MATRICES

AND

MATRIX OPERATIONS

I

Definition

A matrix is a rectangular array of numbers denoted by

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & & . \\ . & & & \\ . & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

***Unless stated otherwise*, we assume that all our matrices *are composed entirely of real numbers*.**

The **ith row** of **A** is

$$\begin{bmatrix} a_{i1} & a_{i2} & \ldots & a_{in} \end{bmatrix} \quad 1 \le \mathrm{i} \le \mathrm{m}$$

while **the jth column** of **A** is

$$\begin{bmatrix} a_{i1} \\ a_{i2} \\ . \\ . \\ . \\ a_{im} \end{bmatrix} \qquad 1 \le j \le n$$

If a matrix A has m rows and n columns, we say that A is an m by n matrix (written $m \times n$).

Square Matrix

A matrix **A** with n rows and n columns is called a **square matrix of order** n and the elements $a_{11}, a_{22}, \ldots, a_{nn}$ are said to be on the **main diagonal** of A.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & & . \\ . & & & \\ . & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

We refer to a_{ij} as the (i, j) **entry** or (i, j) **element**

and we often write

$$A = [a_{ij}].$$

We shall also write $A_{m \times n}$ indicate that A has m rows and n columns. If A is a square matrix we merely write A_n.

Definition (**Equality of Matrices**)

Two matrices are defined to be equal if they have the same size and their corresponding entries are equal.

In matrix notation if $A = [a_{ij}]$ and $B = [b_{ij}]$ have the same size, then

$$A=B$$

if and only if $(A)_{ij} = (B)_{ij}$, or equivalently, $a_{ij} = b_{ij}$ for all i and j.

$$A = \begin{bmatrix} 2 & 1 \\ 3 & x \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 5 & 0 \end{bmatrix}$$

- If x=5 then A=B, but all other values of x A and B are not equal since not all of their corresponding entries are equal.
- There is no value of x for which A=C since A and C have different sizes.

Definition (Addition and Subtraction)

If A and B are matrices of the same size, the sum A+B is the matrix obtained by adding the entries of B to the corresponding entries of A, and the difference A-B is the matrix obtained by subtracting the entries of B from the corresponding entries of A. Matrices of different sizes cannot be added or subtracted.

In matrix notation if $A=[a_{ij}]$ and $B=[b_{ij}]$ have the same size, then

$$(A+B)_{ij}=(A)_{ij}+(B)_{ij}=a_{ij}+b_{ij}$$

and

$$(A-B)_{ij}=(A)_{ij}-(B)_{ij}=a_{ij}-b_{ij}$$

Example: (Addition and Subtraction)

Consider the matrices

$$A = \begin{bmatrix} 2 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 4 & -2 & 7 & 0 \end{bmatrix} \quad B = \begin{bmatrix} -4 & 3 & 5 & 1 \\ 2 & 2 & 0 & -1 \\ 3 & 2 & -4 & 5 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

Then

$$A + B = \begin{bmatrix} -2 & 4 & 5 & 4 \\ 1 & 2 & 2 & 3 \\ 7 & 0 & 3 & 5 \end{bmatrix} \quad A - B = \begin{bmatrix} 6 & -2 & -5 & 2 \\ -3 & -2 & 2 & 5 \\ 1 & -4 & 11 & -5 \end{bmatrix}$$

The expressions A+C, B+C, A-C, and B-C are undefined.

```
>> A=[2 1 0 3 ; -1 0 2 4; 4 -2 7 0]
A =
     2     1     0     3
    -1     0     2     4
     4    -2     7     0

>> B=[-4 3 5 1; 2 2 0 -1; 3 2 -4 5]
B =
    -4     3     5     1
     2     2     0    -1
     3     2    -4     5

>> C=[1 1 ; 2 2 ]
C =
     1     1
     2     2
>> A+B
ans =
    -2     4     5     4
     1     2     2     3
     7     0     3     5
```

```
>> A-B
ans =
     6    -2    -5     2
    -3    -2     2     5
     1    -4    11    -5
>> A+C
??? Error using ==> +
Matrix dimensions must agree.

>> B+C
??? Error using ==> +
Matrix dimensions must agree.

>>
```

