SOLVING LINEAR SYSTEMS BY FACTORING

(LU FACTORIZATION)

With Gaussian elimination and Gauss-Jordan elimination, a linear system is solved by operating systematically on the augmented matrix. Now we introduce a different organization of this approach, one based on the factoring the coefficient matrix into a product of lower and upper triangular matrices. This method is well suited for computers and is the basis for many practical computer programs.

Definition:

A factorization of a nonsingular square matrix A as $A = LU$,where L is lower triangular and U is upper triangular, is called an LU-decomposition or triangular decomposition of the matrix A.

Example:

$$A=\begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix}$$

$$A=LU$$

$$A=\begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix}=\begin{bmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{bmatrix}\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$L=\begin{bmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{bmatrix} \qquad U=\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

If an $n\times n$ matrix A can be factored into a product of $n\times n$ matrices as

$$A=LU$$

where L is lower triangular and U is upper triangular, then the linear system

$$Ax=b$$

can be solved as follows:

After finding L and U, the solution X is computed in two steps:

1. ***Solve LY=b for Y using forward substitution.***
2. ***Solve UX=Y for X using back substitution.***

Upper-Triangular Linear Systems

Definition: ***$A_{n \times n}$ matrix $A=[a_{ij}]$ is called <u>upper-triangular</u> provided that the elements satisfy $a_{ij}=0$ whenever $i > j$. The $n \times n$ matrix $A=[a_{ij}]$ is called <u>lower-triangular</u> provided that the elements satisfy $a_{ij}=0$ whenever $i< j$.***

If **A** is an upper triangular matrix, then **AX=B** is said to be an <u>**upper-triangular system of linear equations**</u> and has the form

Theorem: (Back Substitution).

Suppose that AX=B *is an* ***<u>upper-triangular system</u>***.

If $a_{kk} \neq 0$ for $k=1,2,\ldots,n$, then there is exist a unique solution to the system.

$$x_k = \frac{b_k - \sum_{j=k+1}^{n} a_{kj} x_j}{a_{kk}} \qquad for \quad k = n-1, n-2, \ldots 1.$$

Example: Solve

$$\begin{aligned} x_1 + 2x_2 + 4x_3 + x_4 &= 21 \\ 2x_1 + 8x_2 + 6x_3 + 4x_4 &= 52 \\ 3x_1 + 10x_2 + 8x_3 + 8x_4 &= 79 \\ 4x_1 + 12x_2 + 10x_3 + 6x_4 &= 82 \end{aligned}$$

```
>> A=[1 2 4 1 ; 2 8 6 4 ; 3 10 8 8 ; 4 12 10 6]
A =
     1     2     4     1
     2     8     6     4
     3    10     8     8
     4    12    10     6
>> inv(A)
ans =

   -0.5000   -1.7500         0    1.2500
   -0.2500    0.4583   -0.1667   -0.0417
    0.5000    0.2500         0   -0.2500
         0   -0.1667    0.3333   -0.1667

>> b=[21 52 79 82]

b =
    21    52    79    82

>> x=inv(A)*b'
x =
    1.0000
    2.0000
    3.0000
    4.0000
```

Use the triangular factorization method to solve and the fact that

$$A=\begin{bmatrix}1 & 2 & 4 & 1\\ 2 & 8 & 6 & 4\\ 3 & 10 & 8 & 8\\ 4 & 12 & 10 & 6\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & 0\\ 2 & 1 & 0 & 0\\ 3 & 1 & 1 & 0\\ 4 & 1 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 4 & 1\\ 0 & 4 & -2 & 2\\ 0 & 0 & -2 & 3\\ 0 & 0 & 0 & -6\end{bmatrix}=LU$$

Use the forward-substitution method to solve LY=B:

$$\begin{aligned} y_1 &= 21\\ 2y_1 + y_2 &= 52\\ 3y_1 + y_2 + y_3 &= 79\\ 4y_1 + y_2 + 2y_3 + y_4 &= 82 \end{aligned}$$

Compute the values

y_1=21,

y_2=52-2(21)=10,

y_3=79-3(21)-10=6,

y_4=82-4(21)-10-2(6) = -24

Next write the system UX=Y:

$$
\begin{aligned}
x_1 + 2x_2 + 4x_3 + x_4 &= 21 \\
4x_2 - 2x_3 + 2x_4 &= 10 \\
-2x_3 + 3x_4 &= 6 \\
-6x_4 &= -24
\end{aligned}
$$

