Friedrich-Alexander-Universität
Erlangen-Nürnberg

# Decision Theory

## Lecture 6

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# Decision Theory

## Recap: What did we do?

- Analytic Hierarchy Process (AHP)
- Multi Attribute Utility Theory (MAUT)

# Decision Theory

## Analytic Hierarchy Process (AHP)

- You want to choose between alternatives, how do you rank them?
- Structure problem, define hierarchy
- Create pairwise comparison matrices

$$R = \begin{pmatrix} r_{11} & \dots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{n1} & \dots & r_{nn} \end{pmatrix}$$

- $r_{ij}$: how much is $i$ better than $j$

# Decision Theory

## Analytic Hierarchy Process (AHP)

$$R = \begin{pmatrix} r_{11} & \dots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{n1} & \dots & r_{nn} \end{pmatrix}$$

- Based on matrices, calculate suitable weight vectors

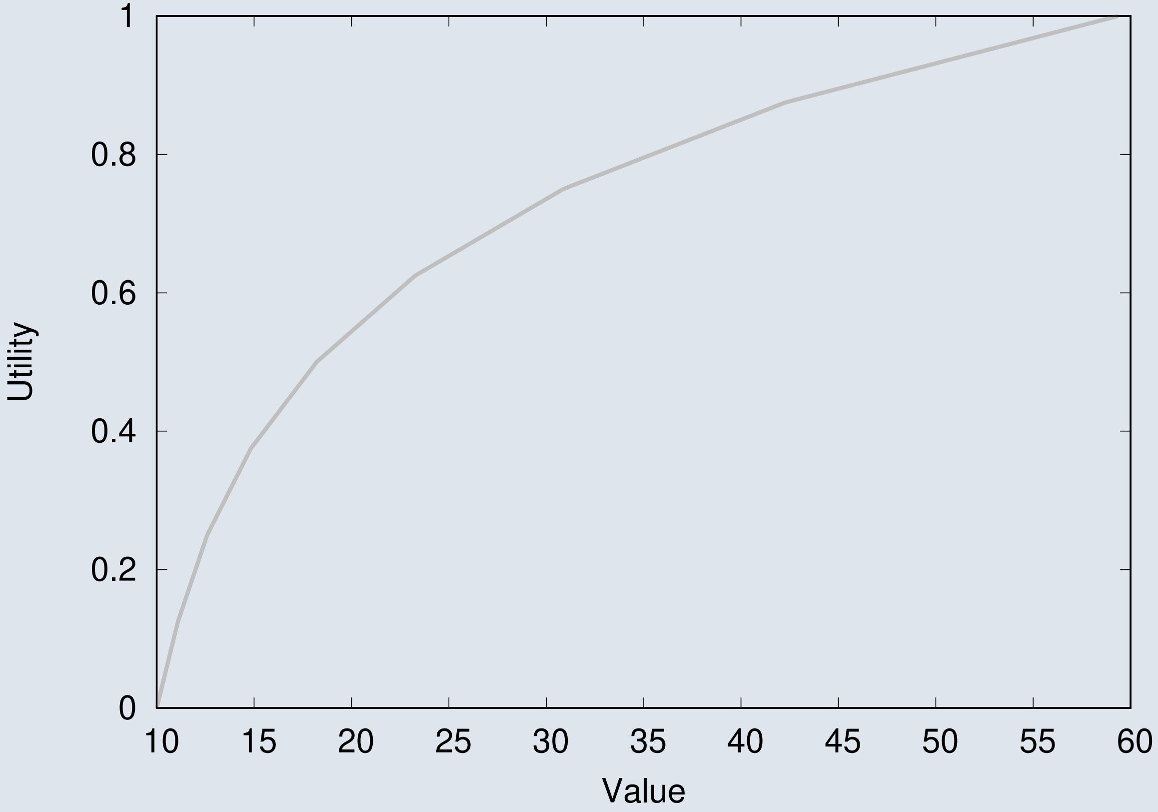
$$r_{ij} \approx \frac{w_i}{w_j}$$

- Comparison matrix and weight vectors for all levels of hierarchy
- Choose alternative with best total weight

# Decision Theory

## MAUT

- Multiple criteria ($K$ many)
- Need to decide
  - Normalised utility function $u_k$ for each criterion
  - Weight $\lambda_k$ of each criterion
- Total utility of $a_i$ is $U(a_i) = \sum_{k \in [K]} \lambda_k u_k(e_i^k)$



# Decision Theory

## MAUT

- To find $u_k$:
  - Direct rating
  - Interval splitting
- To find $\lambda_k$:
  - Trade-off method

# Decision Theory

## Direct Rating

- Take results from table
- Assign points directly
- Normalise to [0, 1]

| Temperature | 13 | 16.5 | 20 | 23.5 | 27 |
|---|---|---|---|---|---|
| Points | 0 | | | | 100 |

# Decision Theory

## Direct Rating

- Take results from table
- Assign points directly
- Normalise to [0, 1]

| Temperature | 13 | 16.5 | 20 | 23.5 | 27 |
|---|---|---|---|---|---|
| Points | 0 | 60 | | | 100 |

# Decision Theory

## Direct Rating

- Take results from table
- Assign points directly
- Normalise to [0, 1]

| Temperature | 13 | 16.5 | 20 | 23.5 | 27 |
|---|---|---|---|---|---|
| Points | 0 | 60 | 90 | | 100 |

# Decision Theory

## Direct Rating

- Take results from table
- Assign points directly
- Normalise to [0, 1]

| Temperature | 13 | 16.5 | 20 | 23.5 | 27 |
|---|---|---|---|---|---|
| Points | 0 | 60 | 90 | 95 | 100 |

# Decision Theory

## Direct Rating

- Take results from table
- Assign points directly
- Normalise to [0, 1]

| Temperature | 13 | 16.5 | 20 | 23.5 | 27 |
|---|---|---|---|---|---|
| Points | 0 | 60 | 90 | 95 | 100 |
| $u_k$ | 0.00 | 0.60 | 0.90 | 0.95 | 1.00 |

# Decision Theory

## Interval Splitting

- Give best and worst the utility 1 and 0, respectively
- Ask questions until desired degree of accuracy has been reached:
  - What value gives you utility $1/2$?
  - What value gives you utility $1/4$?
  - What value gives you utility $3/4$?
  - ...
- For missing values in the table, use interpolation

| | $x_i^0$ | | | | $x_i^1$ |
|---|---|---|---|---|---|
| Utility | 0 | - | - | - | 1 |
| Temperature | 15 | - | - | - | 27 |

# Decision Theory

## Interval Splitting

- Give best and worst the utility 1 and 0, respectively
- Ask questions until desired degree of accuracy has been reached:
  - What value gives you utility $1/2$?
  - What value gives you utility $1/4$?
  - What value gives you utility $3/4$?
  - ...
- For missing values in the table, use interpolation

| | $x_i^0$ | | $x_i^{0.5}$ | | $x_i^1$ |
|---|---|---|---|---|---|
| Utility | 0 | - | 0.5 | - | 1 |
| Temperature | 15 | - | ? | - | 27 |

