Hypothesis Testing III

A LARGE-SAMPLE TEST ON THE MEAN OF A NORMAL DISTRIBUTION, VARIANCE UNKNOWN

We consider hypothesis testing about the mean μ of a single normal population where the variance of the population σ^2 is unknown.

Since the sampling distribution of the sample mean $\bar{X}$ is approximately normal when n is large , the standardized test statistic,

$$Z = \frac{\bar{X} - \mu_0}{s/\sqrt{n}}$$

which has approximate standard normal distribution when H_0 is true and $\mu = \mu_0$.

- **Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from a normal distribution with unknown mean μ and unknown variance σ^2.**

- **We know the sample mean $\bar{X}$ is normally distributed with mean μ and variance s^2/n.**

Hypothesis Tests on the Mean

Suppose that we wish to test the hypotheses

$$H_0: \quad \mu = \mu_0$$
$$H_1: \quad \mu \neq \mu_0$$

Test Statistic

$$Z = \frac{\bar{X} - \mu_0}{s/\sqrt{n}}$$

Distribution of Z when $H_0: \quad \mu = \mu_0$ is true

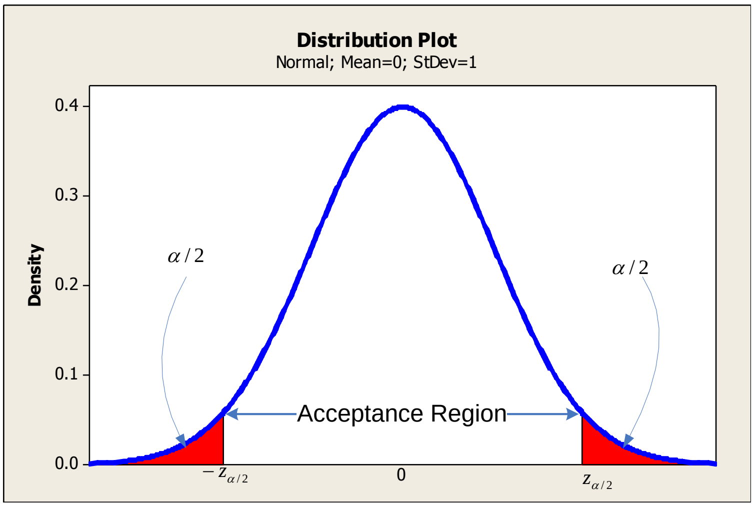
If the null hypothesis $H_0: \quad \mu = \mu_0$ is true, $E(\bar{X}) = \mu_0$, and it follows that the distribution of Z is the standard normal distribution. Consequently, if hypothesis $H_0: \quad \mu = \mu_0$ is true, the probability is 1-α that the test statistic Z falls between $-z_{\alpha/2}$ and $z_{\alpha/2}$, where $z_{\alpha/2}$is the $100\alpha/2$ percentage point of the standard normal distribution.

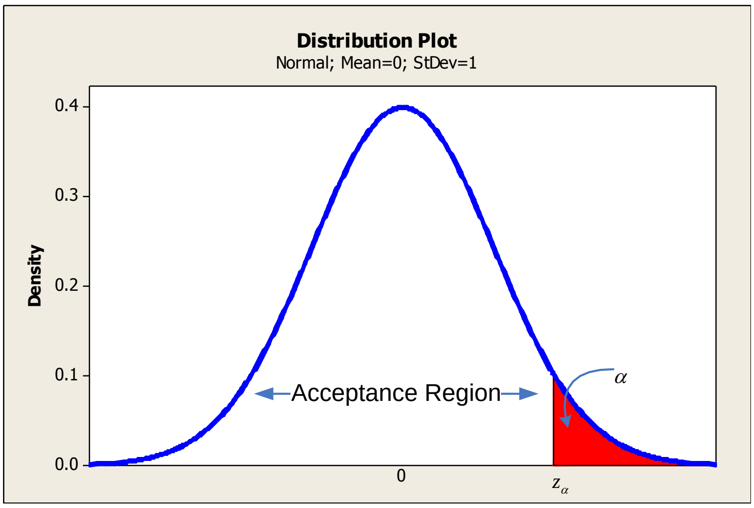
A sample producing a value of the test statistic that falls in the tails of the distribution of Z would be unusual if $H_0: \quad \mu = \mu_0$, therefore, it is an indication that H_0 is false . Thus we should reject H_0 if the observed value of the test statistic Z is either

