

# Linear Equations

## Definitions: Linear Equations

*The simplest type of a linear equation is given by*

$$a x = b$$

*where  $a$  and  $b$  are given and known, and  $x$  is unknown variable to be determined. This equation is linear as it contains only  $x$  and nothing else.*

**It is simple to see that the equation has:**

- a unique solution if  $a \neq 0$
- no solution if  $a = 0$  and  $b \neq 0$
- multiple solutions if  $a = 0$  and  $b = 0$

**(The relationship between the voltage and current of a resistor,  $IR = V$ .)**

Any straight line in the  $xy$  plane can be represented algebraically by an equation of the form

$$a_1x + a_2y = b$$

where  $a_1, a_2$  and  $b$  are real constants, and  $a_1, a_2$  are not both zero. An equation of this form is called a linear equation in the variables  $x$  and  $y$ .

$$3x + 5y = 15 \quad \text{is a linear equation.}$$

But

$$3x - \cos(y) = 1,$$

$$e^{x^2} + 4y = 10,$$

$$5x + xy^2 = 8$$

$$x + \sqrt{y} = 5$$

are not linear equations.

All variables occur only to the first power and do not appear as arguments for trigonometric, logarithmic or exponential functions.

Generally we define a linear equation in the  $n$  variables  $x_1, x_2, \dots, x_n$  to be one that can be expressed in the form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$$

where  $a_1, a_2, \dots, a_n$  and  $b$  are real constants. The **variables** in a linear equation are sometimes called **unknowns**.

## Solution of a linear equation:

A solution of a linear equation  $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$  is a sequence of  $n$  numbers  $s_1, s_2, \dots, s_n$  such that the equation is satisfied when we substitute

$$x_1 = s_1,$$

$$x_2 = s_2$$

•

•

•

$$x_n = s_n.$$

The set of all solutions of the equation is called its **solution set** or the **general solution** of the equation.

If we set specific values for  $s_i$  we obtain **particular solution**.

## Example

Find the solution of  $4x - 2y = 1$

If we assign  $x$  an arbitrary value  $t$ , we obtain

$$x = t, \quad y = 2t - 0.5$$

Particular numerical solutions can be obtained by substituting specific values for the parameter  $t$ .

$t = 3$  Yields the solution  $x = 3, y = 5.5$

$t = -0.5$  Yields the solution  $x = -0.5, y = -1.5$

If we assign  $y$  an arbitrary value  $t$ , we obtain

$$y = t, \quad x = 0.5t + 0.25$$

Although the formulas are different from those obtained above, they yield the same solution set as  $t$  varies over all possible real numbers. For example, the previous formulas gave the solution  $x = 3, y = 5.5$  when  $t = 3$ , whereas the formulas immediately above yield the solution when  $t = 5.5$ .

### Example

Find the solution of  $x_1 - 4x_2 + 7x_3 = 5$

We can assign arbitrary values to any two variables and solve for the third variable.

If we assign arbitrary values  $s$  and  $t$  to  $x_2$  and  $x_3$ , respectively, and solve for  $x_1$ , we obtain

$$x_1 = 5 + 4s - 7t, \quad x_2 = s, \quad x_3 = t.$$

# Introduction to Linear Systems



## Definition

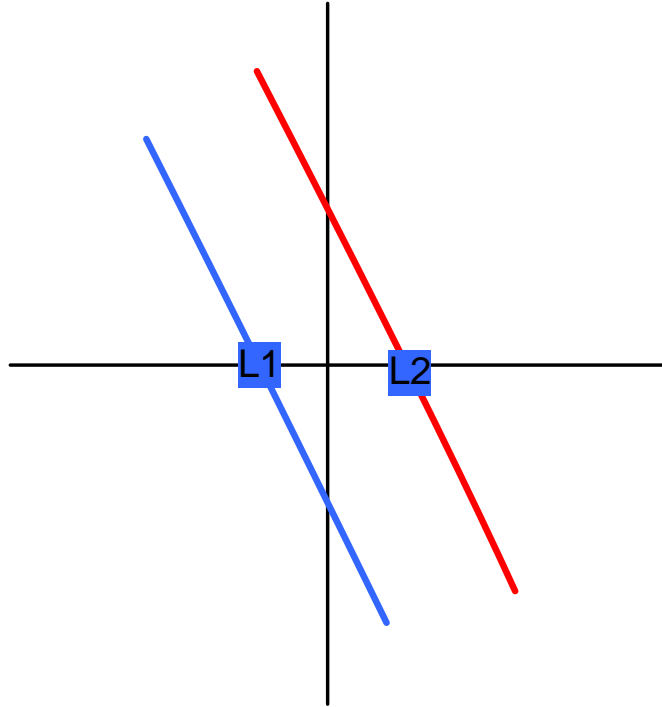
A finite set of linear equations in the variables  $x_1, x_2, \dots, x_n$  is called a **system of linear equations** or a **linear system**. A sequence of numbers  $s_1, s_2, \dots, s_n$  is called a solution of the system if

$x_1 = s_1, x_2 = s_2, \dots, x_n = s_n$  is a solution of every equation in the system.

## Two-equations-two-unknowns linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

where  $a_{11}, a_{12}, a_{21}, a_{22}, b_1$  and  $b_2$  are known constants and  $x_1$  and  $x_2$  are unknown variables to be solved.

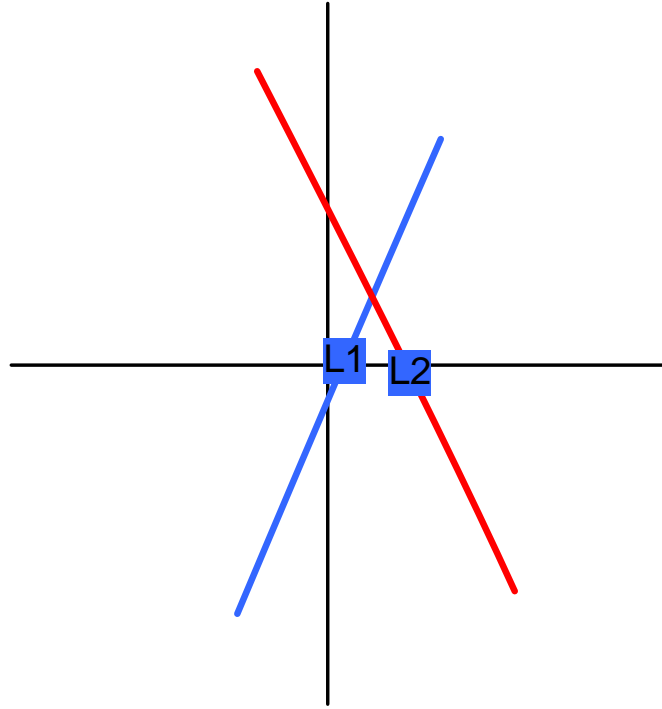


1. The lines  $L_1$  and  $L_2$  may be **parallel**, in which case there is **no intersection** and consequently **no solution** to the system.

