Integer Programming

An integer programming problem in which all variables are required to be integers is called a **pure integer programming problem (All integer linear program).**

$$\text{Max } Z = 2x_1 + 3X_2$$
$$\text{s.t } 3x_1 + 3x_2 \leq 12$$
$$x_1 + 2x_2 \leq 6$$
$$x_1, x_2 \geq 0, \mathbf{x_1, x_2 \text{ integer}}$$

An integer programming in which only some of variables are required to be integers is called a **mixed integer programming problem**.

$$\text{Max } Z = 3x_1 + 4X_2$$
$$\text{s.t } -x_1 + x_2 \leq 8$$
$$x_1 + 2x_2 \leq 12$$
$$2x_1 + x_2 \leq 16$$
$$x_1, x_2 \geq 0, \text{ and } \mathbf{x_2 \text{ integer}}$$

An integer programming problem in which all the variables must equal 0 or 1 is called a **0-1 binary integer linear programming problem.**

$$\text{Min } Z = x_1 + x_2 + x_3$$
$$\text{s.t } x_1 + x_2 \geq 1$$
$$x_2 + x_3 \geq 1$$
$$x_1 + x_3 \geq 1$$
$$\mathbf{x_1, x_2 = 0 \text{ or } 1}$$

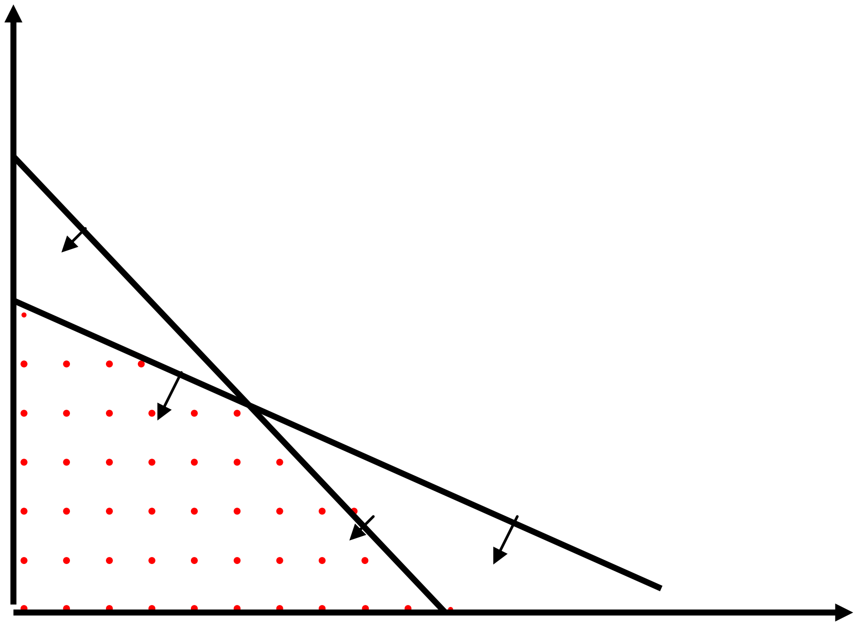
Definition:

The LP obtained by omitting all integer or 0-1 constraints on variables is called the LP relaxation of the problem.

Property:

The value of the optimal solution to any integer or mixed-integer linear program involving maximization (minimization) yields a value less than or equal to (greater than or equal to) the value of the optimal solution to its LP relaxation.

Graphical Solution:



- **Feasible points for original LP.**

The Branch-and-Bound Method for Solving Integer Programming Problems

$$\text{Max } Z = 2x_1 + 3x_2$$
$$195x_1 + 273x_2 \le 1365$$
$$4x_1 + 40x_2 \le 140$$
$$x_1 \le 4$$
$$x_1, x_2 \ge 0 \text{ and } x_1, x_2 \text{ integer}$$

Relaxation solution of the LP:

$x_1 = 2.44$ $x_2 = 3.26$ $Z_{max} = 14.66$
(This is not all integer solution)

