Proof of the Pumping Lemma, PDA $\Rightarrow$ CFG, and Turing Machines

204213 Theory of Computation

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Outline

1. Proof of the Pumping Lemma
2. PDAs $\Rightarrow$ CFGs
3. Turing Machines

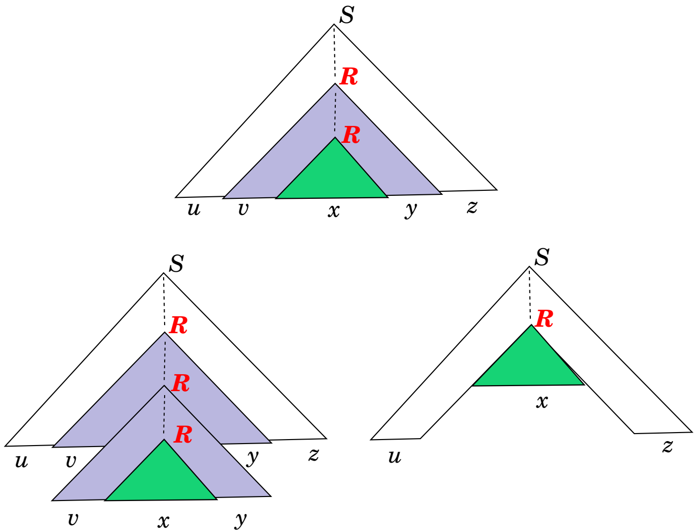
Pumping lemma for CFL

Theorem 1 (pumping lemma for CFL)

If A is a context-free language, then there is a pumping length p such that for any string $s \in A$ of length at least p, s can be divided into 5 pieces $s = uvxyz$ satisfying the following conditions

1. *for each $i \geq 0$, $uv^ixy^iz \in A$,*
2. *$|vy| > 0$, and*
3. *$|vxy| \leq p$.*

Parse tree for s



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    - What?
    - Since we have $|V|$ variables, if, on the parse tree, the path from the start variable to some terminal is long enough (how long?) we should see the same variable twice.
    - How can we make sure that the parse tree is very tall?

Tall parse tree: example

G_1

$$
\begin{aligned}
S &\rightarrow AB \\
A &\rightarrow 1A0|0A1|\varepsilon \\
B &\rightarrow BB|0|1
\end{aligned}
$$

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- What is the longest string whose longest path from S to any terminal is ≤ 4?
- Any bound on the length of the string generated by G_2 that guarantees that the height of its parse tree is at least 5?

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- The length of a string is the number of leaves on a parse tree.
- Therefore a parse tree of a string of length $b^h + 1$ must be at least $h + 1$ high.

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- From PHP, since we have $|V|$ variables, some variable R appears more than once on this path.

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  - For (2), it can be the case that between the first R and the second R, nothing gets generated.

The proof (3): using minimality

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  - Note that the tree τ' that generates uxz has fewer number of nodes than τ. (why?) But it also generates s; this **contradicts** the assumption that τ is the tree with minimum number of nodes.
  - Thus, $uxz \neq uvxyz$, and $|vy| > 0$.

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- HW: show that this implies (3).

Equivalence

Theorem 2

A language is context-free if and only if some pushdown automaton recognizes it.

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  - Given a PDA P, construct a CFG G that generates a language recognized by P. **SKIPPED.**

Skipped...

The second part is quite technical, so we decide to skip the proof of the second part.

Models of computation

- **Finite automata and regular expressions.**
  - Devices with small, limited memory.
- **Push-down automata and context-free languages**
  - Devices with unlimited memory, but have restricted access.

Turing Machines

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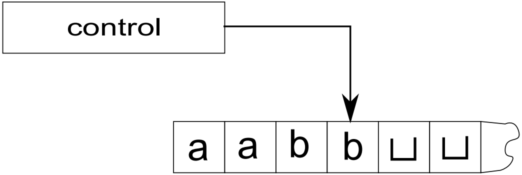
- A finite automaton with an unlimited memory with unrestricted access.
- Can perform any tasks that a computer can. (we'll see)
- However, there are problems that TM can't solve. These problems are beyond the limit of computation.

Components

- An infinite **tape**.
- A tape head that can
  - **read and write** to the tape, and
  - **move** around the tape.

Schematic



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- It can go on forever (not entering any accept or reject states).

Example: M_1

- We'll design a TM that recognizes

$$B = \{w\#w \mid w \in \{0,1\}^*\}.$$

Example: M_1 — strategy

- M_1 works by comparing two copies of w.
- M_1 compares two symbols on the corresponding positions.
- It write marks on the tape to keep track of the position.

