Friedrich-Alexander-Universität
Technische Fakultät

Machine Learning
Data Analytics

FAU

Machine Learning for Time Series

(MLTS or MLTS-Deluxe Lectures)

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Topics overview

- Time series fundamentals and definitions (2 lectures)
- Bayesian Inference (1 lecture)
- Gussian processes (2 lectures) ←
- State space models (2 lectures)
- Autoregressive models (1 lecture)
- Data mining on time series (1 lecture)
- Deep learning on time series (4 lectures)
- Domain adaptation (1 lecture)

In this lecture…

1. Gaussian process classification (GPC) formulation

2. Gaussian process classification (GPC) prediction

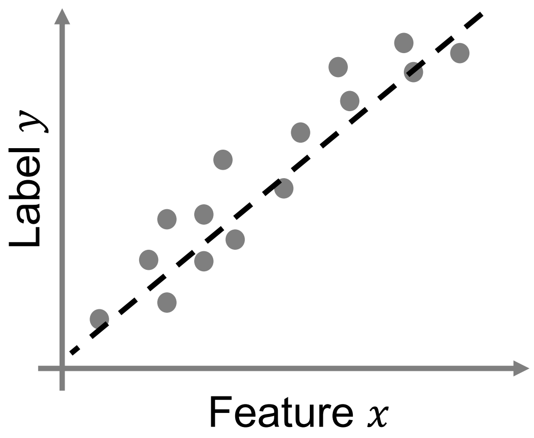
Gaussian process classification (GPC)

GPC formulation

Gaussian Process Classification (GPC)

Regression vs. Classification

Regression



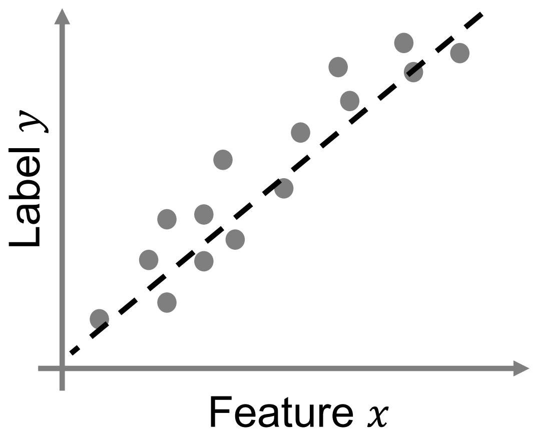
For regression we typically have:

- $x \in \mathbb{R}^d$
- $y_R \in \mathbb{R}$
- $y_R = f(x)$

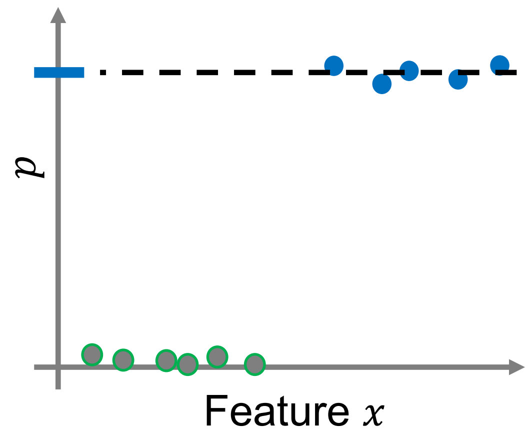
Gaussian Process Classification (GPC)

Regression vs. Classification

Regression



Classification



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For (binary) classification, instead:

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- Task: $y_C \in \{-1, +1\}$
- $p \in [0, 1]$

Gaussian Process Classification (GPC)

Regression vs. Classification

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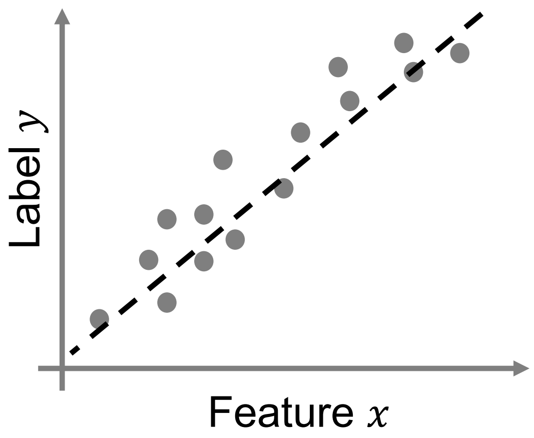

