Sets Cheatsheet

Contents

1 Geometric Series

Geometric Series are a special type of Power Series that can be rewritten in the form

$$f(x) = \frac{a_1}{1-r} = \sum_{n=0}^{\infty} a_1 * r^n$$

To find the interval of convergence, just remember that $|r| < 1$

1.1 Examples

$$f(x) = \frac{1}{2-x}, c = 0 \Longrightarrow f(x) = \frac{1}{2} * \frac{1}{1-\frac{x}{2}} \Longrightarrow f(x) = \sum_{n=0}^{\infty} \frac{1}{2} * (\frac{x}{2})^n$$

$$\Longrightarrow f(x) = \sum_{n=0}^{\infty} (\frac{1}{2})^{n+1} x^n$$

$$|r| < 1 \Longrightarrow |\frac{x}{2}| < 1 \Longrightarrow |x| < 2 \Longrightarrow x \in (-2, 2)$$

If we change the center:

$$f(x) = \frac{1}{2-x}, c = 5 \Longrightarrow f(x) = \frac{1}{2-5-(x-5)} \Longrightarrow \frac{1}{-3-(x-5)}$$

$$\Longrightarrow -\frac{1}{3} * \frac{1}{1-\frac{-(x-5)}{3}} \Longrightarrow \sum_{n=0}^{\infty} -\frac{1}{3} * (\frac{-(x-5)}{3})^n$$

$$\Longrightarrow \sum_{n=0}^{\infty} (-\frac{1}{3})^{n+1} (-(x-5))^n$$

$$|r| < 1 \Longrightarrow r \in (-1, 1) \Longrightarrow \frac{-x+5}{3} \in (-1, 1) \Longrightarrow -(x-5) \in (-3, 3) \Longrightarrow x \in (2, 8)$$

Now let's completely change it up.

$$f(x) = \frac{3}{2x-1}, c = 0 \Longrightarrow f(x) = -3 * \frac{1}{1-2x} \Longrightarrow * \sum_{n=0}^{\infty} -3 * (2x)^n, c \in (-\frac{1}{2}, \frac{1}{2})$$

Let's move the center

$$f(x) = \frac{3}{2x-1}, c = -3 \Longrightarrow f(x) = \frac{1}{2(x+3)-5-6} \Longrightarrow f(x) = -\frac{1}{11} * \frac{1}{1-\frac{2}{11}(x+3)}$$

$$f(x) = \sum_{n=0}^{\infty} -\frac{1}{11} * (\frac{2}{11}(x+3))^n$$

$$\frac{2}{11}(x+3) \in (-1, 1) \Longrightarrow (x+3) \in (-\frac{11}{2}, \frac{11}{2}) \Longrightarrow x \in (-\frac{17}{2}, \frac{5}{2})$$

What if there is no $-x$? You can just do $-(-x)$. Remember: Basic Algebra will take you a long way.

$$f(x) = \frac{3}{x+2}, c = 0 \Longrightarrow f(x) = \frac{3}{2} * \frac{1}{1 - \frac{1}{2}(-x)}$$

What if your function looks a bit more complicated?

$$f(x) = \frac{4x-7}{2x^2+3x-2}, c = 0 \Longrightarrow f(x) = \frac{4x-7}{(2x-1)(x+2)} \Longrightarrow f(x) = \frac{A}{2x-1} + \frac{B}{x+2}$$

$$A(x+2) + B(2x-1) = 4x - 7 \Longrightarrow Ax + 2Bx = 4x, 2A - B = -7 \Longrightarrow B = 2A + 7$$

$$\Longrightarrow A + 4A + 14 = 4 \Longrightarrow 5A = -10 \Longrightarrow A = -2, B = 3$$

$$f(x) = \frac{-2}{2x-1} + \frac{3}{x+2}$$

Solve it normally from here.

2 General Power Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

2.1 Derivation

We take general power series to be something like the following:

$$f(x) = 1 + x + 2x^2 + 3x^3 + ... + nx^n + ...$$

or

$$f(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + ... + \frac{1}{n}x^n + ...$$

So generally:

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$$

Instead of a_i because the coefficient changes for each element in the series. Let's expand this series:

$$f(x) = a_0 + c + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + ... + a_n(x-c)^n + ...$$

Take its derivative:

$$f'(x) = 0 + a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + ... + na_n(x-c)^{n-1} + ...$$

Notice how $f'(0) = a_1$.

$$f''(x) = 0 + 0 + 2a_2 + 6a_3(x-c) + ... + n(n-1)a_n(x-c)^{n-2} + ...$$

Notice how $f''(0) = 2a_2$ and $f'''(0) = 6a_3$. In order to get the a_n term, you just need to take $f^{(n)}(c)$ and divide it by n!
Put it together, and you'll get the equation we started with.

