THE TWO-PHASE SIMPLEX METHOD
(FULL SIMPLEX METHOD)

Recall that the simplex algorithm requires a starting bfs (initial basic feasible solution). In all the problems we have solved so far, we found a *initial basic feasible solution* by using the slack variables as our basic variables.

If an LP has any ≥ or equality (=) constraints, however, a starting bfs may not be readily apparent.

When a basic feasible solution is not readily available, the two-phase simplex method may be used. In the two-phase simplex method, we add **artificial variables** to the ≥ or = constraints. Then we find a bfs to the original LP by solving the Phase I LP. In the Phase I LP, the **objective function** is simply to minimize the **sum of all artificial variables.** At the completion of Phase I, we reintroduce the original LP's objective function and determine the optimal solution to the original LP.

Steps of the two-phase simplex method

Step 1 Modify the constraints so that the **right-hand side of each constraint is nonnegative.** This requires that each constraint with a negative right-hand side be multiplied through by –1.

Step 1’ **Identify** each constraint that is now(after step 1) an **equality or ≥ constraint**. In Step 3, we will add an artificial variable to each of these constraints.

Step 2 Convert each inequality constraint to the standard form . If constraint i is a ≤ constraint , **add** a **slack variable** s_i. If constraint i is a ≥ constraint , **subtract** an **excess variable** e_i.

Step 3 If(after step 1’) constraint i is a ≥ constraint or equality (=) constraint, add an **artificial variable** a_i to constraint i. Also add the sign restriction $a_i \geq 0$.

Step 4 For the time being, ignore the original LP’s objective function. Instead solve an LP whose objective function is

min w=(sum of all artificial variables).

This is called the **Phase I LP**.

The act of solving the Phase I LP will force the artificial variables to be zero.

Since each $a_i \geq 0$, solving the Phase I LP will result in one of the following three cases.

Case 1 The optimal value of w is greater than zero. In this case, the original LP **has no feasible solution (infeasible).**

Case 2 The optimal value of w is equal to zero, and no artificial variables are in the optimal Phase I basis. In this case, we **drop all columns** in the optimal Phase I tableau that correspond to the artificial variables. We now combine the original objective function with the constraints from the optimal Phase I tableau. This yields the **Phase II LP**. The optimal solution to the Phase II LP is the optimal solution to the original LP.

Case 3 The optimal value of w is equal to zero, and no artificial variables are in the optimal Phase I basis Sometimes, we can find the optimal solution to the original LP if at the end of Phase I we drop from the optimal Phase I Tableau all non-basic artificial variables and any variable from the original problem which has a negative coefficient in row 0 of the optimal Phase I tableau.

Example

$$\textbf{Max} \quad \mathbf{Z} = 3x_1 + 4x_2$$

s.t

$$\begin{aligned} x_1 \quad & \geq 10 \\ x_2 \quad & \geq 5 \\ x_1 + x_2 \quad & \leq 20 \\ -x_1 + 4x_2 \quad & \leq 20 \\ x_1, x_2 \quad & \geq 0 \end{aligned}$$

$$\begin{aligned} \mathbf{x_1} \quad & \mathbf{-e_1} && \mathbf{= 10} \\ \mathbf{x_2} \quad & \mathbf{-e_2} && \mathbf{= 5} \\ \mathbf{x_1 + x_2} \quad & \mathbf{+ s_1} && \mathbf{= 20} \\ \mathbf{-x_1 + 4x_2} \quad & \mathbf{+ s_2} && \mathbf{= 20} \end{aligned}$$

NBV=(x_1,x_2) BV=(e_1,e_2,s_1,s_2)= -10,-5,20,20 **no feasible starting solution**

