All Integrals Cheatsheet

Contents

$C \in \mathbb{R}$

1 Basic Integrals

$$\int x^n = \frac{x^{n+1}}{n+1} + C$$
$$\int \frac{1}{x} dx = \ln|x| + C$$

2 U-subsitution

Recall
$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$
Therefore by identifying a function and it's derivative, we can use u-subsitution.
$$\int f'(g(x))g'(x)dx = f(g(x)) + C$$
What does this mean?

First identify a $g(x)$ and $g'(x)$ modified by a $f'(x)$. Say $f'(x) = \frac{1}{x}$:
$$\int \frac{g'(x)}{g(x)} dx$$
$$u = g(x) \ du = g'(x) \ \frac{du}{g'(x)} = dx$$
$$\int \frac{g'(x)}{g(x)} dx = \int \frac{1}{u} du = \ln|u| + C = \ln|g(x)| + C$$
Similarly, if $f'(x) = x$
$$\int g'(x)g(x)dx \ u = g(x) \ du = g'(x) \ \frac{du}{g'(x)} = dx \ \int udu = \frac{1}{2}u^2 + C = \frac{1}{2}g(x) + C$$

3 Integration by Parts

3.1 Derivation

Recall product rule:
$$\frac{d}{dx}[uv] = u\frac{dv}{dx} + \frac{du}{dx}v$$
Where u and v are $u(x)$ and $v(x)$ respectively.
$$u\frac{dv}{dx} = \frac{d}{dx}[uv] - v\frac{du}{dx}$$

$$\int u\frac{dv}{dx}dx = \int \frac{d}{dx}[uv] - v\frac{du}{dx}dx$$

Simplify:

$$\int udv = uv - \int vdu$$

3.2 When to Use

Notice that u is **never integrated.** Only u and $\int vdu$ are used. In exchange, dv is integrated twice; once in uv and once in $\int vdu$.

3.2.1 Traditional usage

$$\int 2xe^x dx \ u = 2x \ dv = e^x dx \ \int udv = uv - \int vdu \implies 2xe^x - \int e^x dx * 2$$

$$\int 2xe^x dx = 2xe^x - 2e^x + C$$

Sometimes, u-subsitution or another method may be necessary after integration by parts. Or you might have to integrate by parts again.

3.2.2 Doing it twice

$$\int e^x \cos(x)dx \ u = e^x \ dv = \cos(x)dx \ \int udv = uv - \int vdu \ e^x \sin(x) - \int \sin(x) * e^x dx$$

Now, $\int e^x \sin(x)dx$ has to be integrated.

$$\int e^x \sin(x)dx \ u = e^x \ dv = \sin(x)dx \ \int udv = uv - \int vdu$$

$$-\int e^x \sin(x)dx = e^x \cos(x) - (-\int \cos(x) * e^x dx)$$

Put it all together:

$$\int e^x \cos(x)dx = e^x \sin(x) - (-e^x \cos(x) - (-\int e^x \cos(x)dx))$$

$$\int e^x \cos(x)dx = e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)dx$$

$$2\int e^x \cos(x)dx = e^x(\sin(x) + \cos(x)) + C$$

$$\int e^x \cos(x)dx = \frac{e^x(\sin(x) + \cos(x))}{2} + C$$

3.2.3 Where $dv = 1$

$$\int \ln(x)dx \; u = \ln(x) \; dv = dx \; \int udv = uv - \int vdu$$

$$\int \ln(x)dx = \ln(x)x - \int x\frac{1}{x}dx + C = x\ln(x) - x + C$$

4 Trig Substitution

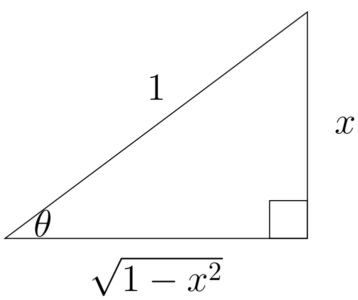
4.1 Inverse Trig Functions

4.1.1 $\arcsin(x)$

$$\int \frac{1}{\sqrt{1-u^2}}dx = \arcsin(x) + C$$

$$\frac{d}{dx}[y] = \frac{d}{dx}[\arcsin(x)] \Longrightarrow \frac{d}{dx}[\sin(y)] = \frac{d}{dx}[x] \Longrightarrow \cos(y) * \frac{dy}{dx} = 1 \Longrightarrow \frac{dy}{dx} = \frac{1}{\cos(y)}$$

$$\frac{dy}{dx} = \frac{1}{\cos(\arcsin(x))} \Longrightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$



