Stochastic Processes

Lecture Notes

Week 03: Ergodic Stochastic Processes

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https://stoch-sut.github.io/

Contents

1 Ergodic Stochastic Processes

1.1 Definition of Ergodicity

- A random process $X(t)$ is ergodic if all of its statistics can be determined from a single sample function (sample path) of the process.
- Mathematically, this means that ensemble averages equal their corresponding time averages with probability one.

1.2 Illustration of Ergodicity

- Consider a single realization of $X(t)$ (a sample path).
- Statistics such as mean, variance, and autocorrelation can be estimated by averaging over time from this single realization.
- Diagrammatic Representation:
  - Sample Path: A single trajectory of $X(t)$ over time.
  - Ensemble: The set of all possible trajectories of $X(t)$.

1.3 Ergodicity and Stationarity

- Wide-Sense Stationary (WSS):
  - Mean $E[X(t)] = \mu_X$ is constant over time.
  - Autocorrelation $R_{XX}(t_1, t_2) = R_{XX}(\tau)$ depends only on the time difference $\tau = t_1 - t_2$.
- Strictly Stationary (SSS):
  - All statistical properties (not just mean and autocorrelation) are invariant to time shifts.
- Relationship:
  - Ergodic processes are generally both SSS and WSS.
  - SSS implies WSS, but WSS does not necessarily imply SSS unless additional conditions (like Gaussianity) are met.

1.4 Weak Forms of Ergodicity

- Complete statistics estimation is often difficult; thus, focus is usually on:
  - Ergodicity in Mean: $E[X(t)] = \langle x(t) \rangle$
  - Ergodicity in Autocorrelation: $R_{XX}(\tau) = \langle x(t+\tau)x(t) \rangle$

2 Ergodicity in Mean

2.1 Definition

- A random process $X(t)$ is ergodic in mean if:

$$E[X(t)] = \langle x(t) \rangle$$

- Where $\langle \cdot \rangle$ denotes time-averaging:

$$\langle x(t) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^T x(t)\, dt$$

- Necessary and Sufficient Condition:

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T C(\tau)\, d\tau = 0$$

  where $C(\tau) = R_{XX}(\tau) - \mu_X^2$ is the autocovariance function.

- This condition ensures that the covariance between $X(t)$ and its time-shifted version $X(t+\tau)$ diminishes as τ increases, leading the time average to converge to the ensemble mean.

2.2 Example 1-a: Ergodic in Mean

Example 1 (Ergodic in Mean)**.** *Consider the stochastic process:*

$$X(t) = a\cos(\omega_0 t + \theta)$$

where:

- θ *is a random variable uniformly distributed over* $[0, 2\pi]$
- t *is the time index*
- a *and* ω_0 *are constant variables*
- $X(t)$ *is a WSS process with mean zero.*
- *Mean is independent of the random variable* θ*.*

Solution:

- **Mean**:

$$E[X(t)] = E[a\cos(\omega_0 t + \theta)] = a \cdot E[\cos(\omega_0 t + \theta)] = a \cdot 0 = 0$$

- **Time Average**:

$$\langle x(t) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^T a\cos(\omega_0 t + \theta)\, dt = 0$$

- Therefore, $E[X(t)] = \langle x(t) \rangle = 0$, satisfying ergodicity in mean.

- **Autocovariance Condition**:

$$C(\tau) = R_{XX}(\tau) - \mu_X^2 = R_{XX}(\tau) - 0 = R_{XX}(\tau)$$

- Since $X(t)$ is WSS and $R_{XX}(\tau)$ for a cosine process with uniformly distributed phase is:

$$R_{XX}(\tau) = \frac{a^2}{2}\cos(\omega_0\tau)$$

- Evaluate the necessary condition:

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{a^2}{2}\cos(\omega_0\tau)\,d\tau = \frac{a^2}{2}\lim_{T\to\infty}\frac{1}{T}\cdot\frac{\sin(\omega_0 T)}{\omega_0} = 0$$

- Hence, the condition is satisfied, confirming that $X(t)$ is ergodic in mean.

$\square$

2.3 Example 1-a Continued: Non-Ergodic in Mean

Example 2 (Non-Ergodic in Mean)**.** *Consider the stochastic process:*

$$X(t) = a\cos(\omega_0 t + \theta) + c_r$$

where:

- θ *is a random variable uniformly distributed over* $[0, 2\pi]$
- c_r *is a random variable*
- a *and* ω_0 *are constant variables*

Solution:

- **Mean**:

$$E[X(t)] = E[a\cos(\omega_0 t + \theta) + c_r] = 0 + \mu_{c_r} = \mu_{c_r}$$

- **Time Average**:

$$\langle x(t)\rangle = \lim_{T\to\infty}\frac{1}{T}\int_0^T (a\cos(\omega_0 t + \theta) + c_r)\,dt = 0 + c_r = c_r$$

