Linear Equations

Definitions: Linear Equations

The simplest type of a linear equation is given by

$$a\ x = b$$

where a ***and*** b ***are given and known, and*** x ***is unknown variable to be determined. This equation is linear as it contains only*** x ***and nothing else.***

It is simple to see that the equation has:

- **a unique solution if** $a \neq 0$
- **no solution if** $a = 0$ **and** $b \neq 0$
- **multiple solutions if** $a = 0$ **and** $b = 0$

(The relationship between the voltage and current of a resistor, $IR = V.$**)**

Any straight line in the xy **plane can be represented algebraically by an equation of the form**

$$a_1x + a_2y = b$$

where a_1, a_2 **and** b **are real constants, and** a_1, a_2 **are <u>not both zero</u>.** ***An equation of this form is <u>called a linear equation in the variables</u>*** x ***<u>and</u>*** y**.**

$3x + 5y = 15$ **<u>is a linear equation.</u>**

But

$3x - \cos(y) = 1$,

$e^{x^2} + 4y = 10,$

$5x + xy^2 = 8$

$x + \sqrt{y} = 5$

<u>are not linear equations</u>.

All variables occur only to the <u>first power</u> and <u>do not appear</u> as arguments for <u>trigonometric, logarithmic or exponential functions</u>.

Generally we define a linear equation in the n variables $x_1, x_2, \ldots, x_n$ to be one that can be expresses in the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

where $a_1, a_2, \ldots, a_n$ and b are real constants. The variables in a linear equation are sometimes called unknowns.

Solution of a linear equation:

A solution of a linear equation $a_1x_1 + a_2x_2 + ... + a_nx_n = b$ is a sequence of n numbers $s_1, s_2, ..., s_n$ such that the equation is satisfied when we substitute

$$x_1 = s_1,$$
$$x_2 = s_2$$
$$\vdots$$
$$x_n = s_n.$$

The set of all solutions of the equation is called its solution set or the general solution of the equation.

If we set specific values for s_i we obtain particular solution.

Example

Find the solution of $4x - 2y = 1$

If we assign x an arbitrary value t, we obtain

$$x = t, \qquad y = 2t - 0.5$$

Particular numerical solutions can be obtained by substituting specific values for the parameter t.

$t = 3$ Yields the solution $x = 3,\ y = 5.5$

$t = -0.5$ Yields the solution $x = -0.5,\ y = -1.5$

If we assign y an arbitrary value t, we obtain

$$y = t, \qquad x = 0.5t + 0.25$$

Although the formulas are different from those obtained above, they yield the same solution set as t varies over all possible real numbers. For example, the previous formulas gave the solution $x = 3,\ y = 5.5$ when $t = 3$, whereas the formulas immediately above yield the solution when $t = 5.5.$

Example

Find the solution of $x_1 - 4x_2 + 7x_3 = 5$

We can assign arbitrary values to any two variables and solve for the third variable.

If we assign arbitrary values s and t to x_2 and x_3, respectively, and solve for x_1, we obtain

$$x_1 = 5 + 4s - 7t, \quad x_2 = s \;, \quad x_3 = t.$$

Introduction to
Linear Systems

Definition

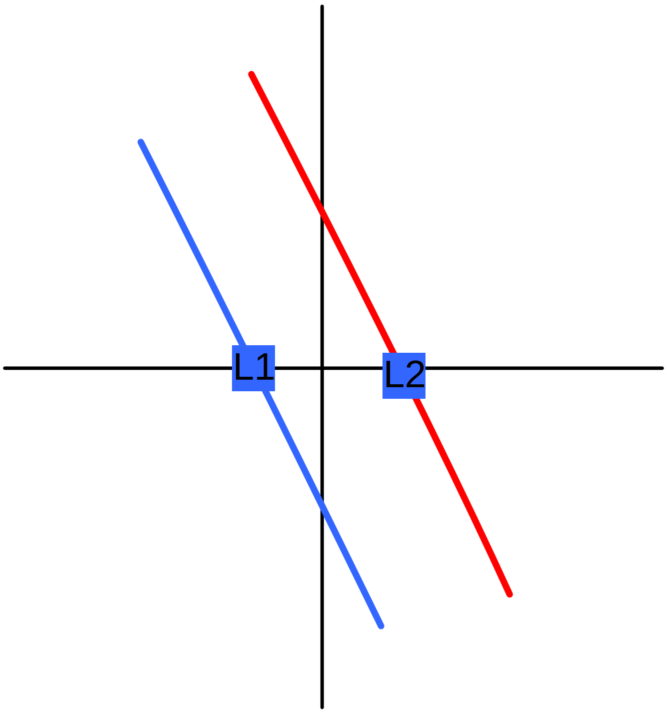
A finite set of linear equations in the variables $x_1, x_2, \ldots, x_n$ ***is called a system of linear equations or a linear system. A sequence of numbers*** $s_1, s_2, \ldots, s_n$ ***is called a solution of the system if***

$x_1 = s_1, \quad x_2 = s_2, \quad \ldots, \quad x_n = s_n$ ***is a solution of every equation in the system.***

Two-equations-two-unknowns linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

where $a_{11}, a_{12}, a_{21}, a_{22}, b_1$ and b_2 are known constants and x_1 and x_2 are unknown variables to be solved.

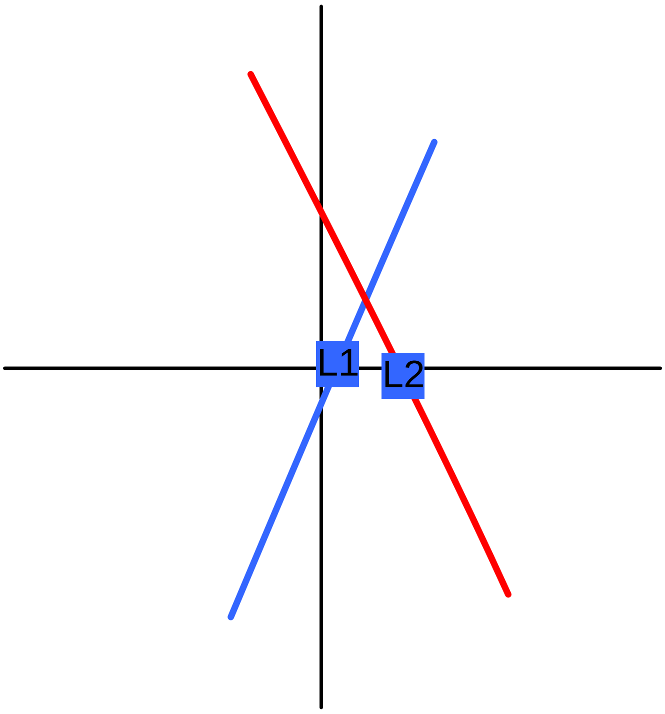


1. The lines L_1 and L_2 may be **parallel**, in which case there is **no intersection** and consequently **no solution** to the system.

