Sets Cheatsheet

Contents

1 Geometric Series

Geometric Series are a special type of Power Series that can be rewritten in the form

$$f(x) = \frac{a_1}{1-r} = \sum_{n=0}^{\infty} a_1 \cdot r^n$$

1.1 Proof

$$S = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \dots + a_1 r^n + \dots$$
$$rS = a_1 r + a_1 r^2 + a_1 r^3 + a_1 r^4 \dots + a_1 r^n + \dots$$
$$S - rS = a_1 + a_1 r - a_1 r + a_1 r^2 - a_1 r^2 + \dots + a_1 r^n - a_1 r^n + a_1 r^{n+1} + \dots$$
$$\implies S(1-r) = a_1 + a_1 r^{n+1}$$

Evaluate the lim as $n \to \infty$

$$S = \lim_{n \to \infty} \frac{a_1 + a_1 r^{n+1}}{1-r}$$

We can only evaluate this equation when $|r| < 1$. Therefore:

$$\frac{a_1}{1-r}$$

Is the sum of the geometric series.

To find the interval of convergence, just remember that $|r| < 1$

1.2 Examples

$$f(x) = \frac{1}{2-x}, c = 0 \implies f(x) = \frac{1}{2} \cdot \frac{1}{1 - \frac{x}{2}} \implies f(x) = \sum_{n=0}^{\infty} \frac{1}{2} \cdot \left(\frac{x}{2}\right)^n$$

$$\implies f(x) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} x^n$$

$$|r| < 1 \implies \left|\frac{x}{2}\right| < 1 \implies |x| < 2 \implies x \in (-2, 2)$$

If we change the center:

$$f(x) = \frac{1}{2-x}, c = 5 \implies f(x) = \frac{1}{2 - 5 - (x-5)} \implies \frac{1}{-3 - (x-5)}$$

$$\implies -\frac{1}{3} \cdot \frac{1}{1 - \frac{-(x-5)}{3}} \implies \sum_{n=0}^{\infty} -\frac{1}{3} \cdot \left(\frac{-(x-5)}{3}\right)^n$$

$$\implies \sum_{n=0}^{\infty} (-\frac{1}{3})^{n+1}(-(x-5))^n$$

$$|r| < 1 \implies r \in (-1, 1) \implies \frac{-x+5}{3} \in (-1, 1) \implies -(x-5) \in (-3, 3) \implies x \in (2, 8)$$

Now let's completely change it up.

$$f(x) = \frac{3}{2x - 1}, c = 0 \implies f(x) = -3 \cdot \frac{1}{1 - 2x} \implies \cdot \sum_{n=0}^{\infty} -3 \cdot (2x)^n, c \in (-\frac{1}{2}, \frac{1}{2})$$

Let's move the center

$$f(x) = \frac{3}{2x - 1}, c = -3 \implies f(x) = \frac{1}{2(x+3) - 5 - 6} \implies f(x) = -\frac{1}{11} \cdot \frac{1}{1 - \frac{2}{11}(x+3)}$$

$$f(x) = \sum_{n=0}^{\infty} -\frac{1}{11} \cdot (\frac{2}{11}(x+3))^n$$

$$\frac{2}{11}(x+3) \in (-1, 1) \implies (x+3) \in (-\frac{11}{2}, \frac{11}{2}) \implies x \in (-\frac{17}{2}, \frac{5}{2})$$

What if there is no $-x$? You can just do $-(-x)$. Remember: Basic Algebra will take you a long way.

$$f(x) = \frac{3}{x + 2}, c = 0 \implies f(x) = \frac{3}{2} \cdot \frac{1}{1 - \frac{1}{2}(-x)}$$

What if your function looks a bit more complicated?

$$f(x) = \frac{4x - 7}{2x^2 + 3x - 2}, c = 0 \implies f(x) = \frac{4x - 7}{(2x - 1)(x + 2)} \implies f(x) = \frac{A}{2x - 1} + \frac{B}{x + 2}$$

$$A(x + 2) + B(2x - 1) = 4x - 7 \implies Ax + 2Bx = 4x, 2A - B = -7 \implies B = 2A + 7$$

$$\implies A + 4A + 14 = 4 \implies 5A = -10 \implies A = -2, B = 3$$

$$f(x) = \frac{-2}{2x - 1} + \frac{3}{x + 2}$$

Solve it normally from here.

2   General Power Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x - c)^n$$

2.1 Derivation

We take general power series to be something like the following:
$$f(x) = 1 + x + 2x^2 + 3x^3 + \dots + nx^n + \dots$$
or
$$f(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + \frac{1}{n}x^n + \dots$$

So generally:
$$f(x) = \sum_{n=0}^{\infty} a_n (x - c)^n$$

Instead of a_i because the coefficient changes for each element in the series. Let's expand this series:

$$f(x) = a_0 + c + a_1(x - c) + a_2(x - c)^2 + a_3(x - c)^3 + \dots + a_n(x - c)^n + \dots$$

Take its derivative:
$$f'(x) = 0 + a_1 + 2a_2(x - c) + 3a_3(x - c)^2 + \dots + na_n(x - c)^{n-1} + \dots$$

Notice how $f'(0) = a_1$.
$$f''(x) = 0 + 0 + 2a_2 + 6a_3(x - c) + \dots + n(n - 1)a_n(x - c)^{n-2} + \dots$$

Notice how $f''(0) = 2a_2$ and $f'''(0) = 6a_3$. In order to get the a_n term, you just need to take $f^{(n)}(c)$ and divide it by $n!$

Put it together, and you'll get the equation we started with.

2.2 Examples

$$f(x) = e^x, c = 0 \implies f(x) = \sum_{n=0}^{\infty} \frac{e^0}{n!}(x)^n \implies f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\implies 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + \frac{1}{n}x^n + \dots$$

Evaluate a geometric series using Taylor series

$$f(x) = \frac{1}{1 - 2x}, c = 0 \implies \frac{\frac{1}{1-2(0)}}{0!} + \frac{\frac{2}{(1-2c)^2}}{1!}x + \frac{\frac{8}{(1-2c)^3}}{2!}x + \frac{\frac{48}{(1-2c)^4}}{3!}x + \dots$$

$$\implies 1 + \frac{2}{(1 - 2c)^2}x + \frac{4}{(1 - 2c)^3}x^2 + \frac{8}{(1 - 2c)^4}x^3 + \dots$$

General term: $2^n x^n$.

