MATRICES

AND

MATRIX OPERATIONS

II

MATRIX INVERSION

Definition (Invertible Matrix)

If A is a square $n \times n$ matrix, and if a matrix B of the same size can be found such that

$$AB = BA = I$$

then A is said to be invertible and B is called an inverse of A. If no such matrix B can be found, then A is said to be singular.

Example: Verifying the Inverse Requirements

The matrix

$B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$ is an inverse of $A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$ since

$$AB = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \text{ and}$$

$$BA = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

```
>> B=[3 5; 1 2]
B =
    3    5
    1    2
>> A=[2 -5;-1 3]
A =
    2   -5
   -1    3
>> A*B
ans =
    1    0
    0    1
>> B*A
ans =
    1    0
    0    1
```

Example: A matrix with No Inverse

The matrix

$$A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$$

is singular. To see why, let

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

be any 3 x 3 matrix. The third column of BA is

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus

$$BA \neq I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Theorem

If B and C are both inverses of the matrix A, then B=C.

(An invertible matrix has **exactly one** *inverse.)*

Theorem

The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ ***is invertible if*** $ad - bc \neq 0$***, in which case the inverse is given by the formula***

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \dfrac{d}{ad-bc} & \dfrac{-b}{ad-bc} \\ \dfrac{-c}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix}$$

$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ $\quad$ $A^{-1} = \frac{1}{3-2}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$

```
>> A=[1 2 ; 1 3]    A =     1     2
                            1     3
>> inv(A)
ans =
    3   -2
   -1    1
>>
```

Theorem

If A and B are invertible matrices of the same size, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

(A product of any number of invertible matrices is invertible, and inverse of the product is the product of the inverses in the reverse order.)

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} \qquad AB = \begin{bmatrix} 7 & 6 \\ 9 & 8 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix} \qquad (AB)^{-1} = \begin{bmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix}$$

Also,

$$B^{-1}A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix}$$

Therefore $(AB)^{-1}=B^{-1}A^{-1}$, as guaranteed by Theorem.

Definition (**Powers of a Matrix**)

If A is a square matrix, then we define the nonnegative integer powers of A to be

$$A^0 = I \quad A^n = AA\ldots A \quad \textit{(n factors n >0)}$$

Moreover, if A is invertible, then we define the negative integer powers to be

$$A^{-n} = (A^{-1})^n = A^{-1}A^{-1}\ldots A^{-1}$$

Theorem (**Laws of Exponents**)

If A is a square matrix and r and s are integers, then

- $A^r A^s = A^{r+s}, \quad (A^r)^s = A^{rs}$

Theorem (**Laws of Exponents)**

If A is an invertible matrix, then;

- A^{-1} ***is invertible and*** $(A^{-1})^{-1} = A$
- A^n ***is invertible and*** $(A^n)^{-1} = (A^{-1})^n$ ***for n=0,1,2..***
- ***For any nonzero scalar k, the matrix kA is invertible and***

$$(kA)^{-1} = \frac{1}{k}A^{-1}$$

Example: (Powers of a Matrix)

Consider the matrices,

$$A=\begin{bmatrix}1 & 2\\ 1 & 3\end{bmatrix} \quad \text{and} \quad A^{-1}=\begin{bmatrix}3 & -2\\ -1 & 1\end{bmatrix}.$$

Then

$$A^3=\begin{bmatrix}1 & 2\\ 1 & 3\end{bmatrix}\begin{bmatrix}1 & 2\\ 1 & 3\end{bmatrix}\begin{bmatrix}1 & 2\\ 1 & 3\end{bmatrix}=\begin{bmatrix}11 & 30\\ 15 & 41\end{bmatrix}$$

$$A^{-3}=(A^{-1})^3=\begin{bmatrix}3 & -2\\ -1 & 1\end{bmatrix}\begin{bmatrix}3 & -2\\ -1 & 1\end{bmatrix}\begin{bmatrix}3 & -2\\ -1 & 1\end{bmatrix}=\begin{bmatrix}41 & -30\\ -15 & 11\end{bmatrix}$$

$$(A^3)^{-1}=\begin{bmatrix}11 & 30\\ 15 & 41\end{bmatrix}^{-1}=\begin{bmatrix}41 & -30\\ -15 & 11\end{bmatrix}$$

```
>> A=[1 2 ;1 3]

A =

     1     2

     1     3

>> A^3

ans =

    11    30

    15    41

>> inv(A)

ans =

     3    -2

    -1     1

>> (inv(A))^3

ans =

    41   -30

   -15    11

>> inv(A^3)

ans =

   41.0000  -30.0000

  -15.0000   11.0000

>>
```

