Friedrich-Alexander-Universität
Erlangen-Nürnberg

Decision Theory

Lecture 7

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Decision Theory

Recap: What did we do?

- Utilité Additive (UTA)
- Intertemporal decision making

Decision Theory

UTA

- Optimization approach
- Given:
  - Criteria
  - Alternatives A
  - Score matrix
  - Ranking information on $A^R \subseteq A$
- Find:
  - Utility function for each criterion
  - Such that rankings are reproduced
  - As close as possible

Decision Theory

UTA

- Let $[\alpha_i, \beta_i]$ be range of criterion i
- Use grid of points $x_i^0, x_i^1, \dots, x_i^{\gamma_i}$ in interval
- Set u_i as a piece-wise linear function
- We only need to find values $u_i(x_i^j)$
- Then $U(x) = \sum_{i \in [n]} u_i(x_i)$ with

$$u_i(a) = u_i(x_i^j) + \frac{u_i(x_i^{j+1}) - u_i(x_i^j)}{x_i^{j+1} - x_i^j}(g_i(a) - x_i^j)$$

Decision Theory

UTA

- Determine $u_i(x_i^j)$ such that

$$\begin{aligned}
U(a_k) &\geq U(a_{k+1}) + \varepsilon && \forall k \in [m-1] : a_k \succ a_{k+1} \\
U(a_k) &= U(a_{k+1}) && \forall k \in [m-1] : a_k \sim a_{k+1} \\
u_i(x_i^{j+1}) - u_i(x_i^j) &\geq 0 && \forall i \in [n], j \in [\gamma_i - 1] \\
u_i(\alpha_i) &= 0 && \forall i \in [n] \\
\sum_{i \in [n]} u_i(\beta_i) &= 1
\end{aligned}$$

Decision Theory

UTA Example

- I want to rank my favourite actors

- Criteria:
  1. Academy awards, including nominations
  2. Voice quality
  3. Acting quality

Decision Theory

UTA Example

	awards	voice	acting
Sam Neill	0	8	8
Sean Bean	0	10	7
Leo diC	5	3	6
Jason Statham	0	7	2
Keanu Reeves	0	6	7
Nic Cage	2	2	10

Partial ranking:

$$\text{Sam Neill} \succ \text{Leo diC} \succ \text{Jason Statham}$$

UTA Example

- I use a simple grid where only α_i and β_i are included (affine linear functions)
- Write as an optimization problem:
  - Find value functions fulfilling my ranking constraints
  - No objective function
- Solution:

	awards	voice	acting
$u_i(\alpha_i)$	0.000	0.000	0.000
$u_i(\beta_i)$	0.334	0.666	0.000

Decision Theory

UTA Example

	awards	voice	acting
$u_i(\alpha_i)$	0.000	0.000	0.000
$u_i(\beta_i)$	0.334	0.666	0.000

	awards	voice	acting	score
Sam Neill	0	8	8	0.4995
Sean Bean	0	10	7	0.6660
Leo diC	5	3	6	0.4173
Jason Statham	0	7	2	0.4163
Keanu Reeves	0	6	7	0.3330
Nic Cage	2	2	10	0.1336

