

Sets Cheatsheet

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1 Geometric Series

Geometric Series are a special type of Power Series that can be rewritten in the form

$$f(x) = \frac{a_1}{1-r} = \sum_{n=0}^{\infty} a_1 \cdot r^n$$

1.1 Proof

$$\begin{aligned} S &= a_1 + a_1r + a_1r^2 + a_1r^3 + \dots + a_1r^n + \dots \\ rS &= a_1r + a_1r^2 + a_1r^3 + a_1r^4 \dots + a_1r^n + \dots \\ S - rS &= a_1 + a_1r - a_1r + a_1r^2 - a_1r^2 + \dots + a_1r^n - a_1r^n + a_1r^{n+1} + \dots \\ &\implies S(1-r) = a_1 + a_1r^{n+1} \end{aligned}$$

Evaluate the lim as $n \rightarrow \infty$

$$S = \lim_{n \rightarrow \infty} \frac{a_1 + a_1r^{n+1}}{1-r}$$

We can only evaluate this equation when $|r| < 1$. Therefore:

$$\frac{a_1}{1-r}$$

Is the sum of the geometric series.

To find the interval of convergence, just remember that $|r| < 1$

1.2 Examples

$$\begin{aligned} f(x) = \frac{1}{2-x}, c=0 &\implies f(x) = \frac{1}{2} \cdot \frac{1}{1-\frac{x}{2}} \implies f(x) = \sum_{n=0}^{\infty} \frac{1}{2} \cdot \left(\frac{x}{2}\right)^n \\ &\implies f(x) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} x^n \end{aligned}$$

$$|r| < 1 \implies \left|\frac{x}{2}\right| < 1 \implies |x| < 2 \implies x \in (-2, 2)$$

If we change the center:

$$\begin{aligned} f(x) = \frac{1}{2-x}, c=5 &\implies f(x) = \frac{1}{2-5-(x-5)} \implies \frac{1}{-3-(x-5)} \\ &\implies -\frac{1}{3} \cdot \frac{1}{1-\frac{-(x-5)}{3}} \implies \sum_{n=0}^{\infty} -\frac{1}{3} \cdot \left(\frac{-(x-5)}{3}\right)^n \end{aligned}$$

$$\implies \sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^{n+1} (-(x-5))^n$$

$$|r| < 1 \implies r \in (-1, 1) \implies \frac{-x+5}{3} \in (-1, 1) \implies -(x-5) \in (-3, 3) \implies x \in (2, 8)$$

Now let's completely change it up.

$$f(x) = \frac{3}{2x-1}, c = 0 \implies f(x) = -3 \cdot \frac{1}{1-2x} \implies \sum_{n=0}^{\infty} -3 \cdot (2x)^n, c \in \left(-\frac{1}{2}, \frac{1}{2}\right)$$

Let's move the center

$$f(x) = \frac{3}{2x-1}, c = -3 \implies f(x) = \frac{1}{2(x+3)-5-6} \implies f(x) = -\frac{1}{11} \cdot \frac{1}{1-\frac{2}{11}(x+3)}$$

$$f(x) = \sum_{n=0}^{\infty} -\frac{1}{11} \cdot \left(\frac{2}{11}(x+3)\right)^n$$

$$\frac{2}{11}(x+3) \in (-1, 1) \implies (x+3) \in \left(-\frac{11}{2}, \frac{11}{2}\right) \implies x \in \left(-\frac{17}{2}, \frac{5}{2}\right)$$

What if there is no $-x$? You can just do $-(-x)$. Remember: Basic Algebra will take you a long way.

$$f(x) = \frac{3}{x+2}, c = 0 \implies f(x) = \frac{3}{2} \cdot \frac{1}{1-\frac{1}{2}(-x)}$$

What if your function looks a bit more complicated?

$$f(x) = \frac{4x-7}{2x^2+3x-2}, c = 0 \implies f(x) = \frac{4x-7}{(2x-1)(x+2)} \implies f(x) = \frac{A}{2x-1} + \frac{B}{x+2}$$

$$A(x+2) + B(2x-1) = 4x-7 \implies Ax + 2Bx = 4x, 2A - B = -7 \implies B = 2A + 7 \\ \implies A + 4A + 14 = 4 \implies 5A = -10 \implies A = -2, B = 3$$

$$f(x) = \frac{-2}{2x-1} + \frac{3}{x+2}$$

Solve it normally from here.

2 General Power Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n$$

2.1 Derivation

We take general power series to be something like the following:

$$f(x) = 1 + x + 2x^2 + 3x^3 + \dots + nx^n + \dots$$

or

$$f(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + \frac{1}{n}x^n + \dots$$

So generally:

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$$

Instead of a_i because the coefficient changes for each element in the series. Let's expand this series:

$$f(x) = a_0 + c + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \dots + a_n(x-c)^n + \dots$$

Take its derivative:

$$f'(x) = 0 + a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \dots + na_n(x-c)^{n-1} + \dots$$

Notice how $f'(0) = a_1$.

$$f''(x) = 0 + 0 + 2a_2 + 6a_3(x-c) + \dots + n(n-1)a_n(x-c)^{n-2} + \dots$$

Notice how $f''(0) = 2a_2$ and $f'''(0) = 6a_3$. In order to get the a_n term, you just need to take $f^{(n)}(c)$ and divide it by $n!$

Put it together, and you'll get the equation we started with.

2.2 Examples

$$\begin{aligned} f(x) = e^x, c = 0 &\implies f(x) = \sum_{n=0}^{\infty} \frac{e^0}{n!}(x)^n \implies f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ &\implies 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + \frac{1}{n}x^n + \dots \end{aligned}$$

Evaluate a geometric series using Taylor series

$$\begin{aligned} f(x) = \frac{1}{1-2x}, c = 0 &\implies \frac{\frac{1}{1-2(0)}}{0!} + \frac{\frac{2}{(1-2c)^2}}{1!}x + \frac{\frac{8}{(1-2c)^3}}{2!}x + \frac{\frac{48}{(1-2c)^4}}{3!}x + \dots \\ &\implies 1 + \frac{2}{(1-2c)^2}x + \frac{4}{(1-2c)^3}x^2 + \frac{8}{(1-2c)^4}x^3 + \dots \end{aligned}$$

General term: $2^n x^n$.