Definition (**Scalar Multiples**)

If A is any matrix and c is any scalar then the product cA is the matrix obtained by multiplying each entry of the matrix A by c. The matrix cA is said to be <u>a scalar multiple</u> of A

In matrix notation, if $A = [a_{ij}]$ then

$$c(A_{ij}) = (cA)_{ij} = ca_{ij}$$

Example: (Scalar Multiplies)

For the matrices

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 3 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 2 & 7 \\ -1 & 3 & -5 \end{bmatrix} \quad C = \begin{bmatrix} 9 & -6 & 3 \\ 3 & 0 & 12 \end{bmatrix}$$

We have

$$2A = \begin{bmatrix} 4 & 6 & 8 \\ 2 & 6 & 2 \end{bmatrix} \qquad (-1)B = -B = \begin{bmatrix} 0 & -2 & -7 \\ 1 & -3 & 5 \end{bmatrix}$$

$$\frac{1}{3}C = \begin{bmatrix} 3 & -2 & 1 \\ 1 & 0 & 4 \end{bmatrix}$$

If $A_1, A_2, \ldots, A_n$ are matrices of **the same size** and $c_1, c_2, \ldots, c_n$ are scalars, then an expression of the form

$$c_1A_1 + c_2A_2 + \ldots + c_nA_n$$

is called a **linear combination** of $A_1, A_2, \ldots, A_n$ with coefficients $c_1, c_2, \ldots, c_n$.

```
>> A=[ 2 3 4; 1 3 1]
A =
    2    3    4
    1    3    1
>> B=[0 2 7; -1 3 -5]
B =
    0    2    7
   -1    3   -5
>> C=[9 -6 3; 3 0 12]
C =
    9   -6    3
    3    0   12
>> 2*A
ans =
    4    6    8
    2    6    2
>> (-1)*B
ans =
    0   -2   -7
    1   -3    5
```

```
>> (1/3)*C
ans =
     3    -2     1
     1     0     4
>> 2*A-B+(1/3)*C
ans =
     7     2     2
     4     3    11
>>
```

Definition (Matrix Multiplication)

If A is an $m\times r$ and B is an $r\times n$ matrix, then the product AB is the $m\times n$ matrix whose entries are determined as follows. To find the entry in row i and column j of AB, single out row i from the matrix A and column j from the matrix B. Multiply the corresponding entries from the row and columns together, and then add up the resulting products.

$$A_{m\times r} \quad B_{r\times n} \quad = AB_{m\times n}$$

The i, j entry of AB is the <u>inner product</u> of the ith row of A and the jth column of B.

$$A_{3\times 4} \quad B_{4\times 8} \quad = AB_{3\times 8}$$

- ***Two vectors, each with the same number of components, may be added or subtracted.***
- ***Two vectors are equal if each component of one equals the corresponding component of the other.***
- ***A very important special case is the multiplication of two vectors. The <u>first must be a row vector</u> if the <u>second is a column vector</u>, <u>and each must have the same number of components.</u>***

$$[1 \quad 3 \quad -2]\begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} = [4-3-6] = [-5]$$ ***gives a***

"matrix" of one row and one column. The result is a pure number, a scalar. This product is called the <u>scalar product</u> of the vectors, also called the <u>inner product.</u>

- ***If we reverse the order of multiplication of these two vectors, we obtain***

$$\begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}[1 \quad 3 \quad -2] = \begin{bmatrix} 4 & 12 & -8 \\ -1 & -3 & 2 \\ 3 & 9 & -6 \end{bmatrix}$$ ***This product is***

called the <u>outer product</u>.

