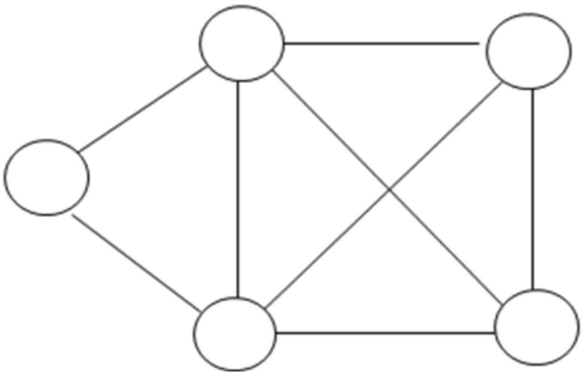
NETWORK OPTIMIZATION MODELS

A graph (or network) consist of

- **a set of points**
- **a set of lines connecting certain pairs of the points.**



The points are called nodes (or vertices).

The lines are called arcs (or edges or links).

What is a node?

Often called a ***vertex***, or ***point***. It is normally represented by circle. In a transportation Network, these might be locations or cities on a map.

What is arc?

Often called an ***edge*** or ***arrow***. It may be either directed or undirected. The head is the destination the tail is the origin. The head and tail are Nodes that are at either end.

Graphs in our daily lives

- **Transportation,**
- **Telephone,**
- **Computer,**
- **Electrical (power),**
- **Pipelines,**
- **Molecular structures in biochemistry.**

The family of network optimization problems includes the following prototype models

- **Transportation Problem**
- **Transshipment Problem**
- **Assignment Problem**
- **Traveling Salesman Problem**
- **Maximal Flow Problem**
- **Minimal Spanning Tree Problem**
- **Shortest Path Problem**
- **Critical Path Problem**
- **Min Cost Flow Problems**

A network problem is one that can be represented by

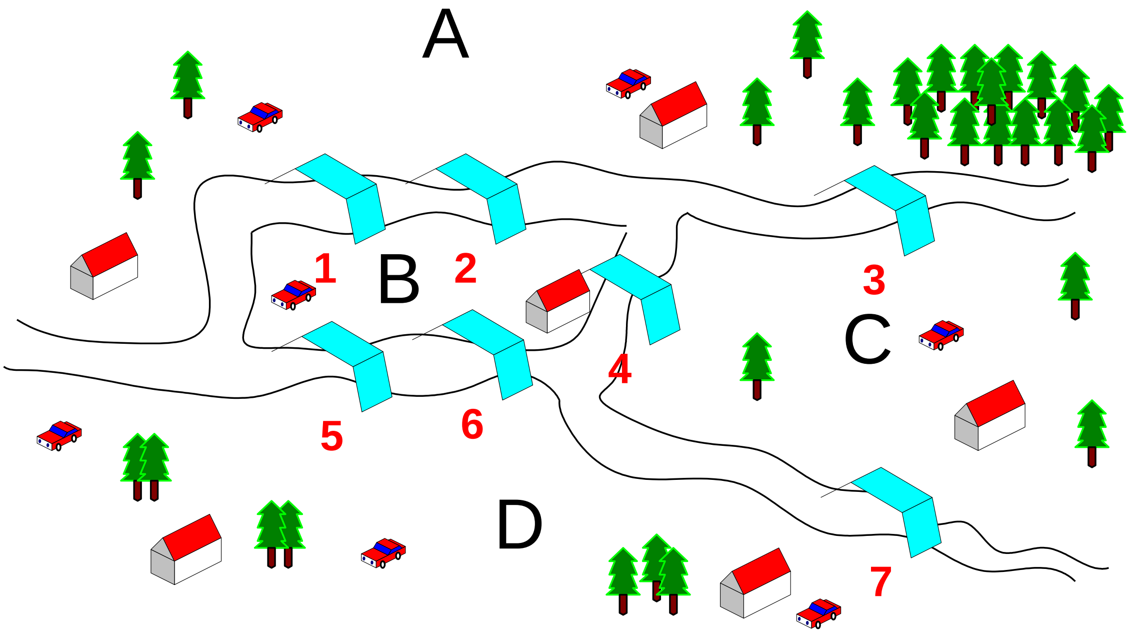
1. **A set of nodes**
2. **A set of arcs**
3. **Functions defined on the nodes and/or arcs**

Network Models are important for three basic reasons:

