THE SIMPLEX ALGORITHM

HOW TO CONVERT AN LP TO STANDARD FORM

$$\begin{aligned} \text{Min } Z &= 2x_1 - 3x_2 \\ \text{s.t} \quad x_1 + x_2 &\leq 4 \\ x_1 - x_2 &\leq 6 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Convert the LP's objective function to the **row 0 format.**

$$-Z + 2x_1 - 3x_2 = 0$$

To put the constraints in standard form, we simply add **slack variables** s_1 and s_2, respectively, to the two constraints.

$$\begin{aligned} x_1 + x_2 + s_1 \quad &= 4 \\ x_1 - x_2 \quad + s_2 &= 6 \end{aligned}$$

Before the simplex algorithm can be used to solve an LP, the LP must be converted into an equivalent problem in which all constraints are equations and all variables are nonnegative. An LP in this form is said to be in **standard form**.

$$
\begin{aligned}
\text{Max } Z = 60\,x_1 + 30\,x_2 + 20\,x_3 & \\
\text{s. t} \quad 8\,x_1 + 6\,x_2 + x_3 &\leq 48 \\
4\,x_1 + 2\,x_2 + 1.5\,x_3 &\leq 20 \\
2\,x_1 + 1.5\,x_2 + 0.5\,x_3 &\leq 8 \\
x_2 &\geq 5
\end{aligned}
$$

$$x_1 , x_2 , x_3 , x_4 \geq 0$$

Row 0 format of the objective function

$$Z - 60\,x_1 - 30\,x_2 - 20\,x_3 = 0$$

$$\begin{array}{lllllll}
8\,x_1 & +\,6\,x_2 & +\,x_3 & +\,s_1 & & & = 48 \\
4\,x_1 & +\,2\,x_2 & +\,1.5\,x_3 & & +\,s_2 & & = 20 \\
2\,x_1 & +\,1.5\,x_2 & +\,0.5\,x_3 & & & +\,s_3 & = 8 \\
 & x_2 & & & & -\,e_1 & = 5
\end{array}$$

- **If the *i*th constraint is a $\leq$ constraint, we convert it to an equality constraint by adding a slack variable s_i and the sign restriction $s_i \geq 0$.**

- **If the *i* th constraint is a $\geq$ constraint, we convert it to an equality constraint by adding a excess variable e_i and the sign restriction $e_i \geq 0$.**

Standard Maximization Problem

A **standard maximization problem** in n unknown is a linear programming problem in which we are required to **minimize** (**not maximize**) the objective function.

This is not unique?

Standard Maximization Problem

A **standard maximization problem** in n unknowns is a linear programming problem in which we are required to maximize (not minimize) the objective function, subject to constraints of the form

$$x \geq 0, y \geq 0, z \geq 0, \ldots,$$

and further constraints of the form

$$Ax + By + Cz + \ldots \leq N,$$

where A, B, C, . . . and N are numbers with N nonnegative.

Note that the inequality here must be a "$\leq$," and not "=" or "$\geq$."

$$Max(orMin)Z = c_1x_1 + c_2x_2 + c_3x_3 + \ldots + c_nx_n$$

s.t

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n \le \; = \; \ge b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n \le \; = \; \ge b_2$$

.

.

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$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n \le \; = \; \ge b_m$$

$$x_i \ge 0$$

If we define

$$A = \begin{bmatrix} a_{11} \; a_{12} \ldots a_{1n} \\ a_{21} \; a_{22} \ldots a_{2n} \\ . \\ . \\ . \\ a_{m1} \; a_{m2} \; a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n . \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_m \end{bmatrix}$$

the constraints may be written as the system of equations **Ax = b**

Consider a system **Ax = b of m linear equations in n variables (assume n≥m).**

Definition

A basic solution to Ax= b is obtained by setting n-m variables equal to 0 and solving for the values of the remaining m variables .

To find a basic solution to Ax= b we choose n –m variables (the nonbasic variables, or NBV) and set each of these variables equal to zero. Then solve for the values of the remaining n-(n-m) = m variables (the basic variables or BV) that satisfy Ax=b.

Basic Solution

To get the **basic solution** corresponding to any tableau in the simplex method, set to zero all variables that do not appear as row labels (these are the **inactive** variables).

The value of a variable that does appear as a row label (an **active variable**) is the number in the rightmost column in that row divided by the number in that row in the column labeled by the same variable.

