THE BIG M METHOD

When a basic feasible solution is not readily apparent, the big M method (or the two-phase simplex method may be used to solve the problem.

- In choosing the entering variable, remember that M is a very large positive number. For example -6M-5 is more negative than –5M –4000, 4M-2 is more positive 3M + 9000.
- Solve the transformed problem by the simplex.
- If all artificial variables are equal to zero in the optimal solution, we have found the optimal solution to the original problem.
- If any **artificial variables are positive** in the optimal solution (RHS value), the original problem is infeasible.

Big-M Simplex Method

Step 1 Modify the constraints so that the right-hand side of each constraint is nonnegative. This requires that each constraint with a negative right-hand side be multiplied through by –1.

Step 1' Identify each constraint that is now (after step 1) an equality or $\geq$ constraint. In Step 3, we will add an artificial variable to each of these constraints.

Step 2 Convert each inequality constraint to the standard form . If constraint i is a $\leq$ constraint , **add** a **slack variable** s_i. If constraint i is a $\geq$ constraint , **subtract** an **excess variable** e_i.

Step 3 If(after step 1') constraint i is a $\geq$ constraint or equality (=) constraint, add an **artificial variable** a_i to constraint i. Also add the sign restriction $a_i \geq 0$.

Step 4 Let M denote a very large positive number. If the LP is a min problem, add (for each artificial variable) Ma_i to the objective function. If the LP is a max problem, add (for each artificial problem) $-Ma_i$ to the objective function.

Step 5 Since each artificial variable will be in the starting basis, all artificial variables must be eliminated from Z (row 0) before beginning the simplex. This ensures that we begin with a canonical form. If all artificial variables are equal to zero in the optimal solution, we have found the optimal solution to the original problem. **If any artificial variables are positive in the optimal solution, the original problem is infeasible.**

Example

$$\textbf{Min Z} = 2x_1 + 3x_2$$

s.t

$$\begin{aligned} 1/2\, x_1 + \tfrac{1}{4} x_2 &\le 4 \\ x_1 + 3x_2 &\ge 20 \\ x_1 + x_2 &= 10 \\ x_1, x_2 &\ge 0 \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 \quad & + s1 = 4 \\ x_1 + 3x_2 \quad & -e_1 + a_1 = 20 \\ x_1 + x_2 \quad & + a_2 = 10 \end{aligned}$$

or

$$\begin{aligned} \tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 \quad & + x_3 = 4 \\ x_1 + 3x_2 \quad & -x_4 + x_5 = 20 \\ x_1 + x_2 \quad & + x_6 = 10 \end{aligned}$$

NBV=(x_1,x_2,x_4) BV=(x_3,x_5,x_6)=4,20,10 basic feasible staring solution.

$$-Z + 2x_1 + 3x_2 + M x_5 + M x_6 = 0$$

$$-Z + 2x_1 + 3x_2 + M(20-x_1-3x_2+x_4) + M(10-x_1-x_2) = 0$$

$$-Z + (2-2M)x_1 + (3-4M)x_2 + M x_4 = -30M$$

Initial Tableau for the Big M Simplex Method

BASIS	x_1	x_2	x_3	x_4	x_5	x_6	RHS	RATIO
x_3	½	¼	1	0	0	0	**4**	16
x_5	1	3<	0	-1	1	0	**20**	20/3<
x_6	1	1	0	0	0	1	**10**	10
-Z	**2-2M**	**3-4M**	**0**	**M**	**0**	**0**	**-30M**	

Initial and First Tableau for the Big M Simplex Method

BASIS	x_1	x_2	x_3	x_4	x_5	x_6	RHS	RATIO
x_3	½	¼	1	0	0	0	**4**	16
x_5	1	3<	0	-1	1	0	**20**	20/3<
x_6	1	1	0	0	0	1	**10**	10
-Z	**2-2M**	**3-4M**	**0**	**0**	**0**	**0**	**0**	
x_3	5/12	0	1	1/12	-1/12	0	**7/3**	28/5
x_2	1/3	1	0	-1/3	1/3	0	**20/3**	20
x_6	2/3<	0	0	1/3	-1/3	1	**10/3**	5<
-Z	**1-2/3M**	**0**	**0**	**1-1/3M**	**4/3M-1**	**0**	**-20-10/3M**	

Initial and the First two Tableau for the Big M Simplex Method

BASIS	x_1	x_2	x_3	x_4	x_5	x_6	RHS	RATIO
x_3	½	¼	1	0	0	0	**4**	16
x_5	1	3<	0	-1	1	0	**20**	20/3<
x_6	1	1	0	0	0	1	**10**	10
-Z	**2-2M**	**3-4M**	**0**	**0**	**0**	**0**	**0**	
x_3	5/12	0	1	1/12	-1/12	0	**7/3**	28/5
x_2	1/3	1	0	-1/3	1/3	0	**20/3**	20
x_6	2/3<	0	0	1/3	-1/3	1	**10/3**	5<
-Z	**1-2/3M**	**0**	**0**	**1-1/3M**	**4/3M-1**	**0**	**-20-10/3M**	
x_3	0	0	1	-1/8	1/8	-5/8	**¼**	
x_2	0	1	0	-1/2	1/2	-1/2	**5**	
x_1	1	0	0	1/2	-1/2	3/2	**5**	
-Z	**0**	**0**	**0**	**1/2**	**M-1/2**	**M-3/2**	**-25**	

$x_1 = 5$ $x_2 = 5$ $x_3 = 1/4$ $x_4 = 0$ $Z_{min} = 25$

$x_5 = x_6 = 0$

All artificial variables are equal to zero ($x_5 = x_6 = 0$) in the optimal solution, we have found the optimal solution to the original problem.

