_i^1
Utility	0	0.25	0.5	-	1
Air Temp.	15	?	18	-	27



Decision Theory

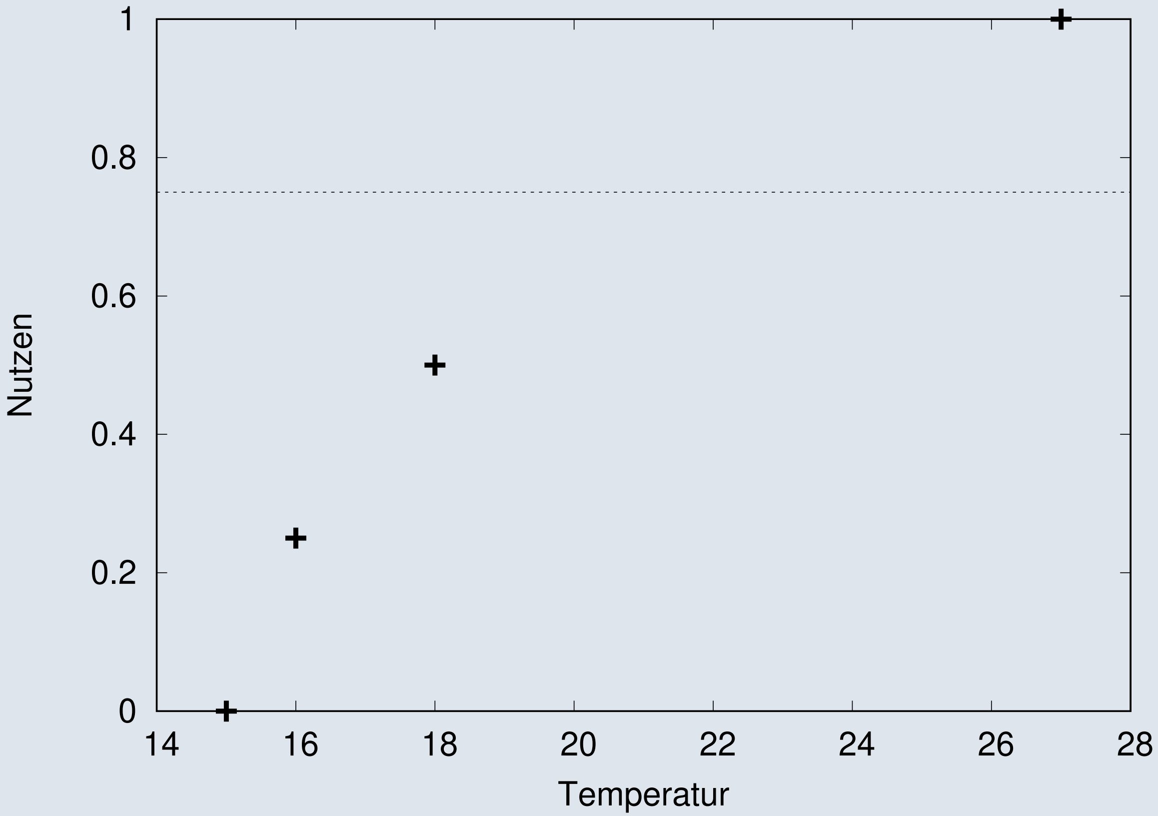
Example

	x_i^0	$x_i^{0.25}$	$x_i^{0.5}$	$x_i^{0.75}$	x_i^1
Utility	0	0.25	0.5	-	1
Air Temp.	15	16	18	-	27



