Friedrich-Alexander-Universität
Erlangen-Nürnberg

Decision Theory

Lecture 11

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Decision Theory

Today

- Game theory

Decision Theory

What is it about?

- So far:
  - Decision under certainty = no opponent
  - Decision under uncertainty = ”against nature”
- Game theory: consciously acting opponent
- Examples:
  - Wage negotiations
  - Auctions
  - Competitive situations, e.g., in an oligopoly
  - Cold War

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Example: Prisoner's Dilemma (see Lecture 1)

- You and your accomplice stole apples and got arrested
- You are in different cells
- Confess or not?
  - You confess
    - Accomplice confesses: 4 years in prison for both
    - Accomplice does not confess: 1 year in prison for you, 6 years for him
  - You do not confess
    - Accomplice confesses: 6 years in prison for you, 1 year for him
    - Accomplice does not confess: 2 years in prison for both

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Example: Prisoner's Dilemma (see Lecture 1)

- Decision matrix:

	He confesses	He does not confess
I confess	(4,4)	(1,6)
I do not confess	(6,1)	(2,2)

- First value: my outcome
- Second value: accomplice's outcome
- Minimize prison years
- What should I do? Decision via survey.

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Example, continued

	He confesses	He does not confess
I confess	(4,4)	(1,6)
I do not confess	(6,1)	(2,2)

- From the perspective of game theory: choose the best strategy assuming that everyone follows that strategy
- So, I do not confess?
- ”Optimality” for game theory usually: ”Nash Equilibria,” more on that later

Decision Theory

Distinctions

- Number of players:
  - Two-person games
  - N-person games ($N > 2$) with or without the possibility of coalition formation
- Rules of the game:
  - Finite games (number of repetitions is fixed)
  - Infinite games (number of repetitions is not fixed)
- Strategy space:
  - Discrete strategy space
  - Continuous strategy space

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Distinctions, continued

- Strategy choice:
  - Pure strategy: Consistent behavior
  - Mixed strategy: Choosing a strategy using a random mechanism
- Timing of strategy choice:
  - Simultaneous, i.e., players have no knowledge (only assumptions) about the opponent's strategy choice
  - Sequential, i.e., players know the previous move or moves of the opponent

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Distinctions, continued

- Coordination between players:
  - Cooperative games (agreements between players)
  - Non-cooperative games (no agreements between players)
- Outcomes:
  - Games with variable gains
  - Constant-sum games
  - Zero-sum games

Terms and Representations

- Strategy:
  - Plan that contains, for each information available to player *i* at the time of executing a move, a (conditional) instruction on how to execute the move.
- Representation of games in normal form (payoff matrix):

		Player 2		
		Strategy b_1	...	Strategy b_n
Player 1	Strategy a_1	(u^1_{11}, u^2_{11})	...	(u^1_{1n}, u^2_{1n})
	$\vdots$	$\vdots$		$\vdots$
	Strategy a_m	(u^1_{m1}, u^2_{m1})	...	(u^1_{mn}, u^2_{mn})

- Representation as a game tree is also possible

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Examples

Player 1		Player 2		
		Strategy b_1	$\dots$	Strategy b_n
	Strategy a_1	(u^1_{11}, u^2_{11})	$\dots$	(u^1_{1n}, u^2_{1n})
	$\vdots$	$\vdots$		$\vdots$
	Strategy a_m	(u^1_{m1}, u^2_{m1})	$\dots$	(u^1_{mn}, u^2_{mn})

	b_1	b_2	b_3
a_1	$(10, 10)$	$(4, 4)$	$(0, 18)$
a_2	$(8, 7)$	$(8, -6)$	$(7, 8)$

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Examples

		Player 2		
		Strategy b_1	...	Strategy b_n
Player 1	Strategy a_1	(u^1_{11}, u^2_{11})	...	(u^1_{1n}, u^2_{1n})
	$\vdots$	$\vdots$		$\vdots$
	Strategy a_m	(u^1_{m1}, u^2_{m1})	...	(u^1_{mn}, u^2_{mn})

	b_1	b_2	b_3
a_1	$(10, 10)$	$(4, 4)$	$(0, 18)$
a_2	$(8, 7)$	$(8, -6)$	$(7, 8)$

Task!

