MATRICES
AND
MATRIX OPERATIONS
IV
Linear Independence
And
Rank of a Matrix

Definition:

Suppose $c_1v_1 + c_2v_2 + \ldots + c_nv_n = 0$ *only happens when* $c_1 = c_2 = \ldots = c_n = 0.$

Then the vectors $v_1, v_2, \ldots, v_n$ *are* **linearly independent.**

If any c's are nonzero, the v's are **linearly dependent.**

One vector is combination of the others.

If say $c_1 \neq 0$ *then we can write*

$$v_1 = (-\frac{c_2}{c_1})v_2 + \ldots + (-\frac{c_n}{c_1})v_n$$

- **Two vectors are <u>dependent</u> if they lie on the same line.**
- **Three vectors are <u>dependent</u> if they lie in the same plane (*coplanar vectors*).**

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix}$$

The columns of the matrix are linearly dependent, since the <u>second column is three times the first.</u>

$$\begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

The columns of the triangular matrix are <u>linearly independent</u>.

Example:

Determining Linear Independence/Dependence

Determine whether the vectors $v_1 = (1,1)$, $v_2 = (-3,2)$ form a linearly dependent set or a linearly independent set.

Let k1 and k2 be two real numbers such that

$$k_1 v_1 + k_2 v_2 = 0$$

$$k_1(1,1) + k_2(-3,2) = (0,0)$$

$$k_1 - 3k_2 = 0$$

$$k_1 + 2k_2 = 0$$

Solving for k_1 and k_2, we find that $k_1 = 0$ and $k_2 = 0$

<u>The system has only trivial solutions</u>

$$A = \begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix}.$$

$$\det A = 1 \cdot 2 - 1 \cdot (-3) = 5 \neq 0.$$

The vectors (1, 1) and (−3, 2) in R^2 are <u>linearly independent.</u>

Example:

Determining Linear Independence/Dependence

Determine whether the vectors

$v_1 = (1,-2,3)$, $v_2 = (5,6,-1)$, $v_3 = (3,2,1)$

form a linearly dependent set or a linearly independent set.

In terms of components, the vector equation

$$k_1 v_1 + k_2 v_2 + k_3 v_3 = 0$$

becomes

$$k_1(1,-2,3) + k_2(5,6,-1) + k_3(3,2,1) = (0,0,0)$$

Or equivalently

$$(k_1 + 5k_2 + 3k_3, -2k_1 + 6k_2 + 2k_3, 3k_1 - k_2 + k_3) = (0,0,0)$$

equating corresponding components gives

$$k_1 + 5k_2 + 3k_3 = 0$$
$$-2k_1 + 6k_2 + 2k_3 = 0$$
$$3k_1 - k_2 + k_3 = 0$$

Thus v_1, v_2, v_3 form a **linearly dependent set if this system has a nontrivial solution** or **linearly independent set if it has only the trivial solution**.

Solving this system using Gaussian elimination yields

$$k_1 = k_2 = -\frac{1}{2}t, k_3 = t$$

Thus the system has nontrivial solutions and v_1, , v_2 and v_3, form a linearly dependent set.

Alternatively, we could show the existence of nontrivial solutions without solving the system by showing that the coefficient matrix has **determinant zero** **and consequently is not invertible.**

```
A=[1 5 3; -2 6 2; 3 -1 1]
A =
     1     5     3
    -2     6     2
     3    -1     1
>> Det (A)
Ans =     0
```

We say again the given system has nontrivial solutions and v_1, , v_2 and v_3, form a linearly dependent set.

Theorem

A set S with two or more vectors is

- ***Linearly dependent if and only if at least one of the vectors in S is expressible a linear combination of the other vectors in S.***
- ***Linearly independent if and only if no vector in S is expressible a linear combination of the other vectors in S.***

Theorem

- ***A finite set of vectors that contains the zero vector is linearly dependent.***
- ***A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.***

For any vectors $v_1, v_2, \ldots, v_r$, the set $S = \{v_1, v_2, \ldots, v_r, 0\}$ is linearly dependent since the equation

$0v_1 + 0v_2 + \ldots + 0v_r + 1(0) = 0$ expresses **0 as** a linear combination of the vectors in S with coefficients **that are not all zero.**

(If any c's are nonzero, the v's are linearly dependent.)

Rank

Definition

The common dimension of the row space and column space of a matrix A is called the rank of A and is denoted by **rank (A).**

Alternative Definitions

- ***The column rank of a matrix A is the maximal number of linearly independent columns of A.***
- ***Likewise, the row rank of a matrix is the maximal number of linearly independent rows of A.***
- ***The number of non-zero rows in the row reduced form of a matrix*** A ***is called the rank of*** A, ***denoted*** $\operatorname{rank}(A)$.

Since the column rank and the row rank are always equal, they are simply called the rank of A. It is commonly denoted by either **rank(A)** ***or rank A.***

Another equivalent definition of the rank of a matrix is the order of the greatest non-vanishing minor in the matrix.

