THE SIMPLEX ALGORITHM

HOW TO CONVERT AN LP TO STANDARD FORM

$$\begin{aligned} &\text{Min } Z = 2x_1 - 3x_2 \\ &\text{s.t} \quad x_1 + x_2 \le 4 \\ &\qquad\ \ x_1 - x_2 \le 6 \\ &\qquad\ \ x_1, x_2 \ge 0 \end{aligned}$$

Convert the LP's objective function to the **row 0 format.**

$$-Z + 2x_1 - 3x_2 = 0$$

To put the constraints in standard form, we simply add **slack variables** s_1 and s_2, respectively, to the two constraints.

$$\begin{aligned} x_1 + x_2 + s_1 \quad &= 4 \\ x_1 - x_2 \quad + s_2 &= 6 \end{aligned}$$

Before the simplex algorithm can be used to solve an LP, the LP must be converted into an equivalent problem in which all constraints are equations and all variables are nonnegative. An LP in this form is said to be in **standard form**.

$$
\begin{aligned}
\text{Max } Z = {} & 60\, x_1 + 30\, x_2 + 20\, x_3 \\
\text{s. t } \quad & 8\, x_1 + 6\, x_2 + x_3 \leq 48 \\
& 4\, x_1 + 2\, x_2 + 1.5\, x_3 \leq 20 \\
& 2\, x_1 + 1.5\, x_2 + 0.5\, x_3 \leq 8 \\
& x_2 \geq 5
\end{aligned}
$$

$$x_1 , x_2 ,x_3,x_4 \geq 0$$

Row 0 format of the objective function

$$Z - 60x_1 - 30x_2 - 20x_3 = 0$$

$$\begin{aligned}
8x_1 + 6x_2 + x_3 + s_1 &= 48 \\
4x_1 + 2x_2 + 1.5x_3 + s_2 &= 20 \\
2x_1 + 1.5x_2 + 0.5x_3 + s_3 &= 8 \\
x_2 - e_1 &= 5
\end{aligned}$$

- **If the *i*th constraint is a ≤ constraint, we convert it to an equality constraint by adding a slack variable s_i and the sign restriction $s_i \geq 0$.**

- **If the *i* th constraint is a ≥ constraint, we convert it to an equality constraint by adding a excess variable e_i and the sign restriction $e_i \geq 0$.**

$$Max(orMin)Z = c_1x_1 + c_2x_2 + c_3x_3 + \ldots + c_nx_n$$

s.t

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n \le \; = \; \ge b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n \le \; = \; \ge b_2$$
$$.$$
$$.$$
$$.$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n \le \; = \; \ge b_m$$

$$x_i \ge 0$$

If we define

$$A = \begin{bmatrix} a_{11} \; a_{12} \ldots a_{1n} \\ a_{21} \; a_{22} \ldots a_{2n} \\ . \\ . \\ . \\ a_{m1} \; a_{m2} \; a_{mn} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n. \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_m \end{bmatrix}$$

the constraints may be written as the system of equations **Ax = b**

Consider a system **Ax = b** **of m linear equations in n variables** **(assume n≥m).**

Definition

A basic solution to Ax= b is obtained by setting n-m variables equal to 0 and solving for the values of the remaining m variables .

To find a basic solution to Ax= b we choose n –m variables (the nonbasic variables, or NBV) and set each of these variables equal to zero. Then solve for the values of the remaining n-(n-m) = m variables (the basic variables or BV) that satisfy Ax=b.

DIFFERENT CHOICES OF NONBASIC VARIABLES WILL LEAD TO DIFFERENT BASIC SOLUTIONS

Definition

Any basic solution to Ax= b in which all variables are nonnegative is a basic feasible solution (or bfs).

Theorem

The feasible region for any linear programming problem is a convex set. Also, if an LP has an optimal solution, there must be an extreme point of the feasible region that is optimal.

Theorem

For any LP, there is a unique extreme point of the LP's feasible region corresponding to each basic feasible solution. Also there is at least one bfs corresponding to each extreme point of the feasible region.