Example

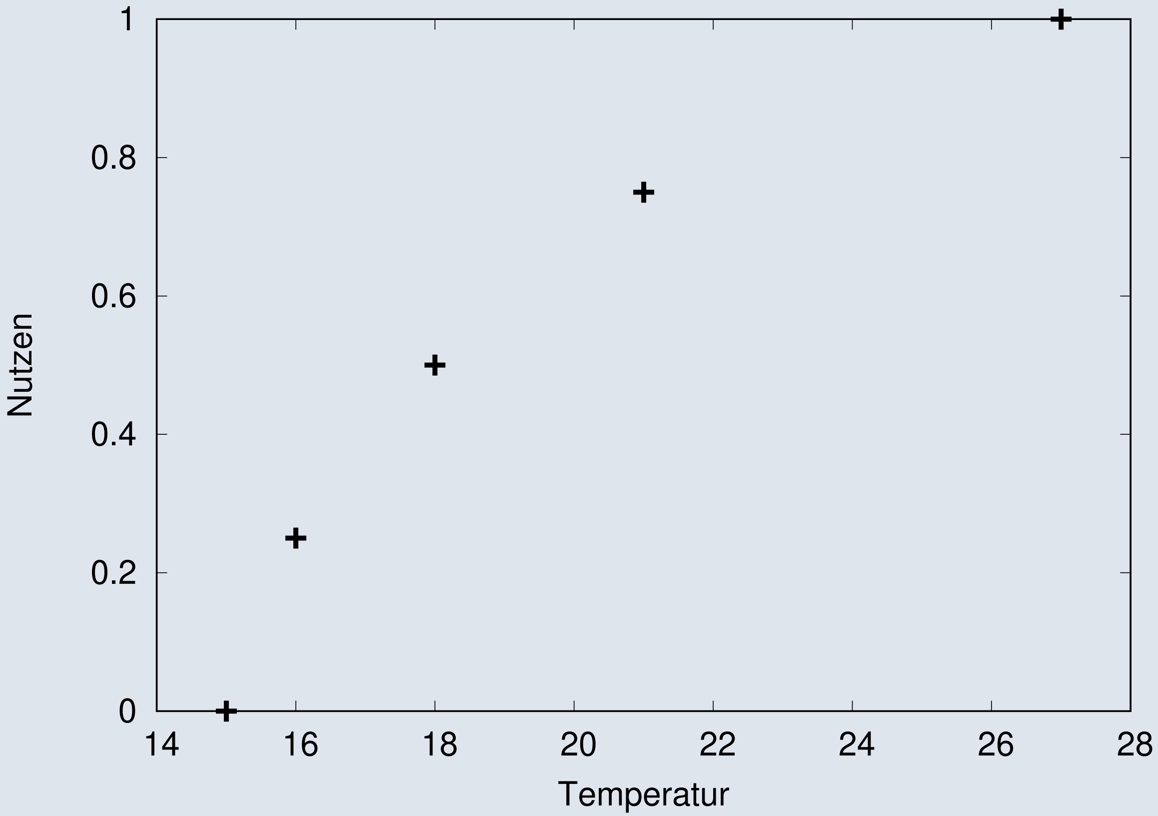
	x_i^0	$x_i^{0.25}$	$x_i^{0.5}$	$x_i^{0.75}$	x_i^1
Utility	0	0.25	0.5	0.75	1
Air Temp.	15	16	18	?	27



Decision Theory

Example

	x_i^0	$x_i^{0.25}$	$x_i^{0.5}$	$x_i^{0.75}$	x_i^1
Utility	0	0.25	0.5	0.75	1
Air Temp.	15	16	18	21	27



Decision Theory

Interpolation

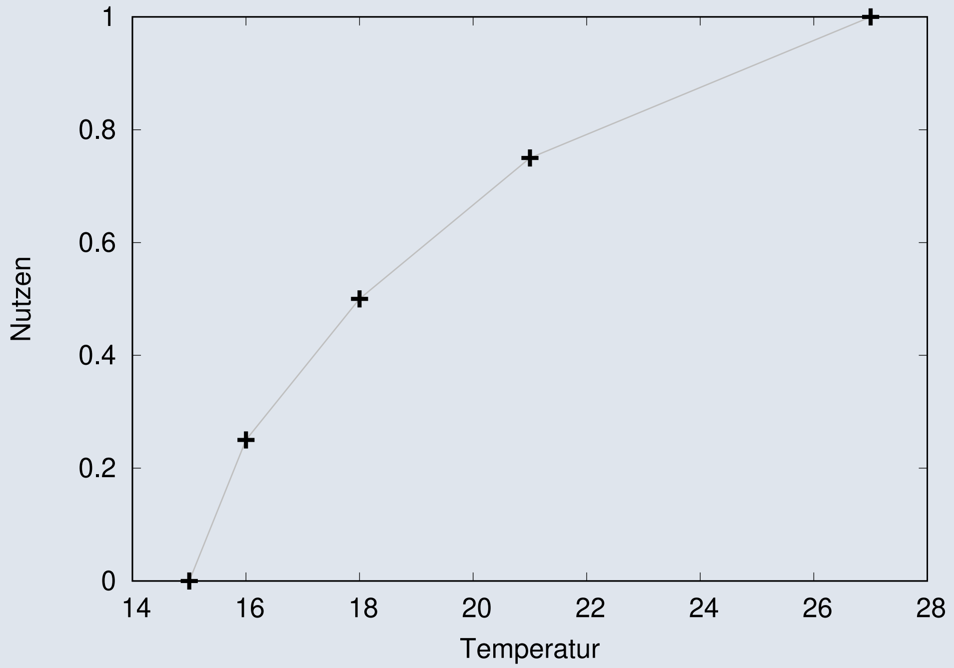
- Given data points (e.g., $(x_1, u(x_1)), \ldots, (x_k, u(x_k))$)
- Find function $u(x)$ that:
  - Satisfies the given data points
  - Provides a mathematical relationship for calculating intermediate values
- Linear interpolation: connect points with straight line segments
- For neighboring points $(x_1, u(x_1))$ and $(x_2, u(x_2))$, for all $x \in [x_1, x_2]$:

$$u(x) = u(x_1) + \frac{u(x_2) - u(x_1)}{x_2 - x_1} \cdot (x - x_1)$$

Decision Theory

Interpolation – Example

	x_i^0	$x_i^{0.25}$	$x_i^{0.5}$	$x_i^{0.75}$	x_i^1
Utility	0	0.25	0.5	0.75	1
Air Temp.	15	16	18	21	27