Now use back substitution and compute the solution

x_4=-24/(-6)=4,

x_3=(6-3(4))/(-2)=3,

x_2=(10-2(4)+2(3))/4=2,

x_1=21-4-4(3)-2(2)=1,

or $X=[\,1 \quad 2 \quad 3 \quad 4]^T$

Example: Solving a System by Factorization

$$A=\begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix}=\begin{bmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{bmatrix}\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Use the given result and the method described above to solve the system

$$\begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$

As specified in Step 2 above, define y_1, y_2, y_3 by the equation

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

So lower triangular system can be written as

$$\begin{bmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}=\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$

Or equivalently,

$$\begin{aligned} 2y_1 \qquad\qquad\qquad &= 2 \\ -3y_1 + y_2 \qquad\quad &= 2 \\ 4y_1 - 3y_2 + 7y_3 &= 3 \end{aligned}$$

The procedure for solving this system is similar to back-substitution except that the equations are solved from the top down instead of from the bottom up. This procedure, which is called forward-substitution, yields

$$y_1 = 1, y_2 = 5, y_3 = 2$$

Substituting these values in

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$$

Or equivalently,

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 1 \\ x_2 + 3x_3 &= 5 \\ x_3 &= 2 \end{aligned}$$

Solving this system by back substitution yields the solution $x_1 = 2, x_2 = -1, x_3 = 2$

LU FACTORIZATION

FROM

GAUSSIAN ELIMINATION

The process of Gaussian elimination forms the basis for finding useful representation of a matrix A known as an LU factorization.

The L and U matrices are initialized as

$$L=I \quad \text{and} \quad U=A.$$

The elimination process transforms the original matrix A into an upper triangular matrix U. The lower triangular matrix L is formed by placing the negatives of the multipliers.

Example: **Use Gaussian elimination to construct the triangular factorization of the given matrix A.**

$$A=\begin{bmatrix}1 & 2 & 3\\ 2 & 6 & 10\\ 3 & 14 & 28\end{bmatrix}$$

The initial matrices

$$L=I_3=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \qquad U=A=\begin{bmatrix}1 & 2 & 3\\ 2 & 6 & 10\\ 3 & 14 & 28\end{bmatrix}$$

Step 1:

For U=A

$$E_2=E_2+(-2)E_1$$

$$E_3=E_3+(-3)E_1$$

For L

Store the negative of the multiplier in the 1st Column, 2nd row, of L.

$$l_{21}=2$$

Store the negative of the multiplier in the 1st Column, 3rd row, of L.

$$l_{31}=3$$

The resulting matrices

$$L=\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 3 & 0 & 1\end{bmatrix} \qquad U=\begin{bmatrix}1 & 2 & 3\\ 0 & 2 & 4\\ 0 & 8 & 19\end{bmatrix}$$

Step 2:

For U

$E_3 = E_3 + (-4)E_2$

For L

Store the negative of the multiplier in the 2nd Column, 3rd row, of L.

$l_{32} = 4$

The matrices that result are

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

Multiply L by U, and verify the result:

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 6 & 10 \\ 3 & 14 & 28 \end{bmatrix} = A$$

```
function [L,U] = LUFACTORGAUSS(A)
%   LU factorization of matrix A using Gaussian Elimination without pivoting
%input   -   A n by n matrix
%Output-   L is lower triangular and U is upper triangular matrix

[n, m ]=size(A);
% initialize matrices
L=eye(n) ;
U=A;
for j=1:n
        for i=j+1 : n
         L(i,j )=U(i,j)/U(j,j) ;
            U( i, : ) =U( i, : )-L( i, j) *U(j, :);
     end
  end
  % display L and U
  L
  U
  %verify result
  B=L*U
  A
```

```
A =
     1     2     3
     2     6    10
     3    14    28
>> LUFACTORGAUSS(A)
L =
     1     0     0
     2     1     0
     3     4     1
U =
     1     2     3
     0     2     4
     0     0     3
B =
     1     2     3
     2     6    10
     3    14    28
A =
     1     2     3
     2     6    10
     3    14    28
```

Example:

$$A=\begin{bmatrix}4 & 12 & 8 & 4\\ 1 & 7 & 18 & 9\\ 2 & 9 & 20 & 20\\ 3 & 11 & 15 & 14\end{bmatrix}$$

