Machine Learning for Time Series

(MLTS or MLTS-Deluxe Lectures)

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Topics overview

- Time series fundamentals and definitions (2 lectures)
- Bayesian Inference (1 lecture)
- Gussian processes (2 lectures)
- State space models (2 lectures)
- Autoregressive models (1 lecture) ←
- Data mining on time series (1 lecture)
- Deep learning on time series (4 lectures)
- Domain adaptation (1 lecture)

Review concept: Stochastic process

Non-deterministic time series can be regarded as manifestations (equiv., realization) of a **stochastic process**, which is defined as a set of random variables $\{X_t\}_{t\in\{1,\ldots,\mathrm{T}\}}$

Even if we were to imagine having observed the process for an infinite period T of time, the infinite sequence

$$S = \{\ldots, s_{t-1}, s_t, s_{t+1,\ldots}\} = \{s_t\}_{t=-\infty}^{+\infty}$$

would still be a single **realization** from that process.

Still, if we had a battery of N computers generating series $S^{(1)}$, …, $S^{(N)}$, and considering selecting the observation at time t from each series,

$$\left\{s_t^{(1)}, \ldots, s_t^{(N)}\right\}$$

this would be described as a sample of N realizations of the random variable X_t

Review concept: Autocovariance

Given any particular realization $S^{(i)}$ of a stochastic process (i.e., a time series), we can define the vector of the $j+1$ most recent observations

$$x_t^{\mathrm{i}} = [s_{t-j}^{(i)}, \ldots, s_t^{(i)}]$$

We want to know the probability distribution of this vector x_t^{i} across realizations. We can calculate the ***j*-th autocovariance**

$$\gamma_{jt} = E(X_t - \mu_t)(X_{t-j} - \mu_{t-j})$$

Review concept: Autocorrelation function (ACF)

We can express the linear predictability of X_t from an adjacent value X_s, using the **autocorrelation function**:

$$\rho(s,t) = \frac{\gamma_{st}}{\sqrt{\gamma_{ss}\gamma_{tt}}}$$

where γ is the autocovariance defined previously.

Review concept: Stationarity

There are two types of stationarity.

A process is said **strictly stationary** if the joint distribution of $X_{t_1:t_2}$ is the same as that of $X_{t_1+h:t_2+h}$.

The term h is called **lag**.

For stricly stationary time series, all statistics do not depend on time.

A process is said **weakly stationary** if it has:

- $\mu = const.$
- $\sigma^2 < \infty$
- $\gamma_{jt} = \gamma_{j+h,t+h}$

A weakly stationary time series has finite variation, constant first moment, and that the second moment only depends on $h = t - j$.

Review concept: Partial autocorrelation function (PACF)

For stationary time series, the **partial autocorrelation function** expresses the correlation between X_t and an adjacent value X_s, but “removes” the effect of all values in between:

$$\phi_{11} = corr(X_{t+1}, X_t) = \rho_1$$

$$\phi_{hh} = corr\left(X_{t+h} - P_{t,h}(X_{t+h}), X_t - P_{t,h}(X_t)\right) = \rho_h$$

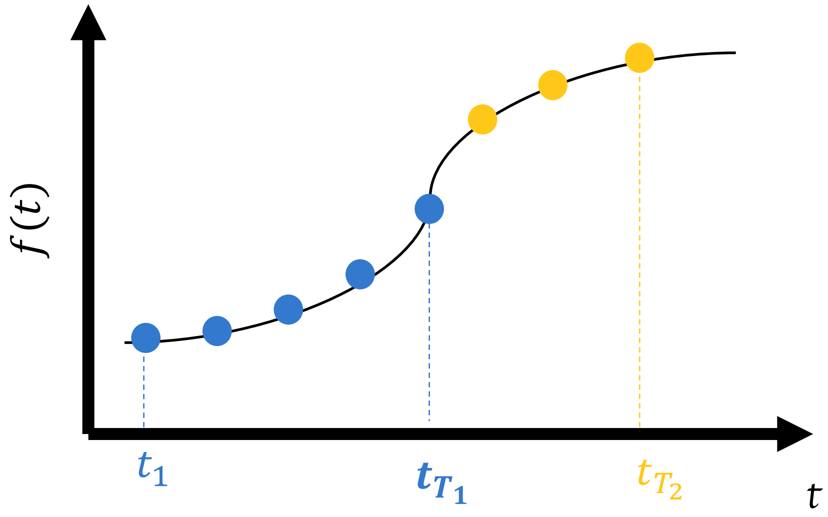
for $h \geq 2$, where $P_{t,h}$ is the surjective operator of orthogonal projection onto the linear subspace spanned by the intermediate values $X_{t+1}, \ldots, X_{t+h-1}$.

Review concept: Time series forecasting

Let $S = \{s_1, \dots, s_{T_1}, s_{T_1+1}, \dots, s_{T_2}\}$ be a time series, with s_i being the i-th observation collected at time t_i, and $t_i < t_j, \forall j$.

Then, a time series forecasting task is about predicting future values of a time series given some past data, i.e.,

$$f(s_1, \dots, s_{T_1}) = (s_{T_1+1}, \dots, s_{T_2})$$



In this lecture…

- Linear processes
    - Autoregressive processes (AR)
    - Moving average processes (MA)
- Combining AR and MA:
    - ARMA
    - ARIMA

Autoregressive models

AR and MA models

White noise

The basic building block for all the processes considered in this lecture is the **white noise**, defined as a sequence

$$\{e_t\}_{t=-\infty}^{+\infty}$$

whose elements have zero mean and variance σ^2, and are *uncorrelated* across time, i.e.,

- $\mathbb{E}(e_t) = 0$ **(zero mean)**
- $\mathbb{E}(e_t^2) = \sigma^2$ **(variance)**
- $\mathbb{E}(e_t e_\tau) = 0$ **(zero autocovariance, i.e., uncorrelated)**

If $e_t \sim \mathcal{N}(0, \sigma^2)$, then we have a so-called **Gaussian white noise**.

