Elementary Mathematical Analysis

Precalculus / Derivatives

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Book cover by Aldo Cortesi, *A portrait of the Hilbert curve as a young fruit salad*, which is "a Hilbert curve traversal of the three-dimensional RGB colour space, projected onto a two-dimensional Hilbert curve covering the plane.
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Introduction

Paul Foerster (1936-, U.S.A.) taught mathematics at Alamo Heights High School in San Antonio, Texas, from 1961 to 2011[**BuckleyFoerste**] . After earning a BS in chemical engineering, he served four years as an engineering duty officer with the Navy's Nuclear Propulsion Program. Tutoring high school students while on active duty, he found his true passion — seeing students get the "Aha!" reaction as they finally grasp new concepts. After completing his Navy service, he went back to college for a spring and a summer to get his teaching certification. His teaching has been interrupted only by a year's leave of absence to earn a master's degree in mathematics through an academic year institute awarded by the National Science Foundation.

0.1 Motivation

This textbook exists to facilitate a class coming after Algebra II and before Calculus II. It is meant to be a coherent mixture of Paul A Foerster's *Precalculus with Trigonometry* and his *Calculus: Concepts and Applications*. Some students can handle moving at a much faster pace through Precalculus topics, and hence cover the first four chapters of *Calculus*. A student should not be expected to do well on the AP exam after this class (without additional side work), because of inadequate time spent on integrals.

Once a school is over a certain size, and the pool of students to draw from is sufficient, there may be the critical mass needed to create an accelerated class for those who can proceed quickly through the skills and understanding needed to appreciate calculus. While it may be possible in such situations to promote kids sooner, it may also make sense to wait until their junior year, in which case an analysis class is a possible solution. Students are introduced to categories of functions, with an eye towards their uses in calculus, at the same time. At best, a student would proceed from this class to a Calculus II class, which would move quickly through derivatives and spend most of its time on integrals. At worst, if a student (barely) made it through this class, they would no doubt do well in an AB class, and have time to go back and master derivatives.

0.2 Prerequisites

Students are expected to have performed well in a strong Algebra II class, and retain excellent arithmetic, graphing, and factoring skills. This textbook contains an extensive set of appendices , some of which should be assigned as summer work. We recommend thorough consultation with the sophomore teacher to see which areas need more work and require additional assignments. It is burdensome, but we also highly recommend making assignments due at regular intervals throughout the summer, perhaps via electronic submission. American summer breaks are too long, and students forget a great deal over three months. Because of the exploratory nature of assignments in the appendix, it is possible to assign exercises which were not covered at all in Algebra II, with little or no instruction.

This textbook contains many calculator lessons for the Texas Instrument 83 or 84 (hereafter **TI-8***). This antiquated device may be taken into the AP, ACT, and SAT exams, and hence should be thoroughly mastered by a student who wishes to maximize his or her tools. Other devices may be used, but are unwise, since the TI-8* is the most powerful tool allowed in the standardized tests.

0.3 Worldview

Immanuel Kant (1724-1804, Prussian) was a German philosopher, and is is considered a central figure in philosophy. He argued that the human mind creates the structure of experience, that reason is the source of morality, and that the world as it is "in-itself" is independent of our concepts of it. He is sometimes credited with being the greatest Modernist philosopher, and yet laying the ground from which Post-Modernism sprung. (Wikipedia)

It is highly remarkable how many mathematics textbooks begin without prolegomena of any kind. Such an oversight belies all of mathematical work, which seeks to establish foundations and ascertain unassailable truths. We will attempt to counter this trend in the shortest space.

The universe in which we find ourselves doing mathematics is a reasonable and orderly one. The Ancient Greek distinction between 'cosmos' and 'chaos', or the Biblical separation of the watery abyss from the peaceable land are metaphors in recognition of this fact. The ethical and personal disorder which we bring to the world does not belie its inherent predictability. God has not absconded from his creation, but has chosen to mediate himself, leaving us with the ability and responsibility to discover what is hidden in the material world (Proverbs 25:2). It is only by suppressing this truth that one could pronounce as Einstein did that, "The most incomprehensible thing about the universe is that it is comprehensible."[1]. Only by *a priori* rejecting the Judeo-Christian worldview – that a reasonable and rational God has made human beings in his image, and that we are therefore capable of thinking his thoughts after him, and making true statements about the world – can we understand comments such as Kant's "[W]ir auch, gleich als ob es ein glücklicher unsre Absicht begünstigender Zufall wäre, erfreuet (eigentlich eines Bedürfnisses entledigt) werden, wenn wir eine solche systematische Einheit unter bloß empirischen Gesetzen antreffen."[2] Rationality in human discourse and orderliness in the natural world are not things which need to be proven but which must be presupposed for any proofs to be made.

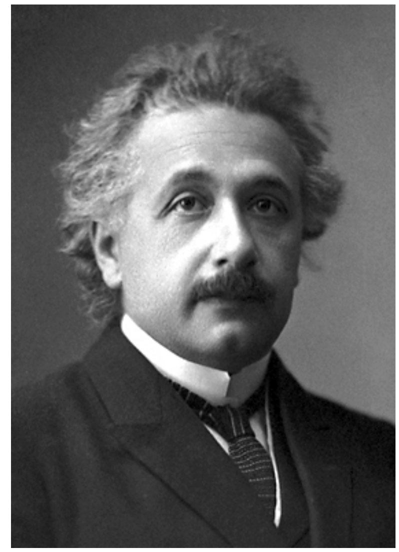
Mathematics is not a construct of the human mind. It is entirely foreseeable and understandable that advances in number theory later yield results in cryptography, or that esoteric processes such as analytic continuation, produce useful dividends in quantum physics, long after they are discovered. Mathematics is the representation and conceptualization of the invisible, created world. Human beings are the representation of the maximally personal God in the impersonal media of time, energy, and matter. 'Number' and other mathematical features are

[1] Albert Einstein, *Journal of the Franklin Institute*, Physics and Reality, p221,1936.

[2] [We rejoice when, just as if it were a lucky chance favoring our aim, we do find such systematic unity among merely empirical laws.] *Critique of Judgment*, 1790.

part of human language: doing math is part of what it means to be human. It is part of *what* we are, it is good, and it is imperative — an integral component of our purpose.

0.4 Method



Albert Einstein (1879-1955, German) was a theoretical physicist, and TIME's person of the century. He is considered to be one of the smartest men of the the modern era for his revolutionary Special Theory of Relativity, and again for General Relativity. He stated on numerous occasions that he could not "imagine some will or goal outside the human sphere" and hence it is a happy coincidence that mathematics and science are so mutually affirming. (Wikipedia)

Human beings are not mere containers of information. Teaching is not plugging-in a download wire from the side of the teacher's head into that of the students. Most things of value are *caught*, not taught. It is expected that a teacher will show the lessons in this book to their students in the context of a safe, loving relationship, which is of necessity personal, and therefore costly. Eventually, students should feel safe to fail in this class, because only failing produces change and ultimately learning. Success only confirms us in our biases. Of course, no one likes to fail, so every class must balance a three-legged stool: structure, support, and challenge.

