Analysis of Algorithms

Chapter 7.1, 7.2, 7.3

ROAD MAP

- **Space-for-time tradeoffs**
  - Sorting by Counting
  - Input Enhancement in String Matching
  - Hashing

Space-for-time tradeoffs

Two varieties of space-for-time algorithms:

- *input enhancement* — preprocess the input (or its part) to store some info to be used later in solving the problem
  - counting sorts
  - string searching algorithms
- *prestructuring* — preprocess the input to make accessing its elements easier
  - hashing
  - indexing schemes (e.g., B-trees)

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Sorting by Counting

ALGORITHM *ComparisonCountingSort*$(A[0..n-1])$

//Sorts an array by comparison counting
//Input: An array $A[0..n-1]$ of orderable elements
//Output: Array $S[0..n-1]$ of A's elements sorted in nondecreasing order
for $i \leftarrow 0$ **to** $n-1$ **do** $Count[i] \leftarrow 0$
for $i \leftarrow 0$ **to** $n-2$ **do**
    for $j \leftarrow i+1$ **to** $n-1$ **do**
        if $A[i] < A[j]$
            $Count[j] \leftarrow Count[j] + 1$
        else $Count[i] \leftarrow Count[i] + 1$
for $i \leftarrow 0$ **to** $n-1$ **do** $S[Count[i]] \leftarrow A[i]$
return S

Sorting by Counting

Array *A*[0..5]		62	31	84	96	19	47
Initially	*Count* []	0	0	0	0	0	0
After pass *i* = 0	*Count* []	3	0	1	1	0	0
After pass *i* = 1	*Count* []		1	2	2	0	1
After pass *i* = 2	*Count* []			4	3	0	1
After pass *i* = 3	*Count* []				5	0	1
After pass *i* = 4	*Count* []					0	2
Final state	*Count* []	3	1	4	5	0	2
Array *S*[0..5]		19	31	47	62	84	96

FIGURE 7.1 Example of sorting by comparison counting.

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Review: String searching by brute force

pattern: a string of *m* characters to search for

text: a (long) string of *n* characters to search in

Brute force algorithm

Step 1 Align pattern at beginning of text

Step 2 Moving from left to right, compare each character of pattern to the corresponding character in text until either all characters are found to match (successful search) or a mismatch is detected

Step 3 While a mismatch is detected and the text is not yet exhausted, realign pattern one position to the right and repeat Step 2

String searching by preprocessing

Several string searching algorithms are based on the input

enhancement idea of preprocessing the pattern

- Boyer -Moore algorithm preprocesses pattern right to left and store information into two tables

- Horspool's algorithm simplifies the Boyer-Moore algorithm by using just one table

Horspool's Algorithm

A simplified version of Boyer-Moore algorithm:

- preprocesses pattern to generate a shift table that determines how much to shift the pattern when a mismatch occurs

- always makes a shift based on the text's character *c* aligned with the <u>last</u> character in the pattern according to the shift table's entry for *c*

Horspool's Algorithm

s_0 . . . S . . . s_{n-1}

$\nparallel$

B A R B E R

B A R B E R

Case 1

s_0 . . . B . . . s_{n-1}

$\nparallel$

B A R B E R

B A R B E R

Case 2

s_0 . . . M E R . . . s_{n-1}

$\nparallel$ $\parallel$ $\parallel$

L E A D E R

L E A D E R

Case 3

s_0 . . . A R . . . s_{n-1}

$\nparallel$ $\parallel$

R E O R D E R

R E O R D E R

Case 4

Shift table

- Shift sizes can be precomputed by the formula

$$t(c) = \begin{cases} \text{the pattern's length } m, \\ \text{if } c \text{ is not among the first } m-1 \text{ characters of the pattern;} \\ \\ \text{the distance from the rightmost } c \text{ among the first } m-1 \text{ characters} \\ \text{of the pattern to its last character, otherwise.} \end{cases} \quad \textbf{(7.1)}$$

  by scanning pattern before search begins and stored in a table called *shift table*

- Shift table is indexed by text and pattern alphabet
  Eg, for `BARBER`:

character c	A	B	C	D	E	F	...	R	...	Z	_
shift $t(c)$	4	2	6	6	1	6	6	3	6	6	6

Example of Horspool's alg. application

character c	A	B	C	D	E	F	...	R	...	Z	_
shift $t(c)$	4	2	6	6	1	6	6	3	6	6	6

```
J I M _ S A W _ M E _ I N _ A _ B A R B E R S H O P
B A R B E R           B A R B E R
        B A R B E R       B A R B E R
          B A R B E R           B A R B E R
```