Random walk

A random walk is a stochastic random process that describes a path started at y_0 and consisting of random steps.

$$y_t = y_{t-1} + e_t$$

Equiv., for $t \geq 1$:

$$y_t = y_0 + \sum_{i=1}^{t} e_i$$

where e_i can be regarded as random variables of a white noise process.

https://www.mit.edu/~kardar/teaching/projects/chemotaxis(AndreaSchmidt)/random.htm

Linear process representation

A linear process can be represented as an **infinite moving average process,** starting from a white noise $\{e_t\}_{t=-\infty}^{+\infty}$, as

$$
\begin{aligned}
y_{\mathrm{t}} &= \mu + e_t + \psi_1 e_{t-1} + \psi_2 e_{t-2} + \cdots \\
&= \mu + e_t + \sum_{j=1}^{+\infty} \psi_j e_{t-j} \\
&= \mu + \Psi(q^{-1}) e_t
\end{aligned}
$$

where,

- ψ_i are constant values
- q^{-m} is the *backshift operator*, such that $q^{-m} e_t = e_{t-m}$
- $\Psi(\mathrm{q}^{-1}) = 1 + \psi_1 q^{-1} + \psi_2 q^{-2} + \cdots = \sum_{j=0}^{+\infty} \psi_j q^{-j}$ is a **linear filter**.

If the sequence $\psi_1, \psi_2, \ldots,$ has finite sum $\sum_i^{\infty} \psi_i < \infty$, then the filter is stable and the process y_{t} is stationary.

Linear process representation

Alternatively, a linear process can be represented with respect to its previous values as an **infinite autoregressive process**:

$$y_t = \mu + e_t + \pi_1 y_{t-1} + \pi_2 y_{t-2} + \cdots$$

$$y_t = \mu + e_t + \sum_{j=1}^{+\infty} \pi_j y_{t-j}$$

$$\Pi(q^{-1}) y_t = \mu + e_t$$

where, similarly,

- π_i are constant values
- q^{-m} is the *backshift operator*, such that $q^{-m} e_t = e_{t-m}$
- $\Pi(\mathrm{q}^{-1}) = 1 + \pi_1 q^{-1} + \pi_2 q^{-2} + \cdots = \sum_{j=0}^{+\infty} \pi_j q^{-j}$ is a **linear filter**.

Linear process representation

The previous two formulations are algebrically equivalent, in fact:

$$y_t = \mu + e_t + \sum_{j=1}^{+\infty} \pi_j y_{t-j} \quad \text{(infinite autoregressive process)}$$

$$y_t = \mu + e_t + \pi_1 q^{-1} y_t + \cdots$$

$$y_t - \pi_1 q^{-1} y_t - \cdots = \mu + e_t$$

$$(1 - \pi_1 q^{-1} - \cdots)\, y_t = \mu + e_t$$

$$\Pi(q^{-1}) y_t = \mu + e_t$$

If the linear filter $\Pi(q^{-1})$ is **invertible**, then:

$$y_t = \bar{\mu} + \frac{1}{\Pi(q^{-1})} e_t \quad \text{(infinite moving average process)}$$

Autoregressive models (AR)

Autoregressive models are based on the idea that the value of a time series at time t can be expressed as a linear combination of n past values, up to a random error:

$$AR(n): y_t = a_1 y_{t-1} + a_2 y_{t-2} + \cdots + a_n y_{t-n} + e_t$$

Where:

- n is the model's order
- $a_1, \dots, a_n$ are the model's parameters, $a_n \neq 0$

In other words, the hyper-parameter n represents how far back to look for dependences with previous values in the time series.

Autoregressive models (AR)

We can simplify the notation for $AR(n)$ using the backshift operator:

$$y_t = a_1 y_{t-1} + a_2 y_{t-2} + \cdots + a_n y_{t-n} + e_t$$

$$y_t - a_1 y_{t-1} - \ldots - a_n y_{t-n} = e_t$$

$$(1 - a_1 q^{-1} - \cdots - a_n q^{-n}) y_t = e_t$$

$$\boldsymbol{A(q^{-1}) y_t = e_t}$$

where $\boldsymbol{A(q^{-1})}$ is called **autoregressive operator**.

Example: AR(0)

The simplest autoregressive model is **AR(0)**, which has no dependences between values in the time series.

$$AR(0)\colon y_t = e_t$$

→ AR(0) is equivalent to a white noise process.

Example: AR(1)

The first order autoregressive model AR(1) can be written as:

$$AR(1)\colon\ y_t = a_1 y_{t-1} + e_t$$

Notice that:

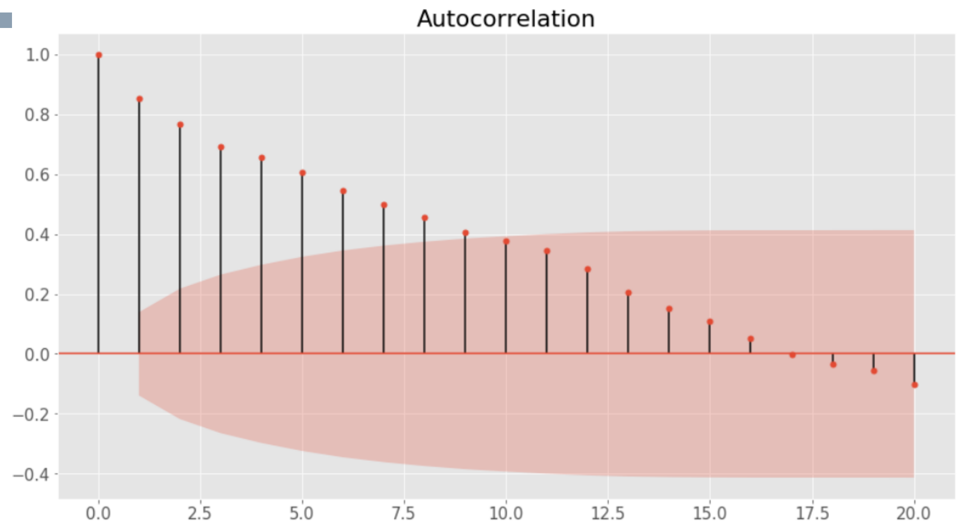
- Only the previous term y_{t-1} and the current noise e_t contribute to the output.
- As $|a_1| \to 0$, the process looks like white noise.
- When $a_1 < 0$, the process oscillates around zero.
- When $a_1 = 1$, the process is equivalent to a *random walk*.

