DUALITY

Associated with any LP is another LP, called the dual. Knowing the relation between an LP and its dual is vital to understanding advanced topics in linear and non-linear programming. This relation is important because it gives us interesting economic insights. Knowledge of duality will also provide additional insights into sensitivity.

FINDING THE DUAL OF AN LP

When taking the **dual** of a given LP, we refer to the given LP as the **primal** .

Two problems are said to be duals to each other (or mutually dual) if they possess the following properties:

- **One is maximization and the other is a minimization problem.**
- **If one has (m) constraints and (n) decision variables, the other has (n) constraints and (m) decision variables.**
- **The [A] matrix of one is the transpose of the other.**
- **The RHS coefficients of one constitute the objective function coefficients of the other.**
- **If the primal is unbounded the dual problem is infeasible.**
- **If the dual is unbounded, the primal is infeasible.**

For convenience, we define the variables for the max problem to be $x_1, x_2, x_3, \ldots, x_n$ and the variables for the min problem $y_1, y_2, y_3, \ldots, y_m$.

A normal max problem (all constraints are ≤) may be written as

$$\text{Max} \quad Z = c_1x_1 + c_2x_2 + \ldots\ldots + c_nx_n$$

$$\begin{array}{lll} \text{s.t.} & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & = b_1 \\ & a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & = b_2 \\ & \vdots & \\ & a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & = b_m \end{array}$$

$$x_j \geq 0 \; (j=1,2,\ldots,n)$$

The dual of a normal max problem is defined to be

$$\text{Min} \quad W = b_1y_1 + b_2y_2 + \ldots\ldots + b_my_m$$

$$\begin{array}{lll} \text{s.t.} & a_{11}y_1 + a_{21}y_2 + \ldots + a_{m1}y_m & = c_1 \\ & a_{12}y_1 + a_{22}y_2 + \ldots + a_{m2}y_m & =c_2 \\ & \vdots & \\ & a_{1n}y_1 + a_{2n}y_2 + \ldots + a_{mn}y_m & = c_n \end{array}$$

$$y_i \geq 0 \; (i=1,2,\ldots,m)$$

A min problem that has all ≥ constraints all variables nonnegative is called <u>normal min problem</u>.

The general structure of the primal and dual problems may be represented as

Primal

$$\mathit{Max}Z = \sum_{j=1}^{n} c_j x_j$$

s.t. $$\sum_{j=1}^{n} a_{ij} x_j \le b_i \qquad i = 1,2,\ldots,m \qquad x_j \ge 0$$

Dual

$$\mathit{Min}W = \sum_{i=1}^{m} b_i yi$$

s.t. $$\sum_{i=1}^{m} a_{ij} y_i \ge c_j \qquad j = 1,2,\ldots,n \qquad y_i \ge 0$$

Finding the dual of a non-normal LP

Unfortunately, many LPs are not normal max problems or normal min problems.
For example,

$$\text{Max } z = 2x_1 + x_2$$

$$\begin{aligned} \text{s.t.} \quad & x_1 + x_2 = 2 \\ & 2x_1 - x_2 \geq 3 \\ & x_1 - x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

- Replace each equality constraint by two inequality constraints (a ≥ constraint and a ≤)
- Multiply each ≥ constraint by –1. This converts each ≥ constraint into a ≤ constraint.

$$\text{Max } z = 2x_1 + x_2$$

$$\begin{aligned} \text{s.t.} \quad & x_1 + x_2 \leq 2 \\ & -x_1 - x_2 \leq -2 \\ & -2x_1 + x_2 \leq -3 \\ & x_1 - x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

After these transformations are complete normal max problem

Primal

$$\text{Max } z = 2x_1 + x_2$$

s.t.

$$x_1 + x_2 \leq 2$$
$$-x_1 - x_2 \leq -2$$
$$-2x_1 + x_2 \leq -3$$
$$x_1 - x_2 \leq 1$$

$$x_1, x_2 \geq 0$$

Dual

$$\text{Min } W = 2y_1 - 2y'_1 - 3y_2 + y_3$$

s. t.

$$y_1 - y'_1 - 2y_2 + y_3 \geq 2$$
$$y_1 - y'_1 + y_2 - y_3 \geq 1$$

$$y_1, y'_1, y_2, y_3 \geq 0$$

We define $y_1 = y_1 - y'_1$. Now the final dual problem is

Dual

$$\text{Min } W = 2y_1 - 3y_2 + y_3$$

s. t.

