

Sets Cheatsheet

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1 Geometric Series

Geometric Series are a special type of Power Series that can be rewritten in the form

$$f(x) = \frac{a_1}{1-r} = \sum_{n=0}^{\infty} a_1 * r^n$$

1.1 Proof

$$\begin{aligned} S &= a_1 + a_1r + a_1r^2 + a_1r^3 + \dots + a_1r^n + \dots \\ rS &= a_1r + a_1r^2 + a_1r^3 + a_1r^4 \dots + a_1r^n + \dots \\ S - rS &= a_1 + a_1r - a_1r + a_1r^2 - a_1r^2 + \dots + a_1r^n - a_1r^n + a_1r^{n+1} + \dots \\ &\implies S(1-r) = a_1 + a_1r^{n+1} \end{aligned}$$

Evaluate the lim as $n \rightarrow \infty$

$$S = \lim_{n \rightarrow \infty} \frac{a_1 + a_1r^{n+1}}{1-r}$$

We can only evaluate this equation when $|r| < 1$. Therefore:

$$\frac{a_1}{1-r}$$

Is the sum of the geometric series.

To find the interval of convergence, just remember that $|r| < 1$

1.2 Examples

$$\begin{aligned} f(x) = \frac{1}{2-x}, c = 0 &\implies f(x) = \frac{1}{2} * \frac{1}{1-\frac{x}{2}} \implies f(x) = \sum_{n=0}^{\infty} \frac{1}{2} * \left(\frac{x}{2}\right)^n \\ &\implies f(x) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} x^n \\ |r| < 1 &\implies \left|\frac{x}{2}\right| < 1 \implies |x| < 2 \implies x \in (-2, 2) \end{aligned}$$

If we change the center:

$$\begin{aligned} f(x) = \frac{1}{2-x}, c = 5 &\implies f(x) = \frac{1}{2-5-(x-5)} \implies \frac{1}{-3-(x-5)} \\ &\implies -\frac{1}{3} * \frac{1}{1-\frac{-(x-5)}{3}} \implies \sum_{n=0}^{\infty} -\frac{1}{3} * \left(\frac{-(x-5)}{3}\right)^n \\ &\implies \sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^{n+1}(-(x-5))^n \end{aligned}$$

$$|r| < 1 \implies r \in (-1, 1) \implies \frac{-x+5}{3} \in (-1, 1) \implies -(x-5) \in (-3, 3) \implies x \in (2, 8)$$

Now let's completely change it up.

$$f(x) = \frac{3}{2x-1}, c=0 \implies f(x) = -3 * \frac{1}{1-2x} \implies * \sum_{n=0}^{\infty} -3 * (2x)^n, c \in (-\frac{1}{2}, \frac{1}{2})$$

Let's move the center

$$f(x) = \frac{3}{2x-1}, c=-3 \implies f(x) = \frac{1}{2(x+3)-5-6} \implies f(x) = -\frac{1}{11} * \frac{1}{1-\frac{2}{11}(x+3)}$$

$$f(x) = \sum_{n=0}^{\infty} -\frac{1}{11} * (\frac{2}{11}(x+3))^n$$

$$\frac{2}{11}(x+3) \in (-1, 1) \implies (x+3) \in (-\frac{11}{2}, \frac{11}{2}) \implies x \in (-\frac{17}{2}, \frac{5}{2})$$

What if there is no $-x$? You can just do $-(-x)$. Remember: Basic Algebra will take you a long way.

$$f(x) = \frac{3}{x+2}, c=0 \implies f(x) = \frac{3}{2} * \frac{1}{1-\frac{1}{2}(-x)}$$

What if your function looks a bit more complicated?

$$f(x) = \frac{4x-7}{2x^2+3x-2}, c=0 \implies f(x) = \frac{4x-7}{(2x-1)(x+2)} \implies f(x) = \frac{A}{2x-1} + \frac{B}{x+2}$$

$$\begin{aligned} A(x+2) + B(2x-1) &= 4x-7 \implies Ax+2Bx=4x, 2A-B=-7 \implies B=2A+7 \\ &\implies A+4A+14=4 \implies 5A=-10 \implies A=-2, B=3 \\ f(x) &= \frac{-2}{2x-1} + \frac{3}{x+2} \end{aligned}$$

Solve it normally from here.

