

On Schemes for Exponential Decay

Hans Petter Langtangen^{1,2}

Center for Biomedical Computing, Simula Research Laboratory¹

Department of Informatics, University of Oslo²

May 28, 2013



The primary goal of this demo talk is to demonstrate how to write talks with doconce and get them rendered in numerous HTML formats.

Layout

This version utilizes beamer slides with the theme hpl2.

Mathematical problem

$$u'(t) = -au(t), \quad (1)$$

$$u(0) = I, \quad (2)$$

- $t \in (0, T]$
- a , I , and T are prescribed parameters
- $u(t)$ is the unknown function



Numerical solution method

- Mesh in time: $0 = t_0 < t_1 \cdots < t_N = T$
- Assume constant $\Delta t = t_n - t_{n-1}$
- u^n : numerical approx to the exact solution at t_n

Numerical scheme:

$$u^{n+1} = \frac{1 - (1 - \theta)a\Delta t}{1 + \theta a\Delta t} u^n, \quad n = 0, 1, \dots, N - 1$$

Numerical solution method

- Mesh in time: $0 = t_0 < t_1 \cdots < t_N = T$
- Assume constant $\Delta t = t_n - t_{n-1}$
- u^n : numerical approx to the exact solution at t_n

Numerical scheme:

$$u^{n+1} = \frac{1 - (1 - \theta)a\Delta t}{1 + \theta a\Delta t} u^n, \quad n = 0, 1, \dots, N - 1$$

Numerical solution method

- Mesh in time: $0 = t_0 < t_1 \cdots < t_N = T$
- Assume constant $\Delta t = t_n - t_{n-1}$
- u^n : numerical approx to the exact solution at t_n

Numerical scheme:

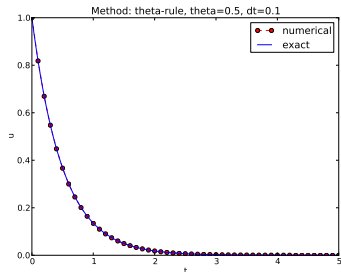
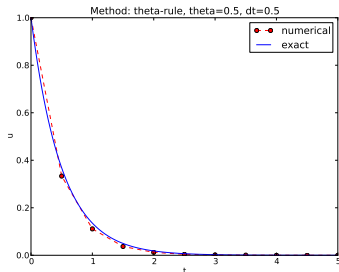
$$u^{n+1} = \frac{1 - (1 - \theta)a\Delta t}{1 + \theta a\Delta t} u^n, \quad n = 0, 1, \dots, N - 1$$

Implementation

The numerical method is implemented in a Python function:

```
def solver(I, a, T, dt, theta):  
    """Solve  $u' = -a*u$ ,  $u(0)=I$ , for  $t$  in  $(0,T]$  with steps of  $dt$ ."""  
    dt = float(dt) # avoid integer division  
    N = int(round(T/dt)) # no of time intervals  
    T = N*dt # adjust T to fit time step dt  
    u = zeros(N+1) # array of  $u[n]$  values  
    t = linspace(0, T, N+1) # time mesh  
  
    u[0] = I # assign initial condition  
    for n in range(0, N): #  $n=0,1,\dots,N-1$   
        u[n+1] = (1 - (1-theta)*a*dt)/(1 + theta*dt*a)*u[n]  
    return u, t
```

The Crank-Nicolson method



The artifacts can be explained by some theory

Exact solution of the scheme:

$$u^n = A^n, \quad A = \frac{1 - (1 - \theta)a\Delta t}{1 + \theta a\Delta t}.$$

- Stability: $|A| < 1$
- No oscillations: $A > 0$
- Always for Backward Euler ($\theta = 1$)
- $\Delta t < 1/a$ for Forward Euler ($\theta = 0$)
- $\Delta t < 2/a$ for Crank-Nicolson ($\theta = 1/2$)

Concluding remarks:

Only the Backward Euler scheme is guaranteed to always give qualitatively correct results.

The artifacts can be explained by some theory

Exact solution of the scheme:

$$u^n = A^n, \quad A = \frac{1 - (1 - \theta)a\Delta t}{1 + \theta a\Delta t}.$$

- Stability: $|A| < 1$
- No oscillations: $A > 0$
- Always for Backward Euler ($\theta = 1$)
- $\Delta t < 1/a$ for Forward Euler ($\theta = 0$)
- $\Delta t < 2/a$ for Crank-Nicolson ($\theta = 1/2$)

Concluding remarks:

Only the Backward Euler scheme is guaranteed to always give qualitatively correct results.

The artifacts can be explained by some theory

Exact solution of the scheme:

$$u^n = A^n, \quad A = \frac{1 - (1 - \theta)a\Delta t}{1 + \theta a\Delta t}.$$

- Stability: $|A| < 1$
- No oscillations: $A > 0$
- Always for Backward Euler ($\theta = 1$)
- $\Delta t < 1/a$ for Forward Euler ($\theta = 0$)
- $\Delta t < 2/a$ for Crank-Nicolson ($\theta = 1/2$)

Concluding remarks:

Only the Backward Euler scheme is guaranteed to always give qualitatively correct results.

The artifacts can be explained by some theory

Exact solution of the scheme:

$$u^n = A^n, \quad A = \frac{1 - (1 - \theta)a\Delta t}{1 + \theta a\Delta t}.$$

- Stability: $|A| < 1$
- No oscillations: $A > 0$
- Always for Backward Euler ($\theta = 1$)
- $\Delta t < 1/a$ for Forward Euler ($\theta = 0$)
- $\Delta t < 2/a$ for Crank-Nicolson ($\theta = 1/2$)

Concluding remarks:

Only the Backward Euler scheme is guaranteed to always give qualitatively correct results.