محمد پیام تابشی ۴۰۰۱۰۴۸۶۷

۱) $X \sim N(0,1)$, $Y \sim N(0,1)$ $\quad U = X \quad V = \frac{X}{Y}$

$$U = g_1(X, Y) = X$$

$$V = g_2(X, Y) = \frac{X}{Y}$$

۱)

$$\begin{cases} u = g_1(x, y) = x \\ v = g_2(x, y) = \frac{x}{y} \end{cases} \longrightarrow$$

این ۲ معادله بطور منحصر به فرد

از هر u و v ، x و y بدست می‌آورد

یعنی $x = u$

$$y = \frac{u}{v}$$

۲)

$$J(x, y) = \begin{vmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \frac{1}{y} & -\frac{x}{y^2} \end{vmatrix}$$

$$= -\frac{x}{y^2} \rightarrow |J(x,y)| = \frac{|x|}{y^2}$$

حال می‌توان گفت که اگر $J(x,y) \neq 0$ باشد:

$$\boxed{f_{UV}(u,v) = \frac{f_{XY}(x,y)}{|J(x,y)|}}$$

اثبات:

$$P(U \le u, V \le v) = \iint_{\substack{g_1(x,y) \le u \\ g_2(x,y) \le v}} f_{XY}(x,y)\, dx\, dy$$

حال با مشتق‌گیری نسبت به u و v به عبارت بالا رسید

الف

$$f_{XY}(x,y) = f_X(x)\, f_Y(y)$$ → چون X, Y مستقل اند

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \times \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}$$

$$f_{UV}(u,v) = \frac{f_{XY}(x,y)}{J(x,y)} = \frac{\frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}}{\frac{|x|}{y^2}}$$

و $x = u$, $y = \frac{u}{v}$

$$\rightarrow f_{UV}(u,v) = \frac{1}{2\pi} e^{-\frac{1}{2}\left(u^2 + \frac{u^2}{v^2}\right)} \times \frac{u^2}{v^2} \times \frac{1}{|u|}$$

$$\rightarrow f_{UV}(u,v) = \frac{1}{2\pi} \times \frac{|u|}{v^2} \times e^{-\frac{1}{2}\left(u^2 + \frac{u^2}{v^2}\right)}$$

ب

$$f_Y(y) = \int_{-\infty}^{\infty} f_{UY}(u,y)\, du$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} \times \frac{|u|}{y^2} \times e^{-\frac{1}{2}\left(u^2 + \frac{u^2}{y^2}\right)}\, du$$

$$= \frac{-1}{2\pi y^2} \int_{-\infty}^{0} \underbrace{u\, e^{-\frac{1}{2}\left(u^2+\frac{u^2}{y^2}\right)}}\, du + \frac{1}{2\pi y^2} \int_{0}^{\infty} \underbrace{u\, e^{-\frac{1}{2}\left(u^2+\frac{u^2}{y^2}\right)}}\, du$$

تابع فرد

$f(u) = -f(-u)$

$$= \frac{1}{\pi y^2} \int_{0}^{\infty} u\, e^{-\frac{u^2}{2}\left(1+\frac{1}{y^2}\right)}\, du$$

$$= \frac{1}{\pi y^2} \left(\frac{-1}{1+\frac{1}{y^2}}\, e^{-\frac{u^2}{2}\left(1+\frac{1}{y^2}\right)} \right)\Bigg|_{0}^{\infty}$$

$$= \frac{-1}{\pi y^2\left(1+\frac{1}{y^2}\right)} \times (0-1) = \frac{1}{\pi(y^2+1)}$$

$$\Rightarrow f_Y(y) = \frac{1}{\pi(y^2+1)} \rightarrow Y \sim \text{cauchy}(0,1)$$