We would have gotten the same thing if we used the power series expansion of the Taylor series expansion. As with most things in algebra, pick the method that's **most convenient for you**.

2.3 Error Checking

In an ideal world, you want to know how far off your estimates are. For alternating series, this process is pretty easy.

$$f(x) = 1 - x + \frac{x}{2!} - \frac{x}{3!} + \frac{x}{4!} - \frac{x}{5!} + \frac{x}{6!} - \frac{1}{7!} + \dots + \frac{1}{n!}(-1)^n + \dots$$

Let's choose the first four terms for our **Taylor Polynomial**, which will be represented by $P_n(x)$ where n is the degree.

$$P_4(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{3}x^3$$

Let's set the remainder terms to $R_5(x)$. We can call the error $E(x)$.

$$E(x) = |P_4(x) - R_5(x)|$$

Since you will never truly know $R_5(x)$ or any $R_n(x)$ for that matter, you will never know the true error. But you can set an upper bound on that error, which is pretty easy in alternating power series.

$$E(x) \leq P_{n+1}(x) - P_n(x)$$

A fancy way of saying, the term right after is the highest the error can be. It's because you will never add back what you subtracted, always less.

2.4 Deriving e^a

We can use a Taylor Series to derive e^a !

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(1 + \frac{a}{n}\right)^n = L &\implies \ln\left(\lim_{n \rightarrow \infty} \left(1 + \frac{a}{n}\right)^n\right) = \ln L \implies n \ln\left(\lim_{n \rightarrow \infty} \left(1 + \frac{a}{n}\right)\right) = \ln L \\ t = \frac{a}{n}, &\implies \frac{a}{t} \left(\lim_{n \rightarrow \infty} \ln(1 + t)\right) = \ln L \implies \frac{a}{t} \left(\lim_{t \rightarrow 0} \ln(1 + t)\right) = \ln L \\ &\implies \ln(1 + t) = t - \frac{t^2}{2} + \frac{t^3}{3} + \dots \approx t \\ &\implies \ln L = a \implies L = e^a \end{aligned}$$

2.5 Lagrange Error Checking

If your series **isn't alternating**, you can use Lagrange Error Checking. Although the proof is complicated, it's pretty simple

$$E(x) \leq \left| \frac{\max_{z \in [x, c]} f^{(n+1)}(z)}{(n+1)!} \right| (x - c)^{n+1}$$

The proof isn't necessary for Calc BC. You can find it online.

2.5.1 Full Example

Find the first four terms about $c = 2$ for $\ln|x + 1|$.

We can start with its derivative, $\frac{1}{x+1}$.

For simplicity, we will use a geometric series. Say we wanted to find $\ln|\frac{3}{2}|$ and approximate error to be 0.05 or less

$$\begin{aligned}
 \frac{1}{x-1} &= \frac{1}{1-(x-2)+2} = \frac{1}{-1-(x-2)} = -\frac{1}{1-(-(x-2))} \\
 \Rightarrow \sum_{n=0}^{\infty} -(-(x-2))^n &= \sum_{n=0}^{\infty} (-1)^{n+1}(x-2)^n = -1 + (x-2) - \frac{1}{2}(x-2)^2 + \frac{1}{3}(x-2)^3 + \dots \\
 \int -\frac{1}{1-(-(x-2))} dx &= \int -1 + (x-2) - \frac{1}{2}(x-2)^2 + \frac{1}{3!}(x-2)^3 + \dots dx \\
 \Rightarrow -\ln|1-(-(x-2))| &= C - x + \frac{(x-2)^2}{2} - \frac{(x-2)^3}{3 \cdot 2!} + \frac{(x-2)^4}{4 \cdot 3!} + \dots \\
 &\rightarrow \ln|1-(-(2-2))| = \ln|1| \\
 &\rightarrow C + x - \frac{(x-2)^2}{2} + \frac{(x-2)^3}{3 \cdot 2!} - \frac{(x-2)^4}{4 \cdot 3!} + \dots \\
 &\rightarrow \ln|1| = C + 2 + 0 + \dots \rightarrow C = -2 \\
 \Rightarrow \ln|1-(-(x-2))| &= -2 + x - \frac{(x-2)^2}{2} + \frac{(x-2)^3}{3 \cdot 2!} - \frac{(x-2)^4}{4 \cdot 3!} + \dots \\
 \Rightarrow \ln|\frac{3}{2}| = \ln|1-(-(\frac{5}{2}-2))| &= (\frac{5}{2}-2) - \frac{(\frac{5}{2}-2)^2}{2} + \frac{(\frac{5}{2}-2)^3}{3 \cdot 2!} - \frac{(\frac{5}{2}-2)^4}{4 \cdot 3!} + \dots \\
 &\Rightarrow \frac{1}{2} - \frac{1}{2} \cdot (\frac{1}{2})^2 + \frac{1}{3} \cdot (\frac{1}{2})^3 - \frac{1}{4} \cdot (\frac{1}{2})^4 + \dots
 \end{aligned}$$

Since it's an alternating series, it should be pretty simple to solve for from here.

2.5.2 Another Problem

$$\sin(5x + \frac{\pi}{4}) = \sin(5(x + \frac{\pi}{20}))$$

3 P Series

NOT Power Series.

A P Series is any series in the form:

$$\sum_{n=0}^{\infty} \frac{1}{n^p}$$

We use Integral Test to prove if it is diverging or converging.

3.1 Harmonic Series

A Harmonic Series is:

$$\sum_{n=0}^{\infty} \frac{1}{n}$$

Or a P Series where $p = 1$.

