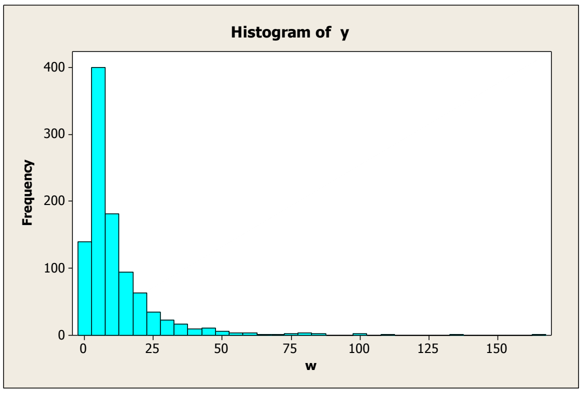
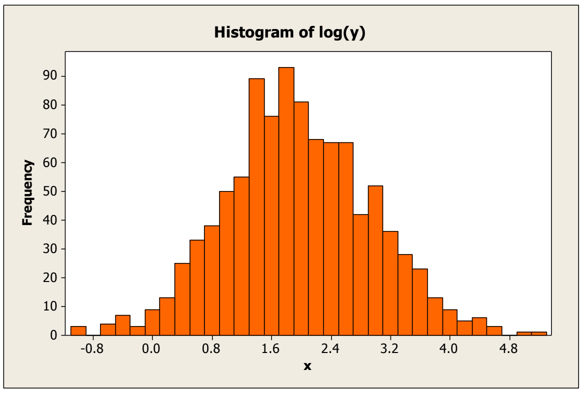
Lognormal Distribution

The **log-normal distribution** is the **single-tailed probability distribution of any random variable whose logarithm is normally distributed.** **If *X* is a random variable with a normal distribution, then *Y* = exp(*X*) has a log-normal distribution; likewise, if *Y* is log-normally distributed, then log(*Y*) is normally distributed.**



Histogram of Y



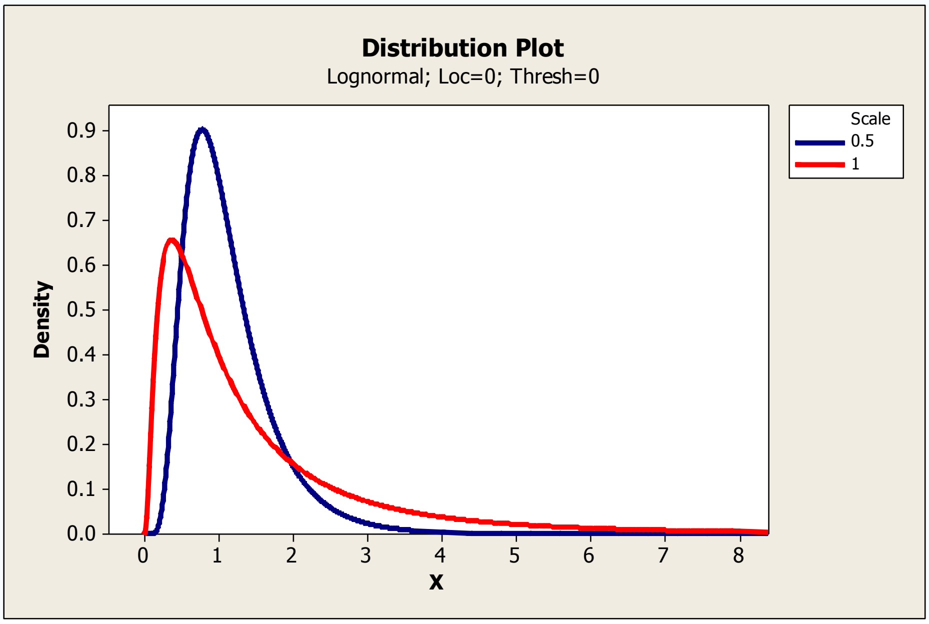
Histogram of log(Y)

Lognormal Distribution

Let Y have a normal distribution with mean μ and variance σ^2; then $X=\exp(Y)$ is a lognormal random variable with probability density function