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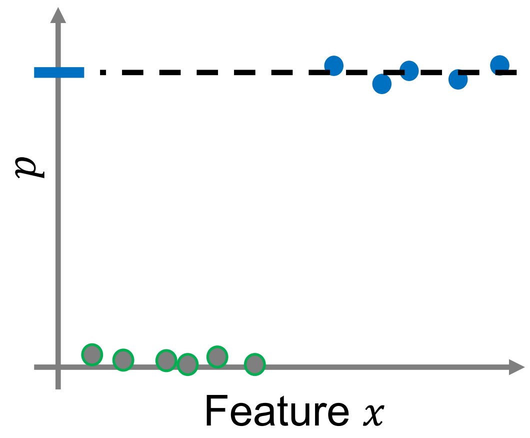

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Gaussian Process Classification (GPC)

Gaussian linear model

We use a Gaussian linear model in order to obtain the likelihood:

$$p(y = \pm 1 \,|x, w) = \sigma(x^T w)$$

where $\sigma(x^T w)$ is called sigmoid function.

Gaussian Process Classification (GPC)

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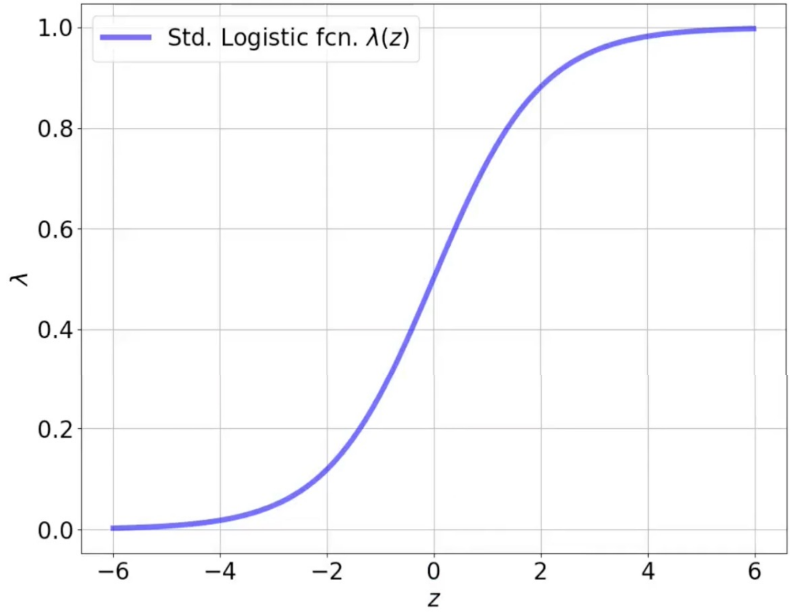
Notice:

- $p(y = \pm 1 \,|x, w)$ is the likelihood.
- Generally, we denote $\pi(x) \coloneqq \sigma(x^T w)$

Gaussian Process Classification (GPC)

The sigmoid function

Common options for the sigmoid functions:



$$\lambda(z) = \frac{1}{1+e^{-z}}$$

(Logistic function)

Gaussian Process Classification (GPC)

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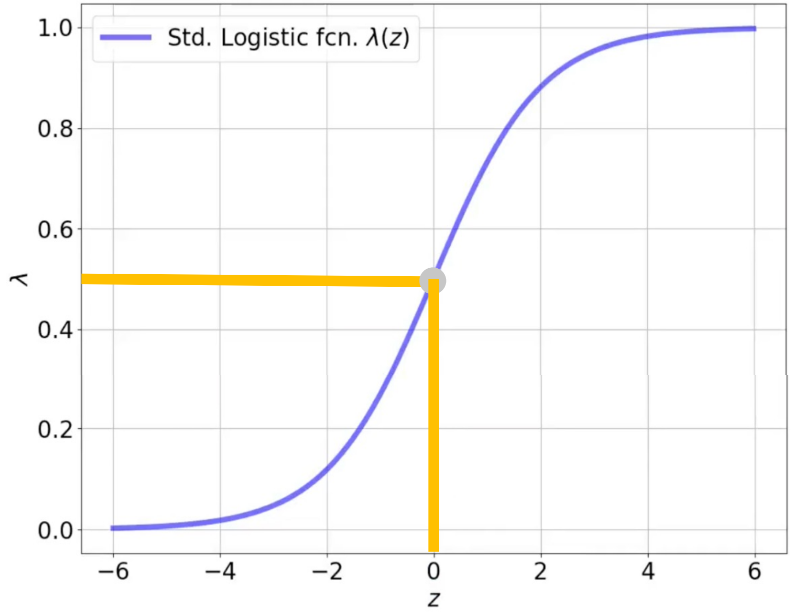

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Gaussian Process Classification (GPC)