Write a payoff matrix for Rock-Paper-Scissors

Decision Theory

Strong and Weak Dominance

- Strong Dominance:
  - Always better outcomes regardless of the opponent's strategy.
- Weak Dominance:
  - At least as good and better in one aspect of the outcome.

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Strong and Weak Dominance

	b_1	b_2	b_3
a_1	(10, 10)	(4, 4)	(0, 18)
a_2	(8, 7)	(8, -6)	(7, 8)

- Strong dominance: b_3 dominates b_1 and b_2
- Assumption: No one chooses strongly dominated strategies
- Then, player 1 will choose strategy a_2
- ”Iterated Dominance”: Repeatedly applying dominance checking

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Example of Iterated Dominance

	b_1	b_2	b_3
a_1	$(3, 6)$	$(19, 27)$	$(15, 15)$
a_2	$(18, 16)$	$(13, 28)$	$(4, 8)$
a_3	$(30, 5)$	$(13, 15)$	$(10, 26)$

Nash Equilibrium: Strategy Profile

- Assuming there are N players
- Player i chooses strategy $s_i \in S_i$
- Obtain strategy profile $s = (s_1, \ldots, s_N)$
- Corresponds to a cell in our payoff matrix

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Nash Equilibrium

- When no one wants to change their strategy in response
- A strategy profile where, for each player, there is no better strategy if others do not change theirs
- Formal: $\hat{s} = (\hat{s}_1, \ldots, \hat{s}_N)$ is a nash equilibrium if, for every player i and $s_i \in S_i$, it holds that;

$$u_i(\hat{s}) \geq u_i(\hat{s}_1, \ldots, \hat{s}_{i-1}, s_i, \hat{s}_{i+1}, \ldots, \hat{s}_N)$$

Decision Theory

Examples

Games with a unique nash equilibrium:

- Example 1

	b_1	b_2	b_3
a_1	(10, 10)	(4, 4)	(0, 18)
a_2	(8, 7)	(8, -6)	(7, 8)

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Examples

Games with a unique nash equilibrium:

- Example 1

	b_1	b_2	b_3
a_1	(10, 10)	(4, 4)	(0, 18)
a_2	(8, 7)	(8, -6)	(7, 8)

Decision Theory

Examples

Games with a unique nash equilibrium:

- Example 1

	b_1	b_2	b_3
a_1	(10, 10)	(4, 4)	(0, 18)
a_2	(8, 7)	(8, -6)	(7, 8)

- Example 2

	b_1	b_2	b_3
a_1	(7, -2)	(0, 2)	(9, 1)
a_2	(4, 4)	(3, 2)	(-2, 4)
a_3	(-2, 7)	(8, 8)	(7, 8)

Decision Theory

Examples

Games with a unique nash equilibrium:

- Example 1

	b_1	b_2	b_3
a_1	(10, 10)	(4, 4)	(0, 18)
a_2	(8, 7)	(8, -6)	(7, 8)

- Example 2

	b_1	b_2	b_3
a_1	(7, -2)	(0, 2)	(9, 1)
a_2	(4, 4)	(3, 2)	(-2, 4)
a_3	(-2, 7)	(8, 8)	(7, 8)

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Examples

- Games with multiple nash equilibria:

	b_1	b_2	b_3
a_1	(1, 1)	(0, 0)	(0, 0)
a_2	(0, 0)	(0, 0)	(100, 100)
a_3	(0, 0)	(10, 10)	(0, 0)

- Game without nash equilibrium:

	b_1	b_2	b_3
a_1	(7, 7)	(4, 4)	(18, 0)
a_2	(8, 8)	(8, 5)	(7, 10)
a_3	(0, 14)	(7, 9)	(-2, 18)

- Is there an equilibrium in the Prisoner's Dilemma?

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How to Find nash equilibria?