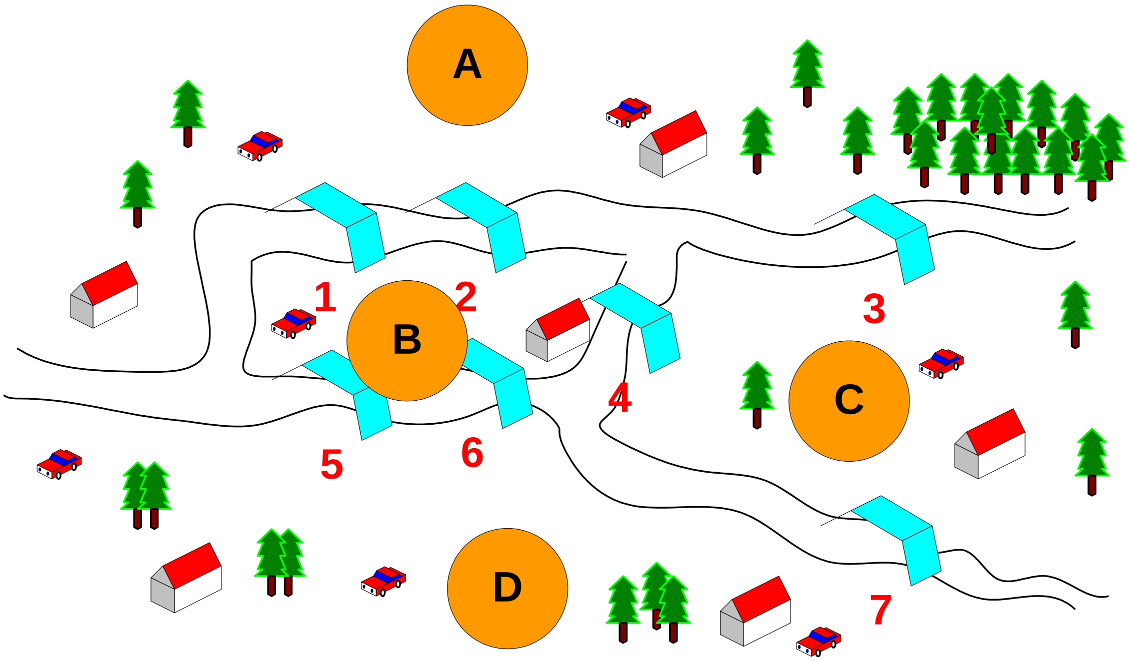
1. **Many business problems naturally lend themselves to a network formulation.**
2. **Network problems are integer mathematical programming problems, but, because of their special structure, under very unrestrictive conditions the integer constraints can be ignored and the optimal solution is guaranteed to be integer valued anyway.**
3. **Network problems can be solved by more compact and efficient algorithms than those used for general linear and integer programming problems.**

- **“Graph Theory” began in 1736**
- **Leonhard Euler**
  - **Visited Koenigsberg**
  - **People wondered whether it is possible to take a walk, end up where you started from, and cross each bridge in Koenigsberg exactly once.**
  - **Generally it was believed to be impossible.**

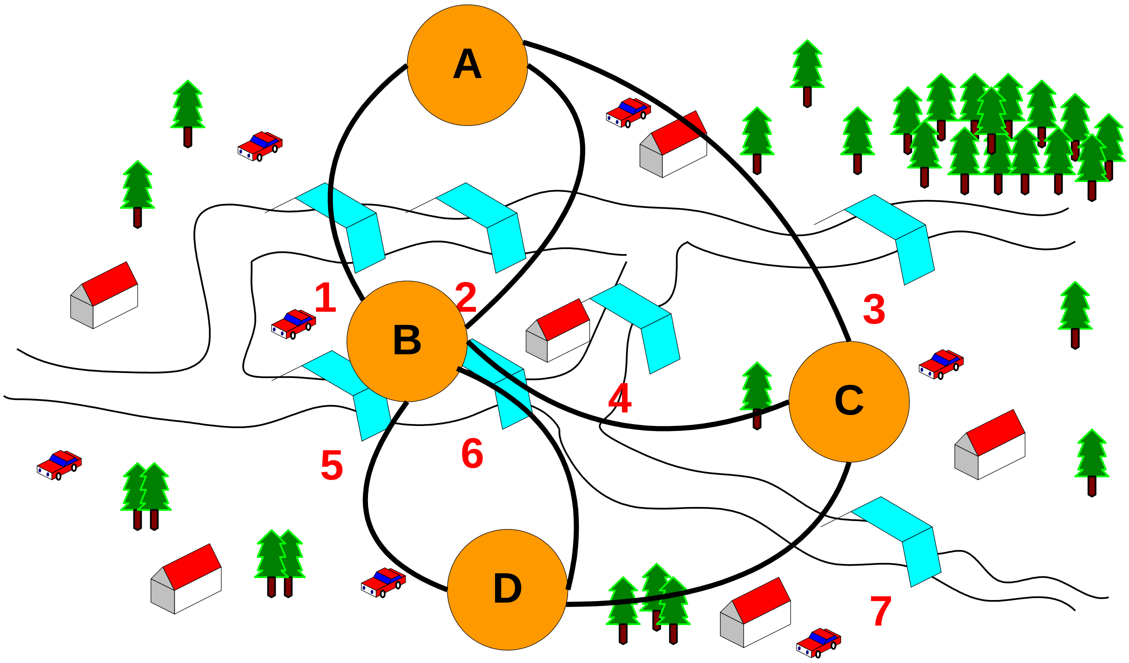
The Bridges of Koenigsberg : Euler 1736



Is it possible to start in A, cross over each bridge exactly once, and end up back in A?

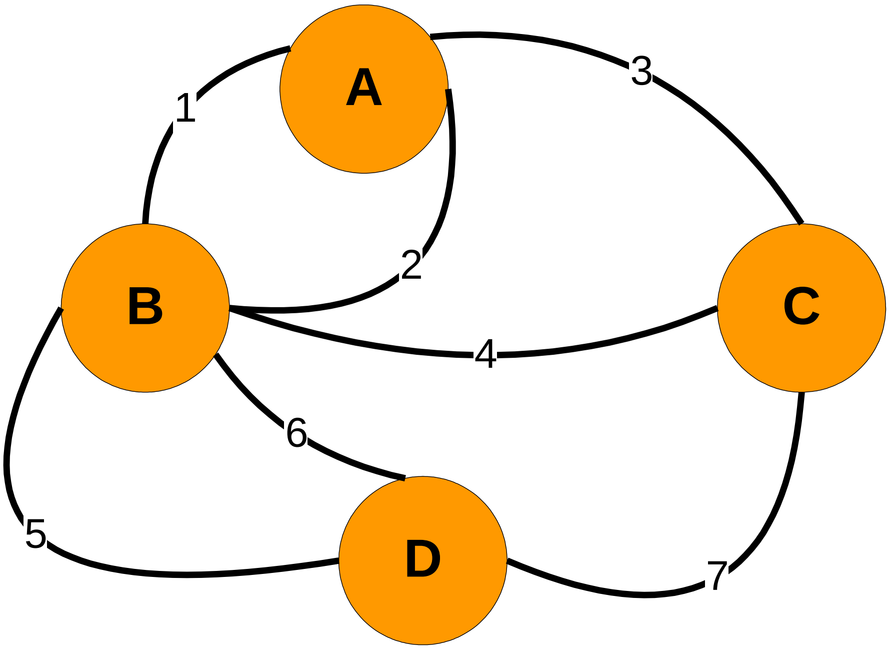


Translation into a graph problem: Land masses are “nodes”.



