Sets Cheatsheet

Contents

1 Geometric Series

Geometric Series are a special type of Power Series that can be rewritten in the form

$$f(x) = \frac{a_1}{1-r} = \sum_{n=0}^{\infty} a_1 * r^n$$

1.1 Proof

$$\begin{aligned} S &= a_1 + a_1 r + a_1 r^2 + a_1 r^3 + ... + a_1 r^n + ... \\ rS &= a_1 r + a_1 r^2 + a_1 r^3 + a_1 r^4 ... + a_1 r^n + ... \\ S - rS &= a_1 + a_1 r - a_1 r + a_1 r^2 - a_1 r^2 + ... + a_1 r^n - a_1 r^n + a_1 r^{n+1} + ... \\ &\implies S(1-r) = a_1 + a_1 r^{n+1} \end{aligned}$$

Evaluate the lim as $n \to \infty$

$$S = \lim_{n\to\infty} \frac{a_1 + a_1 r^{n+1}}{1-r}$$

We can only evaluate this equation when $|r| < 1$. Therefore:

$$\frac{a_1}{1-r}$$

Is the sum of the geometric series.

To find the interval of convergence, just remember that $|r| < 1$

1.2 Examples

$$f(x) = \frac{1}{2-x}, c = 0 \implies f(x) = \frac{1}{2} * \frac{1}{1-\frac{x}{2}} \implies f(x) = \sum_{n=0}^{\infty} \frac{1}{2} * (\frac{x}{2})^n$$

$$\implies f(x) = \sum_{n=0}^{\infty} (\frac{1}{2})^{n+1} x^n$$

$$|r| < 1 \implies |\frac{x}{2}| < 1 \implies |x| < 2 \implies x \in (-2, 2)$$

If we change the center:

$$f(x) = \frac{1}{2-x}, c = 5 \implies f(x) = \frac{1}{2-5-(x-5)} \implies \frac{1}{-3-(x-5)}$$

$$\implies -\frac{1}{3} * \frac{1}{1-\frac{-(x-5)}{3}} \implies \sum_{n=0}^{\infty} -\frac{1}{3} * (\frac{-(x-5)}{3})^n$$

$$\implies \sum_{n=0}^{\infty} (-\frac{1}{3})^{n+1} (-(x-5))^n$$

$$|r| < 1 \Longrightarrow r \in (-1,1) \Longrightarrow \frac{-x+5}{3} \in (-1,1) \Longrightarrow -(x-5) \in (-3,3) \Longrightarrow x \in (2,8)$$

Now let's completely change it up.

$$f(x) = \frac{3}{2x-1}, c = 0 \Longrightarrow f(x) = -3 * \frac{1}{1-2x} \Longrightarrow * \sum_{n=0}^{\infty} -3 * (2x)^n, c \in (-\frac{1}{2}, \frac{1}{2})$$

Let's move the center

$$f(x) = \frac{3}{2x-1}, c = -3 \Longrightarrow f(x) = \frac{1}{2(x+3)-5-6} \Longrightarrow f(x) = -\frac{1}{11} * \frac{1}{1-\frac{2}{11}(x+3)}$$

$$f(x) = \sum_{n=0}^{\infty} -\frac{1}{11} * (\frac{2}{11}(x+3))^n$$

$$\frac{2}{11}(x+3) \in (-1,1) \Longrightarrow (x+3) \in (-\frac{11}{2}, \frac{11}{2}) \Longrightarrow x \in (-\frac{17}{2}, \frac{5}{2})$$

What if there is no $-x$? You can just do $-(-x)$. Remember: Basic Algebra will take you a long way.

$$f(x) = \frac{3}{x+2}, c = 0 \Longrightarrow f(x) = \frac{3}{2} * \frac{1}{1-\frac{1}{2}(-x)}$$

What if your function looks a bit more complicated?

$$f(x) = \frac{4x-7}{2x^2+3x-2}, c = 0 \Longrightarrow f(x) = \frac{4x-7}{(2x-1)(x+2)} \Longrightarrow f(x) = \frac{A}{2x-1} + \frac{B}{x+2}$$

$$A(x+2) + B(2x-1) = 4x-7 \Longrightarrow Ax + 2Bx = 4x, 2A - B = -7 \Longrightarrow B = 2A+7$$

$$\Longrightarrow A + 4A + 14 = 4 \Longrightarrow 5A = -10 \Longrightarrow A = -2, B = 3$$

$$f(x) = \frac{-2}{2x-1} + \frac{3}{x+2}$$

Solve it normally from here.

