Decision theory

Exercise 5

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Decision Theory - Exercise

Exercise 1

Maximize. Which of these functions are risk-seeking, which are risk-averse?

1. $\Phi(\mu, \sigma) = \mu - \sigma$
2. $\Phi(\mu, \sigma) = \mu + \sigma$
3. $\Phi(\mu, \sigma) = (\sigma + 1)^2$
4. $\Phi(\mu, \sigma) = 1/\sigma$

Decision Theory – Exercise

1. $\Phi(\mu, \sigma) = \mu - \sigma$
2. $\Phi(\mu, \sigma) = \mu + \sigma$
3. $\Phi(\mu, \sigma) = (\sigma + 1)^2$
4. $\Phi(\mu, \sigma) = 1/\sigma$

Solution 1

1. $\partial\Phi/\partial\sigma = -1 < 0 \to$ averse.
2. $\partial\Phi/\partial\sigma = 1 > 0 \to$ seeking.
3. $\partial\Phi/\partial\sigma = 2\sigma + 2 > 0 \to$ seeking.
4. $\partial\Phi/\partial\sigma = -1/\sigma^2 < 0 \to$ averse.

Decision Theory – Exercise

Exercise 2

Maximize.

p	0.2	0.2	0.4	0.2
a_1	2	5	3	2
a_2	2	2	2	2
a_3	3	4	2	4

Determine the best solution for the decision criterion

$$\Phi(\mu, \sigma) = \mu - \sigma^2$$

Decision Theory – Exercise

Solution 2

p	0.2	0.2	0.4	0.2	Φ
a_1	2	5	3	2	1.8
a_2	2	2	2	2	2
a_3	3	4	2	4	2.2

- a_1:
  - $\mu_1 = 0.2 \cdot 2 + 0.2 \cdot 5 + 0.4 \cdot 3 + 0.2 \cdot 2 = 3$
  - $\sigma_1^2 = 0.2 \cdot (2-3)^2 + 0.2 \cdot (5-3)^2 + 0.4 \cdot (3-3)^2 + 0.2 \cdot (2-3)^2 = 1.2$
- a_2:
  - $\mu_2 = 0.2 \cdot 2 + 0.2 \cdot 2 + 0.4 \cdot 2 + 0.2 \cdot 2 = 2$
  - $\sigma_2^2 = 0.2 \cdot (2-2)^2 + 0.2 \cdot (2-2)^2 + 0.4 \cdot (2-2)^2 + 0.2 \cdot (2-2)^2 = 0$
- a_3:
  - $\mu_3 = 0.2 \cdot 3 + 0.2 \cdot 4 + 0.4 \cdot 2 + 0.2 \cdot 4 = 3$
  - $\sigma_3^2 = 0.2 \cdot (3-3)^2 + 0.2 \cdot (4-3)^2 + 0.4 \cdot (2-3)^2 + 0.2 \cdot (4-3)^2 = 0.8$
- best alternative: a_3

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Exercise 3

Choose 2 out of 4 items such that $\Phi(\mu, \sigma) = \sigma^2$ is minimized.

p	0.5	0.5
1	8	3
2	5	7
3	9	8
4	10	5

Decision Theory – Exercise

p	0.5	0.5
1	8	3
2	5	7
3	9	8
4	10	5

Solution 3

- 1+2: $(13, 10)$, σ^2 = 2.25
- 1+3: $(17, 11)$, σ^2 = 9
- 1+4: $(18, 8)$, σ^2 = 25
- 2+3: $(14, 15)$, σ^2 = 0.25
- 2+4: $(15, 12)$, σ^2 = 2.25
- 3+4: $(19, 13)$, σ^2 = 9
- best solution: 2+3

Decision Theory – Exercise

Remark

For $p_1 = p_2 = 0.5$ it holds:

$$\begin{aligned}
\sigma^2 &= \frac{1}{2}(e_1 - \mu)^2 + \frac{1}{2}(e_2 - \mu)^2 \\
&= \frac{1}{2}(e_1 - \frac{1}{2}e_1 - \frac{1}{2}e_2)^2 + \frac{1}{2}(e_2 - \frac{1}{2}e_1 - \frac{1}{2}e_2)^2 \\
&= \frac{1}{2}(\frac{e_1 - e_2}{2})^2 + \frac{1}{2}(\frac{e_2 - e_1}{2})^2 \\
&= \frac{1}{4}(e_1 - e_2)^2
\end{aligned}$$

- To minimize σ^2 in this case, choose the combination for which $|e_1 - e_2|$ is minimal.

