Row and Column Vectors of an $m \times n$ Matrix

For an $m \times n$ ***matrix***

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & & . \\ . & & & \\ . & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

the vectors

$$r_1 = \langle a_{11} \quad a_{12} \quad \cdots \quad a_{1n} \rangle$$

$$r_2 = \langle a_{21} \quad a_{22} \quad \cdots \quad a_{2n} \rangle$$

.

.

$r_m = \langle a_{m1} \quad a_{m2} \quad \cdots \quad a_{mn} \rangle$ in R^n formed

from the rows of A are called the row vectors of A,

The vectors

$$c_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ . \\ . \\ . \\ a_{m1} \end{bmatrix}, c_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ . \\ . \\ . \\ a_{m2} \end{bmatrix}, \dots, c_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ . \\ . \\ . \\ a_{mn} \end{bmatrix},$$

in R^m ***formed from the columns of A are called the column vectors of A.***

Example Row and Column Vectors in a 2x3 Matrix

Let $$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \end{bmatrix}$$

The row vectors of A are

$$r_1 = \langle 2 \;\; 1 \;\; 0 \rangle \quad \text{and} \quad r_2 = \langle 3 \;\; -1 \;\; 4 \rangle$$

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> with(VectorCalculus):
> PlotVector([<2,1,0>, <3,-1,4>], color=[red, blue]);
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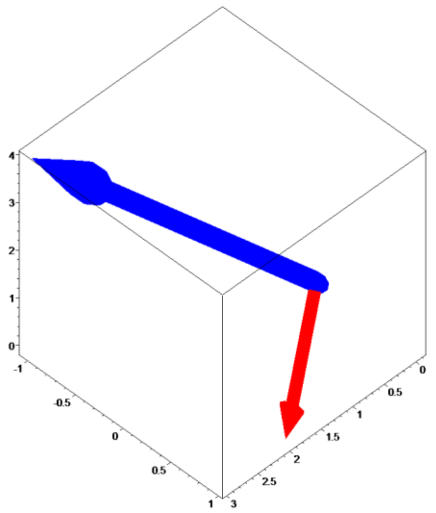

