EngMath - ps_1

A. Q. Snyder

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1 HW template

For more details, the entire template and its supplementary materials can be found on GitHub: here. Under the ps_1 directory exist the necessary files for this LaTeX generation, the compiled plots, MATLAB exports, Python exports, the virtual Jupyter kernels used to prototype Python code, and all MATLAB files.

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2 Hand Calcs

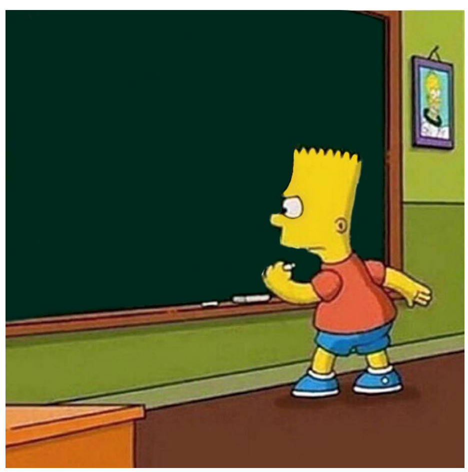


Figure 1: Hand Calculations

1.

$$\boxed{m\left(\frac{dv}{dt}\right) = mg - c(v)^2 \ ; \quad v(0) = 0} \sim \frac{dv}{dt} = 0 \ @ \ V_s$$

$$\frac{dv}{dt} = g - \frac{c(v)^2}{m}$$

Variables: v, t
parameters: g, c, m

$$\tilde{v} = \frac{v - \cancel{v_r}}{v_s} \qquad \tilde{t} = \frac{t - \cancel{t_r}}{t_s}$$

$\_{r}$ = reference value
$\_{s}$ = scaling factor

$v(0) = 0 \rightarrow v_r, t_r = 0$

$$\frac{dv}{dt} = \frac{d(\tilde{v} \cdot v_s)}{d(\tilde{t} \cdot t_s)} \longrightarrow \frac{v_s}{t_s} \frac{d\tilde{v}}{d\tilde{t}}$$

$$m \frac{v_s}{t_s} \frac{d\tilde{v}}{d\tilde{t}} = mg - c v_s^2 \tilde{v}^2$$

non-dim

$$\left[m \frac{v_s}{t_s} = mg \qquad t_s = \frac{v_s}{g} \right]$$

$$m \frac{v_s}{\frac{v_s}{g}} \frac{d\tilde{v}}{d\tilde{t}} = mg - c v_s^2 \tilde{v}^2$$

— divide by mg

$$\frac{d\tilde{v}}{d\tilde{t}} = 1 - \frac{c}{mg} v_s^2 \tilde{v}^2$$

— choose v_c to be terminal velocity

$$\frac{dv}{dt} = 0 = g - \frac{cv^2}{m}$$

$$v_c^2 = \frac{mg}{c}$$

$$\frac{d\tilde{v}}{d\tilde{t}} = 1 - \tilde{v}^2$$

- plot with Runge-kutta
- set $v = 0.95$
- check $v = 1 \rightarrow \frac{dv}{dt}$ should $= 0$

2. $y'' + 9y = 0$ $\quad y(5) + 14y'(5) = 10 \quad y(1) = 0$ $\quad K = 9$, $I = 14$

$$y = \sum_{n=0}^{\infty} a_n x^n \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 9 \sum_{n=0}^{\infty} a_n x^n = 0$$

$p = n-2$, $n = p+2$ $\qquad$ $p = n$, $n = p$

$$\sum_{p=0}^{\infty} (p+2)(p+1) a_{p+2} x^p + 9 \sum_{p=0}^{\infty} a_p x^p = 0$$

$$\sum_{p=0}^{\infty} \left[(p+2)(p+1) a_{p+2} + 9 a_p \right] x^p = 0 \implies p = 0, 1, 2, 3 \ldots$$

