Friedrich-Alexander-Universität
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Decision theory

Exercise 3

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Decision theory – Exercise

DEA – Model 1

$$
\begin{aligned}
\min\ & \theta_j \\
\text{s.t.}\ & \sum_{i \in I} \lambda_i x_{ik} \le \theta_j x_{jk} && \forall k \in [N] \\
& \sum_{i \in I} \lambda_i y_{ik} \ge y_{jk} && \forall k \in [M] \\
& \theta_j \in \mathbb{R} \\
& \lambda_i \in \mathbb{R}_+ && \forall i \in I
\end{aligned}
$$

Data

Branch	(1)	(2)	(3)
A	125	50	18
B	44	20	16
C	80	55	17
D	23	12	11

with:

(1) Private transactions

(2) Business transactions

(3) Employees

Exercise 1

Write the DEA LP for DMU A. Which branch is efficient?

Decision theory – Exercise

Solution 1, Part 1

(1), (2) are Output, (3) is Input

$$\begin{aligned} &\min \theta_1 \\ \text{s.t. } & 18\lambda_1 + 16\lambda_2 + 17\lambda_3 + 11\lambda_4 \leq 18\theta_1 \\ & 125\lambda_1 + 44\lambda_2 + 80\lambda_3 + 23\lambda_4 \geq 125 \\ & 50\lambda_1 + 20\lambda_2 + 55\lambda_3 + 12\lambda_4 \geq 50 \\ & \theta_1, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0 \end{aligned}$$

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Solution 1, Part 2

Have only one input $\rightarrow$ can normalize

Branch	(1)	(2)	(3)
A	6.9	2.8	1
B	2.8	1.3	1
C	4.7	3.2	1
D	2.1	1.1	1

Only A and C are efficient in terms of DEA (supported Pareto efficient).

Decision Theory - Exercise

AHP: Normalized Columns

How do I compute suitable weights w_1, w_2, w_3 from a comparison matrix

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

such that $r_{ij} \approx w_i / w_j$?

- Using eigenvectors of R
- Through geometric mean
- Via the method of least squares
- Now: normalized columns

Decision Theory - Exercise

AHP: Normalized Columns

1. Normalize R such that the sum of each column equals 1
2. w_i is the mean of each row

The method is simple but lacks a theoretical foundation.

Example

$$R = \begin{pmatrix} 1 & 3 & 1/2 \\ 1/3 & 1 & 2 \\ 2 & 1/2 & 1 \end{pmatrix} \to \overline{R} = \begin{pmatrix} 3/10 & 6/9 & 1/7 \\ 1/10 & 2/9 & 4/7 \\ 6/10 & 1/9 & 2/7 \end{pmatrix} \to w \approx \begin{pmatrix} 0.37 \\ 0.30 \\ 0.33 \end{pmatrix}$$