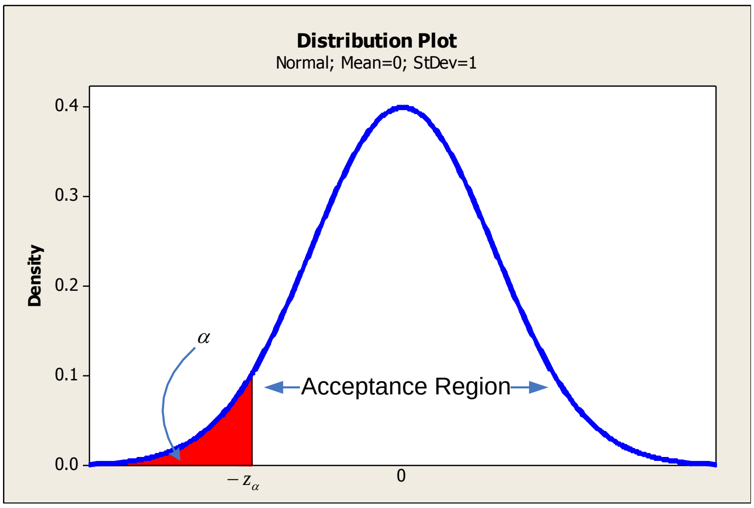
$$z > z_{\alpha/2} \ \textbf{or} \ \ z < -z_{\alpha/2}$$

and we should fail to reject H_0 if

$$-z_{\alpha/2} \le z \le z_{\alpha/2}$$


Distribution Plot
Normal; Mean=0; StDev=1
Density
0.4
0.3
0.2
0.1
0.0
α / 2
α / 2
Acceptance Region
$-z_{\alpha/2}$
0
$z_{\alpha/2}$


Distribution Plot
Normal; Mean=0; StDev=1
Density
0.4
0.3
0.2
0.1
0.0
α
Acceptance Region
0
z_{α}


Distribution Plot
Normal; Mean=0; StDev=1
Density
0.4
0.3
0.2
0.1
0.0
α
Acceptance Region
$-z_{\alpha}$
0

A SMALL-SAMPLE TEST ON THE MEAN OF A NORMAL DISTRIBUTION, VARIANCE UNKNOWN

We again consider hypothesis testing about the mean μ of a single normal population where the variance of the population σ^2 is unknown but our sample size is small. The situation is analogous to confidence interval on the mean for the same situation.

Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from a normal distribution with unknown mean μ and unknown variance σ^2, then the random variable (statistic)

$$T = \frac{\bar{X} - \mu}{s/\sqrt{n}}$$

has a t distribution with n-1 degrees of freedom.

We now consider testing the hypotheses

$$\begin{aligned} H_0 &: \quad \mu = \mu_0 \\ H_1 &: \quad \mu \neq \mu_0 \end{aligned}$$

