Context-Free Grammar and Push-Down Automata

204213 Theory of Computation

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Outline

Quiz

1. Given 2 DFAs $M_1 = (Q, \Sigma, \delta_1, q_0, F_1)$ and $M_2 = (R, \Sigma, \delta_2, r_0, F_2)$, show how to construct a DFA M_3 that recognizes language $A_1 \cap A_2$ where A_i is a language recognized by M_i for each $i = 1, 2$.
2. Show that if A_1 and A_2 are regular languages, $A_1 - A_2$ is also regular.

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  - Let M simulates N by keeping the set of states that copies of N are in.

Thus, the sets of languages recognized by DFAs and NFAs are equal.

(D/N)FAs and regular expressions

An example

Grammar G_1

$$
\begin{array}{lcl}
A & \rightarrow & 0A1 \\
A & \rightarrow & B \\
B & \rightarrow & \#
\end{array}
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Start with $A \Rightarrow 0A1$ (rule 1) $\Rightarrow 00A11$ (rule 1) $\Rightarrow 00B11$ (rule 2)

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Start with $A \Rightarrow 0A1$ (rule 1) $\Rightarrow 00A11$ (rule 1) $\Rightarrow 00B11$ (rule 2) $\Rightarrow 00\#11$ (rule 3).

This sequence of substitution is called a **derivation**.

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To obtain a derivation, we also need a start variable.

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To obtain a derivation, we also need a start variable. (If not specified otherwise, it is the left-hand side of the top rule.)

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- The grammar G_1 **generates** the string 000#111.

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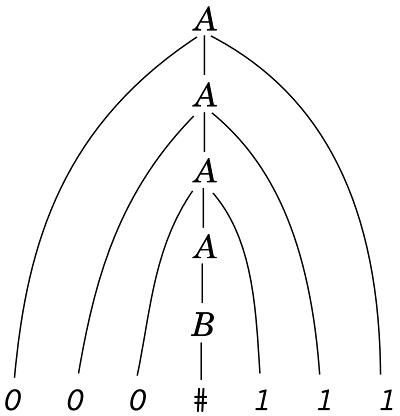
- How to use the grammar to generate a string:
  - Begin with start variable.
  - Find a variable in the string and a rule that starts with that variable. Replace the variable with the right-hand side of the rule.
  - Repeat.

A parse tree



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- For a grammar G, let $L(G)$ denote the language of G.
- $L(G_1) = \{0^n\#1^n | n \geq 0\}$

A context-free language

A language described by some context-free grammar is called a context-free language.

More example

Grammar G_2

$$
\begin{aligned}
S &\rightarrow NP\ VP \\
NP &\rightarrow CN | CN\ PP \\
VP &\rightarrow CV | CV\ PP \\
PP &\rightarrow PREP\ CN \\
CN &\rightarrow ART\ N \\
CV &\rightarrow V | V\ NP \\
ART &\rightarrow \texttt{a} | \texttt{the} \\
N &\rightarrow \texttt{boy} | \texttt{girl} | \texttt{flower} \\
V &\rightarrow \texttt{touches} | \texttt{likes} | \texttt{sees} \\
PREP &\rightarrow \texttt{with}
\end{aligned}
$$

Small English grammar

- Examples of strings in $L(G_2)$ are:
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  - `a boy sees`
  - `the boy sees a flower`
  - `a girl with a flower likes the boy`

Derivation

- Show the derivation of string "`a boy sees`".

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- Show the derivation of string "`a boy sees`".
- Try to generate more strings from G_2 and find their parse trees.

Definition [context-free grammar]

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A **context-free grammar** is a 4-tuple (V, Σ, R, S), where

1. V is a finite set called the **variables**,
2. Σ is a finite set, disjoint from V, called the **terminals**,
3. R is a finite set of **rules**, with each rule being a variable and a string of variables and terminals, and
4. $S \in V$ is the **start variable**.

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- We say that u **derives** v, written as $u \stackrel{*}{\Rightarrow} v$,
  - if $u = v$, or
  - if a sequence $u_1, u_2, \ldots, u_k$ exists for $k \geq 0$ and

$$u \Rightarrow u_1 \Rightarrow u_2 \Rightarrow \cdots \Rightarrow u_k \Rightarrow v.$$

Example: G_3

$G_3 = (\{S\}, \{a, b\}, R, S)$, where R is

$$S \rightarrow aSb|SS|\varepsilon.$$

Practice

Find a CFG that describes the following language

$$\{\mathtt{a}^i\mathtt{b}^j\mathtt{c}^k \mid i, j, k \geq 0 \text{ and } i = j \text{ or } j = k\}$$

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- the rules are

$$EXPR \rightarrow EXPR + EXPR \mid EXPR \times EXPR \mid (EXPR) \mid a$$

Generate some string from G_4'.

Ambiguity

Find a parse tree for $a + a \times a$ in grammar G_4'.

