Gamma Distribution

An exponential random variable describes the length until the first count is obtained in a Poisson process. Consider the following example.

Example: The failures of the central processor units of large computer systems are often modeled as a Poisson process. Typically, failures are not caused by components wearing out, but by more random failures of the large number of semiconductors circuits in the units. Assume that the units that fail are immediately repaired, and assume that the mean number of failures per hour is 0.0001.

Let X denote the time until the four (4) failures occur in a system.

Determine the probability that X exceeds 40000 hours.

Let the random variable N denote the number of failures in 40000 hours of operation.

The time until 4 failures occur exceeds 40000 hours if and only if the number of failures in 40000 hours is 3 or less.

$$P(X > 40000) = P(N \le 3)$$

$E(N) = 40000(0.0001) = 4$ **Failures per 40000 hours**

Therefore

$$P(X > 40000) = P(N \le 3) = \sum_{k=0}^{3} \frac{e^{-4} 4^k}{k!} = 0.433$$

The results from the previous example can be generalized to show that if X is the time until the rth event in a Poisson process then

$$P(X > x) = \sum_{k=0}^{r-1} \frac{e^{-\lambda x} (\lambda x)^k}{k!}$$

Because P(X>x)=1-F(x), the probability density function of X equals the negative of the derivative of the right-hand side of the equation.

After algebraic simplifications, the probability density function of X can be shown to equal

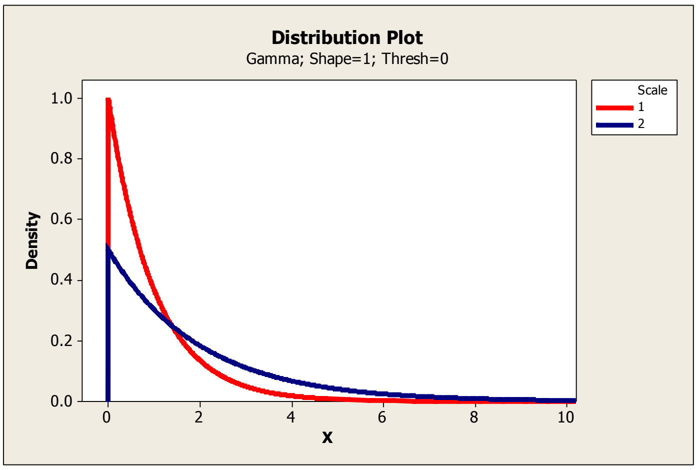
$$f(x) = \frac{\lambda^r}{(r-1)!} x^{r-1} e^{-\lambda x}, x > 0, r = 1,2,3,\ldots$$

Gamma Distribution

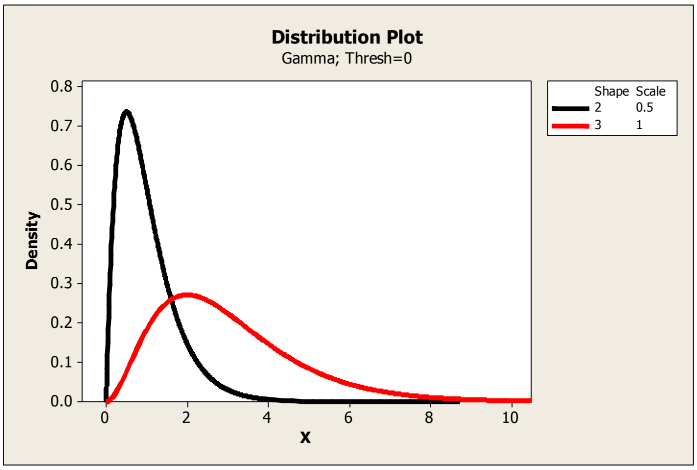
The random variable X with probability density function

$$f(x)=\frac{\lambda^r}{\Gamma(r)}x^{r-1}e^{-\lambda x}, x>0$$

has a gamma random variable with parameters λ>0 and r>0.



The parameters λ and r are often called the scale and shape parameters, respectively. Minitab defines the scale parameter as 1/ λ.



Gamma Function:

$$\Gamma(r)=\int_{0}^{\infty} x^{r-1}e^{-x}dx, r>0$$

This integral is finite.

$$\Gamma(r)=(r-1)\Gamma(r-1)$$

If r is a positive integer

$$\Gamma(r)=(r-1)!$$

Mean and Variance:

If X is a gamma random variable with parameters λ and r,

$$E(X) = r/\lambda \quad \textbf{\textit{and}} \quad V(X) = r/\lambda^2$$

Erlang Distribution

If r is integer in gamma distribution then gamma random variable X has an Erlang distribution.

$$f(x) = \frac{\lambda^r}{(r-1)!} x^{r-1} e^{-\lambda x}, x > 0$$

Example:

The time to prepare micro-array slide for high-throughput genomics is a Poisson process with a mean of two hours per slide.

What is the probability that 10 slides require more than 25 hours to prepare?

Let X denote the time to prepare 10 slides.

X has a gamma distribution with λ=1/2, r=10, and required probability is P(X>25).

Solution with Poisson probabilities

`Poisson with mean = 25(1/2)=12.5`

$$P(X>25)=\sum_{k=0}^{9}\frac{e^{-12.5}(12.5)^k}{k!}=0.2014$$

Cumulative Distribution Function

`Poisson with mean = 25(1/2)=12.5`

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x   P( X <= x )
9      0.201431
```

Solution with Gamma probabilities

Cumulative distribution function

The cumulative distribution function is the regularized gamma function, which can be expressed in terms of the incomplete gamma function,

$$F(x;k,\theta) = \int_0^x f(u;k,\theta)\,du = \frac{\gamma(k, x/\theta)}{\Gamma(k)}$$

It can also be expressed as follows, if k is an integer (i.e., the distribution is an Erlang distribution)[1]:

$$F(x;k,\beta) = 1 - \sum_{i=0}^{k-1} \frac{(\beta x)^i}{i!} e^{-\beta x}$$

where $\beta = 1/\theta$.

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MTB > CDF 25;
SUBC> GAMMA 10 2.
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Cumulative Distribution Function

Gamma with shape = 10 and scale = 2(?)

x	P(X <= x)
25	0.798569

$$P(X > 25) = 1 - F(25) = 1 - 0.798569 = 0.2014$$