Duality between Total Variation and Equivalence Couplings

Adam Quinn Jaffe

April 12, 2023

I. Stochastic processes

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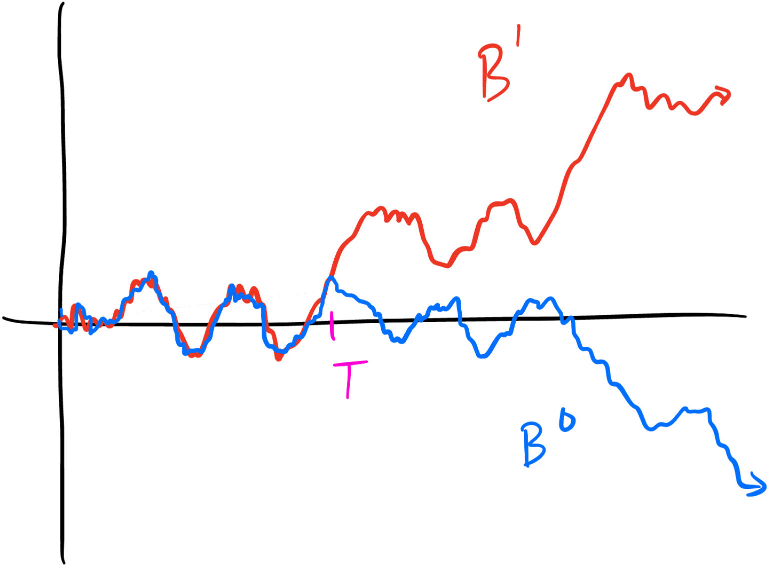
such that $B_t^{\theta_1} = B_t^{\theta_2}$ for all $0 \leq t \leq T$ almost surely.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Explicit construction based on Itô excursion theory.

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Know that $(\mathbb{W}^{\theta_1}, \mathbb{W}^{\theta_2})$ has the GCP for all $\theta_1, \theta_2 \in \mathbb{R}$, where $\mathbb{W}^{\theta}$ denotes the law of BM with drift $\theta \in \mathbb{R}$.

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Which other pairs have the GCP?

II. Some vignettes

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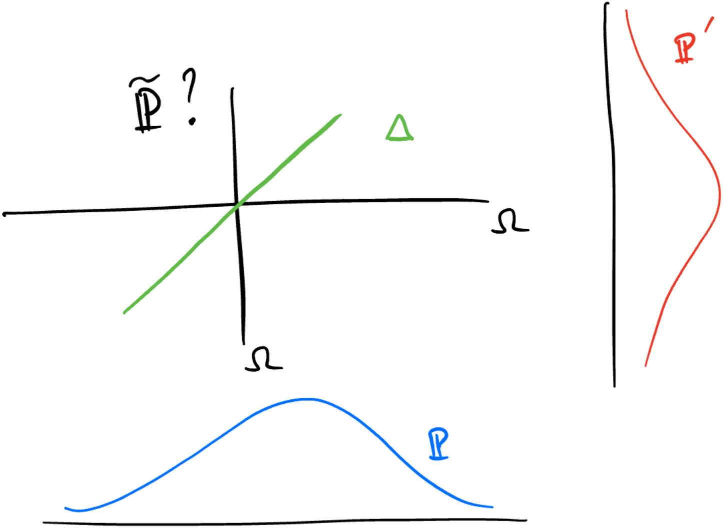

$\tilde{\mathbb{P}}$?
Δ
Ω
Ω
$\mathbb{P}'$
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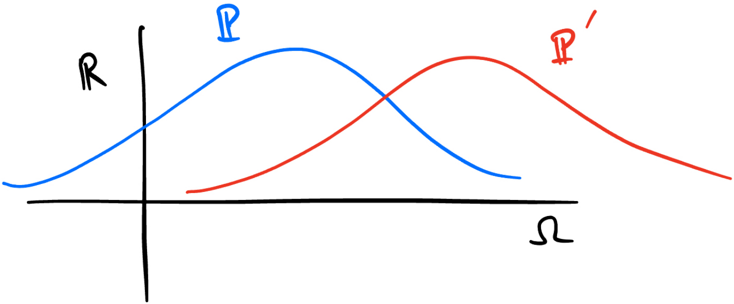