Ex:

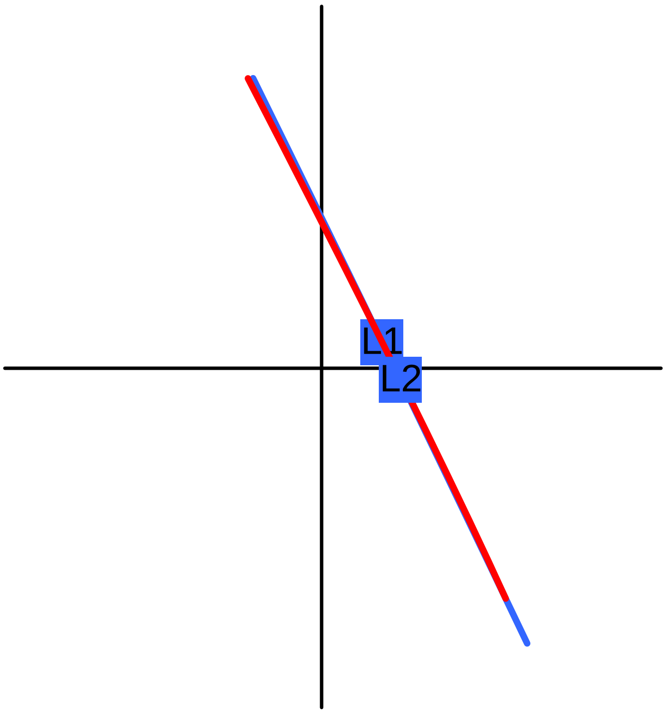
$$x + 2y = 3$$
$$4x + 8y = 6$$

A system of equations that has no solutions is said to be inconsistent; if there is at least one solution of the system, it is called consistent.



2. The lines L_1 and L_2 may **intersect at only one point**, in which case the system has **exactly one solution**.

Ex: $$\begin{aligned} x + 2y &= 3 \\ 4x + 5y &= 6 \\ (x, y) &= (-1, 2) \end{aligned}$$



3. ***The lines L_1 and L_2 may coincide, in which case there are infinitely many points of intersection and consequently infinitely many solutions to the system.***

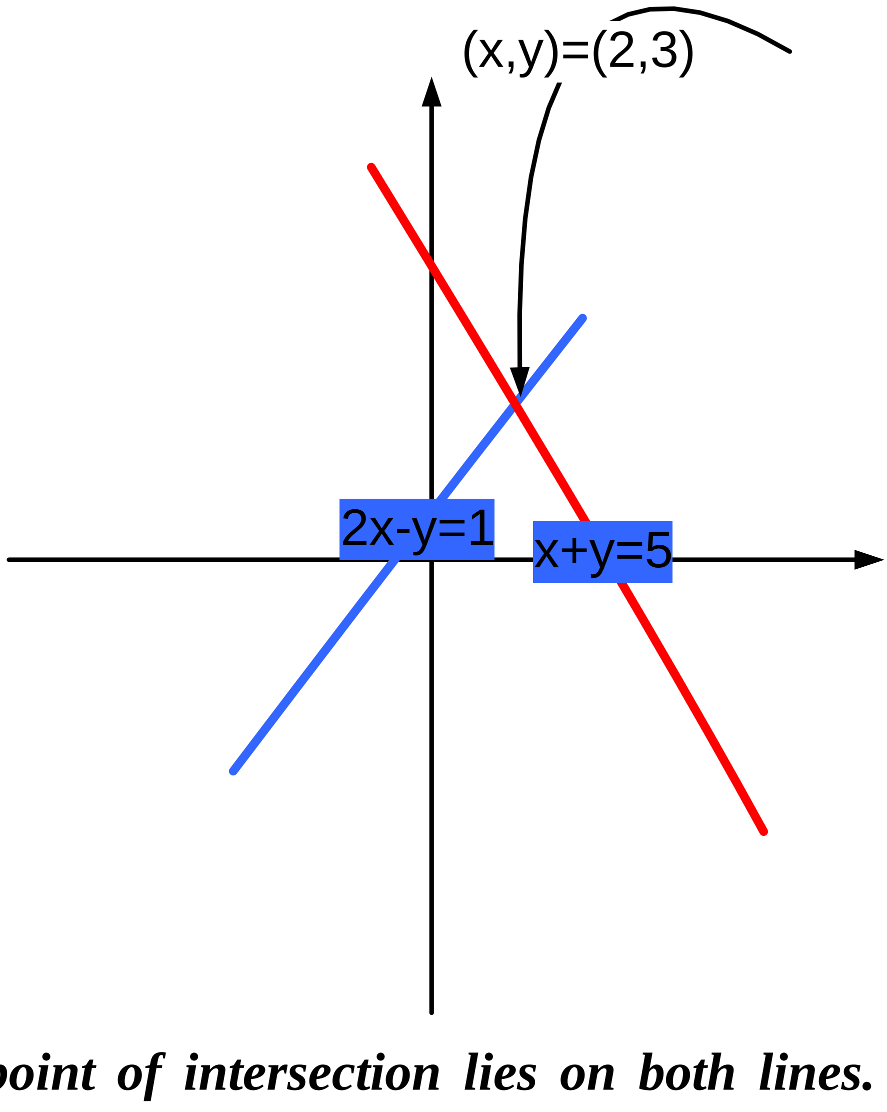
Ex: $$\begin{aligned} x + 2y &= 3 \\ 4x + 8y &= 12 \end{aligned}$$

Every system of linear equations has no solution or has exactly one solution or has infinitely many solutions.