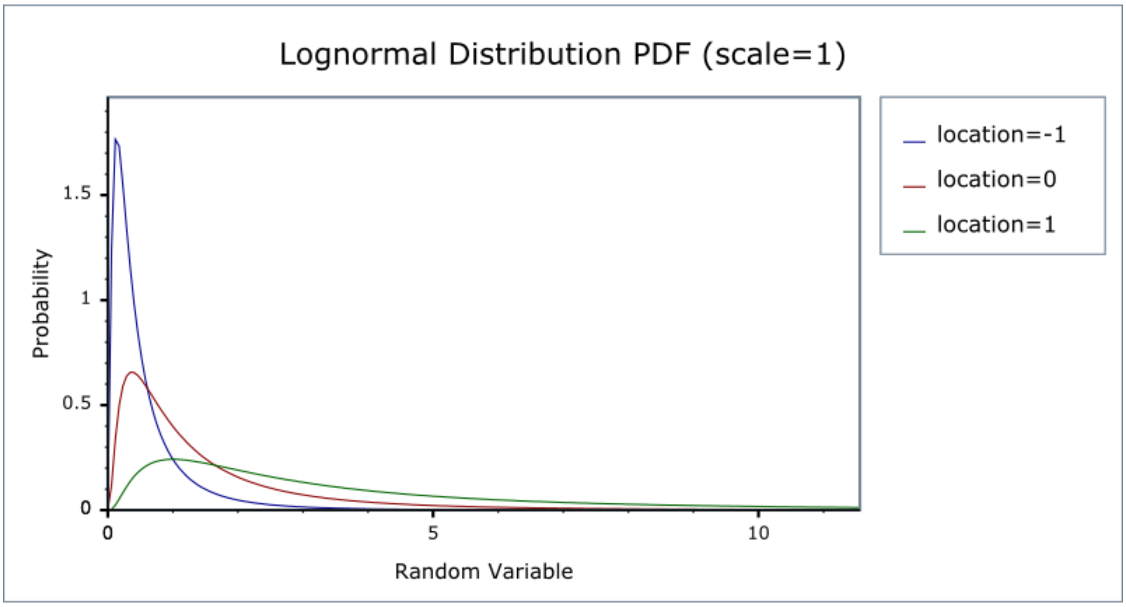
$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left[-\frac{(\ln(x)-\mu)^2}{2\sigma^2} \right], \qquad x > 0$$

The location and scale parameters are equivalent to the mean and standard deviation of the logarithm of the random variable.

