Sets Cheatsheet

Contents

1 Geometric Series

Geometric Series are a special type of Power Series that can be rewritten in the form

$$f(x) = \frac{a_1}{1-r} = \sum_{n=0}^{\infty} a_1 \cdot r^n$$

1.1 Proof

$$\begin{aligned} S &= a_1 + a_1 r + a_1 r^2 + a_1 r^3 + ... + a_1 r^n + ... \\ rS &= a_1 r + a_1 r^2 + a_1 r^3 + a_1 r^4 ... + a_1 r^n + ... \\ S - rS &= a_1 + a_1 r - a_1 r + a_1 r^2 - a_1 r^2 + ... + a_1 r^n - a_1 r^n + a_1 r^{n+1} + ... \\ &\implies S(1-r) = a_1 + a_1 r^{n+1} \end{aligned}$$

Evaluate the lim as $n \to \infty$

$$S = \lim_{n\to\infty} \frac{a_1 + a_1 r^{n+1}}{1-r}$$

We can only evaluate this equation when $|r| < 1$. Therefore:

$$\frac{a_1}{1-r}$$

Is the sum of the geometric series.
To find the interval of convergence, just remember that $|r| < 1$

1.2 Examples

$$f(x) = \frac{1}{2-x}, c = 0 \implies f(x) = \frac{1}{2} \cdot \frac{1}{1 - \frac{x}{2}} \implies f(x) = \sum_{n=0}^{\infty} \frac{1}{2} \cdot \left(\frac{x}{2}\right)^n$$

$$\implies f(x) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} x^n$$

$$|r| < 1 \implies \left|\frac{x}{2}\right| < 1 \implies |x| < 2 \implies x \in (-2, 2)$$

If we change the center:

$$f(x) = \frac{1}{2-x}, c = 5 \implies f(x) = \frac{1}{2 - 5 - (x-5)} \implies \frac{1}{-3 - (x-5)}$$

$$\implies -\frac{1}{3} \cdot \frac{1}{1 - \frac{-(x-5)}{3}} \implies \sum_{n=0}^{\infty} -\frac{1}{3} \cdot (\frac{-(x-5)}{3})^n$$

$$\Longrightarrow \sum_{n=0}^{\infty}(-\frac{1}{3})^{n+1}(-(x-5))^n$$

$$|r|<1 \Longrightarrow r\in(-1,1) \Longrightarrow \frac{-x+5}{3}\in(-1,1) \Longrightarrow -(x-5)\in(-3,3) \Longrightarrow x\in(2,8)$$

Now let's completely change it up.

$$f(x)=\frac{3}{2x-1}, c=0 \Longrightarrow f(x)=-3\cdot\frac{1}{1-2x} \Longrightarrow \cdot\sum_{n=0}^{\infty}-3\cdot(2x)^n, c\in(-\frac{1}{2},\frac{1}{2})$$

Let's move the center

$$f(x)=\frac{3}{2x-1}, c=-3 \Longrightarrow f(x)=\frac{1}{2(x+3)-5-6} \Longrightarrow f(x)=-\frac{1}{11}\cdot\frac{1}{1-\frac{2}{11}(x+3)}$$

$$f(x)=\sum_{n=0}^{\infty}-\frac{1}{11}\cdot(\frac{2}{11}(x+3))^n$$

$$\frac{2}{11}(x+3)\in(-1,1) \Longrightarrow (x+3)\in(-\frac{11}{2},\frac{11}{2}) \Longrightarrow x\in(-\frac{17}{2},\frac{5}{2})$$

What if there is no $-x$? You can just do $-(-x)$. Remember: Basic Algebra will take you a long way.

$$f(x)=\frac{3}{x+2}, c=0 \Longrightarrow f(x)=\frac{3}{2}\cdot\frac{1}{1-\frac{1}{2}(-x)}$$

What if your function looks a bit more complicated?

$$f(x)=\frac{4x-7}{2x^2+3x-2}, c=0 \Longrightarrow f(x)=\frac{4x-7}{(2x-1)(x+2)} \Longrightarrow f(x)=\frac{A}{2x-1}+\frac{B}{x+2}$$

$$A(x+2)+B(2x-1)=4x-7 \Longrightarrow Ax+2Bx=4x, 2A-B=-7 \Longrightarrow B=2A+7$$

$$\Longrightarrow A+4A+14=4 \Longrightarrow 5A=-10 \Longrightarrow A=-2, B=3$$

$$f(x)=\frac{-2}{2x-1}+\frac{3}{x+2}$$

Solve it normally from here.

2 General Power Series

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n$$

2.1 Derivation

We take general power series to be something like the following:

$$f(x) = 1 + x + 2x^2 + 3x^3 + ... + nx^n + ...$$

or

$$f(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + ... + \frac{1}{n}x^n + ...$$

So generally:

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

Instead of a_i because the coefficient changes for each element in the series. Let's expand this series:

$$f(x) = a_0 + c + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + ... + a_n(x-c)^n + ...$$

Take its derivative:

$$f'(x) = 0 + a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + ... + na_n(x-c)^{n-1} + ...$$

Notice how $f'(0) = a_1$.

$$f''(x) = 0 + 0 + 2a_2 + 6a_3(x-c) + ... + n(n-1)a_n(x-c)^{n-2} + ...$$

Notice how $f''(0) = 2a_2$ and $f'''(0) = 6a_3$. In order to get the a_n term, you just need to take $f^{(n)}(c)$ and divide it by $n!$

Put it together, and you'll get the equation we started with.

2.2 Examples

$$f(x) = e^x, c = 0 \implies f(x) = \sum_{n=0}^{\infty} \frac{e^0}{n!}(x)^n \implies f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\implies 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + ... + \frac{1}{n}x^n + ...$$

Evaluate a geometric series using Taylor series

$$f(x) = \frac{1}{1-2x}, c = 0 \implies \frac{\frac{1}{1-2(0)}}{0!} + \frac{\frac{2}{(1-2c)^2}}{1!}x + \frac{\frac{8}{(1-2c)^3}}{2!}x + \frac{\frac{48}{(1-2c)^4}}{3!}x + ...$$

$$\implies 1 + \frac{2}{(1-2c)^2}x + \frac{4}{(1-2c)^3}x^2 + \frac{8}{(1-2c)^4}x^3 + ...$$

General term: $2^n x^n$.

