NFA, DFA, and regular expressions

204213 Theory of Computation

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Outline

Regular operations

Last time, we defined 3 regular operations:

- Union: $A \cup B = \{x | x \in A \text{ or } x \in B\}$,
- Concatenation: $A \circ B = \{xy | x \in A \text{ and } y \in B\}$,
- Star: $A^* = \{x_1 x_2 \cdots x_k | k \geq 0 \text{ and each } x_i \in A\}$,

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and proved the following theorem.

Theorem 1

The class of regular languages is closed under the union operation.

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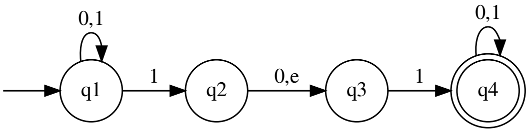
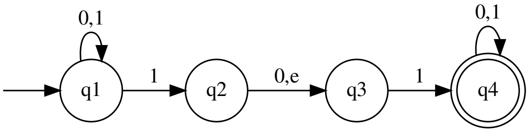
Example: Nondeterministic Finite Automaton N_1



Note: e in the figure is ε.

Differences



- Duplicate symbols
- Missing symbols
- Empty string: ε

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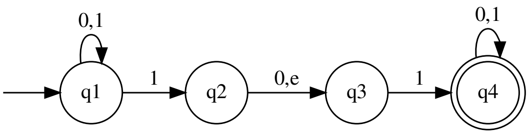
- Computation where each next step is fully determined is called **deterministic** computation.
- On the other hand, in **nondeterministic** computation, many choices may exist.
- Therefore, we have deterministic finite automata (**DFA**) and nondeterministic finite automata (**NFA**).

How does N_1 compute?



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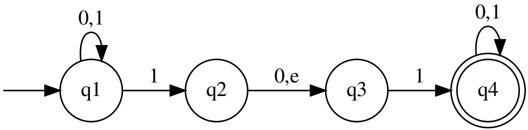

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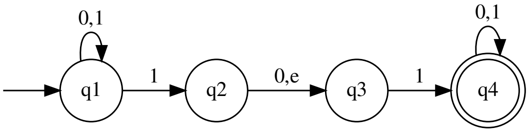

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See simulation.

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- When to accept a string:
    - At the end of the input, if **any** of the copies is in an accept state, it **accept** the input.

N_1 on 010110

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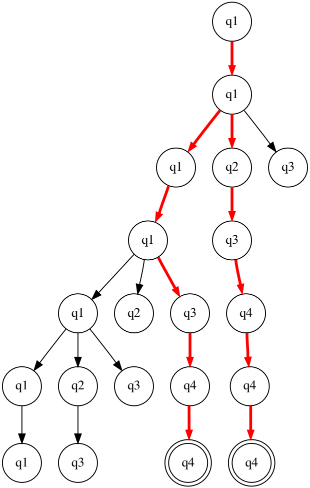
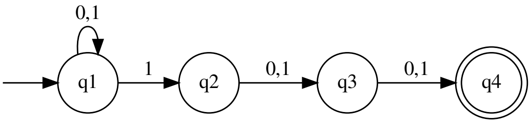
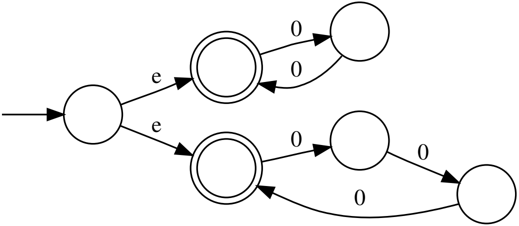


NFA N_2: what are the strings accepted by N_2?



NFA N_3: what are the strings accepted by N_3?

Let $\{0\}$ be the alphabet for N_3.



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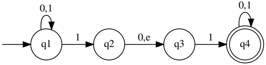
Definition [NFA]

A **nondeterministic finite automaton** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set of states,
2. Σ is a finite alphabet,
3. $\delta : Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ is the transition function,
4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.

Example: N_1



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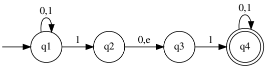

N_1 is $(Q, \Sigma, \delta, q_1, F)$ where

1. $Q = \{q_1, q_2, q_3, q_4\}$,
2. $\Sigma = \{0, 1\}$,
3. δ is defined as

	0	1	$\emptyset$
q_1	$\{q_1\}$	$\{q_1, q_2\}$	$\emptyset$
q_2	$\{q_3\}$	$\emptyset$	$\{q_3\}$
q_3	$\emptyset$	$\{q_4\}$	$\emptyset$
q_4	$\{q_4\}$	$\{q_4\}$	$\emptyset$

4. q_1 is the start state, and
5. $F = \{q_4\}$.

