Friedrich-Alexander-Universität
Erlangen-Nürnberg

Decision Theory

Lecture 2

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Recap: Organisation

- StudOn: Decision Theory
- Two bi-weekly tutorial groups (start this week)

Decision Theory

Recap: Types of Decision Theory

- Normative DT
  - Start from axioms (fundamental assumptions) to determine rational decisions
  - More philosophical
- Descriptive DT
  - Observe how decisions are being made
  - More psychology or sociology
- Prescriptive DT
  - Also called decision analysis, focus on methods for decision making
  - Focus not on discussing axioms

Decision Theory

Recap: Why are decisions hard?

Examples:

- Uncertainty
- Multiple criteria
- Too many alternatives
- Difficult evaluation
- Unknown choices and unknown consequences

Decision Theory

Recap: What do we require from a rational decision?

- How we formulate the problem should not affect decision
  - Will not always be true in reality
- Previous decisions irrelevant (future orientation)
- Relation: transitivity and completeness

Decision Theory

Recap: Omelette Example

Build a decision matrix to model the problem

	s_1 (good)	s_2 (foul)
a_1	$(6, 0, 0)$	$(0, 5, 0)$
a_2	$(6, 0, 1)$	$(5, 0, 1)$
a_3	$(5, 1, 0)$	$(5, 0, 0)$

with

- a_1, a_2, a_3 possible alternatives
- s_1, s_2 possible scenarios
- (x^1, x^2, x^3) outcomes with three criteria

Decision Theory

Today

- Modelling
- Types of models
- Types of scales
- Basic models
- Probabilities

Decision Theory

What kind of models do you know?

Decision Theory

Model

- Simplified representation of a real situation
- Tool for describing, explaining, predicting, and designing real situations
- Errors are made by simplifying
- Requires:
  - Structural equality (isomorphism), or
  - Structural similarity (homomorphism) with reality

Decision Theory

Modelling

- Reduction of complexity and focus on relevant aspects:
  - Aggregation of individual variables
  - Omission of irrelevant features
  - Formation of interfaces to the relevant model environment (partial models); keyword: digital twin
  - Hierarchization, decomposition of decision problems

Decision Theory

Reasons for Abstraction

- Information not available
- Information costs too high
- Other features not relevant
- Impossibility of the model or too high methodical effort in relation to the expected improvement of results

Decision Theory

Modelling

Step 1

Formulation of the question to be investigated (verbal problem description)

$\downarrow$

Step 2

Selection of relevant variables and their relationships (1st abstraction level)

$\downarrow$

Step 3

Search for a structure-preserving mapping through their coordination (2nd abstraction level)

Decision Theory

Types of Models: Descriptive Model

- Also known as a capture or investigation model
- For the descriptive capture and simplified (selective) representation of real situations
- Examples:
  - Map
  - Business accounting
  - National economic accounting
  - Determination of price floors

Decision Theory

Types of Models: Explanatory Model

- For theory-based representation and explanation of real situations based on existing/suspected causalities
- Special form: forecast model
- Basis for means-end analyses and for forecasting the consequences of events and actions
- Examples:
  - COVID models
  - Climate models
  - Economic models
  - Economic cycle model
  - Price-sales function of a product