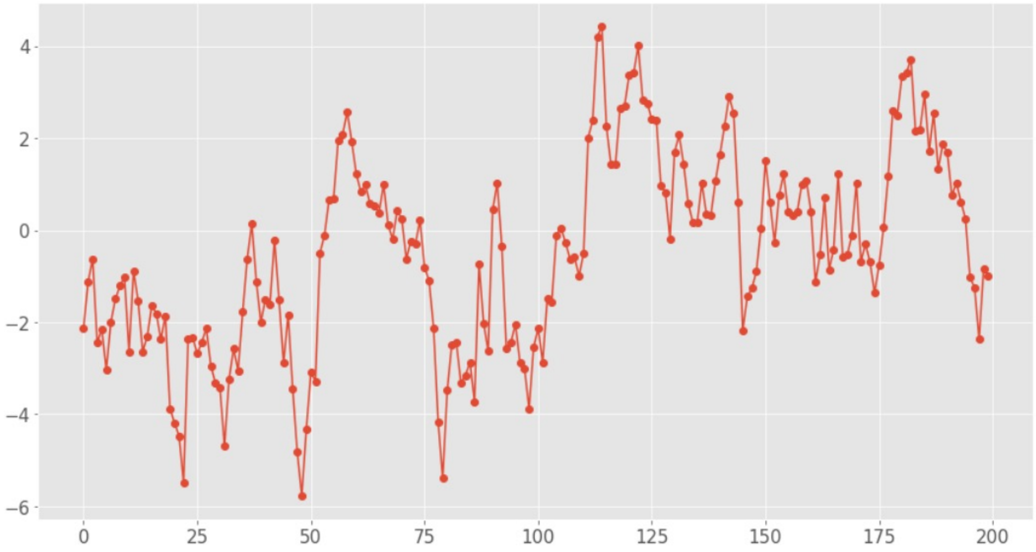
Example: AR(1)

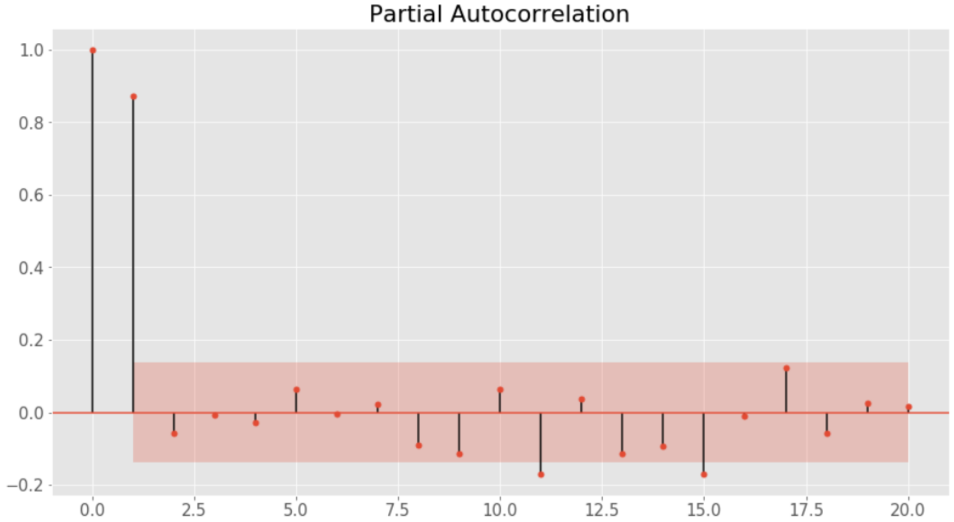
A numerical example could be given by

$$y_t = \mathbf{0.9}\, y_{t-1} + e_t$$

Where, e.g., the Gaussian white noise $e_t = \mathcal{N}(0, 1)$.