If at all possible, teachers should present the material from a given lesson from a different perspective than it is shown herein. This gives the student the opportunity to read the book, should they wish for another angle on the matter. If students know how to learn, they should be able to do the **problems** on their own, and spend class time presenting their thoughts to others, or working through the **exercises** and discussing the book. The first section of each chapter (after the first) is somewhat informal and self-explanatory, and should be assigned as homework after the previous test without instruction.

Math assignments should never consist of 10 easy exercises, rotely completing the task defined in the lesson, followed by 10 harder exercises, followed by 5 word-problems. Problem sets should be a journey, that is, should possess a narrative arc. Class periods should do the same. Students must learn to ask themselves questions, a process they should see modeled in the teacher. "What did we just learn? How can I summarize it in my own words? How do I think it will be applied? How does it relate to what I already know? When was this encountered for the first time, and in what context (history)? How can it help me understand more of the myriad things above my level? How do I solve difficult problems involving this principle? What might this become in my life and in the lives of my neighbors? How can I communicate this to others? What are some common misunderstandings people have when encountering this material?"

Finally, while we do have a volume of material we wish students to learn and retain, it is irresponsible to allow students to think they have "mastered" mathematics. Anyone with a Masters degree in any field has been dislodged from his or her notion that one can know *everything* about anything. When the Ph.Ds tell us they do not know all there is to know about even a sliver of their discipline, we did not believe them ... at first. But slowly, over years of paper-writing and drilling down into the subject, we see even they can not know how much there is to know about one topic within their discipline! This is only discouraging if we are delusional enough to think that our achievement is the *summum bonum*, the highest good. We can indeed learn true things and asymptotically approach full understanding, but no amateur should be so deluded as to think they know all there is to know about some field within mathematics. Hence, many lessons should include a taste of just how much more there is to know, including problems that cannot be fully solved.

0.5 To the Student

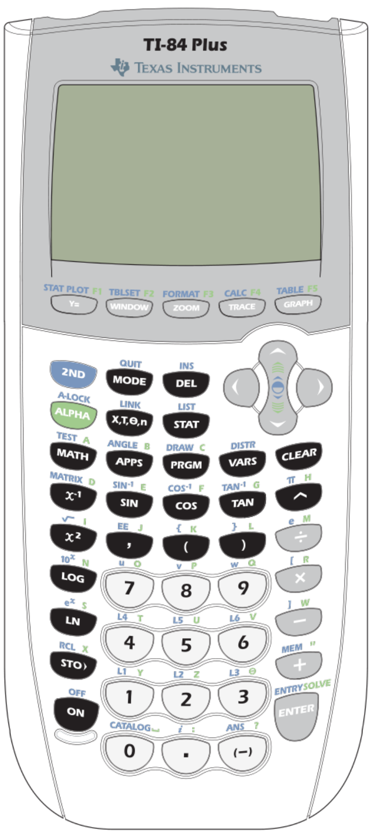


Figure 0.1 The TI-84+.

This is both a textbook and a workbook. “Math is not a spectator sport”, and that means you do not understand something until you’ve tried it. Don’t shortchange yourself by reading a section before you’ve attempted the problems. If you want to read ahead, that’s fine, but try the assignment before the reading. Even if you don’t get it all, consider it part of the reading. Unless specifically asked to, we recommend you do not Google the answers. Do you want to understand the world? Do you know that you are in a protected space, a safe time to explore and make mistakes, a time that you will not be afforded later? Take the time to learn as you go, and you will learn how to learn, a lesson you can use for the rest of your existence. If you get desperate, cultivate relationships with human beings who can ask you questions that allow you to discover the answer yourself, not be spoon-fed.

Always look out for **bold** words, in the problems, in the readings, and in the exercises. Copy them down into your notes and try to define them for yourself, before you look in the glossary or online. We encourage you to take notes in the Cornell style. This simply involves you drawing a large, left-of-center capital ‘I’ over your entire note page before you begin, taking notes as usual in the right section, adding subject and paragraph headings on the left, and making succinct summaries at the bottom of the page, perhaps a few before the test. Headings, your name, and the date should go across the top. There are many videos on the internet about ‘Cornell Notes’.

0.6 To the Teacher

All the material in the prerequisites needs to be covered, ideally before the year begins. Chapters 1-12 and 14 are the minimum. Chapter 16 and 18-20 exist for the teacher’s sake, to allow variety over the years. Ideally, that next class would use Foerster’s excellent *Calculus* textbook, taking a month to review chapters 1-4, and then moving on to the rest of the book, perhaps augmented the material with some lessons from multivariable calculus and/or differential equations.

In a perfect world, every capable student would take Calculus II and Statistics before graduating high school, but we don’t live in that world. Time permitting, discrete topics should be broached, especially with students who will not take Stats. The other discrete topics are here as aides for those who have never seen them before. Appendices B, C, and D exist for the teachers who love those topics and wish to expand or vary the chapters from year to year. There is always one student who can tell they are not getting the whole story and will not let it rest!

Classes should consist of a brief introduction or grabber, the “problem” set, a succinct lesson, and then discussion as much as possible. Ideally, every class period will end with students summarizing what they have learned and speculating about what it could become, or what they will do next. Homework should consist of careful note-taking and selected exercises.

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II Appendices

I Linear Transformations

1. Functions

Functions are the central objects of investigation in many subdivision of mathematics today, and are very effective means of representing real-life phenomena. For example, in any given year, there is one population of deer in a certain forest. Would it make sense for a forest ranger to answer you two numbers when you ask how many *Rangifer tarandus* (caribou) exist in a given range, at a certain time?

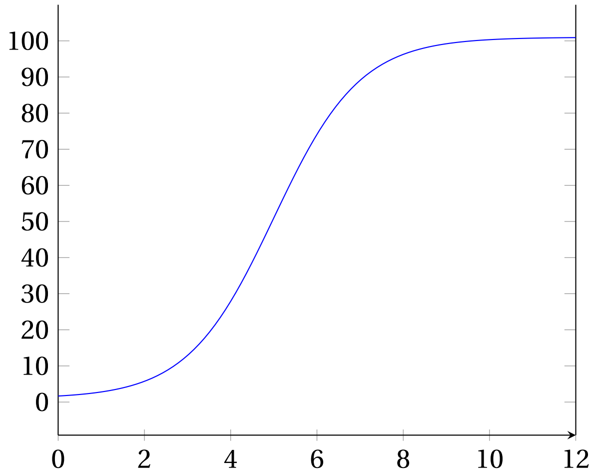


Figure 1.1 Population of caribou over time

How do we model phenomena in the world? Is it reasonable to expect caribou population to follow a curve such as the above figure? How can we extract information from a graph like this? If actual measurements diverge some from the expectation established by the curve, are they *wrong*? Are functions better tools than relations? What else could there be?

