

# Row and Column Vectors of an $m \times n$ Matrix

For an  $m \times n$  matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & & & \\ \cdot & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

the vectors

$$r_1 = \langle a_{11} \ a_{12} \ \dots \ a_{1n} \rangle$$

$$r_2 = \langle a_{21} \ a_{22} \ \dots \ a_{2n} \rangle$$

$\cdot$

$\cdot$

$$r_m = \langle a_{m1} \ a_{m2} \ \dots \ a_{mn} \rangle \text{ in } \mathbb{R}^n \text{ formed}$$

from the rows of  $A$  are called **the row vectors** of  $A$ ,

*The vectors*

$$c_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \cdot \\ \cdot \\ \cdot \\ a_{m1} \end{bmatrix}, c_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \cdot \\ \cdot \\ \cdot \\ a_{m2} \end{bmatrix}, \dots, c_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \cdot \\ \cdot \\ \cdot \\ a_{mn} \end{bmatrix},$$

in  $R^m$  formed from the columns of  $A$  are *called the column vectors* of  $A$ .

## Example Row and Column Vectors in a 2x3 Matrix

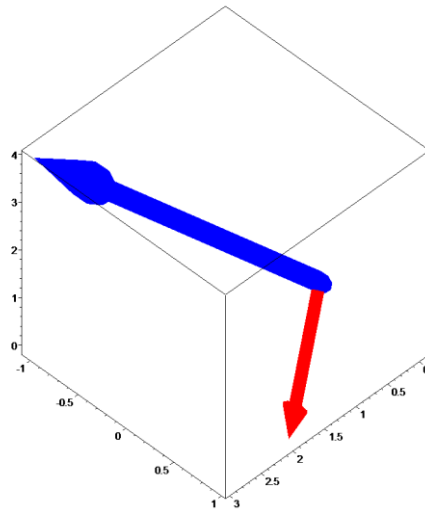
Let

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \end{bmatrix}$$

The row vectors of A are

$$r_1 = \langle 2 \ 1 \ 0 \rangle \quad \text{and} \quad r_2 = \langle 3 \ -1 \ 4 \rangle$$

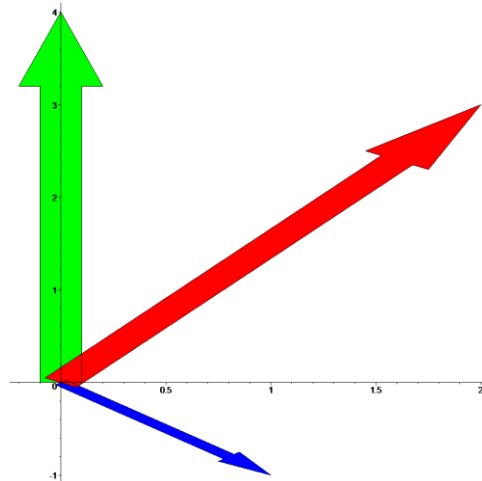
```
> with(VectorCalculus):  
> PlotVector([<2,1,0>, <3,-1,4>], color=[red, blue]);
```



and the column vectors of A are

$$c_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad c_3 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

```
> PlotVector([<2,3>, <1,-1>,<0 ,4>], color=[red, blue,green]);
```



## Row Space and Column Space

- Let  $A$  be an  $m \times n$  matrix, with row vectors  $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_m$ . A linear combination of these vectors is any vector of the form
- $c_1\mathbf{r}_1 + c_2\mathbf{r}_2 + \dots + c_m\mathbf{r}_m$ ,
- where  $c_1, c_2, \dots, c_m$  are constants. **The set of all possible linear combinations of  $\mathbf{r}_1, \dots, \mathbf{r}_m$  is called the row space of  $A$ .** That is, the row space of  $A$  is the span of the vectors  $\mathbf{r}_1, \dots, \mathbf{r}_m$ .
- For example, if
- $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$ ,
- then the row vectors are  $\mathbf{r}_1 = (1, 0, 2)$  and  $\mathbf{r}_2 = (0, 1, 0)$ . A linear combination of  $\mathbf{r}_1$  and  $\mathbf{r}_2$  is any vector of the form
- $c_1(1, 0, 2) + c_2(0, 1, 0) = (c_1, c_2, 2c_1)$ .

