Elementary Row Operations

The basic method for solving a system of linear equations is to replace the given system by a new system that has the same solution set but is easier to solve. This new system is generally obtained by applying the following three types of operations to eliminate unknowns systematically:

1. **Multiply an equation through by a nonzero constant.**
2. **Interchange two equations.**
3. **Add a multiple of one equation to another row.**

Augmented Matrix

General Linear Form

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \ldots. + a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + \ldots. + a_{2n}x_n = b_2 \\
&. \\
&. \\
&. \\
&a_{m1}x_1 + a_{m2}x_2 + \ldots. + a_{mn}x_n = b_m
\end{aligned}
$$

The matrix form of a linear system

$$
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
. & . & & . \\
. & & & \\
. & & & \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
\begin{bmatrix}
x_1 \\ x_2 \\ . \\ . \\ . \\ x_n
\end{bmatrix}
=
\begin{bmatrix}
b_1 \\ b_2 \\ . \\ . \\ . \\ b_m
\end{bmatrix}
$$

$$Ax = b$$

Corresponded Augmented Matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ . & & & & \\ . & . & & . & \\ . & & & & \\ . & & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

This is called the ***augmented matrix*** for the system

Example: The augmented matrix for the system of equations

$$\begin{aligned} x_1 + \quad x_2 + 2x_3 &= 9 \\ 2x_1 + 4x_2 - 3x_3 &= 1 \\ 3x_1 + 6x_2 - 5x_3 &= 0 \end{aligned}$$

is

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Example: (MATLAB)

Use MATLAB to construct the augmented matrix for the linear system

$$\begin{aligned} x_1 + \quad x_2 + 2x_3 &= 9 \\ 2x_1 + 4x_2 - 3x_3 &= 1 \\ 3x_1 + 6x_2 - 5x_3 &= 0 \end{aligned}$$

```
>> A=[1 1 2; 2 4 -3;3 6 -5]
A =
   1   1   2
   2   4  -3
   3   6  -5
>> B=[9 1 0]'
B =
   9
   1
   0
>> Aug=[A B]
Aug =
   1   1   2   9
   2   4  -3   1
   3   6  -5   0
>>
```

Since the rows (the horizontal lines) of an *augmented* matrix correspond to the equations in the associated system, these three operations correspond to the following operations on the rows of the augmented matrix:

1. **Multiply a row through by a nonzero constant.**
2. **Interchange two rows.**
3. **Add a multiple of one row to another row.**

These are called elementary row operations.

1.Scaling

2.Interchanging

3.Replacement

Example:

$$\begin{aligned} x \;+y+2z &= 9 \\ 2x \;+4y-3z &= 1 \\ 3x \;+6y-5z &= 0 \end{aligned} \qquad \begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Add -2 times the ***first equation*** to the ***second***

Add -2 times the ***first row*** to ***the second***

$$\begin{aligned} x \;+y+2z &= 9 \\ 2y-7z &= -17 \\ 3x \;+6y-5z &= 0 \end{aligned} \qquad \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Add -3 times the ***first equation*** to the **third**

Add -3 times the ***first row*** to the ***third***

$$\begin{aligned} x \;+y+2z &= 9 \\ 2y-7z &= -17 \\ 3y-11z &= -27 \end{aligned} \qquad \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

Multiply the ***second equation* by ½**

Multiply the ***second row* by ½**

$$\begin{aligned} x \;+ y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ 3y - 11z &= -27 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

Add -3 times the ***second equation*** to the ***third***

Add -3 times the ***second row*** to the ***third***

$$\begin{aligned} x \;+ y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ -\tfrac{1}{2}z &= -\tfrac{3}{2} \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{bmatrix}$$

Multiply the ***third equation*** **by -2**

Multiply the ***third row*** **by -2**

$$\begin{aligned} x \;+ y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ z &= 3 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Row Echelon Form

