MATRICES

AND

MATRIX OPERATIONS

III

DETERMINANTS

Determinants

Recall that the 2x2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if $ad - bc \neq 0.$

The expressions $ad - bc$ occurs so frequently in mathematics that it has a name; it is called the determinant of the matrix A and is denoted by the symbol $\det(A)$ or $|A|$. With this notation, the formula A^{-1}*is*

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The determinant of a 1 by 1 matrix is $det(a) = a$.

Definition: (Minors and Cofactors)

If A square matrix, then the minor of entry a_{ij} is denoted by M_{ij} and is defined to be determinant of the submatrix that remains after the <u>ith row and jth column are deleted</u> from A.

The number

$$(-1)^{i+j}M_{ij}$$

is denoted by C_{ij}

and is called cofactor of entry a_{ij}.

$$C_{ij} = (-1)^{i+j}M_{ij}$$

Example: (Finding Minors and Cofactors)

Let

$$A=\begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$

The minor of entry a_{11} is

$$M_{11}=\begin{vmatrix} 1 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix}=\begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix}=16$$

The cofactor of a_{11} is $C_{11}=(-1)^{1+1}M_{11}=M_{11}=16$

Similarly the minor of entry a_{32} is

$$M_{32}=\begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix}=\begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix}=26$$

The cofactor of a_{32} is $C_{32}=(-1)^{3+2}M_{32}=-M_{32}=-26$

The cofactor and minor of an element a_{ij} ***differ only in sign; that is,*** $C_{ij} = \pm M_{ij}$

$$\begin{bmatrix} +-+-+-+-......+........ \\ -+-+-+-+......-........ \\ +-+-+-+-......+........ \\ -+-+-+-+......-........ \\ ... \\ ... \\ +-+-+-+-......+........ \\ -+-+-+-+......-........ \\ ... \\ ... \\ .. \end{bmatrix}$$

Check board Array

The definition of a 3x3 determinant in terms of minors and cofactors is

$$\begin{aligned} \det(A) &= a_{11}M_{11} + a_{12}(-M_{12}) + a_{13}M_{13} \\ &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \end{aligned}$$

Evaluating det(A) is called cofactor expansion along the first row of A.

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \ldots + a_{1n}C_{1n}$$

Example: (Cofactor expansion along the first row)

Let

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$

Evaluate det(A) by cofactor expansion along the first row of A.

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - 1\begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + 0\begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix}$$

$$= 3(-4) - 1(-11) + 0 = -1$$

If A is a 2x2 matrix, then its determinant is

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\det(A) \ = a_{11}a_{22} - a_{12}a_{21}$$

If A is a 3x3 matrix, then its determinant is

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{11}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{11}(a_{21}a_{33} - a_{31}a_{23}) +$$
$$a_{11}(a_{21}a_{32} - a_{31}a_{22})$$
$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$- a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{Bmatrix}$$

$$\begin{aligned} \det(A) &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\quad - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \end{aligned}$$

Example: (Evaluating Determinants)

$$A = \begin{bmatrix} 3 & 1 \\ 4 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix}$$

$$\det(A) = (3)(-2) - (1)(4) = -10$$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix} \begin{Bmatrix} 1 & 2 \\ -4 & 5 \\ 7 & -8 \end{Bmatrix}$$

$$\det(B) = (45) + (84) + (96) - (105) - (-48) - (-72) = 240$$

There should be no trouble checking that all of the following are correct

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$
$$= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}$$
$$= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$
$$= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$$
$$= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$$
$$= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33}$$

Theorem (Expansion by Cofactors)

The determinant of nxn matrix A can be computed by multiplying the entries in any row (or column) by their cofactors and adding the resulting products; that is, for each $1 \le i \le n$ ***and*** $1 \le j \le n$**.**

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}$$

(Cofactor expansion along the jth column)

$$\det(A) = \sum_{i=1}^{n} a_{ij}C_{ij} \qquad \text{for any } 1 \le \mathrm{j} \le \mathrm{n}$$

And

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$$

(Cofactor expansion along the ith row)

$$\det(A) = \sum_{j=1}^{n} a_{ij}C_{ij} \qquad \text{for any } 1 \le i \le \mathrm{n}$$

