Equivalence between FA and Regular Expression, Nonregular languages, and the Pumping Lemma

204213 Theory of Computation

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November 22, 2008

Outline

Short review: NFA and DFA

- For a **deterministic** finite automaton, given its current state and an input symbol from the alphabet, the next state is determined.

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- For a **deterministic** finite automaton, given its current state and an input symbol from the alphabet, the next state is determined.
- For a **nondeterministic** finite automaton, given its current state and an input symbol from the alphabet, there can be many possible states (or none).

Proof of the NFA-DFA Equivalence

Given an NFA $N = (Q, \Sigma, \delta, q_0, F)$, we shall construct an equivalence DFA $M = (Q', \Sigma, \delta', q_0', F')$ that recognizes the same language.

- Note that both automata take the same alphabet Σ.

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- Let $Q' = \mathcal{P}(Q)$.
- Define δ' so that M correctly simulates many copies of N.
- Carefully handle ε.
- M accepts any state $R \in Q'$ such that $R \cap F \neq \emptyset$.

Definition [regular expression]

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- R is a **regular expression** if R is
  1. a for some $a \in \Sigma$,
  2. ε,
  3. $\emptyset$,
  4. $(R_1 \cup R_2)$ where R_1 and R_2 are regular expressions,
  5. $(R_1 \circ R_2)$ where R_1 and R_2 are regular expressions, and
  6. (R_1^*) where R_1 is a regular expression.

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There are two directions to prove the theorem:

- If a language is described by a regular expression, then it is regular. **Proved last time by considering how regular expressions can be constructed.**
- **Today:** If a language is regular, then it can be described by a regular expression.

The second part

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Any regular language can be described with a regular expression.

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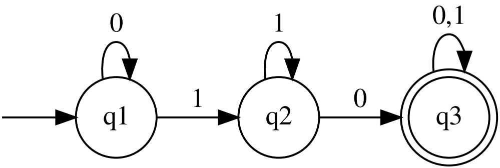
- What do we know?
  - A is a regular language.
- What does that mean? ummm... (hint: use definition)
  - There is a DFA M that recognizes A.

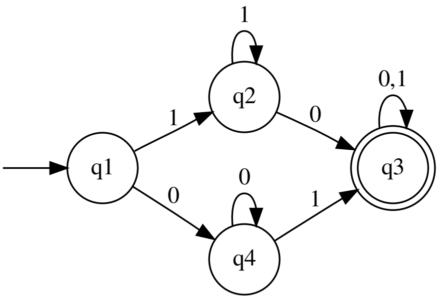
Practice: M_1

Okay, let's do some practice.



What is a regular expression describing the language recognized by M_1.

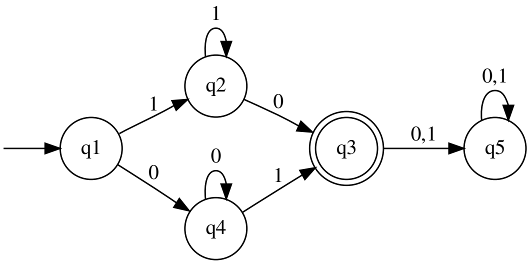
Practice: M_2



What is a regular expression describing the language recognized by M_2.

While you're trying to figure out the regular expression, try to think about a "mechanical" method for constructing it from a DFA.

Practice: M_3



What is a regular expression describing the language recognized by M_3.

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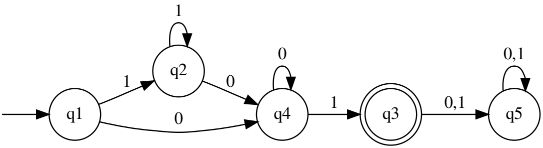
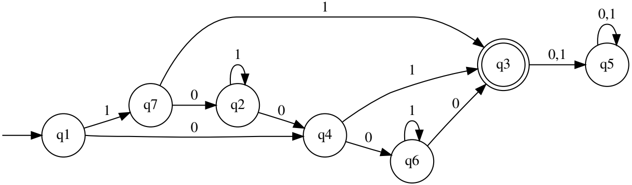
Practice: M_4



What is a regular expression describing the language recognized by M_4.

While you're trying to figure out the regular expression, try to think about a "mechanical" method for constructing it from a DFA.

Practice: M_5



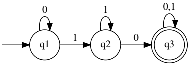
What is a regular expression describing the language recognized by M_5.

While you're trying to figure out the regular expression, try to think about a "mechanical" method for constructing it from a DFA.

Easy rules?

Just like a way to calculating resistances:

- Series



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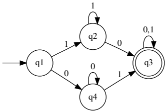

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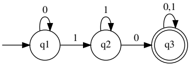

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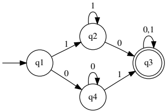

But things could mess up really quickly. (Think about M_5.)