Second case:

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \int \frac{1}{\sqrt{a^2(1-\frac{x^2}{a^2})}}dx = \frac{1}{a}\int \frac{1}{\sqrt{1-(\frac{x}{a})^2}}dx$$

$$u = \frac{x}{a}; du = \frac{1}{a}dx \Longrightarrow adu = dx$$

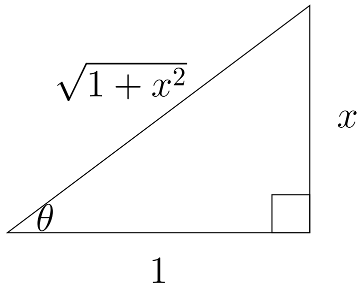
$$\frac{1}{a}\int a\frac{1}{1-u^2}du = \arcsin(u) + C = \arcsin(\frac{x}{a}) + C$$

4.1.2 $\arctan(x)$

$$\int \frac{1}{1+x^2}dx = \arctan(x) + C$$

$$\frac{d}{dx}[y] = \frac{d}{dx}[\arctan(x)] \Longrightarrow \frac{d}{dx}[\tan(y)] = \frac{d}{dx}[x] \Longrightarrow \sec^2 y * \frac{dy}{dx} = 1 \Longrightarrow \frac{dy}{dx} = \cos^2 y$$

$$\frac{dy}{dx} = \cos^2(\arctan(x)) = \cos^2(\theta) = (\frac{1}{\sqrt{1+x^2}})^2 = \frac{1}{1+x^2}$$



Second case:

$$\int \frac{1}{a^2+x^2}dx = \int \frac{1}{a^2(1+(\frac{x}{a})^2)}dx = \frac{1}{a}\int \frac{1}{1+(\frac{x}{a})^2}dx$$

$$u = \frac{x}{a}; du = \frac{1}{a}dx \Longrightarrow adu = dx$$

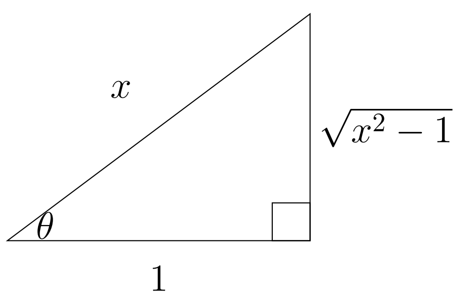
$$\frac{1}{a}\int \frac{1}{1+u^2}adu = \arctan(u) + C = \arctan(\frac{x}{a}) + C$$

4.1.3 $arcsec(x)$

$$\int \frac{1}{x\sqrt{x^2-1}}dx = arcsec(|x|) + C$$

$$\frac{d}{dx}[y] = \frac{d}{dx}[arcsec(|x|)] \Longrightarrow \frac{d}{dx}[\sec(y)] = \frac{d}{dx}[x] \Longrightarrow \sec(x)\tan(x) * \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec x \tan x} = \frac{1}{x\sqrt{x^2-1}}$$



Second case:

$$\int \frac{1}{x\sqrt{x^2-a^2}}dx = \int \frac{1}{ax\sqrt{(\frac{x}{a})^2-1}}dx$$

$$u = \frac{x}{a}; du = \frac{1}{a}dx \Longrightarrow adu = dx$$

$$\frac{1}{a}\int a\frac{1}{au\sqrt{u^2-1}}du = \frac{1}{a}arcsec(|u|) + C = \frac{1}{a}arcsec(|\frac{x}{a}|) + C$$

