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A global index of riskiness

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ABSTRACT

We extend the pioneering work of Aumann–Serrano by presenting an index of riskiness for gambles with either positive or negative expectations. It can be of use for a variety of abstract behaviors, when adapting the framework of either Expected-Utility Theory or Prospect Theory.

0. Introduction

What is Inherent Risk? It is the riskiness of a gamble defined independently of either the utility or the wealth of the individual contemplating taking the gamble. In other words, the index is unconcerned with the attitude of the individual towards risk, but attempts to capture that risk which is inherent to the gamble itself. The first such index has been presented by Aumann and Serrano (2008) (hereafter [AS]), and was simplified by Hart (2011). However, it is restricted to gambles with positive expected return which only a risk-averse agent would accept.

The AS-index is uniquely characterized by two properties. First is what [AS] term the *Duality Axiom*: As they put it, "Duality says that if the more risk-averse of two agents accepts the riskier of two gambles, then a fortiori the less risk-averse agent accepts the less risky gamble". The second axiom used in the [AS] cardinal characterization of their index is *positive homogeneity*.

In this paper, we extend the AS-index providing a measure of *inherent* riskiness for gambles with either positive or negative expectation. The extended index is applicable to all agents who may be risk lovers, risk-averse individuals or agents behaving with respect to S-shaped utility functions. Our index satisfies the duality axiom, while positive homogeneity is naturally replaced by an appropriate notion. When restricted to gambles with positive expectations, then our index is a monotonic increasing function of the AS-index.

1. The index of riskiness—axiomatic characterization

A *generalized utility function* $u(x)$ is assumed here to be a strictly monotonically increasing, twice continuously differentiable function, defined over the entire line. We normalize u so that

$$u(0) = 0 \quad \textit{and} \quad u'(0) = K \geq 1. \tag{1}$$

Remark. In prospect theory, the value functions are not assumed to be differentiable at 0. The prototypical example in Kahneman and Tversky (1979), is given below:

$$u(x) = \begin{cases} x^{\alpha} & x \geq 0 \\ -\lambda(-x)^{\beta} & x < 0 \end{cases}$$

where $0 < \alpha, \beta < 1, \lambda > 1$.

However, their examples can be approximated in a neighborhood of 0 by continuously twice differentiable functions. For instance, take a specific example where $\alpha = \beta = 0.5, \lambda = 1.5$. We

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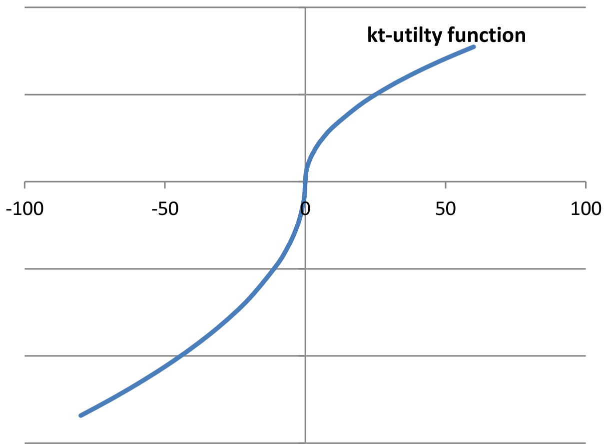
can modify only the small neighborhood $-0.25 < x < 0.25$ to obtain a continuously twice differentiable utility function. The new formula is given explicitly by:

$$u(x) = \begin{cases} \sqrt{x} & 0.25 \le x \\ 2x + 60x^3 - 416x^4 + 704x^5 & 0 \le x < 0.25 \\ 2x + 186x^3 + 1136x^4 + 1824x^5 & -0.25 \le x < 0 \\ -1.5\sqrt{-x} & x < -0.25. \end{cases}$$

As can be seen from the following graph, the modification has almost no effect on the final results.



Let agent i have a generalized utility function u_i that can be concave, convex, or winding from convex to concave (usually at 0)[1] or vice versa.

A gamble g is a random variable with real values, some of which are negative and some positive. If g has an infinite number of possible outcomes then we assume also that its moment generating function exists for all x. Note if g is truncated, or if its tail is sufficiently small, as, e.g., the normal distribution, then its moment generating function is defined for all x.

We say that i accepts g at w if

$$Eu_i(w + g) > u_i(w),$$

where E stands for "expectation"; otherwise, i rejects g at w. We say that i accepts g if i accepts g at 0.

