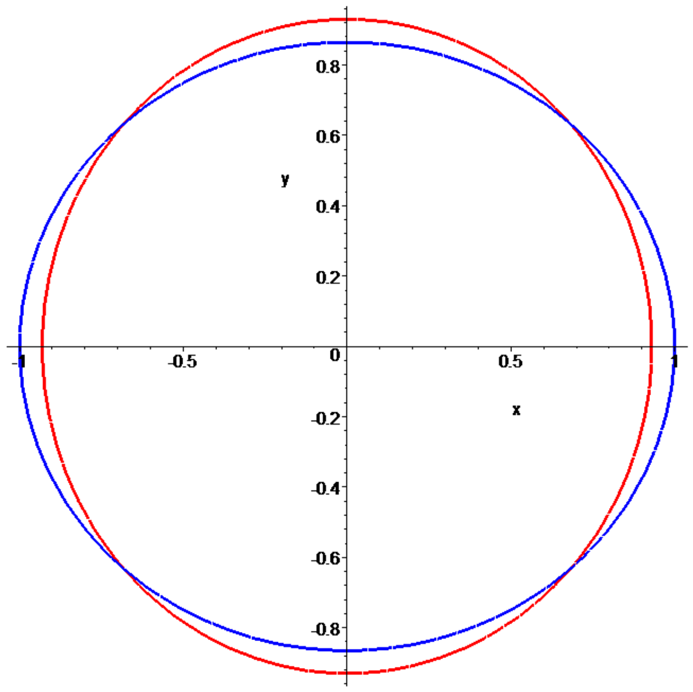
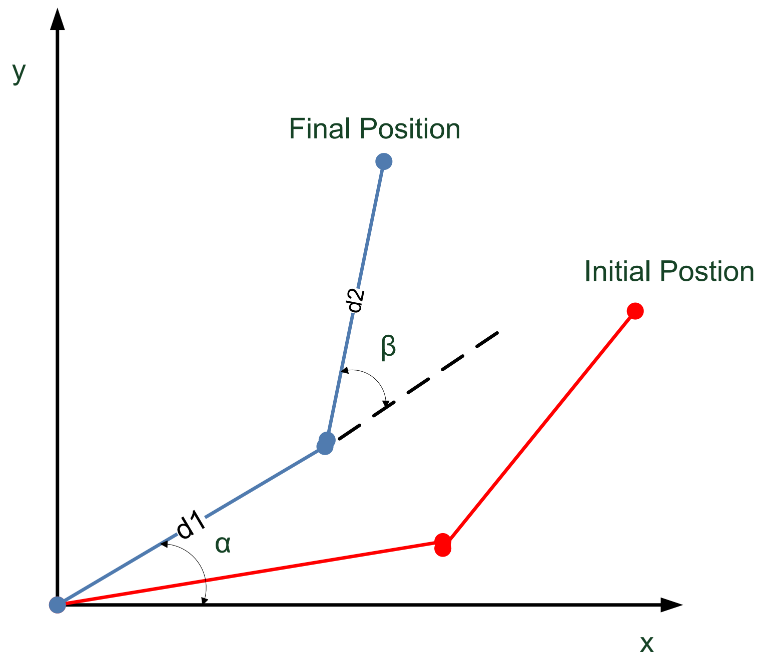
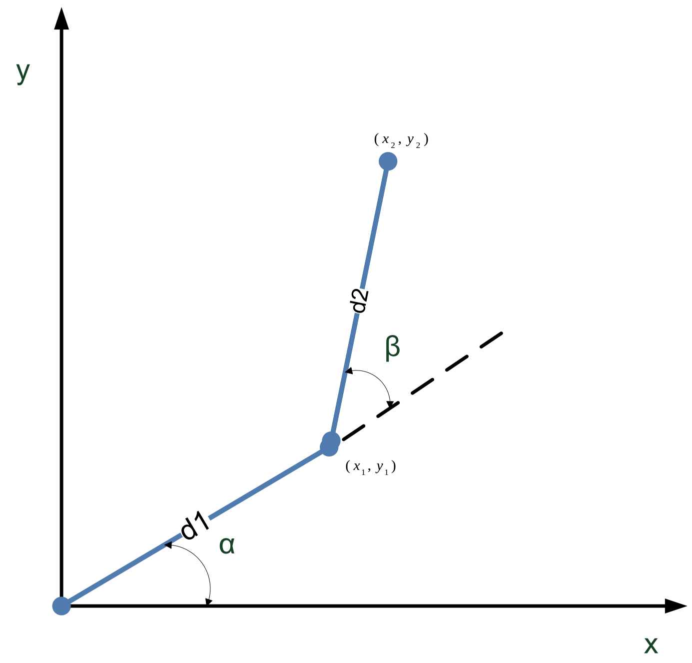
Solution of Nonlinear Systems