DIFFERENT CHOICES OF NONBASIC VARIABLES WILL LEAD TO DIFFERENT BASIC SOLUTIONS

Definition

Any basic solution to Ax= b in which all variables are nonnegative is a basic feasible solution (or bfs).

Theorem

The feasible region for any linear programming problem is a convex set. Also, if an LP has an optimal solution, there must be an extreme point of the feasible region that is optimal.

Theorem

For any LP, there is a unique extreme point of the LP's feasible region corresponding to each basic feasible solution. Also there is at least one bfs corresponding to each extreme point of the feasible region.

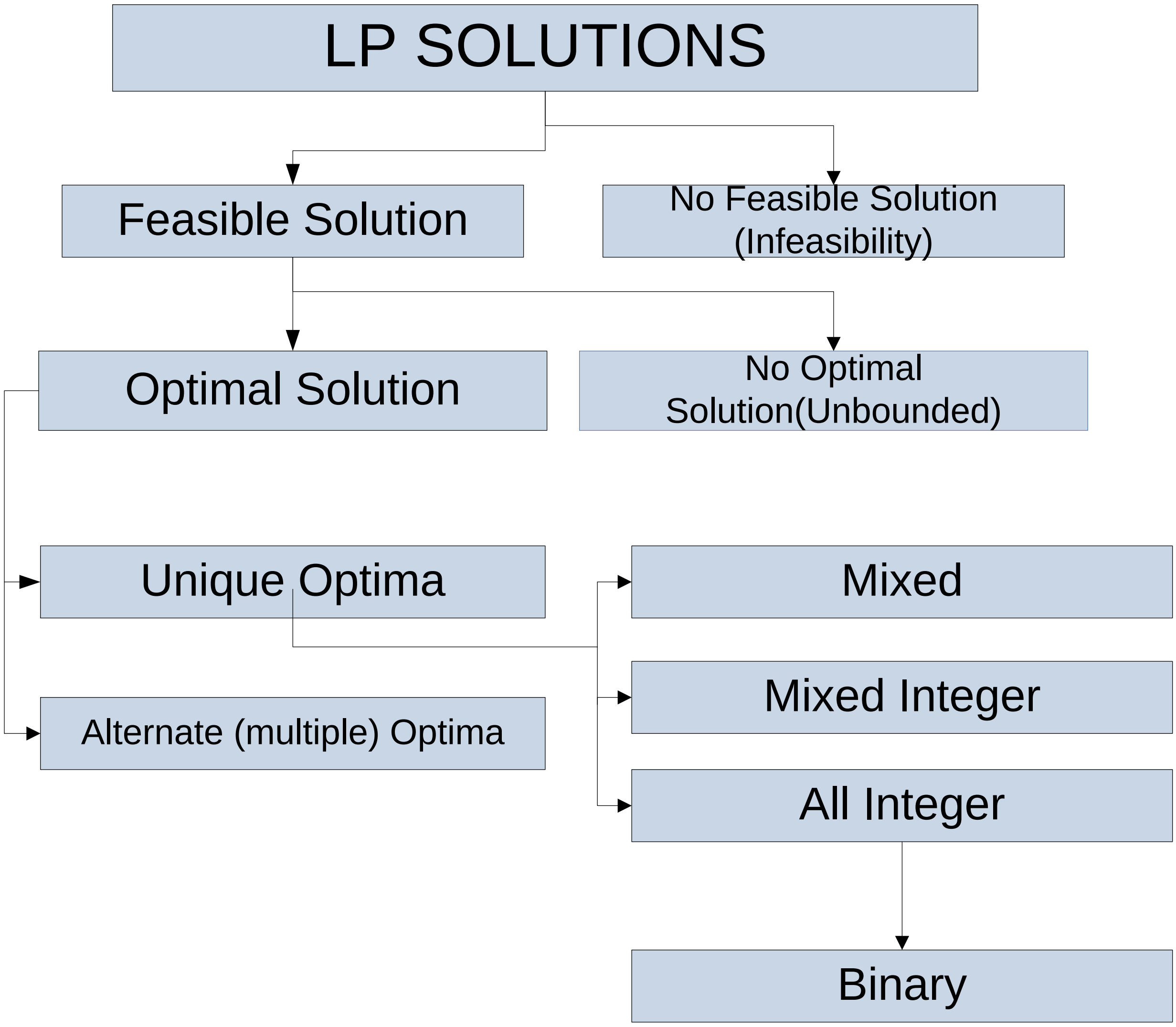
If an LP in standard form has m constraints and n variables, there may be a basic solution for each choice of non-basic variables. Since some basic solutions may be not be feasible, an LP can have at most

$$\binom{n}{m}$$

basic feasible solutions. If we were to proceed from the current bfs to a better bfs we would surely find the optimal bfs after examining at most

$$\binom{n}{m}$$

basic feasible solutions. This means that the simplex algorithm will find the optimal bfs after a finite number of calculations.