Ex:

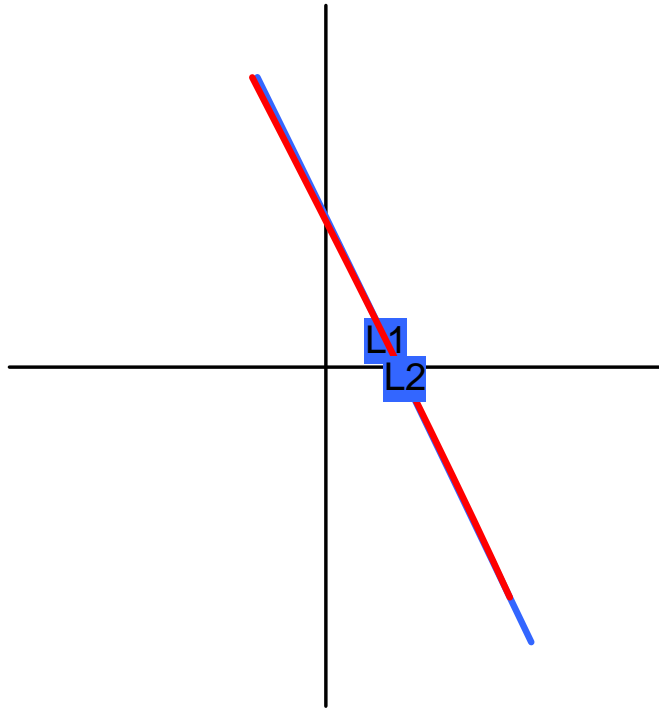
$$\begin{aligned}x + 2y &= 3 \\4x + 8y &= 6\end{aligned}$$

*A system of equations that has no solutions is said to be inconsistent; if there is at least one solution of the system, it is called consistent.*



2. The lines  $L_1$  and  $L_2$  may **intersect at only one point**, in which case the system has **exactly one solution**.

$$\begin{aligned} &x + 2y = 3 \\ \text{Ex: } &4x + 5y = 6 \\ &(x, y) = (-1, 2) \end{aligned}$$



3. The lines  $L_1$  and  $L_2$  may coincide, in which case there are infinitely many points of intersection and consequently infinitely many solutions to the system.

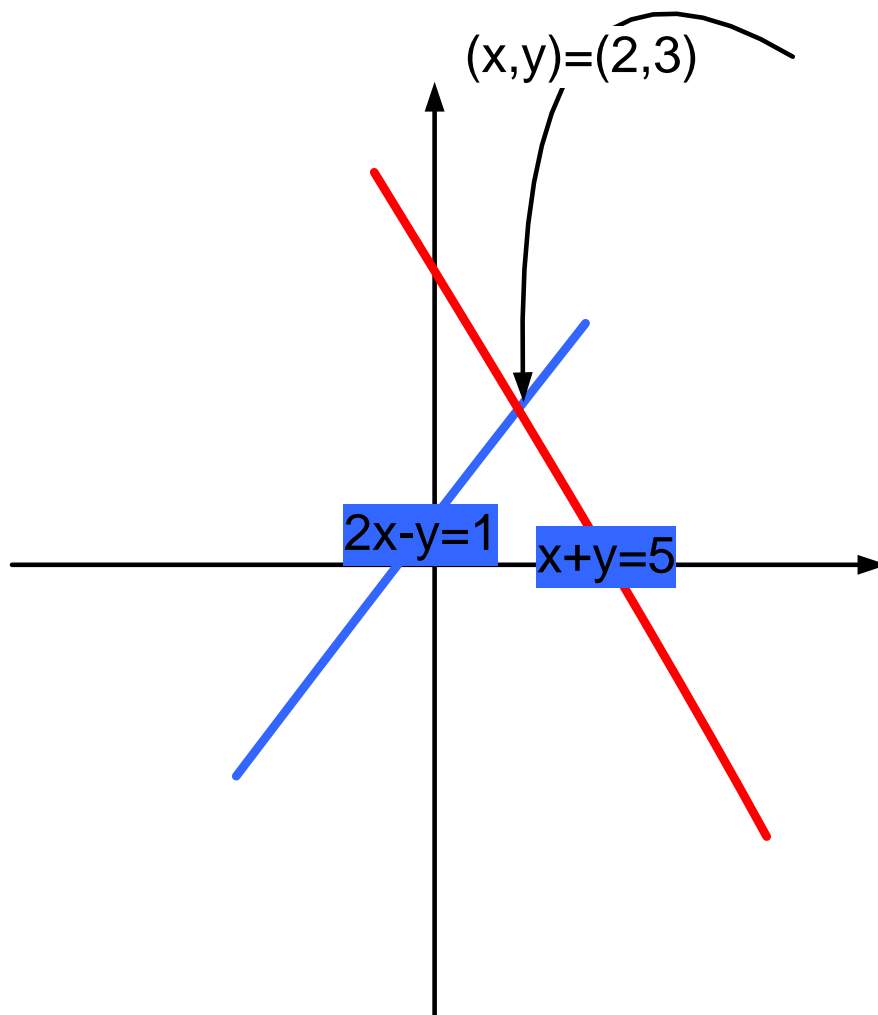
Ex: 
$$\begin{aligned} x + 2y &= 3 \\ 4x + 8y &= 12 \end{aligned}$$

**Every system of linear equations has no solution or has exactly one solution or has infinitely many solutions.**

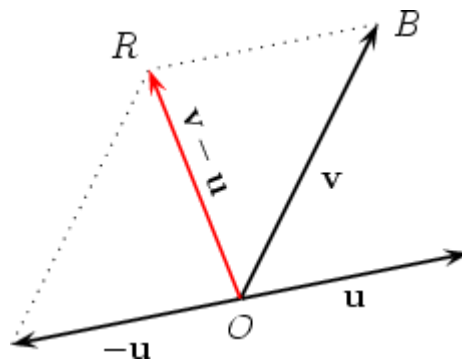
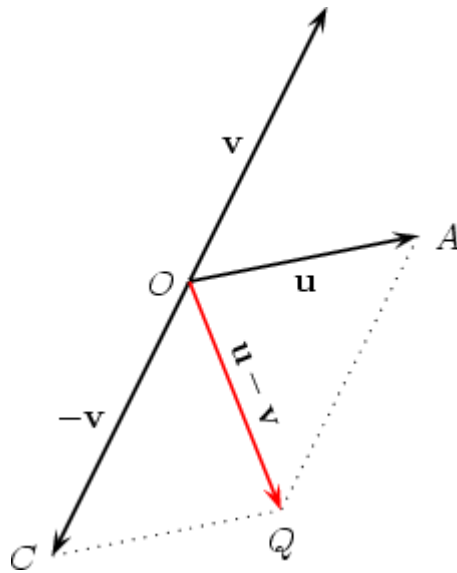
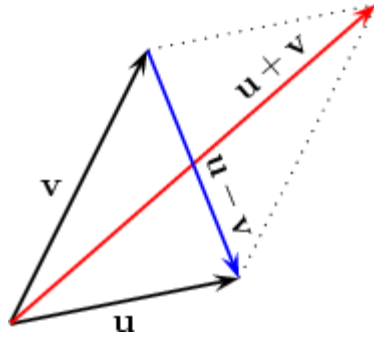
# **The Geometry of Linear Equations**

$$\begin{aligned}2x - y &= 1 \\ x + y &= 5\end{aligned}$$

**First Approach:**



*The point of intersection lies on both lines. It is the only solution to both equations. That point  $x=2$  and  $y=3$  will soon be found by “elimination”.*