The initial matrices

$$L=I_4=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \qquad U=A=\begin{bmatrix}4 & 12 & 8 & 4\\ 1 & 7 & 18 & 9\\ 2 & 9 & 20 & 20\\ 3 & 11 & 15 & 14\end{bmatrix}$$

Step 1:

For U=A

$$E_2=E_2+(-\frac{1}{4})E_1 \qquad E_3=E_3+(-\frac{1}{2})E_1 \qquad E_4=E_4+(-\frac{3}{4})E_1$$

For L (use negative of the multipliers)

$$l_{21}=\frac{a_{21}}{a_{11}}=\frac{1}{4} \qquad l_{31}=\frac{a_{31}}{a_{11}}=\frac{1}{2} \qquad l_{41}=\frac{a_{41}}{a_{11}}=\frac{3}{4}$$

The resulting matrices

$$L=\begin{bmatrix}1 & 0 & 0 & 0\\ 1/4 & 1 & 0 & 0\\ 1/2 & 0 & 1 & 0\\ 3/4 & 0 & 0 & 1\end{bmatrix} \qquad U=\begin{bmatrix}4 & 12 & 8 & 4\\ 0 & 4 & 16 & 8\\ 0 & 3 & 16 & 18\\ 0 & 2 & 9 & 11\end{bmatrix}$$

Step 2:

For U

$$E_3 = E_3 + (-\frac{3}{4})E_2 \qquad E_4 = E_4 + (-\frac{1}{2})E_2$$

For L

$$l_{32} = \frac{a_{32}}{a_{22}} = \frac{3}{4} \qquad l_{42} = \frac{a_{42}}{a_{22}} = \frac{1}{2}$$

The resulting matrices

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 \\ 1/2 & 3/4 & 1 & 0 \\ 3/4 & 1/2 & 0 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 4 & 12 & 8 & 4 \\ 0 & 4 & 16 & 8 \\ 0 & 0 & 4 & 12 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

Step 3:

For U

$$E_4 = E_4 + (-\frac{1}{4})E_3$$

For L

$$l_{43} = \frac{a_{43}}{a_{33}} = \frac{1}{4}$$

The resulting matrices

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1 & 0 & 0 \\ 1/2 & 3/4 & 1 & 0 \\ 3/4 & 1/2 & 1/4 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 4 & 12 & 8 & 4 \\ 0 & 4 & 16 & 8 \\ 0 & 0 & 4 & 12 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Multiply L by U to verify the result:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & 1 & 0 & 0 \\ \frac{1}{2} & \frac{3}{4} & 1 & 0 \\ \frac{3}{4} & \frac{1}{2} & \frac{1}{4} & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 & 12 & 8 & 4 \\ 0 & 4 & 16 & 8 \\ 0 & 0 & 4 & 12 \\ 0 & 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 12 & 8 & 4 \\ 1 & 7 & 18 & 9 \\ 2 & 9 & 20 & 20 \\ 3 & 11 & 15 & 14 \end{bmatrix} = A$$

```
A =
     4    12     8     4
     1     7    18     9
     2     9    20    20
     3    11    15    14
>> LUFACTORGAUSS(A)
L =
    1.0000         0         0         0
    0.2500    1.0000         0         0
    0.5000    0.7500    1.0000         0
    0.7500    0.5000    0.2500    1.0000
U =
     4    12     8     4
     0     4    16     8
     0     0     4    12
     0     0     0     4
B =
     4    12     8     4
     1     7    18     9
     2     9    20    20
     3    11    15    14
A =
     4    12     8     4
     1     7    18     9
     2     9    20    20
     3    11    15    14
```

DIRECT
LU
FACTORIZATION

LU Factorization (when it exists) is not unique

An alternative approach to Gaussian elimination for finding the LU factorization of matrix A is based on equating the elements of the product LU with the corresponding elements of A, in a systematic manner.

- **Doolittle's Method**

  (Diagonal elements of L are 1's)

$$l_{11} = l_{22} = \ldots = l_{nn} = 1$$

- **Crout's Method**

  (Diagonal elements of U are 1's)

$$u_{11} = u_{22} = \ldots = u_{nn} = 1$$

- **Choleski's Method**

  (For each value of i $l_{ii} = u_{ii}$)

$$l_{ii} = u_{ii} \qquad for \quad i = 1 \ldots n$$

Doolittle LU Factorization

The Doolittle form of LU factorization assumes that the diagonal elements of matrix L are 1's.

