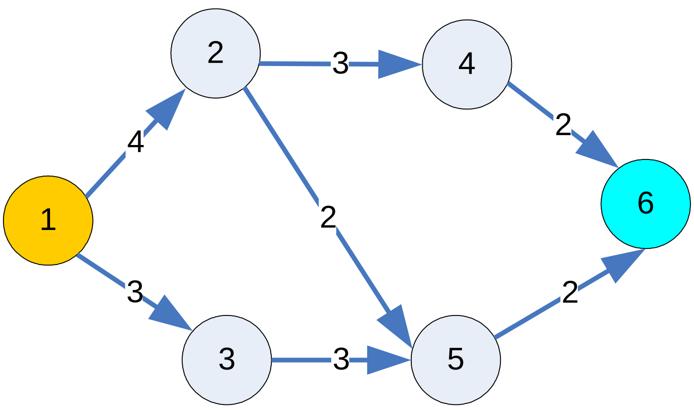
THE SHORTEST PATH (SHORTEST ROUTE) PROBLEM

Assume that each arc in the network has a length associated with it. Suppose we start at a particular node (say, node 1). The problem of finding the shortest path (path of minimum length) from node 1 to any other node in the network is called a shortest path problem.



THE SHORTEST PATH PROBLEM AS A TRANSSHIPMENT PROBLEM

Finding the shortest path between node *i* and node *j* in a network may be viewed as a transshipment problem. Simply try to minimize the cost of sending 1 unit from node *i* to node j (with all other nodes in the network being transshipment points), where the cost of sending 1 unit from node *k* to node *k'* is the length of arc (*k*, *k'*) if such an arc exists and M (a large positive number) if such an arc does not exists. As in transshipment section the cost of shipping 1 unit from a node to itself is zero. This transshipment problem may be transformed into a balanced transportation problem.

Model:

$$\text{Min} \quad 4x_{12} + 3x_{13} + 3x_{24} + 2x_{25} + 3x_{35} + 2x_{46} + 2x_{56}$$

s.t.

$$\begin{array}{lll}
\text{node-1} & x_{12} + x_{13} & = 1 \\
\text{node-2} & x_{24} + x_{25} - x_{12} & = 0 \\
\text{node-3} & x_{35} - x_{13} & = 0 \\
\text{node-4} & x_{46} - x_{24} & = 0 \\
\text{node-5} & x_{56} - x_{35} - x_{25} & = 0 \\
\text{node-6} & x_{46} + x_{56} & = 1
\end{array}$$

$$x_{ij} \geq 0, \quad x_{ij} = 0, 1$$

Transshipment Representation of Shortest Path Problem and One Optimal Solution

From/To	2		3		4		5		6		Supply
1	1	4		3		M		M		M	1
2		0		M		3	1	2		M	1
3		M	1	0		M		3		M	1
4		M		M	1	0		M		2	1
5		M		M		M		0	1	2	1
Demand	1		1		1		1		1		

This transportation problem has two optimal solutions:

1. **$Z = 4 + 2 + 2 = 8$,**
   $x_{11} = x_{25} = x_{56} = x_{33} = x_{44} = 1$ (all other variables equal zero). This solution corresponds to the path $1 \rightarrow 2 \rightarrow 5 \rightarrow 6$.

2. **$Z = 3 + 3 + 2 = 8$,**
   $x_{13} = x_{35} = x_{56} = x_{22} = x_{44} = 1$ (all other variables equal zero). This solution corresponds to the path $1 \rightarrow 3 \rightarrow 5 \rightarrow 6$.

Shortest Path Algorithms

The Dijkstra Algorithm

Floyd Shortest Path Algorithm (All Shortest Path)

Dantzig Shortest Path Algorithm

Ford Algorithm

K-th Shortest Path Algorithm

The Dijkstra Algorithm

Initialization Step

Assign a temporary label of 0 to the start node and +∞ to all other nodes. (These are the minimum distances found thus far from the start node to all other nodes; we do not put the + ∞ values on the network.)