2.2 Examples

$$f(x) = e^x, c = 0 \Longrightarrow f(x) = \sum_{n=0}^{\infty} \frac{e^0}{n!}(x)^n \Longrightarrow f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\Longrightarrow 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + ... + \frac{1}{n}x^n + ...$$

Evaluate a geometric series using Taylor series

$$f(x) = \frac{1}{1-2x}, c = 0 \Longrightarrow \frac{\frac{1}{1-2(0)}}{0!} + \frac{\frac{2}{(1-2c)^2}}{1!}x + \frac{\frac{8}{(1-2c)^3}}{2!}x + \frac{\frac{48}{(1-2c)^4}}{3!}x + ...$$

$$\Longrightarrow 1 + \frac{2}{(1-2c)^2}x + \frac{4}{(1-2c)^3}x^2 + \frac{8}{(1-2c)^4}x^3 + ...$$

General term: $2^n x^n$.

We would have gotten the same thing if we used the power series expansion of the Taylor series expansion. As with most things in algebra, pick the method that's **most conveinient for you**.

2.3 Error Checking

In an ideal world, you want to know how far off your estimates are. For alternating series, this process is pretty easy.

$$f(x) = 1 - x + \frac{x}{2!} - \frac{x}{3!} + \frac{x}{4!} - \frac{x}{5!} + \frac{x}{6!} - \frac{1}{7!} + ... + \frac{1}{n!}(-1)^n + ...$$

Let's choose the first four terms for our **Taylor Polynomial**, which will be represented by $P_n(x)$ where n is the degree.

$$P_4(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{3}x^3$$

Let's set the remainder terms to $R_5(x)$. We can call the error $E(x)$.

$$E(x) = |P_4(x) - R_5(x)|$$

Since you will never truly know $R_5(x)$ or any $R_n(x)$ for that matter, you will never know the true error. But you can set an upper bound on that error, which is pretty easy in alternating power series.

$$E(x) \leq P_{n+1}(x) - P_n(x)$$

A fancy way of saying, the term right after is the highest the eror can be. It's because you will never add back what you subtracted, always less.

2.4 Lagrange Error Checking

If your series **isn't alternating**, you can use Lagrange Error Checking. Although the proof is complicated, it's pretty simple

$$E(x) \leq |\frac{\max_{z\in[x,c]} f^{(n+1)}(z)}{(n+1)!}|(x-c)^{n+1}$$

The proof isn't necessary for Calc BC. You can find it online.

2.4.1 Full Example

Find the first four terms about $c = 2$ for $ln|x + 1|$.

We can start with its derivative, $\frac{1}{x+1}$.

For simplicity, we will use a geometric series. Say we wanted to find $\ln|\frac{3}{2}|$ and approximate error to be 0.05 or less

$$\frac{1}{x-1} = \frac{1}{1-(x-2)+2} = \frac{1}{-1-(x-2)} = -\frac{1}{1-(-(x-2))}$$

$$\implies \sum_{n=0}^{\infty} -(-(x-2))^n = \sum_{n=0}^{\infty} (-1)^{n+1}(x-2)^n = -1 + (x-2) - \frac{1}{2}(x-2)^2 + \frac{1}{3}(x-2)^3 + ...$$

$$\int -\frac{1}{1-(-(x-2))}dx = \int -1 + (x-2) - \frac{1}{2}(x-2)^2 + \frac{1}{3!}(x-2)^3 + ...dx$$

$$\implies -\ln|1-(-(x-2))| = C - x + \frac{(x-2)^2}{2} - \frac{(x-2)^3}{3*2!} + \frac{(x-2)^4}{4*3!} + ...$$

$$\longrightarrow \ln|1-(-(2-2))| = \ln|1|$$

$$\longrightarrow C + x - \frac{(x-2)^2}{2} + \frac{(x-2)^3}{3*2!} - \frac{(x-2)^4}{4*3!} + ...$$

$$\longrightarrow \ln|1| = C + 2 + 0 + ... \longrightarrow C = -2$$

$$\implies \ln|1-(-(x-2))| = -2 + x - \frac{(x-2)^2}{2} + \frac{(x-2)^3}{3*2!} - \frac{(x-2)^4}{4*3!} + ...$$

$$\implies \ln|\frac{3}{2}| = \ln|1-(-(\frac{5}{2}-2))| = (\frac{5}{2}-2) - \frac{(\frac{5}{2}-2)^2}{2} + \frac{(\frac{5}{2}-2)^3}{3*2!} - \frac{(\frac{5}{2}-2)^4}{4*3!} + ...$$

$$\implies \frac{1}{2} - \frac{1}{2}*(\frac{1}{2})^2 + \frac{1}{3}*(\frac{1}{2})^3 - \frac{1}{4}*(\frac{1}{2})^4 + ...$$

Since it's an alternating series, it should be pretty simple to solve for from here.

2.4.2 Another Problem

$$\sin(5x + \frac{\pi}{4}) = \sin(5(x + \frac{\pi}{20}))$$