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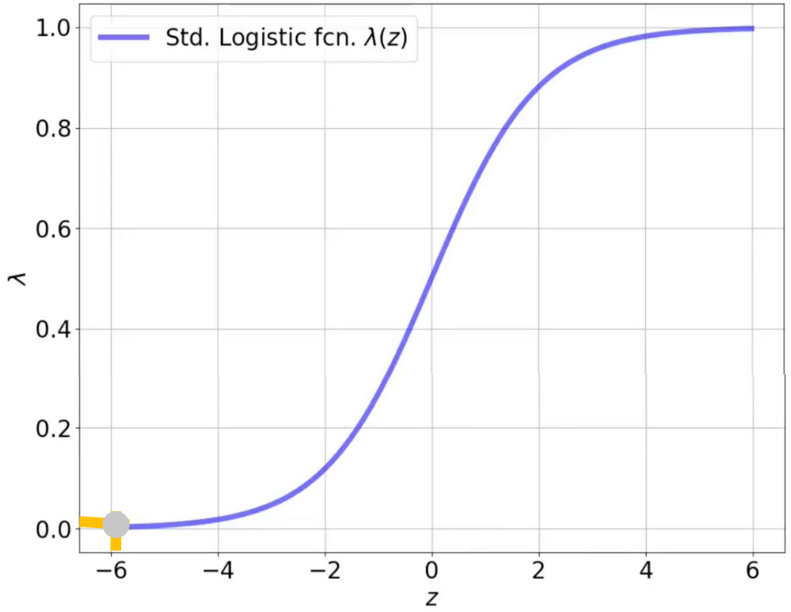

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Gaussian Process Classification (GPC)

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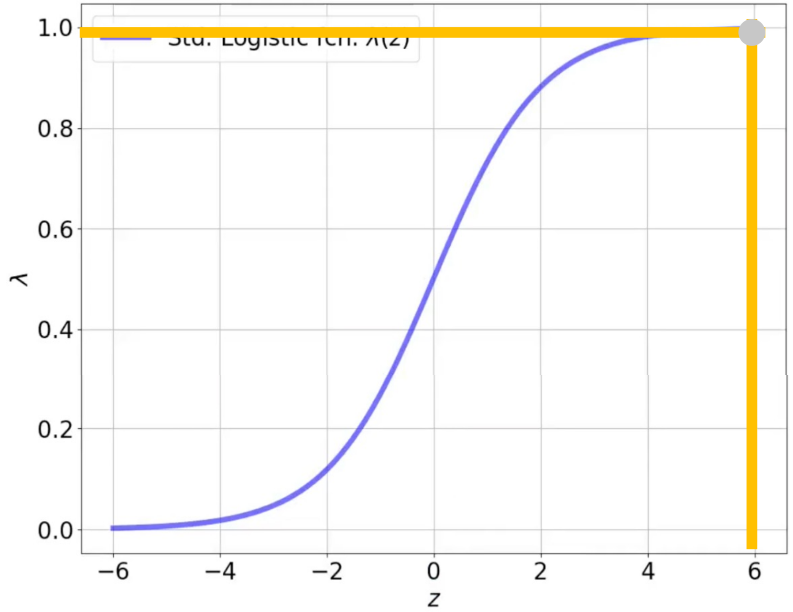

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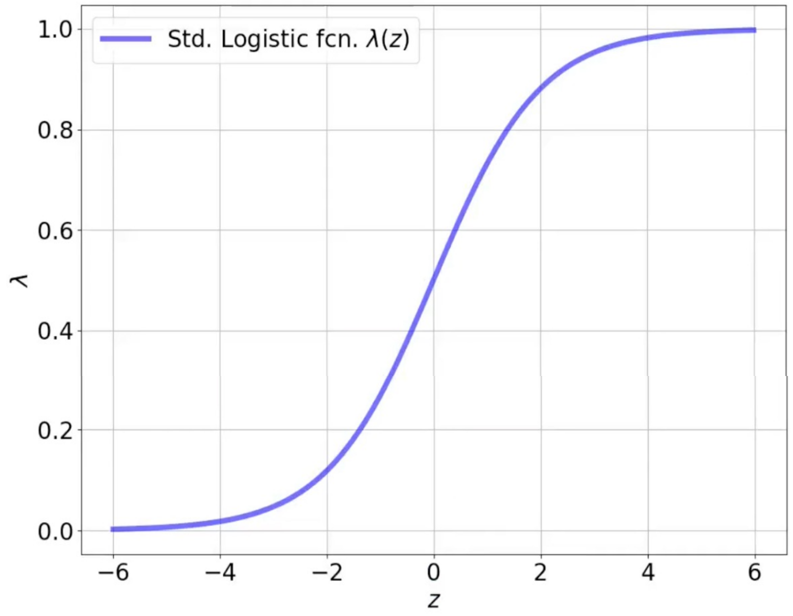