We would have gotten the same thing if we used the power series expansion of the Taylor series expansion. As with most things in algebra, pick the method that's **most conveinient for you**.

Note: if a problem asks for the nth term, that will always match the power on a Taylor polynomial.

2.3 Error Checking

In an ideal world, you want to know how far off your estimates are. For alternating series, this process is pretty easy.

$$f(x) = 1 - x + \frac{x}{2!} - \frac{x}{3!} + \frac{x}{4!} - \frac{x}{5!} + \frac{x}{6!} - \frac{1}{7!} + ... + \frac{1}{n!}(-1)^n + ...$$

Let's choose the first four terms for our **Taylor Polynomial**, which will be represented by $P_n(x)$ where n is the degree.

$$P_4(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{3}x^3$$

Let's set the remainder terms to $R_5(x)$. We can call the error $E(x)$.

$$E(x) = |P_4(x) - R_5(x)|$$

Since you will never truly know $R_5(x)$ or any $R_n(x)$ for that matter, you will never know the true error. But you can set an upper bound on that error, which is pretty easy in alternating power series.

$$E(x) \leq P_{n+1}(x) - P_n(x)$$

A fancy way of saying, the term right after is the highest the eror can be. It's because you will never add back what you subtracted, always less.

2.4 Deriving e^a

We can use a Taylor Series to derive e^a!

$$\lim_{n \to \infty} (1 + \frac{a}{n})^n = L \implies \ln(\lim_{n \to \infty} (1 + \frac{a}{n})^n) = \ln L \implies n \ln(\lim_{n \to \infty} (1 + \frac{a}{n})) = \ln L$$

$$t = \frac{a}{n}, \implies \frac{a}{t}(\lim_{n \to \infty} \ln(1 + t)) = \ln L \implies \frac{a}{t}(\lim_{t \to 0} \ln(1 + t)) = \ln L$$

$$\longrightarrow \ln(1 + t) = t - \frac{t^2}{2} + \frac{t^3}{3} + ... \approx t$$

$$\implies \ln L = a \implies L = e^a$$

2.5 Lagrange Error Checking

If your series **isn't alternating**, you can use Lagrange Error Checking. Although the proof is complicated, it's pretty simple

$$E(x) \leq |\frac{\max_{z \in [x,c]} f^{(n+1)}(z)}{(n+1)!}|(x - c)^{n+1}$$

The proof isn't necessary for Calc BC. You can find it online.

2.5.1 Full Example

Find the first four terms about $c = 2$ for $ln|x+1|$.

We can start with its derivative, $\frac{1}{x+1}$.

For simplicity, we will use a geometric series. Say we wanted to find $\ln|\frac{3}{2}|$ and approximate error to be 0.05 or less

$$\frac{1}{x-1} = \frac{1}{1-(x-2)+2} = \frac{1}{-1-(x-2)} = -\frac{1}{1-(-(x-2))}$$

$$\implies \sum_{n=0}^{\infty} -(-(x-2))^n = \sum_{n=0}^{\infty} (-1)^{n+1}(x-2)^n = -1 + (x-2) - \frac{1}{2}(x-2)^2 + \frac{1}{3}(x-2)^3 + ...$$

$$\int -\frac{1}{1-(-(x-2))}dx = \int -1 + (x-2) - \frac{1}{2}(x-2)^2 + \frac{1}{3!}(x-2)^3 + ...dx$$

$$\implies -\ln|1-(-(x-2))| = C - x + \frac{(x-2)^2}{2} - \frac{(x-2)^3}{3 \cdot 2!} + \frac{(x-2)^4}{4 \cdot 3!} + ...$$

$$\longrightarrow \ln|1-(-(2-2))| = \ln|1|$$

$$\longrightarrow C + x - \frac{(x-2)^2}{2} + \frac{(x-2)^3}{3 \cdot 2!} - \frac{(x-2)^4}{4 \cdot 3!} + ...$$

$$\longrightarrow \ln|1| = C + 2 + 0 + ... \longrightarrow C = -2$$

$$\implies \ln|1-(-(x-2))| = -2 + x - \frac{(x-2)^2}{2} + \frac{(x-2)^3}{3 \cdot 2!} - \frac{(x-2)^4}{4 \cdot 3!} + ...$$

$$\implies \ln|\frac{3}{2}| = \ln|1-(-(\frac{5}{2}-2))| = (\frac{5}{2}-2) - \frac{(\frac{5}{2}-2)^2}{2} + \frac{(\frac{5}{2}-2)^3}{3 \cdot 2!} - \frac{(\frac{5}{2}-2)^4}{4 \cdot 3!} + ...$$

$$\implies \frac{1}{2} - \frac{1}{2} \cdot (\frac{1}{2})^2 + \frac{1}{3} \cdot (\frac{1}{2})^3 - \frac{1}{4} \cdot (\frac{1}{2})^4 + ...$$

Since it's an alternating series, it should be pretty simple to solve for from here.

2.5.2 Another Problem

$$\sin(5x + \frac{\pi}{4}) = \sin(5(x + \frac{\pi}{20}))$$

3 P Series

NOT Power Series.

A P Series is any series in the form:

$$\sum_{n=0}^{\infty} \frac{1}{n^p}$$

We use Integral Test to prove if it is diverging or converging.

3.1 Harmonic Series

A Harmonic Series is:

$$\sum_{n=0}^{\infty} \frac{1}{n}$$

Or a P Series where $p = 1$.

