LINEAR PROGRAMMING

The General
Mathematical Programming Problem

Optimize $F(x)$

Subject to (s.t) $G(x) \in S_1$

$\mathbf{X} \in S_2$

X: Vector of Decision Variables

F(X) : Objective Function

G(X) : Constraints

A mathematical Program is an optimization problem in which the <u>objective</u> and <u>constraints</u> are given as mathematical functions and function relationship.

Optimize

$$Z = f(x_1, x_2, \ldots, x_n)$$

Subject to

$$\left.\begin{array}{l} g_1(x_1, x_2, \ldots, x_n) \\ g_2(x_1, x_2, \ldots, x_n) \\ . \\ . \\ . \\ g_m(x_1, x_2, \ldots, x_n) \end{array}\right\} \leq, \geq, = \left\{\begin{array}{l} b_1 \\ b_2 \\ . \\ . \\ . \\ b_m \end{array}\right.$$

Definition

A function $f(x_1, x_2, ..., x_n)$ of $x_1, x_2, ..., x_n$ is a linear function if and only if for some set of constants $c_1, c_2, ..., c_n$, $f(x_1, x_2, ..., x_n) = c_1x_1 + c_2x_2 + ... + c_nx_n$.

For example

$f(x_1,x_2)=2x_1 + x_2$ ***is a linear function of x_1 and x_2 but***
$f(x_1,x_2)=x_1^2x_2$ ***is not a linear function of x_1 and x_2.***

Definition

For any linear function $f(x_1, x_2, ..., x_n)$ and any number b, the inequalities $f(x_1, x_2, ..., x_n) \le b$ and $f(x_1, x_2, ..., x_n) \ge b$ are linear inequalities.

Definition

A linear programming problem (LP) is an optimization problem for which we do the following:

1. ***We attempt to maximize (or minimize) a linear function of the decision variables. The function that is to be maximized or minimized is called the objective function.***
2. ***The values of the decision variables must satisfy a set of constraints. Each constraint must be a linear equation or linear equality.***
3. ***A sign restriction is associated with each variable. For any variable x_i, the sign restriction specifies either that x_i must be nonnegative ($x_i \geq 0$) or that x_i may be unrestricted in sign (urs).***

$$Max(or Min) Z = c_1x_1 + c_2x_2 + c_3x_3 + ... + c_nx_n$$

s.t

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1n}x_n \leq \ = \ \geq b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + ... + a_{2n}x_n \leq \ = \ \geq b_2$$
$$.$$
$$.$$
$$.$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + ... + a_{mn}x_n \leq \ = \ \geq b_m$$

$$x_i \geq 0$$

$$\begin{aligned} &Max(Min) = Z = CX \\ &s.t \qquad AX \leq b \\ &\qquad\quad X \geq 0 \end{aligned}$$

Where

$$A = \begin{bmatrix} a_{11}a_{12}...a_{1n} \\ a_{21}a_{22}...a_{2n} \\ . \\ . \\ . \\ a_{m1}a_{m2}a_{mn} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n. \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_m \end{bmatrix} \qquad C = [c_1, c_2, ..., c_n]$$

Assumptions

The proportionality and additive assumptions

- ***Objective Function***

1. ***The contribution to the objective function from each decision variable is proportional to the value of the decision variable.***
2. ***The contribution to the objective function for any variable is independent of the values of the other decision variables.***

- ***Constraints***

1. ***The contribution of each variable to the left-hand side of each constraint is proportional to the value of the variable.***
2. ***The contribution of a variable to the left-hand side of each constraint is independent of the values of other variables.***

The divisibility assumption

The Divisibility Assumption requires that each decision variable be allowed to assume fractional values.

A linear programming problem in which some or all of the variables must be nonnegative integers is called an integer- programming problem.

The Certainty Assumption

The Certainty Assumption is that each parameter

- ***Objective function coefficient,***
- ***Right-hand side and***
- ***Technological coefficient***

is known with certainty.

