ALTERNATIVE SIMPLEX METHOD

(DUAL SIMPLEX METHOD)

Recall that the method deals with situations where we have a simplex tableau with the following features:

- **Some of the right-hand side values (b_i) are negative**
- **All the reduced costs satisfy the optimality condition.**

The method attempts to restore feasibility (make the right-hand side values non-negative) without forcing the reduced costs to violate the optimality condition.

Dual simplex algorithm is just the opposite of the primal simplex algorithm.

If it fails, the conclusion is that the problem is infeasible.

Procedure:

1. **Find a negative basic variable.**

If there is none we have the optimal solution; if there is more than one find the most negative. Suppose this variable is the basic variable in the r. constraint. This gives the variable to come out of the basis.

2. **In row r look for negative coefficients a'_{rj}.**

If there are none there is no feasible solution to the problem. For negative coefficients a'_{rj} in this row find the

$$\min \left| \frac{c'_j}{a'_{rj}} \right| .$$

3. **Carry out the usual Simplex Transformation with a'_{rs} as pivot.**

Dual simplex method differs from the Simplex Method only in the way in which it selects the variables to leave and enter (in that order) the basis.

Example :

$$\textbf{Min Z} = 4\,x_1 + 6\,x_2 + 18\,x3$$

$$\begin{aligned} \text{s.t.} \quad x_1 + \qquad\quad 3\,x_3 &\geq 3 \\ x_2 \quad + 2\,x_3 &\geq 5 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

$$\textbf{Min Z} = 4\,x_1 + 6\,x_2 + 18\,x3$$

$$\begin{aligned} \text{s.t.} \quad -x_1 - \qquad\quad 3\,x_3 &\leq -3 \\ -\,x_2 \quad -2\,x_3 &\leq -5 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

$$\begin{aligned} -x_1 - \qquad\quad 3\,x_3 \quad + x_4 \qquad &= -3 \\ -x_2 \qquad - 2x_3 \qquad + x_5 &= -5 \end{aligned}$$

NBV=(x_1,x_2,x_3) BV(x_4,x_5)=-3,-5 **(No feasible)**

Full Simplex Solution

$$\textbf{Min Z} = 4x_1 + 6x_2 + 18x_3$$

$$\begin{aligned} \text{s.t.} \quad x_1 + \quad & 3x_3 \geq 3 \\ x_2 \quad & +2x_3 \geq 5 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

$$\begin{aligned} \mathbf{x_1 + \quad 3x_3 \quad -x_4 + x_6 \quad = 3} \\ \mathbf{x_2 \quad + 2x_3 \quad - x_5 \quad + x_7 \ = 5} \end{aligned}$$

$$\mathbf{w = x_6 + x_7}$$

$$\mathbf{-w - x_1 - x_2 - 5x_3 + x_4 + x_5 = -8}$$

Dual Problem

$$\mathbf{Max\ Z' = 3y_1 + 5y_2}$$

$$\begin{aligned} \text{s.t.} \quad y_1 \quad & \leq 4 \\ y_2 \quad & \leq 6 \\ 3y_1 + 2y_2 \quad & \leq 18 \end{aligned}$$

$$y_1, y_2 \geq 0$$

Two-phase(Full) simplex

BASIS	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS	RATIO
x_6	1	0	3<	-1	0	1	0	**3**	1
x_7	0	1	2	0	-1	0	1	**5**	2.5<
-w	**-1**	**-1**	**-5<**	**1**	**1**	**0**	**0**	**-8**	
-Z	**4**	**6**	**18**	**0**	**0**	**0**	**0**	**0**	
x_3	1/3	0	1	-1/3	0	**1/3**	**0**	**1**	
x_7	2/3	1	0	2/3	-1	**-2/3**	**1**	**3**	3<
-w	**2/3**	**-1**	**0**	**-2/3**	**1**	**5/3**	**0**	**-3**	
-Z	**-2**	**6**	**0**	**6**	**0**	**-6**	**0**	**-18**	
x_3	1/3	0	1	-1/2	0	1/3	0	**1**	
x_2	-2/3	1	0	2/3	-1	2/3	1	**3**	
-w	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**0**	
-Z	**2**	**0**	**0**	**2**	**6**	**-2**	**-6**	**-36**	