***If  $A$  is a  $m \times n$  matrix, then***

- ***the subspace of  $R^n$  spanned by the row vectors called the row space of  $A$ .***

Let  $A$  be an  $m \times n$  matrix, with column vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ . A linear combination of these vectors is any vector of the form

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_n \mathbf{v}_n,$$

where  $c_1, c_2, \dots, c_n$  are scalars. **The set of all possible linear combinations of  $\mathbf{v}_1, \dots, \mathbf{v}_n$  is called the column space of  $A$ .** That is, the column space of  $A$  is the span of the vectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$ .

Example

If  $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}$ , then the column vectors are  $\mathbf{v}_1 = (1, 0, 2)$  and  $\mathbf{v}_2 = (0, 1, 0)$ .

A linear combination of  $\mathbf{v}_1$  and  $\mathbf{v}_2$  is any vector of the form

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ 2c_1 \end{bmatrix}$$

The set of all such vectors is the column space of  $A$ .

*the subspace of  $R^m$  spanned by the column vectors of  $A$  is called the column space of  $A$ .*

**What relationships exist between the solutions of a linear system  $Ax=b$  and the row space and column space of the coefficient matrix  $A$ ?**

**Suppose that**

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & & & \\ \cdot & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$$

If  $c_1, c_2, \dots, c_n$  denote the column vectors of  $A$ , then the product  $Ax$  can be expressed as a linear combinations of these column vectors with coefficients from  $x$ ; that is,

$$Ax = x_1 c_1 + x_2 c_2 + \dots + x_n c_n$$

$$Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Thus a linear system,  $Ax=b$  of  $m$  equations in  $n$  unknowns can be written as

$$x_1c_1 + x_2c_2 + \dots + x_nc_n = b$$

from which we conclude that  $Ax=b$  is consistent if and only if  $b$  is **expressible** as a linear combination of the column vectors of  $A$  or, **equivalently**, if and only if  $b$  is in the column space of  $A$ .

**Theorem:** A system of linear equation  $Ax=b$  is consistent if and only if  $b$  is in the column space of  $A$ .



**Example (A Vector  $\mathbf{b}$  in the Column Space of  $\mathbf{A}$ )**

Let  $\mathbf{Ax}=\mathbf{b}$  be the linear system

$$\begin{aligned} -x_1 + 3x_2 + 2x_3 &= 1 \\ x_1 + 2x_2 - 3x_3 &= -9 \\ 2x_1 + x_2 - 2x_3 &= -3 \end{aligned}$$

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

**Show that  $\mathbf{b}$  is in the column space of  $\mathbf{A}$ , and express  $\mathbf{b}$  as linear combination of the column vectors of  $\mathbf{A}$ .**

**Solving the system by Gaussian elimination yields**

$$x_1 = 2, x_2 = -1, x_3 = 3$$

Since the system is consistent,  $b$  is in the column space of  $A$ .

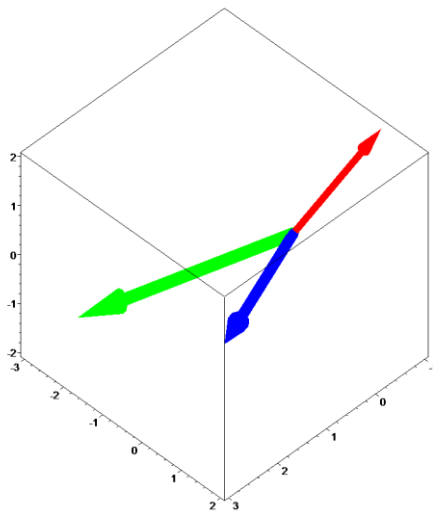
$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

