Statistical Intervals for a Single Sample

An ***interval estimator*** **is a rule for calculating two numbers – say L and U- to create an interval that you are** ***fairly certain*** **contains the parameter of interest. The concept of** ***fairly certain*** **means "with high probability". We measure this probability using the confidence coefficient, designed by 1- α.**

The basic ideas of a confidence interval (CI) are most easily understood by initially considering a simple situation. Suppose that we have a normal population with unknown mean μ and known variance σ^2. This is a somewhat unrealistic scenario because typically both the mean and variance are unknown. However, in subsequent sections we will present confidence intervals for more general situations.

Confidence Interval on the Mean of a Normal Distribution, Variance Known

- **Suppose that $X_1, X_2, ..., X_n$ is a random sample from a normal distribution with unknown mean μ and known variance σ^2.**

- **We know the sample mean $\overline{X}$ is normally distributed with mean μ and variance σ^2/n.**

- **We may standardize $\overline{X}$ as**

$$Z = \frac{\overline{X} - \mu}{\sigma / \sqrt{n}}$$

- **The random variable Z has a standard normal distribution.**

A *confidence interval* estimate for μ is an interval of the form $l \le \mu \le u$, where the endpoints l and u are computed from the sample data. Because different samples will produce different values of l and u, these endpoints are values of random variables L and U, respectively. Suppose that we can determine values of L and U such that the following probability statement is true:

$$P(L \le \mu \le \mathrm{U}) = 1 - \alpha$$

where $0 \le \alpha \le 1$.
There is a probability of $1-\alpha$ of selecting a sample for which the CI will contain the true value of μ.

Once we have selected the sample, so that

$$X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$$

and computed l and u, the resulting confidence interval for μ is

$$l \le \mu \le u$$

The end-points l and u are called the lower and upper-confidence limits, respectively, and $1-\alpha$ is called the confidence coefficient.

Confidence Interval on the Mean, Variance Known

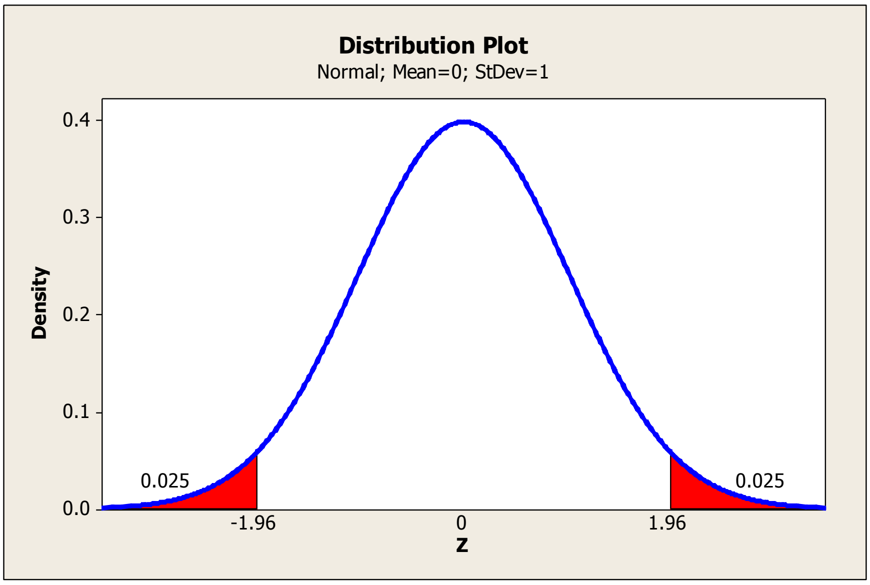
If $\bar{x}$ is the sample mean of a random sample of size n from a normal population with known variance σ^2, a $100(1-\alpha)\%$ ***CI on μ is given by***

$$\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

where $z_{\alpha/2}$ is the upper $100\,\alpha/2$ ***percentage point of the standard normal distribution.***

