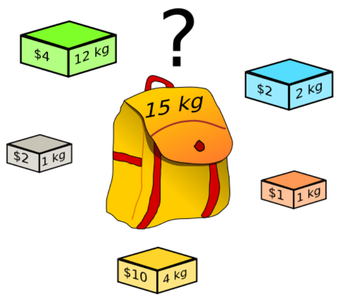
Knapsack Problem



Mathematically the 0-1-knapsack problem can be formulated as:

Maximize $\sum_{i=1}^{n} p_i x_i$

subject to

$$\sum_{i=1}^{n} w_i x_i \leq c, \qquad x_i = 0,1 \quad i = 1,\ldots,n$$

Example:

Stocko is considering four investments; Investment 1 will yield a net present value (NPV) of $16000; investment 2, NPV of $22000; investment 3, an NPV of $12000; and investment 4, an NPV of $8000. Each investment requires a certain cash flows at the present time; Investment-1: $5000; Investment-2: $7000; Investment-3: $4000 and Investment-4: $3000. At present $14000 is available for investments. Formulate an IP whose solution will tell Stocko how to maximize the NPV obtained from investments 1-4.

$$x_i\,(i=1,2,3,4)=\begin{cases}1 & \text{if investment i is made}\\ 0 & \text{otherwise}\end{cases}$$

$$\begin{aligned}&MaxZ = 16x_1+22x_2+12x_3+8x_4\\ &s.t \quad 5x_1+7x_2+4x_3+3x_4 \le 14\\ &\qquad\qquad x_i = 0\; or\; 1 \;\; (i=1,2,3,4)\end{aligned}$$

$$Z - 16x_1 - 22x_2 - 12x_3 - 8x_4 = 0$$
$$5x_1 + 7x_2 + 4x_3 + 3x_4 + x_5 = 14$$

NBV= (x_1,x_2,x_3,x_4) BV=(x_5) =14

LP Relaxation Solution

Initial Tableau

BASIS	x_1	x_2	x_3	x_4	x_5	RHS	RATIO
x_5	5	7	4	3	1	14	2<
Z	-16	-22<	-12	-8	0	0	

Entering variable: x_2
Leaving variable: x_5

First Improved Simplex Tableau

BASIS	x_1	x_2	x_3	x_4	x_5	RHS	RATIO
x_2	0.7143	1	0.5714	0.4286	0.1429	2	2.8
Z	-0.2857	0	0.5714	1.4286	3.1429	44.0	

Entering variable: x_1
Leaving variable: x_2

Final Simplex Tableau

BASIS	x_1	x_2	x_3	x_4	x_5	RHS	RATIO
x_1	1	1.4	0.8	0.6	0.2	2.8	
Z	0	0.4	0.8	1.6	3.2	44.8	