0.8
0.6
y
0.4
0.2
-1
-0.5
0
0.5
1
x
-0.2
-0.4
-0.6
-0.8


y
Final Position
Initial Postion
d2
β
d1
α
x

Example:

The position of a two-link robot arm can be described in terms of the angle α that the first link makes with the horizontal axis and the angle β that the second link makes with the first link. The problem is to find the angles α and β that allow the end of the second link to be at a specified point, with coordinates (x, y).



First Link $x_1 = d_1 \cos(\alpha)$ $y_1 = d_1 \sin(\alpha)$

Second Link $x_2 = x_1 + d_2 \cos(\alpha + \beta)$ $y_2 = y_1 + d_2 \sin(\alpha + \beta)$

Thus, we need to solve for the unknown angles α **and** β

$$x = d_1 \cos(\alpha) + d_2 \cos(\alpha + \beta)$$
$$y = d_1 \sin(\alpha) + d_2 \sin(\alpha + \beta)$$

Example:

When we have a system of simultaneous nonlinear equations, the situation is more difficult. In fact, some sets have no real solutions.

Consider this example of pair of equations:

$$x^2 + y^2 = 4,$$
$$e^x + y = 1.$$

Graphically, the solution to this system is represented by the intersection of the circle $x^2 + y^2 = 4$ with the curve $e^x + y = 1.$

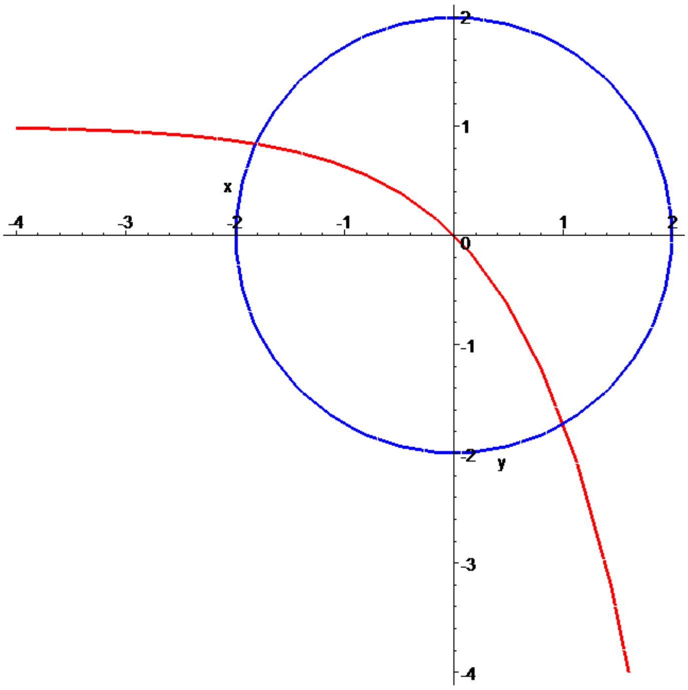


Figure shows that these are near (-1.8, 0.8) and (1,-1.8).

Fixed Point Iteration for Nonlinear Systems

The fixed point iteration can be modified to solve two simultaneous, nonlinear equations. We know how to solve a single nonlinear equation by fixed iterations. We rearrange it to solve for the variable in a way that successive computations may reach a solution. Sometimes we can do the same for a system

Let

$$F(x_1,x_2,\ldots,x_n)=\begin{pmatrix} f_1(x_1,x_2,\ldots,x_n) \\ f_2(x_1,x_2,\ldots,x_n) \\ f_3(x_1,x_2,\ldots,x_n) \\ . \\ . \\ . \\ f_n(x_1,x_2,\ldots,x_n) \end{pmatrix}$$

Suppose G is a continuous function generated from the F with continuous first partial derivatives and

$$\left|\frac{\partial g_i(x)}{\partial x_j}\right| \leq 1 \qquad j = 1,2,\ldots,,n$$

$$\text{and each component function } g_i.$$

Then the sequence $\left\{x^{(k)}\right\}_{k=0}^{\infty}$ **defined by an arbitrarily selected** $\mathbf{x}^{(0)}$ **generated by**

$$x^{(k)} = G(x^{(k-1)}).$$

This approach will be illustrated in the following example.

