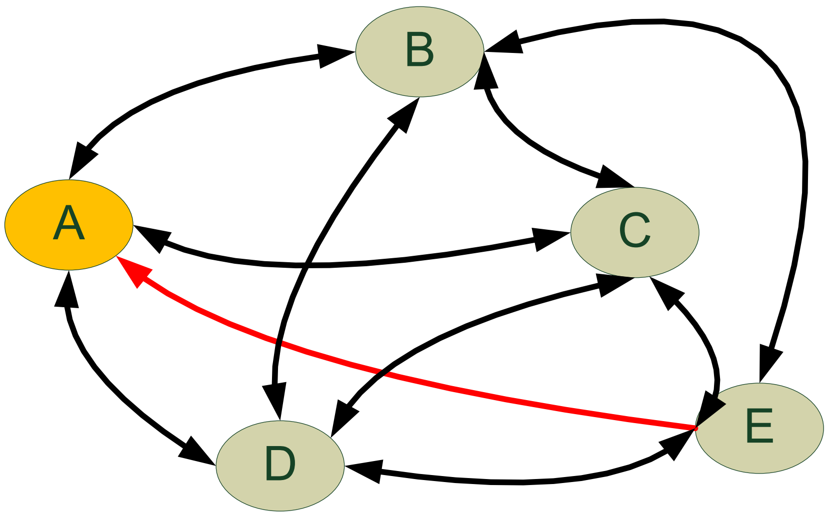
THE TRAVELING SALESMAN (SALESPERSON) PROBLEM

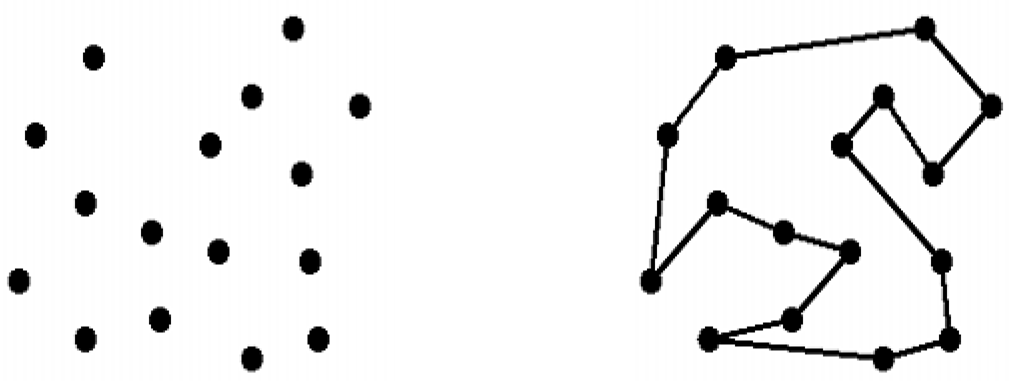



Sample of TSP Applications
Lincoln's Circuit
Soil Surveys
School Bus Routes
Telecom Networks
Space Telescope
Drilling PCB Boards
Pick Paths
Hiking Trails
Blue Claw Crabs

1. There are m nodes.
2. Unit costs c_{ij} are associated with utilizing the arc from node i to node j.

Goal:

Find the cycle that minimizes the total distance required to visit all nodes without visiting any node twice.



INPUT	OUTPUT

A problem that involves 20 nodes requires over 500,000 linear constraints, 50 nodes requires over 500 trillion constraints.

Numerous algorithms have been proposed, but few are efficient when the number of nodes is large.

Input Description: A weighted graph *G*.

Problem: Find the cycle of minimum cost visiting all of the vertices of *G* exactly once.

The Algorithm Design Manual: The traveling salesman problem is the most notorious **NP-complete problem.** A problem is assigned to the **NP** (**nondeterministic polynomial time)** class if it is solvable in polynomial time by a nondeterministic Turing machine.