1.1.1

Patterns All Around Us

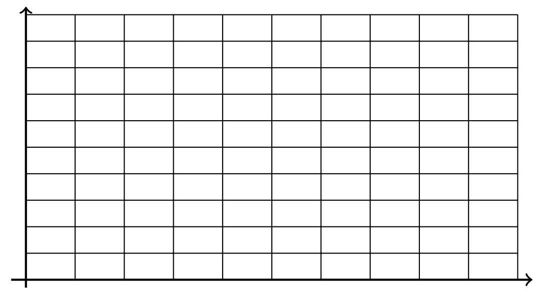
L1.1 Pick a feature of the room you are in or one nearby. It should be a repeated, adjacent pattern, such as carpet squares or ceiling tiles. In a complete sentence, tell which feature you chose:

L1.2 Measure the length of one, a few, and several of your items. There should be some number(s) you skip over (e.g. 1, 3, 4, and 6). Record your data below, including units.

Number	**Length**

L1.3 In order to construct a graph, you must decide on the relationship between the two variables. Does the number of items depend on the length, or visa versa? In other words, decide which one is the **independent variable** and which is the **dependent variable**. Justify your choice.

L1.4 Plot the data points you just observed on a **Cartesian plane**, putting the independent variable as the left-to-right, and the dependent as the bottom-to-top. Label your axes and indicate a scale. Connect your points in a smooth, continuous graph.



L1.5 Find a **function** that describes your data, and record it in **function notation**. Call the function L and the argument (i.e. input) n. What kind of function is it?

L1.6 Not all numbers make sense, either as outputs or inputs to your function. Conjecture a reason **domain** and **range** and tell how you arrived at the intervals you did. Record them below.

L1.7 When using a **mathematical model**, there are two words to describe going beyond recorded data: **interpolation** and **extrapolation**. Which prefix means 'within' and which means 'without'? Which word is appropriate to describe finding the missing data point you skipped over from your observations? Which would is appropriate to describe finding how long 100 objects would be? Explain your choice.

L1.8 Describe what you think the point of this problem set is, using technical vocabulary in complete sentences.

1.1.2 Numerical, Graphical, Algebraic, and Verbal

Objective

Use functions defined numerically, graphically, algebraically, or verbally.

A **function** is a relation that uniquely associates members of one **set** with members of another set. More simply, a function connects members of a **domain** to members of a **range**, with the proviso that each element in the domain maps onto only one element in the range. When graphing a function, it is customary to plot the *input* or **independent variable** on the x-axis (going left to right), and the *output* or **dependent variable** on the y-axis (going down to up). The independent variable is so-called because it may be arbitrarily chosen from within the domain. The dependent variable is then forced to be a particular value (by the function).

Numerically

Typically, information is gathered about the physical world as **discrete** numbers, occurring at fixed points. Such information may be represented as in table 1.1.

x (months since)	**y** (100's of $)
0	1200.00
1	840.00
10	1680.24
16	2072.98
23	2264.72
36	4303.93
48	6770.53

Table 1.1 Monthly income represented numerically

Graphical

A helpful way to visualize relationships between input and output sets is a **graph** of the function. Similar techniques include diagrams (see Figure for an example) and maps (see Figure for an example). "Mapping diagrams" blur the line between numerical and graphical data (see Figure 1.2 for an example).

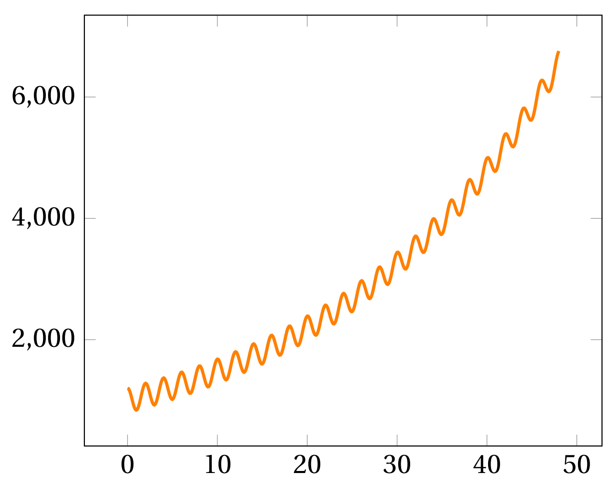


Figure 1.2 Monthly Income - Graphically

The income (y) may be modeled by the number of months since inception (x) as a sinusoid with an amplitude of 200, a period of 2, and a midline of an exponential growth rate of 4% beginning at $1,000.

Figure 1.4 Monthly Income - Verbally

Algebraically

Typically, graphs come from humans or machines plotting many, many points derived from an **algebraic** formula. Algebra is a powerful tool for describing natural phenomena, but it cannot do everything. Important numbers — such as π or e — cannot be defined via algebraic techniques. Worse, many confuse manipulation of the symbols with actually mathematics. However, the symbols are very concise, as you might see in Tab. 1.2.

$y = 1000 \cdot 1.04^x + 200\cos(\pi x)$.

Table 1.2 Monthly Income - Algebraically

Verbally

Ordinary language serves us well most of the time, with an average level of precision and an average level of emotional content. Sometimes, however, what we wish to convey is more meaningful and deep than regular wording can describe. Conversely, we might wish to exclude a whole host of meanings, and narrowly zoom in on precise terminology, in order to avoid error to the fullest extent possible (see Fig. 1.3).

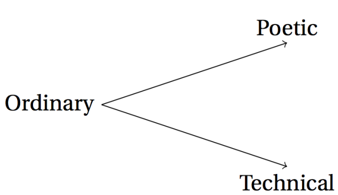


Figure 1.3 Levels of language

Most of us encounter poetic language in the form of songs, since ours is an age sadly bereft of poetry, something that was not true a century ago or more. Another common phenomenon is to meet technical language when it is not expected or welcome, such as when we read the instruction manual for new hardware or legal documents. However, such documents are typically not aiming at easy comprehension, but precision. Like most jargon, they tend to use mostly ordinary words in extraordinary ways, similar but opposite to poetry.

Mathematics is no difference. This textbook is a formal setting, and as such, it uses technical language, carefully choosing each word according the conventions and definitions "in house". As you cross the threshold into the beginning stages of advanced mathematics, you must cultivate the skill of using this "jargon", something you do not ordinarily do. It may help your comprehension to paraphrase what you are learning into the "vernacular", but you should not consider a topic concluded until you are able to re-articulate the mater in mathematical terminology. For example, the equation above might be described as in Fig. 1.4.