3.2 Proof

Where $p = 1$ (Harmonic Series)

$$\begin{aligned} \int_1^{\infty} \frac{1}{x} dx &< \sum_{n=1}^{\infty} \frac{1}{n} < \int_1^{\infty} \frac{1}{x} dx + 1 \implies \int_1^{\infty} \frac{1}{x} dx = \ln |x| \Big|_1^{\infty} \\ &\implies \lim_{x \rightarrow \infty} \ln x = \infty \end{aligned}$$

Diverges

Where $p \neq 1$

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^p} dx &< \sum_{n=1}^{\infty} \frac{1}{n^p} < \int_1^{\infty} \frac{1}{x^p} dx + 1 \longrightarrow \int_1^{\infty} x^{-p} dx = \frac{x^{-p+1}}{-p+1} \Big|_1^{\infty} \\ &\longrightarrow \lim_{x \rightarrow \infty} \frac{x^{-p+1}}{-p+1} \end{aligned}$$

If $p > 1$ it converges, $p < 1$ (or $p \leq 1$) it diverges.

3.3 Examples

$$\sum_{n=1}^{\infty} \frac{1}{(2n+3)^2} \implies \int_1^{\infty} \frac{1}{(2x+3)^2} dx \longrightarrow u = 2x+3 \quad du = 2dx \implies \int_3^{\infty} \frac{1}{2u^3} du$$

Convergent

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{n+2}{n+1} &\implies \int_1^{\infty} \frac{x+2}{x+1} dx = \int_1^{\infty} \left(1 + \frac{1}{x+1}\right) dx = (x + \ln |x+1|) \Big|_1^{\infty} \\ &= \lim_{x \rightarrow \infty} x + \ln |x+1| = \infty \end{aligned}$$

Divergent

4 Convergent or Divergent

4.1 nth Term Test for Divergence

If

$$\lim_{x \rightarrow \infty} a_x \neq 0$$

then f is divergent. If

$$\lim_{x \rightarrow \infty} a_x = 0$$

then use another test, it's inconclusive.

4.2 Ratio Test

Commonly used for geometric series, if you just check to make sure $|r| < 1$ then it's going to converge.

$$\lim_{n \rightarrow \infty} \left| \frac{a^{n+1}}{a^n} \right|$$

Basically checking that for some far off value, the common ratio remains.
Example:

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^{n-1} n^2}{2^n} &\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{\frac{(-1)^n (n+1)^2}{2^{n+1}}}{\frac{(-1)^{n-1} n^2}{2^n}} \right| = 0 \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(-1)^n (n+1)^2 \cdot 2^n}{2^n \cdot 2 \cdot (-1)^{n-1} n^2} \right| \\ &\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(n+1)^2}{2 \cdot -1 \cdot n^2} \right| = \frac{1}{2} \end{aligned}$$

If $|r| = 1$, we have a problem. Recall the limit from the geometric series proof:

$$\lim_{n \rightarrow \infty} \frac{a - ar^{n+1}}{1 - r}$$

That's going to be undefined. So we have to use another test.

This test is best used for rapidly growing questions like $n!$.

4.2.1 More Examples

$$\sum_{n=0}^{\infty} \frac{e^n}{n^3} \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{\frac{e^n \cdot e}{(n+1)^3}}{\frac{e^n}{n^3}} \right| \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{e \cdot n^3}{(n+1)^3} \right| = e \not< 1$$

Divergent

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(n!)^2}{(3n)!} &\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{\frac{((n+1)!)^2}{(3n+3)!}}{\frac{(n!)^2}{(3n)!}} \right| \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{((n+1)!)^2 \cdot (3n)!}{(3n+3)! \cdot (n!)^2} \right| \\
&\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(n+1)^2 \cdot (n!)^2 \cdot (3n)!}{(3n+3) \cdot (3n+2) \cdot (3n+1) \cdot (3n)! \cdot (n!)^2} \right| \\
&\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(n+1)^2}{(3n+3) \cdot (3n+2) \cdot (3n+1)} \right| = 0 < 1
\end{aligned}$$

Convergent

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{n!}{n^{n+1}} &\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{\frac{(n+1)!}{(n+1)^{n+2}}}{\frac{n!}{n^{n+1}}} \right| \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(n+1)! \cdot n^{n+1}}{n! \cdot (n+1)^{n+2}} \right| \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{n! \cdot n^{n+1}}{n! \cdot (n+1)^{n+1}} \right| \\
&\Rightarrow \lim_{n \rightarrow \infty} \left| \left(\frac{n}{n+1} \right)^{n+1} \right| = \lim_{n \rightarrow \infty} \left| \left(\frac{n+1-1}{n+1} \right)^{n+1} \right| = \lim_{n \rightarrow \infty} \left| \left(1 - \frac{1}{n+1} \right)^{n+1} \right| = \\
&\lim_{n \rightarrow \infty} \left| \left(1 + \frac{(-1)}{n+1} \right)^{n+1} \right| = e^{-1} < 1
\end{aligned}$$

Convergent

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{3^{1-2n}}{n^2+1} &\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{\frac{3^{1-2n-2}}{(n+1)^2+1}}{\frac{3^{1-2n}}{n^2+1}} \right| \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{3^{-1-2n} \cdot (n^2+1)}{(n+1)^2+1 \cdot 3^{1-2n}} \right| \\
&\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{n^2+1}{(n+1)^2+1 \cdot 3^{1-2n} \cdot 3^{1+2n}} \right| \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{n^2+1}{(n+1)^2+1 \cdot 3 \cdot 3^{-2n} \cdot 3 \cdot 3^{2n}} \right| \\
&\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{n^2+1}{(n+1)^2+1 \cdot 3 \cdot 3} \right| = \frac{1}{9}
\end{aligned}$$