Now, we add artificial variables;

$$\begin{aligned} \mathbf{x_1} \quad & \mathbf{-e_1 + a_1} && \mathbf{= 10} \\ \mathbf{x_2} \quad & \mathbf{-e_2 + a_2} && \mathbf{= 5} \\ \mathbf{x_1 + x_2} \quad & \mathbf{+ s_1} && \mathbf{= 20} \\ \mathbf{-x_1 + 4x_2} \quad & \mathbf{+ s_2} && \mathbf{= 20} \end{aligned}$$

e: **excess variable**

s : **slack variable**

a : **artificial variable**

or

$$\begin{aligned} \mathbf{x_1} \quad & \mathbf{-x_3 + x_7} && \mathbf{= 10} \\ \mathbf{x_2} \quad & \mathbf{-x_4 + x_8} && \mathbf{= 5} \\ \mathbf{x_1 + x_2} \quad & \mathbf{+ x_5} && \mathbf{= 20} \\ \mathbf{-x_1 + 4x_2} \quad & \mathbf{+ x_6} && \mathbf{= 20} \end{aligned}$$

NBV=(x_1, x_2, x_3, x_4) BV=(x_7, x_8, x_5, x_6)=10,5,20,20 basic feasible staring solution.

Artificial objective function

$$\mathbf{Min\ w = a_1 + a_2 = x_7 + x_8}$$

Note, however w contains the **basic variables** x_7 , x_8

- x_7, x_8 **must be eliminated** from w before we can solve Phase I.

$\mathbf{x_7 = 10 - x_1 + x_3}$

$\mathbf{x_8 = 5 - x_2 + x_4}$

$$w = \mathbf{10 - x_1 + x_3 + 5 - x_2 + x_4}$$

$$\mathbf{- w - x_1 - x_2 + x_3 + x_4 = -15}$$

$$\mathbf{Z - 3 x_1 - 4 x_2 = 0}$$

Initial Tableau for the Two-Phase Simplex Method

Leaving Variable **Entering variable**

BASIS	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS	RATIO
x_7	1<	0	-1	0	0	0	1	0	**10**	10<
x_8	0	1	0	-1	0	0	0	1	**5**	∞
x_5	1	1	0	0	1	0	0	0	**20**	20
x_6	-1	4	0	0	0	1	0	0	**20**	
-w	**-1<**	**-1**	**1**	**1**	**0**	**0**	**0**	**0**	**-15**	
Z	**-3**	**-4**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	

PHASE I

BASIS	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS	RATIO
x_7	1<	0	-1	0	0	0	1	0	**10**	10<
x_8	0	1	0	-1	0	0	0	1	**5**	∞
x_5	1	1	0	0	1	0	0	0	**20**	20
x_6	-1	4	0	0	0	1	0	0	**20**	
-w	**-1<**	**-1**	**1**	**1**	**0**	**0**	**0**	**0**	**-15**	
Z	**-3**	**-4**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	
x_1	1	0	-1	0	0	0	1	0	**10**	
x_8	0	1	0	-1	0	0	0	1	**5**	5<
x_5	0	1	1	0	1	0	-1	0	**10**	10
x_6	0	4	-1	0	0	1	1	0	**30**	7.5
-w	**0**	**-1<**	**0**	**1**	**0**	**0**	**1**	**0**	**-5**	
Z	**0**	**-4**	**-3**	**0**	**0**	**0**	**3**	**0**	**30**	
x_1	1	0	-1	0	0	0	1	0	**10**	
x_2	0	1	0	-1	0	0	0	1	**5**	
x_5	0	0	1	1	1	0	-1	-1	**5**	5
x_6	0	0	-1	4<	0	1	1	-4	**10**	2.5<
~~**-w**~~	~~**0**~~	~~**0**~~	~~**0**~~	~~**0**~~	~~**0**~~	~~**0**~~	~~**1**~~	~~**1**~~	~~**0**~~	
Z	**0**	**0**	**-3**	**-4<**	**0**	**0**	**3**	**4**	**50**	