Boyer-Moore algorithm

Based on same two ideas:

- comparing pattern characters to text from right to left
- precomputing shift sizes in two tables
  - *bad-symbol table* indicates how much to shift based on text's character causing a mismatch
  - *good-suffix table* indicates how much to shift based on matched part (suffix) of the pattern

Bad-symbol shift in Boyer-Moore algorithm

- If the rightmost character of the pattern doesn't match, BM algorithm acts as Horspool's
- If the rightmost character of the pattern does match, BM compares preceding characters right to left until either all pattern's characters match or a mismatch on text's character *c* is encountered after *k* > 0 matches

$$
\begin{array}{ccccccccccc}
s_0 & \ldots & & c & s_{i-k+1} & \ldots & s_i & \ldots & s_{n-1} & \text{text} \\
 & & & \not\parallel & \parallel & & \parallel & & & \\
 & p_0 & \ldots & p_{m-k-1} & p_{m-k} & \ldots & p_{m-1} & & & \text{pattern}
\end{array}
$$

bad-symbol shift $d_1 = \max\{t_1(c) - k,\ 1\}$

Good-suffix shift in Boyer-Moore algorithm

- Good-suffix shift d_2 is applied after $0 < k < m$ last characters were matched
- $d_2(k)$ = the distance between matched suffix of size k and its rightmost occurrence in the pattern that is not preceded by the same character as the suffix

  Example: `CABABA` $d_2(1) = 4$

- If there is no such occurrence, match the longest part of the k-character suffix with corresponding prefix;
  if there are no such suffix-prefix matches, $d_2(k) = m$

  Example: `WOWWOW` $d_2(2) = 5$, $d_2(3) = 3$, $d_2(4) = 3$, $d_2(5) = 3$

Good-suffix shift in the Boyer-Moore alg. (cont.)

After matching successfully $0 < k < m$ characters, the algorithm shifts the pattern right by

$$d = \max\{d_1, d_2\}$$

where $d_1 = \max\{t_1(c) - k, 1\}$ is bad-symbol shift

$d_2(k)$ is good-suffix shift

Boyer-Moore Algorithm (cont.)

Step 1 Fill in the bad-symbol shift table

Step 2 Fill in the good-suffix shift table

Step 3 Align the pattern against the beginning of the text

Step 4 Repeat until a matching substring is found or text ends:

Compare the corresponding characters right to left.

If no characters match, retrieve entry $t_1(c)$ from the bad-symbol table for the text's character c causing the mismatch and shift the pattern to the right by $t_1(c)$.
If $0 < k < m$ characters are matched, retrieve entry $t_1(c)$ from the bad-symbol table for the text's character c causing the mismatch and entry $d_2(k)$ from the good-suffix table and shift the pattern to the right by

$$d = \max\{d_1, d_2\}$$

where $d_1 = \max\{t_1(c) - k, 1\}$.

Example of Boyer-Moore alg. application

c	A	B	C	D	. . .	O	. . .	Z	_
$t_1(c)$	1	2	6	6	6	3	6	6	6

B E S S _ K N E W _ A B O U T _ B A O B A B

B A O B A B

$d_1 = t_1(K) - 0 = 6$ B A O B A B

$d_1 = t_1(\_) - 2 = 4$ B A O B A B

$d_2 = 5$ $d_1 = t_1(\_) - 1 = 5$

$d = \max\{4, 5\} = 5$ $d_2 = 2$

$d = \max\{5, 2\} = 5$

B A O B A B

k	**pattern**	d_2
1	$\mathrm{BAO\overline{B}A\underline{B}}$	2
2	$\mathrm{\overline{B}AOB\underline{AB}}$	5
3	$\mathrm{\overline{B}AO\underline{BAB}}$	5
4	$\mathrm{\overline{B}A\underline{OBAB}}$	5
5	$\mathrm{\overline{B}\underline{AOBAB}}$	5

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  - **Hashing**

Hashing

- A very efficient method for implementing a *dictionary,* i.e., a set with the operations:
  - find
  - insert
  - delete

- Based on representation-change and space-for-time tradeoff ideas

- Important applications:
  - symbol tables
  - databases (*extendible hashing*)

Hash tables and hash functions

The idea of *hashing* is to map keys of a given file of size n into a table of size m, called the *hash table*, by using a predefined function, called the *hash function*,

$$h: K \rightarrow \text{ location (cell) in the hash table}$$

Example: student records, key = SSN. Hash function:
$h(K) = K \bmod m$ where m is some integer (typically, prime)
If $m = 1000$, where is record with SSN= 314159265 stored?