This is a function of its general usefulness, and because it is easy to explain to the public at large. Imagine a traveling salesman who has to visit each of a given set of cities by car. Although the problem arises in transportation applications, its most important applications arise in optimizing the tool paths for manufacturing equipment. For example, consider a robot arm assigned to solder all the connections on a printed circuit board. The shortest tour that visits each solder point exactly once defines the most efficient path for the robot. A similar application arises in minimizing the amount of time taken by a graphics plotter to draw a given figure. The best book available for this problem is The Traveling Salesman Problem : A Guided Tour of Combinatorial Optimization by E.L. Lawler (Editor) and A. H. Rinnooy-Kan.

Implementations

- TSP solvers (C) (rating 8)
- Netlib / TOMS -- Collected Algorithms of the ACM (FORTRAN) (rating 6)
- Discrete Optimization Methods (Pascal) (rating 5)
- GA Playground (Java) (rating 4)
- Xtango and Polka Algorithm Animation Systems (C++) (rating 3)
- Combinatorica (Mathematica) (rating 3)

Related Problems

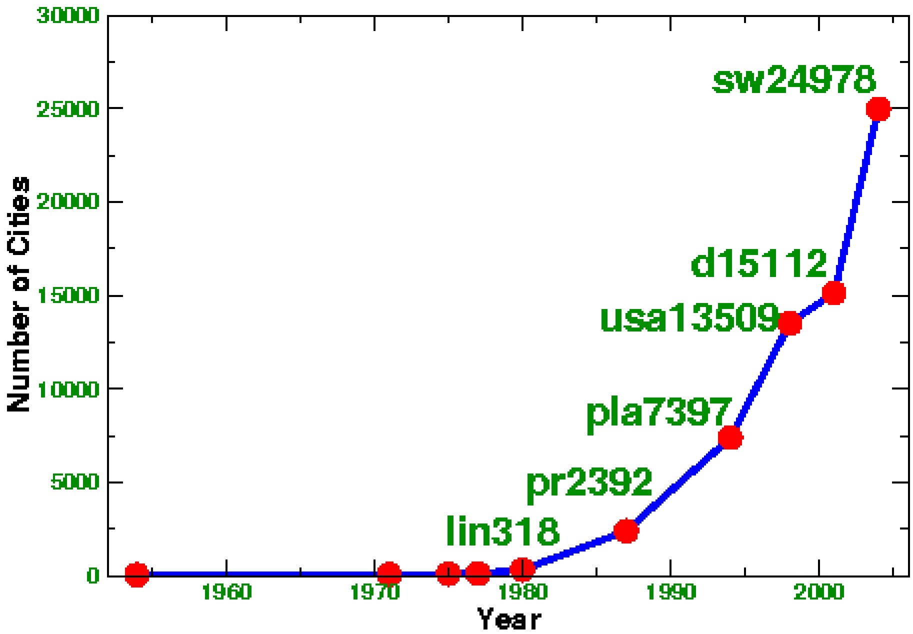
- Convex Hull
- Hamiltonian Cycle
- Minimum Spanning Tree
- Satisfiability