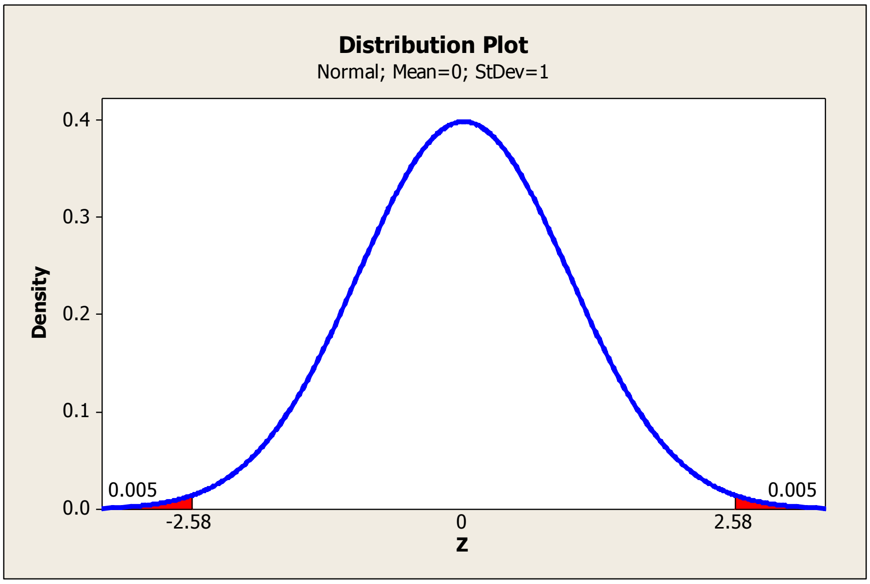
$$P\left\{\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right\} = 1-\alpha$$

Assume that the confidence coefficient is 0.95



Estimator ± 1.96 SE

Assume that the confidence coefficient is 0.99



Estimator ± 2.58 SE

Example: Ten measurements of impact energy (J) on specimens of A238 steel cut at 60°C are as follows:

64.1, 64.7, 64.5 64.6 64.5 64.3 64.6 64.8 64.2 64.3

Assume that impact energy is **normally distributed with** **σ=1J.**

Find 95% CI for μ, the mean impact energy.

$z_{\alpha/2} = z_{0.025} = 1.96$, n=10, σ=1, $\bar{x} = 64.46$

$$\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

$$64.46 - 1.96\frac{1}{\sqrt{10}} \le \mu \le 64.46 + 1.96\frac{1}{\sqrt{10}}$$

$$63.84 \le \mu \le 65.08$$

```
MTB > zint 95 1 c1
```

One-Sample Z: Impact Energy

```
The assumed standard deviation = 1

Variable         N    Mean   StDev  SE Mean        95% CI
Impact Energy   10  64.460   0.227    0.316  (63.840; 65.08)
```

Interpreting a Confidence Interval

In the impact energy estimation problem the 95% CI is $63.84 \le \mu \le 65.08$, so it is tempting to conclude that μ is within this interval with probability 0.95. However, with a little reflection, it's easy to see that this **cannot be correct**; the true value of μ is unknown and the statement $63.84 \le \mu \le 65.08$ is either correct (***true with probability 1***) or incorrect (***false with probability 1***). **The correct interpretation lies in the realization that CI is a random interval because in the probability statement defining the end-points of the interval L and U are random variables.**

- **Consequently, the correct interpretation of a**

100(1-α)%CI

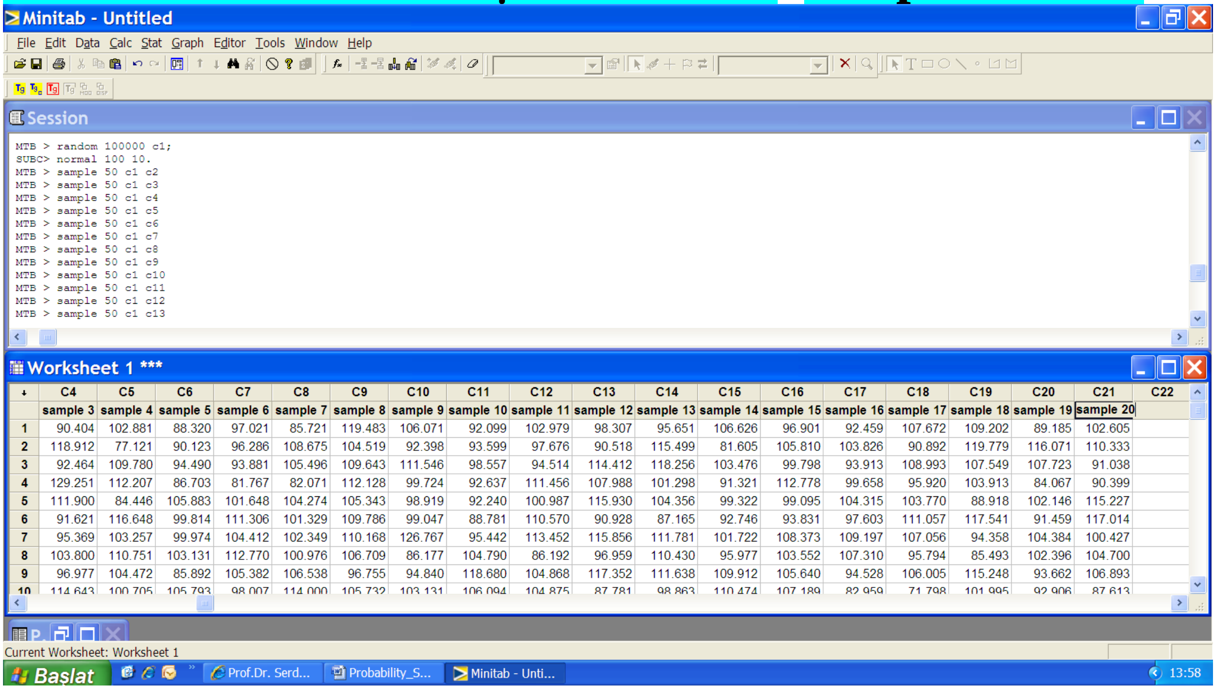
depends on the relative frequency view of probability.

