Friedrich-Alexander-Universität
Erlangen-Nürnberg

Decision Theory

Lecture 4

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Decision Theory

Recap: What did we do?

- Domination criteria
- Preference functions
- Excursion: descriptive decision theory

Decision Theory

Domination Criteria

- Three criteria:
  - Absolute dominance: worst is better than best
  - State dominance: better in each scenario
  - Probabilistic dominance: cumulative distribution function is better
- Example:

	s_1	s_2	s_3	s_4
p	0.25	0.25	0.25	0.25
a_1	100	100	50	50
a_2	100	75	75	110

Decision Theory

Preference Functions

- Three types of preferences:
  - Extremization
  - Satisficing
  - Fixation
- Lexicographic rule

Decision Theory

Descriptive Decision Theory

- Anchoring effect
- Estimating probabilities
- Altruism and revenge in games
- Two systems of decision making
- Heuristics in system one, connection to evolutionary preferences

Decision Theory

Today

- Decision under certainty
- Multicriteria problems
- Data Envelopment Analysis

Decision Theory

Decision Under Certainty

Multiple scenarios, multiple criteria $\rightarrow$ uncertainty

	s_1	$\dots$	s_n
a_1	$(e_{11}^1, \dots, e_{11}^k)$	$\dots$	$(e_{1n}^1, \dots, e_{1n}^k)$
a_2	$(e_{21}^1, \dots, e_{21}^k)$	$\dots$	$(e_{2n}^1, \dots, e_{2n}^k)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_m	$(e_{m1}^1, \dots, e_{m1}^k)$	$\dots$	$(e_{mn}^1, \dots, e_{mn}^k)$

Decision Theory

Decision Under Certainty

One scenario, multiple criteria $\rightarrow$ certainty

	s_1
a_1	$(e_1^1, \ldots, e_1^k)$
a_2	$(e_2^1, \ldots, e_2^k)$
$\vdots$	$\vdots$
a_m	$(e_m^1, \ldots, e_m^k)$

Decision Theory

Decision Under Certainty

One scenario, one criterion $\rightarrow$ certainty

$$\begin{array}{c|c} & s_1 \\ \hline a_1 & e_1 \\ a_2 & e_2 \\ \vdots & \vdots \\ a_m & e_m \end{array}$$

Decision Theory

Decision Under certainty

- Why should this be difficult?
- If the decision matrix is given, the problem is trivial: choose the best alternative
- Not so easy when the alternatives are only implicitly known

Decision Theory

Example: Fuel Consumption

The fuel consumption F of a car traveling at constant speed v (in mph) is

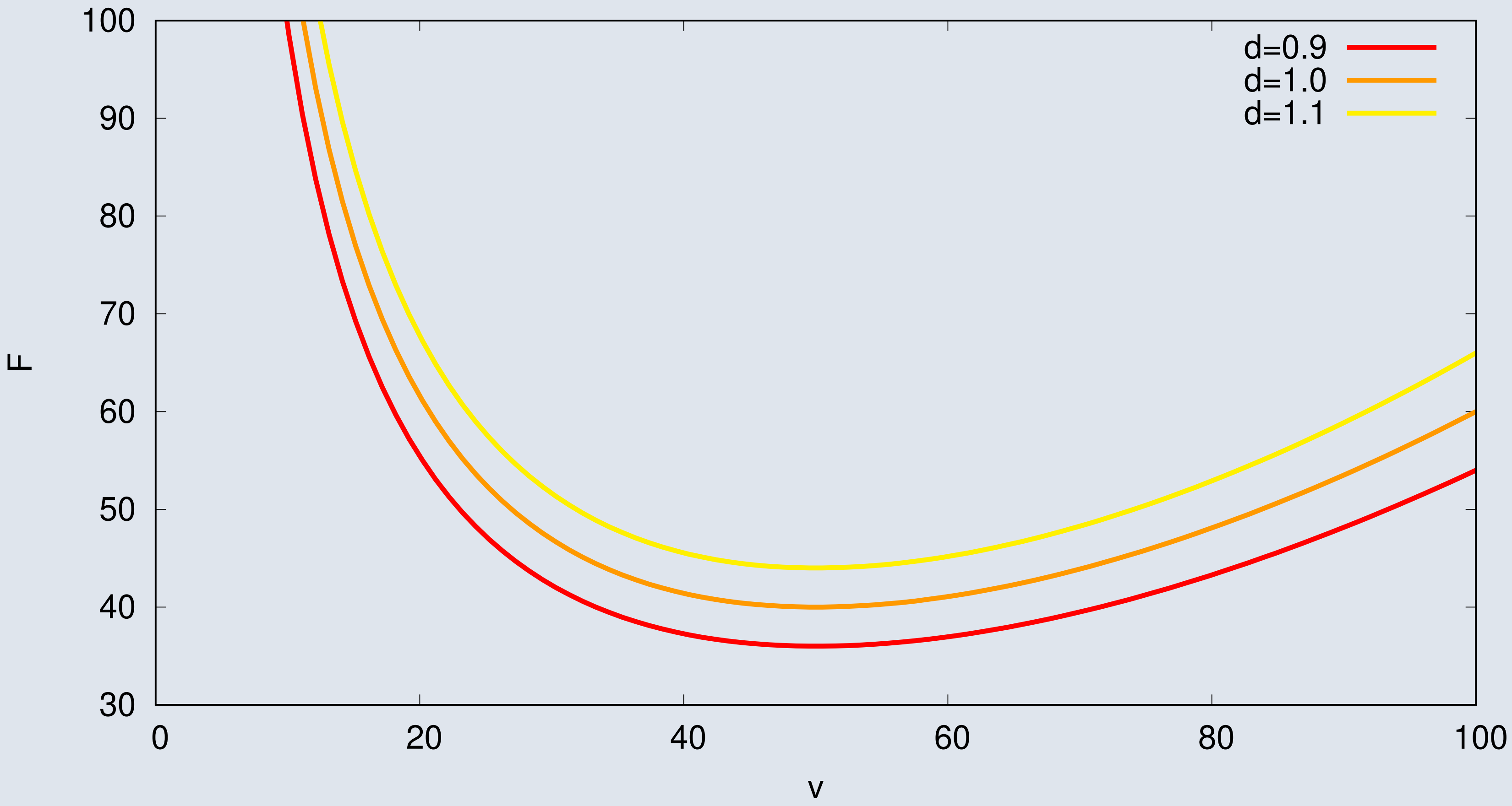
$$F = \frac{ad}{v} + bdv^2 + cd$$

where d is the distance traveled, and a, b, c are parameters depending on the car.

What is the most cost-effective speed for $a = 1000$, $b = 0.004$, and $c = 10$?

Decision Theory

Solution



Decision Theory

Solution

$$F = \frac{ad}{v} + bdv^2 + cd$$

The first derivative with respect to v is

$$F' = \frac{-ad}{v^2} + 2bdv$$

Setting the derivative to zero gives $v = 50$. The second derivative is

$$F'' = \frac{2ad}{v^3} + 2bd \geq 0$$

Thus, $v = 50$ is a minimum. Also, test $v \to \infty$.

Decision Theory

Decision Under Certainty

- Some decision problems are solvable using school methods (curve sketching)
- Approach does not work for problems with constraints

Decision Theory

Example

- Produce two chemicals P_1 and P_2
- Yield the same profit
- Must go through two machines
- 100 operating hours per machine available
- Processing time per unit is:

	M_1	M_2
P_1	1	$1/2$
P_2	$1/3$	1

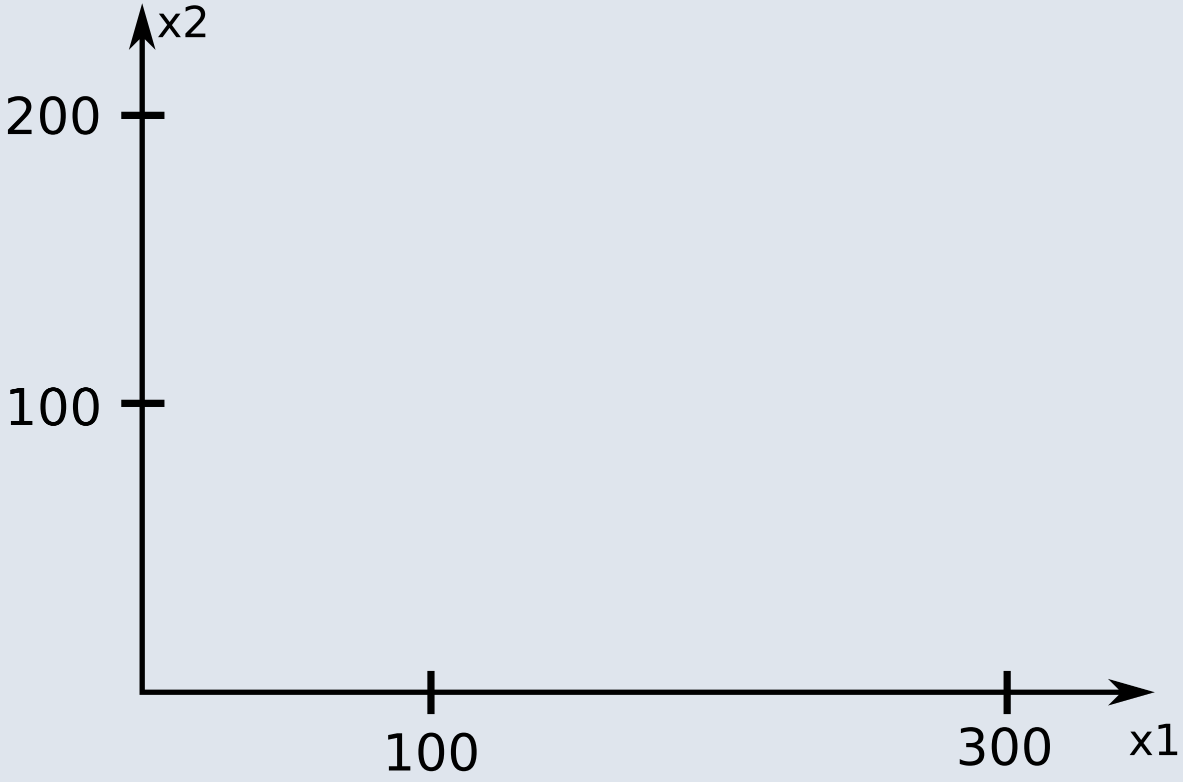
Decision Theory

Model

- Variables:
  - x_1: amount of chemical P_1
  - x_2: amount of chemical P_2
- Profit: $x_1 + x_2$
- Constraints:
  - Machine M_1: $x_1 + 1/2x_2 \leq 100$
  - Machine M_2: $1/3x_1 + x_2 \leq 100$