ℙ
ℙ'
ℝ
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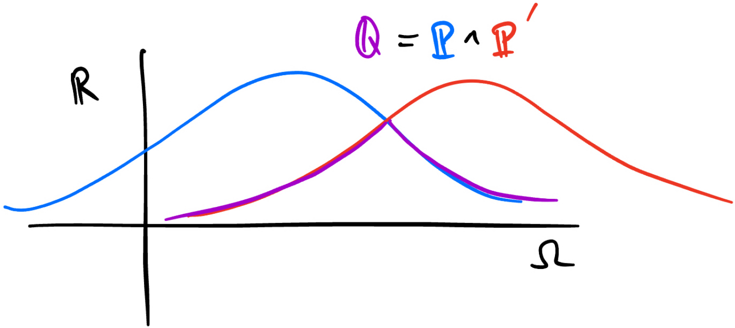
$\mathbb{Q} = \mathbb{P} \wedge \mathbb{P}'$
$\mathbb{R}$
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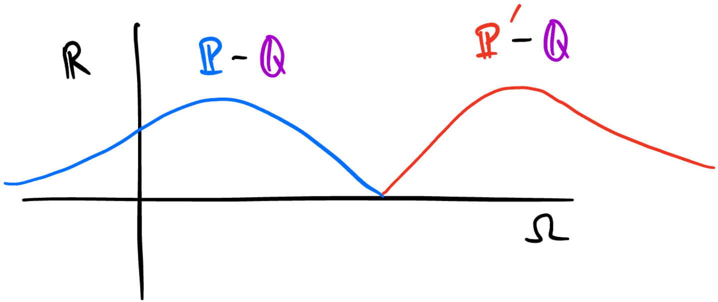
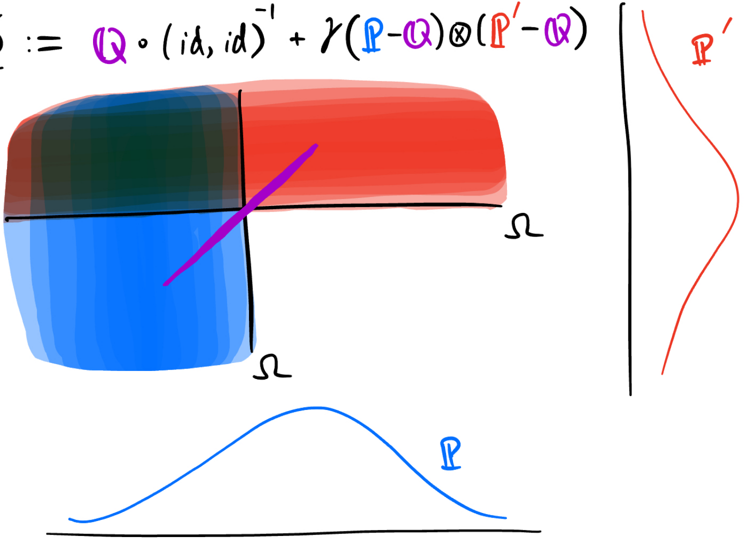


$\mathbb{R}$
$\mathbb{P} - \mathbb{Q}$
$\mathbb{P}' - \mathbb{Q}$
Ω

$$\tilde{\mathbb{P}} := \mathbb{Q} \circ (id, id)^{-1} + \gamma(\mathbb{P} - \mathbb{Q}) \otimes (\mathbb{P}' - \mathbb{Q})$$



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I. **Stochastic processes**

II. **Some vignettes**

III. **Problem statement**

IV. **Results**

V. **Applications**

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Roughly speaking, $(\Delta, \mathcal{B}(\Omega)), (E_0, \mathcal{T})$, and $(E_{\mathbb{Z}}, \mathcal{I}_{\mathbb{Z}})$ are strongly dual.

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Roughly speaking, $(\Delta, \mathcal{B}(\Omega)), (E_0, \mathcal{T})$, and $(E_{\mathbb{Z}}, \mathcal{I}_{\mathbb{Z}})$ are strongly dual.

Which pairs $(E, \mathcal{G})$ are strongly dual?

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Say that E is strongly dualizable if (E, E^*) is strongly dual.

Connection to optimal transport?

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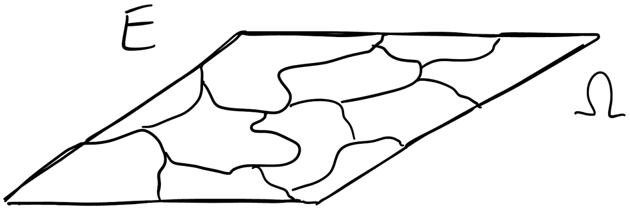
$$c(x, x') = 1 - \mathbb{1}\{(x, x') \in E\}.$$

In words: cost 0 to move within an equivalence class, and cost 1 to move between equivalence classes.



E
Ω

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However, most interesting equivalence relations, from the point of view of probability, are F_σ (countable union of closed) in $\Omega \times \Omega$.

Idea: One can exchange topological regularity (lower semi-continuity of the cost function) with algebraic regularity (the cost function is supported on an equivalence relation) while maintaining a form of duality.

IV. Results

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- There exists $\tilde{\mathbb{P}} \in \Pi(\mathbb{P}, \mathbb{P}')$ and $N \in \mathcal{F} \otimes \mathcal{F}$ with $\tilde{\mathbb{P}}(N) = 0$ and $(\Omega \times \Omega) \setminus E \subseteq N$.

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Then E is strongly dualizable if and only if it is measurable and quasi-strongly dualizable.

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Roughly speaking, E is smooth if and only if the quotient Ω/E can be given a natural standard Borel structure.

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The equivalence between (i) and (ii) is classical, but the equivalence with (iii) appears to be novel.