# Decision Theory

## Interval Splitting

- Give best and worst the utility 1 and 0, respectively
- Ask questions until desired degree of accuracy has been reached:
  - What value gives you utility $1/2$?
  - What value gives you utility $1/4$?
  - What value gives you utility $3/4$?
  - ...
- For missing values in the table, use interpolation

| | $x_i^0$ | | $x_i^{0.5}$ | | $x_i^1$ |
|---|---|---|---|---|---|
| Utility | 0 | - | 0.5 | - | 1 |
| Temperature | 15 | - | 18 | - | 27 |

# Decision Theory

## Interval Splitting

- Give best and worst the utility 1 and 0, respectively
- Ask questions until desired degree of accuracy has been reached:
  - What value gives you utility $1/2$?
  - What value gives you utility $1/4$?
  - What value gives you utility $3/4$?
  - ...
- For missing values in the table, use interpolation

| | $x_i^0$ | $x_i^{0.25}$ | $x_i^{0.5}$ | | $x_i^1$ |
|---|---|---|---|---|---|
| Utility | 0 | 0.25 | 0.5 | - | 1 |
| Temperature | 15 | ? | 18 | - | 27 |

# Decision Theory

## Interval Splitting

- Give best and worst the utility 1 and 0, respectively
- Ask questions until desired degree of accuracy has been reached:
  - What value gives you utility $1/2$?
  - What value gives you utility $1/4$?
  - What value gives you utility $3/4$?
  - ...
- For missing values in the table, use interpolation

| | $x_i^0$ | $x_i^{0.25}$ | $x_i^{0.5}$ | $x_i^{0.75}$ | $x_i^1$ |
|---|---|---|---|---|---|
| Utility | 0 | 0.25 | 0.5 | - | 1 |
| Temperature | 15 | 16 | 18 | - | 27 |

# Decision Theory

## Interval Splitting

- Give best and worst the utility 1 and 0, respectively
- Ask questions until desired degree of accuracy has been reached:
  - What value gives you utility $1/2$?
  - What value gives you utility $1/4$?
  - What value gives you utility $3/4$?
  - ...
- For missing values in the table, use interpolation

| | $x_i^0$ | $x_i^{0.25}$ | $x_i^{0.5}$ | $x_i^{0.75}$ | $x_i^1$ |
|---|---|---|---|---|---|
| Utility | 0 | 0.25 | 0.5 | 0.75 | 1 |
| Temperature | 15 | 16 | 18 | ? | 27 |

# Decision Theory

## Interval Splitting

- Give best and worst the utility 1 and 0, respectively
- Ask questions until desired degree of accuracy has been reached:
  - What value gives you utility $1/2$?
  - What value gives you utility $1/4$?
  - What value gives you utility $3/4$?
  - ...
- For missing values in the table, use interpolation

| | $x_i^0$ | $x_i^{0.25}$ | $x_i^{0.5}$ | $x_i^{0.75}$ | $x_i^1$ |
|---|---|---|---|---|---|
| Utility | 0 | 0.25 | 0.5 | 0.75 | 1 |
| Temperature | 15 | 16 | 18 | 21 | 27 |

# Decision Theory

## Trade-Off Method

- Have all utility functions $u_k$
- Find the one criterion $k$ where the utility difference between $x_k^0$ and $x_k^1$ feels largest
- Let us call this criterion number 1
- We compare the other criteria against criterion 1:
- For all $k = 2, \ldots, K$:
  - Find $\overline{x}_1^k$ for criterion 1 such that the two options $(\overline{x}_1^k, x_k^0)$ and $(x_1^0, x_k^1)$ are equally good
  - This gives equation $\lambda_k = u_1(\overline{x}_1^k)\lambda_1$
- Solve system of equations also using $\sum_{k\in[K]} \lambda_k = 1$

# Decision Theory

## Today

- UTA
- Decisions with multiple time steps

# Decision Theory

## UTA

- UTA = ”UTilité Additive” (Jacquet-Lagreze, Siskos 1981)
- Setting as in MAUT:
  - List of alternatives
  - Multiple criteria
  - Find a utility function
- In MAUT: Asking for relevant comparisons to find utility functions and weights

# Decision Theory

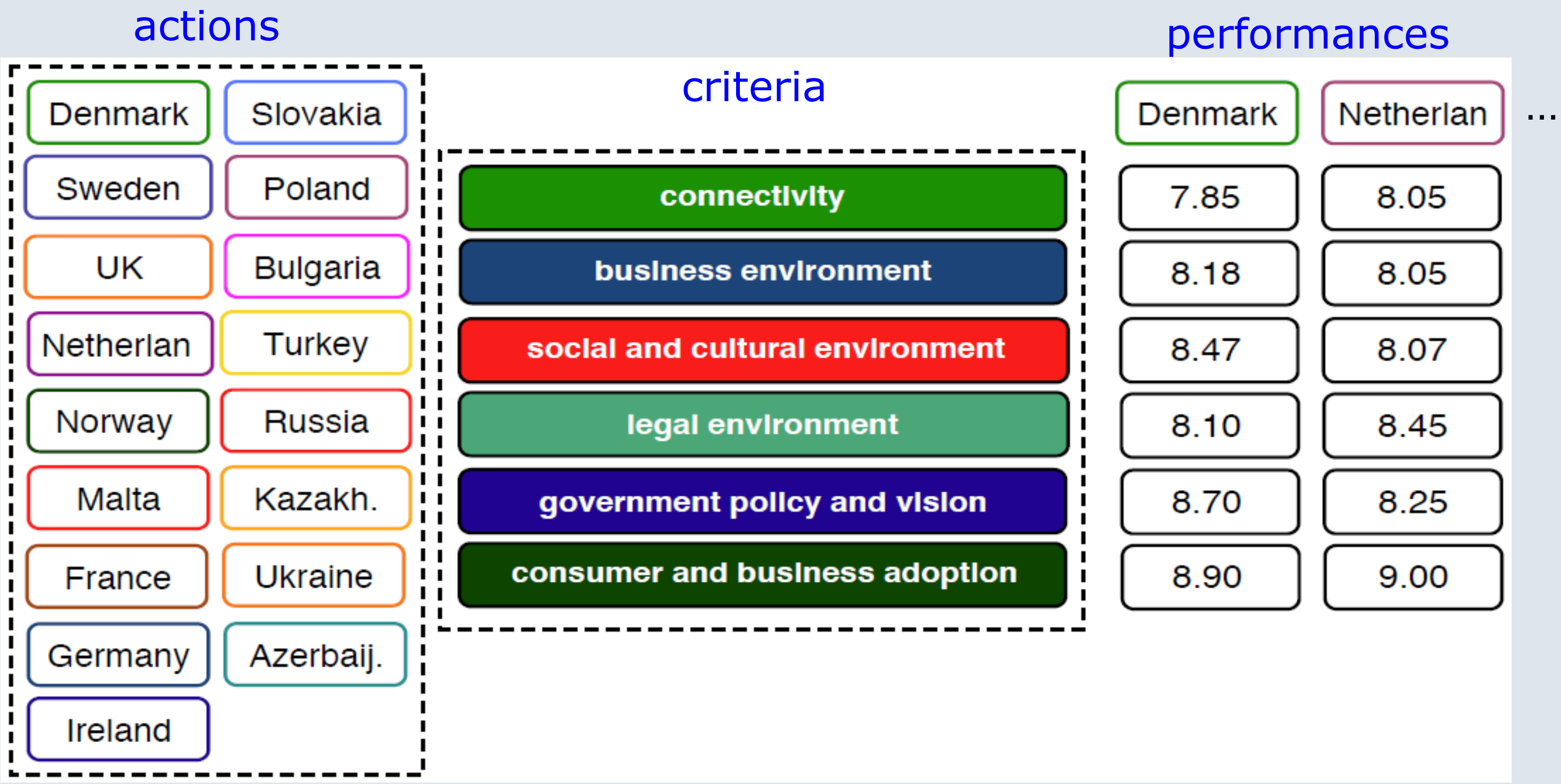
## UTA Approach

- Alternatives $A = \{a_1, a_2, a_3, \ldots\}$
- $n$ criteria, $g_i : A \to X_i$ with $X_i = [\alpha_i, \beta_i]$ range in criterion $i$
- Assumption: the larger $g_i(a)$, the better $a$
- Search for (additive) utility function

