Sensitivity Analysis

Sensitivity analysis is concerned with how changes in an LP's parameters affect the LP's optimal solution.

- **Changing the Objective Function Coefficient of a Basic Variable,**
- **Changing the Right-Hand Side of a Constraint,**
- **Adding a New Variable,**
- **Adding a New Constraint.**

Changing the Objective Function Coefficient of a Basic Variable

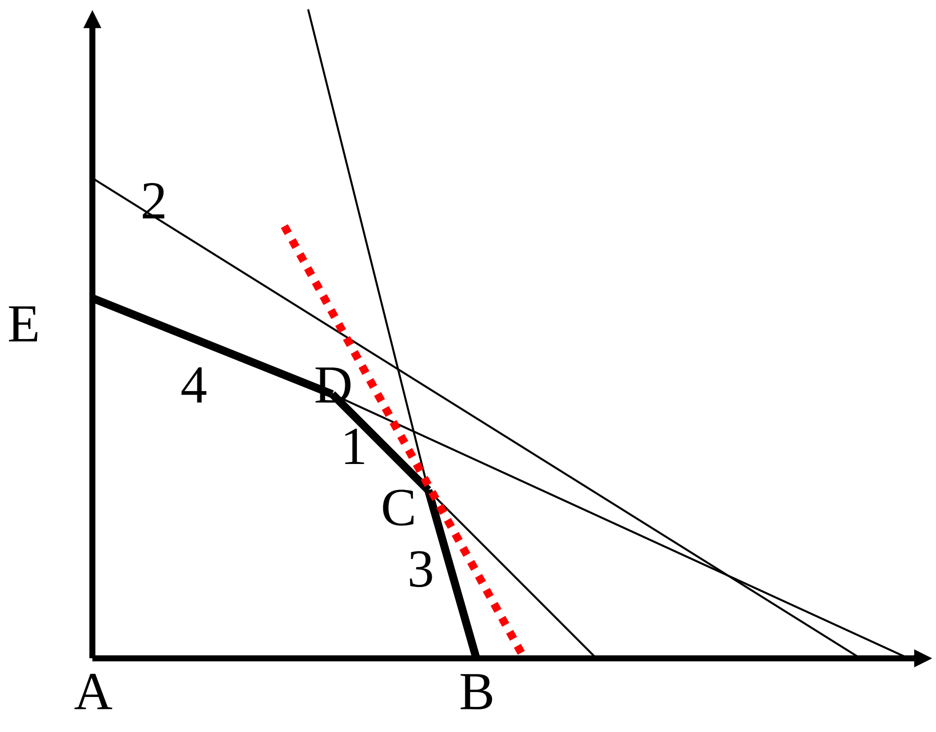
Range of Optimality

The range of values over which an objective function coefficient <u>may vary without causing any change in the optimal solution</u> (that is, t<u>he values of all variables will remain the same</u>, <u>but the value of the objective function will change</u>).

Graphical Sensitivity Analysis

Max $Z = 10\,x_1 + 9\,x_2$

Constraint-1	$7/10\,x_1 + x_2 \le 630$
Constraint-2	$½\,x_1 + 5/6\,x_2 \le 600$
Constraint-3	$x_1 + 2/3\,x_2 \le 708$
Constraint-4	$1/10\,x_1 + 1/4\,x_2 \le 135$
	$x_1, x_2 \ge 0$



The Optimal Solution to this problem is Z_{max}= 7668 x_1=540, x_2= 252   Point C (540,252).

Slope of the equation (1):

$7/10\ x_1 + x_2 = 630$ $x_2 = -7/10\ x_1 + 630$ $m_1 = -7/10$

Slope of the equation (3):

$x_1 + 2/3\ x_2 = 708$ $x_2 = -3/2\ x_1 + 1062$ $m_3 = -3/2$

Slope of the Objective Function

$Z = c_1 x_1 + c_2 x_2$ $x_2 = -c_1/c_2\ x_1 + Z/c_2$ $m_z = -c_1/c_2$

The range of optimality

$$m_3 \leq m_z \leq m_1$$

$$-3/2 \leq -c_1/c_2 \leq -7/10$$

or

$$\mathbf{7/10 \leq c_1/c_2 \leq 3/2}$$

Simplex-Based Sensitivity Analysis

$$\begin{aligned}
\text{Max } Z &= 10x_1 + 9x_2 \\
\tfrac{7}{10}x_1 + x_2 &\le 630 \\
\tfrac{1}{2}x_1 + \tfrac{5}{6}x_2 &\le 600 \\
x_1 + \tfrac{2}{3}x_2 &\le 708 \\
\tfrac{1}{10}x_1 + \tfrac{1}{4}x_2 &\le 135 \\
x_1, x_2 &\ge 0
\end{aligned}$$

$$Z - 10x_1 - 9x_2 = 0$$

$$\begin{aligned}
\tfrac{7}{10}x_1 + x_2 + x_3 &= 630 \\
\tfrac{1}{2}x_1 + \tfrac{5}{6}x_2 + x_4 &= 600 \\
x_1 + \tfrac{2}{3}x_2 + x_5 &= 708 \\
\tfrac{1}{10}x_1 + \tfrac{1}{4}x_2 + x_6 &= 135
\end{aligned}$$

