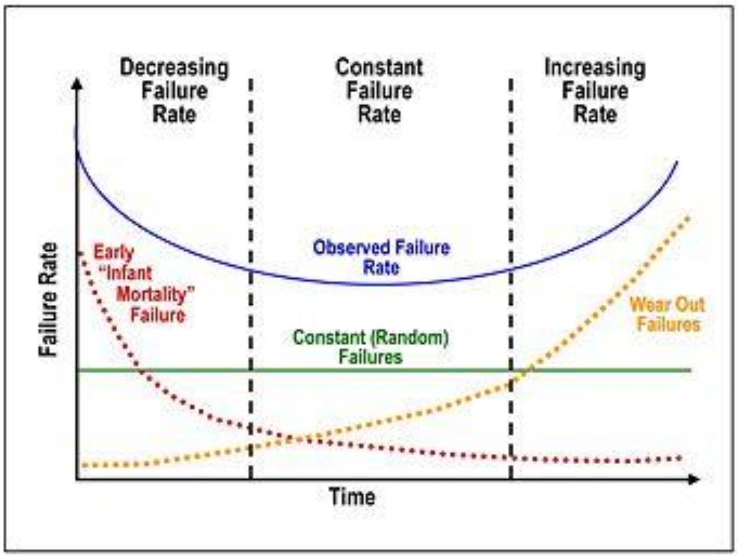
Weibull Distribution

The Weibull distribution is often used to model the time until failure of many different physical systems. The parameters in the distribution provide a great deal of flexibility to model systems in which the number of failures increases with time (bearing wear), decreases with time (some semiconductors), or remain constant with time(failures caused by external shocks to the system).



Weibull Distribution

The random variable X with probability density function

$$f(x)=\frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1}\exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right],\qquad x>0,\alpha,\beta>0$$

is a Weibull random variable with scale parameter α and shape parameter β.

Cumulative Distribution Function

If X has a Weibull distribution with parameters α and β, then the cumulative distribution function of X is

$$F(x)=1-\exp\left[-\left(\frac{x}{\alpha}\right)^{\beta}\right],\qquad x>0,\alpha,\beta>0$$

Mean and Variance

If X has a Weibull distribution with parameters α and β then,

$$E(X) = \alpha\Gamma\left(1+\frac{1}{\beta}\right)$$

$$V(X) = \alpha^2\Gamma\left(1+\frac{2}{\beta}\right) - \alpha^2\left[\Gamma\left(1+\frac{2}{\beta}\right)\right]^2$$

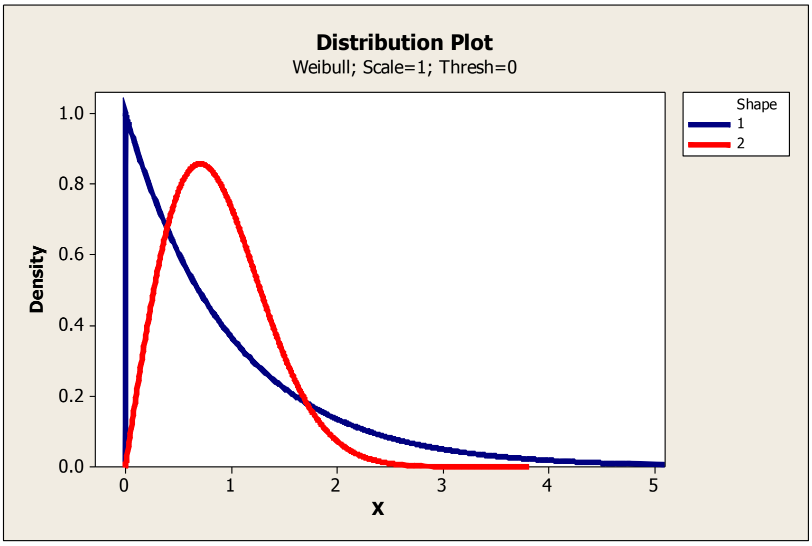
Gamma Function

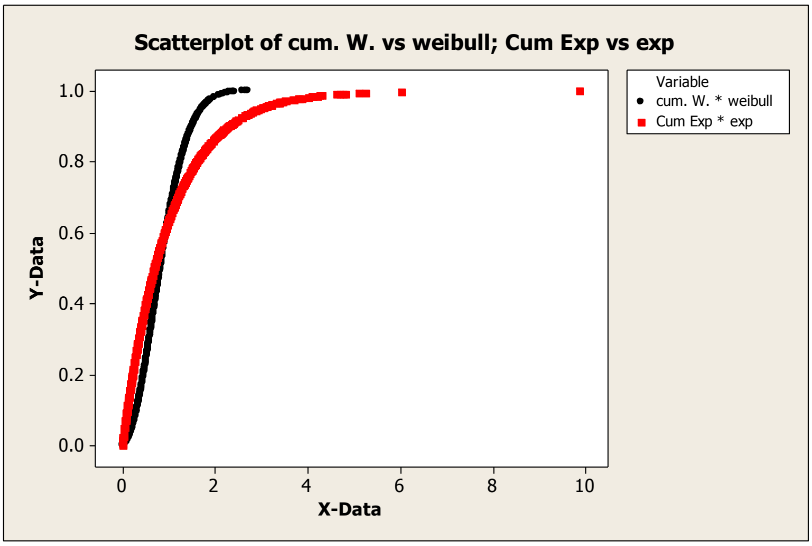
$$\Gamma(r) = \int_0^\infty x^{r-1}e^{-x}dx, r > 0$$