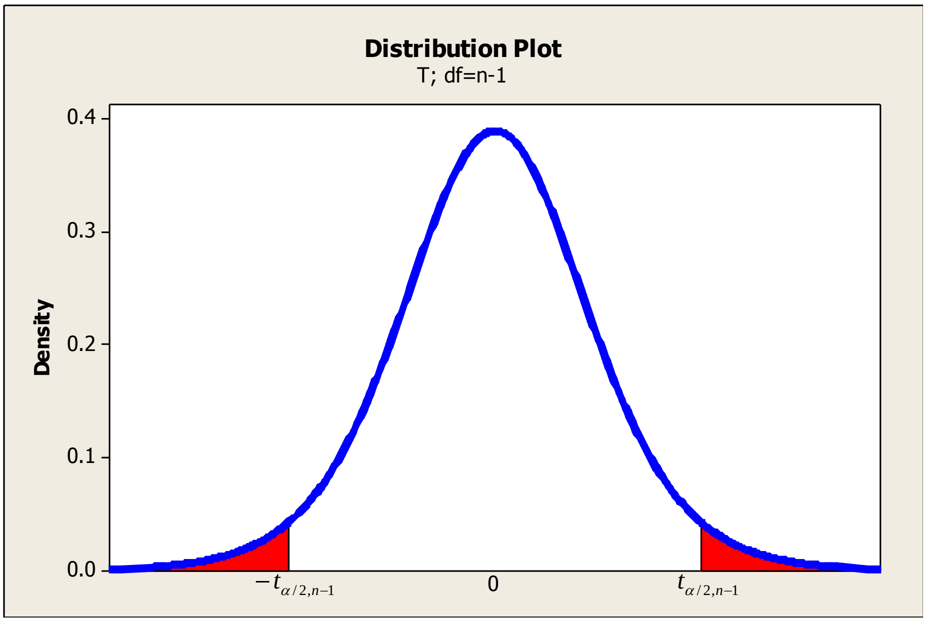
We will use the test statistic

$$T_0 = \frac{\bar{X} - \mu_0}{s/\sqrt{n}}$$

If the null hypothesis is true, T_0 has a T distribution with n-1 degrees of freedom.

In this case we would use the t percentage points $-t_{\alpha/2,n-1}$ and $t_{\alpha/2,n-1}$ as the boundaries of the critical region so that we would reject $H_0: \quad \mu = \mu_0$ if

$t_0 > t_{\alpha/2,n-1}$ or $t_0 < -t_{\alpha/2,n-1}$ where t_0 is the observed value of the test statistic T_0.

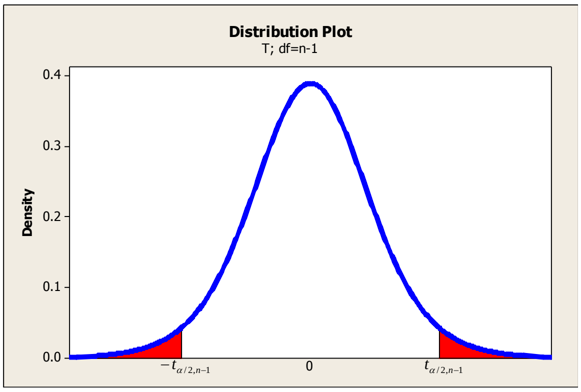


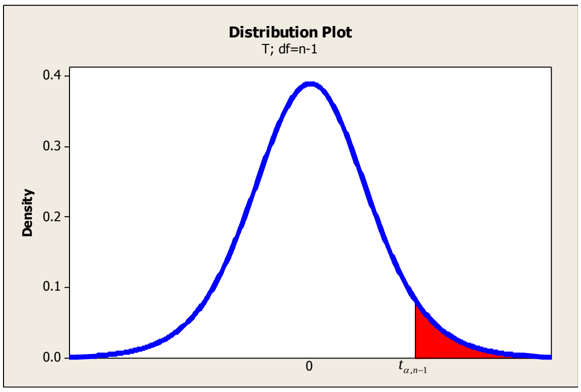
A summary of the test procedures for both two- and one-sided alternative hypotheses follows:

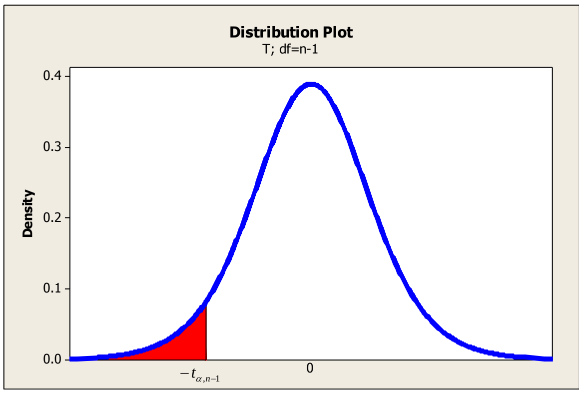
Null hypothesis: $H_0: \quad \mu = \mu_0$

Test statistic: $$t_0 = \frac{\overline{X} - \mu_0}{s/\sqrt{n}}$$

Alternative hypothesis	Rejection criteria
$H_1: \quad \mu \neq \mu_0$	$t_0 > t_{\alpha/2,n-1}$ **or** $t_0 < -t_{\alpha/2,n-1}$
$H_1: \quad \mu > \mu_0$	$t_0 > t_{\alpha,n-1}$
$H_1: \quad \mu < \mu_0$	$t_0 < -t_{\alpha,n-1}$