2 General Power Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n$$

2.1 Derivation

We take general power series to be something like the following:

$$f(x) = 1 + x + 2x^2 + 3x^3 + \dots + nx^n + \dots$$

or

$$f(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + \frac{1}{n}x^n + \dots$$

So generally:

$$f(x) = \sum_{n=0}^{\infty} a_n(x - c)^n$$

Instead of a_i because the coefficient changes for each element in the series. Let's expand this series:

$$f(x) = a_0 + c + a_1(x - c) + a_2(x - c)^2 + a_3(x - c)^3 + \dots + a_n(x - c)^n + \dots$$

Take its derivative:

$$f'(x) = 0 + a_1 + 2a_2(x - c) + 3a_3(x - c)^2 + \dots + na_n(x - c)^{n-1} + \dots$$

Notice how $f'(0) = a_1$.

$$f''(x) = 0 + 0 + 2a_2 + 6a_3(x - c) + \dots + n(n - 1)a_n(x - c)^{n-2} + \dots$$

Notice how $f''(0) = 2a_2$ and $f'''(0) = 6a_3$. In order to get the a_n term, you just need to take $f^{(n)}(c)$ and divide it by $n!$

Put it together, and you'll get the equation we started with.

2.2 Examples

$$\begin{aligned} f(x) = e^x, c = 0 \implies f(x) &= \sum_{n=0}^{\infty} \frac{e^0}{n!}(x)^n \implies f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ &\implies 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + \frac{1}{n}x^n + \dots \end{aligned}$$

Evaluate a geometric series using Taylor series

$$\begin{aligned} f(x) = \frac{1}{1 - 2x}, c = 0 \implies \frac{\frac{1}{1-2(0)}}{0!} + \frac{\frac{2}{(1-2c)^2}}{1!}x + \frac{\frac{8}{(1-2c)^3}}{2!}x + \frac{\frac{48}{(1-2c)^4}}{3!}x + \dots \\ \implies 1 + \frac{2}{(1 - 2c)^2}x + \frac{4}{(1 - 2c)^3}x^2 + \frac{8}{(1 - 2c)^4}x^3 + \dots \end{aligned}$$

General term: $2^n x^n$.

We would have gotten the same thing if we used the power series expansion of the Taylor series expansion. As with most things in algebra, pick the method that's **most convenient for you**.

2.3 Error Checking

In an ideal world, you want to know how far off your estimates are. For alternating series, this process is pretty easy.

$$f(x) = 1 - x + \frac{x}{2!} - \frac{x}{3!} + \frac{x}{4!} - \frac{x}{5!} + \frac{x}{6!} - \frac{1}{7!} + \dots + \frac{1}{n!}(-1)^n + \dots$$

Let's choose the first four terms for our **Taylor Polynomial**, which will be represented by $P_n(x)$ where n is the degree.

$$P_4(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{3}x^3$$

Let's set the remainder terms to $R_5(x)$. We can call the error $E(x)$.

$$E(x) = |P_4(x) - R_5(x)|$$

Since you will never truly know $R_5(x)$ or any $R_n(x)$ for that matter, you will never know the true error. But you can set an upper bound on that error, which is pretty easy in alternating power series.

$$E(x) \leq P_{n+1}(x) - P_n(x)$$

A fancy way of saying, the term right after is the highest the error can be. It's because you will never add back what you subtracted, always less.

2.4 Deriving e^a

We can use a Taylor Series to derive e^a !

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(1 + \frac{a}{n}\right)^n &= L \implies \ln\left(\lim_{n \rightarrow \infty} \left(1 + \frac{a}{n}\right)^n\right) = \ln L \implies n \ln\left(\lim_{n \rightarrow \infty} \left(1 + \frac{a}{n}\right)\right) = \ln L \\ t = \frac{a}{n}, \implies \frac{a}{t} \left(\lim_{n \rightarrow \infty} \ln\left(1 + \frac{a}{n}\right)\right) &= \ln L \implies \frac{a}{t} \left(\lim_{t \rightarrow 0} \ln(1 + t)\right) = \ln L \\ \implies \ln(1 + t) &= t - \frac{t^2}{2} + \frac{t^3}{3} + \dots \approx t \\ \implies \ln L &= a \implies L = e^a \end{aligned}$$

2.5 Lagrange Error Checking

If your series **isn't alternating**, you can use Lagrange Error Checking. Although the proof is complicated, it's pretty simple

$$E(x) \leq \left| \frac{\max_{z \in [x, c]} f^{(n+1)}(z)}{(n+1)!} \right| (x - c)^{n+1}$$

The proof isn't necessary for Calc BC. You can find it online.