Example: (Matrix Polynomial)

If $p(x) = 2x^2 - 3x + 4$ and $A = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}$

then

$$p(A) = 2A^2 - 3A + 4I =$$

$$2\begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}^2 - 3\begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 2 & 8 \\ 0 & 18 \end{bmatrix} - \begin{bmatrix} -3 & 6 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ 0 & 13 \end{bmatrix}$$

```
> A=[-1 2; 0 3]
A =
   -1     2
    0     3
>> I=[1 0; 0 1]
I =
    1     0
    0     1
>> 2*A^2-3*A+4*I
ans =
    9     2
    0    13
>>
```

Theorem

If A is an invertible matrix, then A^T is also invertible and

$$(A^T)^{-1} = (A^{-1})^T$$

We can prove the invertibility of A^T and obtain $(A^T)^{-1} = (A^{-1})^T$ by showing that

$$A^T (A^{-1})^T = (A^{-1})^T A^T = I$$

We know $I^T = I$ and $(AB)^T = B^T A^T$, we have

$$A^T (A^{-1})^T = (A^{-1}A)^T = I^T = I$$

$$(A^{-1})^T A^T = (A\ A^{-1})^T = I^T = I$$

this completes the proof.

Example:

Consider the matrices,

$$A = \begin{bmatrix} -5 & -3 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad A^T = \begin{bmatrix} -5 & 2 \\ -3 & 1 \end{bmatrix}.$$

$$A^{-1} = \begin{bmatrix} 1 & 3 \\ -2 & -5 \end{bmatrix}, \qquad (A^{-1})^T = \begin{bmatrix} 1 & -2 \\ 3 & -5 \end{bmatrix}$$

$$(A^T)^{-1} = \begin{bmatrix} 1 & -2 \\ 3 & -5 \end{bmatrix}$$

Definition (Elementary Matrix)

An nxn matrix is called an elementary matrix if it can be obtained from the nxn identity matrix I_n by performing a single elementary row operation.

Example: (Elementary Matrices and Row Operations)

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Multiply the second row of I_2 by -1.

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Interchange the second and fourth rows of I_4.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Add 3 times the third row of I_3 to the first row.

Theorem (Row operations by Matrix Multiplication)

If the elementary matrix E results from performing a certain row operation on I_m and if A is an $m \times n$ matrix, then the product EA is the matrix that results when this same row operation is performed on A.

Example: (Using Elementary Matrices)

Consider the matrix,

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix}$$ and the matrix

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$ which results from adding 3

times the first row of I_3 to the third row.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The product EA is

$$EA = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 4 & 4 & 10 & 9 \end{bmatrix}.$$

This is precisely the same matrix that results when we add the times the fist row of A to the third row.

Example

```
A=[1 2 3;3 4 5;7 8 9]

A =
     1     2     3
     3     4     5
     7     8     9
>> k=1
k =
     1
>> r=2
r =
     2
>> A([r k],:)=A([k r],:)
A =
     3     4     5
     1     2     3
     7     8     9
```

Row Operations and Inverse Row Operations

Row operations on I that produces E.	Row operations on E that reproduces I.
$\textit{Multiply row i by } c \neq 0$	$\textit{Multiply row i by } 1/c$
$\textit{Interchange rows i and j}$	$\textit{Interchange rows i and j}$
$\textit{Add c times row i to row } j$	$\textit{Add } -c \textit{ times row i to row } j$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Multiply the second row by 7.   Multiply the second row by 1/7.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Interchange the first and second rows. Interchange the first and second rows.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Add 5 times the second row to the first. Add -5 times the second row to the first.