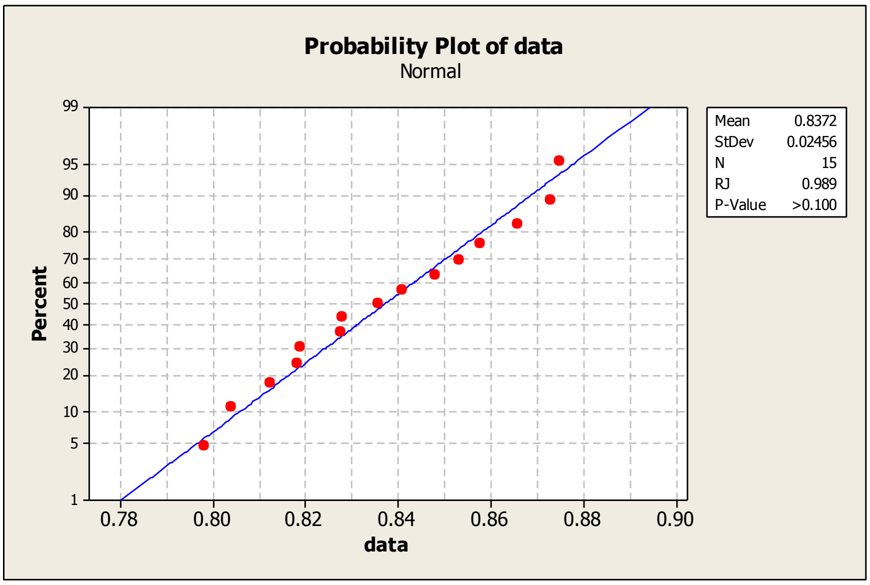
Example: **It is of interest to determine if there is evidence (with α=0.05) to support a claim that the mean coefficient of restitution exceeds 0.82. The observations follow:**

```
data
   0.8411 0.8191 0.8182 0.8125 0.8750 0.8580
   0.8532 0.8483 0.8276 0.7983 0.8042 0.8730
   0.8282 0.8359 0.8660
```

The sample mean and sample standard deviation are

$\bar{X} = 0.83725$ **and** $S = 0.02456$.

Assume that the distribution is normal.
The following normal probability plot of the data

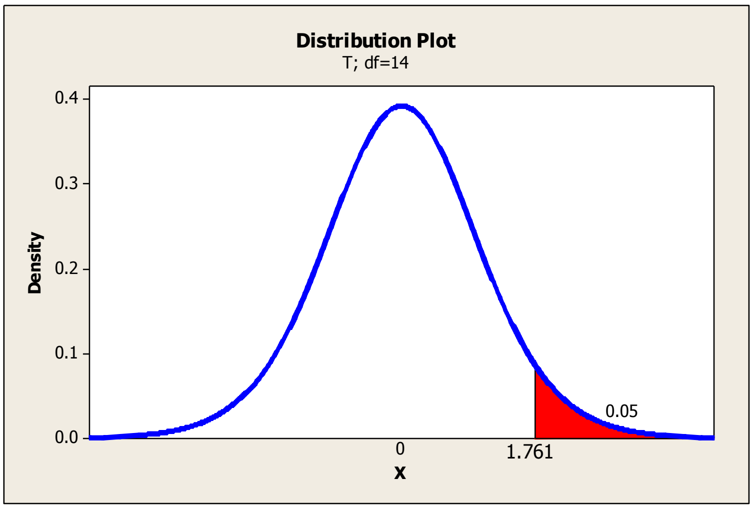


Since the objective of the experimenter is to demonstrate that the mean coefficient of restitution exceeds 0.82, a one-sided alternative hypothesis is appropriate.

- **The parameter of interest is the mean coefficient of restitution, μ.**
- $H_0: \quad \mu = 0.82$
- $H_1: \quad \mu > 0.82$
- **α=0.05**
- **The test statistic is**

