# Newton-Raphson Method
# for
# Nonlinear Systems

**Recall that Newton-Raphson method was predicated on employing the derivative of a function to estimate its intercept with the axis of the independent variable-that is the root. This estimate was based on the first order Taylor Series expansion**

$$f(x_{i+1}) = f(x_i) + (x_{i+1} - x_i) f^{'}(x_i)$$

**Where $x_i$ is the initial guess at the root and $x_{i+1}$ is the point at which the slope intercepts the x axis. At this intercept equating the zero yields**

$$x_{i+1} = x_i - \frac{f(x_i)}{f^{'}(x_i)}$$

**which is the single-equation form of the Newton's method.**

**We begin with the forms**

$$f(x, y) = 0,$$
$$g(x, y) = 0.$$

**The multiequation form is derived in an identical fashion. However, a multivariable Taylor series must be used to account for the fact that more than one variable contributes to the determination of the root. For the two variable cases, a <u>first order</u> Taylor Series can be written for <u>each nonlinear equation</u> as**

$$f(x_{i+1}, y_{i+1}) = f(x_i, y_i) + (x_{i+1} - x_i)\frac{\partial f}{\partial x} + (y_{i+1} - y_i)\frac{\partial f}{\partial y}$$

**and**

$$g(x_{i+1}, y_{i+1}) = g(x_i, y_i) + (x_{i+1} - x_i)\frac{\partial g}{\partial x} + (y_{i+1} - y_i)\frac{\partial g}{\partial y}$$

**Just as for the single equation version, the root estimate corresponds to the values of x and y, where** $f(x_{i+1}, y_{i+1})$ **and** $g(x_{i+1}, y_{i+1})$ **<u>equal zero</u>.**

**Equations can be rearranged to give**

$$\frac{\partial f}{\partial x} x_{i+1} + \frac{\partial f}{\partial y} y_{i+1} = -f(x_i, y_i) + x_i \frac{\partial f}{\partial x} + y_i \frac{\partial f}{\partial y}$$

$$\frac{\partial g}{\partial x} x_{i+1} + \frac{\partial g}{\partial y} y_{i+1} = -g(x_i, y_i) + x_i \frac{\partial g}{\partial x} + y_i \frac{\partial g}{\partial y}$$

**Thus we obtain is a set of two linear equations with two unknowns.**

**We convert nonlinear system solution to the linear system solution**

**From these equations we obtain**

$$x_{i+1} = x_i - \frac{f(x_i, y_i)\left\{\dfrac{\partial g}{\partial y}\right\}_{x_i, y_i} - g(x_i, y_i)\left\{\dfrac{\partial f}{\partial y}\right\}_{x_i, y_i}}{\left|\begin{matrix} \left\{\dfrac{\partial f}{\partial x}\right\} & \left\{\dfrac{\partial f}{\partial y}\right\} \\ \left\{\dfrac{\partial g}{\partial x}\right\} & \left\{\dfrac{\partial g}{\partial y}\right\} \end{matrix}\right|_{x_i, y_i}}$$

**and**

$$y_{i+1} = y_i - \frac{g(x_i, y_i)\left\{\dfrac{\partial f}{\partial x}\right\}_{x_i, y_i} - f(x_i, y_i)\left\{\dfrac{\partial g}{\partial x}\right\}_{x_i, y_i}}{\left|\begin{matrix} \left\{\dfrac{\partial f}{\partial x}\right\} & \left\{\dfrac{\partial f}{\partial y}\right\} \\ \left\{\dfrac{\partial g}{\partial x}\right\} & \left\{\dfrac{\partial g}{\partial y}\right\} \end{matrix}\right|_{x_i, y_i}}$$

**The denominator of each of these equations is formally referred to as the determinant of the *Jacobian* of the system.**