Theorem

Every elementary matrix is invertible, <u>and the inverse is also an elementary matrix.</u>

$$E_0E = I \quad \textit{and} \quad EE_0 = I$$

Thus the elementary matrix E_0 ***is the inverse of*** E.

Example

```
>> E1=[1 0 0; 0 1 0; 3 0 1]
E1 =
     1     0     0
     0     1     0
     3     0     1
>> E0=inv(E1)
E0 =
     1     0     0
     0     1     0
    -3     0     1
```

```
>> E0*E1
ans =
     1     0     0
     0     1     0
     0     0     1
>> E1*E0
ans =
     1     0     0
     0     1     0
     0     0     1
>>
```

Properties of Invertible Matrices

Theorem: ***Let A be a square matrix***

(a) If B is a square matrix satisfying BA=I, then $B=A^{-1}$.

(b) If B is a square matrix satisfying AB=I, then $B=A^{-1}$.

Theorem: ***Equivalent Statements***

If A is $n \times n$ matrix, then the following are equivalent.

(a) A is invertible.

(b) Ax=0 has only the trivial solution.

(c) The reduced row-echelon form of A is I_n.

(d) A is expressible as a product of elementary matrices.

(e) Ax=b is consistent for every $n \times 1$ matrix b .

(f) Ax=b has exactly one solution for every $n \times 1$ matrix b.

Orthogonal Matrices

A real orthogonal matrix is a square matrix Q whose transpose is its inverse.

$$Q^T Q = QQ^T = I$$

Examples

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0.96 & -0.28 \\ 0.28 & 0.96 \end{bmatrix} \qquad \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} 0 & -0.80 & -0.60 \\ 0.80 & -0.36 & 0.48 \\ 0.60 & 0.48 & -0.64 \end{bmatrix}$$

If Q has orthonormal columns, then

$$Q^T Q = QQ^T = I$$

Any permutation matrix P is an orthogonal matrix. The columns are certainly unit vectors and certainly orthogonal- because the 1 appears in a different place in each column.

If $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ **then** $P^{-1} = P^T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Using Row Operations to Find A^{-1}

To find the inverse of an invertible matrix A, we must find a sequence of elementary row operations that reduces A to the identity and then perform this same sequence of operations on I_n to obtain A^{-1}.

Example:

Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

First produce a matrix of the form

$$[A \mid I]$$

Then apply row operations to this matrix until the left side is reduced to I ; these operations will convert the right side to A^{-1}, so the final matrix will have the form

$$[I \mid A^{-1}]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

Add -2 times the first row to the second and -1 times the first row to the third.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$$

Add 2 times the second row to the third.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

Multiply the third row by -1.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

Add 3 times the third row to the second and -3 times the third row to the first.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

Add -2 times the second row to the first.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

Thus

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

We showed that $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$ is invertible matrix. It follows that the homogenous system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 2x_1 + 5x_2 + 3x_3 &= 0 \\ x_1 \qquad + 8x_3 &= 0 \end{aligned}$$

has only **trivial** solution.

Example: (Showing that a Matrix is not Invertible)

Consider the matrix

$$A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$$

Add -2 times the first row to the second and add the first row to the third.

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array}\right]$$

Add the second row to the third.

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right]$$

Since we have obtained a row of zeros on the left side, A is not invertible.

Diagonal Matrices

Definition: (Diagonal Matrices)

A square matrix in which all the entries off the main diagonal are zero is called a diagonal matrix.