GLOSSARY

Decision making: The process of defining the problem, identifying the alternatives, determining the criteria, evaluating the alternatives and choosing an alternative.

Decision: The alternative selected.

Single criterion decision problem: A problem in which the objective is to find the best solution with respect to just one criterion.

Multi-criteria decision problem: A problem that involves more than one criterion; the objective is to find the best solution.

Model: A representation of a real object or situation.

Mathematical Model: Mathematical symbols and expressions used to represent a real situation.

Objective Function: A mathematical expression used to represent the criterion for evaluating solutions to a problem.

Constraints: **Restrictions or limitations imposed on a problem.**

Deterministic Model: **A model in which all uncontrollable inputs are known and cannot vary.**

Stochastic Model: **A model in which at least one uncontrollable input is uncertain and subject to variation; stochastic model are also referred to as probabilistic models.**

Feasible solution: **A decision alternative or solution that satisfies all constraints.**

Infeasible solution: **A decision alternative or solution that violates one or more constraints.**

The Graphical Solution of two-variable Linear Programming Problems

A linear programming problem involving only two decision variables can be solved using a graphical solution procedure.

Any point on the graph can be identified by the x_1 and x_2 values, which indicate the position of the point along the x_1 and x_2 axes respectively. Since every point (x_1,x_2) corresponds to a possible solution, every point on the graph is called a solution point. The solution point where $x_1=x_2=0$ is referred to as a origin.

Definition

The feasible region (search space, solution space) for an LP is the set of all points satisfying all the LP's constraints and all the LP' sign restrictions.

Definition

For a maximization problem, an optimal solution to an LP is a point in the feasible region with the largest objective function value. Similarly, for a minimization problem, an optimal solution is a point in the feasible region with the smallest objective function value.

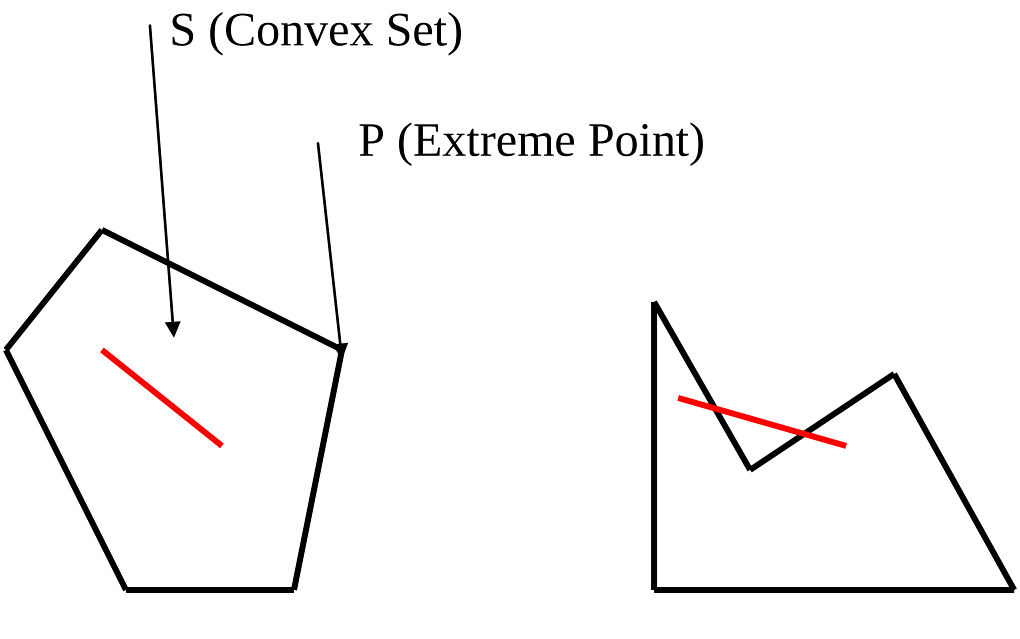
CONVEX SETS, EXTREME POINTS AND LP

Definition

A set of points S is a <u>convex set</u> if the line segment joining any pair of points in S wholly contained in S

Definition

For any convex set S, a point P in S is an <u>extreme point</u> if each line segment that lies completely in S and contains the point P has P as endpoint of the line segment.