$$U : \prod_{i \in [n]} X_i \to \mathbb{R}, \;\; U(x) = \sum_{i \in [n]} u_i(x_i)$$

- Short: $U(a)$ instead of $U(g_1(a), \ldots, g_n(a))$

## Example: Ranking by IT Infrastructure Quality

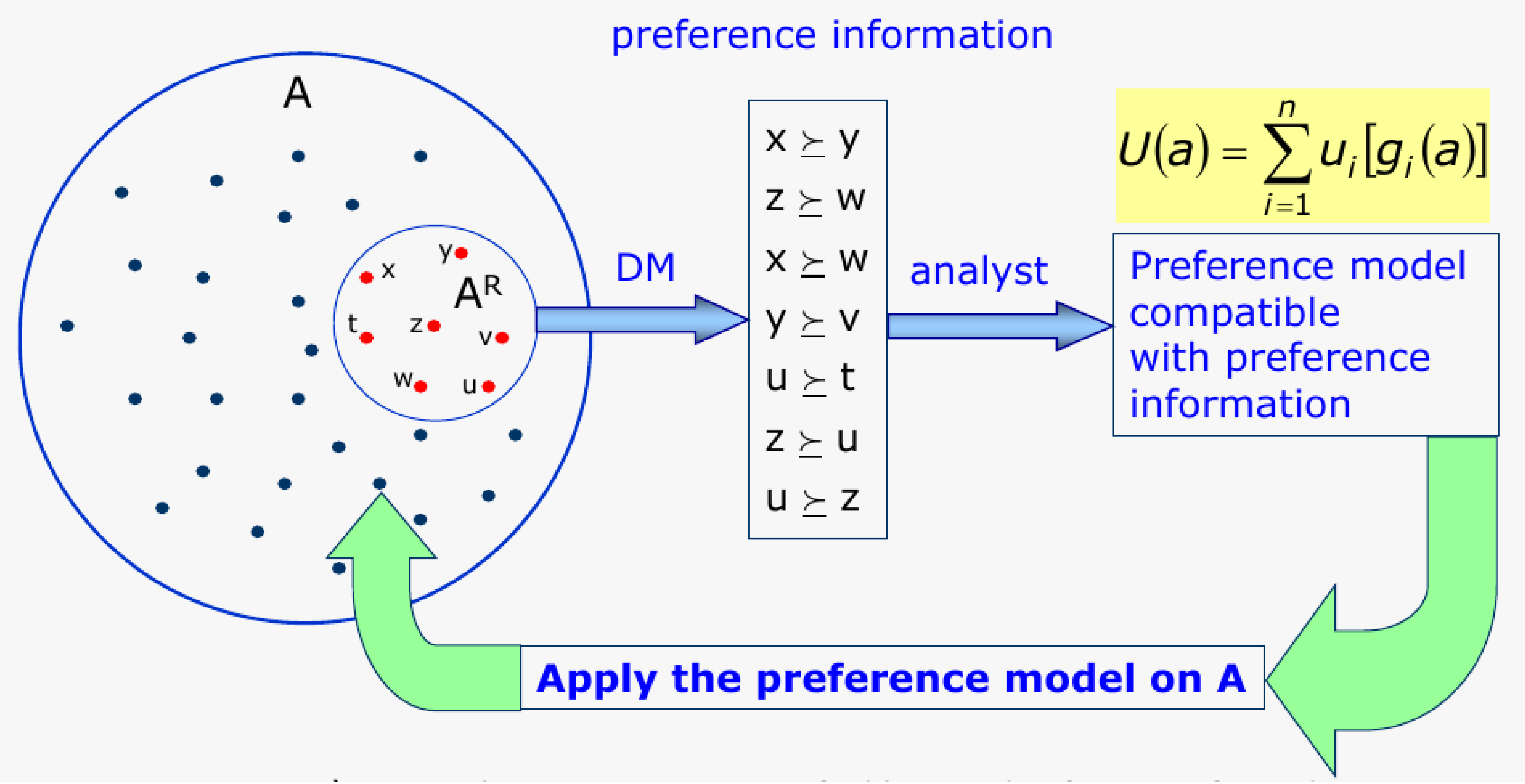


# Decision Theory

## UTA Approach

- Know values $g_i$
- Receive information once and derive utility functions
- No iterative questioning like in MAUT
- Receive information on reference alternatives $A^R \subseteq A$
- On $A^R$, we obtain a (complete or partial) order like $a \succeq b$ ($a$ at least as good as $b$)

# Decision Theory



# Decision Theory

## UTA Approach

- Decompose relation $a \succeq b$ into:
  - $a \succ b \Leftrightarrow a \succeq b$ and not $b \succeq a$
  - $a \sim b \Leftrightarrow a \succeq b$ and $b \succeq a$
- Sort $A^R = \{a_1, \ldots, a_m\}$ so that $a_k \succeq a_{k+1}$
- Seek ”compatible” utility function $U$, i.e.,
  - $a_k \succ a_{k+1} \Leftrightarrow U(a_k) > U(a_{k+1})$
  - $a_k \sim a_{k+1} \Leftrightarrow U(a_k) = U(a_{k+1})$