Convergent

$$\begin{aligned}
&\sum_{n=2}^{\infty} \frac{(-2)^{1+3n}(n+1)}{n^2 5^{1+n}} \\
&\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{\frac{(-2)^{4+3n}(n+2)}{(n+1)^2 5^{2+n}}}{\frac{(-2)^{1+3n}(n+1)}{n^2 5^{1+n}}} \right| \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(-2)^{4+3n}(n+2)n^2 5^{1+n}}{(n+1)^2 5^{2+n} (-2)^{1+3n}(n+1)} \right| \\
&\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(-2)^3(n+2)n^2}{(n+1)^3 5} \right| = \frac{(-2)^3}{5^5} = -\frac{8}{5}
\end{aligned}$$

Diverges

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{6n+7} &\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{\frac{(-1)^{n+2}}{6n+6+7}}{\frac{(-1)^{n+1}}{6n+7}} \right| \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+2}(6n+7)}{(6n+6+7)(-1)^{n+1}} \right| \\
&\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{6n+7}{6n+6+7} \right|
\end{aligned}$$

Inconclusive: But if you used the Divergence Test first:

$$\lim_{n \rightarrow \infty} \frac{(-1)^{n+1}}{6n+7} \approx \frac{(-1)^\infty}{\infty} = 0$$

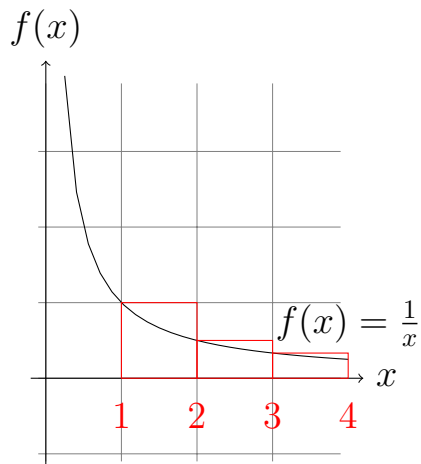
Inconclusive, but the alternating series test proves it right.

$$\begin{aligned} & \sum_{n=3}^{\infty} \frac{e^{4n}}{(n-2)!} \\ & \longrightarrow \lim_{n \rightarrow \infty} \frac{e^{4n}}{(n-2)!} = 0 \\ \implies & \lim_{n \rightarrow \infty} \left| \frac{\frac{e^{4n+4}}{(n-1)!}}{\frac{e^{4n}}{(n-2)!}} \right| \implies \lim_{n \rightarrow \infty} \left| \frac{e^{4n+4}(n-2)!}{(n-1)!e^{4n}} \right| \implies \lim_{n \rightarrow \infty} \left| \frac{e^4(n-2)!}{(n-1)(n-2)!} \right| \implies \lim_{n \rightarrow \infty} \left| \frac{e^4}{(n-1)} \right| \\ & \implies 0 \end{aligned}$$

Convergent

4.3 Integral Test

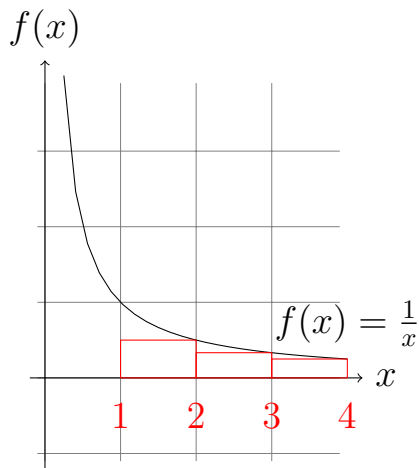
If $f(x)$ is decreasing, then for $x \geq 1$:



The Left Riemann Sum is $>$ than $f(x)$:

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots \implies \sum_{n=1}^{\infty} \frac{1}{n} > \int_1^{\infty} \frac{1}{x} dx$$

Similarly,



The Right Riemann Sum is $<$ than $f(x)$:

$$\begin{aligned} \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots &\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n} < \int_1^{\infty} \frac{1}{x} dx \\ &\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n} < \int_1^{\infty} \frac{1}{x} dx + 1 \\ &\Rightarrow \int_1^{\infty} \frac{1}{x} dx < \sum_{n=1}^{\infty} \frac{1}{n} < \int_1^{\infty} \frac{1}{x} dx + 1 \end{aligned}$$

Using a method analogous to the Squeeze Theorem, if the bounds diverge, so does the series, and vice versa.

4.4 Direct Comparison Test

The direct comparison test allows for **no negative terms** in the series.

Let a_n and b_n represent an unknown and known sequence, respectively.

If $a_n < b_n$ for all n (according to CollegeBoard, however the first few terms are an exception) and $\lim_{n \rightarrow \infty} \sum_i^n b_i = L$ then $\lim_{n \rightarrow \infty} \sum_i^n a_i = L$

If $a_n > b_n$ for all n and $\lim_{n \rightarrow \infty} \sum_i^n b_i = L$ then $\lim_{n \rightarrow \infty} \sum_i^n a_i$ could equal L or ∞ .

If $a_n < b_n$ for all n and $\lim_{n \rightarrow \infty} \sum_i^n b_i = \infty$ then $\lim_{n \rightarrow \infty} \sum_i^n a_i$ could equal L or ∞ .

If $a_n > b_n$ for all n and $\lim_{n \rightarrow \infty} \sum_i^n b_i = \infty$ then $\lim_{n \rightarrow \infty} \sum_i^n a_i = \infty$.

It is recommended to use this test only if the degree is the same.

4.4.1 Examples

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{1}{n^2+1} &= \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \dots \\ \sum_{n=2}^{\infty} \frac{1}{n^2} &= \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots \\ \Rightarrow \sum_{n=2}^{\infty} \frac{1}{n^2} &> \sum_{n=2}^{\infty} \frac{1}{n^2+1} \Rightarrow \sum_{n=2}^{\infty} \frac{1}{n^2} = L \therefore \sum_{n=2}^{\infty} \frac{1}{n^2+1} = L_2\end{aligned}$$

Convergent

4.5 Limit Comparison Test

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$$

For $0 < n < \infty$ (if it is 0 or ∞ anyway, the previous test would suffice).

