

All Integrals Cheatsheet

Contents

1	Basic Integrals	2
2	U-substitution	2
3	Integration by Parts	2
3.1	Derivation	2
3.2	When to Use	3
3.2.1	Traditional usage	3
3.2.2	Doing it twice	3
3.2.3	Where $dv = 1$	4
4	Trig Substitution	4
4.1	Inverse Trig Functions	4
4.1.1	$\arcsin(x)$	4
4.1.2	$\arctan(x)$	4
4.1.3	$\operatorname{arcsec}(x)$	5
4.2	Esoteric $\sqrt{\quad}$ Simplification	6
4.2.1	$\sqrt{a^2 - x^2}$	6
4.2.2	$\sqrt{a^2 + x^2}$	6
4.2.3	$\sqrt{x^2 - a^2}$	6
4.3	Completing the Square	6
5	Trig Identity Integrals	6
5.1	Self Explanatory	6
5.2	$\int \sec^2(x)dx = \tan(x) + C$	7
5.3	$\int -\csc^2(x)dx = \cot(x) + C$	7
5.4	$\int \sec(x)\tan(x)dx = \sec(x) + C$	7
5.5	$\int \csc(x)\cot(x)dx = \int \frac{\cos(x)}{\sin^2(x)}dx = -\csc(x) + C$	8
5.6	$\int \tan(x)dx = \ln \sec(x) + C$	8
5.7	Again for $\int \cot(x)dx$	9
5.8	Algebraic: $\int \sec(x)dx = \ln \sec(x) + \tan(x) + C$	9
5.9	Similarly, $\int \csc(x)dx = -\ln \csc(x) + \cot(x) + C$	9
5.10	$\int \sin^2(x)dx = \frac{1}{2}x - \frac{\sin(2x)}{4} + C$	9
5.11	$\int \cos^2(x)dx = \frac{1}{2}x + \frac{\sin(2x)}{4} + C$	10
6	Polynomials	10
6.1	Long Division	10
6.2	Partial Fraction Decomposition	10
7	Improper Integrals	11

8	Riemann Sum	11
8.1	Left Riemann Sum	12
8.2	Right Riemann Sum	12
8.3	Midpoint Riemann Sum	13
8.4	Trapezoidal Riemann Sum	13
9	Distance	14
9.1	Example	14

$$C \in \mathbb{R}$$

1 Basic Integrals

$$\int x^n = \frac{x^{n+1}}{n+1} + C$$

$$\int \frac{1}{x} dx = \ln |x| + C$$

2 U-substitution

Recall

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$

Therefore by identifying a function and it's derivative, we can use u-substitution.

$$\int f'(g(x))g'(x)dx = f(g(x)) + C$$

What does this mean?

First identify a $g(x)$ and $g'(x)$ modified by a $f'(x)$. Say $f'(x) = \frac{1}{x}$:

$$\int \frac{g'(x)}{g(x)} dx$$

$$u = g(x) \quad du = g'(x) \frac{du}{g'(x)} = dx$$

$$\int \frac{g'(x)}{g(x)} dx = \int \frac{1}{u} du = \ln |u| + C = \ln |g(x)| + C$$

Similarly, if $f'(x) = x$

$$\int g'(x)g(x)dx \quad u = g(x) \quad du = g'(x) \frac{du}{g'(x)} = dx \quad \int u du = \frac{1}{2}u^2 + C = \frac{1}{2}g(x) + C$$

3 Integration by Parts

3.1 Derivation

Recall product rule:

$$\frac{d}{dx}[uv] = u \frac{dv}{dx} + \frac{du}{dx}v$$

Where u and v are $u(x)$ and $v(x)$ respectively.

$$u \frac{dv}{dx} = \frac{d}{dx}[uv] - v \frac{du}{dx}$$

$$\int u \frac{dv}{dx} dx = \int \frac{d}{dx}[uv] - v \frac{du}{dx} dx$$

Simplify:

$$\int u dv = uv - \int v du$$

3.2 When to Use

Notice that u is **never integrated**. Only u and $\int v du$ are used. In exchange, dv is integrated twice; once in uv and once in $\int v du$.

3.2.1 Traditional usage

$$\begin{aligned} \int 2xe^x dx \quad u = 2x \quad dv = e^x dx \quad \int u dv = uv - \int v du &\implies 2xe^x - \int e^x dx * 2 \\ \int 2xe^x dx = 2xe^x - 2e^x + C \end{aligned}$$

Sometimes, u-substitution or another method may be necessary after integration by parts. Or you might have to integrate by parts again.