۳) الف) $Y_1 \sim exp(\lambda_1)$ و $Y_2 \sim exp(\lambda_2)$

u و v را فرض کنید : $U = \dfrac{Y_1}{Y_2}$ و $V = Y_1$

$$U = g_1(Y_1, Y_2) = \frac{Y_1}{Y_2} \quad \text{و} \quad V = g_2(Y_1, Y_2) = Y_1$$

$$J(Y_1, Y_2) = \begin{vmatrix} \dfrac{\partial g_1}{\partial Y_1} & \dfrac{\partial g_1}{\partial Y_2} \\ \dfrac{\partial g_2}{\partial Y_1} & \dfrac{\partial g_2}{\partial Y_2} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{Y_2} & -\dfrac{Y_1}{Y_2^2} \\ 1 & 0 \end{vmatrix}$$

$$= \frac{Y_1}{Y_2^2} \rightarrow |J_{(Y_1, Y_2)}| = \frac{U^2}{|V|}$$

Y_1, Y_2 مستقل اند

$$\text{و} \quad f_{UV}(u, v) = \frac{f_{Y_1 Y_2}(Y_1, Y_2)}{|J(Y_1, Y_2)|} = \frac{f_{Y_1}(y_1) \times f_{Y_2}(y_2)}{|J(Y_1, Y_2)|}$$

$$= \frac{\lambda_1 e^{-\lambda_1 y_1} \times \lambda_2 e^{-\lambda_2 y_2}}{v^2} \times |r|$$

$$= \frac{\lambda_1 \lambda_2 \times e^{-\left(\lambda_1 r + \lambda_2 \frac{r}{u}\right)} \times |r|}{u^2}$$

$$\boxed{\Rightarrow f_{UR}(u,r) = \frac{\lambda_1 \lambda_2}{u^2} |r| \, e^{-r\left(\lambda_1 + \frac{\lambda_2}{u}\right)}}$$

$u, r \geqslant 0$

و برای $u, r < 0$ ← $f_{(u,r)} = 0$

$$f_U(u) = \int_{-\infty}^{\infty} f_{UR}(u,r)\, dr$$

$$= \frac{\lambda_1 \lambda_2}{u^2} \left(\cancel{-\int_{-\infty}^{0} 0 \, dr} + \int_{0}^{\infty} r e^{-r\left(\lambda_1 + \frac{\lambda_2}{u}\right)} dr \right)$$

$$\boxed{, \int z e^{az} dz = \frac{e^{az}(az-1)}{a^2} + C}$$

$z = r$

$a = -\left(\lambda_1 + \frac{\lambda_2}{u}\right)$

$$\Rightarrow f_U(u) = \frac{\lambda_1 \lambda_2}{u^2} \times \left. \frac{e^{-r(\lambda_1 + \frac{\lambda_2}{u})}\left(-r(\lambda_1 + \frac{\lambda_2}{u}) - 1\right)}{(\lambda_1 + \frac{\lambda_2}{u})^2} \right|_0^{\infty}$$

$$= \frac{\lambda_1 \lambda_2}{u^2} \times \left(0 - \frac{-1}{(\lambda_1 + \frac{\lambda_2}{u})^2}\right) = \lambda_1 \lambda_2 \times \frac{1}{(u\lambda_1 + \lambda_2)^2}$$

$$\rightarrow \boxed{f_U(u) = \frac{\lambda_1 \lambda_2}{(u\lambda_1 + \lambda_2)^2}} \quad U = \frac{Y_1}{Y_2}$$

تابع چگالی احتمال:

$u \geq 0$

$$F_U(u) = \int_{-\infty}^{u} f_U(u)\, du$$

$$= \int_0^u \frac{\lambda_1 \lambda_2}{(u\lambda_1 + \lambda_2)^2}\, du = \left. \frac{-\lambda_2}{u\lambda_1 + \lambda_2} \right|_0^u$$

$$= \frac{-\lambda_2}{u\lambda_1 + \lambda_2} - -1 = 1 - \frac{\lambda_2}{u\lambda_1 + \lambda_2} = \frac{u\lambda_1}{u\lambda_1 + \lambda_2}$$

$$\Rightarrow F_U(u) = \frac{u\lambda_1}{u\lambda_1 + \lambda_2}$$

ب)