Translation into a graph problem: Bridges are “arcs”.

The Bridges of Koenigsberg



Is there a "walk" starting at A and ending at A and passing through each arc exactly once?

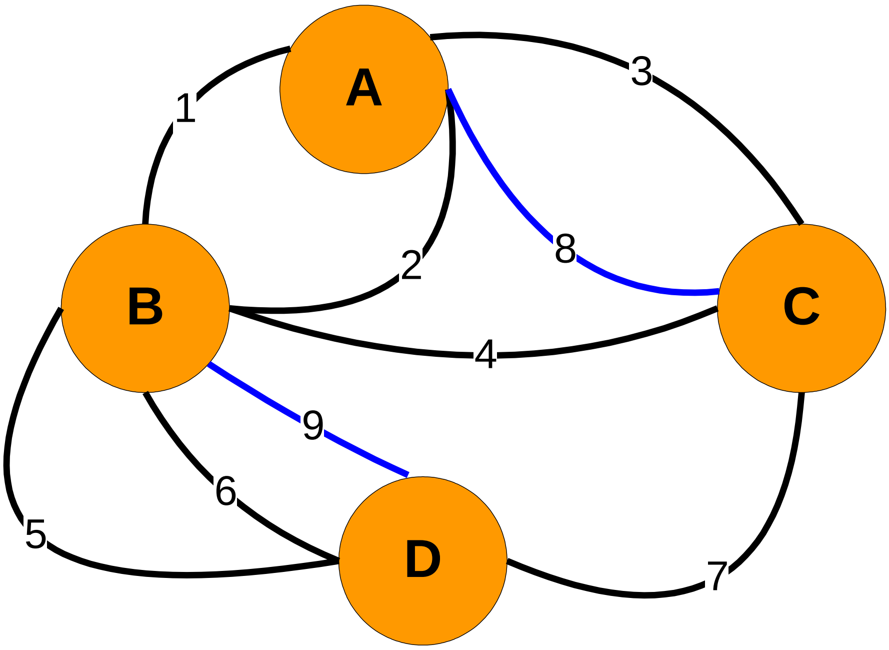
Such a walk is called an *Eulerian* cycle.

Theorem: An undirected graph has an Eulerian cycle if and only if

1. **Every node degree is even and**
2. **The graph is connected (that is, there is a path from each node to each other node).**

The degree of a node in an undirected graph is the number of incident arcs.

Adding two bridges creates such a walk



Here is the walk (*Eulerian* cycle):

A , 1 , B , 5 , D , 6 , B , 4 , C , 8 , A, 3 ,C, 7, D , 9 , B , 2 , A

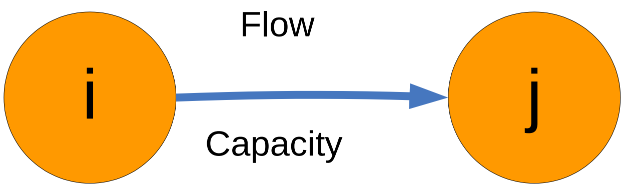
Hamiltonian Cycle:

A *Hamiltonian* cycle is a cycle that passes through each node of the graph exactly once.

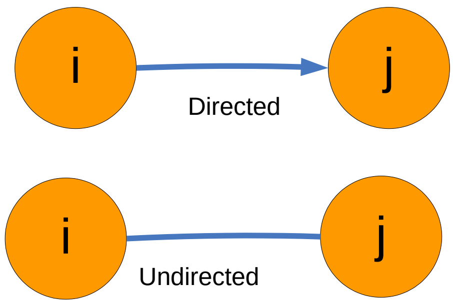
This is often called a traveling salesman tour.

What is a flow?

When an arc connects two nodes *a flow* of some kind (current, traffic, micro-waves, time, etc.) can occur directly between them.