Add -1 times the ***second equation*** to the ***first***

Add -1 times the ***second row*** to the ***first***

$$\begin{aligned} x + \tfrac{11}{2}z &= \tfrac{35}{2} \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ z &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & \frac{11}{2} & \frac{35}{2} \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Add -11/2 times the ***third equation*** to the ***first***

Add -11/2 times the ***third row*** to the ***first***

$$\begin{aligned} x + \quad &= 1 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ z &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Add 7/2 times the ***third equation*** to the ***second***

Add 7/2 times the ***third row*** to the ***second***

$$\begin{aligned} x + \quad &= 1 \\ y \quad &= 2 \\ z &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The solution is $x = 1, y = 2, z = 3$.

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Reduced Row Echelon Form

Reduced Row Echelon Form (Row Canonical Form)

In mathematics, a matrix is in reduced row echelon form (also known as row canonical form - the resulting matrix is sometimes called a Hermite matrix) if it satisfies the following requirements:

- ***All nonzero rows are above any rows of all zeroes.***
- ***The leading coefficient of a row is always to the right of the leading coefficient of the row above it.***
- ***All leading coefficients are 1.***
- ***All entries above a leading coefficient in the same column are zero.***

Gaussian Elimination

Karl Friedrich Gauss (1777-1835) ***was a German mathematician and scientist.***

The way to understand elimination is by example. We begin in three dimensions:

Original System $$\begin{aligned} 2x \; + y + z \; &= 5 \\ 4x \; - 6y \; &= -2 \\ -2x \; + 7y + 2z &= 9 \end{aligned}$$

The problem is to find the unknown values of x, y and z, and we shall apply Gaussian elimination.

$$\begin{aligned} 2x \; + y + z \; &= 5 \\ 4x \; - 6y \; &= -2 \\ -2x \; + 7y + 2z &= 9 \end{aligned}$$

The method starts by subtracting multipliers of the first equation from the other equations. <u>The goal is to eliminate x from the last two equations</u>.

$$
\begin{aligned}
2x \quad + y + z \quad &= 5 \\
4x \quad - 6y \quad &= -2 \\
-2x \quad + 7y + 2z &= 9
\end{aligned}
$$

This requires that we

- **Subtract 2 times the first equation from the second,**
- **Subtract -1 times the first equation from the third.**

Equivalent System

$$
\begin{aligned}
2x \quad + y + z \quad &= 5 \\
-8y - 2z \quad &= -12 \\
8y + 3z \quad &= 14.
\end{aligned}
$$

$$
\begin{aligned}
2x \ + y + z \quad &= 5 \\
-8y - 2z \ &= -12 \\
8y + 3z \ &= 14.
\end{aligned}
$$

The coefficient **2** is the **first pivot.**

We know <u>ignore the first equation.</u>

The **pivot for the second stage of elimination** is **-8**.

- **Subtract -1 times the second equation from the third.**

The elimination process is now complete, at least in the "**forward**" direction:

<u>Triangular System</u>

$$
\begin{aligned}
2x \ + y + z \quad &= 5 \\
-8y - 2z \ &= -12 \\
z \ &= 2.
\end{aligned}
$$

$$\begin{aligned} 2x \;+ y + z \quad &= 5 \\ -8y - 2z \;&= -12 \\ z \;&= 2. \end{aligned}$$

This system is solved backward, **bottom to top**.

The last equation gives $z = 2$. Substituting into the second equation, we find $y = 1$. Then the first equation gives $x = 1$. This process is called **back-substitution**.

$$\begin{bmatrix} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & 12 \\ 0 & 8 & 3 & 14 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

At the end is the **triangular** system, ready for **back-substitution**.

In a larger problem, forward elimination takes most of the effort.