The Geometry of Linear Equations

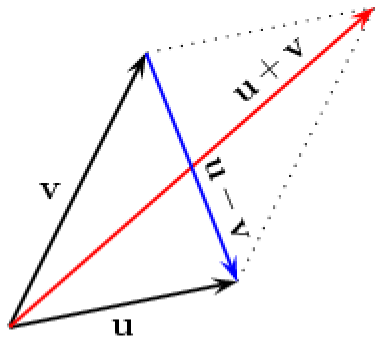
$$2x - y = 1$$
$$x + y = 5$$

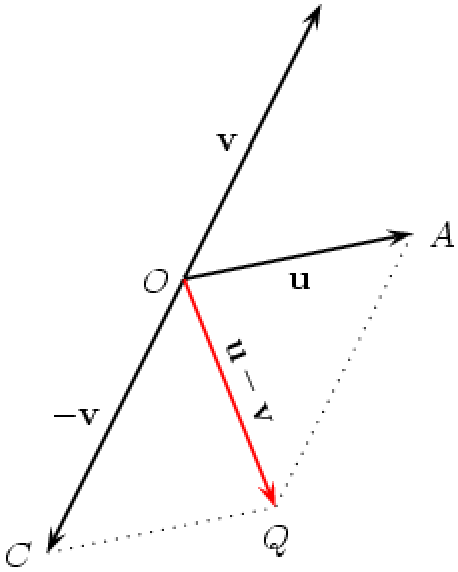
First Approach:

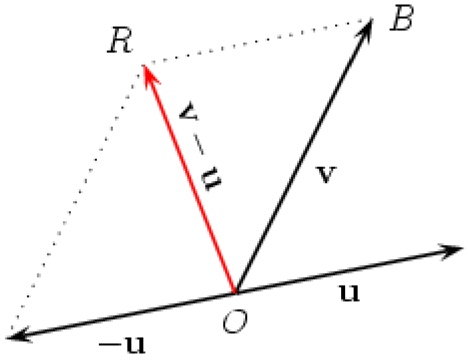


The point of intersection lies on both lines. It is the only solution to both equations. That point x=2 and y=3 will soon be found by "elimination".

Addition and subtraction of Vectors





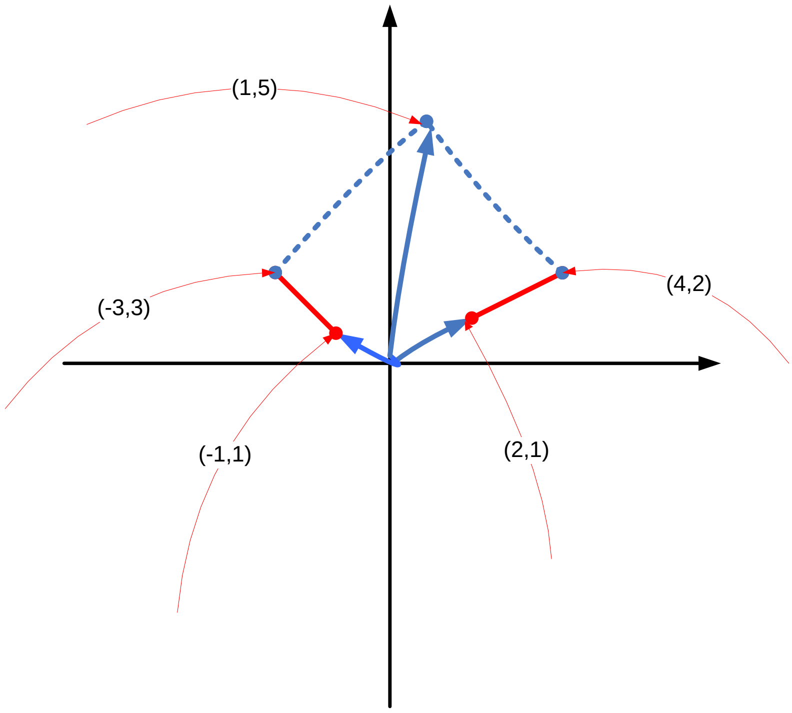


Addition and subtraction of Vectors

Second Approach:

The two separate equations are really ***one vector equation:***

$$x\begin{bmatrix}2\\1\end{bmatrix}+y\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}1\\5\end{bmatrix}$$



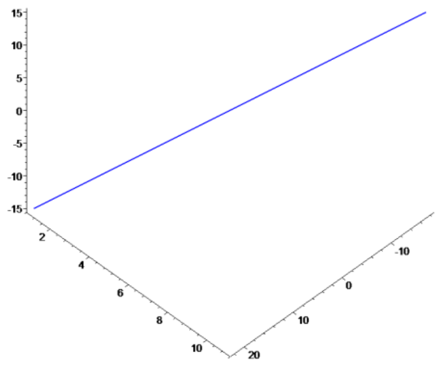
The problem is to find **<u>the combination of the column vectors on the left side that produces the vector on the right side.</u>**

The column picture confirms that x=2 and y=3.

```
> v:=PositionVector([2-4*t,6+t,3*t]);
```

$$v := \begin{bmatrix} 2-4\,t \\ 6+t \\ 3\,t \end{bmatrix}$$

```
> PlotPositionVector(v,t=-5..5);
```



Parametric equation

A line through point $\mathbf{r_0} = (x_0, y_0, z_0)$ parallel to vector $\mathbf{v}$ = <a, b, c> is given by

$$\boxed{\begin{aligned} x &= x_0 + at, \\ y &= y_0 + bt, \\ z &= z_0 + ct. \end{aligned}}$$

Vector-Valued Functions

When a particle moves along a curve in the xy plane defined parametrically by

$$x = f(t) \quad \text{and} \quad y = g(t)$$

If the coordinates of the moving point at time t are given by the parametric equations $x = f(t)$ and $y = g(t)$ then the position vector

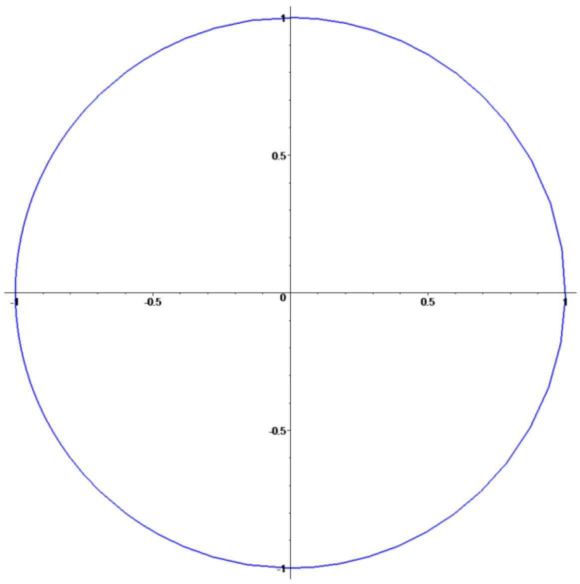
$$r(t) = f(t)i + g(t)j$$

determines a **vector valued** function.

```
> v:=PositionVector([(1-t^2)/(1+t^2),2*t/(1+t^2)]);
```