Example: Consider this pair of equations:

$$x^2 + y^2 = 4,$$
$$e^x + y = 1.$$

Use fixed point iteration to determine the roots of the given system.
If we will try this rearrangement:

$$y = -\sqrt{(4 - x^2)},$$
$$x = \ln(1 - y).$$

and

$$y_{i+1} = -\sqrt{(4 - x_i^{\,2})},$$
$$x_{i+1} = \ln(1 - y_i).$$

For simplification

$$y_{i+1} = -\sqrt{(4 - x_i^{\,2})},$$
$$x_{i+1} = \ln(1 - y_{i+1}).$$

And begin with $x_0=1.0$, the successive values for y

$$y_1 = -\sqrt{(4 - 1^2)} = -1.7321,$$
$$x_1 = \ln(1 + 1.7321) = 1.0051$$

$$y_2 = -\sqrt{(4 - 1.0051^2)} = -1.7291,$$
$$x_2 = \ln(1 + 1.7291) = 1.00397$$

Other rearrangements are possible.

$$y = (4 - x^2)/y,$$
$$x = \ln(1 - y).$$

and

$$y_{i+1} = (4 - x_i{}^2)/y_i,$$
$$x_{i+1} = \ln(1 - y_i).$$

Try these arrangements to solve the system and compare results. Begin with x_0=1.0 and y_0=-1.7

MATLAB SOLUTION

```
>> [x,y]=solve('x^2+y^2=4','exp(x)+y=1')
x =-1.8162640688251505742443123715859
y =1-exp(-1.8162640688251505742443123715859)+1
>>
```

BUT THIS IS THE LEFTMOST INTERSECTION. WE CAN GET ONE NEAR (1,-1.7) WITH

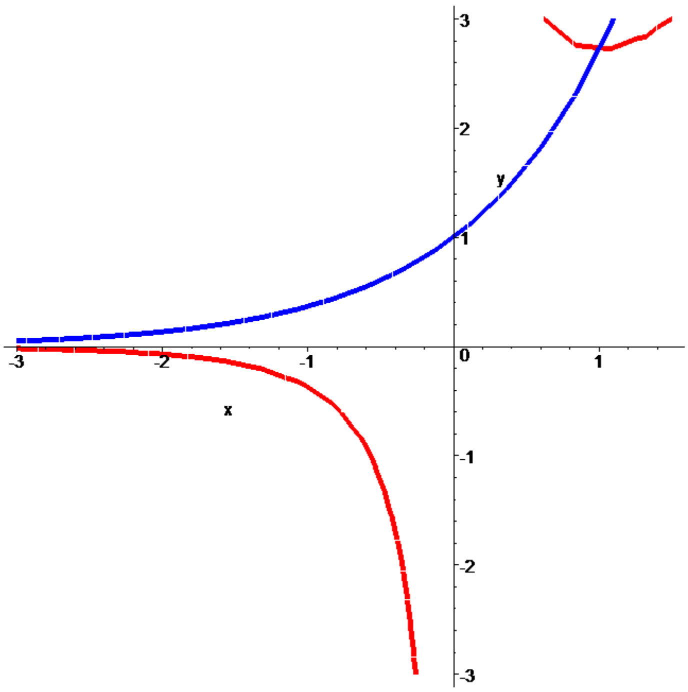
```
>> [x,y]=solve('abs(x^2)+y^2=4','exp(x)+y=1')
x = 1.0041687384746591657874315472901
y = -1.7296372870258699313633129362508
>>
```

Example:

Consider this pair of equations:

$$e^x - y = 0,$$
$$xy - e^x = 0.$$

Use fixed point iteration to determine the roots of the given system.