3.2 Proof

Where $p = 1$ (Harmonic Series)

$$\int_{1}^{\infty} \frac{1}{x} dx < \sum_{n=1}^{\infty} \frac{1}{n} < \int_{1}^{\infty} \frac{1}{x} dx + 1 \implies \int_{1}^{\infty} \frac{1}{x} dx = \ln|x| \Big|_{1}^{\infty}$$

$$\implies \lim_{x \to \infty} \ln x = \infty$$

Diverges

Where $p \neq 1$

$$\int_{1}^{\infty} \frac{1}{x^p} dx < \sum_{n=1}^{\infty} \frac{1}{n^p} < \int_{1}^{\infty} \frac{1}{x^p} dx + 1 \longrightarrow \int_{1}^{\infty} x^{-p} dx = \frac{x^{-p+1}}{-p+1} \Big|_{1}^{\infty}$$

$$\longrightarrow \lim_{x \to \infty} \frac{x^{-p+1}}{-p+1}$$

If $p > 1$ it converges, $p < 1$ (or $p \leq 1$) it diverges.

3.3 Examples

$$\sum_{n=1}^{\infty} \frac{1}{(2n+3)^2} \implies \int_{1}^{\infty} \frac{1}{(2x+3)^2} dx \longrightarrow u = 2x+3 \quad du = 2dx \implies \int_{3}^{\infty} \frac{1}{2u^3} du$$

Convergent

$$\sum_{n=1}^{\infty} \frac{n+2}{n+1} \implies \int_{1}^{\infty} \frac{x+2}{x+1} dx = \int_{1}^{\infty} (1 + \frac{1}{x+1}) dx = (x + \ln|x+1|) \Big|_{1}^{\infty}$$

$$= \lim_{x \to \infty} x + \ln|x+1| = \infty$$

Divergent

4 Convergent or Divergent

4.1 nth Term Test for Divergence

If
$$\lim_{x \to \infty} a_x \neq 0$$
then f is divergent. If
$$\lim_{x \to \infty} a_x = 0$$
then use another test, it's inconclusive.

4.2 Ratio Test

Commonly used for geometric series, if you just check to make sure $|r| < 1$ then it's going to converge.

$$\lim_{n \to \infty} |\frac{a^{n+1}}{a^n}|$$

Basically checking that for some far off value, the common ratio remains.
Example:

$$\sum_{n=0}^{\infty} \frac{(-1)^{n-1}n^2}{2^n} \implies \lim_{n \to \infty} |\frac{\frac{(-1)^n(n+1)^2}{2^{n+1}}}{\frac{(-1)^{n-1}n^2}{2^n}}| = 0 \implies \lim_{n \to \infty} |\frac{(-1)^n(n+1)^2 \cdot 2^n}{2^n \cdot 2 \cdot (-1)^{n-1}n^2}|$$

$$\implies \lim_{n \to \infty} |\frac{(n+1)^2}{2 \cdot -1 \cdot n^2}| = \frac{1}{2}$$

If $|r| = 1$, we have a problem. Recall the limit from the geometric series proof:

$$\lim_{n \to \infty} \frac{a - ar^{n+1}}{1 - r}$$

That's going to be undefined. So we have to use another test.
This test is best used for rapidly growing questions like $n!$.

4.2.1 More Examples

$$\sum_{n=0}^{\infty} \frac{e^n}{n^3} \implies \lim_{n \to \infty} |\frac{\frac{e^n \cdot e}{(n+1)^3}}{\frac{e^n}{n^3}}| \implies \lim_{n \to \infty} |\frac{e \cdot n^3}{(n+1)^3}| = e \nless 1$$

Divergent

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(3n)!} \implies \lim_{n\to\infty} |\frac{\frac{((n+1)!)^2}{(3n+3)!}}{\frac{(n!)^2}{(3n)!}}| \implies \lim_{n\to\infty} |\frac{((n+1)!)^2 \cdot (3n)!}{(3n+3)! \cdot (n!)^2}|$$

$$\implies \lim_{n\to\infty} |\frac{(n+1)^2 \cdot (n!)^2 \cdot (3n)!}{(3n+3) \cdot (3n+2) \cdot (3n+1) \cdot (3n)! \cdot (n!)^2}|$$

$$\implies \lim_{n\to\infty} |\frac{(n+1)^2}{(3n+3) \cdot (3n+2) \cdot (3n+1)}| = 0 < 1$$

Convergent

$$\sum_{n=0}^{\infty} \frac{n!}{n^{n+1}} \implies \lim_{n\to\infty} |\frac{\frac{(n+1)!}{(n+1)^{n+2}}}{\frac{n!}{n^{n+1}}}| \implies \lim_{n\to\infty} |\frac{(n+1)! \cdot n^{n+1}}{n! \cdot (n+1)^{n+2}}| \implies \lim_{n\to\infty} |\frac{n! \cdot n^{n+1}}{n! \cdot (n+1)^{n+1}}|$$

$$\implies \lim_{n\to\infty} |(\frac{n}{n+1})^{n+1}| = \lim_{n\to\infty} |(\frac{n+1-1}{n+1})^{n+1}| = \lim_{n\to\infty} |(1 - \frac{1}{n+1})^{n+1}| =$$

$$\lim_{n\to\infty} |(1 + \frac{(-1)}{n+1})^{n+1}| = e^{-1} < 1$$

Convergent

$$\sum_{n=1}^{\infty} \frac{3^{1-2n}}{n^2+1} \implies \lim_{n\to\infty} |\frac{\frac{3^{1-2n-2}}{(n+1)^2+1}}{\frac{3^{1-2n}}{n^2+1}}| \implies \lim_{n\to\infty} |\frac{3^{-1-2n} \cdot (n^2+1)}{(n+1)^2+1 \cdot 3^{1-2n}}|$$

$$\implies \lim_{n\to\infty} |\frac{n^2+1}{(n+1)^2+1 \cdot 3^{1-2n} \cdot 3^{1+2n}}| \implies \lim_{n\to\infty} |\frac{n^2+1}{(n+1)^2+1 \cdot 3 \cdot 3^{-2n} \cdot 3 \cdot 3^{2n}}|$$

$$\implies \lim_{n\to\infty} |\frac{n^2+1}{(n+1)^2+1 \cdot 3 \cdot 3}| = \frac{1}{9}$$