$$P(Y_1 < Y_2) = P\left(\frac{Y_1}{Y_2} < 1\right) = P(U < 1)$$

$$= F_U(1) = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

$$X = \sum_{i=1}^{n} X_i \longrightarrow X \sim \text{binomial}(n, p)$$ (٤

جمع n متغیر برنولی با پارامتر p که مستقل اند یک متغیر باینومال با پارامتر p و n را می سازد،

$$E_{[X]} = np$$

$$Z = \frac{X - np}{\sqrt{np(1-p)}} \qquad \text{var}_{[X]} = np(1-p)$$

$$M_X(t) = E\left[e^{tX}\right] = \sum_x e^{tx} P_{(X=x)}$$

$$\Rightarrow M_Z(t) = E\left[e^{tZ}\right] = \sum_z e^{tz} P_{(z)}$$

$$= \sum_{x=0}^{n} e^{t\left(\frac{x-np}{\sqrt{np(1-p)}}\right)} P_{(X=x)} = \sum_{x=0}^{n} e^{t\left(\frac{x-np}{\sqrt{np(1-p)}}\right)} \times \binom{n}{x} p^{x} (1-p)^{n-x}$$

$$= e^{\frac{-npt}{\sqrt{np(1-p)}}} \sum_{x=0}^{n} e^{\frac{tx}{\sqrt{np(1-p)}}} \times \binom{n}{x} \times p^{x} \times (1-p)^{n-x}$$

$$= e^{\frac{-npt}{\sqrt{np(1-p)}}} \sum_{x=0}^{n} \left(p e^{\frac{t}{\sqrt{np(1-p)}}}\right)^{x} \times (1-p)^{n-x} \times \binom{n}{x}$$

$$(u+v)^n = \sum_{x=0}^{n} u^x v^{n-x} \binom{n}{x} \quad \star$$

$$= e^{\frac{-npt}{\sqrt{np(1-p)}}} \left(p e^{\frac{t}{\sqrt{np(1-p)}}} + 1 - p \right)^n$$

$\Rightarrow$

$$M_Z(t) = e^{\frac{-npt}{\sqrt{np(1-p)}}} \left(1 - p\left(1 - e^{\frac{t}{\sqrt{np(1-p)}}}\right)\right)^n$$

(a)

$$\text{cor}(X, Y) = E[(X - E[X])(Y - E[Y])]$$

$$= E[XY - XE[Y] - YE[X] + E[X]E[Y]]$$

$$= E[XY] - E[X]E[Y]$$

$$\text{cor}(X+A, Y+B) = E[(X+A)(Y+B)] - E[X+A]E[Y+B]$$

$$= E[XY + XB + AY + AB] - (E[X] + E[A])(E[Y] + E[B])$$

$$= E[XY] + E[XB] + E[AY] + E[AB] - E[X]E[Y] - E[X]E[B]$$

$$- E[A]E[Y] - E[A]E[B]$$

و $\mathrm{Cov}(A,B) = E[AB] - E[A]E[B]$

$$\rightarrow \mathrm{cov}(X+A, Y+B) - \mathrm{cov}(A,B) =$$

$$= E[XY] + E[XB] + E[AY] - E[X]E[Y] - E[X]E[B] - E[A]E[Y]$$

$$\rightarrow \mathrm{cov}(X+A, Y+B) - \mathrm{cov}(A,B) - 3\,\mathrm{cov}(X,Y)$$

$$= E[XY] + E[XB] + E[AY] - E[X]E[Y] - E[X]E[B] - E[A]E[Y]$$

$$- 3E[XY] + 3E[X]E[Y]$$

$$= -2E[XY] + E[XB] + E[AY] + 2E[X]E[Y] - E[X]E[B] - E[A]E[Y]$$

و می دانیم:

$$\boxed{E[E[X|Y]] = E[X]}$$

در آخر اثبات میکنیم

$$\boxed{E_{[XY]} = E_{[X E_{[Y|X]}]}}$$ و به دنبال آن می‌دانیم :