- **Autocovariance Condition**:

$$C(\tau) = R_{XX}(\tau) - \mu_X^2 = R_{XX}(\tau) - \mu_{c_r}^2$$

- The autocorrelation function $R_{XX}(\tau)$ becomes:

$$R_{XX}(\tau) = \frac{a^2}{2}\cos(\omega_0\tau) + \mathrm{Cov}(c_r, a\cos(\omega_0 t + \theta) + c_r)$$

$$= \frac{a^2}{2}\cos(\omega_0\tau) + \mathrm{Cov}(c_r, c_r) = \frac{a^2}{2}\cos(\omega_0\tau) + \mathrm{Var}(c_r)$$

- Therefore:

$$C(\tau) = \frac{a^2}{2}\cos(\omega_0\tau) + \mathrm{Var}(c_r) - \mu_{c_r}^2$$

- Evaluate the necessary condition:

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\left(\frac{a^2}{2}\cos(\omega_0\tau)+\mathrm{Var}(c_r)-\mu_{c_r}^2\right)d\tau=\lim_{T\to\infty}\left(\frac{a^2}{2T}\int_0^T\cos(\omega_0\tau)\,d\tau+\frac{1}{T}(\mathrm{Var}(c_r)-\mu_{c_r}^2)\cdot T\right)$$

$$=\lim_{T\to\infty}\left(\frac{a^2}{2T}\cdot\frac{\sin(\omega_0 T)}{\omega_0}+\mathrm{Var}(c_r)-\mu_{c_r}^2\right)=0+\mathrm{Var}(c_r)-\mu_{c_r}^2$$

- Unless $\mathrm{Var}(c_r)-\mu_{c_r}^2=0$ (i.e., c_r is deterministic), the condition is not satisfied.
- Therefore, $X(t)$ is **not ergodic in mean**.

□

2.4 Example 2: Non-Ergodic in Mean

Example 3 (Non-Ergodic in Mean)**.** *Let C be a random variable, and define the process:*

$$X(t)=C$$

- *Is $X(t)$ mean ergodic?*

Solution:

- **Ensemble Average**:

$$E[X(t)]=E[C]=\mu_C$$

- **Time Average**:

$$\langle X(t)\rangle=\lim_{T\to\infty}\frac{1}{T}\int_0^T X(t)\,dt=\lim_{T\to\infty}\frac{1}{T}\int_0^T C\,dt=C$$

- **Autocovariance Condition**:

$$C(\tau)=R_{XX}(\tau)-\mu_X^2=E[X(t)X(t+\tau)]-\mu_C^2=E[C^2]-\mu_C^2=\mathrm{Var}(C)$$

- Evaluate the necessary condition:

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\mathrm{Var}(C)\,d\tau=\mathrm{Var}(C)\neq 0$$

- Since $\mathrm{Var}(C)>0$, the condition is not satisfied.
- Therefore, $X(t)$ is **not ergodic in mean**.

□

3 Ergodicity in Autocorrelation

3.1 Definition

- A process is ergodic in autocorrelation if the autocorrelation function can be determined by time averaging a single realization:

$$R_{XX}(\tau) = \langle x(t+\tau)x(t)\rangle$$

- Where:

$$\langle x(t+\tau)x(t)\rangle = \lim_{T\to\infty} \frac{1}{T}\int_0^T x(t+\tau)x(t)\,dt$$

- Necessary and Sufficient Condition:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T C(\tau)\,d\tau = 0$$

where $C(\tau) = R_{XX}(\tau) - \mu_X^2$

3.2 Example 3: Ergodic in Autocorrelation

Example 4 (Ergodic in Autocorrelation)**.** *Consider the stochastic process:*

$$X(t) = A\cos(2\pi f_c t + \theta)$$

where:

- *A and f_c are constants*
- *θ is a random variable uniformly distributed over* $[0, 2\pi]$
- *The autocorrelation of $X(t)$ is:*

$$R_{XX}(\tau) = \frac{A^2}{2}\cos(2\pi f_c \tau)$$

- *Determine if $X(t)$ is ergodic in autocorrelation.*

Solution:

- **Autocorrelation of Sample Function**:

$$\langle X(t+\tau)X(t)\rangle = \lim_{T\to\infty} \frac{1}{T}\int_0^T A\cos(2\pi f_c(t+\tau)+\theta)\cdot A\cos(2\pi f_c t+\theta)\,dt$$

- Using trigonometric identities:

$$\cos(a)\cos(b) = \frac{1}{2}[\cos(a-b)+\cos(a+b)]$$

- Substitute $a = 2\pi f_c(t+\tau)+\theta$ and $b = 2\pi f_c t + \theta$:

$$\cos(2\pi f_c(t+\tau)+\theta)\cos(2\pi f_c t+\theta) = \frac{1}{2}\left[\cos(2\pi f_c \tau) + \cos(4\pi f_c t + 2\theta + 2\pi f_c \tau)\right]$$

- Therefore:

$$\begin{aligned}\langle X(t+\tau)X(t)\rangle &= \lim_{T\to\infty} \frac{A^2}{2T} \int_0^T \left[\cos(2\pi f_c \tau) + \cos(4\pi f_c t + 2\theta + 2\pi f_c \tau)\right] dt \\ &= \frac{A^2}{2T} \left[\cos(2\pi f_c \tau)\cdot T + \int_0^T \cos(4\pi f_c t + 2\theta + 2\pi f_c \tau)\, dt\right] \\ &= \frac{A^2}{2}\cos(2\pi f_c \tau) + \frac{A^2}{2T}\cdot \left[\frac{\sin(4\pi f_c t + 2\theta + 2\pi f_c \tau)}{4\pi f_c}\right]_0^T\end{aligned}$$