PHASE II

BASIS	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS	RATIO
x_1	1	0	-1	0	0	0			**10**	
x_2	0	1	-1/4	0	0	1/4			**15/2**	
x_5	0	0	5/4<	0	1	-1/4			**5/2**	2<
x_4	0	0	-1/4	1	0	1/4			**5/2**	
Z	**0**	**0**	**-4<**	**0**	**0**	**1**			**60**	
x_1	1	0	0	0	4/5	-1/5			**12**	
x_2	0	1	0	0	1/5	1/5			**8**	5<
x_3	0	0	1	0	4/5	-1/5			**2**	10
x_4	0	0	0	1	1/5	1/5			**3**	7.5
Z	**0**	**0**	**0**	**0**	**16/5**	**1/5**			**68**	

x_1= 12 ; x_2= 8; x_3=2; x_4=3; x_5=x_6=x_7=x_8=0 Z_{max}=68

Two-Phase Simplex Method (Phase I and Phase II)

BASIS	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS	RATIO
x_7	1<	0	-1	0	0	0	1	0	**10**	10<
x_8	0	1	0	-1	0	0	0	1	**5**	∞
x_5	1	1	0	0	1	0	0	0	**20**	20
x_6	-1	4	0	0	0	1	0	0	**20**	
-w	**-1<**	**-1**	**1**	**1**	**0**	**0**	**0**	**0**	**-15**	
Z	**-3**	**-4**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	
x_1	1	0	-1	0	0	0	1	0	**10**	
x_8	0	1	0	-1	0	0	0	1	**5**	5<
x_5	0	1	1	0	1	0	-1	0	**10**	10
x_6	0	4	-1	0	0	1	1	0	**30**	7.5
-w	**0**	**-1<**	**0**	**1**	**0**	**0**	**1**	**0**	**-5**	
Z	**0**	**-4**	**-3**	**0**	**0**	**0**	**3**	**0**	**30**	
x_1	1	0	-1	0	0	0	1	0	**10**	
x_2	0	1	0	-1	0	0	0	1	**5**	
x_5	0	0	1	1	1	0	-1	-1	**5**	5
x_6	0	0	-1	4<	0	1	1	-4	**10**	2.5<
-w	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**0**	
Z	**0**	**0**	**-3**	**-4<**	**0**	**0**	**3**	**4**	**50**	
x_1	1	0	-1	0	0	0			**10**	
x_2	0	1	-1/4	0	0	1/4			**15/2**	
x_5	0	0	5/4<	0	1	-1/4			**5/2**	2<
x_4	0	0	-1/4	1	0	1/4			**5/2**	
Z	**0**	**0**	**-4<**	**0**	**0**	**1**			**60**	
x_1	1	0	0	0	4/5	-1/5			**12**	
x_2	0	1	0	0	1/5	1/5			**8**	5<
x_3	0	0	1	0	4/5	-1/5			**2**	10
x_4	0	0	0	1	1/5	1/5			**3**	7.5
Z	**0**	**0**	**0**	**0**	**16/5**	**1/5**			**68**	

$x_1=12$; $x_2=8$; $x_3=2$; $x_4=3$; $x_5=x_6=x_7=x_8=0$ $Z_{max}=68$

Example

$$\textbf{Min } \mathbf{Z} = 2x_1 + 3x_2$$

s.t

$$\frac{1}{2}x_1 + \frac{1}{4}x_2 \leq 4$$
$$x_1 + 3x_2 \geq 20$$
$$x_1 + x_2 = 10$$
$$x_1, x_2 \geq 0$$

$$\begin{aligned}
\tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 \quad + s1 &= 4 \\
x_1 + 3x_2 \quad - e_1 + a_1 &= 20 \\
x_1 + x_2 \quad + a_2 &= 10
\end{aligned}$$

or

$$\begin{aligned}
\tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 \quad + x_3 &= 4 \\
x_1 + 3x_2 \quad - x_4 + x_5 &= 20 \\
x_1 + x_2 \quad + x_6 &= 10
\end{aligned}$$

NBV=(x_1,x_2,x_4) BV=(x_3,x_5,x_6)=4,20,10 basic feasible staring solution.