Iterative Steps

1. **Find the node with the smallest temporary label and make it *permanent*. This node is the *assigned node*. If all nodes have permanent labels STOP, the minimum distances have been found.**
2. **From the assigned node, consider all arcs to its adjacent nodes with temporary labels. For these adjacent nodes calculate:**

D= (Permanent label at assigned node) + (Arc distance)

Replace the temporary label at the adjacent node by D *only if* the current label at the adjacent node is *greater than D*. If the label is replaced, record the assigned node that generated the label(shown next to the label valve).

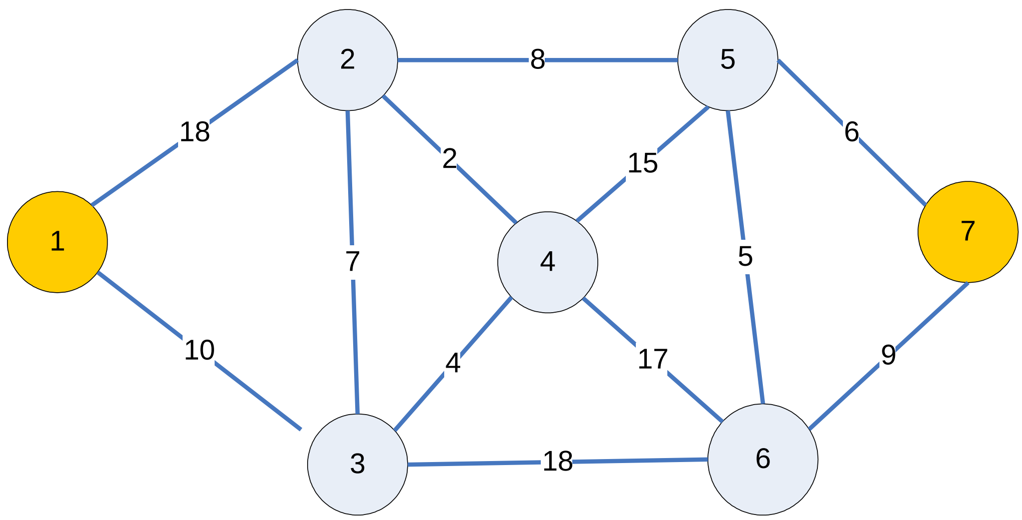
GO TO STEP 1

Comments

1. **Many applications of the shortest path algorithm involve criteria such as time or cost instead of distance . In these cases the shortest path algorithm provides the minimum-time or minimum-cost solution. However, since the shortest path algorithm always identifies a minimum value solution, it would not make sense to apply the algorithm to problem that involves a profit criterion.**

2. **In some applications the value associated with an arc may be negative. For example, in situations where cost is the criterion, a negative arc value would denote a negative cost; in other words, a profit would be realized by traversing the arc. The shortest path algorithm presented in the course can only be applied to networks with nonnegative arc values. More advanced texts discuss algorithms that can be used to solve problems having negative arc values.**

Example: (Dijkstra Algorithm)



