Confidence Interval on the Variance and Standard Deviation of a Normal Distribution

Sometimes confidence intervals on the population variance or standard deviation are needed. The following result provides the basis of constructing these confidence intervals.

Chi-Square (χ^2) Distribution

Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal distribution with mean μ and variance σ², and let S² be the sample variance. Then the random variable

$$\chi^2 = \frac{(n-1)S^2}{\sigma^2}$$

has chi-square distribution with n-1 degrees of freedom.

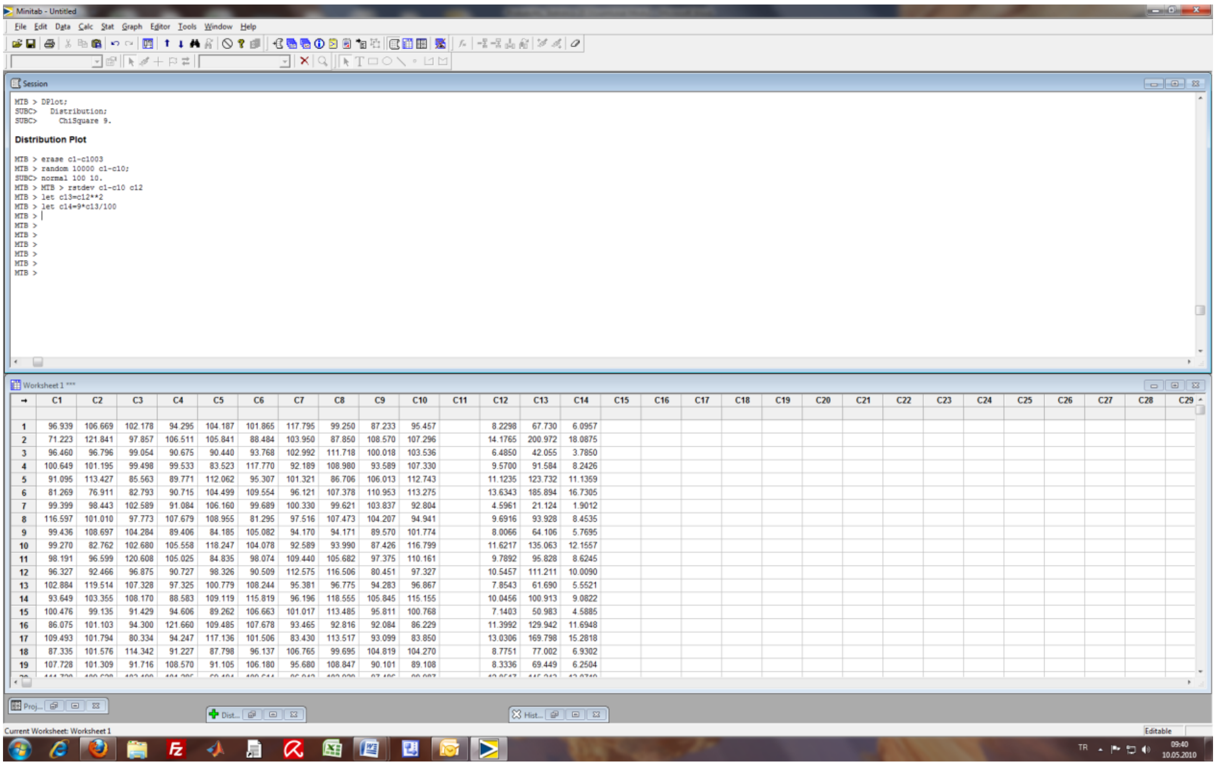
The probability density function of a χ^2 random variable is

$$f(x) = \frac{1}{2^{k/2}\Gamma(k/2)} x^{(k/2)-1} e^{-x/2} \qquad x > 0$$

where k is the number of degrees of freedom.

The mean and variance of the χ^2 distribution are k and $2k$ respectively.

```
MTB > random 10000 c1-c10;
SUBC> normal 100 10.