$$a_{p+2} = \frac{-9 a_p}{(p+2)(p+1)}$$ recurrence

code
$\left(\begin{array}{l} idx = p+2 \\ p = idx - 2 \end{array} \right)$

$a_2 = \dfrac{-9a_0}{2} = 0 \quad p = 0$

$a_3 = \dfrac{-9a_1}{6} = \dfrac{-9a_1}{6} \quad p = 1$

$a_4 = \dfrac{-9a_2}{12} = 0 \quad p = 2$

$a_5 = \dfrac{-9a_3}{20} \quad p = 3$

$a_6 = 0 \quad p = 4$

$a_7 = \dfrac{-9a_5}{42} \quad p = 5$

3. $x^2y'' + xy' - y = 9x^2 + 41x \qquad y(1)=0 \quad y(5)=14$

find two linearly independent soln. $(y_1 \& y_2)$ for the homogenous part

$$x^2y'' + xy' - y = 0$$

Cauchy-Euler equation!!

$\hookrightarrow$ soln. $y(x) = x^m$

$$\Downarrow$$

$$m(m-1) + m - 1 = 0$$

roots: $m_1 = 1 \quad m_2 = -1 \longrightarrow$ linearly independent solutions $\begin{bmatrix} y_1(x) = x \\ y_2(x) = 1/x \end{bmatrix}$

Wronskian: $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$$W(y_1, y_2) = \begin{vmatrix} x & 1/x \\ 1 & -1/x^2 \end{vmatrix} = -2$$

particular soln: $y_p(x) = y_1(x)\int \frac{y_2(x)(kx^2+4x)}{W(y_1,y_2)}dx - y_2(x)\int \frac{y_1(x)(kx^2+4x)}{W(y_1,y_2)}dx$

expanding:

$$u_1(x) = \int \frac{1/x\,(kx^2+4x)}{-2}\,dx$$

$$u_2(x) = -\int \frac{x(kx^2+4x)}{-2}\,dx$$

$$y_p(x) = x \cdot u_1(x) + \frac{1}{x} \cdot u_2(x)$$

complete solution with Simpson integration

4. $xy'' + 2y' + y = 0 \qquad y(1) = -10 \qquad y(5) = -18$

- Frobenius: $y = x^{\lambda}\sum_{j=0}^{\infty} c_j x^j = \sum_{j=0}^{\infty} (c_j) x^{(\lambda+j)}$

$$\left[\begin{aligned} y' &= \sum_{j=0}^{\infty} (\lambda+j)(c_j)\, x^{\lambda+j-1} \\ y'' &= \sum_{j=0}^{\infty} (c_j)(\lambda+j)(\lambda+j-1) x^{\lambda+j-2} \end{aligned}\right]$$ (first and second deriv.)

- Solution: (by substitution into orig. ODE)

$$(x)\left[\sum_{j=0}^{\infty}(c_j)(\lambda+j)(\lambda+j-1)x^{\lambda+j-2}\right] + (2)\left[\sum_{j=0}^{\infty}(\lambda+j)(c_j)\,x^{\lambda+j-1}\right] + \left[\sum_{j=0}^{\infty}(c_j)x^{\lambda+j}\right] = 0$$

$$\cancel{x^{\lambda}}\left[\sum_{j=0}^{\infty}(c_j)(\lambda+j)(\lambda+j-1)x^{j-1} + \sum_{j=0}^{\infty}(\lambda+j)(c_j)2x^{j-1} + \sum_{j=0}^{\infty}(c_j)x^{j}\right] = 0$$

(get powers to match) $k=j \qquad k=j \qquad k=j+1$

$$\sum_{k=0}^{\infty}(c_k)(\lambda+k)(\lambda+k-1)x^{k-1} + \sum_{k=0}^{\infty}(\lambda+k)(c_k)2x^{k-1} + \sum_{k=1}^{\infty}(c_{k-1})x^{k-1}$$

$$c_k \sum_{k=0}^{\infty}(\lambda+k)(\lambda+k-1)x^{k-1} + (\lambda+k)2x^{k-1} + \sum_{k=1}^{\infty}(c_{k-1})x^{k-1}$$