Theorem (AQJ)

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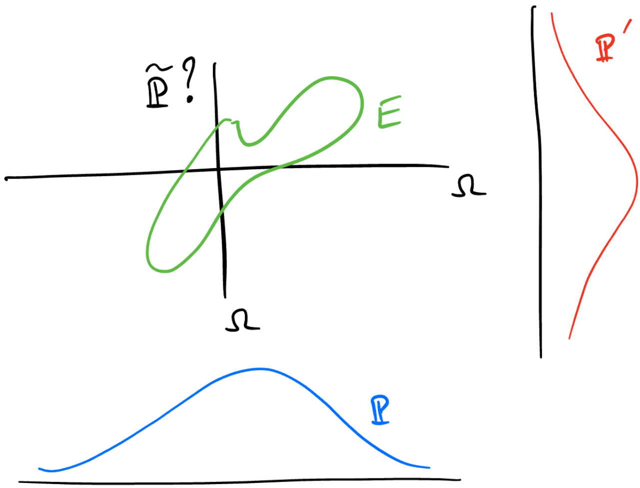
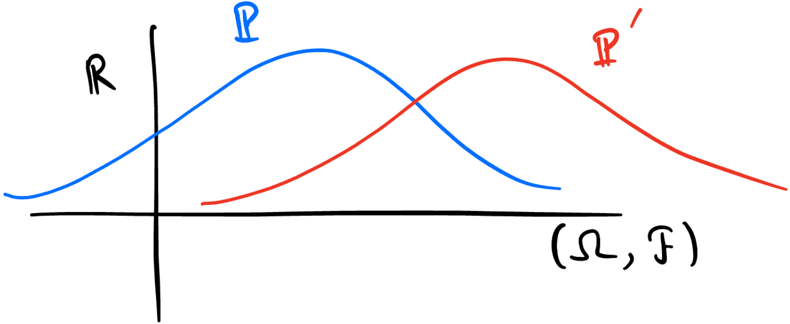
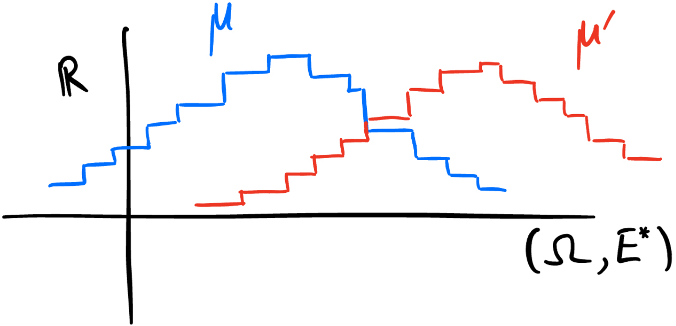
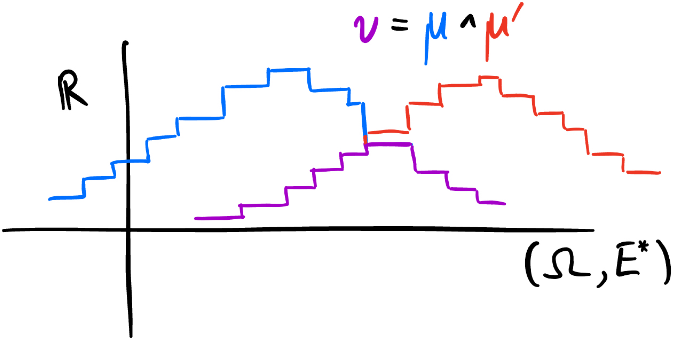
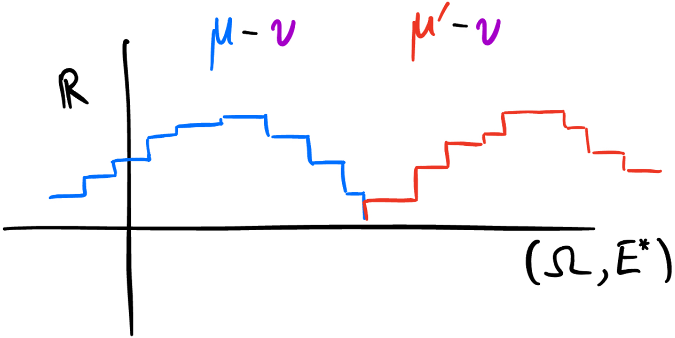
Idea of proof: Do the Doeblin coupling in (Ω, E^*) then pull back.


$\tilde{\mathbb{P}}$?
E
Ω
Ω
$\mathbb{P}'$
$\mathbb{P}$


$\mathbb{P}$
$\mathbb{P}'$
$\mathbb{R}$
$(\Omega, \mathcal{F})$

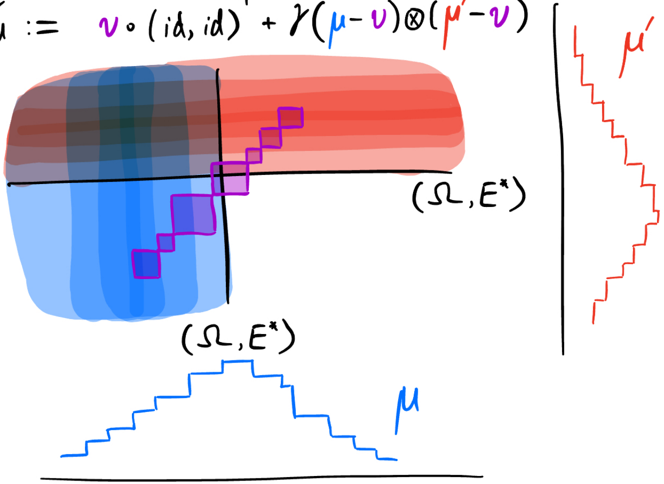

μ
μ'
$\mathbb{R}$
(Ω, E^*)

