Engr 5011: Homework #2

Sturm-Liouville theory and orthogonal-function decomposition in cartesian coordinates

Please scan the entire homework submission as a single pdf , with this page first, and the problems attached in order.

Name (print): ____________________ **TUID:**

I	**H**	**G**	**F**	**E**	**D**	**C**	**B**	**A**

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1. [**3 pt**] Problem 1 score/comments:
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2. [**7 pt**] Problem 2 score/comments
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3. [**15 pt**] Problem 3 score/comments.
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4. [**25 pt**] Problem 4 score/comments.
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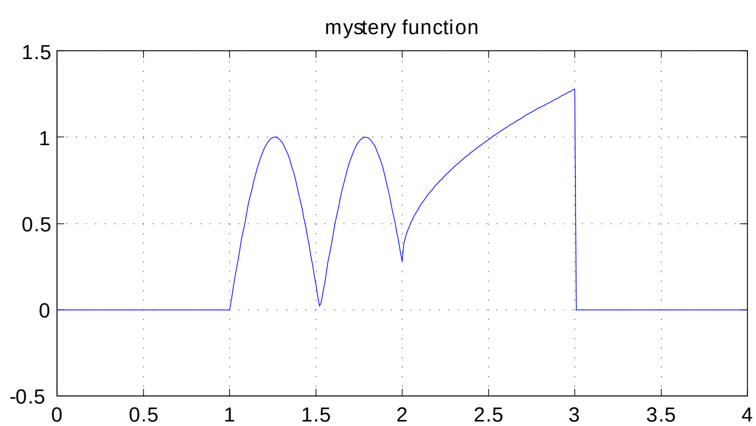
   ...

The following function $Q(x)$ will be used in problems 1-3:

$$Q(x) = \left\{ \begin{array}{ll} 0 & 0 \le x \le 1 \\ |\sin(6(x-1))| & 1 \le x \le 2 \\ \sqrt{x-2} - \sin(6) & 2 \le x \le 3 \\ 0 & 3 \le x \le 4 \end{array} \right\}$$



All integrals in for problems 1-3 should be carried out numerically. (Use Simpson's rule. The matlab built-in `integral` function won't be "happy" with these weird discontinuities in the integrand.)

Engr-5011: Homework #2 Problem #1

Assemble submission for this problem in the following order:

(1) this page on top, followed by
(2) your handwritten solution, followed by
(3) listing of your matlab scripts/functions
(4) plots

Name (print): ____________________

1. Represent $Q(x)$ as a series expansion using the eigenfunctions of:

$$\boxed{y'' + \lambda^2 y = 0; \quad y(0) = y'(4) = 0}$$

2. Summarize here:

   (a) What are the eigenfunctions $y_n(x)$ and the eigenvalues λ_n?

   (b) What is the weighting function $w(x)$?

   (c) What is the expression for the normalizing constant N_n and the expansion of $Q(x)$ as a series in the eigenfunctions $y_n(x)$?

3. Plot $Q(x)$ and the eigenfunction series expansion of $Q(x)$ in $y_n(x)$ for 5,20, and 40 terms.

Engr-5011: Homework #2 Problem #2

Assemble submission for this problem in the following order:
(1) this page on top, followed by
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(4) plots

Name (print): ______________________________

1. Represent $Q(x)$ as a series expansion using the eigenfunctions of:

$$\boxed{xy'' + 3y' + \lambda^2 xy = 0;\ y(0) \text{ finite},\quad y'(4) + 3y(4) = 0}$$

2. Summarize here:

   (a) What are the eigenfunctions $y_n(x)$ and the eigenvalues λ_n?

   (b) What is the weighting function $w(x)$?

   (c) What is the expression for the normalizing constant N_n and the expansion of $Q(x)$ as a series in the eigenfunctions $y_n(x)$?