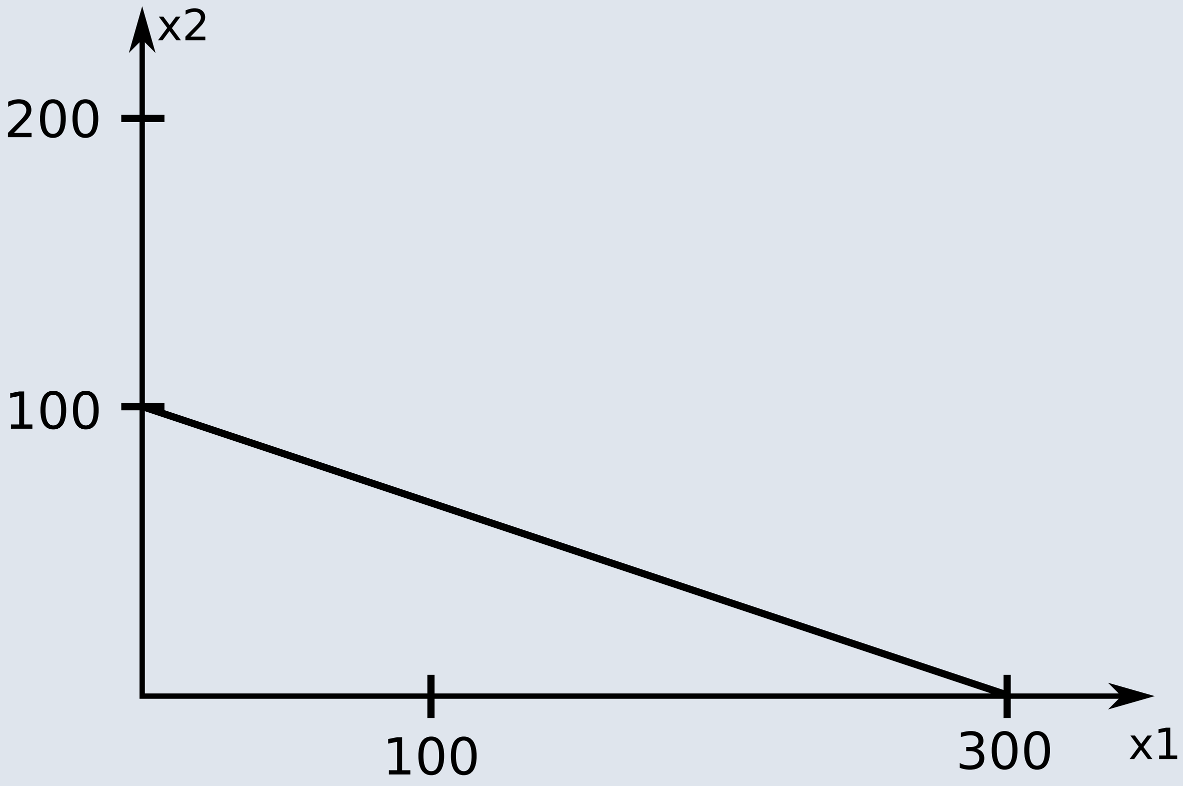
Decision Theory

Graphical Solution



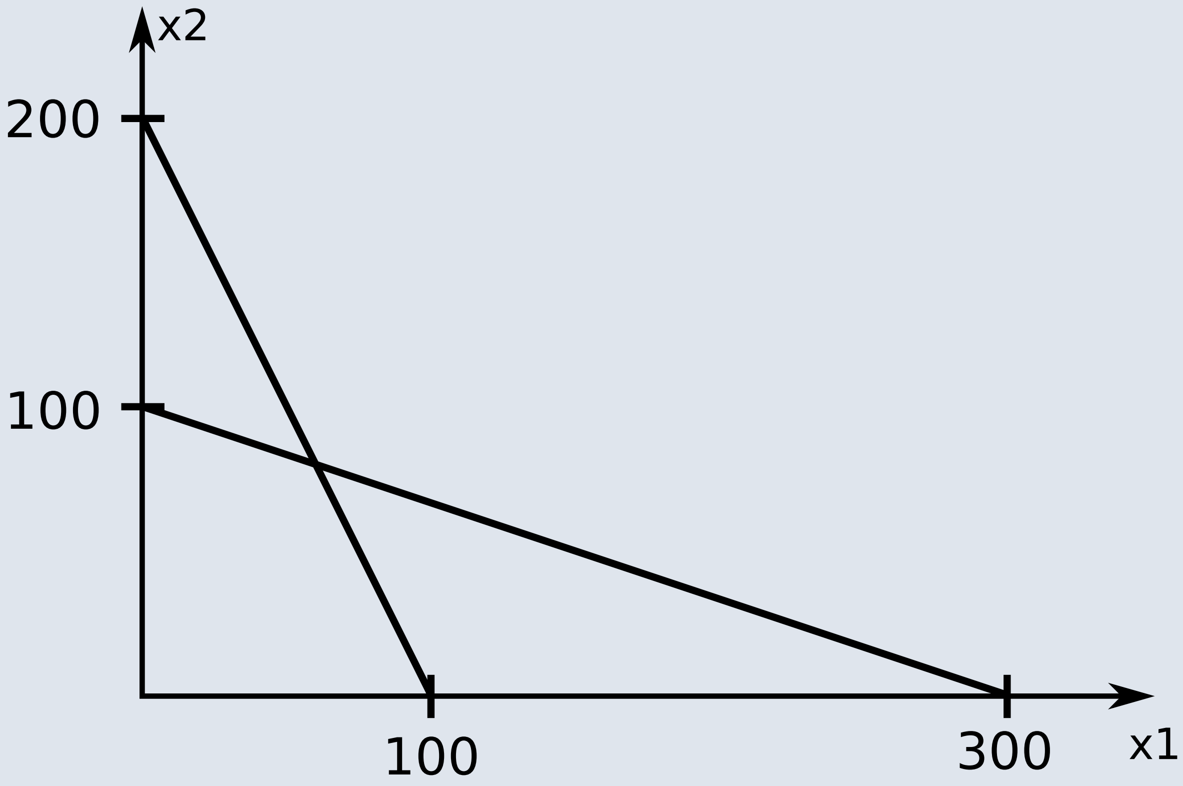
Coordinate system

Graphical Solution



Machine M_2

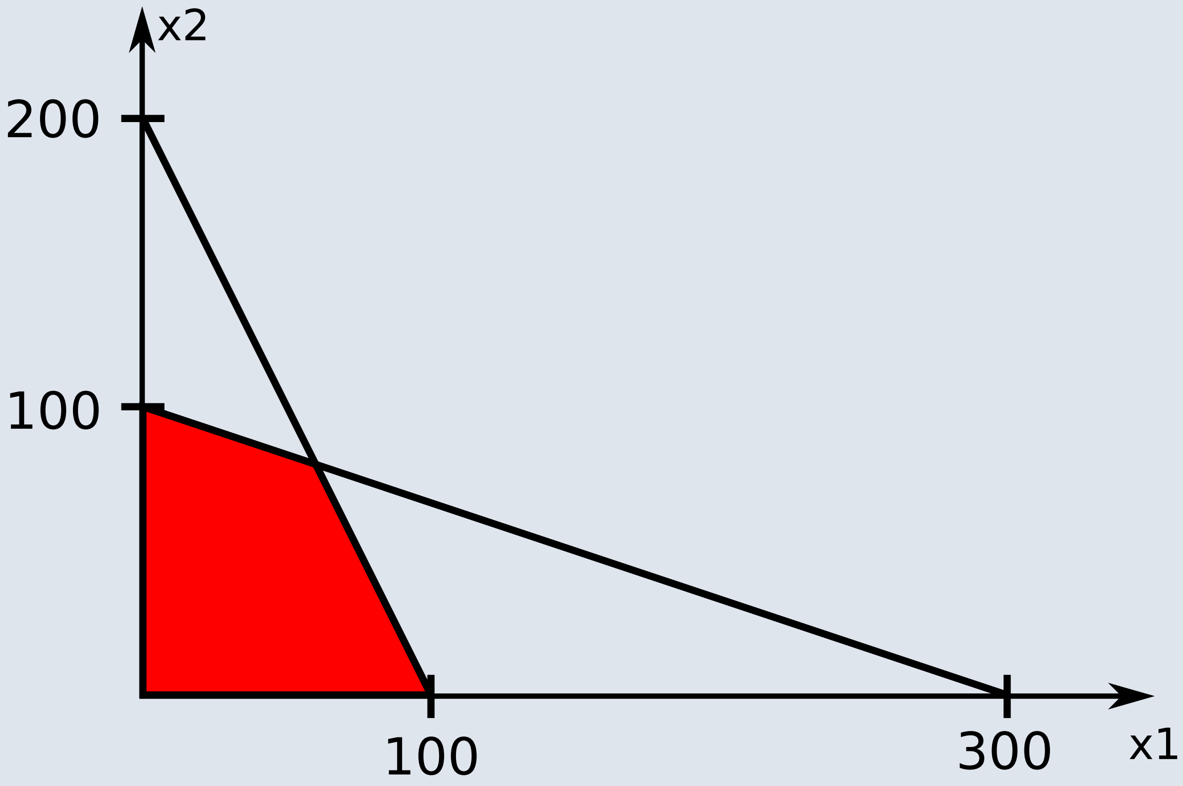
Graphical Solution



Machine M_1

Decision Theory

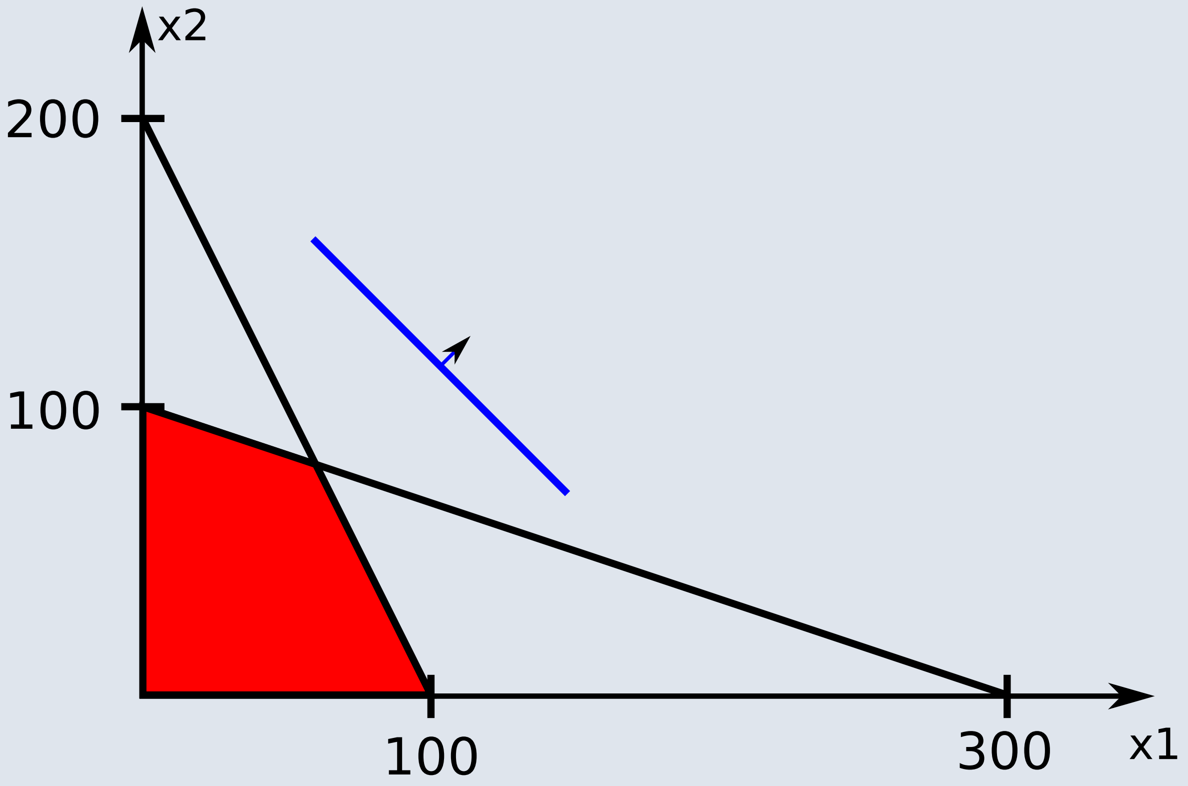
Graphical Solution



Feasible region

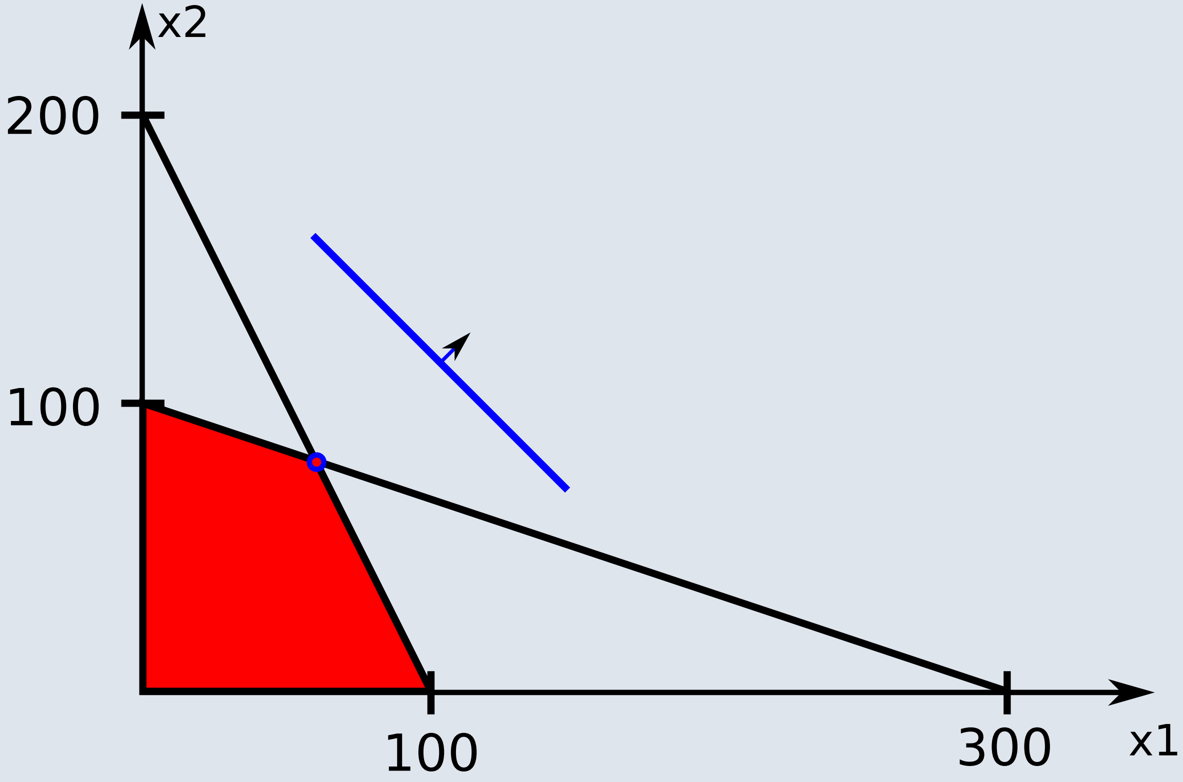
Decision Theory

Graphical Solution



Objective function

Graphical Solution



Optimal solution

Decision Theory

Linear Programming

- Linear program: all constraints and objective function are linear
- Fundamental theorem of linear optimization: there is always an optimal solution among the extreme points
- Can be ”quickly” searched (in polynomial time)
- Only possible when all variables are continuous

Decision Theory

Example: Knapsack Problem

- n possible projects
- Profits p_i, $i = 1, \ldots, n$
- Investment costs w_i, $i = 1, \ldots, n$
- Investment budget is B
- Which projects should be chosen?

Decision Theory

Knapsack problem: Model

- Use variables x_i, $i = 1, \dots, n$
  - $x_i = 1$ if project i is chosen
  - $x_i = 0$ otherwise
- Model:

$$
\begin{aligned}
\max \quad & \sum_{i \in [m]} p_i x_i \\
\text{subject to} \quad & \sum_{i \in [n]} w_i x_i \leq B \\
& x_i \in \{0, 1\} \qquad \text{for all } i \in [n]
\end{aligned}
$$

Decision Theory

Knapsack Problem

- Variables are no longer continuous, but discrete
- In the worst case, up to 2^n possible combinations to check
- Solutions can be searched ”slowly” (exponentially many candidates)
- Heuristics are used for this purpose

Decision Theory

Knapsack problem

- Example heuristic:
  - Sort all items i by p_i / w_i (benefit per cost)
  - Pack the item with the best ratio
  - Pack the item with the second-best ratio...
  - Until item number k no longer fits
  - Then stop
  - ”Greedy” heuristic

Knapsack Problem: Example

- Problem: budget $B = 6$ with

i	1	2	3
p_i	7	5	8
w_i	3	2	4
p_i/w_i	2.33	2.5	2

- Sort:

i	2	1	3
p_i	5	7	8
w_i	2	3	4
p_i/w_i	2.5	2.33	2

- Heuristic: pack items 1 and 2, profit 12
- Best solution: pack items 2 and 3, profit 13

Decision Theory

Knapsack Problem: Heuristic

- How bad can the so-called greedy heuristic be?
- Can you find an example where it performs poorly?
- With a small modification: it always achieves at least half of the optimal profit

Decision Theory

Multicriteria Problems

- Now:

$$
\begin{array}{c|c}
 & s_1 \\
\hline
a_1 & (e_1^1, \ldots, e_1^k) \\
a_2 & (e_2^1, \ldots, e_2^k) \\
\vdots & \vdots \\
a_m & (e_m^1, \ldots, e_m^k)
\end{array}
$$

Decision Theory

Example

- We want to buy a car
- We are only interested in:
  - Costs (e^1, minimize)
  - Environmental impact (e^2, minimize)
- Criteria cannot be converted into each other
- No clear preference between criteria

Decision Theory

Example

- Five cars are available for selection:

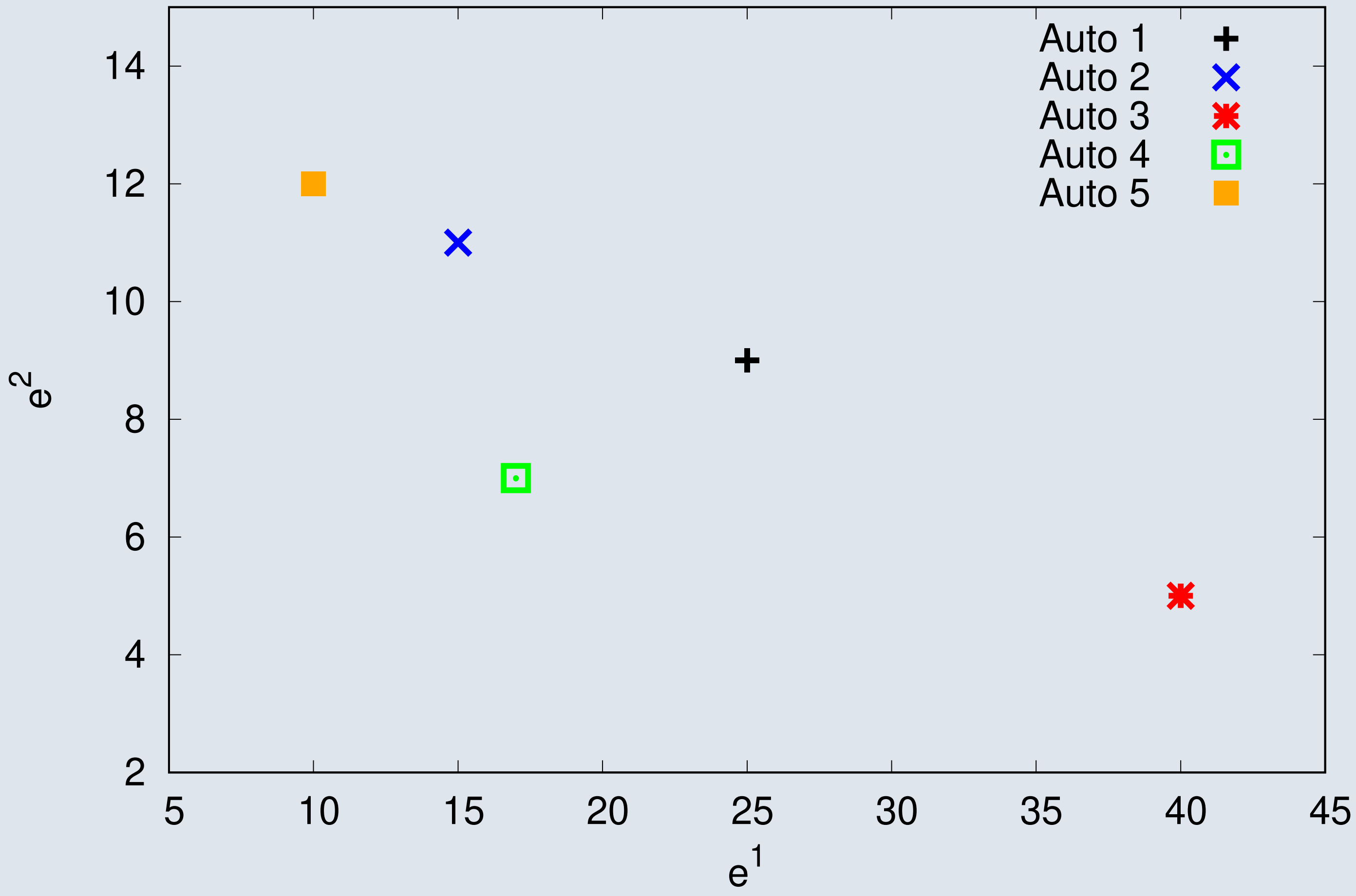
Car	1	2	3	4	5
e^1	25	15	40	17	10
e^2	9	11	5	7	12

- Which one do you choose? (we minimize)

Decision Theory

Visualization

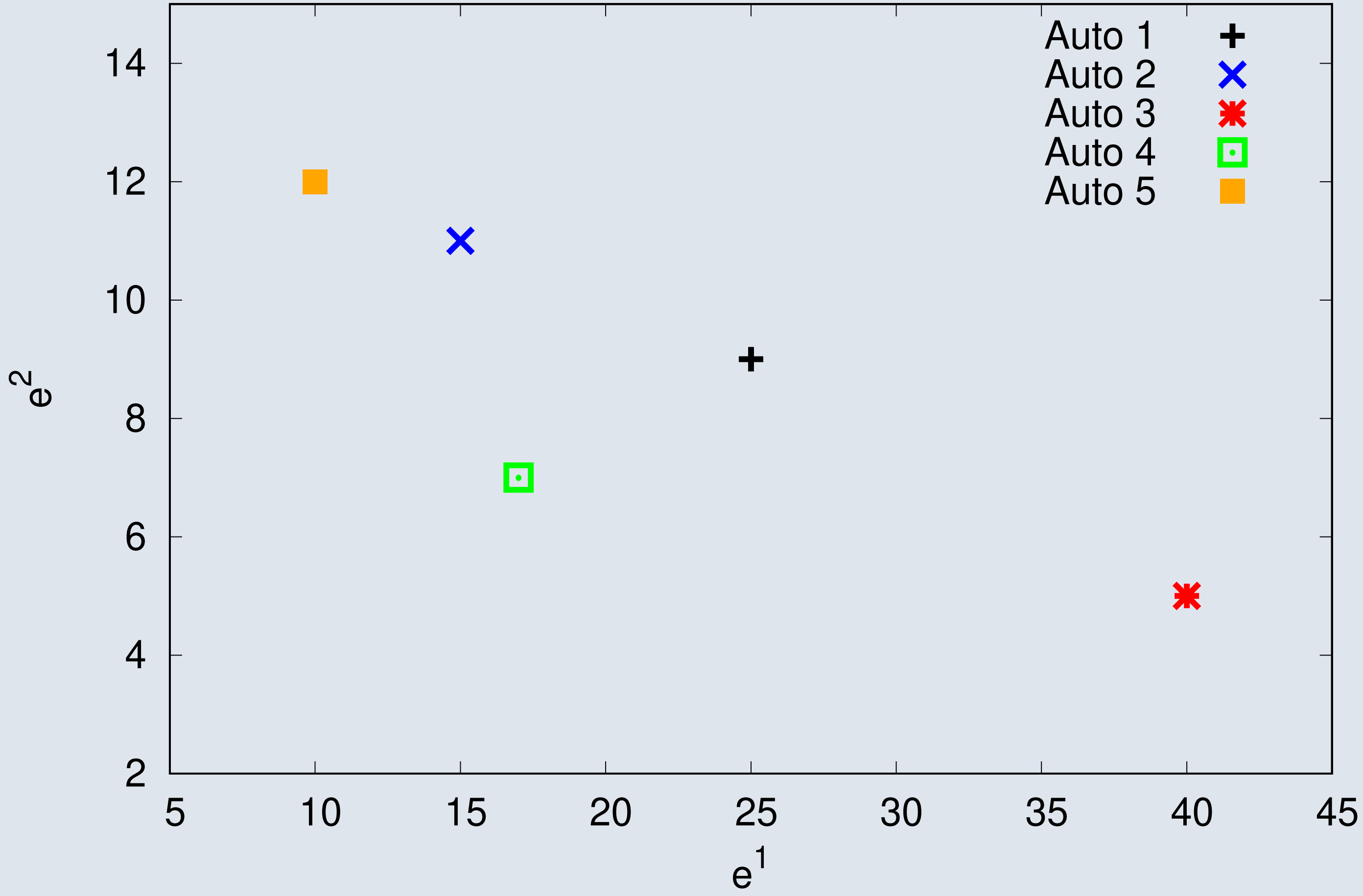
Car	1	2	3	4	5
e^1	25	15	40	17	10
e^2	9	11	5	7	12



Decision Theory

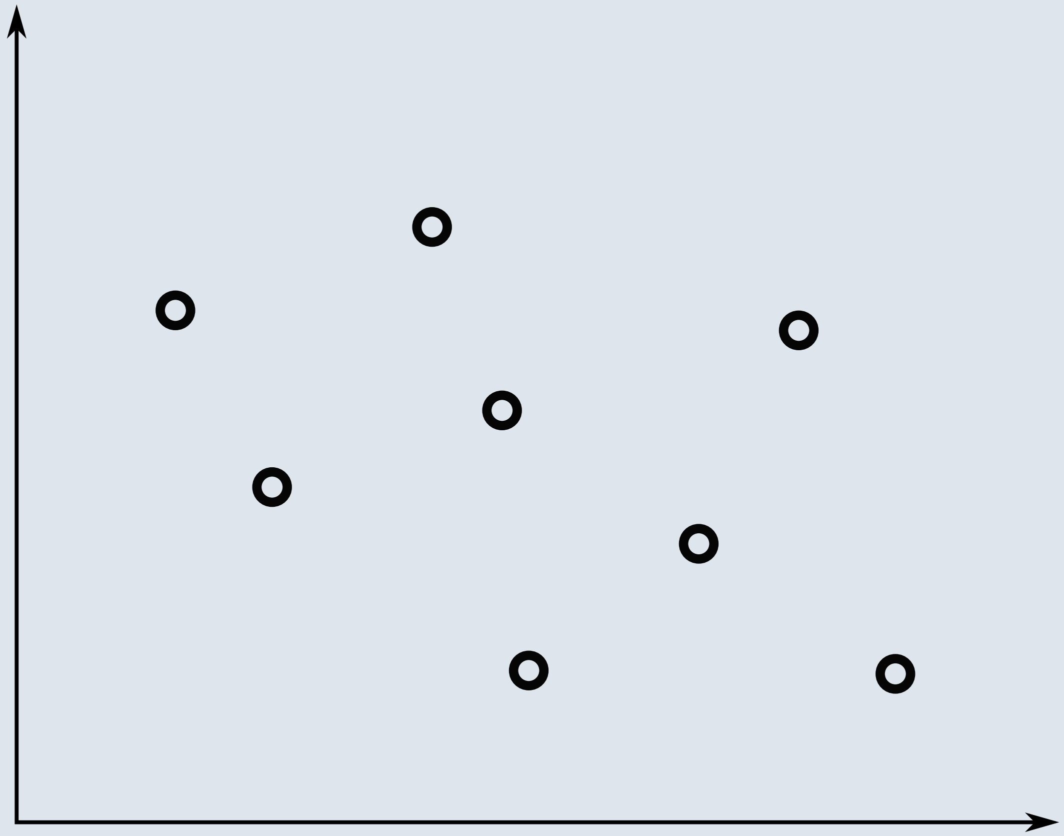
Which solution is optimal?

- What is NOT optimal: Car 1 (as Car 4 is always better)
- A solution is (Pareto) efficient if:
  - There is no other solution that is always at least as good and strictly better in one criterion



Decision Theory

What is efficient? (We maximize)



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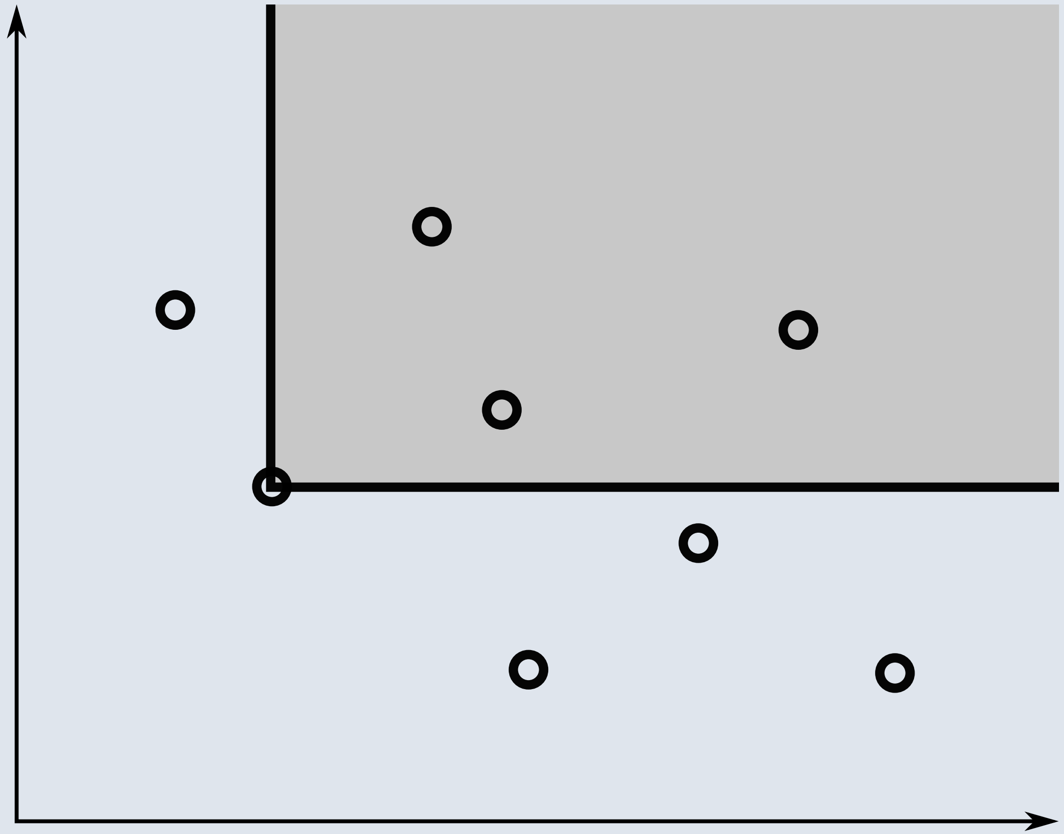
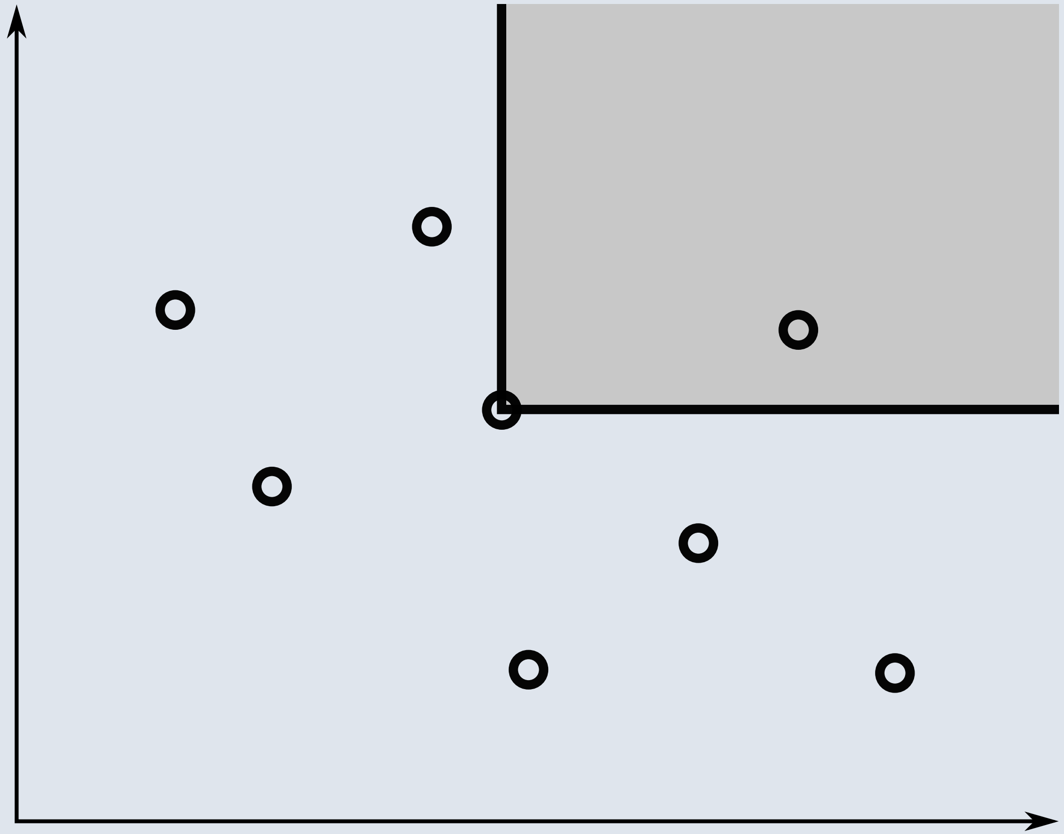