$$AB=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1r} \\ a_{21} & a_{22} & \dots & a_{2r} \\ . & & & \\ . & & & \\ a_{i1} & a_{i2} & \dots & a_{ir} \\ . & & & \\ . & & & \\ a_{m1} & a_{m2} & \dots & a_{mr} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \dots b_{1j} & \dots & b_{1n} \\ b_{11} & b_{12} & \dots b_{2j} & \dots & b_{1n} \\ . & & & & \\ . & & & & \\ . & & & & \\ b_{r1} & b_{r2} & \dots b_{rj} & \dots & b_{rn} \end{bmatrix}$$

$$(AB)_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+a_{i3}b_{3j}+\dots+a_{ir}b_{rj}$$

$$AB=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \end{bmatrix} = \begin{bmatrix} * & * \\ * & * \\ * & (AB)_{32} \end{bmatrix}$$

$$(AB)_{32}=a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32}+a_{34}b_{42}$$

Example:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

(1.4) + (2.0) + (4.2) = 12

(1.1) - (2.1) + (4.7) = 27

(1.4) + (2.3) + (4.5) = 30

(1.3) + (2.1) + (4.2) = 13

(2.4) + (6.0) + (0.2) = 8

(2.1) - (6.1) + (0.7) = -4

(2.4) + (6.3) + (0.5) = 26

(2.3) + (6.1) + (0.2) = 12

$$AB = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

```
>> A=[1 2 4;2 6 0]

A =

     1     2     4

     2     6     0

>> B=[4 1 4 3; 0 -1 3 1; 2 7 5 2]

B =

     4     1     4     3

     0    -1     3     1

     2     7     5     2

>> AB=A*B

AB =

    12    27    30    13

     8    -4    26    12

>>

BA=B*A
```

??? Error using ==> *

Inner matrix dimensions must agree.

Partitioned Matrices

A matrix can be subdivided or *partitioned* into smaller matrices by inserting horizontal and vertical rules between selected rows and columns.

$$A=\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right]=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

Partition of A into four **submatrices** $A_{11}, A_{12}, A_{21}, A_{22}$

$$A=\left[\begin{array}{cccc} a_{11} & a_{12} & a_{13} & a_{14} \\ \hline a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right]=\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$$

Partition of A into its **row matrices** r_1, r_1, r_3.

$$A=\left[\begin{array}{c|c|c|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right]=\begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix}$$

Partition of A into its **column matrices** c_1, c_2, c_3, c_4.

Block Multiplication

If A and B are partitioned into submatrices, for example,

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \quad \textbf{and} \quad B = \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right]$$

then AB can be expressed as

$$AB = \left[\begin{array}{c|c} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ \hline A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{array}\right]$$

Provided the sizes of the submatrices of A and B are such that the indicated operations can be performed. This method of multiplying partitioned matrices is called block multiplication.

Example:

$$A = \left[\begin{array}{cc|cc} -1 & 2 & 1 & 5 \\ 0 & -3 & 4 & 2 \\ \hline 1 & 5 & 6 & 1 \end{array}\right] \qquad B = \left[\begin{array}{cc|c} 2 & 1 & 4 \\ -3 & 5 & 2 \\ \hline 7 & -1 & 5 \\ 0 & 3 & -3 \end{array}\right]$$