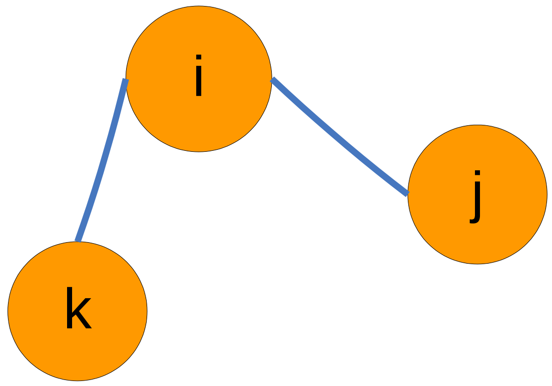
Direct/Undirected Arcs



- **Flow may only be from i to j**
- **Flow may be in either direction**

Adjacent Nodes

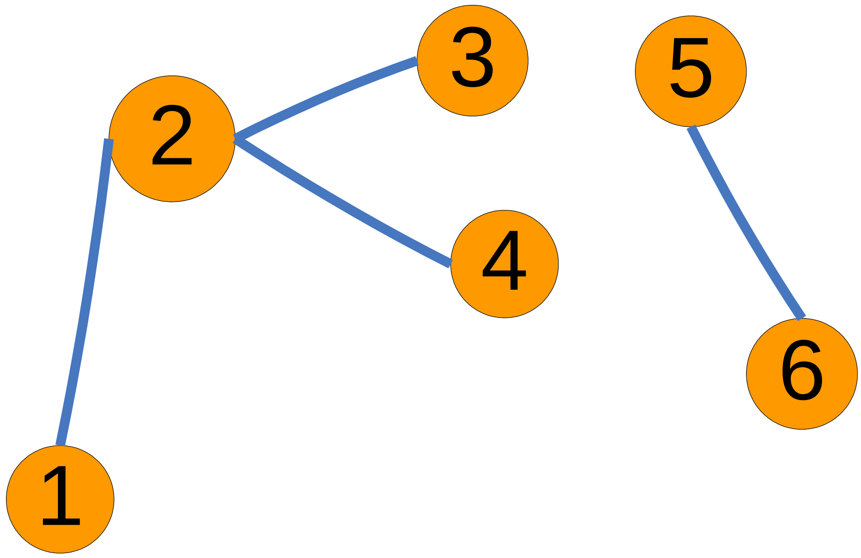
A node (i) is adjacent to another node (j), if an arc joins node i to node j.



Node i is adjacent node j and node k. However node j is not adjacent to node k.

Paths/Connected Nodes

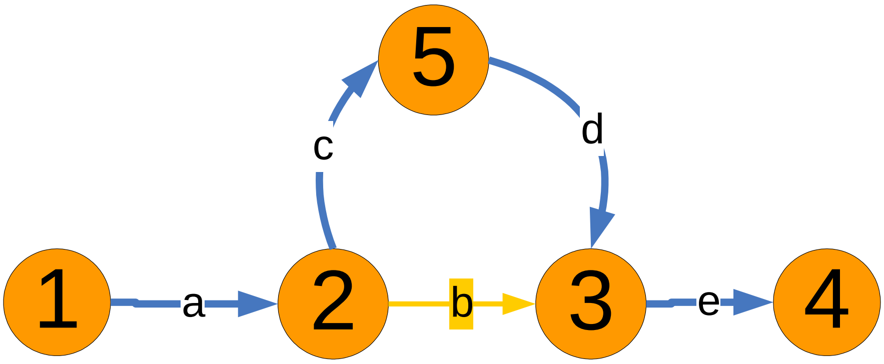
The collection of arcs formed by a series of adjacent nodes. *When a path exists between two nodes these nodes said to be connected*.



There is a path between node 1 to node 3; node 1 is connected to node 3.
There is no path between node1 to node 5; node 1 is not connected to node 5.

Directed Path

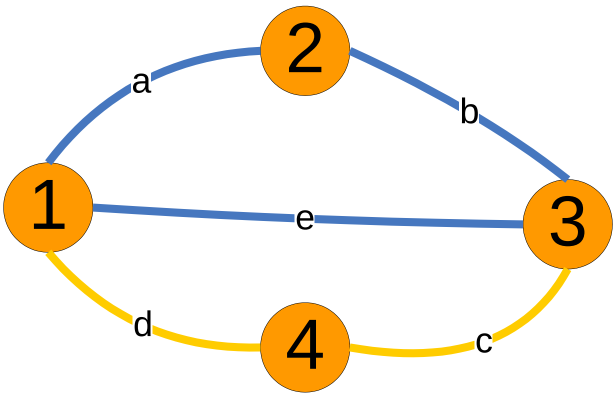
- Directions are important
- No node is repeated



Directed Path 1,2,5,3,4 or
(1,a,2,c,5,d,3,e,4) , (a,c,d,e)

Cycle (or circuit or loop)

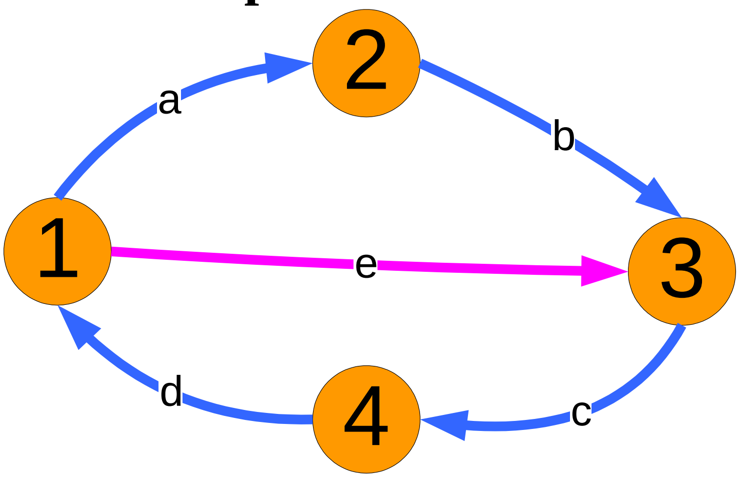
- **Directions are ignored.**
- **Cycle is a path such that the start node and end node are the same.**



Cycle 1,2,3,1. or (1,a,2,b,3,e)

Directed Cycle

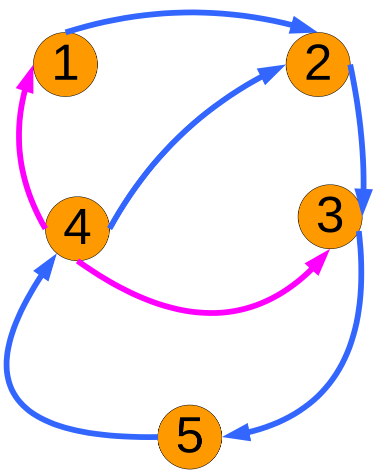
- **Directions are important.**



Directed Cycle 1,2,3,4,1. or
(1,a,2,b,3,c,4,d,1)