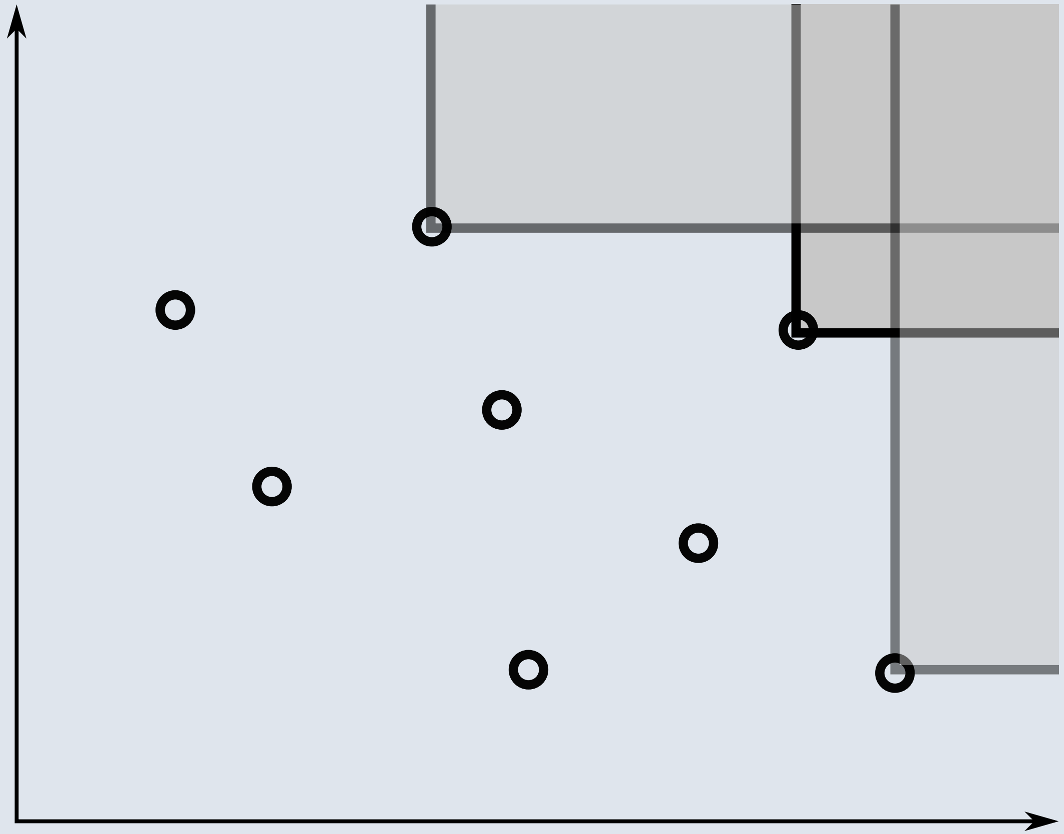
Decision Theory

What is efficient? (We maximize)



Decision Theory

What is efficient? (We maximize)



Decision Theory

What is efficient?

- Want to maximize

	e^1	e^2	e^3
a_1	9	10	6
a_2	10	10	10
a_3	13	7	5
a_4	7	11	12
a_5	12	5	5

Decision Theory

What is efficient?

- Want to maximize

	e^1	e^2	e^3
a_1	9	10	6
a_2	10	10	10
a_3	13	7	5
a_4	7	11	12
a_5	12	5	5

- The same principle as in state dominance

Decision Theory

Cases

Two significantly different problem cases:

1. Solutions are many, indirectly given (linear program case)
   - Find all that are relevant (efficient)
2. Given list (decision matrix case)
   - According to which principle do I choose one?

Decision Theory

Approach 1

- How do I find all efficient solutions?
- Method: Weighted sums
- Objective functions $e^1, \ldots, e^k$
- Weights $w^1, \ldots, w^k$
- Find a decision that maximizes $w^1 e^1 + \ldots + w^k e^k$
- Repeat with new weights

Decision Theory

Weighted Sums

- Special case: we want to find solutions on the boundary
- Weights with a 1 at one point and 0 elsewhere are not enough
- Need a lexicographic approach (last lecture)

Decision Theory

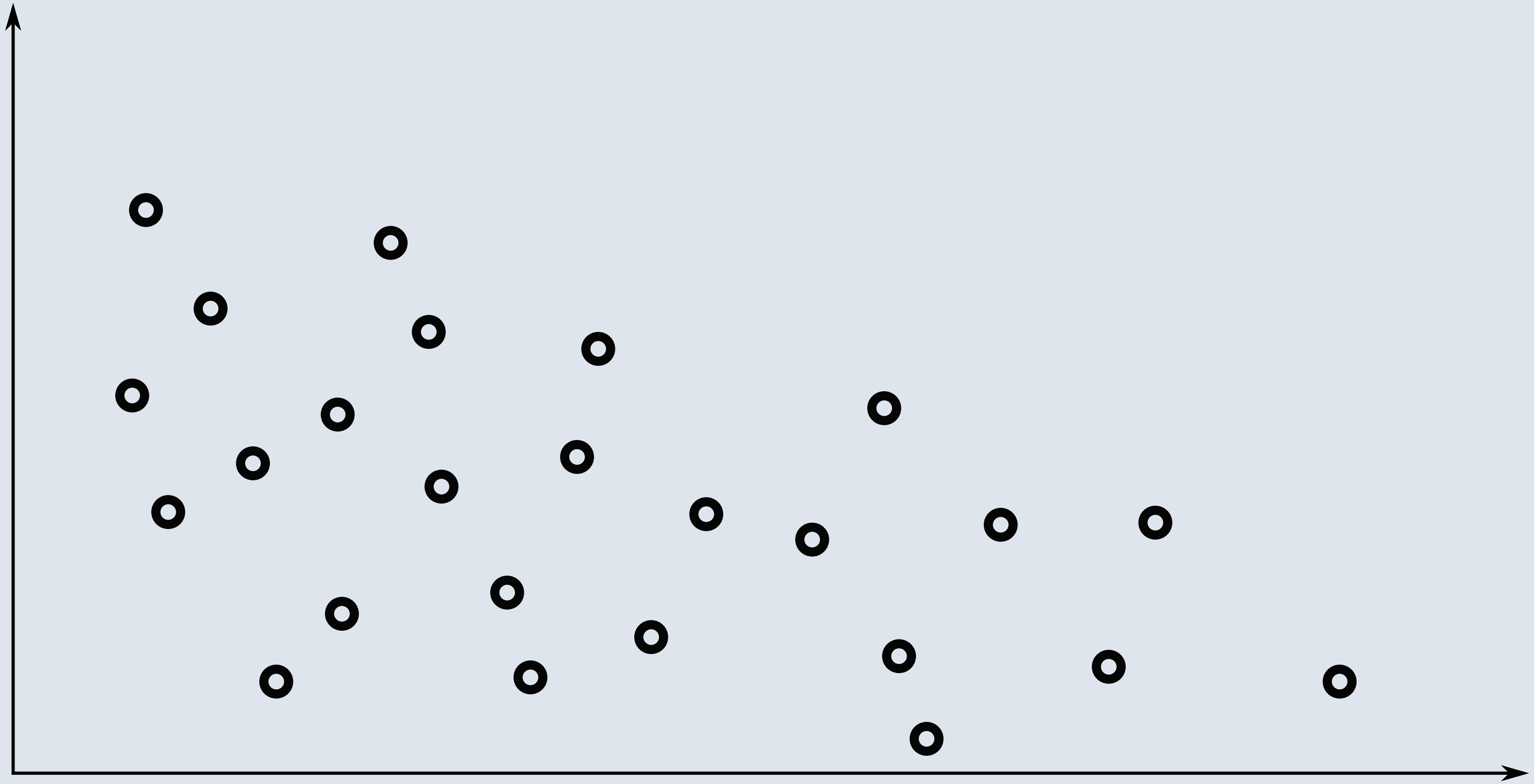
Weighted Sums

- Can I find all efficient solutions this way? (We maximize both criteria)

Decision Theory

Weighted Sums

- Can I find all efficient solutions this way? (We maximize both criteria)



Weighted Sums

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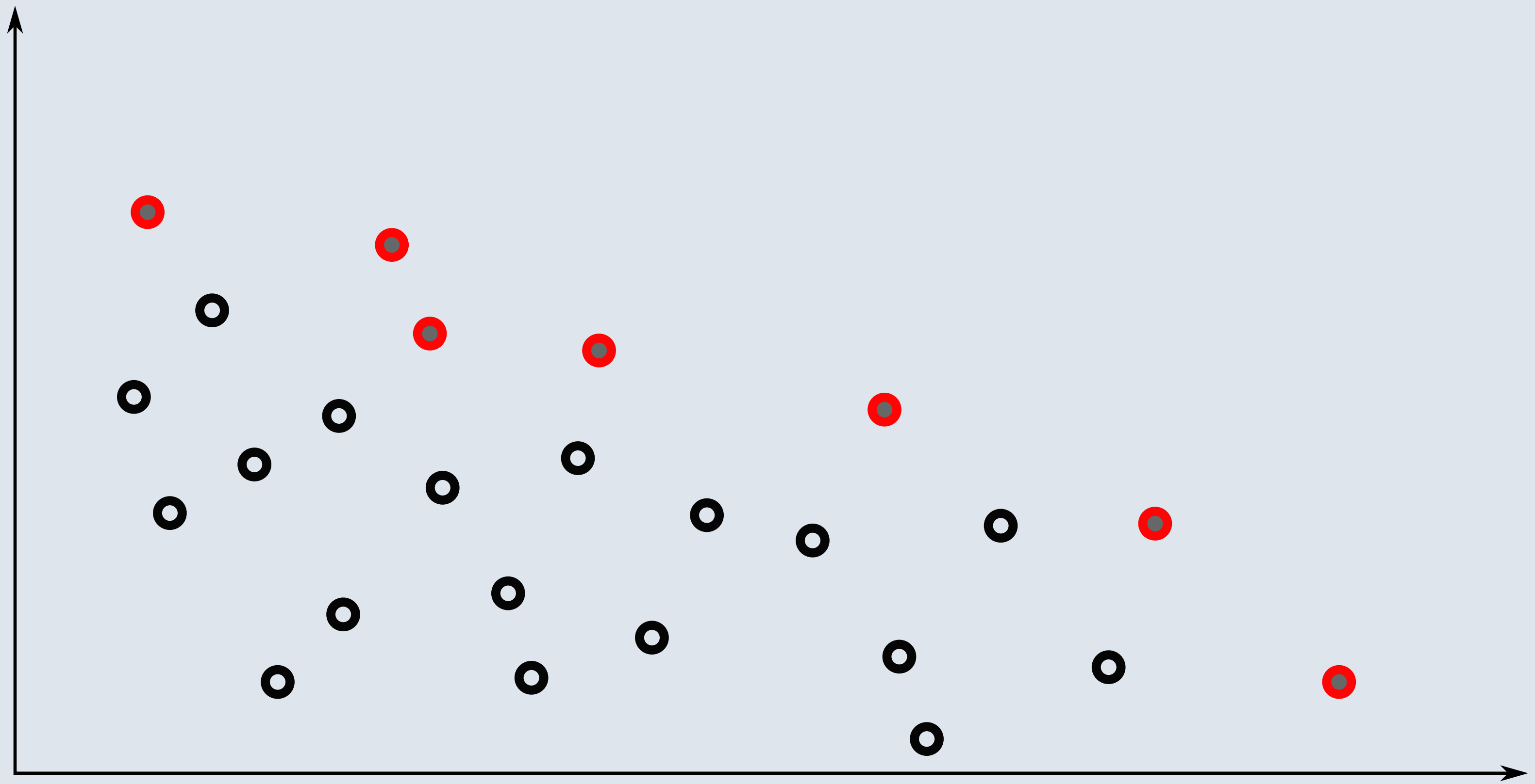
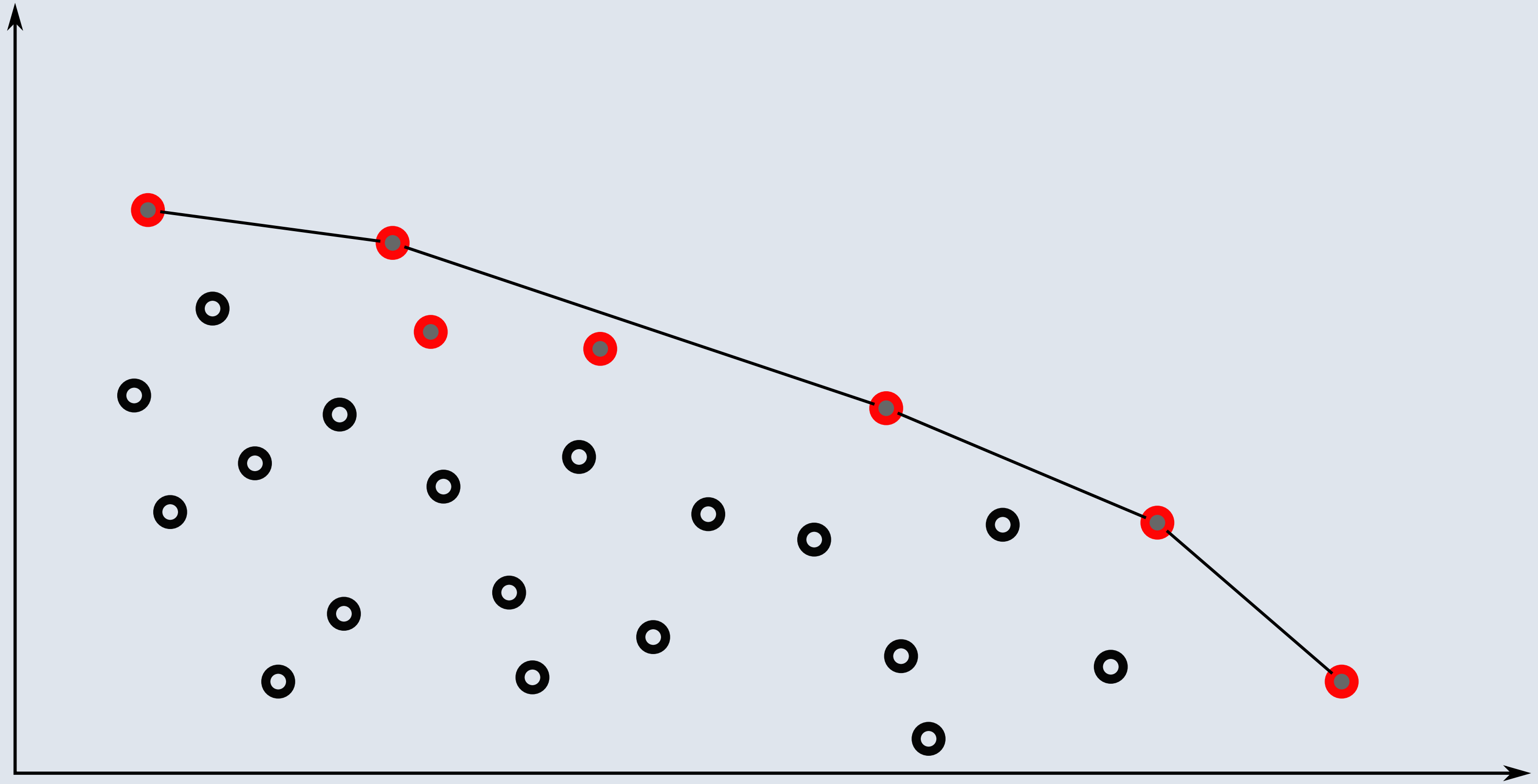


Decision Theory

Weighted Sums

- Can I find all efficient solutions this way? (We maximize both criteria)



Weighted Sums

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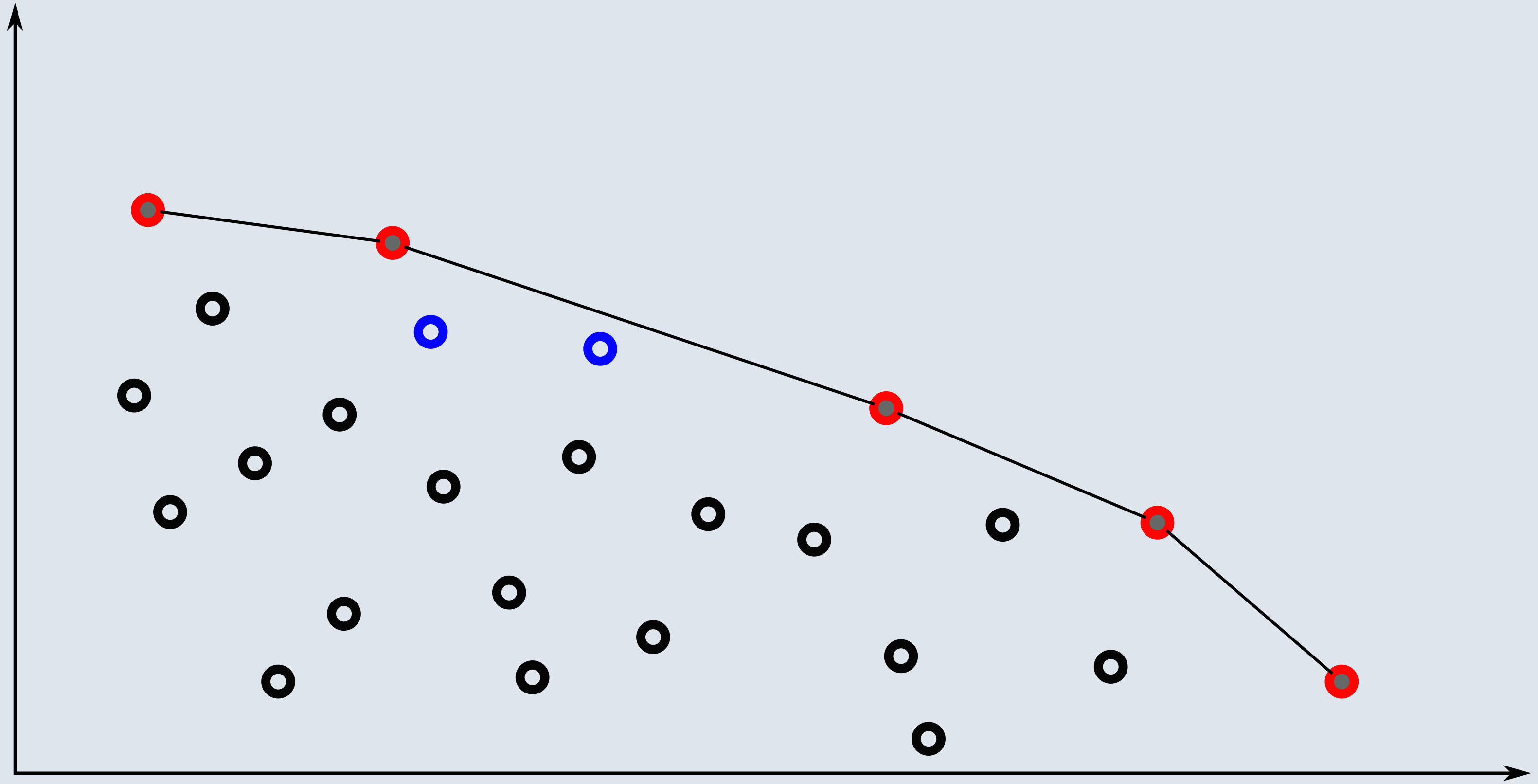