- **Specifically, if an infinite number of random samples are collected and**

100(1-α)%CI

for μ is computed from each sample, 100(1-α)% of these intervals will contain the true value of μ.

Assume Normal >> μ=100 σ=10 sample size=50



Descriptive Statistics: Population; sample 1; sample 2; sample 3; sample 4; ...

Variable	Mean	SE Mean	StDev
Population	100.02	0.0317	10.02
sample 1	99.10	1.27	8.96
sample 2	100.21	1.46	10.33
sample 3	100.36	1.57	11.10
sample 4	98.80	1.42	10.05
sample 5	97.21	1.42	10.02
sample 6	100.95	1.21	8.57
sample 7	99.62	1.18	8.33
sample 8	102.32	1.27	8.98
sample 9	98.65	1.49	10.53
sample 10	99.25	1.55	10.99
sample 11	100.28	1.26	8.93
sample 12	101.86	1.34	9.48
sample 13	102.19	1.48	10.48
sample 14	100.99	1.30	9.20
sample 15	100.19	1.32	9.34
sample 16	102.01	1.46	10.33
sample 17	99.63	1.54	10.90
sample 18	100.38	1.75	12.39
sample 19	97.60	1.24	8.80
sample 20	101.25	1.16	8.19

```
MTB > zint 95 10 c2-c21
```

One-Sample Z: sample 1; sample 2; sample 3; sample 4; sample 5; sample 6; ...

The assumed standard deviation = 10

Variable	N	Mean	StDev	SE Mean	95% CI
sample 1	50	99.10	8.96	1.41	(96.32; 101.87)
sample 2	50	100.21	10.33	1.41	(97.44; 102.98)
sample 3	50	100.36	11.10	1.41	(97.59; 103.13)
sample 4	50	98.80	10.05	1.41	(96.03; 101.57)
sample 5	50	97.21	10.02	1.41	(94.44; 99.98)
sample 6	50	100.95	8.57	1.41	(98.18; 103.72)
sample 7	50	99.62	8.33	1.41	(96.85; 102.39)
sample 8	50	102.32	8.98	1.41	(99.55; 105.10)
sample 9	50	98.65	10.53	1.41	(95.88; 101.42)
sample 10	50	99.25	10.99	1.41	(96.47; 102.02)
sample 11	50	100.28	8.93	1.41	(97.51; 103.05)
sample 12	50	101.86	9.48	1.41	(99.09; 104.63)
sample 13	50	102.19	10.48	1.41	(99.42; 104.96)
sample 14	50	100.99	9.20	1.41	(98.22; 103.76)
sample 15	50	100.19	9.34	1.41	(97.42; 102.97)
sample 16	50	102.01	10.33	1.41	(99.24; 104.78)
sample 17	50	99.63	10.90	1.41	(96.86; 102.40)
sample 18	50	100.38	12.39	1.41	(97.61; 103.15)
sample 19	50	97.60	8.80	1.41	(94.82; 100.37)
sample 20	50	101.25	8.19	1.41	(98.48; 104.02)

Confidence level and Precision of Estimation

Notice in the previous example that our choice of the 95% level of confidence was essentially arbitrary. What would have happened if we had chosen a higher level of confidence, say, 99%?
At α=0.01, we find $z_{\alpha/2}=z_{0.005}=2.58$ **, while for α=0.01** $z_{\alpha/2}=z_{0.025}=1.96$**. Thus, the length of the 95% confidence interval is**

$$2(1.96\sigma/\sqrt{n})=3.92\sigma/\sqrt{n}$$

whereas the length of the 99% CI is

$$2(2.58\sigma/\sqrt{n})=5.16\sigma/\sqrt{n}$$

Thus the 99% CI is longer that 95% CI. This is why we have a higher level of confidence in the 99% confidence interval.