LP SOLUTIONS
Feasible Solution
No Feasible Solution (Infeasibility)
Optimal Solution
No Optimal Solution(Unbounded)
Unique Optima
Alternate (multiple) Optima
Mixed
Mixed Integer
All Integer
Binary

Example

$$\text{Max } Z = 2x_1 + 4x_2$$
$$\text{s.t} \quad 3x_1 + 4x_2 \le 1700$$
$$2x_1 + 5x_2 \le 1600$$
$$x_1, x_2 \ge 0$$

standard form

$$Z - 2x_1 - 4x_2 = 0$$

$$3x_1 + 4x_2 + s_1 = 1700$$
$$2x_1 + 5x_2 + s_2 = 1600$$

or

$$3x_1 + 4x_2 + x_3 = 1700$$
$$2x_1 + 5x_2 + x_4 = 1600$$

n= 4 m= 2

All basic solutions

Basic solutions	x_1	x_2	x_3	x_4	z
1	**0**	**0**	**1700**	**1600**	0
2	**0**	**425**	**0**	**-525**	not a bfs
3	**0**	**320**	**420**	**0**	1280
4	**566.66**	**0**	**0**	**466.66**	1133.4
5	**800**	**0**	**-700**	**0**	not a bfs
6	**300**	**200**	**0**	**0**	1400

bfs: basic feasible solution

The Simplex Algorithm

Step1 Convert the LP to standard form

Step2 Obtain a bfs (if possible) from the standard form

Step3 Determine whether the current bfs is optimal

Step4 If the current bfs is not optimal, determine which non-basic variable should become a basic variable and which basic variable should become a non-basic variable in order to find a new bfs with a better objective function value and go to step 3.

Step5 Use ero's to find the new bfs with better objective value and go to step 3.

$$\text{Max } Z = 2x_1 + 4x_2$$
$$\text{s.t} \quad 3x_1 + 4x_2 \leq 1700$$
$$2x_1 + 5x_2 \leq 1600$$
$$x_1, x_2 \geq 0$$

$$Z - 2x_1 - 4x_2 = 0$$

$$3x_1 + 4x_2 + s_1 = 1700$$
$$2x_1 + 5x_2 + s_2 = 1600$$

or

$$3x_1 + 4x_2 + x_3 = 1700$$
$$2x_1 + 5x_2 + x_4 = 1600$$

BV=(x_3,x_4) =1700,1600   NBV=(x_1,x_2)=0

BV=(x_3,x_4) =1700,1600 NBV=(x_1,x_2)=0

<u>Initial Simplex Tableau</u> PIVOT

BASIS	x_1	x_2	x_3	x_4	RHS	RATIO
x_3	3	4	1	0	**1700**	425
x_4	2	5<	0	1	**1600**	320<
Z	-2	-4<	0	0	**0**	

- **Is the current basic feasible solution optimal?**
- **Determine the entering variable (Pivot Column)**
- **The ratio test**

When entering a variable into the basis, compute the ratio <u>Right-hand side of row</u> Coefficient of entering variable in row for every constraint in which the entering variable has a positive coefficient. The constraint with the smallest ratio is called winner of the ratio test.