Comparison of the Two-Phase and Big M Simplex

Example:

$$\textbf{Max Z} = x_1 + 2\,x_2$$

s.t. $3\,x_1 + x_2 \leq 6$

$2x1 + x_2 = 5$

$x_1, x_2 \geq 0$

Two-Phase Simplex

$$\mathbf{Z - x_1 - 2x_2 = 0}$$
$$\mathbf{3x_1 + x_2 + x_3 = 6}$$
$$\mathbf{2x_1 + x_2 + x_4 = 5}$$
$$\mathbf{W = x_4}$$
$$\mathbf{-W - 2x_1 - x_2 = -5}$$

Big-M Simplex

$$\mathbf{Z - x_1 - 2x_2 + M\,x_4 = 0}$$
$$\mathbf{Z - x_1 - 2x_2 + M\,(5 - 2x_1 - x_2) = 0}$$
$$\mathbf{Z + (-2M-1)x_1 + (-M-2)x_2 = -5M}$$
$$\mathbf{3x_1 + x_2 + x_3 = 6}$$
$$\mathbf{2x_1 + x_2 + x_4 = 5}$$

Initial Tableau for the two-phase simplex method

BASIS	x_1	x_2	x_3	x_4	RHS	RATIO
x_3	3<	1	1	0	**6**	2<
x_4	2	1	0	1	**5**	2.5
-W	**-2<**	**-1**	**0**	**0**	**-5**	
Z	**-1**	**-2**	**0**	**0**	**0**	

Initial Tableau for the Big M Simplex Method

BASIS	x_1	x_2	x_3	x_4	RHS	RATIO
x_3	3<	1	1	0	**6**	2<
x_4	2	1	0	1	**5**	2.5
Z	**-2M-1**	**-M-2**	**0**	**0**	**-5M**	

Initial and the first tableau for the two-phase simplex method

BASIS	x_1	x_2	x_3	x_4	RHS	RATIO
x_3	3<	1	1	0	**6**	2<
x_4	2	1	0	1	**5**	2.5
-W	**-2<**	**-1**	**0**	**0**	**-5**	
Z	**-1**	**-2**	**0**	**0**	**0**	
x_1	1	1/3	1/3	0	**2**	6
x_4	0	1/3<	-2/3	1	**1**	3<
-W	**0**	**-1/3<**	**2/3**	**0**	**-1**	
Z	**0**	**-5/3**	**1/3**	**0**	**2**	

Initial and the first tableau for the big M simplex method

BASIS	x_1	x_2	x_3	x_4	RHS	RATIO
x_3	3<	1	1	0	**6**	2<
x_4	2	1	0	1	**5**	2.5
Z	**-2M-1**	**-M-2**	**0**	**0**	**-5M**	
x_1	1	1/3	1/3	0	**2**	6
x_4	0	1/3<	-2/3	1	**1**	3<
Z	**0**	**-M/3-5/3**	**2M/3+1/3**	**0**	**-M+2**	

Initial and the first two table for the two-phase simplex method (Phase I)

BASIS	x_1	x_2	x_3	x_4	RHS	RATIO
x_3	3<	1	1	0	**6**	2<
x_4	2	1	0	1	**5**	2.5
-W	**-2<**	**-1**	**0**	**0**	**-5**	
Z	**-1**	**-2**	**0**	**0**	**0**	
x_1	1	1/3	1/3	0	**2**	6
x_4	0	1/3<	-2/3	1	**1**	3<
-W	**0**	**-1/3<**	**2/3**	**0**	**-1**	
Z	**0**	**-5/3**	**1/3**	**0**	**2**	
x_1	1	0	1	-1	**1**	1<
x_2	0	1	-2	3	**3**	
-W	**0**	**0**	**0**	**1**	**0**	
Z	**0**	**0**	**-3**	**5**	**7**	

Initial and the first two table for the big M simplex method

BASIS	x_1	x_2	x_3	x_4	RHS	RATIO
x_3	3<	1	1	0	**6**	2<
x_4	2	1	0	1	**5**	2.5
Z	**-2M-1**	**-M-2**	**0**	**0**	**-5M**	
x_1	1	1/3	1/3	0	**2**	6
x_4	0	1/3<	-2/3	1	**1**	3<
Z	**0**	**-M/3-5/3**	**2M/3+1/3**	**0**	**-M+2**	
x_1	1	0	1	-1	**1**	1<
x_2	0	1	-2	3	**3**	
Z	**0**	**0**	**-3**	**M+5**	**7**	