3.2.2 Doing it twice

$$\int e^x \cos(x) dx \quad u = e^x \quad dv = \cos(x) dx \quad \int u dv = uv - \int v du \quad e^x \sin(x) - \int \sin(x) * e^x dx$$

Now, $\int e^x \sin(x) dx$ has to be integrated.

$$\begin{aligned} \int e^x \sin(x) dx \quad u = e^x \quad dv = \sin(x) dx \quad \int u dv = uv - \int v du \\ - \int e^x \sin(x) dx = e^x \cos(x) - (- \int \cos(x) * e^x dx) \end{aligned}$$

Put it all together:

$$\begin{aligned} \int e^x \cos(x) dx &= e^x \sin(x) - (-e^x \cos(x) - (- \int e^x \cos(x) dx)) \\ \int e^x \cos(x) dx &= e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x) dx \\ 2 \int e^x \cos(x) dx &= e^x (\sin(x) + \cos(x)) + C \\ \int e^x \cos(x) dx &= \frac{e^x (\sin(x) + \cos(x))}{2} + C \end{aligned}$$

3.2.3 Where $dv = 1$

$$\int \ln(x) dx \quad u = \ln(x) \quad dv = dx \quad \int u dv = uv - \int v du$$

$$\int \ln(x) dx = \ln(x)x - \int x \frac{1}{x} dx + C = x \ln(x) - x + C$$

4 Trig Substitution

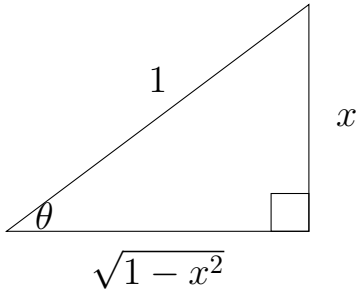
4.1 Inverse Trig Functions

4.1.1 $\arcsin(x)$

$$\int \frac{1}{\sqrt{1-u^2}} dx = \arcsin(x) + C$$

$$\frac{d}{dx}[y] = \frac{d}{dx}[\arcsin(x)] \implies \frac{d}{dx}[\sin(y)] = \frac{d}{dx}[x] \implies \cos(y) * \frac{dy}{dx} = 1 \implies \frac{dy}{dx} = \frac{1}{\cos(y)}$$

$$\frac{dy}{dx} = \frac{1}{\cos(\arcsin(x))} \implies \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$



Second case:

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \int \frac{1}{\sqrt{a^2(1-\frac{x^2}{a^2})}} dx = \frac{1}{a} \int \frac{1}{\sqrt{1-(\frac{x}{a})^2}} dx$$

$$u = \frac{x}{a}; du = \frac{1}{a} dx \implies a du = dx$$

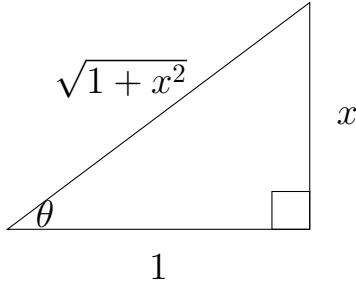
$$\frac{1}{a} \int a \frac{1}{1-u^2} du = \arcsin(u) + C = \arcsin\left(\frac{x}{a}\right) + C$$

4.1.2 $\arctan(x)$

$$\int \frac{1}{1+x^2} dx = \arctan(x) + C$$

$$\frac{d}{dx}[y] = \frac{d}{dx}[\arctan(x)] \implies \frac{d}{dx}[\tan(y)] = \frac{d}{dx}[x] \implies \sec^2 y * \frac{dy}{dx} = 1 \implies \frac{dy}{dx} = \cos^2 y$$

$$\frac{dy}{dx} = \cos^2(\arctan(x)) = \cos^2(\theta) = \left(\frac{1}{\sqrt{1+x^2}}\right)^2 = \frac{1}{1+x^2}$$



