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ST – Homework 2

Problem 1

Consider 200 independent coin flips. We wish to find an upper bound on the probability that the number of heads is greater or equal than 150. Use Markov's, Chebyshev's and Chernoff inequality and compare.

Problem 2

We explore some elementary aspects of concentration.

(a) Prove that if Z is a non-negative random variable with expectation $\mathrm{E}[Z]$, then for all $t > 0$, we have $\mathrm{P}[Z \geq t] \leq \mathrm{E}[Z]/t$.

(b) A zero-mean random variable is said to be sub-Gaussian with parameter $\sigma > 0$ if $\mathrm{E}[\exp(tX)] \leq \exp(\sigma^2 t^2/2)$ for all $t \in \mathbb{R}$. Show that $X \sim \mathcal{N}(0, \sigma^2)$ is sub-Gaussian.

(c) Suppose that X is Bernoulli with $\mathrm{P}[X = +1] = \mathrm{P}[X = -1] = 1/2$. Show that X is sub-Gaussian.

(d) Show that any sub-Gaussian random variable X satisfies the two-sided tail bound

$$\mathrm{P}[|X| > t] \leq 2\exp(-\frac{t^2}{2\sigma^2})$$

Problem 3

Poisson distribution: Let $X \sim Pois(\lambda)$, that is,

$$\mathrm{P}[X = k] = \frac{\lambda^k \exp(-\lambda)}{k!}.$$

The following limit theorem holds:

Theorem 1.3.4 (Poisson Limit Theorem). *Let $X_{N,i}$, $1 \leq i \leq N$, be independent random variables $X_{N,i} \sim \mathrm{Ber}(p_{N,i})$, and let $S_N = \sum_{i=1}^N X_{N,i}$. Assume that, as $N \to \infty$,*

$$\max_{i \leq N} p_{N,i} \to 0 \quad \textit{and} \quad \mathbb{E}\, S_N = \sum_{i=1}^{N} p_{N,i} \to \lambda < \infty.$$

Then, as $N \to \infty$,

$$S_N \to \mathrm{Pois}(\lambda) \quad \textit{in distribution.}$$

(a) Show that for any $t > \lambda$,

$$\mathrm{P}[X > t] \leq \exp(-\lambda)\left(\frac{\exp(1)\lambda}{t}\right)^t$$

(b) Show that for $t \in (0, \lambda]$

$$\mathrm{P}[|X - \lambda| > t] \leq 2\exp\left(-c\frac{t^2}{\lambda}\right)$$

For (b) use the following Chernoff type inequality for small deviations: For $\delta \in (0, 1]$, we have $\mathrm{P}(|S_N - \mu| \geq \delta\mu) \leq 2\exp(-c\mu\delta^2)$, where $c > 0$ is some constant.

Problem 4

Let $Z \sim \mathcal{N}(0, 1)$ and use the identity $\int_0^\infty (1 - 3x^{-4})e^{-\frac{x^2}{2}}\,dx = \left(\frac{1}{t} - \frac{1}{t^3}\right)e^{-\frac{t^2}{2}}$.

(a) (Tails of the normal distribution). Then for all $t > 0$, we have

$$\left(\frac{1}{t} - \frac{1}{t^3}\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}} \leq \mathrm{P}(Z \geq t) \leq \frac{1}{t}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}.$$

In particular, for $t \geq 1$ the tail is bounded by the density:

$$\mathrm{P}(Z \geq t) \leq \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}.$$

(b) Show that for all $t \geq 0$

$$\mathrm{P}(|Z| \geq t) \leq 2e^{-\frac{t^2}{2}}.$$

(c) (Truncated normal distribution) Show that for all $t \geq 1$,

$$\mathrm{E}[Z^2 1_{Z>t}] = t\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}} + \mathrm{P}(Z \geq t) \leq \left(t + \frac{1}{t}\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}.$$

Problem 5

Coding problem

(a) Let X be a Poisson random variable with $\lambda = 0.8$. Visually compare the Markov bound, Chernoff bound, and the theoretical probabilities for $x = 1, \ldots, 12$. Use Problem 3 (a).