Compute the product by block multiplication. Check our results by multiplying directly.

```
A11 =

   -1     2

    0    -3

>> A12=[1 5 ; 4 2]

A12 =

    1     5

    4     2

>> A21=[ 1 5]

A21 =

    1     5

>> A22=[6 1]

A22 =

    6     1
```

```
>> B11=[2 1; -3 5]

B11 =

     2     1
    -3     5

>> B12=[4 2]

B12 =

     4     2

>> B12=B12'

B12 =

     4
     2

>> B21=[7 -1; 0 3]

B21 =

     7    -1
     0     3

>> B22=[5 -3]

B22 =

     5    -3

>> B22=B22'

B22 =

     5
    -3
```

```
>> AB11=A11*B11+A12*B21;
>> AB12=A11*B12+A12*B22;
>> AB21=A21*B11+A22*B21;
>> AB22=A21*B12+A22*B22;
>> AB11
AB11 =
    -1    23
    37   -13
>> AB12
AB12 =
   -10
     8
>> AB21
AB21 =
    29    23
>> AB22
AB22 =
    41
                    AB =
                  -1    23   -10
                  37   -13     8
                  29    23    41
```

```
>> A=[-1 2 1 5;0 -3 4 2;1 5 6 1]
A =
   -1    2    1    5
    0   -3    4    2
    1    5    6    1
>> B=[2 1 4;-3 5 2; 7 -1 5; 0 3 -3]
B =
    2    1    4
   -3    5    2
    7   -1    5
    0    3   -3
>> AB=A*B
AB =

   -1   23  -10
   37  -13    8
   29   23   41

>>
```

Matrix Products as Linear Combinations

Row and column matrices provide an alternative way of thinking about matrix multiplication.
Suppose that

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & & . \\ . & & & \\ . & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \text{ and } x = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}$$

Then

$$Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ . \quad . \quad . \\ . \\ . \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ . \\ . \\ . \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ . \\ . \\ . \\ a_{m2} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ . \\ . \\ . \\ a_{mn} \end{bmatrix}$$

Example:

The matrix product

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

can be written as the linear combination of **column matrices**

$$2\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - 1\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

Example:

The matrix product

$$\begin{bmatrix} 1 & -9 & -3 \end{bmatrix}\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} = \begin{bmatrix} -16 & -18 & 35 \end{bmatrix}$$

can be written as the linear combination of **row matrices**

$$1\begin{bmatrix} -1 & 3 & 2 \end{bmatrix} - 9\begin{bmatrix} 1 & 2 & -3 \end{bmatrix} - 3\begin{bmatrix} 2 & 1 & -2 \end{bmatrix} = \begin{bmatrix} -16 & -18 & 35 \end{bmatrix}$$

Example: Columns of a Product AB as a Linear Combinations

$$AB = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

The column matrices of AB can be expressed as linear combinations of the column matrices of A as follows:

$$\begin{bmatrix} 12 \\ 8 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 6 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 27 \\ -4 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 1\begin{bmatrix} 2 \\ 6 \end{bmatrix} + 7\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 30 \\ 26 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 6 \end{bmatrix} + 5\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 13 \\ 12 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 6 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

Matrix Form a Linear System

Consider any system of m linear equations in n unknowns.

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \ldots . + a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + \ldots . + a_{2n}x_n = b_2 \\
&. \\
&. \\
&. \\
&a_{m1}x_1 + a_{m2}x_2 + \ldots . + a_{mn}x_n = b_m
\end{aligned}
$$

Since two matrices are equal if and only if their corresponding entries are equal, we can replace the m equations in this system by the single matrix

$$
\begin{bmatrix}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\
. \quad\quad . \quad\quad\quad\quad . \\
. \\
. \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n
\end{bmatrix}
=
\begin{bmatrix}
b_1 \\ b_2 \\ . \\ . \\ . \\ b_m
\end{bmatrix}
$$

$$
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
. & . & & . \\
. & & & \\
. & & & \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
\begin{bmatrix}
x_1 \\ x_2 \\ . \\ . \\ . \\ x_n
\end{bmatrix}
=
\begin{bmatrix}
b_1 \\ b_2 \\ . \\ . \\ . \\ b_m
\end{bmatrix}
$$

$$Ax = b$$

A: Coefficient matrix of the system

Properties of Matrix Arithmetic

Assuming that the sizes of the matrices are such that the individual operations can be performed, the following rules of matrix arithmetic are valid.