Upper Bound=UB=14.66

Any integer feasible solution $x_1 = 2$ $x_2 = 3$ $Z = 13$

Lower Bound=LB=13.00

UB=14.66
LB = 13.0

UB=14.66 LB = 13.0

Sub problem 1

x_1 = 2.44 x_2 = 3.26
Z_{max} = 14.66

$x_1 \leq 2$
Sub Problem-2

$x_1 \geq 3$
Sub Problem-3

Sub Problem-2

$$\text{Max } Z = 2x_1 + 3x_2$$
$$195\,x_1 + 273\,x_2 \leq 1365$$
$$4x_1 + 40\,x_2 \leq 140$$
$$x_1 \leq 4$$
$$x_1 \leq 2$$
(this makes $x_1 \leq 4$ redundant)
$$x_1, x_2 \geq 0$$

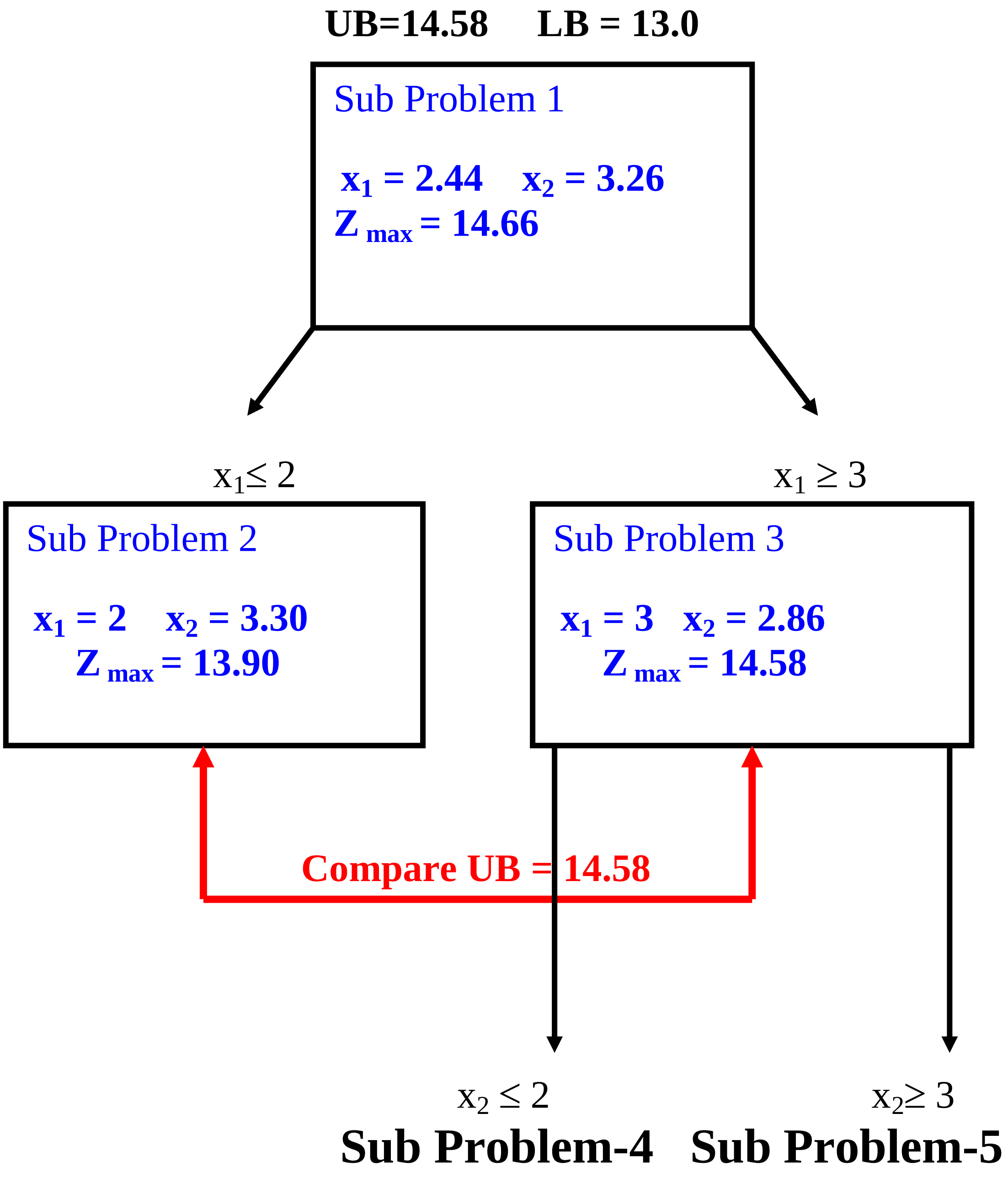
$x_1 = 2$ $x_2 = 3.30$ $Z_{max} = 13.90$