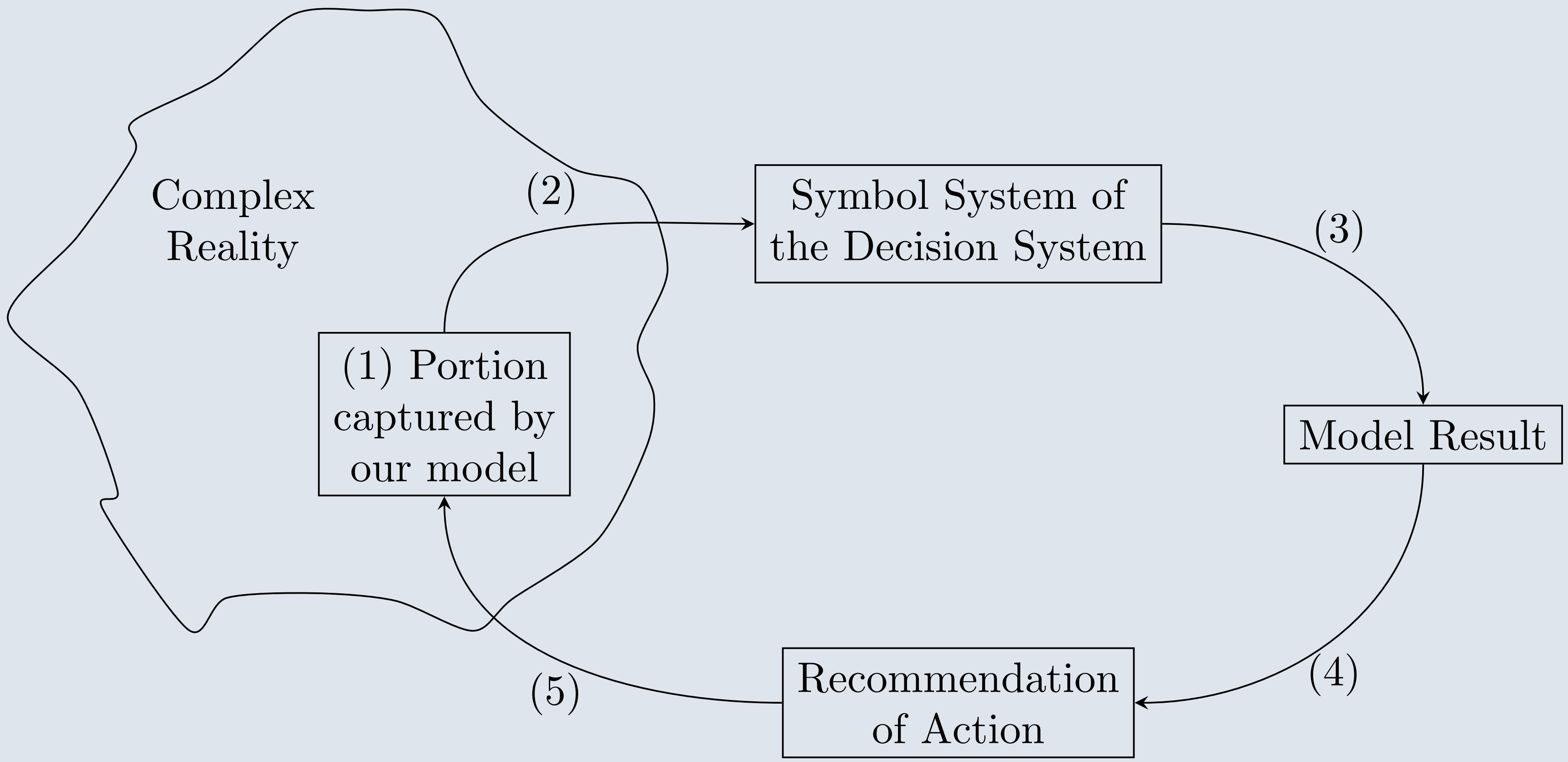
Decision Theory

Types of Models: Decision Model

- To support a decision-maker in choosing among several decision alternatives
- To gain insights into how upcoming decisions can be optimally made
- Complete achievement of a given goal
- Directed towards deriving a recommendation for action
- Examples:
  - Investment models
  - Contribution margin maximization model
  - Classical lot-sizing model

Decision Theory

Reality, Model, Model Analysis: Five Steps



Decision Theory

Reality, Model, Model Analysis: Five steps

1. Determination of the portion of reality to be captured by the model
2. Abstract transfer of the portion of reality into the symbol system of a formalized decision model
3. Analysis/Solution of the model, provides computational model result
4. Model result includes a recommendation for action; translation from formal language to verbal statement
5. Conclusions are drawn from the recommendation for action and implemented in decisions that affect reality

What scales do you know?

Decision Theory

Scale Forms

- How do I capture data and information?
- Nominal scale
- Ordinal scale
- Interval scale
- Ratio scale
- Absolute scale

Decision Theory

Nominal Scale

- Differences can be named
- But cannot be otherwise valued
- For example:
  - Colors
  - Shapes
  - Gender

Decision Theory

Ordinal Scale

- Ranking possible
- For example:
  - Creditworthiness: "good", "medium", "bad"
  - Military rank: "General", "Major", "Lieutenant"

Decision Theory

Interval Scale

- Arbitrary but fixed unit of measurement
- Distances can be compared and interpreted
- Zero point is arbitrarily chosen, conversions possible
- Ratios not meaningful
- For example:
  - Temperature: celsius and fahrenheit