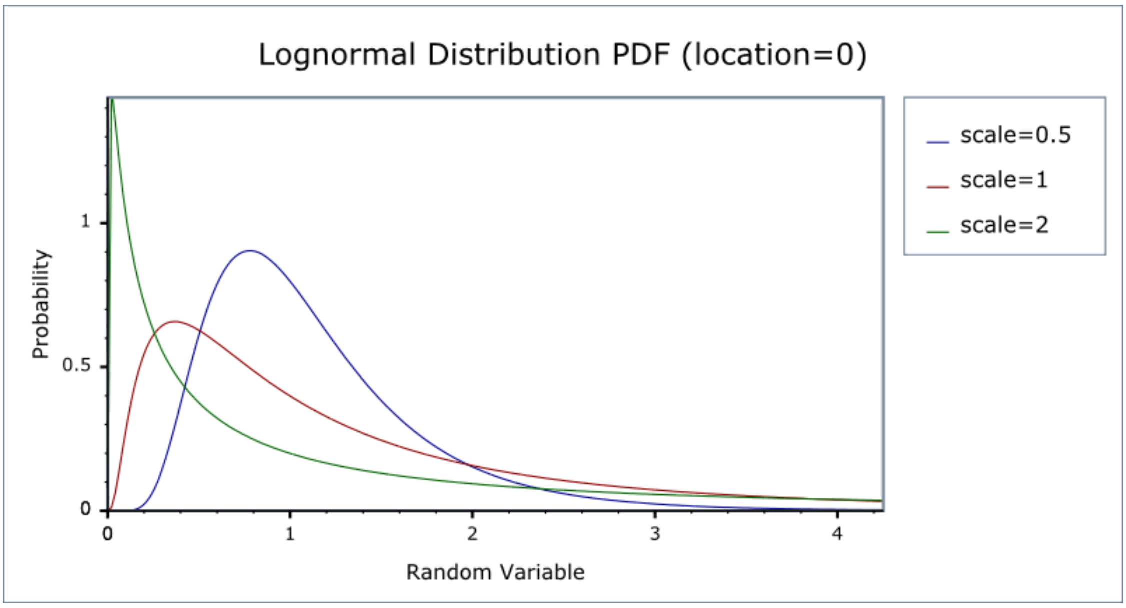


μ: Location   σ: Scale

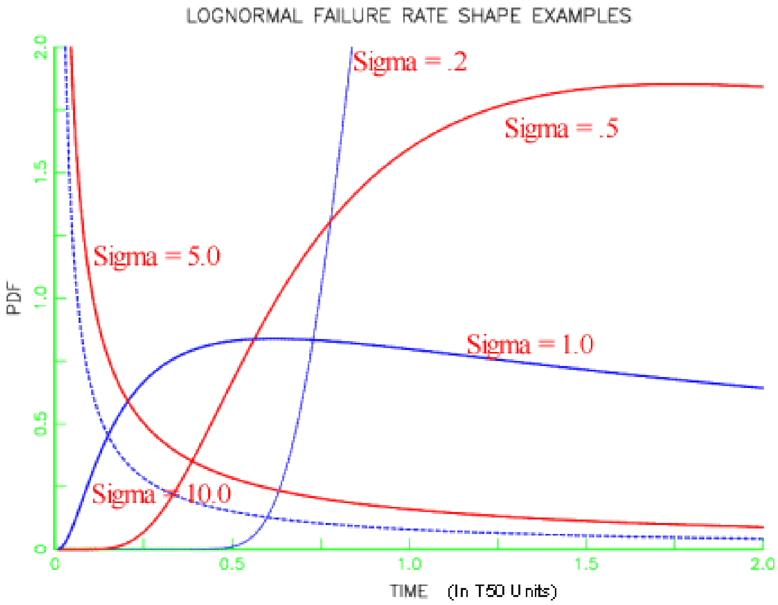
The following graph illustrates the effect of the location parameter on the PDF, note that the range of the random variable remains [0,+∞] irrespective of the value of the location parameter:



The next graph illustrates the effect of the scale parameter on the PDF:



The lifetime of a product that degrades over time is often modeled by a lognormal random variable.



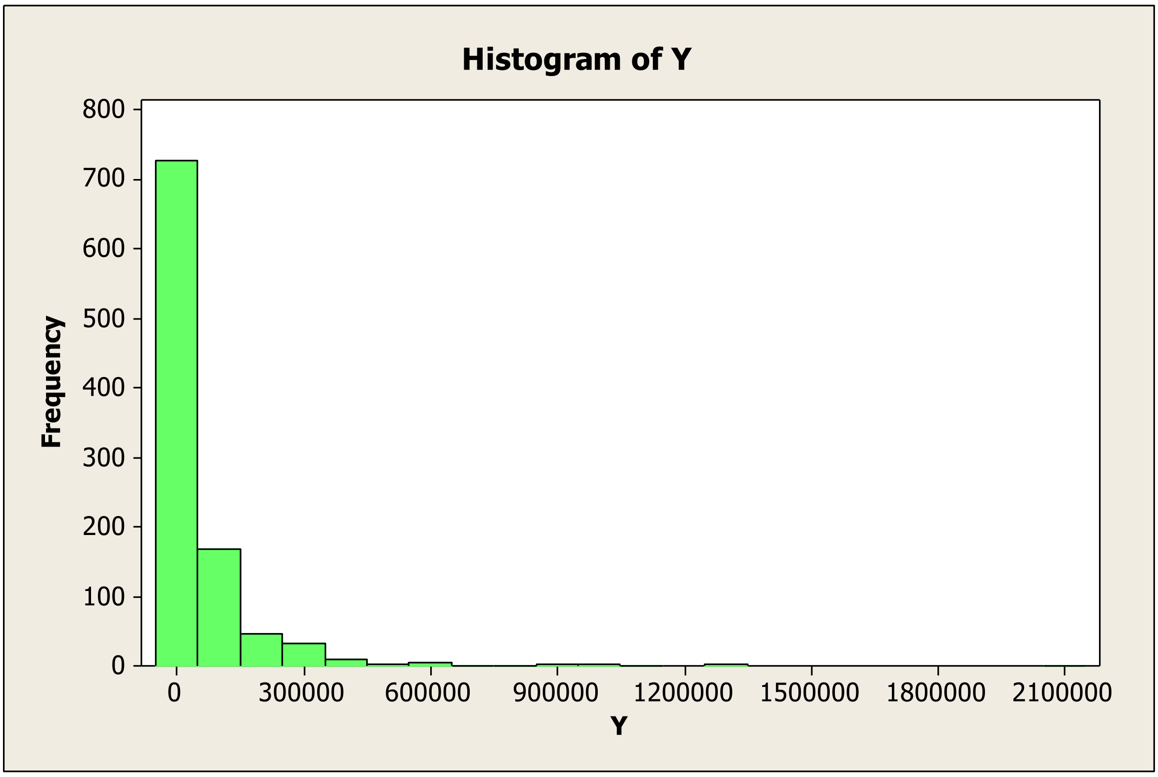
Example: The **lifetime of a semiconductor laser** has a **[lognormal distribution]** with μ=10 hours and σ=1.5 hours. **What is the probability the lifetime exceeds 10000 hours?**

```
MTB > Random 1000 c1;
SUBC>   LNormal 10 1.5.
MTB > desc c1
```