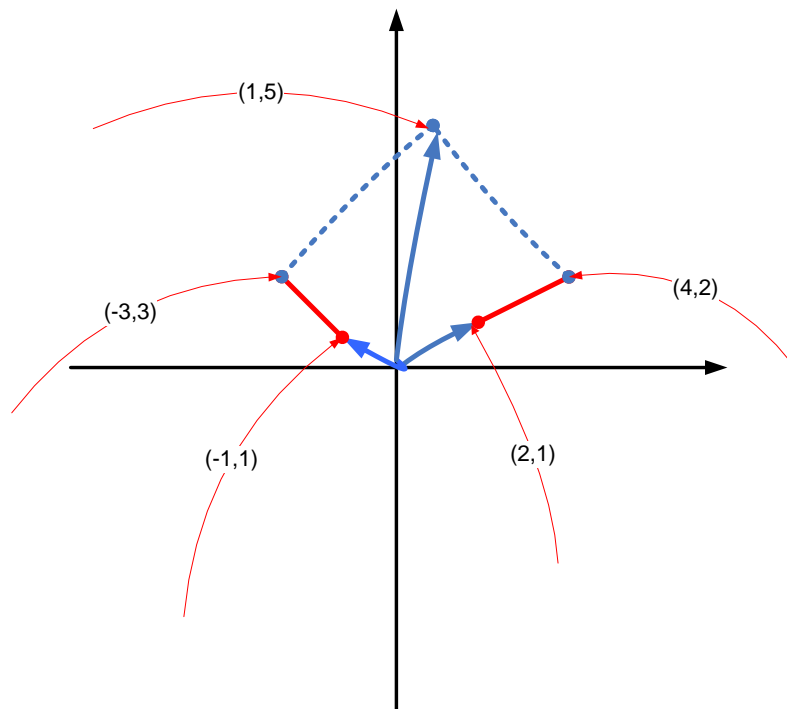


## Addition and subtraction of Vectors

## Second Approach:

The two separate equations are really *one vector equation*:

$$x \begin{bmatrix} 2 \\ 1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$



The problem is to find the combination of the column vectors on the left side that produces the vector on the right side.

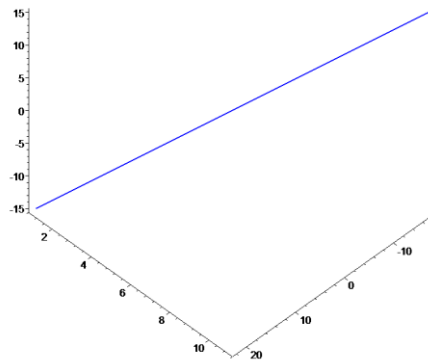
**The column picture confirms that  $x=2$  and  $y=3$ .**



```
> v:=PositionVector([2-4*t,6+t,3*t]);
```

$$v := \begin{bmatrix} 2-4t \\ 6+t \\ 3t \end{bmatrix}$$

```
> PlotPositionVector(v,t=-5..5);
```



## Parametric equation

A line through point  $\mathbf{r}_0 = (x_0, y_0, z_0)$  parallel to vector

$\mathbf{v} = \langle a, b, c \rangle$  is given by

$$\begin{aligned} x &= x_0 + at, \\ y &= y_0 + bt, \\ z &= z_0 + ct. \end{aligned}$$

## Vector-Valued Functions

When a particle moves along a curve in the xy plane defined parametrically by

$$x = f(t) \quad \text{and} \quad y = g(t)$$

If the coordinates of the moving point at time  $t$  are given by the parametric equations  $x = f(t)$  and  $y = g(t)$  then the position vector

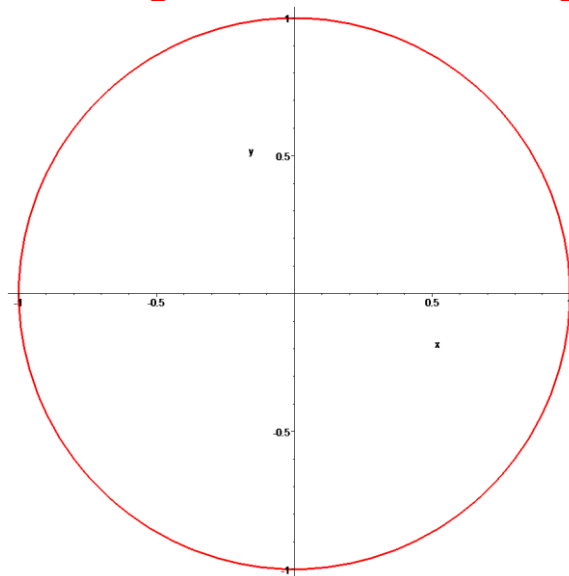
$$r(t) = f(t)i + g(t)j$$

determines a **vector valued** function.

```
> with(plots, implicitplot);
```

```
[implicitplot]
```

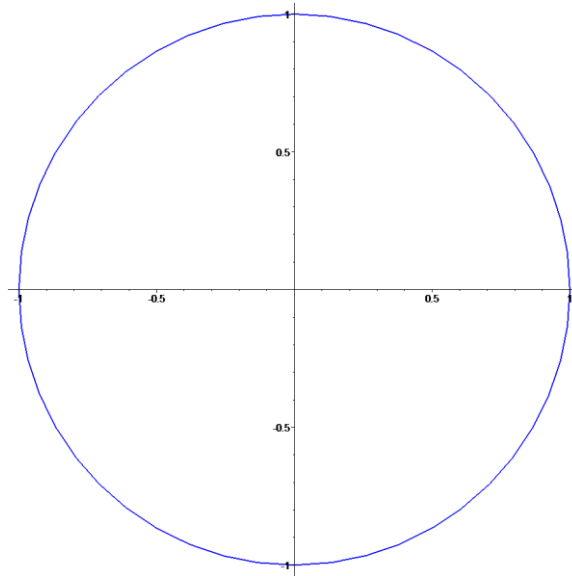
```
> implicitplot(x^2+y^2=1, x=-1..1, y=-1..1);
```



```
> v:=PositionVector([cos(t),sin(t)]);
```

$$v := \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$$

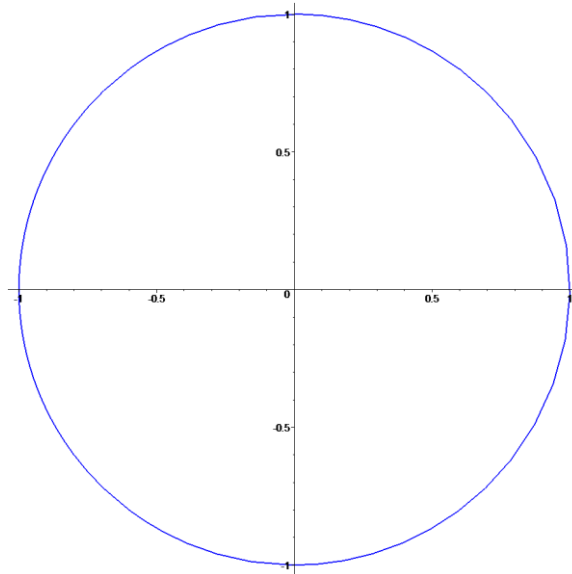
```
> PlotPositionVector(v,t=-Pi..Pi);
```



```
> v:=PositionVector([(1-t^2)/(1+t^2),2*t/(1+t^2)]);
```

$$v := \begin{bmatrix} \frac{1-t^2}{1+t^2} \\ \frac{2t}{1+t^2} \end{bmatrix}$$

```
> PlotPositionVector(v,t=-infinity..infinity);
```