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- But what kind of progress?
  - It maybe better to start by asking what kind of finishing line that we want.

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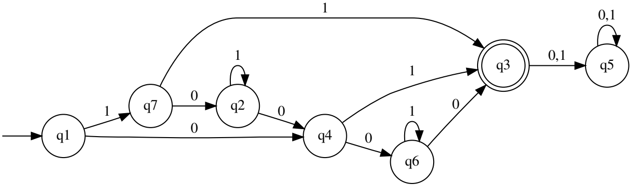
- Try to reduce the number of states.
- Each step decreases the number of states by one.

Let's try with M_5



- Try to remove q_7.

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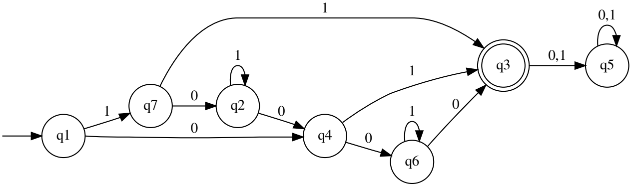

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- Start over. Try to remove q_4.

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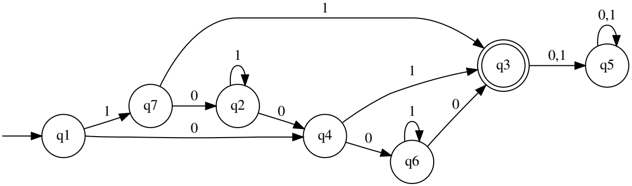

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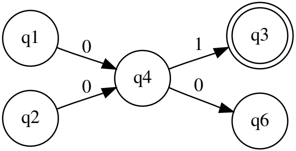
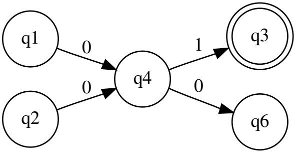
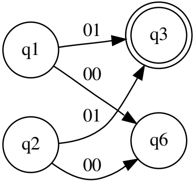
Now you get an idea.

Removing q_4

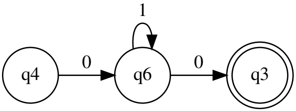


Removing q_4





Removing state q_6



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- This is fine: we shall define the generalized nondeterministic finite automata.

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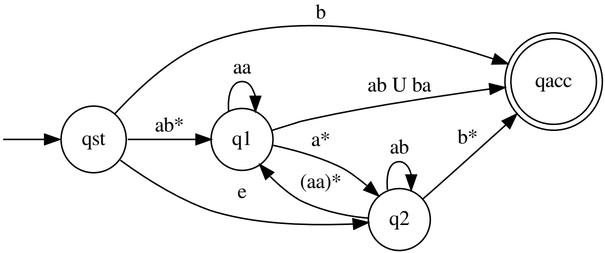
- A **generalized nondeterministic finite automata** are nondeterministic finite automata where we allow regular expressions as labels on transition arrows.
- A GNFA can move to a new state only if it can read a block of input symbols that is described by the regular expression on the arrow.

An example of a GNFA



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This form of GNFA will be easy to be converted into a regular expression.

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- Magic helpers:
  - Arrows with ϵ
  - Arrows with $\emptyset$

DFA $\Rightarrow$ GNFA: Construction

Given $M = (Q, \Sigma, \delta, q_0, F)$:

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- Add all other arrows labelled with $\emptyset$.

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    - Build an equivalent G' by removing q_{rip}
    - Repeat.

Definition [GNFA]

A **generalized nondeterministic finite automaton** is a 5-tuple, $(Q, \Sigma, \delta, q_{start}, q_{accept})$, where

1. Q is the finite set of states,
2. Σ is the input alphabet,
3. $\delta : (Q - \{q_{accept}\}) \times (Q - \{q_{start}\}) \rightarrow \mathcal{R}$ is the transition function,
4. q_{start} is the start state, and
5. q_{accept} is the accept state.

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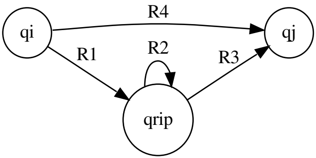
- For GNFA, we do not care that much how it actually works, but we want to use it to do the conversion.
- We focus more on "what's the regular expression on this arrow?". That's how the definition of the transition function δ in this case is defined quite differently.