2.5.1 Full Example

Find the first four terms about $c = 2$ for $\ln|x + 1|$.

We can start with its derivative, $\frac{1}{x+1}$.

For simplicity, we will use a geometric series. Say we wanted to find $\ln|\frac{3}{2}|$ and approximate error to be 0.05 or less

$$\begin{aligned}
 \frac{1}{x-1} &= \frac{1}{1-(x-2)+2} = \frac{1}{-1-(x-2)} = -\frac{1}{1-(-(x-2))} \\
 \Rightarrow \sum_{n=0}^{\infty} -(-(x-2))^n &= \sum_{n=0}^{\infty} (-1)^{n+1}(x-2)^n = -1 + (x-2) - \frac{1}{2}(x-2)^2 + \frac{1}{3}(x-2)^3 + \dots \\
 \int -\frac{1}{1-(-(x-2))} dx &= \int -1 + (x-2) - \frac{1}{2}(x-2)^2 + \frac{1}{3!}(x-2)^3 + \dots dx \\
 \Rightarrow -\ln|1-(-(x-2))| &= C - x + \frac{(x-2)^2}{2} - \frac{(x-2)^3}{3*2!} + \frac{(x-2)^4}{4*3!} + \dots \\
 &\rightarrow \ln|1-(-(2-2))| = \ln|1| \\
 &\rightarrow C + x - \frac{(x-2)^2}{2} + \frac{(x-2)^3}{3*2!} - \frac{(x-2)^4}{4*3!} + \dots \\
 &\rightarrow \ln|1| = C + 2 + 0 + \dots \rightarrow C = -2 \\
 \Rightarrow \ln|1-(-(x-2))| &= -2 + x - \frac{(x-2)^2}{2} + \frac{(x-2)^3}{3*2!} - \frac{(x-2)^4}{4*3!} + \dots \\
 \Rightarrow \ln|\frac{3}{2}| &= \ln|1-(-(\frac{5}{2}-2))| = (\frac{5}{2}-2) - \frac{(\frac{5}{2}-2)^2}{2} + \frac{(\frac{5}{2}-2)^3}{3*2!} - \frac{(\frac{5}{2}-2)^4}{4*3!} + \dots \\
 &\Rightarrow \frac{1}{2} - \frac{1}{2} * (\frac{1}{2})^2 + \frac{1}{3} * (\frac{1}{2})^3 - \frac{1}{4} * (\frac{1}{2})^4 + \dots
 \end{aligned}$$

Since it's an alternating series, it should be pretty simple to solve for from here.

2.5.2 Another Problem

$$\sin(5x + \frac{\pi}{4}) = \sin(5(x + \frac{\pi}{20}))$$

3 Convergent or Divergent

3.1 Divergence Test

If

$$\lim_{x \rightarrow \infty} f(x) \neq 0$$

then f is divergent. If

$$\lim_{x \rightarrow \infty} f(x) = 0$$

then use another test, it's inconclusive.

3.2 Ratio Test

Commonly used for geometric series, if you just check to make sure $|r| < 1$ then it's going to converge.

$$\lim_{n \rightarrow \infty} \left| \frac{a^{n+1}}{a^n} \right|$$

Basically checking that for some far off value, the common ratio remains.
Example:

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^{n-1} n^2}{2^n} &\implies \lim_{n \rightarrow \infty} \left| \frac{\frac{(-1)^n (n+1)^2}{2^{n+1}}}{\frac{(-1)^{n-1} n^2}{2^n}} \right| = 0 \implies \lim_{n \rightarrow \infty} \left| \frac{(-1)^n (n+1)^2 * 2^n}{2^n * 2 * (-1)^{n-1} n^2} \right| \\ &\implies \lim_{n \rightarrow \infty} \left| \frac{(n+1)^2}{2 * -1 * n^2} \right| = \frac{1}{2} \end{aligned}$$

If $|r| = 1$, we have a problem. Recall the limit from the geometric series proof:

$$\lim_{n \rightarrow \infty} \frac{a - ar^{n+1}}{1 - r}$$

That's going to be undefined. So we have to use another test.