Convergent

$$\sum_{n=2}^{\infty} \frac{(-2)^{1+3n}(n+1)}{n^2 5^{1+n}}$$

$$\implies \lim_{n\to\infty} |\frac{\frac{(-2)^{4+3n}(n+2)}{(n+1)^2 5^{2+n}}}{\frac{(-2)^{1+3n}(n+1)}{n^2 5^{1+n}}}| \implies \lim_{n\to\infty} |\frac{(-2)^{4+3n}(n+2)n^2 5^{1+n}}{(n+1)^2 5^{2+n}(-2)^{1+3n}(n+1)}|$$

$$\implies \lim_{n\to\infty} |\frac{(-2)^3(n+2)n^2}{(n+1)^3 5}| = \frac{(-2)^3}{5^5} = -\frac{8}{5}$$

Diverges

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{6n+7} \implies \lim_{n\to\infty} |\frac{\frac{(-1)^{n+2}}{6n+6+7}}{\frac{(-1)^{n+1}}{6n+7}}| \implies \lim_{n\to\infty} |\frac{(-1)^{n+2}(6n+7)}{(6n+6+7)(-1)^{n+1}}|$$

$$\implies \lim_{n\to\infty} |\frac{6n+7}{6n+6+7}|$$

Inconclusive: But if you used the Divergence Test first:

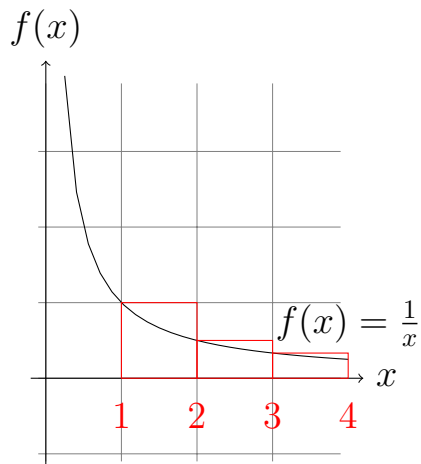$$\lim_{n \to \infty} \frac{(-1)^{n+1}}{6n+7} \approx \frac{(-1)^{\infty}}{\infty} = 0$$

Inconclusive, but the alternating series test proves it right.

$$\sum_{n=3}^{\infty} \frac{e^{4n}}{(n-2)!}$$

$$\longrightarrow \lim_{n \to \infty} \frac{e^{4n}}{(n-2)!} = 0$$

$$\implies \lim_{n \to \infty} \left| \frac{\frac{e^{4n+4}}{(n-1)!}}{\frac{e^{4n}}{(n-2)!}} \right| \implies \lim_{n \to \infty} \left| \frac{e^{4n+4}(n-2)!}{(n-1)!e^{4n}} \right| \implies \lim_{n \to \infty} \left| \frac{e^{4}(n-2)!}{(n-1)(n-2)!} \right| \implies \lim_{n \to \infty} \left| \frac{e^{4}}{(n-1)} \right|$$

$$\implies 0$$

Convergent

4.3 Integral Test

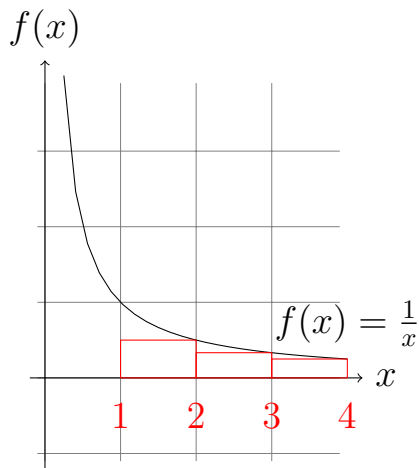If $f(x)$ is decreasing, then for $x \geq 1$:

$f(x)$

$f(x) = \frac{1}{x}$

x

1 2 3 4

The Left Riemann Sum is $>$ than $f(x)$:

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + ... \implies \sum_{n=1}^{\infty} \frac{1}{n} > \int_{1}^{\infty} \frac{1}{x} dx$$

Similarly,

$f(x) = \frac{1}{x}$

The Right Riemann Sum is $<$ than $f(x)$:

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... \implies \sum_{n=2}^{\infty} \frac{1}{n} < \int_1^\infty \frac{1}{x} dx$$

$$\implies \sum_{n=1}^{\infty} \frac{1}{n} < \int_1^\infty \frac{1}{x} dx + 1$$

$$\implies \int_1^\infty \frac{1}{x} dx < \sum_{n=1}^{\infty} \frac{1}{n} < \int_1^\infty \frac{1}{x} dx + 1$$

Using a method analogous to the Squeeze Theorem, if the bounds diverge, so does the series, and vice versa.

4.4 Direct Comparison Test

The direct comparison test allows for **no negative terms** in the series.

Let a_n and b_n represent an unknown and known sequence, respectively.

If $a_n < b_n$ for all n (according to CollegeBoard, however the first few terms are an exception) and $\lim_{n \to \infty} \sum_i^n b_i = L$ then $\lim_{n \to \infty} \sum_i^n a_i = L_2$

If $a_n > b_n$ for all n and $\lim_{n \to \infty} \sum_i^n b_i = L$ then $\lim_{n \to \infty} \sum_i^n a_i$ could equal L_2 or ∞.

If $a_n < b_n$ for all n and $\lim_{n \to \infty} \sum_i^n b_i = \infty$ then $\lim_{n \to \infty} \sum_i^n a_i$ could equal L_2 or ∞.

If $a_n > b_n$ for all n and $\lim_{n \to \infty} \sum_i^n b_i = \infty$ then $\lim_{n \to \infty} \sum_i^n a_i = \infty$.

It is recommended to use this test only if the degree is the same.