We would have gotten the same thing if we used the power series expansion of the Taylor series expansion. As with most things in algebra, pick the method that's **most conveinient for you**.

Note: if a problem asks for the nth term, that will always match the power on a Taylor polynomial.

2.3 Error Checking

In an ideal world, you want to know how far off your estimates are. For alternating series, this process is pretty easy.

$$f(x) = 1 - x + \frac{x}{2!} - \frac{x}{3!} + \frac{x}{4!} - \frac{x}{5!} + \frac{x}{6!} - \frac{1}{7!} + ... + \frac{1}{n!}(-1)^n + ...$$

Let's choose the first four terms for our **Taylor Polynomial**, which will be represented by $P_n(x)$ where n is the degree.

$$P_4(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{3}x^3$$

Let's set the remainder terms to $R_5(x)$. We can call the error $E(x)$.

$$E(x) = |P_4(x) - R_5(x)|$$

Since you will never truly know $R_5(x)$ or any $R_n(x)$ for that matter, you will never know the true error. But you can set an upper bound on that error, which is pretty easy in alternating power series.

$$E(x) \leq P_{n+1}(x) - P_n(x)$$

A fancy way of saying, the term right after is the highest the eror can be. It's because you will never add back what you subtracted, always less.

2.4 Deriving e^a

We can use a Taylor Series to derive e^a!

$$\begin{aligned} \lim_{n\to\infty}(1+\frac{a}{n})^n = L \Longrightarrow \ln(\lim_{n\to\infty}(1+\frac{a}{n})^n) = \ln L \Longrightarrow n\ln(\lim_{n\to\infty}(1+\frac{a}{n})) = \ln L \\ t = \frac{a}{n}, \Longrightarrow \frac{a}{t}(\lim_{n\to\infty}\ln(1+t)) = \ln L \Longrightarrow \frac{a}{t}(\lim_{t\to 0}\ln(1+t)) = \ln L \\ \longrightarrow \ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} + ... \approx t \\ \Longrightarrow \ln L = a \Longrightarrow L = e^a \end{aligned}$$

2.5 Lagrange Error Checking

If your series **isn't alternating**, you can use Lagrange Error Checking. Although the proof is complicated, it's pretty simple

$$E(x) \leq |\frac{\max_{z\in[x,c]} f^{(n+1)}(z)}{(n+1)!}|(x-c)^{n+1}$$

The proof isn't necessary for Calc BC. You can find it online.

2.5.1 Full Example

Find the first four terms about $c = 2$ for $ln|x+1|$.

We can start with its derivative, $\frac{1}{x+1}$.

For simplicity, we will use a geometric series. Say we wanted to find $\ln|\frac{3}{2}|$ and approximate error to be 0.05 or less

$$\frac{1}{x-1} = \frac{1}{1-(x-2)+2} = \frac{1}{-1-(x-2)} = -\frac{1}{1-(-(x-2))}$$

$$\implies \sum_{n=0}^{\infty} -(-(x-2))^n = \sum_{n=0}^{\infty} (-1)^{n+1}(x-2)^n = -1 + (x-2) - \frac{1}{2}(x-2)^2 + \frac{1}{3}(x-2)^3 + ...$$

$$\int -\frac{1}{1-(-(x-2))}dx = \int -1 + (x-2) - \frac{1}{2}(x-2)^2 + \frac{1}{3!}(x-2)^3 + ...dx$$

$$\implies -\ln|1-(-(x-2))| = C - x + \frac{(x-2)^2}{2} - \frac{(x-2)^3}{3 \cdot 2!} + \frac{(x-2)^4}{4 \cdot 3!} + ...$$

$$\longrightarrow \ln|1-(-(2-2))| = \ln|1|$$

$$\longrightarrow C + x - \frac{(x-2)^2}{2} + \frac{(x-2)^3}{3 \cdot 2!} - \frac{(x-2)^4}{4 \cdot 3!} + ...$$

$$\longrightarrow \ln|1| = C + 2 + 0 + ... \longrightarrow C = -2$$

$$\implies \ln|1-(-(x-2))| = -2 + x - \frac{(x-2)^2}{2} + \frac{(x-2)^3}{3 \cdot 2!} - \frac{(x-2)^4}{4 \cdot 3!} + ...$$

$$\implies \ln|\frac{3}{2}| = \ln|1-(-(\frac{5}{2}-2))| = (\frac{5}{2}-2) - \frac{(\frac{5}{2}-2)^2}{2} + \frac{(\frac{5}{2}-2)^3}{3 \cdot 2!} - \frac{(\frac{5}{2}-2)^4}{4 \cdot 3!} + ...$$

$$\implies \frac{1}{2} - \frac{1}{2} \cdot (\frac{1}{2})^2 + \frac{1}{3} \cdot (\frac{1}{2})^3 - \frac{1}{4} \cdot (\frac{1}{2})^4 + ...$$

Since it's an alternating series, it should be pretty simple to solve for from here.

2.5.2 Another Problem

$$\sin(5x + \frac{\pi}{4}) = \sin(5(x + \frac{\pi}{20}))$$

3 P Series

NOT Power Series.

A P Series is any series in the form:

$$\sum_{n=0}^{\infty} \frac{1}{n^p}$$

We use Integral Test to prove if it is diverging or converging.

3.1 Harmonic Series

A Harmonic Series is:

$$\sum_{n=0}^{\infty} \frac{1}{n}$$

Or a P Series where $p = 1$.