Second case:

$$\int \frac{1}{a^2 + x^2} dx = \int \frac{1}{a^2(1 + (\frac{x}{a})^2)} dx = \frac{1}{a} \int \frac{1}{1 + (\frac{x}{a})^2} dx$$

$$u = \frac{x}{a}; du = \frac{1}{a} dx \implies a du = dx$$

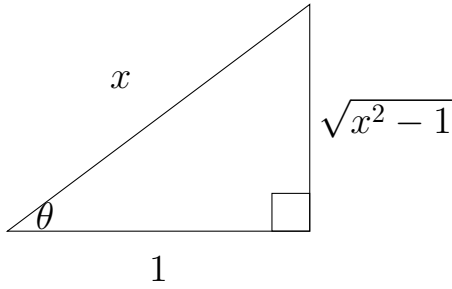
$$\frac{1}{a} \int \frac{1}{1 + u^2} a du = \arctan(u) + C = \arctan\left(\frac{x}{a}\right) + C$$

4.1.3 $\operatorname{arcsec}(x)$

$$\int \frac{1}{x\sqrt{x^2 - 1}} dx = \operatorname{arcsec}(|x|) + C$$

$$\frac{d}{dx}[y] = \frac{d}{dx}[\operatorname{arcsec}(|x|)] \implies \frac{d}{dx}[\sec(y)] = \frac{d}{dx}[x] \implies \sec(x) \tan(x) * \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec x \tan x} = \frac{1}{x\sqrt{x^2 - 1}}$$



Second case:

$$\int \frac{1}{x\sqrt{x^2 - a^2}} dx = \int \frac{1}{ax\sqrt{(\frac{x}{a})^2 - 1}} dx$$

$$u = \frac{x}{a}; du = \frac{1}{a} dx \implies a du = dx$$

$$\frac{1}{a} \int a \frac{1}{au\sqrt{u^2 - 1}} du = \frac{1}{a} \operatorname{arcsec}(|u|) + C = \frac{1}{a} \operatorname{arcsec}\left(\left|\frac{x}{a}\right|\right) + C$$

The absolute value remains because of:

$$\int \frac{1}{|x|\sqrt{x^2 - 1}} dx = \operatorname{arcsec}(x) + C \implies \frac{|x|}{x} \int \frac{1}{|x|\sqrt{x^2 - 1}} dx = (\operatorname{arcsec}(x) + C) \frac{|x|}{x}$$

$$\implies \int \frac{1}{x\sqrt{x^2 - 1}} dx = \operatorname{arcsec}(|x|) + C$$

4.2 Esoteric $\sqrt{\quad}$ Simplification

The same methods can be used to simplify $\sqrt{\quad}$ when **u-substitution and all other methods have failed..** Like the title suggests, these are esoteric at best.

4.2.1 $\sqrt{a^2 - x^2}$

$$x = a \sin(\theta); dx = a \cos(\theta) d\theta$$
$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2 \sin^2(\theta)} = \sqrt{a^2 \cos^2(\theta)} = a \cos(\theta)$$

4.2.2 $\sqrt{a^2 + x^2}$

$$x = a \tan(\theta); dx = a \sec^2(\theta) d\theta$$
$$\sqrt{a^2 + x^2} = \sqrt{a^2 + a^2 \tan^2(\theta)} = \sqrt{a^2 \sec^2(\theta)} = a \sec(\theta)$$

4.2.3 $\sqrt{x^2 - a^2}$

$$x = a \sec(\theta); dx = a \sec(\theta) \tan(\theta) d\theta$$
$$\sqrt{x^2 - a^2} = \sqrt{a^2 \sec^2(\theta) - a^2} = \sqrt{a^2 \tan^2(\theta)} = a \tan(\theta)$$

4.3 Completing the Square

Completing the square can facilitate usage of the methods outlined above. An example would best illustrate this point:

$$\int \sqrt{5 - 4x - x^2} dx = \int \sqrt{5 + 4 - (x^2 + 4x + 4)} dx = \int \sqrt{9 - (x + 2)^2} dx$$

Since $\frac{d}{dx}[x + 2] = 1$, we can use the second-case derivative of $\arctan(x)$

$$= \arctan\left(\frac{x + 2}{3}\right) + C$$

5 Trig Identity Integrals

5.1 Self Explanatory

$$\int \cos(x) dx = \sin(x) + C$$
$$\int \sin(x) dx = -\cos(x) + C$$

5.2 $\int \sec^2(x)dx = \tan(x) + C$

We can derive it using Quotient Rule:

$$\begin{aligned}\frac{d}{dx}[\tan(x)] &= \frac{d}{dx}\left[\frac{\sin(x)}{\cos(x)}\right] \\ \frac{d}{dx}\left[\frac{\sin(x)}{\cos(x)}\right] &= \frac{\cos(x) * \frac{d}{dx}[\sin(x)] - \sin(x) * \frac{d}{dx}[\cos(x)]}{\cos^2(x)} \\ \frac{\cos(x) * \frac{d}{dx}[\sin(x)] - \sin(x) * \frac{d}{dx}[\cos(x)]}{\cos^2(x)} &= \frac{\cos^2(x) - (-\sin^2(x))}{\cos^2(x)}\end{aligned}$$