## Curve in three dimension

We define a **space curve** or a curve in three dimensions as a set  $C$  of ordered triples of the form

$$(f(t), g(t), h(t))$$

The equations

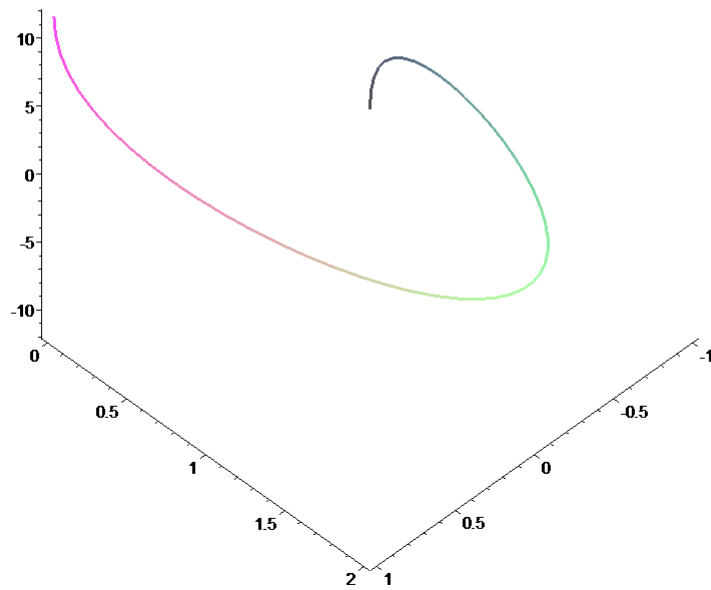
$$\begin{aligned}x &= f(t), \\y &= g(t), \\z &= h(t)\end{aligned}$$

are called parametric equations for space curve.

## Corresponding position vector

$$r(t) = f(t)i + g(t)j + h(t)k$$

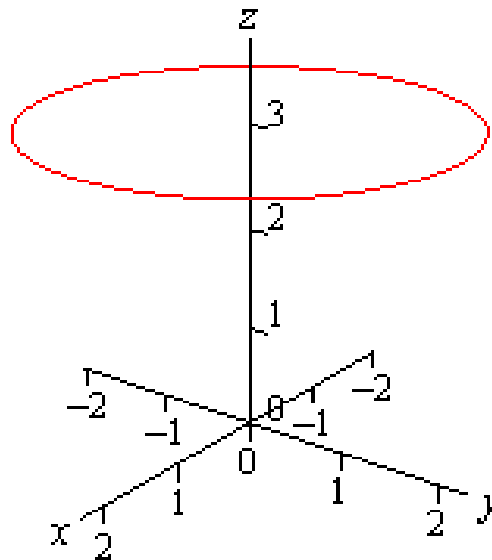
```
> with(plots):  
> spacecurve([sin(t), 2*cos(t)^2, 3*t^3], t=-  
Pi/2..Pi/2, axes=frame);
```



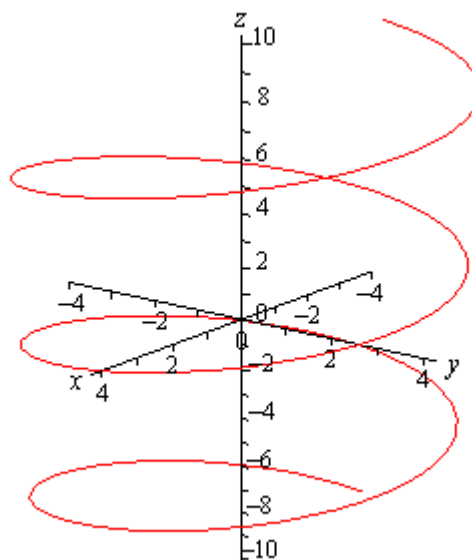
Sketch the graph of the following vector function.

$$\vec{r}(t) = \langle 2 \cos t, 2 \sin t, 3 \rangle$$

$$\vec{r}(t) = \langle 2 \cos t, 2 \sin t, 3 \rangle$$

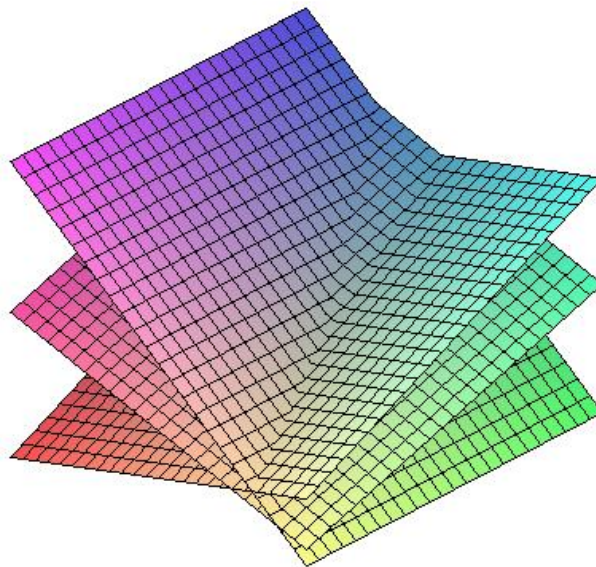


$$\vec{r}(t) = \langle 4 \cos t, 4 \sin t, t \rangle$$



## Three Planes

$$\begin{aligned}2x + y + z &= 5 \\4x - 6y &= -2 \\-2x + 7y + 2z &= 9\end{aligned}$$



# Column Vectors and Linear Combinations

## Column form

$$x \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + y \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} = b$$

**The result is called a linear combination and this combination solves our equation.**

## Linear combination

$$\begin{aligned} & 1 \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \\ & \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} = b \end{aligned}$$

**$x=1, y=1, z=2$  give the correct combination of the columns. They also give (1,1,2) where the three planes intersect.**



# General Linear Form

In general, a linear system with **m equations** and **n unknown** can be written as the following form:

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\&\cdot \\&\cdot \\&\cdot \\a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m\end{aligned}$$

where  $x_1, x_2, \dots, x_n$  are the unknowns. The double subscripting on the coefficients of the unknowns is a useful device that is used to specify the location of the coefficient in the system.

- The **first subscript** on the coefficient  $a_{ij}$  indicates the **equation** in which the coefficient occurs,
- The **second subscript** indicates **which unknown** it multiplies.

