Decision theory

Exercise 1

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Decision theory – Exercise

Definition: Relation

A relation R on a set M is a subset

$$R \subseteq M \times M = \{(m_1, m_2) : m_1, m_2 \in M\}.$$

For $(m_1, m_2) \in R$, we also write $m_1 R m_2$.

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- $M \times M = \{(m_1, m_1), (m_1, m_2), (m_1, m_3), (m_2, m_1), (m_2, m_2), (m_2, m_3), (m_3, m_1), (m_3, m_2), (m_3, m_3)\}$

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- There are 2^9 different relations possible

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- ”exactly one triangle” $R = \{(m_1, m_2), (m_1, m_3), (m_2, m_1), (m_3, m_1)\}$

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- There are 2^9 different relations possible
- ”exactly one triangle” $R = \{(m_1, m_2), (m_1, m_3), (m_2, m_1), (m_3, m_1)\}$
- ”no triangle” $R = \{(m_1, m_1)\}$

Decision theory – Exercise

Definition: Transitivity

A relation R on a set M is transitive, when for all $m_1, m_2, m_3 \in M$ with $m_1 R m_2$, $m_2 R m_3$ also $m_1 R m_3$ holds.

Decision theory – Exercise

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- To disapprove a property, we only need a counterexample.

Decision theory – Exercise

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A relation R on a set M is transitive, when for all $m_1, m_2, m_3 \in M$ with $m_1 R m_2$, $m_2 R m_3$ also $m_1 R m_3$ holds.

- To disapprove a property, we only need a counterexample.
- To prove a property, we need to find a statement about all elements in $M \times M$.

Decision theory – Exercise

Exercise 1: Transitive or not?

- $M = \mathbb{R}$
  a) $xRy \Leftrightarrow x \leq y + 1$
  b) $xRy \Leftrightarrow x \leq y - 1$
  c) $xRy \Leftrightarrow x^2 \leq y^2$
- $M = \{\text{Students in Erlangen}\}$
  d) aRb, if a and b have attended at least one lecture together
  e) aRb, if the student ID of a is smaller than of b

Decision theory – Exercise

Solution for $M = \mathbb{R}$

a) $xRy \Leftrightarrow x \leq y + 1$

Decision theory – Exercise

Solution for $M = \mathbb{R}$

a) $xRy \Leftrightarrow x \leq y + 1$

- not transitive
- Counterexample: $(1.9, 1)$, $(1, 0.1)$, $(1.9, 0.1)$
- even for $M = \mathbb{Z}$ not transitive

Decision theory – Exercise

Solution for $M = \mathbb{R}$

b) $xRy \Leftrightarrow x \leq y - 1$

Decision theory – Exercise

Solution for $M = \mathbb{R}$

b) $xRy \Leftrightarrow x \leq y - 1$

- transitive
- Let $x \leq y - 1$ and $y \leq z - 1$.Then $x \leq (z - 1) - 1$ and also $x \leq z - 1$.

Decision theory – Exercise

Solution for $M = \mathbb{R}$

c) $xRy \Leftrightarrow x^2 \leq y^2$

Decision theory – Exercise

Solution for $M = \mathbb{R}$

c) $xRy \Leftrightarrow x^2 \leq y^2$
- transitive
- Let $x^2 \leq y^2$ and $y^2 \leq z^2$. Then $x^2 \leq z^2$.

Decision theory – Exercise

Solution for $M = \{Students\}$

d) aRb, if a and b have attended at least one lecture together

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Decision theory – Exercise

Solution for $M = \{Students\}$

d) aRb, if a and b have attended at least one lecture together
   - not transitive

e) aRb, if the student ID of a is smaller than that of b
   - transitive, since equivalent to $\leq$ on $\mathbb{R}$

Decision theory – Exercise

Exercise 2: Transitive or not?

a) $(\mathbb{R}, <)$

b) $(\mathbb{R}, \leq)$

c) $(\mathbb{R}, =)$

d) $(\mathbb{R}, \neq)$

e) (Subset of $\mathbb{R}$, $\subseteq$)

f) $M = \{1, 2, 3\}$, $R = \{(1, 1), (2, 2), (3, 3)\}$

g) $M = \{1, 2, 3, 4, 5\}$, $R = \{(3, 1), (1, 5), (2, 2), (2, 5), (3, 5)\}$

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   - yes

Decision theory – Exercise

Basic Model

A system consisting of:

- $A = \{a_1, \ldots,\}$ a non-empty set of alternatives
- $S = \{s_1, \ldots,\}$ a non-empty set of scenarios,possibly also with probabilities p_i
- E a set of outcomes with $g : A \times S \to E$

Decision Tree

- Decision: $\square$
- Event: $\bigcirc$
- State: $\triangleleft$

Decision Theory – Exercise

Exercise 3: Lottery

A numbered ball is randomly drawn from a drum containing 100 balls in a lottery.Create a decision matrix and a decision tree.

Lottary A

- 100% Chance of 1 Mio EUR

Lottery B

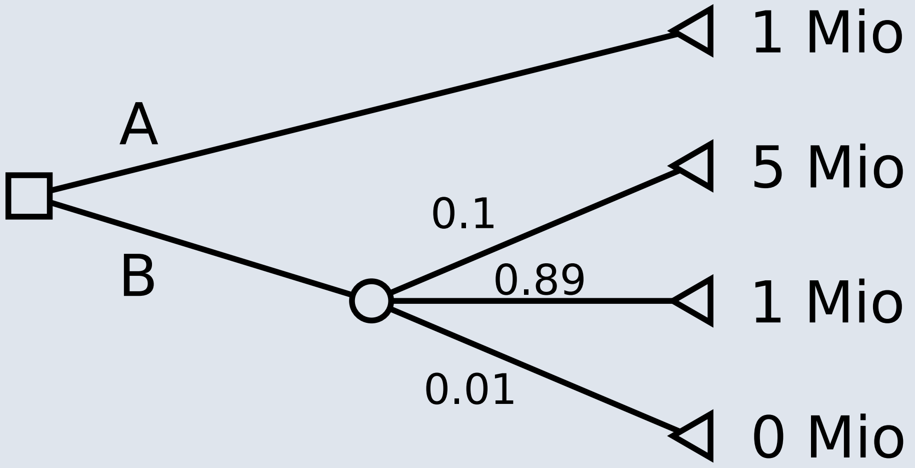
- 10% Chance of 5 Mio EUR
- 89% Chance of 1 Mio EUR
- 1% Chance of 0 EUR

Decision theory – Exercise

Solution

- Set of alternatives: $A = \{a_A, a_B\}$
- Set of scenarios: $S = \{s_1, s_2, s_3\}$
- Probabilities: $p_1 = 0.1$, $p_2 = 0.89$ and $p_3 = 0.01$

Decision tree



Decision matrix

	s_1	s_2	s_3
	0.1	0.89	0.01
a_A	1 Mio	1 Mio	1 Mio
a_B	5 Mio	1 Mio	0 Mio

Decision Theory – Exercise

Exercise 4: Lottery

A numbered ball is randomly drawn from a drum containing 100 balls in a lottery.Create a decision matrix and a decision tree.

Lottery C

- 11% Chance of 1 Mio EUR
- 89% Chance of 0 EUR

Lotterie D

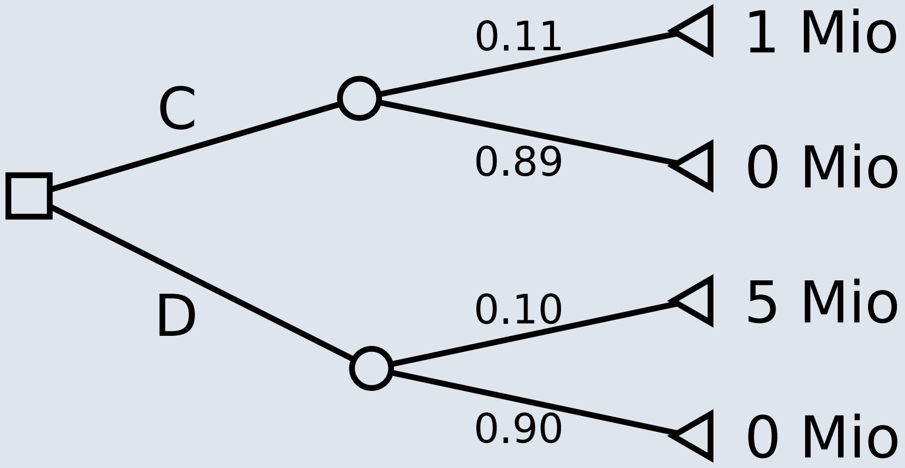
- 10% Chance of 5 Mio EUR
- 90% Chance of 0 EUR

