Stochastic Processes

Lecture Notes

Week 04: Poisson Processes

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https://stoch-sut.github.io/

Contents

1 Introduction to Poisson Processes

1.1 Outline of Week 04 Lectures

- Poisson Process
- Point Process (Ignored as per instructions)

2 Binomial and Poisson Distributions

2.1 Binomial Distribution

- A random variable X follows a Binomial distribution with parameters n and p, denoted as $X \sim \text{Binomial}(n, p)$, if:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, 2, \ldots, n$$

- This represents the probability of exactly k successes in n independent Bernoulli trials with success probability p.

2.2 Relation to Poisson Distribution

- The Poisson distribution can be derived as the limiting case of the Binomial distribution when:

$$n \to \infty, \quad \text{and} \quad \lambda = np \quad \text{remains constant}$$

- Under these conditions, $X \sim \text{Binomial}(n, p)$ converges to $X \sim \text{Poisson}(\lambda)$.

Example 1 (Poisson Approximation of Binomial)**. *Poisson Approximation of Binomial Random Variables***

Problem: *Show that as $n \to \infty$ and $p \to 0$ such that $\lambda = np$ remains constant, the Binomial distribution Binomial(n, p) converges to the Poisson distribution Poisson(λ).*

Solution:

$$\begin{aligned}
\lim_{n\to\infty} P(X = k) &= \lim_{n\to\infty} \binom{n}{k} p^k (1-p)^{n-k} \\
&= \lim_{n\to\infty} \frac{n!}{k!(n-k)!} p^k (1-p)^n (1-p)^{-k} \\
&\approx \lim_{n\to\infty} \frac{n^k}{k!} p^k e^{-\lambda} \quad (\text{using } (1-p)^n \approx e^{-\lambda}) \\
&= \frac{\lambda^k}{k!} e^{-\lambda} \\
&= P(Y = k), \quad Y \sim \text{Poisson}(\lambda)
\end{aligned}$$

Thus, $X \sim \text{Binomial}(n, p)$ converges to $Y \sim \text{Poisson}(\lambda)$ as $n \to \infty$ and $p \to 0$ with $\lambda = np$ constant. □

3 Poisson Processes

3.1 Definition of a Poisson Process

A stochastic process $X(t) = n(0, t)$ represents a *Poisson process* if it satisfies the following properties:

1. *Independent Increments:* The number of arrivals $n(t_1, t_2)$ in the interval (t_1, t_2) of length $\tau = t_2 - t_1$ is a Poisson random variable with parameter $\lambda\tau$, where $\lambda > 0$ is the rate parameter.
$$P(n(t_1, t_2) = k) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}, \quad k = 0, 1, 2, \ldots$$
2. *Stationary Increments:* The number of arrivals in any interval of length τ depends only on τ and not on the location of the interval on the time axis.
3. *No Multiple Events:* In any infinitesimally small interval $(t, t + dt)$, at most one arrival can occur. The probability of more than one arrival in such an interval is negligible.

3.2 Properties of Poisson Processes

- *Mean and Variance:*
$$E[X(t)] = \lambda t$$
$$\mathrm{Var}(X(t)) = \lambda t$$
- *Autocorrelation Function:* For $t_2 > t_1$,
$$R_{XX}(t_1, t_2) = \lambda^2 t_1 t_2 + \lambda \min(t_1, t_2)$$
- *Autocorrelation Simplification:*
$$R_{XX}(t_1, t_2) = \lambda^2 t_1 t_2 + \lambda \min(t_1, t_2)$$

3.3 Autocorrelation Function Derivation

To determine the autocorrelation function $R_{XX}(t_1, t_2)$, assume without loss of generality that $t_2 > t_1$. Then, from the properties of the Poisson process:

$$X(t_1) = n(0, t_1) \quad \text{and} \quad X(t_2) = n(0, t_2)$$

$$n(t_1, t_2) = X(t_2) - X(t_1)$$

Since $n(0, t_1)$ and $n(t_1, t_2)$ are independent Poisson random variables with parameters λt_1 and $\lambda(t_2 - t_1)$, respectively:

$$E[n(0, t_1) \cdot n(t_1, t_2)] = E[n(0, t_1)] \cdot E[n(t_1, t_2)] = \lambda t_1 \cdot \lambda(t_2 - t_1) = \lambda^2 t_1 (t_2 - t_1)$$