NP-hard; NP-problem

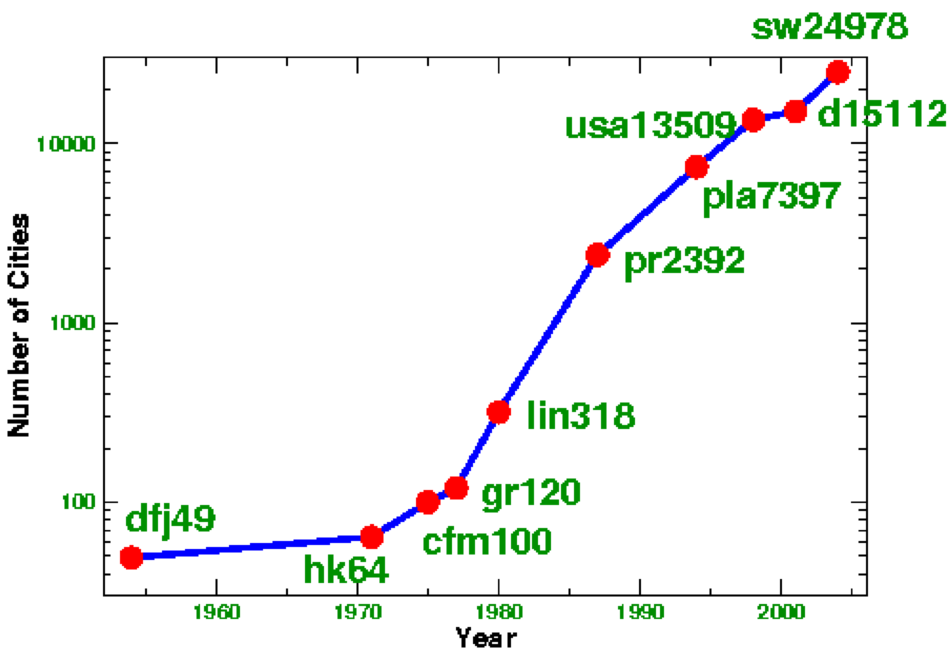
A **problem** which is both **NP** (verifiable in **nondeterministic polynomial time**) and **NP**-hard (any other **NP-problem** can be translated into this **problem**). **...**

http://www.tsp.gatech.edu/

http://www.tsp.gatech.edu/methods/talks/block/slide1.html



A second plot of the TSP milestones, using a log-scale for the number of cities, is given below.



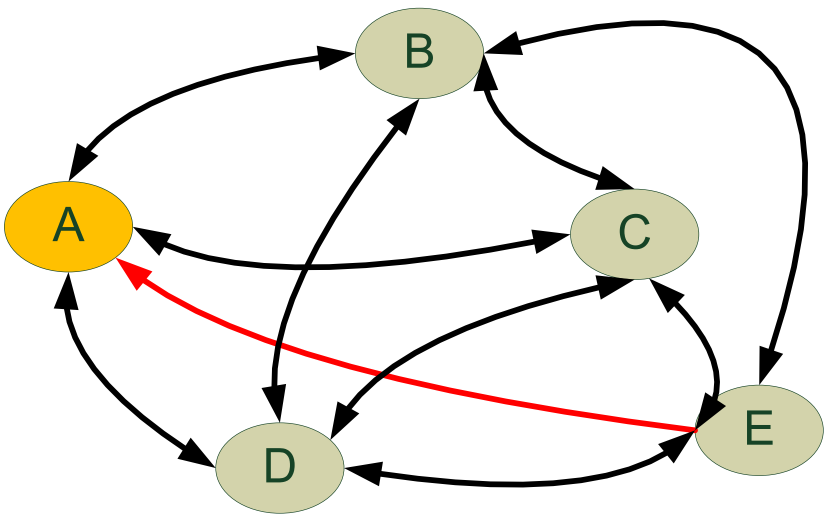
Solution by assignment problem

$x_{ij} = 1$ if the arc is on the tour.
$x_{ij} = 0$ if it is not.

Also for $i \neq j$

d_{ij}= distance between nodes (cities) i and j

d_{ij}= M , where M is a large positive number



Example

Distance between Cities in Traveling Salesperson Problem

From/To	A	B	C	D	E
A	0	1	4	2	**-**
B	1	0	3	1	3
C	5	4	0	4	2
D	2	1	5	0	1
E	3	2	3	1	0

Modified Matrix

From/To	A	B	C	D	E
A	**M**	1	4	2	**M**
B	1	**M**	3	1	3
C	5	4	**M**	4	2
D	2	1	5	**M**	1
E	3	2	3	1	**M**

Assignment Problem Solution

Row Subtraction (Row Reduction)