Mathematical Models

Functions that can be used to make predictions and interpretations of real-world phenomena are called **mathematical models**. It is important to be clear which variable is being used as input and which is output, so that a meaningful relationship between independent and dependent variables can be ascertained. In the model of income above, time does not depend upon money, but rather money upon time. The domain of the function is the initial time index of zero until four years later, so in months $0 \le x \le 48$. The income varies from as low as \$840 but generally goes up from there, so the range is $y \ge 840$.

Example 1.1

There are only so many hours in a day. If you decide to stay up *all* night studying, you will surely get a bad grade on the test tomorrow, due to sleep deprivation. Sketch a reasonable graph showing hours spent studying the day before a test and the grade you will get on said test. Give the domain and range of the function.

Solution: One cannot possibly get less than zero hours of sleep, and presumedly around 12 or so hours of sleep is long enough to have slept through the exam and angered one's parents! Test scores are typically measured in percents, ranging from 0 to 100.

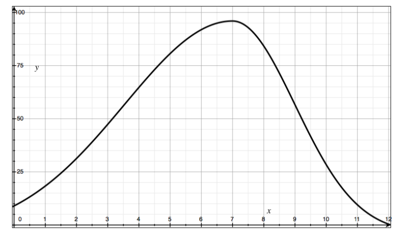


Figure 1.5 Possible graph of sleep vs. grade

1.1.3 Exercises

Problem Set 1-1

1. *Archery Problem 1:* An archer climbs a tree near the edge of a cliff, then shoots an arrow high into the air. The arrow goes up, then comes back down, going over the cliff and landing in the valley, 30 m below the top of the cliff. The arrow's height, *y,* in meters above the top of the cliff depends on the time, *x,* in seconds, since the archer released it. Figure 1-1g shows the height as a function of time.

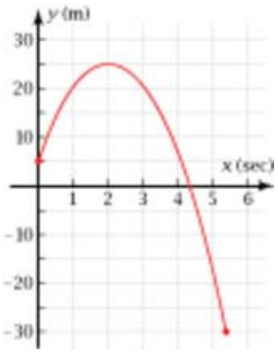


Figure 1-1g

a. What was the approximate height of the arrow at 1 second? At 5 seconds? How do you explain the fact that the height is negative at 5 seconds?

b. At what *two* times was the arrow at 10 m above the ground? At what time does the arrow land in the valley below the cliff?

c. How high was the archer above the ground at the top of the cliff when she released the arrow?

d. Why can you say that altitude is a *function* of time? Why is time *not* a function of altitude?

e. What is the domain of the function? What is the corresponding range?

2. *Gas Temperature and Volume Problem:* When you heat a fixed amount of gas, it expands, increasing its volume. In the late 1700s, French chemist Jacques Charles used numerical measurements of the temperature and volume of a gas to find a quantitative relationship between these two variables. Suppose that these temperatures and volumes had been recorded for a fixed amount of oxygen.

Pietro Longhi's painting, The Alchemists, *depicts a laboratory setting from the middle of the 18th century.*

T (°C)	**V (l)**
0	9.5
50	11.2
100	12.9
150	14.7
200	16.4
250	18.1
300	19.9

a. On graph paper, plot V as a function of T. Choose scales that go at least from $T = -300$ to $T = 400$.

b. You should find, as Charles did, that the points lie in a straight line! Extend the line backward until it crosses the T-axis. The temperature you get is called *absolute zero*, the temperature at which, supposedly, all molecular motion stops. Based on your graph, what temperature in degrees Celsius is absolute zero? Is this the number you recall from science courses?

c. Extending a graph beyond all given data, as you did in 2b, is called **extrapolation.** "*Extra-*" means "beyond," and "*-pol-*" comes from "pole," or end. Extrapolate the graph to $T = 400$ and predict what the volume would be at 400°C.

d. Predict the volume at $T = 30$°C. Why do you suppose this prediction is an example of **interpolation?**

e. At what temperature would the volume be 5 liters? Which do you use, interpolation or extrapolation, to find this temperature?

e. At what temperature would the volume be 5 liters? Which do you use, interpolation or extrapolation, to find this temperature?

f. Why can you say that the volume is a *function* of temperature? Is it also true that the temperature is a function of volume? Explain.

g. Considering volume to be a function of temperature, write the domain and the range for this function.

h. See if you can write an algebraic equation for V as a function of T.

i. In this problem, the temperature is the independent variable and the volume is the dependent variable. This implies that you can change the volume by changing the temperature. Is it possible to change the *temperature* by changing the *volume,* such as you would do by pressing down on the handle of a tire pump?

3. *Mortgage Payment Problem:* People who buy houses usually get a loan to pay for most of the house and pay on the resulting *mortgage* each month. Suppose you get a \$50,000 loan and pay it back at \$550.34 per month with an interest rate of 12% per year (1% per month). Your balance, B dollars, after n monthly payments is given by the algebraic equation

$$B = 50{,}000(1.01^n) + \frac{550.34}{0.01}(1 - 1.01^n)$$

a. Make a table of your balances at the end of each 12 months for the first 10 years of the mortgage. To save time, use the table feature of your grapher to do this.

b. How many months will it take you to pay off the entire mortgage? Show how you get your answer.

c. Plot on your grapher the graph of B as a function of n from $n = 0$ until the mortgage is paid off. Sketch the graph on your paper.

d. True or false: "After half the payments have been made, half the original balance remains to be paid." Show that your conclusion agrees with your graph from part c.

e. Give the domain and range of this function. Explain why the domain contains only *integers*.

4. *Stopping Distance Problem:* The distance your car takes to stop depends on how fast you are going when you apply the brakes. You may recall from driver's education that it takes more than twice the distance to stop your car if you double your speed.

a. Sketch a reasonable graph showing your stopping distance as a function of speed.

b. What is a reasonable domain for this function?

c. Consult a driver's manual, the Internet, or another reference source to see what the stopping distance is for the maximum speed you stated for the domain in part b.

d. When police investigate an automobile accident, they estimate the speed the car was going by measuring the length of the skid marks. Which are they considering to be the independent variable, the speed or the length of the skid marks? Indicate how this would be done by drawing arrows on your graph from part a.

5. *Stove Heating Element Problem:* When you turn on the heating element of an electric stove, the temperature increases rapidly at first, then levels off. Sketch a reasonable graph showing temperature as a function of time. Show the horizontal asymptote. Indicate on the graph the domain and range.

6. In mathematics you learn things in four ways—algebraically, graphically, numerically, and verbally.

a. In which of the five problems above was the function given algebraically? Graphically? Numerically? Verbally?

b. In which problem or problems of the five above did you go from verbal to graphical? From algebraic to numerical? From numerical to graphical? From graphical to algebraic? From graphical to numerical? From algebraic to graphical?