Dual Solution (with simplex)

BASIS	y_1	y_2	y_3	y_4	y_5	RHS	RATIO
y_3	1	0	1	0	0	**4**	-
y_4	0	1<	0	1	0	**6**	6<
y_5	3	2	0	0	1	**18**	9
Z'	**-3**	**-5<**	**0**	**0**	**0**	**0**	
y_3	1	0	1	0	0	**4**	4
y_2	0	1	0	1	0	**6**	-
y_5	3	0	0	-2	1	**6**	2
Z'	**-3**	**0**	**0**	**5**	**0**	**30**	
y_3	0	0	1	2/3	-1/3	2	
y_2	0	1	0	1	0	6	
y_1	1	0	0	-2/3	1/3	2	
Z'	0	0	0	3	1	36	

NEW SOLUTION (Alternative Simplex)

Initial tableau for the alternative simplex method

BASIS	x_1	x_2	x_3	x_4	x_5	RHS
x_4	-1	0	-3	1	0	**-3**
x_5	0	-1<	-2	0	1	**-5<**
-Z	**4**	**6**	**18**	**0**	**0**	**0**
RATIO		**6<**	**9**			

Leaving variable x_5

Entering variable x_2

Initial and the first tableau for the alternative simplex method

BASIS	x_1	x_2	x_3	x_4	x_5	RHS
x_4	-1	0	-3	1	0	**-3**
x_5	0	-1<	-2	0	1	**-5<**
-Z	**4**	**6**	**18**	**0**	**0**	**0**
RATIO		**6<**	**9**			
x_4	-1	0	-3<	1	0	**-3<**
x_2	0	1	2	0	-1	**5**
-Z	**4**	**0**	**6**	**0**	**6**	**-30**
RATIO	**4**		**2<**			

Leaving variable x_4

Entering variable x_3

Initial and the first two table for the alternative simplex method

BASIS	x_1	x_2	x_3	x_4	x_5	RHS
x_4	-1	0	-3	1	0	-3
x_5	0	-1<	-2	0	1	-5<
-Z	4	6	18	0	0	0
RATIO		6<	9			
x_4	-1	0	-3<	1	0	-3<
x_2	0	1	2	0	-1	5
-Z	4	0	6	0	6	-30
RATIO	4		2<			
x_3	1/3	0	1	-1/3	0	1
x_2	-2/3	1	0	2/3	-1	3
-Z	2	0	0	2	6	-36

Solution :

$X_1 = 0$; $X_2 = 3$; $X_3 = 1$ $Z_{min} = 36$

Example :

$$\textbf{Min Z} = x_1 + x_2$$

s. t. $x_1 + 2x_2 \geq 6$

$2x1 + x_2 \geq 6$

$7x_1 + 8x_2 \leq 56$

$x_1, x_2 \geq 0$

$$-Z + x1 + x2 = 0$$

$$-x_1 - 2x_2 + x_3 = -6$$
$$2x_1 - x_2 + x_4 = -6$$
$$7x_1 + 8x_2 + x_5 = 56$$

BASIS	x_1	x_2	x_3	x_4	x_5	RHS
x_3	-1	-2	1	0	0	**-6**
x_4	-2	-1	0	1	0	**-6<**
x_5	7	8	0	0	1	**56**
-Z	**1**	**1**	**0**	**0**	**0**	**0**
RATIO	**0.5<**	**1**				
x_3	0	-3/2<	1	-1/2	0	**-3**
x_1	1	½	0	-1/2	0	**3**
x_5	0	9/2	0	7/2	1	**35**
-Z	**0**	**1/2**	**0**	**1/2**	**0**	**-3**
RATIO		**1/3**				
x_2	0	1	-2/3	1/3	0	**2**
x_1	1	0	1/3	-2/3	0	**2**
x_5	0	0	3	2	1	**26**
-Z	**0**	**0**	**1/3**	**1/3**	**0**	**-4**

