4. $xy'' + 2y' + y = 0$ $\qquad y(1) = -10 \qquad y(5) = -18$

- Frobenius: $y = x^{\lambda}\sum_{j=0}^{\infty} c_j x^j = \sum_{j=0}^{\infty} (c_j) x^{(\lambda+j)}$

$$\left[\begin{aligned} y' &= \sum_{j=0}^{\infty} (\lambda+j)(c_j)\, x^{\lambda+j-1} \\ y'' &= \sum_{j=0}^{\infty} (c_j)(\lambda+j)(\lambda+j-1) x^{\lambda+j-2} \end{aligned}\right]$$ (first and second deriv.)

- Solution: (by substitution into orig. ODE)

$$(x)\left[\sum_{j=0}^{\infty}(c_j)(\lambda+j)(\lambda+j-1)x^{\lambda+j-2}\right] + (2)\left[\sum_{j=0}^{\infty}(\lambda+j)(c_j)\,x^{\lambda+j-1}\right] + \left[\sum_{j=0}^{\infty}(c_j)x^{\lambda+j}\right] = 0$$

$$\cancel{x^{\lambda}}\left[\sum_{j=0}^{\infty}(c_j)(\lambda+j)(\lambda+j-1)x^{j-1} + \sum_{j=0}^{\infty}(\lambda+j)(c_j)2x^{j-1} + \sum_{j=0}^{\infty}(c_j)x^{j}\right] = 0$$

(get powers to match) $k=j$ $\qquad k=j$ $\qquad k=j+1$

$$\sum_{k=0}^{\infty}(c_k)(\lambda+k)(\lambda+k-1)x^{k-1} + \sum_{k=0}^{\infty}(\lambda+k)(c_k)2x^{k-1} + \sum_{k=1}^{\infty}(c_{k-1})x^{k-1}$$

$$c_k \sum_{k=0}^{\infty}(\lambda+k)(\lambda+k-1)x^{k-1} + (\lambda+k)2x^{k-1} + \sum_{k=1}^{\infty}(c_{k-1})x^{k-1}$$

(evaluate first two groups at $k=0$ to get matching indices

$$c_0\left[\underbrace{\lambda(\lambda-1)x^{-1} + 2\lambda x^{-1}}_{=0}\right] + \sum_{k=1}^{\infty}\left[c_k(\lambda+k)(\lambda+k-1) + (\lambda+k)(2) + (c_{k-1})\right]x^{k-1} = 0$$

indicial eqn for λ $\qquad$ recursion relation for c_j

$$\lambda^2 + \lambda = 0 \longrightarrow \frac{-b \pm \sqrt{b^2-4ac}}{2a} \qquad \lambda_1 = 0, \quad \lambda_2 = -1$$

given $y = x^{\lambda}\left[\sum_{j=0}^{\infty} c_j x^j\right]$

$$c_k = \frac{-c_{k-1}}{(\lambda+k)(\lambda+k-1)(2\lambda+2k}$$

$$y_1 = x^0\left[c_0 + c_1 x + c_2 x^2 + \ldots\right]$$

$$y_2 = x^{-1}\left[d_0 + d_1 x + d_2 x^2 + \ldots\right]$$

$$c_k = \frac{-c_{k-1}}{(k)(k-1)(2k)}$$

$$c_k = \frac{-c_{k-1}}{(k-1)(k-2)(2k-2)}$$

- · -2

4. <u>Bessel Trick</u> – $\frac{d}{dx}(x^a y') + bx^c y = 0$

$$xy'' + 2y' + y = 0$$

$$x^2y'' + 2xy' + xy = 0$$ —— multiply by x

$$\frac{d}{dx}\left(x^2y'\right) = x^2y'' + 2xy'$$ — rearrang

$$\frac{d}{dx}\left(x^2y'\right) + xy = 0$$ —— bessel trig gen. form

$$\begin{pmatrix} a=2 \\ b=1 \\ c=1 \end{pmatrix} \quad \nu = \frac{1-a}{c-a+2} = \frac{1-2}{1-2+2} = \frac{-1}{1} = -1$$

$$\alpha = \frac{2}{c-a+2} = \frac{2}{1-2+2} = \frac{2}{1} = 2$$

$$y = x^{\frac{\nu}{\alpha}} Z_{\nu}\left(\alpha\sqrt{b}\, x^{\left(\frac{1}{\alpha}\right)}\right) = \frac{1}{\sqrt{x}}\begin{bmatrix} I_0(2\sqrt{x}) \\ K_0(2\sqrt{x}) \end{bmatrix} = \begin{bmatrix} u_1(x) \\ u_2(x) \end{bmatrix}$$

$$y = Au_1(x) + Bu_2(x)$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{bmatrix} u_1(1) & u_2(1) \\ u_2(5) & u_2(5) \end{bmatrix} \Big\backslash \begin{bmatrix} -10 \\ -18 \end{bmatrix}$$