Example: (Cofactor expansion along the first column)

Let

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix}$$

$$= 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix} + 5\begin{vmatrix} 1 & 0 \\ -4 & 3 \end{vmatrix}$$

$$= 3(-4) - (-2)(-2) + 5(3) = -1$$

Smart choice of row or column

Example:

If A is the 4x4 matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

Then to find det(A) it will be ***easiest*** to use cofactor expansion along the ***second column***, since it has the most zeros:

$$\det(A) = 1 \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix}$$

For the 3x3 determinant, it will be ***easiest*** to use cofactor expansion along its ***second column*** since it has the most zeros:

$$\det(A) = 1.(-2) \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = (-2)(1-(-2)) = -6$$

We would have found the same answer if we had used any other row or column.

Adjoint of a Matrix

Definition:

If A is any $n \times n$ matrix and C_{ij} is the cofactor of a_{ij}, then the matrix

$$C = \begin{bmatrix} C_{11} C_{12} \ldots\ldots C_{1n} \\ C_{21} C_{22} \ldots\ldots C_{2n} \\ .. \\ . \\ C_{n1} C_{n2} \ldots\ldots C_{nn} . \end{bmatrix}$$

is called the matrix of cofactors from A. The transpose of this matrix is called the adjoint of A and is denoted by adj(A).

$$adj(A) = C^T$$

$$adj(A) = C^T = \begin{bmatrix} C_{11} C_{21} \ldots\ldots C_{n1} \\ C_{12} C_{22} \ldots\ldots C_{n2} \\ .. \\ . \\ C_{1n} C_{2n} \ldots\ldots C_{nn} . \end{bmatrix}$$

Example: **(Adjoint of a matrix)**

Let

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

The cofactors of A are

$$C_{11} = 12, C_{12} = 6, C_{13} = -16$$
$$C_{21} = 4, C_{22} = 2, C_{23} = 16$$
$$C_{31} = 12, C_{32} = -10, C_{33} = 16.$$

So the matrix of cofactors is

$$\begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & 12 \\ 12 & -10 & 16 \end{bmatrix}$$

and the adjoint of A is

$$adj(A) = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

MATLAB ADJOINT

```
function [B] = adj(A,mode)
%ADJ Matrix adjoint.
%
%   ADJ(A) is the adjoint matrix of matrix A.
%   A may be complex and any size, RxC
%
%   usage: B = adj(A)
%
%   If R <= C then the Right Adjoint is returned
%   If R >= C then the Left  Adjoint is returned
%
%   The inverse of A is: INV(A) = ADJ(A)/det(A).
%   Matrices that are not invertable still have an
adjoint.
%
%   See also SVD, PINV, INV, RANK, SLASH, DETT

%Paul Godfrey, October, 2006

[r,c]=size(A);
[u,s,v]=svd(A);

k=det(u)*det(v');
if r==1 | c==1, s=s(1); end
s=diag(s);

if exist('mode','var')
    B=k*prod(s)*pinv(A);
else
    for n=1:length(s)
        p=s;
        p(n)=[];
        V(:,n)=k*prod(p)*v(:,n);
    end
    B=V*u(:,1:length(s))';
end

return
```

Theorem (Inverse of a Matrix Using Its Adjoint)

If A is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

Example: (Using the Adjoint to find an inverse matrix)

Let

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

det(A)=64 (Check). Thus

$$A^{-1} = \frac{1}{\det(A)} adj(A) = \frac{1}{64}\begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

Theorem: (Determinant of a triangular matrix)

If A is any $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal),then det(A) is the product of the entries on the main diagonal of the matrix; that is,

$$\det(A) = a_{11}a_{22}\ldots a_{nn}$$
$$= \prod_{i=1}^{n} a_{ii}$$

Example: (Determinant of a triangular matrix)