Formal definition of computation of NFAs

Let $N = (Q, \Sigma, \delta, q_0, F)$ be an NFA and let w be a string over alphabet Σ.
We say that N **accepts** w if we can write $w = w_1 w_2 \cdots w_n$ where each w_i is a member of Σ_ε and

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1. $r_0 = q_0$,
2. $r_{i+1} \in \delta(r_i, w_{i+1})$ for $i = 0, \ldots, n-1$, and
3. $r_n \in F$.

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- In fact, DFAs and NFAs recognize the **same** class of languages!
- We say that two machines are *equivalent* if they recognize the same language.

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Two directions:

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  - No that easy.

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How can we prove that?

- Recall "reader as automaton"?
- Given an NFA N, think of a DFA M as a manager who operates N.
- What does M have to **remember** in order to simulate N correctly?

Proof of Theorem 2

Let $N = (Q, \Sigma, \delta, q_0, F)$ be an NFA.

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Let $N = (Q, \Sigma, \delta, q_0, F)$ be an NFA. We shall construct a DFA $M = (Q', \Sigma, \delta', q'_0, F')$ recognizing the same language.

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4. $F' = \{R \in Q' | R \text{ contains an accept state of } N\}$.

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- If N is at q, it can move freely to any states in $D(q)$.

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- Fix start states $q_0' = E(\{q_0\})$.

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- M correctly simulates N.
- Thus, our proof is complete.

Note on the correctness proof

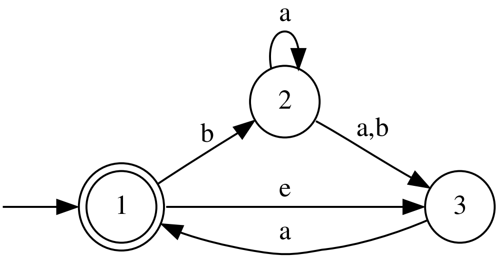
- Our previous proof of Theorem 2 is quite short and does not give out all the details.
- This is okay for now, since our construction is simple enough so that it is quite obvious that it is correct.
- For more complicated constructions, we need to be more formal.

A more general definition of regular languages

Corollary 3

A language is regular iff some nondeterministic finite automaton recognizes it.

Example



Closure under the regular operations

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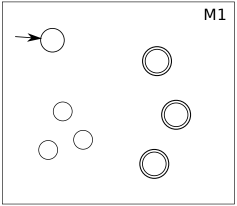
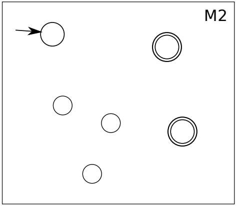
- We'll look at each operation.

Union





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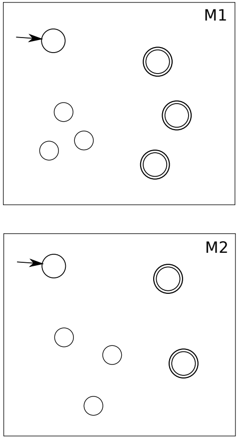
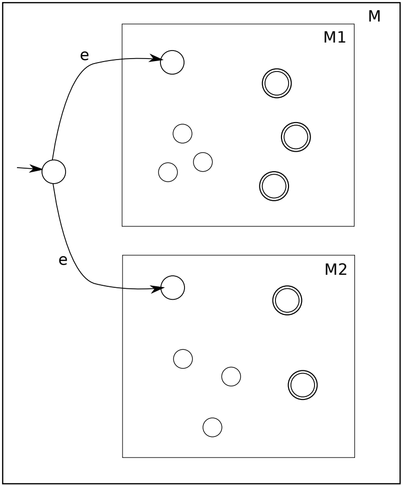
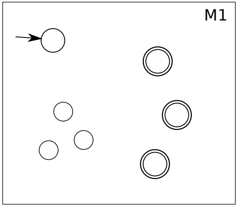
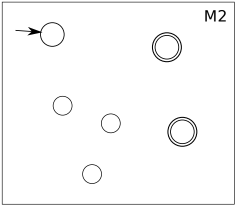




Concatenation





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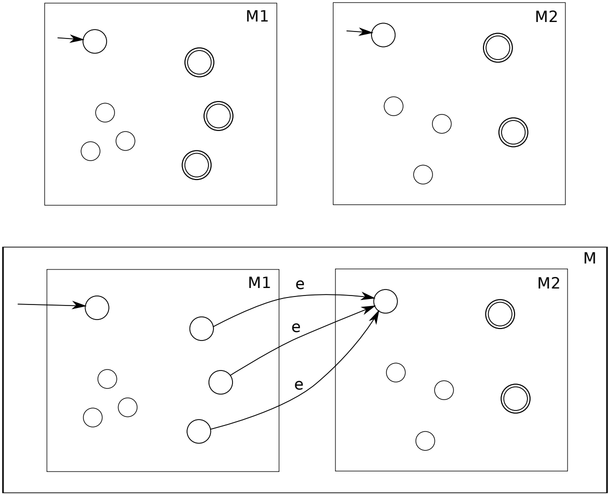
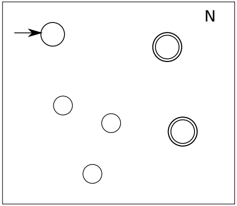