We extend the ideas in [AS] to gambles with either positive or negative expectations by showing:

Theorem (*Boundary*). *Let g be a gamble so that $Eg \neq 0$. Then there exists a unique number $\alpha_g \neq 0$, satisfying,*

$$Ee^{-\alpha_g g} = 1. \tag{2}$$

Moreover, α_g is positive (negative) if and only if Eg is positive (resp. negative). If $Eg = 0$ then $\alpha_g \neq 0$ is the only number satisfying (2).

Call the nonzero root α_g obtained in (2)— *the bound of taking g*. (When $E(g) = 0$ set $\alpha_g = 0$). The notion will be justified in the sequel.

The following is a small refinement of Arrow (1965, 1971) and Pratt (1964) which describes attitude towards risk in terms of the first and second derivatives. We extend their definition to generalized utility functions.

Definition. Given an agent i with utility function u_i, the coefficient of absolute risk at w is given by:

$$\rho_i(w) = \rho_i(w, u_i) = -u_i''(w)/u_i'(w).$$

Note $u_i(x)$ is concave in a neighborhood of w if and only if $\rho_i(w) > 0$, while it is convex if and only if $\rho_i(w) < 0$.

Fix a range of wealth W including all possible outcomes of g. For an agent i with a utility function u_i define

$$\rho_{\max}(u_i) = \sup_{w \in W} \rho_i(w) \qquad \rho_{\min}(u_i) = \inf_{w \in W} \rho_i(w). \tag{3}$$

We show: Let α_g be the bound of taking a gamble g. If $\rho_{\max}(u_i) < \alpha_g$ then i accepts g at any wealth level $w \in W$. Similarly, if $\rho_{\min}(u_i) > \alpha_g$ then i rejects g at any level $w \in W$.

Thus, if α_g is high then there are more agents who accept g at any wealth level w. Moreover, a gamble with negative expectation will be rejected by all agents who are risk averse in the entire selected range W, while a gamble with a positive expectation will be accepted by all gamblers that are risk lovers in the entire range W.

Note that if $\alpha_g > 0$ then α_g^{-1} is the index of riskiness of g as defined in [AS]. At this point we could extend [AS] directly and define the index of riskiness as $R = \alpha^{-1}$. If we do so, the index will be positive for positive expected-value gambles, and negative for negative expected-value gambles. It will be ∞ for even gambles. We find it counter-intuitive that gambles with a positive expected return have an index of riskiness higher than that of gambles with a negative expected return. Moreover, the index will no longer be continuous. To avoid this confusion, we propose the following:

Definition. Given a gamble g and its bound α_g define its index of riskiness $Q(g)$ by

$$Q(g) = e^{-\alpha_g}.$$

The index is continuous in the sense that whenever $g_n \to g$ uniformly then $Q(g_n) \to Q(g)$.

It is straightforward to see that the generalized index $Q(g)$ satisfies the following:

The inverse power-law property:

(i) $Q(g) > 0$ for all g.
(ii) If $Eg > 0$ then $Q(g) < 1$ and if $Eg < 0$ then $Q(g) > 1$.
(iii) $Q(Ng) = Q(g)^{1/N}$. In particular $Q(-g) = Q(g)^{-1}$.

The *inverse power-law* is the appropriate replacement for positive homogeneity of the AS-index. Observe that if $Eg > 0$ then by (ii) $Q(g) < 1$ hence for $N > 1$ we have by (iii):

$$Q(Ng) = Q(g)^{1/N} > Q(g).$$

That is, increasing the investment in a gamble with a positive expectation increases the riskiness, not by the same factor though. In fact, first and second order stochastic dominance is saved for gambles with positive expected return and reversed for gambles with negative expected return.

The crucial property of the index is duality, which is described below.

Definition. Call i at least as risk averse or no more risk loving than j (written $i \trianglerighteq j$) if j accepts at any level w a gamble g that i accepts at some level w_1. Call i more risk averse or less risk loving than j (written $i \triangleright j$) if $i \trianglerighteq j$ and $j \not\trianglerighteq i$.

It follows from (3) that

Corollary. *If $i \trianglerighteq j$ then $\rho_{\max}(u_j) \le \rho_{\min}(u_i)$.*

Hence:

Theorem (*Duality*). *Let g be a gamble and let i and j be agents so that $i \trianglerighteq j$. If i accepts g at some wealth level w_1 and $Q(g) > Q(h)$ then j accepts h at all levels w.*

The *inverse-power-law* and *duality* determine the generalized index up to a fixed power. That is,

Theorem (*Uniqueness*). *Any index satisfying duality and inverse-power-law has the form Q^t for some positive real number t.*

[1] An exception is Friedman and Savage (1948) behavior, where the switch from convex to concave occurs above 0.