- As $T \to \infty$, the second term averages out to zero:

$$\lim_{T\to\infty} \frac{\sin(4\pi f_c T + 2\theta + 2\pi f_c \tau) - \sin(2\theta + 2\pi f_c \tau)}{8\pi f_c T} = 0$$

- Therefore:

$$\langle X(t+\tau)X(t)\rangle = \frac{A^2}{2}\cos(2\pi f_c \tau)$$

- Comparing with the ensemble autocorrelation:

$$R_{XX}(\tau) = \frac{A^2}{2}\cos(2\pi f_c \tau)$$

- Hence, $X(t)$ is **ergodic in autocorrelation**.

□

3.3 Example 4: Non-Ergodic in Mean

Example 5 (Non-Ergodic in Mean)**.** *Let $Y(t) = X(t) + A$, where:*

- *$X(t)$ is a WSS Gaussian process with $E[X(t)] = 0$ and $R_{XX}(\tau) = e^{-\tau}$*
- *A is a Gaussian random variable, $A \sim \mathcal{N}(0,1)$, independent of $X(t)$*
- *Is $Y(t)$ mean ergodic?*

Solution:

- **Ensemble Average**:

$$E[Y(t)] = E[X(t) + A] = E[X(t)] + E[A] = 0 + 0 = 0$$

- **Time Average**:

$$\langle Y(t)\rangle = \lim_{T\to\infty} \frac{1}{T}\int_0^T (X(t) + A)\, dt = \lim_{T\to\infty} \frac{1}{T}\int_0^T X(t)\, dt + A = 0 + A = A$$

- **Autocovariance Condition**:

$$C(\tau) = R_{YY}(\tau) - \mu_Y^2 = R_{YY}(\tau) - 0 = R_{YY}(\tau)$$

- The autocorrelation function $R_{YY}(\tau)$ is:

$$R_{YY}(\tau) = R_{XX}(\tau) + R_{AA}(\tau) = e^{-\tau} + \delta(\tau)$$

where $R_{AA}(\tau) = E[A^2]\delta(\tau) = 1 \cdot \delta(\tau)$

- Evaluate the necessary condition:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T R_{YY}(\tau)\, d\tau = \lim_{T\to\infty} \frac{1}{T} \left(\int_0^T e^{-\tau}\, d\tau + \int_0^T \delta(\tau)\, d\tau \right) = \lim_{T\to\infty} \left(\frac{1 - e^{-T}}{T} + \frac{1}{T} \right) = 0$$

- However, the time average $\langle Y(t) \rangle = A$ remains a random variable with $E[A] = 0$ and $\mathrm{Var}(A) = 1$.
- Since $\langle Y(t) \rangle$ does not converge to the ensemble mean $E[Y(t)] = 0$ almost surely (due to the randomness of A), the condition is not satisfied in practical terms.
- Therefore, $Y(t)$ is **not mean ergodic**.

□

4 Fourier Transforms

4.1 Definitions

- Fourier Transform (FT) of a function $x(t)$:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

- Inverse Fourier Transform (IFT):

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega$$

4.2 Properties of Fourier Transforms

- Linearity:

$$\mathcal{F}\{ax(t) + by(t)\} = aX(\omega) + bY(\omega)$$

- Time Shifting:

$$\mathcal{F}\{x(t - t_0)\} = X(\omega) e^{-j\omega t_0}$$

- Frequency Shifting:

$$\mathcal{F}\{x(t) e^{j\omega_0 t}\} = X(\omega - \omega_0)$$

- Scaling:

$$\mathcal{F}\{x(at)\} = \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

- Conjugation:

$$\mathcal{F}\{x^*(t)\} = X^*(-\omega)$$

- Duality:
$$\mathcal{F}\{X(t)\} = 2\pi x(-\omega)$$
- Convolution:
$$\mathcal{F}\{x(t) * y(t)\} = X(\omega)Y(\omega)$$
- Multiplication:
$$\mathcal{F}\{x(t)y(t)\} = \frac{1}{2\pi}\left(X(\omega) * Y(\omega)\right)$$
- Parseval's Theorem:
$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega$$

4.3 Table of Important Fourier Transforms

Function	**Fourier Transform**	**Inverse FT**
Delta Function	$\delta(t)$	1
Constant	1	$2\pi\delta(\omega)$
Rectangular Pulse	$\text{rect}(t)$	$2\pi\text{sinc}(\omega)$
Exponential Decay	$e^{-at}u(t)$	$\frac{1}{a+j\omega}$
Cosine	$\cos(\omega_0 t)$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
Sine	$\sin(\omega_0 t)$	$j\pi[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$

4.4 Example 5: Fourier Transform Application

Example 6 (Fourier Transform of a Sine Wave)**.** *Find the Fourier transform of* $x(t) = \sin(\omega_0 t)$*.*

Solution:

- Using Euler's formula:
$$\sin(\omega_0 t) = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$
- Fourier transform:
$$\mathcal{F}\{\sin(\omega_0 t)\} = \frac{1}{2j}\left[\mathcal{F}\{e^{j\omega_0 t}\} - \mathcal{F}\{e^{-j\omega_0 t}\}\right]$$
- Using the Fourier transform of exponentials:
$$\mathcal{F}\{e^{j\omega_0 t}\} = 2\pi\delta(\omega-\omega_0)$$
$$\mathcal{F}\{e^{-j\omega_0 t}\} = 2\pi\delta(\omega+\omega_0)$$
- Substituting back:
$$\mathcal{F}\{\sin(\omega_0 t)\} = \frac{1}{2j}\left[2\pi\delta(\omega-\omega_0) - 2\pi\delta(\omega+\omega_0)\right] = \pi\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]\frac{1}{j}$$
- Simplifying using $j = \sqrt{-1}$:
$$\mathcal{F}\{\sin(\omega_0 t)\} = j\pi\left[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right]$$

□

4.5 Example 6: Inverse Fourier Transform Application

Example 7 (Inverse Fourier Transform of a Rectangular Pulse). *Find the inverse Fourier transform of* $X(\omega) = 2\pi rect\left(\frac{\omega}{2B}\right)$.

Solution:

- The inverse Fourier transform of $X(\omega) = 2\pi\text{rect}\left(\frac{\omega}{2B}\right)$ is:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\text{rect}\left(\frac{\omega}{2B}\right)e^{j\omega t}\,d\omega = \int_{-\infty}^{\infty}\text{rect}\left(\frac{\omega}{2B}\right)e^{j\omega t}\,d\omega$$

- The rectangular function $\text{rect}\left(\frac{\omega}{2B}\right)$ is defined as:

$$\text{rect}\left(\frac{\omega}{2B}\right) = \begin{cases} 1 & |\omega| \le B \\ 0 & \text{otherwise} \end{cases}$$

- Therefore:

$$x(t) = \int_{-B}^{B} e^{j\omega t}\,d\omega = \frac{e^{jBt} - e^{-jBt}}{jt} = \frac{2\sin(Bt)}{t} = 2\cdot\text{sinc}\left(\frac{Bt}{\pi}\right)$$

- Thus:

$$x(t) = 2\cdot\text{sinc}\left(\frac{Bt}{\pi}\right)$$

□

4.6 Fourier Transform Properties

- Linearity:

$$\mathcal{F}\{ax(t) + by(t)\} = aX(\omega) + bY(\omega)$$

- Time Shifting:

$$\mathcal{F}\{x(t - t_0)\} = X(\omega)e^{-j\omega t_0}$$

- Frequency Shifting:

$$\mathcal{F}\{x(t)e^{j\omega_0 t}\} = X(\omega - \omega_0)$$

- Scaling:

$$\mathcal{F}\{x(at)\} = \frac{1}{|a|}X\left(\frac{\omega}{a}\right)$$

- Conjugation:

$$\mathcal{F}\{x^*(t)\} = X^*(-\omega)$$

- Duality:

$$\mathcal{F}\{X(t)\} = 2\pi x(-\omega)$$

- Convolution:

$$\mathcal{F}\{x(t) * y(t)\} = X(\omega)Y(\omega)$$

- Multiplication:

$$\mathcal{F}\{x(t)y(t)\} = \frac{1}{2\pi}\left(X(\omega) * Y(\omega)\right)$$

- Parseval's Theorem:

$$\int_{-\infty}^{\infty} |x(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\,d\omega$$

5 Power Spectrum

5.1 Definition for Deterministic Signals

- For a deterministic signal $x(t)$, the Fourier transform $X(\omega)$ is defined as:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt$$

- The energy spectrum is given by:

$$|X(\omega)|^2$$

- By Parseval's Theorem:

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega$$

- Therefore, $|X(\omega)|^2$ represents the energy distribution across frequencies.

5.2 Power Spectrum for Stochastic Processes

- For stochastic processes, directly applying the Fourier transform to $X(t)$ yields a sequence of random variables for each frequency ω.
- To obtain a meaningful spectral distribution, focus on the autocorrelation function $R_{XX}(\tau)$.

5.3 Power Spectral Density (PSD)

- The power spectral density $S_{XX}(\omega)$ of a WSS process $X(t)$ is the Fourier transform of its autocorrelation function $R_{XX}(\tau)$:

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-j\omega\tau}\, d\tau$$

- Wiener-Khinchin Theorem: The autocorrelation function and the power spectrum form a Fourier transform pair.
- Inverse Fourier Transform:

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega)e^{j\omega\tau}\, d\omega$$

- The total power of the process is:

$$P = R_{XX}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega = E[|X(t)|^2]$$

- Note: The division by 2π is essential to maintain consistency between time and frequency domains.

5.4 Power of $X(t)$

- The power of $X(t)$ can be expressed in various forms:

$$\begin{aligned} P &= E[|X(t)|^2] \\ &= R_{XX}(0) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega \end{aligned}$$

- These representations highlight the relationship between the time-domain autocorrelation and the frequency-domain power spectrum.

5.5 Example 4: Power Spectrum of a WSS Process

Example 8 (Power Spectrum)**.** *Let $X(t)$ be a WSS process with autocorrelation function:*

$$R_{XX}(\tau) = e^{-|\tau|}$$

Find the power spectral density $S_{XX}(\omega)$.

