Assignment3 – Decisions

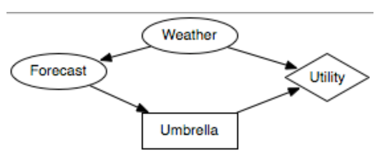
Given: May 11 Due: May 21

Problem 3.1 (Decision Network) 0 pt

You try to decide on whether to take an umbrella to Uni. Obviously, it is useful to do so if it rains when you go back home, but it is annoying to carry around if it does not even rain. You decide based on the weather forecast F, which does or does not predict rain.

1. Draw the decision network for bringing/leaving an umbrella depending on the weather forecast and the actual weather.
2. Explain formally how to compute whether or not to take an umbrella, assuming you know the probability that the forecast is correct.

Solution:



We use four random variables:

- Boolean F for the weather forecast (true if rainy).
- Boolean W for the weather (true if rainy).
- Number U for the utility U is a utility node and as such is a deterministic variable. Instead of a probability table $P(U = u|W, M)$, we store the function $U(W, M)$ that yields the utility of (not) having an umbrella if it is (not) rainy.
- Boolean M for whether we took an umbrella. Normally, M would be influenced by F but because M is a decision node, we do not need to store its probability table. Its incoming arrows become evidence variables.

The probability of the forecast being correct gives us $P(W = b|F = b)$ for booleans b.

Now given evidence $F = b$, we plug in each possible value for the decision variable M and compute the value of the utility variable.

- $M = true$

$$P(W = true|F = b)U(W = true, M = true)+$$
$$P(W = false|F = b)U(W = false, M = true)$$

- $M = false$

$$P(W = true|F = b)U(W = true, M = false)+$$
$$P(W = false|F = b)U(W = false, M = false)$$

We choose the decision that yields higher utility. Note that we may arrive at different decisions for different values of b.

Problem 3.2 (Expected Utility) 30 pt

1. State the formal definition of *expected utility* of an action in the current state of an agent? Explain the meaning of every variable in the defining equation.

2. How do we use expected utility to make decisions?

Solution:

1. The expected utility EU is defined as

$$EU(a|e) = \sum_{s'} P(R(a) = s'|a, e) \cdot U(s')$$

   where

   (a) a is the action for which we want to find out the expected utility, given the evidence e.

   (b) $U(s')$ is the utility of a state s'.

   (c) $R(a)$ is the result of the action a.

2. The principle of maximum expected utility says that a rational agent should choose the action that *maximizes* the agent's expected utility.

Problem 3.3 (Decision Theory) 30 pt

You are offered the following game: You pay x dollars to play. A fair coin is then tossed repeatedly until it comes up heads for the first time. Your payout is 2^n, where n is the number of tosses that occurred.

1. Assume your utility function is exactly the monetary value. How much should you, as a rational agent, be willing to pay to play? Use the formal definition of "expected utility" from the lecture.

2. Assume now, that your utility function for having k dollars is $U(k) = m \log_n k$ for some $m, n \in \mathbb{N}^+$. How does this change the result?

3. What is wrong with the result from the first exercise? Which implicit assumption leads to the apparently nonsensical result? How could it be fixed?

Hint: The series $\sum_{k=1}^{\infty} \frac{k}{2^k}$ is convergent with limit 2.

Solution:

1. We have

$$EU(\text{Play}) = \sum_{s'} (P(\text{Payout} = s' | \text{Play}) \cdot U(s') = \sum_{k \in \mathbb{N}^+} P(\text{Payout} = k) \cdot 2^k$$
$$= \sum_{k \in \mathbb{N}^+} \frac{1}{2^k} \cdot 2^k = \sum_{k \in \mathbb{N}^+} 1 \to \infty$$

   We should be willing to pay any amount for a chance to play.

2. $$EU(\text{Play}) = \sum_{k \in \mathbb{N}^+} \frac{1}{2^k} \cdot m \log_n(2^k) = m \log_n(2) \sum_{k \in \mathbb{N}^+} \frac{k}{2^k} = 2m \log_n(2)$$

   Of course, this is wrong insofar as the utility, being logarithmic, is in particular not linear, i.e. the actual utility depends on our original capital K, but this just makes everything more complicated. The point is that using a logarithmic utility for money yields a finite result.

3. This model assumes that the payout is potentially infinite (which is unrealistic), as well as that we have an unlimited amount of money at our disposal. One way to fix this is to calculate our overall utility as the difference of $EU(\text{Play})$ and the utility $U(k)$ of paying the cost k. Then we can set an upper limit of how much we can afford to pay by setting $U(k) = -\infty$ when k is greater than the amount in our bank account.

Problem 3.4 (Decision Network) 40 pt

You need a new car. Your local dealership has two models on offer

- C_1 for \$1500 with market value \$2000
- C_2 for \$1150 with market value \$1400

Either car can be of good quality or bad quality, and you have no information about that. If C_1 is of bad quality, repairing it will cost \$700, if C_2 is of bad quality repairing it will cost \$150.