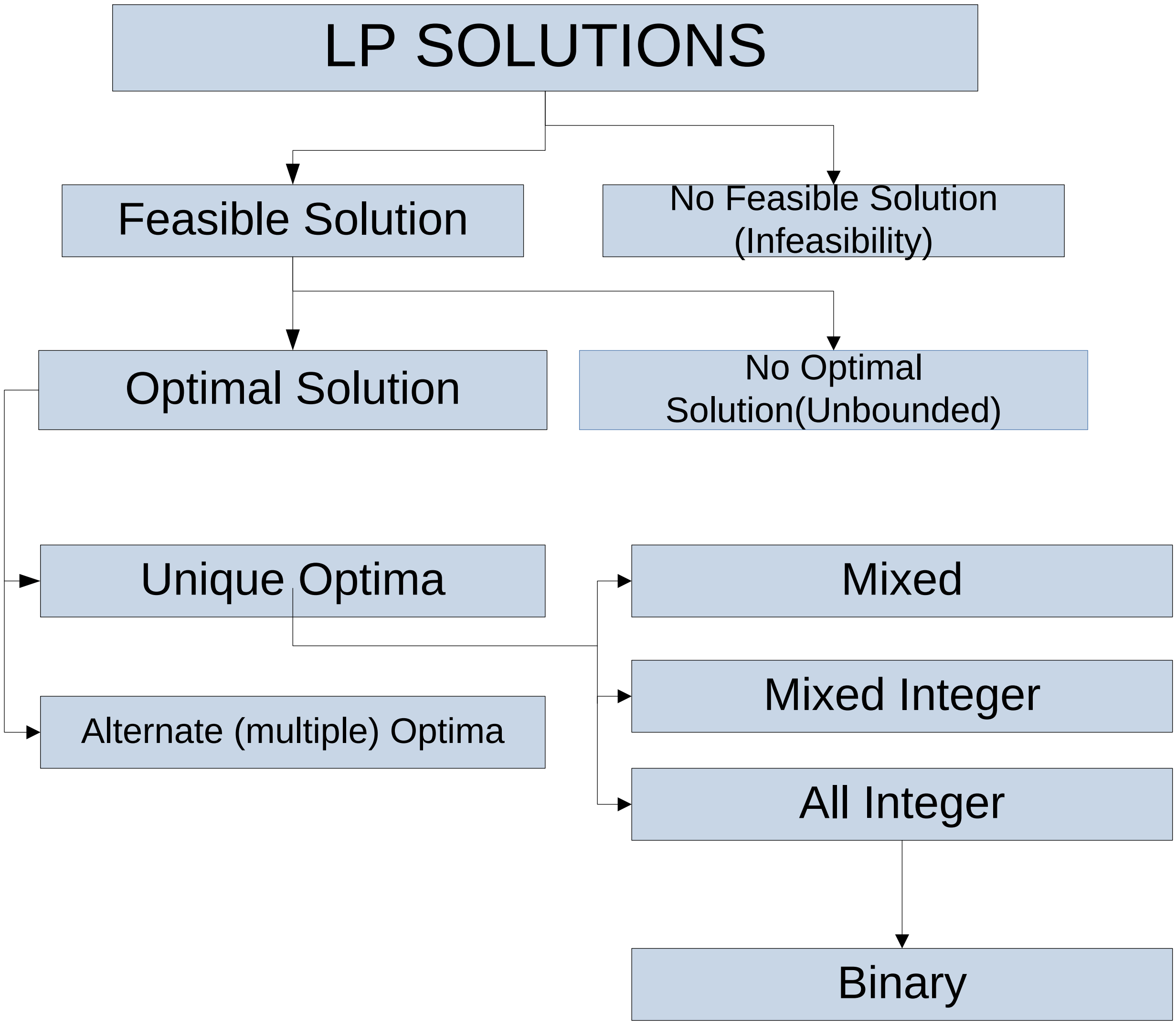
If an LP in standard form has m constraints and n variables, there may be a basic solution for each choice of non-basic variables. Since some basic solutions may be not be feasible, an LP can have at most

$$\binom{n}{m}$$

basic feasible solutions. If we were to proceed from the current bfs to a better bfs we would surely find the optimal bfs after examining at most

$$\binom{n}{m}$$

basic feasible solutions. This means that the simplex algorithm will find the optimal bfs after a finite number of calculations.


LP SOLUTIONS
Feasible Solution
No Feasible Solution (Infeasibility)
Optimal Solution
No Optimal Solution(Unbounded)
Unique Optima
Alternate (multiple) Optima
Mixed
Mixed Integer
All Integer
Binary

Example

$$\text{Max } Z = 2x_1 + 4x_2$$

$$\text{s.t} \quad 3x_1 + 4x_2 \le 1700$$

$$2x_1 + 5x_2 \le 1600$$

$$x_1, x_2 \ge 0$$

standard form

$$Z - 2x_1 - 4x_2 = 0$$

$$3x_1 + 4x_2 + s_1 = 1700$$

$$2x_1 + 5x_2 + s_2 = 1600$$

or

$$3x_1 + 4x_2 + x_3 = 1700$$

$$2x_1 + 5x_2 + x_4 = 1600$$

n= 4 m= 2

All basic solutions

Basic solutions	x_1	x_2	x_3	x_4	z
1	**0**	**0**	**1700**	**1600**	0
2	**0**	**425**	**0**	**-525**	not a bfs
3	**0**	**320**	**420**	**0**	1280
4	**566.66**	**0**	**0**	**466.66**	1133.4
5	**800**	**0**	**-700**	**0**	not a bfs
6	**300**	**200**	**0**	**0**	1400