1.2.1 Pick x, Find y

Name: ______________________ Group Members: ______________________

Exploration 1-2a: Names of Functions

Date: ____________

Objective: Recall the names of certain types of functions.

1. $f(x) = 2x + 3$ is the equation for a **linear function.** Plot the graph and sketch the result here. Give a reason for the name *linear.*

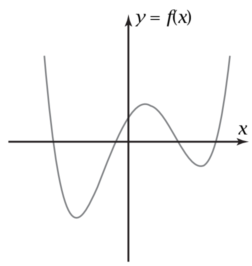
2. $f(x) = x^2 - 6x + 10$ is the equation for a **quadratic function.** Plot the graph and sketch the result. Explain how the word *quadratic* is related to the word *quadrangle.*

3. $f(x) = 3x^{0.7}$ is the equation for a **power function.** Plot the graph and sketch the result. Why do you think it is called a *power* function?

4. $f(x) = 3 \cdot 0.7^x$ is the equation for an **exponential function.** Plot the graph and sketch the result. How does an exponential function differ from a power function algebraically? graphically?

5. $f(x) = \frac{24}{x}$ is the equation for an **inverse variation** power function. Plot the graph for $x > 0$ and sketch the result. Why do the words "y varies inversely with x" make sense for this function? Why can the function be called a *power* function?

6. $f(x) = x^4 - 4x^3 - 43x^2 + 130x + 168$ is the equation of this **quartic function.** Why do you think the name *quartic* is used for this function? Use your grapher to find the largest value of x at which the graph crosses the x-axis.



7. $f(x) = \frac{x - 4}{x - 3}$ is the equation of a **rational function.** Plot the graph and sketch the result. Why do you think it is called a *rational* function? What happens to the graph at $x = 3$?

8. What did you learn as a result of doing this Exploration that you did not know before?

Objective

Connect names of types of equations with graphs.

We have said that a function is a relation of inputs to outputs, with no more than one output per input. We have learned that functions can be defined graphically, algebraically, numerical, or verbally. A function that is defined by an algebra equation usually has a descriptive name. In this section, we will look at several groupings of various functions, some of which you should be very familiar with, and some of which may be new.

You might be used to seeing a **graph** of certain equations, written algebraically with y in terms of x, like $y = x$. These have then been graphed on the Cartesian plane, as a continuous curve, with each point corresponding to an ordered pair, (x, y). Leonhard Euler created much of the notation we use today, including *function notation*: $y = f(x)$.

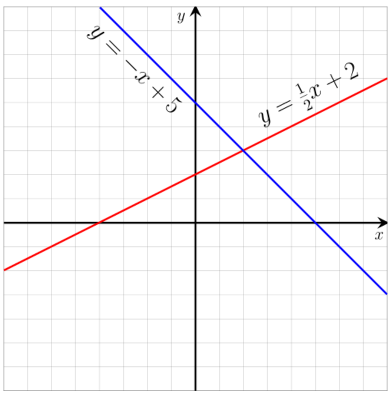


Figure 1.6 Linear functions.

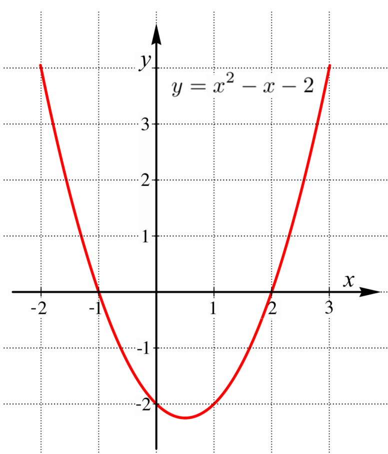


Figure 1.7 A quadratic function is a polynomial of degree two.

Linear

One of the most obvious things to see is a straight line. Humans create straight lines seemingly more often than anything else. Hence, lines can feel unnatural or reassuring. What are some of the properties of lines? How might two lines be the same? How might they differ?

Lines could have the same slope, and therefore never run into each other. They would only be distinguished by their heights. For convenience, we measure the height of a **linear** function in Analytic Geometry by its starting value, its y when x equals zero. Conversely, these starting locations might be the same and slope might be different. You have learned to distinguish these two different variables as m and b, as in $y = mx + b$, and we shall see that it is expedient to distinguish them **constant** functions, $y = k$.

Quadratic

Many things in our world operate over two dimensions, such as gravity. Hence, Newton find that the force of two objects upon each other is proportional to the *square* of their distance. Squares graphed make a **parabola**, a word reference the path of a falling or thrown object. Algebraically, we can see all such shapes have an equation of the form $y = ax^2 + bx + c$, which is called **quadratic**. You should already know a great deal about quadratics from previous classes.

Power As more dimension interact, the exponent on x can become very complicated, and even fractional. We can generalize from $y = x$ to $y = x^2$ and $y = x^3$ to $y = a \cdot x^b$. We shall study them in more depth in chapter 5.

Polynomial A sum of power functions with whole number exponents is called a **polynomial**. Such equations are among the most well-studied areas in mathematics. A polynomials divided by a polynomial is called a **rational function**. Both are the subject of chapter 6.

Exponential

Quantities that experience the same percentage growth or decay year over year look similar in the algebra: x is in the exponent, and hence such an equation is called an **exponential** function. The general form is $y = a \cdot b^x$. The "opposite"[1] of such a function is a called a **logarithm**, and we will follow the TI-8* for now and use the generic equation $y = a + b \ln x$.

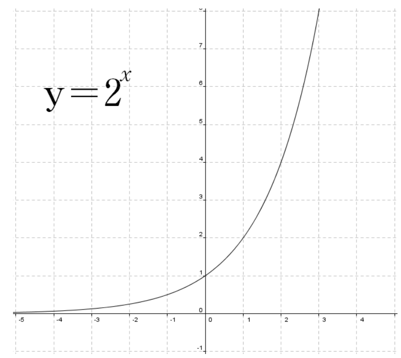


Figure 1.8 2^x is an exponential function

Periodic

Many phenomena is nature reoccur the same way at regular intervals. Such functions are said to be **periodic**. We will study 'simple harmonic motion', which comes from components of motion in circles, in section III, Trigonometric Functions. For now[2], use the general equation $y = a \cdot \sin(bx + c) + d$.

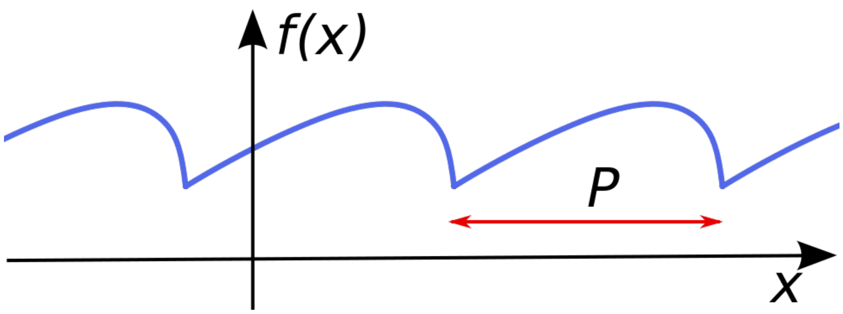


Figure 1.9 A periodic function is so called because it repeats at a given interval, called the period (P).