BASIS	x_1	x_2	x_3	x_4	x_5	x_6	RHS	RATIO
x_3	7/10	1	1	0	0	0	**630**	900
x_4	½	5/6	0	1	0	0	**600**	1200
x_5	**1**<	2/3	0	0	1	0	**708**	**708**
x_6	1/10	1/4	0	0	0	1	**135**	1350
Z	**-10**	**-9**	**0**	**0**	**0**	**0**	**0**	
x_3	0	**16/30**	1	0	-7/10	0	**134.4**	**252**
x_4	0	½	0	1	-12	0	**246**	492
x_1	1	2/3	0	0	1	0	**708**	1062
x_6	0	22/120	0	0	-1/10	1	**64.2**	350.18
Z	**0**	**-7/3**	**0**	**0**	**10**	**0**	**7080**	
x_2	0	1	30/16	0	-21/16	0	**252**	
x_4	0	0	-15/16	1	5/32	0	**120**	
x_1	1	0	-20/16	0	30/16	0	**540**	
x_6	0	0	-11/32	0	9/64	1	**18**	
Z	0	0	70/16	0	111/16	0	**7668**	

$$(c_2 c_4 c_1 c_6)\begin{pmatrix} 30/16 \\ -15/16 \\ -20/16 \\ -11/32 \end{pmatrix} \geq 0$$

$$(c_2 c_4 c_1 c_6)\begin{pmatrix} -21/16 \\ 5/32 \\ 30/16 \\ 9/64 \end{pmatrix} \geq 0$$

$-20/16\ c_1 + 30/16\ c_2 \geq 0$ $-20/16\ c_1 \geq -\ 30/16\ c_2$

$20/16 c_1 \leq 30/16\ c_2$

$c_1/c_2 \leq (30x16)/(20x16)$ $\mathbf{c_1/c_2 \leq 3/2}$

$-21/16\ c_2 + 30/16\ c_1 \geq 0$ $30/16\ c_1 \geq 21/16\ c_2$

$c_1/c_2 \geq (21x16)/(30x16)$ $\mathbf{c_1/c_2 \geq 7/10}$

The range of optimality:

$$\mathbf{7/10 \leq c_1/c_2 \leq 3/2}$$

Set $c_2 = 9$

$7/10 \leq c_1/9 \leq 3/2$

$c_1 \leq 27/2$ $c_1 \geq 63/10$

The range of optimality for c_1

$$\mathbf{6.3 \leq c_1 \leq 13.5}$$

Set $c_1=10$

$7/10 \leq 10/c_2 \leq 3/2$

$3c_2 \geq 20$ $c_2 \geq 20/3$ $7c_2 \leq 100$ $c_2 \leq 100/7$

The range of optimality for c_2

$$\mathbf{6.67 \leq c_2 \leq 14.286}$$

Simultaneous Changes
The 100% Rule
For Objective Function Coefficients

For all objective function coefficients changed, sum the percentages of allowable increases and allowable decreases. <u>If the sum of percentages does not exceed 100 percent, then the optimal solution will not change</u>.

Δc_j : change in c_j

If $\Delta c_j \geq 0$ then $r_j = \Delta c_j / I_j$

If $\Delta c_j \leq 0$ then $r_j = -\Delta c_j / D_j$

Where I_J : Max allowable increase in c_j

D_j : Max allowable decrease in c_j

Lower	$\mathbf{D_j}$	$\mathbf{c_j}$	$\mathbf{I_j}$	**Upper**
6.3	3.7	**10**	3.5	13.5
6.67	2.33	**9**	5.286	14.286

Assume $c_1 \rightarrow 12$ and $c_2 \rightarrow 8.5$

$\Delta c_1 = 2$ $r_1 = (12-10)/3.5 = 0.57$

$\Delta c_2 = -0.5$ $r_2 = -(8.5-9)/2.33 = 0.2145$

$\Sigma\ r_i = r_1 + r_2 = 0.57 + 0.2145 = 0.7845$ or 78.45%

We can be sure that the current basis remains optimal ($\Sigma\ r_i \leq 1$).