UTA Example

1.  0.6660
2.  0.4995
3.  0.4173
4.  0.4163
5.  0.3330
6.  0.1336

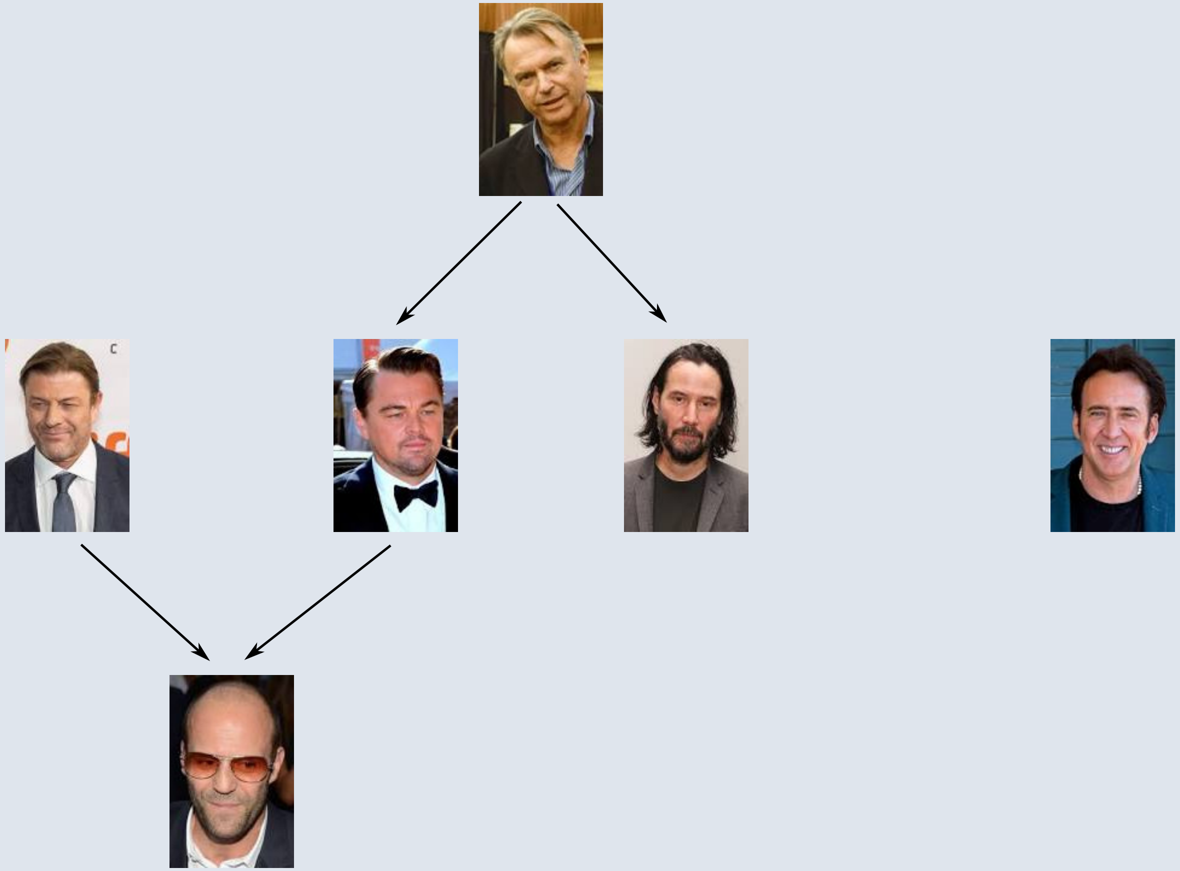
Decision Theory

UTA

- Is this the end of the story?
- There can be many possible value functions fulfilling the criteria
- We have chosen an arbitrary one
- Necessary preference relations $\succeq^N$:
  - Hold for every compatible value function
  - Found by minimizing value difference over compatible value functions
- Possible preference relations $\succeq^P$
  - Hold for at least one compatible value function
  - Found by maximizing value difference over compatible value functions
- Solve LP for each pair of alternatives, checking $\succeq^N$ and $\succeq^P$

Decision Theory

UTA Example



Decision Theory

UTA

- How to find a representative ranking?
- Maximize value difference for alternatives where necessary preference is known
  - Max difference of Neill over diCaprio and Reeves
  - Max difference of Bean and diCaprio over Statham
- Lexicographically, minimize value difference for alternatives where this is not the case
  - Min difference between Bean, diCaprio, and Reeves

Decision Theory

UTA Example

	awards	voice	acting
$u_i(\alpha_i)$	0.000	0.000	0.000
$u_i(\beta_i)$	0.111	0.000	0.889

	awards	voice	acting	score
Sam Neill	0	8	8	0.6667
Sean Bean	0	10	7	0.5556
Leo diC	5	3	6	0.5556
Jason Statham	0	7	2	0.0000
Keanu Reeves	0	6	7	0.5556
Nic Cage	2	2	10	0.9333

Decision Theory

UTA Example

1.		0.9333
2.		0.6667
3.		0.5556
3.		0.5556
3.		0.5556
6.		0.0000

Decision Theory

Intertemporal Decision Making

- Decisions over multiple time steps
- Utility at time τ:

$$U^\tau(a_i) = \delta^\tau u_\tau(e_i^\tau) + \beta \sum_{t=\tau+1}^{T} \delta^t u_t(e_i^t)$$