MTB > rstdev c1-c10 c12
MTB > let c13=c12**2
MTB > let c14=9*c13/100
```




```
MTB > mean c14
```

Mean of C14

```
Mean of C14 = 8.93715
MTB > stdev c14
```

Standard Deviation of C14

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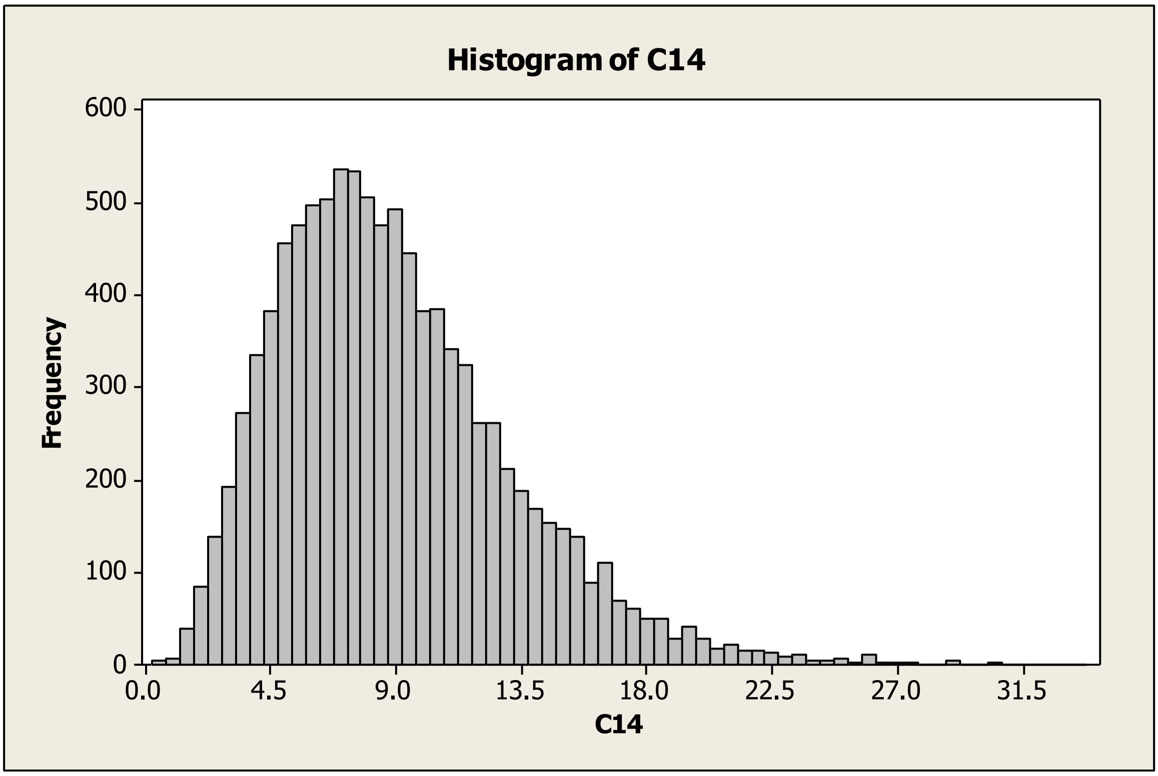