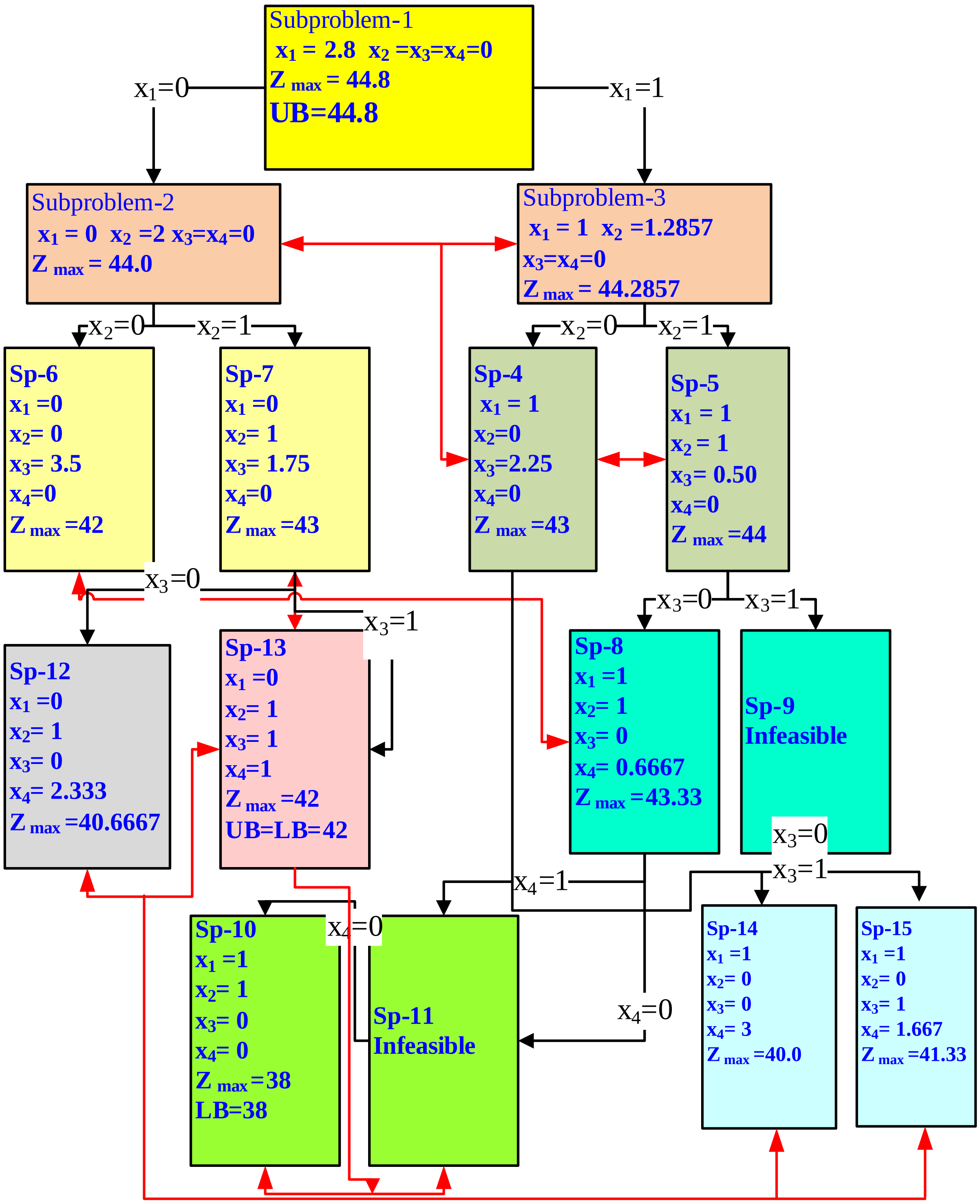
LP Relaxation Solution

BASIS	x_1	x_2	x_3	x_4	x_5	RHS	RATIO
x_5	5	7	4	3	1	14	2<
Z	-16	-22<	-12	-8	0	0	
x_2	0.7143	1	0.5714	0.4286	0.1429	2	2.8
Z	-0.2857	0	0.5714	1.4286	3.1429	44.0	
x_1	1	1.4	0.8	0.6	0.2	2.8	
Z	0	0.4	0.8	1.6	3.2	44.8	

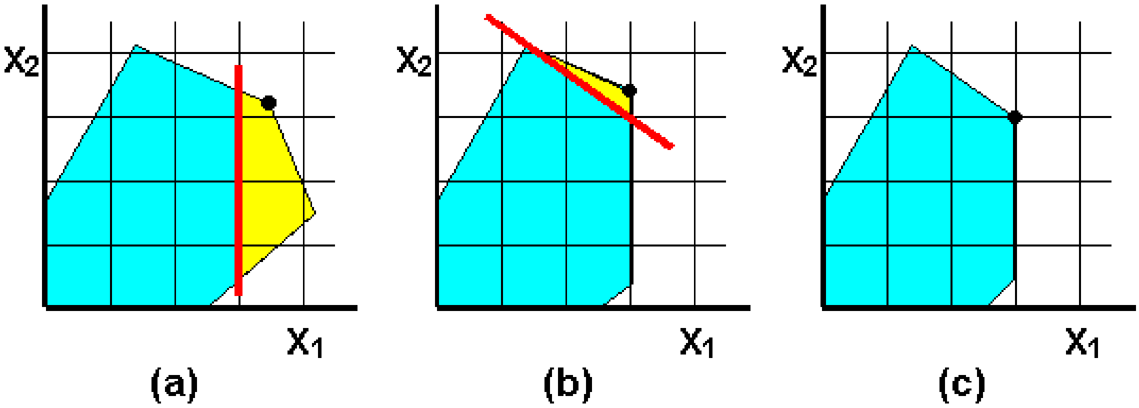
LP Relaxation Solution

$x_1=2.8 \quad x_2=x_3=x_4=0 \quad Z_{max}=44.8$

Branch and Bound Solution of the problem



Cutting Plane Algorithm



Cutting Plane Example

Cutting Plane Algorithm

Step1 ***Find the optimal tableau for the IP's linear programming relaxation. If all variables in the optimal solution assume integer values, we have found an optimal solution to the IP; otherwise, we proceed to Step 2.***

Step2. ***Pick a constraint in the LP relaxation optimal tableau whose right-hand side has the fractional part closest to ½. This constraint will be used to generate a cut.***

Step2a. ***For the constraint identified in Step 2, write the constraint's right-hand side and each variable's coefficient in the form [x] +f, where $0 \le f \le 1$.***

Step2b. ***Rewrite the constraint used to generate the cut as*** ***All terms with integer coefficients=all terms with fractional coefficients***