Upper-Triangular Linear Systems

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots. + a_{1n}x_n &= b_1 \\
a_{22}x_2 + \ldots. + a_{2n}x_n &= b_2 \\
&\;\cdot \\
a_{nn}x_n &= b_n
\end{aligned}
$$

Theorem: (Back Substitution) *Suppose that AX=B is an upper-triangular system.*

If $a_{kk} \neq 0$ ***for n then there exists a unique solution to the system.***

The last equation involves only x_n***, so we solve it first:***

$$x_n = \frac{b_n}{a_{nn}}.$$

Now x_n ***is known and it can be used in the next-to-last equation:***

$$x_{n-1} = \frac{b_{n-1} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$

Similarly

$$x_{n-2} = \frac{b_{n-2} - a_{n-2,n-1}x_{n-1} - a_{n-2,n}x_n}{a_{n-2,n-2}}$$

The general step is

$$x_k = \frac{b_k - \sum_{j=k+1}^{n} a_{kj}x_j}{a_{kk}} \quad k = n-1, n-2, \ldots, 1$$

Example

Use back substitution to solve the linear system

$$\begin{aligned} 4x_1 - x_2 + 2x_3 + 3x_4 &= 20 \\ -2x_2 + 7x_3 - 4x_4 &= -7 \\ 6x_3 + 5x_4 &= 4 \\ 3x_4 &= 6 \end{aligned}$$

$$x_4 = \frac{6}{3} = 2$$

$$x_3 = \frac{4 - 5(2)}{6} = -1$$

$$x_2 = \frac{-7 - 7(-1) + 4(2)}{-2} = -4$$

$$x_1 = \frac{20 + 1(-4) - 2(-1) - 3(2)}{4} = 3$$

Example

Show that there is <u>no solution</u> to the linear system

$$\begin{aligned} 4x_1 - x_2 + 2x_3 + 3x_4 &= 20 \\ 0x_2 + 7x_3 - 4x_4 &= -7 \\ 6x_3 + 5x_4 &= 4 \\ 3x_4 &= 6 \end{aligned}$$

Using the last equation, we must have $x_4 = 2$, which is substituted into the second and third equations to obtain

$$\begin{aligned} 7x_3 - 8 &= -7 \\ 6x_3 + 10 &= 4. \end{aligned}$$

The first equation implies that $x_3 = 1/7$, and the second equation implies that $x_3 = -1$. This ***contradiction*** leads to the conclusion that there is no solution to the given linear system. $a_{kk} \neq 0$ **?**

Example

Show that there are <u>infinitely many solutions</u> to the linear system

$$
\begin{aligned}
4x_1 - x_2 + 2x_3 + 3x_4 &= 20 \\
0x_2 + 7x_3 + 0x_4 &= -7 \\
6x_3 + 5x_4 &= 4 \\
3x_4 &= 6
\end{aligned}
$$

Using the last equation, we must have $x_4 = 2$, which is substituted into <u>the second and third equations</u> to get $x_3 = -1$, which checks out in both equations. When x_3 and x_4 are substituted into the first equation, the result is $x_2 = 4x_1 - 16$ whish has infinitely many solutions; hence the given system has ***<u>infinitely many solutions</u>***.

Example (Singular, Unsolvable)

Original System

$$\begin{aligned} x &+ y + z &&= - \\ 2x &+ 2y + 5z &&= - \\ 4x &+ 4y + 8z &&= - \end{aligned}$$

Equivalent System

$$\begin{aligned} x + y + z &= - \\ 3z &= - \\ 4z &= - \end{aligned}$$

- If the last two equations are $3z = 6$ and $4z = 7$, there is ***no solution***.
- If those two equations happen to be ***consistent*** as in $3z = 6$ and $4z = 8$ then this singular case has an infinity of solutions. We know that $z = 2$, but the first equation ***cannot decide both*** x and y.