[1] There are *many* things which could be called 'opposite' in mathematics, so this is not technical language. We will define 'inverses' of functions in 4.4.

[2] Later, we will factor the "inside", but this first kind is the sort produced by your grapher.

1.2.2 Exercises

in Kuta

1.3.1 Multiplying and Adding Functions

Name: ______________________ Group Members: ______________________________

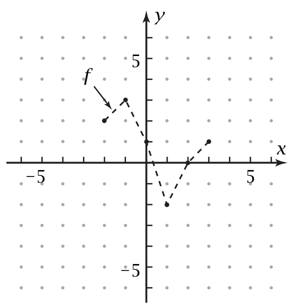
Exploration 1-3a: Translations and Dilations, Numerically

Date: ____________

Objective: By calculating values and plotting points, discover the effect on a function graph of adding and multiplying by constants.

1. The table shows values of a **pre-image** function $y = f(x)$. The graph of f is a set of line segments connecting the points, shown dashed in the figure. Find values of the **image** function $g(x) = f(x) + 3$. For instance, $g(-2) = 2 + 3 = 5$. Plot the graph of this transformed function.

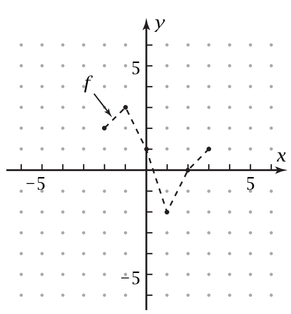
x	$f(x)$	$g(x)$
−2	2	
−1	3	
0	1	
1	−2	
2	0	
3	1	



2. The transformation in Problem 1 is a **vertical translation** by 3 units. Give the meaning of a vertical translation.

3. Use the values of $f(x)$ in Problem 1 to make a table of values of a new image function, $g(x) = f(x - 3)$. For instance, $g(1) = f(1 - 3) = f(-2) = 2$. Plot the image of this transformed function.

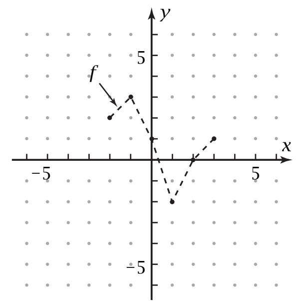
x	$g(x) = f(x - 3)$
1	
2	
3	
4	
5	
6	



4. Describe the transformation in Problem 3.

5. Use the values of $f(x)$ in Problem 1 to make a table of values of a new image function, $g(x) = 2f(x)$. For instance, $g(-1) = 2f(-1) = 2 \cdot 3 = 6$. Plot the image of this transformed function.

x	$g(x) = 2f(x)$
−2	
−1	
0	
1	
2	
3	



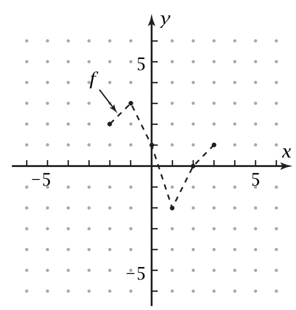
6. The transformation in Problem 5 is a **vertical dilation** by a factor of 2. Give the meaning of a vertical dilation, and explain how it differs from a vertical translation.

7. Use the values of $f(x)$ in Problem 1 to make a table of values of a new image function, $g(x) = f\left(\frac{1}{2}x\right)$. For instance,

$$g(-2) = f\left(\frac{1}{2} \cdot (-2)\right) = f(-1) = 3$$

Plot the image of this transformed function.

x	$g(x) = f\left(\frac{1}{2}x\right)$
−4	
−2	
0	
2	
4	
6	



8. The transformation in Problem 7 is a **horizontal dilation.** By what factor is the graph dilated? How is that factor related to the $\frac{1}{2}$ in $f\left(\frac{1}{2}x\right)$?

9. What did you learn as a result of doing this Exploration that you did not know before?

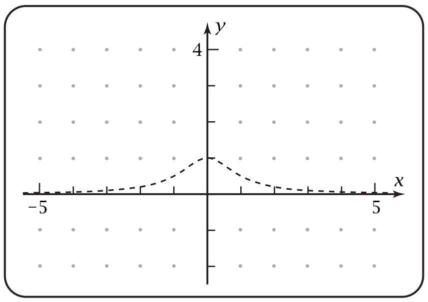
Name: ____________________ Group Members: ____________________

Exploration 1-3b: Translations and Dilations, Algebraically

Date: __________

Objective: Find the effect on a function graph of adding and multiplying by constants.

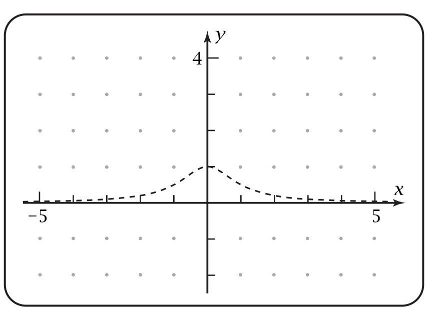
1. The graph shows the **pre-image** function $f(x) = \dfrac{1}{1 + x^2}$. Plot this graph as y_1 on your grapher. Use the window shown, using GRID ON format.



2. Plot the graph of $y_2 = f(x) + 3$. Sketch the result on the graph in Problem 1.

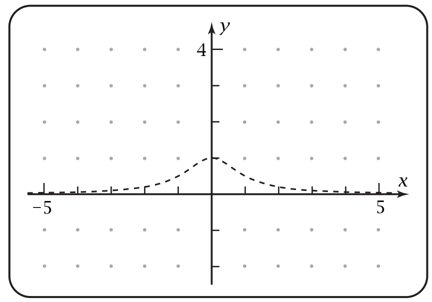
3. The transformation in Problem 2 is a **vertical translation** of 3. Give the meaning of a vertical translation.

4. Deactivate y_2 from Problem 2. Then plot $y_3 = f(x - 3)$. Sketch the result here.



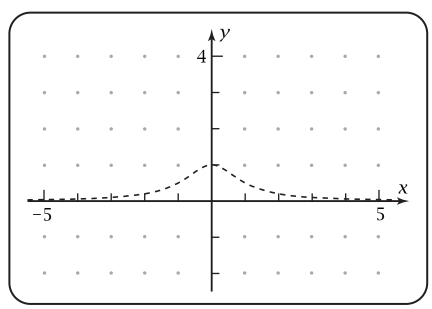
5. What words describe the transformation in Problem 4?