- Approach 1: check cell by cell
- Approach 2:
  - Mark highest values on the left per column
  - Mark highest values on the right per row
  - Check for overlap

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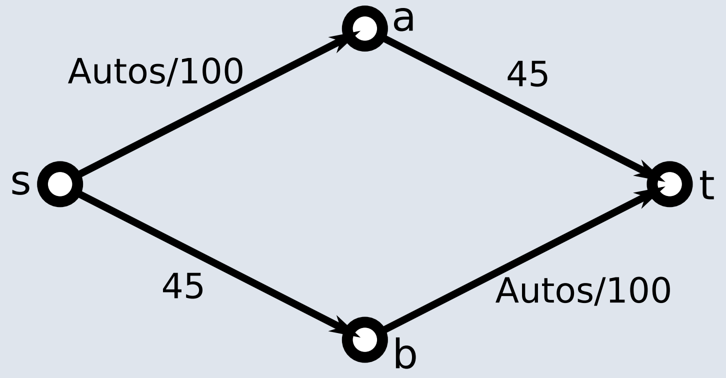
Braess's Paradox

- Introducing a new alternative can make everyone worse off



- 4000 drivers want to travel from s to t
- A: drivers going above. Travel time A: $A/100 + 45$
- B: drivers going below. Travel time B: $B/100 + 45$
- $A + B = 4000$
- Nash equilibrium: $A = B = 2000$, each $2000/100 + 45 = 65$ minutes

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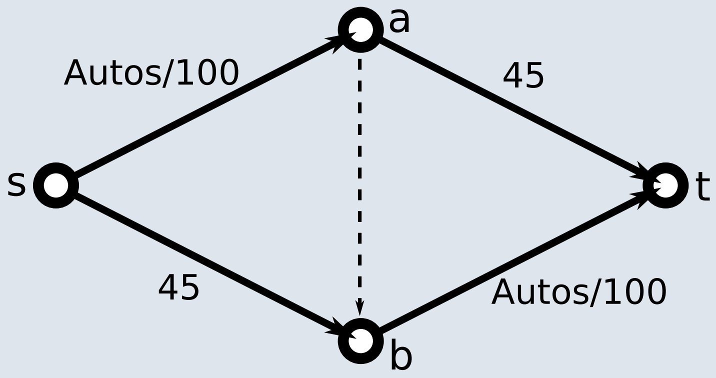
Braess's Paradox



- Build new road $a \to b$ with travel time 0
- For someone currently in A or B: travel time $s - a - b - t$ is now $2000/100 + 2001/100 = 40.01$ minutes
- Players switch, switch, switch, ...
- $s - a - b - t$ remains the best alternative but gets worse over time
- In the end: everyone uses $s - a - b - t$
- Duration: $4000/100 + 4000/100 = 80$ minutes

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Games of the Prisoner's Dilemma Type

	b_1	b_2
a_1	(4, 4)	(0, 6)
a_2	(6, 0)	(2, 2)

- In matrix: payoff instead of damage
- Wide application: two companies want to set prices
- a_1/b_1: keep the price
- a_2/b_2: reduce the price
- Nash equilibrium at the ”bad” point (a_2, b_2)

Games of the ”Battle of the Sexes” Type

- A and B want to go out
- Theater or boxing match
- Independently choose either theater ticket (a_1/b_1) or ticket for boxing match (a_2/b_2)
- A prefers a joint theater evening
- B prefers a joint boxing match
- Separated worse than being together

	b_1	b_2
a_1	(2, 1)	(-1, -1)
a_2	(-1, -1)	(1, 2)

- Nash equilibria are at (2, 1) and (1, 2)

Decision Theory

Cooperation

- Simultaneous, independent decision-making is difficult to lead to the desired outcome
- Assume both flip a coin (mixed strategy)
- A: π_A for a_1, $(1 - \pi_A)$ for a_2
- B: π_B for b_1, $(1 - \pi_B)$ for b_2
- Expected outcomes:
  - For A:

$$u_1(\pi_A, \pi_B) = 2\pi_A\pi_B - \pi_A(1 - \pi_B) - (1 - \pi_A)\pi_B + (1 - \pi_A)(1 - \pi_B)$$