Descriptive Statistics: C1

Variable	N	N*	Mean	SE Mean	StDev	Minimum	Q1	Median	Q3
C1	1000	0	64150	4526	143112	154	7523	21964	56087

Variable	Maximum
C1	2087225

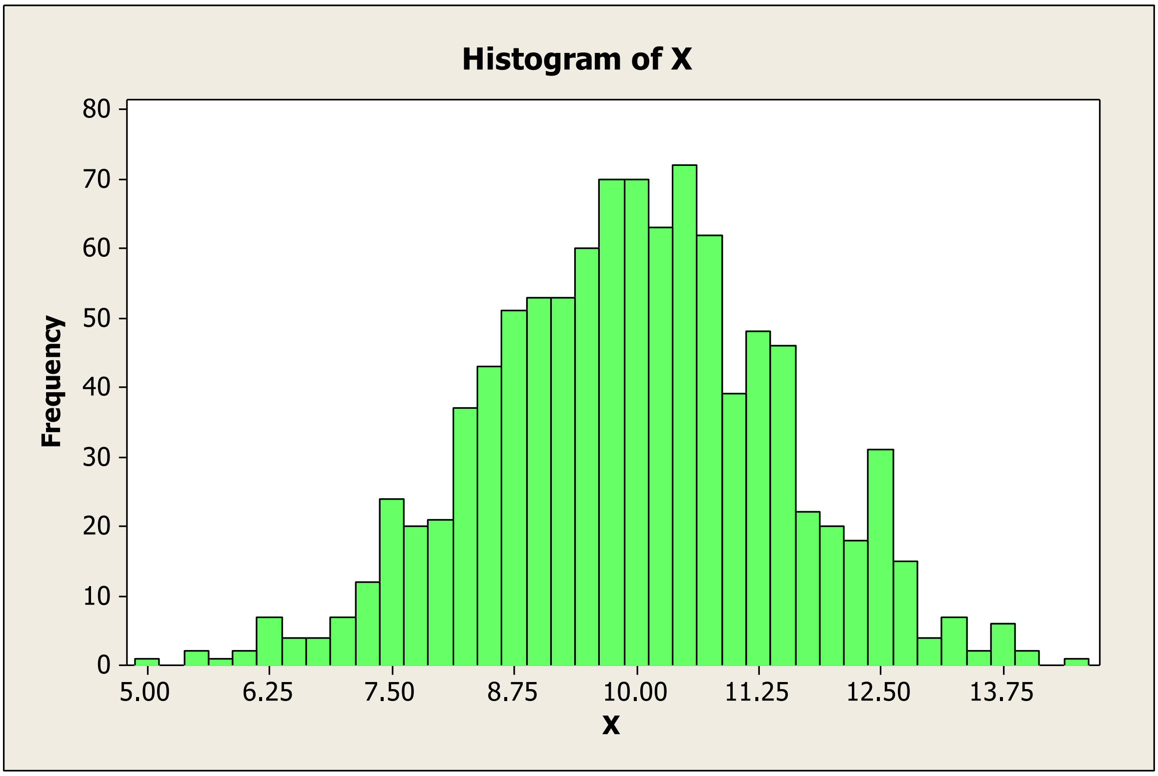


```
MTB > let c2=loge(c1)
MTB > desc c2
```