$$v := \begin{bmatrix} \frac{1-t^2}{1+t^2} \\ \frac{2\,t}{1+t^2} \end{bmatrix}$$

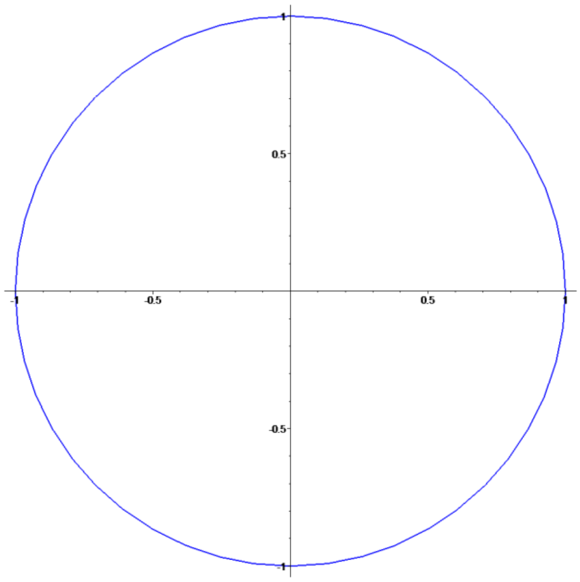
```
> PlotPositionVector(v,t=-infinity..infinity);
```



```
> v:=PositionVector([cos(t),sin(t)]);
```

$$v := \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$$

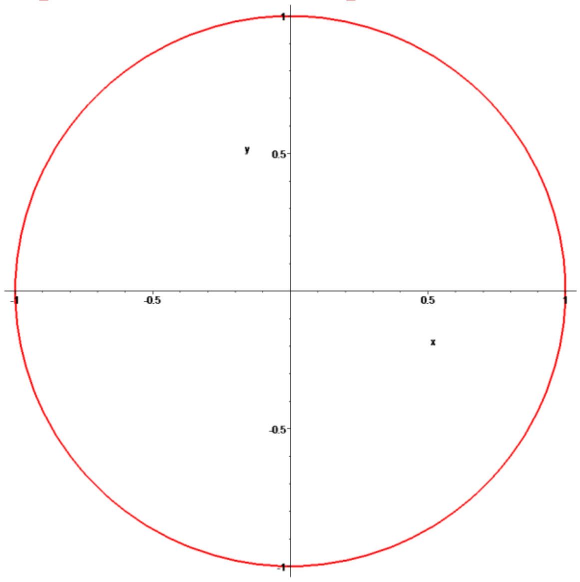
```
> PlotPositionVector(v,t=-Pi..Pi);
```




```
> with(plots,implicitplot);
```

$$[\,implicitplot\,]$$

```
> implicitplot(x^2+y^2=1, x=-1..1, y=-1..1);
```



Curve in three dimension

We define a **space curve** or a curve in three dimensions as a set C of ordered triples of the form

$$(f(t), g(t), h(t))$$

The equations

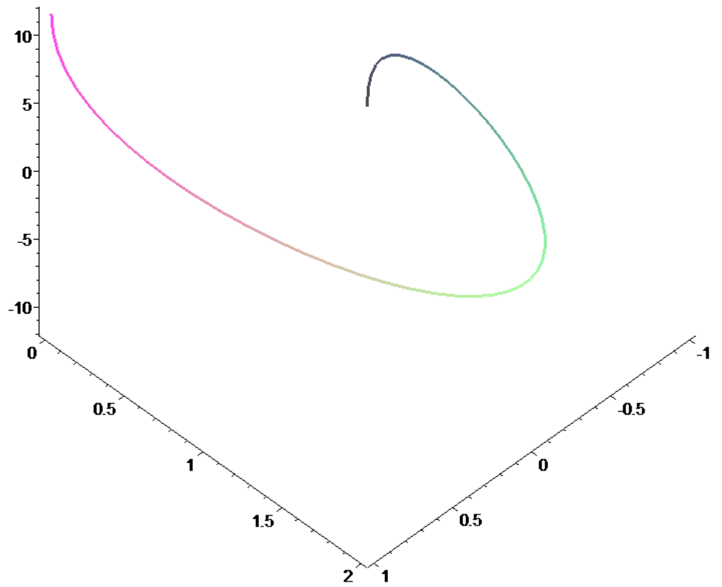
$$\begin{aligned} x &= f(t), \\ y &= g(t), \\ z &= h(t) \end{aligned}$$

are called parametric equations for space curve.

Corresponding position vector

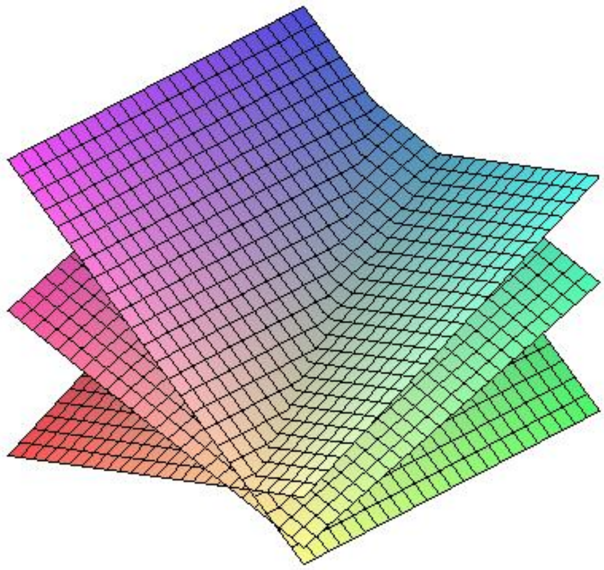
$$r(t) = f(t)i + g(t)j + h(t)k$$

```
> with(plots):
> spacecurve([sin(t),2*cos(t)^2,3*t^3],t=-
Pi/2..Pi/2,axes=frame);
```



Three Planes

$$\begin{aligned} 2x + y + z &= 5 \\ 4x - 6y &= -2 \\ -2x + 7y + 2z &= 9 \end{aligned}$$



Column Vectors and Linear Combinations

Column form

$$x\begin{bmatrix}2\\4\\-2\end{bmatrix}+y\begin{bmatrix}1\\-6\\7\end{bmatrix}+z\begin{bmatrix}1\\0\\2\end{bmatrix}=\begin{bmatrix}5\\-2\\9\end{bmatrix}=b$$

The result is called a linear combination and this combination solves our equation.