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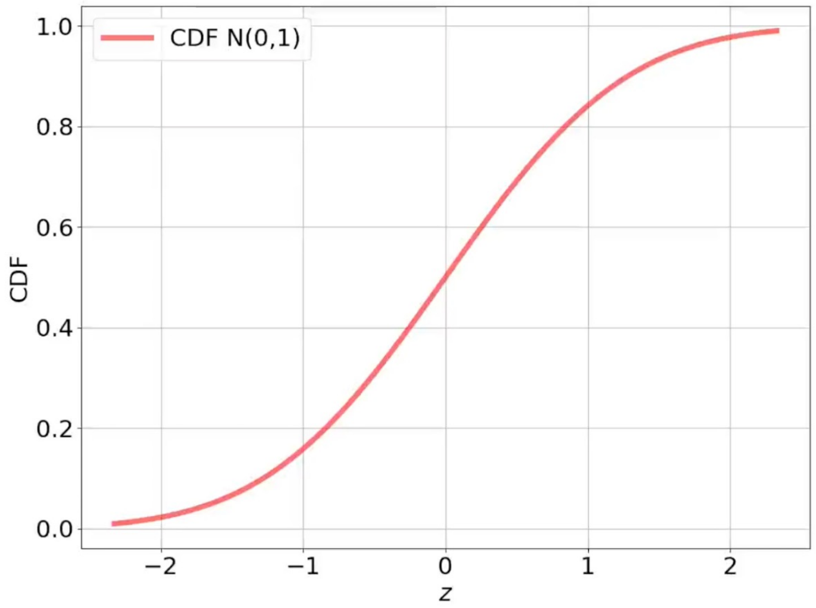


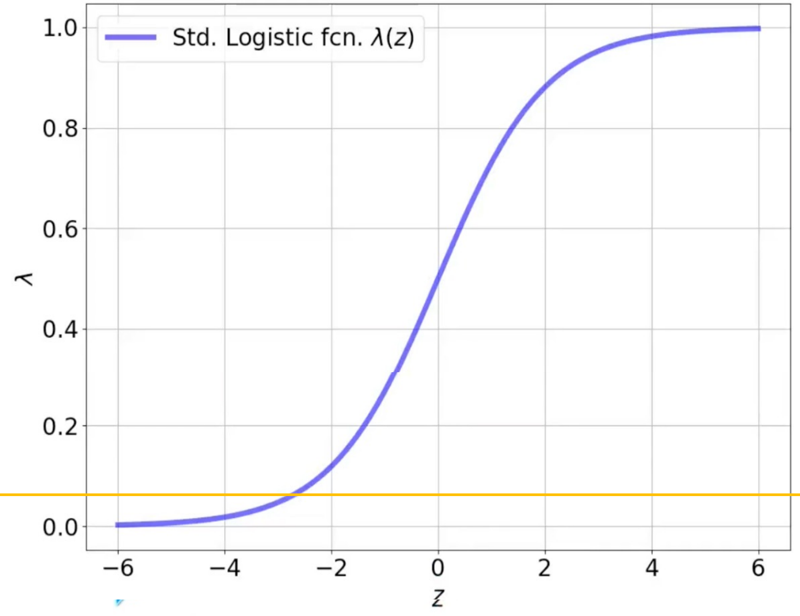
$$\phi(z) = \int_{-\infty}^{z} \mathcal{N}(x|0,1)\, dz$$

(Cumulative distribution function - CDF)

Gaussian Process Classification (GPC)

The sigmoid function

Common options for the sigmoid functions:



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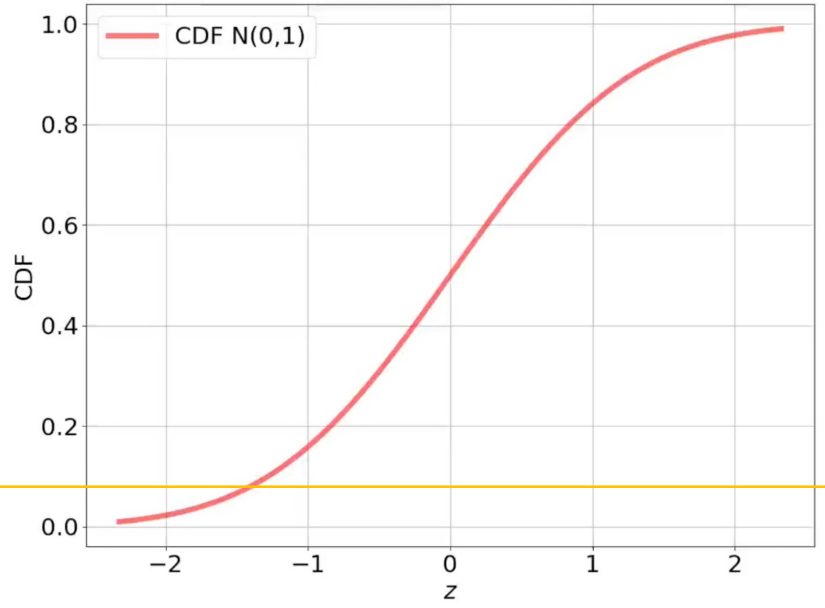
(Logistic function)