- δ: the further in the future, the less important
- β: inconsistency, today is more important than the future

Decision Theory

Intertemporal Decision Making

- Three types of decision makers
- Time consistent (TC)
  - $\beta = 1$
  - Does what is overall best
- Naive
  - x$\beta < 1$
  - Is not aware of being naive
  - Optimistic about the future
- Sophisticated
  - $\beta < 1$
  - Aware of own decision making
  - Realistic about the future

Decision Theory

Today

- Decision under Uncertainty: Basics

Decision Theory

Problem

$$
\begin{array}{c|cccc}
 & s_1 & s_2 & \dots & s_n \\
\hline
a_1 & e_{11} & e_{12} & \dots & e_{1n} \\
a_2 & e_{21} & e_{22} & \dots & e_{2n} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
a_m & e_{m1} & e_{m2} & \dots & e_{mn}
\end{array}
$$

- Know possible consequences for each decision
- Do not know which state will occur
- Do not know probabilities

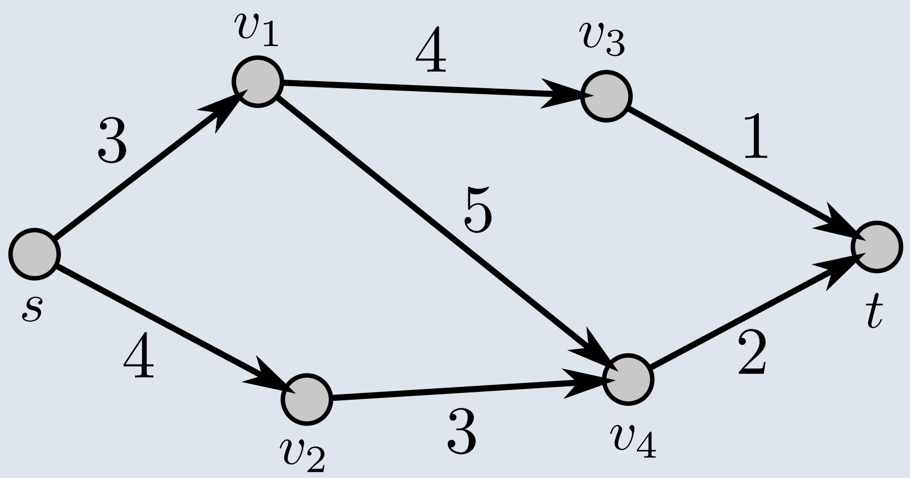
Decision Theory

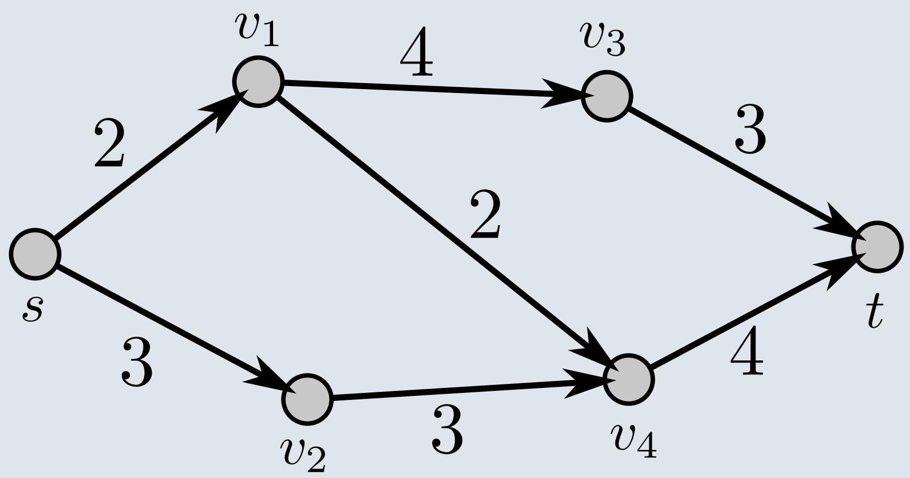
Criteria

- As with multi-criteria problems: there is no ”best” solution
- Can only decide if we set a criterion
- In this lecture:
  - Minimax/Maximin
  - Minimax Regret
  - Hurwicz
  - Average

Decision Theory

Example





What is a good path?