Phase II

BASIS	x_1	x_2	x_3	x_4	RHS	RATIO
x_3	1	0	1		**1**	
x_2	2	1	0		**5**	
Z	**3**	**0**	**0**		**10**	

Initial and the first two table for the big M simplex method

BASIS	x_1	x_2	x_3	x_4	RHS	RATIO
x_3	3<	1	1	0	**6**	2<
x_4	2	1	0	1	**5**	2.5
Z	**-2M-1**	**-M-2**	**0**	**0**	**-5M**	
x_1	1	1/3	1/3	0	**2**	6
x_4	0	1/3<	-2/3	1	**1**	3<
Z	**0**	**-M/3-5/3**	**2M/3+1/3**	**0**	**-M+2**	
x_1	1	0	1	-1	**1**	1<
x_2	0	1	-2	3	**3**	
Z	**0**	**0**	**-3**	**M+5**	**7**	
x_3	1	0	1	**-1**	**1**	
x_2	2	1	0	**-2**	**5**	
Z	**3**	**0**	**0**	**M+2**	**10**	

Two-Phase Simplex

BASIS	x_1	x_2	x_3	x_4	RHS	RATIO
x_3	3<	1	1	0	**6**	2<
x_4	2	1	0	1	**5**	2.5
-W	**-2<**	**-1**	**0**	**0**	**-5**	
Z	**-1**	**-2**	**0**	**0**	**0**	
x_1	1	1/3	1/3	0	**2**	6
x_4	0	1/3<	-2/3	1	**1**	3<
-W	**0**	**-1/3<**	**2/3**	**0**	**-1**	
Z	**0**	**-5/3**	**1/3**	**0**	**2**	
x_1	1	0	1	-1	**1**	1<
x_2	0	1	-2	3	**3**	
-W	**0**	**0**	**0**	**1**	**0**	
Z	**0**	**0**	**-3**	**5**	**7**	
x_3	1	0	1		**1**	
x_2	2	1	0		**5**	
Z	**3**	**0**	**0**		**10**	

$x_1 = 0 \quad x_2 = 5 \quad Z_{max} = 10$

Big-M Simplex

BASIS	x_1	x_2	x_3	x_4	RHS	RATIO
x_3	3<	1	1	0	**6**	2<
x_4	2	1	0	1	**5**	2.5
Z	**-2M-1**	**-M-2**	**0**	**0**	**-5M**	
x_1	1	1/3	1/3	0	**2**	6
x_4	0	1/3<	-2/3	1	**1**	3<
Z	**0**	**-M/3-5/3**	**2M/3+1/3**	**0**	**-M+2**	
x_1	1	0	1	-1	**1**	1<
x_2	0	1	-2	3	**3**	
Z	**0**	**0**	**-3**	**M+5**	**7**	
x_3	1	0	1	**-1**	**1**	
x_2	2	1	0	**-2**	**5**	
Z	**3**	**0**	**0**	**M+2**	**10**	

$x_1 = 0 \quad x_2 = 5 \quad Z_{max} = 10$

RULE

If any artificial variables are positive in the optimal solution, the original problem is infeasible.

Example

$$\textbf{Min Z} = 2x_1 + 3\,x_2$$

s.t

$$x_1 + x_2 \geq 10$$
$$3\,x_1 + 5\,x_2 \leq 15$$
$$x_1\;,\, x_2 \geq 0$$

$$\begin{aligned} x_1 + x_2 \quad - e_1 + a_1 \quad &= 10 \\ 3\,x_1 + 5\,x_2 \quad + s_1 &= 15 \end{aligned}$$

or

$$\begin{aligned} x_1 + x_2 \quad - x_3 + x_5 \quad &= 10 \\ 3\,x_1 + 5\,x_2 \quad + x_4 &= 15 \end{aligned}$$

NBV=(x_1,x_2,x_3) BV=(x_4,x_5)=10,15 basic feasible staring solution.

$$-Z + 2x_1 + 3x_2 + Mx_5 = 0$$

$$x_5 = 10 - x_1 - x_2 + x_3$$

$$-Z + 2x_1 + 3x_2 + M\,(10 - x_1 - x_2 + x_3) = 0$$

$$-Z + (2-M)\,x_1 + (3-M)\,x_2 + Mx_3 = 0$$

Simplex Tableau for the Big M Method

BASIS	x_1	x_2	x_3	x_4	x_5	RHS	RATIO
x_5	1	1	-1	0	1	**10**	10
x_4	3<	5	0	1	0	**15**	5<
-Z	**2-M<**	**3-M**	**M**	**0**	**0**	**-10M**	
x_5	0	-2/3	-1	-1/3	1	**5**	
x_1	1	5/3	0	1/3	0	**5**	
-Z	**0**	**2/3M-1/3**	**M**	**1/3M-2/3**	**0**	**-5M-10**	

If any **artificial variables are positive (x_5)** in the optimal solution (RHS value), **the original problem is infeasible.**