# Jacobian of the system

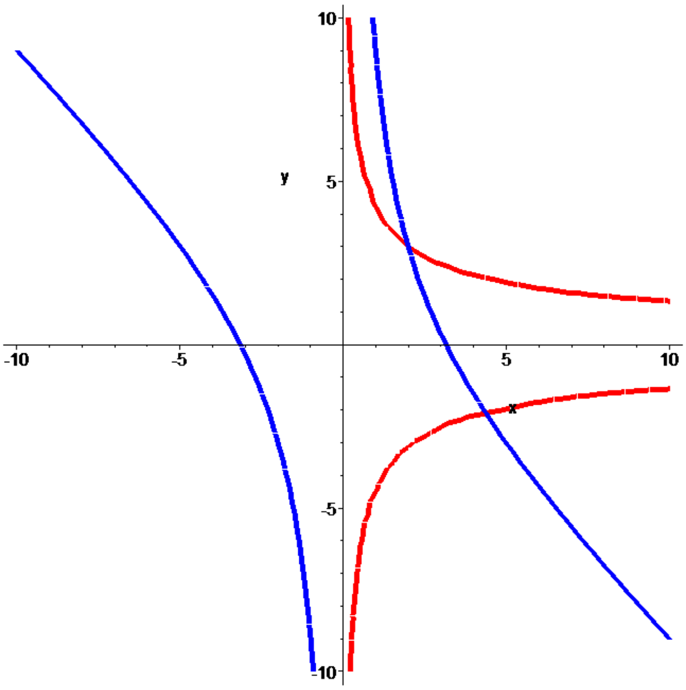
$$\begin{bmatrix} \left\{\dfrac{\partial f}{\partial x}\right\} & \left\{\dfrac{\partial f}{\partial y}\right\} \\ \left\{\dfrac{\partial g}{\partial x}\right\} & \left\{\dfrac{\partial g}{\partial y}\right\} \end{bmatrix}$$

**Example:**

$$f(x,y) = x^2 + xy - 10 = 0$$

$$g(x,y) = y + 3xy^2 - 57 = 0$$

**Obtain the first iterates with guesses of x=1.5 and y=3.5.**



**Solution:**

**First compute the partial derivatives and evaluate them at the initial value**

$$\left\{\frac{\partial f}{\partial x}\right\}_{x_0,y_0} = \{2x+y\}_{x_0,y_0} = \{2x+y\}_{1.5,3.5} = 6.5$$

$$\left\{\frac{\partial f}{\partial y}\right\}_{x_0,y_0} = \{x\}_{x_0,y_0} = \{x\}_{1.5,3.5} = 1.5$$

$$\left\{\frac{\partial g}{\partial x}\right\}_{x_0,y_0} = \{3y^2\}_{x_0,y_0} = \{3y^2\}_{1.5,3.5} = 36.75$$

$$\left\{\frac{\partial g}{\partial y}\right\}_{x_0,y_0} = \{1+6xy\}_{x_0,y_0} = \{1+6xy\}_{1.5,3.5} = 32.5$$

Thus the determinant of the **Jacobian** for the first iteration is

$$\left|\begin{matrix}\left\{\frac{\partial f}{\partial x}\right\} & \left\{\frac{\partial f}{\partial y}\right\} \\ \left\{\frac{\partial g}{\partial x}\right\} & \left\{\frac{\partial g}{\partial y}\right\}\end{matrix}\right|_{(x_0,y_0)} = \left|\begin{matrix}6.5 & 1.5 \\ 36.75 & 32.5\end{matrix}\right|$$

$$= 6.5(32.5) - 1.5(36.75) = 156.25$$

**The values of the functions can be evaluated at $x_0$, $y_0$ as**

$$f(x_0, y_0) = -2.5$$
$$g(x_0, y_0) = 1.625$$

**Then**

$$x_1 = 1.5 - \frac{-2.5(32.5) - 1.625(1.5)}{156.25} = 2.03603$$

$$y_1 = 3.5 - \frac{1.625(6.5) - (-2.5)(36.75)}{156.25} = 2.84388$$

**Compute the second iterations $x_2$ and $y_2$.**

$$\left| \begin{matrix} \left\{ \dfrac{\partial f}{\partial x} \right\} & \left\{ \dfrac{\partial f}{\partial y} \right\} \\ \left\{ \dfrac{\partial g}{\partial x} \right\} & \left\{ \dfrac{\partial g}{\partial y} \right\} \end{matrix} \right|_{(x_1, y_1)} = ?$$