The absolute value remains because of:

$$\int \frac{1}{|x|\sqrt{x^2-1}}dx = arcsec(x) + C \Longrightarrow \frac{|x|}{x}\int \frac{1}{|x|\sqrt{x^2-1}}dx = (arcsec(x) + C)\frac{|x|}{x}$$

$$\Longrightarrow \int \frac{1}{x\sqrt{x^2-1}}dx = arcsec(|x|) + C$$

4.2 Esoteric $\sqrt{}$ Simplification

The same methods can be used to simplify $\sqrt{}$ when **u-subsitution and all other methods have failed.**. Like the title suggests, these are esoteric at best.

4.2.1 $\sqrt{a^2 - x^2}$

$$x = a\sin(\theta); dx = a\cos(\theta)d\theta$$

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2(\theta)} = \sqrt{a^2\cos^2(\theta)} = a\cos(\theta)$$

4.2.2 $\sqrt{a^2 + x^2}$

$$x = a\tan(\theta); dx = a\sec^2(\theta)d\theta$$

$$\sqrt{a^2 + x^2} = \sqrt{a^2 + a^2\tan(\theta)} = \sqrt{a^2\sec^2(\theta)} = a\sec(\theta)$$

4.2.3 $\sqrt{x^2 - a^2}$

$$x = a\sec(\theta); dx = a\sec(\theta)\tan(\theta)d\theta$$

$$\sqrt{x^2 - a^2} = \sqrt{a^2\sec^2(\theta) - a^2} = \sqrt{a^2\tan^2(\theta)} = atan(\theta)$$

4.3 Completing the Square

Completing the square can facilitate usage of the methods outlined above. An example would best illustrate this point:

$$\int \sqrt{5 - 4x - x^2}dx = \int \sqrt{5 + 4 - (x^2 + 4x + 4)}dx = \int \sqrt{9 - (x+2)^2}dx$$

Since $\frac{d}{dx}[x+2] = 1$, we can use the second-case derivative of $\arctan(x)$

$$= \arctan(\frac{x+2}{3}) + C$$

5 Trig Identity Integrals

5.1 Self Explanatory

$$\int \cos(x)dx = \sin(x) + C$$

$$\int \sin(x)dx = -\cos(x) + C$$

5.2 $\int sec^2(x)dx = tan(x) + C$

We can derive it using Quotient Rule:

$$\frac{d}{dx}[tan(x)] = \frac{d}{dx}[\frac{\sin(x)}{\cos(x)}]$$

$$\frac{d}{dx}[\frac{\sin(x)}{\cos(x)}] = \frac{\cos(x) * \frac{d}{dx}[\sin(x)] - \sin(x) * \frac{d}{dx}[\cos(x)]}{\cos^2(x)}$$

$$\frac{\cos(x) * \frac{d}{dx}[\sin(x)] - \sin(x) * \frac{d}{dx}[\cos(x)]}{\cos^2(x)} = \frac{\cos^2(x) - (-\sin^2(x))}{\cos^2(x)}$$

Use the trig identity $\cos^2(x) + \sin^2(x) = 1$:

$$\frac{\cos^2(x) - (-\sin^2(x))}{\cos^2(x)} = \frac{1}{\cos^2(x)} = sec^2(x)$$

5.3 $\int -csc^2(x)dx = cot(x) + C$

We can also derive it using Quotient Rule:

$$\frac{d}{dx}[cot(x)] = \frac{d}{dx}[\frac{\cos(x)}{\sin(x)}]$$

$$\frac{d}{dx}[\frac{\cos(x)}{\sin(x)}] = \frac{\sin(x) * \frac{d}{dx}[\cos(x)] - \cos(x) * \frac{d}{dx}[\sin(x)]}{\sin^2(x)}$$

$$\frac{\sin(x) * \frac{d}{dx}[\cos(x)] - \cos(x) * \frac{d}{dx}[\sin(x)]}{\sin^2(x)} = \frac{-\cos^2(x) - \sin^2(x)}{\sin^2(x)}$$

