Equilibrated Residual Error Estimates

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1 General framework

Equilibrated residual error estimators provide upper bounds for the discretization error in energy norm without any generic constant. We consider the standard problem: find $u \in V := H_0^1(\Omega)$ such that

$$\int_\Omega \lambda \nabla u \cdot \nabla v = \int_\Omega f v \qquad \forall\, v \in V$$

The left hand side defines the bilinear-form $A(\cdot,\cdot)$, the right hand side the linear-form $f(\cdot)$. We define a finite element sub-space $V_h \subset V$ of order k, and the finite element solution

$$\text{find } u_h \in V_h : \quad A(u_h, v_h) = f(v_h) \qquad \forall\, v_h \in V_h.$$

We assume that f is element-wise polynomial of order $k-1$, and λ is element-wise constant and positive.

The residual $r(\cdot) \in V^*$ is

$$r(v) = f(v) - A(u_h, v) \qquad v \in V$$

Since

$$\|u - u_h\|_A = \sup_{v \in V} \frac{A(u - u_h, v)}{\|v\|_A} = \sup_{v \in V} \frac{r(v)}{\|v\|_A},$$

we aim in estimating $\|r\|$ in the norm dual to $\|\cdot\|_A$, which is essentially the H^{-1}-norm. In general, the direct evaluation of this norm is not feasible. Using the structure of the problem, we can represent the residual as

$$r(v) = \sum_{T \in \mathcal{T}} \int_T r_T v + \sum_{E \in \mathcal{E}} \int_E r_E v,$$

where r_T and r_E are given asgiven as

$$r_T = f_T + \operatorname{div} \lambda_T \nabla u_{h|T} \qquad \text{and} \qquad r_E = \left[\lambda \frac{\partial u_h}{\partial n} \right]_E$$

The element-residual r_T is a polynomial of order $k-1$ on the element T, and the edge residual (the normal jump) is a polynomial of order $k-1$ on the edge E.

The *residual error estimator* estimates the residual in terms of weighted L_2-norms:

$$\|r\|^2 \simeq \eta^{res}(u_h, f)^2 := \sum_T \frac{h_T^2}{\lambda_T} \|r_T\|^2_{L_2(T)} + \sum_E \frac{h_E}{\lambda_E} \|r_E\|^2_{L_2(E)}$$

Here, λ_E is some averaging of the coefficients on the two elements containing the edge E. The equivalence holds with constants depending on the shape of elements, the relative jump of the coefficient, and the polynomial order k.

The *equilibrated residual error estimator* η^{er} is defined in terms of the same data r_T and r_E. It satisfies

$$\begin{array}{rcll} \|u-u_h\|_A & \leq & \eta^{er} & \text{reliable with constant 1} \\ \|u-u_h\|_A & \geq & c\,\eta^{er} & \text{efficient with a generic constant } c \end{array}$$

The lower bound depends on the shape of elements and the coefficient λ, but is robust with respect to the polynomial order k.

The main idea is the following: Instead of calculating the H^{-1}-norm of r, we compute a lifting σ^Δ such that $\operatorname{div} \sigma^\Delta = r$, and calculate the L_2-norm of σ^Δ. Since r is not a regular function, the equation must be posed in distributional form:

$$\int_\Omega \sigma^\Delta \cdot \nabla \varphi = -r(\varphi) \qquad \forall\, \varphi \in V$$

Then, the residual can be estimated without envolving any generic constant:

$$\begin{aligned} \|r\|_{A^*} &= \sup_{v \in V} \frac{r(v)}{\|v\|_A} = \sup_v \frac{\int \sigma^\Delta \cdot \nabla v}{\|v\|_A} \\ &= \sup_v \frac{\int \lambda^{-1/2} \sigma^\Delta \cdot \lambda^{1/2} \nabla v}{\|v\|_A} \leq \sup_v \frac{\sqrt{\int \lambda^{-1} |\sigma^\Delta|^2} \sqrt{\int \lambda |\nabla v|^2}}{\|v\|_A} = \|\sigma^\Delta\|_{L_2, 1/\lambda} \end{aligned}$$

The norm $\|\sigma^\Delta\| := \int \lambda^{-1} |\sigma^\Delta|^2$ can be evaluated easily.

Remark: The flux-postprocessing $\sigma := \lambda \nabla u_h + \sigma^\Delta$ provides a flux $\sigma \in H(\text{div})$ such that $\operatorname{div} \sigma = f$, i.e. the flux is in exact equilibrium with the source f. Thus the name.

2 Computation of the lifting $\|\sigma^\Delta\|$

The residual is a functional of the form

$$r(v) = \sum_T (r_T, v)_{L_2(T)} + \sum_E (r_E, v)_{L_2(E)},$$

where r_T and r_E are polynomials of order $k-1$. We search for σ^Δ which is element-wise a vector-valued polynomial of order k, and not continuous across edges. Element-wise integration by parts gives

$$\int_\Omega \sigma \cdot \nabla \varphi = -\sum_T \int_T \operatorname{div} \sigma_{|T} \varphi + \sum_E \int_E [\sigma \cdot n]_E \varphi.$$

Thus $\operatorname{div} \sigma = r$ in distributional sense reads as

$$\operatorname{div} \sigma | T = r_T \qquad \text{and} \qquad [\sigma \cdot n]_E = -r_E$$

for all elements T and edges E. We could now pose the problem

$$\min_{\substack{\sigma \in P^k(\mathcal{T})^2 \\ \operatorname{div} \sigma = r}} \|\sigma\|_{L_2, 1/\lambda}$$

We minimize the weighted-L_2 norm since we want to find the smallest possible upper bound for the error. This is already a computable approach. But, the problem is global, and its solution is of comparable cost as the solution of the original finite element system. The existence of a σ such that div $\sigma = r$ also needs a proof.