3. Plot $Q(x)$ and the eigenfunction series expansion of $Q(x)$ in $y_n(x)$ for 5,10,20, and 40 terms.

"Bessel trick." If: $\boxed{\dfrac{d}{dx}\left(x^a y'\right) + bx^c y = 0}$ then the solution is of form:

$$\begin{array}{|ccl|}
\hline
y & = & x^{\left(\frac{\nu}{\alpha}\right)} Z_\nu\left(\alpha\sqrt{b}x^{\left(\frac{1}{\alpha}\right)}\right) \\
\hline
Z & = & \text{"appropriate Bessel function"} \\
\nu & = & \left(\dfrac{1-a}{c-a+2}\right) \\
\alpha & = & \left(\dfrac{2}{c-a+2}\right) \\
\hline
\end{array}$$

"Bessel function derivative"

$$\begin{array}{|ccl|}
\hline
\dfrac{d\ J_0(x)}{dx} & = & -J_1(x) \\
\dfrac{d\ J_m(x)}{dx} & = & \frac{1}{2}\left(J_{m-1}(x) - J_{m+1}(x)\right) \\
\hline
\end{array}$$

Engr-5011: Homework #2   Problem #3

Assemble submission for this problem in the following order:
(1) this page on top, followed by
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(4) plots

Name (print): ______________________________

1. Suppose $Z(x)$ is a function to be found and its governing equation is:

$$\boxed{Z''(x) + \alpha^2 Z(x) = Q(x); \quad Z(0) = \text{A}, \quad Z'(4) = \text{B}; \qquad \alpha^2 = \max(\,2,\text{C})}$$

   Here, the values of A,B,and C are taken from your TUID.

   Derive the solution method and plot (on $0 \le x \le 4$) three ways:

   (a) Using the two-point finite-difference solution method;
   (b) Using eigenfunction expansion (using the eigens from problem 1);
   (c) Using the Wronskian exact solution (the "2^{nd}-order recipe") and integrating the integrals numerically using Simpsons rule.

2. Of course you will plot all three solutions on the same plot for comparison!

Engr-5013: Homework #2 Problem #4

Assemble submission for this problem in the following order:
(1) this page on top, followed by
(2) your handwritten solution, followed by
(3) listing of your matlab scripts/functions
(4) plots

Name (print): ______________________________

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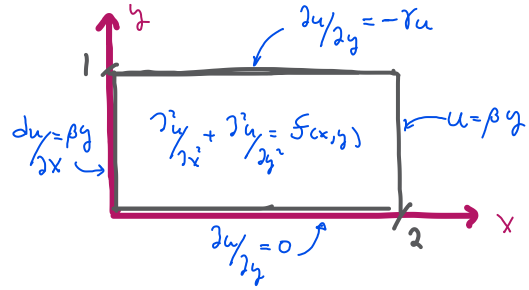
1. The schematic on the right represents:

$$\begin{array}{|rcl|}\hline \nabla^2 u &=& f(x,y) = \left\{ \begin{array}{ll} 0 & \text{if } x < 1 \\ 40\sin(\alpha\, y)(1-x) & \text{if } x > 1 \end{array} \right\} \\ u_y(y=0) &=& 0 \\ u_x(x=0) &=& \beta\, y \\ u\ (x=2) &=& \beta\, y \\ u_y(y=1) &=& -\,\gamma\, u \\ \hline \end{array}$$



Here, $\alpha = \max(\mathrm{A}, 2)$, $\beta = \max(\mathrm{B}, 2)$, and $\gamma = \max(\mathrm{G}, 4)$, where A,B, and G are taken from your TUID.