2 General Power Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

2.1 Derivation

We take general power series to be something like the following:

$$f(x) = 1 + x + 2x^2 + 3x^3 + ... + nx^n + ...$$

or

$$f(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + ... + \frac{1}{n}x^n + ...$$

So generally:

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

Instead of a_i because the coefficient changes for each element in the series. Let's expand this series:

$$f(x) = a_0 + c + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + ... + a_n(x-c)^n + ...$$

Take its derivative:

$$f'(x) = 0 + a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + ... + na_n(x-c)^{n-1} + ...$$

Notice how $f'(0) = a_1$.

$$f''(x) = 0 + 0 + 2a_2 + 6a_3(x-c) + ... + n(n-1)a_n(x-c)^{n-2} + ...$$

Notice how $f''(0) = 2a_2$ and $f'''(0) = 6a_3$. In order to get the a_n term, you just need to take $f^{(n)}(c)$ and divide it by $n!$
Put it together, and you'll get the equation we started with.

2.2 Examples

$$f(x) = e^x, c = 0 \Longrightarrow f(x) = \sum_{n=0}^{\infty} \frac{e^0}{n!}(x)^n \Longrightarrow f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\Longrightarrow 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + ... + \frac{1}{n}x^n + ...$$

Evaluate a geometric series using Taylor series

$$f(x) = \frac{1}{1-2x}, c = 0 \Longrightarrow \frac{\frac{1}{1-2(0)}}{0!} + \frac{\frac{2}{(1-2c)^2}}{1!}x + \frac{\frac{8}{(1-2c)^3}}{2!}x + \frac{\frac{48}{(1-2c)^4}}{3!}x + ...$$

$$\Longrightarrow 1 + \frac{2}{(1-2c)^2}x + \frac{4}{(1-2c)^3}x^2 + \frac{8}{(1-2c)^4}x^3 + ...$$

General term: $2^n x^n$.

We would have gotten the same thing if we used the power series expansion of the Taylor series expansion. As with most things in algebra, pick the method that's **most conveinient for you**.

2.3 Error Checking

In an ideal world, you want to know how far off your estimates are. For alternating series, this process is pretty easy.

$$f(x) = 1 - x + \frac{x}{2!} - \frac{x}{3!} + \frac{x}{4!} - \frac{x}{5!} + \frac{x}{6!} - \frac{1}{7!} + ... + \frac{1}{n!}(-1)^n + ...$$

Let's choose the first four terms for our **Taylor Polynomial**, which will be represented by $P_n(x)$ where n is the degree.

$$P_4(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{3}x^3$$

Let's set the remainder terms to $R_5(x)$. We can call the error $E(x)$.

$$E(x) = |P_4(x) - R_5(x)|$$

Since you will never truly know $R_5(x)$ or any $R_n(x)$ for that matter, you will never know the true error. But you can set an upper bound on that error, which is pretty easy in alternating power series.

$$E(x) \leq P_{n+1}(x) - P_n(x)$$

A fancy way of saying, the term right after is the highest the eror can be. It's because you will never add back what you subtracted, always less.