- **Generally, for a fixed sample size n and standard deviation σ, the higher the confidence level, the longer the resulting CI.**
- **The length of a confidence interval is a measure of the precision of estimation.**
- **It is desirable to obtain a confidence interval that is short enough for decision-making purposes and that also has adequate confidence.**
- **One way to achieve this is by choosing the sample size n to be large enough to give a CI of specified length of precision with prescribed confidence.**

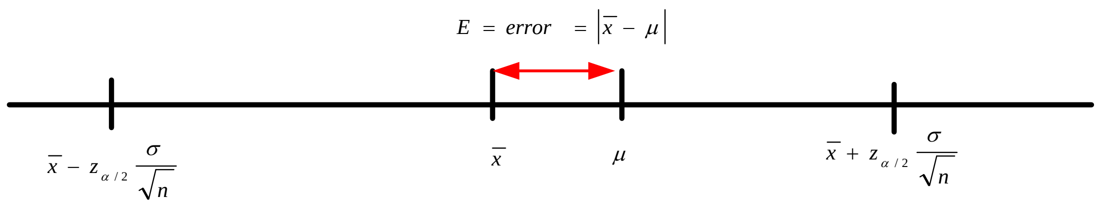
Choice of Sample Size

A $100(1-\alpha)\%$ CI on μ is given by

$$\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

The precision of the confidence interval in the previous equation is

$$2z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$



This means that in using $\bar{x}$ to estimate μ **,** the error $E = |\bar{x} - \mu|$ is less than or equal to $z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$ with confidence 100(1-α).

In situations where the sample size can be controlled, we can choose n so that we are 100(1-α) percent confident that the error in estimating μ **is less that a specified bound on the error** E.

The appropriate sample size is found by choosing n such that

$$z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = E.$$

Solving this equation gives the following formula for n.

Sample Size for Specified Error on the Mean, Variance Known

If $\bar{x}$ is used as an estimate of μ , we can be 100(1-α)% confident that the error $|\bar{x}-\mu|$ will not exceed a specified amount E when the sample size is

$$n = \left(\frac{z_{\alpha/2}\sigma}{E} \right)^2$$

Example: Consider the previous example, and suppose that we wanted to determine how many specimens must be tested to ensure that the 95% CI on μ for A238 steel cut has a length of **at most 1.0J**.

Before we obtain for n equals to 10

$$64.46 - 1.96\frac{1}{\sqrt{10}} \le \mu \le 64.46 + 1.96\frac{1}{\sqrt{10}}$$

$$63.84 \le \mu \le 65.08$$

65-08-63.84=1.24

Since the bound on error in estimation E is one-half of the length of the CI.

E=0.5, σ=1, and $z_{\alpha/2} = 1.96$

The required sample size is $n = \left(\frac{1.96}{0.5}\right)^2 = 15.37$ **and because n must be an integer, the required sample size is n=16.**

Notice the general relationship between sample size, desired length of the confidence interval 2E, confidence level 100(1-α), and standard deviation σ:

- **As the desired length of the interval 2E decreases, the required sample size n increases for a fixed value of σ and specified confidence.**
- **As σ increases, the required sample size n increases for a fixed desired length 2E and specified confidence.**
- **As the level of confidence increases, the required sample size n increases for fixed desired length 2E and standard deviationσ.**

Large-Sample Confidence Interval for μ

We have assumed that the population distribution is normal with unknown mean and known standard deviation σ. However, the standard deviation is unknown. It turns out that when n is large, replacing σ by the sample standard deviation S has little effect on the distribution of Z. This leads to the following useful result.

Large-Sample Confidence Interval on the Mean

When n is large, the quantity

$$\frac{\bar{x}-\mu}{S/\sqrt{n}}$$

has an approximate standard normal distribution. Consequently,

$$\bar{x}-z_{\alpha/2}\frac{s}{\sqrt{n}} \le \mu \le \bar{x}+z_{\alpha/2}\frac{s}{\sqrt{n}}$$

is a large sample confidence interval for μ, with confidence level of approximately 100(1-α)%.

```
MTB > random 100 c1;
SUBC> normal 10 2.
MTB > print c1
```