# MATRIX FORM

We can re-write this set of linear equations in a compact form, i.e., a matrix form:

$$Ax = b$$
$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & & & \\ \cdot & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{bmatrix}$$

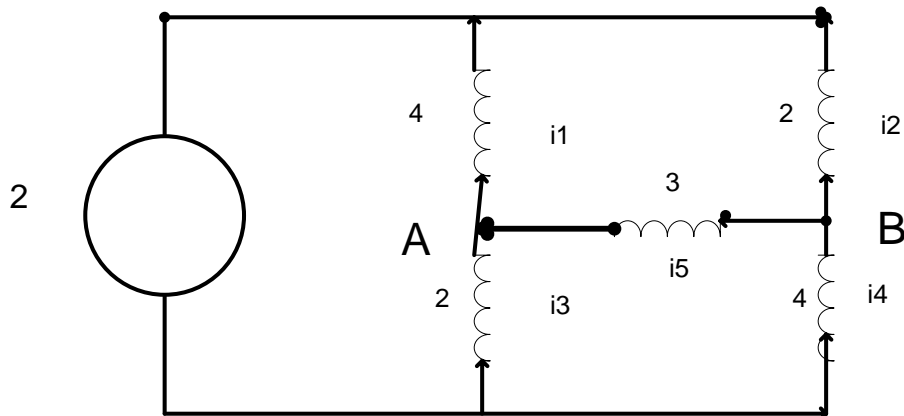
*A: Coefficient Matrix*

*x: Vector of unknowns*

*b: Data vector(RHS vector)*

- Matrix  $A$  has  $m$  rows and  $n$  columns. **Such a matrix** is called  $m \times n$  matrix.
- Each of the sitting on the  $i$ -th row and  $j$ -th column, or simply the  $(i, j)$ -th element is denoted by  $a_{ij}$ .

## Example:



**Kirchoff's Current Law (KCL)**

**Kirchoff's Voltage Law (KVL)**

**KCL at Node A**  $i_1 - i_3 - i_5 = 0$

**KCL at Node B**  $i_2 - i_4 + i_5 = 0$

**KCL to left loop**  $4i_1 + 2i_3 = 2$

**KVL to right upper loop**  $4i_1 - 2i_2 + 3i_5 = 0$

**KVL to right lower loop**  $2i_3 - 4i_4 - 3i_5 = 0$

**In a matrix form**

$$\begin{bmatrix} 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 \\ 4 & 0 & 2 & 0 & 0 \\ 4 & -2 & 0 & 0 & 3 \\ 0 & 0 & 2 & -4 & -3 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

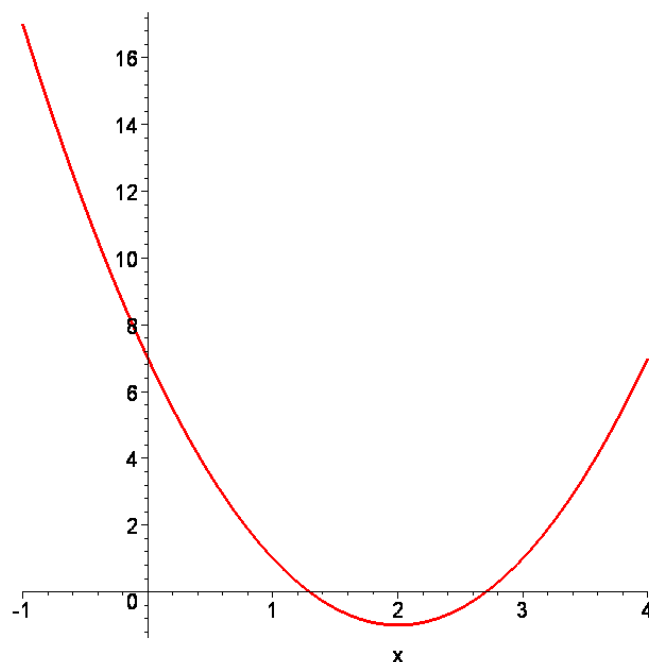
### Example:

Find the parabola  $y = A + Bx + Cx^2$  that passes through the three points (1, 1), (2, -1) and (3, 1).

For each point we obtain an equation relating the value of x to the value y. The result is the linear system

$$\begin{aligned}A + B + C &= 1 \\A + 2B + 4C &= -1 \\A + 3B + 9C &= 1\end{aligned}$$

The solution of this system  $A=7$ ,  $B=-8$  and  $C=2$  and the equation of the parabola is  $y = 7 - 8x + 2x^2$ .



### **Example: (Equation of a Line)**

Find the equation of the line that passes through the two points (2,) and (3,7).

**Substituting the coordinates of the two points into Equation**

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} x & y & 1 \\ 2 & 1 & 1 \\ 3 & 7 & 1 \end{vmatrix} = 0$$

**The cofactor expansion of this determinant along the first row then gives**

$$-6x + 7 + 11 = 0$$

### **Example: (A Circle through Three Points)**

Find the equation of the circle that passes the three points (1,7) , (6,2), and (4,6).

**Substituting the coordinates of the three points into Equation**

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ 50 & 1 & 7 & 1 \\ 40 & 6 & 2 & 1 \\ 52 & 4 & 6 & 1 \end{vmatrix} = 0$$

**which reduces to**

$$10(x^2 + y^2) - 20x - 40y - 200 = 0$$

**In standard form this is**

$$(x - 1)^2 + (y - 2)^2 = 5^2$$

**Thus the circle has center (1,2) and radius 5.**

### Example: (Equation of a Plane)

Find the equation of the plane that passes through the three noncolinear points (1,1,0),(2,0,-1) and (2,9,2).

**Substituting the coordinates of the two points into Equation**

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} x & y & z & 1 \\ 1 & 1 & 0 & 1 \\ 2 & 0 & -1 & 1 \\ 2 & 9 & 2 & 1 \end{vmatrix} = 0$$

**which reduces to**

$$2x - y + 3z - 1 = 0$$

### Example: (Equation of Sphere)

Find the equation of the sphere through the four points (0,3,2), (1,-1,1), (2,1,0), and (5,1,3) is

$$\begin{vmatrix} x^2 + y^2 + z^2 & x & y & z & 1 \\ 13 & 0 & 3 & 2 & 1 \\ 3 & 1 & -1 & 1 & 1 \\ 5 & 2 & 1 & 0 & 1 \\ 35 & 5 & 1 & 3 & 1 \end{vmatrix} = 0$$