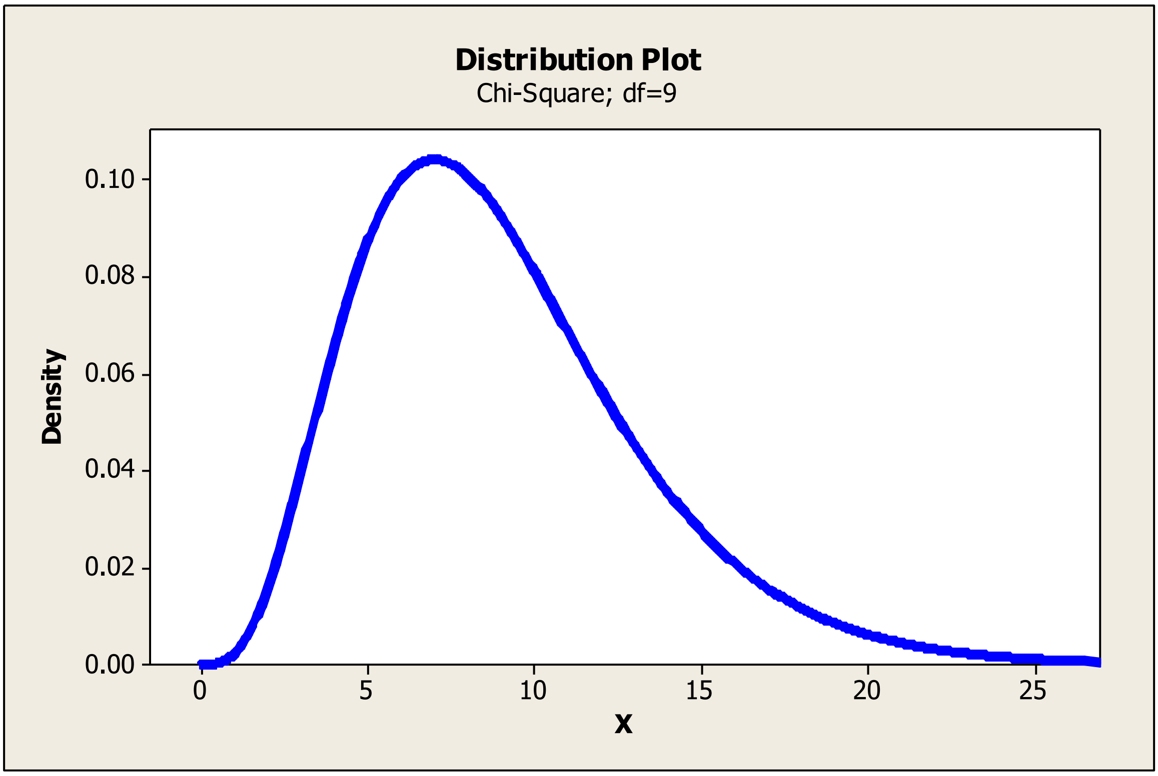
Standard Deviation of C14 = 4.19430
variance 17.58
```

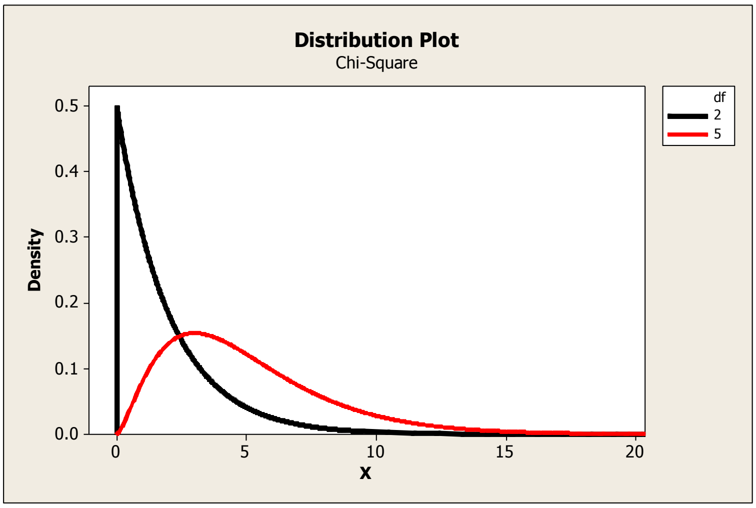
The mean and variance of the χ^2 distribution are k and $2k$ respectively.