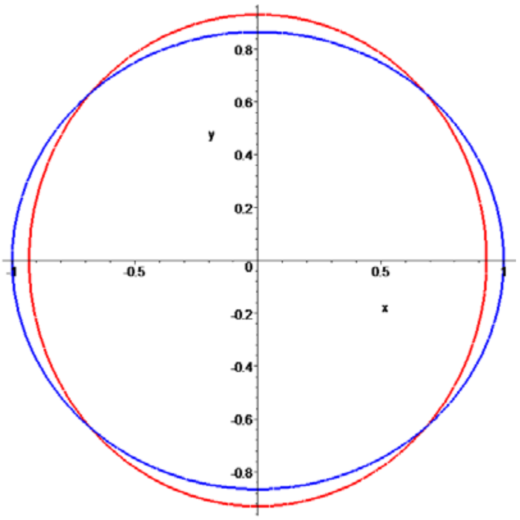
# MAPLE SOLUTION!!!

```
>> [x,y]=solve('x^2+x*y-10=0','y+3*x*y^2-57=0')
x = [
2][
1/6*(4340+4*581717^(1/2))^(1/3)+106/3/(4340+4*581717^(1/2))^(1/3)-2/3]
[ -1/12*(4340+4*581717^(1/2))^(1/3)-53/3/(4340+4*581717^(1/2))^(1/3)-
2/3+1/2*i*3^(1/2)*(1/6*(4340+4*581717^(1/2))^(1/3)-
106/3/(4340+4*581717^(1/2))^(1/3))]
[ -1/12*(4340+4*581717^(1/2))^(1/3)-53/3/(4340+4*581717^(1/2))^(1/3)-2/3-
1/2*i*3^(1/2)*(1/6*(4340+4*581717^(1/2))^(1/3)-
106/3/(4340+4*581717^(1/2))^(1/3))]
y =[  3]
[6/31*(1/6*(4340+4*581717^(1/2))^(1/3)+106/3/(4340+4*581717^(1/2))^(1/3
)-2/3)^2-256/93-19/186*(4340+4*581717^(1/2))^(1/3)-
2014/93/(4340+4*581717^(1/2))^(1/3)]
[ 6/31*(-1/12*(4340+4*581717^(1/2))^(1/3)-
53/3/(4340+4*581717^(1/2))^(1/3)-
2/3+1/2*i*3^(1/2)*(1/6*(4340+4*581717^(1/2))^(1/3)-
106/3/(4340+4*581717^(1/2))^(1/3)))^2-
256/93+19/372*(4340+4*581717^(1/2))^(1/3)+1007/93/(4340+4*581717^(1/2
))^(1/3)-19/62*i*3^(1/2)*(1/6*(4340+4*581717^(1/2))^(1/3)-
106/3/(4340+4*581717^(1/2))^(1/3))]
[ 6/31*(-1/12*(4340+4*581717^(1/2))^(1/3)-
53/3/(4340+4*581717^(1/2))^(1/3)-2/3-
1/2*i*3^(1/2)*(1/6*(4340+4*581717^(1/2))^(1/3)-
106/3/(4340+4*581717^(1/2))^(1/3)))^2-
256/93+19/372*(4340+4*581717^(1/2))^(1/3)+1007/93/(4340+4*581717^(1/2
))^(1/3)+19/62*i*3^(1/2)*(1/6*(4340+4*581717^(1/2))^(1/3)-
106/3/(4340+4*581717^(1/2))^(1/3))]
>>
```

**Example:**

$$3x^2 + 4y^2 - 3 = 0,$$
$$x^2 + y^2 - \sqrt{3}/2 = 0.$$

The first equation represents an ellipse.
The second equation represents a circle.
Both curves are centered at the origin.