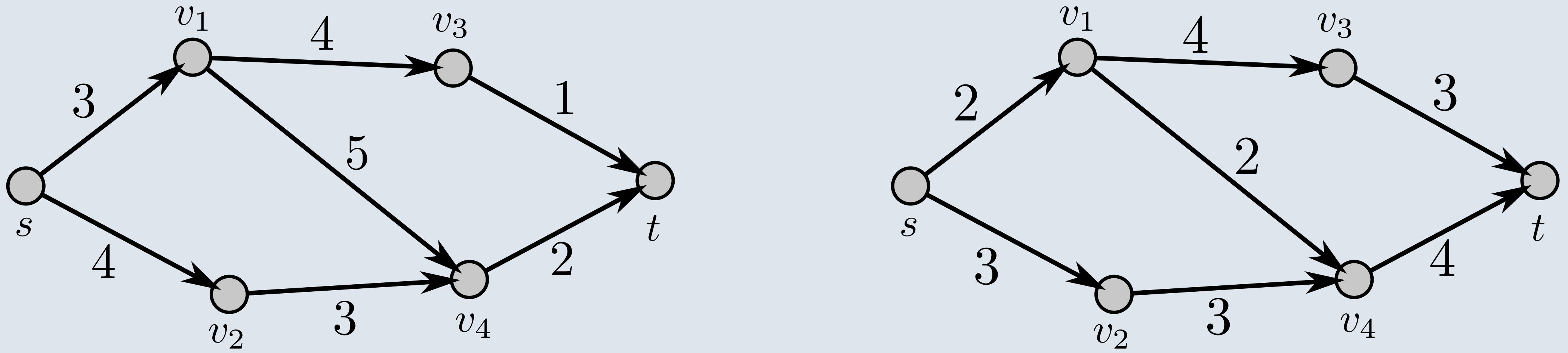
Path	Scenario 1	Scenario 2
Top	8	9
Middle	10	8
Bottom	9	10

Decision Theory

Maximin/Minimax

- Initial approach, aka "worst-case"
- Choose a solution so that the worst case is as good as possible
- For maximization problems: ”maximin” (maximize the minimum)
  - Formally: $\max_{i \in [m]} \min_{j \in [n]} e_{ij}$
- For minimization problems: ”minimax” (minimize the maximum)
  - Formally: $\min_{i \in [m]} \max_{j \in [n]} e_{ij}$

Example



What is a good path? (minimax)

Path	Scenario 1	Scenario 2	Max
Top	8	9	9
Middle	10	8	10
Bottom	9	10	10

Decision Theory

Interpretation

- Against ”devilish nature”
- Against ”saboteur/attacker”
- Conservative
  - A solution can always be good, except for once
  - We evaluate only based on the one bad case
- Provides guarantee:
  - Bridge should not collapse
  - Airplane should not crash
- In practice: choose meaningful scenarios

Minimax Regret

- Variant of Minimax/Maximin
- Always minimax; aims to minimize "regret" always
- Evaluate not based on the worst case but through the difference to the best case in each scenario (what could I have achieved?)
- For maximization problems:

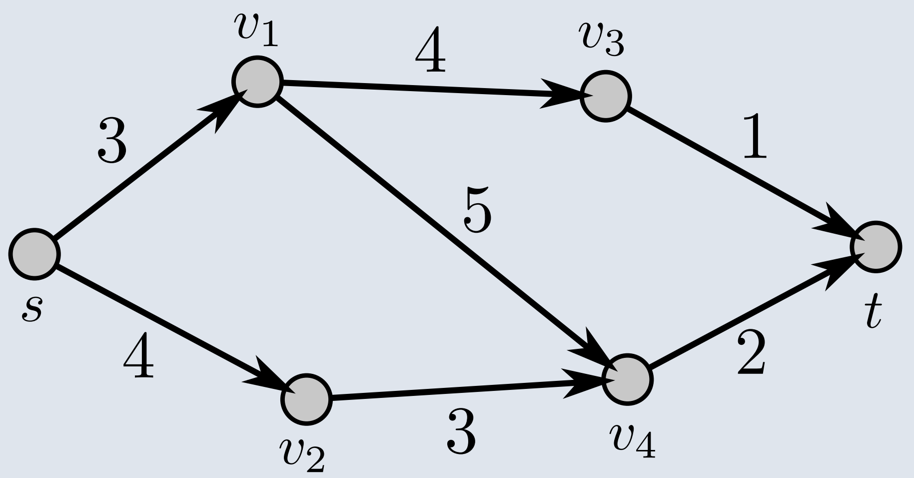
$$\min_{i\in[m]} \max_{j\in[n]} \left(\max_{\ell\in[m]} e_{\ell j} - e_{ij} \right)$$