Some examples

$$\begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

A general $n \times n$ ***diagonal matrix D can be written as***

$$D = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ . & & & \\ . & & & \\ 0 & 0 & \dots & d_n \end{bmatrix}$$

A diagonal matrix is invertible if and only if all of its diagonal entries are nonzero; in this case the inverse of D is

$$D^{-1} = \begin{bmatrix} 1/d_1 & 0 & \dots & 0 \\ 0 & 1/d_2 & \dots & 0 \\ . & & & \\ . & & & \\ 0 & 0 & \dots & 1/d_n \end{bmatrix}$$

$$DD^{-1} = D^{-1}D = I$$

If D is the diagonal matrix and k is a positive integer; then

$$D^{k} = \begin{bmatrix} d_1^k & 0 & \dots & 0 \\ 0 & d_2^k & \dots & 0 \\ . & & & \\ . & & & \\ 0 & 0 & \dots & d_n^k \end{bmatrix}$$

Example: **(Inverses and Powers of Diagonal Matrices)**

If

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

then

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \qquad A^{-5} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{243} & 0 \\ 0 & 0 & \frac{1}{32} \end{bmatrix}$$

$$A^{5} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -243 & 0 \\ 0 & 0 & 32 \end{bmatrix}$$

```
>> A=[1 0 0; 0 -3 0; 0 0 2]
A =
     1     0     0
     0    -3     0
     0     0     2

>> inv(A)
ans =
    1.0000         0         0
         0   -0.3333         0
         0         0    0.5000
>> (inv(A))^5
ans =
    1.0000         0         0
         0   -0.0041         0
         0         0    0.0313
>> A^5
ans =
     1     0     0
     0  -243     0
     0     0    32
>>
```

RULE

To multiply a matrix A on the left by a diagonal matrix D, one can multiply <u>successive rows</u> of A by the successive diagonal entries of D, and to multiply A on the right by D, one can multiply <u>successive columns</u> of A by the successive diagonal entries of D.

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} d_1a_{11} & d_1a_{12} & d_1a_{13} & d_1a_{14} \\ d_2a_{21} & d_2a_{22} & d_2a_{23} & d_2a_{24} \\ d_3a_{31} & d_3a_{32} & d_3a_{33} & d_3a_{34} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} = \begin{bmatrix} d_1a_{11} & d_2a_{12} & d_3a_{13} \\ d_1a_{21} & d_2a_{22} & d_3a_{23} \\ d_1a_{31} & d_2a_{32} & d_3a_{33} \\ d_1a_{41} & d_2a_{42} & d_3a_{43} \end{bmatrix}$$

```
>> V=[1 2 3]
V =
    1    2    3
>> D=diag(V)
D =
    1    0    0
    0    2    0
    0    0    3
>> A=[1 1 1 1;2 2 2 2 ; 3 3 3 3 ]
A =
    1    1    1    1
    2    2    2    2
    3    3    3    3
>> D*A
ans =
    1    1    1    1
    4    4    4    4
    9    9    9    9
> B=[1 1 1 ; 2 2 2 ;3  3 3;4 4 4 ]
B =
    1    1    1
    2    2    2
    3    3    3
    4    4    4
>> B*D
ans =
    1    2    3
    2    4    6
    3    6    9
    4    8   12
```

Triangular Matrices

A square matrix in which all the entries <u>above</u> the main diagonal are zero is called <u>lower triangular</u>, and a square matrix in which all the entries <u>below</u> the main diagonal are zero is called <u>upper triangular.</u>

A matrix that is either upper triangular or lower triangular is called <u>triangular</u>.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

- **A square matrix $A=[a_{ij}]$ is <u>upper triangular</u> if and only if ith row starts with at least $i-1$ zeros.**
- **A square matrix $A=[a_{ij}]$ is <u>lower triangular</u> if and only if jth column starts with at least $j-1$ zeros.**
- **A square matrix $A=[a_{ij}]$ is <u>upper triangular</u> if and only if $a_{ij}=0$ for $i>j$.**
- **A square matrix $A=[a_{ij}]$ is <u>lower triangular</u> if and only if $a_{ij}=0$ for $i<j$.**

Theorem:

(a) The transpose of a lower triangular matrix is upper triangular, and the transpose of an upper triangular matrix is lower triangular.

(b) The product of lower triangular matrices is lower triangular, and the product of upper triangular matrices is upper triangular.

(c) A triangular matrix is invertible if and only its diagonal entries are all nonzero.

(d) The inverse of an invertible lower triangular matrix is lower triangular, and the inverse of an invertible upper triangular matrix is upper triangular.