3.2.1 More Examples

$$\sum_{n=0}^{\infty} \frac{e^n}{n^3} \implies \lim_{n \rightarrow \infty} \left| \frac{\frac{e^n * e}{(n+1)^3}}{\frac{e^n}{n^3}} \right| \implies \lim_{n \rightarrow \infty} \left| \frac{e * n^3}{(n+1)^3} \right| = e \not< 1$$

Divergent

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(n!)^2}{(3n)!} &\implies \lim_{n \rightarrow \infty} \left| \frac{\frac{((n+1)!)^2}{(3n+3)!}}{\frac{(n!)^2}{(3n)!}} \right| \implies \lim_{n \rightarrow \infty} \left| \frac{((n+1)!)^2 * (3n)!}{(3n+3)! * (n!)^2} \right| \\ &\implies \lim_{n \rightarrow \infty} \left| \frac{(n+1)^2 * (n!)^2 * (3n)!}{(3n+3) * (3n+2) * (3n+1) * (3n)! * (n!)^2} \right| \\ &\implies \lim_{n \rightarrow \infty} \left| \frac{(n+1)^2}{(3n+3) * (3n+2) * (3n+1)} \right| = 0 < 1 \end{aligned}$$

Convergent

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{n!}{n^{n+1}} &\implies \lim_{n \rightarrow \infty} \left| \frac{\frac{(n+1)!}{(n+1)^{n+2}}}{\frac{n!}{n^{n+1}}} \right| \implies \lim_{n \rightarrow \infty} \left| \frac{(n+1)! * n^{n+1}}{n! * (n+1)^{n+2}} \right| \implies \lim_{n \rightarrow \infty} \left| \frac{n! * n^{n+1}}{n! * (n+1)^{n+1}} \right| \\ &\implies \lim_{n \rightarrow \infty} \left| \left(\frac{n}{n+1} \right)^{n+1} \right| = \lim_{n \rightarrow \infty} \left| \left(\frac{n+1-1}{n+1} \right)^{n+1} \right| = \lim_{n \rightarrow \infty} \left| \left(1 - \frac{1}{n+1} \right)^{n+1} \right| = \\ &\qquad \qquad \qquad \lim_{n \rightarrow \infty} \left| \left(1 + \frac{(-1)}{n+1} \right)^{n+1} \right| = e^{-1} < 1 \end{aligned}$$

Convergent

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{3^{1-2n}}{n^2 + 1} &\implies \lim_{n \rightarrow \infty} \left| \frac{\frac{3^{1-2n-2}}{(n+1)^2+1}}{\frac{3^{1-2n}}{n^2+1}} \right| \implies \lim_{n \rightarrow \infty} \left| \frac{3^{-1-2n} * (n^2 + 1)}{(n+1)^2 + 1 * 3^{1-2n}} \right| \\ &\implies \lim_{n \rightarrow \infty} \left| \frac{n^2 + 1}{(n+1)^2 + 1 * 3^{1-2n} * 3^{1+2n}} \right| \implies \lim_{n \rightarrow \infty} \left| \frac{n^2 + 1}{(n+1)^2 + 1 * 3 * 3^{-2n} * 3 * 3^{2n}} \right| \\ &\implies \lim_{n \rightarrow \infty} \left| \frac{n^2 + 1}{(n+1)^2 + 1 * 3 * 3} \right| = \frac{1}{9} \end{aligned}$$

Convergent

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{(-2)^{1+3n}(n+1)}{n^2 5^{1+n}} \\ \implies \lim_{n \rightarrow \infty} \left| \frac{\frac{(-2)^{4+3n}(n+2)}{(n+1)^2 5^{2+n}}}{\frac{(-2)^{1+3n}(n+1)}{n^2 5^{1+n}}} \right| \implies \lim_{n \rightarrow \infty} \left| \frac{(-2)^{4+3n}(n+2)n^2 5^{1+n}}{(n+1)^2 5^{2+n}(-2)^{1+3n}(n+1)} \right| \\ \implies \lim_{n \rightarrow \infty} \left| \frac{(-2)^3(n+2)n^2}{(n+1)^3 5} \right| = \frac{(-2)^3}{5^5} = -\frac{8}{5} \end{aligned}$$