## The Jacobian of this system

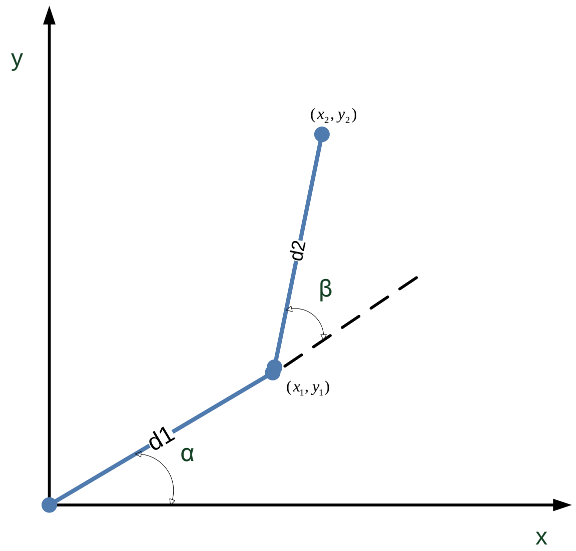
$$\begin{bmatrix} 6x & 8y \\ 2x & 2y \end{bmatrix}$$

$$X^{(0)} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

| $k$ | $x^{(k)}$ | $y^{(k)}$ |
|---|---|---|
| **0** | **0.5** | **0.5** |
| **1** | **0.7141** | **0.65192** |
| **2** | **0.68201** | **0.63422** |
| **3** | **0.68125** | **0.63397** |
| **4** | **0.68125** | **0.63397** |

**Example:**

**Two-link robot arm.**




**We need to solve for the unknown angles $\alpha$ and $\beta$.**

$$x = d_1\cos(\alpha) + d_2\cos(\alpha+\beta)$$
$$y = d_1\sin(\alpha) + d_2\sin(\alpha+\beta)$$

**Let** $d_1 = 5, d_2 = 6$**.**

**We wish to find the angles so that the arm will move to the point (10, 4).**

**Initial angles** $\alpha^{(0)} = 0.7$ $\beta^{(0)} = 0.7$

**The system of equations in this case is**

$$5\cos(\alpha) + 6\cos(\alpha+\beta) - 10 = 0,$$
$$5\sin(\alpha) + 6\sin(\alpha+\beta) - 4 = 0.$$

## Obtain Jacobian of the given system and check the following table.

| $k$ | $\alpha^{(k)}$ | $\beta^{(k)}$ | $\lVert\Delta\rVert$ |
|---|---|---|---|
| **0** | **0.7** | **0.7** | |
| **1** | **-0.59855** | **1.8339** | **1.724** |
| **2** | **-0.10782** | **0.89987** | **1.0551** |
| **3** | **0.086882** | **0.53893** | **0.4101** |
| **4** | **0.14791** | **0.426** | **0.12837** |
| **5** | **0.155585** | **0.41139** | **0.016621** |
| **6** | **0.15598** | **0.41114** | **0.00029053** |

$\Delta = x_{new} - x_{old}$.

# The General Form of a System of Nonlinear Equations

**Let**

$$F(x_1,x_2,\ldots,x_n)=\begin{pmatrix} f_1(x_1,x_2,\ldots,x_n) \\ f_2(x_1,x_2,\ldots,x_n) \\ f_3(x_1,x_2,\ldots,x_n) \\ . \\ . \\ . \\ f_n(x_1,x_2,\ldots,x_n) \end{pmatrix}$$

**Defining the Jacobian matrix J(x) by**

$$J(x)=\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ . & & & \\ . & & & \\ . & & & \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