INITIALIZATION

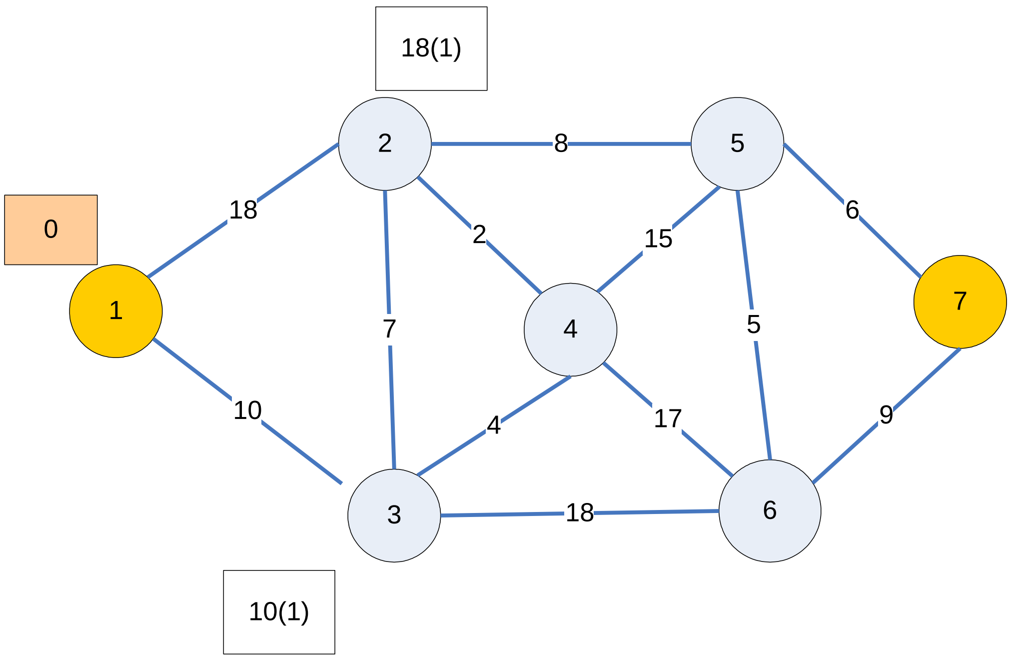
All nodes have temporary labels of +∞ (not shown), except for node-1 which has a temporary label of 0.

ITERATION 1

Minimum Temporary Label (Made Permanent): 0 at Node1

Temporary Nodes Adjacent to Node 1: Nodes 2 and 3

Adjacent Node	Distance from Assigned Node	New Temporary Label At Adjacent Node?
2 3	**0 + 18 = 18 < ∞** **0 + 10 = 10 < ∞**	**Yes - 18 (1)** **Yes - 10 (1)**



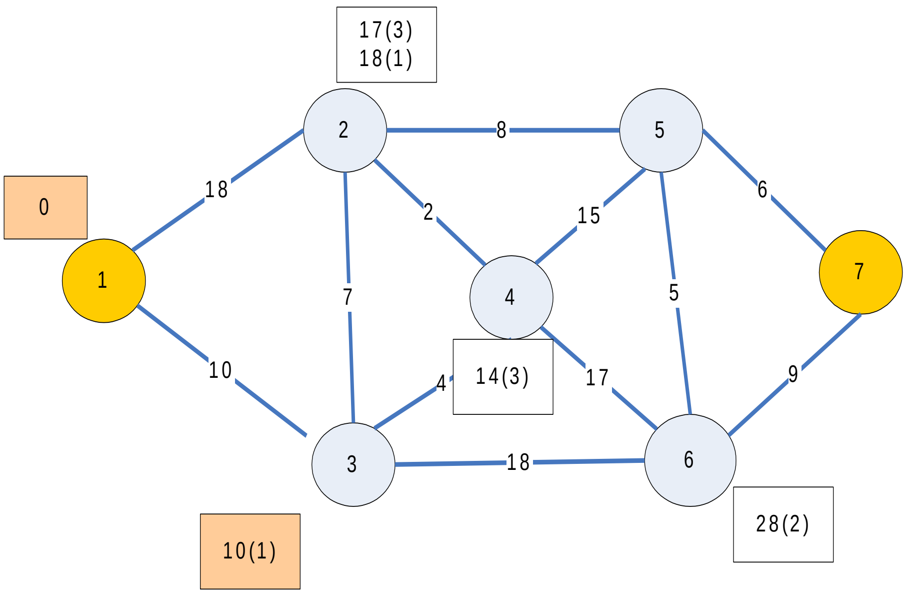
Shortest path solution: Iteration 1

ITERATION 2

Minimum Temporary Label (Made Permanent): 10 at Node3

Temporary Nodes Adjacent to Node 3: Nodes 2, 4, and 6

Adjacent Node	Distance from Assigned Node	New Temporary Label At Adjacent Node?
2	**10 + 7 = 17 < 18**	**Yes - 17 (3)**
4	**10 + 4 = 14 < ∞**	**Yes - 14 (3)**
6	**10 + 18 = 28 < ∞**	**Yes – 28 (3)**



