4. $xy'' + 2y' + y = 0$ $\quad y(1) = -10 \quad y(5) = -18$

<u>method of Frobenius</u>: $y = x^{\lambda}(c_0 + c_1 x + c_2 x^2 + c_3 x^3 \ldots)$

$$y = x^{\lambda} \sum_{j=0}^{\infty} c_j x^j = \sum_{j=0}^{\infty} c_j x^{(j+\lambda)}$$

<u>Derivatives</u>: $y' = \sum_{j=0}^{\infty} (j+\lambda) c_j x^{(j+\lambda-1)} = x^{\lambda} \sum_{j=0} c_j (j+\lambda) x^{j-1}$

$$y'' = \sum_{j=0}^{\infty} (j+\lambda-1)(j+\lambda) c_j x^{(j+\lambda-2)}$$

$$= x^{\lambda} \sum_{j=0}^{\infty} c_j (j+\lambda)(j+\lambda-1) x^{j-2}$$

<u>Substitution</u>:

$$y = x\left[\sum_{j=0}^{\infty}(j+\lambda-1)(j+\lambda) c_j x^{(j+\lambda-2)}\right] + 2\left[\sum_{j=0}^{\infty}(j+\lambda) c_j x^{(j+\lambda-1)}\right] + \left[\sum_{j=0}^{\infty} c_j x^{(j+\lambda)}\right]$$

$$y = \sum_{j=0}^{\infty} c_j (j+\lambda-1)(j+\lambda) x^{(j-1)} + 2\sum_{j=0}^{\infty} c_j (j+\lambda) x^{(j-1)} + \sum_{j=0}^{\infty} c_j x^j = 0$$

$$y = \sum_{j=0}^{\infty}\left(\left[c_j (j+\lambda)\right] x^{j-1} (j+\lambda+1)\right) + \sum_{j=0}^{\infty} c_j x^j = 0$$

$k-1 = j-1 \qquad k-1 = j \qquad$; <u>match powers</u>

$$y = \sum_{k=0}^{\infty}\left(c_k (k+\lambda) x^{k-1} (k+\lambda+1)\right) + \sum_{k=1}^{\infty} c_{k-1} x^{k-1}$$

match k param:

$$y = \sum_{k=0}^{\infty}\left(c_k(k+\lambda)x^{k-1}(k+\lambda+1)\right) + \sum_{k=1}^{\infty} c_{k-1}x^{k-1}$$

$$y = \left(c_0(\lambda)x^{-1}(\lambda+1)\right) + \sum_{k=1}^{\infty}\left(c_k(k+\lambda)x^{k-1}(k+\lambda+1) + c_{k-1}x^{k-1}\right) = 0$$

$\lambda^2 + \lambda = 0$

$\lambda = 0, -1$

$$+ \sum_{k=1}^{\infty}\left(c_k(k+\lambda)(k+\lambda+1) + c_{k-1}\right)x^{k-1}$$

$$c_k = \frac{-c_{k-1}}{(k+\lambda)(k+\lambda+1)}$$

for $\lambda = 0$ $\quad c_k = \dfrac{-c_{k-1}}{k(k+1)}$

recursion relation: $\quad c_k = \dfrac{-1}{(k+\lambda)(k+\lambda+1)}\, c_{k-1}$

for $\lambda = 0$

$$c_k = \frac{-1}{(k)(k+1)}\, c_{k-1}$$

$$c_1 = \left(-1/2\right)c_0$$

$$c_2 = \left(-1/6\right)c_1$$

$$\therefore \frac{c_k}{c_{k-1}} = f_k = \frac{-1}{k(k+1)}$$

for $\lambda = -1$ $\quad c_k = \dfrac{-1}{k(k+1)} c_{k-1}$

$$c_1 = \frac{-1}{(-1)(0)+0} c_0 = 0 \cdot c_0$$

$$y_2 = x^{-1}$$

$$xy'' + 2y' + y = 0 \longleftarrow\!\!\longrightarrow \left\{ \begin{array}{l} y = \frac{1}{x} \\ y' = -1x^{-2} \\ y'' = 2x^{-3} \end{array} \right\}$$

$$x\left(\frac{2}{x^3}\right) + 2\left(\frac{-1}{x^2}\right) + \frac{1}{x} \neq 0$$

because $\lambda_1 = \lambda_2 + n$ (integer) $\quad$ (★ Need to find another soln.)