Decision Theory – Exercise

Solution

- Set of alternatives : $A = \{a_C, a_D\}$
- Set of scenarios : $S = \{s_1, s_2, s_3\}$
- Probabilities : $p_1 = 0.01$, $p_2 = 0.10$ and $p_3 = 0.89$

Decision tree



Decision matrix

	s_1	s_2	s_3
	0.01	0.10	0.89
a_C	1 Mio	1 Mio	0
a_D	0	5 Mio	0

Decision Theory – Exercise

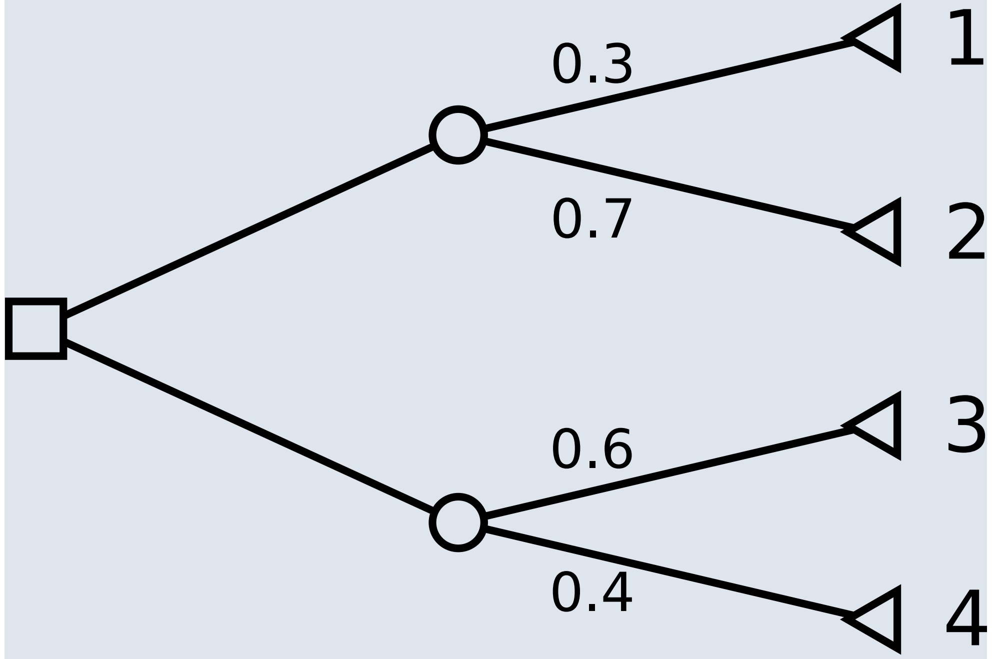
Exercise 5

Convert the following decision matrix into a decision tree:

	s_1	s_2
	0.5	0.5
a_1	5	6
a_2	7	4

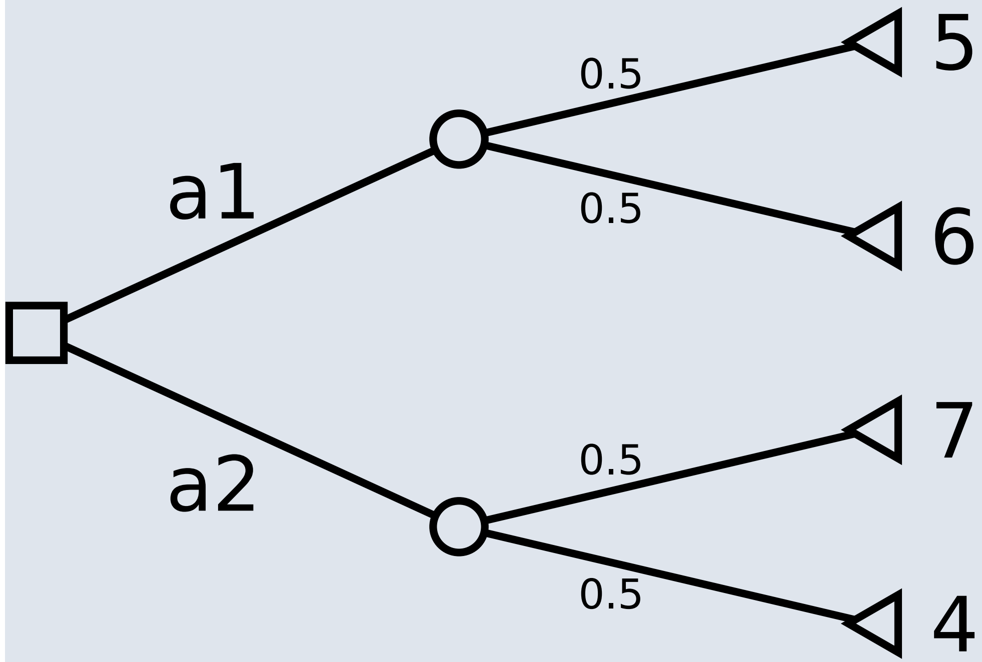
Exercise 6

Convert the following decision tree into a decision matrix:



Decision Theory – Exercise

Solution



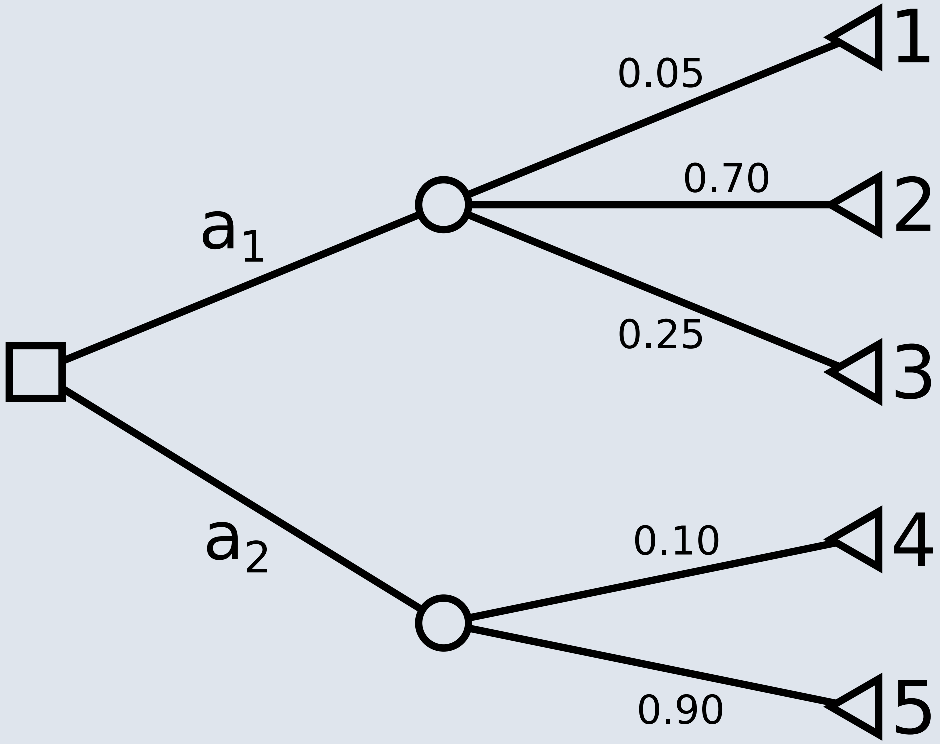
Solution

	s_1	s_2	s_3
p	0.3	0.1	0.6
a_1	1	2	2
a_2	4	4	3

Decision Theory – Exercise

Exercise 7

Convert the following decision tree into a decision matrix



Decision Theory – Exercise

Solution

	s_1	s_2	s_3	s_4
p	0.05	0.05	0.20	0.70
a_1	1	3	3	2
a_2	4	4	5	5