From/To	A	B	C	D	E
A	**M**	0	3	1	**M**
B	0	**M**	2	0	2
C	3	2	**M**	2	0
D	1	0	4	**M**	0
E	2	1	2	0	**M**

Column Subtraction (Column Reduction)

From/To	A	B	C	D	E
A	**M**	0	1	1	**M**
B	0	**M**	0	0	2
C	3	2	**M**	2	0
D	1	0	2	**M**	0
E	2	1	0	0	**M**

From/To	A	B*	C	D	E*
A*	**M**	**0**	1	1	**M**
B	0	**M**	0	**0**	2
C*	3	2	**M**	2	**0**
D*	1	0	2	**M**	0
E	2	1	**0**	0	**M**

Minimum uncovered value (1)

From/To	A	B	C	D	E
A	**M**	0	0	0	**M**
B	0	**M**	0	0	3
C	2	2	**M**	1	0
D	0	0	1	**M**	0
E	2	2	0	0	**M**

Solution:

From/To	A	B	C	D	E
A	**M**	**0**	0	0	**M**
B	0	**M**	**0**	0	3
C	2	2	**M**	1	**0**
D	**0**	0	1	**M**	0
E	2	2	0	**0**	**M**

Tour A→B→C→ E→D→A

(An itinerary that begins and ends at the same city (node) and visits each city once is called a tour.)

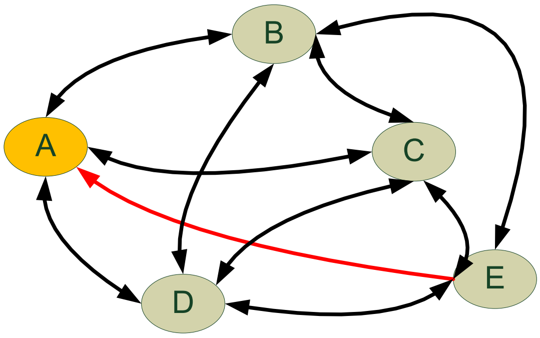
Minimum Traveling Cost (Distance, Time): 9

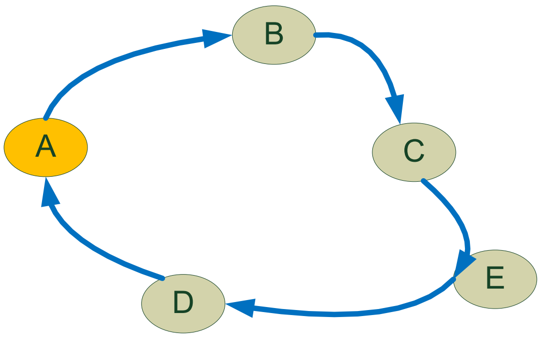
Alternative solution:

From/To	A	B	C	D	E
A	**M**	0	**0**	0	**M**
B	**0**	**M**	0	0	3
C	2	2	**M**	1	**0**
D	0	**0**	1	**M**	0
E	2	2	0	**0**	**M**

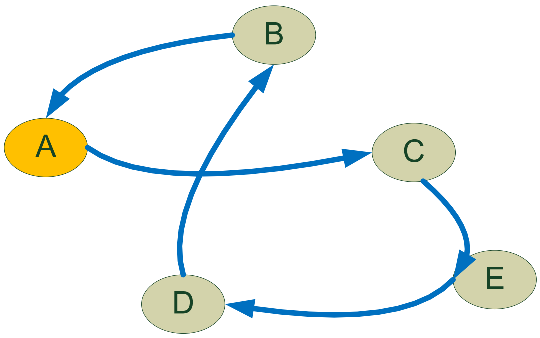
Tour **A→C→E → D→B →A**

Minimum Traveling Cost (Distance, Time):9





Tour **A→B→C→ E→D→A**



Tour **A→C→E → D→B →A**

Branch and Bound Algorithm
For the Traveling Salesman Problem

Sub-problems are assignment problems. If the optimal solution to a sub-problem contains no sub-tours, it is a feasible solution to the traveling salesperson problem. Create new sub-problems by branching to exclude a sub-tour. A sub-problem can be eliminated if its optimal z-value is inferior to the best previously found feasible solution.

