Parametric and Polar Cheatsheet

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1 Parametric Intro

When you have parameters in terms of t to represent $\langle x, y\rangle$ You represent the function as $\langle x(t), y(t)\rangle$ and the derivative as $\langle x'(t), y'(t)\rangle$.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$
$$\frac{d^2y}{dx^2} = \frac{d}{dx}[f'(t)] = \frac{d}{dt}f'(t) * \frac{dt}{dx} = \frac{\frac{d}{dt}[\frac{dy}{dx}]}{\frac{dx}{dt}}$$

Speed:

$$\Delta r = \sqrt{x'(t)^2 + y'(t)^2}$$

If there is no change in $y(t)$:

$$\Delta r = \sqrt{x'(t)^2} = |x'(t)|$$

Distance:

$$r = \int \sqrt{x'(t)^2 + y'(t)^2} dt$$

2 Parametric Examples

1. Find the speed, acceleration, and distance of the following:

$$\vec{v} = \langle (t-1)e^{t^2}, \sin(t^{1.25})\rangle$$
$$\text{speed} = \sqrt{((t-1)e^{t^2})^2 + \sin^2(t^{1.25})}$$
$$\text{distance} = \int \sqrt{((t-1)e^{t^2})^2 + \sin^2(t^{1.25})} dt$$

2. The parametric equation of a curve is given by $\langle 2t-1, 3t^2+1\rangle$. What is the slope of the tangent line to the curve at $t = 2$.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dx}[3t^2+1]}{\frac{d}{dx}[2t-1]} = \frac{6t}{2} = \frac{12}{2} = 6$$

3. A parametric is given by $\langle \cos t, \sin t\rangle$. What is the speed of the particle at $t = \frac{\pi}{4}$?

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dx}[\sin t]}{\frac{d}{dx}[\cos t]} = \frac{\cos t}{-\sin t} = -1$$
$$\text{speed} = \sqrt{\cos^2 t + \sin^2 t} = 1$$

4. A parametric is represneted by $\langle 5t, 4-t^2 \rangle$. At what value of t does the curve have a horizontal tangent line?

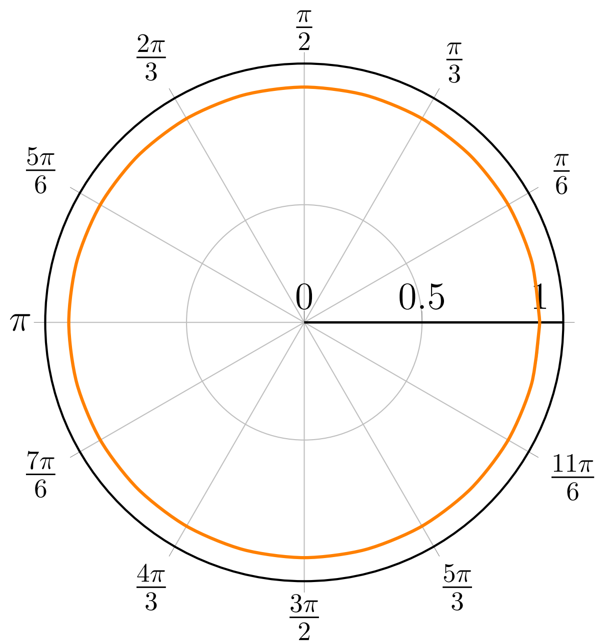
$$\frac{dy}{dt} = 0 = \frac{d}{dt}[4-t^2] = -2t \Longrightarrow t = 0$$

5. A parametric is represented $\vec{v} = \langle 2t+1, 3t^2-4t \rangle$. What is the second derivative of y with respect to x at $t = 2$?

$$\frac{\vec{v}'(t)}{\frac{dx}{dt}} = \frac{\frac{6t}{2}}{2} = 3$$

3 Polar

The polar axis defines functions for $r(\theta)$ where radius is the distance from $(0,0)$ and θ is the angle with the horizontal.