Walks

- **Walks are paths that can repeat nodes and arcs.**
- **A walk is <u>closed</u> if its <u>first and last nodes</u> are the same.**
- **A closed walk is a <u>cycle</u> except that it can repeat nodes and arcs.**



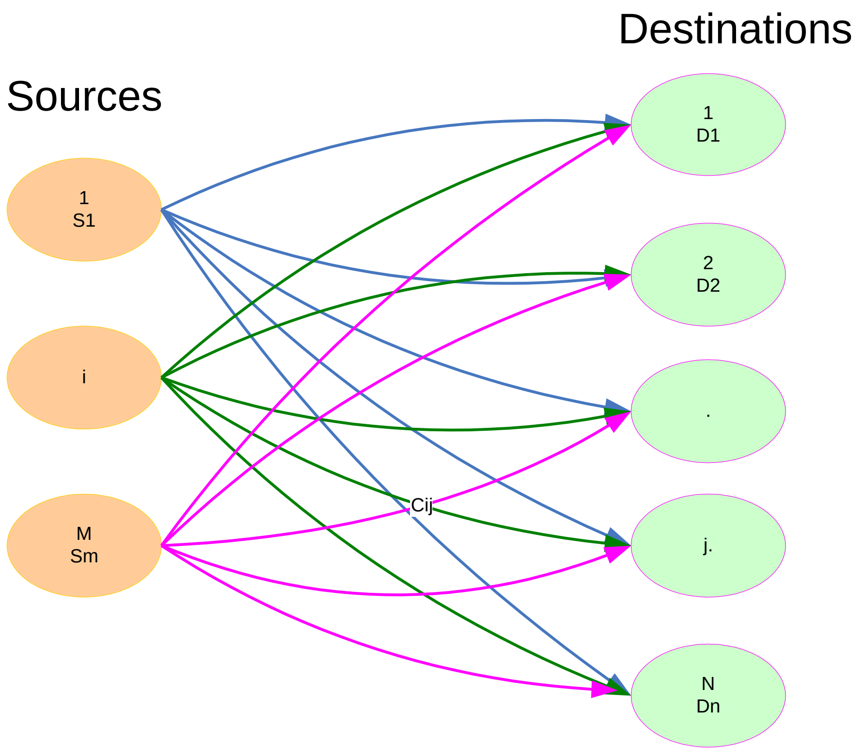
Directed walk 1-2-3-5-4-2-3-5

THE TRANSPORTATION PROBLEM

The transportation problem arises frequently in planning for the distribution of goods and services from several supply locations to several demand locations. Usually the quantity of goods available at each supply location *(origin)* is limited, and the goods are needed at each of several *demand* locations (*destinations*). The objective in a transportation problem is to minimize (maximize) the cost of shipping goods from the origins to destinations.

Transportation Problem

- **There are m sources.**
- **The supply of a resource at source i is S_i.**
- **There are n destinations.**
- **The demand for the resource at destination j is D_j.**
- **The unit shipping cost between nodes i and j is c_{ij}.**



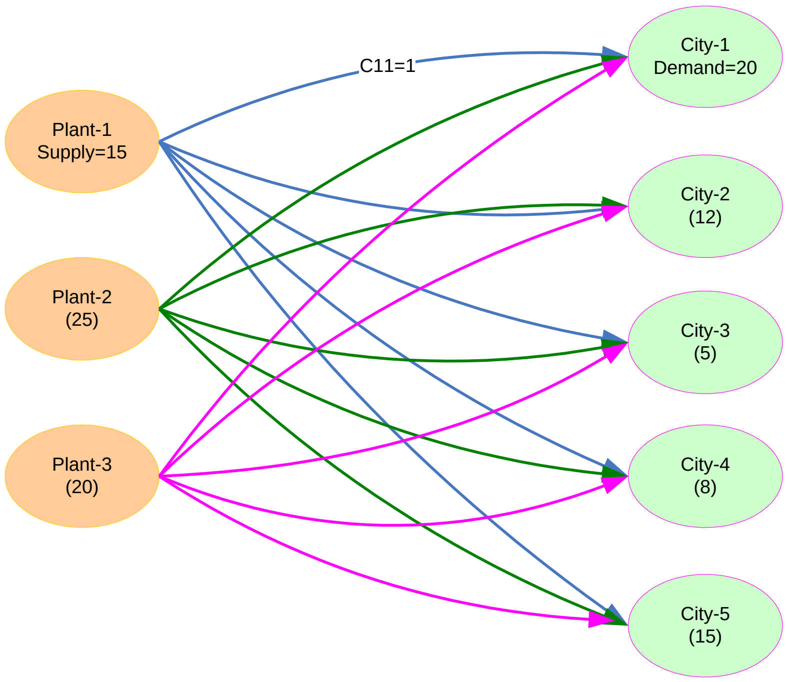
Example

Powerco has three electric power plants that supply the power needs of five cities. Each power plant can supply the following numbers of kilowatt-hours (kwh) of electricity:

- Plant-1 15 million
- Plant-2 25 million
- Plant-3 20 million

The peak power demands in these cities which occur at the same time (2 p.m) are as follows (in kwh):

- City-1 20 million
- City-2 12 million
- City-3 5 million
- City-4 8 million
- City-5 15 million



The costs of sending 1 million kwh of electricity from plant to city depend on the distance the electricity must travel (see Table).

From/To	City-1	City-2	City-3	City-4	City-5	Supply
Plant-1	1	0	3	4	2	15
Plant-2	5	1	2	3	3	25
Plant-3	4	8	1	4	3	20
Demand	20	12	5	8	15	

Formulate an LP to minimize the cost of meeting each city's peak power demand.

Solution

Since Powerco must determine how much power is sent from each plant to each city, we define (for I = 1,2,3, and j= 1,2,3,4,5)

x_{ij} = number of(million) kwh produced at plant I and sent to city j.