Example:

From/To	A-1	B-2	C-3	D-4	E-5
A-1	**M**	132	217	164	58
B-2	132	**M**	290	201	79
C-3	217	290	**M**	113	303
D-4	164	201	113	**M**	196
E-5	58	79	303	196	**M**

The optimal solution: $x_{15} = x_{21} = x_{34} = x_{43} = x_{52} = 1$ $\quad z_{min} = 495$

From/To	A-1	B-2	C-3	D-4	E-5
A-1	**M**	132	217	164	**58**
B-2	**132**	**M**	290	201	79
C-3	217	290	**M**	**113**	303
D-4	164	201	**113**	**M**	196
E-5	58	**79**	303	196	**M**

If the solution to the preceding assignment problem yields a tour, it is the optimal solution to the traveling salesperson problem.(Why?)

Unfortunately, the optimal solution to the assignment problem need not be a tour. For example, the optimal solution to the assignment problem might be $x_{15} = x_{21} = x_{34}= x_{43}= x_{52}=1$.

This solution suggests going from node-1 to node-5, then to node-2, and then back to node-1. This solution also suggests that if someone is in node-3 he should go to node-4 and then return to node-3. This happens because the optimal solution to the assignment problem contains two sub-tours. A sub-tour is a round trip that does not pass through all nodes (cities).

The sub tours could be eliminated by the imposition of the following constraints

Two city (nodes):

$X_{ij} + X_{ji} \leq 1$ for all i and j where $i \neq j$

Three city (nodes):

$X_{ij} + X_{jk} + X_{ki} \leq 2$ for all i, j and k where $i \neq j \neq k$

Four city(nodes):

$X_{ij} + X_{jk} + X_{kl} + X_{li} \leq 3$ for all i, j, k and l where $i \neq j \neq k \neq l$

In a practical problem this way of dealing with sub tours would produce a very large constraint set. For example, with 30 cities there would be 870 constraints for the prevention of the two city sub tour alone. In general, constraints would be required precluding sub tours from size 2 up through the greatest integer number not exceeding half the number of cities. Other formulations exist which preclude sub tours in a more compact fashion. Miller, Tucker and Zemlin show that the following constraints eliminate sub tours in a N city problem

$$U_i - U_j + N X_{ij} \leq N - 1 \quad i=2 \ldots N,\ j=2 \ldots N;\ i \neq j.$$

