Stochastic Processes

Lecture Notes

Chapter 2: First and Higher Order Statistics of Stochastic Processes

Payam Taebi

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https://stoch-sut.github.io/

Contents

1 Introduction

This chapter explores the first and higher-order statistics of stochastic processes. We will delve into distribution functions, autocorrelation, autocovariance, and the properties of stationary processes. Understanding these concepts is essential for analyzing and modeling stochastic systems effectively.

2 Stochastic Processes

2.1 Definition

- Let ξ denote the random outcome of an experiment.
- To every such outcome ξ, a waveform is assigned.
- The collection of such waveforms or sample paths forms a stochastic process.
- The set of ξ and the time index t can be continuous or discrete (countably infinite or finite).

$$X(t, \xi)$$

2.2 Interpretation

- For fixed ξ (the set of all experimental outcomes), $X(t, \xi)$ is a specific time function.
- For fixed t, $X(t, \xi)$ is a random variable (RV).
- The ensemble of all such realizations over time represents the stochastic process $X(t)$.

$$X(t, \xi), \quad t \in T, \quad \xi \in \Xi$$

3 First Order Statistics of $X(t)$

3.1 Distribution Function

- If $X(t)$ is a stochastic process, then for fixed t, $X(t)$ represents a random variable.
- Its distribution function is given by:
$$F_X(x, t) = P\{X(t) \leq x\}$$
- Notice that $F_X(x, t)$ depends on t, since for a different t, we obtain a different random variable.
- $f_X(x, t)$ represents the first-order probability density function (pdf) of the process $X(t)$.

$$F_X(x, t) = P\{X(t) \leq x\} = \int_{-\infty}^{x} f_X(x', t)\, dx'$$

3.2 Joint Distribution Function

- For $t = t_1$ and $t = t_2$, $X(t)$ represents two different random variables $X_1 = X(t_1)$ and $X_2 = X(t_2)$ respectively.
- Their joint distribution is given by $F_{X_1,X_2}(x_1, x_2, t_1, t_2) = P\{X(t_1) \leq x_1, X(t_2) \leq x_2\}$.
- $f_{X_1,X_2}(x_1, x_2, t_1, t_2)$ represents the second-order density function of the process $X(t)$.

$$f_{X_1,X_2}(x_1, x_2, t_1, t_2) = \frac{\partial^2 F_{X_1,X_2}(x_1, x_2)}{\partial x_1 \partial x_2}$$

3.3 Nth Order Density Function

- Similarly, $f_{X_1,X_2,\dots,X_n}(x_1, x_2, \dots, x_n, t_1, t_2, \dots, t_n)$ represents the n-th order density function of the process $X(t)$.
- Complete specification of the stochastic process $X(t)$ requires the knowledge of $f_{X_1,X_2,\dots,X_n}(x_1, x_2, \dots,$ for all $t_1, t_2, \dots, t_n$ and for all n.

$$f_{X_1,X_2,\dots,X_n}(x_1, x_2, \dots, x_n, t_1, t_2, \dots, t_n) = \frac{\partial^n F_{X_1,X_2,\dots,X_n}(x_1, x_2, \dots, x_n)}{\partial x_1 \partial x_2 \dots \partial x_n}$$

4 Mean, Autocorrelation, and Autocovariance of $X(t)$

4.1 Mean (Expected Value)

$$\mu_X(t) = E\{X(t)\} = \int_{-\infty}^{\infty} x\, f_X(x, t)\, dx$$

- Represents the mean value of the process $X(t)$.
- In general, the mean of a process can depend on the time index t.

4.2 Autocorrelation Function

$$R_{XX}(t_1, t_2) = E\{X(t_1)X(t_2)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2\, f_{X(t_1),X(t_2)}(x_1, x_2)\, dx_1 dx_2$$

- Represents the interrelationship between the random variables $X(t_1)$ and $X(t_2)$ generated from the process $X(t)$.

4.3 Autocovariance Function

$$C_{XX}(t_1, t_2) = R_{XX}(t_1, t_2) - \mu_X(t_1)\mu_X(t_2)$$

- Represents the autocovariance function of the process $X(t)$.