Changing Right-Hand Side Values

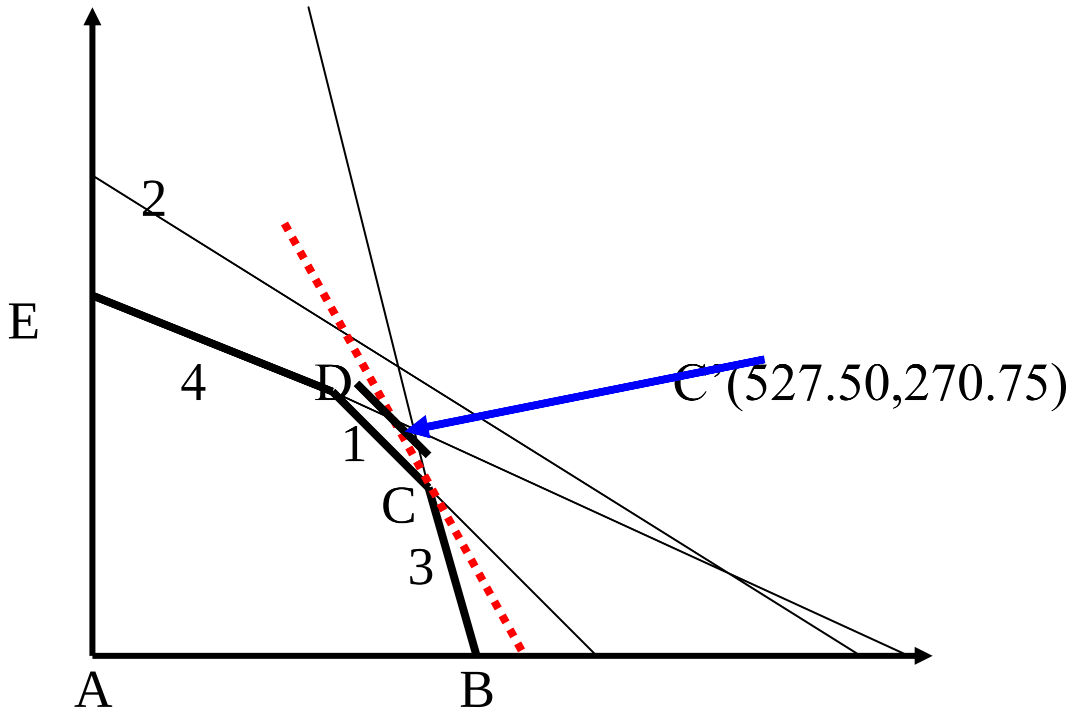
Old Problem

Max $Z = 10 x_1 + 9 x_2$
$7/10\, x_1 + x_2 \leq 630$
$\frac{1}{2} x_1 + 5/6\, x_2 \leq 600$
$x_1 + 2/3\, x_2 \leq 708$
$1/10\, x_1 + 1/4\, x_2 \leq 135$
$x_1, x_2 \geq 0$

New Problem

Max $Z = 10 x_1 + 9 x_2$
$7/10\, x_1 + x_2 \leq$ **640**
$\frac{1}{2} x_1 + 5/6\, x_2 \leq 600$
$x_1 + 2/3\, x_2 \leq 708$
$1/10\, x_1 + 1/4\, x_2 \leq 135$
$x_1, x_2 \geq 0$

The Optimal Solution to original problem was Z_{max}= 7668 x_1=540, x_2= 252 Point C (540,252).



The New Optimal Solution is Z_{max}=7711.75 x1=527.50, x_2= 270.75 (Point C' (527.50, 270.75)).

Shadow Price:

The change in value of the objective function per-unit increase in the value of the right-hand side associated with a linear programming constraint.

Final Simplex Tableau

BASIS	x_1	x_2	x_3	x_4	x_5	x_6	RHS	RATIO
x_2	0	1	30/16	0	-21/16	0	**252**	
x_4	0	0	-15/16	1	5/32	0	**120**	
x_1	1	0	-20/16	0	30/16	0	**540**	
x_6	0	0	-11/32	0	9/64	1	**18**	
Z	0	0	70/16	0	111/16	0	**7668**	

Shadow Price for the first RHS value

New optimal value : 7711.75

Old optimal value : 7668

Difference :43.75

Compare 43.75 with the shadow price for b_1(70/16)

New solution from the simplex tableau

$$new\ solution = \begin{bmatrix} x_2 \\ x_4 \\ x_1 \\ x_6 \\ Z \end{bmatrix} = \begin{bmatrix} 252 \\ 120 \\ 540 \\ 18 \\ 7668 \end{bmatrix} + (640 - 630) \begin{bmatrix} 30/16 \\ -15/16 \\ -20/16 \\ -11/32 \\ 70/16 \end{bmatrix} = \begin{bmatrix} 270.75 \\ 110.625 \\ 527.50 \\ 14.5625 \\ 7711.75 \end{bmatrix}$$