**We know from the fixed point iteration**

$$x^{(k)} = G(x^{(k-1)})$$

**The function G is defined[*] by**

$$G(x) = x - J(x)^{-1}F(x)$$

**And the functional iteration procedure evolves from selecting** $x^{(0)}$ **and generating for** $k \geq 1$ ,

$$x^{(k)} = G(x^{(k-1)}) = x^{(k-1)} - J(x^{(k-1)})^{-1}F(x^{(k-1)})$$

**This method is called, Newton's method for nonlinear systems and is generally expected to give quadratic converge, provided that a sufficiently accurate starting value is known and inverse of the Jacobian matrix exists.**

*** Burden, R. Numerical Analysis p.498**

- **First calculate**

$$F(x) \text{ and } J(x)$$

- **Then solve n x n linear system**

$$J(x)y = -F(x)$$

- **And set**

$$x = x + y$$

# Newton's Method for Systems Algorithm

**To approximate the solution of the nonlinear system F(x)=0 given an initial approximation x:**

**INPUT number n of equations and unknowns; initial approximation**
**$x=(x_1,x_2\ldots,x_n)^t$ , Tolerance TOL, Maximum iterations.**
**OUTPUT approximate solution $x=(x_1,x_2\ldots,x_n)^t$ or a message that**
***number of iteration was exceeded*.**

**Step 1 Set k=1**
**Step 2 While(k≤) do Steps 3-7.**
**Step 3 Calculate F(x) and J(x)**
$J(x)_{i,j} = (\partial f_i(x)/\partial x_j)$ **for $1 \le i, j \le n$**
**Step 4 Solve n x n linear system J(x)y=-F(x)**
**Step 5 Set x=x + y**
**Step 6 If||y|| < TOL Then Output (x)**
**(Procedure completed successfully)**
**Step 7 Set k= k+1,**
**Step 8 OUTPUT('Maximum number of iterations exceeded');**
**STOP**

## Example:

**Given the nonlinear system**

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 - 1 &= 0, \\ x_1^2 \qquad + x_3^2 - 1/4 &= 0, \\ x_1^2 + x_2^2 - 4x_3 &= 0. \end{aligned}$$

$$F(x) = \begin{Bmatrix} x_1^2 + x_2^2 + x_3^2 - 1 \\ x_1^2 + x_3^2 - 1/4 \\ x_1^2 + x_2^2 - 4x_3 \end{Bmatrix}$$

**Newton's method to obtain the first seven iterates with the initial approximation is**

$$x^{(0)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

**Solution:**

**The Jacobian matrix J(x) for this system is given by**

$$J(x) = \begin{bmatrix} 2x_1 & 2x_2 & 2x_3 \\ 2x_1 & 0 & 2x_3 \\ 2x_1 & 2x_2 & -4 \end{bmatrix}$$

**and**

$$\begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ x_3^{(k)} \end{bmatrix} = \begin{bmatrix} x_1^{(k-1)} \\ x_2^{(k-1)} \\ x_3^{(k-1)} \end{bmatrix} + \begin{bmatrix} y_1^{(k-1)} \\ y_2^{(k-1)} \\ y_3^{(k-1)} \end{bmatrix}$$

**Where**

$$\begin{bmatrix} y_1^{(k-1)} \\ y_2^{(k-1)} \\ y_3^{(k-1)} \end{bmatrix} = -(J(x_1^{(k-1)}, x_2^{(k-1)}, x_3^{(k-1)}))^{-1} F(x_1^{(k-1)}, x_2^{(k-1)}, x_3^{(k-1)})$$

$$J(x^{(0)}) = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & -4 \end{bmatrix}$$