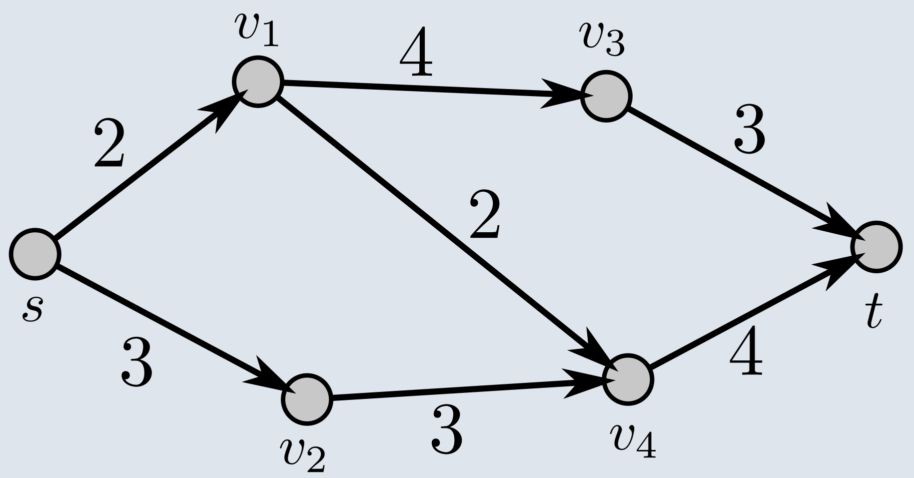
- For minimization problems:

$$\min_{i\in[m]} \max_{j\in[n]} \left(e_{ij} - \min_{\ell\in[m]} e_{\ell j} \right)$$

Decision Theory

Example





What is a good path?

Path	Scenario 1	Scenario 2	Max Regret
Top	8-8=0	9-8=1	1
Middle	10-8=2	8-8=0	2
Bottom	9-8=1	10-8=2	2

Decision Theory

Minimax Regret, Example

- Maximize

	s_1	s_2
a_1	0	100
a_2	1	1

$\rightarrow$

Difference to the best case:

	s_1	s_2
a_1	1	0
a_2	0	99

- Maximin: choose a_2
- Minimax regret: choose a_1

Decision Theory

Hurwicz

- Mix of pessimism and optimism
- Let m_i be the worst value for a_i
- Let M_i be the best value for a_i
- Let $\alpha \in [0, 1]$ be a fixed value, "optimism parameter"
- Choose the alternative for which $\alpha m_i + (1 - \alpha)M_i$ is best
- For $\alpha = 1$: Maximin/Minimax (worst case)
- For $\alpha = 0$: Maximax/Minimin (best case)

Decision Theory

Hurwicz – Problem

- Maximize, with $\alpha = 1/4$

	s_1	s_2	s_3
a_1	0	1	0
a_2	1	0	0
$\frac{1}{2}(a_1 + a_2) = a_3$	1/2	1/2	0

- Hurwicz of a_1 is $1/4 \cdot 0 + 3/4 \cdot 1 = 3/4$
- Hurwicz of a_2 is $1/4 \cdot 0 + 3/4 \cdot 1 = 3/4$
- Hurwicz of a_3 is $1/4 \cdot 0 + 3/4 \cdot 1/2 = 3/8$
- a_1 and a_2 are optimal, but not their combination

Decision Theory

Problem for all Methods

	s_1	s_2	s_3	$\dots$	s_{100}
a_1	0	1	1	$\dots$	1
a_2	1	0	0	$\dots$	0

- Intuitively, we find a_1 better
- According to Maximin, Regret, Hurwicz: both equally good
- Equivalent to

	s_1	s_2
a_1	0	1
a_2	1	0

Decision Theory

Principle of Insufficient Information

When one does not know the probabilities of the states $s_1, \ldots, s_n$, one should behave as if they are equally probable.