$$y_1 = x^0\left[c_0 + c_1 x + c_2 x^2 + \dots\right]$$

$$y_2 = y_1 \log(x) + x^{\lambda_2} \sum_{j=0} d_j x^j \longrightarrow xy'' + 2y' + y = 0$$

$$y = y_1 \log(x) + \sum_{j=0}^{\infty} d_j x^j$$

$$y' = y' \log(x) + \frac{y}{x} + \sum_{j=0}^{\infty} d_j (j) x^{j-1}$$

$$y'' = y'' \log(x) + \frac{2y'}{x} - \frac{y_1}{x^2} + \sum_{j=0}^{\infty} d_j (j)(j-1) x^{j-2}$$

substitution: (derivatives)

$xy'' + 2y' + y$

$$x\left[y_1''\log(x) + \frac{2y_1'}{x} - \frac{y_1}{x^2} + \sum_{j=0}^{\infty} d_j(j)(j-1)x^{j-2}\right] + 2\left[y_1'\log(x) + \frac{y_1}{x} + \sum_{j=0}^{\infty} d_j(j)x^{j-1}\right] + \left[y_1\log(x) + \sum_{j=0}^{\infty} d_0 x^j\right] = 0$$

Simplifications: $\left(xy_1''\log(x) + 2y_1' - \frac{y_1}{x}\right) + \left(2y_1'\log(x) + \frac{y_1}{x}\right) + \left(y_1\log(x)\right)$

$$\cancel{\log(x)\left[xy_1'' + 2y_1' + y_1\right]} + 2y_1' \cancel{- \frac{y_1}{x} + \frac{y_1}{x}}$$

and $2y_1' = 2\sum_{j=0}^{\infty} c_j(j)x^{j-1}$

$$2\sum_{j=0}^{\infty} c_j(j)x^{j-1} + \sum_{j=0}^{\infty} d_j(j^2 - j)x^{j-1} + 2\sum_{j=0}^{\infty} d_j(j)x^{j-1} + \sum_{j=0}^{\infty} d_0 x^j = 0$$

$$2\sum_{j=0}^{\infty}\left[c_j(j) + d_j(j)\right]x^{j-1} + \sum_{j=0}^{\infty}\left[d_j(j^2 - j)\right]x^{j-1} + \sum_{j=0}^{\infty} d_0 x^j = 0$$

$$\sum_{j=0}^{\infty}\left[2c_j(j) + d_j(j) + d_j(j^2)\right]x^{j-1} + \sum_{j=0}^{\infty} d_0 x^j = 0$$

$k = j$ $\quad$ $k-1 = j$

$$\sum_{k=0}^{\infty}\left[2c_k(k) + d_k(k) + d_k(k^2)\right]x^{k-1} + \sum_{k=1} d_{k-1}x^{k-1} = 0$$

$$\sum_{k=1}^{\infty}\left[2c_k(k) + d_k(k^2 + k) + d_{k-1}\right]x^{k-1} = 0$$

$$d_k = \frac{-d_{k-1} - 2c_k(k)}{k^2 + k} \longrightarrow y_2 = y_1\log(x) + \sum_{0} d_k x^k$$

$$y_1 = x^0\left[c_0 + c_1x + c_2x^2 + \dots\right]$$

$$y_2 = y_1\log(x) + \sum_{k=0}^{\infty} d_k x^k \qquad \text{let} \quad d_k = \frac{-d_{k-1} - 2c_k(k)}{k^2 + k}$$

4. <u>Bessel Trick</u> – $\frac{d}{dx}(x^a y') + bx^c y = 0$

$$xy'' + 2y' + y = 0$$

$$x^2y'' + 2xy' + xy = 0$$ ——— multiply by x

$$\frac{d}{dx}\left(x^2y'\right) = x^2y'' + 2xy'$$ — rearrang

$$\frac{d}{dx}(x^2y') + xy = 0$$ ——— bessel trig gen. form

$$\begin{pmatrix} a=2 \\ b=1 \\ c=1 \end{pmatrix} \quad \nu = \frac{1-a}{c-a+2} = \frac{1-2}{1-2+2} = \frac{-1}{1} = -1$$

$$\alpha = \frac{2}{c-a+2} = \frac{2}{1-2+2} = \frac{2}{1} = 2$$

$$y = x^{\frac{\nu}{\alpha}} Z_{\pm\nu}\left(\alpha\sqrt{b}\, x^{\left(\frac{1}{\alpha}\right)}\right) = \frac{1}{\sqrt{x}}\begin{bmatrix} I_0(2\sqrt{x}) \\ K_0(2\sqrt{x}) \end{bmatrix} = \begin{bmatrix} u_1(x) \\ u_2(x) \end{bmatrix}$$

$$y = A u_1(x) + B u_2(x)$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{bmatrix} u_1(1) & u_2(1) \\ u_2(5) & u_2(5) \end{bmatrix} \Bigg\backslash \begin{bmatrix} -10 \\ -18 \end{bmatrix}$$