For 3x3 matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

At the first stage we begin by finding $u_{11} = a_{11}$ **and then solving for**

- **the <u>remaining elements in the first row of</u> U**
- **<u>and the first column of</u> L.**

At the second stage, we find u_{22} **and then**

- **the remainder of the second row of U and**
- **the second column of L.**

Continuing in this manner, we determine all of the elements of U and L.

Example: Doolittle form of LU factorization of A

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 20 & 32 \\ 5 & 32 & 64 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 20 & 32 \\ 5 & 32 & 64 \end{bmatrix}$$

$(1)u_{11} = a_{11} = 1$ $(1)u_{12} = a_{12} = 4$ $(1)u_{13} = a_{13} = 5$ **First Row of U**

$u_{11} = 1$ $u_{12} = 4$ $u_{13} = 5$

$l_{21}u_{11} = a_{21} = 4$ $l_{31}u_{11} = a_{31} = 5$ **First Column of L**

$l_{21} = 4$ $l_{31} = 5$

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 5 & l_{32} & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 20 & 32 \\ 5 & 32 & 64 \end{bmatrix}$$

$(4)(4) + u_{22} = 20 \Rightarrow u_{22} = 20 - 16 = 4$ **Second Row of U**

$(4)(5) + u_{23} = 32 \Rightarrow u_{23} = 32 - 20 = 12$

$(5)(4) + l_{32}u_{22} = 32 \Rightarrow l_{32} = (32 - 20)/4 = 3$ **Second Column of L**

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 5 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \\ 0 & 4 & 12 \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 20 & 32 \\ 5 & 32 & 64 \end{bmatrix}$$

Finally,

$(5)(5) + (3)(12) + u_{33} = 64 \Rightarrow u_{33} = 64 - 25 - 36 = 3$ **Third Row of U**

The factorization is

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 5 & 3 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 4 & 12 \\ 0 & 0 & 3 \end{bmatrix}$$

Exercise: Find the Doolittle factorization of the given matrix A.

$$A = \begin{bmatrix} 6 & 2 & 1 & -1 \\ 2 & 4 & 1 & 0 \\ 1 & 1 & 4 & -1 \\ -1 & 0 & -1 & 3 \end{bmatrix}$$

Check your results

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/3 & 1 & 0 & 0 \\ 1/6 & 1/5 & 1 & 0 \\ -1/6 & 1/10 & -9/37 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 6 & 2 & 1 & -1 \\ 0 & 10/3 & 2/3 & 1/3 \\ 0 & 0 & 37/10 & -9/10 \\ 0 & 0 & 0 & 191/74 \end{bmatrix}$$

MATLAB Function for DOLITLE LU FACTORIZATION

```
function [L,U] = DOOLITLE(A)
%  LU factorization of matrix A using DOOLITLE METHOD
%nput   -  A n by n matrix
%Output-   L is lower triangular and U is upper triangular matrix
[n, m ]=size(A);
% initialize matrices
L = eye(n) ;
U = zeros(n, n);
 for k=1 : n
          U(k, k)= A(k, k)- L(k, 1:k-1)*U(1:k-1, k);
           for j=k+1  : n
          U( k , j)= A( k , j) - L( k, 1: k-1)*U(1:k-1 , j);
          L ( j , k ) =(A( j , k ) - L ( j, 1:k -1)* U(1: k-1, k))/U(k, k );
      end
  end
  % display L and U
  L
  U
  %verify result
  B=L*U
  A
```

```
A=[ 1 4 5; 4 20 32; 5 32 64]
A =
     1     4     5
     4    20    32
     5    32    64
>> DOOLITLE(A)
L =
     1     0     0
     4     1     0
     5     3     1
U =
     1     4     5
     0     4    12
     0     0     3
B =
     1     4     5
     4    20    32
     5    32    64
A =
     1     4     5
     4    20    32
     5    32    64
>>
```

Crout LU Factorization

The Crout form of LU factorization assumes that the diagonal elements of matrix U are 1's.

For 3x3 matrix

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Crout factorization follows a similar procedure Doolittle, but assumes that there are 1's on the diagonal of U rather than L.

Exercise: Find the Crout factorization of the given matrix A.