- **Determine the leaving variable (Pivot Row)**

Initial and First Simplex Tableau

BASIS	x_1	x_2	x_3	x_4	RHS	RATIO
x_3	3	4	1	0	**1700**	425
x_4	2	5<<	0	1	**1600**	320<
Z	-2	-4<	0	0	**0**	
x_3	7/5<<	0	1	-4/5	**420**	300<
x_2	2/5	1	0	1/5	**320**	800
Z	-2/5	0	0	4/5	**1280**	

Initial, First and Second Simplex Tableau

BASIS	x_1	x_2	x_3	x_4	RHS	RATIO
x_3	3	4	1	0	**1700**	425
x_4	2	5<<	0	1	**1600**	320<
Z	-2	-4<	0	0	**0**	
x_3	7/5<<	0	1	-4/5	**420**	300<
x_2	2/5	1	0	1/5	**320**	800
Z	-2/5	0	0	4/5	**1280**	
x_1	1	0	5/7	-4/7	**300**	
x_2	0	1	-2/7	3/7	**200**	
Z	**0**	**0**	**2/7**	**4/7**	**1400**	

$$x_1 = 300$$
$$x_2 = 200$$
$$Z = 1400 \ \mathbf{max}$$

DIFFERENT SIMPLEX TABLEAU

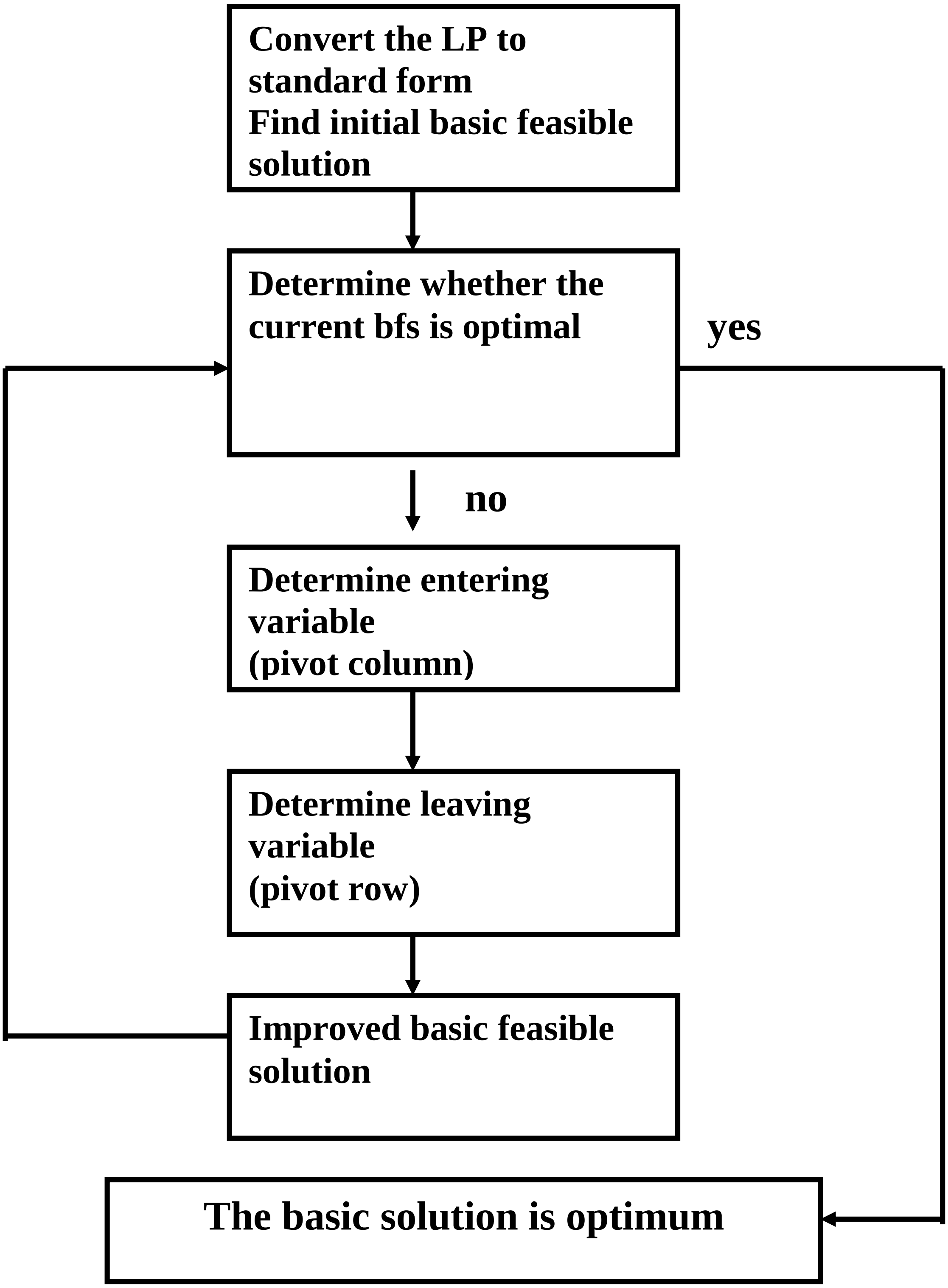
		x_1	x_2	x_3	x_4		
BASIS	c_j	2	4	0	0	**RHS**	**RATIO**
x_3	0	3	4	1	0	**1700**	425
x_4	0	2	5<<	0	1	**1600**	320<
	Z_j	0	0	0	0		
	C_j-Z_j	2	**4<**	0	0		
x_3	0	7/5<<	0	1	-4/5	**420**	300<
x_2	4	2/5	1	0	1/5	**320**	800
	Z_j	8/5	4	0	4/5	**1280**	
	C_j-Z_j	2/5<	0	0	-4/5		
x_1	2	1	0	5/7	-4/7	**300**	
x_2	4	0	1	-2/7	3/7	**200**	
	Z_j	2	4	2/7	4/7	**1400**	
	C_j-Z_j	**0**	**0**	**-2/7**	**-4/7**	**1400**	