## UTA Approach

- $U(x) = \sum_{i \in [n]} u_i(x_i)$
- Assume each function $u_i$ is piecewise linear
- Interval $[\alpha_i, \beta_i]$ divided into $\gamma_i$ equally sized subintervals:

$$[x_i^0, x_i^1], [x_i^1, x_i^2], \ldots, [x_i^{\gamma_i - 1}, x_i^{\gamma_i}]$$

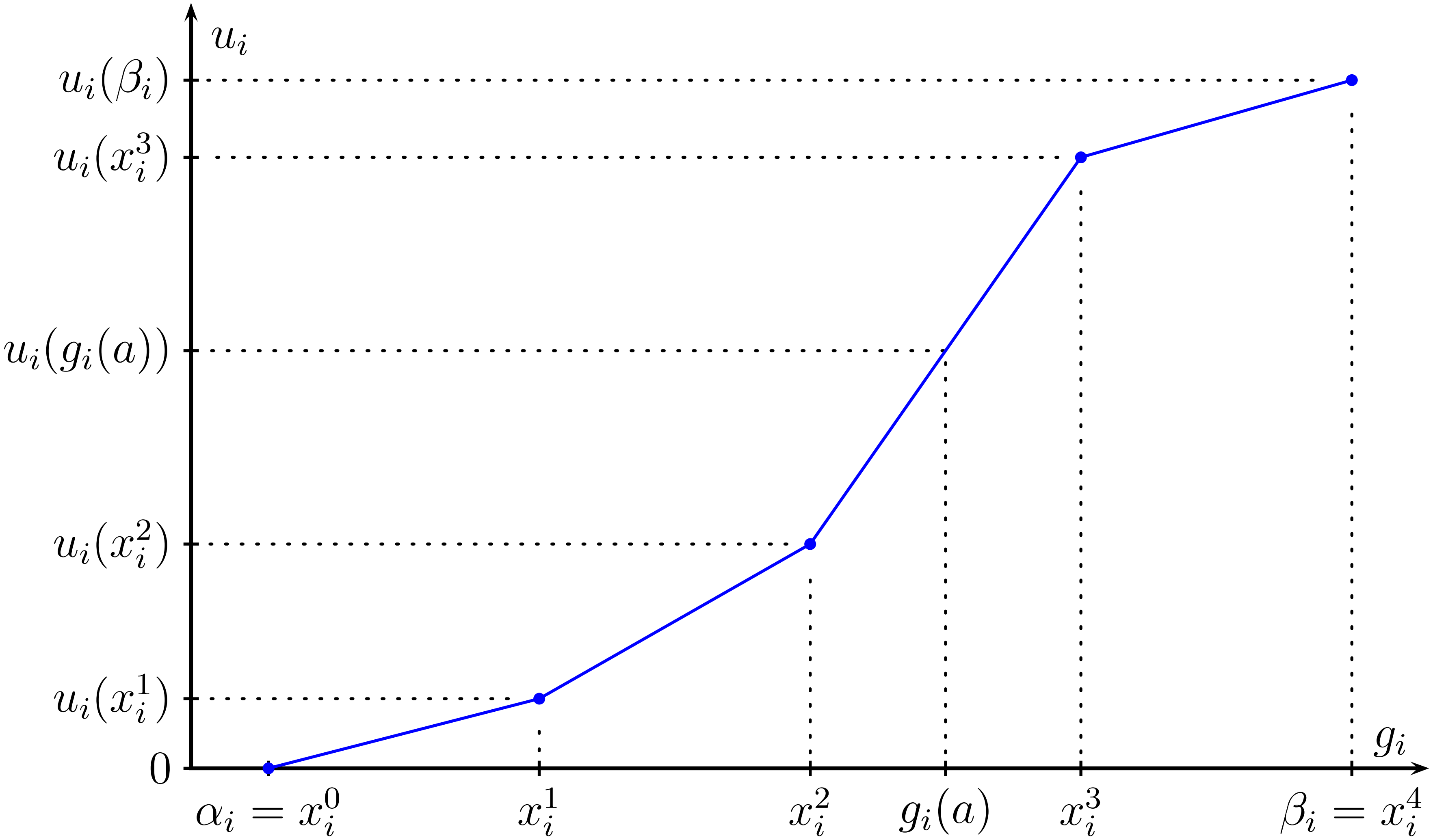
with

$$x_i^j = \alpha_i + \frac{(\beta_i - \alpha_i)}{\gamma_i} j$$

- Linear interpolation in between: for $g_i(a) \in [x_i^j, x_i^{j+1}]$

$$u_i(a) = u_i(x_i^j) + \frac{u_i(x_i^{j+1}) - u_i(x_i^j)}{x_i^{j+1} - x_i^j}(g_i(a) - x_i^j)$$

# Decision Theory



# Decision Theory

## UTA Approach

- Also want normalization of $U$ to interval $[0, 1]$
- Conditions on utility function can now be expressed as "linear" conditions:

$$\begin{aligned}
&U(a_k) > U(a_{k+1}) && \forall k \in [m-1] : a_k \succ a_{k+1} \\
&U(a_k) = U(a_{k+1}) && \forall k \in [m-1] : a_k \sim a_{k+1} \\
&u_i(x_i^{j+1}) - u_i(x_i^j) \geq 0 && \forall i \in [n], j \in [\gamma_i - 1] \\
&u_i(\alpha_i) = 0 && \forall i \in [n] \\
&\sum_{i \in [n]} u_i(\beta_i) = 1 &&
\end{aligned}$$

# Decision Theory

## UTA Approach

- Also want normalization of $U$ to interval $[0, 1]$
- Conditions on utility function can now be expressed as "linear" conditions:

$$
\begin{aligned}
&U(a_k) > U(a_{k+1}) && \forall k \in [m-1] : a_k \succ a_{k+1} \\
&U(a_k) = U(a_{k+1}) && \forall k \in [m-1] : a_k \sim a_{k+1} \\
&u_i(x_i^{j+1}) - u_i(x_i^j) \geq 0 && \forall i \in [n], j \in [\gamma_i - 1] \\
&u_i(\alpha_i) = 0 && \forall i \in [n] \\
&\sum_{i \in [n]} u_i(\beta_i) = 1 &&
\end{aligned}
$$

- Problem 1: "$>$" not allowed in linear programming
- Problem 2: What if there is no compatible function $U$?

# Decision Theory

## UTA Approach

- ”$>$” not allowed in linear program
  - Write $U(a_k) \geq U(a_{k+1}) + \varepsilon$ for a small constant $\varepsilon$
- What if there is no compatible function $U$?
  - Find the most suitable utility function
  - Minimize the violation of conditions
  - Allow slack variables
  - Example:
    $$U(a_k) = U(a_{k+1})$$
    becomes
    $$U(a_k) + \sigma^+(a_k) - \sigma^-(a_k) = U(a_{k+1}) + \sigma^+(a_{k+1}) - \sigma^-(a_{k+1})$$

# Decision Theory

## UTA Approach

- Overall, the following LP (Linear Program):

$$
\begin{aligned}
\min \quad & \sum_{k\in[m]} (\sigma^+(a_k) + \sigma^-(a_k)) & \\
\text{s.t.} \quad & U(a_k) + \sigma^+(a_k) - \sigma^-(a_k) & \\
& \quad \geq U(a_{k+1}) + \sigma^+(a_{k+1}) - \sigma^-(a_{k+1}) + \varepsilon & \forall k \in [m-1] : a_k \succ a_{k+1} \\
& U(a_k) + \sigma^+(a_k) - \sigma^-(a_k) & \\
& \quad = U(a_{k+1}) + \sigma^+(a_{k+1}) - \sigma^-(a_{k+1}) & \forall k \in [m-1] : a_k \sim a_{k+1} \\
& u_i(x_i^{j+1}) - u_i(x_i^j) \geq 0 & \forall i \in [n], j \in [\gamma_i - 1] \\
& u_i(\alpha_i) = 0 & \forall i \in [n] \\
& \sum_{i\in[n]} u_i(\beta_i) = 1 & \\
& \sigma^+(a_k), \sigma^-(a_k) \geq 0 & \forall k \in [m]
\end{aligned}
$$

