MATRICES
AND
MATRIX OPERATIONS
V

EIGENVALUES
AND
EIGENVECTORS

Linear Systems of the Form

$$Ax = \lambda x$$

Many applications of linear algebra are concerned with systems of n linear equations in n unknowns that are expressed in the form

$$Ax = \lambda x$$

where λ is a scalar. Such systems are really homogeneous linear systems in disguise, since the given system can be written as

$\lambda x - Ax = 0$ **or, by inserting an identity matrix and factoring, as**

$$(\lambda I - A)x = 0$$

Example: Finding $(\lambda I - A)$

The linear system

$$x_1 + 3x_2 = \lambda x_1$$
$$4x_1 + 2x_2 = \lambda x_2$$

can be written in matrix form as

$$\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

This system can be written as

$$\lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} \lambda - 1 & -3 \\ -4 & \lambda - 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\lambda I - A = \begin{bmatrix} \lambda - 1 & -3 \\ -4 & \lambda - 2 \end{bmatrix}$$

The primary problem of interest for linear systems of the form $(\lambda I - A)x = 0$ is to determine those values of λ for which the system has a nontrivial solution; such a value of λ is called **characteristic value** or an **eigenvalue** of A. If λ is eigenvalue of A , then the nontrivial solution of $(\lambda I - A)x = 0$ are called **eigenvectors** of A corresponding to λ.

Definition

If A is $n \times n$ matrix, then a nonzero vector x in R^n is called an **eigenvector** of A if Ax is a scalar multiple of x; that is if

$$Ax = \lambda x$$

for some scalar λ. The scalar λ is called an **eigenvalue** of A, and x said to be an **eigenvector** of A corresponding to λ.

Example: Eigenvector of a 2x2 Matrix

The vector $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$$

corresponding to the eigenvalue $\lambda = 3$, since

$$Ax = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3x$$

To find the eigenvalues of $n \times n$ matrix A, we rewrite

$Ax = \lambda x$ as $Ax = \lambda I x$ or equivalently,

$$(\lambda I - A)x = 0$$

For λ to be an eigenvalue, there must be a nonzero solution of this equation.

Theorem ***Equation*** $(\lambda I - A)x = 0$ ***has a nonzero solution if and only if***

$$\det(\lambda I - A) = 0$$

This is called the characteristic equation of A; the scalars satisfying this equation are the eigenvalues of A.

Remark ***A nxn homogeneous system of linear equations has a unique solution (the trivial solution) <u>if and only if its determinant is non-zero</u>. If this determinant is zero, then the system has an infinite number of solutions.***

When expanded the determinant, $\det(\lambda I - A) = 0$ ***is always a polynomial p in*** λ***, called the characteristic polynomial of A:***

If A is an $n \times n$ ***matrix, then the characteristic polynomial of A has degree n and the coefficient of*** λ^n ***is 1; that is, the characteristic polynomial*** $p(\lambda)$ ***of an*** $n \times n$ ***matrix has the form***

$$p(\lambda) = \det(\lambda I - A) = \lambda^n + c_1\lambda^{n-1} + \ldots + c_n$$

It follows from the <u>Fundamental Theorem of Algebra</u> that the characteristic equation

$$\lambda^n + c_1\lambda^{n-1} + \ldots + c_n = 0$$

has at most n distinct solutions, so an $n \times n$ ***matrix has at most n distinct eigenvalues.***

Example: Eigen values and Eigenvectors

Find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$$

The characteristic equation of A is

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix} - \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} = \begin{vmatrix} \lambda - 1 & -3 \\ -4 & \lambda - 2 \end{vmatrix} = 0$$

or

$$\lambda^2 - 3\lambda - 10 = 0$$

The factored form of this equation is

$(\lambda + 2)(\lambda - 5) = 0$, so the eigenvalues of A are $\lambda = -2, \lambda = 5$.