Decision Theory

Ratio Scale

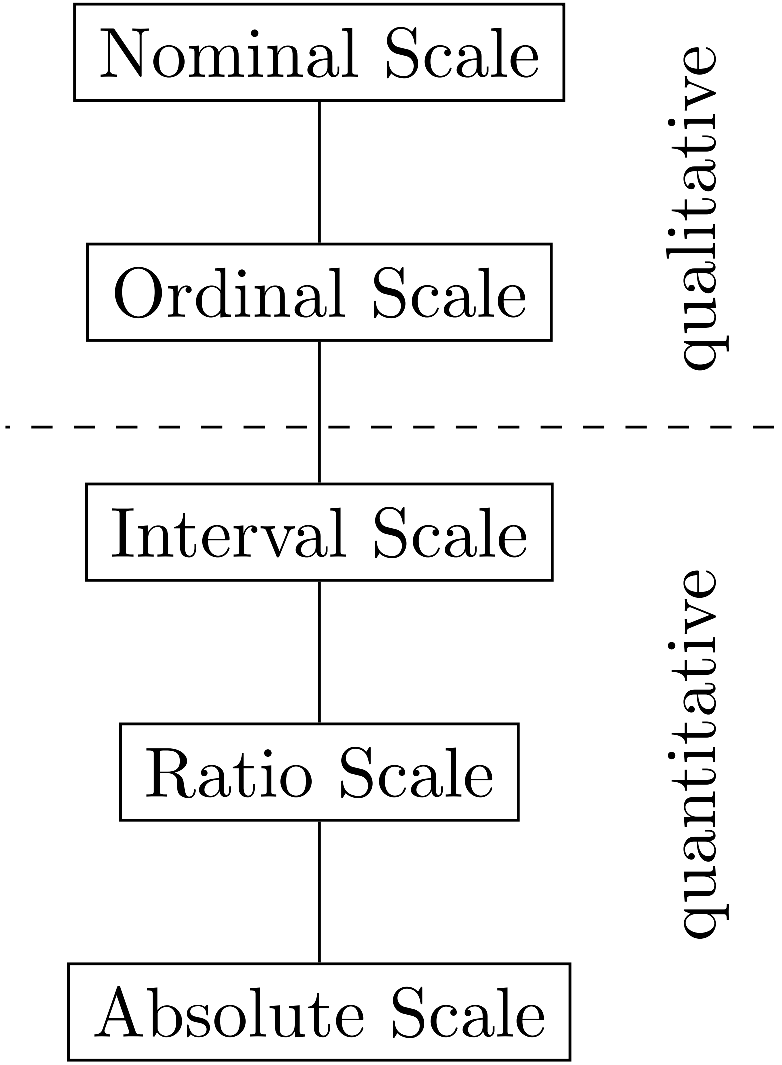
- Absolute zero point exists
- Unit is therefore irrelevant
- Ratios meaningful
- For example:
  - Area
  - Sales quantity
  - Fuel consumption

Decision Theory

Absolute Scale

- Ratio scale with a fixed unit
- Ratios and differences possible
- For example:
  - Income

Decision Theory



Decision Theory

Basic Models of Decision Theory

- Now learning two ways to formalize a decision problem
- result matrix / decision matrix
- Decision tree (not the same as in AI)

Decision Theory

Formulation of a basic model

Action Field or Action Space

- Finite or infinite set of mutually exclusive action alternatives a_i, $i = 1, 2, \ldots, m$, = Action field

Environmental States or State Space

- Finite number of outcome-influencing factors, the so-called environmental states s_j, $j = 1, 2, \ldots, n$
- Determine the decision-maker's action consequences

Decision Theory

Requirements for Action Space

- Must include all alternatives (including non-action alternative, if possible)
- The decision-maker must and may choose exactly one alternative
- Finite or infinite number of actions possible

Decision Theory

What does n mean?

- n is the number of environmental states
- Two types of decision situations:
  - $n = 1$: Decisions under certainty
    - Future development can be precisely predicted
    - We already know the occurring environmental state
  - $n > 1$: Decisions under risk or uncertainty
    - Future cannot be predicted with certainty
    - Have an idea of the alternatively possible environmental states

