HW 9 solutions

Bayesian models

Problem 1

$\{y_i, i = 1 \ldots n\}$ where $(y_i | \theta) \sim Unif(0, \theta)$

a.) $p(y_1 \ldots y_n | \theta) = \theta^{-n} I(\theta \geq \max(y_1 \ldots y_n))$

For a uniform distribution,

$$p(y_i | \theta) = \begin{cases} \frac{1}{\theta} & 0 \leq y_i \leq \theta \\ 0 & \text{otherwise} \end{cases}$$

so $p(y_i | \theta) = \theta^{-1} I(\theta \geq y_i \geq 0)$

$$p(\{y_1 \ldots y_n\} | \theta) = \prod_{i=1}^{n} p(y_i | \theta) \quad \text{(Assume iid)}$$

$$p(\{y_1 \ldots y_n\} | \theta) = \prod_{i=1}^{n} \theta^{-1} I(\theta \geq y_i \geq 0)$$

$$= \theta^{-n} \underbrace{\left[I(\theta \geq y_1) I(\theta \geq y_2) \ldots I(\theta \geq y_n) \right]}_{\text{These will all be true if it is true for the max value of } y_i.}$$

$$\boxed{p(\{y_1 \ldots y_n\} | \theta) = \theta^{-n} I(\theta \geq \max\{y_1 \ldots y_n\})}$$

This is the likelihood, also denoted $\ell(\theta | y)$

b.) Pareto(α, β) is written as (where $\theta \sim (\alpha, \beta)$)

$$p(\theta) = \begin{cases} \alpha \beta^{\alpha} \theta^{-(\alpha+1)} & \theta \geq \beta \\ 0 & \text{otherwise} \end{cases}$$

* Show likelihood from part a. corresponds to Pareto$(n-1, m)$

Considering $\alpha = n-1$

$\beta = m = \max(y_1 \dots y_n)$

then $$\text{Pareto}(n-1, m) = \begin{cases} \alpha \beta^{\alpha} \theta^{-(n-1+1)} & \theta \geq \max(y_1 \dots y_n) \\ 0 & \text{else} \end{cases}$$

$$\text{Pareto}(n-1, m) = \begin{cases} c \; \theta^{-n} & \theta \geq \max(y_1 \dots y_n) \\ 0 & \text{else} \end{cases}$$

where c is the normalizer. We can see that the likelihood is the unnormalized Pareto distribution.

* Let the prior for θ be taken to be (α, β) and derive the posterior distribution $p(\theta | y)$

"Is proportional to" This is so we can ignore the normalizers

$$p(\theta | y) \propto \underbrace{p(y|\theta)}_{\text{likelihood}} \overbrace{p(\theta | \alpha, \beta)}^{\text{prior}}$$

$$\propto \theta^{-n} I(\theta \geq \overbrace{\max(y_1 \dots y_n)}^{m}) \cdot \theta^{-(\alpha+1)} I(\theta \geq \beta)$$

$$\propto \theta^{-(n+\alpha+1)} \; I(\theta \geq \max(m, \beta))$$

$$= \text{Pareto}(\alpha+n, \max(m, \beta))$$

Therefore the Pareto is conjugate to the uniform. Yay