This reduces to

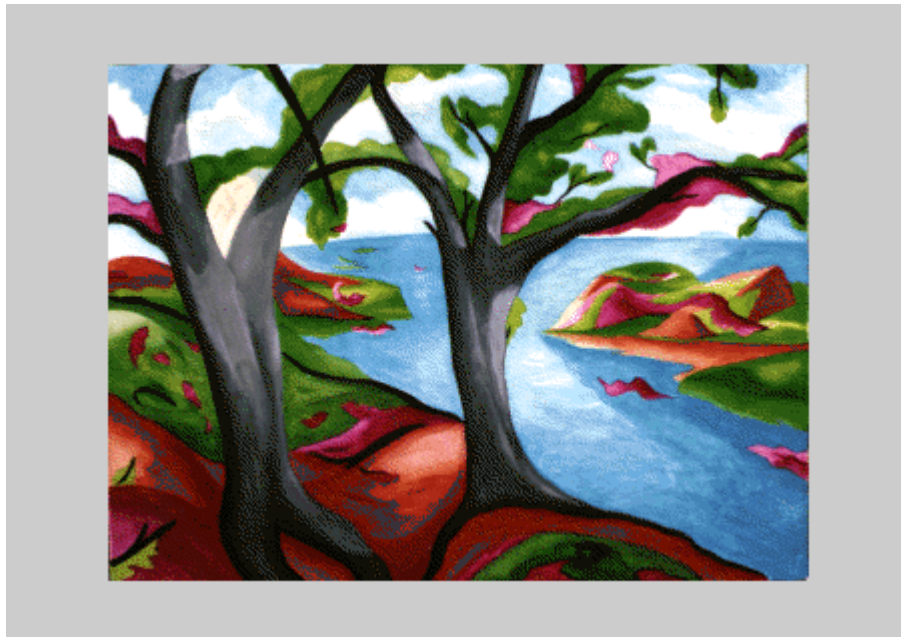
$$x^2 + y^2 + z^2 - 4x - 2y - 6z + 5 = 0 - y + 3z - 1 = 0$$

which in standard form is

$$(x-2)^2 + (y-1)^2 + (z-3)^2 = 3^2$$



load trees (MATLAB)



Imshow(R,G,B)

**R**

0.1922    0.1922    0.1922    0.1922    0.1922    0.1922

0.1922    0.2588

0.1922    0.1922    0.2235    0.2588    0.2588    0.1922

**G**

0.1922    0.1922    0.1922    0.1922    0.2235    0.1922

0.2235    0.1922    0.2235    0.2588

**B**

0.2235    0.1922    0.1922    0.1922    0.1922    0.2235

0.4824    0.7098

```
>> n=10
```

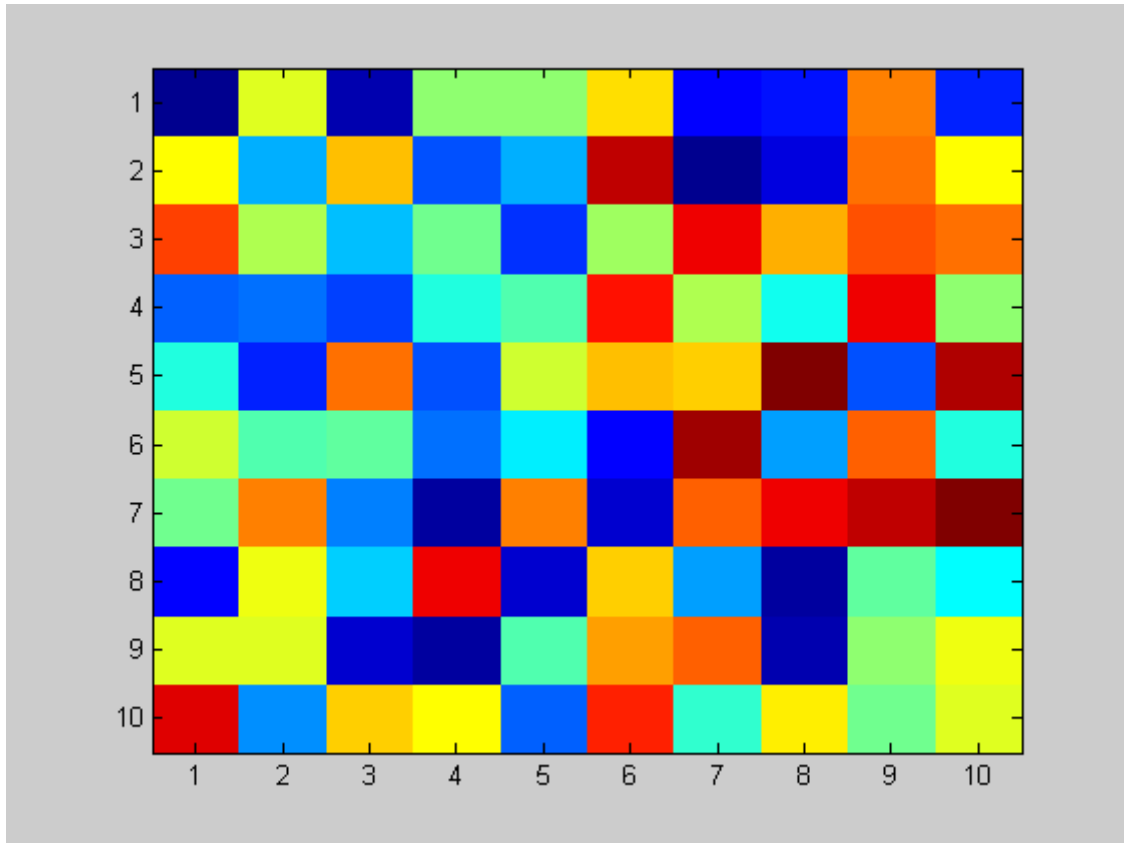
```
n = 10
```

```
>> A=rand(n)
```

```
A =
```

|        |        |        |        |        |        |        |        |        |        |
|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| 0.0159 | 0.5943 | 0.0550 | 0.5165 | 0.5074 | 0.6594 | 0.1353 | 0.1503 | 0.7455 | 0.1551 |
| 0.6283 | 0.3020 | 0.6756 | 0.2126 | 0.2919 | 0.9234 | 0.0285 | 0.1032 | 0.7627 | 0.6196 |
| 0.8125 | 0.5442 | 0.3081 | 0.4850 | 0.1834 | 0.5238 | 0.8892 | 0.6944 | 0.7827 | 0.7581 |
| 0.2176 | 0.2362 | 0.1911 | 0.4113 | 0.4514 | 0.8555 | 0.5510 | 0.3858 | 0.8807 | 0.5107 |
| 0.4054 | 0.1605 | 0.7659 | 0.2088 | 0.5771 | 0.6777 | 0.6685 | 0.9967 | 0.2052 | 0.9383 |
| 0.5699 | 0.4533 | 0.4741 | 0.2452 | 0.3589 | 0.1297 | 0.9616 | 0.2912 | 0.7732 | 0.4046 |
| 0.4909 | 0.7413 | 0.2572 | 0.0396 | 0.7385 | 0.0921 | 0.7800 | 0.8826 | 0.9212 | 0.9974 |
| 0.1294 | 0.6009 | 0.3252 | 0.8854 | 0.0861 | 0.6624 | 0.2836 | 0.0404 | 0.4695 | 0.3764 |
| 0.5909 | 0.5841 | 0.0845 | 0.0348 | 0.4469 | 0.7109 | 0.7718 | 0.0542 | 0.5072 | 0.6043 |
| 0.8985 | 0.2706 | 0.6618 | 0.6223 | 0.2273 | 0.8290 | 0.4244 | 0.6427 | 0.4903 | 0.5947 |

```
>> imagesc(A)
```