4.4.1 Examples

$$\sum_{n=2}^{\infty} \frac{1}{n^2 + 1} = \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \dots$$

$$\sum_{n=2}^{\infty} \frac{1}{n^2} = \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots$$

$$\implies \sum_{n=2}^{\infty} \frac{1}{n^2} > \sum_{n=2}^{\infty} \frac{1}{n^2 + 1} \implies \sum_{n=2}^{\infty} \frac{1}{n^2} = L \therefore \sum_{n=2}^{\infty} \frac{1}{n^2 + 1} = L_2$$

Convergent

4.5 Limit Comparison Test

$$\lim_{n \to \infty} \frac{a_n}{b_n}$$

For $0 < n < \infty$ (if it is 0 or ∞ anyway, the previous test would suffice).

4.5.1 Examples

$$\sum_{n=1}^{\infty} \frac{n}{n^2 + 1} = \frac{1}{2} + \frac{2}{5} + \frac{3}{10} + \dots$$

$$\sum_{n=1}^{\infty} \frac{n}{n^2} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots$$

$$\implies \sum_{n=1}^{\infty} \frac{1}{n} > \sum_{n=1}^{\infty} \frac{n}{n^2 + 1} = \lim_{n \to \infty} \frac{1}{n} > \lim_{n \to \infty} \frac{n}{n^2 + 1} = \infty > \lim_{n \to \infty} \frac{n}{n^2}$$

Inconclusive

$$\lim_{n \to \infty} \frac{\frac{n}{n^2 + 1}}{\frac{1}{n}} = \lim_{n \to \infty} \frac{n^2}{n^2 + 1} = 1$$

Convergent

$$\sum_{n=1}^{\infty} \frac{5n - 3}{n^2 - 2n + 5} = \frac{-3}{5} + \frac{2}{4} + \frac{7}{5} + \frac{12}{8} + \frac{17}{13}\dots$$

$$\sum_{n=1}^{\infty} \frac{n}{n^2} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}\dots$$

$$\implies \lim_{n \to \infty} \frac{\frac{5n-3}{n^2-2n+5}}{\frac{1}{n}} = \lim_{n \to \infty} \frac{5n^2 - 3n}{n^2 - 2n + 5} = 5$$

$$\lim_{n \to \infty} \frac{1}{n} = \infty \therefore \lim_{n \to \infty} \frac{5n - 3}{n^2 - 2n + 5} = \infty$$

Divergent

4.6 Alternating Series ~~Property~~ Theorem

Recall the Alternating Error Bound for a series. Take this example:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

Since the sum is **decreasing** and $\lim_{n \to \infty} \frac{1}{n} = 0$ (since n is increasing), there is an error bound, meaning this alternating series is convergent. Note that this "test" doesn't apply to oscillating serieses like $sin(x)$.

4.6.1 Example

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{2n+1}$$

$$\lim_{n \to \infty} \frac{n}{2n+1} = \frac{1}{2} \Longrightarrow \approx \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2}$$

This sum does not reach 0, it osciilates. It is not convergent.

4.7 Absolute Convergence Test

Let a_n represent a sequence with positive and negative terms.

Using some common sense, we can derive the following:

$$n \in \mathbb{R} \Longrightarrow n \leq |n|$$
$$\therefore a_n \leq |a_n|$$
$$\Longrightarrow a_n + |a_n| \leq 2|a_n|$$
$$\longrightarrow |a_n| + a_n = 0 \quad \text{or} \quad 2a_n \therefore 0 \leq a_n + |a_n| \leq 2a_n$$

By the Direct Comparsion test:

$$\sum_{n=1}^{\infty} |a_n| = L \therefore \sum_{n=1}^{\infty} a_n = L_2$$

In this case it is absolutely convergent. In the scenario that:

$$\sum_{n=1}^{\infty} |a_n| = \infty$$

It is inconclusive. If the other alternating series tests passes, it is **Conditionally Convergent**.

4.8 Examples

$$\sum_{n=2}^{\infty} \frac{n}{\ln n} \implies \lim_{n \to \infty} \frac{n}{\ln n} = \infty$$

Divergent by nth Term Test for Divergence

$$\sum_{n=1}^{\infty} \frac{2^n}{n^{99}} \implies \lim_{n \to \infty} \frac{2^n}{n^{99}} = \infty$$

Divergent by nth Term Test for Divergence

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^2+1}}{n} \implies \lim_{n \to \infty} \frac{\sqrt{n^2+1}}{n} \approx \lim_{n \to \infty} \frac{\sqrt{n^2}}{n} = \lim_{n \to \infty} \frac{n}{n} = 1$$

Divergent by nth Term Test for Divergence

$$\sum_{n=2}^{\infty} \frac{n^e}{n^\pi} = \sum_{n=2}^{\infty} \frac{1}{n^{\pi-e}} \implies 1 \geq \pi - e$$

Divergent by property of P-Series

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+8}} = \frac{1}{\sqrt{8}} + \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+8}} = \frac{1}{\sqrt{8}} + \frac{1}{3} + \frac{1}{\sqrt{10}} + \dots$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots$$

$$\implies \lim_{n \to \infty} \frac{\frac{1}{\sqrt{n+8}}}{\frac{1}{\sqrt{n}}} = \lim_{n \to \infty} \frac{\sqrt{n+8}}{\sqrt{n}} = 1$$

Divergent by Limit Comparison Test

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{1} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots$$

Convergent by Properties of Alternating Series:

1. It is decreasing

2. It is approaching 0

3. It is alternating

$$\sum_{n=3}^{\infty} \frac{4}{n\sqrt{\ln n}} = \frac{4}{3\sqrt{\ln 3}} + \frac{4}{4\sqrt{\ln 4}} + \frac{4}{5\sqrt{\ln 5}} + \ldots$$

$$\sum_{n=3}^{\infty} \frac{4}{n} = \frac{4}{3} + \frac{4}{4} + \frac{4}{5} + \ldots$$

$$\implies \sum_{n=3}^{\infty} \frac{4}{n\sqrt{\ln n}} < \sum_{n=3}^{\infty} \frac{4}{n}, \sum_{n=3}^{\infty} \frac{4}{n} = \infty$$

$$\implies \lim_{n\to\infty} \frac{\frac{4}{n\sqrt{\ln n}}}{\frac{4}{n}} = 0$$

$$\lim_{n\to\infty} \frac{\frac{4}{n\sqrt{\ln(n+1)}+\sqrt{\ln(n+1)}}}{\frac{4}{n\sqrt{\ln n}}} = ???$$