2. Proofs

The first lemma is based on [AS, 2, 3]. In order to adapt fully their proof, we need only to replace u_i by $K^{-1}u_i$.

Lemma 2.1. *Let agents i and j have normalized utility functions u_i, u_j and Arrow–Pratt coefficients ρ_i, ρ_j. For each $\delta > 0$, suppose that $\rho_i(w) > \rho_j(w)$ at each w with $0 \neq |w| < \delta$. Then $u_i(w) < u_j(w)$ whenever $0 \neq |w| < \delta$. If $\rho_i(w) \leq \rho_j(w)$ for all w, then $u_i(w) \geq u_j(w)$ for all w.*

The following fundamental theorem and its corollary relate the shape of the generalized utility functions to accepting/rejecting gambles. The proofs are NOT based on [AS], as the assumptions regarding the utility functions are different.

Theorem 2.2. *Given agents i and j, if there exists w_1 so that $u_i(w_1) > u_j(w_1)$ then there exists a gamble g that i accepts and j rejects at 0.*

Proof. Assume first $w_1 > 0$. By assumptions on utility functions u we have by (1):

$$\lim_{y\to 0}\frac{u_i(y)}{u_j(y)} = \lim_{y\to 0}\frac{u_i'(y)}{u_j'(y)} = 1.$$

Since $u_i(w_1) > u_j(w_1)$, we have $\frac{u_i(w_1)}{u_j(w_1)} > 1$, hence one can choose $\delta > 0$ so that $\frac{u_i(w)}{u_j(w)} < \frac{u_i(w_1)}{u_j(w_1)}$ for all $0 \neq w \in (-\delta, \delta)$. Take $0 < y < \delta$, then we have $\frac{u_i(-y)}{u_j(-y)} < \frac{u_i(w_1)}{u_j(w_1)}$.

Multiplying both sides of the inequality by the positive value $\frac{-u_j(-y)}{u_i(w_1)}$ yields

$$\frac{-u_i(-y)}{u_i(w_1)} < \frac{-u_j(-y)}{u_j(w_1)},$$

hence one can choose $a > 0$ so that

$$\frac{-u_i(-y)}{u_i(w_1)} < a < \frac{-u_j(-y)}{u_j(w_1)}.$$

Thus:

$$-u_i(-y) < au_i(w_1) \quad \text{and} \quad au_j(w_1) < -u_j(-y).$$

Set $0 < \alpha = \frac{a}{a+1} < 1$, and let g be the gamble yielding a loss of y with probability $1-\alpha$ and a gain of w_1 with probability α. Then g satisfies:

$$Eu_i(g) = (1-\alpha)u_i(-y) + \alpha u_i(w_1) > 0,$$
$$Eu_j(g) = (1-\alpha)u_j(-y) + \alpha u_j(w_1) < 0.$$

Thus i accepts g at 0 while j rejects it.

If $w_1 < 0$ replace u_k by $\tilde{u}_k$ for $k = i, j$, where $\tilde{u}_k(x) = -u_k(-x)$ to obtain the desired result. □

Corollary 2.3. *Assume agents i and j have normalized utility functions u_i, u_j, with corresponding Arrow–Pratt coefficient ρ_i and ρ_j. If $\rho_i(w_1) < \rho_j(w_2)$ for some wealth levels w_1, w_2, then there is a gamble g that i accepts at w_1 and j rejects at w_2.*

Proof. Define

$$\tilde{u}_i(x) = u_i'(w_1)^{-1}(u_i(x+w_1) - u_i(w_1))$$
$$\tilde{u}_j(x) = u_j'(w_2)^{-1}(u_j(x+w_2) - u_i(w_2)).$$

Then $\tilde{u}_i$, $\tilde{u}_j$ are utility functions so that $\tilde{\rho}_i(0) = \rho_i(w_1) < \rho_j(w_2) = \tilde{\rho}_j(0)$. Hence there exists $\delta > 0$ so that $\tilde{\rho}_i(w) < \tilde{\rho}_j(w)$ for all $w \in (-\delta, \delta)$. By 2.1 $\tilde{u}_i(w) > \tilde{w}_j(w)$ for all $0 \neq w \in (-\delta, \delta)$. By 2.2 there is a gamble g that $\tilde{i}$ accepts and $\tilde{j}$ rejects at 0. It follows i accepts g at w_1 and j rejects g at w_2. □

The following is a standard extension of results in [AS].