Average (Laplace)

Choose decision *i* that maximizes/minimizes

$$\frac{e_{i1} + e_{i2} + \ldots + e_{in}}{n}$$

Decision Theory

Example

Maximize:

	s_1	s_2	s_3	s_4
a_1	8	6	0	10
a_2	3	5	2	4
a_3	4	5	3	7
a_4	2	11	2	3

Determine the best solution according to Maximin, Average, Hurwicz with $\alpha = 3/4$, and Regret

Decision Theory

Example

Maximize:

	s_1	s_2	s_3	s_4	Min	Max	Average	Hurwicz	Regret
a_1	8	6	0	10	0	10	24/4	10/4	5
a_2	3	5	2	4	2	5	14/4	11/4	6
a_3	4	5	3	7	3	7	19/4	16/4	6
a_4	2	11	2	3	2	11	18/4	17/4	7
Max	8	11	3	10					

Determine the best solution according to Maximin, Average, Hurwicz with $\alpha = 3/4$, and Regret

Result: Maximin: a_3, Average: a_1, Hurwicz: a_4, Regret: a_1

Decision Theory

Example

Maximize:

	s_1	s_2	s_3
a_1	2	12	-3
a_2	5	5	-1
a_3	0	10	-2

Determine rankings according to Maximin, Hurwicz with $\alpha = 3/4$, and Average

Decision Theory

Example

Maximize:

	s_1	s_2	s_3
a_1	2	12	-3
a_2	5	5	-1
a_3	0	10	-2

Determine rankings according to Maximin, Hurwicz with $\alpha = 3/4$, and Average

- Maximin: a_2, a_3, a_1
- Hurwicz: a_3, a_1, a_2
- Average: a_1, a_2, a_3

Decision Theory

Axiomatic Approach

- We now know four decision criteria
  - Maximin (aka ”worst case”)
  - Minimax Regret (aka ”Savage”)
  - Hurwicz
  - Average (aka ”Laplace”)
- Which one to use depends on the decision maker
- Axiomatic approach: what properties do these criteria fulfill?
- Introduce 8 desirable properties (axioms)

Decision Theory

Axiom 1 (Complete Ranking)

A decision rule should provide a complete ranking of alternatives.

Explanation

- Transitive, complete order
- We can build a list from 1st to last place

Axiom 2 (Independence of Order)

For an $m \times n$ outcome matrix, a decision rule should yield the same result for any permutation of columns or rows.

Explanation

- These problems should yield the same solution:

	s_1	s_2
a_1	10	0
a_2	8	3

	s_1	s_2
a_2	8	3
a_1	10	0

	s_2	s_1
a_1	0	10
a_2	3	8

Decision Theory

Axiom 3 (Independence of Scaling)

For an $m \times n$ decision matrix with entries e_{ij}, a decision rule should give the same result if we use instead

$$e'_{ij} = \alpha e_{ij} + \beta$$

for fixed $\alpha > 0$, β.

Explanation

- Can double the entire table
- Can add +10 to all values
- Can convert meters to kilometers
- Can convert degrees Celsius to Fahrenheit

Axiom 4 (Strong Dominance)

If for two alternatives a_i, a_j it holds that $e_{ik} > e_{jk}$ for all $k \in [n]$, then a decision rule should prefer a_i over a_j.

Explanation

- If one alternative is always better, it must also be ranked as better
- Example:

	s_1	s_2	s_3
a_1	8	9	4
a_2	3	1	3

- We must strictly prefer a_1 over a_2

Axiom 5 (Irrelevant Alternatives)

If an additional row is added to a decision matrix, it should not change the order within the previous rows.