$$x_1 c_1 + x_2 c_2 + \dots + x_n c_n = b$$

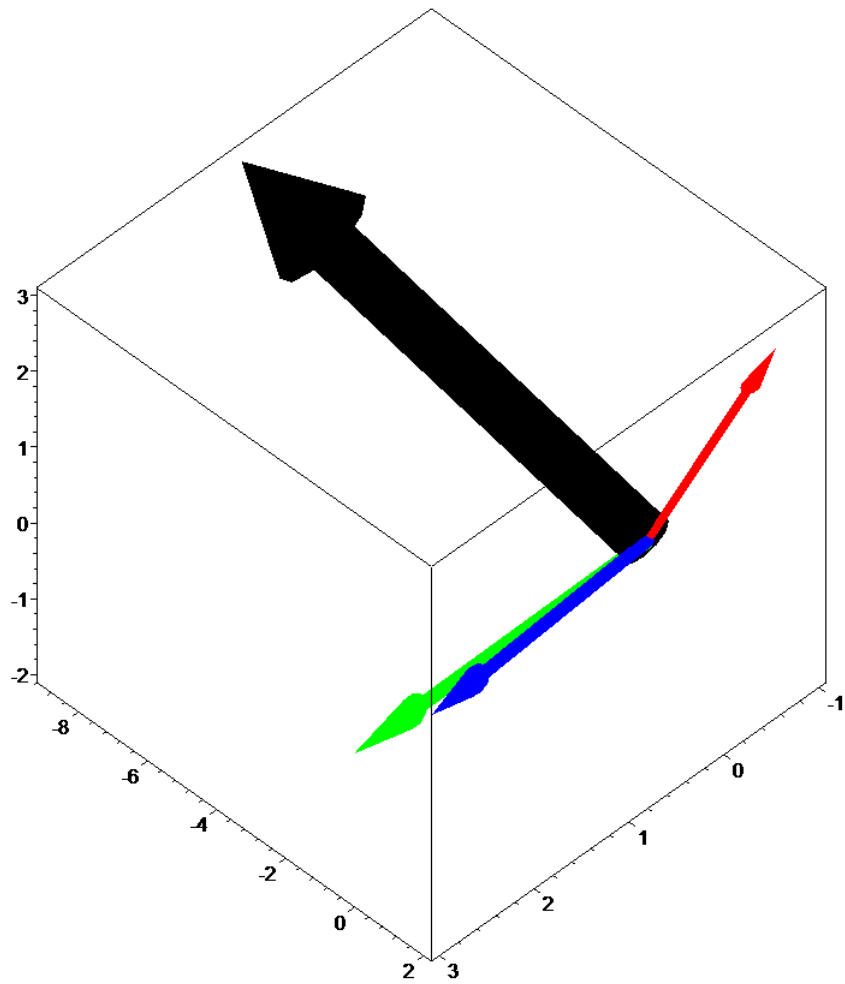
It follows that

$$2 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

```
> PlotVector([<-1,1,2>, <3,2,1>,<2,-3,-2>], color=[red,  
blue,green]);
```



```
> PlotVector([<-1,1,2>, <3,2,1>,<2,-3,-2>,<1,-9,3>],  
color=[red, blue,green,black]);
```



**b is in the column space of A.**

## Example

$$\begin{aligned}x_1 + 3x_2 - 2x_3 + 2x_5 &= 0 \\2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1 \\5x_3 + 10x_4 + 15x_6 &= 5 \\2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 &= 6\end{aligned}$$

## General Solution

$$\begin{aligned}x_1 &= -3r - 4s - 2t, \\x_2 &= r, \\x_3 &= -2s, \\x_4 &= s, \\x_5 &= t, \\x_6 &= \frac{1}{3}.\end{aligned}$$

This result can be written in vector form as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -3r - 4s - 2t \\ r \\ -2s \\ s \\ t \\ 1/3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1/3 \end{bmatrix} + r \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$\xleftarrow{\mathbf{x}_0} \quad \xleftarrow{\mathbf{x}}$

The vector  $\mathbf{x}_0$  is a particular solution of the given non homogeneous system.

The linear combination  $\mathbf{x}$  is the general solution of the following homogeneous system.

$$\begin{aligned}
 x_1 + 3x_2 - 2x_3 + 2x_5 &= 0 \\
 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0 \\
 5x_3 + 10x_4 + 15x_6 &= 0 \\
 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 &= 0
 \end{aligned}$$

## General and Particular Solutions

The vector  $x_0$  is called a *particular solution* of  $Ax=b$ .

The expression

$$x_0 + c_1v_1 + c_2v_2 + \dots + c_kv_k$$

is called **the general solution of  $Ax=b$** .

The expression

$$c_1v_1 + c_2v_2 + \dots + c_kv_k$$

is called the **general solution of  $Ax=0$** .