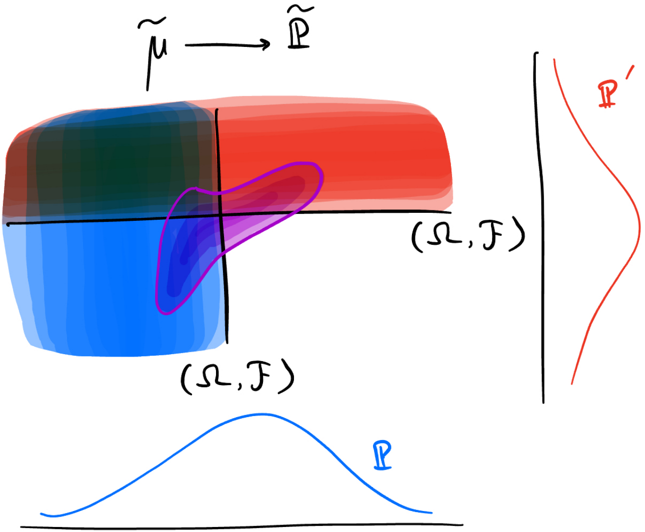

$\nu = \mu \wedge \mu'$
$\mathbb{R}$
(Ω, E^*)



$\mu - \nu$
$\mu' - \nu$
$\mathbb{R}$
(Ω, E^*)

$$\tilde{\mu} := \nu \circ (id, id)^{-1} + \gamma(\mu - \nu) \otimes (\mu' - \nu)$$





$\tilde{\mu} \longrightarrow \tilde{\mathbb{P}}$
$(\Omega, \mathcal{F})$
$(\Omega, \mathcal{F})$
$\mathbb{P}'$
$\mathbb{P}$

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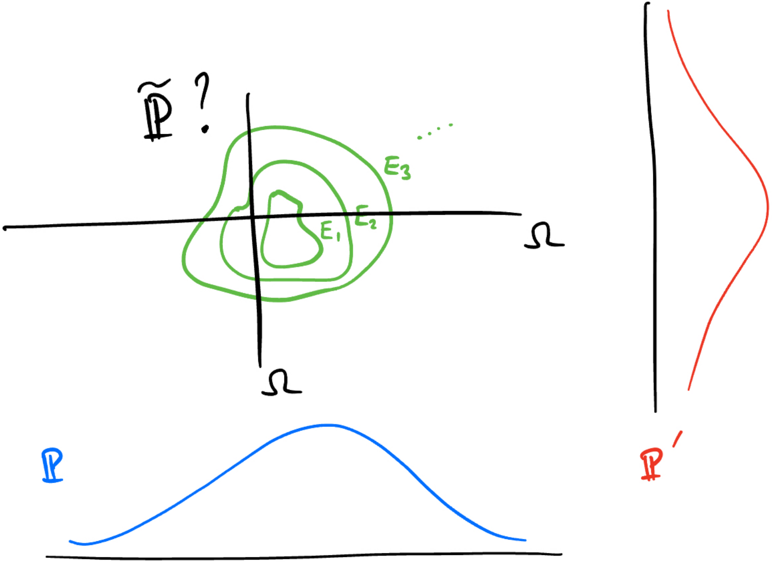
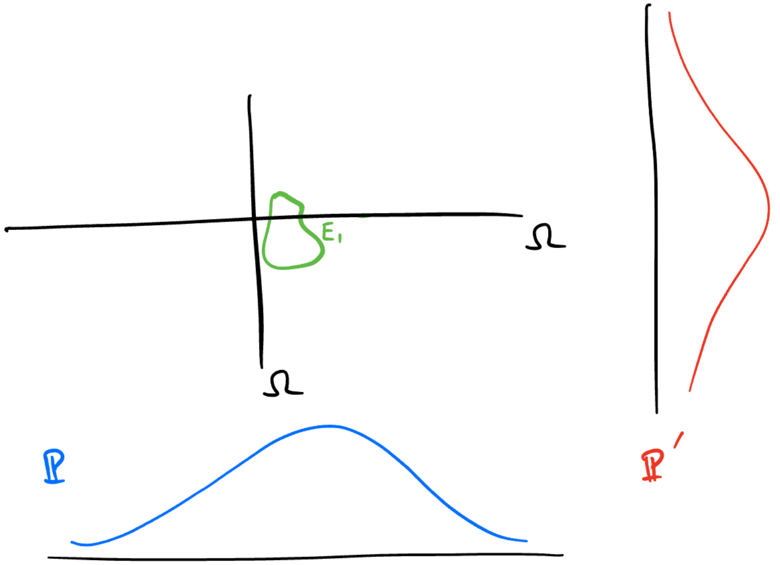
A countable increasing union of strongly dualizable equivalence relations is strongly dualizable.