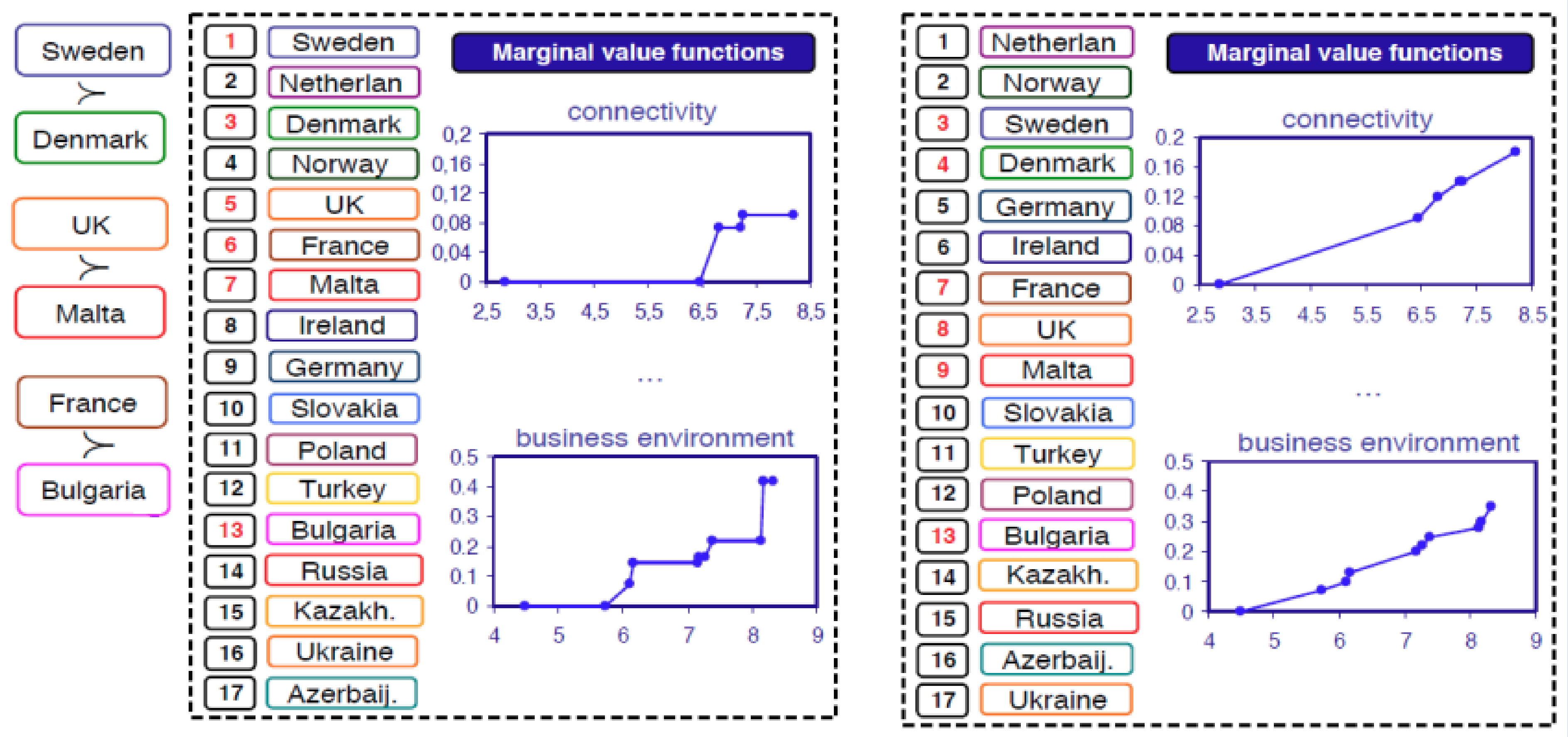
# Decision Theory

## UTA Approach

- The solution gives us the utility function
- Optimal objective value = 0:
    - There is a utility function compatible with all specifications in $A^R$
- Optimal objective value $> 0$:
    - There is no compatible utility function
    - Increase $\gamma_i$ for a more detailed model, or
    - Change $A^R$, or
    - Accept a compromise
- In both cases, the solution is not necessarily unique!

# Decision Theory

## Example



# Decision Theory

## UTA$^{GMS}$

- Assume there is a compatible utility function
- Idea: identify
  - necessary relations $a \succeq^N b$
  - potential relations $a \succeq^P b$
- Necessary relations hold in all compatible utility functions
- Potential relations hold in at least one
- GMS = Greco, Mousseau, Słowiński

# Decision Theory

## UTA$^{GMS}$

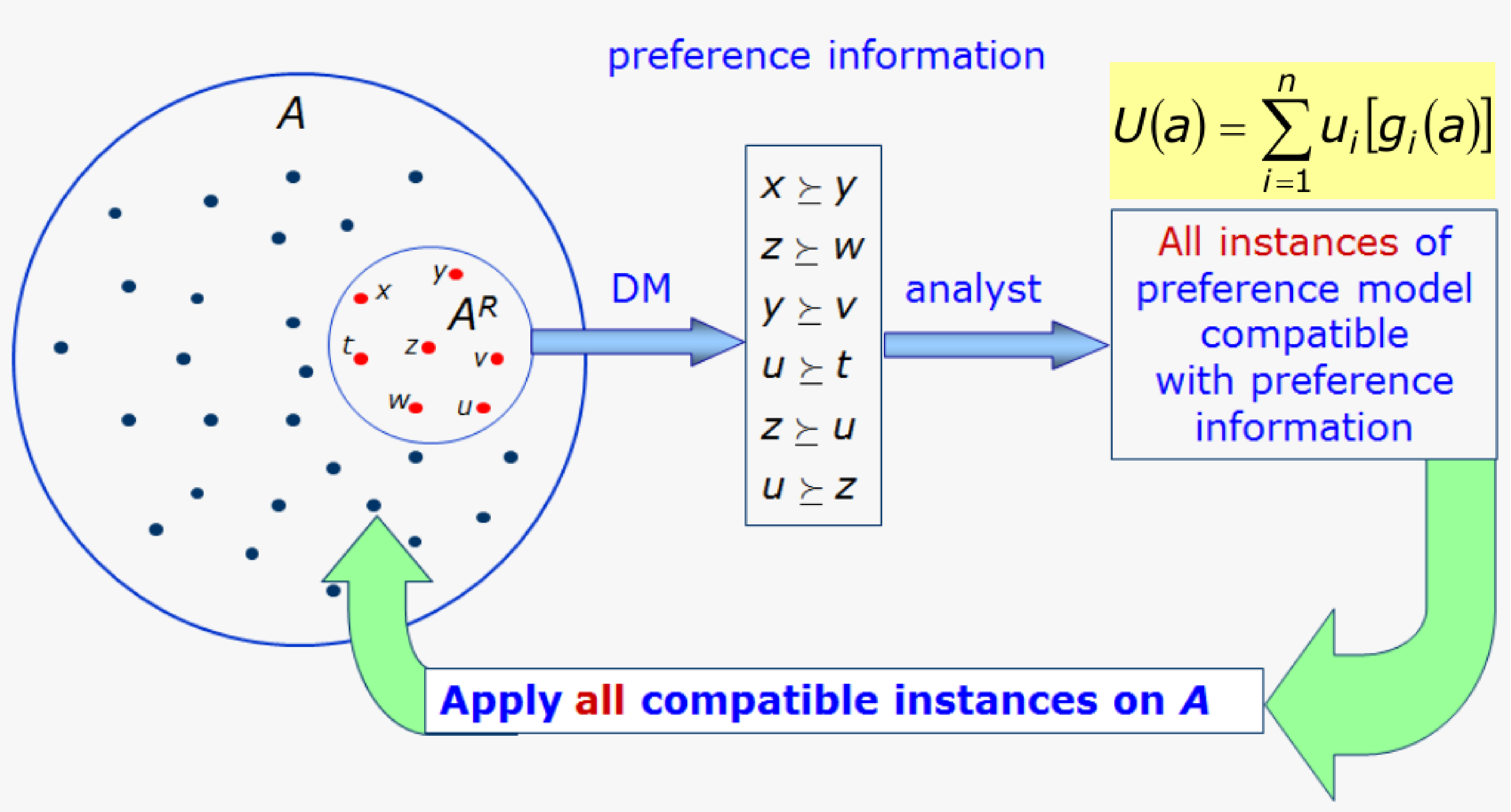
- Observation:
  - $a \succeq b$ for $a, b \in A^R \Rightarrow a \succeq^N b$
  - $a \succ b$ for $a, b \in A^R \Rightarrow$ not $(b \succeq^P a)$
- Properties
  - $\succeq^P \supseteq \succeq^N$
  - $\succeq^N$ is transitive