Use the trig identity $\cos^2(x) + \sin^2(x) = 1$:

$$\frac{-\cos^2(x) - \sin^2(x)}{\cos^2(x)} = \frac{-1}{\sin^2(x)} = -csc^2(x)$$

5.4 $\int \sec(x)tan(x)dx = \sec(x) + C$

OR

$$\int \frac{\sin(x)}{\cos^2(x)}dx = \sec(x) + C$$

There are two methods to arrive at the following. One is to use u-subsitution and the other is to use the power rule and chain. Let's start with the easier one.

$$\frac{d}{dx}[\sec(x) + C] = \frac{d}{dx}[\frac{1}{\cos(x)} + C] = \frac{d}{dx}[(\cos(x))^{-1}] = -(\cos(x))^{-2} * \frac{d}{dx}[\cos(x)]$$

$$-\frac{-\sin(x)}{\cos^2(x)} = \frac{\sin(x)}{\cos^2 x} = \sec(x)tan(x)$$

We can also just directly undo the chain rule.

$$\int \frac{\sin(x)}{\cos^2(x)}dx$$

$$u = \cos(x)$$

$$du = -\sin(x)dx$$

$$-\frac{du}{\sin(x)} = dx$$

$$\int \frac{\sin(x)}{u^2} * -\frac{du}{\sin(x)}dx = -\int \frac{du}{u^2} = u^{-1} + C = \sec(x) + C$$

5.5 $\int csc(x)cot(x)dx = \int \frac{\cos(x)}{\sin^2(x)}dx = -csc(x) + C$

$$\frac{d}{dx}[-csc(x)] = -\frac{d}{dx}[-\frac{1}{\sin(x)}] = \frac{1}{\sin^2(x)} * \frac{d}{dx}[\sin(x)] = \frac{\cos(x)}{\sin^2(x)}$$

$$u = \sin(x) \; du = \cos(x)dx \; \frac{du}{\cos(x)} = dx$$

$$\int \frac{\cos(x)}{u^2} * \frac{du}{\cos(x)} = -u^1 + C = -csc(x) + C$$

5.6 $\int tan(x)dx = \ln|\sec(x)| + C$

Let's break it into components and use u-subsitution:

$$\int tan(x)dx = \int \frac{\sin(x)}{\cos(x)}dx$$

Let's set u to be $\cos(x)$

$$u = \cos(x)$$

$$du = -\sin(x)dx$$

$$\frac{du}{-\sin(x)} = dx$$

Now we have:

$$\int \frac{\sin(x)}{\cos(x)}dx = \int \frac{\sin(x)}{u} * \frac{du}{-\sin(x)}$$

$$-\int \frac{du}{u} = -[\ln|u| + C]$$

$$-\ln|\cos(x)| + C_2$$

5.7 Again for $\int cot(x)dx$

$$\int \frac{\cos(x)}{\sin(x)}dx$$

If we use u subsitution on the denominator:

$$u = \sin(x)$$

$$du = \cos(x)dx$$

$$\frac{du}{cox(x)} = dx$$

Sub it back in:

$$\int \frac{\cos(x)}{u} * \frac{du}{\cos(x)} = \int \frac{du}{u} = \ln|u| + C = \ln|\sin(x)| + C$$

5.8 Algebraic: $\int \sec(x)dx = \ln|\sec(x) + tan(x)| + C$

If you multiply this expression by a certain term, you will get an expression in the form $\int \frac{du}{u}$. Can you guess what it is?