and the column vectors of A are

$$c_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad c_3 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

```
> PlotVector([<2,3>, <1,-1>,<0 ,4>], color=[red, blue,green]);
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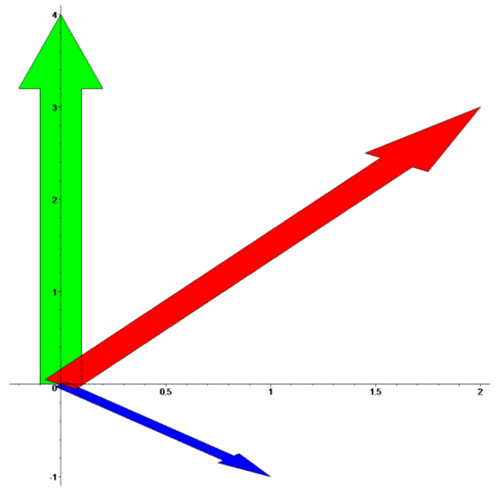

Row Space and Column Space

- Let A be an $m \times n$ matrix, with row vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$. A linear combination of these vectors is any vector of the form
- $c_1\mathbf{r}_1 + c_2\mathbf{r}_2 + \cdots + c_m\mathbf{r}_m,$
- where $c_1, c_2, \ldots, c_m$ are constants. **The set of all possible linear combinations of $\mathbf{r}_1,\ldots,\mathbf{r}_m$ is called the row space of A.** That is, the row space of A is the span of the vectors $\mathbf{r}_1,\ldots,\mathbf{r}_m$.
- For example, if
- $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix},$
- then the row vectors are $\mathbf{r}_1 = (1, 0, 2)$ and $\mathbf{r}_2 = (0, 1, 0)$. A linear combination of $\mathbf{r}_1$ and $\mathbf{r}_2$ is any vector of the form
- $c_1(1,0,2) + c_2(0,1,0) = (c_1, c_2, 2c_1).$

If A is a $m \times n$ matrix, then

- ***the subspace of R^n spanned by the row vectors called the row space of A.***

Let A be an $m \times n$ matrix, with column vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. A linear combination of these vectors is any vector of the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n,$$

where $c_1, c_2, \ldots, c_n$ are scalars. **The set of all possible linear combinations of $\mathbf{v}_1,\ldots,\mathbf{v}_n$ is called the column space of A.** That is, the column space of A is the span of the vectors $\mathbf{v}_1,\ldots,\mathbf{v}_n$.

Example

If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}$, then the column vectors are $\mathbf{v}_1 = (1, 0, 2)$ and $\mathbf{v}_2 = (0, 1, 0)$.
A linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ is any vector of the form

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ 2c_1 \end{bmatrix}$$

The set of all such vectors is the column space of A.

the subspace of R^m spanned by the column vectors of A is called the column space of A.

What relationships exist between the solutions of a linear system Ax=b and the <u>row space</u> and <u>column space</u> of the coefficient matrix A?

Suppose that

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & & . \\ . & & & \\ . & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad x=\begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}$$

If $c_1, c_2, \ldots, c_n$ denote the column vectors of A, then the *<u>product Ax</u>* can be expressed as a linear combinations of these column vectors with coefficients from x; that is,

$$Ax = x_1c_1 + x_2c_2 + \ldots + x_nc_n$$

$$Ax=\begin{bmatrix} a_{11}x_1+a_{12}x_2+ & \dots+a_{1n}x_n \\ a_{21}x_1+a_{22}x_2+ & \dots+a_{2n}x_n \\ . \quad . & . \\ . & \\ . & \\ a_{m1}x_1+a_{m2}x_2+ & \dots+a_{mn}x_n \end{bmatrix} = x_1\begin{bmatrix} a_{11} \\ a_{21} \\ . \\ . \\ . \\ a_{m1} \end{bmatrix} + x_2\begin{bmatrix} a_{12} \\ a_{22} \\ . \\ . \\ . \\ a_{m2} \end{bmatrix} + \dots + x_n\begin{bmatrix} a_{1n} \\ a_{2n} \\ . \\ . \\ . \\ a_{mn} \end{bmatrix}$$

Thus a linear system, Ax=b of m equations in n unknowns can be written as

$$x_1c_1+x_2c_2+\dots+x_nc_n=b$$

from which we conclude that Ax=b is consistent if and only if b is expressible as a linear combination of the column vectors of A or, equivalently, if and only if <u>b is in the column space of A</u>.

Theorem: ***A system of linear equation Ax=b is consistent if and only if b is in the column space of A.***

Example (A Vector b in the Column Space of A)

Let Ax=b be the linear system

$$\begin{aligned} -x_1 + 3x_2 + 2x_3 &= 1 \\ x_1 + 2x_2 - 3x_3 &= -9 \\ 2x_1 + x_2 - 2x_3 &= -3 \end{aligned}$$

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

Show that b is in the column space of A, and express b as linear combination of the column vectors of A. Solving the system by Gaussian elimination yields

$$x_1 = 2, x_2 = -1, x_3 = 3$$

Since the system is consistent, b is in the column space of A.

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

$$x_1c_1 + x_2c_2 + \ldots + x_nc_n = b$$

It follows that

$$2\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