Decision Theory

Consequences

- Outcome e_{ij} when the decision-maker chooses action alternative a_i and state s_j occurs
- The decision-maker aims, for example, for the highest possible value
- Problems can be modeled by a decision matrix:

$$
\begin{array}{c|ccccccc}
 & s_1 & s_2 & \dots & s_j & \dots & s_n \\
\hline
a_1 & e_{11} & e_{12} & \dots & e_{1j} & \dots & e_{1n} \\
a_2 & e_{21} & e_{22} & \dots & e_{2j} & \dots & e_{2n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_i & e_{i1} & e_{i2} & \dots & e_{ij} & \dots & e_{in} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_m & e_{m1} & e_{m2} & \dots & e_{mj} & \dots & e_{mn}
\end{array}
$$

Decision Theory

General form of the Decision Matrix

$$
\begin{array}{c|ccccccc}
 & s_1 & s_2 & \dots & s_j & \dots & s_n \\
\hline
a_1 & e_{11} & e_{12} & \dots & e_{1j} & \dots & e_{1n} \\
a_2 & e_{21} & e_{22} & \dots & e_{2j} & \dots & e_{2n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_i & e_{i1} & e_{i2} & \dots & e_{ij} & \dots & e_{in} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_m & e_{m1} & e_{m2} & \dots & e_{mj} & \dots & e_{mn}
\end{array}
$$

- $e = g(a, s)$ is the assignment that indicates the action consequence e for a combination (a, s).
- Assumptions for the occurrence of individual s_j independent of the choice of alternative

Decision Theory

Example (finite number of actions)

- Investor has a budget of 200,000 EUR, which he can use as follows:
  - Financial investment in any amount
  - Investment in a maximum of one machine of type A (Investment amount 120,000 EUR)
  - Investment in one or more machines of type B (Investment amount 90,000 EUR)
  - Investment in one or more machines of type C (Investment amount 70,000 EUR)
- Action space includes more than 4 alternatives

Decision Theory

Example (finite number of actions)

index	actions
1	financial investment of 200,000€
2	invest once in A and financial investment of 80,000€
3	invest once in A and once in C and financial investment of 10,000€
4	invest once in B and financial investment of 110,000€
5	invest twice in B and financial investment of 20,000€
6	invest once in B and once in C and financial investment of 40,000€
7	invest once in C and financial investment of 130,000€
8	invest twice in C and financial investment of 60,000€

Decision Theory

Example (infinite number of actions)

A company can produce two chemicals. Both chemicals go through laboratory 1, laboratory 2, and quality control. The following capacities are used:

product	lab 1	lab 2	quality control
chemical 1	10 time units	5 tu	15 tu
chemical 2	8 tu	10 tu	12 tu
capacity	2400 tu	1500 tu	3000 tu

Decision Theory

Example (infinite number of actions)

Action space:

$$
\begin{aligned}
10x_1 + 8x_2 &\leq 2,400 \\
5x_1 + 10x_2 &\leq 1,500 \\
15x_1 + 12x_2 &\leq 3,000 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

Decision Theory

State

Possible constellation of factors relevant in a situation (value combination of different relevant environmental data)

State Space

The set $S = \{s_1, s_2, \dots, s_n\}$ of all relevant environmental states or situations $s_1, s_2, \dots, s_n$ is called the state space.
Cases regarding knowledge of the true environmental states:

- Certainty: today's or future true environmental state is known
- Risk: probabilities can be assigned to states
- Uncertainty: no probabilities can be assigned to states

Decision Theory

Example

- Retailer can increase the selling price by 18% to 11.80 EUR (a_1) or not (a_2)
- In the first case, sales quantities (x) of 100 or 120 are possible
- In the second case, quantities of 120 or 140 are possible
- The following approach is intuitive but **incorrect**:

	s_1 ($x = 100$)	s_2 ($x = 120$)	s_3 ($x = 140$)
a_1	$1,180$	$1,416$	$1,652$
a_2	$1,000$	$1,200$	$1,400$

Decision Theory

Incorrect Example

- here, a_1 is clearly the better alternative
- but state s_3 cannot occur when choosing a_1
- state s_1 cannot occur when choosing a_2

	s_1 $(x = 100)$	s_2 $(x = 120)$	s_3 $(x = 140)$
a_1	1.180	1.416	1.652
a_2	1.000	1.200	1.400

Decision Theory

Incorrect Example

- here, a_1 is clearly the better alternative
- but state s_3 cannot occur when choosing a_1
- state s_1 cannot occur when choosing a_2

	s_1 ($x = 100$)	s_2 ($x = 120$)	s_3 ($x = 140$)
a_1	1.180	1.416	~~1.652~~
a_2	~~1.000~~	1.200	1.400

