Variance process:

$$dr(t) = a\left(b - r(t)\right)dt + \sigma\sqrt{r(t)}dW(t)$$

Define:

$$n = \frac{4ab}{\sigma^2}$$

Then for time to expiration T we have the representation:

$$r(T) = \sum_{i=1}^{n-1} X_i^2(T) + Y^2(T)$$

$$\rho^2 = \frac{\sigma^2}{4a}\left(1 - e^{-aT}\right)$$
$$\mu = e^{-\frac{1}{2}aT}\sqrt{r(0)}$$

Where $X_i(T)$ is normally distributed with mean 0 and standard deviation ρ and $Y(T)$ is normally distributed with mean μ and standard deviation ρ.

$$X_i(T) \sim \mathcal{N}(0, \rho^2)$$
$$Y(T) \sim \mathcal{N}(\mu, \rho^2)$$

Then we have:

$$\begin{aligned} r(T) &= \sum_{i=1}^{n-1} \rho^2 \frac{X_i^2(T)}{\rho^2} + \rho^2 \frac{Y^2(T)}{\rho^2} \\ &= \rho^2 \sum_{i=1}^{n-1} \left(\frac{X_i(T)}{\rho}\right)^2 + \left(\frac{Y(T)}{\rho}\right)^2 \\ &= \rho^2 q(T) \end{aligned}$$

Where $q(T)$ is noncentral chi-squared distributed with n degrees of freedom and mean λ:

$$q(T) \sim \chi^2(n, \lambda)$$
$$\lambda = \left(\frac{\mu}{\rho}\right)^2$$

We then have the probability density function of the Q variable:

$$
\begin{aligned}
f_Q(q) &= \sum_{i=0}^{\infty} \frac{e^{-\frac{\lambda}{2}} \left(\frac{\lambda}{2}\right)^i}{i!} \frac{q^{\frac{k}{2}-1} e^{-\frac{q}{2}}}{2^{\frac{k}{2}} \Gamma\left(\frac{k}{2}\right)} \\
k &= n + 2i
\end{aligned}
$$

Since we have the relation $R = \rho^2 Q$ we get the probability density function of the R variable:

$$
\begin{aligned}
f_R(r) &= f_Q(\frac{r}{\rho^2}) \frac{dq}{dr} \\
&= \frac{1}{\rho^2} f_Q(\frac{r}{\rho^2})
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
f_R(r) &= \sum_{i=0}^{\infty} \frac{e^{-\frac{\lambda}{2}} \left(\frac{\lambda}{2}\right)^i}{i!} \frac{1}{\rho^2} \frac{\left(\frac{r}{\rho^2}\right)^{\frac{k}{2}-1} e^{-\frac{\left(\frac{r}{\rho^2}\right)}{2}}}{2^{\frac{k}{2}} \Gamma\left(\frac{k}{2}\right)} \\
k &= n + 2i
\end{aligned}
$$

So we have:

$$
\begin{aligned}
f_R(r) &= \sum_{i=0}^{\infty} \frac{e^{-\frac{\lambda}{2}} \left(\frac{\lambda}{2}\right)^i}{i!} \frac{1}{\rho^k} \frac{r^{\frac{k}{2}-1} e^{-\frac{r}{2\rho^2}}}{2^{\frac{k}{2}} \Gamma\left(\frac{k}{2}\right)} \\
k &= n + 2i
\end{aligned}
$$

Volatility process:

$$
v(t) = \sqrt{r(t)}
$$

And the relations:

$$
\begin{aligned}
r(t) &= v^2(t) \\
dr(t) &= 2v(t)dv(t) \\
\frac{dr(t)}{dv(t)} &= 2v(t)
\end{aligned}
$$

Probability density function of the V variable:

$$f_V(v) = f_R(v^2)\frac{dr}{dv}$$
$$f_V(v) = 2vf_R(v^2)$$

Therefore:

$$\begin{aligned} f_V(v) &= \sum_{i=0}^{\infty} \frac{e^{-\frac{\lambda}{2}}\left(\frac{\lambda}{2}\right)^i}{i!} 2v \frac{1}{\rho^k} \frac{v^{2\left(\frac{k}{2}-1\right)} e^{-\frac{v^2}{2\rho^2}}}{2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)} \\ k &= n + 2i \\ f_V(v) &= \sum_{i=0}^{\infty} \frac{e^{-\frac{\lambda}{2}}\left(\frac{\lambda}{2}\right)^i}{i!} \frac{v^{k-1} e^{-\frac{v^2}{2\rho^2}}}{\rho^k 2^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)} \end{aligned}$$