$$\Rightarrow E_{[XY]} = E_{[XB]} = E_{[YA]}$$

$$, E_{[A]} = E_{[X]} , E_{[B]} = E_{[Y]}$$

$$\Rightarrow -2E_{[XY]} + \overset{+E_{[XY]}}{E_{[XB]}} + \overset{+E_{[XY]}}{E_{[AY]}} + 2E_{[X]}E_{[Y]} - \overset{-E_{[X]}E_{[Y]}}{E_{[X]}E_{[B]}} - \overset{-E_{[X]}E_{[Y]}}{E_{[A]}E_{[Y]}}$$

$$= 0$$

$$\Rightarrow \operatorname{cov}(X+A, Y+B) - \operatorname{cov}(A,B) - 2\operatorname{cov}(X,Y) = 0$$

$$\Rightarrow \operatorname{cov}(X+A, Y+B) - \operatorname{cov}(A,B) = 2\operatorname{cov}(X,Y)$$

اثبات : $$\boxed{E_{[E_{[X|Y]}]} = E_{[X]}}$$

در یک حالت پیوسته پیوسته اثبات می‌کنیم :

$$E_{[E_{[X|Y]}]} = \sum_{y} E_{[X|Y=y]}\, P_{(Y=y)}$$

$$= \sum_{y} \sum_{x} x\, P_{(X=x | Y=y)}\, P_{(Y=y)}$$

$$= \sum_{y} \sum_{x} x \frac{P_{(X=x, Y=y)}}{\cancel{P_{(Y=y)}}}\, \cancel{P_{(Y=y)}}$$

$$= \sum_{x} x\, P_{(X=x)}$$

$$= E_{[X]} \rightarrow \boxed{E_{[E_{[X|Y]}]} = E_{[X]}}$$

در حالت پیوسته نیز تقریبا همین روند را باید طی کرد

$$E_{[XY]} = E_{[X E_{[Y|X]}]}$$ ☆ حالا اثبات می‌کنیم که :

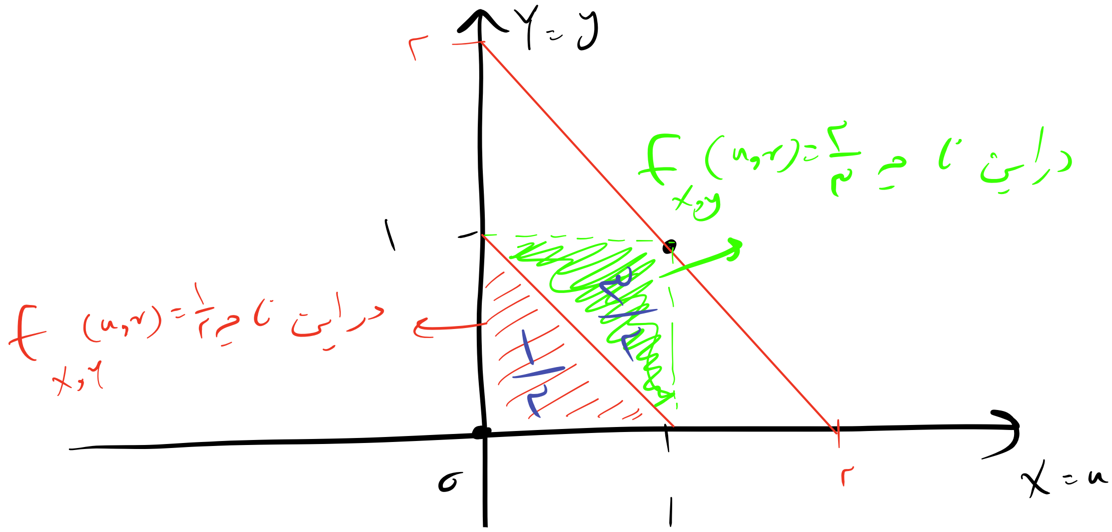
$$E[Y] = E\left[E[Y \mid X]\right]$$

$$\rightarrow E[XY] = E\left[E[XY \mid X]\right]$$

$$= E\left[X E[Y \mid X]\right]$$

۶) تلاشی در رسم

توزیع حاکم :