(evaluate first two groups at $k=0$ to get matching indices

$$c_0\Big[\underbrace{\lambda(\lambda-1)x^{-1} + 2\lambda x^{-1}}_{=0}\Big] + \sum_{k=1}^{\infty}\Big[c_k(\lambda+k)(\lambda+k-1)+(\lambda+k)(2)+(c_{k-1})\Big]x^{k-1} = 0$$

indicial eqn for λ — recursion relation for c_j

$$\lambda^2+\lambda = 0 \longrightarrow \frac{-b\pm\sqrt{b^2-4ac}}{2a} \qquad \lambda_1 = 0 \qquad \lambda_2 = -1$$

given $y = x^{\lambda}\left[\sum_{j=0}^{\infty} c_j x^j\right]$

$$c_k = \frac{-c_{k-1}}{(\lambda+k)(\lambda+k-1)(2\lambda+2k}$$

$$y_1 = x^0\left[c_0 + c_1 x + c_2 x^2 + \dots\right]$$

$$y_2 = x^{-1}\left[d_0 + d_1 x + d_2 x^2 + \dots\right]$$

$$c_k = \frac{-c_{k-1}}{(k)(k-1)(2k)}$$

$$c_k = \frac{-c_{k-1}}{(k-1)(k-2)(2k-2)}$$

3 Python Outputs

Figure 2: Python Logo

1. **[10 pt]** If a mass is dropped from a very large height the governing equation of motion for the (downward) velocity V is:

$$\boxed{m\ \frac{dV}{dt} = m\ g - c\,(V)^2\,; \quad V\,(0) = 0}$$

Here, c is the coefficient of drag in air.

(a) Put this governing equation in dimensionless form. (Hint, think about the terminal velocity V_T.)
(b) Solve the problem in its dimensionless form.
(c) Determine when the mass achieves 95% of its terminal velocity

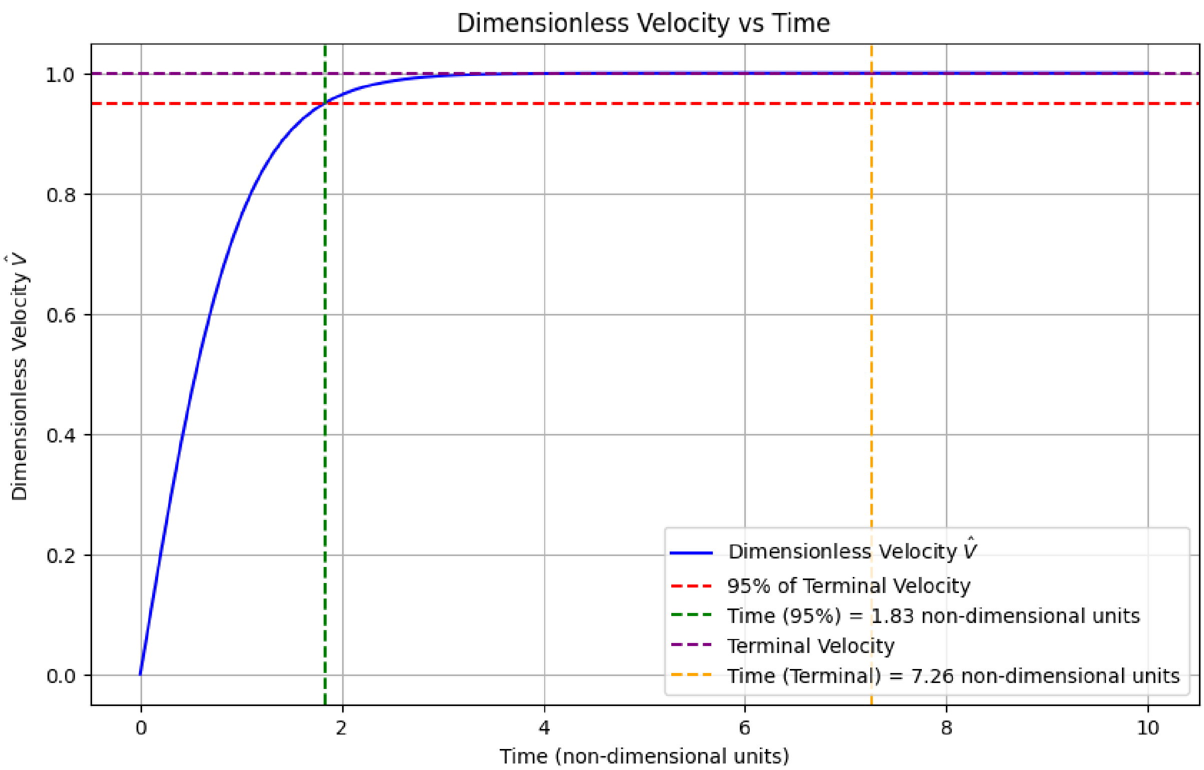
In [ ]:
```
import numpy as np
import matplotlib.pyplot as plt
```