Gaussian Elimination Algorithm

```
To solve a system of n linear equations Ax=b.
For j= to (n-1)
             pvt=|a(j,j)|
             pivot[j] = j
             ivpt_temp=j
     For i=j+1 to n (Find pivot row)
        IF  |a(i,j)| > pvt Then
          pvt=|a(i,j)|
          ivpt_temp=I
          END IF
      End For i
(Switch rows if necessary)
    IF pivot[j]<>ipvt_temp
   [switch_rows(rows j and ipvt_temp)]
    For i=j+1 to n (store multipliers)
         a[i,j]=a[i,j]/a[j,j]
    End For I
(Create zeros below the main diagonal)
For i=j+1 to n
           For k=j+1 to n
             a[i,k]= a[i,k]- a[i,j]*a[j,k]
            End For k

      b[i]= b[i]-a[i,j]*b[j]
End For i
End For j
(Back Substitution Part)
x[n]= b[n]/a[n,n]
    For j=n-1 Down to 1
          x[j]=b[j]
    For k=n Down to j+1
             x[j]=x[j]-x[k]*a[j,k]
    End For k
          x[j]=x[j]/a[j,j]
     End For j
```

MATLAB M-File
(Back Substitution)

```
function X=backsub(A,B)
%Input    - A is an n x n upper-triangular nonsingular matrix
%         - B is an n x 1 matrix
%Output - X is the solution to the linear system AX = B
%Find the dimension of B and initialize X
 n=length(B);
 X=zeros(n,1);
 X(n)=B(n)/A(n,n);

for k=n-1:-1:1
 X(k)=(B(k)-A(k,k+1:n)*X(k+1:n))/A(k,k);
End
```

$$x_k = \frac{b_k - \sum_{j=k+1}^{n} a_{kj} x_j}{a_{kk}} \qquad k = n-1, n-2, \ldots, 1$$

Example

Use back substitution to solve the linear system

$$
\begin{aligned}
4x_1 - x_2 + 2x_3 + 3x_4 &= 20 \\
-2x_2 + 7x_3 - 4x_4 &= -7 \\
6x_3 + 5x_4 &= 4 \\
3x_4 &= 6
\end{aligned}
$$

```
>> A=[4,-1,2,3;0,-2,7,-4;0 0 6 5; 0 0 0 3]

A =
     4    -1     2     3
     0    -2     7    -4
     0     0     6     5
     0     0     0     3

>> B=[20 -7  4  6]'

B =
    20
    -7
     4
     6

>> X=backsub(A,B)

X =
     3
    -4
    -1
     2
```

MATLAB M-File

(Upper Triangularization Followed by Back Substitution)

```
function X = uptrbk(A,B)

%Input    - A is an N x N nonsingular matrix
%         - B is an N x 1 matrix
%Output - X is an N x 1 matrix containing the solution to AX=B.

%Initialize X and the temporary storage matrix C
[N N]=size(A);
X=zeros(N,1);
C=zeros(1,N+1);
%Form the augmented matrix: Aug=[A|B]
Aug=[A B];

for p=1:N-1
   %Partial pivoting for column p
   [Y,j]=max(abs(Aug(p:N,p)));
   %Interchange row p and j
   C=Aug(p,:);
   Aug(p,:)=Aug(j+p-1,:);
   Aug(j+p-1,:)=C;
     if Aug(p,p)==0
      'A was singular.  No unique solution'
     break
  end
  %Elimination process for column p
  for k=p+1:N
     m=Aug(k,p)/Aug(p,p);
     Aug(k,p:N+1)=Aug(k,p:N+1)-m*Aug(p,p:N+1);
  end
end
%Back Substitution on [U|Y] using Program 3.1
X=backsub(Aug(1:N,1:N),Aug(1:N,N+1));
```