Solution:

- Compute the Fourier transform of $R_{XX}(\tau) = e^{-|\tau|}$:

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} e^{-|\tau|} e^{-j\omega\tau}\, d\tau$$

- Split the integral into two parts:

$$\begin{aligned} S_{XX}(\omega) &= \int_{-\infty}^{0} e^{\tau} e^{-j\omega\tau}\, d\tau + \int_{0}^{\infty} e^{-\tau} e^{-j\omega\tau}\, d\tau \\ &= \int_{-\infty}^{0} e^{(1-j\omega)\tau}\, d\tau + \int_{0}^{\infty} e^{-(1+j\omega)\tau}\, d\tau \end{aligned}$$

- Evaluate the integrals:

$$\int_{-\infty}^{0} e^{(1-j\omega)\tau}\, d\tau = \frac{1}{1-j\omega}$$

$$\int_{0}^{\infty} e^{-(1+j\omega)\tau}\, d\tau = \frac{1}{1+j\omega}$$

- Therefore:

$$S_{XX}(\omega) = \frac{1}{1-j\omega} + \frac{1}{1+j\omega} = \frac{(1+j\omega)+(1-j\omega)}{1+\omega^2} = \frac{2}{1+\omega^2}$$

- Thus:

$$S_{XX}(\omega) = \frac{2}{1+\omega^2}$$

□

5.6 Wiener-Khinchin Theorem

- The autocorrelation function and the power spectral density of a WSS process form a Fourier transform pair:

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau}\, d\tau$$

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) e^{j\omega\tau}\, d\omega$$

- This theorem is fundamental in connecting time-domain and frequency-domain analyses.

5.7 Nonnegative-Definiteness of Autocorrelation Function

- The autocorrelation function $R_{XX}(\tau)$ must satisfy:

$$\int_{-\infty}^{\infty} S_{XX}(\omega) |A(\omega)|^2\, d\omega \geq 0 \quad \forall A(\omega)$$

- This implies that the power spectral density $S_{XX}(\omega)$ is nonnegative for all ω:

$$S_{XX}(\omega) \geq 0 \quad \forall \omega$$

6 Power Spectra and LTI Systems

6.1 Input-Output Relations in LTI Systems

- If a WSS process $X(t)$ with autocorrelation function $R_{XX}(\tau)$ is applied to an LTI system with impulse response $h(t)$, then:
  - Output Mean:
$$\mu_Y(t) = \mu_X(t) * h(t) = \mu_X * h(t)$$
  - Cross-Correlation Function:
$$R_{XY}(\tau) = R_{XX}(\tau) * h(\tau)$$
  - Autocorrelation of the Output Process:
$$R_{YY}(\tau) = R_{XY}(\tau) * h(-\tau) = R_{XX}(\tau) * h(\tau) * h(-\tau)$$
- Frequency Domain Representation:
  - Fourier transform relates autocorrelation functions to power spectra:
$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$

6.2 Transfer Function and Spectrum

- The transfer function $H(\omega)$ of the system is the Fourier transform of the impulse response $h(t)$:
$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\,dt$$
- The output power spectral density $S_{YY}(\omega)$ is related to the input power spectral density $S_{XX}(\omega)$ by:
$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$
- Note: The cross-spectrum $S_{XY}(\omega)$ need not be real or nonnegative, but the output spectrum $S_{YY}(\omega)$ is real and nonnegative.

6.3 Example 5: Power Spectrum of Output through LTI System

Example 9 (Power Spectrum of Output)**.** *A WSS white noise process $W(t)$ with $S_{WW}(\omega) = \sigma^2$ is passed through a low-pass filter (LPF) with transfer function:*

$$H(\omega) = \begin{cases} 1 & |\omega| \le B/2 \\ 0 & \textit{otherwise} \end{cases}$$

Find the autocorrelation function of the output process $Y(t)$.

Solution:

- **Input PSD**:
$$S_{WW}(\omega) = \sigma^2$$
- **Transfer Function of LPF**:
$$H(\omega) = \begin{cases} 1 & |\omega| \le B/2 \\ 0 & \text{otherwise} \end{cases}$$
- **Output PSD**:
$$S_{YY}(\omega) = |H(\omega)|^2 S_{WW}(\omega) = \begin{cases} \sigma^2 & |\omega| \le B/2 \\ 0 & \text{otherwise} \end{cases}$$
- **Autocorrelation Function via IFT**:
$$R_{YY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)e^{j\omega\tau}\,d\omega = \frac{\sigma^2}{2\pi}\int_{-B/2}^{B/2} e^{j\omega\tau}\,d\omega$$
$$R_{YY}(\tau) = \frac{\sigma^2}{2\pi}\cdot\frac{2\sin(B\tau/2)}{\tau} = \frac{\sigma^2}{\pi}\frac{\sin(B\tau/2)}{\tau} = \sigma^2\cdot\operatorname{sinc}\left(\frac{B\tau}{2\pi}\right)$$
- Therefore, the autocorrelation function of the output process $Y(t)$ is:
$$R_{YY}(\tau) = \sigma^2\cdot\operatorname{sinc}\left(\frac{B\tau}{2\pi}\right)$$