If r is a positive integer, then

$$\Gamma(r) = (r-1)!$$

Flexibility of the Weibull Distribution

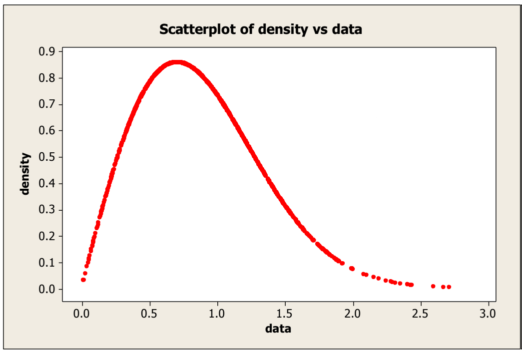


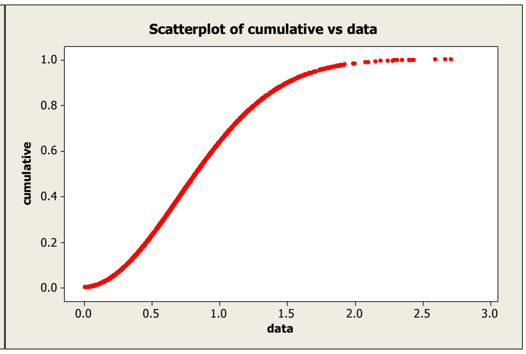


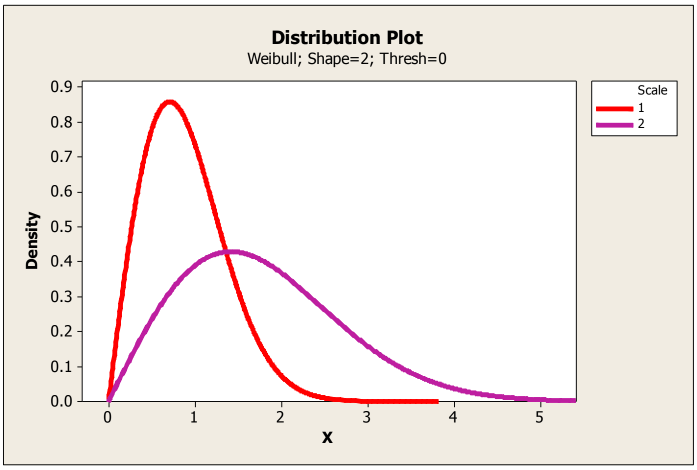
Exponential and Weibull

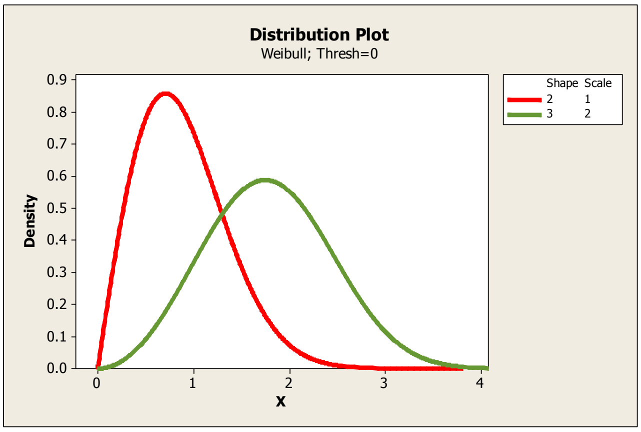
By inspecting the probability density function, it is easily seen that when β=1, the Weibull distribution is identical to the exponential distribution.

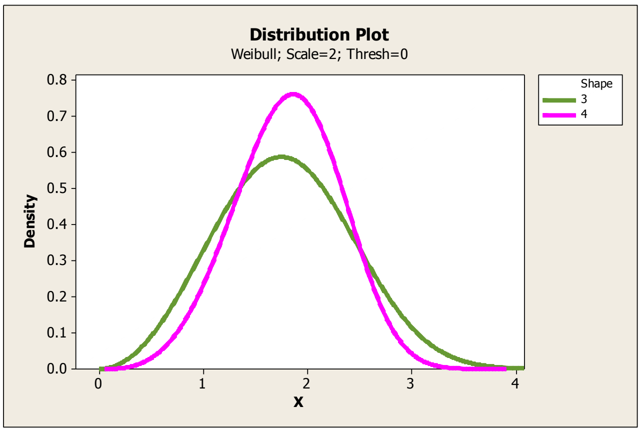
Weibull Distribution with different parameters











Example:

The time to failure (in hours) of a bearing in a mechanical shaft is satisfactorily modeled as a β=0.5 α=5000 hours.

i. Determine the mean time until failure.

$$\mu = E(X) = \alpha\Gamma\left(1+\frac{1}{\beta}\right) = 5000\Gamma(3) = 10000 hours$$

ii. What is the probability that a bearing lasts at least 6000 hours?

$$P(X > 6000) = 1 - F(6000) = \exp\left[-\left(\frac{6000}{5000}\right)^{1/2}\right]$$

$$= \exp(-1.095) = 0.334$$