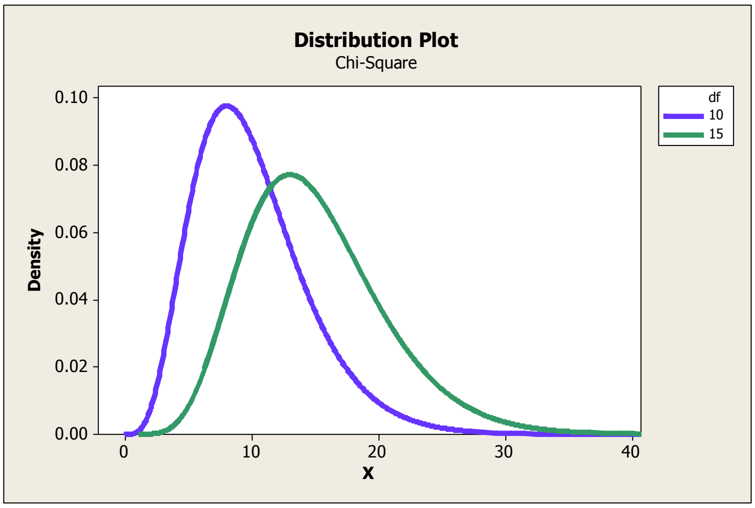


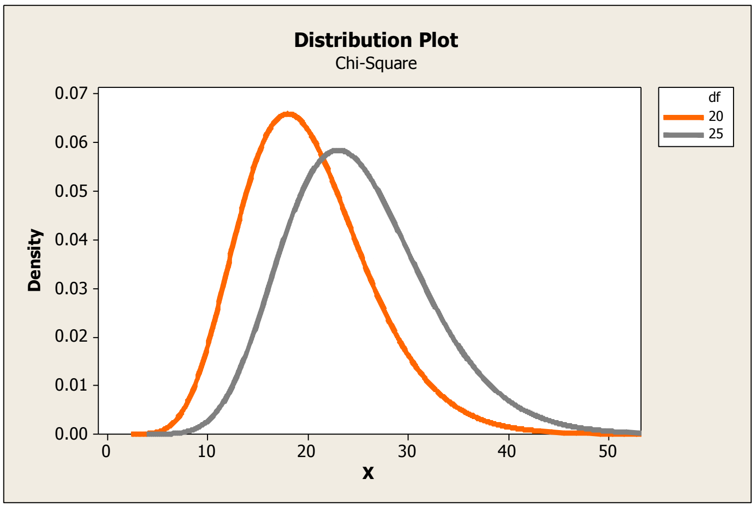
Mean =k=9 Variance=2k=18



Probability Density Functions of Several χ^2 Distributions



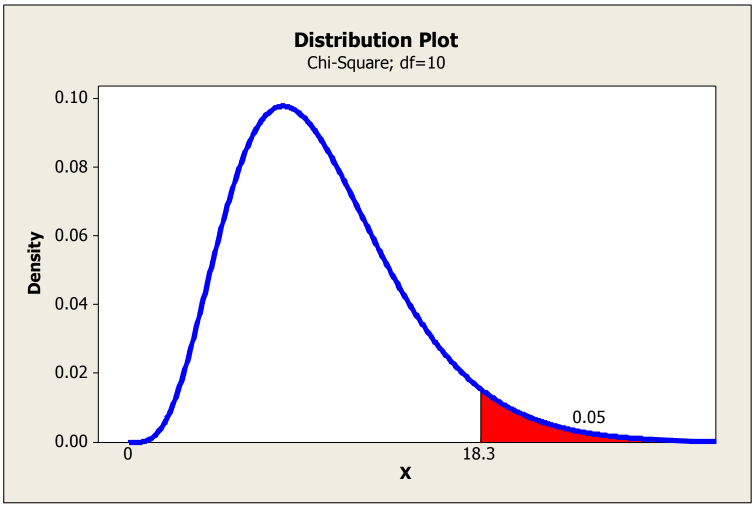




Percentage points of the chi-square distribution

Define $\chi^2_{\alpha,k}$ as the percentage point or value of the chi-square random variable with k degrees of freedom such that the probability that χ^2 exceeds this value is α.