Decision Theory - Exercise

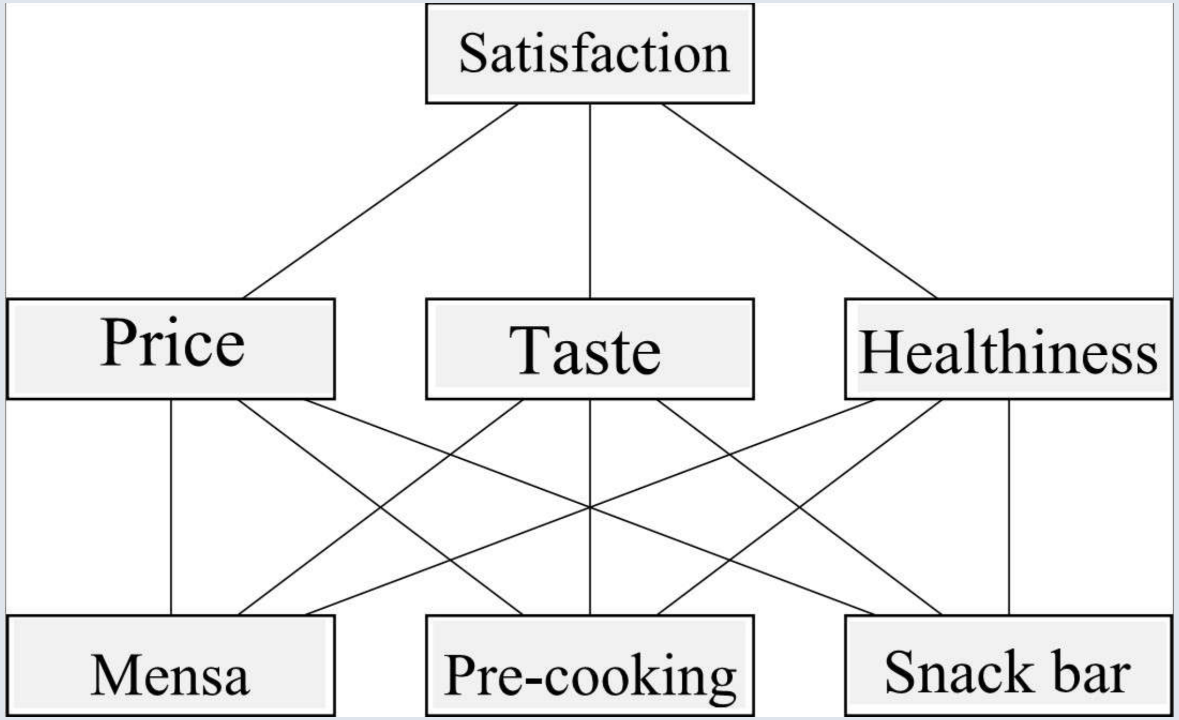
Exercise 2

Model and solve the following problem using AHP:

- For lunch break, you have three options:
  - Going to the cafeteria
  - Pre-cooking for the next day in the evening
  - Going to a snack bar
- Relevant criteria are:
  - Price
  - Taste
  - Healthiness
- Maximize your overall satisfaction
- Use normalized columns

Decision Theory - Exercise

Solution 2 (Example)



Solution 2 (Example)

Comparison matrices and weight vectors:

$$R^{\text{Price}} = \begin{pmatrix} 1 & 1/2 & 1 \\ 2 & 1 & 2 \\ 1 & 1/2 & 1 \end{pmatrix},$$

,

,

Decision Theory - Exercise

Solution 2 (Example)

Comparison matrices and weight vectors:

$$R^{\text{Price}} = \begin{pmatrix} 1 & 1/2 & 1 \\ 2 & 1 & 2 \\ 1 & 1/2 & 1 \end{pmatrix},$$

$$R^{\text{Taste}} = \begin{pmatrix} 1 & 3/2 & 1 \\ 2/3 & 1 & 2/3 \\ 1 & 3/2 & 1 \end{pmatrix},$$

,

Decision Theory - Exercise

Solution 2 (Example)

Comparison matrices and weight vectors:

$$R^{\text{Price}} = \begin{pmatrix} 1 & 1/2 & 1 \\ 2 & 1 & 2 \\ 1 & 1/2 & 1 \end{pmatrix},$$

$$R^{\text{Taste}} = \begin{pmatrix} 1 & 3/2 & 1 \\ 2/3 & 1 & 2/3 \\ 1 & 3/2 & 1 \end{pmatrix},$$

$$R^{\text{Healthiness}} = \begin{pmatrix} 1 & 1/2 & 4/3 \\ 2 & 1 & 7/3 \\ 3/4 & 3/7 & 1 \end{pmatrix},$$

Decision Theory - Exercise

Solution 2 (Example)

Comparison matrices and weight vectors:

$$R^{\text{Price}} = \begin{pmatrix} 1 & 1/2 & 1 \\ 2 & 1 & 2 \\ 1 & 1/2 & 1 \end{pmatrix}, \qquad w^{\text{Price}} = \begin{pmatrix} 0.25 \\ 0.5 \\ 0.25 \end{pmatrix}$$

$$R^{\text{Taste}} = \begin{pmatrix} 1 & 3/2 & 1 \\ 2/3 & 1 & 2/3 \\ 1 & 3/2 & 1 \end{pmatrix},$$

$$R^{\text{Healthiness}} = \begin{pmatrix} 1 & 1/2 & 4/3 \\ 2 & 1 & 7/3 \\ 3/4 & 3/7 & 1 \end{pmatrix},$$

Decision Theory - Exercise

Solution 2 (Example)