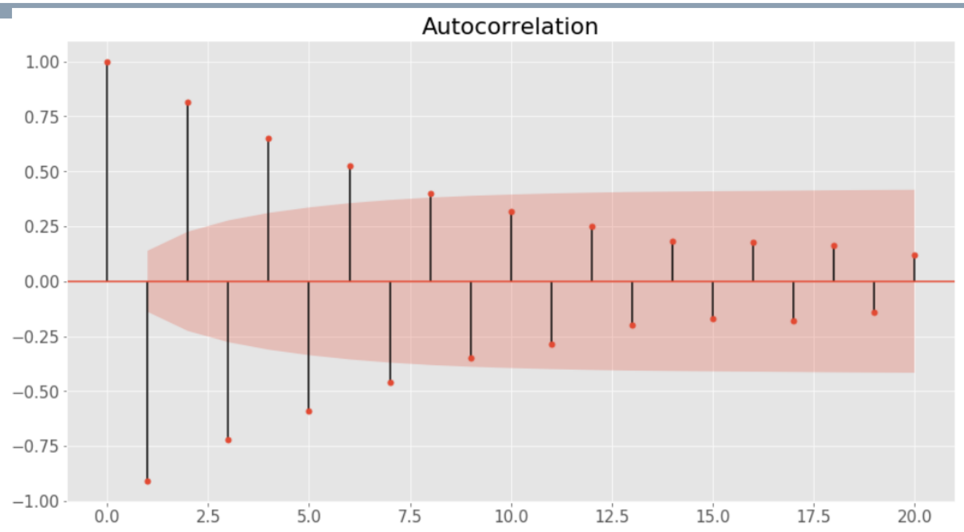


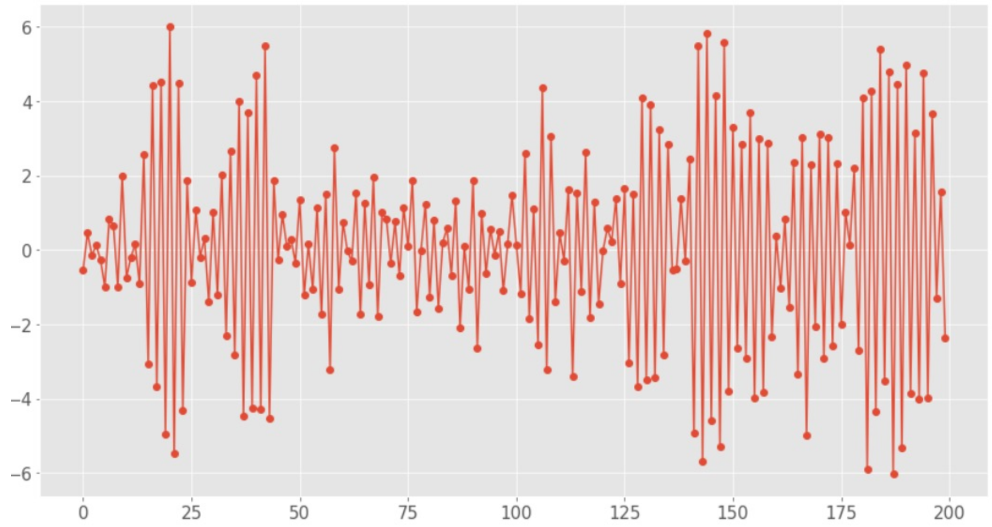
Example: AR(1)

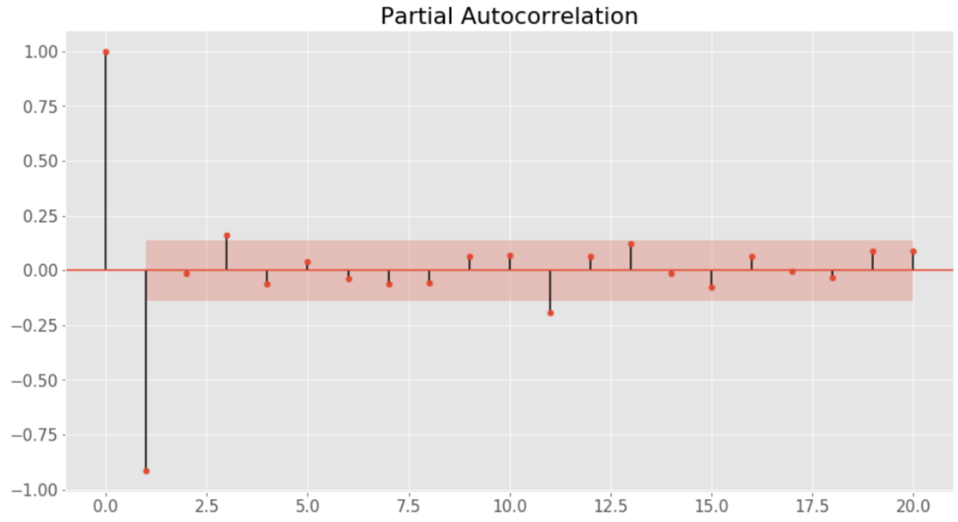
A numerical example could be given by

$$y_t = \mathbf{-0.9}\, y_{t-1} + e_t$$

Where, e.g., the Gaussian white noise $e_t = \mathcal{N}(0, 1)$.







Chosing an AR(n)

In concrete applications, the value of n is an hyper-parameter to optimize.

It is possible to identify the $AR(n)$ model by looking at the partial autocorrelation function (PACF). In fact:

- The theoretical partial autocorrelation for lags $h > n$ is zero.
  → For concrete experimental data, it might be small but non-zero.

- For $h = n$ the partial autocorrelation ϕ_n is not zero.
  → For all lag values in between, it is not necessarily zero

Moving Average models (MA)

Moving average models (MA) are based on the idea that the value of a time series at time t can be expressed as a linear combination of n past input random shock (or white noise).

$$MA(m): y_t = e_t + b_1 e_{t-1} + \cdots + b_m e_{t-m}$$

Where:

- m is the model‘s order
- $b_1, \ldots, b_m$ are the model‘s parameters, $b_m \neq 0$

In other words, the hyper-parameter m, again, represents how far back to look for dependecies with previous noise values.