3.2 Proof

Where $p = 1$ (Harmonic Series)

$$\int_1^{\infty} \frac{1}{x} dx < \sum_{n=1}^{\infty} \frac{1}{n} < \int_1^{\infty} \frac{1}{x} dx + 1 \implies \int_1^{\infty} \frac{1}{x} dx = \ln|x| \Big|_1^{\infty}$$
$$\implies \lim_{x \to \infty} \ln x = \infty$$

Diverges

Where $p \neq 1$

$$\int_1^{\infty} \frac{1}{x^p} dx < \sum_{n=1}^{\infty} \frac{1}{n^p} < \int_1^{\infty} \frac{1}{x^p} dx + 1 \longrightarrow \int_1^{\infty} x^{-p} dx = \frac{x^{-p+1}}{-p+1} \Big|_1^{\infty}$$
$$\longrightarrow \lim_{x \to \infty} \frac{x^{-p+1}}{-p+1}$$

If $p > 1$ it converges, $p < 1$ (or $p \leq 1$) it diverges.

3.3 Examples

$$\sum_{n=1}^{\infty} \frac{1}{(2n+3)^2} \implies \int_1^{\infty} \frac{1}{(2x+3)^2} dx \longrightarrow u = 2x+3 \quad du = 2dx \implies \int_3^{\infty} \frac{1}{2u^3} du$$

Convergent

$$\sum_{n=1}^{\infty} \frac{n+2}{n+1} \implies \int_1^{\infty} \frac{x+2}{x+1} dx = \int_1^{\infty} (1 + \frac{1}{x+1}) dx = (x + \ln|x+1|) \Big|_1^{\infty}$$
$$= \lim_{x \to \infty} x + \ln|x+1| = \infty$$

Divergent

4 Convergent or Divergent

4.1 nth Term Test for Divergence

If

$$\lim_{x\to\infty} a_x \neq 0$$

then f is divergent. If

$$\lim_{x\to\infty} a_x = 0$$

then use another test, it's inconclusive.

4.2 Ratio Test

Commonly used for geometric series, if you just check to make sure $|r| < 1$ then it's going to converge.

$$\lim_{n\to\infty} |\frac{a^{n+1}}{a^n}|$$

Basically checking that for some far off value, the common ratio remains.
Example:

$$\sum_{n=0}^{\infty} \frac{(-1)^{n-1}n^2}{2^n} \Longrightarrow \lim_{n\to\infty} |\frac{\frac{(-1)^n(n+1)^2}{2^{n+1}}}{\frac{(-1)^{n-1}n^2}{2^n}}| = 0 \Longrightarrow \lim_{n\to\infty} |\frac{(-1)^n(n+1)^2 \cdot 2^n}{2^n \cdot 2 \cdot (-1)^{n-1}n^2}|$$
$$\Longrightarrow \lim_{n\to\infty} |\frac{(n+1)^2}{2 \cdot -1 \cdot n^2}| = \frac{1}{2}$$

If $|r| = 1$, we have a problem. Recall the limit from the geometric series proof:

$$\lim_{n\to\infty} \frac{a - ar^{n+1}}{1-r}$$

That's going to be undefined. So we have to use another test.
This test is best used for rapidly growing questions like $n!$.

4.2.1 More Examples

$$\sum_{n=0}^{\infty} \frac{e^n}{n^3} \Longrightarrow \lim_{n\to\infty} |\frac{\frac{e^n \cdot e}{(n+1)^3}}{\frac{e^n}{n^3}}| \Longrightarrow \lim_{n\to\infty} |\frac{e \cdot n^3}{(n+1)^3}| = e \not< 1$$

Divergent

$$\sum_{n=0}^{\infty}\frac{(n!)^2}{(3n)!} \implies \lim_{n\to\infty}|\frac{\frac{((n+1)!)^2}{(3n+3)!}}{\frac{(n!)^2}{(3n)!}}| \implies \lim_{n\to\infty}|\frac{((n+1)!)^2\cdot(3n)!}{(3n+3)!\cdot(n!)^2}|$$

$$\implies \lim_{n\to\infty}|\frac{(n+1)^2\cdot(n!)^2\cdot(3n)!}{(3n+3)\cdot(3n+2)\cdot(3n+1)\cdot(3n)!\cdot(n!)^2}|$$

$$\implies \lim_{n\to\infty}|\frac{(n+1)^2}{(3n+3)\cdot(3n+2)\cdot(3n+1)}| = 0 < 1$$

Convergent

$$\sum_{n=0}^{\infty}\frac{n!}{n^{n+1}} \implies \lim_{n\to\infty}|\frac{\frac{(n+1)!}{(n+1)^{n+2}}}{\frac{n!}{n^{n+1}}}| \implies \lim_{n\to\infty}|\frac{(n+1)!\cdot n^{n+1}}{n!\cdot(n+1)^{n+2}}| \implies \lim_{n\to\infty}|\frac{n!\cdot n^{n+1}}{n!\cdot(n+1)^{n+1}}|$$

$$\implies \lim_{n\to\infty}|(\frac{n}{n+1})^{n+1}| = \lim_{n\to\infty}|(\frac{n+1-1}{n+1})^{n+1}| = \lim_{n\to\infty}|(1-\frac{1}{n+1})^{n+1}| =$$

$$\lim_{n\to\infty}|(1+\frac{(-1)}{n+1})^{n+1}| = e^{-1} < 1$$

Convergent

$$\sum_{n=1}^{\infty}\frac{3^{1-2n}}{n^2+1} \implies \lim_{n\to\infty}|\frac{\frac{3^{1-2n-2}}{(n+1)^2+1}}{\frac{3^{1-2n}}{n^2+1}}| \implies \lim_{n\to\infty}|\frac{3^{-1-2n}\cdot(n^2+1)}{(n+1)^2+1\cdot 3^{1-2n}}|$$

$$\implies \lim_{n\to\infty}|\frac{n^2+1}{(n+1)^2+1\cdot 3^{1-2n}\cdot 3^{1+2n}}| \implies \lim_{n\to\infty}|\frac{n^2+1}{(n+1)^2+1\cdot 3\cdot 3^{-2n}\cdot 3\cdot 3^{2n}}|$$

$$\implies \lim_{n\to\infty}|\frac{n^2+1}{(n+1)^2+1\cdot 3\cdot 3}| = \frac{1}{9}$$