$$\textbf{Min } \mathbf{W} = \mathbf{x_5 + x_6}$$

$$\mathbf{-W - 2x_1 - 4x_2 + x_4 = -30}$$

$$\mathbf{-Z + 2x_1 + 3x_2 = 0}$$

Phase I

BASIS	x_1	x_2	x_3	x_4	x_5	x_6	RHS	RATIO
x_3	½	¼	1	0	0	0	**4**	16
x_5	1	3<	0	-1	1	0	**20**	20/3<
x_6	1	1	0	0	0	1	**10**	10
-w	**-2**	**-4<**	**0**	**1**	**0**	**0**	**-30**	
-Z	**2**	**3**	**0**	**0**	**0**	**0**	**0**	
x_3	5/12	0	1	1/12	-1/12	0	**7/3**	28/5
x_2	1/3	1	0	-1/3	1/3	0	**20/3**	20
x_6	2/3<	0	0	1/3	-1/3	1	**10/3**	5<
-w	**-2/3**	**0**	**0**	**-1/3**	**4/3**	**0**	**-10/3**	
-Z	**1**	**0**	**0**	**1**	**-1**	**0**	**-20**	
x_3	0	0	1	-1/8	1/8	-5/8	¼	
x_2	0	1	0	-1/2	1/2	-1/2	**5**	
x_1	1	0	0	1/2	-1/2	3/2	**5**	
-w	**0**	**0**	**0**	**0**	**1**	**1**	**0**	
-Z	**0**	**0**	**0**	**1/2**	**-1/2**	**-3/2**	**-25**	

Phase I and Phase II completed at the same time. CASE 3

$x_1 = 5$ $x_2 = 5$ $x_3 = 1/4$ $x_4 = 0$ $Z_{min} = 25$

Example

$$\textbf{Max } \mathbf{Z} = 50\,x_1 + 40\,x_2$$

s.t

$$3\,x_1 + 5\,x_2 \le 150$$
$$x_2 \le 20$$
$$8\,x_1 + 5x_2 \le 300$$
$$x_1 + x_2 \ge 25$$
$$x_1\,,\,x_2 \ge 0$$

$$\begin{aligned}
\mathbf{3x_1 + 5\,x_2 + x_3} &\mathbf{= 150}\\
\mathbf{x_2 + x_4} &\mathbf{= 20}\\
\mathbf{8x_1 + 5x_2 + x_5} &\mathbf{= 300}\\
\mathbf{x_1 + x_2 - x_6 + x_7} &\mathbf{= 25}
\end{aligned}$$

$$\mathbf{Z - 50\,x_1 - 40\,x_2 = 0}$$

$$\mathbf{W = x_7}$$

$$\mathbf{-\,W - x_1 - x_2 + x_6 = -15}$$

Phase I

BASIS	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS	RATIO
x_3	3	5	1	0	0	0	0	**150**	50
x_4	0	1	0	1	0	0	0	**20**	-
x_5	8	5	0	0	1	0	0	**300**	37.5
x_7	1	1	0	0	0	-1	1	**25**	25<
-w	**-1**	**-1**	**0**	**0**	**0**	**1**	**0**	**-25**	
Z	**-50**	**-40**	**0**	**0**	**0**	**0**	**0**	**0**	
x_3	0	2	1	0	0	3	-3	**75**	25
x_4	0	1	0	1	0	0	0	**20**	
x_5	0	-3	0	0	1	8<	-8	**100**	12.5<
x_1	1	1	0	0	0	-1	1	**25**	
-w	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	
Z	**0**	**10**	**0**	**0**	**0**	**-50<**	**50**	**1250**	