Comparison matrices and weight vectors:

$$R^{\text{Price}} = \begin{pmatrix} 1 & 1/2 & 1 \\ 2 & 1 & 2 \\ 1 & 1/2 & 1 \end{pmatrix}, \qquad w^{\text{Price}} = \begin{pmatrix} 0.25 \\ 0.5 \\ 0.25 \end{pmatrix}$$

$$R^{\text{Taste}} = \begin{pmatrix} 1 & 3/2 & 1 \\ 2/3 & 1 & 2/3 \\ 1 & 3/2 & 1 \end{pmatrix}, \qquad w^{\text{Taste}} = \begin{pmatrix} 0.375 \\ 0.25 \\ 0.375 \end{pmatrix}$$

$$R^{\text{Healthiness}} = \begin{pmatrix} 1 & 1/2 & 4/3 \\ 2 & 1 & 7/3 \\ 3/4 & 3/7 & 1 \end{pmatrix},$$

Decision Theory - Exercise

Solution 2 (Example)

Comparison matrices and weight vectors:

$$R^{\text{Price}} = \begin{pmatrix} 1 & 1/2 & 1 \\ 2 & 1 & 2 \\ 1 & 1/2 & 1 \end{pmatrix}, \qquad w^{\text{Price}} = \begin{pmatrix} 0.25 \\ 0.5 \\ 0.25 \end{pmatrix}$$

$$R^{\text{Taste}} = \begin{pmatrix} 1 & 3/2 & 1 \\ 2/3 & 1 & 2/3 \\ 1 & 3/2 & 1 \end{pmatrix}, \qquad w^{\text{Taste}} = \begin{pmatrix} 0.375 \\ 0.25 \\ 0.375 \end{pmatrix}$$

$$R^{\text{Healthiness}} = \begin{pmatrix} 1 & 1/2 & 4/3 \\ 2 & 1 & 7/3 \\ 3/4 & 3/7 & 1 \end{pmatrix}, \qquad w^{\text{Healthiness}} = \begin{pmatrix} 0.27 \\ 0.52 \\ 0.21 \end{pmatrix}$$

Decision Theory - Exercise

Solution 2 (Example)

Comparison matrices and weight vectors:

$$R^{\text{Satisfaction}} = \begin{pmatrix} 1 & 2/3 & 3/4 \\ 3/2 & 1 & 4/3 \\ 4/3 & 3/4 & 1 \end{pmatrix} \qquad w^{\text{Satisfaction}} = \begin{pmatrix} 0.26 \\ 0.41 \\ 0.33 \end{pmatrix}$$

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Solution 2 (Example)

$$w_1^{\text{Satisfaction}} w^{\text{Price}} + w_2^{\text{Satisfaction}} w^{\text{Taste}} + w_3^{\text{Satisfaction}} w^{\text{Healthiness}}$$

Decision Theory - Exercise

Solution 2 (Example)

$$w_1^{\text{Satisfaction}} w^{\text{Price}} + w_2^{\text{Satisfaction}} w^{\text{Taste}} + w_3^{\text{Satisfaction}} w^{\text{Healthiness}}$$

$$= 0.26 \begin{pmatrix} 0.25 \\ 0.5 \\ 0.25 \end{pmatrix} + 0.41 \begin{pmatrix} 0.375 \\ 0.25 \\ 0.375 \end{pmatrix} + 0.33 \begin{pmatrix} 0.27 \\ 0.52 \\ 0.21 \end{pmatrix}$$

Decision Theory - Exercise

Solution 2 (Example)

$$
\begin{aligned}
& w_1^{\text{Satisfaction}} w^{\text{Price}} + w_2^{\text{Satisfaction}} w^{\text{Taste}} + w_3^{\text{Satisfaction}} w^{\text{Healthiness}} \\
=& 0.26 \begin{pmatrix} 0.25 \\ 0.5 \\ 0.25 \end{pmatrix} + 0.41 \begin{pmatrix} 0.375 \\ 0.25 \\ 0.375 \end{pmatrix} + 0.33 \begin{pmatrix} 0.27 \\ 0.52 \\ 0.21 \end{pmatrix} \\
\approx & \begin{pmatrix} 0.31 \\ 0.40 \\ 0.29 \end{pmatrix}
\end{aligned}
$$