Sub Problem-3

$$\text{Max } Z = 2x_1 + 3x_2$$
$$195\,x_1 + 273\,x_2 \leq 1365$$
$$4x_1 + 40\,x_2 \leq 140$$
$$x_1 \leq 4$$
$$x_1 \geq 3$$
$$x_1, x_2 \geq 0$$

$x_1 = 3$ $x_2 = 2.86$ $Z_{max} = 14.58$

UB =14.58   (max of 13.90 and 14.58)



Sub Problem-4

$$\text{Max } Z = 2x_1 + 3x_2$$
$$195x_1 + 273x_2 \leq 1365$$
$$4x_1 + 40x_2 \leq 140$$
$$x_1 \leq 4$$
$$x_1 \geq 3$$
$$x_2 \leq 2$$
$$x_1, x_2 \geq 0$$

$x_1 = 4$ $x_2 = 2$ $Z_{max} = 14$

(Candidate solution)

Sub Problem-5

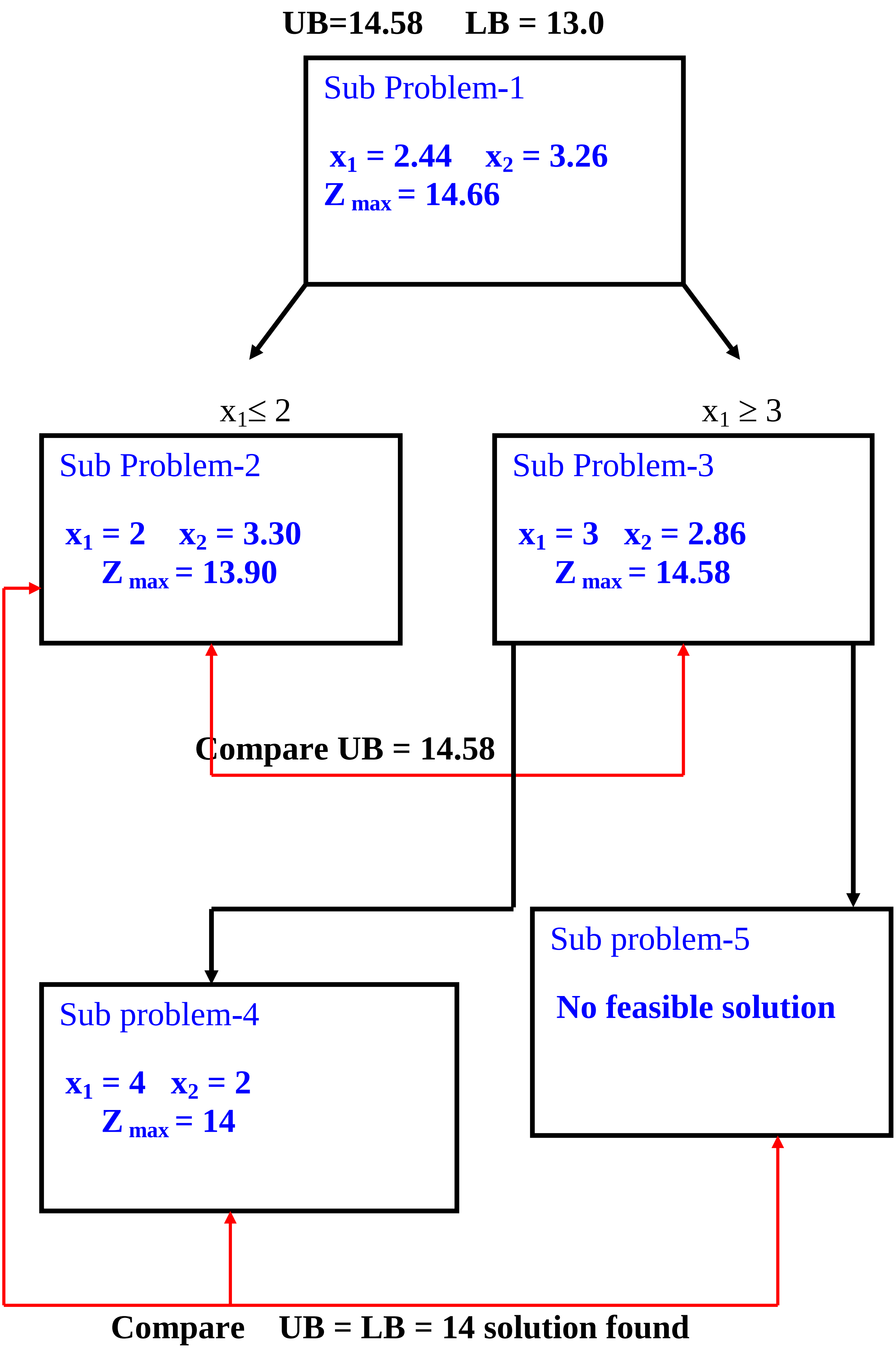
$$\text{Max } Z = 2x_1 + 3x_2$$
$$195x_1 + 273x_2 \leq 1365$$
$$4x_1 + 40x_2 \leq 140$$
$$x_1 \leq 4$$
$$x_1 \geq 3$$
$$x_2 \geq 3$$
$$x_1, x_2 \geq 0$$