The formula

$$x = x_0 + c_1v_1 + c_2v_2 + \dots + c_kv_k$$

*states that the general solution of  $Ax=b$  is the sum of any particular solution of  $Ax=b$  and the general solution of  $Ax=0$ .*

# Homogenous Linear Systems

A system of linear equations is said to be *homogenous* if the constant terms are all zero; that is, the system has the form

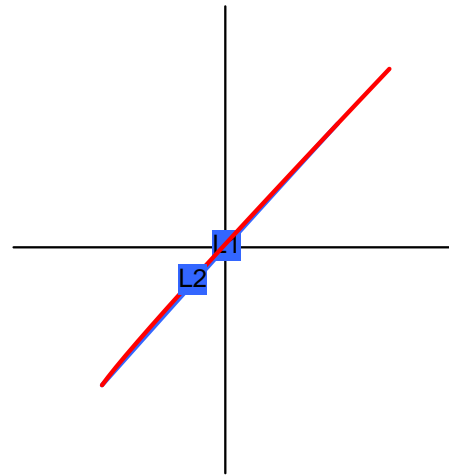
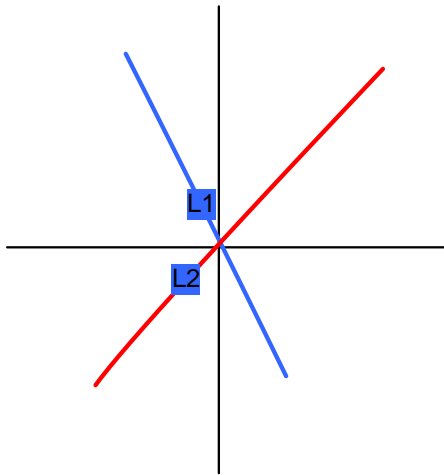
$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0 \\a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= 0 \\&\cdot \\&\cdot \\&\cdot \\a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= 0\end{aligned}$$



- Every homogeneous system of linear equation is consistent, since all such systems have  $x_1 = 0, x_2 = 0, \dots, x_m = 0$  as a solution. This solution is called the trivial solution; if there are other solutions, they are called nontrivial solutions.
- Because a homogeneous linear system always has the trivial solution, there are only two possibilities for its solutions:
  - The system has only the trivial solution.
  - The system has infinitely many solutions in addition to the trivial solution.

**In the special case of a homogenous linear system of two equations in two unknowns, say**

$$\begin{aligned}a_1x + b_1y &= 0 \quad (a_1, b_1 \text{ not both zero}) \\a_2x + b_2y &= 0 \quad (a_2, b_2 \text{ not both zero})\end{aligned}$$



**Only the trivial solution.      Infinitely many solutions**

## Example:

## Gauss-Jordan Elimination for homogenous system

Solve the following homogenous system of linear equations by using Gauss-Jordan elimination,

$$\begin{array}{rcl} 2x_1 + 2x_2 - x_3 & + x_5 & = 0 \\ -x_1 - x_2 + 2x_3 - 3x_4 + x_5 & & = 0 \\ x_1 + x_2 - 2x_3 & - x_5 & = 0 \\ & x_3 + x_4 + x_5 & = 0 \end{array}$$

**The augmented matrix for the system is**

$$\begin{bmatrix} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Reducing this matrix to **reduced row-echelon form**, we obtain

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding system of equations is

$$\begin{aligned} x_1 + x_2 - x_3 + x_5 &= 0 \\ x_3 + x_5 &= 0 \\ x_4 &= 0 \end{aligned}$$

Solving for the leading variables yields

$$\begin{aligned} x_1 &= -x_2 - x_5, \\ x_3 &= -x_5, \\ x_4 &= 0. \end{aligned}$$

Thus, the general solution is

$$x_1 = -s - t,$$

$$x_2 = s,$$

$$x_3 = -t$$

$$x_4 = 0$$

$$x_5 = t.$$

**Note:** The trivial solution is obtained when  $s = t = 0$ .

The solution vectors can be written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -s - t \\ s \\ -t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -s \\ s \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -t \\ 0 \\ -t \\ 0 \\ t \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

The solution space is **two dimensional** ( $v_1$  and  $v_2$ ) and the vectors

$$v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

form a **basis for nullspace**.

**Theorem:** *A homogeneous system of linear equations with more unknowns than equations has infinitely many solutions.*

Theorem applies only to homogeneous systems.

A nonhomogeneous system with more unknowns than equations need not be consistent; however, if the system is consistent, it will have infinitely many solutions.