2.4 Deriving e^a

We can use a Taylor Series to derive e^a!

$$\begin{aligned}
\lim_{n\to\infty}(1 + \frac{a}{n})^n = L \Longrightarrow \ln(\lim_{n\to\infty}(1 + \frac{a}{n})^n) = \ln L \Longrightarrow n\ln(\lim_{n\to\infty}(1 + \frac{a}{n})) = \ln L \\
t = \frac{a}{n}, \Longrightarrow \frac{a}{t}(\lim_{n\to\infty}\ln(1+t)) = \ln L \Longrightarrow \frac{a}{t}(\lim_{t\to 0}\ln(1+t)) = \ln L \\
\longrightarrow \ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} + ... \approx t \\
\Longrightarrow \ln L = a \Longrightarrow L = e^a
\end{aligned}$$

2.5 Lagrange Error Checking

If your series **isn't alternating**, you can use Lagrange Error Checking. Although the proof is complicated, it's pretty simple

$$E(x) \leq |\frac{\max_{z\in[x,c]} f^{(n+1)}(z)}{(n+1)!}|(x-c)^{n+1}$$

The proof isn't necessary for Calc BC. You can find it online.

2.5.1 Full Example

Find the first four terms about $c = 2$ for $ln|x + 1|$.

We can start with its derivative, $\frac{1}{x+1}$.

For simplicity, we will use a geometric series. Say we wanted to find $\ln|\frac{3}{2}|$ and approximate error to be 0.05 or less

$$\frac{1}{x-1} = \frac{1}{1-(x-2)+2} = \frac{1}{-1-(x-2)} = -\frac{1}{1-(-(x-2))}$$

$$\implies \sum_{n=0}^{\infty} -(-(x-2))^n = \sum_{n=0}^{\infty} (-1)^{n+1}(x-2)^n = -1 + (x-2) - \frac{1}{2}(x-2)^2 + \frac{1}{3}(x-2)^3 + ...$$

$$\int -\frac{1}{1-(-(x-2))}dx = \int -1 + (x-2) - \frac{1}{2}(x-2)^2 + \frac{1}{3!}(x-2)^3 + ...dx$$

$$\implies -\ln|1-(-(x-2))| = C - x + \frac{(x-2)^2}{2} - \frac{(x-2)^3}{3*2!} + \frac{(x-2)^4}{4*3!} + ...$$

$$\longrightarrow \ln|1-(-(2-2))| = \ln|1|$$

$$\longrightarrow C + x - \frac{(x-2)^2}{2} + \frac{(x-2)^3}{3*2!} - \frac{(x-2)^4}{4*3!} + ...$$

$$\longrightarrow \ln|1| = C + 2 + 0 + ... \longrightarrow C = -2$$

$$\implies \ln|1-(-(x-2))| = -2 + x - \frac{(x-2)^2}{2} + \frac{(x-2)^3}{3*2!} - \frac{(x-2)^4}{4*3!} + ...$$

$$\implies \ln|\frac{3}{2}| = \ln|1-(-(\frac{5}{2}-2))| = (\frac{5}{2}-2) - \frac{(\frac{5}{2}-2)^2}{2} + \frac{(\frac{5}{2}-2)^3}{3*2!} - \frac{(\frac{5}{2}-2)^4}{4*3!} + ...$$

$$\implies \frac{1}{2} - \frac{1}{2}*(\frac{1}{2})^2 + \frac{1}{3}*(\frac{1}{2})^3 - \frac{1}{4}*(\frac{1}{2})^4 + ...$$

Since it's an alternating series, it should be pretty simple to solve for from here.

2.5.2 Another Problem

$$\sin(5x + \frac{\pi}{4}) = \sin(5(x + \frac{\pi}{20}))$$

3 P Series

NOT Power Series.

A P Series is any series in the form:

$$\sum_{n=0}^{\infty} \frac{1}{n^p}$$

We use Integral Test to prove if it is diverging or converging.

3.1 Harmonic Series

A Harmonic Series is:

$$\sum_{n=0}^{\infty} \frac{1}{n}$$

Or a P Series where $p = 1$.