By definition,

$$x = \begin{vmatrix} x_1 \\ x_2 \end{vmatrix}$$

is an eigenvector of A if and only if **x** is a nontrivial solution of $(\lambda I - A)x = 0$; that is,

$$\begin{bmatrix} \lambda - 1 & -3 \\ -4 & \lambda - 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If $\lambda = -2$, then

$$\begin{bmatrix} -3 & -3 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system yields $x_1 = -x_2$ $x_2 = t$ so the eigenvectors corresponding to $\lambda = -2$ are the nonzero solutions of the form

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

If $\lambda = 5$, then

$$\begin{bmatrix} 4 & -3 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving this system yields $x_1 = \frac{3}{4}t, x_2 = t$ so the eigenvectors corresponding to $\lambda = 5$ are the nonzero solutions of the form

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{4}t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix}$$

EIGEN VALUES AND EIGENVECTORS IN MATLAB

```
>> A=[1 3; 4 2]
A =
     1     3
     4     2
>> [v,d]=eig(A)
v =
   -0.7071   -0.6000
    0.7071   -0.8000
d =
    -2     0
     0     5
>> vv=[-1 1]
vv =    -1     1
>> vv=vv'
vv =
    -1
     1
>> nor=norm(vv)
nor =
    1.4142
>> vv=vv/nor
vv =
   -0.7071
    0.7071
>> vv'*vv
ans =
    1.0000
```

Example: Eigen values and Eigenvectors

Find the eigenvalues and corresponding eigenvectors of the matrix

$$A=\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

$$\det(\lambda I - A)=\begin{vmatrix}\lambda-2 & 1\\ 1 & \lambda\text{-}2\end{vmatrix}=0$$

or

$$(\lambda-2)^2-1=0$$

The roots of this equation $\lambda=3, \lambda=1$.

Consider the first Eigen value

$\lambda=3$

$$\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

Both rows of this matrix equation reduce to

$x_1=-x_2$

$$x=\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}-t\\ t\end{bmatrix}=t\begin{bmatrix}-1\\ 1\end{bmatrix}$$

For t=1 the corresponding eigenvector is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Consider the second Eigen value

$\lambda = 1$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Similar process leads to $x_1 = x_2$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For t=1 the corresponding eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

```
>> A=[2 1; 1 2]
A =
     2     1
     1     2
>> [v,d]=eig(A)
v =
   -0.7071    0.7071
    0.7071    0.7071

d =
     1     0
     0     3
```

Example: Eigen values of a 3x3 Matrix

Find the eigenvalues of

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$$

The characteristic polynomial of A is

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -4 & 17 & \lambda - 8 \end{bmatrix} = \lambda^3 - 8\lambda^2 + 17\lambda - 4$$

The Eigenvalues of A must therefore satisfy the cubic equation

$$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$$

$$(\lambda - 4)(\lambda^2 - 4\lambda + 1) = 0$$

$$\lambda = 4, \lambda = 2 + \sqrt{3}, \lambda = 2 - \sqrt{3}$$

```
>> A=[0 1 0;0 0 1;4 -17,8]
A =
     0     1     0
     0     0     1
     4   -17     8
>> [v,d]=eig(A)
v =
   -0.9636   -0.0692   -0.0605
   -0.2582   -0.2582   -0.2421
   -0.0692   -0.9636   -0.9684
d =
    0.2679         0         0
         0    3.7321         0
         0         0    4.0000
>>
```

If $\lambda = 4$

$$\begin{bmatrix} 4 & -1 & 0 \\ 0 & 4 & -1 \\ -4 & 17 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 4 & -1 & 0 \\ 0 & 4 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$4x_1 - x_2 = 0$$
$$4x_2 - x_3 = 0$$