6. Deactivate y_3 from Problem 4. Then plot $y_4 = 3f(x)$. Sketch the result here.



7. The transformation in Problem 6 is a **vertical dilation** by a factor of 3. Give the meaning of a vertical dilation, and explain how it differs from a vertical translation.

8. Deactivate y_4 from Problem 7. Then plot $y_5 = f(3x)$. Sketch the result here.



9. The transformation in Problem 8 is a **horizontal dilation.** By what factor is the graph dilated?

10. What did you learn as a result of doing this Exploration that you did not know before?

1.3.2 Inside vs. Outside

Objective

Produce and decipher translations and dilations of functions.

"Outside" Operations

How would you triple the output of a function? How would you add four the output of a function? Because function notation means "perform the operation of the function upon the input", we must write the operators we described *to the right* of the function operation, written $f(x)\cdot 3$ and $f(x)+4$ respectively. Normally, multiplicative operators are written on the left, without an intervening symbol. Because addition is commutative, it may be written on the left as well.

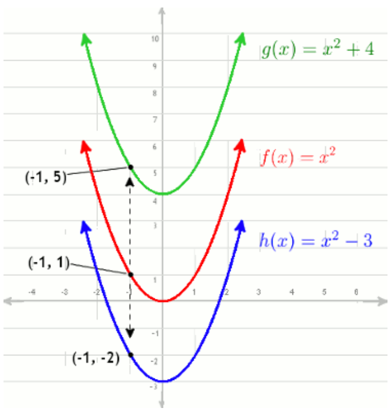


Figure 1.10 Vertical translation is outside addition.

Addition What is the graphical effect of the algebraic operation, $f(x)+4$? Let us build up a visual picture numerically at first.

adding moves it up, negatives move it down. Multiplication by > 1 *makes it taller.* $(0,1)$ *makes it shorter. All relative to the x-axis.*

Example 1.2 Outside Transformations

$r(x) = \sqrt[3]{x}$ and $s(x) = \frac{1}{2}r(x)+3$. In what ways does the graph of $s(x)$ differ from $r(x)$?

Solution: up 3, half as tall

Multiplication Multiplication also behave as one might expect, effecting y in a directly-proportional way. For example, regardless of what $f(x)$ is (excepting 0), then the graph of $3\cdot f(x)$ with be three times taller, a vertical dilation of 3.

"Inside" Operations

Things done "inside" are done *before* the function operates on the domain. This means graphically they will effect x. However, their effect is quite curious, typically being the *opposite* of what one might expect.

For example, consider the quadratic function $f(x) = x^2$, and a transformation of it, $g(x) = f(x+4)$. We can see that the +4 is on the "inside", and we know that 4 is added to members of the domain before they are plugged in to the function. This means -4 will become 0 before it is squared. Another way to say this that what used to be outputted at $x = 0$ will now be outputted at $x = -4$.

This opposite effect also applies to dilations. When we multiply the inside of a function by 2, it does not produce a graph twice as wide, but *half*.

Example 1.3 Inside Transformation

Consider the continuous function $f(x)$ given by the graph (looks like a radical sign). Describe and graph the continuous $g(x)$, when $g(x) = f(2x-2)$. What would be a more informative way to write $g(x)$ in terms of $f(x)$?

Solution: It's half as wide and left 1. $f(2(x-1))$ would be more transparent.

In summary, $f(x)+d$ will shift the graph d units to the right. $f(x+c)$ will shift the graph c units to the left. $a \cdot f(x)$ will make the graph a times taller. $f(b \cdot x)$ will make the graph b times skinnier.

1.3.3 Exercises

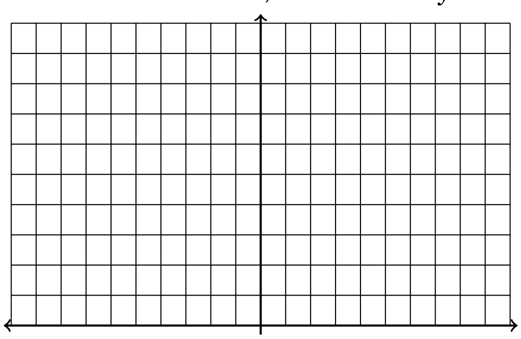
in Kuta

1.4.1 Through the Looking Glass

1. In the four, blank squares below, fill in the correct sign (positive or negative) of the answer. Take a number of the sign given on the left to a power of the kind given above. For example, top-left square is asking for the sign of a positive number to an even power (e.g. 31^{60} is positive).

	even	odd
positive		
negative		

2. Sketch the graph $y = x^2$ without using a graphing calculator. Leave x 1:1, but scale the y-values 1:10.

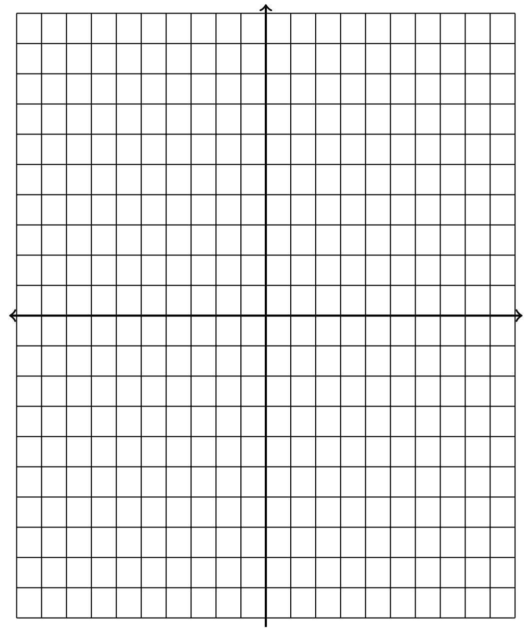


3. On the same graph above, graph $y = x^4$, $y = x^6$, and $y = x^8$. Do not use the graphing function of your calculator!

4. Continue not to use a graphing aide and stick to the same scale, but on the graph at the top of the next column, plot $y = x^3$.

5. On the same graph with the cubic equation, add $y = x^5$, $y = x^7$, and the very easy $y = x^1$.

6. In technical language, describe the two different symmetries displayed by **even** powers of x and **odd** powers of x.



7. Next, create two algebra sentences for the two symmetries, each in the form $f(x) = \ldots f(-x)$.

8. Do an image search on the internet for "natural symmetry". Classify what you find as odd, or even, or rotational. Taking odd and rotational together and contrasting them with even, which kind(s) do you see as more *human* than the other? Why do you think that is?

9. Describe what you think the point of this problem set is, using technical vocabulary in complete sentences.

1.4.2 Reflection

Objective

Produce reflections of functions and describe their symmetry.