Let us solve the first of the pair for and the second for y:

$$x = \ln(y), \quad y = e^x / x. \qquad \text{and} \qquad x_{i+1} = \ln(y_i), \quad y_{i+1} = e^{x_i} / x_i.$$

Rewrite the formula as

$$x_{i+1} = \ln(y_i),$$
$$y_{i+1} = e^{x_{i+1}} / x_{i+1}.$$

To start, **we guess at a value for y, say, y_0=2.**

$$x_1 = \ln(y_0) = 0.69315,$$
$$y_1 = e^{0.69315} / 0.69315 = 2.88539$$

i	y-value	x-value
1	2.88539	0.69315
2	2.72294	1.05966
3	2.71829	1.00171
4	2.71828	1.00000

which are precisely the correct results. (Look the figure)

MATLAB SOLUTION

```
>> [x,y]=solve('exp(x)-y=0','x*y-exp(x)=0')
 x = 1
 y = exp(1)
```

Example:

Given the nonlinear system

$$3x_1 - \cos(x_2 x_3) - \frac{1}{2} = 0,$$
$$x_1^2 - 81(x_2 + 0.1)^2 + \sin(x_3) + 1.06 = 0,$$
$$e^{-x_1 x_2} + 20x_3 + \frac{10\pi - 3}{3} = 0$$

Use Fixed Point Iteration method to obtain the first five iterates with the initial approximation is

$$x^{(0)} = \begin{bmatrix} 0.1 \\ 0.1 \\ -0.1 \end{bmatrix}$$

Solution:

If the ith equation is solved for x_i, the system can be changed into the fixed point problem

$$x_1 = \frac{1}{3}\cos(x_2 x_3) + \frac{1}{6},$$
$$x_2 = \frac{1}{9}\sqrt{x_1^2 + \sin(x_3) + 1.06} - 0.1,$$
$$x_3 = -\frac{1}{20}e^{-x_1 x_2} - \frac{10\pi - 3}{60}$$

Then $G(x)=(g_1(x),g_2(x),g_3(x))^T$ where

$$g_1(x_1,x_2,x_3)=\frac{1}{3}\cos(x_2x_3)+\frac{1}{6},$$
$$g_2(x_1,x_2,x_3)=\frac{1}{9}\sqrt{x_1^2+\sin(x_3)+1.06}-0.1,$$
$$g_3(x_1,x_2,x_3)=-\frac{1}{20}e^{-x_1x_2}-\frac{10\pi-3}{60}$$

First we obtain partial derivatives for g_i to check convergence.

$$\left|\frac{\partial g_1}{\partial x_1}\right|=0,\quad \left|\frac{\partial g_1}{\partial x_2}\right|=\frac{1}{3}|x_3||\sin x_2x_3|=0.0003334,\quad \left|\frac{\partial g_1}{\partial x_3}\right|=\frac{1}{3}|x_2||\sin x_2x_3|=0.0003334,$$

$$\left|\frac{\partial g_2}{\partial x_1}\right|=\frac{|x_1|}{9\sqrt{x_1^2+\sin x_3+1.06}}=0.01028,\quad \left|\frac{\partial g_2}{\partial x_2}\right|=0,$$

$$\left|\frac{\partial g_2}{\partial x_3}\right|=\frac{|\cos x_3|}{18\sqrt{x_1^2+\sin x_3+1.06}}=0.057,$$

$$\left|\frac{\partial g_3}{\partial x_1}\right|=\frac{|x_2|}{20}e^{-x_1x_2}=0.00496,\quad \left|\frac{\partial g_3}{\partial x_2}\right|=\frac{|x_1|}{20}e^{-x_1x_2}=0.00496,\quad \left|\frac{\partial g_2}{\partial x_3}\right|=0,$$

$$\left|\frac{\partial g_i(x)}{\partial x_j}\right|\le 0.057 \quad for\ \ each\ \ i=1,2,3\ \ and\ \ \ j=1,2,3.$$

$$g_1(x_1,x_2,x_3) = \frac{1}{3}\cos(x_2 x_3) + \frac{1}{6},$$
$$g_2(x_1,x_2,x_3) = \frac{1}{9}\sqrt{x_1^2 + \sin(x_3) + 1.06} - 0.1,$$
$$g_3(x_1,x_2,x_3) = -\frac{1}{20}e^{-x_1 x_2} - \frac{10\pi - 3}{60}$$

Then the sequence of vectors generated by

$$x_1^{(k)} = \frac{1}{3}\cos(x_2^{(k-1)} x_3^{(k-1)}) + \frac{1}{6},$$
$$x_2^{(k)} = \frac{1}{9}\sqrt{(x_1^{(k-1)})^2 + \sin(x_3^{(k-1)}) + 1.06} - 0.1,$$
$$x_3^{(k)} = -\frac{1}{20}e^{-x_1^{(k-1)} x_2^{(k-1)}} - \frac{10\pi - 3}{60}$$