Removing q_{rip}



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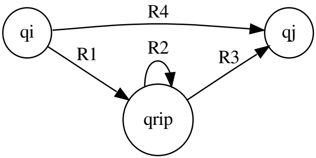

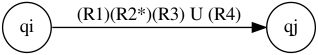

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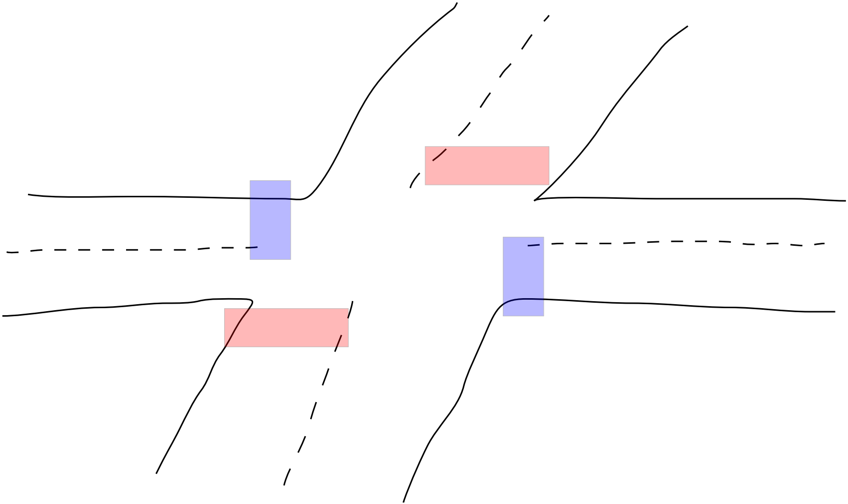
- Fix q_{rip}. Let $Q' = Q - \{q_{rip}\}$.
- for any $q_i \in Q' - \{q_{accept}\}$ and $q_j \in Q' - \{q_{start}\}$, let

$$\delta'(q_i, q_j) = (R_1)(R_2)^*(R_3) \cup (R_4),$$

where $R_1 = \delta(q_i, q_{rip})$, $R_2 = \delta(q_{rip}, q_{rip})$, $R_3 = \delta(q_{rip}, q_j)$, and $R_4 = \delta(q_i, q_j)$.

Traffic light control



Extracting string constants

```
#include <stdio.h>

main()
{
  int a, b;
  scanf("%d %d",&a,&b);
  printf("Hello!  \"welcome\" %d\n",a+b);
}
```

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  - Some intuition: any DFA M recognizing B seems to have to remember the number of 0, but since M has finite state it will remember incorrectly when n is very large.

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  - Some intuition: any DFA M recognizing B seems to have to remember the number of 0, but since M has finite state it will remember incorrectly when n is very large.
  - Again, that's **not** a proof.

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Solution: C is not regular, but D is!

Main tool: the pumping lemma

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  For any regular language, there is a string length, called the pumping length, such that for any string as long as the pumping length can be "pumped".

- "pumped" — the string contains a section that can be repeated any number of times while the resulting string remains in the language.

Theorem [Pumping Lemma]

Theorem 3 (Pumping lemma)

If A is a regular language, then there is a number p (the pumping length) where, if s is any string in A of length at least p, then s maybe divided into three pieces $s = xyz$, satisfying the following conditions:

1. *for each $i \geq 0$, $xy^iz \in A$,*
2. *$|y| > 0$, and*
3. *$|xy| \leq p$.*

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- Let $s = 0^p 1^p$. We know that $s \in B$, and $|s| \geq p$.
- Now applying the pumping lemma, we have that s can be split into $s = xyz$, and for any i, $xy^i z$ is also in B.

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- Thus, B is not regular.

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- **Happy!**

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- **Hint:** don't forget condition 3.

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- **Hint:** choose the right $s \in F$.

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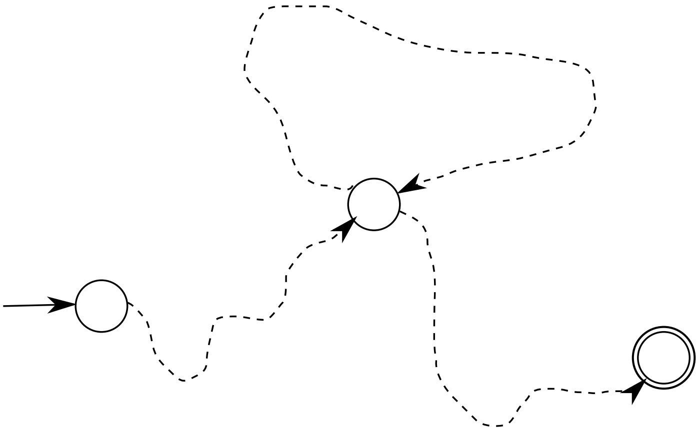
- Think about what happens when M **accepts** a really long string.
- Since Q is finite, when taking a really long string, you'll see some state on the sequence of states from q_0 to some accept state (remember?) repeats.

Proving the pumping lemma: idea (2)



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- (Now you try to fill the rest.)