Convergent

$$\sum_{n=2}^{\infty}\frac{(-2)^{1+3n}(n+1)}{n^2 5^{1+n}}$$

$$\implies \lim_{n\to\infty}|\frac{\frac{(-2)^{4+3n}(n+2)}{(n+1)^2 5^{2+n}}}{\frac{(-2)^{1+3n}(n+1)}{n^2 5^{1+n}}}| \implies \lim_{n\to\infty}|\frac{(-2)^{4+3n}(n+2)n^2 5^{1+n}}{(n+1)^2 5^{2+n}(-2)^{1+3n}(n+1)}|$$

$$\implies \lim_{n\to\infty}|\frac{(-2)^3(n+2)n^2}{(n+1)^3 5}| = \frac{(-2)^3}{5^5} = -\frac{8}{5}$$

Diverges

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{6n+7} \implies \lim_{n\to\infty}|\frac{\frac{(-1)^{n+2}}{6n+6+7}}{\frac{(-1)^{n+1}}{6n+7}}| \implies \lim_{n\to\infty}|\frac{(-1)^{n+2}(6n+7)}{(6n+6+7)(-1)^{n+1}}|$$

$$\implies \lim_{n\to\infty}|\frac{6n+7}{6n+6+7}|$$

Inconclusive: But if you used the Divergence Test first:

$$\lim_{n\to\infty} \frac{(-1)^{n+1}}{6n+7} \approx \frac{(-1)^{\infty}}{\infty} = 0$$

Inconclusive, but the alternating series test proves it right.

$$\sum_{n=3}^{\infty} \frac{e^{4n}}{(n-2)!}$$

$$\longrightarrow \lim_{n\to\infty} \frac{e^{4n}}{(n-2)!} = 0$$

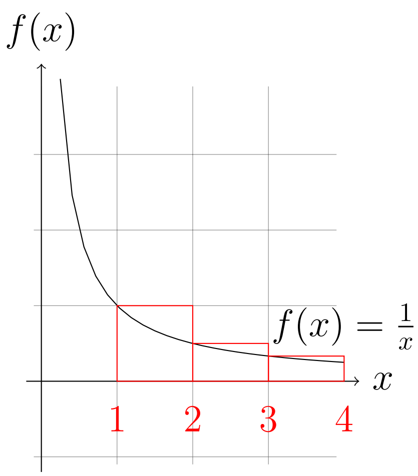
$$\Longrightarrow \lim_{n\to\infty} \left|\frac{\frac{e^{4n+4}}{(n-1)!}}{\frac{e^{4n}}{(n-2)!}}\right| \Longrightarrow \lim_{n\to\infty} \left|\frac{e^{4n+4}(n-2)!}{(n-1)!e^{4n}}\right| \Longrightarrow \lim_{n\to\infty} \left|\frac{e^{4}(n-2)!}{(n-1)(n-2)!}\right| \Longrightarrow \lim_{n\to\infty} \left|\frac{e^{4}}{(n-1)}\right|$$

$$\Longrightarrow 0$$

Convergent

4.3 Integral Test

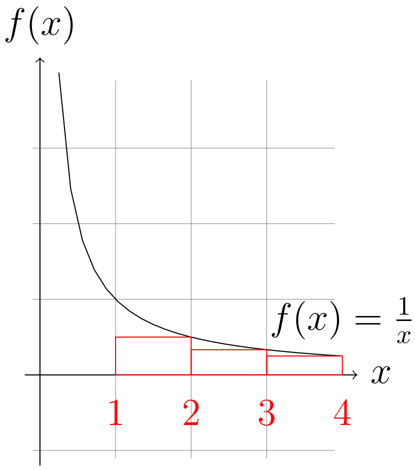
If $f(x)$ is decreasing, then for $x \geq 1$:



The Left Riemann Sum is > than $f(x)$:

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + ... \Longrightarrow \sum_{n=1}^{\infty} \frac{1}{n} > \int_{1}^{\infty} \frac{1}{x} dx$$

Similarly,



The Right Riemann Sum is < than $f(x)$:

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... \implies \sum_{n=2}^{\infty} \frac{1}{n} < \int_{1}^{\infty} \frac{1}{x} dx$$

$$\implies \sum_{n=1}^{\infty} \frac{1}{n} < \int_{1}^{\infty} \frac{1}{x} dx + 1$$

$$\implies \int_{1}^{\infty} \frac{1}{x} dx < \sum_{n=1}^{\infty} \frac{1}{n} < \int_{1}^{\infty} \frac{1}{x} dx + 1$$

Using a method analogous to the Squeeze Theorem, if the bounds diverge, so does the series, and vice versa.

4.4 Direct Comparison Test

The direct comparison test allows for **no negative terms** in the series.

Let a_n and b_n represent an unknown and known sequence, respectively.

If $a_n < b_n$ for all n (according to CollegeBoard, however the first few terms are an exception) and $\lim_{n\to\infty} \sum_i^n b_i = L$ then $\lim_{n\to\infty} \sum_i^n a_i = L_2$

If $a_n > b_n$ for all n and $\lim_{n\to\infty} \sum_i^n b_i = L$ then $\lim_{n\to\infty} \sum_i^n a_i$ could equal L_2 or ∞.

If $a_n < b_n$ for all n and $\lim_{n\to\infty} \sum_i^n b_i = \infty$ then $\lim_{n\to\infty} \sum_i^n a_i$ could equal L_2 or ∞.

If $a_n > b_n$ for all n and $\lim_{n\to\infty} \sum_i^n b_i = \infty$ then $\lim_{n\to\infty} \sum_i^n a_i = \infty$.

It is recommended to use this test only if the degree is the same.