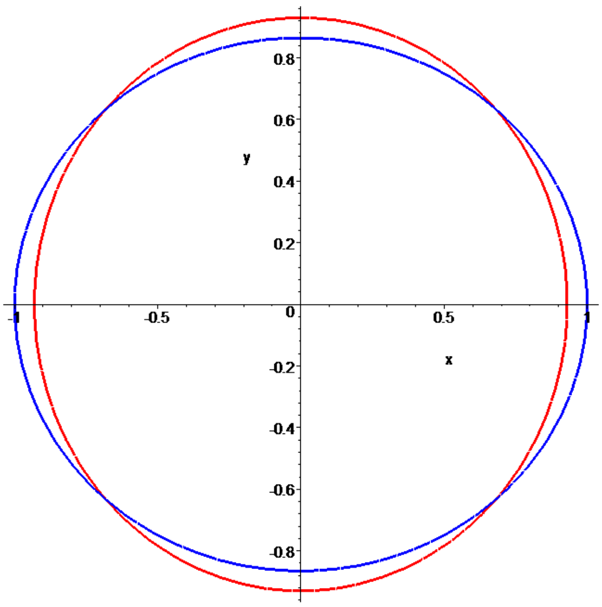
The results are given in the following Table.

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
0	**0.10000000**	**0.10000000**	**-0.10000000**
1	**0.49998333**	**0.00944115**	**-0.52310127**
2	**0.49999593**	**0.00002557**	**-0.52336331**
3	**0.50000000**	**0.00001234**	**-0.52359814**
4	**0.50000000**	**0.00000003**	**-0.52359847**
5	**0.50000000**	**0.00000002**	**-0.52359877**

$$\left\| x^{(5)} - x^{(4)} \right\| = 3.1\ 10^{-7}$$

Exercise: Fixed Point Iteration

Consider the problem of finding the points of intersection of two curves. The first equation represents an ellipse of eccentricity 0.5. The second equation represents a circle with the same area as ellipse. Both curves are centered at the origin.



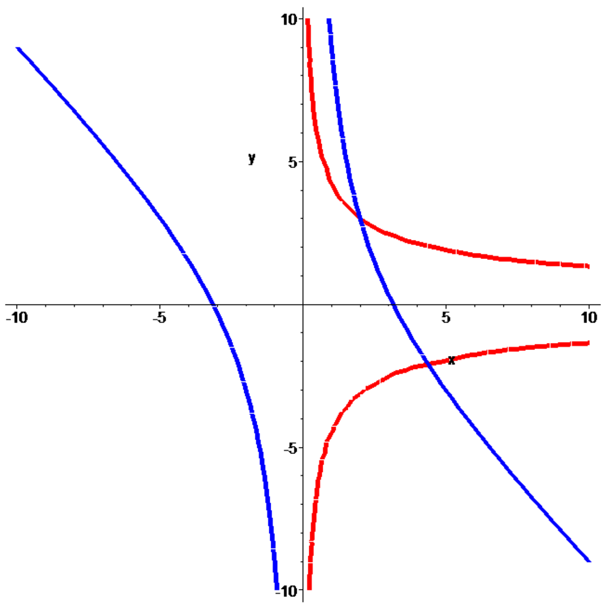
$$3x^2 + 4y^2 - 3 = 0,$$
$$x^2 + y^2 - \sqrt{3}/2 = 0.$$

Use fixed point iteration to determine the roots of the given system.

Exercise:

Use fixed point iteration for the following system.

$$x^2 + xy = 10,$$
$$y + 3xy^2 = 57.$$



MATLAB M-File (Nonlinear Fixed Point Systems-Nonlinear Seidel Iteration)

```
function [P,iter]= seideldim(G,P,delta, maxit)
%Input    -  G is the nonlinear fixed-point system
%             saved as an M-file function
%           -  P is the initial guess at the solution
%           -  delta is the error bound
%           - maxit is the number of iterations
%Output - P is the seidel approximation to the solution
%          - iter is the number of iterations required
%Use the @ notation call [P,iter]=seidel(@G, P, delta, maxit).
N=length(P);
for k=1:maxit
   X=P;
    % X is the kth approximation to the solution
   for j=1:N
      A=G(X);
      % Update the terms of X as they are calculated
      X(j)=A(j);
   end
   err=abs(norm(X-P));
   relerr=err/(norm(X)+eps);
   P=X;
   iter=k;
   if (err<delta)|(relerr<delta)
     break
   end
end
```