Let

$$A = \begin{bmatrix} 2 & 7 & -3 & 8 & 3 \\ 0 & -3 & 7 & 5 & 1 \\ 0 & 0 & 6 & 7 & 6 \\ 0 & 0 & 0 & 9 & 8 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 2 & 7 & -3 & 8 & 3 \\ 0 & -3 & 7 & 5 & 1 \\ 0 & 0 & 6 & 7 & 6 \\ 0 & 0 & 0 & 9 & 8 \\ 0 & 0 & 0 & 0 & 4 \end{vmatrix} = (2)(-3)(6)(9)(4) = -1296$$

```
>> A=[2 7 -3 8 3; 0 -3 7 5 1;0 0 6 7 6; 0 0 0 9 8; 0 0 0 0 4]

A =

     2     7    -3     8     3
     0    -3     7     5     1
     0     0     6     7     6
     0     0     0     9     8
     0     0     0     0     4

>> det(A)

ans =

       -1296
>> B=[2 0 0 0; -3 4 0 0; 1 2 5 0; 3 4 5 6]

B =
     2     0     0     0
    -3     4     0     0
     1     2     5     0
     3     4     5     6

>> det(B)
ans =

   240

>> C=[1 0 0;0 2 0; 0 0 3]
C =
     1     0     0
     0     2     0
     0     0     3
>> det(C)
ans =
     6
>>
```

Theorem:

Let A be a square matrix. If A has a row of zeros or a column of zeros, then det(A)=0.

Theorem:

Let A be a square matrix. Then det(A)=det(A^T).

```
>> A=[1 2 5; 7 6 4; 0 0 0]

A =
     1     2     5
     7     6     4
     0     0     0

>> det(A)
ans =     0
>> B=[1 2 5; 7 6 4; 9 8 7]
B =
     1     2     5
     7     6     4
     9     8     7
>> C=B'
C =
     1     7     9
     2     6     8
     5     4     7
>> det(B)
ans =    -6
>> det(C)

ans =    -6

>>
```

Theorem:

Let A be an $n \times n$ ***matrix.***

1. ***If B is the matrix that results when a single row or single column of A is multiplied by a scalar k, then***

$$\det(B) = k\det(A)$$

2. ***If B is the matrix that results when two rows or two columns of A are interchanged, then***

$$\det(B) = -\det(A)$$

3. ***If B is the matrix that results when a multiple of one row or when a multiple of one column is added to another column is added to another column then***

$$\det(B) = \det(A)$$

Example:

$$\det(B) = \begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= ka_{11}C_{11} + ka_{12}C_{12} + ka_{13}C_{13}$$

$$= k(a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13})$$

$$= k\det(A)$$

$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\det(B) = k\det(A)$$

The first row of A is multiplied by k

$$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\det(B) = -\det(A)$$

The first and second rows of A are interchanged.

$$\begin{vmatrix} a_{11} + ka_{21} & a_{12} + ka_{22} & a_{13} + ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\det(B) = \det(A)$$

A multiple of the second row of A is added to the first row.

Elementary Matrices: ***Recall that an elementary matrix results from performing a single elementary row operation on an identity matrix.***

Theorem:

Let E be an $n \times n$ matrix.

1. If E results from multiplying a row of I_n by k, then

$$\det(E) = k$$

2. If E results from interchanging two rows of I_n, then

$$\det(E) = -1$$

3. If E results from adding a multiple of one row of I_n to another, then

$$\det(E) = 1$$

Example: (Determinants of Elementary Matrices)

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 3$$

The second row of I_4 was multiplied by 3.

$$\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = -1$$

The first and last rows of I_4 were interchanged.

$$\begin{vmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 7$$

7 times the last row of I_4 was added to the first row.

Matrices with Proportional Rows or Columns:

If a square matrix A has two proportional rows, then a row of zeros can be introduced by adding a suitable multiple of one of the rows to other. Similarly for columns.

Theorem:

If A is a square matrix with two proportional rows or two proportional columns, then

$$\det(A) = 0$$

Example: (Introducing Zero Rows)

$$\begin{vmatrix} 1 & 3 & -2 & 4 \\ 2 & 6 & -4 & 8 \\ 3 & 9 & 1 & 5 \\ 1 & 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -2 & 4 \\ 0 & 0 & 0 & 0 \\ 3 & 9 & 1 & 5 \\ 1 & 1 & 4 & 8 \end{vmatrix} = 0$$

The second row is 2 times the first, so we added -2 times the first row to the second to introduce a row of zeros.