$x_1 = 2 \quad x_2 = 2 \quad x_3 = x_4 = 0 \quad x_5 = 26 \quad Z_{min} = 4$

Example: Different Approximations

$$\textbf{Max } Z = x_1 + 2x_2$$

$$\begin{aligned} \text{s.t.} \quad & 3x_1 + x_2 \le 6 \\ & 2x1 + x_2 = 5 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Two-Phase Simplex

$$\begin{aligned} & Z - x_1 - 2x_2 = 0 \\ & 3x_1 + x_2 + x_3 = 6 \\ & 2x_1 + x_2 + x_4 = 5 \\ & W = x_4 \\ & -W - 2x_1 - x_2 = -5 \end{aligned}$$

Big-M Simplex

$$\begin{aligned} & Z - x_1 - 2x_2 + M\,x_4 = 0 \\ & Z - x_1 - 2x_2 + M\,(5 - 2x_1 - x_2) = 0 \\ & Z + (-2M-1)x_1 + (-M-2)x_2 = -5M \\ & 3x_1 + x_2 + x_3 = 6 \\ & 2x_1 + x_2 + x_4 = 5 \end{aligned}$$

Dual Model

$$\textbf{Min } Z' = 6y_1 + 5y_2$$

$$\begin{aligned} \text{s. t.} \quad & 3y_1 + 2y_2 \ge 1 \\ & y1 + y_2 \ge 2 \\ & y_1, y_2 \ge 0 \end{aligned}$$

Alternative Simplex for Dual Model

$$\begin{aligned} & -3y_1 - 2y_2 \le -1 \\ & -y_1 - y_2 \le -2 \end{aligned}$$

$$\begin{aligned} & -Z' + 6y_1 + 5y_2 = 0 \\ & -3y_1 - 2y_2 + y_3 = -1 \\ & -y_1 - y_2 + y_4 = -2 \end{aligned}$$

Two-Phase

BASIS	x_1	x_2	x_3	x_4	RHS	RATIO
x_3	3<	1	1	0	**6**	2<
x_4	2	1	0	1	**5**	2.5
-w	**-2<**	**-1**	**0**	**0**	**-5**	
Z	**-1**	**-2**	**0**	**0**	**0**	
x_1	1	1/3	1/3	0	**2**	6
x_4	0	1/3<	-2/3	1	**1**	3<
-w	**0**	**-1/3<**	**2/3**	**0**	**-1**	
Z	**0**	**-5/3**	**1/3**	**0**	**2**	
x_1	1	0	1	-1	**1**	1<
x_2	0	1	-2	3	**3**	
-W	**0**	**0**	**0**	**1**	**0**	
Z	**0**	**0**	**-3**	**5**	**7**	
x_3	1	0	1		**1**	
x_2	2	1	0		**5**	
Z	**3**	**0**	**0**		**10**	

$x_1 = 0 \quad x_2 = 5 \quad Z_{max} = 10$

Big-M

BASIS	x_1	x_2	x_3	x_4	RHS	RATIO
x_3	3<	1	1	0	**6**	2<
x_4	2	1	0	1	**5**	2.5
Z	**-2M-1**	**-M-2**	**0**	**0**	**-5M**	
x_1	1	1/3	1/3	0	**2**	6
x_4	0	1/3<	-2/3	1	**1**	3<
Z	**0**	**-M/3-5/3**	**2M/3+1/3**	**0**	**2**	
x_1	1	0	1	-1	**1**	1<
x_2	0	1	-2	3	**3**	
Z	**0**	**0**	**-3**	M+5	**7**	
x_3	1	0	1	**-1**	**1**	
x_2	2	1	0	**-2**	**5**	
Z	**3**	**0**	**0**	M+2	**10**	

$x_1 = 0$ $x_2 = 5$ $Z_{max} = 10$

Alternative Simplex for Dual Model

BASIS	y_1	y_2	y_3	y_4	RHS	RATIO
y_3	-3	-2	1	0	**-1**	
y_4	-1	-1<	0	1	**-2<**	
-Z'	**6**	**5**	**0**	**0**	**0**	
RATIO	6	5<				
y_3	**-1**	**0**	**1**	**2**	**3**	
y_2	**1**	**1**	**0**	**-1**	**2**	
-Z'	**1**	**0**	**0**	**5**	**-10**	

$$x_1 = 0 \qquad x_2 = 5$$

$$-Z'_{min} = Z_{max} = 10$$