The elements in the Zj row are the sum of the products obtained by multiplying the elements in the Ci column of the Simplex tableau by the corresponding elements in the columns of the A matrix.

Stopping Criterion

The optimal solution to a linear programming problem has been reached when all of the entries in the net evaluation row (C_j-Z_j) are zero or negative. In such cases the optimal solution is the current basic feasible solution.

The Simplex Algorithm



Standard maximization problem

A linear programming (LP) problem is called a **standard maximization problem** if:

- We are to find the *maximum* (not minimum) value of the objective function.
- All the decision variables $x_1, x_2, \ldots, x_n$ are constrained to be *non-negative*.
- All further constraints have the form $bx_1 + bx_2 + .. + bx_n \leq c$ (and not $\geq$) with c nonnegative.

Example

The following is a standard maximization problem:

Maximize $p = 2x - 3y + z$ Objective function

subject to
$$x + y + z \leq 10$$
$$4x - 3y + z \leq 3$$
$$2x + y - z \leq 10$$
$$x \geq 0, y \geq 0, z \geq 0$$

Constraints

Step 1: *Convert the LP problem to a system of linear equations.*

Step 1: *Convert the LP problem to a system of linear equations.*
We do this by turning each constraint inequality into a linear equation by adding new variables we call **slack variables,** and rewriting the objective function as illustrated in the following example:

Example

Start with the following LP problem:

Maximize $p = 2x - 3y + z$ Objective function

subject to
$$\begin{aligned} x + y + z &\le 10 \\ 4x - 3y + z &\le 3 \\ 2x + y - z &\le 10 \\ x \ge 0, y \ge 0, z &\ge 0 \end{aligned}$$
Constraints

This is the LP problem we will be using throughout this tutorial to explain the steps. We will refer it to throughout the tutorial as the demonstration LP problem. For practice, you will be given a different problem to work on below.

To turn the constraints into equalities, we will add new nonnegative "slack variables" to the left-hand side "take up the slack." In addition we will rewrite the objective function by bringing all the unknowns to the left-hand side:

$$\begin{aligned} x + y + z \quad + s \qquad\qquad\qquad &= 10 \\ 4x - 3y + z \qquad + t \qquad\qquad &= 3 \\ 2x + y - z \qquad\qquad + u \qquad &= 10 \\ -2x + 3y - z \qquad\qquad\qquad + p \; &= 0 \end{aligned}$$
Objective function rewritten

Note that we omit the constraints $x \ge 0, y \ge 0, z \ge 0$; the simplex method assumes that all variables are nonnegative.

Step 2: *Represent the system of equations obtained above in the form of an augmented matrix.*

This matrix is called the **first tableau,** and we set it up as shown in the example below.