# Decision Theory

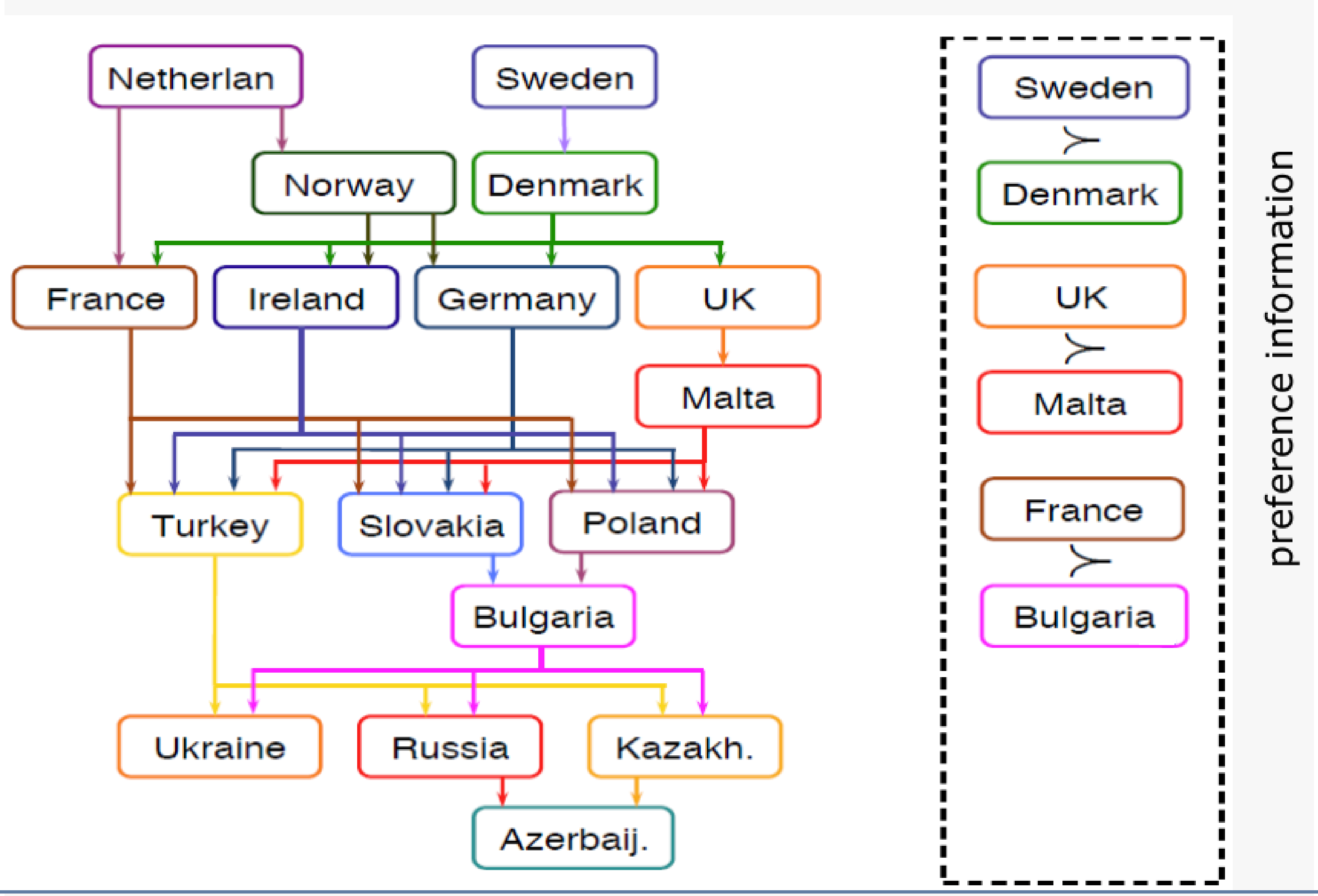
## UTA$^{GMS}$

- How to calculate $\succeq^N$ and $\succeq^P$?
- Let $d(a, b) = \min(U(a) - U(b))$ over all compatible $U$
- Let $D(a, b) = \max(U(a) - U(b))$ over all compatible $U$
- It holds:
  - $a \succeq^N b \Leftrightarrow d(a, b) \geq 0$
  - $a \succeq^P b \Leftrightarrow D(a, b) \geq 0$