4.5.1 Examples

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{n}{n^2+1} &= \frac{1}{2} + \frac{2}{5} + \frac{3}{10} + \dots \\ \sum_{n=1}^{\infty} \frac{n}{n^2} &= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots \\ \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n} &> \sum_{n=1}^{\infty} \frac{n}{n^2+1} = \lim_{n \rightarrow \infty} \frac{1}{n} > \lim_{n \rightarrow \infty} \frac{n}{n^2+1} = \infty > \lim_{n \rightarrow \infty} \frac{n}{n^2}\end{aligned}$$

Inconclusive

$$\lim_{n \rightarrow \infty} \frac{\frac{n}{n^2+1}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2+1} = 1$$

Convergent

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{5n-3}{n^2-2n+5} &= \frac{-3}{5} + \frac{2}{4} + \frac{7}{5} + \frac{12}{8} + \frac{17}{13} \dots \\ \sum_{n=1}^{\infty} \frac{n}{n^2} &= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \dots \\ \Rightarrow \lim_{n \rightarrow \infty} \frac{\frac{5n-3}{n^2-2n+5}}{\frac{1}{n}} &= \lim_{n \rightarrow \infty} \frac{5n^2-3n}{n^2-2n+5} = 5 \\ \lim_{n \rightarrow \infty} \frac{1}{n} &= \infty \therefore \lim_{n \rightarrow \infty} \frac{5n-3}{n^2-2n+5} = \infty\end{aligned}$$

Divergent

4.6 Alternating Series Property Theorem

Recall the Alternating Error Bound for a series. Take this example:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

Since the sum is **decreasing** and $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ (since n is increasing), there is an error bound, meaning this alternating series is convergent. Note that this "test" doesn't apply to oscillating serieses like $\sin(x)$.

4.6.1 Example

$$\lim_{n \rightarrow \infty} \frac{n}{2n+1} = \frac{1}{2} \implies \sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{2n+1}$$

This sum does not reach 0, it osciilates. It is not convergent.

4.7 Absolute Convergence Test

Let a_n represent a sequence with positive and negative terms.

Using some common sense, we can derive the following:

$$\begin{aligned} n \in \mathbb{R} &\implies n \leq |n| \\ &\therefore a_n \leq |a_n| \\ &\implies a_n + |a_n| \leq 2|a_n| \\ \longrightarrow |a_n| + a_n &= 0 \quad \text{or} \quad 2a_n \therefore 0 \leq a_n + |a_n| \leq 2a_n \end{aligned}$$

By the Direct Comparsion test:

$$\sum_{n=1}^{\infty} |a_n| = L \therefore \sum_{n=1}^{\infty} a_n = L_2$$

In this case it is absolutely convergent. In the scenario that:

$$\sum_{n=1}^{\infty} |a_n| = \infty$$

It is inconclusive. If the other alternating series tests passes, it is **Conditionally Convergent**.

4.8 Examples

$$\sum_{n=2}^{\infty} \frac{n}{\ln n} \Rightarrow \lim_{n \rightarrow \infty} \frac{n}{\ln n} = \infty$$

Divergent by nth Term Test for Divergence

$$\sum_{n=1}^{\infty} \frac{2^n}{n^{99}} \Rightarrow \lim_{n \rightarrow \infty} \frac{2^n}{n^{99}} = \infty$$

Divergent by nth Term Test for Divergence

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^2 + 1}}{n} \Rightarrow \lim_{n \rightarrow \infty} \frac{\sqrt{n^2 + 1}}{n} \approx \lim_{n \rightarrow \infty} \frac{\sqrt{n^2}}{n} = \lim_{n \rightarrow \infty} \frac{n}{n} = 1$$

Divergent by nth Term Test for Divergence

$$\sum_{n=2}^{\infty} \frac{n^e}{n^{\pi}} = \sum_{n=2}^{\infty} \frac{1}{n^{\pi-e}} \Rightarrow 1 \geq \pi - e$$

Divergent by property of P-Series

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+8}} &= \frac{1}{\sqrt{8}} + \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+8}} = \frac{1}{\sqrt{8}} + \frac{1}{3} + \frac{1}{\sqrt{10}} + \dots \\ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} &= \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots \\ &\Rightarrow \lim_{n \rightarrow \infty} \frac{\frac{1}{\sqrt{n+8}}}{\frac{1}{\sqrt{n}}} = \lim_{n \rightarrow \infty} \frac{\sqrt{n+8}}{\sqrt{n}} = 1 \end{aligned}$$

Divergent by Limit Comparison Test

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{1} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots$$

Convergent by Properties of Alternating Series:

1. It is decreasing
2. It is approaching 0
3. It is alternating

$$\begin{aligned}
\sum_{n=3}^{\infty} \frac{4}{n\sqrt{\ln n}} &= \frac{4}{3\sqrt{\ln 3}} + \frac{4}{4\sqrt{\ln 4}} + \frac{4}{5\sqrt{\ln 5}} + \dots \\
\sum_{n=3}^{\infty} \frac{4}{n} &= \frac{4}{3} + \frac{4}{4} + \frac{4}{5} + \dots \\
\Rightarrow \sum_{n=3}^{\infty} \frac{4}{n\sqrt{\ln n}} &< \sum_{n=3}^{\infty} \frac{4}{n}, \sum_{n=3}^{\infty} \frac{4}{n} = \infty \\
&\Rightarrow \lim_{n \rightarrow \infty} \frac{\frac{4}{n\sqrt{\ln n}}}{\frac{4}{n}} = 0 \\
&\lim_{n \rightarrow \infty} \frac{\frac{4}{n\sqrt{\ln(n+1)+\sqrt{\ln(n+1)}}}}{\frac{4}{n\sqrt{\ln n}}} = ???
\end{aligned}$$