3.2 Proof

Where $p = 1$ (Harmonic Series)

$$\int_1^{\infty} \frac{1}{x} dx < \sum_{n=1}^{\infty} \frac{1}{n} < \int_1^{\infty} \frac{1}{x} dx + 1 \Longrightarrow \int_1^{\infty} \frac{1}{x} dx = \ln|x| \Big|_1^{\infty}$$
$$\Longrightarrow \lim_{x \to \infty} \ln x = \infty$$

Diverges

Where $p \neq 1$

$$\int_1^{\infty} \frac{1}{x^p} dx < \sum_{n=1}^{\infty} \frac{1}{n^p} < \int_1^{\infty} \frac{1}{x^p} dx + 1 \longrightarrow \int_1^{\infty} x^{-p} dx = \frac{x^{-p+1}}{-p+1} \Big|_1^{\infty}$$
$$\longrightarrow \lim_{x \to \infty} \frac{x^{-p+1}}{-p+1}$$

If $p > 1$ it converges, $p < 1$ (or $p \leq 1$) it diverges.

3.3 Examples

$$\sum_{n=1}^{\infty} \frac{1}{(2n+3)^2} \Longrightarrow \int_1^{\infty} \frac{1}{(2x+3)^2} dx \longrightarrow u = 2x + 3 \quad du = 2dx \Longrightarrow \int_3^{\infty} \frac{1}{2u^3} du$$

Convergent

$$\sum_{n=1}^{\infty} \frac{n+2}{n+1} \Longrightarrow \int_1^{\infty} \frac{x+2}{x+1} dx = \int_1^{\infty} (1 + \frac{1}{x+1}) dx = (x + \ln|x+1|) \Big|_1^{\infty}$$
$$= \lim_{x \to \infty} x + \ln|x+1| = \infty$$

Divergent

4 Convergent or Divergent

4.1 Divergence Test

If

$$\lim_{x\to\infty} f(x) \neq 0$$

then f is divergent. If

$$\lim_{x\to\infty} f(x) = 0$$

then use another test, it's inconclusive.

4.2 Ratio Test

Commonly used for geometric series, if you just check to make sure $|r| < 1$ then it's going to converge.

$$\lim_{n\to\infty} |\frac{a^{n+1}}{a^n}|$$

Basically checking that for some far off value, the common ratio remains. Example:

$$\sum_{n=0}^{\infty} \frac{(-1)^{n-1}n^2}{2^n} \implies \lim_{n\to\infty} |\frac{\frac{(-1)^n(n+1)^2}{2^{n+1}}}{\frac{(-1)^{n-1}n^2}{2^n}}| = 0 \implies \lim_{n\to\infty} |\frac{(-1)^n(n+1)^2 * 2^n}{2^n * 2 * (-1)^{n-1}n^2}|$$

$$\implies \lim_{n\to\infty} |\frac{(n+1)^2}{2 * -1 * n^2}| = \frac{1}{2}$$

If $|r| = 1$, we have a problem. Recall the limit from the geometric series proof:

$$\lim_{n\to\infty} \frac{a - ar^{n+1}}{1-r}$$

That's going to be undefined. So we have to use another test.

4.2.1 More Examples

$$\sum_{n=0}^{\infty} \frac{e^n}{n^3} \implies \lim_{n\to\infty} |\frac{\frac{e^n * e}{(n+1)^3}}{\frac{e^n}{n^3}}| \implies \lim_{n\to\infty} |\frac{e * n^3}{(n+1)^3}| = e \not< 1$$

Divergent

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(3n)!} \implies \lim_{n\to\infty} |\frac{\frac{((n+1)!)^2}{(3n+3)!}}{\frac{(n!)^2}{(3n)!}}| \implies \lim_{n\to\infty} |\frac{((n+1)!)^2 * (3n)!}{(3n+3)! * (n!)^2}|$$

$$\implies \lim_{n\to\infty} |\frac{(n+1)^2 * (n!)^2 * (3n)!}{(3n+3) * (3n+2) * (3n+1) * (3n)! * (n!)^2}|$$

$$\implies \lim_{n\to\infty} |\frac{(n+1)^2}{(3n+3) * (3n+2) * (3n+1)}| = 0 < 1$$