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(Cumulative distribution function - CDF)

Gaussian Process Classification (GPC)

Likelihood

For a 2-class problem, we can write the likelihood of the value pair (x_i, y_i):

$$\rightarrow \sigma\left(x_i^T w\right) \qquad \text{if } y_i = +1$$

$$\rightarrow 1 - \sigma\left(x_i^T w\right) \quad \text{if } y_i = -1$$

Gaussian Process Classification (GPC)

Likelihood

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For symmetric sigmoid functions: $\sigma(-z) = 1 - \sigma(z)$

Thus: $p\left(y_i \middle| x_i^T w\right) = \sigma(y_i\, f_i)$

- ➢ $y_i = \pm 1$ **(Sign)**
- ➢ $f_i = f(x_i) = x_i^T w$ **(Gaussian Process)**

Gaussian Process Classification (GPC)

Posterior

Let's assume the prior on w:

$$w \sim \mathcal{N}(0, \sigma_p) \quad \text{or} \quad w \sim \mathcal{N}(0, \Sigma_p)$$

Then, we can write the posterior over weights:

$$p(w|y, X) = \frac{p(y|X, w)\ p(w)}{p(y|X)}$$

The marginal likelihood can be written as:

$$p(y|X) = \int p(y|X, w)\ p(w)\, dw$$

Gaussian Process Classification (GPC)

A two-steps approach

Step 1: Gaussian Process (GP) over latent funtion $f(x)$

Step 2: Filter f through a sigmoid function to obtain

$$\pi(x) = p(y = +1|x) = \sigma(f(x))$$

Gaussian process classification

GPC prediction

GPC prediction

Predict a new point x^*:

$$p(y^* = +1|x^*, D) = \int p(y *= +1|w, x^*)\ p(w|D)\ dw$$

GPC prediction

Predict a new point $\boldsymbol{x}^*$:

$$p(y^* = +1|x^*, D) = \int \boxed{p(y^* = +1|w, x^*)}\, p(w|D)\ dw$$

Prediction = $\sigma(x^T w)$

GPC prediction

Predict a new point x^*:

$$\boxed{p(y^* = +1|x^*, D)} = \int \boxed{p(y^* = +1|w, x^*)}\; p(w|D)\; dw$$

Unimodal Non-Gaussian

Prediction = $\sigma(x^T w)$

Implementing prediction

Step 1: Compute the distribution of f^* at case x^*.

$$p(f^*|X, y, x^*) = \int p(f^*|X, x^*, f)\ p(f|X, y)\ df$$

The posterior on $f(x)$ can be written as:

$$p(f|X, y) = \frac{p(y|f)\ p(f|X)}{p(y|X)}$$

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The posterior on $f(x)$ can be written as:

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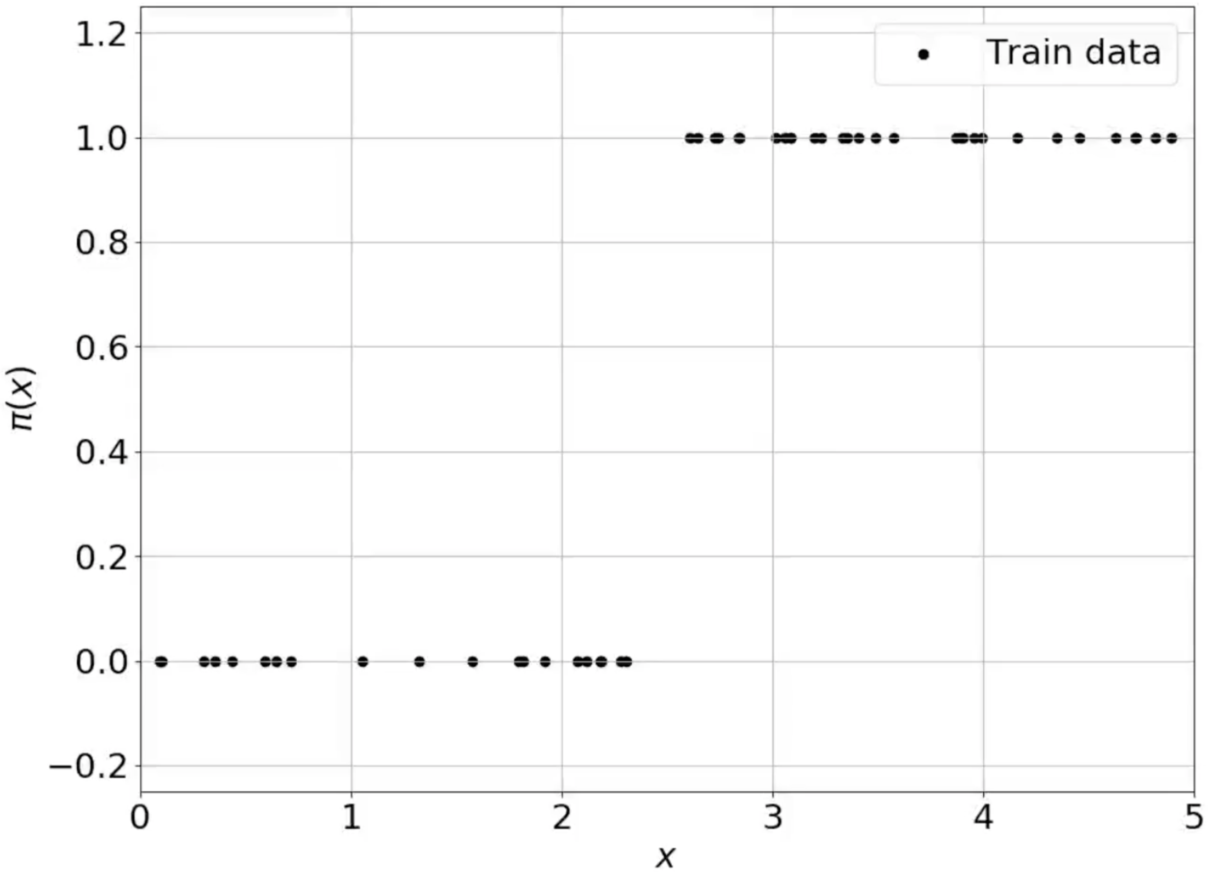
Implementing prediction

Step 2: Produce a probabilistic prediction π^*.

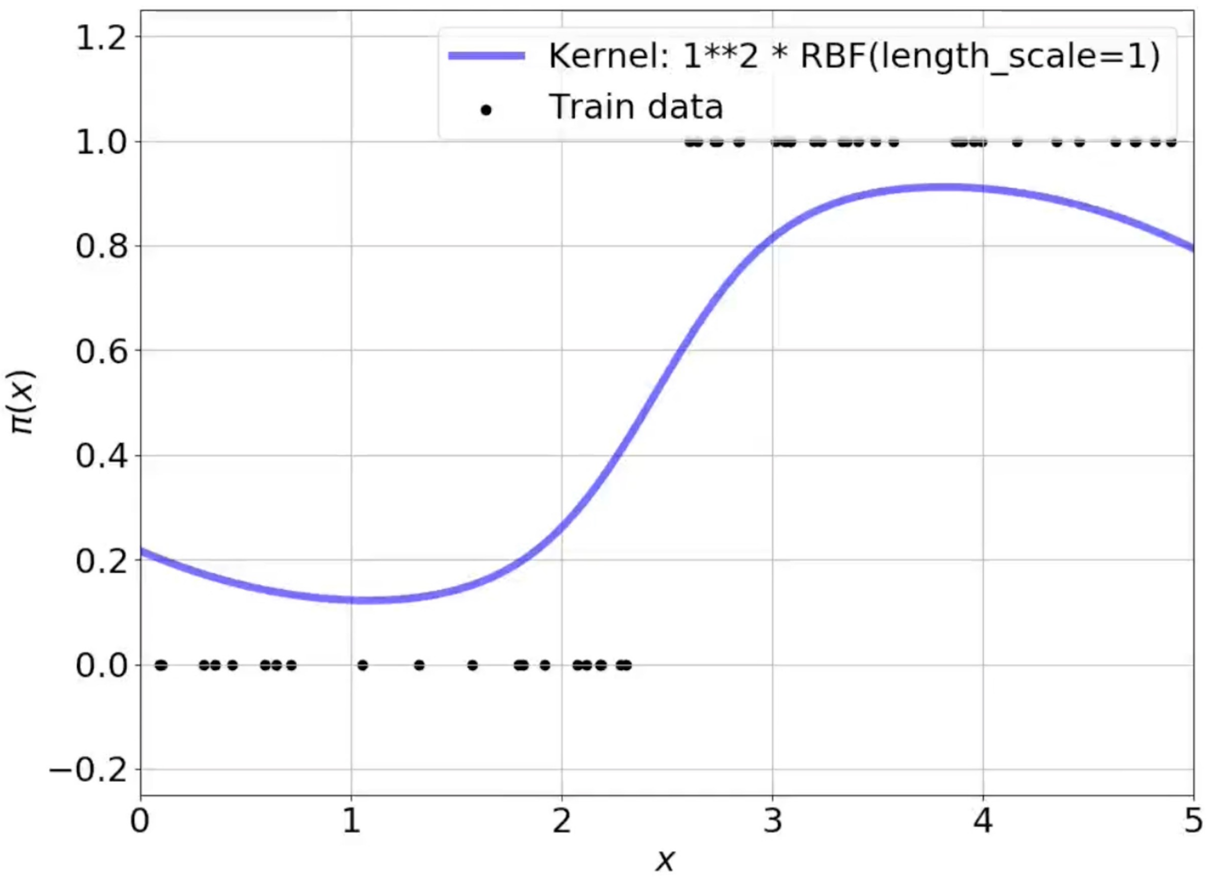
$$\pi^* \triangleq p(y^* = +1 \mid X, y, x^*) = \int \sigma(f^*)\ p(f^*|X, y, x^*)\ df^*$$