Example: G_4

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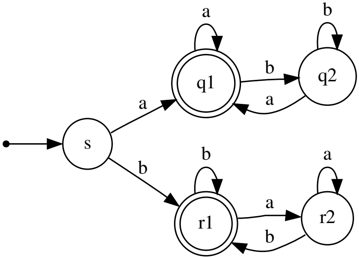
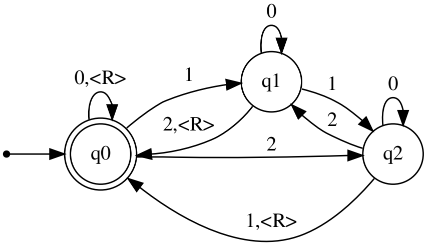
CFGs and regular languages (1)

Can you find a context-free grammar that describes the language recognized by the following DFA?



CFGs and regular languages (2)

Can you find a context-free grammar that describes the language recognized by the following DFA?



Again, think about a “mechanical” procedure for constructing a CFG.

CFGs and regular languages (3)

Any general procedure?

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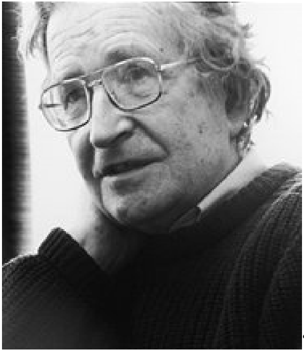
- Again, CFGs is quite general and sometimes we want them to be in a simpler form.
- One of the forms is called Chomsky normal form.

Noam Chomsky

[a]

Avram Noam Chomsky is an American linguist, philosopher, cognitive scientist, political activist, author, and lecturer. [from wikipedia]

[a] From wikipedia. URL:
`http://en.wikipedia.org/wiki/Image:Noam_chomsky_cropped.jpg`

Definition [Chomsky normal form]

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where a is any terminal and A, B, and C are any variables, except that B and C cannot be the start variable. We also permit the rule $S \longrightarrow \varepsilon$, where S is the start variable.

Theorem 1

Any context-free grammar is generated by a context-free grammar in Chomsky normal form.

Pushdown automata

- NFAs power-up

Pushdown automata

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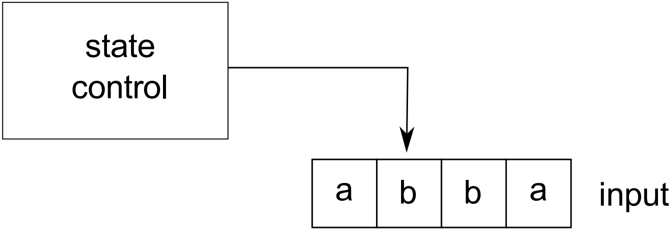
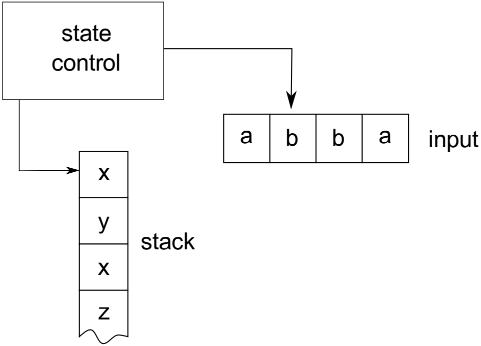
- Think of them as NFAs with extra memory, called **stack**.

NFAs



PDas



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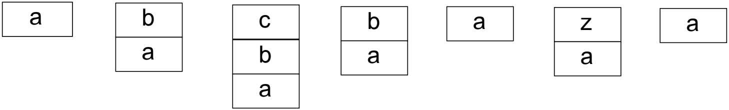

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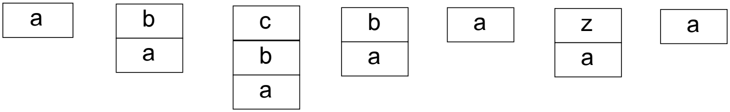

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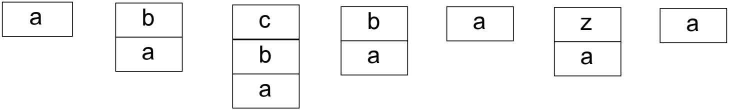

A stack is an infinite memory but you can only access the topmost element.
You can pop (put something on top) and push (remove the topmost).

Informally

Can you find an NFA with a stack that recognizes $\{0^n 1^n | n \geq 0\}$?

Transition function with stack (1)

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- Thus, the transition function accepts (q, x, s) where q is a state, x is an input symbol, and s is the top of the stack.

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  - It changes the state and writes something to the top of the stack.
- Thus, the transition function accepts (q, x, s) where q is a state, x is an input symbol, and s is the top of the stack.
- The transition function returns a set of pairs (q', s') where q' is a new state and s' is the stack symbol written to the stack.

Transition function with stack (2)

- Transition function δ:
  - Domain: $Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon$
  - Range: $\mathcal{P}(Q \times \Gamma_\varepsilon)$

Definition [pushdown automaton]

A **pushdown automaton** is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$, where Q, Σ, Γ, and F are finite sets, and

1. Q is the set of states,
2. Σ is the input alphabet,
3. Γ is the stack alphabet,
4. $\delta : Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \longrightarrow \mathcal{P}(Q \times \Gamma_\varepsilon)$ is the transition function,
5. $q_0 \in Q$ is the start state, and
6. $F \subseteq Q$ is the set of accept states.