```
> PlotVector([<-1,1,2>, <3,2,1>,<2,-3,-2>], color=[red,
blue,green]);
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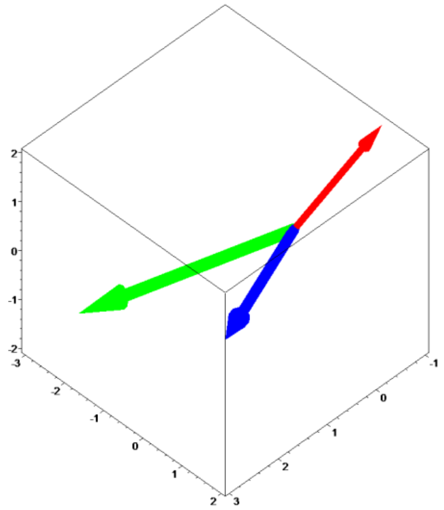

```
> PlotVector([<-1,1,2>, <3,2,1>,<2,-3,-2>,<1,-9,3>],
color=[red, blue,green,black]);
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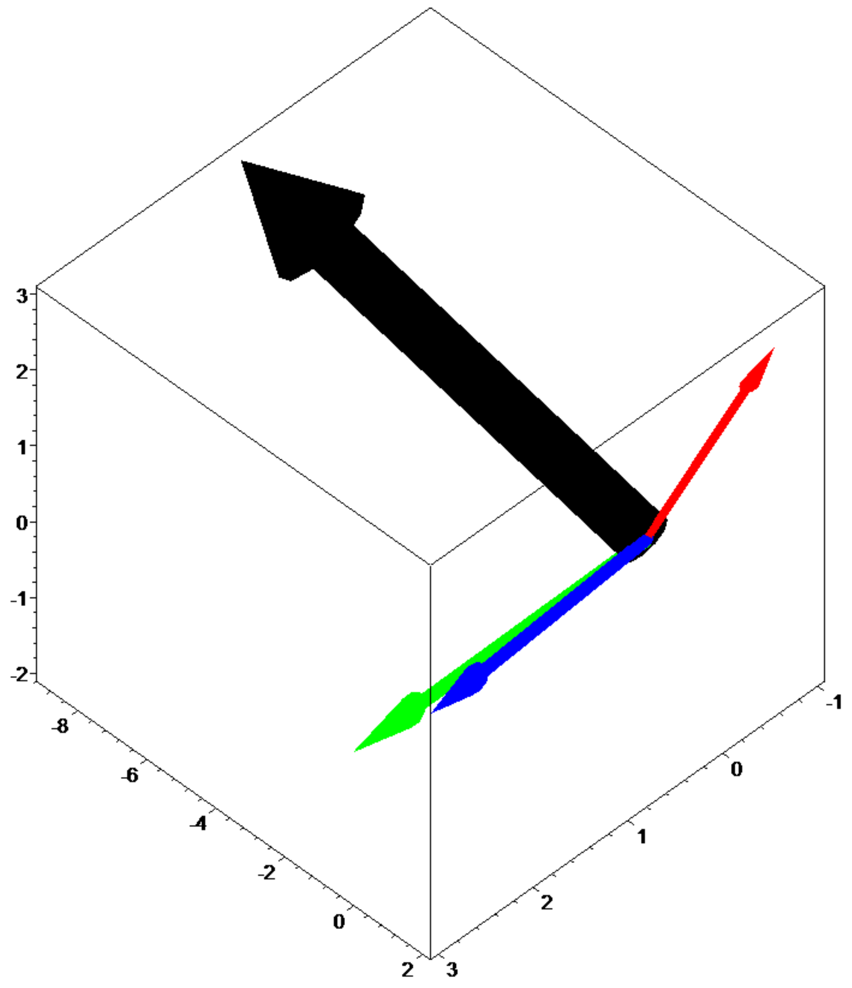


b is in the column space of A.

Example

$$\begin{aligned}
x_1 + 3x_2 - 2x_3 \qquad\quad + 2x_5 \qquad\qquad\qquad &= 0 \\
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 \qquad &= -1 \\
5x_3 + 10x_4 \qquad\qquad + 15x_6 &= 5 \\
2x_1 + 6x_2 \qquad\qquad + 8x_4 + 4x_5 + 18x_6 &= 6
\end{aligned}$$

General Solution

$$\begin{aligned}
x_1 &= -3r - 4s - 2t, \\
x_2 &= r, \\
x_3 &= -2s, \\
x_4 &= s, \\
x_5 &= t, \\
x_6 &= \tfrac{1}{3}.
\end{aligned}$$

This result can be written in vector form as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -3r-4s-2t \\ r \\ -2s \\ s \\ t \\ 1/3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1/3 \end{bmatrix}}_{\mathbf{x_0}} + \underbrace{r\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{x}}$$

The vector $\mathbf{x_0}$ is a *particular solution* of the given non homogeneous system.

The linear combination $\mathbf{x}$ is the *general solution* of the following homogeneous system.

$$\begin{aligned}
x_1 + 3x_2 - 2x_3 \qquad + 2x_5 \qquad\qquad &= 0 \\
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0 \\
5x_3 + 10x_4 \qquad + 15x_6 &= 0 \\
2x_1 + 6x_2 \qquad + 8x_4 + 4x_5 + 18x_6 &= 0
\end{aligned}$$

General and Particular Solutions

The vector x_0 is called a *particular solution* of Ax=b.

The expression

$$x_0 + c_1 v_1 + c_2 v_2 + \ldots + c_k v_k$$

is called the general solution of Ax=b.

The expression

$$c_1 v_1 + c_2 v_2 + \ldots + c_k v_k$$

is called the general solution of Ax=0.

The formula

$$x = x_0 + c_1 v_1 + c_2 v_2 + \ldots + c_k v_k$$

states that the general solution of Ax=b is the sum of any particular solution of Ax=b and the general solution of Ax=0.

Homogenous Linear Systems

A system of linear equations is said to be ***homogenous*** **if the** **<u>constant terms are all zero</u>; that is, the system has the form**

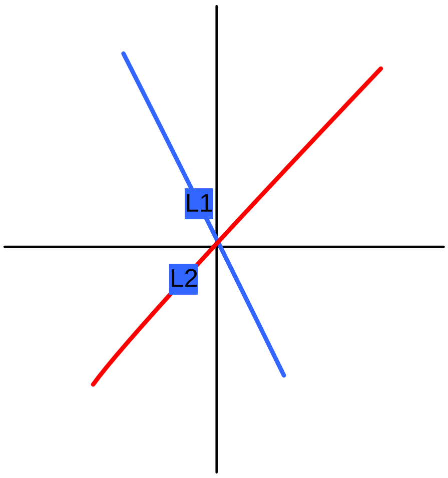
$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \ldots . + a_{1n}x_n = 0 \\
&a_{21}x_1 + a_{22}x_2 + \ldots . + a_{2n}x_n = 0 \\
&. \\
&. \\
&. \\
&a_{m1}x_1 + a_{m2}x_2 + \ldots . + a_{mn}x_n = 0
\end{aligned}
$$

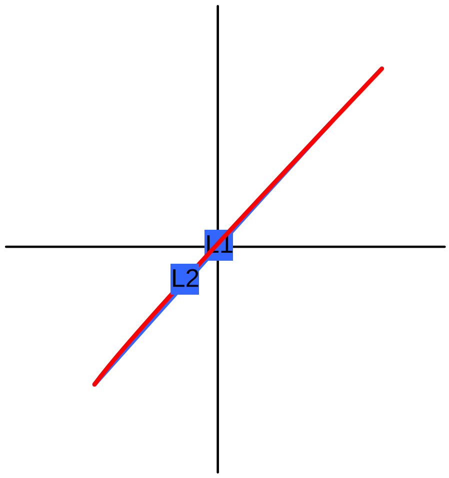
- **Every homogeneous system of linear equation is *consistent*, since all such systems have $x_1 = 0, x_2 = 0, \ldots, x_m = 0$ as a solution. This solution is called the *trivial solution*; if there are other solutions, they are called *nontrivial solutions*.**