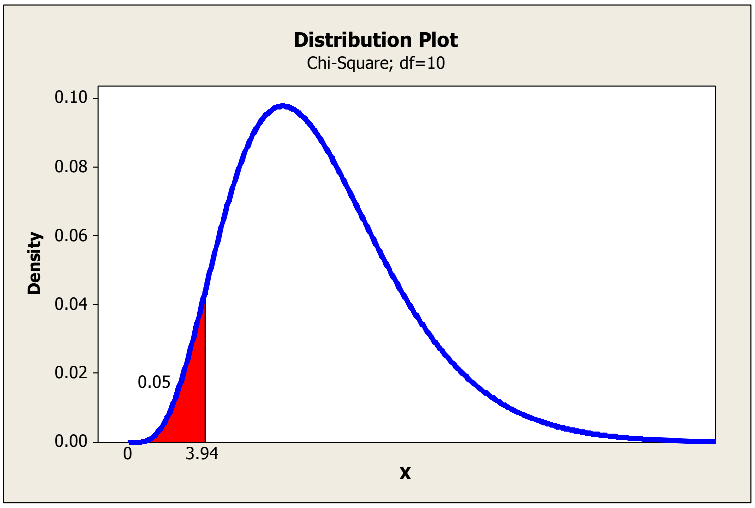
$$P(\chi^2 > \chi^2_{\alpha,k}) = \int_{\chi^2_{\alpha,k}}^{\infty} f(u)du = \alpha$$

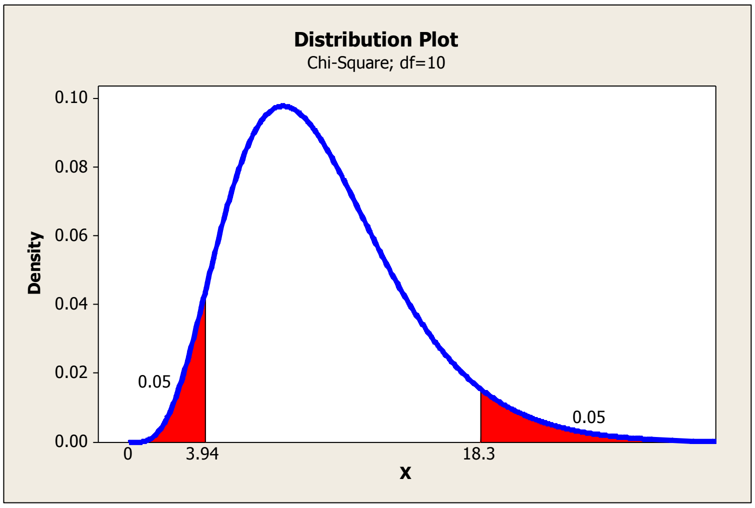


$$P(\chi^2 > \chi^2_{0.05,10}) = P(\chi^2 > 18.31) = 0.05$$

This value is often called an upper 5% point of chi-square with 10 degrees of freedom.

Conversely, a lower 5% point of chi-square with 10 degrees of freedom would be $\chi^2_{0.95,10} = 3.94$.





The upper percentage point $\chi^2_{0.05,10} = 18.31$

The lower percentage point $\chi^2_{0.95,10} = 3.94$

Confidence Interval on the Variance

The construction of the 100(1-α)% CI for σ^2 is straightforward. Because

$$\chi^2 = \frac{(n-1)S^2}{\sigma^2}$$

is a chi-square with (n-1) degrees of freedom, we may write

$$P(\chi^2_{1-\alpha/2,n-1} \le \chi^2 \le \chi^2_{\alpha/2,n-1}) = 1-\alpha$$

So that

$$P(\chi^2_{1-\alpha/2,n-1} \le \frac{(n-1)S^2}{\sigma^2} \le \chi^2_{\alpha/2,n-1}) = 1-\alpha$$

This equation can be arranged as

$$P\left(\frac{(n-1)S^2}{\chi^2_{\alpha/2,n-1}} \le \sigma^2 \le \frac{(n-1)S^2}{\chi^2_{1-\alpha/2,n-1}}\right) = 1-\alpha$$

Confidence Interval on the Variance

If S^2 is the sample variance from a random sample of n observations from a normal distribution with unknown variance σ^2, then a 100(1-α)% confidence interval on σ^2 is

$$\frac{(n-1)S^2}{\chi^2_{\alpha/2,n-1}} \le \sigma^2 \le \frac{(n-1)S^2}{\chi^2_{1-\alpha/2,n-1}}$$

where $\chi^2_{\alpha/2,n-1}$ and $\chi^2_{1-\alpha/2,n-1}$ ***are the upper and lower 100α/2 percentage points of the chi-square distribution with (n-1) degrees of freedom, respectively.***

A confidence interval for σ has lower and upper limits that are the square roots of the corresponding limits. That is

$$\sqrt{\frac{(n-1)S^2}{\chi^2_{\alpha/2,n-1}}} \le \sigma \le \sqrt{\frac{(n-1)S^2}{\chi^2_{1-\alpha/2,n-1}}}$$

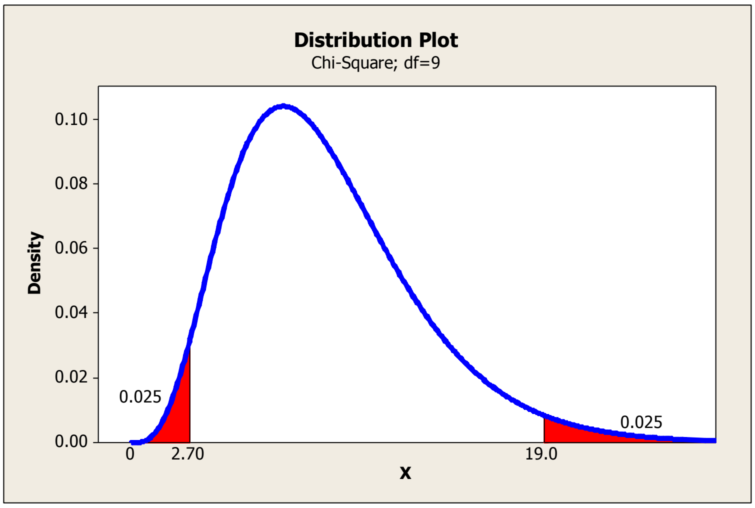
Example: Ten measurements of impact energy (J) on specimens of A238 steel cut at 60°C are as follows:

64.1, 64.7, 64.5 64.6 64.5 64.3 64.6 64.8 64.2 64.3

Assume that impact energy is normally distributed.