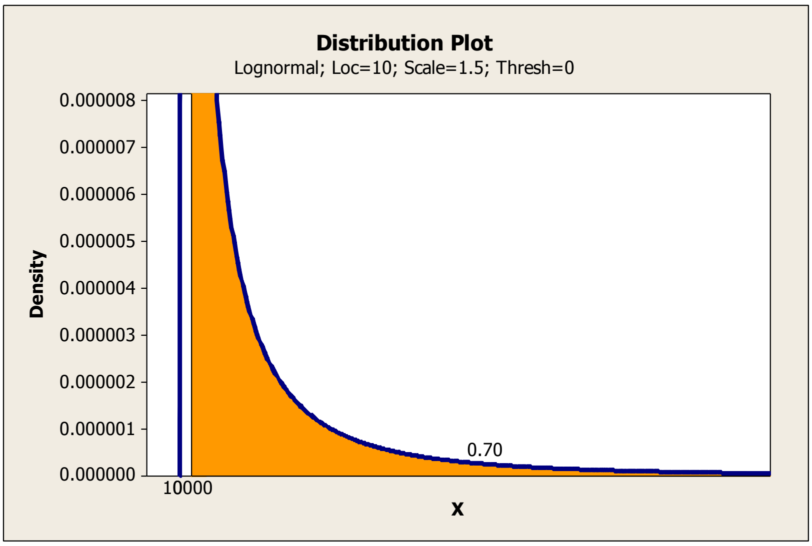
Descriptive Statistics: C2

```
Variable      N  N*    Mean  SE Mean   StDev  Minimum      Q1  Median
Q3
C2         1000   0  9.9651   0.0477  1.5098   5.0340  8.9257  9.9971
10.9347

Variable  Maximum
C2        14.5513
```

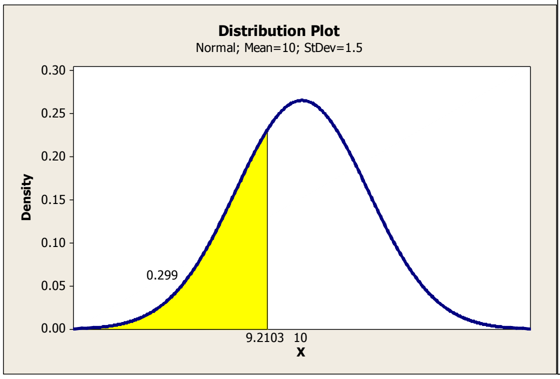


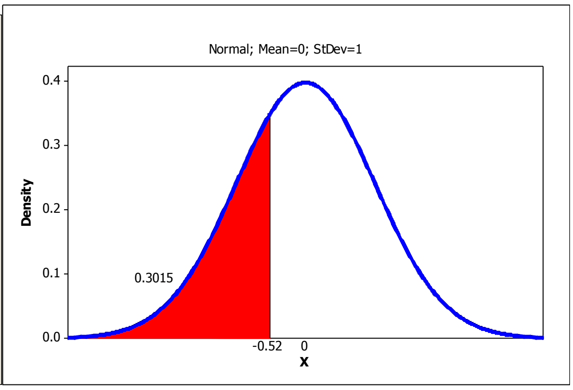
What is the probability the lifetime exceeds 10000 hours?



$$P(X > 10000) = 0.70$$

$$\begin{aligned} P(X > 10000) &= 1 - P(X \le 10000) \\ &= 1 - P(\exp(Y) \le 10000) \\ &= 1 - P(Y \le \ln(10000)) = 1 - P(Y \le 9.21034) \\ &= 1 - \Phi\left(\frac{9.211034 - 10}{1.5}\right) \\ &= 1 - \Phi(-0.52) = 1 - 0.30 = 0.70 \end{aligned}$$





