Solution of Geometric Brownian Motion SDE

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1 The SDE

Geometric Brownian motion SDE:

$$\begin{aligned} \frac{dS(t)}{S(t)} &= \mu dt + \sigma \sqrt{dt} z \\ z &\sim \mathcal{N}(0,1) \end{aligned} \tag{1}$$

Raise to square:

$$\begin{aligned} dS(t)^2 &= \left(\mu S(t) dt + \sigma S(t) \sqrt{dt} z\right)^2 \\ &= \mu^2 S(t)^2 dt^2 + 2\mu\sigma S(t)^2 dt \sqrt{dt} + \sigma^2 S(t)^2 \left(\sqrt{dt}\right)^2 z^2 \end{aligned} \tag{2}$$

When $dt \to 0$ we have $dt^2 = 0$ and $dt\sqrt{dt} = 0$. Also since $z \sim \mathcal{N}(0,1)$ one can verify that on average $z^2 = 1$.

Therefore:

$$dS(t)^2 = \sigma^2 S(t)^2 dt \tag{3}$$

2 Stochastic Integrals

Take the following integral:

$$\begin{aligned} I &= \int_0^t dW(t) \\ &= \int_0^t \sigma \sqrt{dt} z \\ z &\sim \mathcal{N}(0,1) \end{aligned} \tag{4}$$

Dividing the interval $[0,t]$ into $n \to \infty$ equal subintervals of equal size dt we have:

$$\begin{aligned} I &= \sum_{i=0}^{n} \left(\sigma \sqrt{dt} z_i\right) \\ &= \sigma \sqrt{dt} \sum_{i=0}^{n} z_i \end{aligned} \tag{5}$$

Since we are adding n normal distributed random variables of mean zero and variance $\sigma^2 dt$, their sum will be still a normal distributed random variable of mean zero and variance $n\sigma^2 dt$. But $ndt = t$ so finally we get the variance $\sigma^t t$ so the deviation $\sigma\sqrt{t}$. Therefore:

$$\begin{aligned} I &= W(t) \\ \int_0^t \sigma\sqrt{dt}z &= \sigma\sqrt{t}z \\ z &\sim \mathcal{N}(0,1) \end{aligned} \tag{6}$$

3 The solution

Take the function $f(S,t) = \ln S(t)$. We have it's Taylor series expansion:

$$df(S,t) \approx \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S}dS + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}dt^2 + \frac{\partial^2 f}{\partial t \partial S}dtdS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}dS^2 \tag{7}$$

We then have the partial derivatives:

$$\begin{aligned} \frac{\partial \ln S(t)}{\partial t} &= 0 \\ \frac{\partial \ln S(t)}{\partial S} &= \frac{1}{S(t)} \\ \frac{\partial^2 \ln S(t)}{\partial t^2} &= 0 \\ \frac{\partial^2 \ln S(t)}{\partial S \partial t} &= 0 \\ \frac{\partial^2 \ln S(t)}{\partial S^2} &= -\frac{1}{S(t)^2} \end{aligned} \tag{8}$$

Replacing in the We Taylor series expansion we get:

$$d\left(\ln S(t)\right) \approx \frac{1}{S(t)}dS(t) - \frac{1}{S(t)^2}dS(t)^2 \tag{9}$$

By equation (3) we then have:

$$\begin{aligned} d\left(\ln S(t)\right) &\approx \frac{dS(t)}{S(t)} - \frac{1}{2}\frac{\sigma^2 S(t)^2}{S(t)^2}dt \Leftrightarrow \\ d\left(\ln S(t)\right) &= \frac{dS(t)}{S(t)} - \frac{\sigma^2}{2}dt \end{aligned} \tag{10}$$

Replacing from (1) $\frac{dS(t)}{S(t)}$ we get:

$$
\begin{aligned}
d\left(\ln S(t)\right) &= \mu dt + \sigma\sqrt{dt}z - \frac{\sigma^2}{2}dt \Leftrightarrow \\
d\left(\ln S(t)\right) &= \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma\sqrt{dt}z \\
d\left(\ln S(t)\right) &= \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dW(t)
\end{aligned}
\tag{11}
$$

Where we have denoted the differential of the Weiner process $dW(t) \approx \sqrt{dt}z$. Integrating:

$$
\begin{aligned}
\int_0^t d\left(\ln S(t)\right) &= \int_0^t \left(\mu - \frac{\sigma^2}{2}\right)dt + \int_0^t \sigma dW(t) \Rightarrow \\
\ln S(t)\Big|_0^t &= \left(\mu - \frac{\sigma^2}{2}\right)t\Big|_0^t + \sigma W(t)\Big|_0^t \Leftrightarrow \\
\ln S(t) - \ln S(0) &= \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t) \Leftrightarrow \\
\ln\frac{S(t)}{S(0)} &= \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}z \\
z &\sim \mathcal{N}(0,1)
\end{aligned}
\tag{12}
$$

Taking the exponential:

$$
\begin{aligned}
e^{\ln\frac{S(t)}{S(0)}} &= e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}z} \Rightarrow \\
\frac{S(t)}{S(0)} &= e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}z}
\end{aligned}
\tag{13}
$$

Which finally gives the solution:

$$
\begin{aligned}
S(t) &= S(0)e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}z} \\
z &\sim \mathcal{N}(0,1)
\end{aligned}
\tag{14}
$$