Eigenvectors of A corresponding to $\lambda = 4$

$$x = t \begin{bmatrix} -1/16 \\ -1/4 \\ -1 \end{bmatrix}$$

Example:

$$A=\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

The characteristic equation of matrix A is

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

Or in factored form,

$$(\lambda - 1)(\lambda - 2)^2 = 0$$

Thus the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_{2,3} = 2$

By definition,

$$x=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

is an eigenvector of A corresponding to λ if and only if **x** is a nontrivial solution of $(\lambda I - A)x = 0$ that is, of

$$\begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If $\lambda = 1$

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system yields (verify)

$$x_1 = -2s, x_2 = s, x_3 = s$$

Thus the eigenvectors of A corresponding to $\lambda = 1$ are the nonzero vectors of the form

$$\begin{bmatrix} -2s \\ s \\ s \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

If $\lambda = 2$

$$\begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system using Gaussian elimination yields (verify)

$$x_1 = -s, x_2 = t, x_3 = s$$

Thus the eigenvectors of A corresponding to $\lambda = 2$ are the nonzero vectors of the form

$$x = \begin{bmatrix} -s \\ t \\ s \end{bmatrix}$$

$$x = \begin{bmatrix} -s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -s \\ 0 \\ s \end{bmatrix} + \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Since $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ are **orthogonal**, these vectors form a basis for the **eigenspace** corresponding to $\lambda = 2$.

```
A=[0 0 -2;1 2 1;1 0 3]

A =

     0     0    -2
     1     2     1
     1     0     3

>> [v,d]=eig(A)

v =

         0   -0.8165    0.7071 UNIT VECTORS
    1.0000    0.4082         0
         0    0.4082   -0.7071

d =

     2     0     0
     0     1     0
     0     0     2

>>
```

Example: Eigenvalues of an Upper Triangular Matrix

Find the eigenvalues of the upper triangular matrix

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

$$\det(\lambda I - A) = \det\begin{bmatrix} \lambda - a_{11} & -a_{12} & -a_{13} & -a_{14} \\ 0 & \lambda - a_{22} & -a_{23} & -a_{24} \\ 0 & 0 & \lambda - a_{33} & -a_{34} \\ 0 & 0 & 0 & \lambda - a_{44} \end{bmatrix}$$
$$= (\lambda - a_{11})(\lambda - a_{22})(\lambda - a_{33})(\lambda - a_{44})$$

Thus, the characteristic equation is

$$(\lambda - a_{11})(\lambda - a_{22})(\lambda - a_{33})(\lambda - a_{44}) = 0$$

and the eigenvalues are

$$\lambda_1 = a_{11}, \lambda_2 = a_{22}, \lambda_3 = a_{33}, \lambda_4 = a_{44}$$

which are precisely the diagonal entries of A.

Theorem: ***If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then the eigenvalues of A are the entries on the main diagonal of A.***

Example: Eigenvalues of a Lower Triangular Matrix

By inspection, the eigenvalues of the lower triangular matrix

$$A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -1 & \frac{2}{3} & 0 \\ 5 & -8 & -\frac{1}{4} \end{bmatrix}$$

are $\lambda = \frac{1}{2}, \lambda = \frac{2}{3}, \lambda = -\frac{1}{4}$.

Complex Eigenvalues

It is possible for the characteristic equation of a matrix with real entries to have complex solutions. In fact, because the eigenvalues of an $n \times n$ matrix are the roots of a polynomial of precise degree n, every $n \times n$ matrix has exactly n eigenvalues if we count them as we count the roots of a polynomial (<u>meaning that they may be repeated, and may occur in complex conjugate pairs</u>).

Example: Complex eigenvalues

$$A = \begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix}$$

The characteristic polynomial of the given matrix is

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda + 2 & 1 \\ -5 & \lambda - 2 \end{bmatrix} = \lambda^2 + 1$$

So the characteristic equation is $\lambda^2 + 1 = 0$, the solutions of which are the imaginary numbers

$$\lambda = i, \lambda = -i.$$

Thus we are forced to consider complex eigenvalues, even for real matrices.