Decision Theory

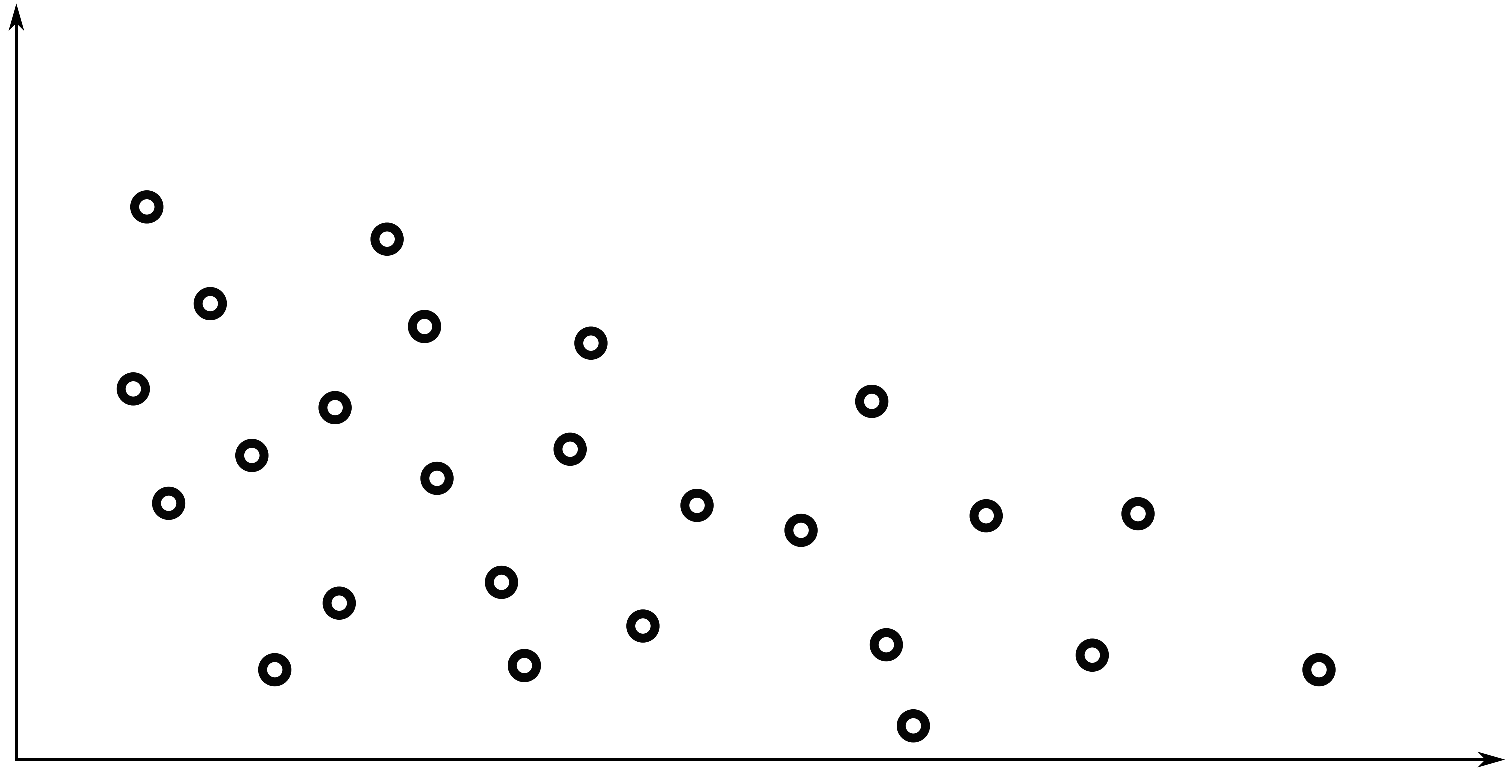
Weighted Sums

- Can I find all efficient solutions this way? (We maximize both criteria)
- No!
- Only ”outer” solutions (”supported”)
- Solutions within the convex hull are not found

Decision Theory

Approach 2

- Add constraints and optimize over one criterion only
- "ϵ-constraint"



Decision Theory

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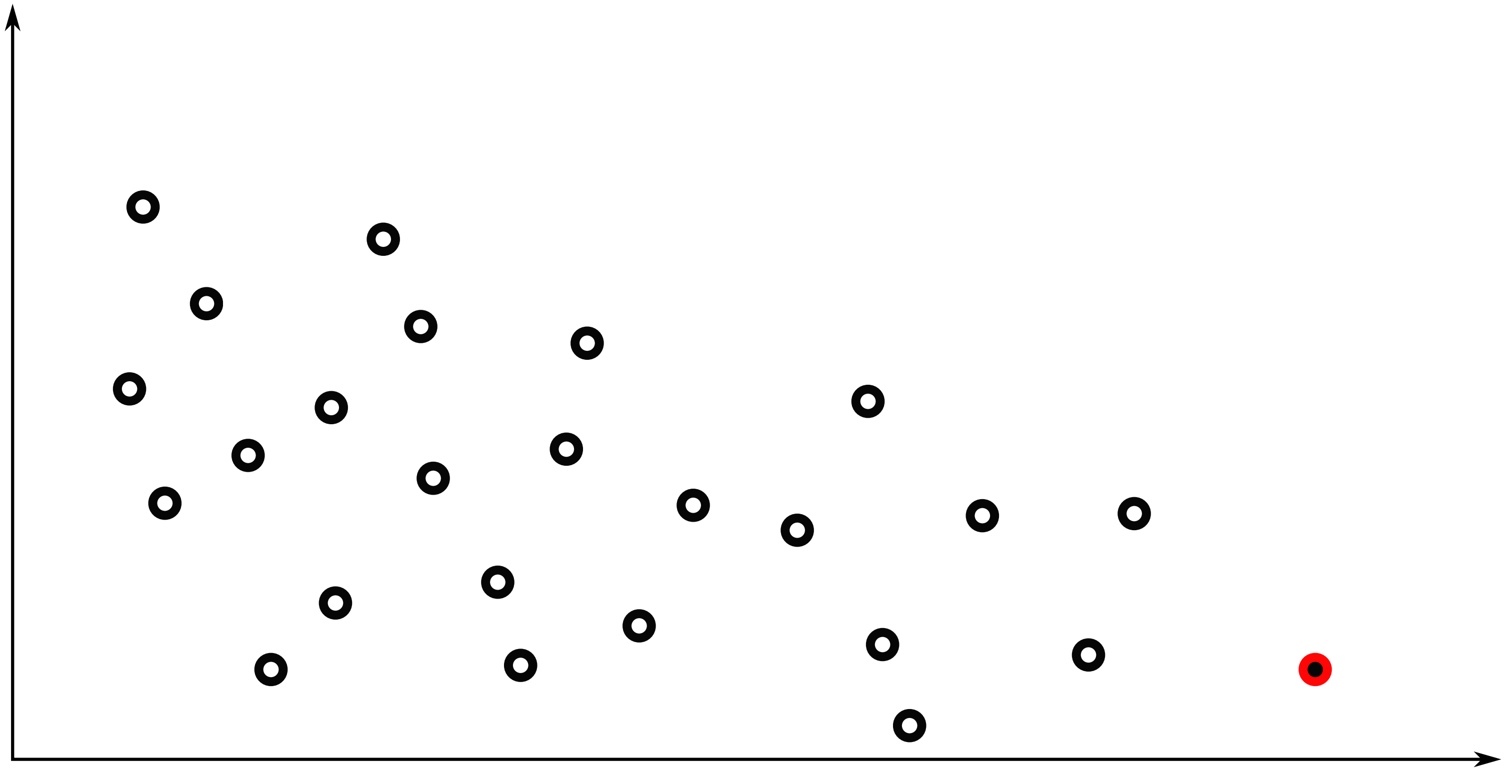

Decision Theory

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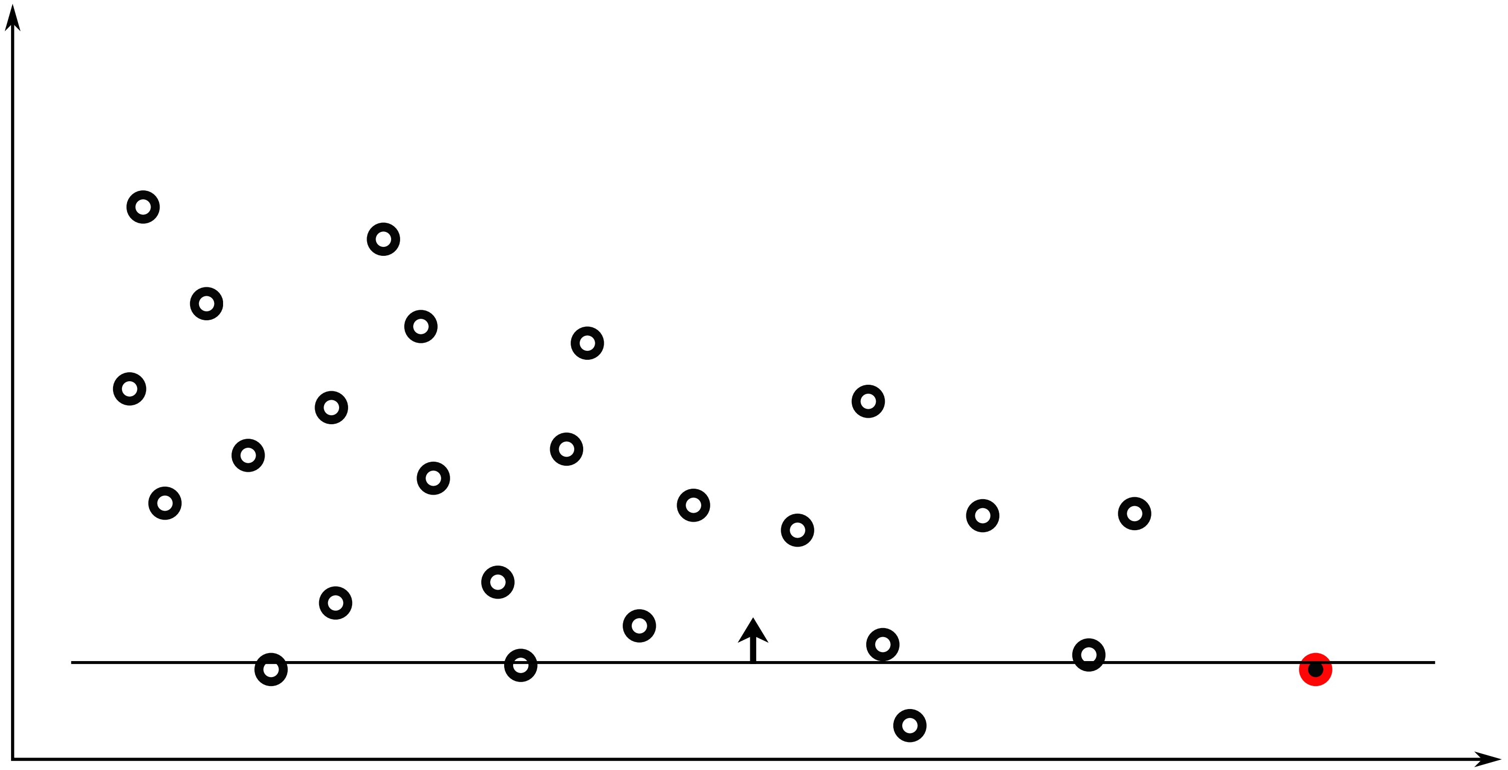

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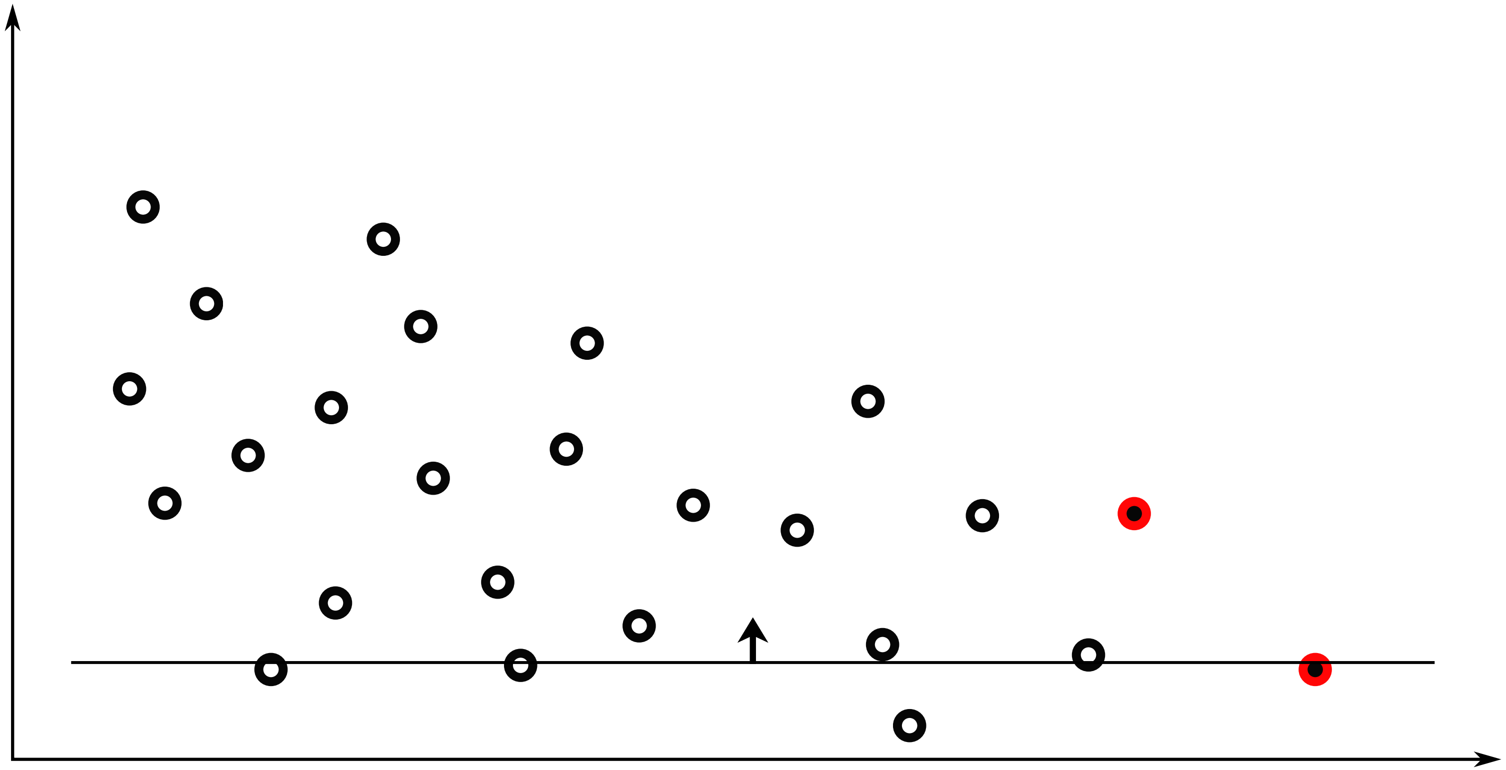

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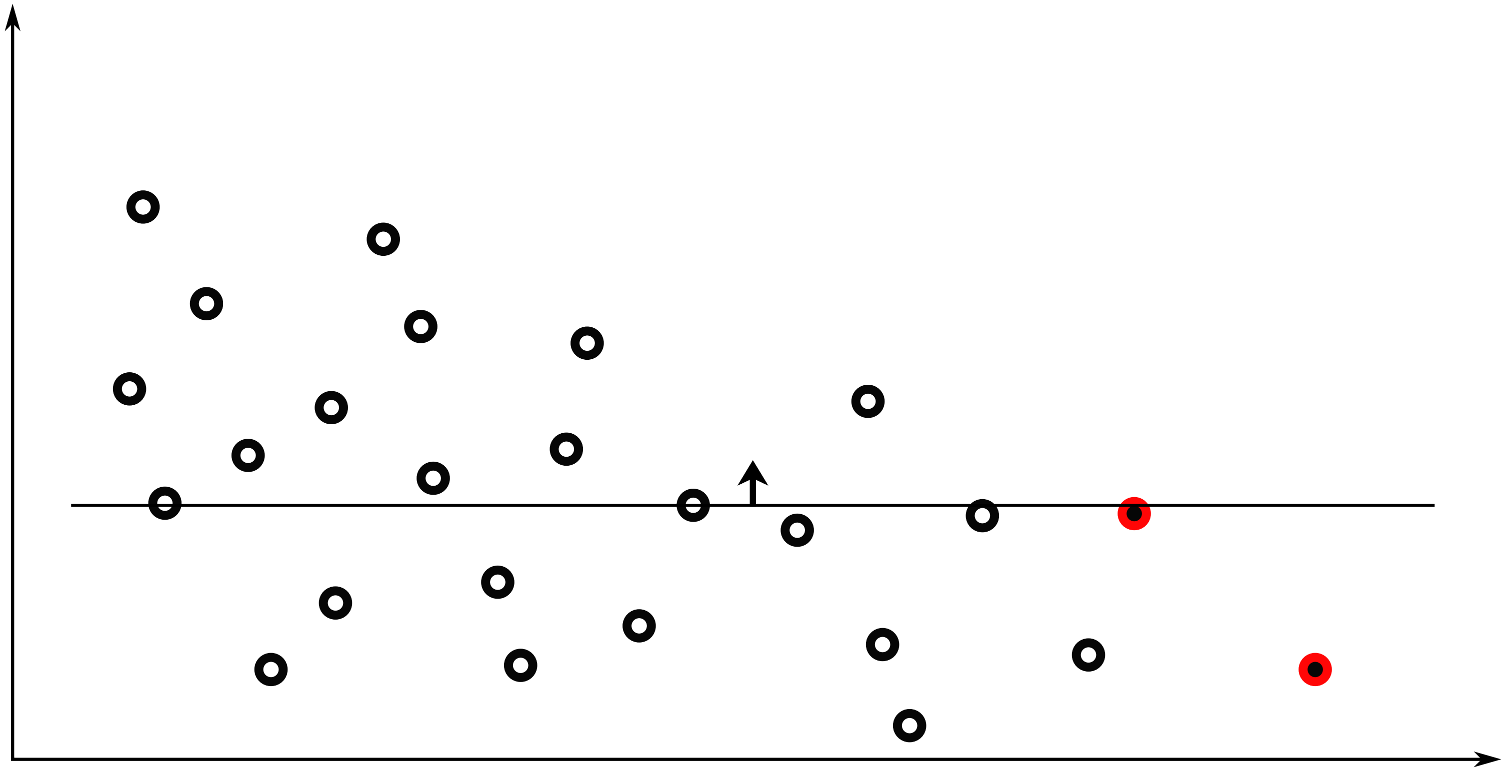

Decision Theory

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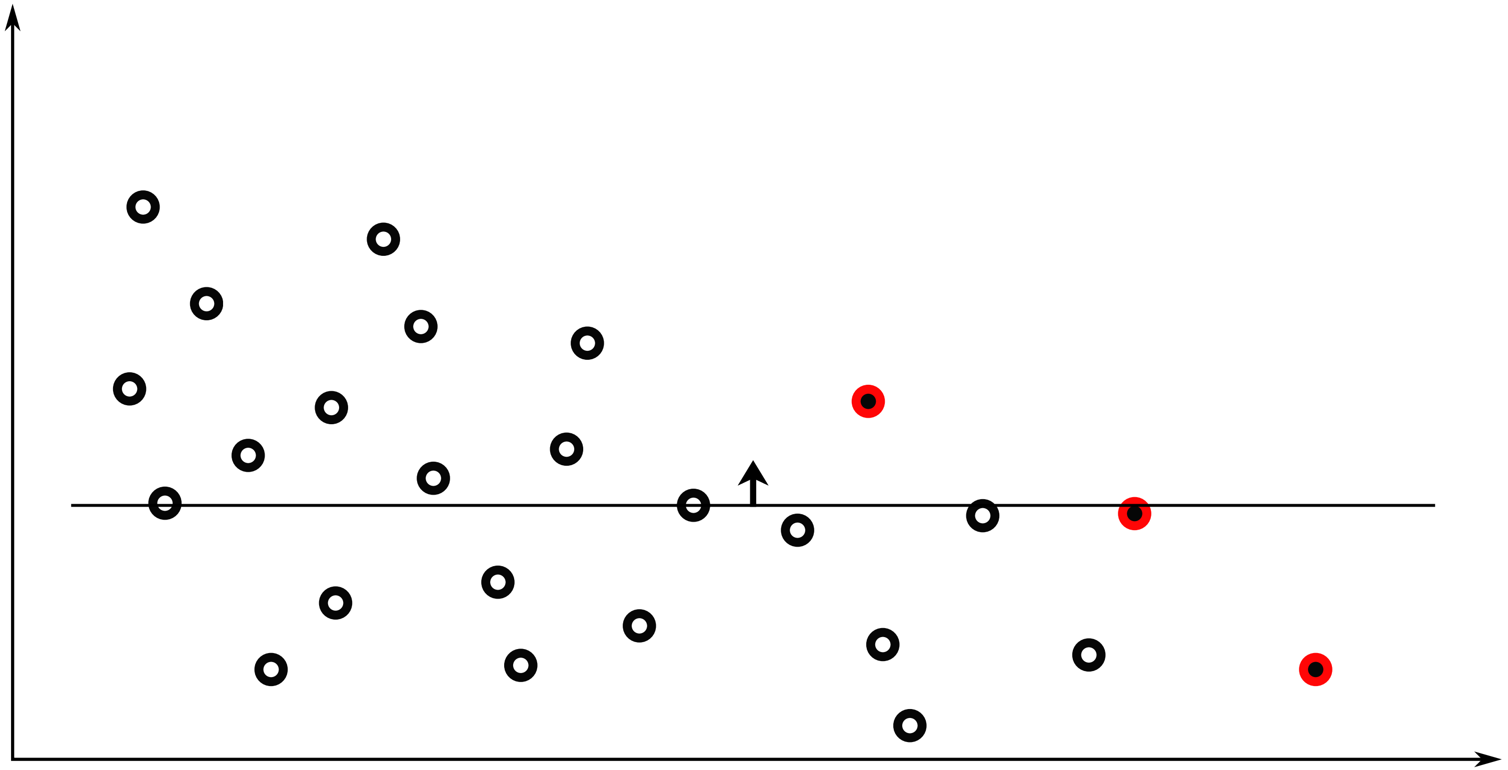