Convex Set (Convex Polygon)

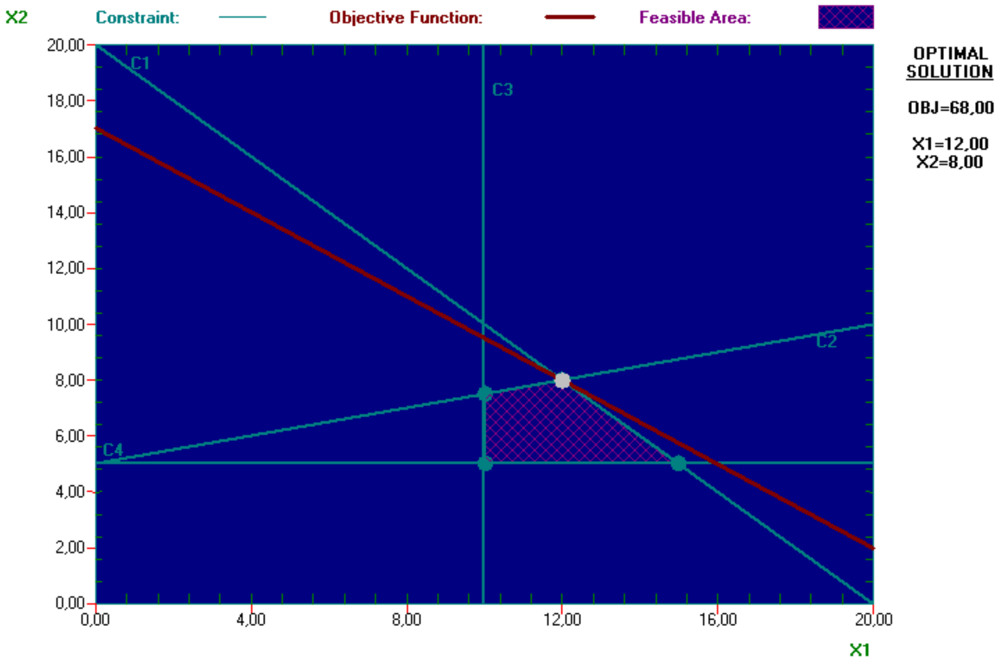
Non-convex set
(Non-convex Polygon)

The graphical solution of LP problems

Example:

$$\max \; Z = 3x_1 + 4x_2$$

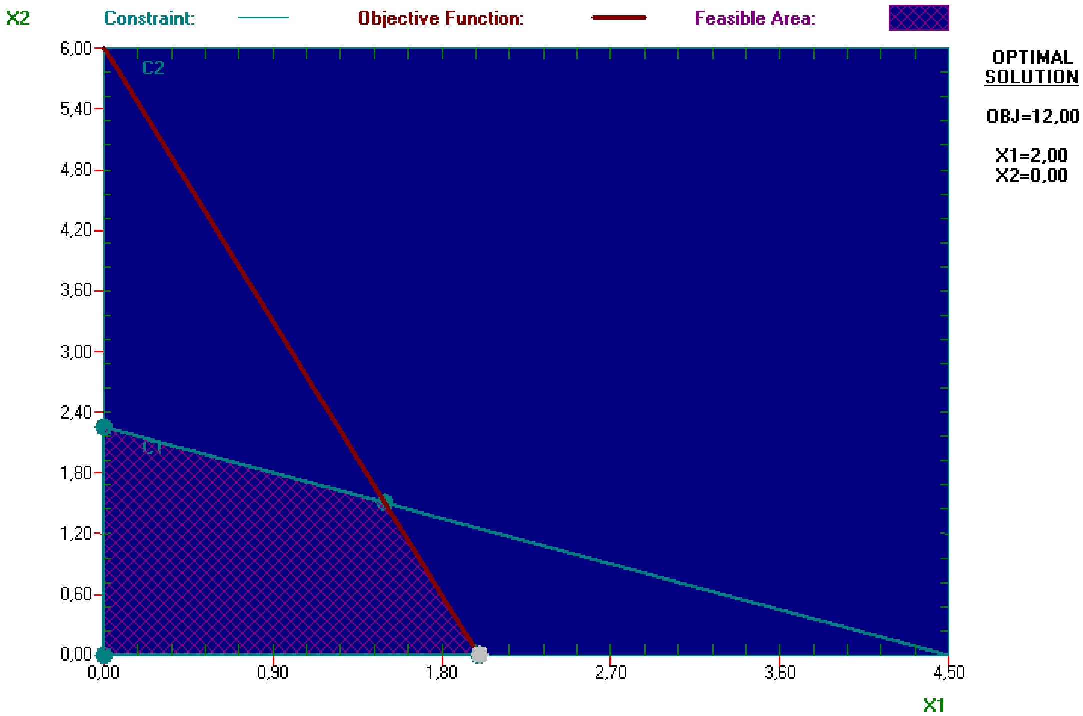
$$\begin{aligned} \text{s.t} \quad x_1 + x_2 &\leq 20 \\ -x_1 + 4x_2 &\leq 20 \\ x_1 &\geq 10 \\ x_2 &\geq 5 \\ x_1, x_2 &\geq 0 \end{aligned}$$



Z increases in the direction $\begin{pmatrix} \dfrac{\partial Z}{\partial x_1} \\ \dfrac{\partial Z}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

Example:

$$\begin{aligned} \max \quad & Z = 6x_1 + 2x_2 \\ \text{s.t} \quad & 2x_1 + 4x_2 \leq 9 \\ & 3x_1 + x_2 \leq 6 \\ & x_1, x_2 \geq 0 \end{aligned}$$

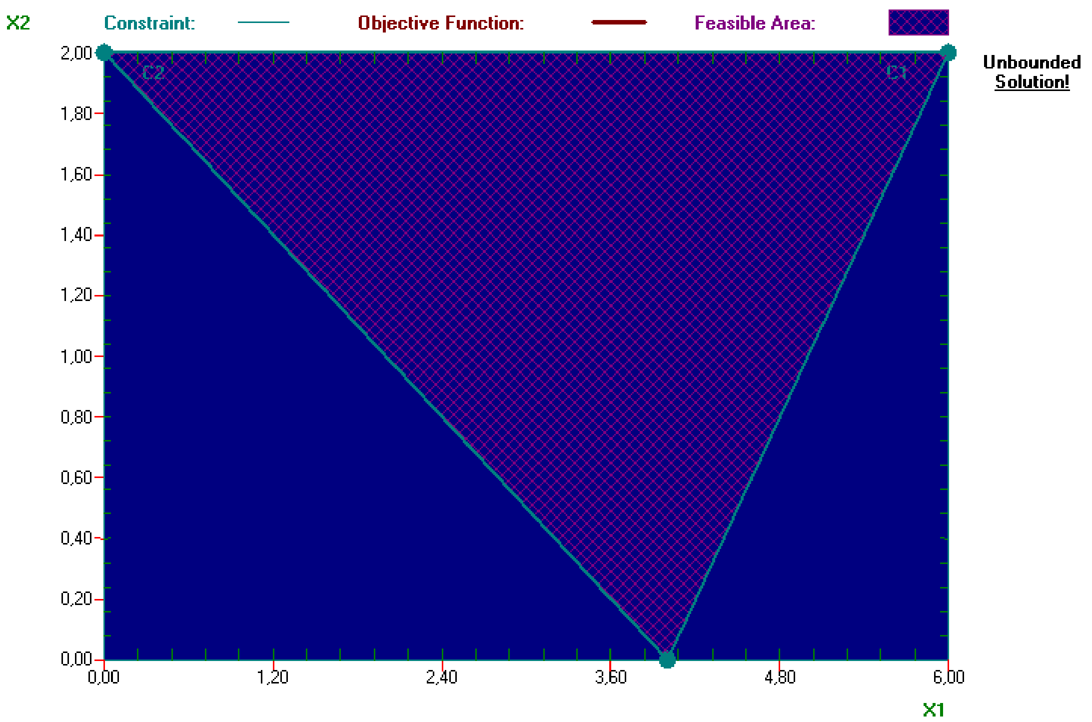


Any point on BC represents an optimal solution.
Problem has multiple (infinite solution).