**Inverse of the Jacobian matrix:**

$$J^{-1}(x^{(0)}) = \begin{bmatrix} -0.1667 & 0.5000 & 0.1667 \\ 0.5000 & -0.5000 & 0 \\ 0.1667 & 0 & -0.1667 \end{bmatrix}$$

$$F(x^{(0)}) = \begin{Bmatrix} 2 \\ 1.75 \\ -2 \end{Bmatrix}$$

$$J^{-1}(x^{(0)})F(x^{(0)}) = \begin{bmatrix} -0.1667 & 0.5000 & 0.1667 \\ 0.5000 & -0.5000 & 0 \\ 0.1667 & 0 & -0.1667 \end{bmatrix} \begin{bmatrix} 2 \\ 1.75 \\ -2 \end{bmatrix} = \begin{bmatrix} 0.2083 \\ 0.1250 \\ 0.6667 \end{bmatrix}$$

$$\begin{bmatrix} y_1^{(0)} \\ y_2^{(0)} \\ y_3^{(0)} \end{bmatrix} = -\begin{bmatrix} 0.20833 \\ 0.1250 \\ 0.6667 \end{bmatrix}$$

$$x = x + y$$

$$\begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{bmatrix} = \begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \end{bmatrix} + \begin{bmatrix} y_1^{(0)} \\ y_2^{(0)} \\ y_3^{(0)} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -0.20833 \\ -0.1250 \\ -0.6667 \end{bmatrix} = \begin{bmatrix} 0.79167 \\ 0.875 \\ 0.3333 \end{bmatrix}$$

$$\Delta x^{(1)} = \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{bmatrix} - \begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \end{bmatrix} = \begin{bmatrix} 0.79167 \\ 0.875 \\ 0.3333 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.20833 \\ -0.125 \\ -0.6667 \end{bmatrix}$$

$$\left\| \Delta x^{(1)} \right\| = \sqrt{0.043401 + 0.015625 + 0.444489} = \sqrt{0.503503} = 0.709588$$

| $k$ | $x_1^{(k)}$ | $x_2^{(k)}$ | $x_3^{(k)}$ | $\lVert\Delta x\rVert$ |
|---|---|---|---|---|
| **0** | **1.00000** | **1.00000** | **1.00000** | |
| **1** | **0.79167** | **0.875** | **0.33333** | **0.70959** |
| **2** | **0.44365** | **0.86607** | **0.42875** | **0.36111** |
| **3** | **0.28927** | **0.86603** | **0.44538** | **0.16405** |
| **4** | **0.2296** | **0.86603** | **0.44705** | **0.0507** |
| **5** | **0.22371** | **0.86603** | **0.4472** | **0.0058853** |
| **6** | **0.22361** | **0.86603** | **0.44721** | **0.00010352** |
| **7** | **0.22361** | **0.86603** | **0.44721** | **2.4665e-06** |

**Example:**

**Given the nonlinear system**

$$\begin{aligned}
&3x_1 - \cos(x_2x_3) - \frac{1}{2} = 0,\\
&x_1^2 - 81(x_2 + 0.1)^2 + \sin(x_3) + 1.06 = 0,\\
&e^{-x_1x_2} + 20x_3 + \frac{10\pi - 3}{3} = 0
\end{aligned}$$

$$F(x) = \begin{Bmatrix} 3x_1 - \cos(x_2x_3) - 0.5 \\ x_1^2 - 81(x_2 + 0.1)^2 + \sin(x_3) + 1.06 \\ e^{-x_1x_2} + 20x_3 + \dfrac{10\pi - 3}{3} \end{Bmatrix}$$

**Use Newton's method to obtain the first five iterates with the initial approximation is**

$$x^{(0)} = \begin{bmatrix} 0.1 \\ 0.1 \\ -0.1 \end{bmatrix}$$

**Solution:**

**The Jacobian matrix J(x) for this system is given by**

$$J(x_1, x_2, x_3) = \begin{bmatrix} 3 & x_3 \sin(x_2 x_3) & x_2 \sin(x_2 x_3) \\ 2x_1 & 162(x_2 + 0.1) & \cos(x_3) \\ -x_2 e^{-x_1 x_2} & -x_1 e^{-x_1 x_2} & 20 \end{bmatrix}$$

**and**

$$\begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ x_3^{(k)} \end{bmatrix} = \begin{bmatrix} x_1^{(k-1)} \\ x_2^{(k-1)} \\ x_3^{(k-1)} \end{bmatrix} + \begin{bmatrix} y_1^{(k-1)} \\ y_2^{(k-1)} \\ y_3^{(k-1)} \end{bmatrix}$$