Use the trig identity $\cos^2(x) + \sin^2(x) = 1$:

$$\frac{\cos^2(x) - (-\sin^2(x))}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)$$

5.3 $\int -\csc^2(x)dx = \cot(x) + C$

We can also derive it using Quotient Rule:

$$\begin{aligned}\frac{d}{dx}[\cot(x)] &= \frac{d}{dx}\left[\frac{\cos(x)}{\sin(x)}\right] \\ \frac{d}{dx}\left[\frac{\cos(x)}{\sin(x)}\right] &= \frac{\sin(x) * \frac{d}{dx}[\cos(x)] - \cos(x) * \frac{d}{dx}[\sin(x)]}{\sin^2(x)} \\ \frac{\sin(x) * \frac{d}{dx}[\cos(x)] - \cos(x) * \frac{d}{dx}[\sin(x)]}{\sin^2(x)} &= \frac{-\cos^2(x) - \sin^2(x)}{\sin^2(x)}\end{aligned}$$

Use the trig identity $\cos^2(x) + \sin^2(x) = 1$:

$$\frac{-\cos^2(x) - \sin^2(x)}{\sin^2(x)} = \frac{-1}{\sin^2(x)} = -\csc^2(x)$$

5.4 $\int \sec(x)\tan(x)dx = \sec(x) + C$

OR

$$\int \frac{\sin(x)}{\cos^2(x)}dx = \sec(x) + C$$

There are two methods to arrive at the following. One is to use u-substitution and the other is to use the power rule and chain. Let's start with the easier one.

$$\begin{aligned}\frac{d}{dx}[\sec(x) + C] &= \frac{d}{dx}\left[\frac{1}{\cos(x)} + C\right] = \frac{d}{dx}[(\cos(x))^{-1}] = -(\cos(x))^{-2} * \frac{d}{dx}[\cos(x)] \\ -\frac{\sin(x)}{\cos^2(x)} &= \frac{\sin(x)}{\cos^2 x} = \sec(x)\tan(x)\end{aligned}$$

We can also just directly undo the chain rule.

$$\begin{aligned} \int \frac{\sin(x)}{\cos^2(x)} dx \\ u = \cos(x) \\ du = -\sin(x) dx \\ -\frac{du}{\sin(x)} = dx \\ \int \frac{\sin(x)}{u^2} * -\frac{du}{\sin(x)} dx = -\int \frac{du}{u^2} = u^{-1} + C = \sec(x) + C \end{aligned}$$

5.5 $\int \csc(x) \cot(x) dx = \int \frac{\cos(x)}{\sin^2(x)} dx = -\csc(x) + C$

$$\begin{aligned} \frac{d}{dx}[-\csc(x)] &= -\frac{d}{dx}\left[-\frac{1}{\sin(x)}\right] = \frac{1}{\sin^2(x)} * \frac{d}{dx}[\sin(x)] = \frac{\cos(x)}{\sin^2(x)} \\ u = \sin(x) \quad du &= \cos(x) dx \quad \frac{du}{\cos(x)} = dx \\ \int \frac{\cos(x)}{u^2} * \frac{du}{\cos(x)} &= -u^{-1} + C = -\csc(x) + C \end{aligned}$$

5.6 $\int \tan(x) dx = \ln |\sec(x)| + C$

Let's break it into components and use u-substitution:

$$\int \tan(x) dx = \int \frac{\sin(x)}{\cos(x)} dx$$

Let's set u to be cos(x)

$$\begin{aligned} u &= \cos(x) \\ du &= -\sin(x) dx \\ \frac{du}{-\sin(x)} &= dx \end{aligned}$$

Now we have:

$$\begin{aligned} \int \frac{\sin(x)}{\cos(x)} dx &= \int \frac{\sin(x)}{u} * \frac{du}{-\sin(x)} \\ &= -\int \frac{du}{u} = -[\ln |u| + C] \\ &= -\ln |\cos(x)| + C_2 \end{aligned}$$