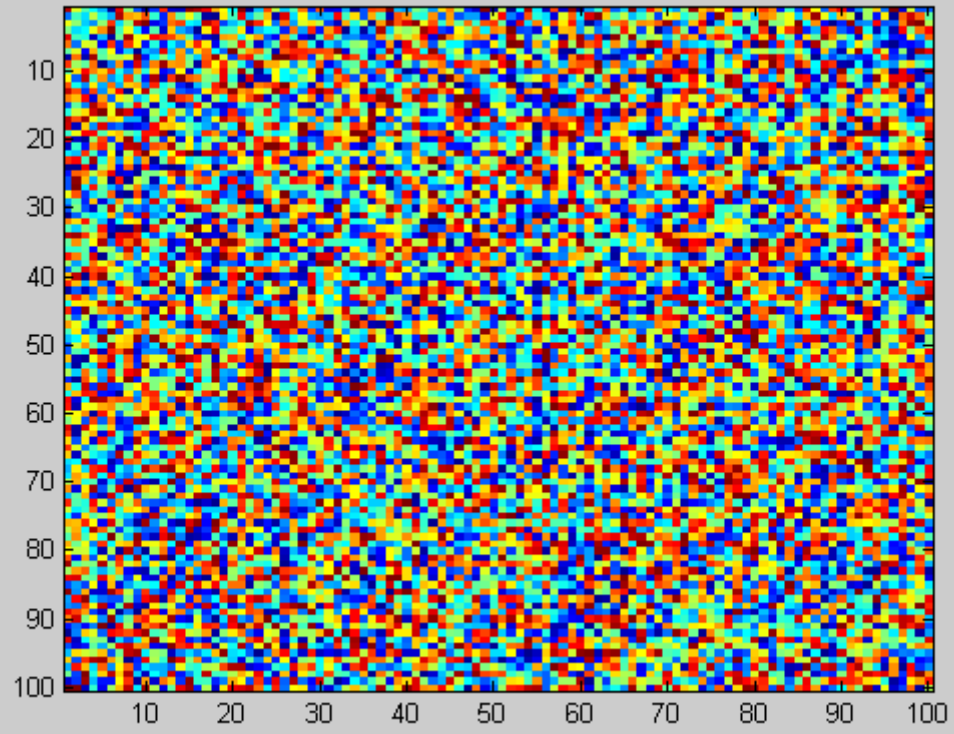
```
>> n = 100;
```

```
B = rand(n)
```

```
0.1848  0.6793  0.2799  0.1424  0.9438  0.4611  0.5755  0.8587  0.8505  0.9715  0.4131  0.5420
0.2894  0.3664  0.1876  0.1536  0.1848  0.5867  0.3250  0.1446  0.7536  0.1251  0.6427  0.4189
0.6639  0.6393  0.7063  0.3120  0.1587  0.0235  0.6651  0.0171  0.9130  0.2630  0.2266  0.5447
0.6036  0.6301  0.5934  0.3417  0.6445  0.5686  .....
```

```
...
```

```
imagesc(B);
```



Maple 12 - [LinearAlgebra,General,LAProgCallSeq]

File Edit View Window Help

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Basic Features... Advanced Features... Connectivity... Physics... Programming... Graphics... Student Package... Science and Engineering... Applications and Example Worksheets... Reference... System... Mathematics... LinearAlgebra... LinearAlgebra

Differential-algebraic Equations... DifferentialEquations... Discrete Mathematics... Evaluation... Factorization and Solving Equations... Fractional Functions... FunctionAdvisor... General Information... Geometry... Group Theory... Linear Functions... LinearAlgebra... LinearAlgebra

LinearAlgebra Overview... LinearAlgebra Details... changecoords... dot... Indexing Functions... LinearAlgebra Commands... LinearAlgebra Computations... LinearAlgebra Matrix... Matrix Polynomial Algebra... OrderBasis... LinearAlgebra

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Efficient Computations... Programming... Submodule Calling Sequences... LinearAlgebra

### LinearAlgebra Programming Submodule Calling Sequences

The following is an alphabetical list of the programming submodule calling sequences for all of the functions available in the LinearAlgebra module.

After the closing parenthesis for each function, the expression "type" indicates the type of object normally returned by the function. (This is not valid Maple syntax.) If the expression "identical(LinearAlgebra:-NoUserValue)" appears in the type specification for a parameter, the name LinearAlgebra:-NoUserValue can be used for the value supplied for that parameter. This special value indicates that the function is to obtain the value for the corresponding parameter by some other means, for example, by a call to indets on a polynomial parameter.

```

Add( A :: {Matrix, Vector, scalar}, B :: {Matrix, Vector, scalar}, c1 :: scalar, c2 :: scalar, ip :: identical(inplace) = truefalse, outopts :: identical(outputoptions) = list ) :: {Matrix, Vector}
Adjoint( M :: Matrix, outopts :: identical(outputoptions) = list ) :: Matrix
BackwardSubstitute( A :: Matrix, B :: {Matrix, Vector}, t :: identical(free) = symbol, ip :: BooleanOpt(inplace), outopts :: identical(outputoptions) = list ) :: {Matrix, Vector}
BandMatrix( data :: (list(list(scalar)), list(scalar)), l :: nonnegint, rows :: nonnegint, cols :: nonnegint, outopts :: identical(outputoptions) = list ) :: Matrix
Basis( V :: {Vector, list(Vector), set(Vector)}, outopts :: identical(outputoptions) = list ) :: {list(Vector), set(Vector)}
BezoutMatrix( p :: polynomial, q :: polynomial, x :: (name, identical(LinearAlgebra:-NoUserValue)), outopts :: identical(outputoptions) = list ) :: Matrix
BidiagonalForm( M :: Matrix, out :: identical(output) = list, ip :: identical(inplace) = truefalse, outoptsB :: identical(outputoptions["B"]) = list, outoptsU :: identical(outputoptions["U"]) = list, outoptsV :: identical(outputoptions["V"]) = list, outoptsNAG :: identical(outputoptions["NAG"]) = list ) :: {Matrix, exprseq}
BilinearForm( V1 :: Vector, V2 :: Vector, M :: Matrix, conj :: identical(conjugate) = truefalse ) :: scalar
CharacteristicMatrix( A :: Matrix(square), lambda :: name, outopts :: identical(outputoptions) = list ) :: Matrix
CharacteristicPolynomial( M :: Matrix, lambda :: name ) :: polynomial
Column( M :: matrix, cols :: MVIndex, outopts :: identical(outputoptions) = list ) :: exprseq(Vector[column])
ColumnDimension( M :: Matrix ) :: nonnegint
ColumnOperation( M :: Matrix, colsel :: [{integer, integer}, integer], expr :: algebraic, ip :: identical(inplace) = truefalse, outopts :: identical(outputoptions) = list ) :: Matrix
ColumnSpace( M :: Matrix, outopts :: identical(outputoptions) = list ) :: list(Vector[column])
CompanionMatrix( p :: polynomial, x :: name, cmp :: identical(compact) = truefalse, outopts :: identical(outputoptions) = list ) :: Matrix
ConditionNumber( M :: Matrix, p :: {positive, identical(Frobenius)}, input :: identical(factors)=list, conj :: identical(conjugate)=truefalse ) :: scalar
ConstantMatrix( const :: algebraic, rows :: nonnegint, cols :: nonnegint, cmp :: identical(compact) = truefalse, outopts :: identical(outputoptions) = list ) :: Matrix

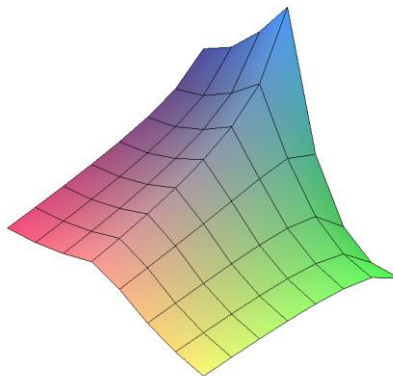
```