Explanation

- A guest enters a new restaurant
- The menu offers salmon for 15 euros or steak for 20 euros
- In a very good restaurant, the guest would choose the steak
- In this case, the guest opts for the cheaper salmon
- The waiter comes and mentions that there are frog legs today
- Then the guest decides on the steak

Axiom 5 – Explanation, Continued

- Action violates axiom 5 (alternative added, optimality of old changes)
- But is it irrational?
- Frog legs were an indication of the restaurant's quality for the guest
- Axiom 5: new alternatives do not change the a priori information about what the true state is
- Availability of alternatives does not change the plausibility of the states

Decision Theory

Axiom 6 (Constants to Columns)

If a constant c is added to a column, the order of decisions should remain the same.

Explanation

- For example:

	s_1	s_2
a_1	6	4
a_2	3	8

	s_1	s_2
a_1	16	4
a_2	13	8

- The bonus +10 is received in every alternative
- Note: different from axiom 3, where all entries are changed

Decision Theory

Axiom 7 (State Permutation)

Let a_i and a_j be two alternatives. If there exists a permutation π of the states such that

$$e_{ik} = e_{j\pi(k)},$$

then a decision rule should evaluate both alternatives equally.

Explanation

- For example:

	s_1	s_2	s_3
a_1	6	0	3
a_2	0	6	3

- a_1 and a_2 should be equally good.

Decision Theory

Axiom 8 (Column Duplication)

We can duplicate columns in a decision matrix without changing the ranking of alternatives.

Explanation

- The following problems should lead to the same evaluation:

	s_1	s_2
a_1	9	4
a_2	2	6

	s_1	s_2	s_3	s_4	s_5
a_1	9	4	4	4	4
a_2	2	6	6	6	6

Decision Theory

Summary

- Axiom 1: Complete ranking
- Axiom 2: Independence of order
- Axiom 3: Independence of scaling
- Axiom 4: Strong dominance
- Axiom 5: Irrelevant alternatives
- Axiom 6: Constants to columns
- Axiom 7: State permutation
- Axiom 8: Column duplication

Decision Theory

Result

	Minimax	Hurwicz	Regret	Average
Ax 1	✓	✓	✓	✓
Ax 2	✓	✓	✓	✓
Ax 3	✓	✓	✓	✓
Ax 4	✓	✓	✓	✓
Ax 5	✓	✓	✗	✓
Ax 6	✗	✗	✓	✓
Ax 7	✓	✓	✗	✓
Ax 8	✓	✓	✓	✗

- Ax 5: Irrelevant alternatives
- Ax 6: Constants to columns
- Ax 7: State permutation
- Ax 8: Column duplication

Decision Theory

Irrelevant Alternatives

- Minimax regret is not independent of irrelevant alternatives:
- Maximize

	s_1	s_2	s_2
a_1	0	10	4
a_2	5	2	10

- Regret for a_1 is 6, regret for a_2 is 8 $\rightarrow$ choose a_1
- Add a new alternative:

	s_1	s_2	s_2
a_1	0	10	4
a_2	5	2	10
a_3	10	5	1

- New regrets are 10 for a_1, 8 for a_2, 9 for a_3 $\rightarrow$ choose a_2

Decision Theory

Theorem

If a criterion satisfies the following axioms:

- Axiom 1: Complete ranking
- Axiom 4: Strong dominance
- Axiom 5: Irrelevant alternatives
- Axiom 6: Constants to columns
- Axiom 7: State permutation

then it is equivalent to the Average criterion.

Consequence

There is no decision rule that satisfies all 8 axioms.

Decision Theory

Quiz

Question 1

True or false?

The Average criterion satisfies axiom 8: duplicating columns does not change the result.

Question 2

Maximize:

	s_1	s_2	s_3	s_4
a_1	0	10	5	5
a_2	9	0	1	0
a_3	3	1	1	10
a_4	5	2	0	5

- A decision maker prefers a_4
- Is this compatible with
  - Maximin
  - Hurwicz for any α
  - Minimax Regret
  - Average

Decision Theory

Quiz

Question 1

True or false?

The Average criterion satisfies axiom 8: duplicating columns does not change the result.

Solution

False: duplicating a column makes it more influential in the evaluation.

Question 2

Maximize:

	s_1	s_2	s_3	s_4
a_1	0	10	5	5
a_2	9	0	1	0
a_3	3	1	1	10
a_4	5	2	0	5

- A decision maker prefers a_4
- Is this compatible with
  ✗ Maximin

  ✗ Hurwicz for any α

  ✓ Minimax Regret

  ✗ Average