After trying three tests, we are left with the Integral Test

$$\sum_{n=3}^{\infty} \frac{4}{n\sqrt{\ln n}}$$

$$\implies 4\int_{3}^{\infty} \frac{1}{n\sqrt{\ln n}} dn, u = \ln n, \quad du = \frac{1}{n} dn$$

$$\implies 4\int_{\ln 3}^{\infty} \frac{1}{\sqrt{u}} du = \int_{\ln 3}^{\infty} u^{-\frac{1}{2}} du = 4 * \left[2u^{\frac{1}{2}}\right]\Big|_{\ln 3}^{\infty}$$

Divergent by Integral Test

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{27n^2}} = \frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{n^{\frac{2}{3}}}$$

Divergent by Property of P-Series

$$\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}} \implies \lim_{x\,to\infty} \frac{x}{\sqrt{x^2+1}} \approx \lim_{x\,to\infty} \frac{x}{x} = 1$$

Divergent by nth Term Test for Divergence

$$\sum_{n=1}^{\infty} \sqrt{\frac{n+1}{n}}$$

Be careful, this is NOT a disguised $\frac{1}{e}$.

$$\approx \sum_{n=1}^{\infty} \sqrt{\frac{n}{n}} = 1$$

Divergent by nth Term Test for Divergence

$$\sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n} = \sum_{n=1}^{\infty} \frac{2^n}{4^n} + \frac{3^n}{4^n} = \sum_{n=1}^{\infty} \frac{2^n}{4^n} + \sum_{n=1}^{\infty} \frac{3^n}{4^n} = \sum_{n=1}^{\infty} \left(\frac{2}{4}\right)^n + \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$$

Convergent by Property of Geometric Series

$$\sum_{n=3}^{\infty} \frac{1}{n(\ln n)(\ln(\ln n))} = \sum_{n=3}^{\infty} \frac{1}{n(\ln n)} \frac{1}{(\ln(\ln n))}$$

$$\implies \sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)} = \frac{1}{\ln(\ln 3)} + \frac{1}{\ln(\ln 4)} + \frac{1}{\ln(\ln 5)} + \dots$$

$$\sum_{n=3}^{\infty} \frac{1}{n} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$$

$$\implies \sum_{n=3}^{\infty} \frac{1}{n} < \sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$$

$$\sum_{n=3}^{\infty} \frac{1}{n} = \infty \therefore \sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)} = \infty$$

A divergent sum times any sum is divergent.

Divergent by Property of P-Series

$$\sum_{n=3}^{\infty} \frac{1}{2^n - n} = \frac{1}{5} + \frac{1}{12} + \frac{1}{27} + \dots$$

$$\sum_{n=3}^{\infty} \frac{1}{2^n} = \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots$$

$$\sum_{n=3}^{\infty} \frac{1}{2^n - n} > \sum_{n=3}^{\infty} \frac{1}{2^n}$$

$$\sum_{n=3}^{\infty} \frac{1}{2^n} = L$$

$$\lim_{n \to \infty} \frac{\frac{1}{2^n - n}}{\frac{1}{2^n}} = \lim_{n \to \infty} \frac{2^n}{2^n - n} = 1$$

Convergent by Limit Comparison Test

$$\sum_{n=1}^{\infty} \frac{5 + 8^n}{2 - 7^n} = \sum_{n=1}^{\infty} \left(\frac{5}{2 - 7^n} + \frac{8^n}{2 - 7^n} \right) = \sum_{n=1}^{\infty} \left(\frac{5}{2 - 7^n} - \frac{8^n}{7^n - 2} \right)$$

$$\implies \sum_{n=1}^{\infty} \frac{8^n}{7^n - 2} = \frac{8}{5} + \frac{64}{47} + \dots$$

$$\lim_{n \to \infty} \frac{8^n}{7^n - 2} \approx \lim_{n \to \infty} \frac{8^n}{7^n} = \lim_{n \to \infty} \left(\frac{8}{7} \right)^n = \infty$$

A divergent series plus any series is divergent.
Divergent by nth Term Test for Divergence

5 More Examples

1.

$$\sum_{n=1}^{\infty} (-1)^n \frac{(x - 2)^n}{n + 1}$$

For P-Series, you can **only use** the Ratio Test to determine Convergence over an Interval

$$\implies \lim_{n \to \infty} \frac{\frac{(-1)^{n+1}(x-2)^{n+1}}{n+2}}{\frac{(-1)^n(x-2)^n}{n+1}} = \lim_{n \to \infty} \left| \frac{(n+1)(-1)^{n+1}(x-2)^{n+1}}{(n+2)(-1)^n(x-2)^n} \right| = \lim_{n \to \infty} \left| \frac{(n+1)(x-2)}{(n+2)} \right|$$

$$= \lim_{n \to \infty} |x - 2| < 1$$

$$\implies 1 < x < 3$$

Now you have to test the values 1 and 3.

$$\sum_{n=0}^{\infty} \left| (-1)^n \frac{(x - 2)^n}{n + 1} \right|$$

$$\longrightarrow \sum_{n=0}^{\infty} \left| (-1)^n \frac{(1 - 2)^n}{n + 1} \right|$$

$$\longrightarrow \sum_{n=0}^{\infty} \left| (-1)^n \frac{(-1)^n}{n + 1} \right| = \sum_{n=0}^{\infty} \left| \frac{1}{n + 1} \right|$$

Divergent by Comparsion Test

$$\sum_{n=0}^{\infty} \left| (-1)^n \frac{(3-2)^n}{n+1} \right|$$

$$\longrightarrow \sum_{n=0}^{\infty} \left| (-1)^n \frac{(1)^n}{n+1} \right|$$

$$\longrightarrow \sum_{n=0}^{\infty} \left| \frac{(-1)^n}{n+1} \right| = \sum_{n=0}^{\infty} \left| \frac{(-1)^n}{n+1} \right|$$