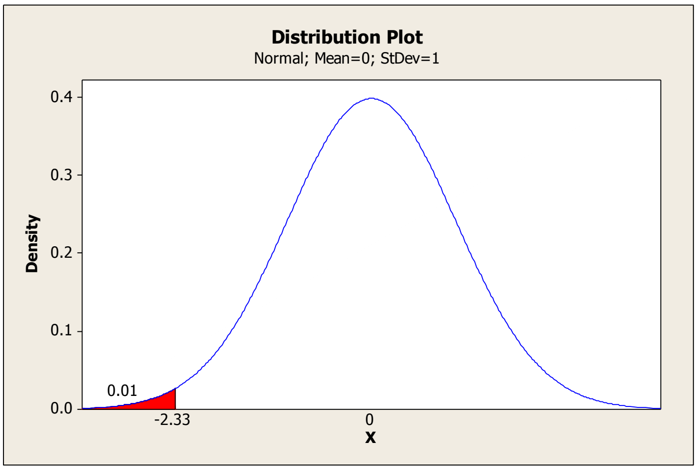
What lifetime is exceeded by 99% of lasers?

The question is to determine x such that P(X>x)=0.99

$$
\begin{aligned}
P(X > x) &= P[\exp(Y) > x] = P[Y > \ln(x)] \\
&= 1 - \Phi\left(\frac{\ln(x) - 10}{1.5}\right) = 0.99
\end{aligned}
$$

From Standart Normal Table

$1 - \Phi(z) = 0.99 \quad \text{when } z = -2.33$



Therefore,

$$
\frac{\ln(x) - 10}{1.5} = -2.33
$$

$$
x = \exp(6.505) = 668.48\,hours
$$

Mean and Variance

If X has a lognormal distribution with parameters μ and σ then,

$$E(X) = e^{\mu+\sigma^2/2}$$

$$V(X) = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right)$$

Determine the mean and standard deviation of lifetime for previous example.

$$E(X) = e^{\mu+\sigma^2/2} = \exp(10 + 1.125) = 67846.3$$

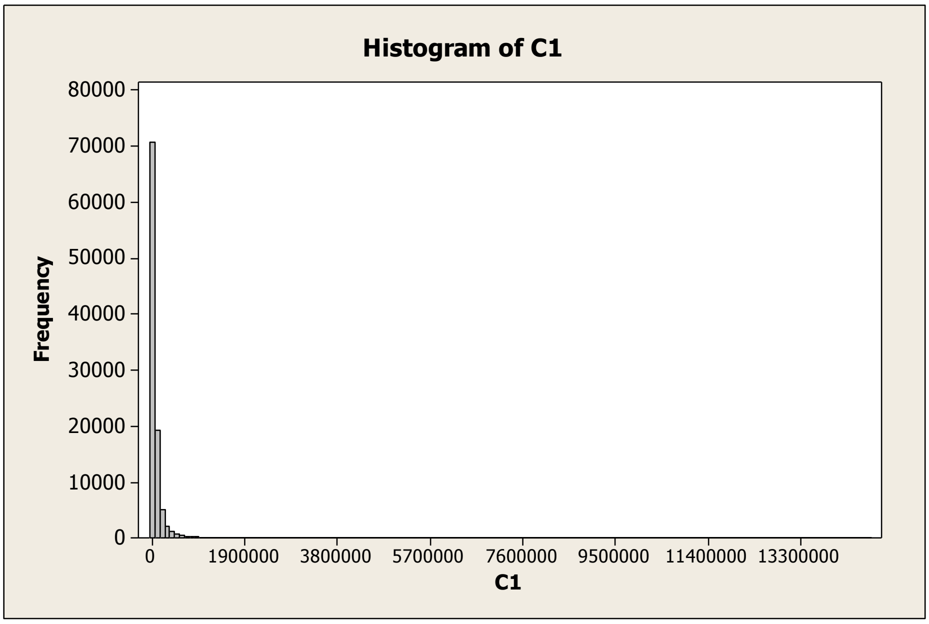
$$V(X) = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right)$$
$$= \exp(20 + 2.25)[\exp(2.25) - 1] = 39070059886.6$$

```
MTB > random 100000 c1;
SUBC> Lnormal 10 1.5.
MTB > desc c1
```

Descriptive Statistics: C1

Variable	N	N*	Mean	SE Mean	StDev	Minimum	Q1	Median	Q3
C1	100000	0	67999	594	187721	47	8046	21977	61068

Variable	Maximum
C1	14675585




```
let c2=loge(c1)
```