□

7 Fourier Transform Formulas and Properties

7.1 Fourier Transform Formulas

- Delta Function:
$$\mathcal{F}\{\delta(t)\} = 1$$
- Constant:
$$\mathcal{F}\{1\} = 2\pi\delta(\omega)$$
- Rectangular Pulse:
$$\mathcal{F}\{\text{rect}(t)\} = 2\pi\text{sinc}(\omega)$$
- Exponential Decay:
$$\mathcal{F}\{e^{-at}u(t)\} = \frac{1}{a + j\omega}, \quad \text{Re}(a) > 0$$
- Cosine:
$$\mathcal{F}\{\cos(\omega_0 t)\} = \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$
- Sine:
$$\mathcal{F}\{\sin(\omega_0 t)\} = j\pi[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$$

7.2 Inverse Fourier Transform Formulas

- Inverse Fourier Transform of Constant:
$$\mathcal{F}^{-1}\{2\pi\delta(\omega)\} = 1$$
- Inverse Fourier Transform of Delta Function:
$$\mathcal{F}^{-1}\{1\} = \delta(t)$$
- Inverse Fourier Transform of Rectangular Pulse:
$$\mathcal{F}^{-1}\{2\pi\text{sinc}(\omega)\} = \text{rect}(t)$$
- Inverse Fourier Transform of Exponential Decay:
$$\mathcal{F}^{-1}\left\{\frac{1}{a + j\omega}\right\} = e^{-at}u(t)$$
- Inverse Fourier Transform of Cosine:
$$\mathcal{F}^{-1}\{\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]\} = \cos(\omega_0 t)$$
- Inverse Fourier Transform of Sine:
$$\mathcal{F}^{-1}\{j\pi[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]\} = \sin(\omega_0 t)$$

8 Additional Formulas and Theorems

8.1 Slutsky's Theorem

- Statement: If X_n converges in distribution to X and Y_n converges in probability to a constant c, then:

$$X_n + Y_n \xrightarrow{d} X + c$$

- Application in Ergodicity: While Slutsky's Theorem is a powerful tool in convergence of random variables, it is not directly applicable in establishing ergodicity in mean. Instead, ergodicity in mean relies on the autocovariance condition:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T C(\tau)\, d\tau = 0$$

where $C(\tau) = R_{XX}(\tau) - \mu_X^2$.

8.2 Cauchy-Schwarz Inequality

$$|E[XY]| \leq \sqrt{E[X^2]E[Y^2]}$$

- Equality Condition: Holds if and only if X and Y are linearly dependent.

8.3 Covariance Matrix for Multivariate RVs

$$\Sigma = \begin{pmatrix} \mathrm{Var}(X) & \mathrm{Cov}(X,Y) \\ \mathrm{Cov}(Y,X) & \mathrm{Var}(Y) \end{pmatrix}$$

8.4 Transformation of Variables

- Single Variable Transformation:

$$Y = g(X), \quad X = g^{-1}(Y)$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

- Multiple Variable Transformation:

$$\mathbf{Y} = \mathbf{g}(\mathbf{X}), \quad \mathbf{X} = \mathbf{g}^{-1}(\mathbf{Y})$$

$$f_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{X}}(\mathbf{g}^{-1}(\mathbf{y}))\, |\det J|$$

where J is the Jacobian matrix of partial derivatives:

$$J = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}$$

9 Power Spectral Density (PSD)

9.1 Definition for WSS Processes

- The power spectral density $S_{XX}(\omega)$ of a WSS process $X(t)$ is the Fourier transform of its autocorrelation function $R_{XX}(\tau)$:

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-j\omega\tau}\, d\tau$$

- According to the Wiener-Khinchin Theorem, $R_{XX}(\tau)$ and $S_{XX}(\omega)$ form a Fourier transform pair.

9.2 Total Power

- The total power of the process is given by:

$$P = R_{XX}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega = E[|X(t)|^2]$$

- This relationship correctly includes the $\frac{1}{2\pi}$ factor to maintain dimensional consistency between time and frequency domains.

9.3 Example 6: Power Spectrum of Discrete-Time Process

Example 10 (Power Spectrum). *A discrete-time WSS process $X[nT]$ has autocorrelation sequence:*

$$R_{XX}[k] = \sigma^2\delta[k]$$

Find the power spectral density $S_{XX}(\omega)$.

Solution:

- Using the definition:

$$S_{XX}(\omega) = \sum_{k=-\infty}^{\infty} R_{XX}[k]e^{-j\omega k} = \sum_{k=-\infty}^{\infty} \sigma^2\delta[k]e^{-j\omega k} = \sigma^2$$

- Therefore, the power spectral density is flat:

$$S_{XX}(\omega) = \sigma^2 \quad \forall\omega$$

- This indicates a white noise process in discrete-time.

□

9.4 Wiener-Khinchin Theorem

- The autocorrelation function and the power spectral density of a WSS process form a Fourier transform pair:

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-j\omega\tau}\, d\tau$$

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{j\omega\tau}\, d\omega$$

- This theorem is fundamental in connecting time-domain and frequency-domain analyses.