Z increases in the direction $\begin{pmatrix} \dfrac{\partial Z}{\partial x_1} \\ \dfrac{\partial Z}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$

Example:

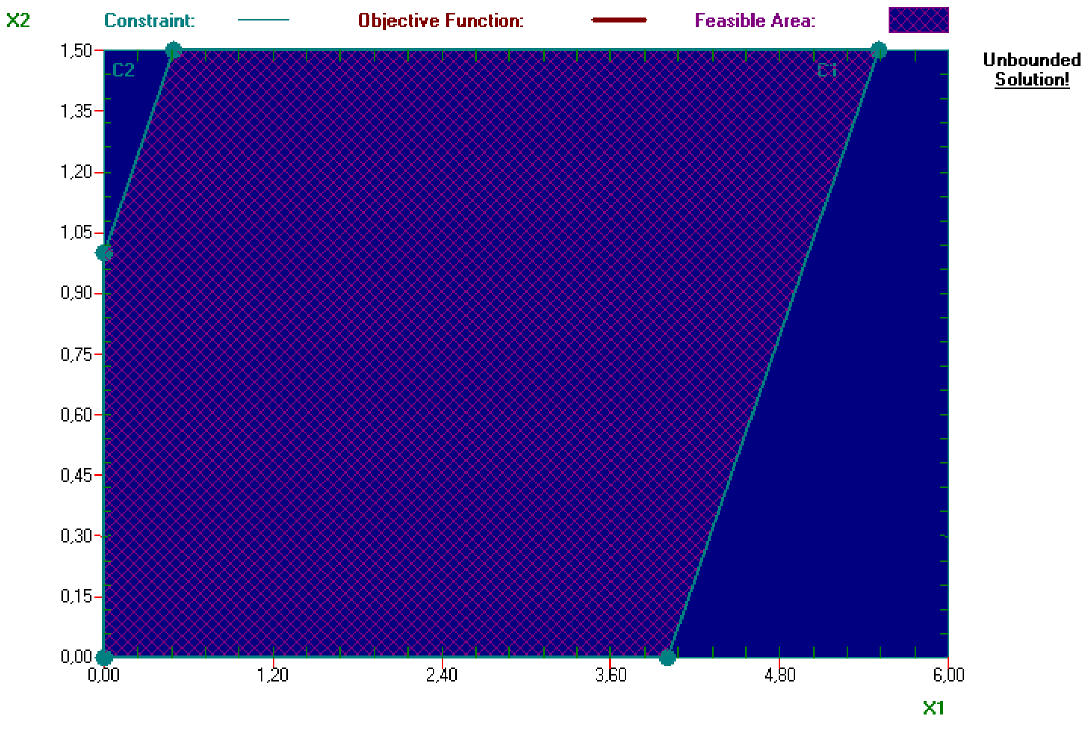
$$\max \; Z = -x_1 + x_2$$
$$\text{s.t} \quad x_1 - x_2 \leq 4$$
$$x_1 + 2x_2 \geq 4$$
$$x_1, x_2 \geq 0$$



UNBOUNDED SOLUTION

Example:

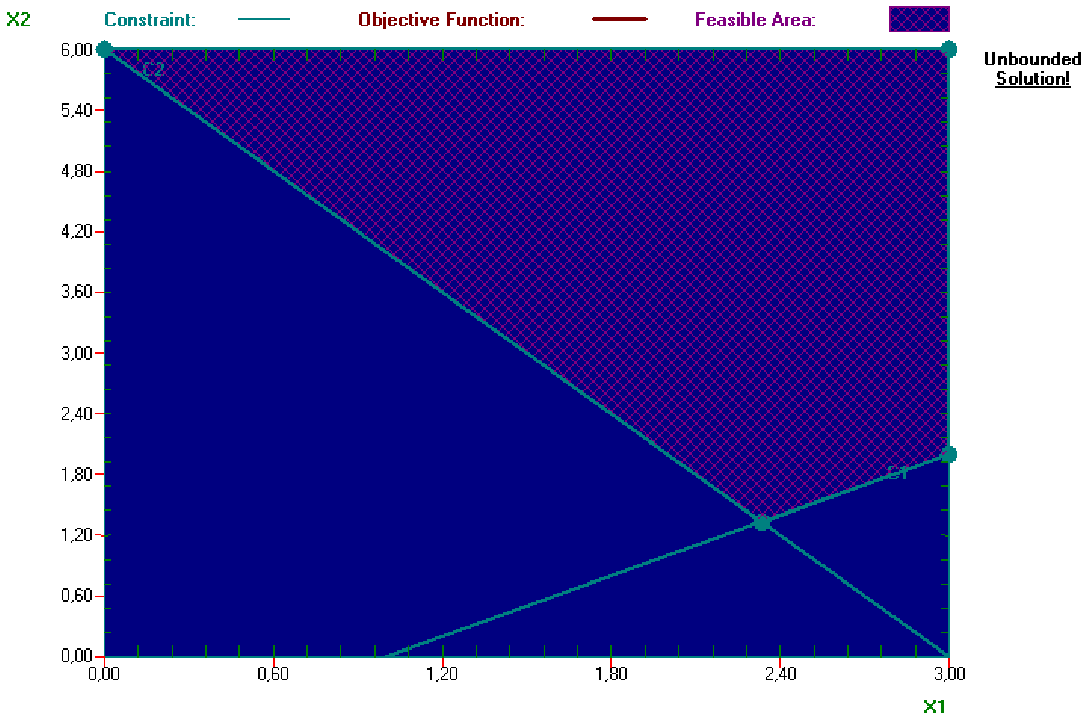
$$\begin{aligned} \max \;\; & Z = x_1 + 2x_2 \\ \text{s.t} \;\; & x_1 - x_2 \leq 4 \\ & -x_1 + x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$



UNBOUNDED SOLUTION

Example:

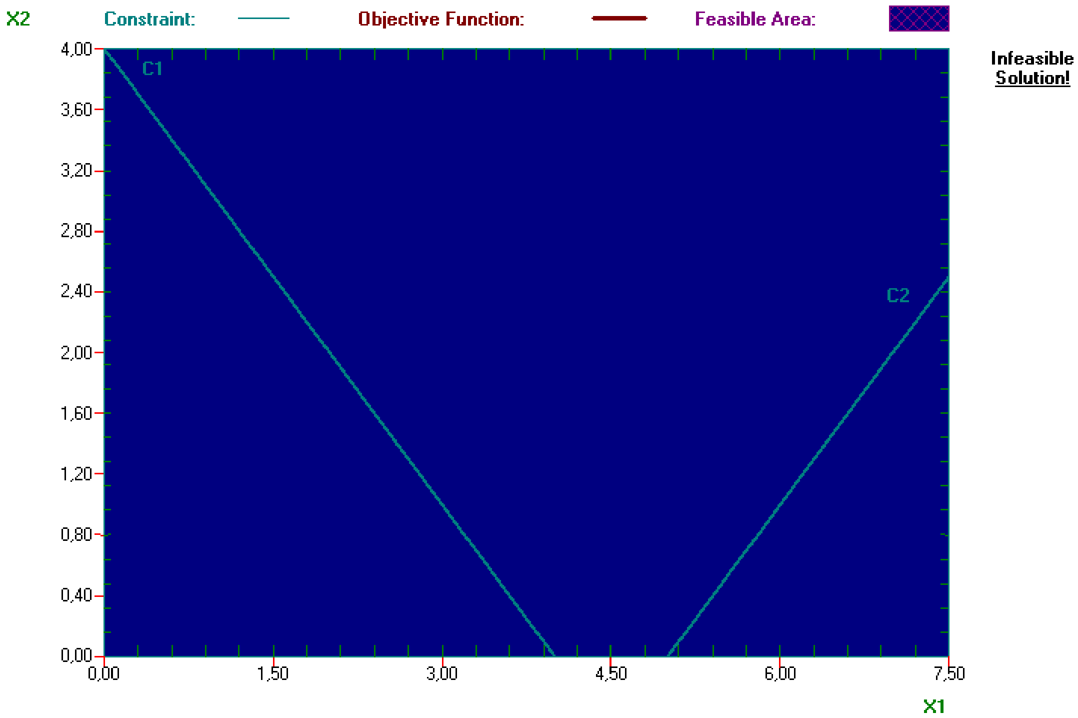
$$\max\ Z = 2x_1 - x_2$$
$$\text{s.t} \quad x_1 - x_2 \le 1$$
$$2x_1 + x_2 \ge 6$$
$$x_1, x_2 \ge 0$$