Dimension Theorem for the Matrices

Theorem:

If A is any matrix, then

$$rank(A) = rank(A^T)$$

Theorem:

If A is an $m \times n$ ***matrix, then***

1. ***rank(A)=<u>the number of leading variables</u> in the solution of* Ax=0.**
2. ***nullity(A)=<u>the number of parameters in the general solution</u> of* Ax=0.**

Theorem:

If A is a matrix with* n *columns, then

$$rank(A) + nullity(A) = n$$

Theorem:

If A is an $m \times n$ matrix, then

rank(A) is at most $\min(m,n)$ A matrix that has a large a rank as possible is said to have full rank; otherwise, the matrix is rank deficient.

Theorem:

The column rank of a matrix A is the maximal number of linearly independent columns of A. Likewise, the row rank is the maximal number of linearly independent rows of A.

Theorem:

In the case of a square matrix A (i.e., m = n), then A is invertible if and only if A has rank n (that is, A has full rank)

Applications

One useful application of calculating the rank of a matrix is the computation of the number of solutions of a system of linear equations. The system is inconsistent if the rank of the augmented matrix is greater than the rank of the coefficient matrix. If, on the other hand, ranks of these two matrices are equal, the system must have at least one solution. The solution is unique if and only if the rank equals the number of variables. Otherwise the general solution has *k* free parameters where *k* is the difference between the number of variables and the rank.

In control theory, the rank of a matrix can be used to determine whether a linear system is controllable, or observable.

Example: Rank and Nullity of a 4 x 6 Matrix

Find the rank and nullity of the matrix

$$A=\begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

The reduced row-echelon form of A is (verify).

$$\begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there are two nonzero rows (or, equivalently, two leading 1's), the row space and column space are both two-dimensional, so **rank(A)=2.**

The number of non-zero rows in the row reduced form of a matrix A is called the rank of A, denoted $\operatorname{rank}(A)$.

```
>> A=[-1 2 0 4;3 -7 2 0;2 -5 2 4;4 -9 2 -4]

A =

    -1     2     0     4

     3    -7     2     0

     2    -5     2     4

     4    -9     2    -4

>> det(A)

ans =  0

>> A=[2 0 4; -7 2 0; -5 2 4]

A =

     2     0     4

    -7     2     0

    -5     2     4

>> det(A)

ans =     0
```

To find the **nullity** of A, we must find the dimension of the solution space of the linear system **Ax=0**.

This system can be solved by reducing the augmented matrix to **reduced row-echelon form**. The resulting matrix will be identical to the matrix given above, except that it will have an additional column of zeros, and the corresponding system of equations will be

$$\begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 & 0 \\ 0 & 1 & -2 & -12 & -16 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - 4x_3 - 28x_4 - 37x_5 + 13x_6 = 0$$
$$x_2 - 2x_3 - 12x_4 - 16x_5 + 5x_6 = 0$$

Solving for the leading variables we obtain,

$$x_1 = 4x_3 + 28x_4 + 37x_5 - 13x_6$$
$$x_2 = 2x_3 + 12x_4 + 16x_5 - 5x_6$$

It follows that the general solution of the system is

$$
\begin{aligned}
x_1 &= 4r + 28\mathrm{s} + 37\mathrm{t} - 13\mathrm{u}, \\
x_2 &= 2r + 12s + 16t - 5u, \\
x_3 &= r, \\
x_4 &= s, \\
x_5 &= t, \\
x_6 &= u
\end{aligned}
$$

Or equivalently

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = r\begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 28 \\ 12 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 37 \\ 16 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u\begin{bmatrix} -13 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
$$

Because the four vectors on the right side of this vector equation form a basis for the solution space, **nullity(A)=4**.

Example:

Consider for example the 4-by-4 matrix

$$A = \begin{bmatrix} 2 & 4 & 1 & 3 \\ -1 & -2 & 1 & 0 \\ 0 & 0 & 2 & 2 \\ 3 & 6 & 2 & 5 \end{bmatrix}$$

We see that the ***second column is twice the first column***, and that the ***fourth column equals the sum of the first and the third***. ***The first and the third columns are linearly independent,*** so **the rank of *A* is two**. This can be confirmed with the Gauss algorithm. It produces the following reduced row echelon form of *A*:

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which has **two non-zero rows**.

Example:

Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 6 & 9 & 12 \\ 5 & 9 & 7 & 7 \end{bmatrix}$$

The first two rows are proportional*. **The rank cannot be 3 but is 2.**

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \\ 5 & 9 & 7 \end{vmatrix} = 0 \ , \quad \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 0 \quad \text{but} \quad \begin{vmatrix} 3 & 6 \\ 5 & 9 \end{vmatrix} = -3$$

r - Rowed **square submatrix** with **non vanishing determinant**. Also, the determinant of any ***r + 1 or more*** **square** submatrix of **A** is zero.

***Theorem:** (Given Before)

If A is a square matrix with two proportional rows or two proportional columns, then

$$\det(A) = 0$$

Example:

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Both matrices have rank 1, but the product has rank 0.

$$AB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$