Data Display

```
  11.7800    8.7355    7.8383    9.8455   10.8566    7.2477   11.1385
  10.6937    8.0952   10.8427    8.0928    9.9364    9.7830    9.5375
   9.0326   10.5664   11.6442    7.0891   10.1029   10.9549    8.0954
  13.0997   12.8814    5.5040    8.2945   13.3737    9.3716    8.2152
   4.6611    9.1933    6.6900   12.8446   11.8177    8.1979    8.6519
   9.7078    9.1291    8.5152   11.7702    9.1775    9.8409    9.1542
   9.4752   10.0042   10.3100   11.7184    9.9753   10.5709    6.5498
   6.7017   10.2139    7.8130    9.7468    8.7328   13.7653    9.6900
   8.7187    5.4559   12.5540    9.1570   11.4057    9.4203    8.5856
   8.5929    9.6257    9.9148   10.4549   10.9034    9.1943    8.2688
  11.1239    6.1162    9.8353   10.1374   10.4322   12.7195   10.2830
   7.7909   14.0790   10.8794   10.7572   10.5939    8.7564    7.9499
  12.4014    6.1800    8.1428   10.7438    8.3990   11.2722   11.9227
   8.6773   11.8154   10.9196   10.7286    9.8517   10.3959   12.1236
   9.2700    9.5138
```

Descriptive Statistics: C1

```
Variable    N  N*   Mean  SE Mean  StDev  Minimum     Q1  Median      Q3
C1        100   0  9.733    0.186  1.861    4.661  8.587   9.809  10.874

Variable  Maximum
C1         14.079

zint 95 1.861 c1
```

One-Sample Z: C1

```
The assumed standard deviation = 1.861

Variable    N   Mean  StDev  SE Mean         95% CI
C1        100  9.733  1.861    0.186  (9.369; 10.098)
```

$$\bar{x} - z_{\alpha/2}\frac{s}{\sqrt{n}} \le \mu \le \bar{x} + z_{\alpha/2}\frac{s}{\sqrt{n}}$$

$$9.733 - 1.96\frac{1.861}{\sqrt{100}} \le \mu \le 9.733 + 1.96\frac{1.861}{\sqrt{100}}$$

$$9.369 \le \mu \le 10.098$$

```
MTB > zint 95 2 c1
```

One-Sample Z: C1

```
The assumed standard deviation = 2
Variable    N   Mean  StDev  SE Mean         95% CI
C1        100  9.733  1.861    0.200  (9.341; 10.125)  COMPARE
```

Confidence Interval on the Mean of a Normal Distribution, Variance Unknown

- **Suppose that the population of interest has a normal distribution with unknown mean μ and unknown variance σ^2.**
- **Assume that a random sample of size n, say $X_1, X_2, \ldots, X_n$ is available and let $\bar{X}$ and S^2 be the sample mean and variance, respectively.**
- **If the variance σ^2 is known, we know that**

$$Z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}}$$

has a standard normal distribution.

- **When σ^2 is unknown, a logical procedure is to replace σ with the sample standard deviation S. The random variable Z now becomes**

$$T = \frac{\bar{X} - \mu}{S / \sqrt{n}}$$

- **A logical question is what effect does replacing σ by S have on the distribution of the random variable T? If n is large, the answer of this question is "very little" and we can proceed to use the confidence interval based on normal distribution. However, n is usually small in most engineering problems, and in this situation a different distribution must be employed to construct the CI.**

t Distribution

Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal distribution is with unknown mean and unknown variance. The random variable

$$T = \frac{\overline{X} - \mu}{S / \sqrt{n}}$$

has a t distribution with n-1 degrees of freedom.

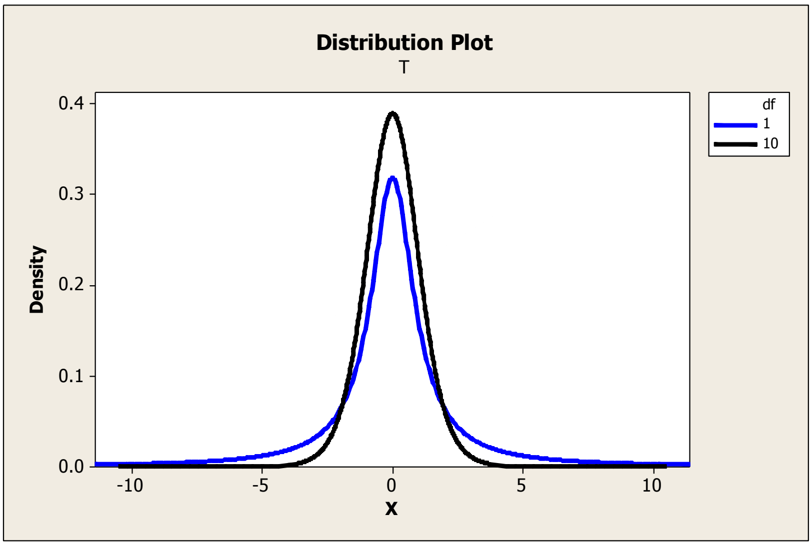
The t probability density function is

$$f(x) = \frac{\Gamma[(k+1)/2]}{\sqrt{\pi k}\,\Gamma(k/2)} \cdot \frac{1}{[(x^2/k)+1]^{(k+1)/2}} \qquad -\infty < x < \infty$$

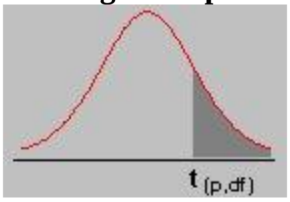
where k is the number of degrees of freedom.