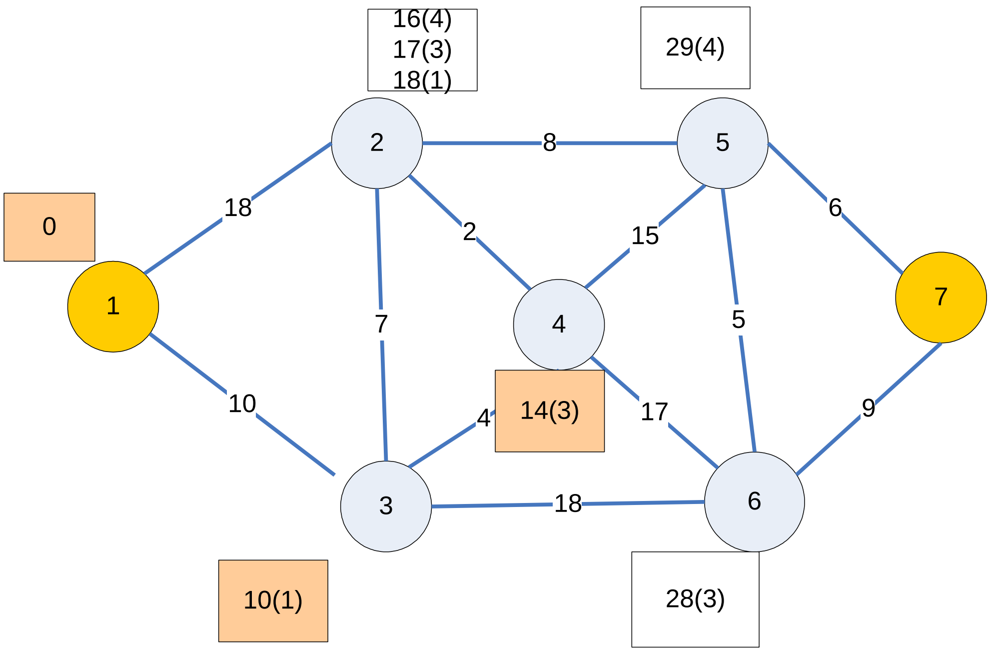
Shortest path solution: Iteration 2

ITERATION 3

Minimum Temporary Label (Made Permanent): 14 at Node4

Temporary Nodes Adjacent to Node 4: Nodes 2, 5, and 6

Adjacent Node	Distance from Assigned Node	New Temporary Label At Adjacent Node?
2	14 + 2 = 16 < 17	Yes - 16 (4)
5	14 + 15 = 29 < ∞	Yes - 29 (4)
6	14 + 17 = 31 < 28	No