5 Properties of Autocorrelation Function for W.S.S Processes

(1) $R_{XX}(\tau) = R_{XX}(-\tau)$ (Even Function)

(2) $R_{XX}(0) \geq R_{XX}(\tau)$ (Non-negative Definite)

(3) $R_{XX}(\tau) \leq R_{XX}(0)$ (Maximum at Zero Lag)

6 Stationary Stochastic Processes

6.1 Definition

- Stationary processes exhibit statistical properties that are invariant to shifts in the time index.
- For example, second-order stationarity implies that the statistical properties of the pairs $\{X(t_1), X(t_2)\}$ and $\{X(t_1 + c), X(t_2 + c)\}$ are the same for any constant c.
- Similarly, first-order stationarity implies that the statistical properties of $X(t_i)$ and $X(t_i + c)$ are the same for any constant c.

6.2 Strict-Sense Stationary (S.S.S) Processes

- In strict terms, the statistical properties are governed by the joint probability density function.
- A process $X(t)$ is n-th order strict-sense stationary if, for any shift c:

$$f_{X(t_1),X(t_2),\ldots,X(t_n)}(x_1, x_2, \ldots, x_n) = f_{X(t_1+c),X(t_2+c),\ldots,X(t_n+c)}(x_1, x_2, \ldots, x_n)$$

6.3 Wide-Sense Stationary (W.S.S) Processes

- A process $X(t)$ is said to be Wide-Sense Stationary if:
  (i) The mean is a constant: $E\{X(t)\} = \mu_X$
  (i) The autocorrelation function depends only on the difference between the time indices: $R_{XX}(t_1, t_2) = R_{XX}(\tau)$ where $\tau = t_1 - t_2$
- **Key Points:**
  - Strict-sense stationarity always implies wide-sense stationarity.
  - The converse is not true in general, except for Gaussian processes.
  - In Gaussian processes, wide-sense stationarity implies strict-sense stationarity.

6.4 Characteristics of Gaussian Processes

- If $X(t)$ is a Gaussian process, then by definition, $\{X(t_i)\}$ are jointly Gaussian random variables for any set $\{t_i\}$.
- The joint characteristic function for Gaussian processes depends only on the mean and covariance, making verification of strict-sense stationarity straightforward.

7 Autocorrelation Function

7.1 Properties for W.S.S Processes

(1) $R_{XX}(\tau) = R_{XX}(-\tau)$ (Even Function)

(2) $R_{XX}(0) = E\{X(t)^2\}$

$$C_{XX}(0) = R_{XX}(0) - \mu_X^2$$

(3) Cauchy-Schwarz Inequality:

$$R_{XX}(\tau)^2 \leq R_{XX}(0)^2$$

8 Cross-Correlation and Marginal Distributions

8.1 Cross-Correlation Function

$$R_{XY}(t_1, t_2) = E\{X(t_1)Y(t_2)\}$$
$$C_{XY}(t_1, t_2) = R_{XY}(t_1, t_2) - \mu_X(t_1)\mu_Y(t_2)$$

- Cross-correlation measures the relationship between two different processes or different time instances.
- For independent processes:

$$R_{XY}(t_1, t_2) = \mu_X(t_1)\mu_Y(t_2)$$

- Covariance:

$$C_{XY}(t_1, t_2) = 0 \quad \text{(for independent processes)}$$

8.2 Marginal Distributions

$$f_{X(t)}(x_1; t_1) = \int_{-\infty}^{\infty} f_{X(t_1),X(t_2)}(x_1, x_2; t_1, t_2)\, dx_2$$
$$f_{Y(t)}(y_1; t_1) = \int_{-\infty}^{\infty} f_{X(t_1),Y(t_2)}(x_1, y_2; t_1, t_2)\, dy_2$$

- For independent processes $X(t)$ and $Y(t)$:

$$f_{X(t),Y(t)}(x, y; t, t) = f_X(x; t) f_Y(y; t)$$

- Thus, $R_{XY}(t_1, t_2) = \mu_X(t_1)\mu_Y(t_2)$

9 Characteristic Function

9.1 Definition

The characteristic function $\phi_X(\omega)$ of a random variable X is defined as:

$$\phi_X(\omega) = \mathcal{F}\{f_X\}(\omega) = E\{e^{j\omega X}\} = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\, dx$$

where j is the imaginary unit and ω is a real number.