The total cost of supplying the peak power demand to cities 1-5 may be written as

$$1x_{11} + 0\,x_{12} + 3\,x_{13} + 4\,x_{14} + 2\,x_{15}$$
$$+5\,x_{21} + 1\,x_{22} + 2\,x_{23} + 3\,x_{24} + 3\,x_{25}$$
$$+4\,x_{31} + 8\,x_{32} + 1\,x_{33} + 4\,x_{34} + 3\,x_{35}$$

The LP formulation of Power co's problem contains the following three supply constraints:

$$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} \leq 15$$
$$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} \leq 25$$
$$x_{31} + x_{32} + x_{33} + x_{34} + x_{35} \leq 20$$

Powerco must satisfy the following five demand constraints:

$$x_{11} + x_{21} + x_{31} \geq 20$$
$$x_{12} + x_{22} + x_{32} \geq 12$$
$$x_{13} + x_{23} + x_{33} \geq 5$$
$$x_{14} + x_{24} + x_{34} \geq 8$$
$$x_{15} + x_{25} + x_{35} \geq 15$$

Since all the x_{ij} must be nonnegative , we add sign restrictions

$x_{ij} \geq 0$ (i=1,2,3; j=1,2,3,4,5)

Combining the objective function, supply constraints, demand constraints and sign restrictions yield the following LP formulation of Power co's problem.

$$\text{Min } Z = 1x_{11} + 0\,x_{12} + 3\,x_{13} + 4\,x_{14} + 2\,x_{15} + 5\,x_{21} + 1\,x_{22} + 2\,x_{23} + 3x_{24} + 3x_{25} + 4x_{31} + 8x_{32} + 1x_{33} + 4x_{34} + 3x_{35}$$

s.t.

$$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} \leq 15$$
$$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} \leq 25$$
$$x_{31} + x_{32} + x_{33} + x_{34} + x_{35} \leq 20$$
$$x_{11} + x_{21} + x_{31} \geq 20$$
$$x_{12} + x_{22} + x_{32} \geq 12$$
$$x_{13} + x_{23} + x_{33} \geq 5$$
$$x_{14} + x_{24} + x_{34} \geq 8$$
$$x_{15} + x_{25} + x_{35} \geq 15$$
$$x_{ij} \geq 0 \text{ (i=1,2,3; j=1,2,3,4,5), all integer}$$

GENERAL DESCRIPTION OF A TRANSPORTATION PROBLEM

x_{ij}* = *number of units shipped from supply point i to demand point j then the general formulation of a transportation problem is

$$\min \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

$$s.t. \sum_{j=1}^{n} x_{ij} \le s_i (i = 1,2,...m)$$

$$\sum_{i=1}^{m} x_{ij} \ge d_j (j = 1,2,...,n)$$

$$x_{ij} \ge 0 (i = 1,...,m; j = 1,...,n)$$

If a problem has the same constraints and is a maximization problem, it is still a transportation problem.

If $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$ ***then total supply equals total demand, and the problem is said to be a balanced transportation problem.***

Balanced transportation problem general linear model maybe written as

$$\min \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$$

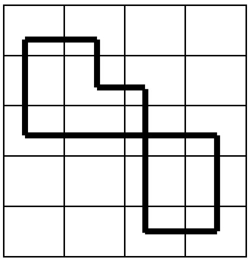
$$s.t. \sum_{j=1}^{n} x_{ij} = s_i \, (i = 1,2,\ldots m)$$

$$\sum_{i=1}^{m} x_{ij} = d_j \, (j = 1,2,\ldots,n)$$

$$x_{ij} \geq 0 (i = 1,\ldots,m; \; j = 1,\ldots,n)$$

Definition: An ordered sequence of at least four different cells is called a **loop** if

1. ***Any two consecutive cells lie in either the same row or same column***
2. ***No three consecutive cells lie in the same row or column***
3. ***The last cell in the sequence has a row or column in common with the first cell in the sequence***



Figure–a : Represent a loop

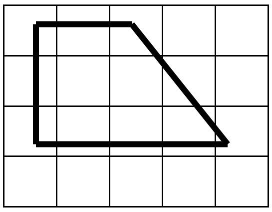


Figure-b: Does not represent a loop

Theorem-1 In a balanced transportation problem with m supply points and n demand points, the cell corresponding to a set of m+n-1 variables contain no loop if and only if the m+n-1 variables yield a basic solution.

Theorem 1 follows from the fact that a set of m + n –1 cells contains no loop if and only if the m + n –1 columns corresponding to these cells are linearly independent. Theorem tells us that figure-a cannot yield a basic solution for the related problem.

Basic Feasible Solution for a Balanced Transportation Problem

1. **Northwest Corner Method**
2. **Minimum Cost Method(Least Cost Method)**
    - **Row minimum**
    - **Column minimum**
    - **Matrix minimum**
3. **Vogel's Method**
4. **Russell's Method**

Northwest Corner Starting Procedure

1. ***Select the remaining variable in the upper left(northwest) corner and note the supply remaining in the row, s, and the demand remaining in the column, d.***
2. ***Allocate the minimum of s or d to this variable. If this minimum is s, eliminate all variables in its row from future consideration and reduce the demand in its column by s; if the minimum is d, eliminate al variables in the column from future consideration and reduce the supply in its row by d.***

REPEATE THESE STEPS UNTIL ALL SUPPLIES HAVE BEEN ALLOCATED.