A unit circle defined in Polar.

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

If we were to convert the unit circle to a parametric equation we would get $\langle r\cos(\theta), r\sin(\theta) \rangle$.

3.1 Tangent Lines

Finding tangent lines of a Polar function is quite simple; just convert it to parametric.

Find the derivative of the following at $\frac{3\pi}{4}$:

$$r(\theta) = 4\sin(2\theta)$$
$$r(\theta)cos(\theta) = 4\sin(2\theta)\cos(\theta) \implies x = 4\sin(2\theta)\cos(\theta)$$
$$y = 4\sin(2\theta)sin(\theta)$$
$$\frac{dy}{dt} = 4\sin(2\theta)\cos(\theta) + 8\sin(\theta)cos(2\theta)$$
$$\frac{dx}{dt} = -4\sin(2\theta)\sin(\theta) + 8\cos(\theta)\cos(2\theta)$$
$$\frac{dy}{dx} = \frac{\sin(2\theta)\cos(\theta) + 2\cos(2\theta)\sin(\theta)}{-\sin(2\theta)\sin(\theta) + 2\cos(2\theta)\cos(\theta)}$$
$$\frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$$

3.2 Arc Length

Using a parametric to start with:

$$\text{distance} = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$
$$\text{distance} = \int \sqrt{\left(\frac{dr}{d\theta} r\cos\theta\right)^2 + \left(\frac{dr}{d\theta} r\sin\theta\right)^2} d\theta$$
$$\text{distance} = \int \sqrt{\Big(-r\sin\theta + \frac{dr}{d\theta}\cos\theta)^2 + \left(r\cos\theta + \frac{dr}{d\theta}\sin\theta\right)^2} d\theta$$
$$\implies \int \sqrt{r^2\sin^2\theta - 2r\frac{dr}{d\theta}\sin\theta\cos\theta + \left(\frac{dr}{d\theta}\right)^2\cos^2\theta + r^2\cos^2\theta + 2r\frac{dr}{d\theta}\cos\theta\sin\theta + \left(\frac{dr}{d\theta}\right)^2\sin^2\theta d\theta}$$
$$\implies \int \sqrt{r^2\sin^2\theta + \left(\frac{dr}{d\theta}\right)^2\cos^2\theta + r^2\cos^2\theta + \left(\frac{dr}{d\theta}\right)^2\sin^2\theta d\theta}$$
$$\implies \int \sqrt{r^2(\sin^2\theta + \cos^2\theta) + \left(\frac{dr}{d\theta}\right)^2(\cos^2\theta + \sin^2\theta) d\theta}$$
$$\implies \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right) d\theta}$$

3.3 $e^{-\theta}$

Find arc length:

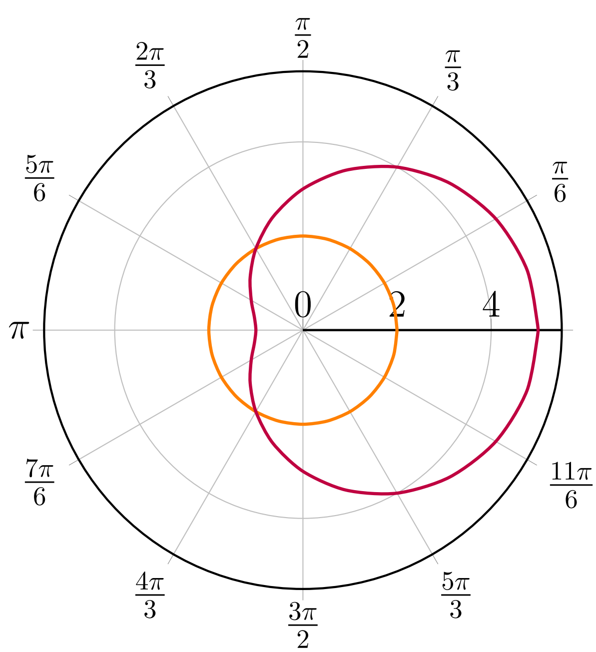
$$r = e^{-\theta} = \frac{1}{e^{\theta}} \Longrightarrow \int_0^{\infty} \sqrt{\left(\frac{1}{e^{\theta}}\right)^2 + \left(\frac{-1}{e^{\theta}}\right)^2} d\theta$$
$$\Longrightarrow \int_0^{\infty} \sqrt{\frac{1}{e^{2\theta}} + \frac{1}{e^{2\theta}}} d\theta$$
$$\Longrightarrow \int_0^{\infty} \sqrt{\frac{2}{e^{2\theta}}} d\theta \Longrightarrow \sqrt{2} \int_0^{\infty} \sqrt{\frac{1}{e^{2\theta}}} d\theta \Longrightarrow \sqrt{2} \int_0^{\infty} (\frac{1}{e^{2\theta}})^{\frac{1}{2}} d\theta \Longrightarrow \sqrt{2} \int_0^{\infty} e^{-\theta} d\theta \Longrightarrow -\sqrt{2}e^{-\theta}\Big|_0^{\infty}$$
$$= \sqrt{2}$$

3.4 Area in the Curve

A polar graph can be thought in terms of mini-sectors in the way an integral is thought of in rectangles.

$$A_{\text{arc}} = \frac{\theta}{2\pi} \pi r^2 = \frac{1}{2} r^2 \theta$$
$$A_{r(\theta)} = \int \frac{1}{2} r^2 d\theta$$

3.5 Examples

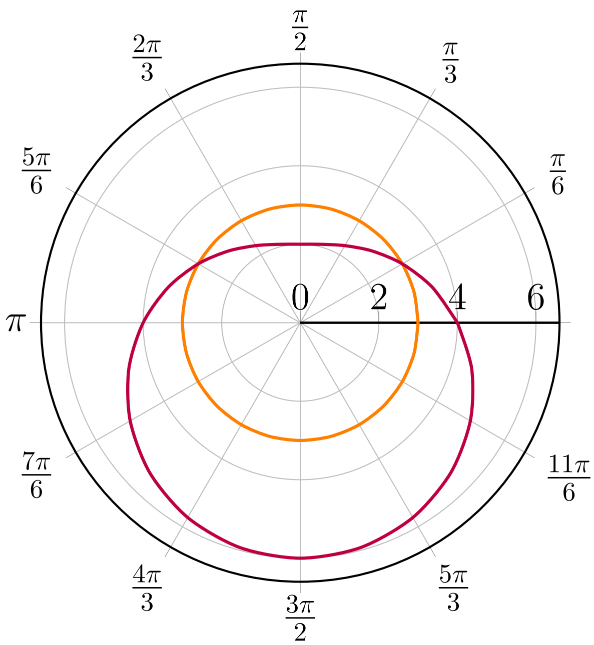


Find the area inside.

$$A = \frac{\pi 2^2}{2} + \frac{1}{2} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (3 + 2\cos\theta)^2 d\theta$$

Solve $\frac{dr}{dt}$ and $\frac{dy}{dt}$ for $t = \theta = \frac{\pi}{3}$.

$$\begin{aligned} \frac{dr}{dt} &= -2\sin\theta = -\sqrt{3} \\ r\sin\theta &= 3\sin\theta + 2\cos\theta\sin\theta \\ \implies y &= 3\sin\theta + \sin 2\theta \\ \frac{dy}{dt} &= 3\cos\theta + 2\sin 2\theta = 3 * \frac{1}{2} + \sqrt{3} \end{aligned}$$



They intersect at $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$.

$$A = \int_0^{\frac{\pi}{6}} 3^2 d\theta + \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (4 - 2\sin\theta)^2 d\theta + \int_{\frac{5\pi}{6}}^{2\pi} 3^2 d\theta$$