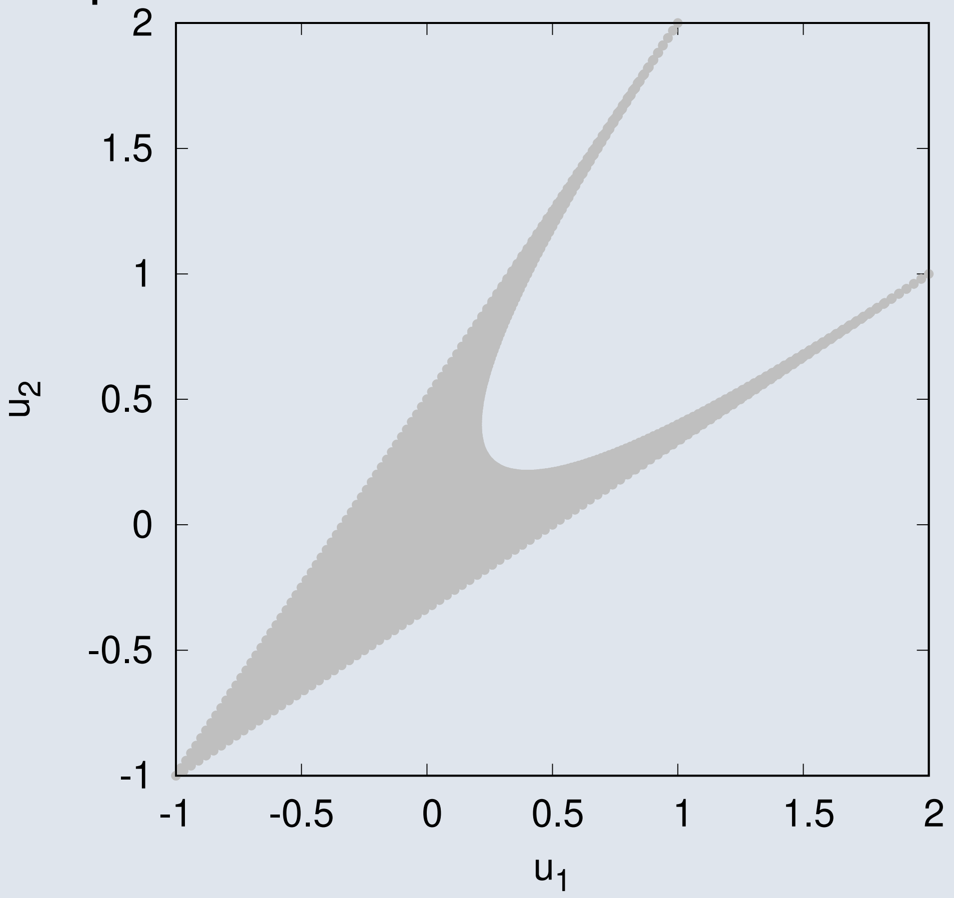
  - For B:

$$u_2(\pi_A, \pi_B) = 1\pi_A\pi_B - \pi_A(1 - \pi_B) - (1 - \pi_A)\pi_B + 2(1 - \pi_A)(1 - \pi_B)$$

Decision Theory

Cooperation

All possible outcomes:



$$u_1\left(\frac{1}{2},\frac{1}{2}\right) = u_2\left(\frac{1}{2},\frac{1}{2}\right) = \frac{1}{4}$$

Decision Theory

Cooperation

- Suppose we allow agreements instead
- Coin toss would lead to utility expectation of (1.5, 1.5)
- Alternating agreement on Boxing/Theater also averages utility value of (1.5, 1.5)
- Point (1.5, 1.5) is outside the previously possible range!
- We call all cooperatively attainable outcomes K

Decision Theory

Nash Solutions

- What are the "optimal solutions" in a cooperative game?
- Let $(\overline{u}_1, \overline{u}_2)$ be a "special value"
- The value one would get without an agreement, e.g.,
  - No trade: $(\overline{u}_1, \overline{u}_2) = (0, 0)$
  - No cooperation: guaranteed payout in the non-cooperative game
- Goal: find point $(\hat{u}_1, \hat{u}_2)$ in data $(K; \overline{u}_1, \overline{u}_2)$

$$F : (K; \overline{u}_1, \overline{u}_2) \mapsto (\hat{u}_1, \hat{u}_2) \in K$$

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Requirements for Nash Solution