We want to localize the construction of the flux. Local problems are associated with vertex-patches $\omega_V = \cup_{T: V \in T} T$. We proceed in two steps:

1. localization of the residual: $r = \sum_V r^V$
2. local liftings: find σ^V such that $\operatorname{div} \sigma^V = r^V$ on the vertex patch

Then, for $\sigma := \sum \sigma^V$ there holds $\operatorname{div} \sigma = r$

The localization is given by multiplication of the P^1 vertex basis functions (hat-functions) ϕ_V:

$$r^V(v) := r(\phi_V v)$$

Since $\sum_V \phi_V = 1$, there holds $\sum r^V(\cdot) = r(\cdot)$. The localized residual has the same structure of element and edge terms:

$$r^V(v) = \sum_{T \subset \omega_V} (r_T^V, v)_{L_2(T)} + \sum_{E \subset \omega_V} (r_E^V, v)_{L_2(E)},$$

with

$$r_T^V = \phi_V r_T \qquad \text{and} \qquad r_E^V = \phi_V r_E$$

The local residual vanishes on constants on the patch:

$$r^V(1) = r(\phi_V\, 1) = A(u - u_h, \phi_V) = 0$$

The last equality follows from the Galerkin-orthogonality.

We give an explicit construction of the lifting σ^V in terms of the Brezzi-Douglas-Marini (BDM) element. The k^{th} order BDM element on a triangle is given by $V_T = [P^k]^2$ and the degrees of freedom:

(i) $\int_E \sigma \cdot n \, q_i$ with q_i a basis for $P^k(E)$

(ii) $\int_T \operatorname{div} \sigma \, q_i$ with q_i a basis for $P^{k-1}(T) \cap L_2^0(T)$

(iii) $\int_T \sigma \cdot \operatorname{curl} q_i$ with q_i a basis for $P_0^{k+1}(T)$

Exercise: Show that these dofs are unisolvent. Count dimensions, and prove that $[\forall i : \psi_i(\sigma) = 0] \Rightarrow \sigma = 0$.

Now, we give an explicit construction of equilibrated fluxes on a vertex patch. Label elements $T_1, T_2, \ldots T_n$ in a counter-clock-wise order. Edge E_i is the common edge between triangle T_{i-1} and T_i (with identifying $T_0 = T_n$). We define σ by specifying the dofs of the BDM element:

1. Start on T_1. We set $\sigma_n = -r_{E_1}^V$ on edge E_1. On the edge on the patch-boundary we set $\sigma_n = 0$, and on E_2 we set $\sigma_n = const$ such that $\int_{\partial T_1} \sigma_n = \int_{T_1} r_T^V$. We use the dofs of type (ii) to specify $\int_T \operatorname{div} \sigma \, q = \int_T r_T^V q \; \forall q \in P^{k-1} \cap L_2^0(T)$. Together with get $\operatorname{div} \sigma = r_T$. Dofs of type (iii) are not needed, and set 0. There holds
$$\int_{E_2} \sigma_n = \int_{T_1} r_T^V - \int_{E_1} \sigma_n = \int_{T_1} r_{T_1}^V + \int_{E_1} r_{E_1}^V$$
2. Continue with element T_2. On edge E_2 common with T_1 set σ_n such that $[\sigma \cdot n]_{E_2} = r_{E_2}$. Otherwise, proceed as on T_1. Thus
$$\int_{E_3} \sigma_n = \int_{T_1} r_{T_1}^V + \int_{E_1} r_{E_1}^V + \int_{T_2} r_{T_2}^V + \int_{E_2} r_{E_2}^V$$
3. Continue to element T_n. Observe that on T_n:
$$\int_{E_1} \sigma_n = \sum_{i=1}^n \int_{T_i} r_{T_i}^V + \sum_{i=1}^n \int_{E_i} r_{E_i}^V = 0,$$
which follows from $r^V(1) = 0$. Thus, also $[\sigma \cdot n]_{E_1} = r_{E_1}^V$ is satisfied.

This explicit construction proves the existence of an equilibrated flux. Instead of this explicit construction, one may solve a local constrained optimization problem
$$\min_{\sigma^V : \operatorname{div} \sigma^V = r^V} \|\sigma\|_{L_2, \lambda^{-1}}$$
This applies also for 3D. Furthoer notes

- mixed boundary conditions are possible
- the efficiency for the h-FEM is shown by scaling arguments, and equivalence to the residual error estimator
- efficiency is also proven to be robust with respect to polynomial order k, examples show overestimation less than 1.5

References

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