Convergent by Alternating Series "Test"

$$1 < x \leq 3$$

2.

$$\sum_{n=1}^{\infty} \frac{x^n n^n}{3^n n!}$$

$$\implies \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}(n+1)^{n+1}}{3^{n+1}(n+1)!}}{\frac{x^n n^n}{3^n n!}} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}(n+1)^{n+1}3^n n!}{x^n n^n 3^{n+1}(n+1)!} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}(n+1)(n+1)^n n!}{x^n n^n 3(n+1)!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x(n+1)^n n!}{n^n 3n!} \right| = \frac{1}{3} \lim_{n\to\infty} \left| x \left(\frac{n+1}{n} \right)^n \right| = \frac{1}{3} \lim_{n\to\infty} \left| x \left(1 + \frac{1}{n} \right)^n \right|$$

$$\implies \left| \frac{e}{3} x \right| < 1 \implies -\frac{3}{e} < x < \frac{3}{e}$$

Plug in the endpoints:

$$\sum_{n=1}^{\infty} \frac{(-\frac{3}{e})^n n^n}{3^n n!} = \sum_{n=1}^{\infty} \frac{(-1)^n (3)^n n^n}{e^n 3^n n!} = \sum_{n=1}^{\infty} \frac{(-1)^n n^n}{e^n n!}$$

Look at the graph and compare it against $\frac{1}{n}$.

3.

$$\sum_{k=0}^{\infty} \frac{2^k x^k}{\ln(k+2)}$$

$$\lim_{k\to\infty} \left| \frac{\frac{2^{k+1}x^{k+1}}{\ln((k+1)+2)}}{\frac{2^k x^k}{\ln(k+2)}} \right| = \lim_{k\to\infty} \left| \frac{2^{k+1}x^{k+1}\ln(k+2)}{2^k x^k \ln((k+1)+2)} \right| = \lim_{k\to\infty} \left| \frac{2x\ln(k+2)}{\ln(k+2+1)} \right| = 2x$$

$$-\frac{1}{2} < x < \frac{1}{2}$$

Plug in the endpoints:

$$\sum_{k=0}^{\infty} \frac{2^k \left(\frac{-1}{2} \right)^k}{\ln(k+2)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{\ln(k+2)}$$

(a) Alternating

(b) Approaching 0

(c) Decreasing

$k = -\frac{1}{2}$ Convergent

$$\sum_{k=0}^{\infty} \frac{2^k \left(\frac{1}{2} \right)^k}{\ln(k+2)} = \sum_{k=0}^{\infty} \frac{1^k}{\ln(k+2)} = \sum_{k=0}^{\infty} \frac{1}{\ln(k+2)}$$

$$\sum_{k=0}^{\infty} \frac{1}{\ln(k+2)} = \frac{1}{\ln 2} + \sum_{k=1}^{\infty} \frac{1}{\ln(k+2)} = \frac{1}{\ln 2} + \frac{1}{\ln 3} + \frac{1}{\ln 4} + \cdots$$

$$\sum_{k=1}^{\infty} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \sum_{k=1}^{\infty} \frac{1}{\ln(k+2)} > \sum_{k=1}^{\infty} \frac{1}{k}$$

First few terms ignored

$$\sum_{k=1}^{\infty} \frac{1}{k} = \infty \therefore \sum_{k=1}^{\infty} \frac{1}{\ln(k+2)} = \infty$$

4. Find the first three terms and the general term for the Maclaurin series for the derivative of the following:

$$f(x) = \frac{1}{1+x^3} = 1 - x^3 + x^6 - x^9 + \cdots + (-1)^n x^{3n} + \cdots$$

$$f'(x) = -\frac{3x^2}{(1+x^3)^2} = -3x^2 + 3x^5 + 3x^8 + \cdots + (-1)^n (3n) x^{3n-1} + \cdots$$

$$-\frac{3}{2^2} + \frac{6}{2^5} - \frac{9}{2^8} + \cdots = -3x^2 + 3x^5 + 3x^8 + \cdots \implies x = \frac{1}{2}$$

$$f'\left(\frac{1}{2} \right) = -\frac{3\left(\frac{1}{2} \right)^2}{(1 + \left(\frac{1}{2} \right)^3)^2} = -\frac{\frac{3}{4}}{\frac{81}{64}} = -\frac{16}{27}$$

5. Consider the following series where $p \geq 0$:

$$\sum_{n=2}^{\infty} \frac{1}{n^p \ln(n)}$$

(a) for $p > 1$

$$\sum_{n=2}^{\infty} \frac{1}{n^p \ln(n)} < \sum_{n=2}^{\infty} \frac{1}{n^p}$$

$$\sum_{n=2}^{\infty} \frac{1}{n^p} = L \therefore \sum_{n=2}^{\infty} \frac{1}{n^p \ln(n)} = L_2$$

(b) for $p = 1$

$$u = \ln n, \quad du = \frac{1}{n}dn$$

$$\sum_{n=2}^{\infty} \frac{1}{n^p \ln(n)} = \int_{\ln 2}^{\infty} \frac{1}{u}du \implies \ln(\ln(n))\Big|_2^{\infty} = \infty$$

(c) for $p < 1$

$$\sum_{n=2}^{\infty} \frac{1}{n^p \ln(n)} > \sum_{n=2}^{\infty} \frac{1}{n^p}$$

$$\sum_{n=2}^{\infty} \frac{1}{n^p} = \infty \therefore \sum_{n=2}^{\infty} \frac{1}{n^p \ln(n)} = \infty$$

6. The Maclaurin series for $f(x)$ is given $1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots + \frac{x^n}{(n+1)!} + \cdots$ where x is a positive real number.

$$f'(0) = \frac{1}{2} \qquad f^{(17)}(0) = \frac{1}{17!}$$

Convergent for what values of x?

$$\lim_{n \to \infty} \frac{x^{n+1}}{(n+2)!} * \frac{(n+1)!}{x^n} < 1$$

$$\implies \lim_{n \to \infty} \left| \frac{x}{(n+2)} \right| < 1 \implies x \in \mathbb{R}$$