4.4.1 Examples

$$\sum_{n=2}^{\infty} \frac{1}{n^2+1} = \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + ...$$

$$\sum_{n=2}^{\infty} \frac{1}{n^2} = \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + ...$$

$$\implies \sum_{n=2}^{\infty} \frac{1}{n^2} > \sum_{n=2}^{\infty} \frac{1}{n^2+1} \implies \sum_{n=2}^{\infty} \frac{1}{n^2} = L \therefore \sum_{n=2}^{\infty} \frac{1}{n^2+1} = L_2$$

Convergent

4.5 Limit Comparison Test

$$\lim_{n\to\infty} \frac{a_n}{b_n}$$

For $0 < n < \infty$ (if it is 0 or ∞ anyway, the previous test would suffice).

4.5.1 Examples

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1} = \frac{1}{2} + \frac{2}{5} + \frac{3}{10} + ...$$

$$\sum_{n=1}^{\infty} \frac{n}{n^2} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + ...$$

$$\implies \sum_{n=1}^{\infty} \frac{1}{n} > \sum_{n=1}^{\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{1}{n} > \lim_{n\to\infty} \frac{n}{n^2+1} = \infty > \lim_{n\to\infty} \frac{n}{n^2}$$

Inconclusive

$$\lim_{n\to\infty} \frac{\frac{n}{n^2+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$$

Convergent

$$\sum_{n=1}^{\infty} \frac{5n-3}{n^2-2n+5} = \frac{-3}{5} + \frac{2}{4} + \frac{7}{5} + \frac{12}{8} + \frac{17}{13} ...$$

$$\sum_{n=1}^{\infty} \frac{n}{n^2} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} ...$$

$$\implies \lim_{n\to\infty} \frac{\frac{5n-3}{n^2-2n+5}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{5n^2-3n}{n^2-2n+5} = 5$$

$$\lim_{n\to\infty} \frac{1}{n} = \infty \therefore \lim_{n\to\infty} \frac{5n-3}{n^2-2n+5} = \infty$$

Divergent

4.6 Alternating Series ~~Property~~ Theorem

Recall the Alternating Error Bound for a series. Take this example:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...$$

Since the sum is **decreasing** and $\lim_{n\to\infty} \frac{1}{n} = 0$ (since n is increasing), there is an error bound, meaning this alternating series is convergent. Note that this "test" doesn't apply to oscillating serieses like $sin(x)$.

4.6.1 Example

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}n}{2n+1}$$
$$\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} \Longrightarrow \approx \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2}$$

This sum does not reach 0, it osciilates. It is not convergent.

4.7 Absolute Convergence Test

Let a_n represent a sequence with positive and negative terms.

Using some common sense, we can derive the following:

$$n \in \mathbb{R} \Longrightarrow n \leq |n|$$
$$\therefore a_n \leq |a_n|$$
$$\Longrightarrow a_n + |a_n| \leq 2|a_n|$$
$$\longrightarrow |a_n| + a_n = 0 \quad \text{or} \quad 2a_n \therefore 0 \leq a_n + |a_n| \leq 2a_n$$

By the Direct Comparsion test:

$$\sum_{n=1}^{\infty} |a_n| = L \therefore \sum_{n=1}^{\infty} a_n = L_2$$

In this case it is absolutely convergent. In the scenario that:

$$\sum_{n=1}^{\infty} |a_n| = \infty$$

It is inconclusive. If the other alternating series tests passes, it is **Conditionally Convergent**.

4.8 Examples

$$\sum_{n=2}^{\infty} \frac{n}{\ln n} \Longrightarrow \lim_{n\to\infty} \frac{n}{\ln n} = \infty$$

Divergent by nth Term Test for Divergence

$$\sum_{n=1}^{\infty} \frac{2^n}{n^{99}} \Longrightarrow \lim_{n\to\infty} \frac{2^n}{n^{99}} = \infty$$

Divergent by nth Term Test for Divergence

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^2+1}}{n} \Longrightarrow \lim_{n\to\infty} \frac{\sqrt{n^2+1}}{n} \approx \lim_{n\to\infty} \frac{\sqrt{n^2}}{n} = \lim_{n\to\infty} \frac{n}{n} = 1$$

Divergent by nth Term Test for Divergence

$$\sum_{n=2}^{\infty} \frac{n^e}{n^{\pi}} = \sum_{n=2}^{\infty} \frac{1}{n^{\pi-e}} \Longrightarrow 1 \geq \pi - e$$

Divergent by property of P-Series

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+8}} = \frac{1}{\sqrt{8}} + \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+8}} &= \frac{1}{\sqrt{8}} + \frac{1}{3} + \frac{1}{\sqrt{10}} + ... \\ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} &= \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... \\ \Longrightarrow \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+8}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n+8}}{\sqrt{n}} &= 1\end{aligned}$$

Divergent by Limit Comparison Test

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{1} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + ...$$

Convergent by Properties of Alternating Series:

1. It is decreasing
2. It is approaching 0
3. It is alternating

$$\sum_{n=3}^{\infty}\frac{4}{n\sqrt{\ln n}}=\frac{4}{3\sqrt{\ln 3}}+\frac{4}{4\sqrt{\ln 4}}+\frac{4}{5\sqrt{\ln 5}}+...$$

$$\sum_{n=3}^{\infty}\frac{4}{n}=\frac{4}{3}+\frac{4}{4}+\frac{4}{5}+...$$

$$\Longrightarrow\sum_{n=3}^{\infty}\frac{4}{n\sqrt{\ln n}}<\sum_{n=3}^{\infty}\frac{4}{n},\sum_{n=3}^{\infty}\frac{4}{n}=\infty$$

$$\Longrightarrow\lim_{n\to\infty}\frac{\frac{4}{n\sqrt{\ln n}}}{\frac{4}{n}}=0$$

$$\lim_{n\to\infty}\frac{\frac{4}{n\sqrt{\ln(n+1)}+\sqrt{\ln(n+1)}}}{\frac{4}{n\sqrt{\ln n}}}=???$$