The autocorrelation function is:

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)] = E\left[n(0, t_1)(n(0, t_1) + n(t_1, t_2))\right]$$
$$= E[n(0, t_1)^2] + E[n(0, t_1)n(t_1, t_2)]$$
$$= \mathrm{Var}(n(0, t_1)) + [E(n(0, t_1))]^2 + \lambda^2 t_1 (t_2 - t_1)$$

$$= \lambda t_1 + (\lambda t_1)^2 + \lambda^2 t_1 (t_2 - t_1)$$
$$= \lambda^2 t_1 t_2 + \lambda t_1$$

Similarly, for $t_1 > t_2$:

$$R_{XX}(t_1, t_2) = \lambda^2 t_1 t_2 + \lambda t_2$$

Thus, combining both cases:

$$R_{XX}(t_1, t_2) = \lambda^2 t_1 t_2 + \lambda \min(t_1, t_2)$$

Example 2 (Autocorrelation Function of a Poisson Process)**.** ***Problem:*** *Derive the autocorrelation function $R_{XX}(t_1, t_2)$ for a Poisson process $X(t)$ with rate λ.*

Solution:

- *Assume $t_2 > t_1$.*
- *Using the definition of autocorrelation:*

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$$

- *Since $X(t_2) = X(t_1) + n(t_1, t_2)$, where $n(t_1, t_2) \sim$ Poisson$(\lambda(t_2 - t_1))$ and independent of $X(t_1)$:*

$$R_{XX}(t_1, t_2) = E[X(t_1)^2] + E[X(t_1)]E[n(t_1, t_2)]$$

- *For a Poisson random variable n with parameter λt:*

$$E[n] = \lambda t \quad \textit{and} \quad \textit{Var}(n) = \lambda t$$

$$E[n^2] = \textit{Var}(n) + [E(n)]^2 = \lambda t + (\lambda t)^2$$

- *Therefore:*

$$R_{XX}(t_1, t_2) = (\lambda t_1 + (\lambda t_1)^2) + \lambda t_1 \cdot \lambda (t_2 - t_1)$$
$$= \lambda^2 t_1 t_2 + \lambda t_1$$

- *Thus, the autocorrelation function is:*

$$R_{XX}(t_1, t_2) = \lambda^2 t_1 t_2 + \lambda \min(t_1, t_2)$$

4 Poisson Distribution vs. Poisson Processes

4.1 Poisson Distribution

- A *Poisson distribution* is a discrete probability distribution that expresses the probability of a given number of events k occurring in a fixed interval of time or space.
- *Characteristics:*
  - Events occur with a known constant mean rate λ.
  - Events occur independently of the time since the last event.
- *Example:* The number of emails received in an hour can be modeled using a Poisson distribution if emails arrive independently and at a constant average rate.

4.2 Poisson Process

- A *Poisson process* is a stochastic process that models a series of events occurring randomly over time or space.
- *Characteristics:*
  - Events occur independently.
  - The average rate λ of events is constant over time.
  - The number of events in non-overlapping intervals are independent.
  - The time between consecutive events follows an exponential distribution.
- *Example:* The arrival of customers at a bank can be modeled as a Poisson process if the arrivals are independent and occur at a constant average rate.

5 Autocorrelation Function Example

Example 3 (Autocorrelation of Poisson Process)**.** ***Problem:*** *Given a Poisson process $X(t)$ with rate λ, derive the autocorrelation function $R_{XX}(t_1, t_2)$.*

Solution:

- *Assume $t_2 > t_1$.*
- *$X(t_1) = n(0, t_1)$ and $X(t_2) = n(0, t_2) = n(0, t_1) + n(t_1, t_2)$.*
- *Since $n(0, t_1)$ and $n(t_1, t_2)$ are independent Poisson random variables with parameters λt_1 and $\lambda(t_2 - t_1)$, respectively:*

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)] = E\left[n(0, t_1)(n(0, t_1) + n(t_1, t_2))\right]$$

$$\begin{aligned} &= E[n(0, t_1)^2] + E[n(0, t_1)]E[n(t_1, t_2)] \\ &= \lambda t_1 + (\lambda t_1)^2 + \lambda^2 t_1(t_2 - t_1) \\ &= \lambda^2 t_1 t_2 + \lambda t_1 \end{aligned}$$

