This first homework is meant to refresh the mathematical background needed for this class. In particular some probability and linear algebra.

Problem 1

Mathematical statistics warm-up I:

(a) For a random variable Z, its mean and variance are defined as $\mathrm{E}[Z]$ and $\mathrm{E}[(Z-\mathrm{E}[Z])^2]$, respectively.

  (1) A random variable $X \sim \mathcal{N}(\mu, \sigma^2)$ is Gaussian distributed with mean μ and variance σ^2. Given that for any $a, b \in \mathbb{R}$, we have that $Y = aX + b$ is also Gaussian, find a, b such that $Y \sim \mathcal{N}(0, 1)$.

  (2) Let $X_1, \ldots, X_n$ be independent and identically distributed random variables, each with mean μ and variance σ^2. If we define $\widehat{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$, what is the mean and variance of $\sqrt{n}(\widehat{X}_n - \mu)$?

(b) Suppose $X \sim Pois(\lambda)$. Show that $\mathrm{Var}(X) = \lambda$.

(c) Suppose $X \sim Exponential(\lambda)$. Show that $\mathrm{Var}(X) = \frac{1}{\lambda^2}$.

(d) Suppose X is a random variable distributed according to F_X with $\mathrm{E}(X) = 0$ and density f_x. Then, $\mathrm{Var}(aX + b) = a^2 \mathrm{Var}(X)$ for constants $a, b \in \mathbb{R}$.

(e) Let $X_n \sim Exp(n)$, show that $X_n \xrightarrow{P} 0$.

(f) Let X be a random variable, and $X_n = X + Y_n$, where

$$\mathrm{E}(Y_n) = \frac{1}{n}, \qquad \mathrm{Var}(Y_n) = \frac{\sigma^2}{n},$$

where $\sigma > 0$ is a constant. Show that $X_n \xrightarrow{P} X$.

Problem 2

Mathematical statistics warm-up II:

(a) Prove that for $X \geq 0$, it holds that $\mathrm{E}(X) = \int_0^\infty \mathrm{P}(X > t)dt$. You may assume that X is continuously distributed and hence has a probability density function.

(b) (p-moments via tails) Prove that for $X \geq 0$ and $p \in (0, \infty)$, it holds that

$$\mathrm{E}(X^p) = \int_0^\infty p t^{p-1}\, \mathrm{P}(X > t) dt$$

whenever the right hand side is finite. You may assume that X is continuously distributed and hence has a probability density function.

Problem 3

Vectors and Matrices:

(a) Consider the matrix X and the vectors y and z below:

$$X = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} \qquad y = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \qquad z = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

  (a) What is the inner product of the vectors y and z?
  (b) What is the product Xy?
  (c) Calculate the determinant, the trace, and the Frobenius and operator norms of the matrix X.
  (d) Is X invertible? If so, give the inverse, and if no, explain why not.
  (e) What is the rank of X? Explain your answer.

(b) Consider $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 1 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$. For each matrix A and B,

  (a) What is its rank?
  (b) What is a (minimal size) basis for its column span?

(c) Assume $A = \begin{bmatrix} 0 & 2 & 4 \\ 2 & 4 & 2 \\ 3 & 3 & 1 \end{bmatrix}$, $b = \begin{bmatrix} -2 & -2 & -4 \end{bmatrix}^T$, and $c = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$.

  (a) What is Ac?
  (b) What is the solution to the linear system $Ax = b$?.

Problem 4

Coding problem I: Use a language of your choice (The course material is in R).

Sampling from a distribution.

(a) Draw 100 samples $x = (x_1\ x_2)^T$ from a 2-dimensional Gaussian distribution with mean $(0, 0)^T$ and identity covariance matrix.

(b) Plot them on a scatter plot.

(c) How does the scatter plot change if the mean is $(1, -1)^T$?

(d) (Change the mean back to $(0, 0)^T$.) Change the covariance matrix as follows

$$\Sigma_1 = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix} \qquad \Sigma_2 = \begin{pmatrix} 1 & -0.5 \\ -0.5 & 1 \end{pmatrix}$$

and plot the corresponding scatter plots.

Problem 5

Coding problem II:

(a) Write down the properties a general covariance matrix has to satisfy.

(b) Write a 2×2 matrix that satisfies those properties.

(c) Draw 100 samples $x = (x_1\ x_2)^T$ from a 2-dimensional Gaussian distribution with mean $(0, 0)'$ and the covariance matrix you chose in (b). Generate a plot of the support region for the Gaussian random variables. Vary the covariance matrix to demonstrate how the shape of the support region changes depending on the nature of the covariance matrix.

(d) Find a way to generate a covariance matrix of arbitrary dimension.

(e) Recover the histogram plots in the lecture notes. Play around with different dimensions and sample sizes.