Decision Theory

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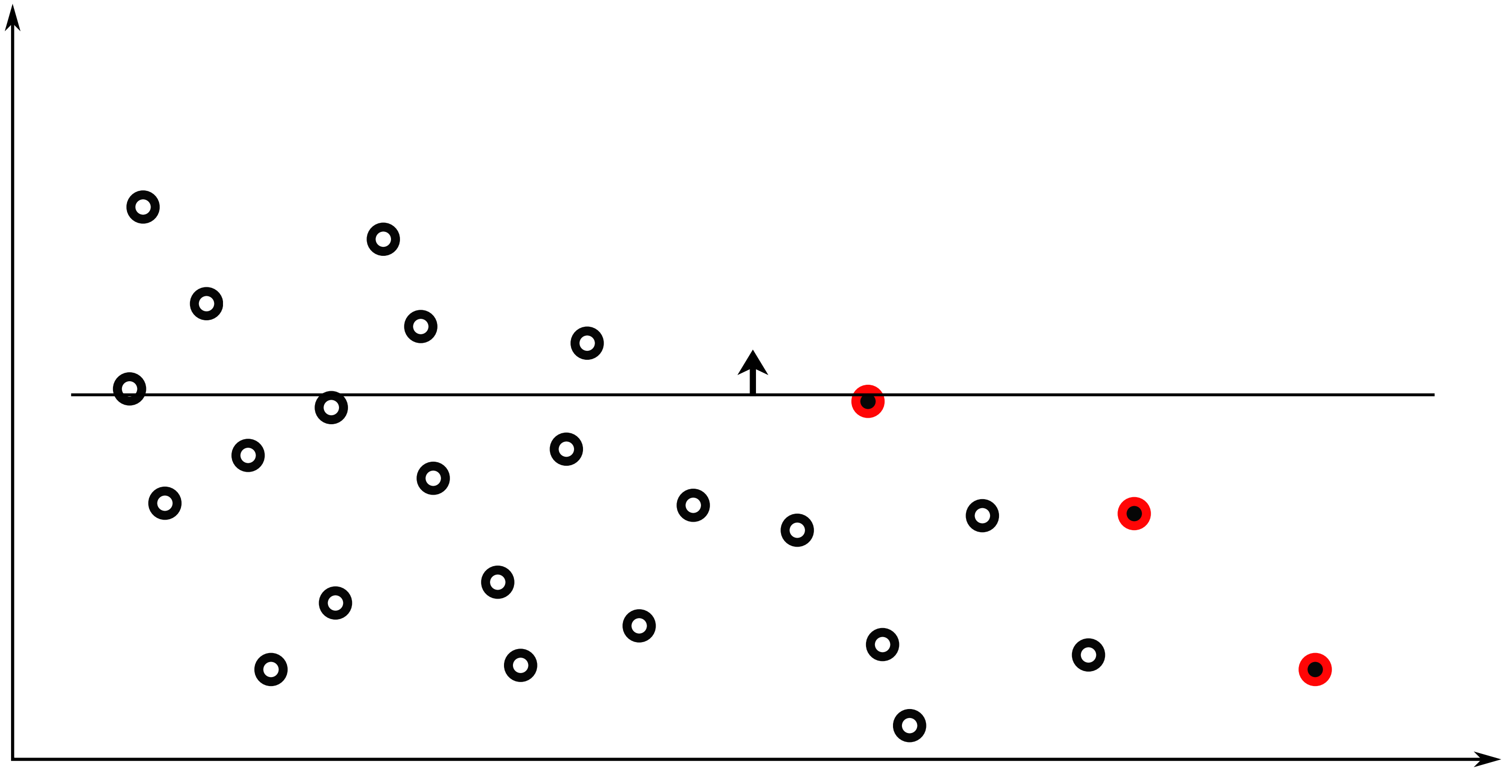

Decision Theory

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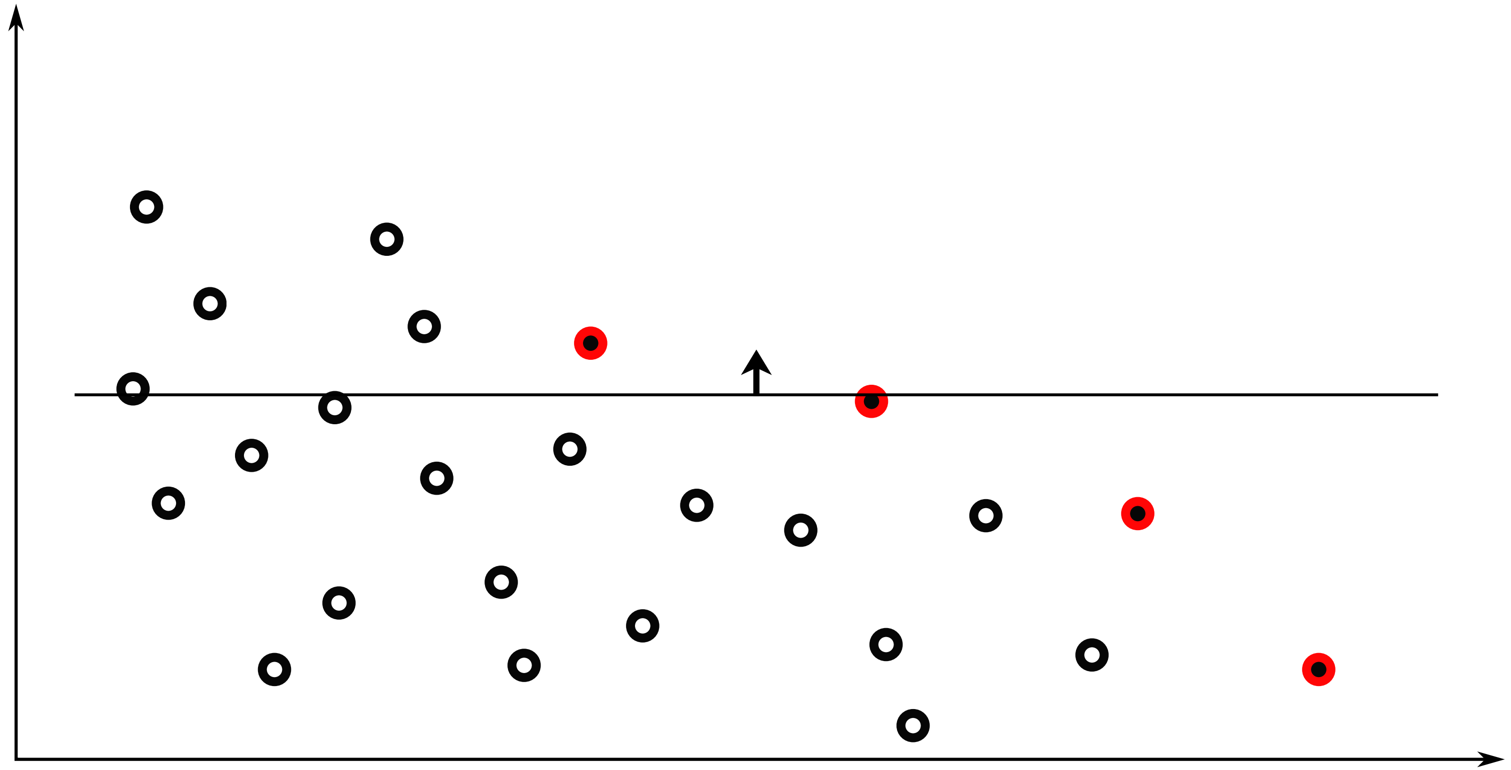

Decision Theory

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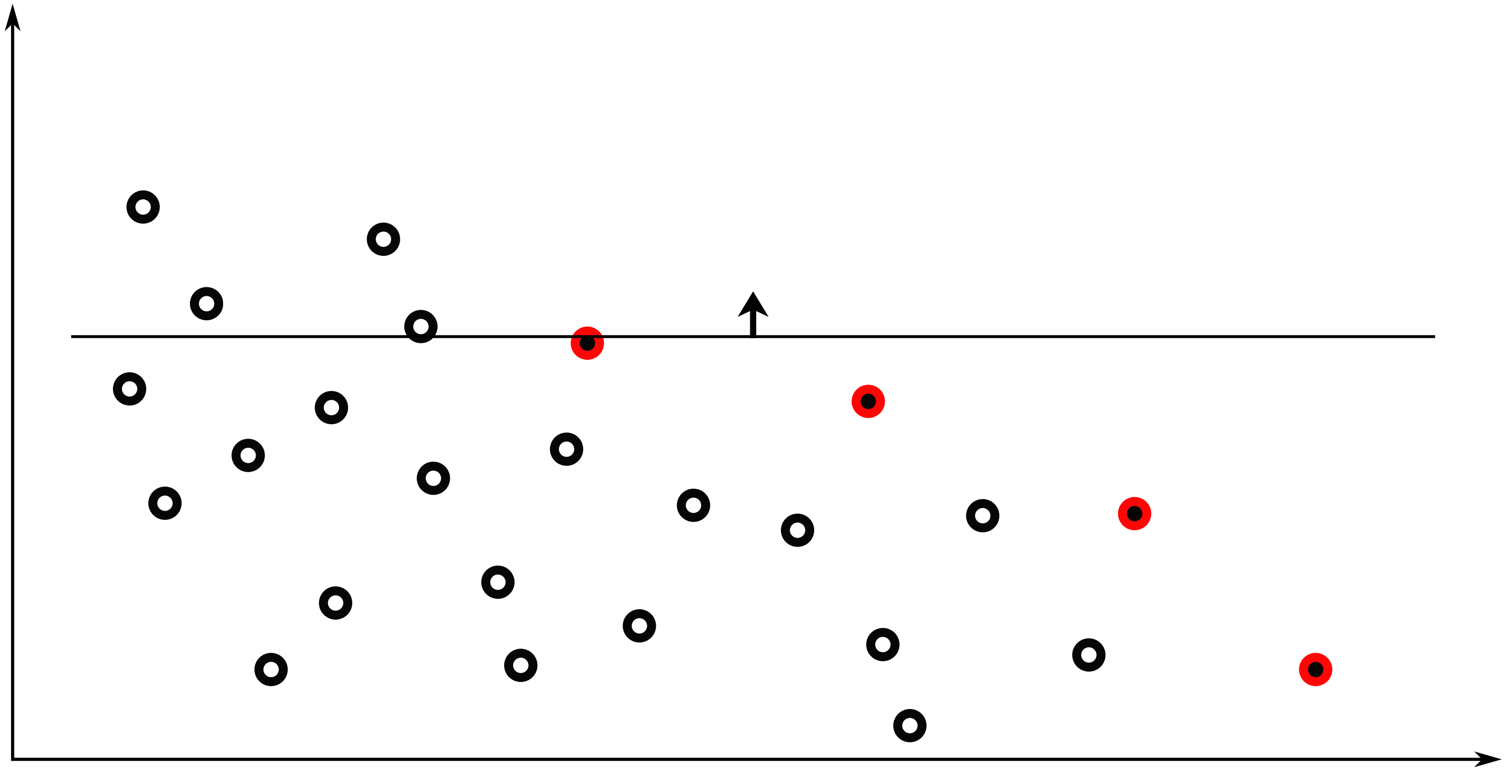

Decision Theory

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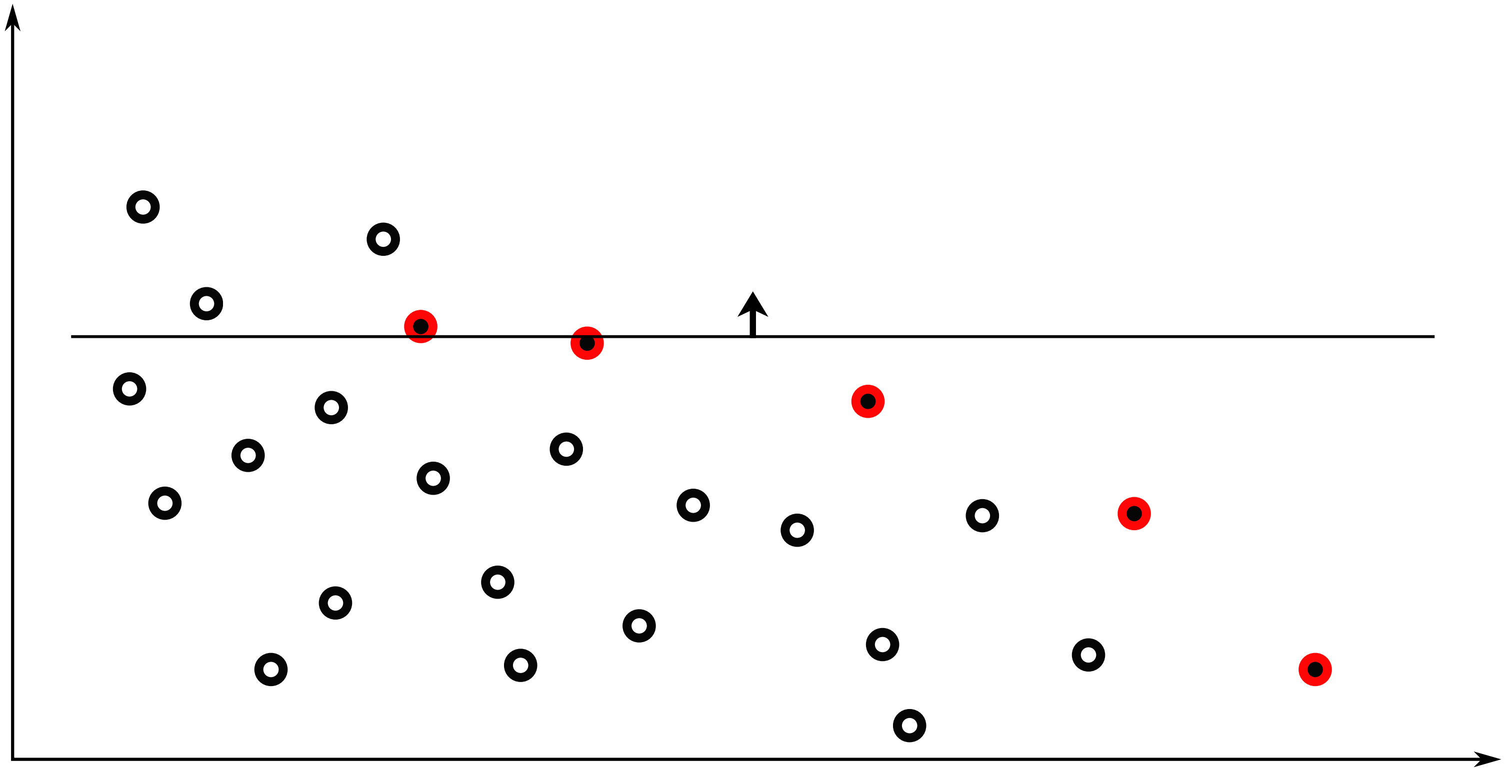

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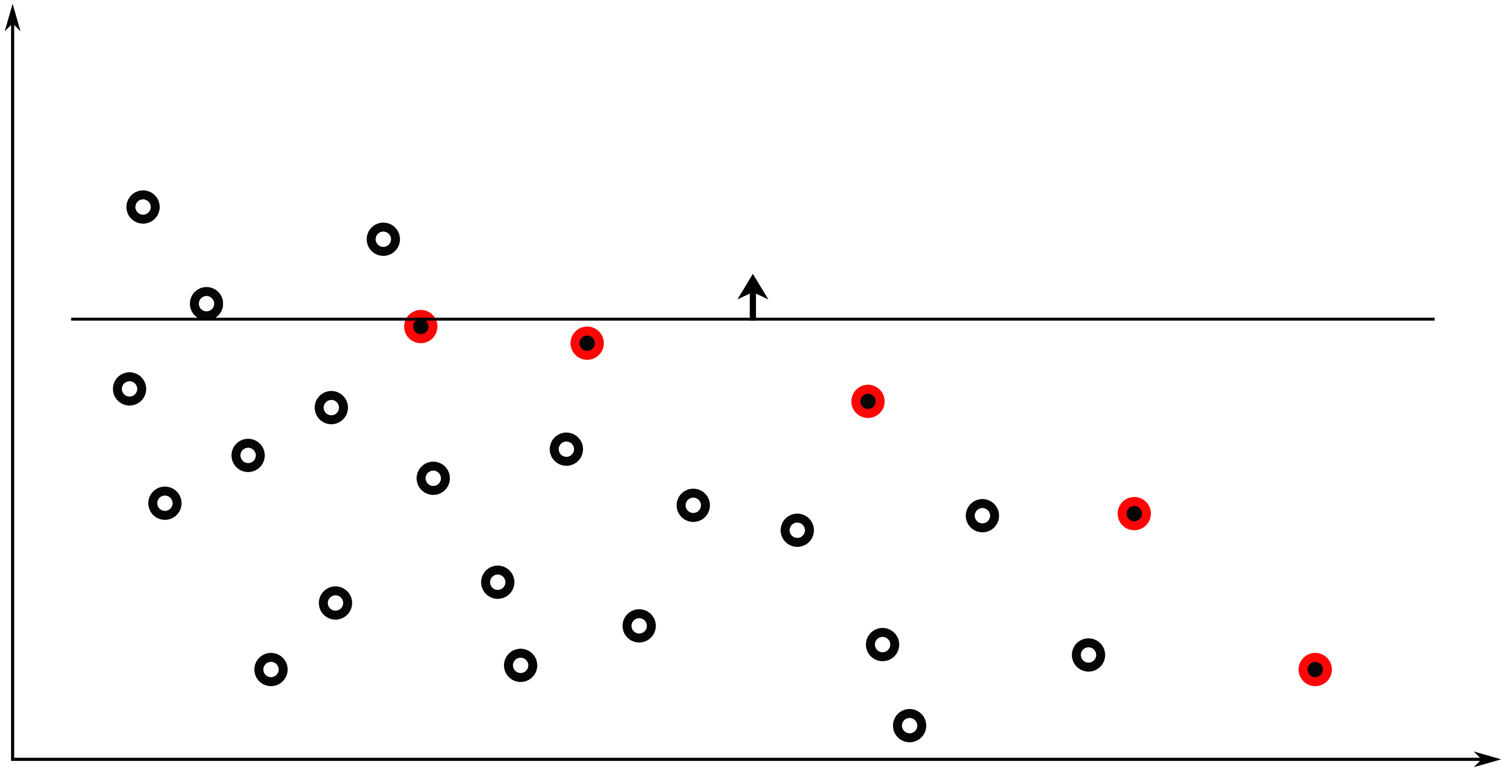

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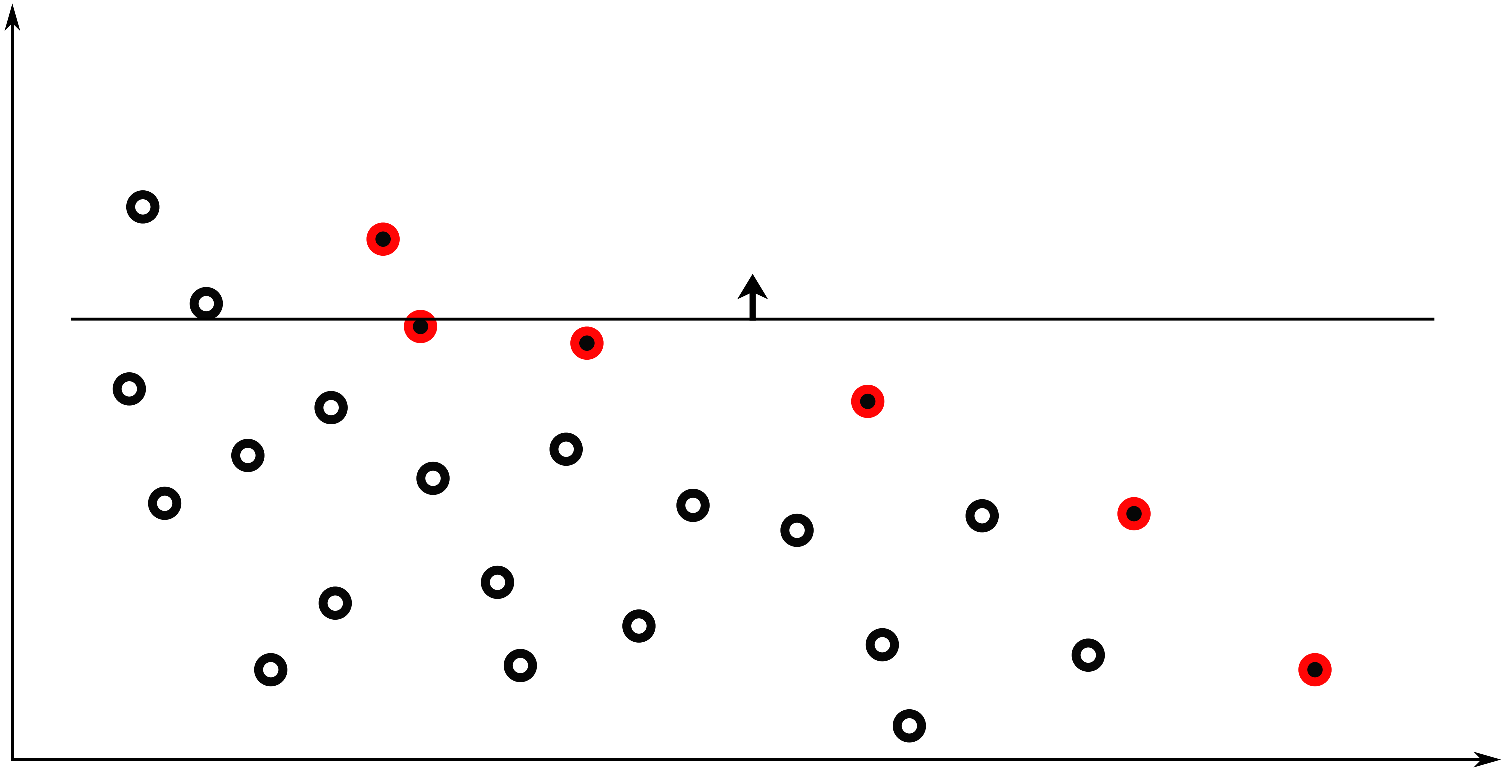