Linear combination

$$1\begin{bmatrix}2\\4\\-2\end{bmatrix}+1\begin{bmatrix}1\\-6\\7\end{bmatrix}+2\begin{bmatrix}1\\0\\2\end{bmatrix}=$$

$$\begin{bmatrix}2\\4\\-2\end{bmatrix}+\begin{bmatrix}1\\-6\\7\end{bmatrix}+\begin{bmatrix}2\\0\\4\end{bmatrix}=\begin{bmatrix}5\\-2\\9\end{bmatrix}=b$$

$x=1, y=1, z=2$ **give the correct combination of the columns. They also give (1,1,2) where the three planes intersect.**

General Linear Form

In general, a linear system with m <u>equations</u> and n <u>unknown</u> can be written as the following form:

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \ldots . + a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + \ldots . + a_{2n}x_n = b_2 \\
&. \\
&. \\
&. \\
&a_{m1}x_1 + a_{m2}x_2 + \ldots . + a_{mn}x_n = b_m
\end{aligned}
$$

where $x_1, x_2, \ldots, x_n$ ***are the unknowns. The double subscripting on the coefficients of the unknowns is a useful device that is used to specify the location of the coefficient in the system.***

- The **<u>first subscript</u>** on the coefficient a_{ij} indicates the **<u>equation</u>** in which the coefficient occurs,
- The **<u>second subscript</u>** indicates **<u>which unknown</u>** it multiplies.

MATRIX FORM

We can re-write this set of linear equations in a compact form, i.e., a matrix form:

$$Ax = b$$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & & . \\ . & & & \\ . & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_m \end{bmatrix}$$

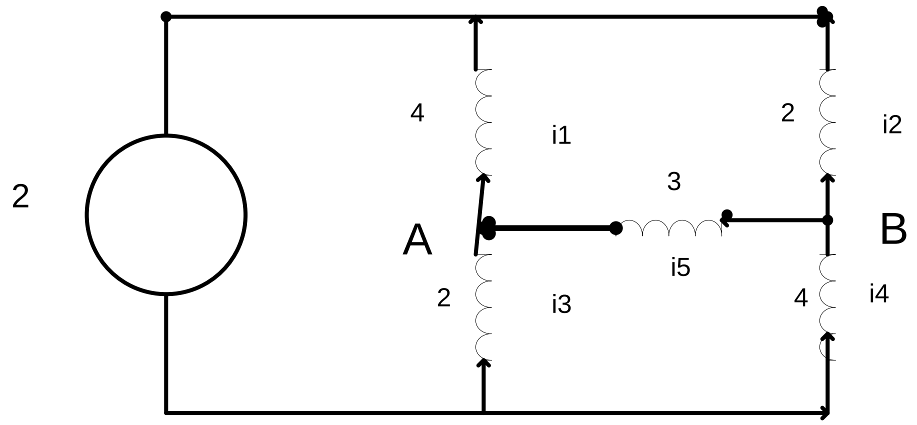
$A: Coefficient\ Matrix$

$x: Vector\ of\ unknowns$

$b: Data\ vector (RHS\ vector)$

- Matrix A has m rows and n columns. **Such a matrix** is called $m \times n$ matrix.
- Each of the sitting on the $i-th$ row and $j-th$ column, or simply the $(i, j)-th$ element is denoted by a_{ij}.

Example:



Kirchoff's Current Law (KCL)

Kirchoff's Voltage Law (KVL)

KCL at Node A $i_1 - i_3 - i_5 = 0$

KCL at Node B $i_2 - i_4 + i_5 = 0$

KCL to left loop $4i_1 + 2i_3 = 2$

KVL to right upper loop $4i_1 - 2i_2 + 3i_5 = 0$

KVL to right lower loop $2i_3 - 4i_4 - 3i_5 = 0$

In a matrix form

$$\begin{bmatrix} 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 \\ 4 & 0 & 2 & 0 & 0 \\ 4 & -2 & 0 & 0 & 3 \\ 0 & 0 & 2 & -4 & -3 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

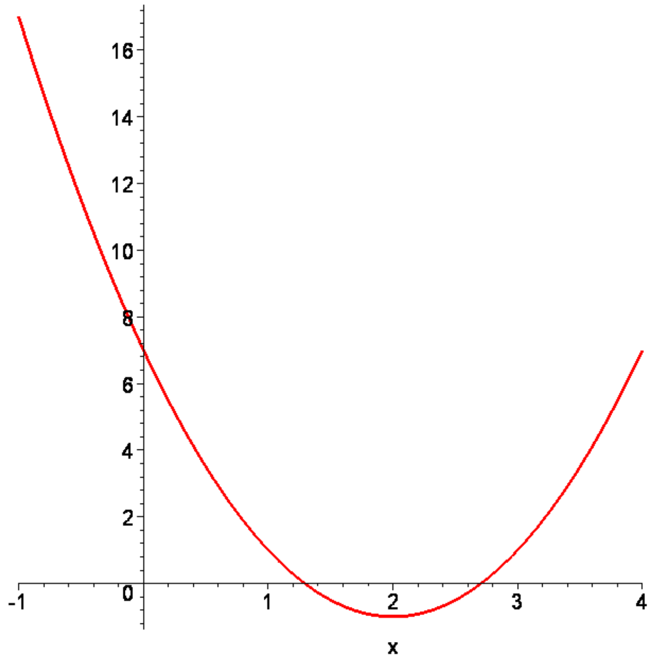
Example:

Find the parabola $y = A + Bx + Cx^2$ that passes through the three points (1, 1), (2, -1) and (3, 1).

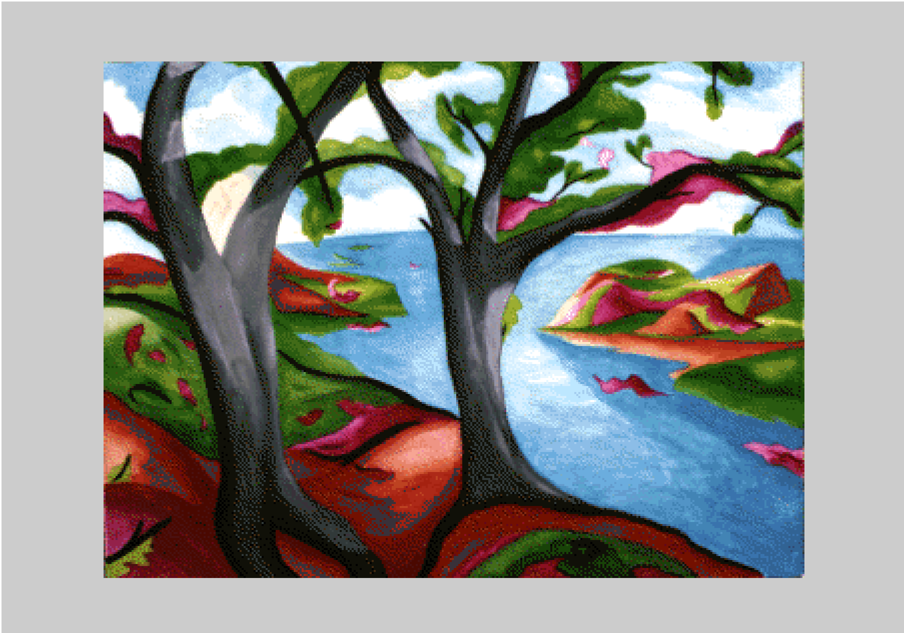
For each point we obtain an equation relating the value of x to the value y. The result is the linear system