$$t_0 = \frac{\bar{X} - \mu_0}{s/\sqrt{n}}$$

- **Reject H_0 if $t_0 > t_{0.05,14} = 1.761$**



- **Computations:**

Descriptive Statistics: data

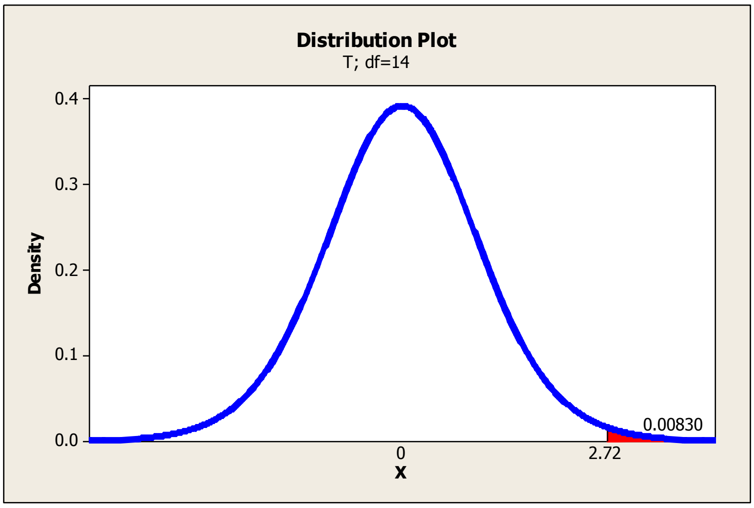
```
Variable    N  N*     Mean  SE Mean     StDev  Minimum       Q1    Median
Q3
data       15   0   0.83724 0.00634  0.02456  0.79830  0.81820  0.83590
0.85800

Variable  Maximum
data      0.87500
```

Since $\bar{X} = 0.83724$ **,** **s=0.02456** **,** $\mu_0 = 0.82$**, and n=15, we have**

$$t_0 = \frac{0.83725 - 0.83}{0.02456 / \sqrt{15}} = 2.72$$

- **Conclusions:** **Since the observed value of the test statistic falls in the rejection region, H_0 is rejected and we conclude that at the 0.05 level of significance that the mean coefficient of restitution exceeds 0.82.**



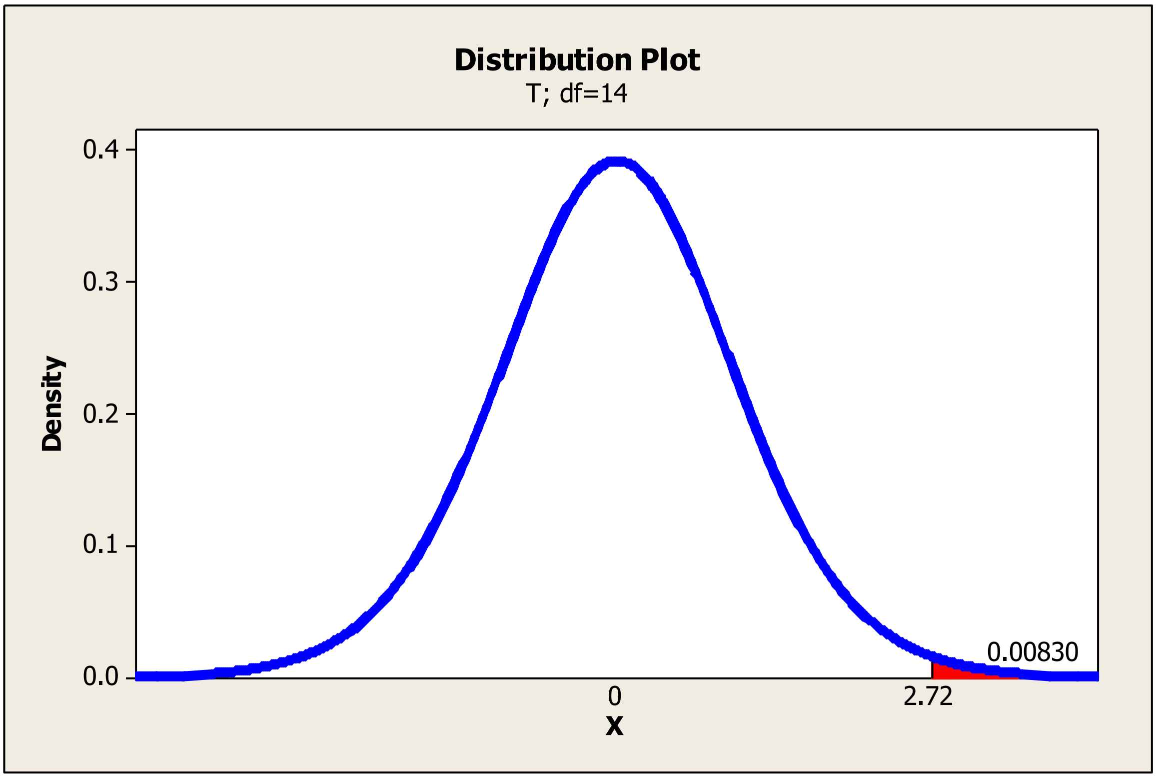
```
MTB > set c1
DATA> 0.8411 0.8191 0.8182 0.8125 0.8750 0.8580
DATA>    0.8532 0.8483 0.8276 0.7983 0.8042 0.8730
DATA>    0.8282 0.8359 0.8660
DATA> end
MTB > Onet 'Restitution Data';
SUBC>   Test 0.82;
SUBC>   Alternative 1.
```