The mean and variance of the t distribution are

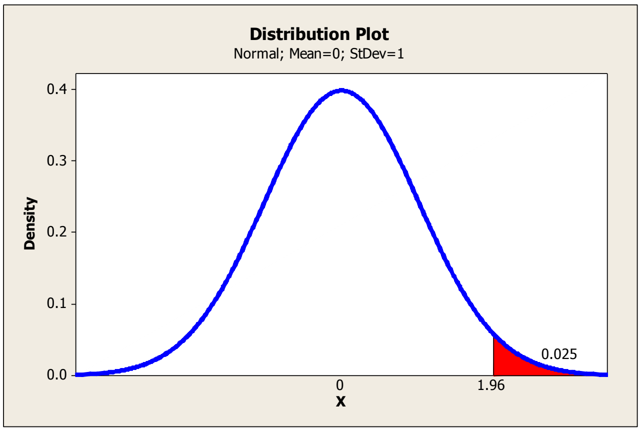
$$E(x) = 0, \qquad Var(x) = k/(k-2) \quad for\ k > 2$$

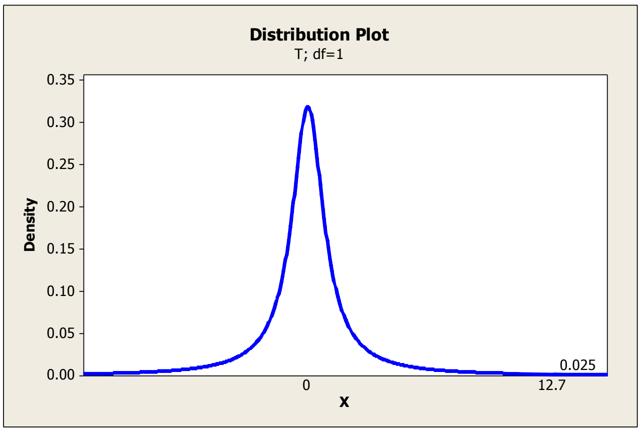


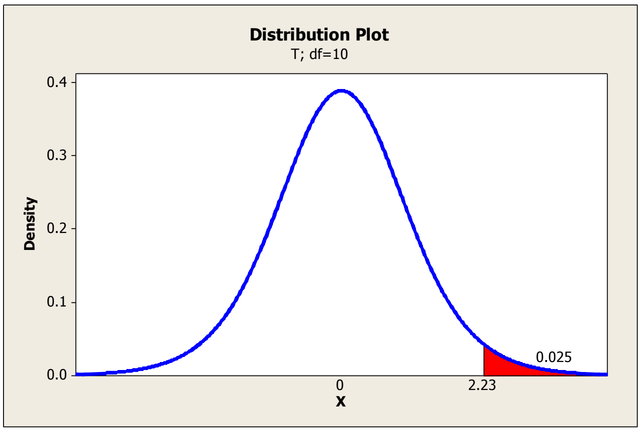
t table with right tail probabilities

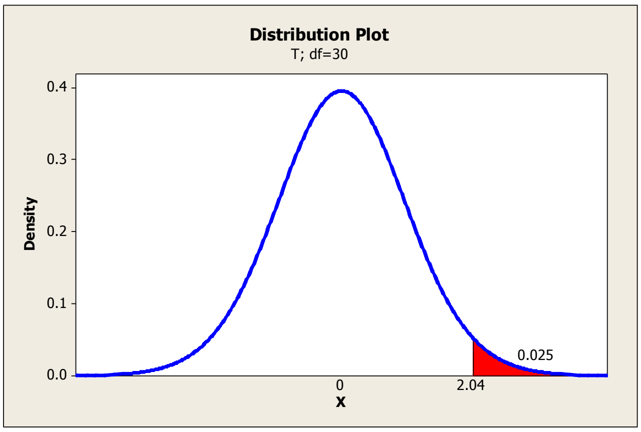


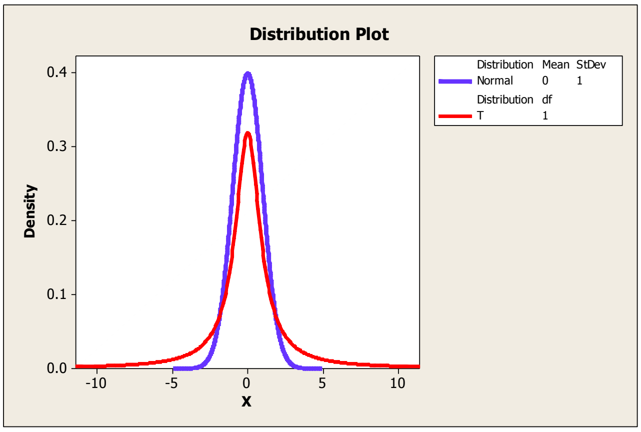
df\p	0.40	0.25	0.10	0.05	0.025	0.01	0.005	0.0005
1	0.324920	1.000000	3.077684	6.313752	12.70620	31.82052	63.65674	636.6192
2	0.288675	0.816497	1.885618	2.919986	4.30265	6.96456	9.92484	31.5991
3	0.276671	0.764892	1.637744	2.353363	3.18245	4.54070	5.84091	12.9240
4	0.270722	0.740697	1.533206	2.131847	2.77645	3.74695	4.60409	8.6103
5	0.267181	0.726687	1.475884	2.015048	2.57058	3.36493	4.03214	6.8688
6	0.264835	0.717558	1.439756	1.943180	2.44691	3.14267	3.70743	5.9588
7	0.263167	0.711142	1.414924	1.894579	2.36462	2.99795	3.49948	5.4079
8	0.261921	0.706387	1.396815	1.859548	2.30600	2.89646	3.35539	5.0413
9	0.260955	0.702722	1.383029	1.833113	2.26216	2.82144	3.24984	4.7809
10	0.260185	0.699812	1.372184	1.812461	2.22814	2.76377	3.16927	4.5869
11	0.259556	0.697445	1.363430	1.795885	2.20099	2.71808	3.10581	4.4370
12	0.259033	0.695483	1.356217	1.782288	2.17881	2.68100	3.05454	4.3178
13	0.258591	0.693829	1.350171	1.770933	2.16037	2.65031	3.01228	4.2208
14	0.258213	0.692417	1.345030	1.761310	2.14479	2.62449	2.97684	4.1405