الف)

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$$

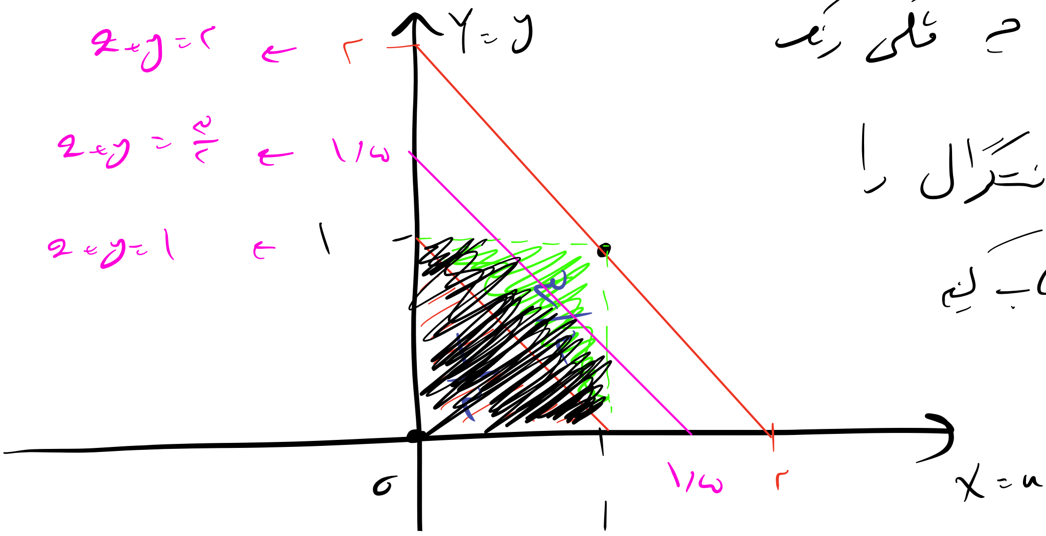
$$= \int_0^{1-x} \frac{1}{2}\, dy + \int_{1-x}^{1} \frac{3}{2}\, dy = \frac{1-x}{2} + \frac{3}{2} - \frac{3}{2}(1-x)$$

$$= x + \frac{1}{2} \rightarrow f_X(x) = \begin{cases} x + \frac{1}{2} & 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

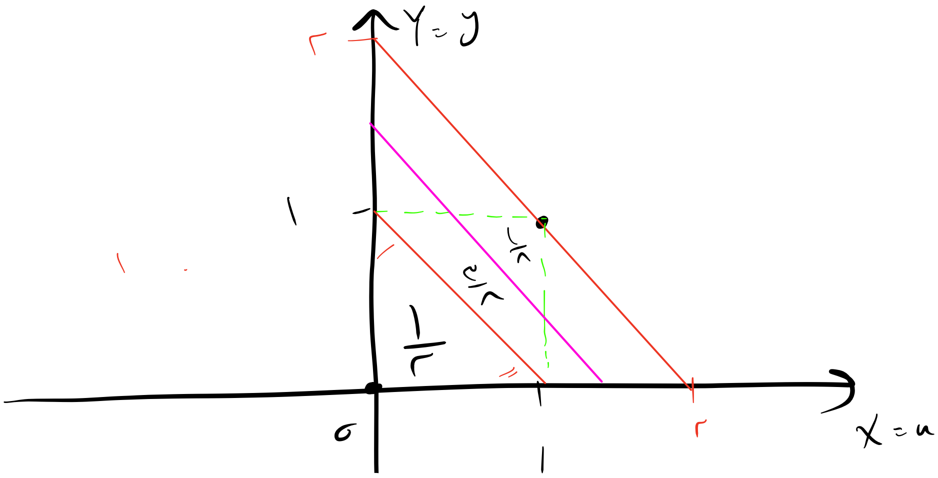
ب)

$$P\left(X+Y \leq \frac{3}{2}\right) = \iint_{\substack{x,y:\\ x+y \leq \frac{3}{2}}} f_{X,Y}(x,y)\, dx\, dy$$