$$\int \sec(x)dx = \int \sec(x)\frac{\sec(x) + tan(x)}{\sec(x) + tan(x)}dx = \int \frac{sec^2(x) + \sec(x)tan(x)}{\sec(x) + tan(x)}dx$$

using $u = \sec(x) + tan(x)$ and $du = sec^2(x) + \sec(x)tan(x)dx$

$$\int \frac{du}{u} = \ln|u| + C = \ln|\sec(x) + tan(x)| + C$$

5.9 Similarly, $\int csc(x)dx = -\ln|csc(x) + cot(x)| + C$

$$\int csc(x)dx = \int csc(x)\frac{csc(x) + cot(x)}{csc(x) + cot(x)}dx = \int \frac{-csc^2(x) - csc(x)cot(x)}{csc(x) + cot(x)}dx$$

using $u = csc(x) + cot(x)$ and $du = -csc^2(x) - csc(x)cot(x)dx$

$$-\int \frac{du}{u} = -\ln|u| + C = -\ln|csc(x) + cot(x)| + C$$

5.10 $\int \sin^2(x)dx = \frac{1}{2}x - \frac{\sin(2x)}{4} + C$

For this one, you need to rewrite $\sin^2(x)$ in a more easily integratable format. Recall that $\cos(2x) = \cos^2(x) - \sin^2(x)$, or $\cos(2x) = 1 - 2\sin^2(x)$. Simplify a bit further and you get $\sin^2(x) = \frac{1-\cos(2x)}{2}$.

$$\int \frac{1 - \cos(2x)}{2}dx = \int \frac{1}{2}dx - \int \frac{\cos(2x)}{2}dx$$

$$u = 2x \ du = 2dx \ \frac{du}{2} = dx$$

$$\int \frac{1}{2}dx - \int \frac{\cos(2x)}{2}dx = \frac{1}{2}x - \frac{1}{2}\int \frac{\cos(u)}{2}dx + C = \frac{1}{2}x - \frac{\sin(u)}{4} + C = \frac{1}{2}x - \frac{\sin(2x)}{4} + C$$

5.11 $\int \cos^2(x)dx = \frac{1}{2}x + \frac{\sin(2x)}{4} + C$

$$\cos(2x) = \cos^2(x) - \sin^2(x) \ \cos(2x) = \cos^2(x) - (1 - \cos^2(x)) \ \cos(2x) = 2\cos^2(x) - 1$$

$$\cos^2(x) = \frac{\cos(2x) + 1}{2} \ \int \cos^2(x)dx = \int \frac{1}{2}dx + \int \frac{\cos(2x)}{2}dx$$

using the same u-sub logic:

$$u = 2x \ du = 2dx \ \frac{du}{2} = dx$$

$$\frac{1}{2}x + \frac{1}{4}\int \sin(u)du + C = \frac{1}{2}x + \frac{\sin(2x)}{4} + C$$

6 Polynomials

These problems assume a polynomial in the form:

$$\int \frac{A_1x^n + A_2x^{n-1} + ... + A_n}{B_1x^m + B_2x^{m-1} + ... + B_n}dx$$

Where $\{n, m\} \in \mathbb{R}$, $\{A_1, A_2, ...A_n\} \in \mathbb{R}$, $\{B_1, B_2, ...B_n\} \in \mathbb{R}$

6.1 Long Division

Where $n \geq m$, simply use long division.

6.2 Partial Fraction Decomposition

Where $n - 1 = m$
Used to simplify:

$$\int \frac{A_1x + A_2}{B_1x^2 + B_2x + B_3}dx$$

Best illustrated with an example:

$$\int \frac{-3x - 7}{x^2 - 5x + 6}dx = \int \frac{-3x - 7}{(x-2)(x-3)}dx$$

We need to find an A and B such that:

$$\frac{A}{x-2} * \frac{B}{x-3} = \frac{-3x - 7}{(x-2)(x-3)}$$

$$\frac{A(x-3)}{(x-2)(x-3)} + \frac{B(x-2)}{(x-2)(x-3)} = \frac{-3x-7}{(x-2)(x-3)} \Longrightarrow A(x-3) + B(x-2) = -3x-7$$