Example: **Solve the following system with upper-triangular system followed back substitution using MATLAB.**

$$
\begin{aligned}
x_1 + 2x_2 + x_3 + 4x_4 &= 13 \\
2x_1 + 0x_2 + 4x_3 + 3x_4 &= 28 \\
4x_1 + 2x_2 + 2x_3 + x_4 &= 20 \\
-3x_1 + x_2 + 3x_3 + 2x_4 &= 6
\end{aligned}
$$

```
>>  A=[1 2 1 4; 2 0 4 3 ;4 2 2 1; -3 1 3 2]

A =
     1     2     1     4
     2     0     4     3
     4     2     2     1
    -3     1     3     2

>> B=[13 28 20 6]'
B =
    13
    28
    20
     6

>> X=uptrbk(A,B)
X =
     3
    -1
     4
     2
>>
```

Gauss-Jordan Elimination

Wilhelm Jordan (1842-1899) ***was a German engineer who specialized in geodesy.***

Example: Gauss-Jordan Elimination

Solve by Gauss-Jordan elimination,

$$
\begin{aligned}
x_1 + 3x_2 - 2x_3 \quad + 2x_5 \quad &= 0 \\
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1 \\
5x_3 + 10x_4 \quad + 15x_6 &= 5 \\
2x_1 + 6x_2 \quad + 8x_4 + 4x_5 + 18x_6 &= 6
\end{aligned}
$$

The augmented matrix for the system is

$$
\begin{bmatrix}
1 & 3 & -2 & 0 & 2 & 0 & 0 \\
2 & 6 & -5 & -2 & 4 & -3 & -1 \\
0 & 0 & 5 & 10 & 0 & 15 & 5 \\
2 & 6 & 0 & 8 & 4 & 18 & 6
\end{bmatrix}
$$

Adding -2 times the first row to the second and fourth rows gives

$$
\begin{bmatrix}
1 & 3 & -2 & 0 & 2 & 0 & 0 \\
0 & 0 & -1 & -2 & 0 & -3 & -1 \\
0 & 0 & 5 & 10 & 0 & 15 & 5 \\
0 & 0 & 4 & 8 & 0 & 18 & 6
\end{bmatrix}
$$

Multiplying the second row by -1 then adding -5 times <u>the new second row</u> to the third row and -4 times <u>the new second row</u> to the fourth row gives

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{bmatrix}$$

Interchanging the third and fourth rows gives

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Multiplying the third row of the resulting matrix by 1/6 gives the row-echelon form

Row-echelon Form

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Adding -3 times the third row to the second row and then adding 2 times the second row of the resulting matrix to the first row yields the <u>reduced row-echelon form</u>

Reduced Row-echelon Form

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding system of equations is

$$
\begin{aligned}
x_1 + 3x_2 \qquad + 4x_4 + 2x_5 \qquad &= 0 \\
x_3 + 2x_4 \qquad &= 0 \\
x_6 &= 1/3
\end{aligned}
$$

(We have discarded the last equation)

Solving the leading variables, we obtain

$$
\begin{aligned}
x_1 &= -3x_2 - 4x_4 - 2x_5 \\
x_3 &= -2x_4 \\
x_6 &= 1/3
\end{aligned}
$$

Example:

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & 1 \end{bmatrix}$$

The first and second rows in the preceding matrix were interchanged.

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & 1 \end{bmatrix}$$

The first row of the preceding matrix was multiplied by ½.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & 1 \end{bmatrix}$$

-2 times the first row of the preceding matrix was added to the third row.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

Second row was multiplied by -½ to introduce a leading 1.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

-5 times the second row of the preceding matrix was added to the third row to introduce a zero below the leading 1.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & 1/2 & 1 \end{bmatrix}$$

Third row was multiplied by 2 to introduce a leading 1.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The entire matrix is now in row-echelon form.
To find the reduced row echelon form we need the following additional step.

7/2 times the third row of the preceding matrix was added to second row.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

-6 times the third row was added to the first row.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

5 times the second row was added to the first row.

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The last matrix is in reduced row-echelon form.