Example:

Each of the following matrices has two proportional rows or columns; thus each has a determinant of zero.

```
>> A=[-1 4 ;-2 8]
A =
   -1     4
   -2     8
>> B=[1 -2 7; -4 8 5; 2 -4 3]
B =
    1    -2     7
   -4     8     5
    2    -4     3
>> C=[3 -1 4 -5; 6 -2 5 2; 5 8 1 4; -9 3 -12 15]
C =
    3    -1     4    -5
    6    -2     5     2
    5     8     1     4
   -9     3   -12    15
>> det(A)
ans =     0
>> det(B)
ans =     0
>> det(C)
ans =     0
```

Evaluating Determinants
by
Row Reduction

The idea of the method is to reduce the given matrix to upper triangular form by elementary row operations, then compute the determinant of the upper triangular matrix (easy computation), and then relate that the determinant to that of the original matrix.

Example: Using row reduction to evaluate a determinant

Evaluate det(A) where

$$A = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} = -\begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

The first and second rows of A were interchanged

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

A common factor of 3 from the first row was taken through the determinant sign.

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix}$$

-2 times the first row was added to the third row.

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix}$$

-10 times the second row was added to the third row.

$$= (-3)(-55)\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{vmatrix}$$

A common factor of -55 from the last row was taken through the determinant sign.

$$= (-3)(-55)(1) = 165$$

Using Column Operations to Evaluate a Determinant

Example:

Compute the determinant of

$$A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 2 & 7 & 0 & 6 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -5 \end{bmatrix}$$

This determinant could be computed as above by using elementary row operations to reduce A to ***row-echelon*** form, but we can put A in **lower triangular form** in one step by adding -3 times the first **column** to the fourth to obtain

$$\det(A) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 7 & 0 & 0 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -26 \end{vmatrix} = (1)(7)(3)(-26) = -546$$

Example: Row Operations and Cofactor Expansion

Compute the determinant of

$$A=\begin{bmatrix}3 & 5 & -2 & 6\\ 1 & 2 & -1 & 1\\ 2 & 4 & 1 & 5\\ 3 & 7 & 5 & 3\end{bmatrix}$$

By adding <u>suitable multiples</u> of the second row to the remaining rows, we obtain

$$\det(A)=\begin{vmatrix}0 & -1 & 1 & 3\\ 1 & 2 & -1 & 1\\ 0 & 0 & 3 & 3\\ 0 & 1 & 8 & 0\end{vmatrix}$$

Cofactor expansion along the first column

$$=-\begin{vmatrix}-1 & 1 & 3\\ 0 & 3 & 3\\ 1 & 8 & 0\end{vmatrix}$$

Add the first row to the third row

$$=-\begin{vmatrix}-1 & 1 & 3\\ 0 & 3 & 3\\ 0 & 9 & 3\end{vmatrix}$$

Cofactor expansion along the first column

$$=-(-1)\begin{vmatrix}3 & 3\\ 9 & 3\end{vmatrix}=-(-1)(-18)=-18$$

Solution to a Linear System Using Determinants: Cramer's Rule

Theorem:

If Ax=b is a system of n linear equations in n unknowns such that

$$\det(A) \neq 0,$$

Then the system has a unique solution. This solution is

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \ldots, \quad x_n = \frac{\det(A_n)}{\det(A)}$$

where A_j is the matrix obtained by replacing the entries in the jth column of A by the entries in the matrix

$$b = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_n \end{bmatrix}$$

Example (Using Cramer's Rule to solve a Linear System)

$$\begin{aligned} x_1 + \quad\quad + 2x_3 &= 6 \\ -3x_1 + 4x_2 + 6x_3 &= 30 \\ -x_1 - 2x_2 + 3x_3 &= 8 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}$$

Therefore

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-40}{44} = \frac{-10}{11} = -0.9091$$

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{72}{44} = \frac{18}{11} = 1.6364$$

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{152}{44} = \frac{38}{11} = 3.4545$$