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Exercise 4

There are two investments with four scenarios each to choose from. All scenarios are equally likely.

	s_1	s_2	s_3	s_4
1	12	15	5	20
2	20	5	10	1

Calculate the correlation between the options. Reminder:

$$\rho_{12} = \frac{cov_{12}}{\sigma_1 \sigma_2}$$

Decision Theory – Exercise

Solution 4

- $\mu_1 = \frac{1}{4}(12 + 15 + 5 + 20) = 13$
- $\mu_2 = \frac{1}{4}(20 + 5 + 10 + 1) = 9$
- $\sigma_1 = \sqrt{\frac{1}{4}((12 - 13)^2 + (15 - 13)^2 + (5 - 13)^2 + (20 - 13)^2)} = 5.43$
- $\sigma_2 = \sqrt{\frac{1}{4}((20 - 9)^2 + (5 - 9)^2 + (10 - 9)^2 + (1 - 9)^2)} = 7.11$
- $cov_{12} = \frac{1}{4}((12 - 13)(20 - 9) + (15 - 13)(5 - 9) + (5 - 13)(10 - 9) + (20 - 13)(1 - 9)) = -20.75$
- $\rho_{12} = cov_{12} / (\sigma_1 \sigma_2) = -0.54$

Problem 5

There are two investments with four scenarios to choose from. All scenarios are equally likely.

	s_1	s_2	s_3	s_4
1	0	10	20	30
2	10	10	10	10

Draw the portfolio line. Reminder:

$$\sigma^2 = x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + 2x_1x_2cov_{12}$$

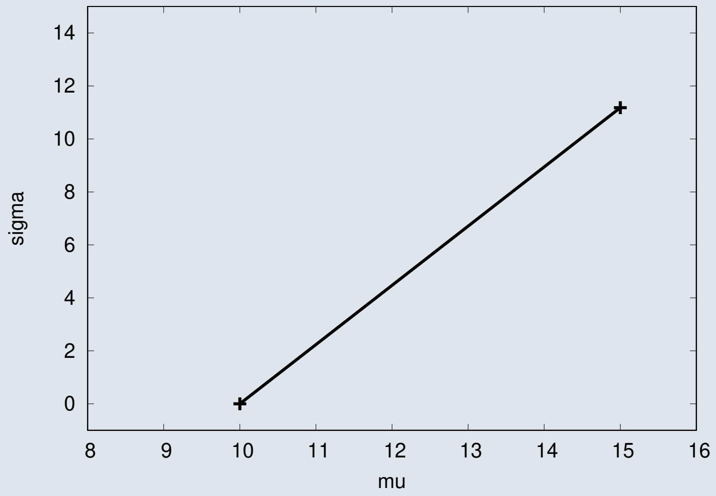
Which portfolio is optimal for $\max \Phi(\mu, \sigma)$ with

$$\Phi(\mu, \sigma) = \mu - 0.1\sigma^2$$

Decision Theory – Exercise

Solution 5

- $x_1 = 1 - x_2$
- $\sigma_1^2 = \frac{1}{4}((0-15)^2 + (10-15)^2 + (20-15)^2 + (30-15)^2) = 125$
- $\mu(x_1) = 15x_1 + 10(1-x_1) = 5x_1 + 10$
- $\sigma^2(x_1) = x_1^2\sigma_1^2 + (1-x_1)^2\sigma_2^2 + 2x_1(1-x_1)cov_{12} = 125x_1^2$
- $\sigma(\mu) = 11.18\frac{\mu-10}{5} = 2.24\mu - 22.36$



Decision Theory – Exercise

Solution 5

$$\begin{aligned}\Phi(\mu, \sigma) &= \mu - 0.1\sigma^2 \\ &= 5x_1 + 10 - 12.5x_1^2 \\ \Phi'(x_1) &= -25x_1 + 5 \\ \Phi'(x_1) &= 0 \Leftrightarrow x_1 = 1/5\end{aligned}$$

Since Φ is a parabola, $x_1 = 1/5$ is thus the maximizer.