Find 95% CI for σ^2, the variance impact energy.

Variable	N	Mean	StDev	SE Mean
Impact Energy	10	64.460	0.227	0.0718

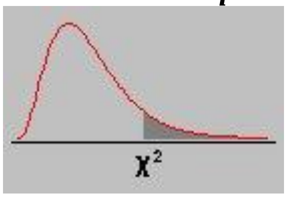


$$\frac{(n-1)S^2}{\chi^2_{\alpha/2,n-1}} \le \sigma^2 \le \frac{(n-1)S^2}{\chi^2_{1-\alpha/2,n-1}}$$

$$\frac{(9)0.0515290}{19.02277} \le \sigma^2 \le \frac{(9)0.0515290}{2.70039}$$

$$0.0243793 \le \sigma^2 \le 0.171739$$
$$0.156139 \le \sigma \le 0.414413$$

Right tail areas for the *Chi-square* Distribution



df\area	.995	.990	.975	.950	.900	.750	.500	.250	.100	.050	.025	.010	.005
1	0.00004	0.00016	0.00098	0.00393	0.01579	0.10153	0.45494	1.32330	2.70554	3.84146	5.02389	6.63490	7.87944
2	0.01003	0.02010	0.05064	0.10259	0.21072	0.57536	1.38629	2.77259	4.60517	5.99146	7.37776	9.21034	10.59663
3	0.07172	0.11483	0.21580	0.35185	0.58437	1.21253	2.36597	4.10834	6.25139	7.81473	9.34840	11.34487	12.83816
4	0.20699	0.29711	0.48442	0.71072	1.06362	1.92256	3.35669	5.38527	7.77944	9.48773	11.14329	13.27670	14.86026
5	0.41174	0.55430	0.83121	1.14548	1.61031	2.67460	4.35146	6.62568	9.23636	11.07050	12.83250	15.08627	16.74960
6	0.67573	0.87209	1.23734	1.63538	2.20413	3.45460	5.34812	7.84080	10.64464	12.59159	14.44938	16.81189	18.54758
7	0.98926	1.23904	1.68987	2.16735	2.83311	4.25485	6.34581	9.03715	12.01704	14.06714	16.01276	18.47531	20.27774
8	1.34441	1.64650	2.17973	2.73264	3.48954	5.07064	7.34412	10.21885	13.36157	15.50731	17.53455	20.09024	21.95495
9	1.73493	2.08790	2.70039	3.32511	4.16816	5.89883	8.34283	11.38875	14.68366	16.91898	19.02277	21.66599	23.58935
10	2.15586	2.55821	3.24697	3.94030	4.86518	6.73720	9.34182	12.54886	15.98718	18.30704	20.48318	23.20925	25.18818
11	2.60322	3.05348	3.81575	4.57481	5.57778	7.58414	10.34100	13.70069	17.27501	19.67514	21.92005	24.72497	26.75685
12	3.07382	3.57057	4.40379	5.22603	6.30380	8.43842	11.34032	14.84540	18.54935	21.02607	23.33666	26.21697	28.29952
13	3.56503	4.10692	5.00875	5.89186	7.04150	9.29907	12.33976	15.98391	19.81193	22.36203	24.73560	27.68825	29.81947
14	4.07467	4.66043	5.62873	6.57063	7.78953	10.16531	13.33927	17.11693	21.06414	23.68479	26.11895	29.14124	31.31935