We can obtain the two following equations:

$$Ax + Bx = -3x \Longrightarrow A + B = -3 \Longrightarrow B = -3 - A$$

$$-3A - 2B = -7$$

Subsitute:

$$-3A - 2(-3 - A) = -7 \Longrightarrow -3A + 6 + 2A = -7 \Longrightarrow -A = -13 \Longrightarrow A = 13$$

$$B = -3 - A \Longrightarrow B = -3 - 13 \Longrightarrow B = -16$$

Now we can integrate the two sections individually:

$$\int \frac{-3x-7}{(x-2)(x-3)} dx = 13 \int \frac{1}{x-2} dx - 16 \int \frac{1}{x-3} dx = 13 \ln|x-2| - 16 \ln|x-3| + C$$

7 Improper Integrals

$$\int_a^\infty f(x)dx = \lim_{b \to \infty} \int_a^b f(x)dx$$

Similarly:

$$\int_{-\infty}^\infty f(x)dx = \int_a^\infty f(x)dx + \int_{-\infty}^a f(x)dx = \lim_{b \to \infty} \int_a^b f(x)dx + \lim_{c \to -\infty} \int_c^a f(x)dx$$

The integral may not always exist. For polynomials,

$$\lim_{x \to \infty} \frac{x^n \ldots}{x^m \ldots} \in \mathbb{R} \iff m \geq n + 2$$

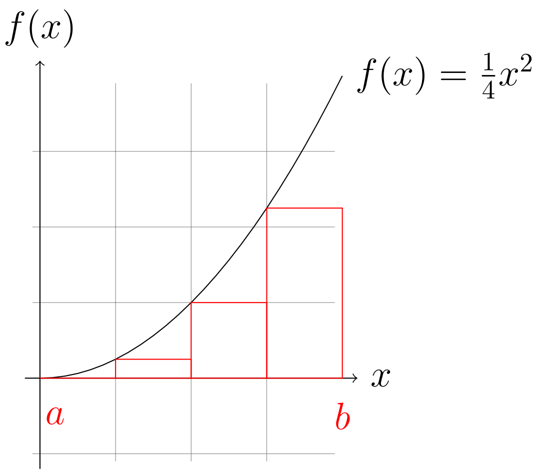
If the degree of the denominator is two higher than the numerator, then the integral will exist.

8 Riemann Sum

A typical Riemann Sum will equal the general expression:

$$\sum f(x_i)\Delta x$$

8.1 Left Riemann Sum



Let total steps be represented by n. Let i represent the current step

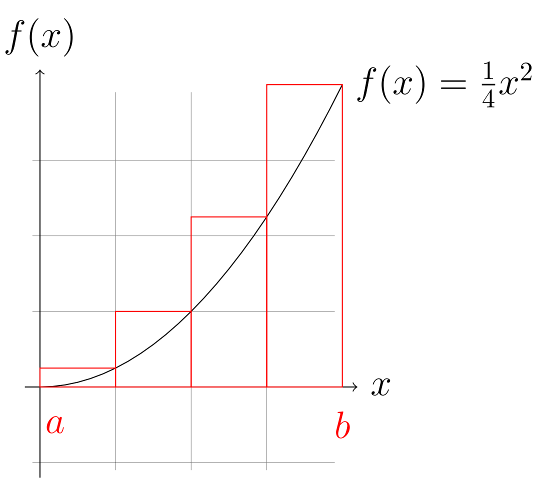
$$\Delta x = \frac{b-a}{n}$$

$$f(x_i) = \frac{f(a + i\Delta x)}{n}$$

Put them together:

$$\sum_{i=0}^{n-1} f(x_i)\Delta x \Longrightarrow \sum_{i=0}^{n-1} \frac{f(a+i\Delta x)}{n}\Delta x \Longrightarrow \sum_{i=0}^{n-1} \frac{f(a+i\frac{b-a}{n}x)(b-a)}{n^2}$$

