4.1

$$\boxed{\nabla^2 u = f(x,y) = \begin{cases} 0 & \text{if } x<1 \\ 40\sin(\alpha y)(1-x) & \text{if } x>1 \end{cases}}$$

$u_y(y=0) = 0$

$u_x(x=0) = \beta y$

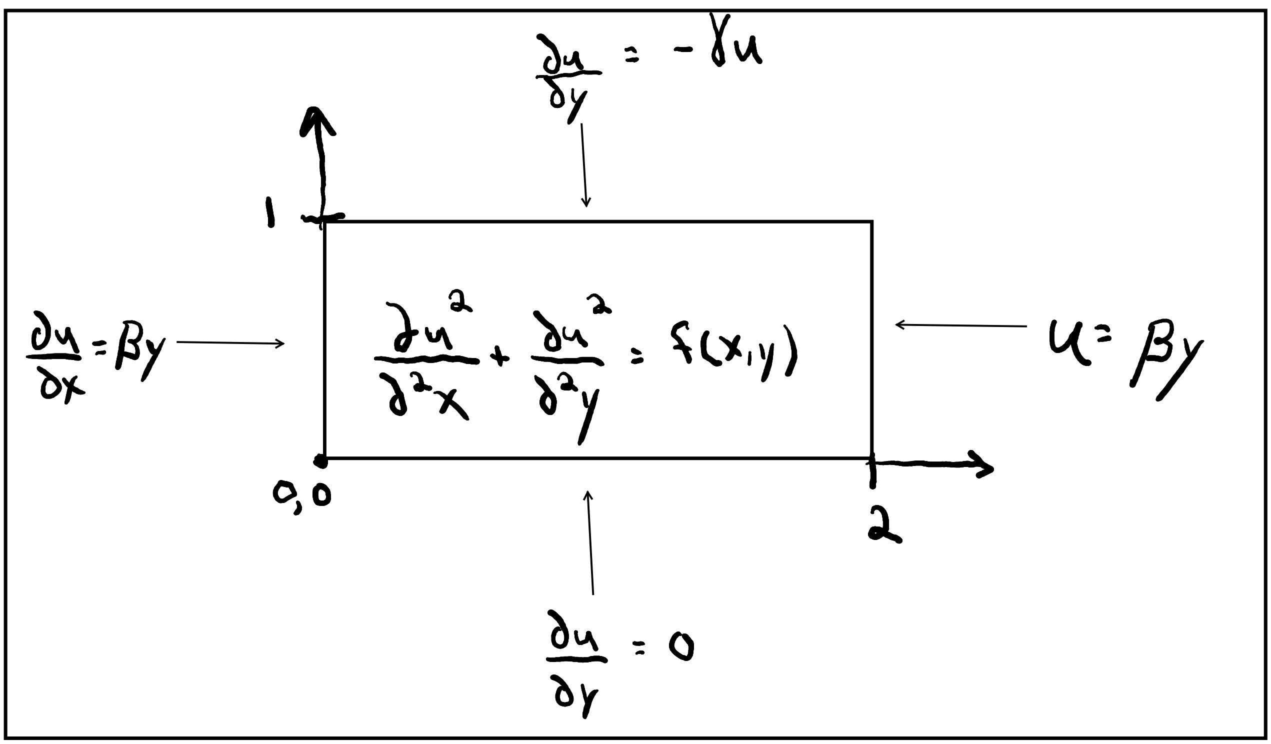
$u_x(x=2) = \beta y$

$u_y(y=1) = -\gamma u$

$\alpha = 9$

$\beta = 8$

$\gamma = 5$

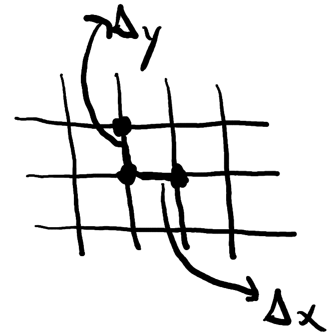


4.a **finite difference method**

① discretize the domain ; $x \; 0 \to 2$

$y \; 0 \to 1$

let $\Delta x = \frac{2}{N-1}$, $\Delta y = \frac{1}{N-1}$



$u_y(y=0) = 0$

$u_x(x=0) = \beta y$

$u_x(x=2) = \beta y$

$\frac{\partial u}{\partial y}(y=1) = -\gamma u$

Stopping condition: $\Delta u < 10^{-5}$

using cent. Diff approx for $\nabla^2 u$

for interior points where $x < 1$, $f(x,y) = 0$

- $u_{i,j} = \frac{1}{4}(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1})$

for $x \geq 1$ where $f(x,y) = 40\sin(\alpha y)(1-x)$

- $u_{i,j} = \frac{1}{4}(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} , u_{i,j-1} - \Delta x^2 \cdot 40\sin(\alpha y_i)(1-x_i))$

At $y=1$

- $u_{i,N} = u_{i,N-1} - \Delta y \cdot \gamma u_{i,N-1}$

4.b.i <u>finite Integral Transform</u> - S-L

$u(0,y) = \beta y$

$u(2,y) = \beta y$

Solution in form $u(x,y) = X_{(x)} Y_{(y)}$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0$$

$$\frac{\partial^2 X}{\partial x^2} - \lambda X = 0 \quad , \quad \frac{\partial^2 Y}{\partial y^2} - \lambda Y = 0$$

for $X(x)$: $X'(0) = \beta Y_{(y)}$ $\quad \lambda = k^2$

$X(2) = \beta Y_{(y)}$

$$X(x) = A\cos(kx) + B\sin(kx)$$

@ Boundary $x = 0$

$$X'(0) = -Ak\sin(0) + Bk\cos(0) = Bk = \beta Y_{(y)}$$

satisfied for $A = 0$, $\sin(2k) = 0$ $\quad k = \frac{n\pi}{2}$

$$\boxed{X_n = \sin\left(\frac{n\pi x}{2}\right) \qquad \lambda_n = \left(\frac{n\pi}{2}\right)^2}$$

4.b.ii

$$\bar{Y}'(0) = 0$$

$$\bar{Y}'(1) = -\gamma \bar{Y}$$

$\mu = -k^2$

$$\bar{Y}(y) = C\cosh(ky) + D\sinh(ky)$$

@ $y=0$

$$\bar{Y}'(0) = Ck\sinh(0) + Dk\cosh(0) = Dk = 0$$

$$\bar{Y}(y) = C\cosh(ky), \quad D = 0$$

$$\bar{Y}'(1) = Ck\sinh(k) = -\gamma C\cosh(k)$$

$$\frac{Ck\sinh(k)}{C\cosh(k)} = \frac{-\gamma C\cosh(k)}{C\cosh(k)}$$

$$\boxed{k\tanh(k) = -\gamma \qquad \mu_m = -k_m^2 \quad \text{need to find roots}}$$

The eigenfunction for $\bar{Y}(y)$ are $\bar{Y}_m(y) = \cosh(k_m y)$ where k_m is the solution to the above.