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 20 & 32 \\ 5 & 32 & 64 \end{bmatrix}$$

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 20 & 32 \\ 5 & 32 & 64 \end{bmatrix}$$

Exercise: Write M FILE for CROUT FACTORIZATION

Cholesky LU Factorization

<u>The Cholesky form of LU factorization assumes that the diagonal elements of L and U are required to be equal</u> $l_{ii} = u_{ii} = x_{ii}, \quad i = 1 \ldots n$

For 3x3 matrix

$$\begin{bmatrix} x_{11} & 0 & 0 \\ l_{21} & x_{22} & 0 \\ l_{31} & l_{32} & x_{33} \end{bmatrix} \begin{bmatrix} x_{11} & u_{12} & u_{13} \\ 0 & x_{22} & u_{23} \\ 0 & 0 & x_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Example: Cholesky form of LU factorization of A

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 20 & 32 \\ 5 & 32 & 64 \end{bmatrix}$$

$$\begin{bmatrix} x_{11} & 0 & 0 \\ l_{21} & x_{22} & 0 \\ l_{31} & l_{32} & x_{33} \end{bmatrix} \begin{bmatrix} x_{11} & u_{12} & u_{13} \\ 0 & x_{22} & u_{23} \\ 0 & 0 & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 20 & 32 \\ 5 & 32 & 64 \end{bmatrix}$$

First Stage

$$x_{11}x_{11} = a_{11} = 1 \Rightarrow x_{11} = 1$$
$$x_{11}u_{12} = a_{12} = 4 \Rightarrow u_{12} = 4$$
$$x_{11}u_{13} = a_{13} = 5 \Rightarrow u_{13} = 5$$
$$l_{21}x_{11} = a_{21} = 4 \Rightarrow l_{21} = 4$$
$$l_{31}x_{11} = a_{31} = 5 \Rightarrow l_{31} = 5$$

From the values computed above, the product is

$$\begin{bmatrix}1 & 0 & 0\\ 4 & x_{22} & 0\\ 5 & l_{32} & x_{33}\end{bmatrix}\begin{bmatrix}1 & 4 & 5\\ 0 & x_{22} & u_{23}\\ 0 & 0 & x_{33}\end{bmatrix}=\begin{bmatrix}1 & 4 & 5\\ 4 & 20 & 32\\ 5 & 32 & 64\end{bmatrix}$$

Second Stage

$$(4)(4)+(x_{22})(x_{22})=20 \Rightarrow x_{22}=\sqrt{(20-16)}=2$$

$$(4)(5)+(x_{22})(u_{23})=32 \Rightarrow u_{23}=(32-20)/2=6$$

$$(5)(4)+(l_{32})(x_{22})=32 \Rightarrow l_{32}=(32-20)/2=6$$

From the values computed above, the product is

$$\begin{bmatrix}1 & 0 & 0\\ 4 & 2 & 0\\ 5 & 6 & x_{33}\end{bmatrix}\begin{bmatrix}1 & 4 & 5\\ 0 & 2 & 6\\ 0 & 0 & x_{33}\end{bmatrix}=\begin{bmatrix}1 & 4 & 5\\ 4 & 20 & 32\\ 5 & 32 & 64\end{bmatrix}$$

Third Stage

$$(5)(5)+(6)(6)+(x_{33})(x_{33})=64 \Rightarrow x_{33}=\sqrt{(64-25-36)}=\sqrt{3}$$

The LU factorization with L and U as follows.

$$L=\begin{bmatrix}1 & 0 & 0\\ 4 & 2 & 0\\ 5 & 6 & \sqrt{3}\end{bmatrix} \qquad U=\begin{bmatrix}1 & 4 & 5\\ 0 & 2 & 6\\ 0 & 0 & \sqrt{3}\end{bmatrix}$$

Note: L = U^T

Properties: ***If the matrix A is symmetric positive definite, then Cholesky factorization yields upper triangular U matrix as transpose of lower triangular matrix L . i. e., $A=LL^T$***

MATLAB Function for CHOLESKY FACTORIZATION

```
function [L,U] = CHOLESKY(A)
%  LU factorization of matrix A using CHOLESKY
METHOD
%nput   -  A n by n matrix a IS ASSUMED TO BE
SYMMETRIC
%Output-   L is lower triangular and U is THE
TRANSPOSE OF L.
[n, m ]=size(A);
% initialize L
L = zeros(n, n);
 for k=1 : n
    L(k,k)=sqrt( A(k,k)- L(k, 1:k-1)* L(k, 1:k-1)'  );
           for i=k+1  : n
    L ( i , k ) =(A( i , k ) - L ( i, 1:k -1)*L(k, 1:k-1)')/L(k, k );
     end
  end
  % display L and U
  L
  %verify result
  B=L*L'
  A
```

```
>> A=[ 1 4 5; 4 20 32; 5 32 64]
A =
     1     4     5
     4    20    32
     5    32    64
>> CHOLESKY(A)
L =
    1.0000         0         0
    4.0000    2.0000         0
    5.0000    6.0000    1.7321
B =
     1     4     5
     4    20    32
     5    32    64
A =
     1     4     5
     4    20    32
     5    32    64
```

