4. $xy'' + 2y' + y = 0 \qquad y(1) = -10 \quad y(5) = -18$

• Frobenius: $y = x^{\lambda}\sum_{j=0}^{\infty} c_j x^j = \sum_{j=0}^{\infty} (c_j) x^{(\lambda+j)}$

$$\left[\begin{array}{l} y' = \sum_{j=0}^{\infty} (\lambda+j)(c_j)\, x^{\lambda+j-1} \\ y'' = \sum_{j=0}^{\infty} (c_j)(\lambda+j)(\lambda+j-1) x^{\lambda+j-2} \end{array}\right]$$ (first and second deriv.)

• Solution: (by substitution into orig. ODE)

$$(x)\left[\sum_{j=0}^{\infty} (c_j)(\lambda+j)(\lambda+j-1)x^{\lambda+j-2}\right] + (2)\left[\sum_{j=0}^{\infty} (\lambda+j)(c_j)\, x^{\lambda+j-1}\right] + \left[\sum_{j=0}^{\infty} (c_j) x^{\lambda+j}\right] = 0$$

$$x^{\lambda}\left[\sum_{j=0}^{\infty} (c_j)(\lambda+j)(\lambda+j-1)x^{j-1} + \sum_{j=0}^{\infty} (\lambda+j)(c_j)2x^{j-1} + \sum_{j=0}^{\infty} (c_j)x^{j}\right] = 0$$

(get powers to match) $k=j$ $\qquad k=j \qquad k=j+1$

$$\sum_{k=0}^{\infty} (c_k)(\lambda+k)(\lambda+k-1)x^{k-1} + \sum_{k=0}^{\infty} (\lambda+k)(c_k)2x^{k-1} + \sum_{k=1}^{\infty} (c_{k-1})x^{k-1}$$

$$c_k \sum_{k=0}^{\infty} (\lambda+k)(\lambda+k-1)x^{k-1} + (\lambda+k)2x^{k-1} + \sum_{k=1}^{\infty} (c_{k-1})x^{k-1}$$

(evaluate first two groups at $k=0$ to get matching indices

$$c_0\underbrace{\left[\lambda(\lambda-1)x^{-1} + 2\lambda x^{-1}\right]}_{=0} + \sum_{k=1}^{\infty}\left[c_k(\lambda+k)(\lambda+k-1) + (\lambda+k)(2) + (c_{k-1})\right]x^{k-1} = 0$$

indicial eqn for λ $\qquad$ recursion relation for c_j

$\lambda^2 + \lambda = 0 \rightarrow \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$ $\qquad \lambda_1 = 0$

$\lambda_2 = -1$

given $y = x^{\lambda}\left[\sum_{j=0}^{\infty} c_j x^j\right]$

$$c_k = \frac{-c_{k-1}}{(\lambda+k)(\lambda+k-1)(2\lambda+2k}$$

$$y_1 = x^0\left[c_0 + c_1 x + c_2 x^2 + \ldots\right]$$

$$y_2 = x^{-1}\left[d_0 + d_1 x + d_2 x^2 + \ldots\right]$$

$$c_k = \frac{-c_{k-1}}{(k)(k-1)(2k)}$$

$$c_k = \frac{-c_{k-1}}{(k-1)(k-2)(2k-2)}$$