After trying three tests, we are left with the Integral Test

$$\sum_{n=3}^{\infty}\frac{4}{n\sqrt{\ln n}}$$

$$\Longrightarrow 4\int_{3}^{\infty}\frac{1}{n\sqrt{\ln n}}dn, u=\ln n, \quad du=\frac{1}{n}dn$$

$$\Longrightarrow 4\int_{\ln 3}^{\infty}\frac{1}{\sqrt{u}}du=\int_{\ln 3}^{\infty}u^{-\frac{1}{2}}du=4*[2u^{\frac{1}{2}}]\Big|_{\ln 3}^{\infty}$$

Divergent by Integral Test

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{27n^2}}=\frac{1}{3}\sum_{n=1}^{\infty}\frac{1}{n^{\frac{2}{3}}}$$

Divergent by Property of P-Series

$$\sum_{n=1}^{\infty}\frac{n}{\sqrt{n^2+1}}\Longrightarrow\lim_{x\ to\infty}\frac{x}{\sqrt{x^2+1}}\approx\lim_{x\ to\infty}\frac{x}{x}=1$$

Divergent by nth Term Test for Divergence

$$\sum_{n=1}^{\infty}\sqrt{\frac{n+1}{n}}$$

Be careful, this is NOT a disguised $\frac{1}{e}$.

$$\approx\sum_{n=1}^{\infty}\sqrt{\frac{n}{n}}=1$$

Divergent by nth Term Test for Divergence

$$\sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n} = \sum_{n=1}^{\infty} \frac{2^n}{4^n} + \frac{3^n}{4^n} = \sum_{n=1}^{\infty} \frac{2^n}{4^n} + \sum_{n=1}^{\infty} \frac{3^n}{4^n} = \sum_{n=1}^{\infty} \left(\frac{2}{4}\right)^n + \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$$

Convergent by Property of Geometric Series

$$\sum_{n=3}^{\infty} \frac{1}{n(\ln n)(\ln(\ln n))} = \sum_{n=3}^{\infty} \frac{1}{n(\ln n)} \frac{1}{(\ln(\ln n))}$$

$$\implies \sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)} = \frac{1}{\ln(\ln 3)} + \frac{1}{\ln(\ln 4)} + \frac{1}{\ln(\ln 5)} + ...$$

$$\sum_{n=3}^{\infty} \frac{1}{n} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + ...$$

$$\implies \sum_{n=3}^{\infty} \frac{1}{n} < \sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$$

$$\sum_{n=3}^{\infty} \frac{1}{n} = \infty \therefore \sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)} = \infty$$

A divergent sum times any sum is divergent.
Divergent by Property of P-Series

$$\sum_{n=3}^{\infty} \frac{1}{2^n - n} = \frac{1}{5} + \frac{1}{12} + \frac{1}{27} + ...$$

$$\sum_{n=3}^{\infty} \frac{1}{2^n} = \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + ...$$

$$\sum_{n=3}^{\infty} \frac{1}{2^n - n} > \sum_{n=3}^{\infty} \frac{1}{2^n}$$

$$\sum_{n=3}^{\infty} \frac{1}{2^n} = L$$

$$\lim_{n\to\infty} \frac{\frac{1}{2^n - n}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{2^n}{2^n - n} = 1$$

Convergent by Limit Comparison Test

$$\sum_{n=1}^{\infty} \frac{5+8^n}{2-7^n} = \sum_{n=1}^{\infty} \left(\frac{5}{2-7^n} + \frac{8^n}{2-7^n} \right) = \sum_{n=1}^{\infty} \left(\frac{5}{2-7^n} - \frac{8^n}{7^n-2} \right)$$

$$\implies \sum_{n=1}^{\infty} \frac{8^n}{7^n-2} = \frac{8}{5} + \frac{64}{47} + ...$$

$$\lim_{n\to\infty} \frac{8^n}{7^n-2} \approx \lim_{n\to\infty} \frac{8^n}{7^n} = \lim_{n\to\infty} \left(\frac{8}{7}\right)^n = \infty$$

A divergent series plus any series is divergent.
Divergent by nth Term Test for Divergence

5 More Examples

1.

$$\sum_{n=1}^{\infty} (-1)^n \frac{(x-2)^n}{n+1}$$

For P-Series, you can **only use** the Ratio Test to determine Convergence over an Interval

$$\implies \lim_{n\to\infty} \frac{\frac{(-1)^{n+1}(x-2)^{n+1}}{n+2}}{\frac{(-1)^n(x-2)^n}{n+1}} = \lim_{n\to\infty} \left| \frac{(n+1)(-1)^{n+1}(x-2)^{n+1}}{(n+2)(-1)^n(x-2)^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)(x-2)}{(n+2)} \right|$$

$$= \lim_{n\to\infty} |x-2| < 1$$

$$\implies 1 < x < 3$$

Now you have to test the values 1 and 3.

$$\sum_{n=0}^{\infty} \left| (-1)^n \frac{(x-2)^n}{n+1} \right|$$

$$\longrightarrow \sum_{n=0}^{\infty} \left| (-1)^n \frac{(1-2)^n}{n+1} \right|$$

$$\longrightarrow \sum_{n=0}^{\infty} \left| (-1)^n \frac{(-1)^n}{n+1} \right| = \sum_{n=0}^{\infty} \left| \frac{1}{n+1} \right|$$

Divergent by Comparsion Test

$$\sum_{n=0}^{\infty}\left|(-1)^n\frac{(3-2)^n}{n+1}\right|$$
$$\longrightarrow\sum_{n=0}^{\infty}\left|(-1)^n\frac{(1)^n}{n+1}\right|$$
$$\longrightarrow\sum_{n=0}^{\infty}\left|\frac{(-1)^n}{n+1}\right|=\sum_{n=0}^{\infty}\left|\frac{(-1)^n}{n+1}\right|$$

Convergent by Alternating Series "Test"

$$1<x\leq 3$$

2.