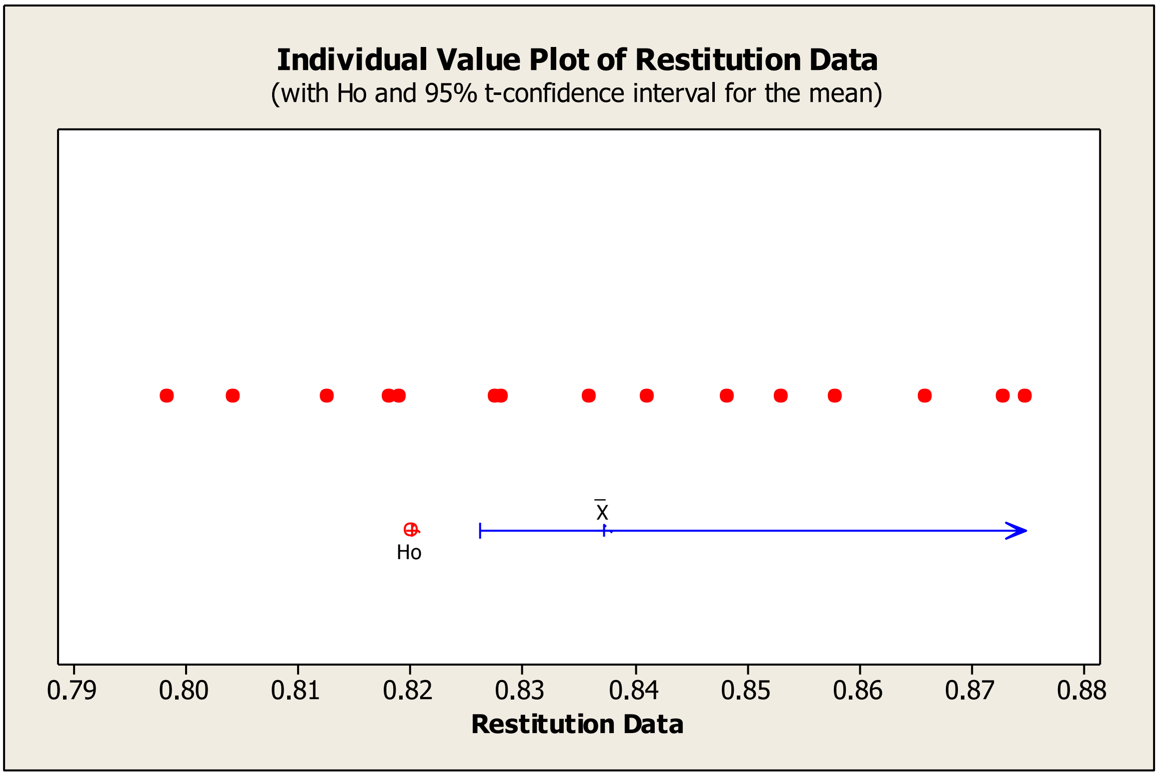
One-Sample T: Restitution Data

```
Test of mu = 0.82 vs > 0.82


                                                     95% Lower
Variable             N     Mean     StDev  SE Mean       Bound     T      P
Restitution Data    15  0.83724   0.02456  0.00634     0.82607  2.72  0.008
```

Minitab computes the p-value for the value $t_0=2.72$ as 0.008.