10 Examples

10.1 Example 1: Integrator

Example 1. *Let*

$$Z = \int_{-T}^{T} X(t)\, dt$$

Compute $E\{Z\}$ *and* $R_{XX}(t_1, t_2)$.

Solution:

$$\begin{aligned}
E\{Z\} &= E\left\{\int_{-T}^{T} X(t)\, dt\right\} = \int_{-T}^{T} E\{X(t)\}\, dt = \int_{-T}^{T} \mu_X(t)\, dt = 2T\mu_X \\
R_{XX}(t_1, t_2) &= E\{X(t_1)X(t_2)\} = E\{(At_1 + b)(At_2 + b)\} \\
&= E\{A^2 t_1 t_2 + Abt_1 + Abt_2 + b^2\} \\
&= t_1 t_2 E\{A^2\} + bt_1 E\{A\} + bt_2 E\{A\} + b^2 \\
&= t_1 t_2 \cdot 1 + bt_1 \cdot 0 + bt_2 \cdot 0 + b^2 \\
&= t_1 t_2 + b^2
\end{aligned}$$

□

10.2 Example 2: Oscillatory Process

Example 2. *Consider the stochastic process*

$$X(t) = a\cos(\omega_0 t) + \phi, \quad \phi \sim U(0, 2\pi)$$

where a *is a constant and* ϕ *is uniformly distributed over* $(0, 2\pi)$.

Compute $\mu_X(t)$ *and* $Var\{X(t)\}$.

Solution:

$$\begin{aligned}
\mu_X(t) &= E\{X(t)\} = E\{a\cos(\omega_0 t) + \phi\} = a\cos(\omega_0 t) + E\{\phi\} = a\cos(\omega_0 t) + \pi \\
\mathrm{Var}\{X(t)\} &= E\{X(t)^2\} - (E\{X(t)\})^2 \\
&= E\{(a\cos(\omega_0 t) + \phi)^2\} - (a\cos(\omega_0 t) + \pi)^2 \\
&= a^2\cos^2(\omega_0 t) + 2a\cos(\omega_0 t)E\{\phi\} + E\{\phi^2\} - (a^2\cos^2(\omega_0 t) + 2a\pi\cos(\omega_0 t) + \pi^2) \\
&= a^2\cos^2(\omega_0 t) + 2a\cos(\omega_0 t)\pi + \frac{(2\pi)^2}{12} - a^2\cos^2(\omega_0 t) - 2a\pi\cos(\omega_0 t) - \pi^2 \\
&= \frac{4\pi^2}{12} - \pi^2 \\
&= \frac{\pi^2}{3}
\end{aligned}$$

□

10.3 Example 3: Linear Stochastic Process

Example 3. *Consider the stochastic process*

$$X(t) = At + b, \quad A \sim \mathcal{N}(0, 1)$$

where A *is a Gaussian random variable with mean 0 and variance 1, and b is a constant.*

Compute $\mu_X(t)$, $Var\{X(t)\}$, *and* $R_{XX}(t_1, t_2)$.

Solution:

$$\begin{aligned}
\mu_X(t) &= E\{X(t)\} = E\{At+b\} = tE\{A\} + b = 0 + b = b \\
\text{Var}\{X(t)\} &= \text{Var}\{At+b\} = t^2\text{Var}\{A\} = t^2 \cdot 1 = t^2 \\
R_{XX}(t_1,t_2) &= E\{X(t_1)X(t_2)\} = E\{(At_1+b)(At_2+b)\} \\
&= E\{A^2t_1t_2 + Abt_1 + Abt_2 + b^2\} \\
&= t_1t_2E\{A^2\} + bt_1E\{A\} + bt_2E\{A\} + b^2 \\
&= t_1t_2 \cdot 1 + bt_1 \cdot 0 + bt_2 \cdot 0 + b^2 \\
&= t_1t_2 + b^2
\end{aligned}$$