Example

For the system in the demonstration LP problem

$$\begin{aligned} x + y + z \quad + s \qquad\qquad\qquad &= 10 \\ 4x - 3y + z \qquad + t \qquad\qquad &= 3 \\ 2x + y - z \qquad\qquad + u \qquad &= 10 \\ -2x + 3y - z \qquad\qquad\qquad + p \; &= 0 \end{aligned}$$

we obtain the following first tableau:

	x	y	z	s	t	u	p	**Ans**
s	1	1	1	1	0	0	0	10
t	4	−3	1	0	1	0	0	3
u	2	1	−1	0	0	1	0	10
p	−2	3	−1	0	0	0	1	0

Step 3: *Select a pivot column.*
The rule for the selecting a pivot column is this: Look at all the numbers in the bottom row, excluding the last entry on the right (in the Answer column).

	x	y	z	s	t	u	p	Ans
s	1	1	1	1	0	0	0	10
t	4	−3	1	0	1	0	0	3
u	2	1	−1	0	0	1	0	10
p	−2	3	−1	0	0	0	1	0

Step 4: *Select the pivot.*
Use the following rules to decide which of the entries in the column you selected to use as the pivot:
1) The pivot must always be a positive number (zeros or negative numbers are not permitted as pivots).
2) For each positive entry b in the pivot column, compute the ratio a/b, where a is the number in the rightmost column in that row. We call this a **test ratio**.
3) Of these ratios, choose the smallest one. The corresponding number b is the pivot.

Example
In the demonstration example we are working with, the pivot will be either the 4, the 1 or the 2 in the x-column. Here are the test ratios for these candidates:

	x	y	z	s	t	u	p	Ans	
s	1	1	1	1	0	0	0	10	Test ratio: 10/1
t	$\boxed{4}$	−3	1	0	1	0	0	3	Test ratio: 3/4 *smallest*
u	2	1	−1	0	0	1	0	10	Test ratio: 10/2
p	−2	3	−1	0	0	0	1	0	

Because the smallest test ratio is in row 2, our pivot will be the 4 as shown.

Step 5: *Use the pivot to clear the pivot column in the normal manner. This gives the next tableau.*

Example

In the demonstration example we are working with, we clear the pivot column using the row operations shown:

	x	y	z	s	t	u	p	Ans	
s	1	1	1	1	0	0	0	10	$4R_1 - R_2$
t	$\boxed{4}$	−3	1	0	1	0	0	3	
u	2	1	−1	0	0	1	0	10	$2R_3 - R_2$
p	−2	3	−1	0	0	0	1	0	$2R_4 + R_2$

and thereby obtain the second tableau:

	x	y	z	s	t	u	p	Ans
s	0	7	3	4	−1	0	0	37
x	4	−3	1	0	1	0	0	3
u	0	5	−3	0	−1	2	0	17
p	0	3	−1	0	1	0	2	3

ü Notice that the label on the left of the pivot row has been changed from "t" to x, which is the heading of the pivot column.

Step 6: *Repeat Steps 3–5 until there are no more negative numbers in the bottom row (with the possible exception of the Answer column).*

Example

Here is the second tableau of the demonstration example we are working with:

	x	y	z	s	t	u	p	Ans
s	0	7	3	4	−1	0	0	37
x	4	−3	1	0	1	0	0	3
u	0	5	−3	0	−1	2	0	17
p	0	3	−1	0	1	0	2	3

Since the third entry of the bottom row is negative, we need to repeat Steps 3–5 to clear its column:

	x	y	z	s	t	u	p	Ans	
s	0	7	3	4	−1	0	0	37	Test ratio: 37/3
x	4	−3	$\boxed{1}$	0	1	0	0	3	Test ratio: 3/1 *smallest*
u	0	5	−3	0	−1	2	0	17	
p	0	3	−1	0	1	0	2	3	

	x	y	z	s	t	u	p	Ans	
s	0	7	3	4	−1	0	0	37	$R_1 - 3R_2$
x	4	−3	$\boxed{1}$	0	1	0	0	3	
u	0	5	−3	0	−1	2	0	17	$R_3 + 3R_2$
p	0	3	−1	0	1	0	2	3	$R_4 + R_2$

	x	y	z	s	t	u	p	Ans
s	−12	16	0	4	−4	0	0	28
z	4	−3	1	0	1	0	0	3
u	12	−4	0	0	2	2	0	26
p	4	0	0	0	2	0	2	6

$x = 0$ (inactive); $y = 0$ (inactive); $z = 3/1 = 3$;
$s = 28/4 = 7$; $t = 0$ (inactive); $u = 26/2 = 13$;
$p = 6/2 = 3$;

Thus, the maximum value of p is 3, and occurs when $(x, y, z) = (0, 0, 3)$.