7. A Taylor series for a function is given about $x = 1$ and converges to $f(x)$ for $|x-1| < R$ where R is the radius of convergence for the series.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n}(x-1)^n \implies \lim_{n\to\infty}\left|\frac{(-1)^{n+2}\frac{2^{n+1}}{n+1}(x-1)^{n+1}}{(-1)^{n+1}\frac{2^n}{n}(x-1)^n}\right|$$

$$\implies \lim_{n\to\infty}\left|\frac{\frac{2}{n+1}(x-1)^{n+1}}{\frac{1}{n}(x-1)^n}\right| \implies \lim_{n\to\infty}\left|\frac{2n}{n+1}(x-1)\right| \implies |2(x-1)| < 1$$

$$\implies -\frac{1}{2} < x-1 < \frac{1}{2}$$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n}(-\frac{1}{2}-1)^n = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n}(-1)^n(\frac{3}{2})^n = \sum_{n=1}^{\infty}-\frac{2^n}{n}(\frac{3}{2})^n = \sum_{n=1}^{\infty}-\frac{3^n}{n}$$

$$\implies \lim_{n\to\infty}-\frac{3^n}{n} = -\infty$$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n}(\frac{1}{2}-1)^n = \sum_{n=1}^{\infty}(-1)^{n+1}(-1)^n\frac{2^n}{n}(\frac{1}{2})^n = \sum_{n=1}^{\infty}-\frac{1^n}{n} = \sum_{n=1}^{\infty}-\frac{1}{n} = \infty \text{ (P series)}$$

$$\implies -\frac{1}{2} < x-1 < \frac{1}{2}$$

Find the derivative series

$$f(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n}(x-1)^n = 2(x-1) - 2(x-1)^2 + \frac{8}{3}(x-1)^3 + \cdots$$

$$f'(x) = 2 - 4(x-1) + 8(x-1)^2 + \cdots + (-1)^{n+1}2^n(x-1)^{n-1} + \cdots$$

Integrate $f'(x)$ to find $f(x)$.

$$f'(x) = \sum_{n=1}^{\infty}2(-2(x-1))^{n-1} = \frac{2}{1--(2(x-1))} = \frac{2}{2x-1}$$

$$f(x) = \int \frac{2}{2x-1}dx = \ln|2x-1| + C$$

$$f(1) = \ln|1| + C = 0 \implies C = 0$$

$$f(x) = \ln|2x-1|, x \in (-\frac{1}{2}, \frac{1}{2})$$

8. Find interval of convergence for $f(x)$ and determine if it fits the differential $xy' - y = \frac{4x^2}{1+2x}$

$$f(x) = \sum_{n=2}^{\infty} \frac{(-1)^n (2x)^n}{n-1} = \sum_{n=2}^{\infty} \frac{(-2)^n (x)^n}{n-1}$$

$$f'(x) = \sum_{n=2}^{\infty} \frac{n(-2)^n (x)^{n-1}}{n-1}$$

$$xy' - y = \sum_{n=2}^{\infty} \frac{n(-2)^n (x)^n}{n-1} - \sum_{n=2}^{\infty} \frac{(-1)^n (2x)^n}{n-1} = \sum_{n=2}^{\infty} \frac{n(-2)^n (x)^n - (-1)^n (2x)^n}{n-1}$$

$$= \sum_{n=2}^{\infty} \frac{n(-2)^n (x)^n - (-1)^n (2x)^n}{n-1} = \sum_{n=2}^{\infty} (-1)^n \frac{n-1}{n-1} (2x)^n = \sum_{n=2}^{\infty} (-2x)^n$$

$$= \sum_{n=0}^{\infty} (4x^2)(-2x)^n = \frac{4x^2}{1+2x}$$

9. The function $f(x)$ is defined by the below series and $g(x)$ too.

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n n x^n}{n+1}$$

$$g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!}$$

For example, here is $f(x)$'s interval of convergence:

$$\lim_{n \to \infty} \left| \frac{\frac{(-1)^{n+1}(n+1)x^{n+1}}{n+2}}{\frac{(-1)^n n x^n}{n+1}} \right| = \lim_{n \to \infty} \left| \frac{\frac{(n+1)x}{n+2}}{\frac{n}{n+1}} \right| \implies -1 < x < 1$$

Test the endpoints as usual.

Find $y = f(x) - g(x)$, passing through $(0, -1)$. Find $y'(0)$ and $y''(0)$ to determine if $y(0)$ is a relative max, min, or neither.

$$y = \sum_{n=0}^{\infty} \frac{(-1)^n n x^n}{n+1} - \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!}$$

$$= \sum_{n=0}^{\infty} \left[\frac{(-1)^n n x^n}{n+1} - \frac{(-1)^n x^n}{(2n)!} \right] = \sum_{n=0}^{\infty} (-x)^n \left[\frac{n(2n)!}{(n+1)(2n)!} - \frac{n+1}{(n+1)(2n)!} \right]$$

$$= \sum_{n=0}^{\infty} (-x)^n \left[\frac{n(2n)! - n - 1}{(n+1)(2n)!} \right]$$

$$= -1 + 0 + \left(\frac{2x^2}{3} - \frac{x^2}{4!} \right) + \cdots = -1 + \frac{4 * 2 * 2x^2 - x^2}{4!} + \cdots = -1 + \frac{15x^2}{4!} + \cdots$$

$$= -1 + \frac{5x^2}{8} + \cdots$$

$$y'(0) = 0 \quad y''(0) = -\frac{5}{4}$$

Minimum

10. Find k for which the following converges.

$$\sum_{n=0}^{\infty} \left((k^3 + 2)e^{-k} \right)^n$$

$$\implies \lim_{n \to \infty} \frac{\left((k^3 + 2)e^{-k} \right)^{n+1}}{\left((k^3 + 2)e^{-k} \right)^n} \implies -1 < \left((k^3 + 2)e^{-k} \right) < 1$$

Use a graphing calculator to cook.