- **Because a homogeneous linear system *always has the trivial solution*, there are only two possibilities for its solutions:**
  - **The system has *only the trivial solution*.**
  - **The system has *infinitely many solutions in addition to the trivial solution*.**

In the special case of a homogenous linear system of two equations in two unknowns, say

$$a_1x + b_1y = 0 \quad (a_1, b_1 \text{ not both zero})$$
$$a_2x + b_2y = 0 \quad (a_2, b_2 \text{ not both zero})$$





Only the trivial solution. **Infinitely many solutions**

Example:

Gauss-Jordan Elimination for homogenous system

Solve the following homogenous system of linear equations by using Gauss-Jordan elimination,

$$\begin{aligned}
2x_1 + 2x_2 - x_3 \qquad + x_5 &= 0 \\
-x_1 - x_2 + 2x_3 - 3x_4 + x_5 &= 0 \\
x_1 + x_2 - 2x_3 \qquad - x_5 &= 0 \\
x_3 + x_4 + x_5 &= 0
\end{aligned}$$

The augmented matrix for the system is

$$\begin{bmatrix} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Reducing this matrix to **reduced row-echelon form**, we obtain

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding system of equations is

$$\begin{aligned} x_1 + x_2 - x_3 \qquad\qquad + x_5 &= 0 \\ x_3 \qquad + x_5 &= 0 \\ x_4 \qquad &= 0 \end{aligned}$$

Solving for the leading variables yields

$$\begin{aligned} x_1 &= -x_2 - x_5, \\ x_3 &= -x_5, \\ x_4 &= 0. \end{aligned}$$

Thus, the general solution is

$$\begin{aligned} x_1 &= -s-t, \\ x_2 &= s, \\ x_3 &= -t \\ x_4 &= 0 \\ x_5 &= t. \end{aligned}$$

Note: The trivial solution is obtained when $s=t=0.$

The solution vectors can be written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -s-t \\ s \\ -t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -s \\ s \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -t \\ 0 \\ -t \\ 0 \\ t \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

The solution space is **two dimensional** (v_1 and v_2) and the vectors

$$v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

form a ***basis for nullspace***.

Theorem: ***A homogeneous system of linear equations with more unknowns than equations has infinitely many solutions.***

Theorem applies only to homogeneous systems.

A nonhomogeneous system with more unknowns than equations need not be consistent; however, if the system is consistent, it will have infinitely many solutions.

The only case in which a system of homogeneous equations additionally admits a set of nontrivial solutions is when A is a singular matrix, i.e.

$$|A| = 0$$

In this case the equations are not all independent and there is an infinite set of nontrivial solutions, dependent on one or more parameters.