One-Sample T: Restitution Data

Variable	N	Mean	StDev	SE Mean	95% Lower Bound
Restitution Data	15	0.83724	0.02456	0.00634	0.82607

Notice that Minitab computes a 95% <u>lower confidence bound</u> for the coefficient of restitution.

$$\bar{X} - t_{\alpha,n-1}\, s/\sqrt{n} \quad *$$

$$0.83724 - (1.761)(0.02456)/\sqrt{15} = 0.82607$$

Because the 95% lower confidence bound exceeds 0.82, we would reject the hypothesis that $H_0{:}\mu=0.82$ and conclude that the alternative hypothesis $H_1{:}\mu>0.82$ is true.

*** Lower confidence Bound**

$$\bar{X} - t_{\alpha,n-1}\, s/\sqrt{n}$$

*** Upper confidence Bound**

$$\bar{X} + t_{\alpha,n-1}\, s/\sqrt{n}$$

$100(1-\alpha)\%$ ***CI on μ***

$$\bar{x} - t_{\alpha/2}\frac{s}{\sqrt{n}} \le \mu \le \bar{x} + t_{\alpha/2}\frac{s}{\sqrt{n}}$$

```
tint 95 c1
```

One-Sample T: C1

```
Variable    N      Mean     StDev   SE Mean
95% CI
C1         15   0.83724   0.02456   0.00634
(0.82364; 0.85084)
```

Power:

If the mean coefficient of restitution exceeds 0.82 by at least 0.02 compute the power of the test.

```
MTB > Power;
SUBC>   TOne;
SUBC>     Sample 15;
SUBC>     Difference 0.02;
SUBC>     Sigma 0.02456;
SUBC>     Alternative 1;
SUBC>   GPCurve.
```

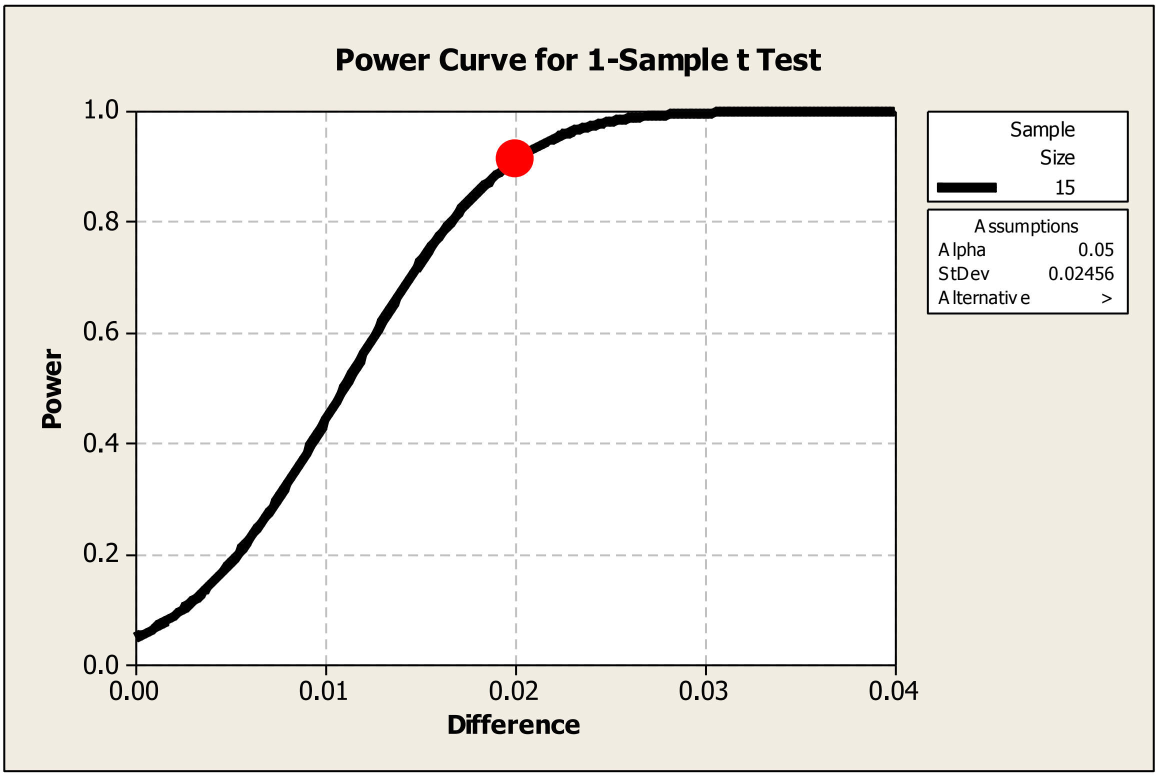
Power and Sample Size

```
1-Sample t Test

Testing mean = null (versus >
null)
Calculating power for mean = null
+ difference
Alpha = 0.05  Assumed standard
deviation = 0.02456

            Sample
Difference    Size      Power
      0.02      15   0.911696
```