5.7 Again for $\int \cot(x)dx$

$$\int \frac{\cos(x)}{\sin(x)} dx$$

If we use u substitution on the denominator:

$$u = \sin(x)$$

$$du = \cos(x)dx$$

$$\frac{du}{\cos(x)} = dx$$

Sub it back in:

$$\int \frac{\cos(x)}{u} * \frac{du}{\cos(x)} = \int \frac{du}{u} = \ln |u| + C = \ln |\sin(x)| + C$$

5.8 Algebraic: $\int \sec(x)dx = \ln |\sec(x) + \tan(x)| + C$

If you multiply this expression by a certain term, you will get an expression in the form $\int \frac{du}{u}$. Can you guess what it is?

$$\int \sec(x)dx = \int \sec(x) \frac{\sec(x) + \tan(x)}{\sec(x) + \tan(x)} dx = \int \frac{\sec^2(x) + \sec(x)\tan(x)}{\sec(x) + \tan(x)} dx$$

using $u = \sec(x) + \tan(x)$ and $du = \sec^2(x) + \sec(x)\tan(x)dx$

$$\int \frac{du}{u} = \ln |u| + C = \ln |\sec(x) + \tan(x)| + C$$

5.9 Similarly, $\int \csc(x)dx = -\ln |\csc(x) + \cot(x)| + C$

$$\int \csc(x)dx = \int \csc(x) \frac{\csc(x) + \cot(x)}{\csc(x) + \cot(x)} dx = \int \frac{-\csc^2(x) - \csc(x)\cot(x)}{\csc(x) + \cot(x)} dx$$

using $u = \csc(x) + \cot(x)$ and $du = -\csc^2(x) - \csc(x)\cot(x)dx$

$$-\int \frac{du}{u} = -\ln |u| + C = -\ln |\csc(x) + \cot(x)| + C$$

5.10 $\int \sin^2(x)dx = \frac{1}{2}x - \frac{\sin(2x)}{4} + C$

For this one, you need to rewrite $\sin^2(x)$ in a more easily integratable format. Recall that $\cos(2x) = \cos^2(x) - \sin^2(x)$, or $\cos(2x) = 1 - 2\sin^2(x)$. Simplify a bit further and you get $\sin^2(x) = \frac{1 - \cos(2x)}{2}$.

$$\int \frac{1 - \cos(2x)}{2} dx = \int \frac{1}{2} dx - \int \frac{\cos(2x)}{2} dx$$

$$u = 2x \quad du = 2dx \quad \frac{du}{2} = dx$$

$$\int \frac{1}{2}dx - \int \frac{\cos(2x)}{2}dx = \frac{1}{2}x - \frac{1}{2} \int \frac{\cos(u)}{2}dx + C = \frac{1}{2}x - \frac{\sin(u)}{4} + C = \frac{1}{2}x - \frac{\sin(2x)}{4} + C$$

$$\mathbf{5.11} \quad \int \cos^2(x)dx = \frac{1}{2}x + \frac{\sin(2x)}{4} + C$$

$$\cos(2x) = \cos^2(x) - \sin^2(x) \quad \cos(2x) = \cos^2(x) - (1 - \cos^2(x)) \quad \cos(2x) = 2\cos^2(x) - 1$$

$$\cos^2(x) = \frac{\cos(2x) + 1}{2} \quad \int \cos^2(x)dx = \int \frac{1}{2}dx + \int \frac{\cos(2x)}{2}dx$$

using the same u-sub logic:

$$u = 2x \quad du = 2dx \quad \frac{du}{2} = dx$$

$$\frac{1}{2}x + \frac{1}{4} \int \sin(u)du + C = \frac{1}{2}x + \frac{\sin(2x)}{4} + C$$

6 Polynomials

These problems assume a polynomial in the form:

$$\int \frac{A_1x^n + A_2x^{n-1} + \dots + A_n}{B_1x^m + B_2x^{m-1} + \dots + B_n}dx$$

Where $\{n, m\} \in \mathbb{R}$, $\{A_1, A_2, \dots, A_n\} \in \mathbb{R}$, $\{B_1, B_2, \dots, B_n\} \in \mathbb{R}$

6.1 Long Division

Where $n \geq m$, simply use long division.