Time: 0.0s | Bytes: 448K | Available: 4.00G

```

> with(plots):
with(LinearAlgebra):
> A:= HilbertMatrix(8): B:= ToeplitzMatrix([1,2,3,4,-4,-3,-2,-1], symmetric):
> matrixplot(A,B);

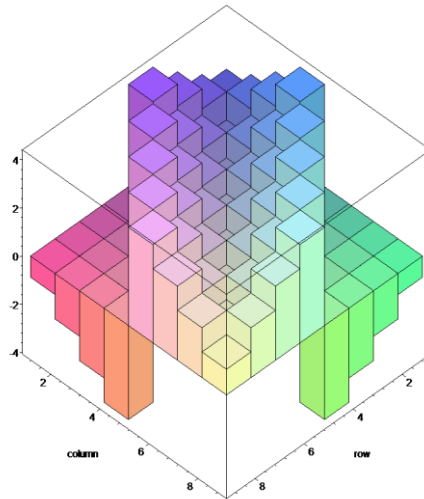
```



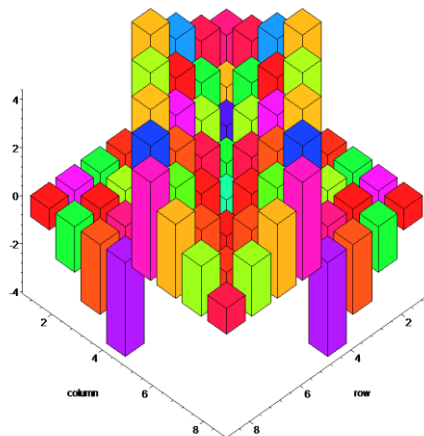
```

> matrixplot(A+B, heights=histogram, axes=boxed);

```

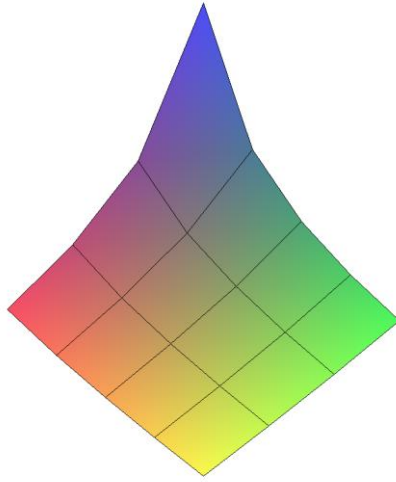


```
> matrixplot(A+B, heights=histogram, axes=frame, gap=0.25,
style=patch);
> F := (x,y) -> sin(x*y):
> matrixplot(A+B, heights=histogram, axes=frame, gap=0.25,
style=patch, color=F);
```

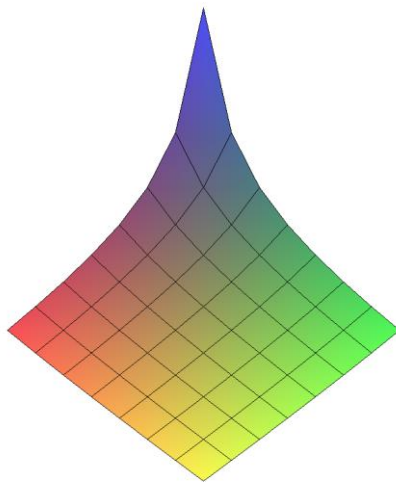


```
> L:=HilbertMatrix(4,5);
```

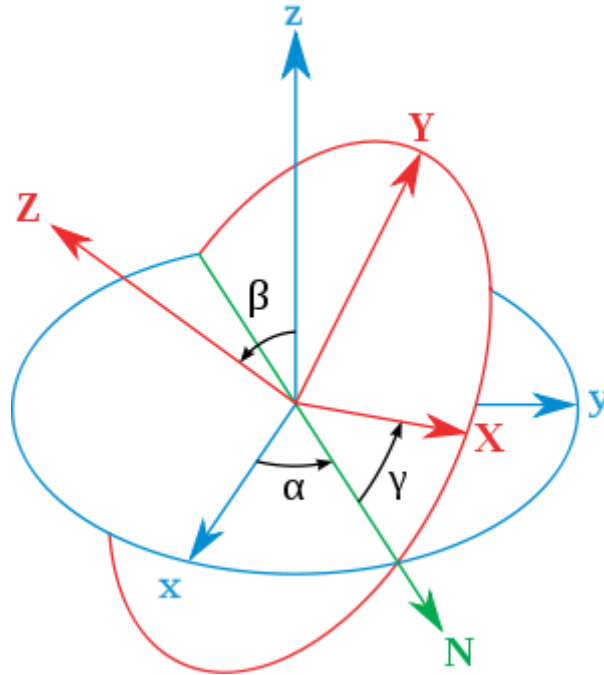
$$L := \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \end{bmatrix}$$



```
> matrixplot(A);
```



## Rotation of Axes



**Euler angles - The xyz (fixed) system is shown in blue, the XYZ (rotated) system is shown in red. The line of nodes, labeled N, is shown in green.**

As consequence of the relationship between Euler angles and Euler rotations, we can find a Matrix expression for any frame given its Euler angles, here named as  $\alpha$ ,  $\beta$ , and  $\gamma$ . Using the **z-x-z** convention, a matrix can be constructed that transforms every vector of the given reference frame in the corresponding vector of the referred frame.

Define three sets of coordinate axes, called **intermediate frames** with their origin in common in such a way that each one of them differs from the previous frame in an elemental rotation, as if they were mounted on a [gimbal](#). In these conditions, any target can be reached performing three simple rotations, because two first rotations determine the new Z axis and the third rotation will obtain all the orientation possibilities that this Z axis allows. These frames could also be defined statically using the reference frame, the referred frame and the line of nodes.



A matrix representing the end result of all three rotations is formed by successive multiplication of the matrices representing the three simple rotations, as in the following transformation equation

$$\mathbf{p}' = \mathbf{R}\mathbf{p}$$

$$\begin{bmatrix} p'_x \\ p'_y \\ p'_z \end{bmatrix} = \mathbf{R} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}, \text{ where}$$

$$\mathbf{R} = \begin{bmatrix} \cos \gamma & \sin \gamma & 0 \\ -\sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \beta & \sin \beta \\ 0 & -\sin \beta & \cos \beta \end{bmatrix} \begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where

- The rightmost matrix represents the  $\alpha$  rotation around the axis '  $\mathbf{z}$  ' of the original reference frame
- The middle matrix represents the  $\beta$  rotation around an intermediate '  $\mathbf{x}$  ' axis which is the "line of nodes".
- The leftmost matrix represents the  $\gamma$  rotation around the axis '  $\mathbf{Z}$  ' of the final reference frame