$$\begin{aligned} A + B + C &= 1 \\ A + 2B + 4C &= -1 \\ A + 3B + 9C &= 1 \end{aligned}$$

The solution of this system A=7, B=-8 and C=2 and the equation of the parabola is $y = 7 - 8x + 2x^2$.



load trees (MATLAB)



Imshow(R,G,B)

R

0.1922 0.1922 0.1922 0.1922 0.1922 0.1922

0.1922 0.2588

0.1922 0.1922 0.2235 0.2588 0.2588 0.1922

G

0.1922 0.1922 0.1922 0.1922 0.2235 0.1922

0.2235 0.1922 0.2235 0.2588

B

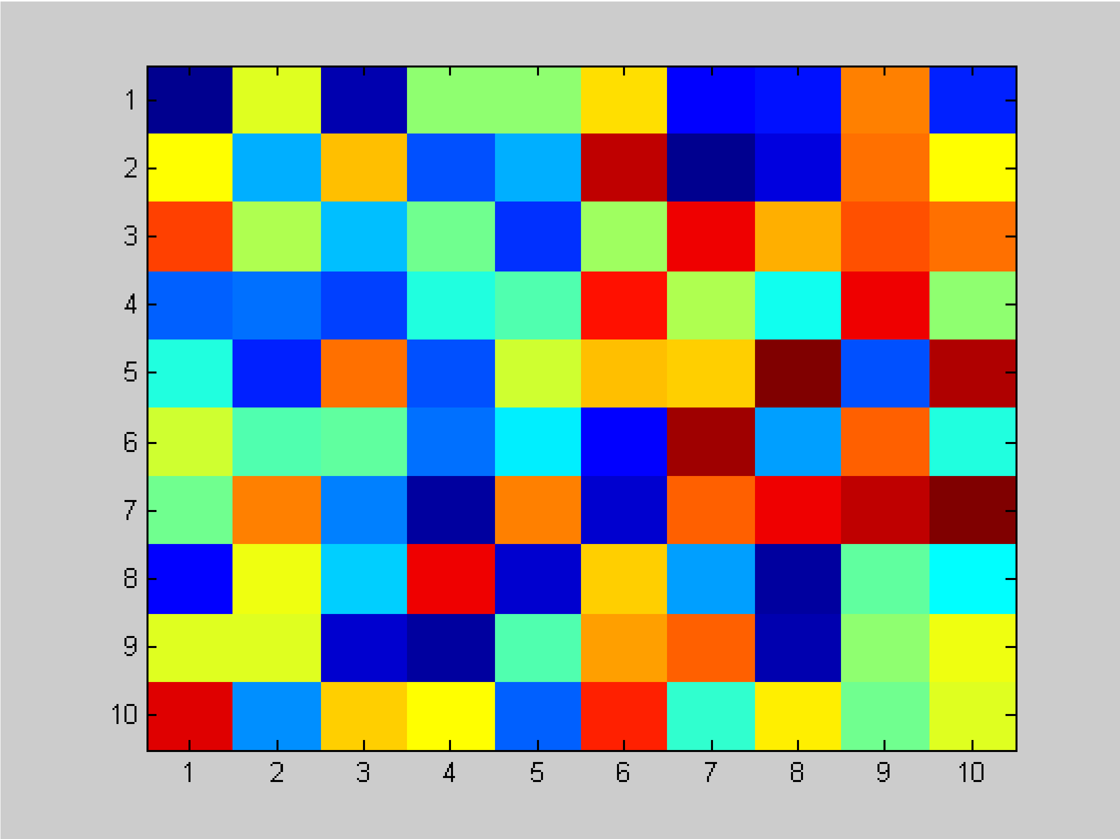
0.2235 0.1922 0.1922 0.1922 0.1922 0.2235

0.4824 0.7098

```
>> n=10

n = 10

>> A=rand(n)

A =
```

0.0159	0.5943	0.0550	0.5165	0.5074	0.6594	0.1353	0.1503	0.7455	0.1551
0.6283	0.3020	0.6756	0.2126	0.2919	0.9234	0.0285	0.1032	0.7627	0.6196
0.8125	0.5442	0.3081	0.4850	0.1834	0.5238	0.8892	0.6944	0.7827	0.7581
0.2176	0.2362	0.1911	0.4113	0.4514	0.8555	0.5510	0.3858	0.8807	0.5107
0.4054	0.1605	0.7659	0.2088	0.5771	0.6777	0.6685	0.9967	0.2052	0.9383
0.5699	0.4533	0.4741	0.2452	0.3589	0.1297	0.9616	0.2912	0.7732	0.4046
0.4909	0.7413	0.2572	0.0396	0.7385	0.0921	0.7800	0.8826	0.9212	0.9974
0.1294	0.6009	0.3252	0.8854	0.0861	0.6624	0.2836	0.0404	0.4695	0.3764
0.5909	0.5841	0.0845	0.0348	0.4469	0.7109	0.7718	0.0542	0.5072	0.6043
0.8985	0.2706	0.6618	0.6223	0.2273	0.8290	0.4244	0.6427	0.4903	0.5947

```
>> imagesc(A)
```



>> n = 100;

B = rand(n)

0.1848 0.6793 0.2799 0.1424 0.9438 0.4611 0.5755 0.8587 0.8505 0.9715 0.4131 0.5420