Moving Average models (MA)

Similarly to the autoregressive case, the moving average model $MA(m)$ can be expressed with a more synthetic notation by using the backshift operator:

$$y_t = e_t + b_1 e_{t-1} + \cdots + b_m e_{t-m}$$

$$y_t = (1 + b_1 q^{-1} + \cdots + b_m q^{-m}) e_t$$

$$\boldsymbol{y_t = B(q^{-1}) e_t}$$

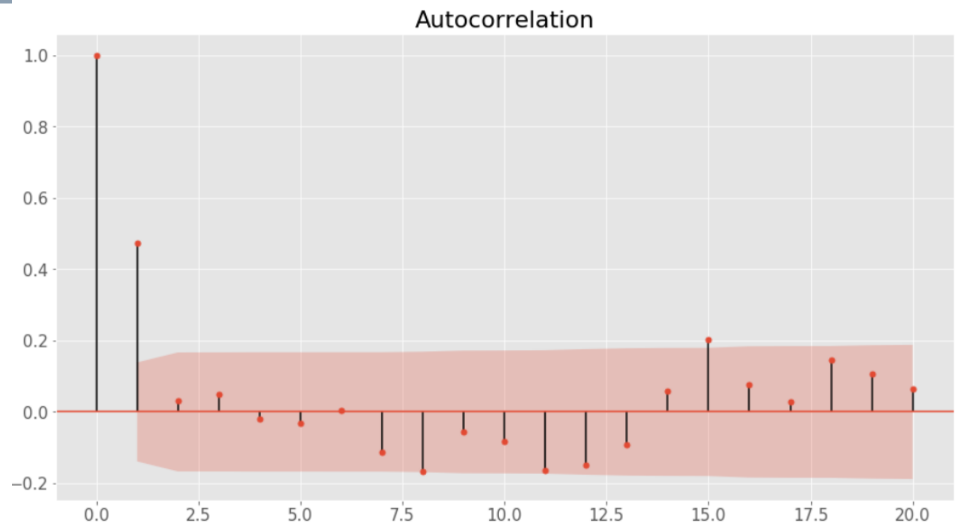
where $\boldsymbol{B\left(q^{-1}\right)}$ is called **moving average operator.**

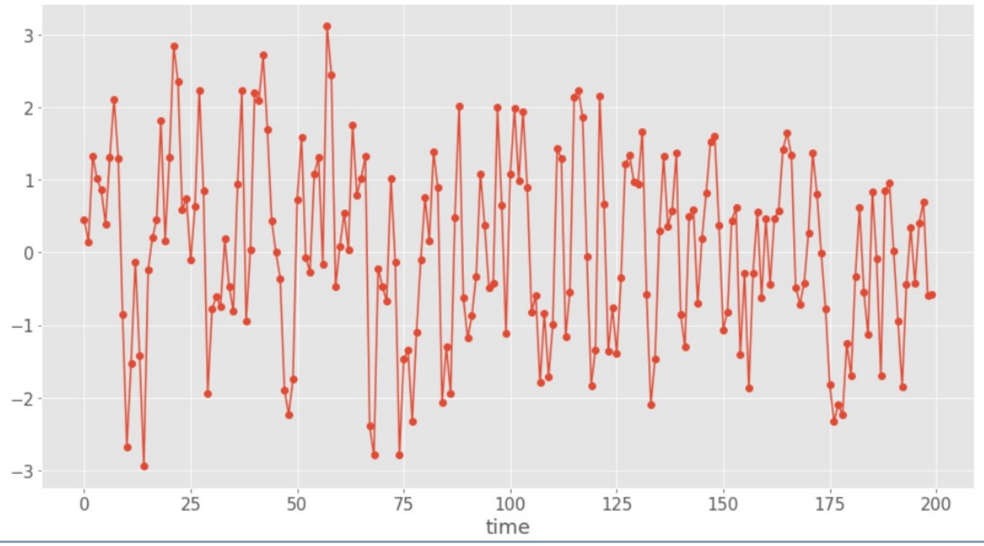
Example: MA(1)

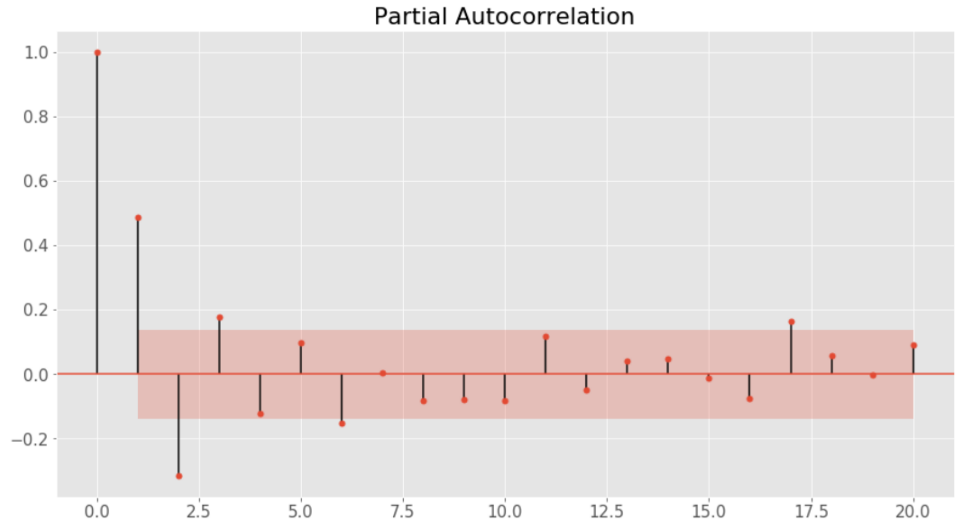
A numerical example could be given by

$$y_t = e_t + \mathbf{0.8}\, e_{t-1}$$

Where, e.g., the Gaussian white noise $e_t = \mathcal{N}(0,1)$.







Chosing an MA(m)

In concrete applications, the value of m is an hyper-parameter to optimize.

It is possible to identify the $MA(m)$ model by looking at the autocorrelation function (ACF). In fact:

- The theoretical autocorrelation for lags $h > m$ is zero.
  → For concrete experimental data, it might be small but non-zero.
- For $h = m$ the autocorrelation ρ_m is not zero.
  → For all lag values in between, it is not necessarily zero

Critical comparison

- Autoregressive models (AR) ignore correlated noise structures in the time series.
- Differently by AR models, finite moving average models (MA) are always stationary.
- It can be proved that:
  - All finite autoregressive processes $AR(n)$ are infinite moving average processes
  - All finite and invertible moving average $MA(m)$ processes are infinite autoregressive processes
- In practice, parameter estimation for MA models is generally more difficult than for AR models.