Then the cut is ***All terms with fractional coefficients ≤ 0***

Step3. ***Use dual simplex method to find the optimal solution to the LP relaxation with the cut as an additional constraint. If all variables assume integer values in the optimal solution, we have found an optimal solution to the IP. Otherwise, we pick the constraint with the most fractional right-hand side and use it to generate another cut, which is added to the tableau. We continue this process until a solution we obtain in which all variables are integers. This will be an optimal solution to the IP.***

Example:

$$MaxZ = 8x_1 + 5x_2$$
$$s.t \quad x_1 + x_2 \leq 6$$
$$9x_1 + 5x_2 \leq 45$$
$$x_i \geq 0 \quad \text{and integer}$$

Standard Form

$$Z - 8x_1 - 5x_2 = 0$$
$$x_1 + x_2 + x_3 = 6$$
$$9x_1 + 5x_2 + x_4 = 45$$

NBV= (x_1,x_2) BV=(x_3,x_4) =6,45

LP Relaxation Solution

BASIS	x_1	x_2	x_3	x_4	RHS	RATIO
x_3	1	1	1	0	**6**	6
x_4	**9**	5	0	1	**45**	**5**
Z	**-8**	-5	0	0	**0**	
x_3	0	**0.4444**	1	-0.111	**1**	2.25<
x_1	1	0.5556	0	0.1111	**5**	9
Z	0	**-0.5556**	0	0.8889	**40**	
x_2	0	1	2.25	-0.25	**2.25**	
x_1	1	0	-1.25	0.25	**3.75**	
Z	**0**	**0**	**1.25**	**0.75**	**41.25**	

x_1 = 3.75
x_2 = 2.25
Z = 41.25 max

To apply the cutting plane method, we begin by choosing any constraint in the LP relaxation's optimal tableau in which a basic variable is fractional.

We arbitrarily choose the second constraint, which is

$$x_1 - 1.25x_3 + 0.25x_4 = 3.75$$

We now define $[x]$ to be the largest integer less than or equal to x. For example [3.75]=3 and [-1.25]=-2.
Any number x can be written in the form $[x]+f$, where $0 \leq f < 1$. We call f the fractional part of x.
For example, 3.75=3+0.75, and -1.25=-2+0.75.
Now $x_1 - 1.25x_3 + 0.25x_4 = 3.75$ **may be written as**

$$x_1 - 2x_3 + 0.75x_3 + 0x_4 + 0.25x_4 = 3 + 0.75$$

Putting all terms with integer coefficients on the left side ans all terms with fractional coefficients on the right side yields

$$x_1 - 2x_3 + 0x_4 - 3 = 0.75 - 0.75x_3 - 0.25x_4$$

The cutting plane algorithm now suggests adding the following constraint to the LP relaxation's optimal tableau:

Right side ≤0

$$0.75 - 0.75x_3 - 0.25x_4 \leq 0$$

This constraint is called a cut.

$$Z - 8x_1 - 5x_2 = 0$$

$$x_1 + x_2 + x_3 = 6$$

$$9x_1 + 5x_2 + x_4 = 45$$

$$3x_1 + 2x_2 + x_5 = 15$$

Putting

$x_3 = 6 - x_1 - x_2$ and $x_4 = 45 - 9x_1 - 5x_2$

in

$$0.75 - 0.75x_3 - 0.25x_4 \leq 0$$

$$0.75 - 0.75(6 - x_1 - x_2) - 0.25(45 - 9x_1 - 5x_2) \leq 0$$

$$0.75 - 4.5 + 0.75x_1 + 0.75x_2 - 11.25 + 2.25x_1 + 1.25x_2 \leq 0$$

Then the cut may be written as

$$3x_1 + 2x_2 \leq 15.$$

Adding this cut to the original problem yields

$$
\begin{aligned}
&MaxZ = 8x_1 + 5x_2 \\
&s.t \quad x_1 + x_2 \le 6 \\
&\quad 9x_1 + 5x_2 \le 45 \\
&\quad 3x_1 + 2x_2 \le 15 \\
&\quad x_i \ge 0
\end{aligned}
$$

Standard Form

$$
\begin{aligned}
&Z - 8x_1 - 5x_2 = 0 \\
&x_1 + x_2 + x_3 = 6 \\
&9x_1 + 5x_2 + x_4 = 45 \\
&3x_1 + 2x_2 + x_5 = 15
\end{aligned}
$$

NBV= (x_1,x_2) BV=(x_3,x_4,x_5) =6,45,15

Addition of a Constraint

Suppose after solving the problem we wish to alter the original problem by the addition of a new constraint. Now it could be that X* satisfies this new constraint. If this is the case, X* is also optimal for the expanded problem, because clearly, by this addition of a constraint, we have not changed the objective function nor increased the set of feasible solutions to the system of constraints. On the other hand, if X* does not satisfy this new constraint, we must find a new optimal solution. Under certain circumstances, however, this problem may be resolved quite easily by creating a new canonical tableau (the new constant column b' may contain some negative entries) from the final tableau solution to the original problem and the application of the **Dual Simplex Algorithm**.