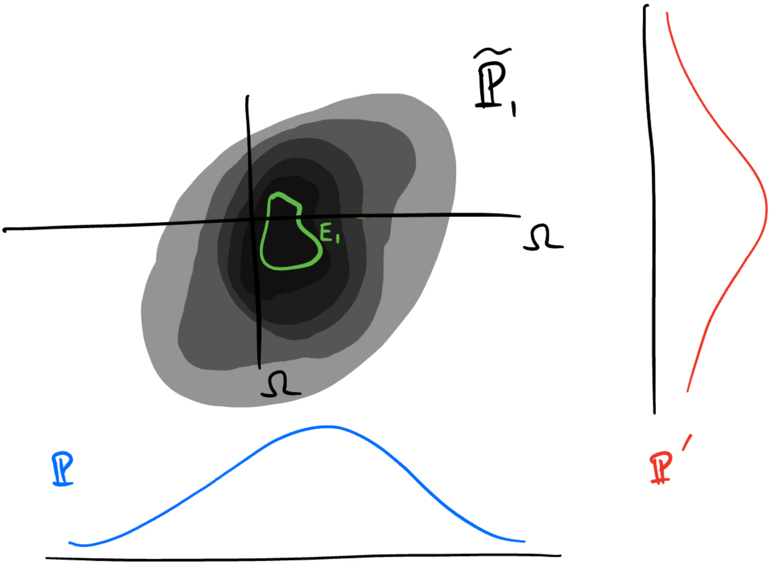
Idea of proof: Iterate greedily.



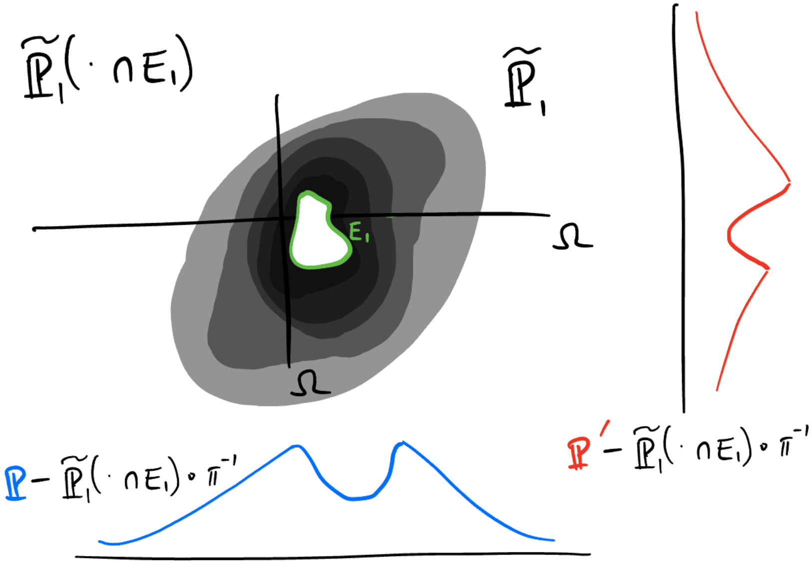
$\tilde{\mathbb{P}}$?
E_3
E_2
E_1
Ω
Ω
$\mathbb{P}$
$\mathbb{P}'$

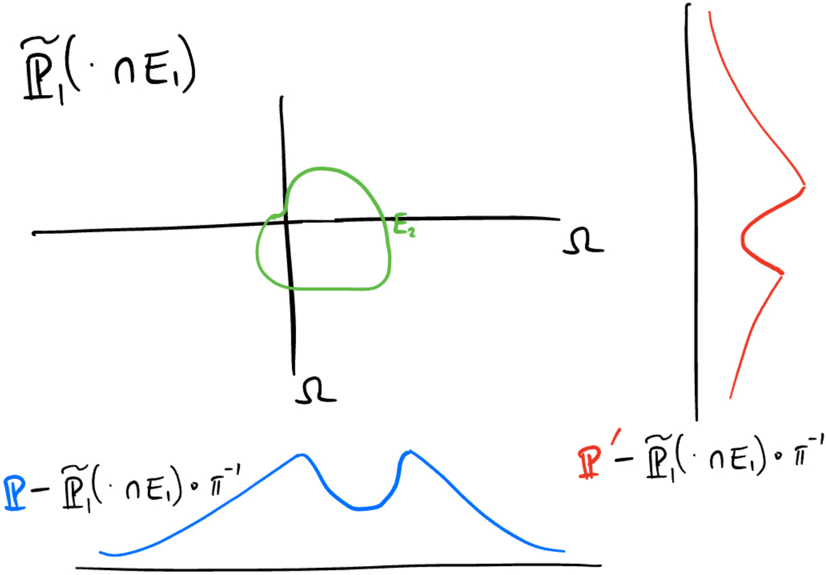


E1
Ω
Ω
ℙ
ℙ′



$\tilde{\mathbb{P}}_1$
E_1
Ω
Ω
$\mathbb{P}$
$\mathbb{P}'$


$\tilde{\mathbb{P}}_1(\cdot \cap E_1)$
$\tilde{\mathbb{P}}_1$
E_1
Ω
Ω
$\mathbb{P} - \tilde{\mathbb{P}}_1(\cdot \cap E_1) \circ \pi^{-1}$
$\mathbb{P}' - \tilde{\mathbb{P}}_1(\cdot \cap E_1) \circ \pi^{-1}$