Decision Theory - Exercise

Solution 2 (Example)

$$
\begin{aligned}
& w_1^{\text{Satisfaction}} w^{\text{Price}} + w_2^{\text{Satisfaction}} w^{\text{Taste}} + w_3^{\text{Satisfaction}} w^{\text{Healthiness}} \\
=& 0.26 \begin{pmatrix} 0.25 \\ 0.5 \\ 0.25 \end{pmatrix} + 0.41 \begin{pmatrix} 0.375 \\ 0.25 \\ 0.375 \end{pmatrix} + 0.33 \begin{pmatrix} 0.27 \\ 0.52 \\ 0.21 \end{pmatrix} \\
\approx & \begin{pmatrix} 0.31 \\ 0.40 \\ 0.29 \end{pmatrix}
\end{aligned}
$$

Thus, the best choice (in my case) is to cook at home!

Decision Theory - Exercise

Exercise 3: MAUT

- You want to rent an apartment
- Four possible alternatives
- Three relevant criteria:
  - Price (Euro)
  - Size (m^2)
  - Distance from the city center (m)
- You determine the following metrics:

	Price	Size	Distance
W1	350	35	200
W2	400	50	1500
W3	500	45	300
W4	600	55	500

Decision Theory - Exercise

Exercise 3: MAUT, Continuation

	Price	Size	Distance
W1	350	35	200
W2	400	50	1500
W3	500	45	300
W4	600	55	500

- Determine a utility function for price using the direct rating method
- Determine utility functions for size and distance using the halving method
- Determine weights for the three objective criteria using the trade-off method

Decision Theory - Exercise

Solution 3: Direct Rating for Price

Price	350	400	500	600
Points				

Decision Theory - Exercise

Solution 3: Direct Rating for Price

Price	350	400	500	600
Points	100			0

Decision Theory - Exercise

Solution 3: Direct Rating for Price

Price	350	400	500	600
Points	100	80		0

Decision Theory - Exercise

Solution 3: Direct Rating for Price

Price	350	400	500	600
Points	100	80	30	0

Decision Theory - Exercise

Solution 3: Direct Rating for Price

Price	350	400	500	600
Points	100	80	30	0
Utility	1.0	0.8	0.3	0.0

Decision Theory - Exercise

Solution 3: Halving Method for Size

	x_2^0	$x_2^{1/4}$	$x_2^{1/2}$	$x_2^{3/4}$	x_2^1
Utility	0	0.25	0.5	0.75	1
Size	35				55

Decision Theory - Exercise

Solution 3: Halving Method for Size

	x_2^0	$x_2^{1/4}$	$x_2^{1/2}$	$x_2^{3/4}$	x_2^1
Utility	0	0.25	0.5	0.75	1
Size	35		43		55

Decision Theory - Exercise

Solution 3: Halving Method for Size

	x_2^0	$x_2^{1/4}$	$x_2^{1/2}$	$x_2^{3/4}$	x_2^1
Utility	0	0.25	0.5	0.75	1
Size	35	38	43		55

Decision Theory - Exercise

Solution 3: Halving Method for Size

	x_2^0	$x_2^{1/4}$	$x_2^{1/2}$	$x_2^{3/4}$	x_2^1
Utility	0	0.25	0.5	0.75	1
Size	35	38	43	48	55

Decision Theory - Exercise

Solution 3: Halving Method for Size

	x_2^0	$x_2^{1/4}$	$x_2^{1/2}$	$x_2^{3/4}$	x_2^1
Utility	0	0.25	0.5	0.75	1
Size	35	38	43	48	55

Interpolation:

$$u_2(45) = u_2(43) + \frac{u_2(48) - u_2(43)}{48 - 43} \cdot (45 - 43) = 0.6$$

$$u_2(50) = u_2(48) + \frac{u_2(55) - u_2(48)}{55 - 48} \cdot (50 - 48) = 0.82$$

Decision Theory - Exercise

Solution 3: Halving Method for Distance

	x_3^0	$x_3^{1/4}$	$x_3^{1/2}$	$x_3^{3/4}$	x_3^1
Utility	0	0.25	0.5	0.75	1
Distance	1500	800	500	250	200