bfs: basic feasible solution

The Simplex Algorithm

Step1 **Convert the LP to standard form**

Step2 **Obtain a bfs (if possible) from the standard form**

Step3 **Determine whether the current bfs is optimal**

Step4 **If the current bfs is not optimal, determine which non-basic variable should become a basic variable and which basic variable should become a non-basic variable in order to find a new bfs with a better objective function value and go to step 3.**

Step5 **Use ero's to find the new bfs with better objective value and go to step 3.**

$$\text{Max } Z = 2x_1 + 4x_2$$
$$\text{s.t} \quad 3x_1 + 4x_2 \leq 1700$$
$$2x_1 + 5x_2 \leq 1600$$
$$x_1, x_2 \geq 0$$

$$Z - 2x_1 - 4x_2 = 0$$

$$3x_1 + 4x_2 + s_1 = 1700$$
$$2x_1 + 5x_2 + s_2 = 1600$$

or

$$3x_1 + 4x_2 + x_3 = 1700$$
$$2x_1 + 5x_2 + x_4 = 1600$$

BV=(x_3,x_4) =1700,1600 NBV=(x_1,x_2)=0

BV=(x_3,x_4) =1700,1600 NBV=(x_1,x_2)=0

Initial Simplex Tableau PIVOT

BASIS	x_1	x_2	x_3	x_4	RHS	RATIO
x_3	3	4	1	0	**1700**	425
x_4	2	5<	0	1	**1600**	320<
Z	-2	-4<	0	0	**0**	

- **Is the current basic feasible solution optimal?**
- **Determine the entering variable (Pivot Column)**
- **The ratio test**

When entering a variable into the basis, compute the ratio Right-hand side of row Coefficient of entering variable in row for every constraint in which the entering variable has a positive coefficient. The constraint with the smallest ratio is called winner of the ratio test.

- **Determine the leaving variable (Pivot Row)**

Initial and First Simplex Tableau

BASIS	x_1	x_2	x_3	x_4	RHS	RATIO
x_3	3	4	1	0	**1700**	425
x_4	2	5<<	0	1	**1600**	320<
Z	-2	-4<	0	0	**0**	
x_3	7/5<<	0	1	-4/5	**420**	300<
x_2	2/5	1	0	1/5	**320**	800
Z	-2/5	0	0	4/5	**1280**	

Initial, First and Second Simplex Tableau

BASIS	x_1	x_2	x_3	x_4	RHS	RATIO
x_3	3	4	1	0	**1700**	425
x_4	2	5<<	0	1	**1600**	320<
Z	-2	-4<	0	0	**0**	
x_3	7/5<<	0	1	-4/5	**420**	300<
x_2	2/5	1	0	1/5	**320**	800
Z	-2/5	0	0	4/5	**1280**	
x_1	1	0	5/7	-4/7	**300**	
x_2	0	1	-2/7	3/7	**200**	
Z	**0**	**0**	**2/7**	**4/7**	**1400**	

$$x_1 = 300$$
$$x_2 = 200$$
$$Z = 1400 \text{ max}$$

DIFFERENT SIMPLEX TABLEAU

		x_1	x_2	x_3	x_4		
BASIS	c_j	2	4	0	0	**RHS**	**RATIO**
x_3	0	3	4	1	0	**1700**	425
x_4	0	2	**5<<**	0	1	**1600**	320<
	Z_j	0	0	0	0		
	C_j-Z_j	2	**4<**	0	0		
x_3	0	7/5<<	0	1	-4/5	**420**	300<
x_2	4	2/5	1	0	1/5	**320**	800
	Z_j	8/5	4	0	4/5	**1280**	
	C_j-Z_j	2/5<	0	0	-4/5		
x_1	2	1	0	5/7	-4/7	**300**	
x_2	4	0	1	-2/7	3/7	**200**	
	Z_j	2	4	2/7	4/7	**1400**	
	C_j-Z_j	**0**	**0**	**-2/7**	**-4/7**	**1400**	

The elements in the Zj row are the sum of the products obtained by multiplying the elements in the Ci column of the Simplex tableau by the corresponding elements in the columns of the A matrix.

Stopping Criterion

The optimal solution to a linear programming problem has been reached when all of the entries in the net evaluation row (C_j-Z_j) are zero or negative. In such cases the optimal solution is the current basic feasible solution.

The Simplex Algorithm