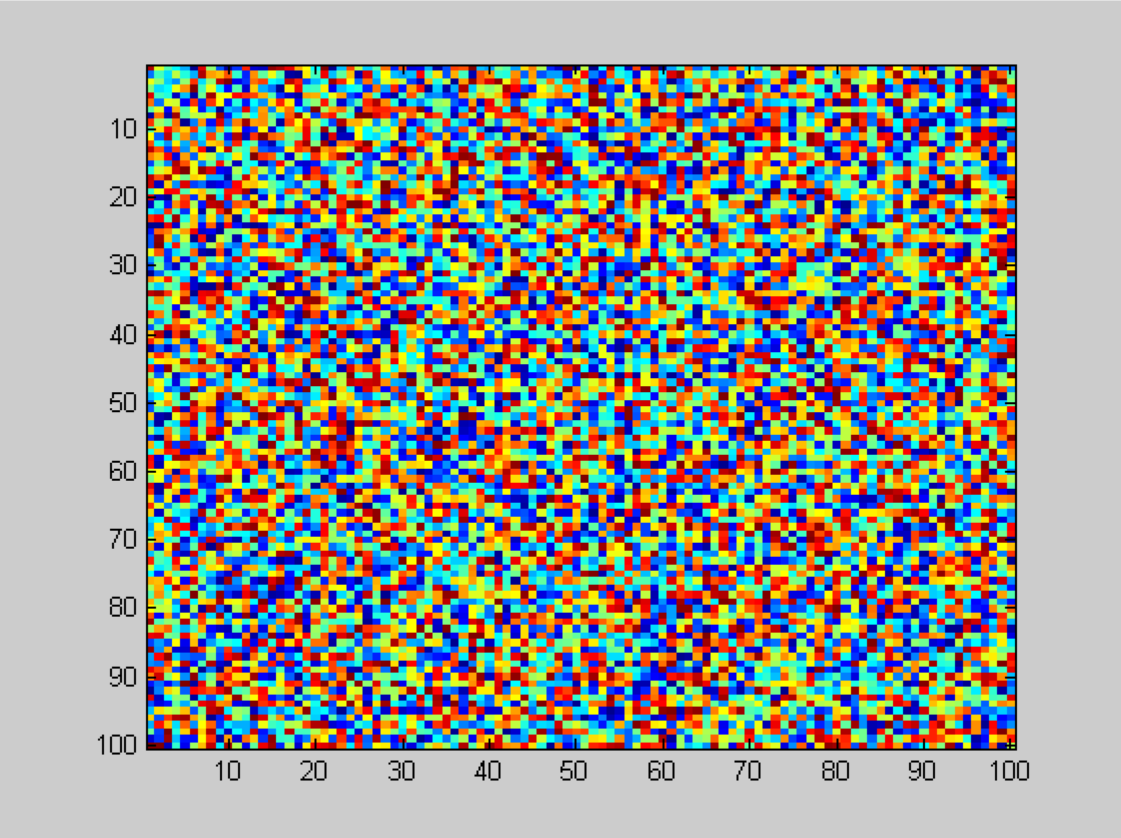
0.2894 0.3664 0.1876 0.1536 0.1848 0.5867 0.3250 0.1446 0.7536 0.1251 0.6427 0.4189

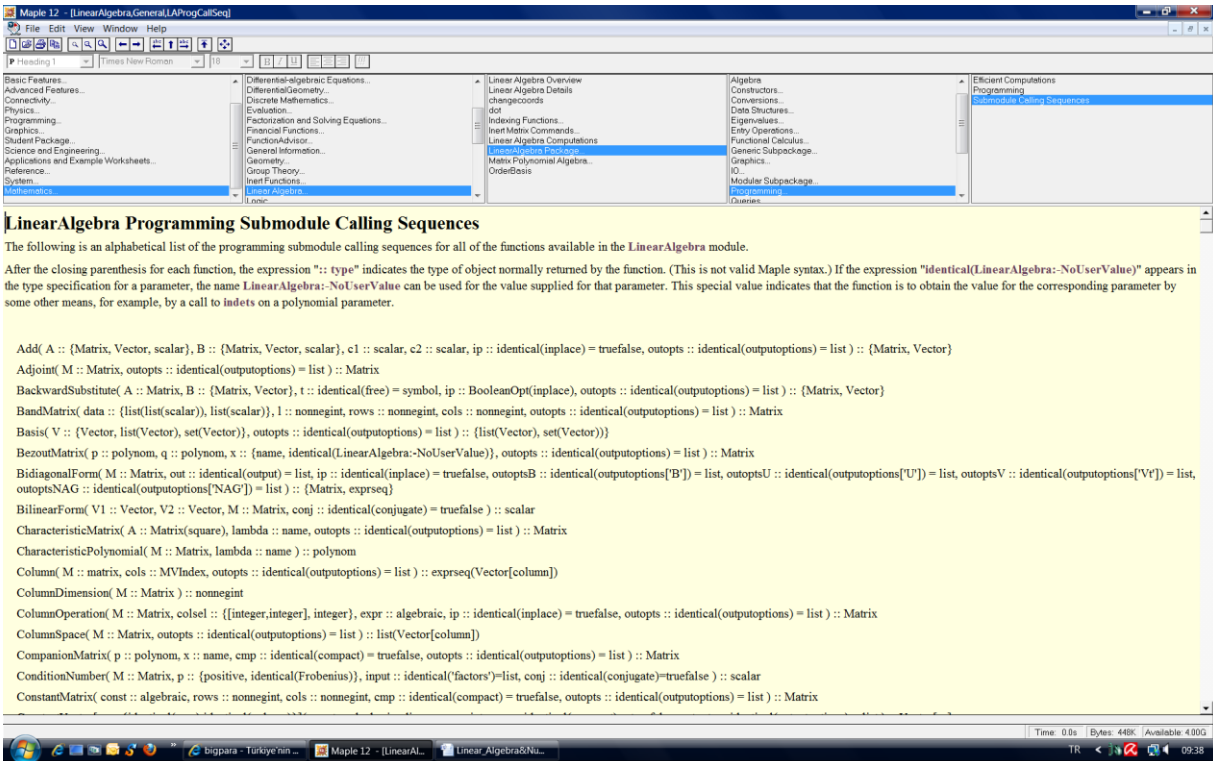
0.6639 0.6393 0.7063 0.3120 0.1587 0.0235 0.6651 0.0171 0.9130 0.2630 0.2266 0.5447

0.6036 0.6301 0.5934 0.3417 0.6445 0.5686 ……

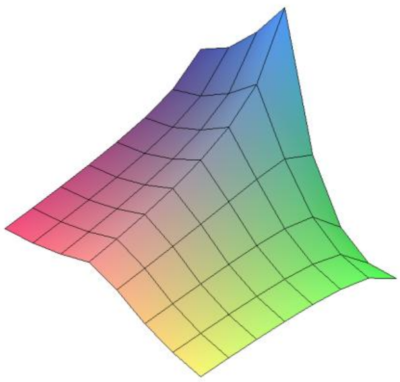
…

imagesc(B);