To solve a system of n equations in n unknowns by Cramer's rule, it is necessary to evaluate n+1 determinant of nxn matrices.

```
>> A=[1 0 2; -3 4 6; -1 -2 3]
A =
     1     0     2
    -3     4     6
    -1    -2     3
>> A1=[6 0 2; 30 4 6; 8 -2 3]
A1 =
     6     0     2
    30     4     6
     8    -2     3
>> A2=[1 6 2;-3 30 6;-1 8 3]
A2 =
     1     6     2
    -3    30     6
    -1     8     3
>> A3=[1 0 6 ;-3 4 30;-1 -2 8]
A3 =
     1     0     6
    -3     4    30
    -1    -2     8
>> x1=det(A1)/det(A)
x1 =   -0.9091
>> x2=det(A2)/det(A)
x2 =    1.6364
>> x3=det(A3)/det(A)
x3 =    3.4545
```

Properties of the Determinant Function

- **Since a common factor of any row of a matrix can be moved through the det sign, and since each of the n rows in kA has common factor of k**

$$\det(kA) = k^n \det(A)$$

$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{vmatrix} = k^3 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

- $$\det(A+B) \neq \det(A) + \det(B)$$

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \qquad B = 9\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 9 & 18 \\ 18 & 45 \end{bmatrix}$$

$$\det(A) = 5 - 4 = 1,$$
$$\det(B) = 405 - 324 = 81$$
$$\det(B) = 9^2 * \det(A) = 81 * 1 = 81$$

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \qquad A + B = \begin{bmatrix} 4 & 3 \\ 3 & 8 \end{bmatrix}$$

$\det(A) = 1, \det(B) = 8, \textit{and} \det(A+B) = 23$ Thus

$$\det(A+B) \neq \det(A) + \det(B)$$

Theorem

Let A, B and C be nxn matrices that differ only in a single row, say the r^{th}, and assume that the r^{th} row of C can be obtained by adding corresponding entries in the r^{th} rows of A and B. Then

$$\det(C) = \det(A) + \det(B)$$

The same result holds for columns.

Example:

$$\det\begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1+0 & 4+1 & 7+(-1) \end{bmatrix} = \det\begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1 & 4 & 7 \end{bmatrix} + \det\begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix}$$

```
>> A=[1 7 5;2 0 3;1 4 7]
A =
    1    7    5
    2    0    3
    1    4    7
>> B=[1 7 5; 2 0 3;0 1 -1]
B =
    1    7    5
    2    0    3
    0    1   -1
>> C=[1 7 5; 2 0 3; 1 5 6]
C =
    1    7    5
    2    0    3
    1    5    6
>> det(A)
ans =   -49
>> det(B)
ans =    21
 >> det(C)ans =   -28  >>
```

Determinant of a Matrix Product

Theorem ***If A and B are square matrices of the same size, then***

$$\det(AB) = \det(A)\det(B)$$

If B is nxn ***matrix and E is an*** nxn ***elementary matrix, then***

$$\det(EB) = \det(E)\det(B)$$

Example: Consider the matrices

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 3 \\ 5 & 8 \end{bmatrix} \qquad AB = \begin{bmatrix} 2 & 17 \\ 3 & 14 \end{bmatrix}$$

$$\det(A) = 1, \det(B) = -23, \quad and \quad \det(AB) = -23$$

```
>> A=[3 1 ; 2 1]
A =
    3    1
    2    1
>> B=[-1 3 ;5 8]
B =
   -1    3
    5    8
>> AB=A*B
AB =
    2   17
    3   14
>> det(A)
ans =     1
>> det(B)
ans =   -23
>> det(AB)
ans =   -23
```

Determinant Test for Invertibility

Theorem

A square matrix A is invertible if and only if

$$\det(A) \neq 0.$$

Example:

Since the **first and third rows of**

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 4 & 6 \end{bmatrix}$$

are proportional, det(A)=0. Thus A is not invertible.

```
>> A=[1 2 3 ;1 0 1; 2 4 6]

A =

     1     2     3

     1     0     1

     2     4     6

>> inv(A)
```

Warning: Matrix is singular to working precision.