Old solution **Change in $b_1 = \Delta b_1$** **New solution**

Old Problem

Max $Z = 10x_1 + 9x_2$

$7/10\,x_1 + x_2 \le 630$

$½\,x_1 + 5/6\,x_2 \le 600$

$x_1 + 2/3\,x_2 \le 708$

$1/10\,x_1 + 1/4\,x_2 \le 135$

$x_1, x_2 \ge 0$

New Problem

Max $Z = 10x_1 + 9x_2$

$7/10\,x_1 + x_2 \le 640$

$½\,x_1 + 5/6\,x_2 \le 600$

$x_1 + 2/3\,x_2 \le 720$

$1/10\,x_1 + 1/4\,x_2 \le 135$

$x_1, x_2 \ge 0$

$$new\ solution = \begin{bmatrix} x_2 \\ x_4 \\ x_1 \\ x_6 \\ Z \end{bmatrix} = \begin{bmatrix} 252 \\ 120 \\ 540 \\ 18 \\ 7668 \end{bmatrix} + 10 \begin{bmatrix} 30/16 \\ -15/16 \\ -20/16 \\ -11/32 \\ 70/16 \end{bmatrix} + 12 \begin{bmatrix} -21/16 \\ 5/32 \\ 30/16 \\ 9/64 \\ 111/16 \end{bmatrix} = \begin{bmatrix} 255 \\ 112.5 \\ 550 \\ 23.125 \\ 7795 \end{bmatrix}$$

Range of Feasibility

The range of values over which a b_i may vary without causing the current basic solution to become infeasible. The values of the variables in solution will change, but the same variables will remain basic.

$$\begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \\ . \\ . \\ \bar{b}_m \end{bmatrix} + \Delta b_i \begin{bmatrix} \bar{a}_{1j} \\ \bar{a}_{2j} \\ . \\ . \\ \bar{a}_{mj} \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \\ . \\ . \\ \bar{b}_m \end{bmatrix} - \Delta b_i \begin{bmatrix} \bar{a}_{1j} \\ \bar{a}_{2j} \\ . \\ . \\ \bar{a}_{mj} \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 0 \\ 0 \end{bmatrix}$$

For the $\leq$ constraint **For the $\geq$ constraint**

Range of feasibility for b_1

$$\begin{bmatrix} 252 \\ 120 \\ 540 \\ 18 \end{bmatrix} + \Delta b_1 \begin{bmatrix} 30/16 \\ -15/16 \\ -20/16 \\ -11/32 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$252 + 30/16\ \Delta b_1 \geq 0$ $\qquad$ $\Delta b_1 \geq (16/30)(-252) = -134.4$

$120 - 15/16\ \Delta b_1 \geq 0$ $\qquad$ $\Delta b_1 \leq (16/15)(-120) = 128$

$540 - 20/16\ \Delta b_1 \geq 0$ $\qquad$ $\Delta b_1 \leq (16/20)(-540) = 432$

$18 - 11/32\ \Delta b_1 \geq 0$ $\qquad$ $\Delta b_1 \leq (32/11)(-18) = 52.364$

Since all the inequalities must be satisfied, the most restrictive on b_1 must be satisfied in order for all the current basic variables to remain nonnegative. Therefore, Δb_1 must satisfy

$$-134.4 \leq \Delta b_1 \leq 52.364$$

or

$$\mathbf{495.6 \leq b_1 \leq 682.364}$$

Range of feasibility for b_3

$$\begin{bmatrix} 252 \\ 120 \\ 540 \\ 18 \end{bmatrix} + \Delta b_3 \begin{bmatrix} -21/16 \\ 5/32 \\ 30/16 \\ 9/64 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$252 - 21/16\ \Delta b_3 \geq 0$ $\quad\quad$ $\Delta b_3 \leq (16/21(252) = 192$

$120 + 5/32\ \Delta b_3 \geq 0$ $\quad\quad$ $\Delta b_3 \geq (32/5)(-120) = -768$

$540 + 30/16\ \Delta b_3 \geq 0$ $\quad\quad$ $\Delta b_3 \geq (16/30)(-540) = -288$

$18 + 9/64\ \Delta b_3 \geq 0$ $\quad\quad$ $\Delta b_3 \geq (64/9)(-18) = -128$

Since all the inequalities must be satisfied, the most restrictive on b_3 must be satisfied in order for all the current basic variables to remain nonnegative. Therefore, Δb_3 must satisfy

$$-128 \leq \Delta b_3 \leq 192$$

or

$$\mathbf{580 \leq b_3 \leq 900}$$

Constraint	**Allowable minimum**	**Allowable maximum**
1	**495.6**	**682.4**
2	**480.0**	**no upper limit ($\propto$)**
3	**580.0**	**900.0**
4	**117.0**	**No upper limit($\propto$)**