- *Therefore, the autocorrelation function is:*

$$R_{XX}(t_1, t_2) = \lambda^2 t_1 t_2 + \lambda \min(t_1, t_2)$$

6 Inter-arrival Distribution for Poisson Processes

6.1 Exponential Distribution of Inter-arrival Times

- Let τ denote the time interval (delay) to the first arrival from a fixed point t_0. The probability distribution of τ is:

$$P(\tau > t) = P(n(t_0, t_0 + t) = 0) = e^{-\lambda t}$$

- The cumulative distribution function (CDF) of τ is:

$$F_\tau(t) = 1 - e^{-\lambda t}, \quad t \geq 0$$

- The probability density function (PDF) of τ is:

$$f_\tau(t) = \frac{d}{dt}F_\tau(t) = \lambda e^{-\lambda t}, \quad t \geq 0$$

- Thus, τ is an exponential random variable with parameter λ:

$$\tau \sim \text{Exponential}(\lambda)$$

- *Expectation:*

$$E[\tau] = \frac{1}{\lambda}$$

Example 4 (Inter-arrival Times)**.** ***Problem:*** *Show that the inter-arrival times of a Poisson process $X(t)$ with rate λ are independent exponential random variables with parameter λ.*
Solution:

- *For the first inter-arrival time τ_1:*

$$P(\tau_1 > t) = P(X(t) = 0) = e^{-\lambda t}$$

$$F_{\tau_1}(t) = 1 - e^{-\lambda t}$$
$$f_{\tau_1}(t) = \lambda e^{-\lambda t}$$

- *For the n-th inter-arrival time τ_n, given the Poisson process has no memory:*

$$P(\tau_n > t \mid \textit{history}) = P(X(t_0, t_0 + t) = 0) = e^{-\lambda t}$$

$$f_{\tau_n}(t) = \lambda e^{-\lambda t}$$

- *Thus, all inter-arrival times $\tau_1, \tau_2, \ldots$ are independent and identically distributed exponential random variables with parameter λ.*

6.2 Gamma Distribution of Waiting Times

- The waiting time until the n-th arrival in a Poisson process follows a *Gamma distribution.*
- Specifically, if T_n is the time until the n-th arrival, then:

$$T_n \sim \text{Gamma}(n, \lambda)$$

  where the Gamma distribution has shape parameter n and rate parameter λ.

- *Probability Density Function (PDF):*

$$f_{T_n}(t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}, \quad t \geq 0$$

- *Expectation:*

$$E[T_n] = \frac{n}{\lambda}$$

- *Variance:*

$$\text{Var}(T_n) = \frac{n}{\lambda^2}$$

Example 5 (Waiting Time Until n-th Arrival)**.** ***Problem:*** *Show that the waiting time T_n until the n-th arrival in a Poisson process $X(t)$ with rate λ follows a Gamma distribution with parameters n and λ.*

Solution:

- *The waiting time T_n is the sum of n independent inter-arrival times $\tau_1, \tau_2, \ldots, \tau_n$, each exponentially distributed with parameter λ:*

$$T_n = \tau_1 + \tau_2 + \cdots + \tau_n$$

- *The sum of n independent exponential random variables with parameter λ follows a Gamma distribution with shape parameter n and rate parameter λ:*

$$T_n \sim Gamma(n, \lambda)$$

- *The PDF of T_n is:*

$$f_{T_n}(t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}, \quad t \geq 0$$

- *Therefore, T_n follows a Gamma distribution with parameters n and λ.*

6.3 Properties of Gamma Distribution in Poisson Processes

- *Sum of Independent Exponentials:* As the inter-arrival times τ_i are independent and exponentially distributed, their sum T_n follows a Gamma distribution.
- *Memoryless Property:* While individual inter-arrival times are memoryless, the sum T_n does not possess the memoryless property unless $n = 1$.