- $\pi^* = \pi(x^*)$ expresses the probability of the class
- The latent f has the role of nuissance function (we do not observe it)

GPC Example

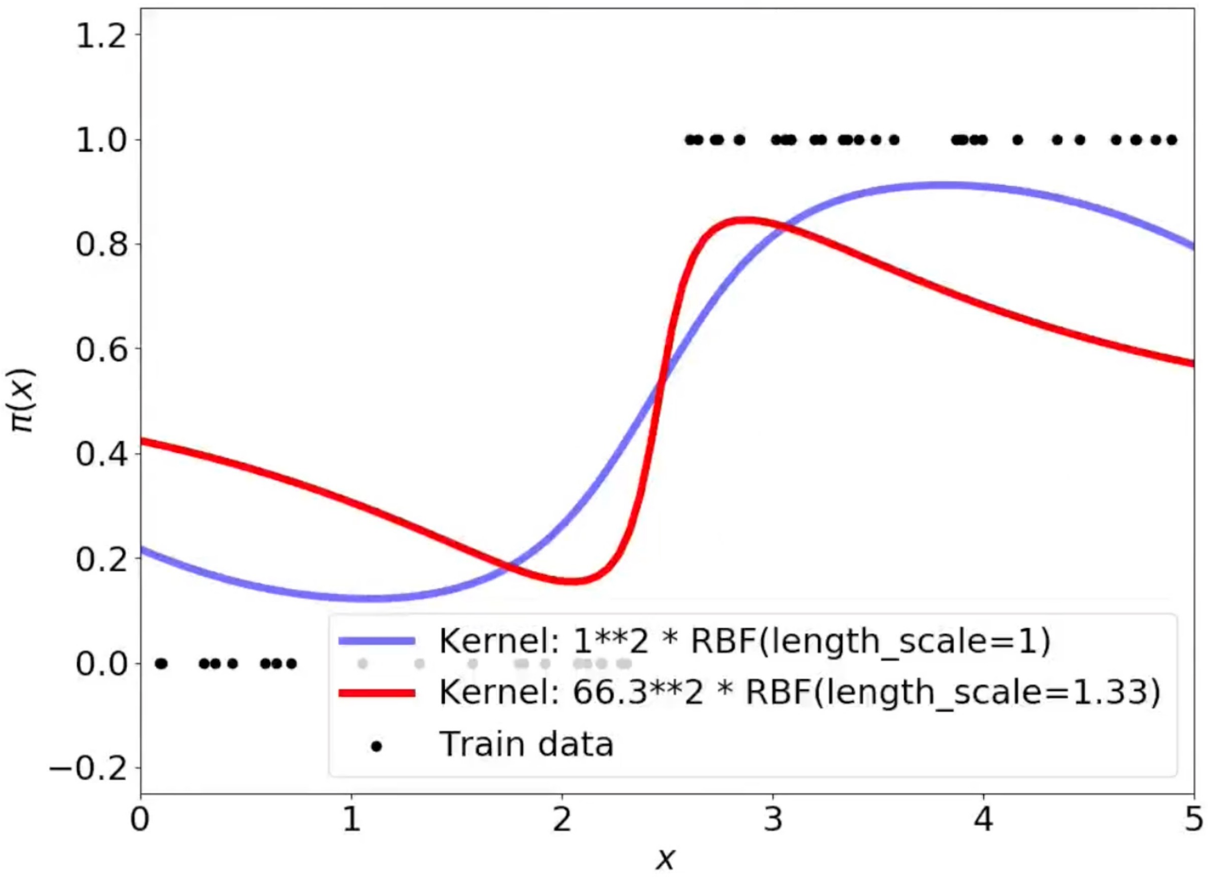


GPC Example



Kernel: 1**2 * RBF(length_scale=1)
Train data
1.2
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
π(x)
0
1
2
3
4
5
x