# Decision Theory



# Entscheidungstheorie

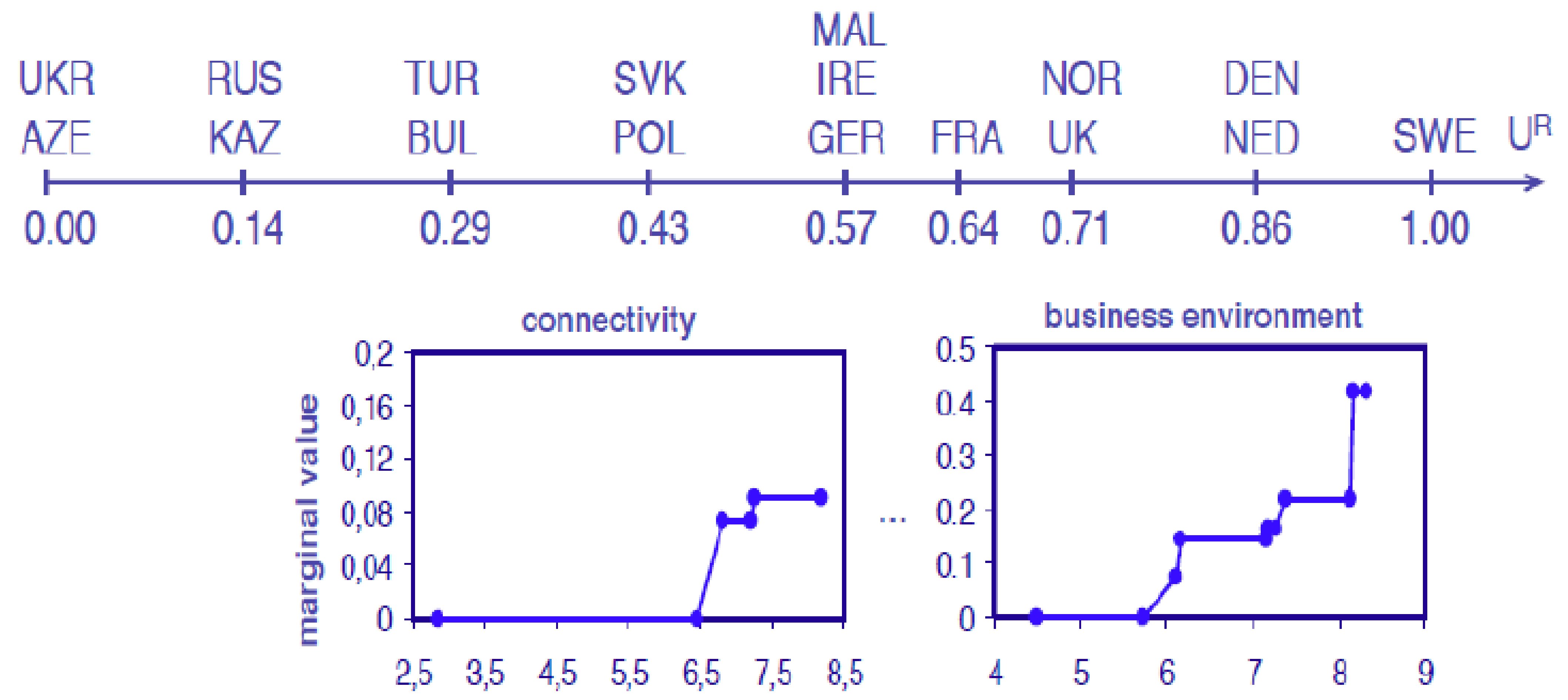


# Decision Theory

## UTA$^{GMS}$

- Still, we might want to choose a single good representative $U$
- For this, a new lexicographic optimization problem:
  1. Represent necessary relations as strongly as possible, i.e., maximize utility difference for $a \succeq^N b$ and not $(b \succeq^N a)$
  2. Minimize utility difference for undecided pairs, i.e., when not $(a \succeq^N b)$ and not $(b \succeq^N a)$

# Decision Theory



# Decision Theory

## UTA$^{GMS}$

- Related approach: extreme ranking analysis
- Determine the highest and lowest possible rank across all compatible rankings

# Decision Theory

Poland
...
Sweden
Netherlan
UK
Norway
...
Germany
Malta
Ireland
Turkey
...
Kazakh.
Ukraine

the highest $P^*(a)$ rank $P_*(a)$ the lowest

assume that *a* is ranked

In the top | in the bottom

identify minimal subset of alternatives that are simultaneously

not worse than a | not better than a

i.e., solve the following MILP problems

$$Minimize: f^{pos}_{\max} - \sum_{b \in A\setminus\{a\}} v_b$$

$$\begin{cases} U(a) \succ U(b) - M \cdot v_b, \;\; for\ all\ \ b \in A \setminus \{a\} \\ E(A^R) \end{cases}$$

$$Minimize: f^{pos}_{\min} - \sum_{b \in A\setminus\{a\}} v_b$$

$$\begin{cases} U(b) \succ U(a) - M \cdot v_b \\ E(A^R) \end{cases}$$

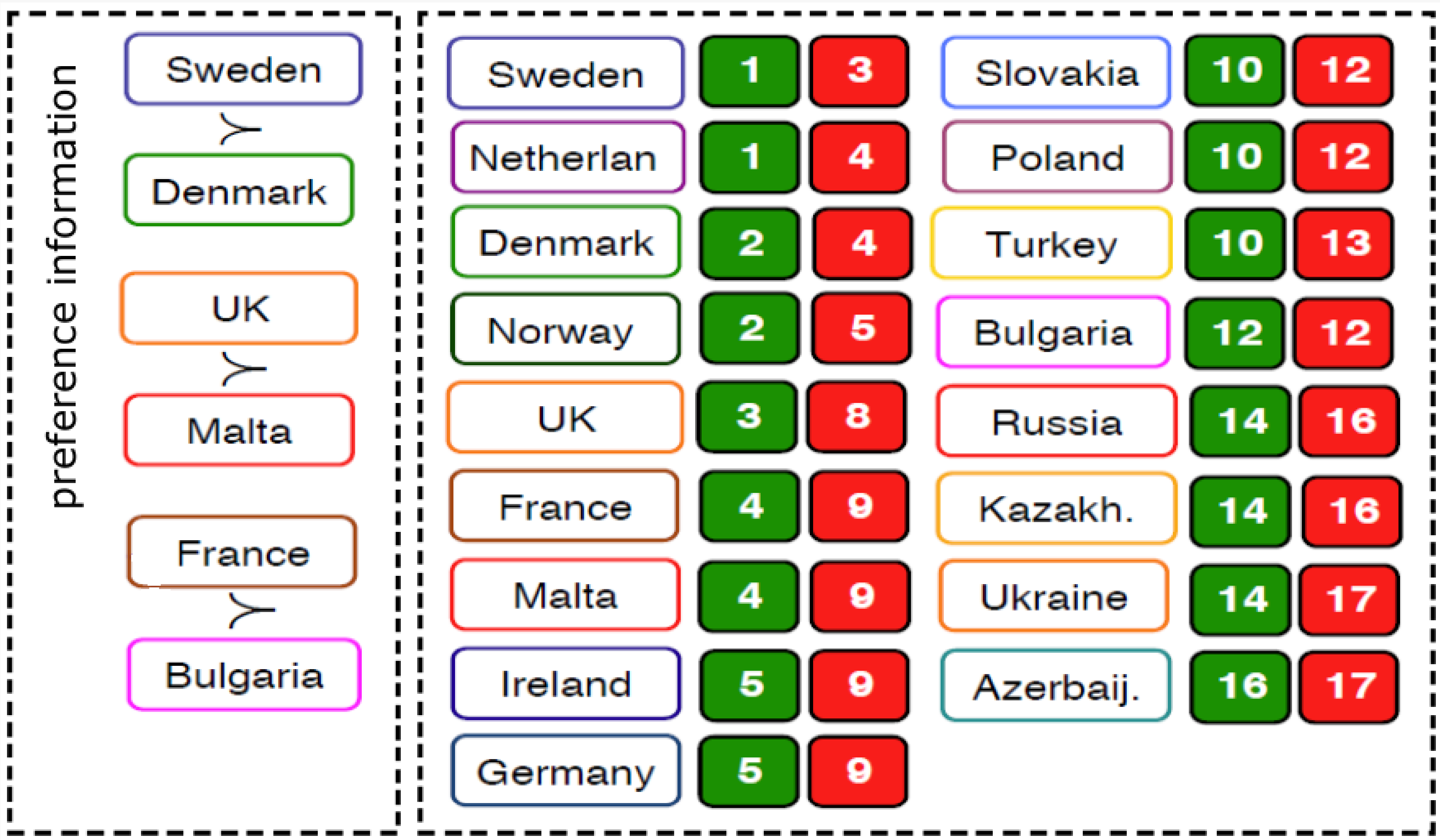
read off the extreme ranks

$$P^*(a) = f^{pos}_{\max} + 1 \qquad P_*(a) = |A| - f^{pos}_{\min}$$

the best case | the worst case

Sweden
Netherlan
UK
Norway
...
Germany
Malta
Ireland
Turkey
...
Kazakh.
Ukraine
...
Poland