7 Sum of Independent Poisson Processes

7.1 Properties

- If $X_1(t)$ and $X_2(t)$ are two independent Poisson processes with rates λ_1 and λ_2, respectively, then their sum $X(t) = X_1(t) + X_2(t)$ is also a Poisson process with rate $\lambda = \lambda_1 + \lambda_2$.
- *Proof:*

$$\begin{aligned}
P(X(t) = k) &= \sum_{m=0}^{k} P(X_1(t) = m) P(X_2(t) = k - m) \\
&= \sum_{m=0}^{k} \frac{(\lambda_1 t)^m e^{-\lambda_1 t}}{m!} \cdot \frac{(\lambda_2 t)^{k-m} e^{-\lambda_2 t}}{(k-m)!} \\
&= e^{-(\lambda_1+\lambda_2)t} \frac{(\lambda_1 + \lambda_2)^k t^k}{k!} \quad \text{where } \lambda = \lambda_1 + \lambda_2 \\
&= \frac{(\lambda t)^k e^{-\lambda t}}{k!}
\end{aligned}$$

- Thus, $X(t) \sim \text{Poisson}(\lambda t)$.

Example 6 (Sum of Independent Poisson Processes)**.** ***Problem:*** *If $X_1(t) \sim Poisson(\lambda_1 t)$ and $X_2(t) \sim Poisson(\lambda_2 t)$ are independent, show that $X(t) = X_1(t) + X_2(t) \sim Poisson((\lambda_1 + \lambda_2)t)$.*

Solution:

- *Using the properties of Poisson distributions and independence:*

$$P(X(t) = k) = \sum_{m=0}^{k} P(X_1(t) = m)P(X_2(t) = k - m)$$

$$= \sum_{m=0}^{k} \frac{(\lambda_1 t)^m e^{-\lambda_1 t}}{m!} \cdot \frac{(\lambda_2 t)^{k-m} e^{-\lambda_2 t}}{(k-m)!}$$

$$= e^{-(\lambda_1+\lambda_2)t} \sum_{m=0}^{k} \frac{(\lambda_1 t)^m (\lambda_2 t)^{k-m}}{m!(k-m)!}$$

$$= e^{-\lambda t} \frac{(\lambda_1 + \lambda_2)^k t^k}{k!} \quad \textit{where } \lambda = \lambda_1 + \lambda_2$$

- *Hence, $X(t) \sim Poisson(\lambda t)$.*

8 Random Selection of Poisson Points

8.1 Splitting a Poisson Process

- Consider a Poisson process $X(t)$ with rate λ. Suppose each arrival in $X(t)$ is independently assigned to one of two new Poisson processes $Y(t)$ and $Z(t)$ with probabilities p and $q = 1 - p$, respectively.
- *Result:* Both $Y(t)$ and $Z(t)$ are independent Poisson processes with rates λp and λq, respectively.

Example 7 (Splitting Poisson Processes)**.** ***Problem:*** *Let $X(t) \sim Poisson(\lambda t)$. Each arrival is independently assigned to $Y(t)$ with probability p and to $Z(t)$ with probability $q = 1 - p$. Show that $Y(t) \sim Poisson(\lambda pt)$ and $Z(t) \sim Poisson(\lambda qt)$, and that $Y(t)$ and $Z(t)$ are independent.*
Solution:

- *Given $X(t) = n$, the number of arrivals in $Y(t)$ follows a Binomial distribution:*

$$Y(t) \mid X(t) = n \sim Binomial(n, p)$$

- *Using the Poisson approximation of the Binomial distribution as $n \to \infty$ and $p \to 0$ with λp constant, we have:*

$$Y(t) \mid X(t) = n \approx Poisson(\lambda pt)$$

- *Since each arrival is independently assigned, $Y(t)$ and $Z(t)$ are independent.*
- *Therefore, $Y(t) \sim Poisson(\lambda pt)$ and $Z(t) \sim Poisson(\lambda qt)$.*

9 Coupon Collecting Example

9.1 Problem Statement

- Suppose a cereal manufacturer randomly inserts a sample of one type of coupon into each cereal box. There are n distinct types of coupons. The question is: How many boxes of cereal should one buy on average to collect at least one coupon of each kind?

9.2 Reformulation Using Poisson Processes

- Let $X_i(t)$ for $i = 1, 2, \dots, n$ represent n independent Poisson processes with a common rate λ.
- Let T_i be the time of the first arrival in $X_i(t)$. Then, $T_i \sim$ Exponential(λ).
- The waiting time until the first coupon of each type is collected corresponds to the maximum of these inter-arrival times.