9.5 Nonnegative-Definiteness of Autocorrelation Function

- The autocorrelation function $R_{XX}(\tau)$ must satisfy:

$$\int_{-\infty}^{\infty} S_{XX}(\omega)|A(\omega)|^2\, d\omega \geq 0 \quad \forall A(\omega)$$

- This implies that the power spectral density $S_{XX}(\omega)$ is nonnegative for all ω:

$$S_{XX}(\omega) \geq 0 \quad \forall \omega$$

10 Power Spectra and LTI Systems

10.1 Input-Output Relations in LTI Systems

- If a WSS process $X(t)$ with autocorrelation function $R_{XX}(\tau)$ is applied to an LTI system with impulse response $h(t)$, then:
  - Output Mean:
$$\mu_Y(t) = \mu_X(t) * h(t) = \mu_X * h(t)$$
  - Cross-Correlation Function:
$$R_{XY}(\tau) = R_{XX}(\tau) * h(\tau)$$
  - Autocorrelation of the Output Process:
$$R_{YY}(\tau) = R_{XY}(\tau) * h(-\tau) = R_{XX}(\tau) * h(\tau) * h(-\tau)$$
- Frequency Domain Representation:
  - Fourier transform relates autocorrelation functions to power spectra:
$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$

10.2 Transfer Function and Spectrum

- The transfer function $H(\omega)$ of the system is the Fourier transform of the impulse response $h(t)$:
$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\, dt$$
- The output power spectral density $S_{YY}(\omega)$ is related to the input power spectral density $S_{XX}(\omega)$ by:
$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$
- Note: The cross-spectrum $S_{XY}(\omega)$ need not be real or nonnegative, but the output spectrum $S_{YY}(\omega)$ is real and nonnegative.

10.3 Example 5: Power Spectrum of Output through LTI System

Example 11 (Power Spectrum of Output)**.** *A WSS white noise process $W(t)$ with $S_{WW}(\omega) = \sigma^2$ is passed through a low-pass filter (LPF) with transfer function:*

$$H(\omega) = \begin{cases} 1 & |\omega| \leq B/2 \\ 0 & otherwise \end{cases}$$

Find the autocorrelation function of the output process $Y(t)$.

Solution:

- **Input PSD**:

$$S_{WW}(\omega) = \sigma^2$$

- **Transfer Function of LPF**:

$$H(\omega) = \begin{cases} 1 & |\omega| \leq B/2 \\ 0 & \text{otherwise} \end{cases}$$

- **Output PSD**:

$$S_{YY}(\omega) = |H(\omega)|^2 S_{WW}(\omega) = \begin{cases} \sigma^2 & |\omega| \leq B/2 \\ 0 & \text{otherwise} \end{cases}$$

- **Autocorrelation Function via IFT**:

$$R_{YY}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{YY}(\omega) e^{j\omega\tau} \, d\omega = \frac{\sigma^2}{2\pi} \int_{-B/2}^{B/2} e^{j\omega\tau} \, d\omega$$

$$R_{YY}(\tau) = \frac{\sigma^2}{2\pi} \cdot \frac{2\sin(B\tau/2)}{\tau} = \frac{\sigma^2}{\pi} \frac{\sin(B\tau/2)}{\tau} = \sigma^2 \cdot \text{sinc}\left(\frac{B\tau}{2\pi}\right)$$

- Therefore, the autocorrelation function of the output process $Y(t)$ is:

$$R_{YY}(\tau) = \sigma^2 \cdot \text{sinc}\left(\frac{B\tau}{2\pi}\right)$$

□

11 Fourier Transforms

11.1 Definitions

- Fourier Transform (FT) of a function $x(t)$:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt$$

- Inverse Fourier Transform (IFT):

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \, d\omega$$

11.2 Properties of Fourier Transforms

- Linearity:
$$\mathcal{F}\{ax(t) + by(t)\} = aX(\omega) + bY(\omega)$$
- Time Shifting:
$$\mathcal{F}\{x(t - t_0)\} = X(\omega)e^{-j\omega t_0}$$
- Frequency Shifting:
$$\mathcal{F}\{x(t)e^{j\omega_0 t}\} = X(\omega - \omega_0)$$
- Scaling:
$$\mathcal{F}\{x(at)\} = \frac{1}{|a|}X\left(\frac{\omega}{a}\right)$$
- Conjugation:
$$\mathcal{F}\{x^*(t)\} = X^*(-\omega)$$
- Duality:
$$\mathcal{F}\{X(t)\} = 2\pi x(-\omega)$$
- Convolution:
$$\mathcal{F}\{x(t) * y(t)\} = X(\omega)Y(\omega)$$
- Multiplication:
$$\mathcal{F}\{x(t)y(t)\} = \frac{1}{2\pi}\left(X(\omega) * Y(\omega)\right)$$
- Parseval's Theorem:
$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega$$

11.3 Table of Important Fourier Transforms

Function	**Fourier Transform**	**Inverse FT**
Delta Function	$\delta(t)$	1
Constant	1	$2\pi\delta(\omega)$
Rectangular Pulse	$\text{rect}(t)$	$2\pi\text{sinc}(\omega)$
Exponential Decay	$e^{-at}u(t)$	$\frac{1}{a+j\omega}$
Cosine	$\cos(\omega_0 t)$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
Sine	$\sin(\omega_0 t)$	$j\pi[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$