# Decision Theory



## UTA – Additional Techniques

- Stochastic ordinal regression
  - Relations with probabilities
  - *a* is 80% better than *b*
- Model over-fitting
  - Want to balance accuracy against complexity
  - Willing to accept that preferences $A^R$ are not precisely represented if a simpler explanation is found
  - Hope: better performance out-of-sample
- Intensity of preference is given

## New Topic! Decisions with Multiple Time Steps: Examples

- You switch to a whole-foods diet
  - Immediate disadvantage: no sweets
  - Later advantage: a healthy life
- Should one invest in basic research?
  - Immediate disadvantage: less money for other areas
  - Later advantage: strengthening other research areas
- ”Intertemporal” decisions (under certainty)

# Decision Theory

## Modeling

- Formally like a multi-criteria problem, e.g.:
  - Criterion 1: profit 2024
  - Criterion 2: profit 2025
  - Criterion 3: profit 2026
  - ...
- Set up a utility function again:

$$U(a_i) = \sum_{t \in [T]} \lambda_t u_t(e_i^t)$$

- Can apply known methods
- Often $\lambda_t = \delta^t$, $\delta < 1$, exponential discounting
- Decision = Decision for all time steps

# Decision Theory

## Variant

- Alternatively: Decision is made anew at each time point

$$U^{\tau}(a_i) = \sum_{t=\tau}^{T} \delta^t u_t(e_i^t)$$

- Again, exponential discount function, $\delta \in [0, 1]$
- $u_t(e_i^t)$: how satisfied am I on day $t$
- $U^{\tau}(a_i)$: on day $\tau$, how satisfied am I with today and the future
- Time-consistent: preferences decrease uniformly

# Decision Theory

## Extension

$$U^{\tau}(a_i) = \delta^{\tau} u_{\tau}(e_i^{\tau}) + \beta \sum_{t=\tau+1}^{T} \delta^t u_t(e_i^t)$$

- $\beta \in [0, 1]$
- $\delta$: long-term impatience
- $\beta$: immediate satisfaction
- $\beta = 1$: as before
- $\beta < 1$: lack of self-control

# Decision Theory

## Example

$$U^{\tau}(a_i) = \delta^{\tau} u_{\tau}(e_i^{\tau}) + \beta \sum_{t=\tau+1}^{T} \delta^{t} u_t(e_i^{t})$$

- $\delta = 1$, $\beta = 0.5$
- On Monday, one person considers the utility of the coming Friday and Saturday equally important
- On Friday, the utility on Friday is twice as important as the utility on Saturday

# Decision Theory

## Types

$$U^\tau(a_i) = \delta^\tau u_\tau(e_i^\tau) + \beta \sum_{t=\tau+1}^{T} \delta^t u_t(e_i^t)$$

- Three types of decision-makers:
- Time-consistent (TC)
  - $\beta = 1$, choose every day $\tau$ so that $U^\tau$ is maximal
  - Formulates a plan and sticks to it
- Naive
  - $\beta < 1$, also choose every day $\tau$ so that $U^\tau$ is maximal
  - Not aware of inconsistency
  - Optimistic: believes he will act consistently in the future
- Sophisticated:
  - Like naive but aware of inconsistency
  - Knows that he will apply this function in the future

# Decision Theory

## Example

- Someone likes to go to the movies on Saturdays
- Program:
  - This week: mediocre movie
  - Next week: good movie
  - In two weeks: very good movie
  - In three weeks: the big hit (Nicolas Cage!)
- Person has to submit term paper within four weeks
- One of the movies has to be skipped

# Decision Theory

## Evaluation

| Week | Movie | Value |
|---|---|---|
| 1 | Medium | 3 |
| 2 | Good | 5 |
| 3 | Very good | 8 |
| 4 | Nic Cage  | 13 |

- TC (with $\delta = 1$): misses the worst movie (Week 1)
- Assuming $\delta = 1$, $\beta = 0.5$
- Naive:
  - Week 1: better utility 3 than $\frac{1}{2} \cdot 5$
  - Week 2: better utility 5 than $\frac{1}{2} \cdot 8$
  - ...
  - Misses Nic Cage

# Decision Theory

## Evaluation

| Week | Movie | Value |
|---|---|---|
| 1 | Medium | 3 |
| 2 | Good | 5 |
| 3 | Very good | 8 |
| 4 | Nic Cage  | 13 |

- Assuming $\delta = 1$, $\beta = 0.5$
- Sophisticated:
  - In Week 3: 8 (watching) vs. 6.5 (not watching) $\rightarrow$ watch
  - In Week 2: 5+4 (watching) vs. 4+6.5 (not watching) $\rightarrow$ not watch
  - In Week 1: 3+10.5 (watching) vs. 2.5+4+6.5 (not watching) $\rightarrow$ watch
  - Watches the movie in Week 1,3, and 4 (a bit of procrastination)

# Decision Theory

## Example, continued

| Week | Movie | Value |
|---|---|---|
| 1 | Medium | 3 |
| 2 | Good | 5 |
| 3 | Very good | 8 |
| 4 | Nic Cage  | 13 |

- The person has only enough money to watch a single movie
- TC (with $\delta = 1$) watches the best movie
- Naive:
  - Week 1: utility 3 worse than utility $\frac{1}{2} \cdot 13$
  - Week 2: utility 5 worse than utility $\frac{1}{2} \cdot 13$
  - Week 3: utility 8 better than utility $\frac{1}{2} \cdot 13$
  - Watches the movie in Week 3
  - Waits because optimistic about the future

# Decision Theory

## Example, continued

| Week | Movie | Value |
|---|---|---|
| 1 | Medium | 3 |
| 2 | Good | 5 |
| 3 | Very good | 8 |
| 4 | Nic Cage  | 13 |

- The person has only enough money to watch a single movie
- Sophisticated:
  - Week 3: 8 (watch) vs. 6.5 (wait) $\rightarrow$ watch
  - Week 2: 5 (watch) vs. 4 (wait) $\rightarrow$ watch
  - Week 1: 3 (watch) vs. 2.5 (wait) $\rightarrow$ watch
  - Watches the movie in Week 1
  - Because aware that future actions won't be correct, the problem gets even worse

# Decision Theory

## Quiz

### Question 1

Given:

- $a \succeq b$
- $b \succeq c$

Is $a \succeq^P c$ true?

### Question 2

How do you decide about the future? Are you of the type:

- TC
- Naive
- Sophisticated

How would you set $\delta$ and $\beta$ in the cinema example?