UNBOUNDED SOLUTION

Example:

$$\begin{aligned} \max\ \ Z &= x_1 + x_2 \\ \text{s.t} \quad x_1 + x_2 &\leq 4 \\ x_1 - x_2 &\geq 5 \\ x_1, x_2 &\geq 0 \end{aligned}$$

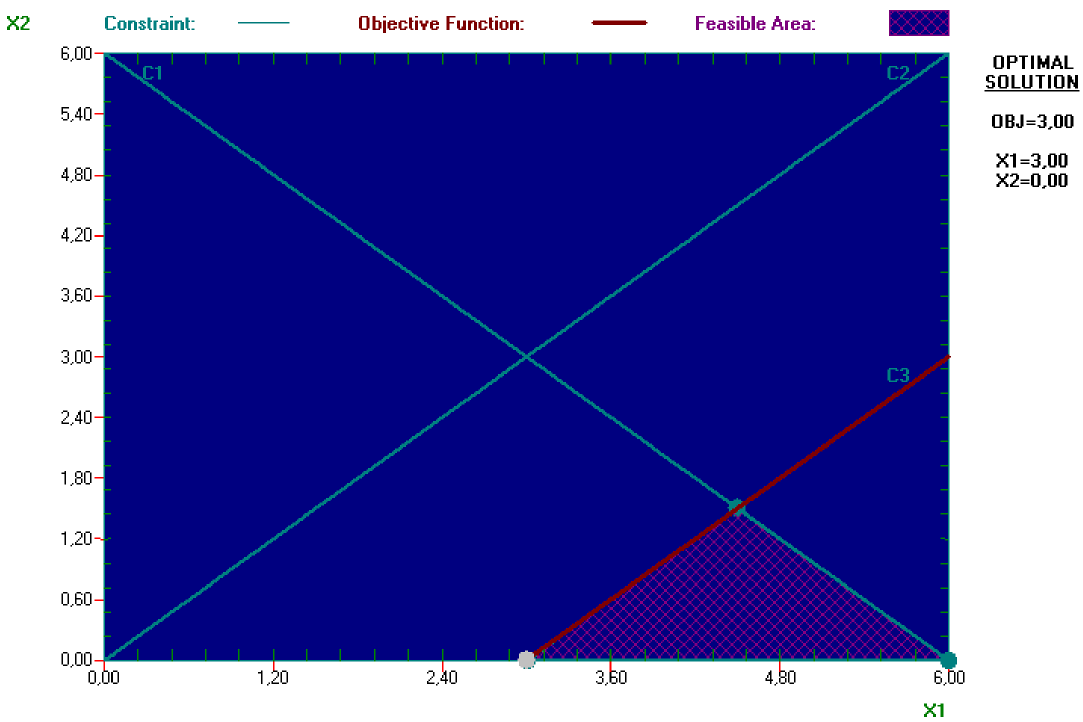


NO FEASIBLE SOLUTION

INFEASIBLE SOLUTION

Example:

$$\begin{aligned} \min \quad & Z = x_1 - x_2 \\ \text{s.t} \quad & x_1 + x_2 \le 6 \\ & x_1 - x_2 \ge 0 \\ & x_1 - x_2 \ge 3 \\ & x_1, x_2 \ge 0 \end{aligned}$$