□

10.4 Example 4: Waiting Time

Example 4. *Taxis are waiting in a queue for passengers who arrive according to a Poisson process with an average rate of $\lambda = 1$ passenger per minute. A taxi departs as soon as two passengers have been collected or 3 minutes have expired since the first passenger has boarded.*

Suppose you are the first passenger. What is your average waiting time for departure?

Hint: *Condition on the first arrival after you get in the taxi.*

Solution: Let S_1 be the arrival time of the second passenger after you have boarded the taxi. S_1 follows an exponential distribution with rate $\lambda = 1$.

Define X as your waiting time for departure. The departure occurs at:

$$X = \begin{cases} S_1 & \text{if } S_1 < 3 \text{ minutes} \\ 3 & \text{if } S_1 \geq 3 \text{ minutes} \end{cases}$$

Thus, the expected waiting time $E\{X\}$ is:

$$\begin{aligned}
E\{X\} &= E\{X|S_1 < 3\}P\{S_1 < 3\} + E\{X|S_1 \geq 3\}P\{S_1 \geq 3\} \\
&= E\{S_1|S_1 < 3\}P\{S_1 < 3\} + 3P\{S_1 \geq 3\}
\end{aligned}$$

For an exponential distribution:

$$P\{S_1 < 3\} = 1 - e^{-\lambda \cdot 3} = 1 - e^{-3}$$

$$E\{S_1|S_1 < 3\} = \frac{1-(3+1)e^{-3}}{1-e^{-3}} = \frac{1-4e^{-3}}{1-e^{-3}}$$

Plugging in the values:

$$\begin{aligned}
E\{X\} &= \frac{1-4e^{-3}}{1-e^{-3}}(1-e^{-3}) + 3e^{-3} = 1 - 4e^{-3} + 3e^{-3} = 1 - e^{-3} \\
&\approx 1 - 0.0498 = 0.9502 \text{ minutes (approximately 57 seconds)}
\end{aligned}$$

□

11 Strict-Sense Stationary (S.S.S) Processes

11.1 Definition

- In strict terms, the statistical properties are governed by the joint probability density function.
- A process $X(t)$ is n-th order strict-sense stationary if, for any shift c:

$$f_{X(t_1),X(t_2),\ldots,X(t_n)}(x_1,x_2,\ldots,x_n) = f_{X(t_1+c),X(t_2+c),\ldots,X(t_n+c)}(x_1,x_2,\ldots,x_n)$$

11.2 Implications for First and Second Order

- **First-Order S.S.S:**
$$f_X(x,t) = f_X(x,t+c) \quad \forall c$$
$$\mu_X(t) = \mu_X \quad \text{(constant)}$$

- **Second-Order S.S.S:**
$$R_{XX}(t_1,t_2) = R_{XX}(t_1+c,t_2+c) \quad \forall c$$
$$R_{XX}(\tau) \quad \text{depends only on} \quad \tau = t_1 - t_2$$

11.3 First-Order S.S.S Implications

- From the first-order condition, for any c, especially $c = -t$, we have:
$$f_X(x,t) = f_X(x,0)$$
$$\mu_X(t) = \mu_X(0) = \mu_X \quad \text{(constant)}$$

11.4 Second-Order S.S.S Implications

- From the second-order condition, for any c, especially $c = -t_2$, we have:
$$R_{XX}(t_1,t_2) = R_{XX}(t_1-t_2,0) = R_{XX}(0,t_1-t_2) = R_{XX}(t_1-t_2)$$

- Thus, the autocorrelation function depends only on the time difference $\tau = t_1 - t_2$.

11.5 Consequences for Gaussian Processes

- For Gaussian processes, strict-sense stationarity and wide-sense stationarity are equivalent.

- This is because Gaussian processes are completely characterized by their first and second moments.