6.2 Partial Fraction Decomposition

Where $n - 1 = m$

Used to simplify:

$$\int \frac{A_1x + A_2}{B_1x^2 + B_2x + B_3}dx$$

Best illustrated with an example:

$$\int \frac{-3x - 7}{x^2 - 5x + 6}dx = \int \frac{-3x - 7}{(x - 2)(x - 3)}dx$$

We need to find an A and B such that:

$$\frac{A}{x - 2} + \frac{B}{x - 3} = \frac{-3x - 7}{(x - 2)(x - 3)}$$

$$\frac{A(x-3)}{(x-2)(x-3)} + \frac{B(x-2)}{(x-2)(x-3)} = \frac{-3x-7}{(x-2)(x-3)} \implies A(x-3) + B(x-2) = -3x-7$$

We can obtain the two following equations:

$$Ax + Bx = -3x \implies A + B = -3 \implies B = -3 - A$$

$$-3A - 2B = -7$$

Substitute:

$$-3A - 2(-3 - A) = -7 \implies -3A + 6 + 2A = -7 \implies -A = -13 \implies A = 13$$

$$B = -3 - A \implies B = -3 - 13 \implies B = -16$$

Now we can integrate the two sections individually:

$$\int \frac{-3x-7}{(x-2)(x-3)} dx = 13 \int \frac{1}{x-2} dx - 16 \int \frac{1}{x-3} dx = 13 \ln |x-2| - 16 \ln |x-3| + C$$

7 Improper Integrals

$$\int_a^\infty f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx$$

Similarly:

$$\int_{-\infty}^\infty f(x) dx = \int_a^\infty f(x) dx + \int_{-\infty}^a f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx + \lim_{c \rightarrow -\infty} \int_c^a f(x) dx$$

The integral may not always exist. For polynomials,

$$\lim_{x \rightarrow \infty} \frac{x^n \dots}{x^m \dots} \in \mathbb{R} \iff m \geq n + 2$$

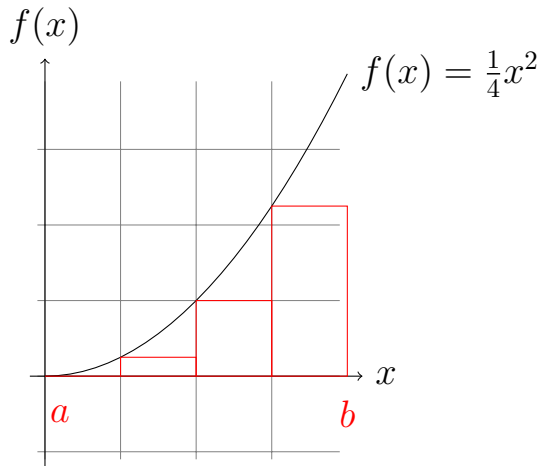
If the degree of the denominator is two higher than the numerator, then the integral will exist.

8 Riemann Sum

A typical Riemann Sum will equal the general expression:

$$\sum f(x_i) \Delta x$$

8.1 Left Riemann Sum



Let total steps be represented by n . Let i represent the current step

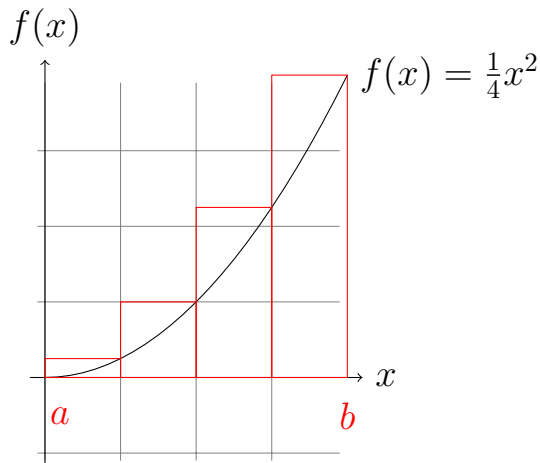
$$\Delta x = \frac{b - a}{n}$$

$$f(x_i) = \frac{f(a + i\Delta x)}{n}$$

Put them together:

$$\sum_{i=0}^{n-1} f(x_i) \Delta x \implies \sum_{i=0}^{n-1} \frac{f(a + i\Delta x)}{n} \Delta x \implies \sum_{i=0}^{n-1} \frac{f(a + i\frac{b-a}{n}x)(b-a)}{n^2}$$