Theorem

If A is invertible, then

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

```
>> A=[1 2 3 ;1 0 1; 3 4 8]
A =
     1     2     3
     1     0     1
     3     4     8
>> B=inv(A)
B =
    2.0000    2.0000   -1.0000
    2.5000    0.5000   -1.0000
   -2.0000   -1.0000    1.0000
>> det(A)
ans =    -2
>> det(B)
ans =   -0.5000
>>
```

Theorem Equivalent Statements

If A is an $n \times n$ matrix, then the following statements are equivalent.

1. ***A is invertible.***
2. ***Ax=0 has only the trivial solution.***
3. ***The reduced row-echelon form of A is I_n.***
4. ***A can be expressed as a product of elementary matrices.***
5. ***Ax=b is consistent for every $n \times 1$ matrix b.***
6. ***Ax=b has exactly one solution for every $n \times 1$ matrix b.***
7. $\det(A) \neq 0$.

Further Results on Systems of Equations and Invertibility

Theorem

Every system of linear equations has no solutions, or has exactly one solution, or has infinitely many solutions.

Every nonhomogeneous system of linear equations with more unknowns than equations either has no solution or infinitely many solutions.

Theorem

A homogeneous system of linear equations with more unknowns than equations has infinitely many solutions.

Theorem

If A is an invertible $n \times n$ ***matrix, then for each*** $n \times 1$ b***, the system of equations*** $Ax = b$ ***has exactly one solution, namely*** $x = A^{-1}b$.

Example: (Solution of a Linear System Using A^{-1})

Consider the system of linear equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 5 \\ 2x_1 + 5x_2 + 3x_3 &= 3 \\ x_1 \qquad\quad + 8x_3 &= 17 \end{aligned}$$

In matrix form this system can be written as $Ax = b$, where

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 5 \\ 2x_1 + 5x_2 + 3x_3 &= 3 \\ x_1 \qquad\quad + 8x_3 &= 17 \end{aligned}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$$

In the preceding section, we showed that A is invertible and

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

By the above Theorem, the solution of the system is

$$x = A^{-1}b = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

or $x_1 = 1, x_2 = -1, x_3 = 2.$

Linear Systems with a Common Coefficient Matrix

Frequently, one is concerned with solving a sequence of systems

$$Ax = b_1,\ Ax = b_2,\ Ax = b_3, \dots, Ax = b_k,$$

each of which has the same **square coefficient matrix** ***A***. If ***A*** is **invertible**, then the solutions

$$\begin{aligned} x_1 &= A^{-1}b_1, \\ x_2 &= A^{-1}b_2, \\ x_3 &= A^{-1}b_3 \\ &. \\ &. \\ &. \\ x_k &= A^{-1}b_k, \end{aligned}$$

can be obtained ***with one matrix inversion*** and k matrix multiplication.

Once again, however, a more efficient method is to form the matrix

$$[A \mid b_1 \mid b_2 \mid b_3 \mid \dots \mid b_k]$$

in which the coefficient matrix ***A*** is ***"augmented"*** by all ***k*** of the matrices $b_1, b_2, b_3, \dots, b_k$ and then reduce to reduced ***row–echelon*** form by ***Gauss-Jordan elimination.***

Example: (Solving Two Linear Systems at Once)

Solve the systems

$$\textbf{a)}\quad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 2x_1 + 5x_2 + 3x_3 &= 5 \\ x_1 \qquad\quad + 8x_3 &= 9 \end{aligned} \qquad \textbf{b)}\quad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 5x_2 + 3x_3 &= 6 \\ x_1 \qquad\quad + 8x_3 &= -6 \end{aligned}$$

If we augment common coefficient matrix with the columns of constants on the right sides of these systems, we obtain

$$\left[\begin{array}{ccc|c|c} 1 & 2 & 3 & 4 & 1 \\ 2 & 5 & 3 & 5 & 6 \\ 1 & 0 & 8 & 9 & -6 \end{array}\right]$$

Reducing this matrix to reduced-row echelon form yields **(verify)**

$$\left[\begin{array}{ccc|c|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{array}\right]$$

The solution of system (a) is $x_1 = 1, x_2 = 0, x_3 = 1.$

The solution of system (b) is $x_1 = 2, x_2 = 1, x_3 = -1.$