Example: M_1 — snapshots

$\textcolor{red}{0}\ 1\ 1\ 0\ 0\ 0\ \#\ 0\ 1\ 1\ 0\ 0\ 0\ \sqcup$

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$\mathtt{x}\ \color{red}{1}\ 1\ 0\ 0\ 0\ \#\ 0\ 1\ 1\ 0\ 0\ 0\ \sqcup$

Example: M_1 — snapshots

```
0 1 1 0 0 0 # 0 1 1 0 0 0 ⊔
x 1 1 0 0 0 # 0 1 1 0 0 0 ⊔
x 1 1 0 0 0 # 0 1 1 0 0 0 ⊔
```

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$$
\begin{array}{cccccccccccccc}
\color{red}{0} & 1 & 1 & 0 & 0 & 0 & \# & 0 & 1 & 1 & 0 & 0 & 0 & \sqcup \\
\mathtt{x} & \color{red}{1} & 1 & 0 & 0 & 0 & \# & 0 & 1 & 1 & 0 & 0 & 0 & \sqcup \\
\mathtt{x} & 1 & 1 & 0 & 0 & 0 & \# & \color{red}{0} & 1 & 1 & 0 & 0 & 0 & \sqcup \\
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$\color{red}{0}$ 1 1 0 0 0 # 0 1 1 0 0 0 ⊔

x $\color{red}{1}$ 1 0 0 0 # 0 1 1 0 0 0 ⊔

x 1 1 0 0 0 # $\color{red}{0}$ 1 1 0 0 0 ⊔

x 1 1 0 0 0 # $\color{red}{x}$ 1 1 0 0 0 ⊔

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0	1	1	0	0	0	#	0	1	1	0	0	0	␣
x	1	1	0	0	0	#	0	1	1	0	0	0	␣
x	1	1	0	0	0	#	0	1	1	0	0	0	␣
x	1	1	0	0	0	#	x	1	1	0	0	0	␣
x	1	1	0	0	0	#	x	1	1	0	0	0	␣
x	1	1	0	0	0	#	x	1	1	0	0	0	␣
x	x	1	0	0	0	#	x	1	1	0	0	0	␣

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x	1	1	0	0	0	#	0	1	1	0	0	0	⊔
x	1	1	0	0	0	#	0	1	1	0	0	0	⊔
x	1	1	0	0	0	#	x	1	1	0	0	0	⊔
x	1	1	0	0	0	#	x	1	1	0	0	0	⊔
x	1	1	0	0	0	#	x	1	1	0	0	0	⊔
x	x	1	0	0	0	#	x	1	1	0	0	0	⊔
x	x	1	0	0	0	#	x	x	1	0	0	0	⊔

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0 1 1 0 0 0 # 0 1 1 0 0 0 ⊔
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x x 1 0 0 0 # x 1 1 0 0 0 ⊔
x x 1 0 0 0 # x x 1 0 0 0 ⊔
x x x x x x # x x x x x x ⊔   accept!
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2. After all symbols on the left of # have been marked, check if there're other unmarked symbols on the right of #, if there's any, reject; otherwise accept."

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- Thus,
    - **Input:** current state and the symbol on the tape
    - **Output:** next state, a symbol to be written to the tape, and the new state.

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- So, δ is in the form: $Q \times \Gamma \rightarrow Q \times \Gamma \times \{\mathrm{L}, \mathrm{R}\}$
- E.g., if $\delta(q, a) = (r, b, \mathrm{L})$, then if the machine is in state q and reads a, it will change its state to r, write b to the tape and move to the left.

Definition

Definition (Turing Machine)

A **Turing machine** is a 7-tuple, $(Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$, where Q, Σ, Γ are finite sets and

1. Q is the set of states,
2. Σ is the input alphabet not containing the **blank symbol** $\sqcup$,
3. Γ is the tape alphabet, where $\sqcup \in \Gamma$ and $\Sigma \subset \Gamma$,
4. $\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{\mathrm{L}, \mathrm{R}\}$ is the transition function,
5. $q_0 \in Q$ is the start state,
6. $q_{accept} \in Q$ is the accept state, and
7. $q_{reject} \in Q$ is the reject state, where $q_{accept} \neq q_{reject}$.

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- The special states for accepting and rejecting take effect immediately.

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A language is called **Turing-recognizable** if some Turing machine recognizes it.

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- We are interested particularly in TM thats halts (does not loop). We call them **decider**.
- A decider that recognizes some language also is said to **decides** that langauge.

Definition

A language is called **Turing-decidable** or **decidable** if some Turing machine decides it.

Example: M_2

Design M_2 that decides the language $A = \{0^{2^n} \mid n \geq 0\}$.

M_2's algorithm

On input string w:

1. Sweep left to right across the tape, crossing off every other 0.

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4. Return the head to the left-hand end of the tape.
5. Go to state 1.

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- We usually write configurate as: $u\ q\ v$, where q is the state, uv is the current content of the tape, and the TM is at the first symbol of v.

Example execution of M_2