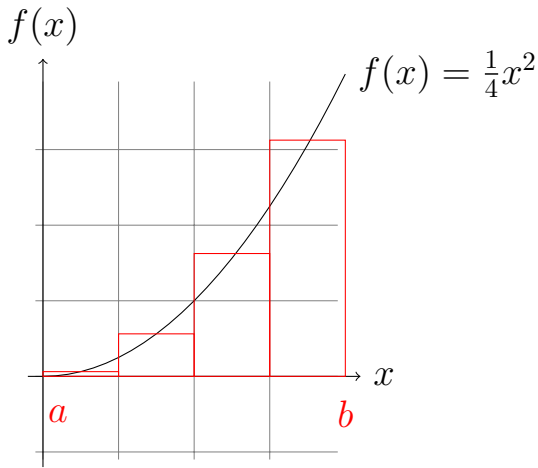
8.2 Right Riemann Sum



Nothing changes, except for the \sum bounds:

$$\sum_{i=1}^n f(x_i) \Delta x \implies \sum_{i=1}^n \frac{f(a + i\Delta x)}{n} \Delta x \implies \sum_{i=1}^n \frac{f(a + i\frac{b-a}{n}x)(b-a)}{n^2}$$

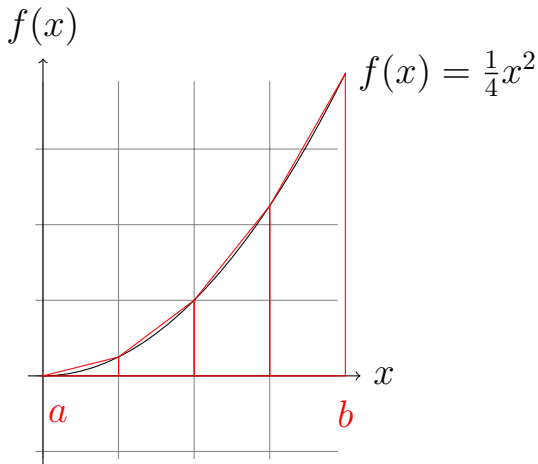
8.3 Midpoint Riemann Sum



As the name suggests, you take the average of $f(x_i)$ and $f(x_{i+1})$. Instead of using $f(x_i)$, use $f(\frac{x_i+x_{i+1}}{2})$

$$\begin{aligned} \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right) \Delta x &\Rightarrow \sum_{i=0}^{n-1} \frac{f\left(\frac{a+i\Delta x + a+(i+1)\Delta x}{2}\right)}{n} \Delta x = \sum_{i=0}^{n-1} \frac{f\left(\frac{2a+(2i+1)\Delta x}{2}\right)}{n} \Delta x \\ &\Rightarrow \sum_{i=0}^{n-1} \frac{f\left(\frac{2a+(2i+1)\frac{b-a}{n}}{2}\right)(b-a)}{n^2} \end{aligned}$$

8.4 Trapezoidal Riemann Sum



Take the average of the two y-values instead. Instead of using $f(x_i)$, use $\frac{f(x_i)+f(x_{i+1})}{2}$

$$\begin{aligned} \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} \Delta x &\Rightarrow \sum_{i=0}^{n-1} \frac{f(a + i\Delta x) + f(a + (i+1)\Delta x)}{2n} \Delta x \\ &\Rightarrow \sum_{i=0}^{n-1} \frac{(f(a + i\frac{b-a}{n}) + f(a + (i+1)\frac{b-a}{n}))(b-a)}{2n^2} \end{aligned}$$

9 Distance

The distance of the curve is represented by the following:

$$\begin{aligned}\text{arc length} &= \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \Delta x \\ &\implies \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx\end{aligned}$$

9.1 Example

These problems are taken from *Calculus for the AP Course* by Michael Sullivan.

1. Let g be the function defined by $g(x) = \int_1^x \sqrt{9t^2 - 1} dt$. Which integral represents the graph of g on the interval $[2, 5]$?

Let $f(x) = \sqrt{9x^2 - 1}$. Let $F(x)$ represent its antiderivative.

$$\begin{aligned}g(x) &= F(x) - F(1) \\ g'(x) &= \sqrt{9x^2 - 1} \\ \int_2^5 \sqrt{1 + 9x^2 - 1} dx &= \int_2^5 3x dx\end{aligned}$$