- Requirement 1: Independence from linear transformations
  - If utility zero points and utility units are changed, the solution $(\hat{u}_1, \hat{u}_2)$ changes in the same proportion
- Requirement 2: Individual Rationality
  - Solution $(\hat{u}_1, \hat{u}_2)$ must satisfy inequalities $\hat{u}_1 \geq \overline{u}_1$, $\hat{u}_2 \geq \overline{u}_2$
  - No rational decision-maker would accept an agreement that puts them in a worse position than a failure of negotiations

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Cooperative Games

- Requirement 3: Pareto-Optimality
  - Solution $(\hat{u}_1, \hat{u}_2)$ is undominated
- Requirement 4: Symmetry
  - If the roles of both players are completely symmetrical, the solution is also symmetrical, i.e., $\hat{u}_1 = \hat{u}_2$.
- Requirement 5: Independence of irrelevant alternatives
  - If $\tilde{K}$ is a subset of K that includes both the point $(\overline{u}_2, \overline{u}_2)$ and the solution $(\hat{u}_1, \hat{u}_2)$ of $(K; \overline{u}_1, \overline{u}_2)$, then $F_0(\tilde{K}; \overline{u}_1, \overline{u}_2) = (\hat{u}_1, \hat{u}_2)$ also holds.

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Summary of Requirements

$F_0(K; \overline{u}_1, \overline{u}_2) = (\hat{u}_1, \hat{u}_2)$ is the maximum of the following optimization problem:

$$\begin{aligned} \max\ & (u_1 - \overline{u}_1) \cdot (u_2 - \overline{u}_2) \\ \text{s.t. } & (u_1, u_2) \in K \\ & u_1 \geq \overline{u}_1 \\ & u_2 \geq \overline{u}_2 \end{aligned}$$

Theorem

There is exactly one arbitration mechanism defined on the set of all bargaining games $(K; \overline{u}_1, \overline{u}_2)$ that simultaneously satisfies all five requirements; this is given by the F_0 described above.