$$\sum_{n=1}^{\infty}\frac{x^n n^n}{3^n n!}$$

$$\Longrightarrow\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}(n+1)^{n+1}}{3^{n+1}(n+1)!}}{\frac{x^n n^n}{3^n n!}}\right|=\lim_{n\to\infty}\left|\frac{x^{n+1}(n+1)^{n+1}3^n n!}{x^n n^n 3^{n+1}(n+1)!}\right|=\lim_{n\to\infty}\left|\frac{x^{n+1}(n+1)(n+1)^n n!}{x^n n^n 3(n+1)!}\right|$$
$$=\lim_{n\to\infty}\left|\frac{x(n+1)^n n!}{n^n 3n!}\right|=\frac{1}{3}\lim_{n\to\infty}\left|x\left(\frac{n+1}{n}\right)^n\right|=\frac{1}{3}\lim_{n\to\infty}\left|x\left(1+\frac{1}{n}\right)^n\right|$$
$$\Longrightarrow\left|\frac{e}{3}x\right|<1\Longrightarrow -\frac{3}{e}<x<\frac{3}{e}$$

Plug in the endpoints:

$$\sum_{n=1}^{\infty}\frac{(-\frac{3}{e})^n n^n}{3^n n!}=\sum_{n=1}^{\infty}\frac{(-1)^n(3)^n n^n}{e^n 3^n n!}=\sum_{n=1}^{\infty}\frac{(-1)^n n^n}{e^n n!}$$

Look at the graph and compare it against $\frac{1}{n}$.

3.

$$\sum_{k=0}^{\infty}\frac{2^k x^k}{\ln(k+2)}$$

$$\lim_{k\to\infty}\left|\frac{\frac{2^{k+1}x^{k+1}}{\ln((k+1)+2)}}{\frac{2^k x^k}{\ln(k+2)}}\right|=\lim_{k\to\infty}\left|\frac{2^{k+1}x^{k+1}\ln(k+2)}{2^k x^k\ln((k+1)+2)}\right|=\lim_{k\to\infty}\left|\frac{2x\ln(k+2)}{\ln(k+2+1)}\right|=2x$$
$$-\frac{1}{2}<x<\frac{1}{2}$$

Plug in the endpoints:

$$\sum_{k=0}^{\infty}\frac{2^k\left(\frac{-1}{2}\right)^k}{\ln(k+2)}=\sum_{k=0}^{\infty}\frac{(-1)^k}{\ln(k+2)}$$

(a) Alternating
(b) Approaching 0
(c) Decreasing

$k = -\frac{1}{2}$ Convergent

$$\sum_{k=0}^{\infty} \frac{2^k \left(\frac{1}{2}\right)^k}{\ln(k+2)} = \sum_{k=0}^{\infty} \frac{1^k}{\ln(k+2)} = \sum_{k=0}^{\infty} \frac{1}{\ln(k+2)}$$

$$\sum_{k=0}^{\infty} \frac{1}{\ln(k+2)} = \frac{1}{\ln 2} + \sum_{k=1}^{\infty} \frac{1}{\ln(k+2)} = \frac{1}{\ln 2} + \frac{1}{\ln 3} + \frac{1}{\ln 4} + \cdots$$

$$\sum_{k=1}^{\infty} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \sum_{k=1}^{\infty} \frac{1}{\ln(k+2)} > \sum_{k=1}^{\infty} \frac{1}{k}$$

First few terms ignored

$$\sum_{k=1}^{\infty} \frac{1}{k} = \infty \therefore \sum_{k=1}^{\infty} \frac{1}{\ln(k+2)} = \infty$$

4. Find the first three terms and the general term for the Maclaurin series for the derivative of the following:

$$f(x) = \frac{1}{1+x^3} = 1 - x^3 + x^6 - x^9 + \cdots + (-1)^n x^{3n} + \cdots$$

$$f'(x) = -\frac{3x^2}{(1+x^3)^2} = -3x^2 + 3x^5 + 3x^8 + \cdots + (-1)^n (3n) x^{3n-1} + \cdots$$

$$-\frac{3}{2^2} + \frac{6}{2^5} - \frac{9}{2^8} + \cdots = -3x^2 + 3x^5 + 3x^8 + \cdots \Longrightarrow x = \frac{1}{2}$$

$$f'\left(\frac{1}{2}\right) = -\frac{3(\frac{1}{2})^2}{(1+(\frac{1}{2})^3)^2} = -\frac{\frac{3}{4}}{\frac{81}{64}} = -\frac{16}{27}$$

5. Consider the following series where $p \geq 0$:

$$\sum_{n=2}^{\infty} \frac{1}{n^p \ln(n)}$$

(a) for $p > 1$

$$\sum_{n=2}^{\infty} \frac{1}{n^p \ln(n)} < \sum_{n=2}^{\infty} \frac{1}{n^p}$$

$$\sum_{n=2}^{\infty} \frac{1}{n^p} = L \therefore \sum_{n=2}^{\infty} \frac{1}{n^p \ln(n)} = L_2$$

(b) for $p = 1$

$$u = \ln n, \quad du = \frac{1}{n} dn$$

$$\sum_{n=2}^{\infty} \frac{1}{n^p \ln(n)} = \int_{\ln 2}^{\infty} \frac{1}{u} du \Longrightarrow \ln(\ln(n)) \Big|_2^{\infty} = \infty$$

(c) for $p < 1$

$$\sum_{n=2}^{\infty} \frac{1}{n^p \ln(n)} > \sum_{n=2}^{\infty} \frac{1}{n^p}$$

$$\sum_{n=2}^{\infty} \frac{1}{n^p} = \infty \therefore \sum_{n=2}^{\infty} \frac{1}{n^p \ln(n)} = \infty$$

6. The Maclaurin series for $f(x)$ is given $1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots + \frac{x^n}{(n+1)!} + \cdots$ where x is a positive real number.

$$f'(0) = \frac{1}{2} \qquad f^{(17)}(0) = \frac{1}{17!}$$

Convergent for what values of x?

$$\lim_{n\to\infty} \frac{x^{n+1}}{(n+2)!} * \frac{(n+1)!}{x^n} < 1$$

$$\Longrightarrow \lim_{n\to\infty} \left| \frac{x}{(n+2)} \right| < 1 \Longrightarrow x \in \mathbb{R}$$