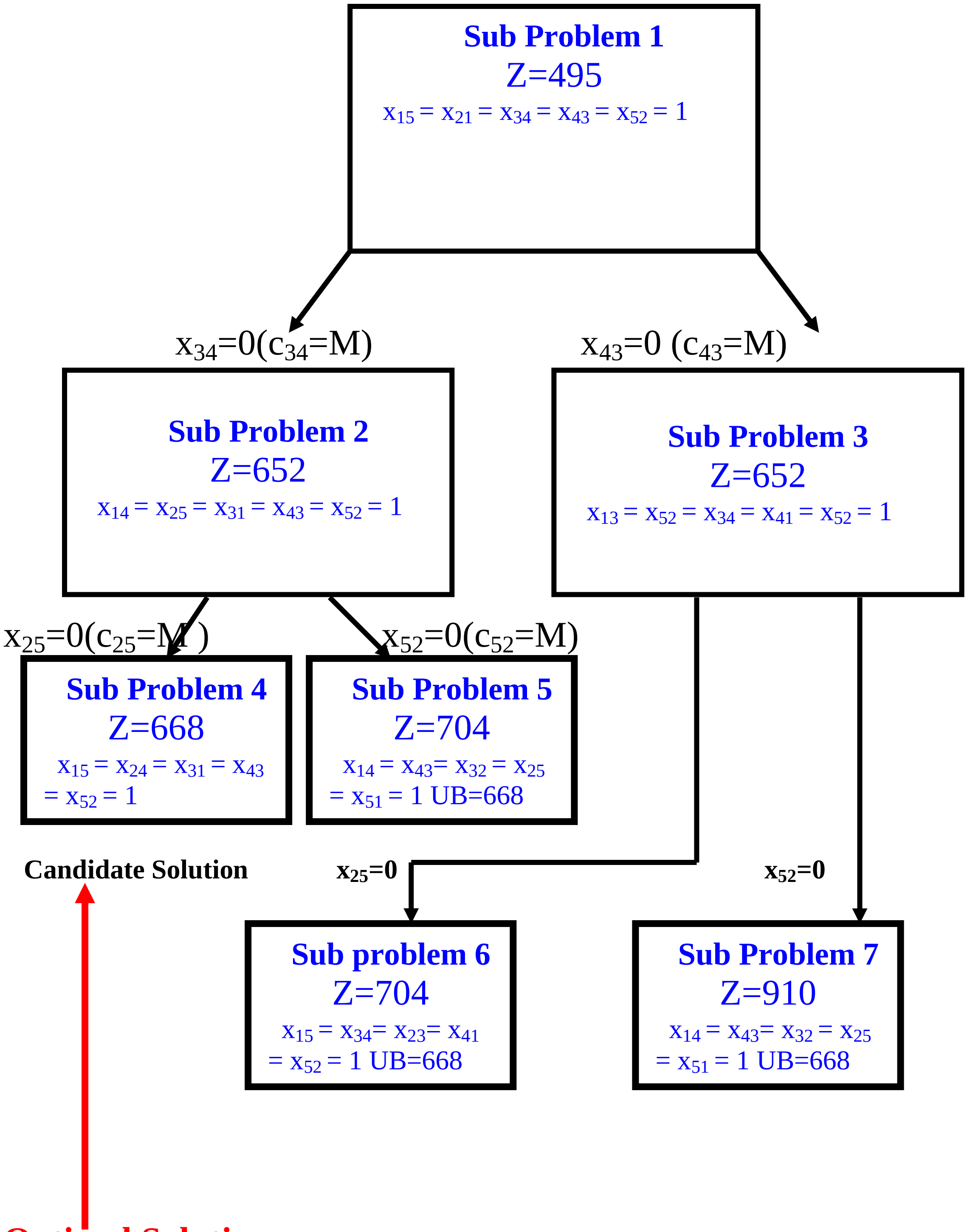
$$U_i \geq 0 \qquad i=2 \ldots N$$

where new continuous variables(U) are introduced. Dantzig, Fulkerson and Johnson(1954) give yet another method.

The present assignment contains the two sub-tours 1-5-2-1 and 3-4-3.

If we could exclude all feasible solutions to the assignment problem that contain sub-tours and then solve the assignment problem, we could obtain the optimal solution to the traveling salesperson problem. This is not easy to do, however. In most cases, **a branch-and-bound approach** **is the most efficient approach** for solving a traveling salesperson problem.

To begin, we solve the preceding assignment problem, in which, for i≠j, the cost c_{ij} is the distance between cities (nodes) i and j and c_{ij}=M (this prevents a person in a city from being assigned to visit that city itself). Since this assignment problem contains no provisions to prevent sub-tours, it is a relaxation (or less constrained problem) of the original traveling salesperson problem. Thus, if the optimal solution to the assignment problem is feasible for the traveling salesperson problem (i.e., if the assignment solution contains no sub-tours), the optimal solutions to the assignment problem are also optimal for the traveling salesperson problem. The results of the branch-and-bound procedure are given.



Optimal Solution

Tour **A(1)→E(5)→B(2) → D(4)→C(3) →A(1)**

Z_{min}= 668 (Traveling Salesperson)