No feasible solution (Infeasible)

UB = 14 LB = 14 solution found.

All integer solution

$x_1 = 4$ $x_2 = 2$ $Z_{max} = 14$


UB=14.58 LB = 13.0
Sub Problem-1
x_1 = 2.44 x_2 = 3.26
Z_{max} = 14.66
$x_1 \leq 2$
$x_1 \geq 3$
Sub Problem-2
x_1 = 2 x_2 = 3.30
Z_{max} = 13.90
Sub Problem-3
x_1 = 3 x_2 = 2.86
Z_{max} = 14.58
Compare UB = 14.58
Sub problem-5
No feasible solution
Sub problem-4
x_1 = 4 x_2 = 2
Z_{max} = 14
Compare UB = LB = 14 solution found

Flowchart of Branch-and –Bound Solution Procedure for the All -Integer Linear Program

Solve the LP relaxation of the integer linear programming problem at node 1. Set the UB equal to the value of the LP solution.

↓

Find feasible integer solution. Set LB equal to the value of the feasible integer solution.

↓

Is UB = LB **Yes IP solution found**

No

Branch from the node with greatest LP value. Find the variable that is furthest from being integral. Create two branches and two descendent nodes one with $x_j \le [x_j]$ and one with $x_j \ge [x_j]$

↓

Solve the LP relaxation at each of the descendent nodes and record its LP value

↓

Recompute the upper bound by finding the maximum over all nodes from which there are no baranches.

↓

Re-compute the lover bound as the maximum value of all feasible solution found to date.

The Branch-and-Bound Method for Solving Mixed Integer Programming Problems

To solve a mixed integer programming by branch and bound modify the method described by branching only on variables that are required to be integers. Also, for a solution to a sub problem to be a candidate solution, it need only assign integer values to those variables that are required to be integers. To illustrate let us solve the following mixed IP:

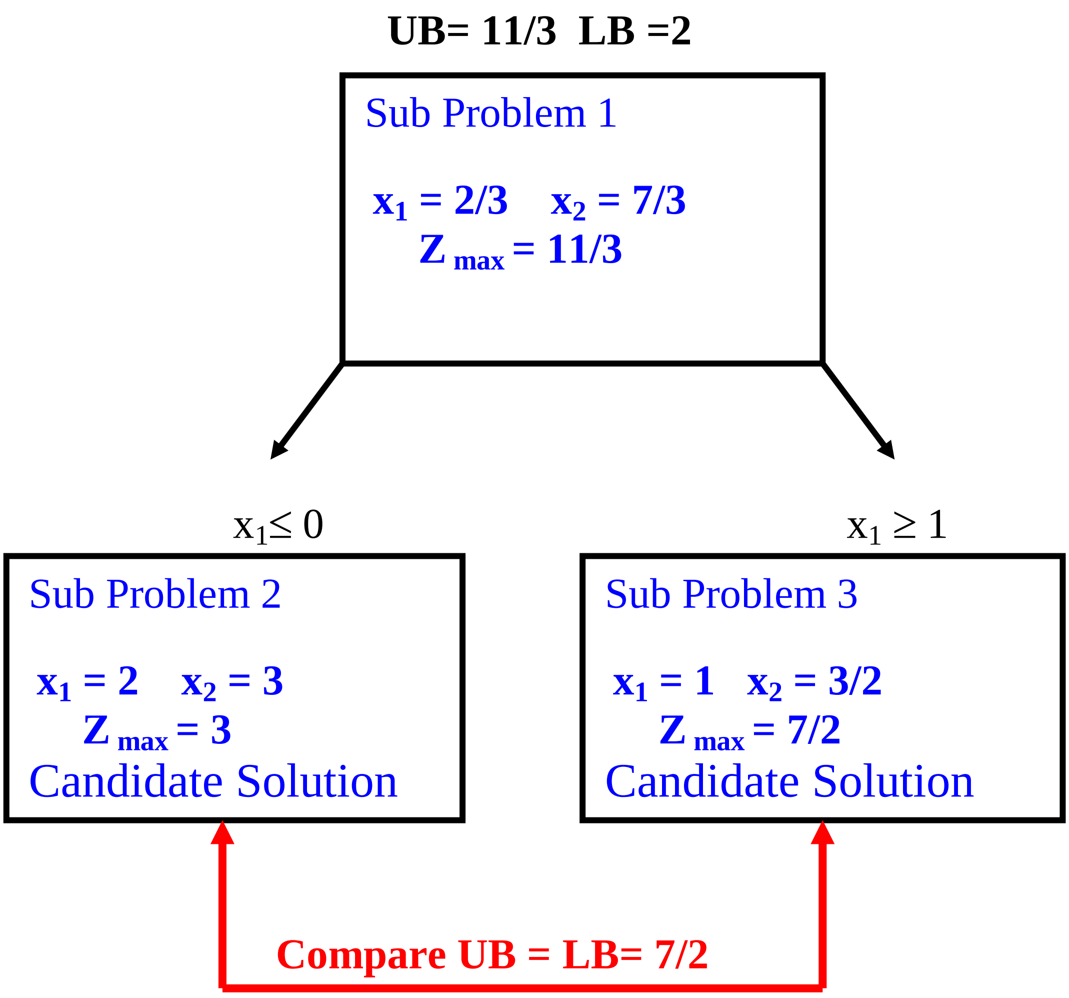
$$\begin{aligned} &\text{Max } Z = 2x_1 + x_2 \\ \text{s.t } \quad & 5x_1 + 2x_2 \le 8 \\ & x_1 + x_2 \le 3 \\ & x_1, x_2 \ge 0 \text{ and } x_1 \text{ integer} \end{aligned}$$

Relaxation solution of the LP:

$\mathbf{x_1 = 2/3 \quad x_2 = 7/3 \qquad Z_{max} = 11/3}$

Since x_2 is allowed to be fractional, we do not branch on $x_{2.}$ Thus we must branch on x_1. This yields sub problems 2 and 3 in the following figure.

UB= 11/3 and LB =2 ($x_1 = 0 \quad x_2 = 2$)



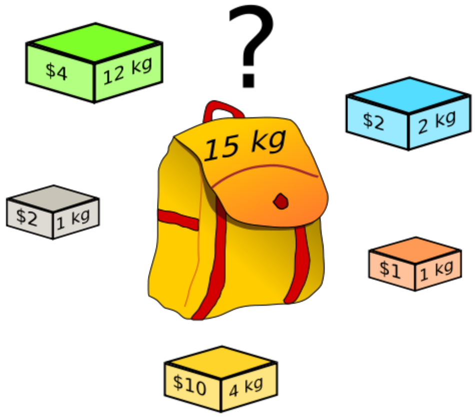
We next choose to solve subproblem 2. The optimal solution to subproblem 2 is the candidate solution $x_1 = 2$ $x_2 = 3$ $Z_{max} = 3$.

We now solve the subproblem 3 and obtain the candidate solution $x_1 = 1$ $x_2 = 3/2$ $Z_{max} = 7/2$. Since z value from the subproblem 3 candidate exceeds the z value for the subproblem 2 candidate, the subproblem 2 can be eliminated from consideration, and the subproblem 3 candidate ($x_1 = 1$ $x_2 = 3/2$ $Z_{max} = 7/2$) is The optimal solution to the mixed IP.

Related Methods:

Knapsack Problems

Any IP which has only one constraint is referred to as a knapsack problem.



