CSC518 **HW #1**

NAME: Alexander Jansing and Brittany Zeo **Mon. 21 September 2015**

The kets $|+\rangle, |-\rangle, |0\rangle$, and $|1\rangle$ are defined as follows:

$$|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |+\rangle := \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad |-\rangle := \tfrac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

1 Watch the video on YouTube (`https://www.youtube.com/watch?v=x6eR2vjdddY`).

2 Consider the 2-qubit state

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle|0\rangle + \frac{1}{\sqrt{2}}|1\rangle|1\rangle.$$

Show that this state is entangled by proving that there are no possible values $\alpha_0, \alpha_1, \beta_0, \beta_1$ such that

$$|\psi\rangle = \big(\alpha_0|0\rangle + \alpha_1|1\rangle\big)\big(\beta_0|1\rangle + \beta_1|1\rangle\big).$$

Proof: If there existed values $\alpha_0, \alpha_1, \beta_0, \beta_1$ such that

$$|\psi\rangle = \big(\alpha_0|0\rangle + \alpha_1|1\rangle\big)\big(\beta_0|1\rangle + \beta_1|1\rangle\big),$$

then

$$|\psi\rangle = \alpha_0\beta_0|0\rangle|0\rangle + \alpha_0\beta_1|0\rangle|1\rangle + \alpha_1\beta_0|1\rangle|0\rangle + \alpha_1\beta_1|1\rangle|1\rangle$$

implies that there exist some numbers such that

$$\alpha_0\beta_0 = \alpha_1\beta_1 = \frac{1}{\sqrt{2}}, \text{ and } \alpha_0\beta_1 = \alpha_1\beta_0 = 0.$$

$$\alpha_0\beta_1 \times \beta_0 = 0 \times \beta_0$$

$$\text{then } \frac{\alpha_0\beta_0\beta_1}{\beta_1} = \frac{0}{\beta_1}.$$

That implies $\alpha_0\beta_0 = \frac{1}{\sqrt{2}} = 0 \rightarrow\leftarrow$.

Therefore, there are no possible values $\alpha_0, \alpha_1, \beta_0, \beta_1$ such that $|\psi\rangle = \big(\alpha_0|0\rangle + \alpha_1|1\rangle\big)\big(\beta_0|1\rangle + \beta_1|1\rangle\big)$. ■

3 Prove that $|\psi\rangle|\beta_{00}\rangle = \frac{1}{2}|\beta_{00}\rangle|\psi\rangle + \frac{1}{2}|\beta_{01}\rangle(X|\psi\rangle) + \frac{1}{2}|\beta_{10}\rangle(Z|\psi\rangle) + \frac{1}{2}|\beta_{00}\rangle(XZ|\psi\rangle)$.

Proof: We know that,

$$\begin{array}{ccccc} & & |\psi\rangle & = & a|0\rangle + b|1\rangle, \\ I & = & |\beta_{00}\rangle & = & \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \\ X & = & |\beta_{01}\rangle & = & \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \\ Z & = & |\beta_{10}\rangle & = & \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \\ XZ & = & |\beta_{11}\rangle & = & \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle). \end{array}$$

So,

$$\begin{array}{rcl} |\psi\rangle|\beta_{00}\rangle & = & \frac{1}{2}|\beta_{00}\rangle|\psi\rangle + \frac{1}{2}|\beta_{01}\rangle(X|\psi\rangle) \\ & + & \frac{1}{2}|\beta_{10}\rangle(Z|\psi\rangle) + \frac{1}{2}|\beta_{00}\rangle(XZ|\psi\rangle), \end{array}$$

$$\begin{array}{rcl} (a|0\rangle + b|1\rangle)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) & = & \frac{1}{2}\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)(a|0\rangle + b|1\rangle) \\ & & +\frac{1}{2}\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)(X[a|0\rangle + b|1\rangle]) \\ & & +\frac{1}{2}\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)(Z[a|0\rangle + b|1\rangle]) \\ & & +\frac{1}{2}\frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)(XZ[a|0\rangle + b|1\rangle]). \end{array}$$

$$\begin{array}{rcl} (a|0\rangle + b|1\rangle)(|00\rangle + |11\rangle) & = & \frac{1}{2}[(|00\rangle + |11\rangle)(a|0\rangle + b|1\rangle) \\ & & (|01\rangle + |10\rangle)(X[a|0\rangle + b|1\rangle]) \\ & & (|00\rangle - |11\rangle)(Z[a|0\rangle + b|1\rangle]) \\ & & (|10\rangle - |01\rangle)(XZ[a|0\rangle + b|1\rangle]). \end{array}$$

$$\begin{array}{rcl} a|000\rangle + b|100\rangle + a|011\rangle + b|111\rangle & = & \frac{1}{2}[(a|000\rangle + b|001\rangle + a|110\rangle + b|111\rangle) \\ & & +(a|011\rangle + b|010\rangle + a|101\rangle + b|100\rangle) \\ & & +(a|000\rangle - b|001\rangle - a|110\rangle + b|111\rangle) \\ & & +(a|011\rangle - b|010\rangle - a|101\rangle + b|100\rangle). \end{array}$$

$$a|000\rangle + b|100\rangle + a|011\rangle + b|111\rangle = \frac{1}{2}(2)(a|000\rangle + b|100\rangle + a|011\rangle + b|111\rangle)$$