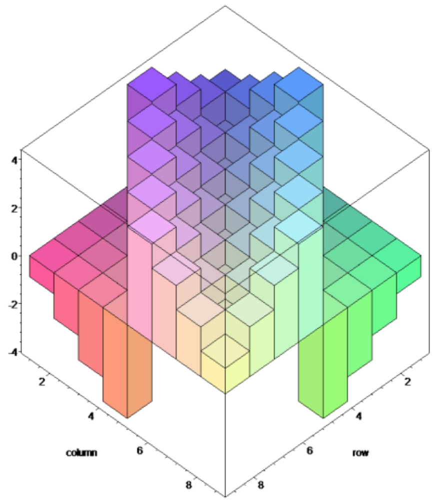




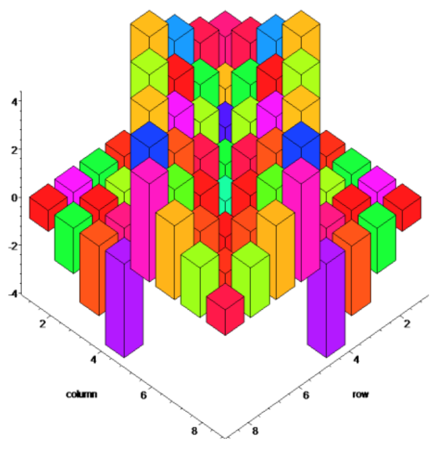

```
> with(plots):
with(LinearAlgebra):
> A:= HilbertMatrix(8): B:= ToeplitzMatrix([1,2,3,4,-4,-3,-2,-
1], symmetric):
> matrixplot(A.B);
```



```
> matrixplot(A+B, heights=histogram, axes=boxed);
```

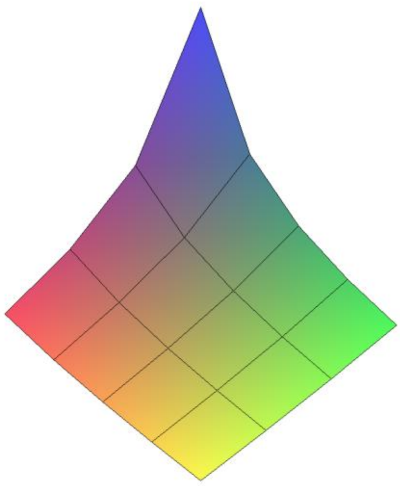



```
> matrixplot(A+B, heights=histogram, axes=frame, gap=0.25,
style=patch);
> F := (x,y) -> sin(x*y):
> matrixplot(A+B, heights=histogram, axes=frame, gap=0.25,
style=patch, color=F);
```

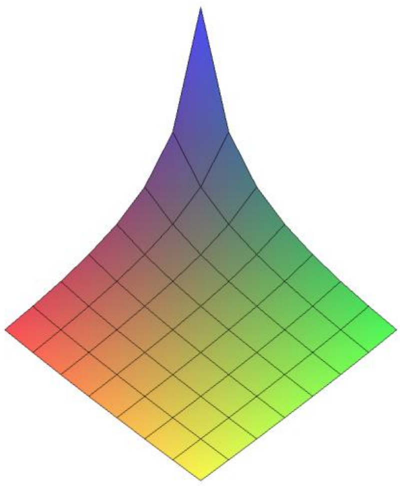


```
> L:=HilbertMatrix(4,5);
```

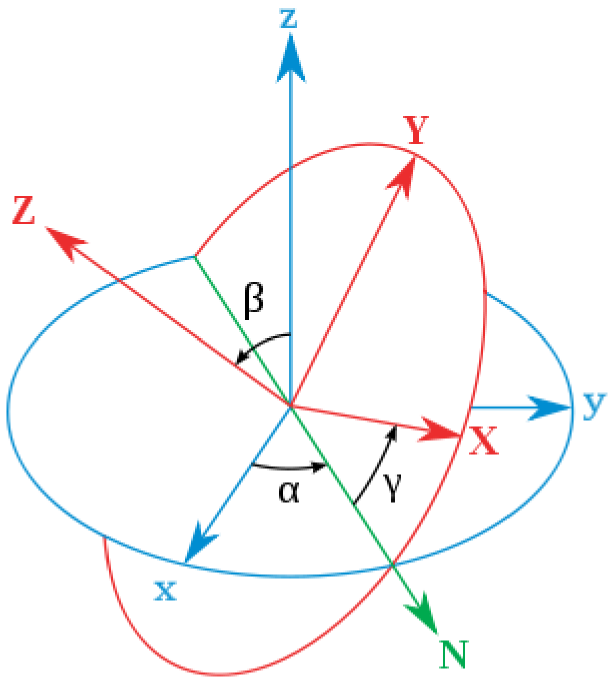
$$L := \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \end{bmatrix}$$



```
> matrixplot(A);
```



Rotation of Axes



Euler angles - The xyz (fixed) system is shown in blue, the XYZ (rotated) system is shown in red. The line of nodes, labeled N, is shown in green.

As consequence of the relationship between Euler angles and Euler rotations, we can find a Matrix expression for any frame given its Euler angles, here named as α, β, and γ. Using the **z-x-z** convention, a matrix can be constructed that transforms every vector of the given reference frame in the corresponding vector of the referred frame.

Define three sets of coordinate axes, called **intermediate frames** with their origin in common in such a way that each one of them differs from the previous frame in an elemental rotation, as if they were mounted on a gimbal. In these conditions, any target can be reached performing three simple rotations, because two first rotations determine the new Z axis and the third rotation will obtain all the orientation possibilities that this Z axis allows. These frames could also be defined statically using the reference frame, the referred frame and the line of nodes.

A matrix representing the end result of all three rotations is formed by successive multiplication of the matrices representing the three simple rotations, as in the following transformation equation

$$\mathbf{p}' = \mathbf{R}\mathbf{p}$$

$$\begin{bmatrix} p'_x \\ p'_y \\ p'_z \end{bmatrix} = \mathbf{R} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix},$$ where

$$\mathbf{R} = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where

- The rightmost matrix represents the α rotation around the axis ' **z** ' of the original reference frame
- The middle matrix represents the β rotation around an intermediate ' **x** ' axis which is the "line of nodes".
- The leftmost matrix represents the γ rotation around the axis ' **Z** ' of the final reference frame