You have the choice between two tests:

1. $Test_1$ at cost \$50: This will confirm that C_1 is of good quality (if it is) with certainty 85% probability, and that it is of bad quality (if it is) with certainty 65%.

2. $Test_2$ at cost \$20: This will confirm that C_2 is of good quality (if it is) with 75% certainty, and that it is of bad quality (if it is) with certainty 70%.

The a priori probability (without any tests) that a car is of good quality is 70% for C_1 and 65% for C_2.

The utility function is the monetary value, i.e., the difference of the market value of the acquired car and the amount of money spent on test, car, and repair.

1. Decision networks in general have three kinds of nodes. Explain the differences regarding the probability tables of the three kinds.

2. Now regarding the concrete network used here, explain the random variables of all nodes and their domains.

3. Draw the decision network for which test to apply and which car to buy in either case. This should include:
   - an action node for the test decision (no test, Test1, or Test2)
   - an action node for which car to buy
   - utility nodes for each of the two cars
   - chance nodes for the quality of the cars and the outcomes of the tests

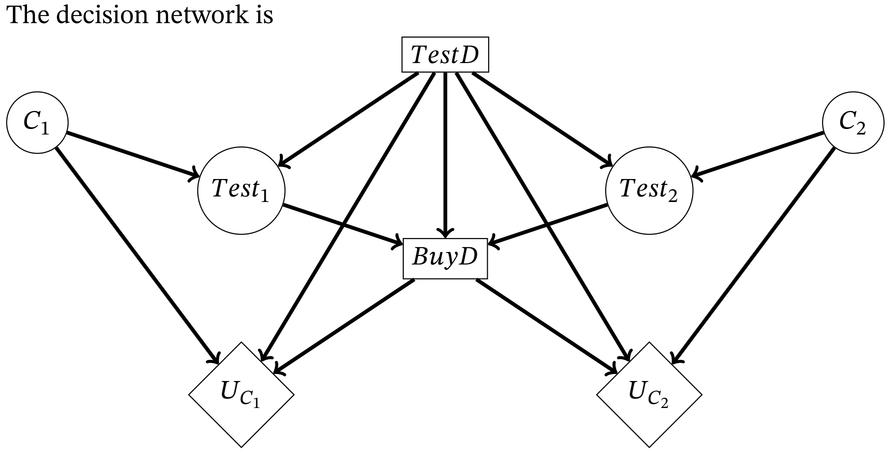
4. Assume we have chosen to do $Test_1$ and the outcome was good. Compute which car to buy.

Solution:

1. Decision nodes have no probability table. Instead, we fix the value for each decision, then calculate the utility. Utility nodes are deterministic: their value can be computed if the values of their parent nodes are known. In particular, we can compute their expected value if the probabilities for their parent nodes are known. Chance nodes have the usual probability tables of Bayesian networks.

2. The variables are
   - $C_1, C_2 \in \{g, b\}$: car 1 resp. car 2 are of good/bad quality
   - $Test_D \in \{\emptyset, 1, 2\}$: Choose no test, Test1, or Test2
   - $Test_1, Test_2 \in \{g, b\}$: test yields that the tested car is of good/bad quality
   - $Buy_D \in \{1, 2\}$: which car to buy
   - $U_{C_1}, U_{C_2} \in Nat$: utility in \$ of buying car 1 resp. car 2

3. The decision network is



4. We have already made one decision: $P(Test_D = 1) = 1$. Our evidence is the event $Test_1 = g$. We need to set $BuyD$ to each of its values and compute the expected utility in each case.

- $BuyD = 1$: We have $P(Test_1 = g|C_1 = g) = .85$ and $P(Test_1 = g|C_1 = b) = .35$. We compute $\langle P(C_1 = g|Test_1 = g), P(C_1 = b|Test_1 = g)\rangle$ using

  $$P(C_1|Test_1 = g) = \alpha P(C_1, Test_1 = g) = \alpha P(C_1) * P(Test_1 = g|C_1)$$

  and obtain

  $$\alpha\langle .7 * .85, .3 * .35\rangle = \langle .7 * .85, .3 * .35\rangle / .7 = \langle .85, .15\rangle$$

  We need to compute $E(U_{C_1}) = P(C_1 = g|Test_1 = g) * (2000 - 1500 - 50) + P(C_1 = b|Test_1 = g) * (2000 - 1500 - 50 - 700) = 345$
- $BuyD = 2$: $Test_1$ is independent of C_2. So we use $P(C_2 = g) = .65$ and $P(C_2 = b) = .35$. We need to compute $E(U_{C_2}) = P(C_2 = g) * (1400 - 1150 - 50) + P(C_2 = b) * (1400 - 1150 - 50 - 150) = 147.5$.

So we should buy car 1.