Decision Theory

Example

- Better model: environmental states are the alternative possible combinations of sales quantities achievable at prices of 10EUR and 11.80EUR
- Thus, four alternative possible combinations

Decision Theory

Example

- With each combination as an environmental state, the result matrix is obtained

	s_1	s_2	s_3	s_4
a_1	1.180	1.416	1.180	1.416
a_2	1.200	1.200	1.400	1.400

- Correct model
- a_1 no longer clearly better

Decision Theory

Reminder: Result Matrix in General Form

$$
\begin{array}{c|ccccccc}
 & s_1 & s_2 & \dots & s_j & \dots & s_n \\
\hline
a_1 & e_{11} & e_{12} & \dots & e_{1j} & \dots & e_{1n} \\
a_2 & e_{21} & e_{22} & \dots & e_{2j} & \dots & e_{2n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_i & e_{i1} & e_{i2} & \dots & e_{ij} & \dots & e_{in} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_m & e_{m1} & e_{m2} & \dots & e_{mj} & \dots & e_{mn}
\end{array}
$$

Extension:

- Multiple criteria
- Multiple time steps

Decision Theory

Extension: Multi-Criteria

- Instead of a one-dimensional result e_{ij}, multiple criteria $1, \ldots, k$

	s_1	$\ldots$	s_n
a_1	$(e_{11}^1, \ldots, e_{11}^k)$	$\ldots$	$(e_{1n}^1, \ldots, e_{1n}^k)$
a_2	$(e_{21}^1, \ldots, e_{21}^k)$	$\ldots$	$(e_{2n}^1, \ldots, e_{2n}^k)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_m	$(e_{m1}^1, \ldots, e_{m1}^k)$	$\ldots$	$(e_{mn}^1, \ldots, e_{mn}^k)$

Extension: Multi-Criteria and Multiple Time Steps

- Instead of a one-dimensional result e_{ij}, multiple criteria $1, \dots, k$ and multiple time steps $1, \dots, t$

$$
\begin{array}{c|c|c|c}
 & s_1 & \dots & s_n \\
\hline
a_1 & \begin{pmatrix} e_{11}^{11} & \dots & e_{11}^{k1} \\ \vdots & \vdots & \vdots \\ e_{11}^{1t} & \dots & e_{11}^{kt} \end{pmatrix} & \dots & \begin{pmatrix} e_{1n}^{11} & \dots & e_{1n}^{k1} \\ \vdots & \vdots & \vdots \\ e_{1n}^{1t} & \dots & e_{1n}^{kt} \end{pmatrix} \\
\vdots & \vdots & \vdots & \vdots \\
a_m & \begin{pmatrix} e_{m1}^{11} & \dots & e_{m1}^{k1} \\ \vdots & \vdots & \vdots \\ e_{m1}^{1t} & \dots & e_{m1}^{kt} \end{pmatrix} & \dots & \begin{pmatrix} e_{mn}^{11} & \dots & e_{mn}^{k1} \\ \vdots & \vdots & \vdots \\ e_{mn}^{1t} & \dots & e_{mn}^{kt} \end{pmatrix}
\end{array}
$$

Preferences

How do I convert results

$$g(a_i, s_j) = \begin{pmatrix} e_{ij}^{11} & \dots & e_{ij}^{k1} \\ \vdots & \vdots & \vdots \\ e_{ij}^{1t} & \dots & e_{ij}^{kt} \end{pmatrix}$$

into a single value $\Phi(a_i)$?

- Height preference relation
- Type preference relation
- Time preference
- Risk or uncertainty preference relation

Decision Theory

Preferences

- Height preference relation
  - Maximum?
  - Minimum?
  - At least a certain value?
  - A more complex function?

Decision Theory

Preferences

- Type preference relation
  - How to weigh the criteria against each other?
  - For example, weights

Decision Theory

Preferences

- Time preference
  - How to weigh the time steps against each other?
  - For example, weights (tomorrow more important than the day after tomorrow)
  - For example, ”prosperity in 100 years”

Decision Theory

Preferences

- Risk or uncertainty preference relation
  - How to weigh different states $s_1, \ldots, s_n$ against each other?
  - For example: Expected value
  - For example: Worst-case