NORTHWEST CORNER STARTING SOLUTION

From/To	City-1		City-2		City-3		City-4		City-5		Supply
Plant-1	15	**1**		**0**		**3**		**4**		**2**	**15**
Plant-2	5	**5**	12	**1**	5	**2**	3	**3**		**3**	**25**
Plant-3		**4**		**8**		**1**	5	**4**	15	**3**	**20**
Demand	**20**		**12**		**5**		**8**		**15**		

The value of the solution =15(1)+0(0)+…+15(3) = 136

Minimum Cost (Least Cost) Starting Procedure

1. ***For the remaining variable with the lowest unit cost, determine the remaining supply left in its row, s, and the remaining demand left in its column, d (break ties arbitrarily).***
2. ***Allocate the minimum of s or d to this variable. If this minimum is s, eliminate all variables in its row from future consideration and reduce the demand in its column by s; if the minimum is d, eliminate al variables in the column from future consideration and reduce the supply in its row by d.***

REPEATE THESE STEPS UNTIL ALL SUPPLIES HAVE BEEN ALLOCATED.

MINIMUM COST STARTING SOLUTION

From/To	City-1		City-2		City-3		City-4		City-5		Supply
Plant-1	3	**1**	12	0		**3**		**4**		**2**	**15**
Plant-2	2	**5**		1		**2**	8	**3**	15	**3**	**25**
Plant-3	15	**4**		8	5	**1**		**4**		**3**	**20**
Demand	**20**		**12**		**5**		**8**		**15**		

The value of the solution =3(1)+12(0)+…+0(3) = 147

Vogel's Approximation Method

1. ***For each remaining row and column, determine the difference between the lowest two remaining costs; these are <u>called the row and column penalties.</u>***
2. ***Select the row or column with the largest penalty found in step 1 and note the supply remaining for its row, s, and the demand remaining in its column, d.***
3. ***Allocate the minimum of s or d to this variable. If this minimum is s, eliminate all variables in its row from future consideration and reduce the demand in its column by s; if the minimum is d, eliminate al variables in the column from future consideration and reduce the supply in its row by d.***

REPEATE THESE STEPS UNTIL ALL SUPPLIES HAVE BEEN ALLOCATED.

VOGEL'S APPROXIMATION STARTING SOLUTION

From/To	City-1		City-2		City-3		City-4		City-5		Supply
Plant-1	15	**1**		**0**		**3**		**4**		**2**	**15**
Plant-2		**5**	12	**1**		**2**	8	**3**	5	**3**	**25**
Plant-3	5	**4**		**8**	5	**1**		**4**	10	**3**	**20**
Demand	**20**		**12**		**5**		**8**		**15**		

The value of the solution =15(1)+0(0)+…+10(3) = 121

TRANSPORTATION SIMPLEX METHOD

1. ***If the problem is unbalanced, balance it.***
2. ***Find an initial basic feasible solution by some starting procedure.***
3. ***Use the fact that u_1=0 (or any u_i=0) and $u_i+v_j=c_{ij}$ for all basic variables to find the $\{u_1,u_2,\ldots,u_m\ ;\ v_1,v_2,\ldots,v_n\}$ for the current bfs.***
4. ***If*** $c'_{ij} = c_{ij} - u_i - v_j \geq 0$ ***for all nonbasic variables, then the current bfs is optimal. If this not the case we enter the variable with the most negative*** $c_{ij} - u_i - v_j$ ***into the basis using the pivoting procedure. This yields a new bfs.***
5. ***Using the new bfs, return to Steps 3 and 4.***

4'.If $c'_{ij} = c_{ij} - u_i - v_j \leq 0$ ***for all nonbasic variables, the current bfs is optimal. Otherwise, enter the variable with the most positive*** $c_{ij} - u_i - v_j$ ***into the basis using the pivoting procedure.***

How to pivot in a transportation problem

- ***Determine the variable that should enter the basis.***
- ***Find the loop (there is only one loop) involving the entering variable and some of the basic variables.***
- ***Counting only cells in the loop, label the cells in the loop found in step 2 that are even number/0,2,4,etc.) of cells away from the entering variable as even cells. Also label those cells in the loop that are an odd number of cells away from the entering variable as odd cells.***
- ***Find the odd cell whose variable assumes the smallest value. Call this value α. The variable corresponding to this odd cell will leave the basis. To perform the pivot, decrease the value of each odd cell by α and increase the value of each even cell by α. The values of variables not in he loop remain unchanged. The pivot is now complete. If α=0, the entering variable will equal zero, and an odd variable that has a current value of zero will leave the basis. In this case, a degenerate bfs existed before and will result after the pivot. If more than one odd cell in the loop equals α, we may arbitrarily choose one of these odd cells to leave the basis; again a degenerate bfs will result.***

Solution

MINIMUM COST STARTING SOLUTION

From/To	City-1	City-2	City-3	City-4	City-5	Supply and u_i
Plant-1	3 1	12 0	3 5	4 5	2 3	15 (0)
Plant-2	2 5	1 -3	2 0	8 3	15 3	25 (4)
Plant-3	15 4	8 5	5 1	4 2	3 1	20 (3)
Demand and v_j	20 (1)	12 (0)	5 (-2)	8 (-1)	15 (-1)	

The value of the solution =3(1)+12(0)+…+0(3) = 147 is not optimal
Because $c_{ij} - u_i - v_j$ are not positive for all nonbasic variables.
The most negative $c_{ij} - u_i - v_j$.

Pivoting

From/To	City-1	City-2	City-3	City-4	City-5	Supply and u_i
Plant-1	3 1	12 0	3 5	4 5	2 3	15 (0)
Plant-2	2 5	1 -3	2 0	8 3	15 3	25 (4)
Plant-3	15 4	8 5	5 1	4 2	3 1	20 (3)
Demand and v_j	20 (1)	12 (0)	5 (-2)	8 (-1)	15 (-1)	

LOOP (α = 2)

The new bfs

From/To	City-1	City-2	City-3	City-4	City-5	Supply and u_i
Plant-1	5 1	10 0	3 5	4 2	2 0	**15** (0)
Plant-2	5 3	2 1	2 3	8 3	15 3	**25** (1)
Plant-3	15 4	8 5	5 1	4 **-1**	3 **-2**	**20** (3)
Demand and v_j	**20** (1)	**12** (0)	**5** (-2)	**8** (2)	**15** (2)	

The value of the solution =147+ c_{22}’ . α = 147-3(2)= 141
This bfs is not optimal.