Phase II

BASIS	x_1	x_2	x_3	x_4	x_5	x_6	RHS	RATIO
x_3	0	25/8<	1	0	-3/8	0	**75/2**	12<
x_4	0	1	0	1	0	0	**20**	20
x_6	0	-3/8	0	0	1/8	1	**25/2**	
x_1	1	5/8	0	0	1/8	0	**75/2**	60
Z	**0**	**-70/8<**	**0**	**0**	**50/8**	**0**	**1875**	
x_2	0	1	8/25	0	-3/25	0	**12**	
x_4	0	0	-8/25	1	3/25	0	**8**	
x_6	0	0	3/25	0	2/25	1	**17**	
x_1	1	0	-5/25	0	5/25	0	**30**	
Z	**0**	**0**	**14/5**	**0**	**26/5**	**0**	**1980**	

$$x_1=30 \quad x_2= 12 \quad Z_{max} = 1980$$

Phase I and II

BASIS	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS	RATIO
x_3	3	5	1	0	0	0	0	**150**	50
x_4	0	1	0	1	0	0	0	**20**	-
x_5	8	5	0	0	1	0	0	**300**	37.5
x_7	1	1	0	0	0	-1	1	**25**	25<
-w	**-1**	**-1**	**0**	**0**	**0**	**1**	**0**	**-25**	
Z	**-50**	**-40**	**0**	**0**	**0**	**0**	**0**	**0**	
x_3	0	2	1	0	0	3	-3	**75**	25
x_4	0	1	0	1	0	0	0	**20**	
x_5	0	-3	0	0	1	8<	-8	**100**	12.5<
x_1	1	1	0	0	0	-1	1	**25**	
-w	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	
Z	**0**	**10**	**0**	**0**	**0**	**-50<**	**50**	**1250**	
x_3	0	25/8<	1	0	-3/8	0		**75/2**	12<
x_4	0	1	0	1	0	0		**20**	20
x_6	0	-3/8	0	0	1/8	1		**25/2**	
x_1	1	5/8	0	0	1/8	0		**75/2**	60
Z	**0**	**-70/8<**	**0**	**0**	**50/8**	**0**		**1875**	
x_2	0	1	8/25	0	-3/25	0		**12**	
x_4	0	0	-8/25	1	3/25	0		**8**	
x_6	0	0	3/25	0	2/25	1		**17**	
x_1	1	0	-5/25	0	5/25	0		**30**	
Z	**0**	**0**	**14/5**	**0**	**26/5**	**0**		**1980**	

$$x_1=30 \quad x_2=12 \quad Z_{max}=1980$$

Example

$$\textbf{Min } \mathbf{Z} = 2x_1 + 3x_2$$

s.t

$$x_1 + x_2 \geq 10$$
$$3x_1 + 5x_2 \leq 15$$
$$x_1, x_2 \geq 0$$

$$\mathbf{x_1 + x_2 - e_1 + a_1 = 10}$$
$$\mathbf{3x_1 + 5x_2 + s_1 = 15}$$

or

$$\mathbf{x_1 + x_2 - x_3 + x_5 = 10}$$
$$\mathbf{3x_1 + 5x_2 + x_4 = 15}$$

NBV=(x_1,x_2,x_3) BV=(x_4,x_5)=10,15 basic feasible starting solution.

$$\text{Min } W = x_5$$

$$-w \; -x_1 - x_2 + x_5 = -10$$

$$-Z \; + 2x_1 + 3x_2 = 0$$

BASIS	x_1	x_2	x_3	x_4	x_5	RHS	RATIO
x_5	1	1	-1	0	1	**10**	10
x_4	3<	5	0	1	0	**15**	5<
-W	**-1<**	**-1**	**1**	**0**	**0**	**-10**	
-Z	**2**	**3**	**0**	**0**	**0**	**0**	
x_5	0	-2/3	-1	-1/3	1	**5**	
x_1	1	5/3	0	1/3	0	**5**	
-W	**0**	**2/3**	**1**	**1/3**	**0**	**-5**	
-Z	**0**	**-1/3**	**0**	**-2/3**	**0**	**-10**	

Case 1 The optimal value of W is greater than zero. In this case, the original LP **has no feasible solution**(**infeasible**).