Autoregressive models
ARMA and ARIMA models

ARMA models

ARMA model is a combination of **autoregressive (AR)** and **moving average (MA)** models.

$$ARMA(n,m): y_t = \boldsymbol{a_1 y_{t-1}} + \boldsymbol{a_2 y_{t-2}} + \cdots + \boldsymbol{a_n y_{t-n}} + \boldsymbol{b_1 e_{t-1}} + \cdots + \boldsymbol{b_m e_{t-m}} + e_t$$

Which can be re-written using the backshift notation as:

$$ARMA(n,m):\ \boldsymbol{A(q^{-1}) y_t} = \boldsymbol{B(q^{-1}) e_t}$$

Where $\boldsymbol{A(q^{-1})}$ is the autoregressive operator and $\boldsymbol{B(q^{-1})}$ is the moving average operator, as defined previously.

Chosing an ARMA(n,m)

We can observe the ACF and PACF to determine the suitable hyper-parameters n and m.

	$AR(n)$	$\boldsymbol{MA(m)}$	$AR\boldsymbol{MA}(n,\boldsymbol{m})$
ACF	Tails off	Cuts off after lag m	Tails off
PACF	Cuts off after lag n	Tails off	Tails off

The choice of n and m is not unique.

How to deal with Nonstationary time series?

A limitation of the ARMA models is the assumption of our time series to be stationary.

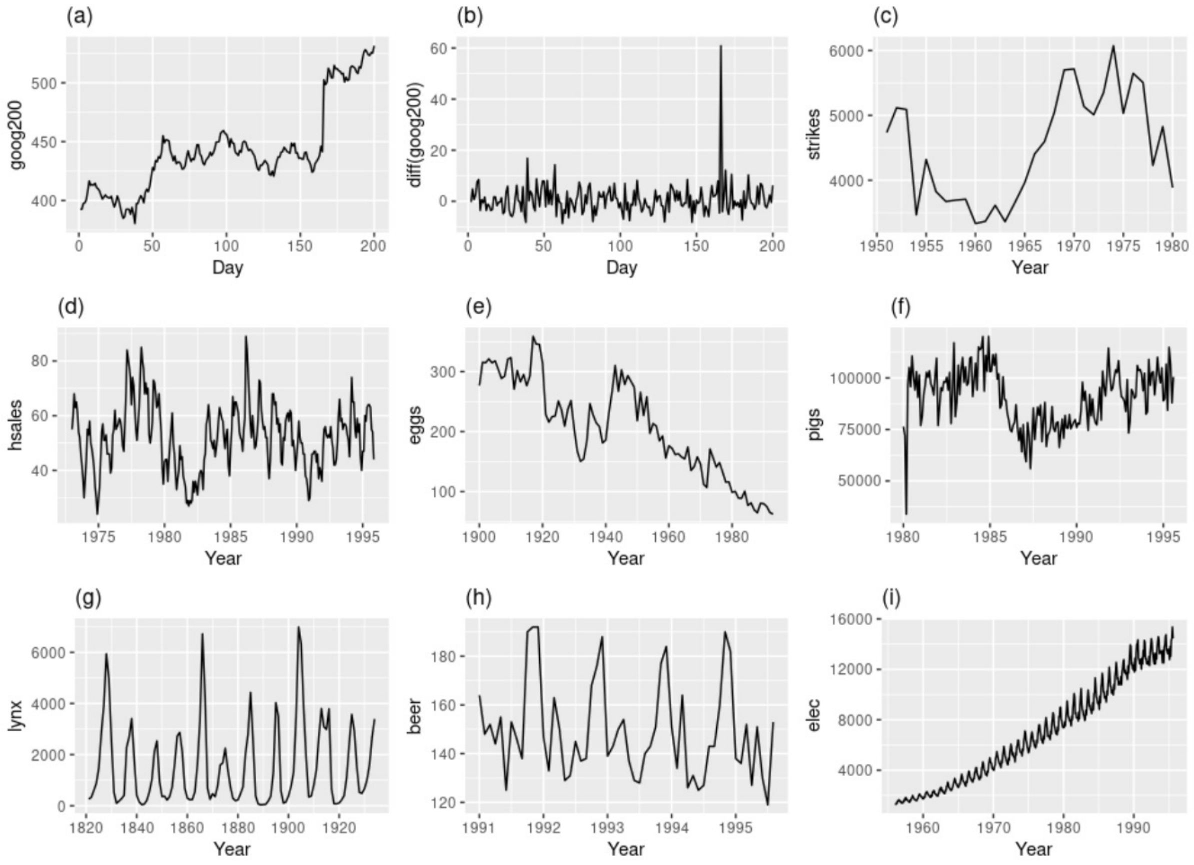
Many times, we can assume the time series to composed by a **non-stationary trend** and a **zero-mean stationary time series**, i.e.,

$$y_t = \mu_t + \phi_t$$

→ We can **„stationarize“** time series.

We can stationarize in two ways:

- Detrending
- Differencing



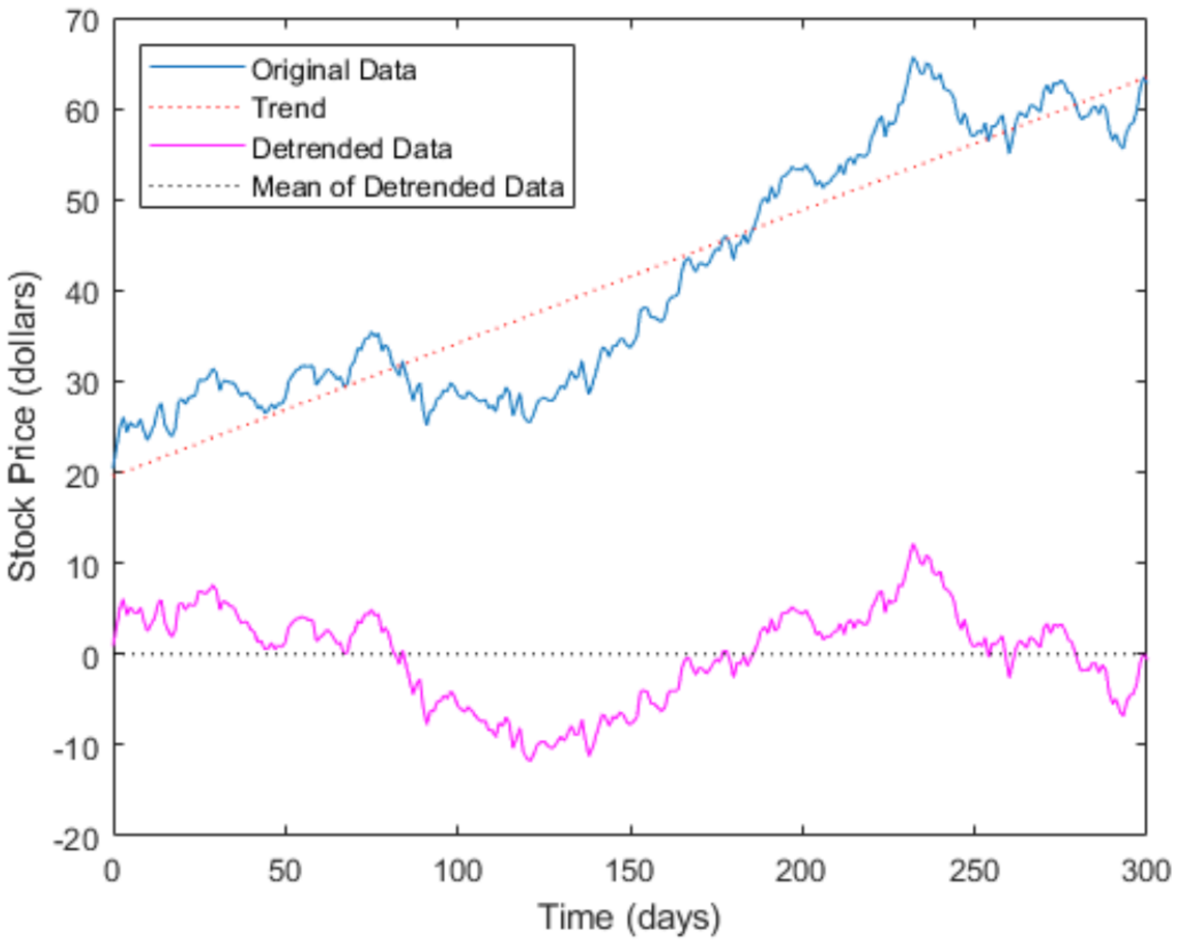
Stationarization: Detrending

By **detrending**, we can subtract an estimate for the time series‘ trend and deal with the remaining terms (i.e., residuals)

In formulas,

$$\hat{y}_t = \mathrm{y_t} - \hat{\mu}_t$$

Detrending needs parameters estimation.



https://www.mathworks.com/help/matlab/data_analysis/detrending-data.html

Stationarization: Differencing

The differencing operator is defined by

$$\nabla y_t = y_t - y_{t-1}$$

By using our backshift operator, the operator can be written as

$$\nabla = 1 - q^{-1}$$

Higher-order d differencing operations are given by

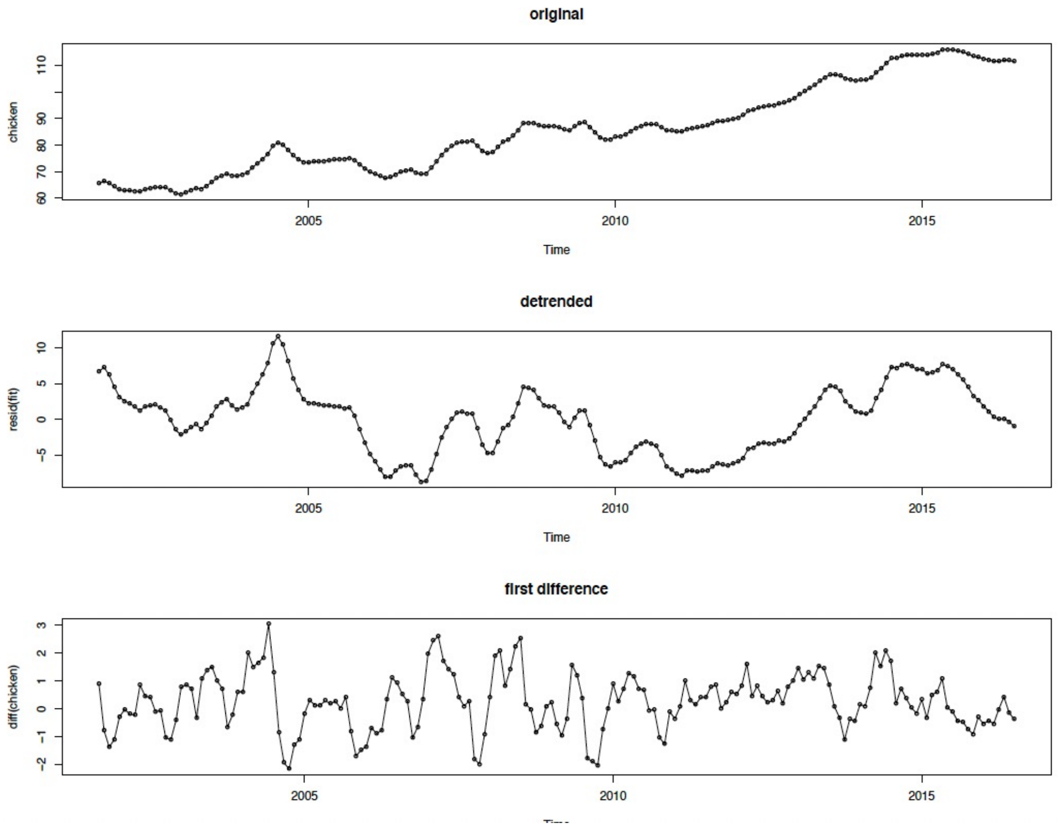
$$\nabla^d = (1 - q^{-1})^d$$

Stationarization: Differencing

By **differencing**, we compute the differences (or higher-order differences) of consecutive observations.