Convergent

$$\sum_{n=0}^{\infty}\frac{n!}{n^{n+1}} \implies \lim_{n\to\infty}|\frac{\frac{(n+1)!}{(n+1)^{n+2}}}{\frac{n!}{n^{n+1}}}| \implies \lim_{n\to\infty}|\frac{(n+1)!*n^{n+1}}{n!*(n+1)^{n+2}}| \implies \lim_{n\to\infty}|\frac{n!*n^{n+1}}{n!*(n+1)^{n+1}}|$$

$$\implies \lim_{n\to\infty}|(\frac{n}{n+1})^{n+1}| = \lim_{n\to\infty}|(\frac{n+1-1}{n+1})^{n+1}| = \lim_{n\to\infty}|(1-\frac{1}{n+1})^{n+1}| =$$

$$\lim_{n\to\infty}|(1+\frac{(-1)}{n+1})^{n+1}| = e^{-1} < 1$$

Convergent

$$\sum_{n=1}^{\infty}\frac{3^{1-2n}}{n^2+1} \implies \lim_{n\to\infty}|\frac{\frac{3^{1-2n-2}}{(n+1)^2+1}}{\frac{3^{1-2n}}{n^2+1}}| \implies \lim_{n\to\infty}|\frac{3^{-1-2n}*(n^2+1)}{(n+1)^2+1*3^{1-2n}}|$$

$$\implies \lim_{n\to\infty}|\frac{n^2+1}{(n+1)^2+1*3^{1-2n}*3^{1+2n}}| \implies \lim_{n\to\infty}|\frac{n^2+1}{(n+1)^2+1*3*3^{-2n}*3*3^{2n}}|$$

$$\implies \lim_{n\to\infty}|\frac{n^2+1}{(n+1)^2+1*3*3}| = \frac{1}{9}$$

Convergent

$$\sum_{n=2}^{\infty}\frac{(-2)^{1+3n}(n+1)}{n^2 5^{1+n}}$$

$$\implies \lim_{n\to\infty}|\frac{\frac{(-2)^{4+3n}(n+2)}{(n+1)^2 5^{2+n}}}{\frac{(-2)^{1+3n}(n+1)}{n^2 5^{1+n}}}| \implies \lim_{n\to\infty}|\frac{(-2)^{4+3n}(n+2)n^2 5^{1+n}}{(n+1)^2 5^{2+n}(-2)^{1+3n}(n+1)}|$$

$$\implies \lim_{n\to\infty}|\frac{(-2)^3(n+2)n^2}{(n+1)^3 5}| = \frac{(-2)^3}{5^5} = -\frac{8}{5}$$

Diverges

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{6n+7} \implies \lim_{n\to\infty}|\frac{\frac{(-1)^{n+2}}{6n+6+7}}{\frac{(-1)^{n+1}}{6n+7}}| \implies \lim_{n\to\infty}|\frac{(-1)^{n+2}(6n+7)}{(6n+6+7)(-1)^{n+1}}|$$

$$\implies \lim_{n\to\infty}|\frac{6n+7}{6n+6+7}|$$

Inconclusive: But if you used the Divergence Test first:

$$\lim_{n\to\infty}\frac{(-1)^{n+1}}{6n+7} \approx \frac{(-1)^{\infty}}{\infty} = 0$$

Inconclusive, but the alternating series test proves it right.