Decision Theory

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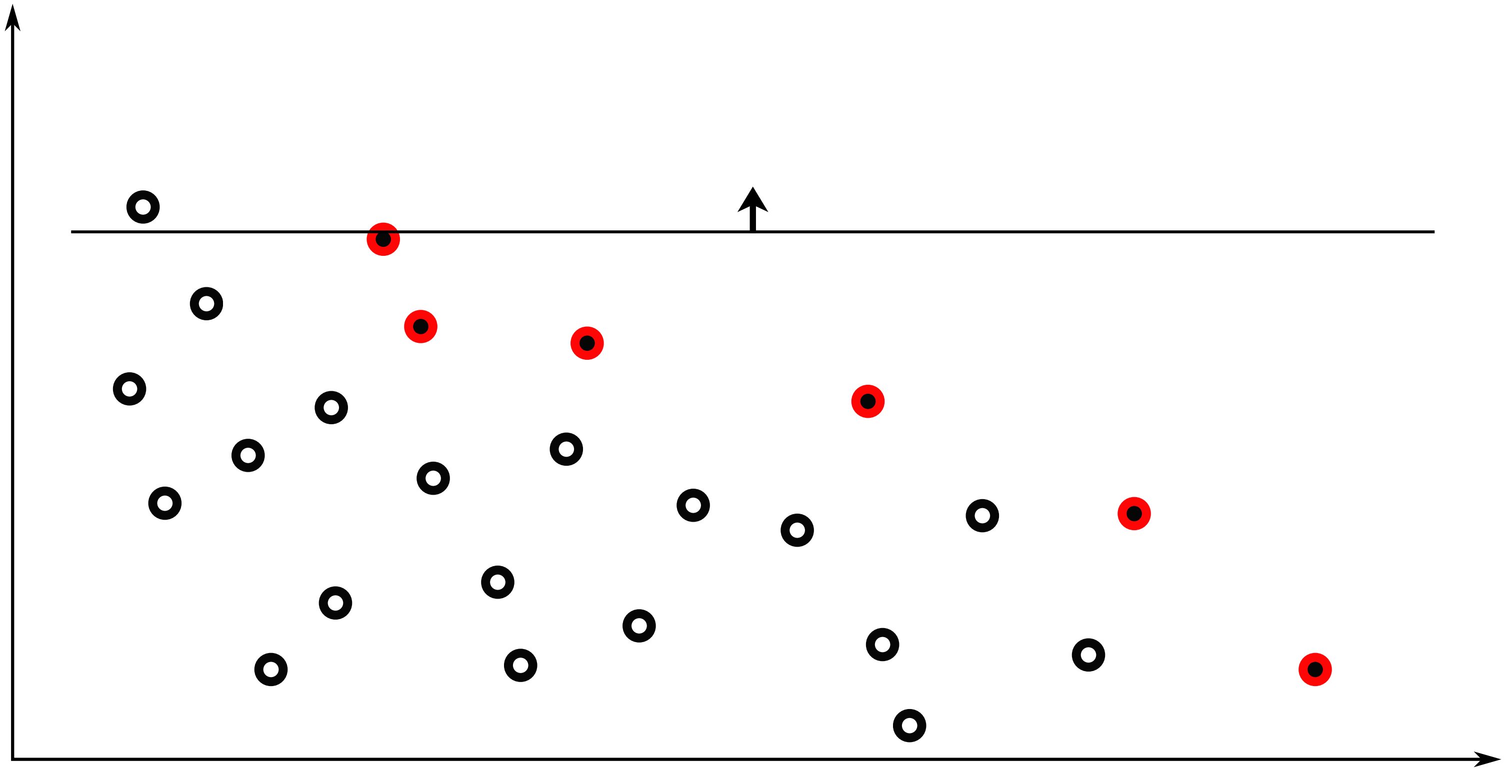

Decision Theory

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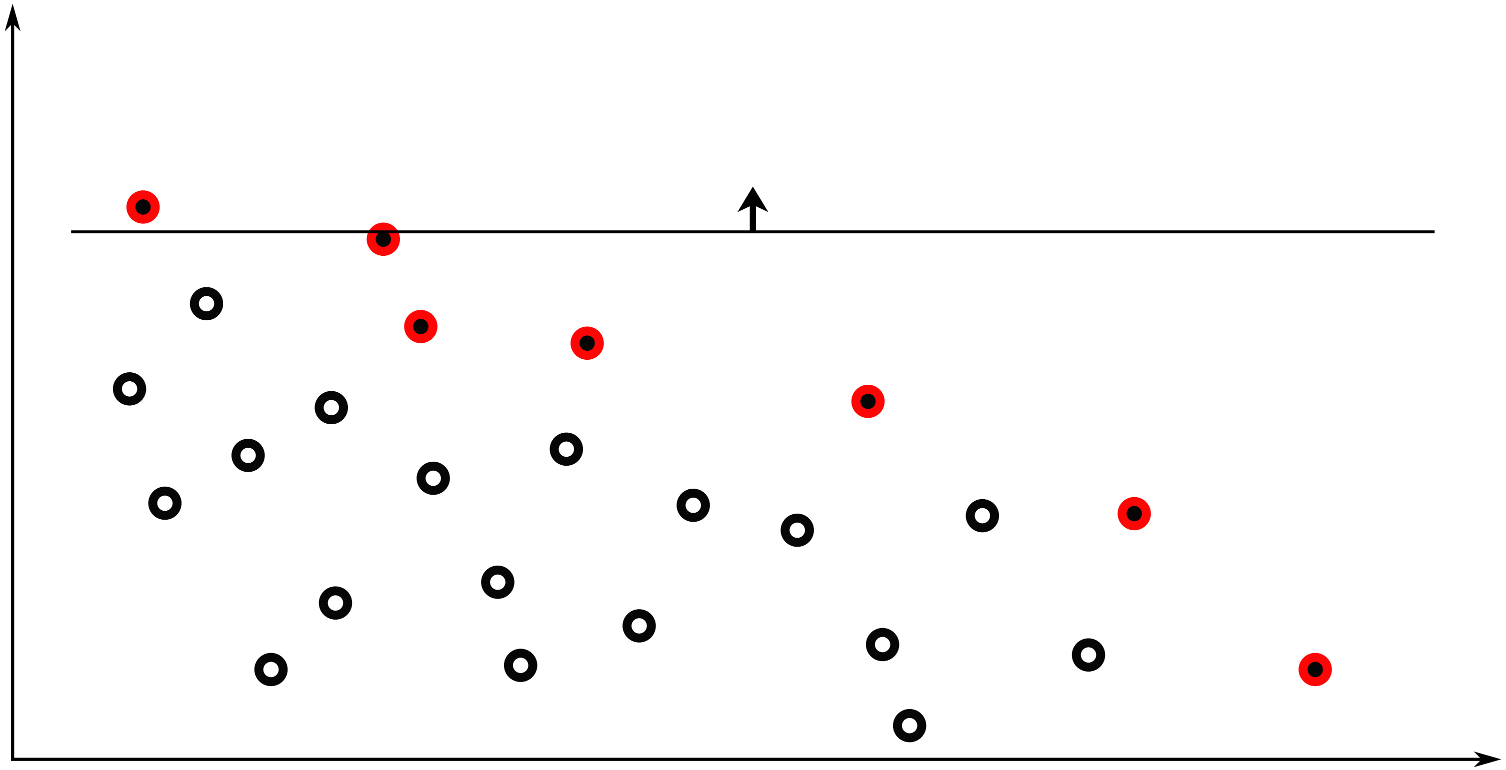


Decision Theory

Discussion

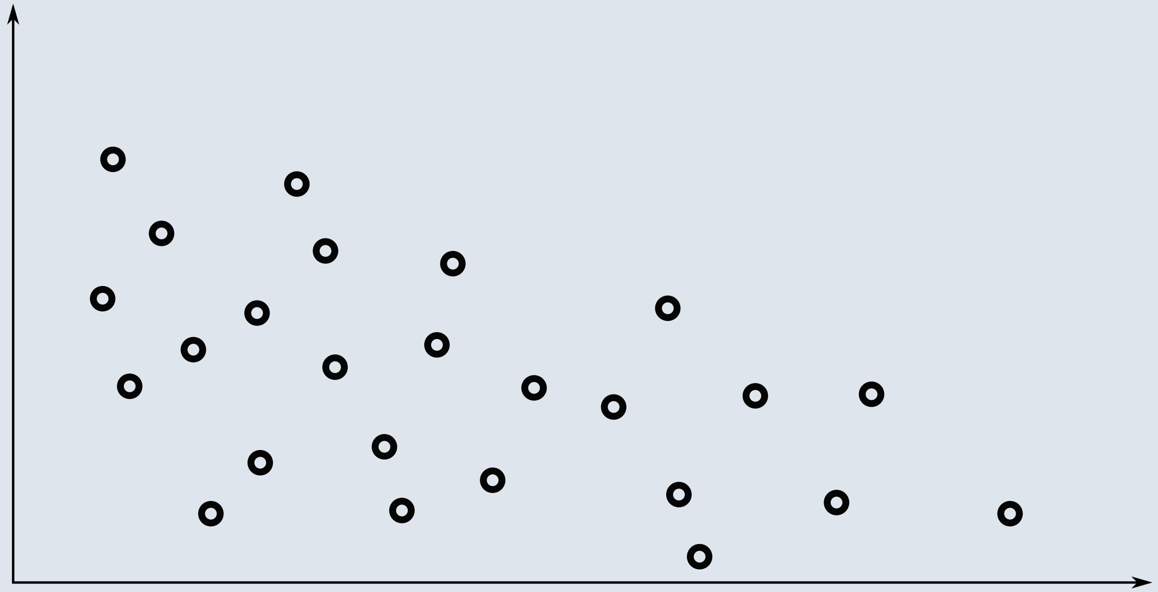
Advantages and disadvantages:

- ϵ-constraint
  - Adds conditions: can make the problem harder
  - Finds unsupported solutions as well
- Weighted sum
  - Without additional conditions, easier to solve
  - Only finds supported solutions

Decision Theory

Selection

- But what if I want to choose a single solution (without knowing much more about user preferences)?
- Example: Goal programming
- Define a distance function, e.g.:
  - Minimize distance to the ideal point
  - Maximize distance to the nadir point



Decision Theory

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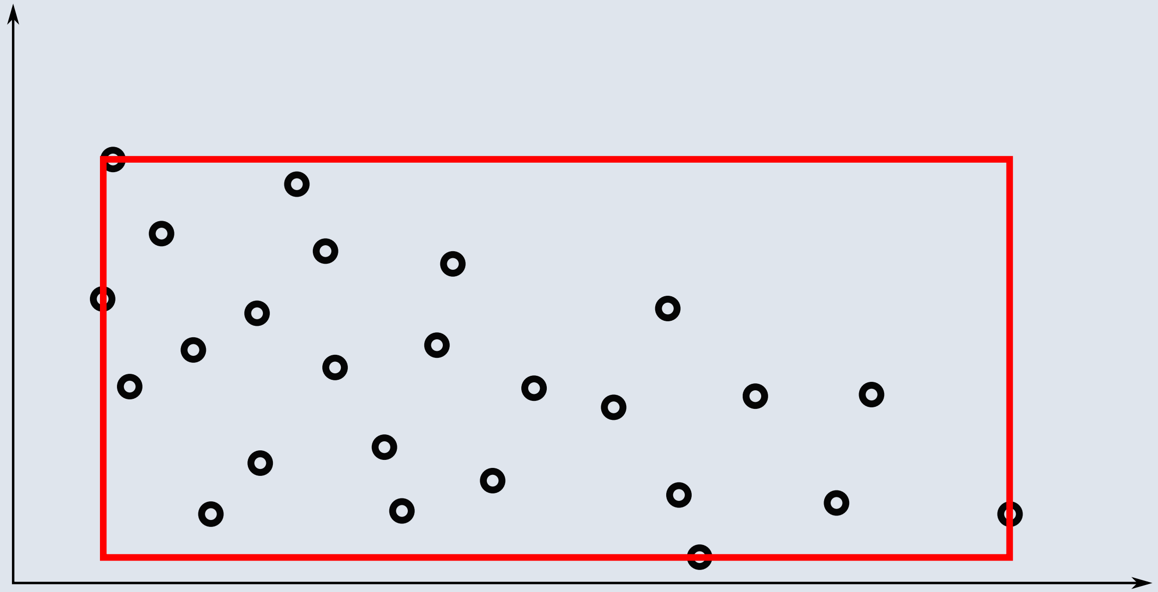

Decision Theory

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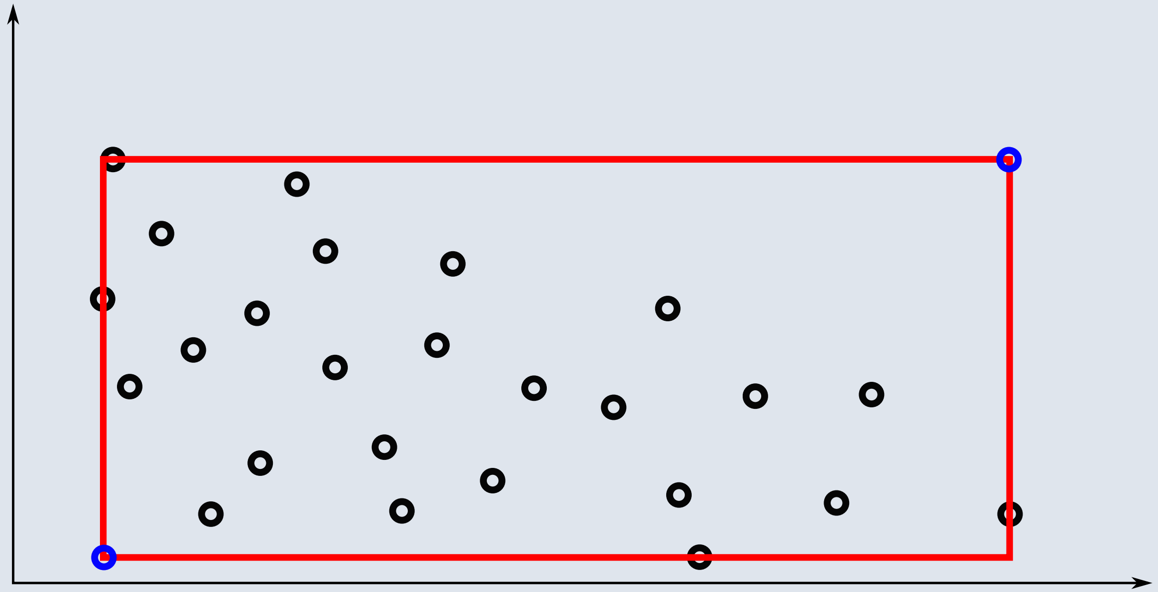

Decision Theory

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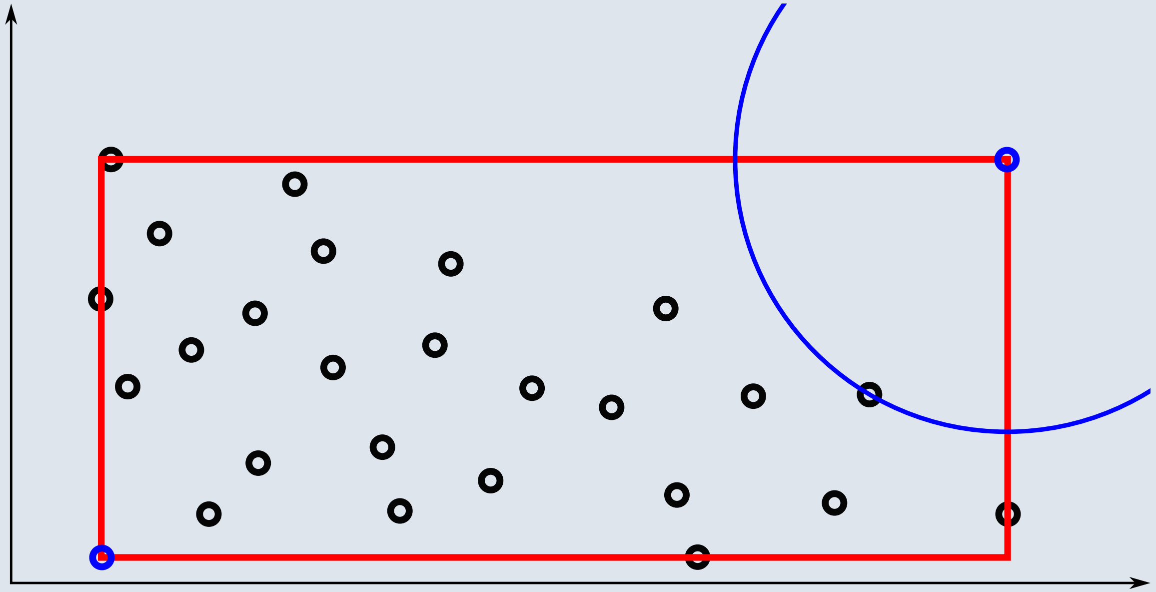


Decision Theory

Now

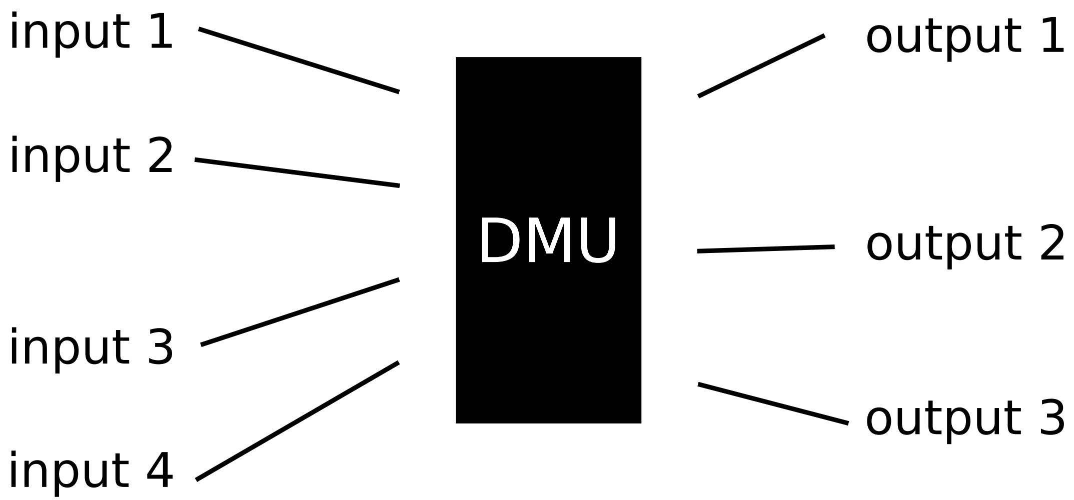
Data Envelopment Analysis (DEA)

Decision Theory

What is it about?

DEA helps assess the efficiency of a unit (Decision Making Unit, DMU)

- Warning: Not Pareto efficiency
- DMU defined very generally
- A set of input variables
- A set of output variables



Decision Theory

DMUs

A DMU can model something different depending on the context:

- Departments of a company
- Stores of a clothing chain
- Prisons
- Banks
- NGOs
- Universities

Decision Theory

DMUs: Examples

DMU	Input	Output
Factory	Raw materials Personnel Energy	Goods
Bank	Counters Managers Branches	Accounts Loans Profit
Professors	Time Salary Personnel	Teaching Papers Funding

Decision Theory

Efficiency

- Want to assess the efficiency of DMUs
- Principle:

$$\text{Efficiency} = \frac{\text{Output}}{\text{Input}}$$

- What is input, what is output?
- How are they quantified?

Decision Theory

Example

- Assess the efficiency of 4 data science departments

University	#Professors	#Degrees	External funding
A	6	132	9600€
B	12	192	26400€
C	10	190	21000€
D	8	144	14400€

- Which university has an efficient department?
- What is input, what is output?

Decision Theory

Example

University	#Professors	#Degrees	External Funding
A	6	132	9600€
B	12	192	26400€
C	10	190	21000€
D	8	144	14400€

- Consider virtual university

Decision Theory

Example

University	#Professors	#Degrees	External Funding
A	6	132	9600€
B	12	192	26400€
C	10	190	21000€
D	8	144	14400€

- Consider virtual university

$$VU = \frac{1}{2}A + \frac{1}{2}B$$

Example

University	#Professors	#Degrees	External Funding
A	6	132	9600€
B	12	192	26400€
C	10	190	21000€
D	8	144	14400€
VU	8	161	15300€

- Consider virtual university

$$VU = \frac{1}{2}A + \frac{1}{2}B$$

Decision Theory

Example

University	#Professors	#Degrees	External Funding
A	6	132	9600€
B	12	192	26400€
C	10	190	21000€
D	8	144	14400€
VU	8	161	15300€

- Consider virtual university

$$VU = \frac{1}{2}A + \frac{1}{2}B$$

Decision Theory

Example

University	#Professors	#Degrees	External Funding
A	6	132	9600€
B	12	192	26400€
C	10	190	21000€
D	8	144	14400€
VU	8	161	15300€

- Consider virtual university

$$VU = \frac{1}{2}A + \frac{1}{2}B$$

- VU achieves better output with the same input!
- University D is inefficient

Decision Theory

Example

Consider output per professor:

University	#Professors	#Degrees	External funding
A	6	132	9600€
B	12	192	26400€
C	10	190	21000€
D	8	144	14400€

University	#Professors	#Degrees	External funding
A	1		
B			
C			
D			

Decision Theory

Example

Consider output per professor:

University	#Professors	#Degrees	External funding
A	6	132	9600€
B	12	192	26400€
C	10	190	21000€
D	8	144	14400€