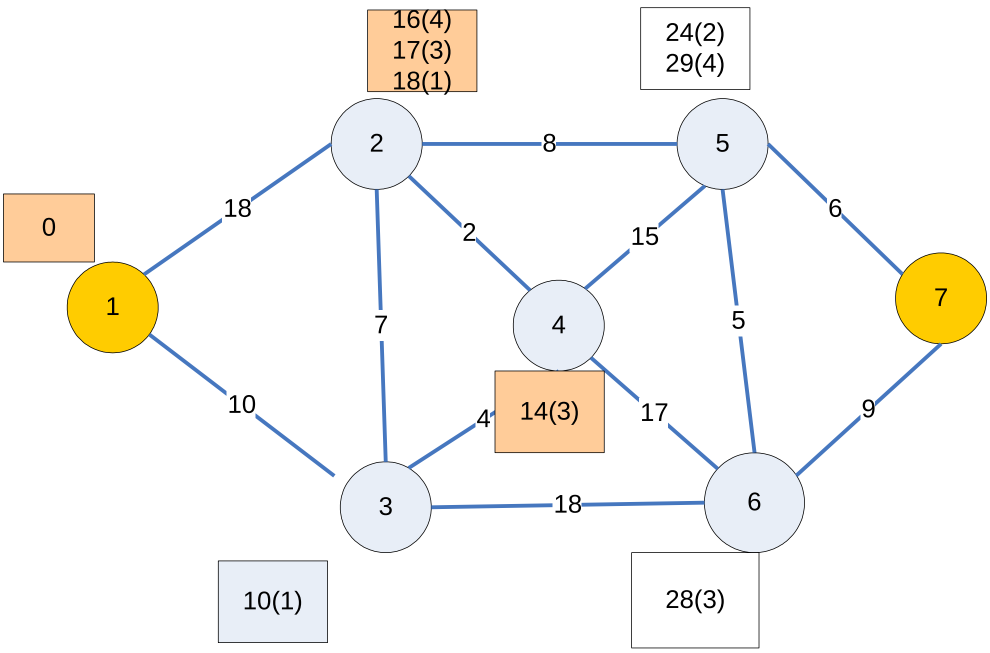
Shortest path solution: Iteration 3

ITERATION 4

Minimum Temporary Label (Made Permanent): 16 at Node2

Temporary Nodes Adjacent to Node 2: Node 5

Adjacent Node	Distance from Assigned Node	New Temporary Label At Adjacent Node?
5	16 + 8 = 24 < 29	Yes - 24 (2)



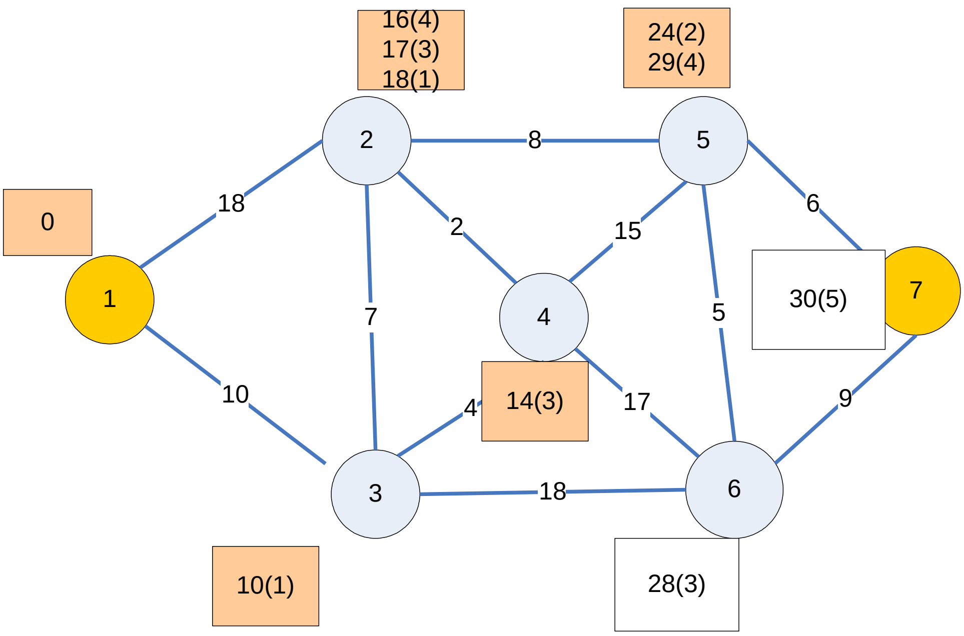
Shortest path solution: Iteration 4

ITERATION 5

Minimum Temporary Label (Made Permanent): 24 at Node 5

Temporary Nodes Adjacent to Node 5: Nodes 6 and 7

Adjacent Node	Distance from Assigned Node	New Temporary Label At Adjacent Node?
6	24 + 5 = 29 < 28	No
7	24 + 6 = 30 < ∞	Yes - 30 (5)