$\tilde{\mathbb{P}}_1(\cdot \cap E_1)$
E_2
Ω
Ω
$\mathbb{P} - \tilde{\mathbb{P}}_1(\cdot \cap E_1) \circ \pi^{-1}$
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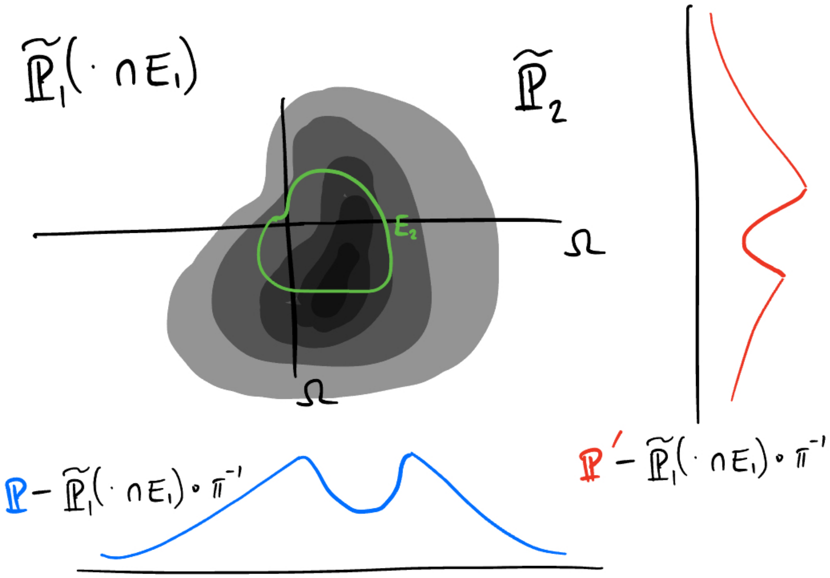
$\tilde{\mathbb{P}}_1(\cdot \cap E_1)$
$\tilde{\mathbb{P}}_2$
E_2
Ω
Ω
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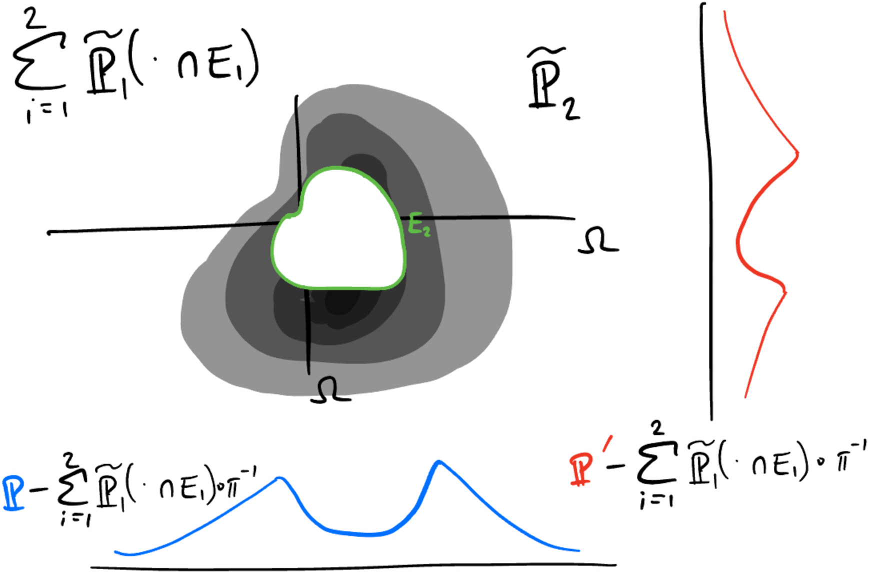

$\sum_{i=1}^{2} \tilde{\mathbb{P}}_1(\cdot \cap E_1)$
$\tilde{\mathbb{P}}_2$
E_2
Ω
Ω
$\mathbb{P} - \sum_{i=1}^{2} \tilde{\mathbb{P}}_1(\cdot \cap E_1) \circ \pi^{-1}$
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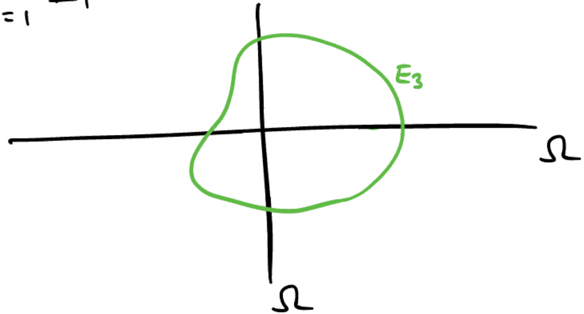

$$\mathbb{P}' - \sum_{i=1}^{2} \tilde{\mathbb{P}}_1(\cdot \cap E_1) \circ \pi^{-1}$$