Solution of the new model

	x_1	x_2	x_3	x_4	x_5	RHS	RATIO
x_3	1	1	1	0	**0**	**6**	6
x_4	**9**	5	0	1	**0**	**45**	5
x_5	**3**	2	0	0	**1**	**15**	5
Z	**-8**	-5	0	0	**0**	**0**	
x_3	0	0.4444	1	-0.111	**0**	**1**	2.25
x_1	1	0.5556	0	0.1111	**0**	**5**	9
x_5	0	**0.3333**	0	-0.333	**1**	**0**	0
Z	0	**-0.5556**	0	0.8889	**0**	**40**	
x_3	0	0	1	0.3333	-1.333	**1**	
x_1	1	0	0	0.6667	-1.667	**5**	
x_2	0	1	0	-1	3	**0**	
Z	**0**	**0**	**0**	**0.3333**	**1.667**	**40**	

$x_1 = 5$

$x_2 = 0$

$Z = 40$ max

We obtain integer solution

By Dual Simplex

$$3x_1 + 2x_2 \leq 15$$

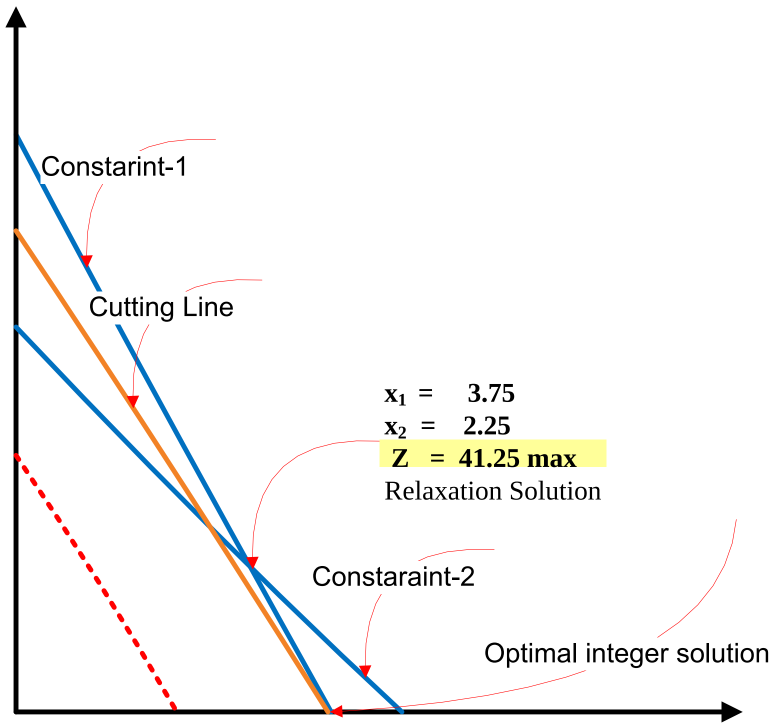
$$-3x_1 - 2x_2 \geq -15$$

$$-3x_1 - 2x_2 - x_5 = -15$$

BASIS	x_1	x_2	x_3	x_4	x_5	RHS	RATIO
x_3	1	1	1	0		6	6
x_4	9	5	0	1		45	5
Z	-8	-5	0	0		0	
x_3	0	0.4444	1	-0.111		1	2.25<
x_1	1	0.5556	0	0.1111		5	9
Z	0	-0.5556	0	0.8889		40	
x_2	0	1	2.25	-0.25		2.25	
x_1	1	0	-1.25	0.25		3.75	
x_5	-3	-2	0	0	-1	-15	<<<
Z	0	0	1.25	0.75	0	41.25	
x_2	0	1	2.25	-0.25	0	2.25	
x_1	0	2/3	-1.25	0.25	1/3	-1.25	<<<
x_1	1	2/3	0	0	1/3	5	
Z	0	0	1.25	0.75	0	41.25	
x_2	0	2.2	0	0.2	0.594	0	
x_3	0	-0.533	1	-0.2	-0.264	1	
x_1	1	2/3	0	0	1/3	5	
Z	0	0	0	1	0	40	

Solution of this problem again gives

x_1 = 5
x_2 = 0
Z = 40 max


Constarint-1
Cutting Line
x_1 = 3.75
x_2 = 2.25
Z = 41.25 max
Relaxation Solution
Constaraint-2
Optimal integer solution