Decision Theory

Decision Trees

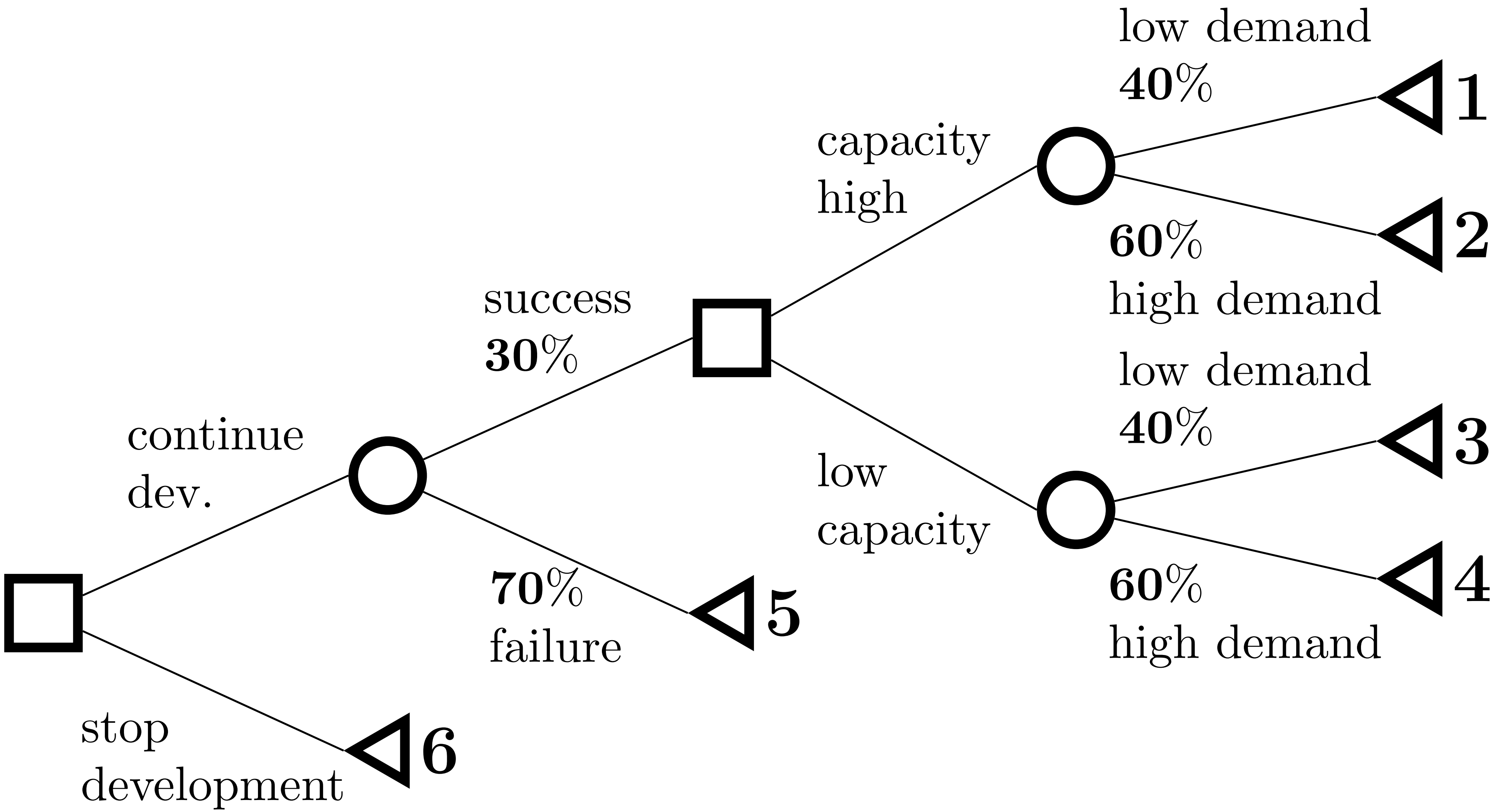
- Alternative to decision matrix
- Easier to show different consequences
- Decision: □
- Event: ○
- State: ◁

Decision Theory

Decision Trees: Example

- Company is faced with the question of whether product development should be continued or abandoned
- Estimated probability of successful completion of development is 30%
- If successful, the decision must be made whether to build large or small production capacity
- Estimated probability of high demand is 60%, for low demand is 40%

Decision Theory



Decision Theory

Conversion: Decision Tree ↔ Decision Matrices

	Successful development high demand $p_1 = 0.18$	Successful development low demand $p_2 = 0.12$	Development fails $p_3 = 0.7$
Continue development. If successful, large capacity	Consequence 1	Consequence 2	Consequence 5
Continue development. If successful, small capacity	Consequence 3	Consequence 4	Consequence 5
Abort development	Consequence 6	Consequence 6	Consequence 6