که x و y در ناحیه مثلثی رنگ قرار می‌گیرد که انتگرال را اگر به شکل هندسی حساب کنیم داریم :



مساحت هر ناحیه :

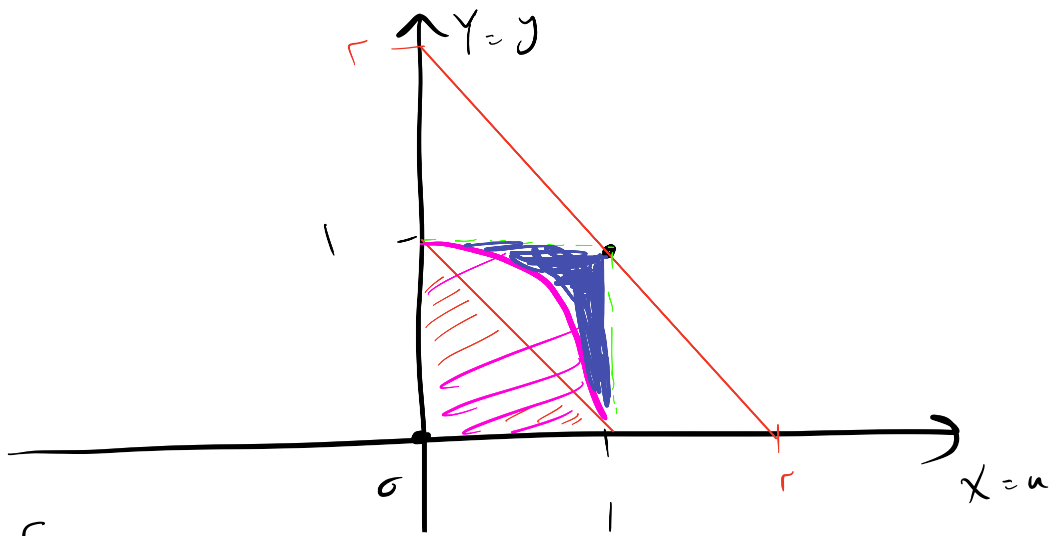


$$\Rightarrow \underbrace{\frac{1}{2}\times\frac{1}{2}}_{\substack{\iint \frac{1}{2} \\ x,y: \\ x+y\le 1}} + \underbrace{\frac{3}{8}\times\frac{3}{2}}_{\substack{\iint \frac{3}{2} \\ x,y: \\ x+y\ge 1 \\ x+y\le \frac{3}{2}}} = \underbrace{\frac{13}{16}}_{\substack{\iint f_{XY}(x,y) \\ x,y: \\ x,y\le \frac{3}{2}}}$$

$$\Rightarrow \boxed{P\left(X+Y\le \frac{3}{2}\right) = \frac{13}{16}}$$

$$P(X^2+Y^2\geq 1)$$

$$=\iint_{\substack{x,y:\\ x^2+y^2\geq 1}} f_{XY}(x,y)\,dx\,dy$$



$$=\iint_{\substack{x,y:\\ \text{x,y در ناحیه آبی باشد}}} \frac{3}{2} = \frac{3}{2}\times S_{\text{آبی رنگ}}$$

$$S_{\text{آبی}} = 1-\frac{\pi}{4}$$

$$\Rightarrow \iint_{\substack{x,y:\\ x^2+y^2\geq 1}} f_{XY}(x,y)\,dx\,dy = \left(1-\frac{\pi}{4}\right)\frac{3}{2}$$

$$\Rightarrow \boxed{P(X^2+Y^2\geq 1) = \frac{3}{2}-\frac{3\pi}{8}} \approx ٠/٣٢١٩$$