15	0.257885	0.691197	1.340606	1.753050	2.13145	2.60248	2.94671	4.0728
16	0.257599	0.690132	1.336757	1.745884	2.11991	2.58349	2.92078	4.0150
17	0.257347	0.689195	1.333379	1.739607	2.10982	2.56693	2.89823	3.9651
18	0.257123	0.688364	1.330391	1.734064	2.10092	2.55238	2.87844	3.9216
19	0.256923	0.687621	1.327728	1.729133	2.09302	2.53948	2.86093	3.8834
20	0.256743	0.686954	1.325341	1.724718	2.08596	2.52798	2.84534	3.8495
21	0.256580	0.686352	1.323188	1.720743	2.07961	2.51765	2.83136	3.8193
22	0.256432	0.685805	1.321237	1.717144	2.07387	2.50832	2.81876	3.7921
23	0.256297	0.685306	1.319460	1.713872	2.06866	2.49987	2.80734	3.7676
24	0.256173	0.684850	1.317836	1.710882	2.06390	2.49216	2.79694	3.7454
25	0.256060	0.684430	1.316345	1.708141	2.05954	2.48511	2.78744	3.7251
26	0.255955	0.684043	1.314972	1.705618	2.05553	2.47863	2.77871	3.7066
27	0.255858	0.683685	1.313703	1.703288	2.05183	2.47266	2.77068	3.6896
28	0.255768	0.683353	1.312527	1.701131	2.04841	2.46714	2.76326	3.6739
29	0.255684	0.683044	1.311434	1.699127	2.04523	2.46202	2.75639	3.6594
30	0.255605	0.682756	1.310415	1.697261	2.04227	2.45726	2.75000	3.6460
inf	0.253347	0.674490	1.281552	1.644854	1.95996	2.32635	2.57583	3.2905

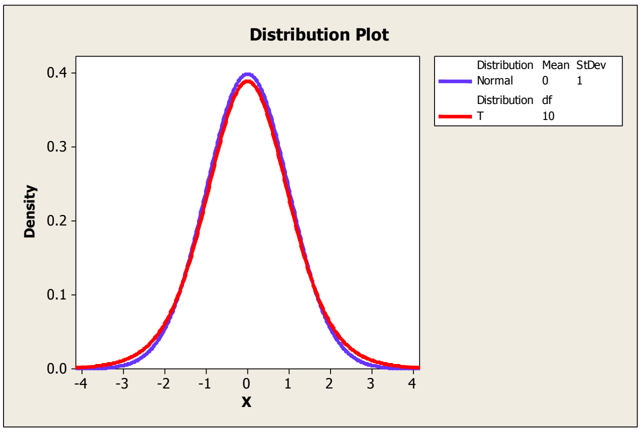

Distribution Plot
Normal; Mean=0; StDev=1
Density
0.4
0.3
0.2
0.1
0.0
0
1.96
0.025
X