15	4.60092	5.22935	6.26214	7.26094	8.54676	11.03654	14.33886	18.24509	22.30713	24.99579	27.48839	30.57791	32.80132
16	5.14221	5.81221	6.90766	7.96165	9.31224	11.91222	15.33850	19.36886	23.54183	26.29623	28.84535	31.99993	34.26719
17	5.69722	6.40776	7.56419	8.67176	10.08519	12.79193	16.33818	20.48868	24.76904	27.58711	30.19101	33.40866	35.71847
18	6.26480	7.01491	8.23075	9.39046	10.86494	13.67529	17.33790	21.60489	25.98942	28.86930	31.52638	34.80531	37.15645
19	6.84397	7.63273	8.90652	10.11701	11.65091	14.56200	18.33765	22.71781	27.20357	30.14353	32.85233	36.19087	38.58226
20	7.43384	8.26040	9.59078	10.85081	12.44261	15.45177	19.33743	23.82769	28.41198	31.41043	34.16961	37.56623	39.99685
21	8.03365	8.89720	10.28290	11.59131	13.23960	16.34438	20.33723	24.93478	29.61509	32.67057	35.47888	38.93217	41.40106
22	8.64272	9.54249	10.98232	12.33801	14.04149	17.23962	21.33704	26.03927	30.81328	33.92444	36.78071	40.28936	42.79565
23	9.26042	10.19572	11.68855	13.09051	14.84796	18.13730	22.33688	27.14134	32.00690	35.17246	38.07563	41.63840	44.18128
24	9.88623	10.85636	12.40115	13.84843	15.65868	19.03725	23.33673	28.24115	33.19624	36.41503	39.36408	42.97982	45.55851
25	10.51965	11.52398	13.11972	14.61141	16.47341	19.93934	24.33659	29.33885	34.38159	37.65248	40.64647	44.31410	46.92789
26	11.16024	12.19815	13.84390	15.37916	17.29188	20.84343	25.33646	30.43457	35.56317	38.88514	41.92317	45.64168	48.28988
27	11.80759	12.87850	14.57338	16.15140	18.11390	21.74940	26.33634	31.52841	36.74122	40.11327	43.19451	46.96294	49.64492
28	12.46134	13.56471	15.30786	16.92788	18.93924	22.65716	27.33623	32.62049	37.91592	41.33714	44.46079	48.27824	50.99338
29	13.12115	14.25645	16.04707	17.70837	19.76774	23.56659	28.33613	33.71091	39.08747	42.55697	45.72229	49.58788	52.33562
30	13.78672	14.95346	16.79077	18.49266	20.59923	24.47761	29.33603	34.79974	40.25602	43.77297	46.97924	50.89218	53.67196

The areas given across the top are the areas to the right of the critical value. To look up an area on the left, subtract it from one, and then look it up (ie: 0.05 on the left is 0.95 on the right).