Theorem 2.4. *Let g be a gamble so that $Eg \neq 0$. Then there exists a unique number $\alpha_g \neq 0$, satisfying,*

$$Ee^{-\alpha_g g} = 1.$$

Moreover, α_g is positive (negative) if and only if Eg is positive (resp. negative). If $Eg = 0$ then $\alpha_g \neq 0$ is the only number satisfying the equation above.

Proof. As in [AS], define a map f_g by

$$f_g(\alpha) = 1 - Ee^{-\alpha g}. \tag{4}$$

Observe that when g is a continuous random variable then $Ee^{\alpha g}$ is the moment generating function of g, hence by our assumptions, $f_g(\alpha)$ is defined for α. Then:

$$\text{(i)} f_g(0) = 0 \qquad \text{(ii)} f_g'(0) = Eg.$$

Since f is concave, it has at most two roots, one of which is zero. If $Eg > 0$ then f increases at 0, and hence the nonzero root is positive. If $Eg < 0$ then f decreases at 0, and hence the nonzero root is negative. If $Eg = 0$ then 0 is the only root. □

Corollary 2.5. *Let α_g be the bound of taking a gamble g. If $\rho_{\max}(u_i) < \alpha_g$ then i accepts g at any wealth level $w \in W$. Similarly, if $\rho_{\min}(u_i) > \alpha_g$ then i rejects g at any level $w \in W$.*

Proof. Assume $\rho_{\max}(u_i) < \alpha_g$. Then at any level $w \in W$, $\rho_i(w) < \alpha_g$. Consider the utility function u_α, where

$$u_\alpha(x) = \alpha^{-1}(1 - e^{-\alpha x}).$$

A direct computation shows that $\rho_{u_\alpha}(w) = \alpha$ for all w. Observe that $E_{u_\alpha}(g) = \alpha^{-1}f(\alpha)$ where $f(\alpha)$ is as in (4). Take $\alpha = \alpha_g$, then by assumption on u_i, $\rho_i(w) < \alpha_g = \rho_{u_{\alpha g}}(w)$, hence by 2.1, $u_i(w) > u_{\alpha^g}(w)$ for all $w \neq 0$. Thus,

$$E_{u_i}(g) > E_{u_{\alpha g}}(g) = \alpha_g^{-1}f(\alpha_g) = 0,$$

implying that i accepts g (at 0). Given any w define (as in the proof of 2.3)

$$\tilde{u}_i(x) = u_i'(w)^{-1}(u_i(x+w) - u_i(w)).$$

Then for all x, $\tilde{\rho}_i(x) = \rho_i(x+w) < \alpha_g$, hence $\tilde{u}_i$ accepts g at 0, which implies that u_i accepts g at w.

The second part follows by the same arguments with the reverse inequalities. □

Corollary 2.6. *If $i \trianglerighteq j$ then $\rho_{\max}(u_j) \leq \rho_{\min}(u_i)$.*

Proof. Assume contrarily that there exist w_1, w_2 so that $\rho_i(w_1) < \rho_j(w_2)$. Then by 2.3 there is a gamble g that i accepts at w_1 and j rejects at w_2. A contradiction. □

We can now prove duality.

Proof of duality. First, since $i \trianglerighteq j$ it follows from 2.6 that $\rho_{\max}(u_j) \leq \rho_{\min}(u_i)$. Furthermore, by the second part of 2.5, if i accepts g at some level w_1 then $\rho_{\min}(u_i) \leq \alpha_g$. On the other hand, if $Q(g) > Q(h)$ then $\alpha_g < \alpha_h$. Thus we obtain:

$$\rho_{\max}(u_j) \leq \rho_{\min}(u_i) \leq \alpha_g < \alpha_h,$$

which implies by the first part of 2.5 that j accepts h at all levels of wealth w. □

Proof of uniqueness. Assume Q' satisfies the inverse-power-law and duality. For any gamble g with $Eg > 0$ set

$$R'(g) = -\frac{1}{\ln(Q'(g))}.$$

Then $R'(g)$ satisfies duality and positive homogeneity, hence by [AS], $R'(g) = tR(g)$, for some $t > 0$, where $R(g)$ is their index. Hence

$$Q'(g) = e^{-\frac{1}{R'(g)}} = e^{-\frac{1}{tR(g)}} = Q(g)^{\frac{1}{t}}.$$

If $E(g) < 0$ then the result follows by replacing g with $-g$. $\square$

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