**Where**

$$\begin{bmatrix} y_1^{(k-1)} \\ y_2^{(k-1)} \\ y_3^{(k-1)} \end{bmatrix} = -(J(x_1^{(k-1)}, x_2^{(k-1)}, x_3^{(k-1)}))^{-1} F(x_1^{(k-1)}, x_2^{(k-1)}, x_3^{(k-1)})$$

$$F(x_1^{k-1}, x_2^{k-1}, x_3^{k-1}) = \begin{bmatrix} 3x_1^{(k-1)} - \cos(x_2^{(k-1)} x_3^{(k-1)} - 0.5 \\ (x_1^2)^{(k-1)} - 81(x_2^{(k-1)} + 0.1)^2 + \sin(x_3^{(k-1)}) + 1.06 \\ e^{x_1^{(k-1)} x_2^{(k-1)}} + 20x_3^{(k-1)} + \dfrac{10\pi - 3}{3} \end{bmatrix}$$

**The results obtained using these iterative procedures are shown in the following table.**

| $k$ | $x_1^{(k)}$ | $x_2^{(k)}$ | $x_3^{(k)}$ | $\|\Delta x\|$ |
|---|---|---|---|---|
| **0** | **0.10000000** | **0.10000000** | **-0.10000000** | |
| **1** | **0.50003702** | **0.01946686** | **-0.52152047** | |
| **2** | **0.50004593** | **0.00158859** | **-0.52355711** | |
| **3** | **0.50000034** | **0.00001244** | **-0.52359845** | |
| **4** | **0.50000000** | **0.00000000** | **-0.52359877** | |
| **5** | **0.50000000** | **0.00000000** | **-0.52359877** | **0.00000** |

$$\left\| x^{(5)} - x^{(4)} \right\| = 0$$

**(Compare results with Fixed Point solution solved before)**

## MATLAB M-File (Newton’s for nonlinear Systems)

```
function X=Newtonsys(F,JF, x0 ,tol, maxit)
%          FAUSETT 5.1.2
%         Solve the nonlinear system F(x)=0 using Newton's
Method
%         vectors x and x0 are rowvectors
%          function F returns a column vector
% stop     if norm of change in solutişon vector is less than tol
value
% solve JF(x) y=-F(x) using Matlab's "backslash operator"
%     y=-feval(JF, x.old) \ feval(F, x.old) ;
% the next approximate solution is x.new=x.old+y';P is the inital
approximation to the solution
x.old=x0;
disp([0 x.old]);
iter=1;
while (iter <=maxit)
       y=-feval(JF, x.old) \ feval(F, x.old) ;
       x.new=x.old+y';
        diff=norm(x.new-x.old);
        disp('Newton method has converged)')
        return;
    else
        x.old=x.new;
    end
    iter=iter+1;
 end
 disp('Newron method did not Converge')
 x=x.new;
```

# ANOTHER FILE

```
function [P,iter,err]=newdim(F,JF,P,delta,epsilon,maxit)
%Input    -F is the system saved as the M-file F.m
%         -JF is the Jacobian of F saved as the M-file JF.M
%         -P is the inital approximation to the solution
%         -delta is the tolerance for P
%         -epsilon is the tolerance for F(P)
%         -maxit is the maximum number of iterations
%Output -P is the approximation to the solution
%         -iter is the number of iterations required
%         -err is the error estimate for P
%Use the @ notation call
%[P,iter,err]=newdim(@F, @JF, P, delta, epsilon, maxit).
Y=F(P);
for k=1:maxit
   J=JF(P);
   Q=P-(J\Y')';
   Z=F(Q);
   err=norm(Q-P);
   relerr=err/(norm(Q)+eps);
   P=Q;
   Y=Z;
   iter=k;
   if (err<delta)|(relerr<delta)|(abs(Y)<epsilon)
     break
   end
end
```