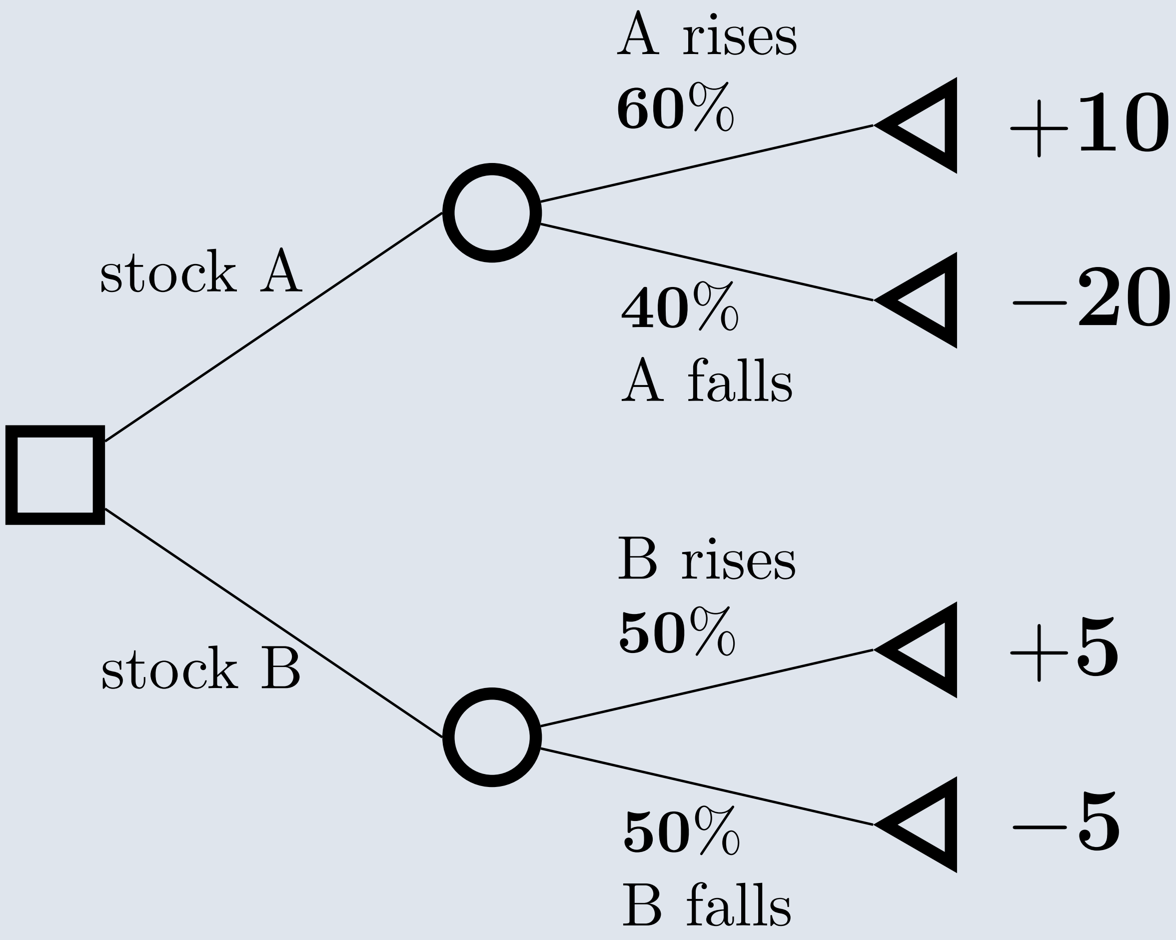
Decision Theory

Example

- We can either invest in Stock A or Stock B
- Probabilities and outcomes for...
  - Stock A rises: 60%, +10 Euro
  - Stock A falls: 40%, -20 Euro
  - Stock B rises: 50%, +5 Euro
  - Stock B falls: 50%, -5 Euro

Decision Theory

Example: Decision Tree



Decision Theory

Example

- **Incorrect** decision matrix:

	0.6	0.4	0.5	0.5
Buy A	10	-20	0	0
Buy B	0	0	5	-5

- What is the problem?
  - Undefined states
  - Probabilities sum up to a value over 1

Decision Theory

Example, correct:

- Correct model:

	0.5	0.4	0.1
Buy A	10	-20	10
Buy B	5	-5	-5

- Also correct:

	$A\nearrow B\nearrow$ 0.3	$A\nearrow B\searrow$ 0.3	$A\searrow B\nearrow$ 0.2	$A\searrow B\searrow$ 0.2
Buy A	10	10	-20	-20
Buy B	5	-5	5	-5

Decision Theory

Example

See example on blackboard

Decision Theory

Decision Under uncertainty

1. Game situation: s_j depends on the decisions of rational (opponent) players (e.g., Prisoner's Dilemma)
2. s_j is determined "by nature" (blind opponent)
   - Uncertainty: no known probabilities
   - Risk: probabilities are known

Decision Theory

Risk Situations

- Probabilities p_j for s_j, $\sum_{j \in [n]} p_j = 1$ known
- Subjective probabilities: formed by speculation or conviction
- Objective probabilities: formed by statistical observation

Decision Theory

Brief Summary: Probabilities

Ingredients:

- Set Ω, the sample space
  - e.g., $\{1, 2, 3, 4, 5, 6\}$ when rolling a dice
- σ-algebra Σ on Ω
  - $\Omega \in \Sigma$
  - $A \in \Sigma \Rightarrow \Omega \setminus A \in \Sigma$
  - $A_1, A_2, \ldots \in \Sigma \Rightarrow \cup_{i=1}^{\infty} A_i \in \Sigma$
  - e.g., Σ includes all subsets of $\{1, 2, 3, 4, 5, 6\}$ when rolling a dice
- $P : \Sigma \to [0, 1]$ is a probability distribution if
  - $P(\Omega) = 1$
  - $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$ for pairwise disjoint $A_i \in \Sigma$
  - e.g., $P(A) = \frac{1}{6}|A|$ when rolling a dice

Decision Theory

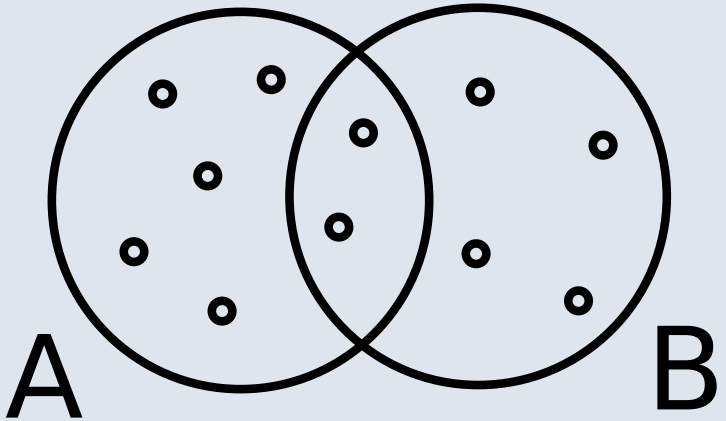
Example: Rolling a Dice

- A ="even number"
  - $A = \{2, 4, 6\}$
  - $P(A) = 3/6$
- B ="odd number"
  - $B = \{1, 3, 5\}$
  - $P(B) = 3/6$
- $A \cup B$ ="even or odd number"
  - $A \cup B = \{1, 2, 3, 4, 5, 6\}$
  - $P(A \cup B) = 1$

Decision Theory

Brief Summary: Probabilities

- Ω = all possible outcomes = all points
- Σ = subsets; $A, B \in \Sigma$
- Rules:
  - $P(\emptyset) = 0$
  - $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
  - if A, B are disjoint: $P(A \cup B) = P(A) + P(B)$
  - $P(A|B) = P(A \cap B) / P(B)$ conditional probability
  - $P(A \cap B) = P(A) \cdot P(B) \Leftrightarrow A, B$ are independent



Decision Theory

Quiz

Question 1 (Yes/No)

When rolling a dice, the outcomes ”even number” and ”4” are independent.

Question 2 (Yes/No)

The omelette problem is a case of decision under risk.

Question 3

Convert this decision matrix into a tree:

	s_1	s_2
	0.5	0.5
a_1	6	4
a_2	8	2

Decision Theory

Quiz

Question 1 (Yes/No)

When rolling a dice, the outcomes ”even number” and ”4” are independent.

No.

Question 2 (Yes/No)

The omelette problem is a case of decision under risk.

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Decision Theory

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Yes: uncertain with probabilities

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Decision Theory

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See drawing

Decision Theory

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Thank you!