Eigenvalues and Eigenvectors of the power of A

Once the eigenvalues and eigenvectors of a matrix A are found, it is simple matter to find the eigenvalues and eigenvectors of any positive integer power of A; for example, if λ is an eigenvalue of A and x is a corresponding eigenvector, then

$$A^2x = A(Ax) = A(\lambda x) = \lambda(\lambda x) = \lambda^2 x$$

which shows that λ^2 is an eigenvalue of A^2 and that x is a corresponding eigenvector.

Theorem: ***If k is a positive integer, λ is an eigenvalue of a matrix A, and x is corresponding eigenvector, then λ^k is an eigenvalue of A^k and x is a corresponding eigenvector.***

Example:

We showed that the eigenvalues of

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

are $\lambda = 2$ and $\lambda = 1$.

$\lambda = 2^5 = 32$ and $\lambda = 1^5 = 1$ are eigenvalues of A^5.

$$A^5 = \begin{bmatrix} -30 & 0 & -62 \\ 31 & 32 & 31 \\ 31 & 0 & 63 \end{bmatrix}$$

```
>> A=[0 0 -2; 1 2 1; 1 0 3]

A =

     0     0    -2
     1     2     1
     1     0     3

>> A5=A^5

A5 =

   -30     0   -62
    31    32    31
    31     0    63
```

```
>> [v,d]=eig(A5)
v =
         0   -0.8165    0.7071
    1.0000    0.4082         0
         0    0.4082   -0.7071
d =
    32     0     0
     0     1     0
     0     0    32
>>
```

We also showed that

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{are eigenvectors of A}$$

corresponding to eigenvalue $\lambda = 2$. They are also eigenvectors of A^5. Similarly, the eigenvector

$$\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

of A corresponding to the eigenvalue $\lambda = 1$ is also an eigenvector of A^5.

Theorem: ***A square matrix A is invertible if and only if*** $\lambda = 0$ ***is <u>not an eigenvalue</u> of A.***

Assume that A is an $n \times n$ matrix and observe first that $\lambda = 0$ is a solution of the characteristic equation

$$\lambda^n + c_1\lambda^{n-1} + \ldots + c_n = 0$$

if and only if the constant term c_n is zero. Thus it suffices to prove that A is invertible if and only if $c_n \neq 0$. But

$$\det(\lambda I - A) = \lambda^n + c_1\lambda^{n-1} + \ldots + c_n$$

Or , on setting $\lambda = 0$,

$$\det(-A) = c_n \quad \text{or} \quad (-1)^n \det(A) = c_n$$

It follows from the last equation that det(A)=0 if and only if $c_n = 0$, and this in turn implies that A is invertible if and only if $c_n \neq 0$.

Example:

We showed that the eigenvalues of

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

are $\lambda_1 = 1$ and $\lambda_{2,3} = 2$

The matrix A is invertible since it has eigenvalues $\lambda = 2$ and $\lambda = 1$ neither of which is zero.

```
>> A=[0 0 -2; 1 2 1; 1 0 3]

A =

     0     0    -2

     1     2     1

     1     0     3

>> inv(A)

ans =

    1.5000         0    1.0000

   -0.5000    0.5000   -0.5000

   -0.5000         0         0
```

Eigenvalues Relationships

If A is a square n-by-n matrix with [real] or [complex] entries and if $\lambda_i, i = 1 \ldots n$ are the (complex and distinct) [eigenvalues] of A, then

$$trace(A) = \sum_{i=1}^{n} \lambda_i .$$

<u>The product of the eigenvalues of a square matrix is equal to the determinant of that matrix.</u>

$$\det(A) = \prod_{i=1}^{n} \lambda_i$$

```
>> A=[0 0 -2;1 2 1;1 0 3]
A =
     0     0    -2
     1     2     1
     1     0     3
>> det(A)
ans =    4
>> eig(A)
ans =
     2
     1
     2
>> trace(A)
ans =     5
```