Price of a call option at strike x:

$$C(x) = \int_x^{\infty} (v - x) f_V(v) dv$$

So we have

$$\begin{aligned} C(x) &= \sum_{i=0}^{\infty} \frac{e^{-\frac{\lambda}{2}}\left(\frac{\lambda}{2}\right)^i}{i!} \int_x^{\infty} (v - x) \frac{v^{k-1} e^{-\frac{v^2}{2\rho^2}}}{\rho^k 2^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)} dv \\ k &= n + 2i \end{aligned}$$

Compute:

$$\begin{aligned} I_1(k) &= \int_x^{\infty} v \frac{v^{k-1} e^{-\frac{v^2}{2\rho^2}}}{\rho^k 2^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)} dv \\ &= \int_x^{\infty} \frac{v^k e^{-\frac{v^2}{2\rho^2}}}{\rho^k 2^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)} dv \end{aligned}$$

Make the change of variable $\frac{v^2}{2\rho^2} = t$, so we have

$$\begin{aligned} v &= \sqrt{2}\rho\sqrt{t} \\ &= 2^{\frac{1}{2}}\rho t^{\frac{1}{2}} \\ dv &= 2^{\frac{1}{2}}\rho\frac{1}{2}t^{-\frac{1}{2}}dt \\ &= 2^{-\frac{1}{2}}\rho t^{-\frac{1}{2}}dt \end{aligned}$$

Therefore

$$
\begin{aligned}
I_1(k) &= \int_{\frac{x^2}{2\rho^2}}^{\infty} \frac{2^{\frac{k}{2}}\rho^k t^{\frac{k}{2}} e^{-t}}{\rho^k 2^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)} 2^{-\frac{1}{2}}\rho t^{-\frac{1}{2}} dt \\
&= \frac{\rho\sqrt{2}}{\Gamma\left(\frac{k}{2}\right)} \int_{\frac{x^2}{2\rho^2}}^{\infty} t^{\frac{k+1}{2}-1} e^{-t} dt \\
&= \rho\sqrt{2}\frac{\Gamma\left(\frac{k+1}{2}, \frac{x^2}{2\rho^2}\right)}{\Gamma\left(\frac{k}{2}\right)}
\end{aligned}
$$

Where upper incomplete gamma function $\Gamma\left(z,x\right)$ is defined as

$$
\Gamma\left(z,x\right) = \int_x^{\infty} t^{z-1}e^{-t}dt
$$

Compute:

$$
I_2(k) = \int_x^{\infty} \frac{v^{k-1}e^{-\frac{v^2}{2\rho^2}}}{\rho^k 2^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)} dv
$$

Make the same change of variable $\frac{v^2}{2\rho^2} = t$, so we have

$$
\begin{aligned}
I_2(k) &= \int_{\frac{x^2}{2\rho^2}}^{\infty} \frac{2^{\frac{k-1}{2}}\rho^{k-1} t^{\frac{k-1}{2}} e^{-t}}{\rho^k 2^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)} 2^{-\frac{1}{2}}\rho t^{-\frac{1}{2}} dt \\
&= \frac{1}{\Gamma\left(\frac{k}{2}\right)} \int_{\frac{x^2}{2\rho^2}}^{\infty} t^{\frac{k}{2}-1} e^{-t} dt \\
&= \frac{\Gamma\left(\frac{k}{2}, \frac{x^2}{2\rho^2}\right)}{\Gamma\left(\frac{k}{2}\right)}
\end{aligned}
$$

Call price

$$
\begin{aligned}
C(x) &= \sum_{i=0}^{\infty} \frac{e^{-\frac{\lambda}{2}}\left(\frac{\lambda}{2}\right)^i}{i!}\left(I_1(k) - xI_2(k)\right) \\
&= \sum_{i=0}^{\infty} \frac{e^{-\frac{\lambda}{2}}\left(\frac{\lambda}{2}\right)^i}{i!}\left(\rho\sqrt{2}\frac{\Gamma\left(\frac{k+1}{2}, \frac{x^2}{2\rho^2}\right)}{\Gamma\left(\frac{k}{2}\right)} - x\frac{\Gamma\left(\frac{k}{2}, \frac{x^2}{2\rho^2}\right)}{\Gamma\left(\frac{k}{2}\right)}\right) \\
k &= n + 2i
\end{aligned}
$$