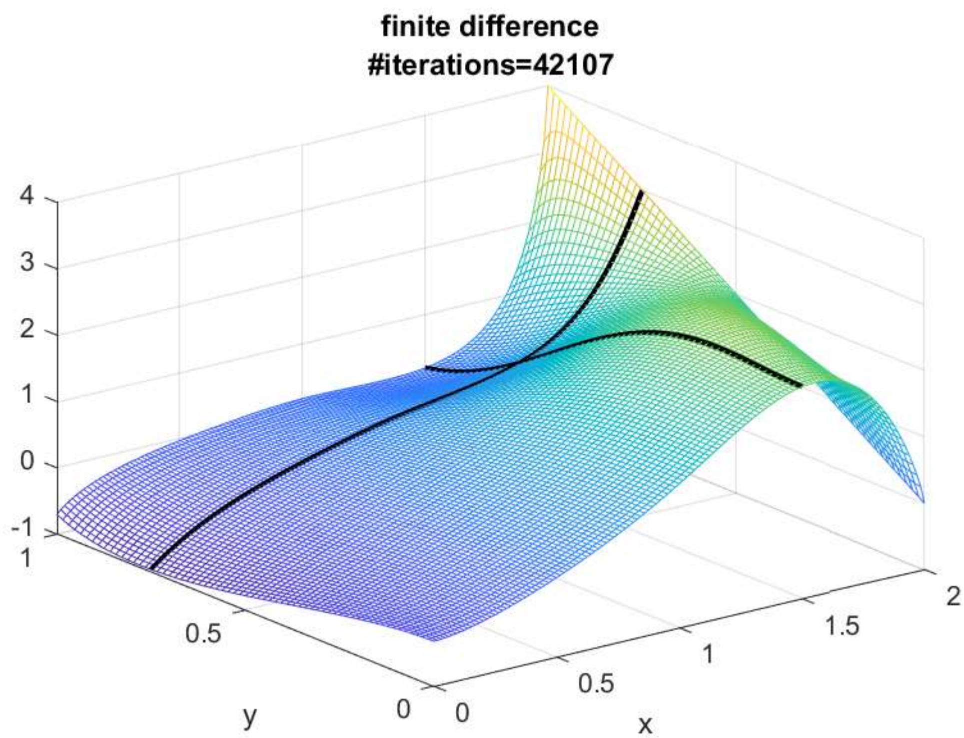
(a) Solve this using a finite-difference method and plot the surface using `mesh()`.

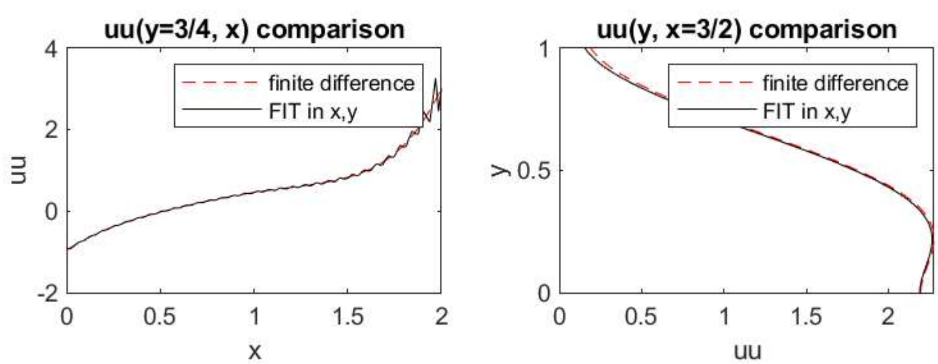
(b) Solve this using the finite-integral transform method in both x and y and plot the surface using `mesh()`. Report here:

i. The Sturm-Liouville problem, the eigenfunctions and eigenvalues, and the transform definition/properties in the x coordinate.

ii. The Sturm-Liouville problem, the eigenfunctions and eigenvalues, and the transform definition/properties in the y coordinate.

(c) From both solutions, plot $u(y)$ on $x = \frac{3}{2}$ and $u(x)$ on $y = \frac{3}{4}$ to compare the results.
See next page for what I have in mind.


finite difference
#iterations=42107
4
3
2
1
0
-1
1
0.5
y
0
0
0.5
1
1.5
2
x


uu(y=3/4, x) comparison
finite difference
FIT in x,y
uu
x
uu(y, x=3/2) comparison
finite difference
FIT in x,y
y
uu