1. **A+B=B+A Commutative law for addition)**
2. **A+(B+C)=(A+B)+C(Associative law for addition)**
3. **A(BC)=(AB)C(Associative law for multiplication)**
4. **A(B+C)=AB+AC (Left Distribution law)**
5. **(B+C)A=BA+CA (Right Distribution law)**
6. **A(B-C)=AB-AC**
7. **(B-C)A=AB-CA**
8. **a(B+C)=aB+aC**
9. **a(B-C)=aB-aC**
10. **(a+b)C=aC+bC**
11. **(a-b)C=aC-bC**
12. **a(bC)=(ab)C**
13. **a(BC)=(aB)C=B(aC)**

Matrix Multiplication is not Commutative

AB and BA Need Not be Equal

Example:

Consider the matrices

$$A = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$$

Multiplying gives

$$AB = \begin{bmatrix} -1 & -2 \\ 11 & 4 \end{bmatrix} \qquad BA = \begin{bmatrix} 3 & 6 \\ -3 & 0 \end{bmatrix}$$

Thus $AB \neq BA$

Most matrices don’t commute.

Consider the matrices

$$C = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Multiplying gives

$$CD = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = DC$$

Example Associativity of Matrix Multiplication

A(BC)=(AB)C

```
A=[1 2 ;3 4;0 1]

A =

     1     2

     3     4

     0     1

>> B=[4 3 ; 2 1]

B =

     4     3

     2     1

>> C=[1 0; 2 3]

C =

     1     0

     2     3

>> A*B

ans =

     8     5

    20    13

     2     1
```

```
>> B*C
ans =
    10     9
     4     3
>> (A*B)*C
ans =
    18    15
    46    39
     4     3
>> A*(B*C)
ans =
    18    15
    46    39
     4     3
>
```

Identity Matrices

Of special interest are square matrices with 1's on the main diagonal and 0's off the main diagonal, such as

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$ **and so on.**

A matrix of this form is called an identity matrix and is denoted by I. If it is important to emphasize the, we shall write I_n for the $n \times n$ identity matrix.

If A is an $m \times n$ matrix, then, as illustrated in the next example,

$$A\,I_n = A \quad \text{and} \quad I_m A = A$$

Example Multiplication by an Identity Matrix

Consider the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Then

$$I_2 A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \mathrm{A}$$

and

$$AI_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \mathrm{A}$$

Definition (Transpose of a Matrix)

If A is any $m \times n$ matrix, then the transpose of A, denoted A^T ,is defined to be $n \times m$ that results from interchanging the rows and columns of A; that is, the first column of A^T is the first row of A, the second column of A^T is the second row of A, and so fort.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & & . \\ . & & & \\ . & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ . & . & & . \\ . & & & \\ . & & & \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

Entries of A^T $(A^T)_{ij} = (A)_{ji}$

Example Some Transposes

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \qquad A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \\ a_{14} & a_{24} & a_{34} \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{bmatrix} \qquad B^T = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 3 & 5 \end{bmatrix} \qquad C^T = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

$$D = [3] \qquad D^T = [3]$$

```
>> B=[2 3; 1 4;5 6]
B =
    2    3
    1    4
    5    6
>> BT=B'
BT =
    2    1    5
    3    4    6
```

In the special case where A is a **square matrix**, the transpose of A can be obtained by interchanging entries that are **symmetrically positioned about the main diagonal**.

$$A=\begin{bmatrix} 1 & -2 & 4 \\ 3 & 7 & 0 \\ -5 & 8 & 6 \end{bmatrix} \rightarrow A^T=\begin{bmatrix} 1 & 3 & -5 \\ -2 & 7 & 0 \\ 4 & 0 & 6 \end{bmatrix}$$

Theorem (Properties of the Transpose)

If the sizes of the matrices are such that the stated operations can be performed, then

- $((A)^T)^T = A$
- $(A+B)^T = A^T + B^T$ **and** $(A-B)^T = A^T - B^T$
- $(kA)^T = kA^T$ k is any scalar
- $(AB)^T = B^T A^T$