$$\sum_{n=3}^{\infty}\frac{e^{4n}}{(n-2)!}$$

$$\longrightarrow \lim_{n\to\infty}\frac{e^{4n}}{(n-2)!}=0$$

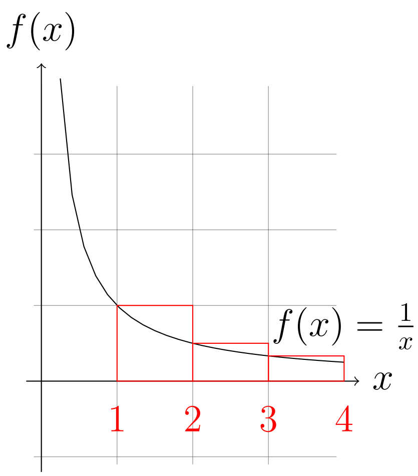
$$\implies \lim_{n\to\infty}\left|\frac{\frac{e^{4n+4}}{(n-1)!}}{\frac{e^{4n}}{(n-2)!}}\right| \implies \lim_{n\to\infty}\left|\frac{e^{4n+4}(n-2)!}{(n-1)!e^{4n}}\right| \implies \lim_{n\to\infty}\left|\frac{e^{4}(n-2)!}{(n-1)(n-2)!}\right| \implies \lim_{n\to\infty}\left|\frac{e^{4}}{(n-1)}\right|$$

$$\implies 0$$

Convergent

4.3 Integral Test

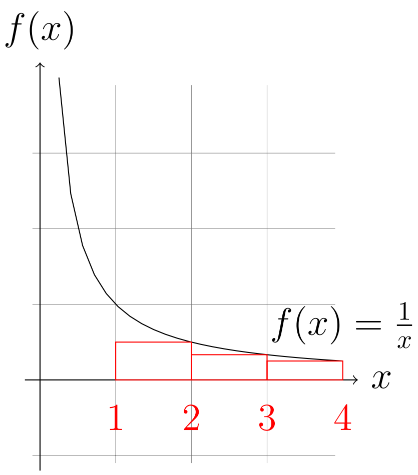
If $f(x)$ is decreasing, then for $x \geq 1$:



The Left Riemann Sum is > than $f(x)$:

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots \implies \sum_{n=1}^{\infty}\frac{1}{n} > \int_{1}^{\infty}\frac{1}{x}dx$$

Similarly,



The Right Riemann Sum is $<$ than $f(x)$:

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... \implies \sum_{n=2}^{\infty} \frac{1}{n} < \int_{1}^{\infty} \frac{1}{x} dx$$

$$\implies \sum_{n=1}^{\infty} \frac{1}{n} < \int_{1}^{\infty} \frac{1}{x} dx + 1$$

$$\implies \int_{1}^{\infty} \frac{1}{x} dx < \sum_{n=1}^{\infty} \frac{1}{n} < \int_{1}^{\infty} \frac{1}{x} dx + 1$$

Using a method analogous to the Squeeze Theorem, if the bounds diverge, so does the series, and vice versa.

4.4 Direct Comparison Test

The direct comparison test allows for **no negative terms** in the series.

Let a_n and b_n represent an unknown and known sequence, respectively.

If $a_n < b_n$ for all n (according to CollegeBoard, however the first few terms are an exception) and $\lim_{n\to\infty} \sum_i^n b_i = L$ then $\lim_{n\to\infty} \sum_i^n a_i = L_2$

If $a_n > b_n$ for all n and $\lim_{n\to\infty} \sum_i^n b_i = L$ then $\lim_{n\to\infty} \sum_i^n a_i$ could equal L_2 or ∞.

If $a_n < b_n$ for all n and $\lim_{n\to\infty} \sum_i^n b_i = \infty$ then $\lim_{n\to\infty} \sum_i^n a_i$ could equal L_2 or ∞.

If $a_n > b_n$ for all n and $\lim_{n\to\infty} \sum_i^n b_i = \infty$ then $\lim_{n\to\infty} \sum_i^n a_i = \infty$.

4.4.1 Examples

$$\sum_{n=2}^{\infty} \frac{1}{n^2+1} = \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + ...$$

$$\sum_{n=2}^{\infty} \frac{1}{n^2} = \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + ...$$

$$\implies \sum_{n=2}^{\infty} \frac{1}{n^2} > \sum_{n=2}^{\infty} \frac{1}{n^2+1} \implies \sum_{n=2}^{\infty} \frac{1}{n^2} = L \therefore \sum_{n=2}^{\infty} \frac{1}{n^2+1} = L_2$$

Convergent

4.5 Limit Comparison Test

$$\lim_{n\to\infty} \frac{b_n}{a_n}$$

For $0 < n < \infty$ (if it is 0 or ∞ anyway, the previous test would suffice).