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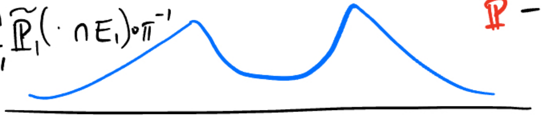

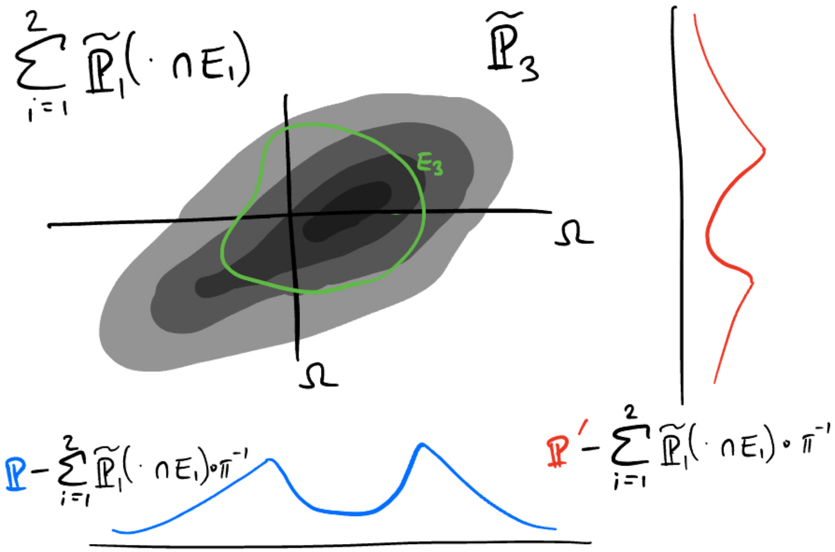
$\sum_{i=1}^{2} \tilde{\mathbb{P}}_i(\cdot \cap E_i)$
$\tilde{\mathbb{P}}_3$
E_3
Ω
Ω
$\mathbb{P} - \sum_{i=1}^{2} \tilde{\mathbb{P}}_i(\cdot \cap E_i) \circ \pi^{-1}$
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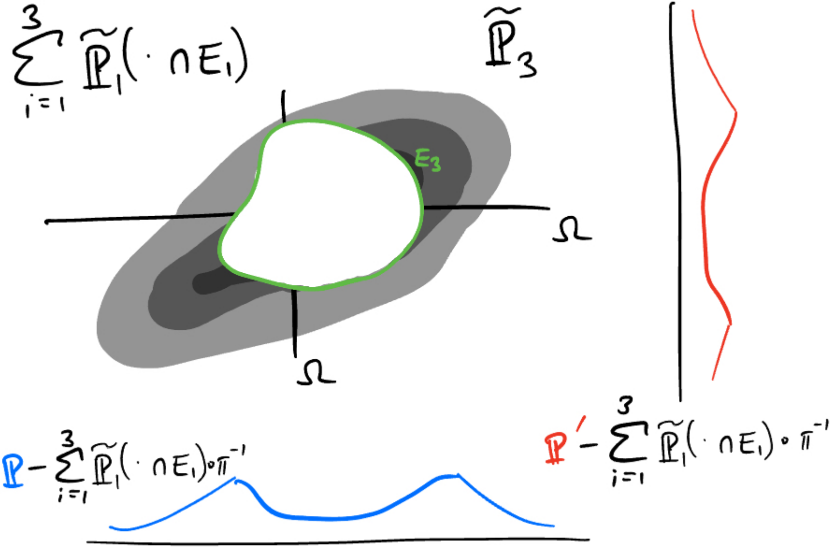

$\sum_{i=1}^{3} \tilde{\mathbb{P}}_1(\cdot \cap E_1)$
$\tilde{\mathbb{P}}_3$
E_3
Ω
Ω
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$\mathbb{P}' - \sum_{i=1}^{3} \tilde{\mathbb{P}}_1(\cdot \cap E_1) \circ \pi^{-1}$

This shows strong dualizability for all *hypersmooth* (countable union of smooth) equivalence relations, This shows strong dualizability for all *hypersmooth* (countable union of smooth) equivalence relations, and this covers most “reasonable” equivalence relations occurring in probability.

V. Applications

If E is strongly dualizable, then the E-coupling problem

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has a minimizer and is equivalent to the E^*-total variation problem:

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Thus, the E-coupling problem can be studied with the help of very many soft arguments.

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Suppose $(X_1, X_2, \ldots)$ is an i.i.d. real-valued sequence with common distribution μ.

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Which random sequences can be simulated in the way, by applying a finite, randomized sorting algorithm to an i.i.d. sequence?

X : 1 -1 2 2 0 3 -2 1 3 . . . .

X' : 2 -1 0 3 2 1 -2 1 3 . . . .

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Then E_∞ is strongly dualizable, and we have $E_\infty^* = \mathcal{E}$, and

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Consequence: If $\mathbb{P} \ll \mu^{\otimes\mathbb{N}}$, then there exists a finite, randomized sorting algorithm which, when given an input consisting of an i.i.d. sequence from μ, returns a sequence with law $\mathbb{P}$.