۷

$$\forall \, k, j \; (k \neq j): \quad Var(X_k) = \sigma^2, \quad cor(X_k, X_j) = \eta$$

ابتدا اثبات می‌کنیم که :

$$Cov\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} Cov(X_i, Y_j)$$

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} \mu_i$$

$$E\left[\sum_{i=1}^{m} Y_i\right] = \sum_{i=1}^{m} E[Y_i] = \sum_{j=1}^{m} r_j$$

$$Cov\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) =$$

$$= E\left[\left(\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \mu_i\right)\left(\sum_{j=1}^{m} Y_j - \sum_{j=1}^{m} r_j\right)\right]$$

$$= E\left[\left(\sum_{i=1}^{n} (X_i - \mu_i)\right)\left(\sum_{j=1}^{m} (Y_j - r_j)\right)\right]$$

$$= E\left[\sum_{i=1}^{n}\sum_{j=1}^{m}(X_i-\mu_i)(Y_j-\nu_j)\right]$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m} E\left[(X_i-\mu_i)(Y_j-\nu_j)\right] = \sum_{i=1}^{n}\sum_{j=1}^{m} \mathrm{cov}(X_i, Y_j)$$

$\Rightarrow$

$$\mathrm{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} \mathrm{Cov}(X_i, Y_j)$$

الف)

$$\mathrm{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{n} X_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{Cov}(X_i, X_j)$$

$$\Rightarrow \mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{cov}(X_i, X_j)$$

$\Rightarrow \mathrm{Var}(X_1+X_2+\dots+X_n) =$ مجموع کواریانس تمام هر ۲ تا از

X_i ها

$$= \sum_{i=1}^{n} cov(x_i, x_i) + \sum\sum_{i \neq j} cov(x_i, x_j)$$

$$= \sum_{i=1}^{n} var(x_i) + \sum\sum_{i \neq j} cov(x_i, x_j)$$

i تا n

$2\times\binom{n}{2}$ تا i و j

$$\Rightarrow Var\left(\sum_{i=1}^{n} X_i\right) = n\sigma^2 + 2\times\binom{n}{2}\eta$$

ب)

$$cov(Y_m, Y_{m+j}) = ?$$

$$= cov(X_m + X_{m+1} + X_{m+2}, X_{m+j} + X_{m+1+j} + X_{m+2+j})$$

طبق چیزی که اثبات کردم

$$= \sum_{i=m}^{m+2} \sum_{j=m+j}^{m+j+2} cov(X_i, X_j)$$

if $j = 0$: $\overset{i=j}{3\, var(X_i, X_i)} + \overset{i \neq j}{6\, cov(X_i, X_j)}$

$$= 3\sigma^2 + 6\eta$$

$x_m \quad x_{m+1} \quad x_{m+2} \quad x_{n+1} \quad x_{n+2} \quad x_{n+3}$

$i \neq j$

if $j=1$: $2\sigma^2 + 7\eta$

$i=j \to x_{m+1} \, x_{n+1}$

$x_{m+2} \, , \, x_{n+2}$

---

if $j=2$: $\sigma^2 + 8\eta$

جز x_{m+3} و x_{m+2} بقیه ۸ ترکیب $i \neq j$

---

if $j \geq 3$: 9η

همه $i=j$ تساوی و هر ترکیبی $i \neq j$ می‌سازد

$$\Rightarrow cov(Y_m , Y_{m+1}) = \begin{cases} 4\sigma^2 + 6\eta & j=0 \\ 2\sigma^2 + 7\eta & j=1 \\ \sigma^2 + 8\eta & j=2 \\ 9\eta & j \geq 3 \end{cases}$$

که چون ذکر شده بود در بخش (ب) متغیرها از هم مستقل

اند پس $\eta = 0$ است پس:

$$\operatorname{Cov}(Y_n, Y_{n+1}) = \begin{cases} 3\sigma^2 & j = 0 \\ 2\sigma^2 & j = 1 \\ \sigma^2 & j = 2 \\ 0 & j \geq 3 \end{cases}$$