- ***If we use only the first five steps, the above procedure produces a <u>row-echelon</u> form and is called Gaussian Elimination.***

- ***Carrying the procedure through to additional steps and producing a matrix in <u>reduced row-echelon</u> form is called Gauss-Jordan elimination.***

What is the difference between the Gauss and Gauss-Jordan elimination methods?

Gauss creates zeros only below the pivot element, while Gauss-Jordan creates zeros <u>both above and below.</u>

Gauss Elimination $\rightarrow$ Row-echelon form

Gauss-Jordan Elimination $\rightarrow$ Reduced Row-echelon form

It is sometimes preferable to solve a system of linear equations by using Gaussian elimination to bring the augmented matrix into <u>row-echelon</u> form without continuing all the way to the reduced row-echelon form. When this is done, the corresponding system of equations can be solved by a technique called <u>back-substitution</u>.

Example:

In the previous example we obtain row-echelon and reduced row-echelon forms respectively

$$\begin{aligned} x_1 + 3x_2 - 2x_3 \qquad + 2x_5 \qquad\qquad &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 \qquad &= -1 \\ 5x_3 + 10x_4 \qquad + 15x_6 &= 5 \\ 2x_1 + 6x_2 \qquad + 8x_4 + 4x_5 + 18x_6 &= 6 \end{aligned}$$

Row-echelon Form

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced Row-echelon Form

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Corresponding system (solved using reduced row echelon form)

$$\begin{aligned} x_1 + 3x_2 \quad\quad + 4x_4 + 2x_5 \quad\quad &= 0 \\ x_3 + 2x_4 \quad\quad &= 0 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

$$\begin{aligned} x_1 &= -3x_2 - 4x_4 - 2x_5 \\ x_3 &= -2x_4 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

From the computation in the previous example, a row-echelon form of the augmented matrix is

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

To solve the corresponding system of equations

$$\begin{aligned} x_1 + 3x_2 - 2x_3 \quad\quad + 2x_5 \quad\quad &= 0 \\ x_3 + 2x_4 \quad\quad + 3x_6 &= 0 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

We proceed as follows:

Step-1 Solve the equations for the leading variables.

$$x_1 = -3x_2 + 2x_3 - 2x_5$$
$$x_3 = 1 - 2x_4 - 3x_6$$
$$x_6 = 1/3$$

Step-2 Beginning with the bottom equation and working upward, successively substitute each equation into all the equation above it.

Substituting $x_6 = \frac{1}{3}$ into the second equation yields

$$x_1 = -3x_2 + 2x_3 - 2x_5$$
$$x_3 = -2x_4$$
$$x_6 = 1/3$$

Substituting $x_3 = -2x_4$ into the first equation yields

$$x_1 = -3x_2 - 4x_4 - 2x_5$$
$$x_3 = -2x_4$$
$$x_6 = 1/3$$

Step-3 Assign arbitrary values to the free variables, if any.

If we assign x_2, x_4 and x_5 the arbitrary values r, s and t respectively, the general solution is given by the formulas

General Solution

$$x_1 = -3r - 4s - 2t,$$
$$x_2 = r,$$
$$x_3 = -2s,$$
$$x_4 = s,$$
$$x_5 = t,$$
$$x_6 = 1/3.$$

This agrees with the solution obtained before.

$$x_1 = -3x_2 - 4x_4 - 2x_5$$
$$x_3 = -2x_4$$
$$x_6 = 1/3$$

Remark

It can be shown that every matrix has a unique reduced row-echelon form; that is, one will arrive at the same reduced row-echelon form for a given matrix no matter how the row operations are varied.

Remark

Since Gauss-Jordan elimination avoids the use of back-substitution, it would seem that this method would be the more efficient of the two methods we have considered. It can be argued that this statement is true for solving small systems by hand since Gauss-Jordan elimination actually involves less writing. However, for large systems of equations, it has been shown that the Gauss-Jordan elimination method requires about 50% more operations than Gaussian elimination.