11.4 Example 5: Fourier Transform Application

Example 12 (Fourier Transform of a Sine Wave)**.** *Find the Fourier transform of* $x(t) = \sin(\omega_0 t)$*.*

Solution:

- Using Euler's formula:
$$\sin(\omega_0 t) = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

- Fourier transform:

$$\mathcal{F}\{\sin(\omega_0 t)\} = \frac{1}{2j}\left[\mathcal{F}\{e^{j\omega_0 t}\} - \mathcal{F}\{e^{-j\omega_0 t}\}\right]$$

- Using the Fourier transform of exponentials:

$$\mathcal{F}\{e^{j\omega_0 t}\} = 2\pi\delta(\omega - \omega_0)$$

$$\mathcal{F}\{e^{-j\omega_0 t}\} = 2\pi\delta(\omega + \omega_0)$$

- Substituting back:

$$\mathcal{F}\{\sin(\omega_0 t)\} = \frac{1}{2j}\left[2\pi\delta(\omega - \omega_0) - 2\pi\delta(\omega + \omega_0)\right] = \pi\left[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\right]\frac{1}{j}$$

- Simplifying using $j = \sqrt{-1}$:

$$\mathcal{F}\{\sin(\omega_0 t)\} = j\pi\left[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)\right]$$

□

11.5 Example 6: Inverse Fourier Transform Application

Example 13 (Inverse Fourier Transform of a Rectangular Pulse)**.** *Find the inverse Fourier transform of* $X(\omega) = 2\pi \mathit{rect}\left(\frac{\omega}{2B}\right)$.

Solution:

- The inverse Fourier transform of $X(\omega) = 2\pi\text{rect}\left(\frac{\omega}{2B}\right)$ is:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\text{rect}\left(\frac{\omega}{2B}\right)e^{j\omega t}\,d\omega = \int_{-\infty}^{\infty}\text{rect}\left(\frac{\omega}{2B}\right)e^{j\omega t}\,d\omega$$

- The rectangular function $\text{rect}\left(\frac{\omega}{2B}\right)$ is defined as:

$$\text{rect}\left(\frac{\omega}{2B}\right) = \begin{cases} 1 & |\omega| \leq B \\ 0 & \text{otherwise} \end{cases}$$

- Therefore:

$$x(t) = \int_{-B}^{B} e^{j\omega t}\,d\omega = \frac{e^{jBt} - e^{-jBt}}{jt} = \frac{2\sin(Bt)}{t} = 2\cdot\text{sinc}\left(\frac{Bt}{\pi}\right)$$

- Thus:

$$x(t) = 2\cdot\text{sinc}\left(\frac{Bt}{\pi}\right)$$

□

11.6 Fourier Transform Properties

- Linearity:
$$\mathcal{F}\{ax(t) + by(t)\} = aX(\omega) + bY(\omega)$$
- Time Shifting:
$$\mathcal{F}\{x(t - t_0)\} = X(\omega)e^{-j\omega t_0}$$
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- Parseval's Theorem:
$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

12 Additional Formulas and Theorems

12.1 Slutsky's Theorem

- Statement: If X_n converges in distribution to X and Y_n converges in probability to a constant c, then:
$$X_n + Y_n \xrightarrow{d} X + c$$
- Application in Ergodicity: While Slutsky's Theorem is a powerful tool in convergence of random variables, it is not directly applicable in establishing ergodicity in mean. Instead, ergodicity in mean relies on the autocovariance condition:
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12.2 Cauchy-Schwarz Inequality

$$|E[XY]| \leq \sqrt{E[X^2]E[Y^2]}$$

- Equality Condition: Holds if and only if X and Y are linearly dependent.

12.3 Covariance Matrix for Multivariate RVs

$$\Sigma = \begin{pmatrix} \mathrm{Var}(X) & \mathrm{Cov}(X,Y) \\ \mathrm{Cov}(Y,X) & \mathrm{Var}(Y) \end{pmatrix}$$

12.4 Transformation of Variables

- Single Variable Transformation:

$$Y = g(X), \quad X = g^{-1}(Y)$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

- Multiple Variable Transformation:

$$\mathbf{Y} = \mathbf{g}(\mathbf{X}), \quad \mathbf{X} = \mathbf{g}^{-1}(\mathbf{Y})$$

$$f_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{X}}(\mathbf{g}^{-1}(\mathbf{y}))\, |\det J|$$

where J is the Jacobian matrix of partial derivatives:

$$J = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}$$

13 Conclusion

This document provides an extensive overview of ergodic stochastic processes, stochastic analysis of LTI systems, and power spectrum analysis, enriched with detailed mathematical formulations, examples, and solutions. The correction of the ergodicity in mean section ensures accurate representation of the necessary and sufficient conditions without the incorrect application of Slutsky's Theorem. Additionally, the power formula includes the essential $\frac{1}{2\pi}$ factor to maintain dimensional consistency and accuracy.

For more in-depth explanations and proofs, refer to your course materials or standard textbooks on stochastic processes and signal analysis.