After trying three tests, we are left with the Integral Test

$$\begin{aligned}
&\sum_{n=3}^{\infty} \frac{4}{n\sqrt{\ln n}} \\
\Rightarrow 4 \int_3^{\infty} \frac{1}{n\sqrt{\ln n}} dn, & u = \ln n, \quad du = \frac{1}{n} dn \\
\Rightarrow 4 \int_{\ln 3}^{\infty} \frac{1}{\sqrt{u}} du &= \int_{\ln 3}^{\infty} u^{-\frac{1}{2}} du = 4 * [2u^{\frac{1}{2}}] \Big|_{\ln 3}^{\infty}
\end{aligned}$$

Divergent by Integral Test

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{27n^2}} = \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{2}{3}}}$$

Divergent by Property of P-Series

$$\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2 + 1}} \Rightarrow \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 + 1}} \approx \lim_{x \rightarrow \infty} \frac{x}{x} = 1$$

Divergent by nth Term Test for Divergence

$$\sum_{n=1}^{\infty} \sqrt{\frac{n+1}{n}}$$

Be careful, this is NOT a disguised $\frac{1}{e}$.

$$\approx \sum_{n=1}^{\infty} \sqrt{\frac{n}{n}} = 1$$

Divergent by nth Term Test for Divergence

$$\sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n} = \sum_{n=1}^{\infty} \frac{2^n}{4^n} + \frac{3^n}{4^n} = \sum_{n=1}^{\infty} \frac{2^n}{4^n} + \sum_{n=1}^{\infty} \frac{3^n}{4^n} = \sum_{n=1}^{\infty} \left(\frac{2}{4}\right)^n + \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$$

Convergent by Property of Geometric Series

$$\begin{aligned} \sum_{n=3}^{\infty} \frac{1}{n(\ln n)(\ln(\ln n))} &= \sum_{n=3}^{\infty} \frac{1}{n(\ln n)} \frac{1}{(\ln(\ln n))} \\ \Rightarrow \sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)} &= \frac{1}{\ln(\ln 3)} + \frac{1}{\ln(\ln 4)} + \frac{1}{\ln(\ln 5)} + \dots \\ &\sum_{n=3}^{\infty} \frac{1}{n} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots \\ \Rightarrow \sum_{n=3}^{\infty} \frac{1}{n} &< \sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)} \\ \sum_{n=3}^{\infty} \frac{1}{n} &= \infty \therefore \sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)} = \infty \end{aligned}$$

A divergent sum times any sum is divergent.

Divergent by Property of P-Series

$$\begin{aligned} \sum_{n=3}^{\infty} \frac{1}{2^n - n} &= \frac{1}{5} + \frac{1}{12} + \frac{1}{27} + \dots \\ \sum_{n=3}^{\infty} \frac{1}{2^n} &= \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots \\ \sum_{n=3}^{\infty} \frac{1}{2^n - n} &> \sum_{n=3}^{\infty} \frac{1}{2^n} \\ \sum_{n=3}^{\infty} \frac{1}{2^n} &= L \\ \lim_{n \rightarrow \infty} \frac{\frac{1}{2^n - n}}{\frac{1}{2^n}} &= \lim_{n \rightarrow \infty} \frac{2^n}{2^n - n} = 1 \end{aligned}$$

Convergent by Limit Comparison Test

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{5+8^n}{2-7^n} &= \sum_{n=1}^{\infty} \left(\frac{5}{2-7^n} + \frac{8^n}{2-7^n} \right) = \sum_{n=1}^{\infty} \left(\frac{5}{2-7^n} - \frac{8^n}{7^n-2} \right) \\
&\Rightarrow \sum_{n=1}^{\infty} \frac{8^n}{7^n-2} = \frac{8}{5} + \frac{64}{47} + \dots \\
\lim_{n \rightarrow \infty} \frac{8^n}{7^n-2} &\approx \lim_{n \rightarrow \infty} \frac{8^n}{7^n} = \lim_{n \rightarrow \infty} \left(\frac{8}{7} \right)^n = \infty
\end{aligned}$$

A divergent series plus any series is divergent.
Divergent by nth Term Test for Divergence

5 More Examples

1.

$$\sum_{n=1}^{\infty} (-1)^n \frac{(x-2)^n}{n+1}$$

For P-Series, you can **only use** the Ratio Test to determine Convergence over an Interval

$$\begin{aligned}
\Rightarrow \lim_{n \rightarrow \infty} \frac{\frac{(-1)^{n+1}(x-2)^{n+1}}{n+2}}{\frac{(-1)^n(x-2)^n}{n+1}} &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)(-1)^{n+1}(x-2)^{n+1}}{(n+2)(-1)^n(x-2)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)(x-2)}{(n+2)} \right| \\
&= \lim_{n \rightarrow \infty} |x-2| < 1 \\
&\Rightarrow 1 < x < 3
\end{aligned}$$

Now you have to test the values 1 and 3.

$$\begin{aligned}
&\sum_{n=0}^{\infty} \left| (-1)^n \frac{(x-2)^n}{n+1} \right| \\
&\rightarrow \sum_{n=0}^{\infty} \left| (-1)^n \frac{(1-2)^n}{n+1} \right| \\
&\rightarrow \sum_{n=0}^{\infty} \left| (-1)^n \frac{(-1)^n}{n+1} \right| = \sum_{n=0}^{\infty} \left| \frac{1}{n+1} \right|
\end{aligned}$$

Divergent by Comparision Test

$$\begin{aligned} & \sum_{n=0}^{\infty} \left| (-1)^n \frac{(3-2)^n}{n+1} \right| \\ & \longrightarrow \sum_{n=0}^{\infty} \left| (-1)^n \frac{(1)^n}{n+1} \right| \\ & \longrightarrow \sum_{n=0}^{\infty} \left| \frac{(-1)^n}{n+1} \right| = \sum_{n=0}^{\infty} \left| \frac{(-1)^n}{n+1} \right| \end{aligned}$$

Convergent by Alternating Series "Test"

$$1 < x \leq 3$$

2.