Example: (Upper Triangular Matrices)

Consider the upper triangular matrices

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix **A is invertible**, since its diagonal entries are nonzero, but the matrix **B is not**.

$$A^{-1} = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{7}{5} \\ 0 & \frac{1}{2} & -\frac{2}{5} \\ 0 & 0 & \frac{1}{5} \end{bmatrix} \qquad AB = \begin{bmatrix} 3 & -2 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

```
>> A=[1 3 -1; 0 2 4; 0 0 5]
A =
   1   3  -1
   0   2   4
   0   0   5
```

```
>> B=[3 -2 2; 0 0 -1; 0 0 1]
B =
     3    -2     2
     0     0    -1
     0     0     1
>> inv(A)
ans =
    1.0000   -1.5000    1.4000
         0    0.5000   -0.4000
         0         0    0.2000
>> inv(B)
Warning: Matrix is singular to working precision.
```

The matrix **A is invertible**, since its diagonal entries are nonzero, but the matrix **B is not**.

```
>> A*B
ans =
     3    -2    -2
     0     0     2
     0     0     5
>> B*A
ans =
     3     5    -1
     0     0    -5
     0     0     5
>>
```

Definition: **(Symmetric Matrices)**

A square matrix A is called Symmetric if $A=A^T$.

Some examples

$$\begin{bmatrix} 7 & -3 \\ -3 & 5 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 4 & 5 \\ 4 & -3 & 0 \\ 5 & 0 & 7 \end{bmatrix},$$

$$\begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}$$

The entries on the main diagonal may be arbitrary, but "mirror images" of entries across the main diagonal must be equal.

Theorem:

If A and are B symmetric matrices with the same size, and if k is any scalar, then;

(a) A^T is symmetric.

(b) A+B and A-B are symmetric.

(c) kA is symmetric.

***Remark** It is not true, in general, that the product of symmetric matrices is symmetric. However, in the special case where AB=BA, then we say that A and B commute. The product of two matrices is symmetric if and only if the matrices commute.*

Theorem

If A is an invertible symmetric matrix, then A^{-1} is symmetric.

$$A = A^T \qquad (A^{-1})^T = (A^T)^{-1} = A^{-1}$$

Products AA^T and A^TA

Matrix products of the form AA^T and A^TA arise in a variety of applications. If A is an $m \times n$, then A^T is an $n \times m$ matrix, so the products AA^T and A^TA are both square matrices the matrix AA^T has size $m \times m$ and the matrix A^TA $n \times n$. <u>Such products are always symmetric since</u>

$$(AA^T)^T = (A^T)^T A^T = AA^T \quad \text{and}$$

$$(A^T A)^T = A^T (A^T)^T = A^T A.$$

(Sum of Squares and Products Matrix)

```
MTB > print m1

 Matrix M1
Weight    Height
53.1269  161.589
58.1104  167.153
59.8191  168.381
63.3447  161.360
55.0703  153.585
53.2396  159.913
66.1300  162.297
66.1616  152.520
58.1226  159.801
59.1607  161.252

MTB > tran m1 m2

MTB > print m2
```

Data Display

```
 Matrix M2

 53.127   58.110   59.819   63.345   55.070   53.240   66.130
66.162   58.123
161.589  167.153  168.381  161.360  153.585  159.913  162.297
152.520  159.801
 59.161
161.252

MTB > mult m2 m1 m3
MTB > print m3

MTB > print m3
```

Data Display

```
 Matrix M3

35286.1    95215
95214.9   258742

MTB > let c3=c1*c1
MTB > sum c3
```

Sum of C3

Sum of C3 = 35286.1

Theorem

If A is an invertible matrix, then AA^T and A^TA are also invertible.

(AA^T and A^TA are invertible since they are products of invertible matrices.)

Example (The product of a Matrix and its Transpose is Symmetric)

Let A be the 2x3 matrix

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix}$$ then

$$A^T A = \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix} = \begin{bmatrix} 10 & -2 & 11 \\ -2 & 4 & -8 \\ -11 & -8 & 41 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} 21 & -17 \\ -17 & 34 \end{bmatrix}$$

AA^T and **A^TA** are symmetric as expected.