Utility for 20 degrees:

Goal Weights λ_k: Tradeoff Method

- Normalized utility functions already found
- Rank goals so that the transition from x_1^0 to x_1^1 is most important
- For all other goals $k = 2, \ldots, K$:
  - What manifestation $\overline{x}_k$ must goal 1 have for the combination $(\overline{x}_k, x_k^0)$ to be as valuable as (x_1^0, x_k^1)
  - In this case:
$$\lambda_1 u_1(\overline{x}_k) + \lambda_k u_k(x_k^0) = \lambda_1 u_1(x_1^0) + \lambda_k u_k(x_k^1)$$
  - So:
$$\lambda_1 u_1(\overline{x}_k) + \lambda_k \cdot 0 = \lambda_1 \cdot 0 + \lambda_k \cdot 1$$
  - Thus:
$$\lambda_k = u_1(\overline{x}_k)\lambda_1$$

Decision Theory

Goal Weights λ_k: Tradeoff Method

- Solve the $K \times K$ system of equations

$$\lambda_k = u_1(\overline{x}_k)\lambda_1 \qquad \forall k = 2, \dots, K$$
$$\sum_{k \in [K]} \lambda_k = 1$$

Decision Theory

Example

- Air temperature: $x_1^0 = 15$, $x_1^1 = 27$
- Water temperature: $x_2^0 = 13$, $x_2^1 = 27$
- Decision-maker finds the transition from 15 to 27 degrees Celsius in air temperature more important than from 13 to 27 degrees Celsius in water temperature
- Correct order already established
- Question: How warm must $\overline{x}_2$ be for the following alternatives to be equally good:
  - Air $\overline{x}_2$, Water 13 degrees
  - Air 15, Water 27 degrees
- Answer: 21 degrees

Decision Theory

Example

- $\overline{x}_2 = 21$
- From utility functions, $u_1(\overline{x}_2) = 0.75$
- Thus, the system of equations:

$$
\begin{aligned}
\lambda_2 &= 0.75\lambda_1 \\
\lambda_1 + \lambda_2 &= 1
\end{aligned}
$$

- Solution: $\lambda_1 = 4/7$, $\lambda_2 = 3/7$

Decision Theory

Example: MAUT Process

	Water Temp.		Air Temp.		Overall Utility
	℃	Utility	℃	Utility	
a_1	13.0	0.00	27.0	1.00	0.43
a_2	16.5	0.60	26.0	0.96	0.75
a_3	20.0	0.90	20.0	0.67	0.80
a_4	23.5	0.95	18.0	0.50	0.76
a_5	27.0	1.00	15.0	0.00	0.57
Importance	0.57		0.43		

Decision Theory

Bandwidth Effect

We want to buy a child car seat:

	Safety		Cost	
	Points	Utility	Euro	Utility
a_1	100	1.0	1200	0.5
a_2	90	0.9	800	1.0
a_3	0	0.0	1600	0.0

Decision Theory

Bandwidth Effect

We want to buy a child car seat:

	Safety		Cost	
	Points	Utility	Euro	Utility
a_1	100	1.0	1200	0.5
a_2	90	0.9	800	1.0
a_3	0	0.0	1600	0.0

- Alternative a_3 is uninteresting
- Can we simply remove it?

Decision Theory

Bandwidth Effect

We want to buy a child car seat:

	Safety		Cost	
	Points	Utility	Euro	Utility
a_1	100	1.0	1200	0.5
a_2	90	0.9	800	1.0

- Alternative a_3 is uninteresting
- Can we simply remove it?

Decision Theory

Bandwidth Effect

We want to buy a child car seat:

	Safety		Cost	
	Points	Utility	Euro	Utility
a_1	100	1.0	1200	0.5
a_2	90	0.9	800	1.0

- Alternative a_3 is uninteresting
- Can we simply remove it?
- Must adjust utility function!

Bandwidth Effect

- When an additional alternative is added (or removed), causing the utility bandwidth to change, we must:
  - Determine new utility functions
  - Determine new goal weights
- When a new goal is added, we must:
  - Determine new goal weights

Quiz

Question 1

Is there a precisely fitting weight vector w?

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 3/2 \\ 1/2 & 2/3 & 1 \end{pmatrix}$$

Question 2

You are looking for a place to live and consider the time it takes to commute. The following alternatives are available:

	Time (h)
a_1	$1/4$
a_2	$1/3$
a_3	1
a_4	2

Determine a utility function using the bisection method.

Quiz

Question 1

Is there a precisely fitting weight vector w?

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 3/2 \\ 1/2 & 2/3 & 1 \end{pmatrix}$$

No.

Question 2

You are looking for a place to live and consider the time it takes to commute. The following alternatives are available:

	Time (h)
a_1	$1/4$
a_2	$1/3$
a_3	1
a_4	2

Determine a utility function using the bisection method.