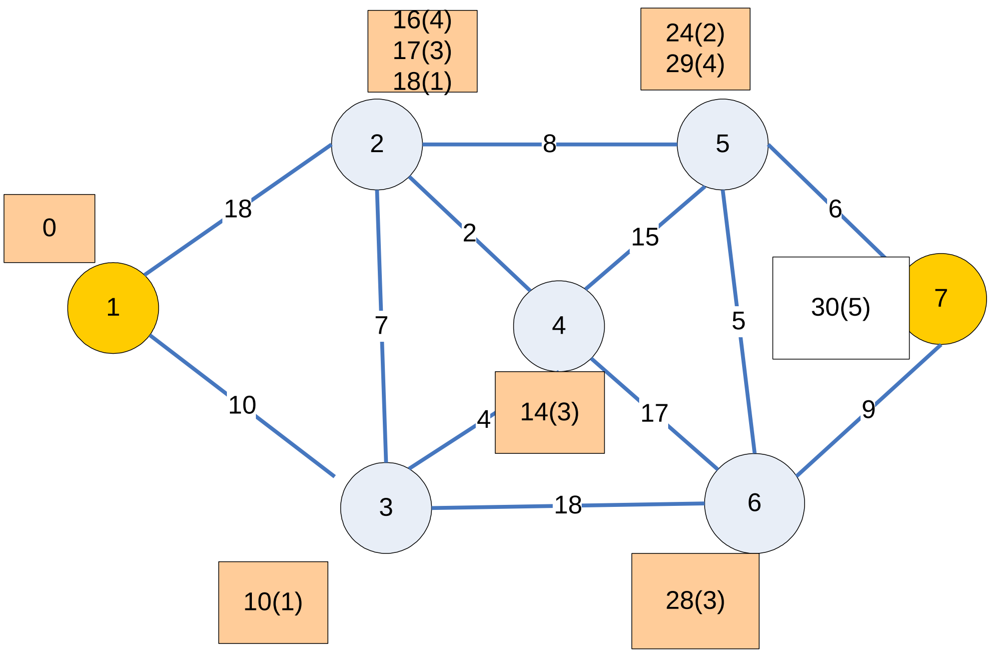
Shortest path solution: Iteration 5

ITERATION 6

Minimum Temporary Label (Made Permanent): 28 at Node6

Temporary Nodes Adjacent to Node 6: Node 7

Adjacent Node	Distance from Assigned Node	New Temporary Label At Adjacent Node?
7	28 + 9 = 37 < 30	No



