Turing Machines and their variants

204213 Theory of Computation

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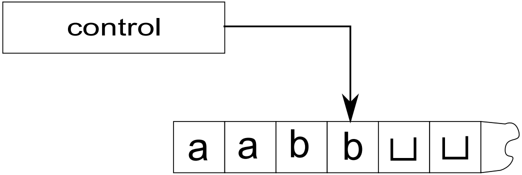
January 6, 2009

Outline

Turing Machines: Components

- An infinite **tape**.
- A tape head that can
  - **read and write** to the tape, and
  - **move** around the tape.

Schematic



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- It can go on forever (not entering any accept or reject states).

Definition

Definition (Turing Machine)

A **Turing machine** is a 7-tuple, $(Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$, where Q, Σ, Γ are finite sets and

1. Q is the set of states,
2. Σ is the input alphabet not containing the **blank symbol** $\sqcup$,
3. Γ is the tape alphabet, where $\sqcup \in \Gamma$ and $\Sigma \subset \Gamma$,
4. $\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{\mathrm{L}, \mathrm{R}\}$ is the transition function,
5. $q_0 \in Q$ is the start state,
6. $q_{accept} \in Q$ is the accept state, and
7. $q_{reject} \in Q$ is the reject state, where $q_{accept} \neq q_{reject}$.

Defining how TM computes

We need two key concepts:

- Configurations: the “states” of the TM
- Transition: how the TM moves.

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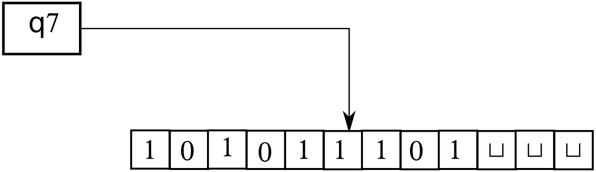
- We usually write configurate as: $u\ q\ v$, where q is the state, uv is the current content of the tape, and the TM is at the first symbol of v.

Configuration: $10101q_7 1101$



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- Are these all case? No. We'll have to deal with the case when the head are a the end of the tape on both sides.

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- the rejecting configurations.
- The accepting and rejecting configurations are called halting configurations.

Computation with TM

A Turing machine M accepts input w if a sequence of configurations $C_1, C_2, \ldots, C_k$ exists, where

1. C_1 is the start configuration of M on input w,
2. each C_i yields C_{i+1}, and
3. C_k is an accepting configuration.

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A language is called **Turing-recognizable** if some Turing machine recognizes it.

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- We are interested particularly in TM thats halts (does not loop). We call them **decider**.
- A decider that recognizes some language also is said to **decides** that langauge.

Definition

A language is called **Turing-decidable** or **decidable** if some Turing machine decides it.

Arithmetics

- Design a TM M_3 that decides the language

$$C = \{\mathrm{a}^i\mathrm{b}^j\mathrm{c}^k \mid i \times j = k \text{ and } i, j, k \geq 1\}.$$

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3. Cross off an $\mathtt{a}$ and scan to the right until $\mathtt{b}$ occurs. Cross each pair of $\mathtt{b}$'s and $\mathtt{c}$'s until all $\mathtt{b}$'s are crossed off.

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    - If all a's are crossed off, check if there is no c's left. If that's the case, **accept**, otherwise **reject**."

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- I.e., design a TM that decides the language

$$E = \{\#x_1\#x_2\#\cdots\#x_l \mid \text{each } x_i \in \{0,1\}^* \text{ and } x_i \neq x_j \text{ for each } i \neq$$

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- We'll see that they all have the same power. This demonstrates the robustness in the definition of TM's. Also, this is an evidence that TM's "capture" the idea of computation (because whatever computing machine we can think of they are all equivalent to TM's).

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  - For any "stay put" transition, we replace with two transitions: "right" and "left".

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- E.g., if $\delta(q_i, a_1, \ldots, a_k) = (q_j, b_1, \ldots, b_k, \mathrm{L}, \mathrm{R}, \ldots, \mathrm{L})$ then if the machine is at state q_i and each head on tape i reads symbol a_i, it'll write b_i on each tape i, change state to q_j and move each head accordingly.

Theorem 1 (equivalence between multitape TM's and TM's)

Every multitape Turing machine has an equivalent single-tape Turing machine.

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- How?

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3. If the content of any simulated tape is overflow (i.e., S has to write over some # on the right), S writes a blank symbol to that delimiter, and shifts all the tape contents to the right."

Corollary 2

A language is Turing-recognizable if and only if some multitape Turing machine recognizes it.

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- Can nondeterminism help?

Theorem 3

Every nondeterministic Turing machine has an equivalent deterministic Turing machine.

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  - D explore all branches at some fixed depth. After exploring all branches, it start exploring at the next depth.
  - This way of exploring the tree is called breadth-first search.