Example 8 (Coupon Collecting Using Poisson Processes)**.** ***Problem:*** *Reformulate the coupon collecting problem in terms of Poisson processes and determine the expected number of cereal boxes one needs to buy to collect at least one of each type of coupon.*

Solution:

- *Consider n independent Poisson processes $X_i(t)$ with rate λ each, representing the arrivals of coupons of type i.*
- *The time until the first coupon of type i is collected is $T_i \sim$ Exponential(λ).*
- *The time until all coupons are collected is $T = \max(T_1, T_2, \dots, T_n)$.*
- *The expected time until all coupons are collected is given by:*

$$E[T] = \sum_{i=1}^{n} \frac{1}{\lambda i}$$

- *Therefore, the expected number of cereal boxes to buy (assuming one box per unit time) is:*

$$E[T] = \frac{1}{\lambda} \sum_{i=1}^{n} \frac{1}{i}$$

- *This is analogous to the classic coupon collector's problem where:*

$$E[T] = \frac{n}{\lambda} \sum_{i=1}^{n} \frac{1}{i} = \frac{n}{\lambda} H_n$$

where H_n is the n-th harmonic number.

10 Compound Poisson Processes

10.1 Definition

- A *Compound Poisson Process* $X(t)$ is defined as the sum of random variables associated with each arrival in a Poisson process. Formally:

$$X(t) = \sum_{i=1}^{N(t)} C_i$$

where:

  - $N(t)$ is a Poisson process with rate λ.
  - C_i are independent and identically distributed (i.i.d.) random variables representing the "marks" or "sizes" associated with each event.

10.2 Properties

- *Mean:*
$$E[X(t)] = E[N(t)]E[C] = \lambda t E[C]$$
- *Variance:*
$$\mathrm{Var}(X(t)) = \lambda t E[C^2]$$
- *Autocorrelation Function:*
$$R_{XX}(t_1, t_2) = \lambda^2 t_1 t_2 E[C]^2 + \lambda \min(t_1, t_2) E[C^2]$$

Example 9 (Compound Poisson Process)**.** ***Problem:*** *Let $X(t)$ be a compound Poisson process with rate λ and i.i.d. marks C_i having mean μ_C and variance σ_C^2. Derive the mean and variance of $X(t)$.*

Solution:

- *Mean:*
$$E[X(t)] = E\left[\sum_{i=1}^{N(t)} C_i\right] = E[N(t)]E[C] = \lambda t \mu_C$$
- *Variance:*
$$Var(X(t)) = E[X(t)^2] - [E(X(t))]^2$$
$$E[X(t)^2] = E\left[\left(\sum_{i=1}^{N(t)} C_i\right)^2\right] = E\left[\sum_{i=1}^{N(t)} C_i^2 + \sum_{i\neq j} C_i C_j\right]$$
$$= E[N(t)]E[C^2] + [E[N(t)^2] - E[N(t)][E[C]]^2$$
$$= \lambda t E[C^2] + (\lambda t)^2 \mu_C^2$$
$$Var(X(t)) = E[X(t)^2] - [E(X(t))]^2 = \lambda t E[C^2] + (\lambda t)^2 \mu_C^2 - (\lambda t \mu_C)^2$$
$$= \lambda t E[C^2] = \lambda t(\sigma_C^2 + \mu_C^2)$$

10.3 Gamma Distribution in Compound Poisson Processes

- The waiting time until the n-th arrival in a Poisson process follows a *Gamma distribution.*
- Specifically, if T_n is the time until the n-th arrival, then:
$$T_n \sim \mathrm{Gamma}(n, \lambda)$$
where the Gamma distribution has shape parameter n and rate parameter λ.

11 Bulk Arrivals and Compound Poisson Processes

11.1 Definition

- In some scenarios, multiple events can occur simultaneously as a cluster at each arrival instant of a Poisson process. Such processes are called *Compound Poisson Processes* or *Bulk Arrival Processes.*

- *Definition:* Let $X(t)$ represent the total number of all occurrences in the interval $(0, t)$. Then:
$$X(t) = \sum_{i=1}^{N(t)} C_i$$
where $N(t) \sim \text{Poisson}(\lambda t)$ and C_i are the number of events in each cluster.