Which boxes should be chosen to maximize the amount of money while still keeping the overall weight under 15 kg?

(The solution in this case is to choose all of the boxes besides the green one.)

$$\text{Max } Z = 4x_1 + 2x_2 + 2x_3 + 2x_4 + 10x_5$$
$$\text{s.t} \quad 12x_1 + 2x_2 + x_3 + x_4 + x_5 \leq 15$$
$$x_i \geq 0 \text{ and } x_i = 0,1$$

Mathematically the 0-1-knapsack problem can be formulated as:

$$\text{Maximize} \quad \sum_{i=1}^{n} p_i x_i$$

subject to

$$\sum_{i=1}^{n} w_i x_i \leq c, \qquad x_i = 0,1 \quad i = 1,\ldots,n$$

Cutting Plane Algorithm

GLOSSARY

Integer Linear Program: A linear program with the additional requirement that some or all of the decision variables must be integer.

All-integer Linear Program: An integer linear program in which all the decision variables are required to be integer**.**

LP Relaxation: The linear program that results from dropping the integer requirements for the decision variables.

Mixed-integer linear program: An integer linear program in which some, but not all, of the decision variables are required to be integer.

0-1 integer linear program: An all-integer or mixed-integer linear program in which the integer variables are only permitted to assume the values 0 or 1.

Upper bound: A value that is known to be greater than or equal to the value of any feasible solution. The solution to the LP relaxation of any integer linear program provides an upper bound for a maximization problem.

Lower bound: A value that is less than or equal to the value of the optimal solution. For a maximization problem the value of any feasible integer solution to an integer linear program is a lower bound.

Branch and bound: A solution procedure for integer linear programs that sequentially partitions the set of feasible solutions into smaller subsets until the optimal solution is found.

Descendent node: A node created in a branch-and-bound solution tree by branching from another node by adding a constraint to the LP Relaxation solved at the previous node.

Software (Wikipedia)

- AIMMS Optimization Modeling AIMMS — include linear programming in industry solutions (free trial license available);
- Calc Optimization Solver — Kohei Yoshida's spreadsheet add-in for OpenOffice.org Calc
- CGAL — The Computational Geometry Algorithms Library includes a linear solver, which is exact and optimized for problems with few constraints or few variables
- The Parma Polyhedra Library includes a LP solver in exact precision
- COIN-OR — COmputational INfrastructure for Operations Research, open-source library
- GAMS — General Algebraic Modeling System
- Cplex — Commercial library for linear programming
- GLPK — GNU Linear Programming Kit; open source LP software
- IMSL Numerical Libraries — Commercial libraries of math and statistical algorithms
- HOPDM — Higher Order Primal Dual Method
- LINDO — LP, IP, Global solver/modeling language
- lp_solve — open-source solver with C library
- MOSEK — Optimization software for LP,IP,QP,SOCP and MIP. Free trial is available. Free for students.
- Mathematica — General technical computing system includes large scale linear programming support
- OptimJ — Java-based algebraic modeling language; free evaluation version.
- Premium Solver — Spreadsheet add-in

- WhatsOP — LP modeling in 6 languages, made practical with easy "what-ifs" for students. Free Trial.
- SAS — Includes optimization modeling language and solvers for LP, MILP, QP, and NLP
- What's Best! — Spreadsheet add-in
- Xpress-MP — Optimization software free to students
- Vanguard Linear Programming Optimization Software
- QSopt Optimization software for LP (free for research purposes).
- Microarray Data Classification Server (MDCS) based on linear programming
- Linear programming and linear goal programming A freeware program for MS-DOS
- Simplex Method Tool A quick-loading web page
- IBM's article on GLPK A technical article on GLPK with an introduction to Linear Programming by IBM
- Orstat2000 — Includes easy-to-use modules for linear and integer programming (free for educational purposes).
- Trilinos is a set of scientific tools with a few linear and non-linear program solvers.
- TOMLAB provides optimization solvers in MATLAB, LabVIEW and .NET.
- SCIP It can be used as a standalone program to solve mixed integer programs given in MPS Format.
- GIPALS32.DLL — Linear programming callable library for Windows (free trial and academic license available).
- CVXOPT — Python library that can be used for linear, second-order cone and semidefinite programming (open source, GPLv3)