In [ ]:
```
def model(v):
    dvdt = 1 - v**2
    return dvdt

def rk4(v, t, dt):
    k1 = dt * model(v)
    k2 = dt * model(v + 0.5 * k1)
    k3 = dt * model(v + 0.5 * k2)
    k4 = dt * model(v + k3)

    v_new = v + (k1 + 2*k2 + 2*k3 + k4) / 6.0
    return v_new

t = np.linspace(0, 10, 5000)
dt = t[1] - t[0]

v = np.zeros_like(t)
v[0] = 0

for i in range(1, len(t)):
    v[i] = rk4(v[i-1], t[i-1], dt)

# Handle for v=0.95
indices_95 = np.where(v >= 0.95)[0]
if indices_95.size > 0:
    idx_95 = indices_95[0]
    time_95 = t[idx_95]
    print(f"Velocity reaches 95% at {time_95:.2f} non-dimensional units.")
else:
    idx_95 = None
    time_95 = None

# Handle for v=1 (terminal velocity)
indices_1 = np.where(v >= 0.999999)[0]
if indices_1.size > 0:
    idx_1 = indices_1[0]
    time_1 = t[idx_1]
    print(f"Velocity reaches terminal velocity at {time_1:.2f} non-dimensional unit
else:
```

```
    idx_1 = None
    time_1 = None
```

```
Velocity reaches 95% at 1.83 non-dimensional units.
Velocity reaches terminal velocity at 7.26 non-dimensional units.
```

In [ ]:
```
plt.figure(figsize=(10, 6))
plt.plot(t, v, label=r"Dimensionless Velocity $\hat{V}$", color='blue')
if time_95:
    plt.axhline(0.95, color='red', linestyle='--', label="95% of Terminal Velocity"
    plt.axvline(time_95, color='green', linestyle='--', label=f"Time (95%) = {time_
if time_1:
    plt.axhline(1.0, color='purple', linestyle='--', label="Terminal Velocity")
    plt.axvline(time_1, color='orange', linestyle='--', label=f"Time (Terminal) = {
plt.xlabel('Time (non-dimensional units)')
plt.ylabel('Dimensionless Velocity $\hat{V}$')
plt.title('Dimensionless Velocity vs Time')
plt.legend()
plt.grid(True)
plt.show()
```