Diverges

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{6n+7} &\implies \lim_{n \rightarrow \infty} \left| \frac{\frac{(-1)^{n+2}}{6n+6+7}}{\frac{(-1)^{n+1}}{6n+7}} \right| \implies \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+2}(6n+7)}{(6n+6+7)(-1)^{n+1}} \right| \\ &\implies \lim_{n \rightarrow \infty} \left| \frac{6n+7}{6n+6+7} \right| \end{aligned}$$

Inconclusive: But if you used the Divergence Test first:

$$\lim_{n \rightarrow \infty} \frac{(-1)^{n+1}}{6n+7} \approx \frac{(-1)^\infty}{\infty} = 0$$

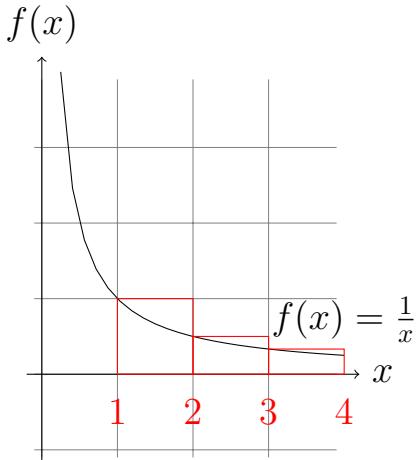
Inconclusive, but the alternating series test proves it right.

$$\begin{aligned} \sum_{n=3}^{\infty} \frac{e^{4n}}{(n-2)!} \\ \longrightarrow \lim_{n \rightarrow \infty} \frac{e^{4n}}{(n-2)!} = 0 \\ \implies \lim_{n \rightarrow \infty} \left| \frac{\frac{e^{4n+4}}{(n-1)!}}{\frac{e^{4n}}{(n-2)!}} \right| \implies \lim_{n \rightarrow \infty} \left| \frac{e^{4n+4}(n-2)!}{(n-1)!e^{4n}} \right| \implies \lim_{n \rightarrow \infty} \left| \frac{e^4(n-2)!}{(n-1)(n-2)!} \right| \implies \lim_{n \rightarrow \infty} \left| \frac{e^4}{(n-1)} \right| \\ \implies 0 \end{aligned}$$

Convergent

3.3 Integral Test

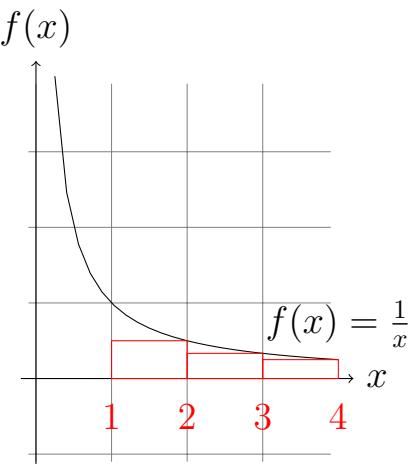
If $f(x)$ is decreasing, then for $x \geq 1$:



The Left Riemann Sum is $>$ than $f(x)$:

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n} > \int_1^{\infty} \frac{1}{x} dx$$

Similarly,



The Right Riemann Sum is $<$ than $f(x)$:

$$\begin{aligned} \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots &\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n} < \int_1^{\infty} \frac{1}{x} dx \\ &\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n} < \int_1^{\infty} \frac{1}{x} dx + 1 \\ &\Rightarrow \int_1^{\infty} \frac{1}{x} dx < \sum_{n=1}^{\infty} \frac{1}{n} < \int_1^{\infty} \frac{1}{x} dx + 1 \end{aligned}$$

Using a method analogous to the Squeeze Theorem, if the bounds diverge, so does the series, and vice versa.

4 P Series

NOT Power Series.

A P Series is any series in the form:

$$\sum_{n=0}^{\infty} \frac{1}{n^p}$$

We use Integral Test to prove if it is diverging or converging.

4.1 Harmonic Series

A Harmonic Series is:

$$\sum_{n=0}^{\infty} \frac{1}{n}$$

Or a P Series where $p = 1$.

$$\int_1^{\infty} \frac{1}{x} dx < \sum_{n=1}^{\infty} \frac{1}{n} < \int_1^{\infty} \frac{1}{x} dx + 1 \Rightarrow \int_1^{\infty} \frac{1}{x} dx = \ln|x| \Big|_1^{\infty}$$