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Consequence: If $\mathbb{P} \ll \mu^{\otimes \mathbb{N}}$, then there exists a finite, randomized sorting algorithm which, when given an input consisting of an i.i.d. sequence from μ, returns a sequence with law $\mathbb{P}$.

How to construct such an algorithm explicitly?

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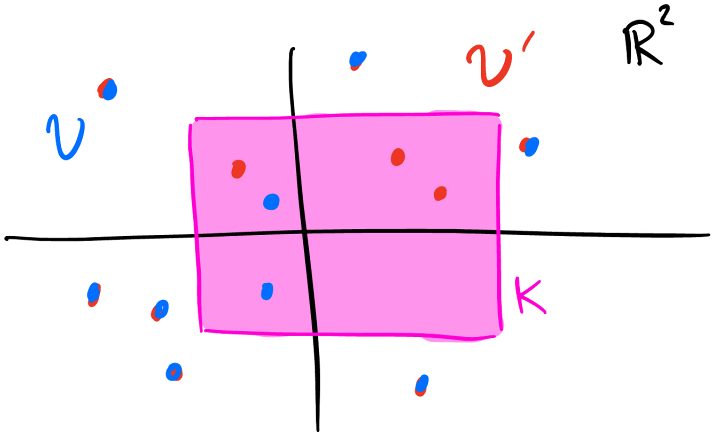
We say that $\mathbf{a}$ and $\mathbf{a}'$ have *similar potential in* $\mathbb{P}$ if there exists a coupling of (ω, ω') of $\mathbb{P}_{\mathbf{a}}$ and $\mathbb{P}_{\mathbf{a}'}$ such that ω and ω' have finitely many disagreements.



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v'
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Consequence: If $\mathbb{P}$ is the law of a DPP with integrable kernel (Ginibre, Gamma, Bessel, Airy) and $|\mathbf{a}| = |\mathbf{a}'|$, then $\mathbf{a}$ and $\mathbf{a}'$ have similar potential in $\mathbb{P}$.

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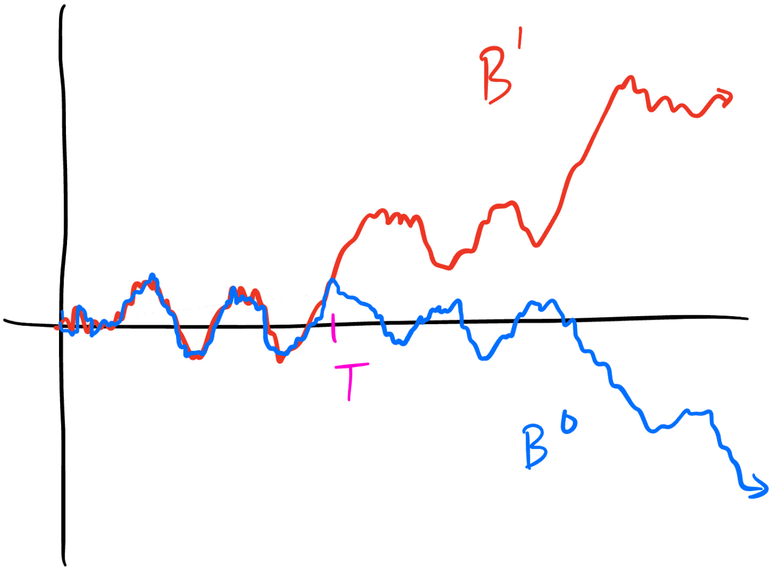
Consequence: $(\mathbb{P},\mathbb{P}')$ has the germ coupling property (GCP) if and only if $\mathbb{P}(A) = \mathbb{P}'(A)$ for all $A \in \mathcal{F}_{0+}$.



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$$\begin{cases} dX_t = \mu(X_t)dt + \sigma(X_t)dB_t \text{ for } 0 \leq t \leq 1 \\ X_0 = x_0, \end{cases} \tag{1}$$

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How to do this explicitly, say, for BM and OU?

For many classes of Markov processes, it turns out that the GCP can be concretely interpreted in terms of some canonical data:

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A pair of Lévy processes in $\mathbb{R}^k$ *with characteristic triplets* (a, Σ, μ) *and* (a', Σ', μ') *has the GCP if and only if*

- $\Sigma = \Sigma'$,
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What about particle systems? Processes from population genetics? For general Feller processes, can one formulate the GCP as an analytic relation between the generators?

Thank you!

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$$\tilde{\mathbb{P}}(\,\cdot \setminus E) \circ \pi^{-1} \qquad \textit{and} \qquad \tilde{\mathbb{P}}(\,\cdot \setminus E) \circ (\pi')^{-1}$$

are mutually singular on (Ω, E^*), where $\pi, \pi : \Omega \times \Omega \to \Omega$ are the coordinate projections.

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Most likely, need a space that is not standard Borel.

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Then, how big is $\mathscr{E}$? How much bigger is (1,2,3) than (1,2)?

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Use strong duality to connect the stories?