Definition (Trace of A)

If A is a square $n \times n$ matrix, then the trace of A, denoted tr(A),is defined to be the sum of entries on the main diagonal of A. The trace of A is not defined if A is not a square matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & & . \\ . & & & \\ . & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$tr(A) = \sum_{i=1}^{n} a_{ii}$$

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 7 & 0 \\ -5 & 8 & -6 \end{bmatrix} \rightarrow tr(A) = 1 + 7 - 6 = 2$$

```
>> A=[1 -2 4;3 7 0;-5 8 -6]
A =
    1   -2    4
    3    7    0
   -5    8   -6
>> trace(A)
ans =     2
```

Norms

Vector Norm

We compute the Euclidian norm of vectors

$$x = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}$$

$$|x|_e = \sum_{i=1}^{n} \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}$$

(Euclidian Norm –Length of the vector)

This is not the only way to compute a vector norm, however. The sum of the absolute values of the x_i can be used as a norm.

Sum of Magnitudes $$|x| = \left(\sum_{i=1}^{n} |x_i| \right)$$

Maximum-Magnitude norm $$|x|_\infty = \max_{1 \le i \le n} |x_i|$$

P-norm $$|x|_p = \left(\sum_{i=1}^{n} |x_i^p| \right)^{1/p}$$

Example **Compute 1,2 and ∝ norms of the vector**

$$x = \begin{bmatrix} 1.25 \\ 0.02 \\ -5.15 \\ 0 \end{bmatrix}$$

$$|x|_1 = |1.25| + |0.02| + |-5.15| + |0| = 6.42$$

$$|x|_2 = [(1.25)^2 + (0.02)^2 + (-5.15)^2 + (0)^2]^{1/2} = 5.2996$$

$$|x|_\infty = |-5.15| = 5.15$$

Matrix Norms

$$|A|_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}| = \max \textit{column sum}$$

$$|A|_\infty = \max_{1 \le j \le n} \sum_{j=1}^{n} |a_{ij}| = \max \textit{row sum}$$

The matrix norm $|A|_2$ that corresponds to the 2-norm of a vector is <u>not readily computed</u>. It is defined terms of the eigenvalues of the matrix $A^T A$. Suppose r is the largest Eigen value of $A^T A$. Then

$$|A|_2 = \sqrt{r}$$

the square root of r. <u>This is called</u> the spectral norm of A, and $|A|_2$ <u>is always less than (or equal to)</u> $|A|_1$ and $|A|_\infty$.

Example **Compute 1 and $\propto$ norms of the matrix**

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 4 \\ 2 & 7 & 5 \end{bmatrix}$$

```
>> A=[ 1 0 2 ; 3 1 4 ; 2 7 5]

A =

     1     0     2

     3     1     4

     2     7     5

>> norm(A,1)

ans =  11
```

$$|A|_1 = \max(1+3+2;0+1+7;2+4+5) = \max(6,8,11) = 11$$

```
>> norm(A,inf)

ans = 14
```

$$|A|_\infty = \max(1+0+2;3+1+4;2+7+5) = \max(3,8,14) = 14$$

>> B=A'*A

B =

14 17 24

17 50 39

24 39 45

>> eig(B)

ans =

0.1918

12.1350

96.6732

>>> norm(A,2)

ans = 9.8323

$|A|_1 = 11$

$|A|_\infty = 14$

$|A|_2 = \sqrt{96.6732} = 9.8323$

$|A|_2$ **is always less than (or equal to)** $|A|_1$ **and** $|A|_\infty$.

Frobenius Norm of the Matrix

For $m \times n$ matrix, the Frobenius norm is defined as

$$|A|_f = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2 \right)^{1/2}$$

>> B

A =

1	0	2
3	1	4
2	7	5

$$|A|_f = \sqrt{(1+0+4+9+1+16+4+49+25} = \sqrt{109} = 10.4403$$

>> norm(B,'fro')

ans = 10.4403