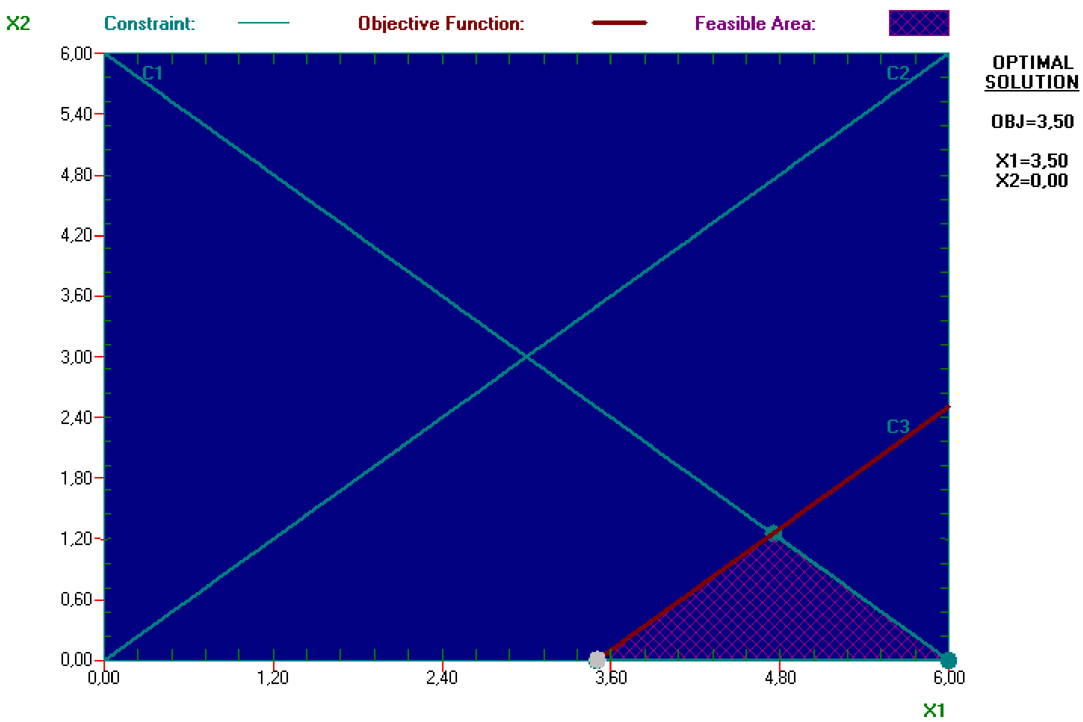


ALTERNATE OPTIMUM SOLUTION

- $X_1 = 3 \; X_2 = 0 \qquad Z_{MİN} = 3.0$
- $X_1 = 4.5 \; X_2 = 1.5 \qquad Z_{MİN} = 3.0$

Example:

$$\min \; Z = x_1 - x_2$$
$$\text{s.t} \quad x_1 + x_2 \leq 6$$
$$x_1 - x_2 \geq 0$$
$$x_1 - x_2 \geq 3.5$$
$$x_1, x_2 \geq 0$$ **AND INTEGER**



ALTERNATE INTEGER OPTIMUM SOLUTION

- $X_1 = 5 \; X_2 = 1 \;\; Z_{MİN} = 4$
- $X_1 = 4 \;\; X_2 = 0 \; Z_{MİN} = \; 4$