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{x^n n^n}{3^n n!} \\ \implies \lim_{n \rightarrow \infty} \left| \frac{\frac{x^{n+1}(n+1)^{n+1}}{3^{n+1}(n+1)!}}{\frac{x^n n^n}{3^n n!}} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}(n+1)^{n+1} 3^n n!}{x^n n^n 3^{n+1} (n+1)!} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}(n+1)(n+1)^n n!}{x^n n^n 3(n+1)!} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{x(n+1)^n n!}{n^n 3 n!} \right| = \frac{1}{3} \lim_{n \rightarrow \infty} \left| x \left(\frac{n+1}{n} \right)^n \right| = \frac{1}{3} \lim_{n \rightarrow \infty} \left| x \left(1 + \frac{1}{n} \right)^n \right| \\ & \implies \left| \frac{e}{3} x \right| < 1 \implies -\frac{3}{e} < x < \frac{3}{e} \end{aligned}$$

Plug in the endpoints:

$$\sum_{n=1}^{\infty} \frac{\left(-\frac{3}{e}\right)^n n^n}{3^n n!} = \sum_{n=1}^{\infty} \frac{(-1)^n (3)^n n^n}{e^n 3^n n!} = \sum_{n=1}^{\infty} \frac{(-1)^n n^n}{e^n n!}$$

Look at the graph and compare it against $\frac{1}{n}$.

3.

$$\begin{aligned} & \sum_{k=0}^{\infty} \frac{2^k x^k}{\ln(k+2)} \\ \lim_{k \rightarrow \infty} \left| \frac{\frac{2^{k+1} x^{k+1}}{\ln((k+1)+2)}}{\frac{2^k x^k}{\ln(k+2)}} \right| &= \lim_{k \rightarrow \infty} \left| \frac{2^{k+1} x^{k+1} \ln(k+2)}{2^k x^k \ln((k+1)+2)} \right| = \lim_{k \rightarrow \infty} \left| \frac{2x \ln(k+2)}{\ln(k+2+1)} \right| = 2x \\ & -\frac{1}{2} < x < \frac{1}{2} \end{aligned}$$

Plug in the endpoints:

$$\sum_{k=0}^{\infty} \frac{2^k \left(\frac{-1}{2} \right)^k}{\ln(k+2)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{\ln(k+2)}$$

- (a) Alternating
 (b) Approaching 0
 (c) Decreasing
 $k = -\frac{1}{2}$ Convergent

$$\sum_{k=0}^{\infty} \frac{2^k \left(\frac{1}{2}\right)^k}{\ln(k+2)} = \sum_{k=0}^{\infty} \frac{1^k}{\ln(k+2)} = \sum_{k=0}^{\infty} \frac{1}{\ln(k+2)}$$

$$\sum_{k=0}^{\infty} \frac{1}{\ln(k+2)} = \frac{1}{\ln 2} + \sum_{k=1}^{\infty} \frac{1}{\ln(k+2)} = \frac{1}{\ln 2} + \frac{1}{\ln 3} + \frac{1}{\ln 4} + \dots$$

$$\sum_{k=1}^{\infty} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots > \sum_{k=1}^{\infty} \frac{1}{\ln(k+2)}$$

First few terms ignored

$$\sum_{k=1}^{\infty} \frac{1}{k} = \infty \therefore \sum_{k=1}^{\infty} \frac{1}{\ln(k+2)} = \infty$$

4. Find the first three terms and the general term for the Maclaurin series for the derivative of the following:

$$f(x) = \frac{1}{1+x^3} = 1 - x^3 + x^6 - x^9 + \dots + (-1)^n x^{3n} + \dots$$

$$f'(x) = -\frac{3x^2}{(1+x^3)^2} = -3x^2 + 3x^5 + 3x^8 + \dots + (-1)^n (3n) x^{3n-1} + \dots$$

$$-\frac{3}{2^2} + \frac{6}{2^5} - \frac{9}{2^8} + \dots = -3x^2 + 3x^5 + 3x^8 + \dots \implies x = \frac{1}{2}$$

$$f'\left(\frac{1}{2}\right) = -\frac{3\left(\frac{1}{2}\right)^2}{\left(1 + \left(\frac{1}{2}\right)^3\right)^2} = -\frac{\frac{3}{4}}{\frac{81}{64}} = -\frac{16}{27}$$

5. Consider the following series where $p \geq 0$:

$$\sum_{n=2}^{\infty} \frac{1}{n^p \ln(n)}$$

- (a) for $p > 1$

$$\sum_{n=2}^{\infty} \frac{1}{n^p \ln(n)} < \sum_{n=2}^{\infty} \frac{1}{n^p}$$

$$\sum_{n=2}^{\infty} \frac{1}{n^p} = L \therefore \sum_{n=2}^{\infty} \frac{1}{n^p \ln(n)} = L_2$$

(b) for $p = 1$

$$u = \ln n, \quad du = \frac{1}{n} dn$$

$$\sum_{n=2}^{\infty} \frac{1}{n^p \ln(n)} = \int_{\ln 2}^{\infty} \frac{1}{u} du \implies \ln(\ln(n)) \Big|_2^{\infty} = \infty$$

(c) for $p < 1$

$$\sum_{n=2}^{\infty} \frac{1}{n^p \ln(n)} > \sum_{n=2}^{\infty} \frac{1}{n^p}$$

$$\sum_{n=2}^{\infty} \frac{1}{n^p} = \infty \therefore \sum_{n=2}^{\infty} \frac{1}{n^p \ln(n)} = \infty$$

6. The Maclaurin series for $f(x)$ is given $1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots + \frac{x^n}{(n+1)!} + \cdots$ where x is a positive real number.

$$f'(0) = \frac{1}{2} \quad f^{(17)}(0) = \frac{1}{17!}$$

Convergent for what values of x ?

$$\lim_{n \rightarrow \infty} \frac{x^{n+1}}{(n+2)!} * \frac{(n+1)!}{x^n} < 1$$

$$\implies \lim_{n \rightarrow \infty} \left| \frac{x}{(n+2)} \right| < 1 \implies x \in \mathbb{R}$$