Pivoting

From/To	City-1	City-2	City-3	City-4	City-5	Supply and u_i
Plant-1	5 1	10 0	3	4	2	**15** (0)
Plant-2	5	2 1	2	8 3	15 3	**25** (1)
Plant-3	15 4	8	5 1	4	3 **-2**	**20** (3)
Demand and v_j	**20** (1)	**12** (0)	**5** (-2)	**8** (2)	**15** (2)	

α = 10

The new bfs

From/To	City-1		City-2		City-3		City-4		City-5		Supply and u_i
Plant-1	15	1		0 2		3 5		4 4		2 2	**15** (0)
Plant-2		5 1	12	1		2 1	8	3	5	3	**25** (3)
Plant-3	5	4		8 7	5	1		4 1	**10**	3	**20** (3)
Demand and v_j	**20** (1)		**12** (-2)		**5** (-2)		**8** (0)		**15** (0)		

The value of the solution =141+ c_{35}' . α = 141-2(10)= 121
This bfs is optimal.

The $c_{ij}^{'} = c_{ij} - u_i - v_j \geq 0$ **for all nonbasic variables, then the current bfs is optimal.**

Thus, the optimal solution to the Powerco problem is

x_{11}=15, x_{22}=12, x_{24}=8, x_{25}=5, x_{31}=5, x_{33}=5, x_{33}=10 and Z_{min} =121.

There is no alternate optima. (<u>There is no $c_{ij}^{'} = 0$</u>)

Solve the same problem by using the northwest starting procedure.

NORTHWEST STARTING SOLUTION

From/To	City-1	City-2	City-3	City-4	City-5	Supply and u_i
Plant-1	15 1	0 3	3 5	4 5	2 4	15 (0)
Plant-2	5 5	12 1	5 2	3 3	3 1	25 (4)
Plant-3	4 -2	8 6	1	5 4	15 3	20 (5)
Demand and v_j	20 (1)	12 (-3)	5 (-2)	8 (-1)	15 (-2)	

The value of the solution =15(1)+5(5)+…+15(3) = 136 is not optimal

The most negative $c_{ij} - u_i - v_j$.

From/To	City-1	City-2	City-3	City-4	City-5	Supply and u_i
Plant-1	15 1	0	3	4	2	15 (0)
Plant-2	5 5	12 1	5 2	3 3	3	25 (4)
Plant-3	4 -2	8	1	5 4	15 3	20 (5)
Demand and v_j	20 (1)	12 (-3)	5 (-2)	8 (-1)	15 (-2)	

α = 5

The new bfs

From/To	City-1	City-2	City-3	City-4	City-5	Supply and u_i
Plant-1	15 1	0 3	3 5	4 5	2 4	15 (0)
Plant-2	**0** 5	12 1	5 2	8 3	3 **-1**	25 (4)
Plant-3	5 4	8 8	1 0	4	15 3	20 (3)
Demand and v_j	20 (1)	12 (-3)	5 (-2)	8 (-1)	15 (-2)	

The value of the solution =136+ c_{31}' . α = 136-2(5)= 126
This bfs is not optimal.

From/To	City-1	City-2	City-3	City-4	City-5	Supply and u_i
Plant-1	15 1	0	3	4	2	15 (0)
Plant-2	**0** 5	12 1	5 2	8 3	3 **-1**	25 (4)
Plant-3	5 4	8	1	4	15 3	20 (3)
Demand and v_j	20 (1)	12 (-3)	5 (-2)	8 (-1)	15 (-2)	

α = 0 (Degeneration ?)

The new bfs

From/To	City-1	City-2	City-3	City-4	City-5	Supply and u_i
Plant-1	15 1	0 2	3 4	4 4	2 2	15 (0)
Plant-2	5 2	12 1	5 2	8 3	0 3	25 (3)
Plant-3	5 4	8 7	1 **-1**	4	15 3	20 (3)
Demand and v_j	20 (1)	12 (-2)	5 (-1)	8 (0)	15 (0)	

The value of the solution =136+ c_{25}' . α = 126-1(0)= 126 (There is no improving)
This bfs is not optimal.

From/To	City-1	City-2	City-3	City-4	City-5	Supply and u_i
Plant-1	15 1	0	3	4	2	15 (0)
Plant-2	5	12 1	5 2	8 3	0 3	25 (3)
Plant-3	5 4	8	1 **-1**	4	15 3	20 (3)
Demand and v_j	20 (1)	12 (-2)	5 (-1)	8 (0)	15 (0)	

α = 5

The new bfs

From/To	City-1		City-2		City-3		City-4		City-5		Supply and u_i
Plant-1	15	1		0 2		3 5		4 4		2 2	15 (0)
Plant-2		5 1	12	1		2 1	8	3	5	3	25 (3)
Plant-3	5	4		8 7	5	1		4 1	10	3	20 (3)
Demand and v_j	20 (1)		12 (-2)		5 (-2)		8 (0)		15 (0)		

The value of the solution =126+ c_{33}' . α = 126-1(5)= 121
This bfs is optimal.

x_{11}=15, x_{22}=12, x_{24}=8, x_{25}=5, x_{31}=5, x_{33}=5, x_{33}=10 and Z_{min} =121.

There is no alternate optima. (There is no $c'_{ij} = 0$)