Power Curve for 1-Sample t Test



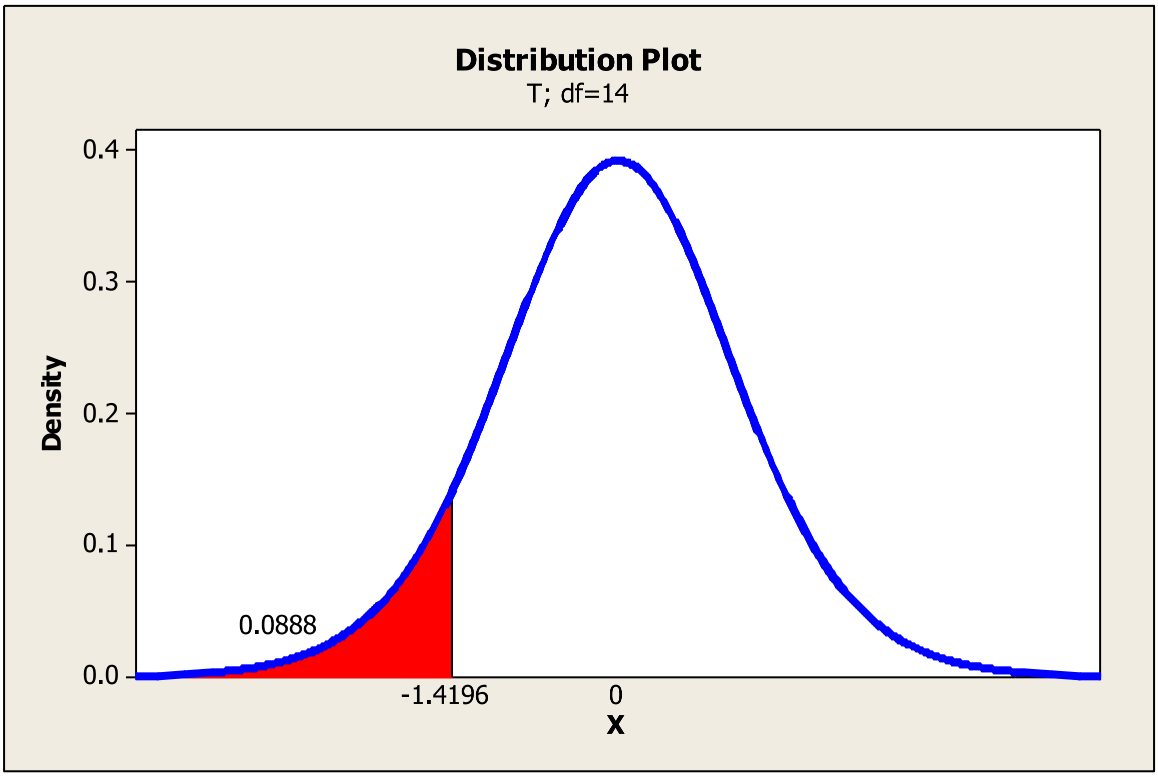
The acceptance region for the example is located $\mu_0 + 1.76\left(\frac{s}{\sqrt{n}}\right)$. Substituting numerical values, we get

$0.82 + 1.76\left(\frac{0.02456}{\sqrt{15}}\right)$ **or** $\cong 0.831$

Then

$$\begin{aligned}\beta &= P(accept\ \ H_0\ \ when\ \ \ \mu = 0.84)\\ &= P(\bar{X} < 0.831)\\ &= P\left(t < \frac{0.831-0.84}{0.02456/\sqrt{15}}\right)\\ &= P(t < -1.4196) \cong 0.0888\end{aligned}$$

$$\beta = 0.0888$$



Hence, the power of the test is

$$1-\beta = 0.911$$