7. A Taylor series for a function is given about $x = 1$ and converges to $f(x)$ for $|x - 1| < R$ where R is the radius of convergence for the series.

$$\begin{aligned}
& \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n} (x-1)^n \implies \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+2} \frac{2^{n+1}}{n+1} (x-1)^{n+1}}{(-1)^{n+1} \frac{2^n}{n} (x-1)^n} \right| \\
& \implies \lim_{n \rightarrow \infty} \left| \frac{\frac{2}{n+1} (x-1)^{n+1}}{\frac{1}{n} (x-1)^n} \right| \implies \lim_{n \rightarrow \infty} \left| \frac{2n}{n+1} (x-1) \right| \implies |2(x-1)| < 1 \\
& \implies -\frac{1}{2} < x-1 < \frac{1}{2} \\
& \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n} \left(-\frac{1}{2} - 1\right)^n = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n} (-1)^n \left(\frac{3}{2}\right)^n = \sum_{n=1}^{\infty} -\frac{2^n}{n} \left(\frac{3}{2}\right)^n = \sum_{n=1}^{\infty} -\frac{3^n}{n} \\
& \implies \lim_{n \rightarrow \infty} -\frac{3^n}{n} = -\infty \\
& \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n} \left(\frac{1}{2} - 1\right)^n = \sum_{n=1}^{\infty} (-1)^{n+1} (-1)^n \frac{2^n}{n} \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} -\frac{1^n}{n} = \sum_{n=1}^{\infty} -\frac{1}{n} = \infty \text{ (P series)} \\
& \implies -\frac{1}{2} < x-1 < \frac{1}{2}
\end{aligned}$$

Find the derivative series

$$\begin{aligned}
f(x) &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n} (x-1)^n = 2(x-1) - 2(x-1)^2 + \frac{8}{3}(x-1)^3 + \dots \\
f'(x) &= 2 - 4(x-1) + 8(x-1)^2 + \dots + (-1)^{n+1} 2^n (x-1)^{n-1} + \dots
\end{aligned}$$

Integrate $f'(x)$ to find $f(x)$.

$$\begin{aligned}
f'(x) &= \sum_{n=1}^{\infty} 2(-2(x-1))^{n-1} = \frac{2}{1 - -(2(x-1))} = \frac{2}{2x-1} \\
f(x) &= \int \frac{2}{2x-1} dx = \ln |2x-1| + C \\
f(1) &= \ln |1| + C = 0 \implies C = 0 \\
f(x) &= \ln |2x-1|, x \in \left(-\frac{1}{2}, \frac{1}{2}\right)
\end{aligned}$$

8. Find interval of convergence for $f(x)$ and determine if it fits the differential $xy' - y = \frac{4x^2}{1+2x}$

$$\begin{aligned}
f(x) &= \sum_{n=2}^{\infty} \frac{(-1)^n (2x)^n}{n-1} = \sum_{n=2}^{\infty} \frac{(-2)^n (x)^n}{n-1} \\
f'(x) &= \sum_{n=2}^{\infty} \frac{n(-2)^n (x)^{n-1}}{n-1} \\
xy' - y &= \sum_{n=2}^{\infty} \frac{n(-2)^n (x)^n}{n-1} - \sum_{n=2}^{\infty} \frac{(-1)^n (2x)^n}{n-1} = \sum_{n=2}^{\infty} \frac{n(-2)^n (x)^n - (-1)^n (2x)^n}{n-1} \\
&= \sum_{n=2}^{\infty} \frac{n(-2)^n (x)^n - (-1)^n (2x)^n}{n-1} = \sum_{n=2}^{\infty} (-1)^n \frac{n-1}{n-1} (2x)^n = \sum_{n=2}^{\infty} (-2x)^n \\
&= \sum_{n=0}^{\infty} (4x^2)(-2x)^n = \frac{4x^2}{1+2x}
\end{aligned}$$

9. The function $f(x)$ is defined by the below series and $g(x)$ too.

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n n x^n}{n+1} \\
g(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!}
\end{aligned}$$

For example, here is $f(x)$'s interval of convergence:

$$\lim_{n \rightarrow \infty} \left| \frac{\frac{(-1)^{n+1} (n+1) x^{n+1}}{n+2}}{\frac{(-1)^n n x^n}{n+1}} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{(n+1)x}{n+2}}{\frac{n}{n+1}} \right| \implies -1 < x < 1$$

Test the endpoints as usual.

Find $y = f(x) - g(x)$, passing through $(0, -1)$. Find $y'(0)$ and $y''(0)$ to determine if $y(0)$ is a relative max, min, or neither.

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} \frac{(-1)^n n x^n}{n+1} - \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!} \\
&= \sum_{n=0}^{\infty} \left[\frac{(-1)^n n x^n}{n+1} - \frac{(-1)^n x^n}{(2n)!} \right] = \sum_{n=0}^{\infty} (-x)^n \left[\frac{n(2n)!}{(n+1)(2n)!} - \frac{n+1}{(n+1)(2n)!} \right] \\
&= \sum_{n=0}^{\infty} (-x)^n \left[\frac{n(2n)! - n - 1}{(n+1)(2n)!} \right] \\
&= -1 + 0 + \left(\frac{2x^2}{3} - \frac{x^2}{4!} \right) + \dots = -1 + \frac{4 * 2 * 2x^2 - x^2}{4!} + \dots = -1 + \frac{15x^2}{4!} + \dots \\
&= -1 + \frac{5x^2}{8} + \dots \\
y'(0) &= 0 \quad y''(0) = -\frac{5}{4}
\end{aligned}$$

Minimum

10. Find k for which the following converges.

$$\begin{aligned}
&\sum_{n=0}^{\infty} ((k^3 + 2)e^{-k})^n \\
\implies \lim_{n \rightarrow \infty} \frac{((k^3 + 2)e^{-k})^{n+1}}{((k^3 + 2)e^{-k})^n} &\implies -1 < ((k^3 + 2)e^{-k}) < 1
\end{aligned}$$

Use a graphing calculator to cook.