

### 3 Regression

#### 3.1 Ridge regression models

i.

$$\text{Let } \mathbf{X} := \begin{bmatrix} x_{11} & \dots & x_{1m} \\ x_{21} & \dots & x_{2m} \\ \vdots & & \vdots \\ x_{n1} & \dots & x_{nm} \end{bmatrix}, \quad \mathbf{w} := \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix} \text{ and } \mathbf{y} := \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.$$

The ordinary least squares regression cost function can be written in matrix notation is given by,

$$J(\mathbf{w}) = \frac{1}{2}(\mathbf{X}\mathbf{w} - \mathbf{y})^\top(\mathbf{X}\mathbf{w} - \mathbf{y}).$$

Adding the quadratic penalty term, we get

$$J(\mathbf{w}) = \frac{1}{2}(\mathbf{X}\mathbf{w} - \mathbf{y})^\top(\mathbf{X}\mathbf{w} - \mathbf{y}) + \frac{\lambda}{2}\mathbf{w}^\top\mathbf{w}$$

ii.  $J(\mathbf{w})$  can be simplified as:

$$\begin{aligned} J(\mathbf{w}) &= \frac{1}{2}(\mathbf{X}\mathbf{w} - \mathbf{y})^\top(\mathbf{X}\mathbf{w} - \mathbf{y}) + \frac{\lambda}{2}\mathbf{w}^\top\mathbf{w} \\ &= \frac{1}{2}[(\mathbf{w}^\top\mathbf{X}^\top - \mathbf{y}^\top)(\mathbf{X}\mathbf{w} - \mathbf{y})] + \frac{\lambda}{2}\mathbf{w}^\top\mathbf{w} \\ &= \frac{1}{2}[\mathbf{w}^\top\mathbf{X}^\top\mathbf{X}\mathbf{w} - \mathbf{y}^\top\mathbf{X}\mathbf{w} - \mathbf{w}^\top\mathbf{X}^\top\mathbf{y} + \mathbf{y}^\top\mathbf{y}] + \frac{\lambda}{2}\mathbf{w}^\top\mathbf{w} \\ &= \frac{1}{2}[\mathbf{w}^\top\mathbf{X}^\top\mathbf{X}\mathbf{w} - 2\mathbf{y}^\top\mathbf{X}\mathbf{w} + \mathbf{y}^\top\mathbf{y}] + \frac{\lambda}{2}\mathbf{w}^\top\mathbf{w} \end{aligned}$$

Using results from task 2.1 [Supplement-2], the gradient of  $J(\mathbf{w})$ :

$$\begin{aligned} \nabla_{\mathbf{w}} J(\mathbf{w}) &= \frac{1}{2} \left[ (\mathbf{X}^\top\mathbf{X} + (\mathbf{X}^\top\mathbf{X})^\top)\mathbf{w} - 2\mathbf{X}^\top\mathbf{y} \right] + \frac{\lambda}{2}[2\mathbf{w}] \\ &= \frac{1}{2}[2\mathbf{X}^\top\mathbf{X}\mathbf{w} - 2\mathbf{X}^\top\mathbf{y}] + \frac{\lambda}{2}[2\mathbf{w}] \\ &= \mathbf{X}^\top\mathbf{X}\mathbf{w} - \mathbf{X}^\top\mathbf{y} + \lambda\mathbf{w} \end{aligned}$$

Setting gradient  $\nabla_{\mathbf{w}} J(\mathbf{w})$  to zero, we obtain:

$$\begin{aligned} 0 &= \mathbf{X}^\top\mathbf{X}\mathbf{w}^* - \mathbf{X}^\top\mathbf{y} + \lambda\mathbf{w}^* \\ \mathbf{w}^* &= (\mathbf{X}^\top\mathbf{X} + \lambda\mathbf{I})^{-1}\mathbf{X}^\top\mathbf{y} \end{aligned}$$