University	#Professors	#Degrees	External funding
A	1	22	1600€
B			
C			
D			

Decision Theory

Example

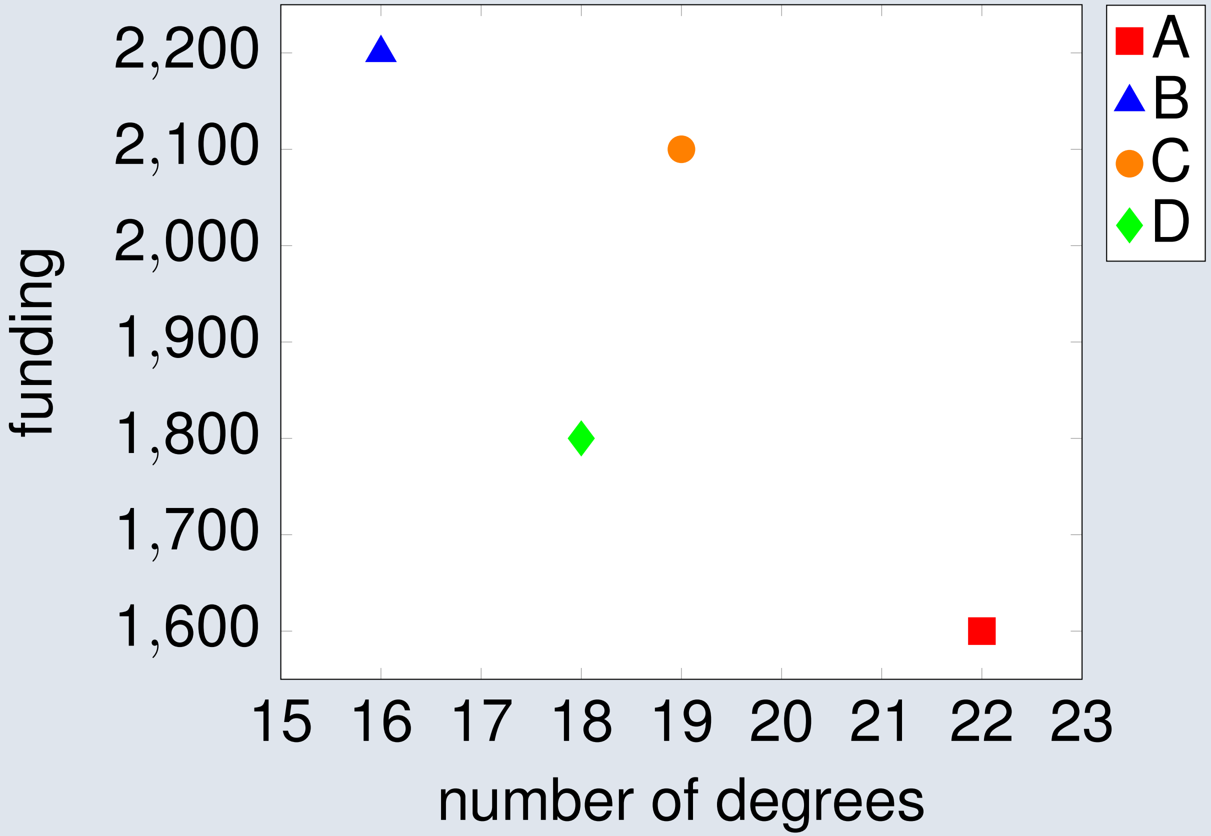
Consider output per professor:

University	#Professors	#Degrees	External funding
A	6	132	9600€
B	12	192	26400€
C	10	190	21000€
D	8	144	14400€

University	#Professors	#Degrees	External funding
A	1	22	1600€
B	1	16	2200€
C	1	19	2100€
D	1	18	1800€

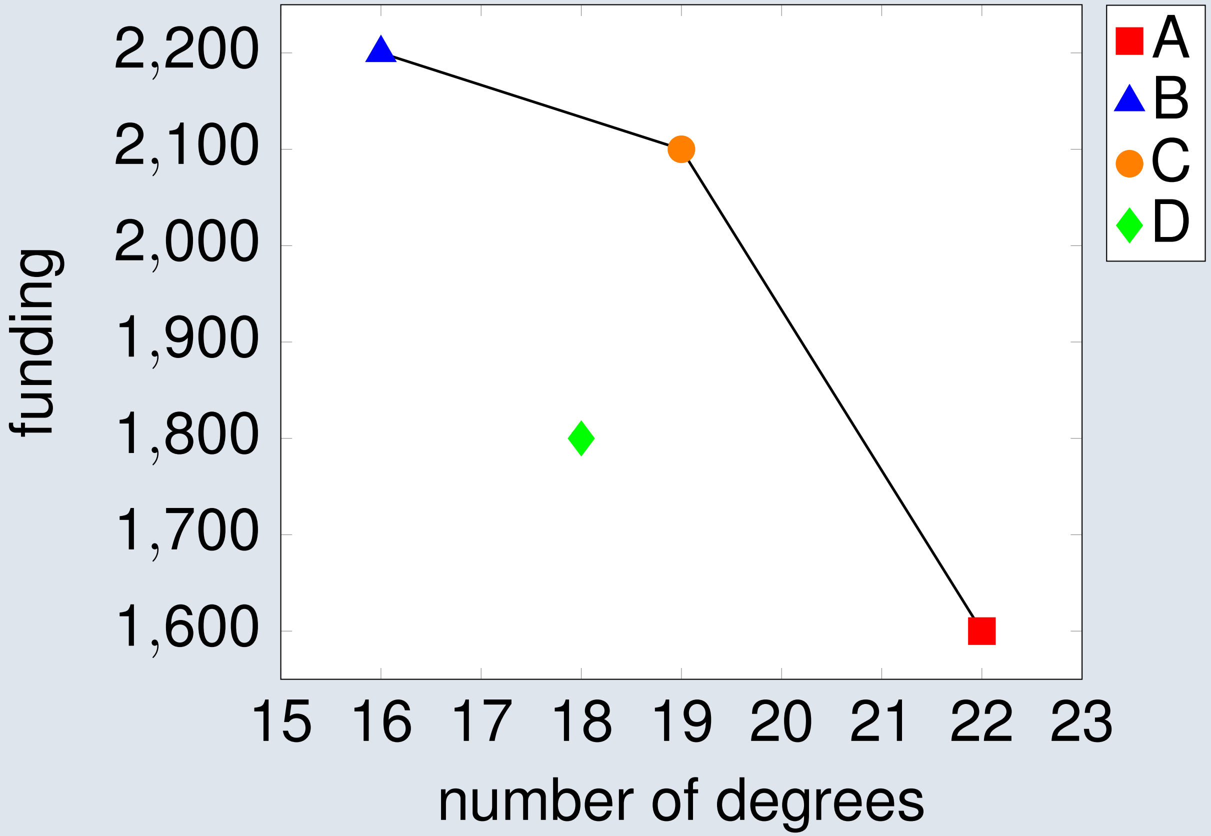
Decision Theory

Visualization

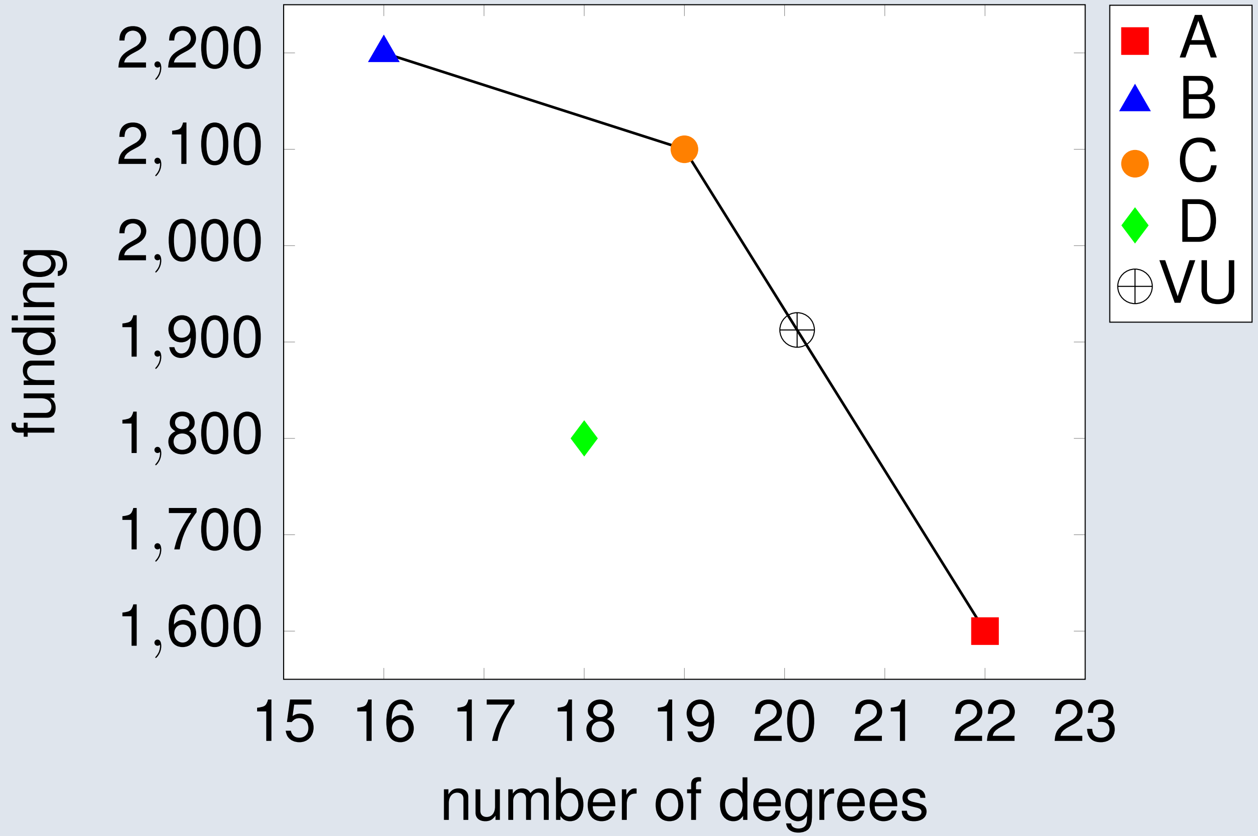


Decision Theory

Visualization



Visualization



Decision Theory

Assumptions

University D is inefficient only under assumptions:

- Outputs scale linearly with inputs
  - 10 professors can teach 1000 students
  - Can 20 professors teach 2000 students?
  - No economy of scale
  - No cost due to additional complexity
  - Constant returns to scale (CRS)
- Each university has the same conditions
  - What if university D moved to tents due to an earthquake?

Decision Theory

Data Envelopment Analysis

- DEA assigns a score to each DMU measuring efficiency
- Result < 1: DMU is inefficient
- Result = 1: DMU is efficient
- Always relative to the DMUs we observe!

Decision Theory

Data Envelopment Analysis

- Let $X_i = (x_{i1}, \dots, x_{iN})$ be the input vector for DMU i
- Let $Y_i = (y_{i1}, \dots, y_{iM})$ be the output vector for DMU i
- What is the efficiency θ_j of DMU j?

Decision Theory

Data Envelopment Analysis

How to define θ_j?

- Want to evaluate DMU j
- Assume to find a linear combination of outputs:

$$\sum_i \lambda_i Y_i = Y_j$$

- Need only $3/4$ of the inputs:

$$\sum_i \lambda_i X_i = \frac{3}{4} X_j$$

- Then θ_j should be at most $3/4$

Data Envelopment Analysis

Why at most?

- Assume to find another linear combination:

$$\sum_i \lambda_i' Y_i = Y_j$$

$$\sum_i \lambda_i' X_i = \frac{1}{2} X_j$$

- Then $\theta_j \leq \frac{1}{2}$

Decision Theory

Data Envelopment Analysis

So looking for

- The smallest θ_j
- Such that a linear combination using θ_j of the inputs produces the same output
- It's its own optimization problem!

First Formulation

Let I be the set of DMUs. We want to evaluate DMU $j \in I$:

$$
\begin{aligned}
\min\ & \theta_j \\
\text{s.t.}\ & \sum_{i \in I} \lambda_i X_i \leq \theta_j X_j \\
& \sum_{i \in I} \lambda_i Y_i \geq Y_j \\
& \theta_j \in \mathbb{R} \\
& \lambda_i \in \mathbb{R}_+ && \forall i \in I
\end{aligned}
$$

Decision Theory

$$\begin{aligned} \min\ & \theta_j \\ \text{s.t.}\ & \sum_{i \in I} \lambda_i X_i \leq \theta_j X_j \\ & \sum_{i \in I} \lambda_i Y_i \geq Y_j \\ & \theta_j \in \mathbb{R} \\ & \lambda_i \in \mathbb{R}_+ && \forall i \in I \end{aligned}$$

First Formulation

Note: X_i, Y_i are vectors! So:

$$\begin{aligned} \min\ & \theta_j \\ \text{s.t.}\ & \sum_{i \in I} \lambda_i x_{ik} \leq \theta_j x_{jk} && \forall k \in [N] \\ & \sum_{i \in I} \lambda_i y_{ik} \geq y_{jk} && \forall k \in [M] \\ & \theta_j \in \mathbb{R} \\ & \lambda_i \in \mathbb{R}_+ && \forall i \in I \end{aligned}$$

Decision Theory

Observations

- $\theta_j \leq 1$, as we can set $\lambda_j = 1$ and $\lambda_i = 0$ for $i \neq j$
- Compared to other DMUs, DMU j wastes

$$X_j - \sum_{i \in I} \lambda_i X_i$$

  units of input
- or: DMU j produces

$$Y_j - \frac{1}{\theta_j} \sum_{i \in I} \lambda_i Y_i$$

  units of output less

Decision Theory

Second Formulation

- Again inputs $[N]$, outputs $[M]$
- Want to set (imaginary) price $u_k > 0$ for inputs $k \in [N]$
- Want to set (imaginary) price $v_k > 0$ for outputs $k \in [M]$
- Cost of inputs for DMU j is then $u^t X_j$
- Profit of outputs is $v^t Y_j$

$$\text{Efficiency} = \frac{\text{Output}}{\text{Input}} = \frac{v^t Y_j}{u^t X_j}$$

Second Formulation

$$\text{Efficiency} = \frac{\text{Output}}{\text{Input}} = \frac{v^t Y_j}{u^t X_j}$$

- Prices are only imaginary
- Which prices u and v maximize the efficiency of DMU j?
- If u is very small and v is very large
- Set a limit:

$$\frac{v^t Y_i}{u^t X_i} \leq 1 \qquad \forall i \in I$$

- Another optimization problem

Decision Theory

Second Formulation

$$
\begin{aligned}
\max\ & \frac{v^t Y_j}{u^t X_j} \\
\text{s.t. } & \frac{v^t Y_i}{u^t X_i} \leq 1 && \forall i \in I \\
& u \in \mathbb{R}_+^N \\
& v \in \mathbb{R}_+^M
\end{aligned}
$$

Decision Theory

Second Formulation

$$
\begin{aligned}
\max \ & \frac{v^t Y_j}{u^t X_j} \\
\text{s.t.} \ & \frac{v^t Y_i}{u^t X_i} \leq 1 && \forall i \in I \\
& u \in \mathbb{R}^N_+ \\
& v \in \mathbb{R}^M_+
\end{aligned}
$$

Equivalent

$$
\begin{aligned}
\max \ & v^t Y_j \\
\text{s.t.} \ & u^t X_j = 1 \\
& v^t Y_i - u^t X_i \leq 0 && \forall i \in I \\
& u \in \mathbb{R}^N_+ \\
& v \in \mathbb{R}^M_+
\end{aligned}
$$

Decision Theory

Second Formulation

$$
\begin{aligned}
\max\ & \frac{v^t Y_j}{u^t X_j} \\
\text{s.t. } & \frac{v^t Y_i}{u^t X_i} \leq 1 && \forall i \in I \\
& u \in \mathbb{R}_+^N \\
& v \in \mathbb{R}_+^M
\end{aligned}
$$

Equivalent

$$
\begin{aligned}
\max\ & v^t Y_j \\
\text{s.t. } & u^t X_j = 1 \\
& v^t Y_i - u^t X_i \leq 0 && \forall i \in I \\
& u \in \mathbb{R}_+^N \\
& v \in \mathbb{R}_+^M
\end{aligned}
$$

Expanded

$$
\begin{aligned}
\max\ & \sum_{k \in [M]} v_k y_{jk} \\
\text{s.t. } & \sum_{k \in [N]} u_k x_{jk} = 1 \\
& \sum_{k \in [M]} v_k y_{ik} - \sum_{k \in [N]} u_k x_{ik} \leq 0 && \forall i \in I \\
& u \in \mathbb{R}_+^N, v \in \mathbb{R}_+^M
\end{aligned}
$$

Decision Theory

Example, continued

University	#Professors	#Degrees	External funding
A	6	132	9600€
B	12	192	26400€
C	10	190	21000€
D	8	144	14400€

Decision Theory

Example, continued

University	#Professors	#Degrees	External funding
A	6	1.32	9.6€
B	12	1.92	26.4€
C	10	1.90	21.0€
D	8	1.44	14.4€

Example, continued

University	#Professors	#Degrees	External funding
A	6	1.32	9.6€
B	12	1.92	26.4€
C	10	1.90	21.0€
D	8	1.44	14.4€

Inputs

$$X_A = (6) \qquad X_B = (12) \qquad X_C = (10) \qquad X_D = (8)$$

Example, continued

University	#Professors	#Degrees	External funding
A	6	1.32	9.6€
B	12	1.92	26.4€
C	10	1.90	21.0€
D	8	1.44	14.4€

Inputs

$$X_A = (6) \qquad X_B = (12) \qquad X_C = (10) \qquad X_D = (8)$$

Outputs

$$Y_A = \begin{pmatrix} 1.32 \\ 9.6 \end{pmatrix} \qquad Y_B = \begin{pmatrix} 1.92 \\ 26.4 \end{pmatrix} \qquad Y_C = \begin{pmatrix} 1.90 \\ 21.0 \end{pmatrix} \qquad Y_D = \begin{pmatrix} 1.44 \\ 14.4 \end{pmatrix}$$

Decision Theory

$$
\begin{aligned}
\min\ & \theta_j \\
\text{s.t.}\ & \sum_{i \in I} \lambda_i X_i \leq \theta_j X_j \\
& \sum_{i \in I} \lambda_i Y_i \geq Y_j \\
& \theta_j \in \mathbb{R} \\
& \lambda_i \in \mathbb{R}_+ && \forall i \in I
\end{aligned}
$$

First Model – DMU 1

$$
\begin{aligned}
\min\ & \theta_1 \\
\text{s.t.}\ &
\end{aligned}
$$

Decision Theory

$$
\begin{aligned}
&\min\ \theta_j \\
&\text{s.t.}\ \sum_{i \in I} \lambda_i X_i \leq \theta_j X_j \\
&\qquad \sum_{i \in I} \lambda_i Y_i \geq Y_j \\
&\qquad \theta_j \in \mathbb{R} \\
&\qquad \lambda_i \in \mathbb{R}_+ \qquad \forall i \in I
\end{aligned}
$$

First Model – DMU 1

$$
\begin{aligned}
&\min\ \theta_1 \\
&\text{s.t.}\ 6\lambda_1 + 12\lambda_2 + 10\lambda_3 + 8\lambda_4 \leq 6\theta_1
\end{aligned}
$$

Inputs (#Professors)

Decision Theory

$$\min \theta_j$$
$$\text{s.t.} \sum_{i \in I} \lambda_i X_i \leq \theta_j X_j$$
$$\sum_{i \in I} \lambda_i Y_i \geq Y_j$$
$$\theta_j \in \mathbb{R}$$
$$\lambda_i \in \mathbb{R}_+ \qquad \forall i \in I$$

First Model – DMU 1

$$\min \theta_1$$
$$\text{s.t. } 6\lambda_1 + 12\lambda_2 + 10\lambda_3 + 8\lambda_4 \leq 6\theta_1$$
$$1.32\lambda_1 + 1.92\lambda_2 + 1.90\lambda_3 + 1.44\lambda_4 \geq 1.32$$

Outputs (#Degrees)

$$\min \theta_j$$
$$\text{s.t.} \sum_{i \in I} \lambda_i X_i \leq \theta_j X_j$$
$$\sum_{i \in I} \lambda_i Y_i \geq Y_j$$
$$\theta_j \in \mathbb{R}$$
$$\lambda_i \in \mathbb{R}_+ \qquad \forall i \in I$$

First Model – DMU 1

$$\min \theta_1$$
$$\text{s.t. } 6\lambda_1 + 12\lambda_2 + 10\lambda_3 + 8\lambda_4 \leq 6\theta_1$$
$$1.32\lambda_1 + 1.92\lambda_2 + 1.90\lambda_3 + 1.44\lambda_4 \geq 1.32$$
$$9.6\lambda_1 + 26.4\lambda_2 + 21\lambda_3 + 14.4\lambda_4 \geq 9.6$$

Outputs (External funding)

Decision Theory

$$\begin{aligned} &\min\ \theta_j \\ &\text{s.t. } \sum_{i \in I} \lambda_i X_i \leq \theta_j X_j \\ &\quad\ \ \sum_{i \in I} \lambda_i Y_i \geq Y_j \\ &\quad\ \ \theta_j \in \mathbb{R} \\ &\quad\ \ \lambda_i \in \mathbb{R}_+ && \forall i \in I \end{aligned}$$

First Model – DMU 1

$$\begin{aligned} &\min\ \theta_1 \\ &\text{s.t. } 6\lambda_1 + 12\lambda_2 + 10\lambda_3 + 8\lambda_4 \leq 6\theta_1 \\ &\quad\ \ 1.32\lambda_1 + 1.92\lambda_2 + 1.90\lambda_3 + 1.44\lambda_4 \geq 1.32 \\ &\quad\ \ 9.6\lambda_1 + 26.4\lambda_2 + 21\lambda_3 + 14.4\lambda_4 \geq 9.6 \end{aligned}$$

Solution: $\theta_1 = 1$ with $\lambda_1 = 1$.

Decision Theory

$$
\begin{aligned}
\min\ & \theta_j \\
\text{s.t.}\ & \sum_{i \in I} \lambda_i X_i \leq \theta_j X_j \\
& \sum_{i \in I} \lambda_i Y_i \geq Y_j \\
& \theta_j \in \mathbb{R} \\
& \lambda_i \in \mathbb{R}_+ && \forall i \in I
\end{aligned}
$$

First Model – DMU 4

$$
\begin{aligned}
\min\ & \theta_4 \\
\text{s.t.}\ & 6\lambda_1 + 12\lambda_2 + 10\lambda_3 + 8\lambda_4 \leq 8\theta_4 \\
& 1.32\lambda_1 + 1.92\lambda_2 + 1.90\lambda_3 + 1.44\lambda_4 \geq 1.44 \\
& 9.6\lambda_1 + 26.4\lambda_2 + 21.0\lambda_3 + 14.4\lambda_4 \geq 14.4
\end{aligned}
$$

Decision Theory

$$
\begin{aligned}
\min\ & \theta_j \\
\text{s.t.}\ & \sum_{i \in I} \lambda_i X_i \leq \theta_j X_j \\
& \sum_{i \in I} \lambda_i Y_i \geq Y_j \\
& \theta_j \in \mathbb{R} \\
& \lambda_i \in \mathbb{R}_+ && \forall i \in I
\end{aligned}
$$

First Model – DMU 4

$$
\begin{aligned}
\min\ & \theta_4 \\
\text{s.t.}\ & 6\lambda_1 + 12\lambda_2 + 10\lambda_3 + 8\lambda_4 \leq 8\theta_4 \\
& 1.32\lambda_1 + 1.92\lambda_2 + 1.90\lambda_3 + 1.44\lambda_4 \geq 1.44 \\
& 9.6\lambda_1 + 26.4\lambda_2 + 21.0\lambda_3 + 14.4\lambda_4 \geq 14.4
\end{aligned}
$$

Solution: $\theta_4 = 0.91$ with $\lambda_1 = 0.30$ and $\lambda_3 = 0.55$.