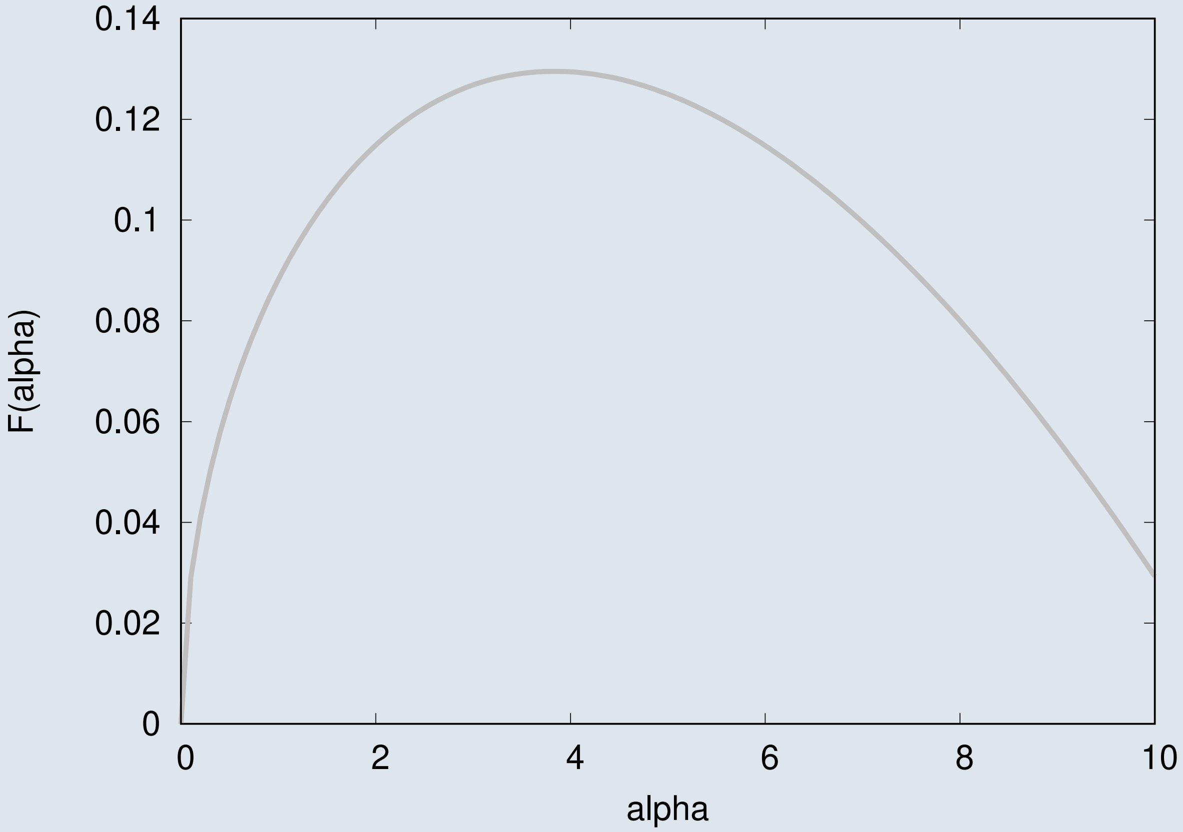
Decision Theory

Example: Resolution of a Wage Dispute

- Unions and employers negotiate a wage increase α in the range from 0 to 10 percent ($0 \leq \alpha \leq 10$)
- Unions: $u_1(\alpha) = \sqrt{\frac{\alpha}{20}}$, $\overline{u}_1 = \frac{1}{5}$
- Employers: $u_2(\alpha) = 1 - \frac{\alpha}{20}$, $\overline{u}_2 = \frac{3}{5}$
- Maximize:

$$\begin{aligned}(u_1 - \overline{u}_2) \cdot (u_2 - \overline{u}_2) &= \left(\sqrt{\frac{\alpha}{20}} - \frac{1}{5}\right) \cdot \left(1 - \frac{\alpha}{20} - \frac{3}{5}\right) \\ &= \frac{2}{5} \cdot \sqrt{\frac{\alpha}{20}} - \frac{2}{25} - \left(\frac{\alpha}{20}\right)^{\frac{3}{2}} + \frac{\alpha}{100}\end{aligned}$$

Example: Resolution of a Wage Dispute



Optimal Solution: α = 3.8 ; u_1 = 0.44 ; u_2 = 0.81

Issues and Applications

- Corresponding data is usually not available
- In reality, negotiation games are often divided into a threat game and a demand game (two-stage game): Threats (e.g., strike and lockout) are only implemented if the negotiation solution fails
- Player behavior often does not meet the rationality postulate (problem applies to the entire analytical game theory)
- However, game theory is very suitable for analyzing conflict and decision situations

Decision Theory

Quiz

Question 1

Is there a nash equilibrium? (non-coop, pure strategies)

	b_1	b_2
a_1	(0,1)	(1,0)
a_2	(1,0)	(0,1)

Question 3

What is the best cooperative solution?

	b_1	b_2
a_1	(0,1)	(1,0)
a_2	(1,0)	(0,1)

Question 3

Is there a nash equilibrium? (non-coop, pure strategies)

	b_1	b_2
a_1	(3,2)	(1,4)
a_2	(2,2)	(4,3)

Decision Theory

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Solution

No.

Decision Theory

Quiz

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Solution

No.

Question 3

What is the best cooperative solution?

	b_1	b_2
a_1	(0,1)	(1,0)
a_2	(1,0)	(0,1)

Solution

Everyone plays $1/2$ with utility $(1/2, 1/2)$.

Question 3

Is there a nash equilibrium? (non-coop, pure strategies)

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Decision Theory

Quiz

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Solution

No.

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Everyone plays $1/2$ with utility $(1/2, 1/2)$.

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Is there a nash equilibrium? (non-coop, pure strategies)

	b_1	b_2
a_1	(3,2)	(1,4)
a_2	(2,2)	(4,3)

Solution

Yes: (a_2, b_2).