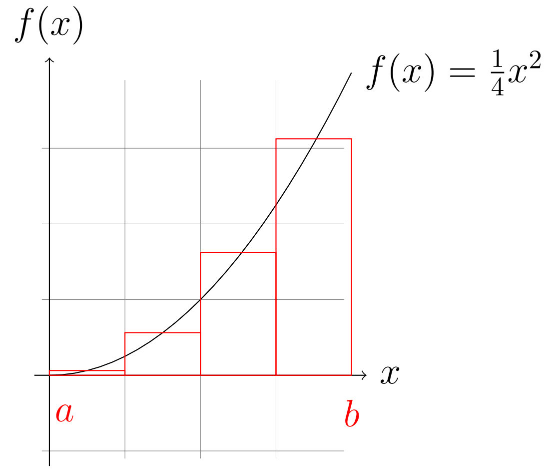
8.2 Right Riemann Sum



Nothing changes, except for the $\sum$ bounds:

$$\sum_{i=1}^{n} f(x_i)\Delta x \Longrightarrow \sum_{i=1}^{n} \frac{f(a+i\Delta x)}{n}\Delta x \Longrightarrow \sum_{i=1}^{n} \frac{f(a+i\frac{b-a}{n}x)(b-a)}{n^2}$$

8.3 Midpoint Riemann Sum

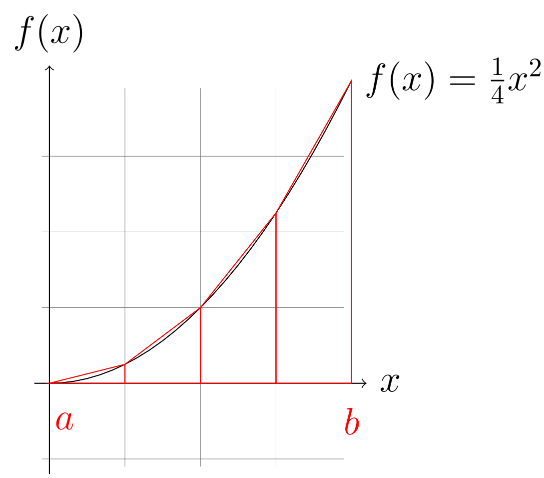


As the name suggests, you take the average of $f(x_i)$ and $f(x_{i+1})$. Instead of using $f(x_i)$, use $f(\frac{x_i+x_{i+1}}{2})$

$$\sum_{i=0}^{n-1} f(\frac{x_i + x_{i+1}}{2})\Delta x \Longrightarrow \sum_{i=0}^{n-1} \frac{f(\frac{a+i\Delta x+a+(i+1)\Delta x}{2})}{n}\Delta x = \sum_{i=0}^{n-1} \frac{f(\frac{2a+(2i+1)\Delta x}{2})}{n}\Delta x$$

$$\Longrightarrow \sum_{i=0}^{n-1} \frac{f(\frac{2a+(2i+1)\frac{b-a}{n}x}{2})(b-a)}{n^2}$$

8.4 Trapezoidal Riemann Sum



Take the average of the two y-values instead. Instead of using $f(x_i)$, use $\frac{f(x_i)+f(x_{i+1})}{2}$

$$\sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2}\Delta x \Longrightarrow \sum_{i=0}^{n-1} \frac{f(a + i\Delta x) + f(a + (i+1)\Delta x)}{2n}\Delta x$$

$$\Longrightarrow \sum_{i=0}^{n-1} \frac{(f(a + i\frac{b-a}{n}x) + f(a + (i+1)\frac{b-a}{n}x)(b-a)}{2n^2}$$

9 Distance

The distance of the curve is represented by the following:

$$\text{arc length} = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\Delta x$$
$$\implies \int \sqrt{1 + (\frac{dy}{dx})^2} dx$$

9.1 Example

These problems are taken from *Calculus for the AP Course* by Michael Sullivan.

1. Let g be the function defined by $g(x) = \int_1^x \sqrt{9t^2 - 1} dt$. Which integral represents the graph of g on the interval $[2, 5]$?

   Let $f(x) = \sqrt{9t^2 - 1}$. Let $F(x)$ represent its antiderivative.

$$g(x) = F(x) - F(1)$$
$$g'(x) = \sqrt{9x^2 - 1}$$
$$\int_2^5 \sqrt{1 + 9x^2 - 1} dx = \int_2^5 3x dx$$