We have the regularized gamma function $Q(z,x) = \frac{\Gamma(z,x)}{\Gamma(z)}$, so

$$\begin{aligned}
C(x) &= \sum_{i=0}^{\infty} \frac{e^{-\frac{\lambda}{2}}\left(\frac{\lambda}{2}\right)^i}{i!}\left(\rho\sqrt{2}\frac{\Gamma\left(\frac{n+1}{2}+i,\frac{x^2}{2\rho^2}\right)}{\Gamma\left(\frac{n}{2}+i\right)} - x\frac{\Gamma\left(\frac{n}{2}+i,\frac{x^2}{2\rho^2}\right)}{\Gamma\left(\frac{n}{2}+i\right)}\right) \\
&= \sum_{i=0}^{\infty} \frac{e^{-\frac{\lambda}{2}}\left(\frac{\lambda}{2}\right)^i}{i!}\left(\rho\sqrt{2}\frac{\Gamma\left(\frac{n+1}{2}+i,\frac{x^2}{2\rho^2}\right)}{\Gamma\left(\frac{n+1}{2}+i\right)}\frac{\Gamma\left(\frac{n+1}{2}+i\right)}{\Gamma\left(\frac{n}{2}+i\right)} - x\frac{\Gamma\left(\frac{n}{2}+i,\frac{x^2}{2\rho^2}\right)}{\Gamma\left(\frac{n}{2}+i\right)}\right) \\
&= \sum_{i=0}^{\infty} \frac{e^{-\frac{\lambda}{2}}\left(\frac{\lambda}{2}\right)^i}{i!}\left(\rho\sqrt{2}Q\left(\frac{n+1}{2}+i,\frac{x^2}{2\rho^2}\right)\frac{\Gamma\left(\frac{n+1}{2}+i\right)}{\Gamma\left(\frac{n}{2}+i\right)} - xQ\left(\frac{n}{2}+i,\frac{x^2}{2\rho^2}\right)\right)
\end{aligned}$$

$$\begin{aligned}
C(x) &= \sum_{i=0}^{\infty} \frac{e^{-\frac{\lambda}{2}}\left(\frac{\lambda}{2}\right)^i}{i!}\left(\rho\sqrt{2}\frac{\Gamma\left(\frac{n+1}{2}+i,\frac{x^2}{2\rho^2}\right)}{\Gamma\left(\frac{n}{2}+i\right)} - x\frac{\Gamma\left(\frac{n}{2}+i,\frac{x^2}{2\rho^2}\right)}{\Gamma\left(\frac{n}{2}+i\right)}\right) \\
&= \sum_{i=0}^{\infty} \frac{e^{-\frac{\lambda}{2}}\left(\frac{\lambda}{2}\right)^i}{i!}\left(\rho\sqrt{2}\frac{\Gamma\left(\frac{n+1}{2}+i,\frac{x^2}{2\rho^2}\right)}{\Gamma\left(\frac{n+1}{2}+i\right)}\frac{\Gamma\left(\frac{n+1}{2}+i\right)}{\Gamma\left(\frac{n}{2}+i\right)} - x\frac{\Gamma\left(\frac{n}{2}+i,\frac{x^2}{2\rho^2}\right)}{\Gamma\left(\frac{n}{2}+i\right)}\right) \\
&= \sum_{i=0}^{\infty} \frac{e^{-\frac{\lambda}{2}}\left(\frac{\lambda}{2}\right)^i}{i!}\left(\rho\sqrt{2}Q\left(\frac{n+1}{2}+i,\frac{x^2}{2\rho^2}\right)\frac{\Gamma\left(\frac{n+1}{2}+i\right)}{\Gamma\left(\frac{n}{2}+i\right)} - xQ\left(\frac{n}{2}+i,\frac{x^2}{2\rho^2}\right)\right)
\end{aligned}$$