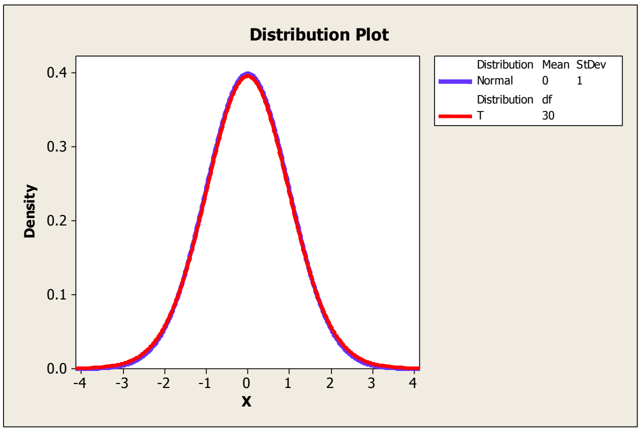

Distribution Plot
T; df=1
Density
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
0
12.7
0.025
X

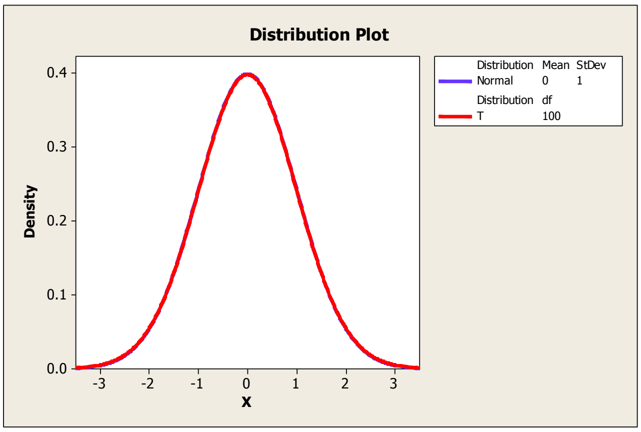

Distribution Plot
T; df=10
Density
0.4
0.3
0.2
0.1
0.0
0
2.23
0.025
X


Distribution Plot
T; df=30
Density
0.4
0.3
0.2
0.1
0.0
0
2.04
0.025
X


Distribution Plot
Distribution Mean StDev
Normal 0 1
Distribution df
T 1
Density
X


Distribution Plot
Distribution Mean StDev
Normal 0 1
Distribution df
T 10
Density
X


Distribution Plot
Distribution Mean StDev
Normal 0 1
Distribution df
T 30
Density
X


Distribution Plot
Distribution Mean StDev
Normal 0 1
Distribution df
T 100
Density
X

t Confidence Interval on μ

If $\bar{x}$ and s are the mean and standard deviation of a random sample from a normal distribution with unknown variance σ^2, a $100(1-\alpha)\%$ CI on μ is given by

$$\bar{x} - t_{\alpha/2}\frac{s}{\sqrt{n}} \le \mu \le \bar{x} + t_{\alpha/2}\frac{s}{\sqrt{n}}$$

where $t_{\alpha/2}$ is the upper $100\alpha/2$ percentage point of the t distribution with (n-1) degrees of freedom.

Example: Find 95% CI for μ for the following data.

18.0463	14.9389	24.7806	15.0582	13.6385	12.0605	23.8229
4.3229	15.3207	15.3834	14.6664	5.7122	11.4338	9.4838
15.8191	8.9311	19.3247	14.1855	13.2899	17.1080	13.5305
13.9155						

Descriptive Statistics: C1

Variable	N	N*	Mean	SE Mean	StDev	Minimum	Q1	Median	Q3	Maximum
C1	22	0	14.31	1.03	4.85	4.32	11.90	14.43	16.14	24.78

$$\bar{x} - t_{\alpha/2}\frac{s}{\sqrt{n}} \le \mu \le \bar{x} + t_{\alpha/2}\frac{s}{\sqrt{n}}$$

$$t_{0.025,21} = 2.080$$

$$14.31 - 2.080\frac{4.85}{\sqrt{22}} \le \mu \le 14.31 + 2.080\frac{4.85}{\sqrt{22}}$$

$$12.16 \le \mu \le 16.46$$

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MTB > TINT 95 C1
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One-Sample T: C1

Variable	N	Mean	StDev	SE Mean	95% CI
C1	22	14.31	4.85	1.03	(12.16; 16.46)