12 Wide-Sense Stationary (W.S.S) Processes

12.1 Definition

- A process $X(t)$ is said to be Wide-Sense Stationary if:
  (i) The mean is a constant: $E\{X(t)\} = \mu_X$
  (i) The autocorrelation function depends only on the difference between the time indices: $R_{XX}(t_1,t_2) = R_{XX}(\tau)$ where $\tau = t_1 - t_2$

- **Key Points:**
  - Strict-sense stationarity always implies wide-sense stationarity.
  - The converse is not true in general, except for Gaussian processes.
  - In Gaussian processes, wide-sense stationarity implies strict-sense stationarity.

12.2 Characteristics of W.S.S Processes

- For W.S.S processes, the mean is constant.

$$E\{X(t)\} = \mu_X \quad \forall t$$

- The autocorrelation function depends only on the time difference:

$$R_{XX}(t_1, t_2) = R_{XX}(\tau), \quad \tau = t_1 - t_2$$

12.3 Relationship with S.S.S Processes

- Strict-sense stationarity implies wide-sense stationarity.
- The converse holds only for Gaussian processes.

13 Characteristic Function of Gaussian Processes

13.1 Definition

$$\phi_X(\omega) = E\{e^{j\omega X}\} = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\, dx$$

where j is the imaginary unit and ω is a real number.

13.2 Characteristic Function of Normal Distribution

Example 5. *For $X \sim \mathcal{N}(\mu, \sigma^2)$, the characteristic function is:*

$$\phi_X(\omega) = e^{j\omega\mu - \frac{1}{2}\omega^2\sigma^2}$$

Solution: The characteristic function of a normal distribution is derived as follows:

$$\phi_X(\omega) = E\{e^{j\omega X}\} = e^{j\omega\mu - \frac{1}{2}\omega^2\sigma^2}$$

This is because:

$$X \sim \mathcal{N}(\mu, \sigma^2) \implies e^{j\omega X} \sim \mathcal{N}(j\omega\mu, \omega^2\sigma^2)$$

Hence,

$$\phi_X(\omega) = e^{j\omega\mu - \frac{1}{2}\omega^2\sigma^2}$$

□

13.3 Characteristic Function for Joint Gaussian Variables

- For Gaussian processes, the joint characteristic function depends only on the mean and covariance.
- If $X(t)$ is a Gaussian process and W.S.S, then S.S.S follows automatically.

14 Additional Notes

14.1 Cauchy-Schwarz Inequality

$$|E\{XY\}| \leq \sqrt{E\{X^2\}E\{Y^2\}}$$

- Equality holds if and only if X and Y are linearly dependent.

14.2 Covariance Matrix for Multivariate RVs

$$\Sigma = \begin{pmatrix} \mathrm{Var}(X) & \mathrm{Cov}(X,Y) \\ \mathrm{Cov}(Y,X) & \mathrm{Var}(Y) \end{pmatrix}$$

14.3 Transformation of Variables

- **Single Variable Transformation:**

$$Y = g(X), \quad X = g^{-1}(Y)$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

- **Multiple Variable Transformation:**

$$\mathbf{Y} = \mathbf{g}(\mathbf{X}), \quad \mathbf{X} = \mathbf{g}^{-1}(\mathbf{Y})$$

$$f_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{X}}(\mathbf{g}^{-1}(\mathbf{y}))\, |\det J|$$

where J is the Jacobian matrix of partial derivatives:

$$J = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}$$

14.4 Jacobians in Transformations

- **Single Variable:**

$$\left| \frac{dy}{dx} \right| \cdot \left| \frac{dx}{dy} \right| = 1$$

- **Multiple Variables:**

$$\det(J) \cdot \det(J^{-1}) = 1$$

15 Conclusion

This chapter provides a comprehensive overview of the first and higher-order statistics of stochastic processes, including distribution functions, autocorrelation, autocovariance, and the properties of stationary processes. Additionally, key examples with solutions illustrate the application of these concepts. For more detailed explanations and proofs, refer to your course materials or standard textbooks on stochastic processes.