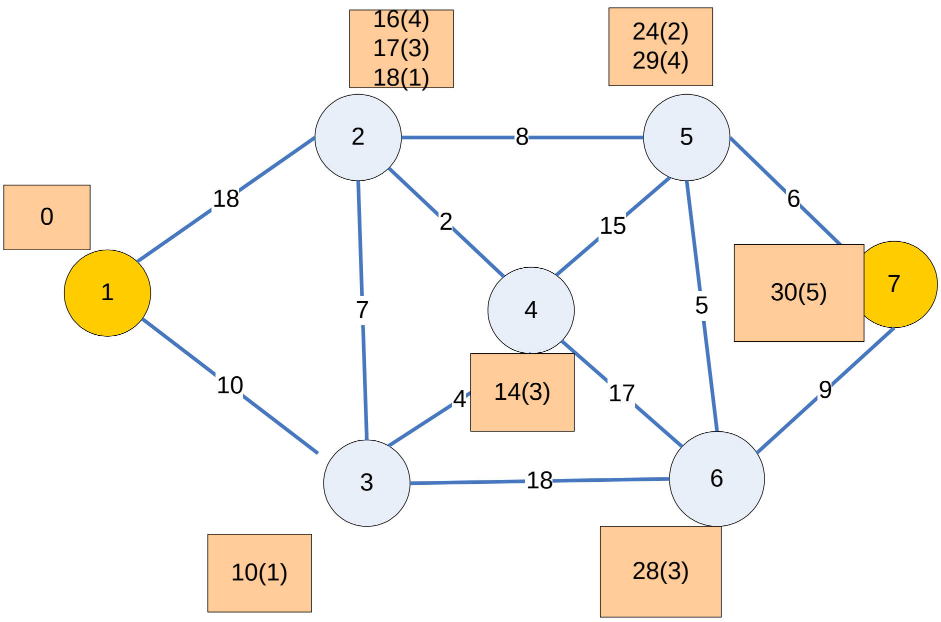
Shortest path solution: Iteration 6

ITERATION 7

Minimum Temporary Label (Made Permanent): 30 at Node7

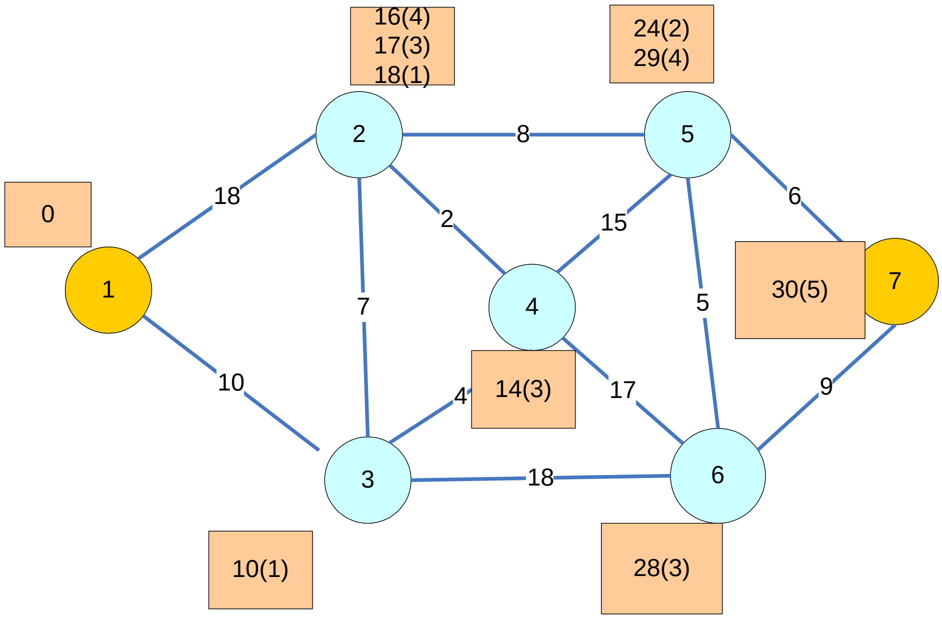
Temporary Nodes Adjacent to Node 7 : None

Adjacent Node	Distance from Assigned Node	New Temporary Label At Adjacent Node?
-	-	-



Shortest path solution: Iteration 7

Solution: **1→3→4→2→5→7**



All nodes have now been assigned a permanent label, so we terminate the algorithm; the shortest distance is 30. Retracing the path from node 7, we see that we got to node 7 from node 5;, to node 5 from node 2; to node 2 from node 4; to node 4 from node 3;and to node 3 from node 1. Thus the shortest path is **1→3→4→2→5→7.**

The shortest distances and paths from node 1 to all other nodes in the network are as follows.

To	Distance	Path
Node 2	**16**	**1→3→4→2**
Node 3	**10**	**1→3**
Node 4	**14**	**1→3→4**
Node 5	**24**	**1→3→4→2→5**
Node 6	**28**	**1→3→6**
Node 7	**30**	**1→3→4→2→5→7**