An advantage over detrending is that we do not need to estimate any parameters.

The differencing operation helps to stabilize the mean of a time series, by removing trends and seasonality.



ARIMA models

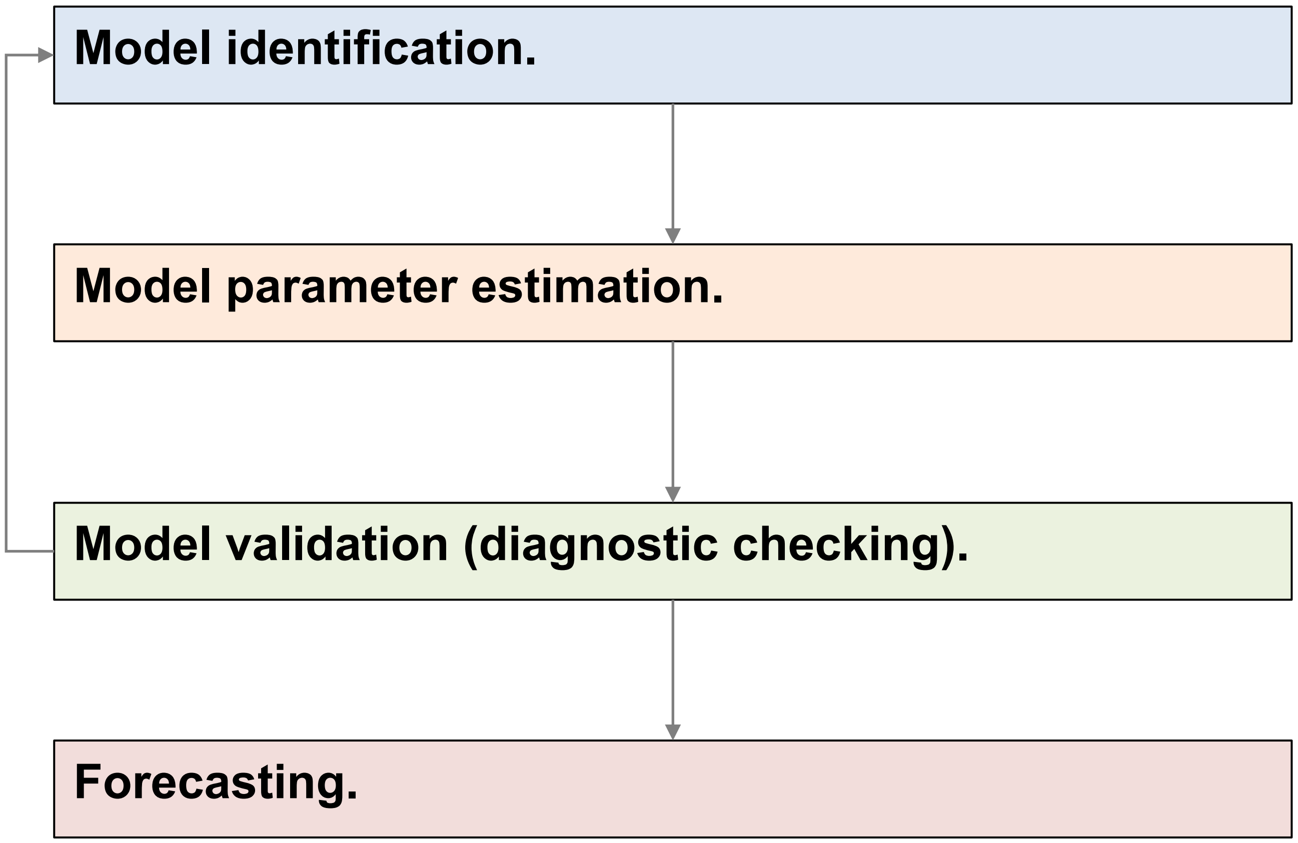
A process y_t is said to be $ARIMA(n, d, m)$ if d-th order differenciation $\nabla^d y_t$ is $ARMA(n, m)$.

Then, the ARIMA model can be written as

$$\boldsymbol{ARIMA}(n, \mathbf{d}, m)\text{: } A(q^{-1})\boldsymbol{\nabla}^{\mathbf{d}} y_t = B(q^{-1}) e_t$$

Notice that, $\mathrm{AR}IMA(n, 0, m)$ is equivalent to $ARMA(n, m)$.

General scheme



- Check stationarity and seasonality, perform differentiation if necessary, to chose ARIMA(n, d, m).
- Determine the model’s parameters that produce the best fitting, e.g., by Least square (LS) or Maximum likelihood estimation (MLE) methods.
- Perform a diagnostic checking, for example, by residual series analysis.
- We use the selected model for forecasting.

Lecture title

Recap

In this lecture...

- Linear processes
  - Autoregressive processes (AR)
  - Moving average processes (MA)
- Combining AR and MA:
  - ARMA
  - ARIMA

ARIMA: Pros and Cons

- Pros:
  - Effective in short-term series forecasting.
    - E.g., short-run inflation forecasts.
  - It is a parametric model and it works better with relatively small number of observations.
- Cons:
  - Techniques for identifying the correct model are difficult to understand and usually computationally expensive.
  - ARIMA models performance are poor at predicting series with turning points.