GPC Example



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Kernel: 66.3**2 * RBF(length_scale=1.33)
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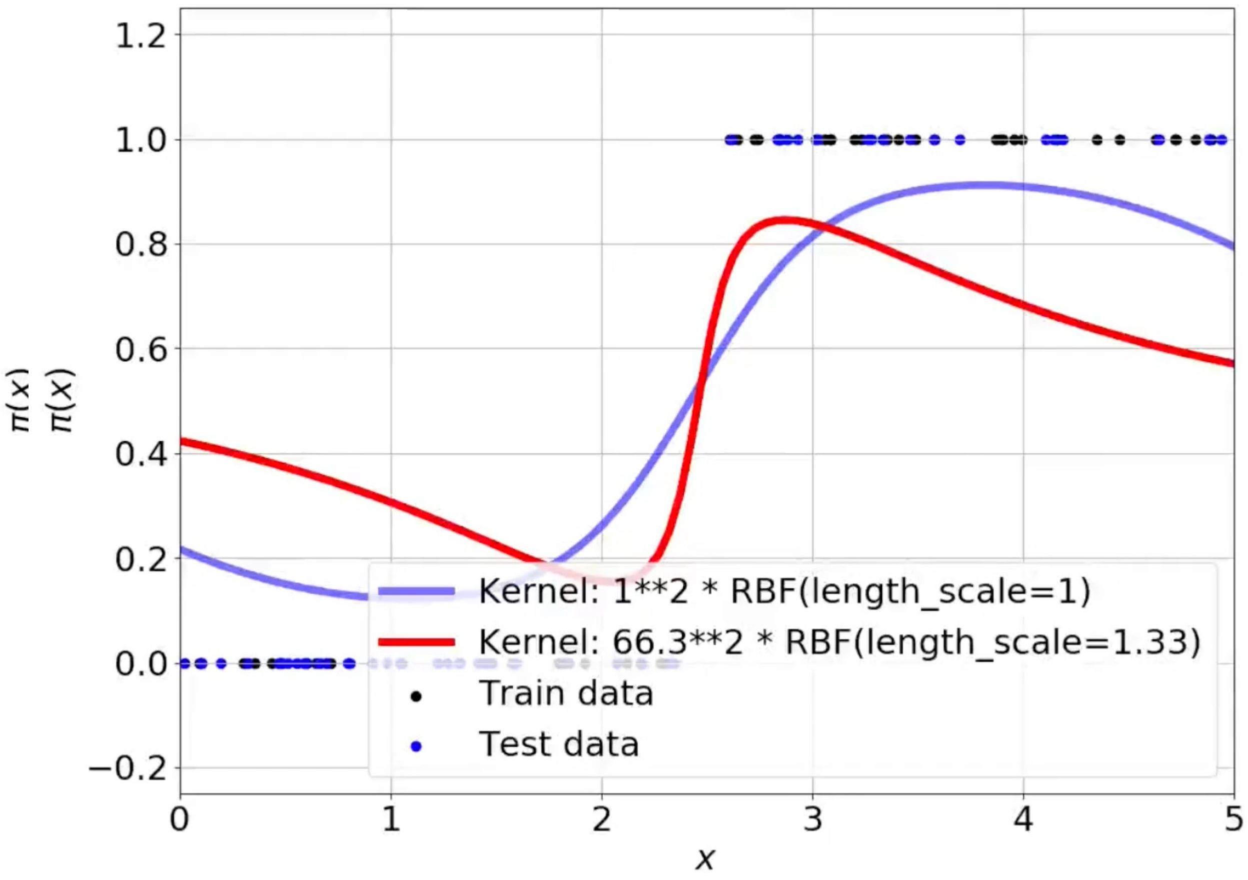
GPC Example



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Train data
Test data

Gaussian process classification (GPC)

Recap

Recap

- Gaussian process classification (GPC) formulation
  - GLM
  - Sigmoid
- Gaussian process classification (GPC) prediction
  - Two step process