7. A Taylor series for a function is given about $x = 1$ and converges to $f(x)$ for $|x - 1| < R$ where R is the radius of convergence for the series.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n}(x-1)^n \Longrightarrow \lim_{n\to\infty}\left|\frac{(-1)^{n+2}\frac{2^{n+1}}{n+1}(x-1)^{n+1}}{(-1)^{n+1}\frac{2^n}{n}(x-1)^n}\right|$$
$$\Longrightarrow \lim_{n\to\infty}\left|\frac{\frac{2}{n+1}(x-1)^{n+1}}{\frac{1}{n}(x-1)^n}\right| \Longrightarrow \lim_{n\to\infty}\left|\frac{2n}{n+1}(x-1)\right| \Longrightarrow |2(x-1)|<1$$
$$\Longrightarrow -\frac{1}{2}<x-1<\frac{1}{2}$$
$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n}(-\frac{1}{2}-1)^n = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n}(-1)^n(\frac{3}{2})^n = \sum_{n=1}^{\infty}-\frac{2^n}{n}(\frac{3}{2})^n = \sum_{n=1}^{\infty}-\frac{3^n}{n}$$
$$\Longrightarrow \lim_{n\to\infty}-\frac{3^n}{n}=-\infty$$
$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n}(\frac{1}{2}-1)^n = \sum_{n=1}^{\infty}(-1)^{n+1}(-1)^n\frac{2^n}{n}(\frac{1}{2})^n = \sum_{n=1}^{\infty}-\frac{1^n}{n} = \sum_{n=1}^{\infty}-\frac{1}{n} = \infty \text{ (P series)}$$
$$\Longrightarrow -\frac{1}{2}<x-1<\frac{1}{2}$$

Find the derivative series

$$f(x)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n}(x-1)^n = 2(x-1)-2(x-1)^2+\frac{8}{3}(x-1)^3+\cdots$$
$$f'(x)=2-4(x-1)+8(x-1)^2+\cdots+(-1)^{n+1}2^n(x-1)^{n-1}+\cdots$$

Integrate $f'(x)$ to find $f(x)$.

$$f'(x)=\sum_{n=1}^{\infty}2(-2(x-1))^{n-1}=\frac{2}{1--(2(x-1))}=\frac{2}{2x-1}$$
$$f(x)=\int\frac{2}{2x-1}dx=\ln|2x-1|+C$$
$$f(1)=\ln|1|+C=0\Longrightarrow C=0$$
$$f(x)=\ln|2x-1|, x\in(-\frac{1}{2},\frac{1}{2})$$

8. Find interval of convergence for $f(x)$ and determine if it fits the differential $xy'-y=\frac{4x^2}{1+2x}$

$$f(x) = \sum_{n=2}^{\infty} \frac{(-1)^n (2x)^n}{n-1} = \sum_{n=2}^{\infty} \frac{(-2)^n (x)^n}{n-1}$$

$$f'(x) = \sum_{n=2}^{\infty} \frac{n(-2)^n (x)^{n-1}}{n-1}$$

$$xy' - y = \sum_{n=2}^{\infty} \frac{n(-2)^n (x)^n}{n-1} - \sum_{n=2}^{\infty} \frac{(-1)^n (2x)^n}{n-1} = \sum_{n=2}^{\infty} \frac{n(-2)^n (x)^n - (-1)^n (2x)^n}{n-1}$$

$$= \sum_{n=2}^{\infty} \frac{n(-2)^n (x)^n - (-1)^n (2x)^n}{n-1} = \sum_{n=2}^{\infty} (-1)^n \frac{n-1}{n-1} (2x)^n = \sum_{n=2}^{\infty} (-2x)^n$$

$$= \sum_{n=0}^{\infty} (4x^2)(-2x)^n = \frac{4x^2}{1+2x}$$

9. The function $f(x)$ is defined by the below series and $g(x)$ too.

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n n x^n}{n+1}$$

$$g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!}$$

For example, here is $f(x)$'s interval of convergence:

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}(n+1)x^{n+1}}{n+2}}{\frac{(-1)^n n x^n}{n+1}} \right| = \lim_{n\to\infty} \left| \frac{\frac{(n+1)x}{n+2}}{\frac{n}{n+1}} \right| \Longrightarrow -1 < x < 1$$

Test the endpoints as usual.

Find $y = f(x) - g(x)$, passing through $(0, -1)$. Find $y'(0)$ and $y''(0)$ to determine if $y(0)$ is a relative max, min, or neither.

$$y = \sum_{n=0}^{\infty} \frac{(-1)^n n x^n}{n+1} - \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!}$$

$$= \sum_{n=0}^{\infty} \left[\frac{(-1)^n n x^n}{n+1} - \frac{(-1)^n x^n}{(2n)!} \right] = \sum_{n=0}^{\infty} (-x)^n \left[\frac{n(2n)!}{(n+1)(2n)!} - \frac{n+1}{(n+1)(2n)!} \right]$$

$$= \sum_{n=0}^{\infty} (-x)^n \left[\frac{n(2n)! - n - 1}{(n+1)(2n)!} \right]$$

$$= -1 + 0 + (\frac{2x^2}{3} - \frac{x^2}{4!}) + \cdots = -1 + \frac{4 * 2 * 2x^2 - x^2}{4!} + \cdots = -1 + \frac{15x^2}{4!} + \cdots$$

$$= -1 + \frac{5x^2}{8} + \cdots$$

$$y'(0) = 0 \quad y''(0) = -\frac{5}{4}$$

Minimum

10. Find k for which the following converges.

$$\sum_{n=0}^{\infty} ((k^3 + 2)e^{-k})^n$$

$$\implies \lim_{n \to \infty} \frac{((k^3 + 2)e^{-k})^{n+1}}{((k^3 + 2)e^{-k})^n} \implies -1 < ((k^3 + 2)e^{-k}) < 1$$

Use a graphing calculator to cook.