Generally, a hash function should:

- be easy to compute
- distribute keys about evenly throughout the hash table

Collisions

If $h(K_1) = h(K_2)$, there is a *collision*

- Good hash functions result in fewer collisions but some collisions should be expected (*birthday paradox*)
- Two principal hashing schemes handle collisions differently:
  - *Open hashing*
    – each cell is a header of linked list of all keys hashed to it
  - *Closed hashing*
    - one key per cell
    - in case of collision, finds another cell by
      - *linear probing:* use next free bucket
      - *double hashing:* use second hash function to compute increment

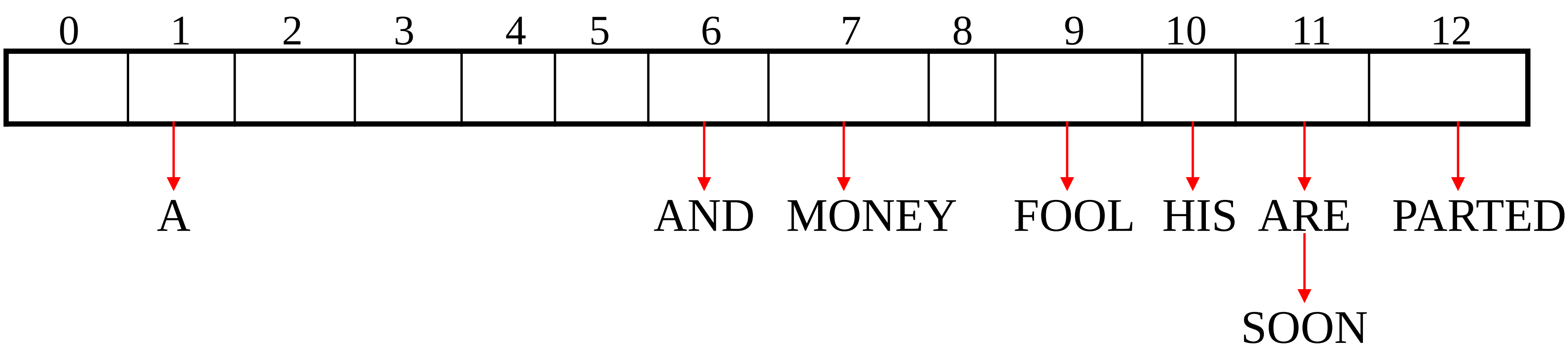
Open hashing (Separate chaining)

Keys are stored in linked lists <u>outside</u> a hash table whose elements serve as the lists' headers.

Example: A, FOOL, AND, HIS, MONEY, ARE, SOON, PARTED

$h(K)$ = sum of K 's letters' positions in the alphabet MOD 13

Key	A	FOOL	AND	HIS	MONEY	ARE	SOON	PARTED
$h(K)$	1	9	6	10	7	11	11	12



Search for KID

Open hashing (cont.)

- If hash function distributes keys uniformly, average length of linked list will be $\alpha = n/m$. This ratio is called *load factor*.

- Average number of probes in successful, S, and unsuccessful searches, U:

$$S \approx 1+\alpha/2, \qquad U = \alpha$$

- Load α is typically kept small (ideally, about 1)

- Open hashing still works if $n > m$

Closed hashing (Open addressing)

Keys are stored <u>inside</u> a hash table.

Key	A	FOOL	AND	HIS	MONEY	ARE	SOON	PARTED
h(*K*)	1	9	6	10	7	11	11	12

0	1	2	3	4	5	6	7	8	9	10	11	12
	A											
	A								FOOL			
	A					AND			FOOL			
	A					AND			FOOL	HIS		
	A					AND	MONEY		FOOL	HIS		
	A					AND	MONEY		FOOL	HIS	ARE	
	A					AND	MONEY		FOOL	HIS	ARE	SOON
PARTED	A					AND	MONEY		FOOL	HIS	ARE	SOON

Closed hashing (cont.)

- Does not work if *n* > *m*
- Avoids pointers
- Deletions are *not* straightforward
- Number of probes to find/insert/delete a key depends on load factor $\alpha = n/m$ (hash table density) and collision resolution strategy. For linear probing:

  $S = (½)\,(1 + 1/(1-\alpha))$ and $U = (½)\,(1 + 1/(1-\alpha)^2)$
- As the table gets filled (α approaches 1), number of probes in linear probing increases dramatically:

α	$\frac{1}{2}(1+\frac{1}{1-\alpha})$	$\frac{1}{2}(1+\frac{1}{(1-\alpha)^2})$
50%	1.5	2.5
75%	2.5	8.5
90%	5.5	50.5