Interpolation:

$$u_3(300) = u_3(500) + \frac{u_3(250) - u_3(500)}{250 - 500} \cdot (300 - 500) = 0.7$$

Decision Theory - Exercise

Solution 3: Trade-Off Method

	Price	Utility	Size	Utility	Distance	Utility
W1	350	1.00	35	0.00	200	1.00
W2	400	0.80	50	0.82	1500	0.00
W3	500	0.30	45	0.60	300	0.70
W4	600	0.00	55	1.00	500	0.50

Decision Theory - Exercise

Solution 3: Trade-Off Method

	Price	Utility	Size	Utility	Distance	Utility
W1	350	1.00	35	0.00	200	1.00
W2	400	0.80	50	0.82	1500	0.00
W3	500	0.30	45	0.60	300	0.70
W4	600	0.00	55	1.00	500	0.50

- First step, rearrange criteria
- Which criterion has the greatest difference in utility from 0 to 1?
- I find: Price
- No rearrangement necessary

Decision Theory - Exercise

Solution 3: Trade-Off Method

	Price	Utility	Size	Utility	Distance	Utility
W1	350	1.00	35	0.00	200	1.00
W2	400	0.80	50	0.82	1500	0.00
W3	500	0.30	45	0.60	300	0.70
W4	600	0.00	55	1.00	500	0.50

- Second step, comparison questions
- What price $\overline{x}_2$ must an apartment have for the following options to be equally good:
  - Price $\overline{x}_2$ Euros, Size $35m^2$
  - Price 600 Euros, Size $55m^2$

Decision Theory - Exercise

Solution 3: Trade-Off Method

	Price	Utility	Size	Utility	Distance	Utility
W1	350	1.00	35	0.00	200	1.00
W2	400	0.80	50	0.82	1500	0.00
W3	500	0.30	45	0.60	300	0.70
W4	600	0.00	55	1.00	500	0.50

- Second step, comparison questions
- What price $\overline{x}_2$ must an apartment have for the following options to be equally good:
  - Price $\overline{x}_2$ Euros, Size $35m^2$
  - Price 600 Euros, Size $55m^2$
- My answer: 350 Euros
- Thus, $\lambda_1 u_1(\overline{x}_2) = \lambda_2$
- So, $\lambda_1 = \lambda_2$

Decision Theory - Exercise

Solution 3: Trade-Off Method

	Price	Utility	Size	Utility	Distance	Utility
W1	350	1.00	35	0.00	200	1.00
W2	400	0.80	50	0.82	1500	0.00
W3	500	0.30	45	0.60	300	0.70
W4	600	0.00	55	1.00	500	0.50

- Second step, comparison questions
- What price $\overline{x}_3$ must an apartment have for the following options to be equally good:
  - Price $\overline{x}_3$ Euros, Distance $1500m$
  - Price 600 Euros, Distance $200m$
- My answer: 400 Euros
- Thus, $\lambda_1 u_1(\overline{x}_3) = \lambda_3$
- So, $0.8\lambda_1 = \lambda_3$

Decision Theory - Exercise

Solution 3: Trade-Off Method

Now solve the following system of equations:

$$\begin{aligned} \lambda_1 &= \lambda_2 \\ 0.8\lambda_1 &= \lambda_3 \\ \lambda_1 + \lambda_2 + \lambda_3 &= 1 \end{aligned}$$

Answer:

$$\lambda_1 = 5/14, \lambda_2 = 5/14, \lambda_3 = 4/14$$

Decision theory – Exercise

Solution 3: Total

	Price	Utility	Size	Utility	Distance	Utility	Total
W1	350	1.00	35	0.00	200	1.00	**0.64**
W2	400	0.80	50	0.82	1500	0.00	0.58
W3	500	0.30	45	0.60	300	0.70	0.52
W4	600	0.00	55	1.00	500	0.50	0.50
λ		5/14		5/14		4/14	

Decision Theory - Exercise

Solution 3: Total

	Price	Utility	Size	Utility	Distance	Utility	Total
W1	350	1.00	35	0.00	200	1.00	**0.64**
W2	400	0.80	50	0.82	1500	0.00	0.58
W3	500	0.30	45	0.60	300	0.70	0.52
W4	600	0.00	55	1.00	500	0.50	0.50
λ		5/14		5/14		4/14	