Therefore, $|\psi\rangle|\beta_{00}\rangle = \frac{1}{2}|\beta_{00}\rangle|\psi\rangle + \frac{1}{2}|\beta_{01}\rangle(X|\psi\rangle) + \frac{1}{2}|\beta_{10}\rangle(Z|\psi\rangle) + \frac{1}{2}|\beta_{00}\rangle(XZ|\psi\rangle)$. ∎

4 Which of the sets are orthonormal basis of $\mathbb{C}^2$?

a $\{|+\rangle, |-\rangle\}$

$$|+\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$
$$|-\rangle := \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$
$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} = \frac{1}{2} - \frac{1}{2} = 0$$

$\perp \in \mathbb{C}$

b $\{|0\rangle, |1\rangle\}$

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

$\perp \in \mathbb{C}$

c $\{|0\rangle - |1\rangle, |1\rangle + |0\rangle\}$

$$|0\rangle - |1\rangle = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
$$|1\rangle + |0\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 - 1 = 0$$

$\perp \in \mathbb{C}$

d $\{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}\}$

$$\frac{1}{\sqrt{2}}\left(\begin{pmatrix} 1 \\ i \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -i \end{pmatrix} \right) = \frac{2}{\sqrt{2}} + \frac{i-i}{\sqrt{2}} = \frac{2}{\sqrt{2}}$$

NOT $\perp \in \mathbb{C}$

e $\{\frac{1}{\sqrt{2}} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}\}$

$$\frac{1}{\sqrt{2}}\left(\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \cdot \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix} \right) = \frac{1}{\sqrt{2}}\big(\cos\theta\sin\theta - \cos\theta\sin\theta\big) = 0$$

$\perp \in \mathbb{C}$

f $\{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}\}$

$$\frac{1}{\sqrt{2}}\left(\begin{pmatrix} 1 \\ i \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) = \frac{1}{\sqrt{2}}(1 - i) \neq 0$$

NOT $\perp \in \mathbb{C}$

5 Let $|\psi\rangle, |\varphi\rangle$ be an orthonormal basis in the Hilbert space $\mathbb{C}^2$.
Let $A := |\psi\rangle\langle\psi| + |\varphi\rangle\langle\varphi|$.
Find the matrix representation of A with respect to the basis $\{|0\rangle, |1\rangle\}$ where $|\psi\rangle$ and $|\varphi\rangle$

are as follows:

a $|\psi\rangle := \binom{1}{0}, |\varphi\rangle := \binom{0}{1}$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

b $|\psi\rangle := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, |\varphi\rangle := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

c $|\psi\rangle := \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, |\varphi\rangle := \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$

$$\begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} + \begin{pmatrix} \sin^2\theta & -\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos^2\theta \end{pmatrix} =$$

$$\begin{pmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \sin^2\theta + \cos^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

6 $T = |0\rangle\langle 0| \otimes (|+\rangle\langle +| + |-\rangle\langle -|) + |1\rangle\langle 1| \otimes (|+\rangle\langle +| - |-\rangle\langle -|)$.
Show that T and $CNOT$ are equal.

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{1}{2}\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.$$

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = CNOT$$

7 Compute:

a $\langle 10||+-\rangle$ and $\langle 1||+\rangle\langle 0||-\rangle$

$$\langle 10||+-\rangle = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} \\ \frac{-1}{2} \\ \frac{1}{2} \\ \frac{-1}{2} \end{pmatrix} = \frac{1}{2}$$

$$\langle 1||+\rangle\langle 0||-\rangle = \begin{pmatrix} 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} \otimes \frac{1}{\sqrt{2}} = \frac{1}{2}$$

b $\langle 01||-+\rangle$ and $\langle 0||-\rangle\langle 1||+\rangle$

$$\langle 01||-+\rangle = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{-1}{2} \\ \frac{-1}{2} \end{pmatrix} = \frac{1}{2}$$

$$\langle 0||-\rangle\langle 1||+\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} \otimes \frac{1}{\sqrt{2}} = \frac{1}{2}$$

c Let $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$ and $|\phi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle$, where $|\psi_1\rangle, |\psi_2\rangle, |\phi_1\rangle, |\phi_2\rangle$ are arbitrary vectors in $\mathbb{C}^2$. Show that $\langle\psi||\phi\rangle = \langle\psi_1||\phi_1\rangle\langle\psi_2||\phi_2\rangle$.
For arbitrary values a, b, c, d, e, f, g, and $h \in \mathbb{C}$,

$$|\psi_1\rangle = a|0\rangle + b|1\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$|\psi_2\rangle = c|0\rangle + d|1\rangle = \begin{pmatrix} c \\ d \end{pmatrix}$$

$$|\phi_1\rangle = e|0\rangle + f|1\rangle = \begin{pmatrix} e \\ f \end{pmatrix}$$

$$|\phi_2\rangle = g|0\rangle + h|1\rangle = \begin{pmatrix} g \\ h \end{pmatrix}$$

$$|\psi\rangle = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix}, |\phi\rangle = \begin{pmatrix} eg \\ eh \\ fg \\ fh \end{pmatrix}$$

$\langle\psi||\phi\rangle = aceg + adeh + bcfg + bdfh$
$\langle\psi_1||\phi_1\rangle\langle\psi_2||\phi_2\rangle = (ae + bf) \otimes (cg + dh) = aecg + aedh + bfcg + bfdh.$