Decision Theory

$$
\begin{aligned}
\max\ & v^t Y_j \\
\text{s.t. } & u^t X_j = 1 \\
& v^t Y_i - u^t X_i \leq 0 \qquad \forall i \in I \\
& u \in \mathbb{R}_+^N \\
& v \in \mathbb{R}_+^M
\end{aligned}
$$

Second Model – DMU 1

$$
\begin{aligned}
\max\ & 1.32 v_1 + 9.6 v_2 \\
\text{s.t. } & 6 u_1 = 1 \\
& 1.32 v_1 + 9.6 v_2 \leq 6 u_1 \\
& 1.92 v_1 + 26.4 v_2 \leq 12 u_1 \\
& 1.90 v_1 + 21.0 v_2 \leq 10 u_1 \\
& 1.44 v_1 + 14.4 v_2 \leq 8 u_1
\end{aligned}
$$

Solution: $v_1 = 0.76$, objective function is $1.32 \cdot 0.76 = 1$

Decision Theory

Comparison

- Both models yield the same result! (duality)
- According to the definition

$$v^t Y_j = \frac{v^t Y_j}{1} = \frac{v^t Y_j}{u^t X_j} = \theta_j$$

- The "profit" $v^t Y_j - u^t X_j$ is negative for inefficient DMUs, and zero otherwise

Decision Theory

DEA – Discussion

Advantages

- Works only quantitatively
  - No functional relationship between input and output
  - DMU is a black-box, no knowledge of the internal workings required
- Can use different units
  - See degrees against external funding
  - Might save us from difficult conversions (money or life)
- Does not estimate weights for input and output
- Efficiency of a DMU is based only on other DMUs
  - Can find best practice or worst practice
  - Identifies a suitable comparison in case of inefficiency

Decision Theory

DEA – Discussion

Disadvantages

- Efficiency of a DMU is based only on other DMUs
  - The one-eyed man is king among the blind
- Must carefully choose inputs and outputs
  - Not tolerant to small errors
  - Not meaningful if too many inputs/outputs (everything efficient)
  - Inputs/Outputs should not be correlated

Examples from the Literature

Decision Theory

ELSEVIER

Socio-Economic Planning Sciences 42 (2008) 151–157

SOCIO-ECONOMIC PLANNING SCIENCES

www.elsevier.com/locate/seps

Evaluation of research in efficiency and productivity: A survey and analysis of the first 30 years of scholarly literature in DEA

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Available online 4 March 2008

The authors wish to dedicate this compendium of DEA's historical accomplishments to one of its founders, Professor William W. Cooper

Decision Theory

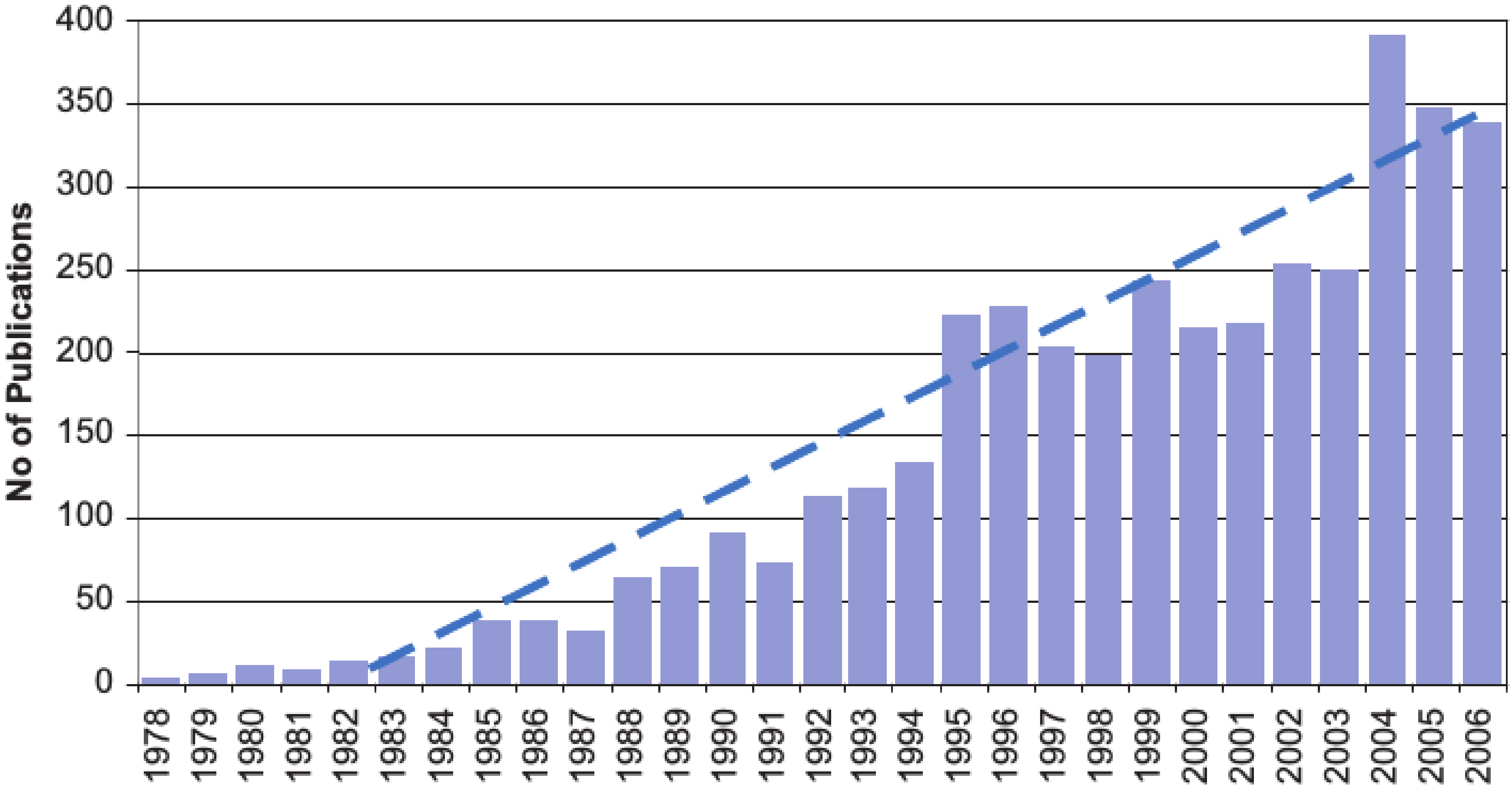

Fig. 1. Distribution of DEA publications by year.

Decision Theory

Available online at www.sciencedirect.com

ScienceDirect

European Journal of Operational Research 189 (2008) 1–18

EUROPEAN JOURNAL OF OPERATIONAL RESEARCH

www.elsevier.com/locate/ejor

Invited Review

A survey of data envelopment analysis in energy and environmental studies

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Received 24 July 2006; accepted 26 April 2007
Available online 6 May 2007

Decision Theory

$$
\begin{aligned}
\min \quad & \theta \\
\text{s.t.} \quad & \sum_{n=1}^{N} x_{nk}\lambda_k \leqslant \theta x_{no}, \quad n = 1, 2, \ldots, N, \\
& \sum_{m=1}^{M} y_{mk}\lambda_k \geqslant y_{mo}, \quad m = 1, 2, \ldots, M, \\
& \lambda_k \geqslant 0, \quad k = 1, \ldots, K.
\end{aligned} \tag{2}
$$

Decision Theory

Table 1
Studies of DEA in E&E with their specific features

Publication	Type of study	Country/region	Methodological aspect					Application scheme
			Reference technology			Efficiency measure	MPI	
			Inputs	Outputs	RTS			
Abbott (2006)	A	Australia	SD	SD	C, V	R	Yes	Electricity distribution utilities
Agrell and Bogetoft (2005)	T + A	Denmark	SD	SD + NC	C, V	SB	Yes	District heating plants
Arcelus and Arocena (2005)	A	14 OECD countries	SD	SD, WD	V	R, DDF	No	Productivity estimation with CO_2 emissions considered
Athanassopoulos et al. (1999)	T + A	UK	SD + B	SD + B	C	NR	No	Electricity generation plants
Bagdadioglu et al. (1996)	A	Turkey	SD	SD	C, V	R	No	Electricity distribution utilities
Barla and Perelman (2005)	A	12 OECD countries	SD	SD	C	R	Yes	Relationship between productivity and SO_2 emissions
Bevilacqua and Braglia (2002)	A	Italy	SD	SD	C	R	No	Environmental performance measurement
Boyd and McClelland (1999)	T + A	US	SD	SD, WD	C	H	No	Impacts of environmental regulations
Boyd and Pang (2000)	A	US	SD	SD	C	R	No	Energy efficiency study
Boyd et al. (2002)	T + A	US	SD	SD, WD	C	DDF	Yes	Impacts of environmental regulations
Brännlund et al. (1998)	T + A	Sweden	SD	WD	NI	Profit	No	Profit estimation with emissions trading
Byrnes et al. (1984)	T + A	US	SD, WD	SD	C, V, NI	R	No	Coal mines
Byrnes et al. (1988)	T + A	US	SD, WD	SD	C, V, NI	R	No	Coal mines
Callens and Tyteca (1999)	T	–	CCR multiplier form with bad outputs considered				No	Environmental performance measurement
Chauhan et al. (2006)	A	India	SD	SD	C, V	R	No	Energy use efficiency study
Chien et al. (2003)	A	China (Taiwan)	SD	SD	C, V	R	No	Electricity distribution districts
Chitkara (1999)	A	India	SD	SD	C	R	Yes	Electricity generation plants
Chung et al. (1997)	T + A	Sweden	SD	WD	C	DDF	Yes	Productivity estimation with pollutants considered
Claggett and Ferrier (1998)	A	US	SD	SD	C, V	Cost	No	Electricity distribution utilities
Cook and Green (2005)	T + A	–	CCR multiplier form + AR method				No	Electricity generation plants
Criswell and Thompson (1996)	A	US	CCR multiplier form + AR method				No	Comparison of different power systems
Dyckhoff and Allen (2001)	T	–	DEA + multi-attribute value theory				No	Environmental performance measurement

Decision Theory

ELSEVIER

Agricultural Systems 59 (1999) 267–282

AGRICULTURAL SYSTEMS

An application of data envelopment analysis to irrigated dairy farms in Northern Victoria, Australia

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Received 10 July 1998; received in revised form 13 November 1998; accepted 15 January 1999

Decision Theory

$$\text{Min}_{\theta,\lambda}\theta \tag{9}$$

subject to

$$Y\lambda - y_i \geq 0 \tag{10}$$

$$\theta x_i - X\lambda \geq 0 \tag{11}$$

$$\lambda \geq 0 \tag{12}$$

Decision Theory

Variable	Type	Units
Total milk fat/protein	Output	Kilograms
Number of cows in the milking herd adjusted for age distribution of herd	Input 1	Number
Milking area—perennial pasture equivalent	Input 2	Hectares
Irrigation water applied	Input 3	Megalitres
Supplementary feeding—grains and pellets	Input 4	MJ ME
Fertiliser	Input 5	Tonnes
Labour	Input 6	Hours

Decision Theory

Table 3
Farm efficiency scores

Farm	VRS 1995/96 output orientated	VRS 1995/96 input orientated	CRS 1995/96 input orientated	VRS 1994/95 output orientated	VRS 1994/95 input orientated	CRS 1994/95 input orientated
1	0.883	0.896	0.864	0.803	0.833	0.79
2	0.876	0.893	0.858	0.952	0.952	0.952
3	1	1	1	1	1	0.986
4	0.845	0.812	0.803	0.683	0.676	0.649
5	0.584	0.63	0.565	0.556	0.574	0.554
6	0.818	0.857	0.801	0.774	0.803	0.756
7	0.882	0.882	0.882	0.888	0.848	0.841
8	0.832	0.851	0.817	0.82	0.847	0.797
9	0.918	0.932	0.87	0.977	0.983	0.849
10	0.788	0.799	0.788	0.754	0.78	0.731
11	1	1	1	1	1	1
12	0.906	0.967	0.637	0.906	0.951	0.787
13	1	1	1	1	1	1
14	1	1	1	1	1	1
15	0.698	0.751	0.695	0.824	0.847	0.798
16	0.963	0.963	0.963	0.957	0.957	0.954
17	0.764	0.762	0.761	0.828	0.776	0.767

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Efficiency measurement of selected Australian and other international ports using data envelopment analysis ☆

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Decision Theory

$$\max_{u,v} \frac{\sum_r u_r y_{r0}}{\sum_i v_i x_{i0}} = \frac{u^{\mathrm{T}} Y_0}{v^{\mathrm{T}} X_0}, \quad \text{where } u = (u_1, \ldots, u_s)^{\mathrm{T}}, \quad v = (v_1, \ldots, v_m)^{\mathrm{T}}$$

subject to

$$\frac{u^{\mathrm{T}} Y_j}{v^{\mathrm{T}} X_j} = \frac{\sum_r u_r y_{rj}}{\sum_i v_i x_{ij}} \leqslant 1$$

$$\text{for } j = 1, 2, \ldots, n; \quad u_r, v_i \geqslant 0 \quad \text{for } r = 1, 2, \ldots, s \quad \text{and} \quad i = 1, 2, \ldots, n,$$

Decision Theory

Output
- TEUs

Inputs
- nocranes
- noberths
- notugs
- termiare
- delaytime
- labor

Decision Theory

Table 1
Relative efficiency measures using the CCR and additive DEA models[a]

(1)	(2)	
Port	CCR	ADDITIVE
	2 Outputs	
Melbourne	0.5885	0.6633
Hong Kong	1.0000	1.0000
Hamburg	1.0000	1.0000
Rotterdam	0.6644	0.8228
Felixstowe	1.0000	1.0000
Yokohama	0.8456	1.0000
Singapore	1.0000	1.0000
Keelung	1.0000	1.0000
Sydney	0.7676	1.0000
Fremantle	0.8251	1.0000
Brisbane	1.0000	1.0000
Tilbury	1.0000	1.0000
Zeebrugge	1.0000	1.0000
La Spezia	1.0000	1.0000
Tanjung Priok	1.0000	1.0000
Osaka	0.6050	0.6023

Decision Theory

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Data envelopment analysis and its application to the measurement of efficiency in higher education

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Decision Theory

Table 1
Definition of input and output variables for the DEA

Variables	Definition[a]
Outputs:	
GRADQUAL[b]	Total number of first degrees awarded weighted by degree classification, i.e. GRADQUAL = (number of firsts × 30) + (number of upper seconds × 25) + (number of lower seconds × 20) + (number of thirds × 15) + (number of unclassifieds × 10).
POSTGRAD[b]	Total number of higher degrees awarded (includes both doctorate and other higher degrees).
RESEARCH[c]	Value of the recurrent grant for research awarded by the Higher Education Funding Council for England (HEFCE) in £.
Inputs:	
UGQUAL[b,e]	Total number of FTE undergraduate students studying for a first degree multiplied by the average A level points for first year full-time undergraduate students (A level score is averaged over 1994/95, 1995/96, 1996/97 and 1997/98. Note that A = 10, B = 8, C = 6, D = 4, E = 2).
PG[b]	Total number of FTE postgraduate students.
STAFF[d]	Total number of full-time academic staff for teaching or teaching and research or research only purposes.
CAPITAL[d]	Total depreciation and interest payable in £.
LIBCOMP[d]	Total expenditure on central libraries and information services, and on central computer and computer networks excluding academic staff costs and depreciation in £.
ADMIN[d]	Expenditure on central administration and central services excluding academic staff costs and depreciation in £.

Decision Theory

Table 4
Efficiency scores for full and preferred models

University name	ID	Full model Technical efficiency Overall mean = 94.61	Preferred model Technical efficiency Overall mean = 92.51	Full model Scale efficiency Overall mean = 96.45	Preferred model Scale efficiency Overall mean = 96.13
Pre-1992 HEIs		*Mean = 96.34*	*Mean = 94.25*	*Mean = 95.69*	*Mean = 95.07*
Aston University	2	87.38	80.69	99.73	99.05
The University of Bath	4	83.12	70.20	97.88	99.81
The University of Birmingham	6	100.00	100.00	100.00	100.00
The University of Bradford	10	100.00	100.00	100.00	100.00
The University of Bristol	13	89.27	88.47	84.62	85.38
Brunel University	14	88.31	77.52	97.81	98.17
The University of Cambridge	16	100.00	100.00	90.83	90.83
City University	24	100.00	100.00	100.00	100.00
Cranfield University	26	100.00	100.00	100.00	100.00
University of Durham	31	100.00	97.94	95.02	93.90
The University of East Anglia	32	80.35	77.72	99.96	99.88
The University of Essex	35	99.96	99.94	99.39	99.39
The University of Exeter	36	93.34	86.57	98.31	95.62
Goldsmiths College	39	100.00	93.95	100.00	99.11
The University of Hull	45	100.00	100.00	100.00	100.00
Imperial College of Science, Technology & Medicine	46	100.00	100.00	100.00	100.00
The University of Keele	49	97.29	85.95	99.77	99.90
The University of Kent at Canterbury	50	88.84	83.01	99.94	99.87
King's College London	53	100.00	100.00	100.00	100.00
The University of Lancaster	55	100.00	100.00	100.00	100.00

Decision Theory

Quiz

Question 1

2 inputs, 2 outputs, 3 DMUs:

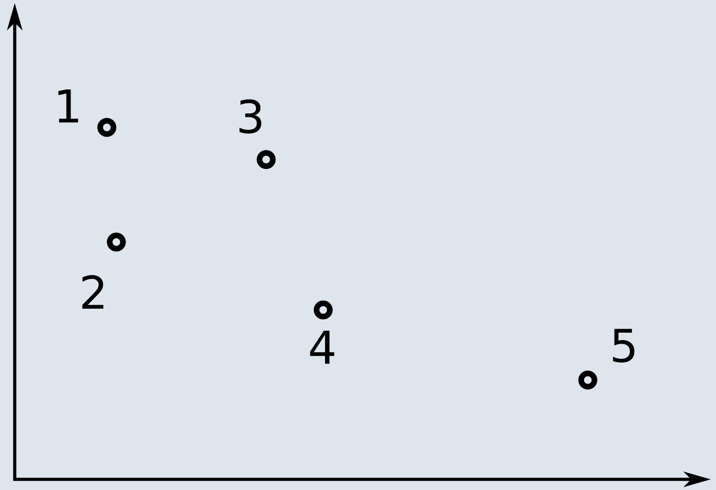
DMU	In 1	In 2	Out 1	Out 2
1	3	3	7	7
2	3	2	8	6
3	2	3	6	8

Write the LP to test the efficiency of DMU 1.

$$
\begin{aligned}
&\min \theta_j \\
\text{s.t. } &\sum_{i \in I} \lambda_i x_{ik} \leq \theta_j x_{jk} && \forall k \in [N] \\
&\sum_{i \in I} \lambda_i y_{ik} \geq y_{jk} && \forall k \in [M] \\
&\theta_j \in \mathbb{R} \\
&\lambda_i \in \mathbb{R}_+ && \forall i \in I
\end{aligned}
$$

Question 2 (maximize)

What is efficient? What is supported/unsupported?



Decision Theory

Quiz

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& \sum_{i \in I} \lambda_i y_{ik} \geq y_{jk} && \forall k \in [M] \\
& \theta_j \in \mathbb{R} \\
& \lambda_i \in \mathbb{R}_+ && \forall i \in I
\end{aligned}
$$

Question 1 – Solution

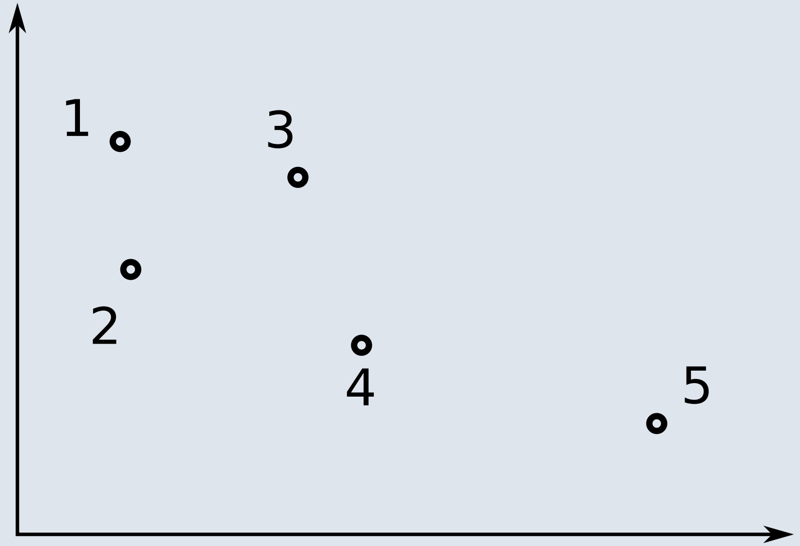
$$
\begin{aligned}
&\min \theta_1 \\
\text{s.t. } & 3\lambda_1 + 3\lambda_2 + 2\lambda_3 \leq 3\theta_1 \\
& 3\lambda_1 + 2\lambda_2 + 3\lambda_3 \leq 3\theta_1 \\
& 7\lambda_1 + 8\lambda_2 + 6\lambda_3 \geq 7 \\
& 7\lambda_1 + 6\lambda_2 + 8\lambda_3 \geq 7 \\
& \theta_1, \lambda_1, \lambda_2, \lambda_3 \geq 0
\end{aligned}
$$

Decision Theory

Quiz

Question 2 (maximize)

What is efficient? What is supported/unsupported?



Question 2 - Solution

- 1,3,5: efficient and supported
- 4: efficient and not supported