Example:

$$A=\begin{bmatrix}1 & 3\\ 4 & 2\end{bmatrix}$$

The characteristic equation of A is

$$\det(\lambda I - A)=\begin{vmatrix}\lambda-1 & -3\\ -4 & \lambda-2\end{vmatrix}=0$$

or

$$\lambda^2-3\lambda-10=0$$

The factored form of this equation is

$(\lambda+2)(\lambda-5)=0$, so the eigenvalues of A are $\lambda=-2, \lambda=5$.

```
>> A=[1 3;4  2]
A =
     1     3
     4     2
>> eig(A)
ans =
    -2
     5
>> det(A)
ans =   -10
>> trace(A)
ans =     3
```

Example:

$$A = \begin{bmatrix} 3 & 6 \\ 1 & 4 \end{bmatrix}.$$

Let's find both of the eigenvalues of the matrix

$$\begin{aligned} A - \lambda I &= \begin{bmatrix} 3-\lambda & 6 \\ 1 & 4-\lambda \end{bmatrix} \\ \det(A - \lambda I) &= (3-\lambda)(4-\lambda) - 6 \\ &= \lambda^2 - 7\lambda + 6 \\ &= (\lambda - 6)(\lambda - 1) \end{aligned}$$

Therefore, $\lambda = 6$ or $\lambda = 1$. We now know our eigenvalues.

$$\begin{aligned} \text{Product of eigenvalues} &= \det(A) \\ 6 * 1 &= 12 - 6 \\ 6 &= 6 \end{aligned}$$

Example

For $$A = \begin{bmatrix} 2 & 6 \\ 2 & -2 \end{bmatrix},$$

$$\begin{aligned} \text{Product of eigenvalues} &= \det(A) \\ 4 * (-4) &= -4 - 12 \\ -16 &= -16 \end{aligned}$$

$$\begin{vmatrix} 2-\lambda & 6 \\ 2 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda) - 12 = 0$$

$$(\lambda^2 - 16) = 0 \qquad \lambda_{1,2} = \pm 4 \qquad \lambda_1 \lambda_2 = -16$$

$$Trace(A) = 0 = \sum_{i=1}^{2} \lambda_i = 4 - 4 = 0$$

Example

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 9 \\ 6 & 8 & 3 & 4 \\ 9 & 8 & 2 & 4 \end{bmatrix}$$

```
A=[1 2 3 4; 5 6 7 9; 6 8 3 4; 9 8 2 4]
A =
     1     2     3     4
     5     6     7     9
     6     8     3     4
     9     8     2     4

>> [V,D]=eig(A)

V =
    0.2499    0.3566    0.4375    0.0725
    0.6496    0.5444   -0.3859   -0.1563
    0.5016   -0.4121    0.6595   -0.7518
    0.5137   -0.6377   -0.4741    0.6365

D =
   20.4460         0         0         0
         0   -6.5683         0         0
         0         0   -0.5759         0
         0         0         0    0.6982
>> trace(D)
ans =   14.0000
>> TRACE(A)
ans =    14
>> det(A)
ans =    54
```

Square root of a matrix

A matrix B is said to be a square root of A if the matrix product $B \cdot B$ is equal to A [1].

[edit] Computation by diagonalization

The square root of a diagonal matrix D is formed by taking the square root of all the entries on the diagonal. This suggests the following methods for general matrices:

An $n \times n$ matrix A is diagonalizable if there is a matrix V such that $D = V^{-1}AV$ is a diagonal matrix. This happens if and only if A has n eigenvectors which constitute a basis for $\mathbf{C}^n$; in this case, V can be chosen to be the matrix with the n eigenvectors as columns.

Now, $A = VDV^{-1}$, and hence the square root of A is

$$A^{1/2} = VD^{1/2}V^{-1}.$$

This approach works only for diagonalizable matrices. For non-diagonalizable matrices one can calculate the Jordan normal form followed by a series expansion, similar to the approach described in logarithm of a matrix.