$$y_1 - 2y_2 + y_3 \geq 2$$
$$y_1 + y_2 - y_3 \geq 1$$

$$y_1, y_2, y_3 \geq 0$$

Example

Non-normal LP minimization problem

$$\textbf{Min } \mathbf{Z} = 6\,x_1 + 10\,x_2$$

$$\text{s.t.} \quad \begin{aligned} 5\,x_1 + 3\,x_2 &\geq 10 \\ x_1 - x_2 &\leq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Normal LP minimization problem

$$\textbf{Min } \mathbf{Z} = 6\,x_1 + 10\,x_2$$

$$\text{s.t.} \quad \begin{aligned} 5\,x_1 + 3\,x_2 &\geq 10 \\ -x_1 + x_2 &\geq -4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Dual Problem

$$\textbf{Max } \mathbf{W} = 10\,y_1 - 4\,y_2$$

$$\text{s.t.} \quad \begin{aligned} 5\,y_1 - y_2 &\leq 6 \\ 3y_1 + y_2 &\leq 10 \\ y_1, y_2 &\geq 0 \end{aligned}$$

Property

If the dual problem has an optimal solution, the primal problem has an optimal solution, and vice versa. Furthermore, the values of the optimal solutions to the dual and primal problems are equal.

How to read the solution?

Comparisons primal and dual solutions.

Primal Problem

$$\textbf{Min } Z = 4x_1 + 6x_2 + 18x_3$$
$$\begin{aligned} \text{s.t.} \quad x_1 + \qquad 3x_3 &\geq 3 \\ x_2 + 2x_3 &\geq 5 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

$$\begin{aligned} \mathbf{x_1 + \quad 3x_3 \quad - x_4 + x_6 \qquad} &\mathbf{= 3} \\ \mathbf{x_2 \quad + 2x_3 \quad - x_5 \quad + x_7} &\mathbf{= 5} \end{aligned}$$

$$\mathbf{w = x_6 + x_7}$$

$$\mathbf{-w - x_1 - x_2 - 5x_3 + x_4 + x_5 = -8}$$

Dual Problem

$$\textbf{Max } Z' = 3y_1 + 5y_2$$
$$\begin{aligned} \text{s.t.} \quad y_1 \qquad &\leq 4 \\ y_2 \quad &\leq 6 \\ 3y_1 + 2y_2 \quad &\leq 18 \\ y_1, y_2 &\geq 0 \end{aligned}$$

Property

Given the simplex tableau corresponding to the optimal dual solution, the optimal values of the primal decision variables are given by the Z_j entries for the surplus variables.

Primal Solution (with two-phase simplex)

BASIS	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS	RATIO
x_6	1	0	3<	-1	0	1	0	3	1
x_7	0	1	2	0	-1	0	1	5	2.5<
-w	-1	-1	-5<	1	1	0	0	-8	
-Z	4	6	18	0	0	0	0	0	
x_3	1/3	0	1	-1/3	0	1/3	0	1	
x_7	2/3	1	0	2/3	-1	-2/3	1	3	3<
-w	2/3	-1	0	-2/3	1	5/3	0	-3	
-Z	-2	6	0	6	0	-6	0	-18	
x_3	1/3	0	1	-1/2	0	1/3	0	1	
x_2	-2/3	1	0	2/3	-1	2/3	1	3	
-w	0	0	0	0	0	1	1	0	
-Z	2	0	0	2	6	-2	-6	-36	

Dual Solution (with simplex)

BASIS	y_1	y_2	y_3	y_4	y_5	RHS	RATIO
y_3	1	0	1	0	0	4	-
y_4	0	1<	0	1	0	6	6<
y_5	3	2	0	0	1	18	9
Z'	-3	-5<	0	0	0	0	
y_3	1	0	1	0	0	4	4
y_2	0	1	0	1	0	6	-
y_5	3	0	0	-2	1	6	2
Z'	-3	0	0	5	0	30	
y_3	0	0	1	2/3	-1/3	2	
y_2	0	1	0	1	0	6	
y_1	1	0	0	-2/3	1/3	2	
Z'	0	0	0	3	1	36	

$x_1=0$ $x_2=3$ $x_3=1$ $Z'_{max} = Z_{min} = 36$