Star



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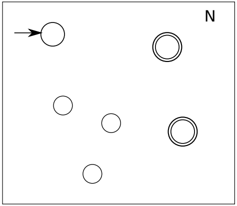

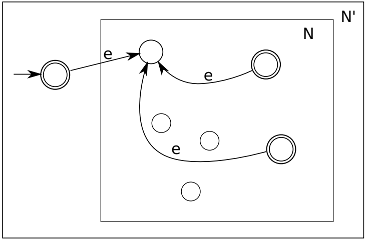

Practice

- Let alphabet $\Sigma = \{0, 1\}$.
- $A_1 = \{w|\ w \text{ contains even number of 1's }\}$.

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Practice

- Let alphabet $\Sigma = \{0, 1\}$.
- $A_1 = \{w \mid w \text{ contains even number of 1's }\}$.
  Find an FA M_1 that recognizes A_1.
- $A_2 = \{w \mid w \text{ contains odd number of 0's }\}$.
  Find an FA M_2 that recognizes A_2.
- Construct an NFA N_1 recognizing $A_1 \cup A_2$.
- Construct an NFA N_2 recognizing $A_1 \circ A_2$.
- Construct an NFA N_3 recognizing $(A_1^*) \circ A_2$.

Building up patterns

- Regular operations can be used to build expressions describing languages.

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- For example,

$$(\{0\} \cup \{1\}) \circ \{0\}^*,$$

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Building up patterns

- Regular operations can be used to build expressions describing languages.
- For example,

$$(\{0\} \cup \{1\}) \circ \{0\}^*,$$

  or in a shorter form $(0 \cup 1)0^*$
- This is called a **regular expression**.
- Note: 0 denotes $\{0\}$, 1 denotes $\{1\}$, and $\circ$ is omitted.

Another examples

$$(0 \cup 1)^*$$

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- All possible strings (including ε).

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- All possible strings (including ε).
- If $\Sigma = \{0, 1\}$, we can write Σ for $(0 \cup 1)$, and write Σ^* for any strings from alphabet Σ.

Definition [regular expression]

R is a **regular expression** if R is

1. a for some $a \in \Sigma$,
2. ε,
3. $\emptyset$,
4. $(R_1 \cup R_2)$ where R_1 and R_2 are regular expressions,
5. $(R_1 \circ R_2)$ where R_1 and R_2 are regular expressions, and
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This is an **inductive definition**.

Precedence

Operations are performed in this order:

- $*$
- $\circ$
- $\cup$

Shorthands

- RR^*

Shorthands

- RR^* can be written as R^+
- $RRRR$

Shorthands

- RR^* can be written as R^+
- $RRRR$ can be written as R^4, in general R^k is the concatenation of R to itself for k times.

Examples of regular expressions

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8. $1^*\emptyset = \emptyset$
9. $\emptyset^*$

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8. $1^*\emptyset = \emptyset$
9. $\emptyset^* = \{\varepsilon\}$

Properties

- $R \cup \emptyset$

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A language is regular iff some regular expression describes it.

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There are two directions to prove the theorem:

- If a language is described by a regular expression, then it is regular.
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Theorem 4

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There are two directions to prove the theorem:

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Today we'll prove only the first direction.

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- Find an FA M_1 that recognizes 01^+
- Find an FA M_2 that recognizes $(10)^*$
- Find an FA M_3 that recognizes $(01^+) \cup (10)^*$
- Find an FA M_4 that recognizes $1^+ \circ ((01^+) \cup (10)^*)$

A regular expression describes a regular language

Lemma 5

If a language is described by a regular expression, then it is regular.

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To prove this, we'll look at how we a regular expression is constructed.

Rule 1

R is a **regular expression** if R is a for some $a \in \Sigma$.

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What is a DFA that recognizes R?

Rule 2

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Rule 6

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R is a **regular expression** if R is (R_1^*) where R_1 is a regular expression.

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Proof by Structural Induction

- In proving Lemma 5, we show how to construct an NFA of a regular expression given NFAs of its subexpressions.

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- In proving Lemma 5, we show how to construct an NFA of a regular expression given NFAs of its subexpressions.
- The proof is again an inductive proof.
- Sometimes, this kind of inductive proofs is called structural induction.
  - Inductive Hypothesis (when considering a regular expression R): Assume that for all smaller regular expressions R', the language described by R' can be recognized by some NFA N'.

Practice

1. Find an NFA recognizing $(01 \cup 0)^*$.

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   Try to build an NFA using the construction discussed in class.
2. Find an NFA recognizing $(0 \cup 1)^* 010$.