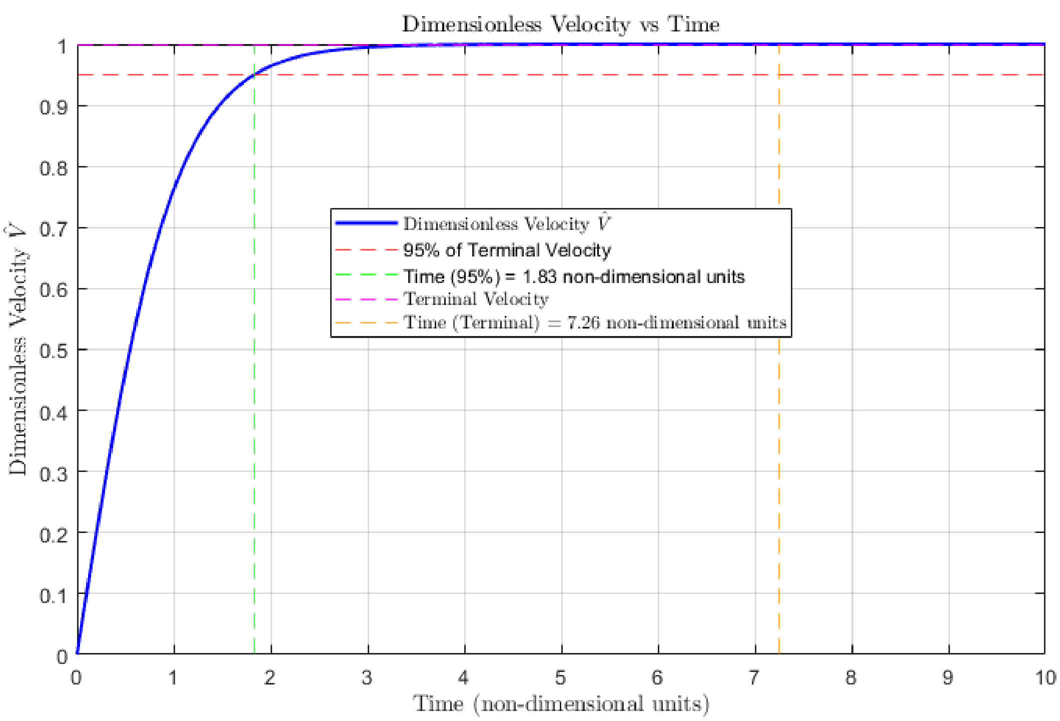
4 MATLAB Outputs

Figure 3: MATLAB Logo

```
function main
    t = linspace(0, 10, 5000);
    dt = t(2) - t(1);

    v = zeros(size(t));
    v(1) = 0;

    for i = 2:length(t)
        v(i) = rk4(v(i-1), t(i-1), dt);
    end

    % Handle for v=0.95
    idx_95 = find(v >= 0.95, 1, 'first');
    if ~isempty(idx_95)
        time_95 = t(idx_95);
        fprintf('Velocity reaches 95%% at %.2f non-dimensional units.\n',
 time_95);
    else
        idx_95 = [];
        time_95 = [];
    end

    % Handle for v=1 (terminal velocity)
    idx_1 = find(v >= 0.999999, 1, 'first');
    if ~isempty(idx_1)
        time_1 = t(idx_1);
        fprintf('Velocity reaches terminal velocity at %.2f non-dimensional
 units.\n', time_1);
    else
        idx_1 = [];
        time_1 = [];
    end

    figure('Position', [100, 100, 800, 480]);
    plot(t, v, 'b-', 'LineWidth', 1.5, 'DisplayName', 'Dimensionless Velocity
 $\hat{V}$');
    xlabel('Time (non-dimensional units)', 'Interpreter', 'latex');
    ylabel('Dimensionless Velocity $\hat{V}$', 'Interpreter', 'latex');
    title('Dimensionless Velocity vs Time', 'Interpreter', 'latex');
    grid on;
    hold on;

    if ~isempty(time_95)
        yline(0.95, 'r--', 'DisplayName', '95% of Terminal Velocity');
        xline(time_95, 'g--', 'DisplayName', sprintf('Time (95%%) = %.2f non-
dimensional units', time_95));
    end

    if ~isempty(time_1)
        yline(1.0, 'm--', 'DisplayName', 'Terminal Velocity');
        xline(time_1, 'Color', [1 0.6 0], 'LineStyle', '--', 'DisplayName',
 sprintf('Time (Terminal) = %.2f non-dimensional units', time_1));
```

```
    end

    legend('show', 'Location', 'best', 'Interpreter', 'latex');
    hold off;
end

function dvdt = model(v)
    dvdt = 1 - v.^2;
end

function v_new = rk4(v, t, dt)
    k1 = dt * model(v);
    k2 = dt * model(v + 0.5 * k1);
    k3 = dt * model(v + 0.5 * k2);
    k4 = dt * model(v + k3);

    v_new = v + (k1 + 2*k2 + 2*k3 + k4) / 6.0;
end
```

```
Velocity reaches 95% at 1.83 non-dimensional units.
Velocity reaches terminal velocity at 7.26 non-dimensional units.
```



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5 Compiled Plots