11.2 Applications

- Inventory orders
- Arrivals at an airport queue
- Ticket purchases for a show
- Any scenario where events occur in batches or clusters

12 Fitting and Sampling from Poisson Processes

12.1 Fitting by Maximum Likelihood

- The maximum likelihood estimation (MLE) for the rate λ of a Poisson process given observed data is straightforward.
- Given N observations $\{t_1, t_2, \ldots, t_N\}$ within a time interval $[0, T]$:
$$L(\lambda) = P(N(T) = N) = \frac{(\lambda T)^N e^{-\lambda T}}{N!}$$
- Taking the natural logarithm:
$$\log L(\lambda) = N \log(\lambda T) - \lambda T - \log(N!)$$
- Differentiating and setting to zero:
$$\frac{d}{d\lambda} \log L(\lambda) = \frac{N}{\lambda} - T = 0 \Rightarrow \hat{\lambda} = \frac{N}{T}$$

12.2 Sampling Using Inversion Sampling

- *Inversion Sampling:* A method to generate random samples from any probability distribution given its cumulative distribution function (CDF).
- *Steps:*
  1. Generate a uniform random sample u from $U(0, 1)$.
  2. Use the inverse CDF $F^{-1}(u)$ to obtain the sample from the target distribution.

12.3 Rejection Sampling (Thinning)

- *Rejection Sampling:* Also known as the acceptance-rejection method, used to generate observations from a target distribution by using a proposal distribution.

- *Steps:*

  1. Choose a proposal distribution $g(x)$ from which it is easy to sample and covers the support of the target distribution $f(x)$.
  2. Generate a sample x from $g(x)$.
  3. Generate a uniform random variable $u \sim U(0, 1)$.
  4. Accept the sample x if $u \leq \frac{f(x)}{Mg(x)}$, where M is a constant such that $f(x) \leq Mg(x)$ for all x.
  5. Repeat until a sample is accepted.

Example 10 (Sampling from an Exponential Distribution)**.** ***Problem:*** *Use inversion sampling to generate a sample from an Exponential distribution with parameter* λ*.*
Solution:

- *Generate* $u \sim U(0, 1)$*.*

- *Use the inverse CDF of the Exponential distribution:*

$$F^{-1}(u) = -\frac{1}{\lambda} \ln(1 - u)$$

- *Since* $1 - u \sim U(0, 1)$*, we can simplify:*

$$X = -\frac{1}{\lambda} \ln(u)$$

- *Thus,* $X \sim$ *Exponential*(λ)*.*

13 Summary of Poisson Processes

- *Definition:* A Poisson process $X(t)$ with rate λ is a counting process with independent and stationary increments, where $X(t) \sim \text{Poisson}(\lambda t)$.

- *Properties:*

  – $E[X(t)] = \lambda t$
  – $\text{Var}(X(t)) = \lambda t$
  – Autocorrelation function:

$$R_{XX}(t_1, t_2) = \lambda^2 t_1 t_2 + \lambda \min(t_1, t_2)$$

- *Inter-arrival Times:* The time between consecutive arrivals follows an exponential distribution with parameter λ. The waiting time until the n-th arrival follows a Gamma distribution with parameters n and λ.

- *Compound Poisson Processes:* Sum of random variables associated with each arrival, leading to more complex stochastic models.

- *Sum of Independent Poisson Processes:* If $X_1(t)$ and $X_2(t)$ are independent Poisson processes with rates λ_1 and λ_2, then $X(t) = X_1(t) + X_2(t)$ is a Poisson process with rate $\lambda = \lambda_1 + \lambda_2$.
- *Splitting Poisson Processes:* Randomly splitting a Poisson process into independent Poisson processes based on a Bernoulli trial preserves the Poisson property.

14 Conclusion

This document provides a detailed overview of **Poisson Processes**, including their definition, properties, relation to the Poisson distribution, autocorrelation function, inter-arrival times with Gamma distribution, compound Poisson processes, and the sum of independent Poisson processes. Comprehensive examples and solutions are included to illustrate key concepts and derivations.

For further reading and more in-depth explanations, refer to standard textbooks on stochastic processes or your course materials.