Application of LU Factorization

Solving systems of Linear Equations

```
function x=SOLVELU(L, U , b)
% Function to solve L U x = b
%input    -  L  Lower  triangular  matrix  with  1's  on
%diagonal
%          U  Upper triangular matrix
%          b  right hand side vector
%Output-   x  solution of the given system
[n, m ]=size(L);
y=zeros(n,1);
x=zeros(n,1);
%
% Solve Ly=b using forward substitution
y(1)=b(1);
 for i=2 : n
    y(i)=b(i) - L(1, 1:i -1) *y(1 : i- 1);
     end
% Solve Ux=y using back  substitution
x(n)=y(n)/U(n, n);
     for i=n-1 : -1 : 1
     x(i)=(y(i) -U( i , i+1 : n) * x(i+1 : n)) / U(i , i );
  end
```

TRY THE M FILE

Exercise: First obtain L and U then solve the given system

$$
\begin{aligned}
x_1 + 2x_2 + 4x_3 + x_4 &= 21 \\
2x_1 + 8x_2 + 6x_3 + 4x_4 &= 52 \\
3x_1 + 10x_2 + 8x_3 + 8x_4 &= 79 \\
4x_1 + 12x_2 + 10x_3 + 6x_4 &= 82
\end{aligned}
$$

Determinant of a Matrix

The determinant of $n \times n$ A can be found as follows:

If A=LU then

$$\det(A) = \prod_{i=1}^{n} l_{ii} \cdot \prod_{i=1}^{n} u_{ii},$$

Example: Find the determinant of the given *three-by-three* matrix using LU factorization

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix}$$

We begin by finding its LU factorization, where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 5 \\ 0 & 0 & 8 \end{bmatrix}$$

$$\det(A) = u_{11}u_{22}u_{33} = (1)(-1)(8) = -8$$

Inverse of a Matrix

The inverse of an $n \times n$ matrix A can be found by solving the system of equations

$$Ax_i = e_i \quad (i = 1,\ldots,n)$$

for the vectors $e_i = [0 \;\; 0 \ldots 1 \ldots 0 \;\; 0]'$,where the 1 appears in the ith position. The matrix X whose columns are the solution vectors $x_1, x_2, \ldots, x_n$ is A^{-1}.

Example: Find the inverse of the given matrix

A=LU, where

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 5 \\ 0 & 0 & 8 \end{bmatrix}$$

First we solve LY=I for Y,

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The first column of Y is the solution vector for the system

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_{11} \\ y_{21} \\ y_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The second and third columns of Y are found in a similar manner, using the second and third columns of I.

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_{12} \\ y_{22} \\ y_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_{13} \\ y_{23} \\ y_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Each column of Y is found by forward substitution, using the corresponding column of I. The solution of Y is

$$Y = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}$$

Finally, we solve UX=Y for X,

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 5 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}$$

The solution is found for each column of X, using <u>back substitution</u> and the corresponding column and the corresponding column of Y.

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 5 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 5 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 5 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The solution is

$$X = A^{-1} = \begin{bmatrix} 1/8 & -5/8 & 3/8 \\ -1/8 & -3/8 & 5/8 \\ 3/8 & 1/8 & 1/8 \end{bmatrix}$$

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>> A=[1 -1 2;-2 1 1;-1 2 1]

A =

     1    -1     2
    -2     1     1
    -1     2     1

>> inv(A)

ans =

    0.1250   -0.6250    0.3750
   -0.1250   -0.3750    0.6250
    0.3750    0.1250    0.1250
```

$$X = A^{-1} = \begin{bmatrix} 1/8 & -5/8 & 3/8 \\ -1/8 & -3/8 & 5/8 \\ 3/8 & 1/8 & 1/8 \end{bmatrix}$$

Exercise: Write M FILE for Matrix Inversion and Matrix determinant using given L and U matrices.