3.1.a

$$Z''(x) + \alpha^2 Z(x) = Q(x)$$

$$Z''(x) + 2Z(x) = Q(x)$$

$Z(0) = 9$

$Z'(4) = 8$

$\alpha^2 = 2$

finite difference method ~ discretize domain on $[0,4]$

$\Delta x = \frac{4}{N}$

($Q(x)$ is piece wise and will require conditional statements to solve)

$$Z''(i) \approx \frac{Z(i-1) - 2Z(i) + Z(i+1)}{\Delta x^2}$$

$$\frac{Z(i-1) - 2Z(i) + Z(i+1)}{\Delta x^2} + \alpha^2 Z(i) = Q(i)$$

$$Z(i-1) - (2 + \alpha^2 \Delta x^2) Z(i) + Z(i+1) = -Q(i)\Delta x^2$$

$Z(0) = A$

$Z'(4) = B \longrightarrow Z'(N) \approx \dfrac{Z(N) - Z(N-1)}{\Delta x} = B$

Coeff Matrix

$M_{i,i-1} = 1$

$M_{i,i} = -(2 + \alpha^2 \Delta x^2)$

$M_{i,i+1} = 1$

$M \cdot Z = b \rightsquigarrow$ linalg solve

3.1.b

$$Z'' + 2Z = Q(x) \qquad Z(0) = 9 = A \qquad Z'(4) = 8 = B$$

from problem #1 : $\boxed{y_n = \sin(\lambda_n x) \qquad \lambda_n = \frac{(2N-1)\pi}{8}}$

$$\hat{N}_n = \int_0^4 y_n^2 \, w(x) \qquad \hat{Q}_n = \int_0^4 Q(x) y_n \qquad Q(x) = \sum \frac{\hat{Q}_n y_n}{\hat{N}_n}$$

$$y_n = \sin(\lambda_n x)$$

$$y'_n = \lambda_n \cos(\lambda_n x)$$

$$y''_n = -\lambda_n^2 \sin(\lambda_n x) = -\lambda_n^2 y_n \qquad \text{with } y_n = 0 = y_n'(4) = 0$$

$$\int_0^4 y_n (Z'' + 2Z) = \int_0^4 Q(x) y_n = \hat{Q}_n$$

$$y_n \cdot Z' \Big|_0^4 - y_n' \cdot Z \Big|_0^4 + \int_0^4 y_n'' \cdot Z + 2\hat{Z}_n = \hat{Q}_n$$

$$8 y_n(4) + 9 y'_n(0) - \lambda_n^2 \hat{Z}_n + 2\hat{Z}_n = \hat{Q}_n$$

$$\boxed{\hat{Z}_n = \frac{\hat{Q}_n}{\alpha^2 - \lambda_n^2}} = \frac{\hat{Q}_n}{2 - \lambda_n^2}$$

$$Z(x) = \sum \frac{\hat{Z}_n y_n}{\hat{N}_n}$$

3.1.c

$$Z''(x) + \alpha^2 Z(x) = Q(x)$$

$$Z_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x)$$

$$W(y_1, y_2)(x) = y_1(x) y_2'(x) - y_1'(x) y_2(x)$$

$$u_1'(x) = \frac{-Q(x) y_2(x)}{W(y_1, y_2)(x)}$$

$$u_2'(x) = \frac{Q(x) y_1(x)}{W(y_1, y_2)(x)}$$

$$Z(x) = C_1 y_1(x) + C_2 y_2(x) + Z_p(x)$$