As you know, multiplying numbers by -1 produces the additive opposite: negative values become positive and positives become negative. Zero is unaffected, being neither positive nor negative. The same principle applies to functions. What would it look like to "flip" all x values? y values?

"Outside" $-f(x)$ is a reflection over the x-axis, leaving x's untouched, and making y's opposite.

We can obtain both the original and reflection across the x-axis by writing $\pm f(x)$. Why is that not a function?

"Inside" $f(-x)$ is a reflection over the y-axis, leaving y's untouched, and making x's opposite.

Does the "opposite" rule of inside transformations apply or not to the negative on the inside?

1.4.3 Symmetry

Even Sometimes, multiplying by a negative makes no difference. In mathematics, this can be very helpful to know. Functions that are the same left-to-right and right-to-left are called **even**. Can you guess why? Aren't only numbers even (or odd)?

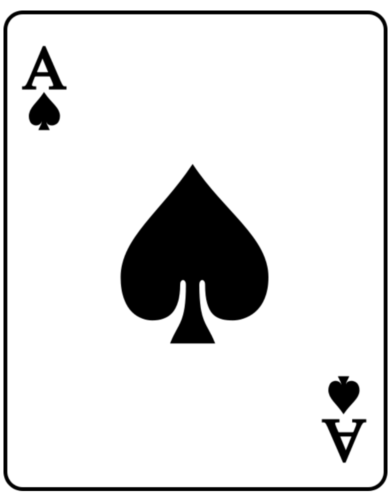


Figure 1.11 The ace of spades (or of clubs or hearts) displays even symmetry about its center.

Even Functions

An even function has the property $f(x) = f(-x)$ for every x in its domain.

Among the more basic functions are power functions. We will study them in great depth in chapter 5. The names "even" and "odd" come from the similar behavior of x^n when n is even or odd.

Notice that evenness has the visual appearance of putting a mirror on the $y-$axis. Did you know human beings are made to find such left-right symmetry appealing? Study the faces of attractive people, and you will find evenness to be a rule of thumb.

Odd Functions

An odd function has the propety $-f(x) = f(-x)$ for every x in its domain.

You might have supposed oddness with be top-to-bottom and bottom-to-top symmetry. Why isn't that possible for functions? Instead, we find that these functions are the same whether we proceed left from the origin, or flip the right half upside-down.

Figure 1.12 The queen of hearts (and many other playing cards) display an odd symmetry about the center.

Rotational Looking at the odd power functions, it becomes clear that there are two ways to regard their symmetry. Either, they are reflections left-right and up-down (in either order), or they are 180° of themselves. Is it possible for a function to have any other angle of symmetry with itself?

Moving on to symmetry *between* functions (or relations), we see a lot of rotational symmetry exists. All lines of the form $y = ax$ are rotations of each other. $f(x) = x^2 + k$ and the relation $y = \pm\sqrt{x-k}$ are 90° rotations of each other.

Relations can be their own rotation. Equilateral triangles with their center at the origin are all 120° rotations of themselves. In fact, every regular $n-$gon (polygon) is its own rotations every $\frac{360}{n}^\circ$. As we take the limit and let n increase, we approach circular symmetry, continuous rotational symmetry.

Rotation is hard to discover under normal algebra, but it very easy with a matrix. You can read in §18.1 how the rotational matrix is $\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$

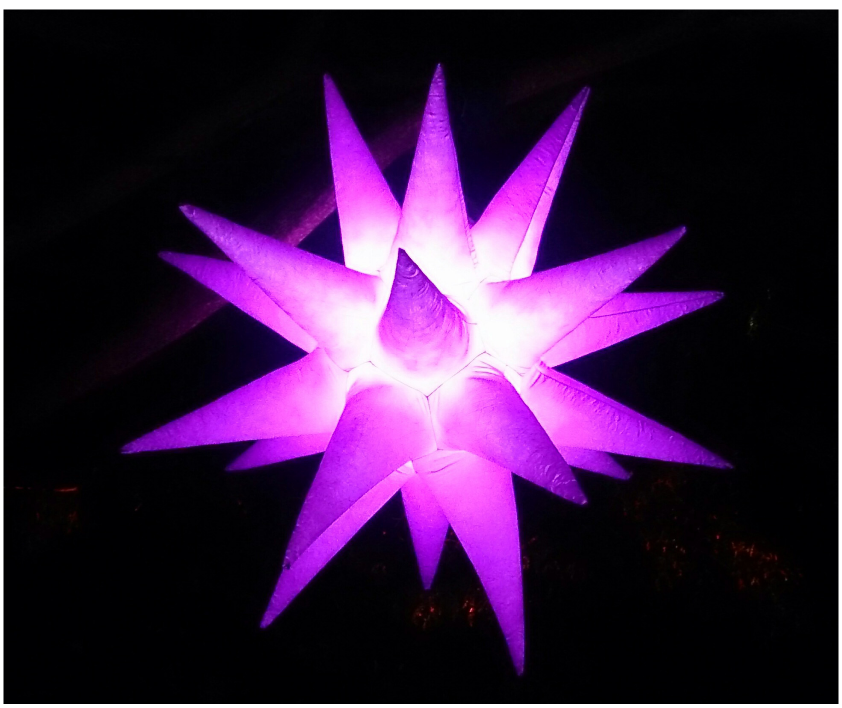


Figure 1.13 Rotational symmetry becomes much more complicated in higher dimensions.

1.4.4 Exercises

in Kuta

1.5.1 Graphical Patterns

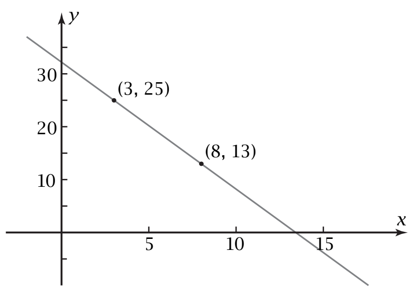
Name: ______________________ Group Members: ______________________

Exploration 7-2a: Graphical Patterns in Functions

Date: __________

Objective: Find the particular equation of a linear, quadratic, power, or exponential function from a given graph.

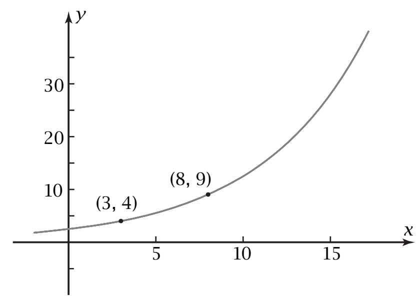
1. Identify what kind of function is graphed, and find its particular equation.



2. Check your answer to Problem 1 graphically. Does your graph agree with the given one?

3. Does the graph in problem 1 have an asymptote? If so, what is it's equation?

4. What graphical evidence do you have that the function graphed is an exponential function, not a power function? Find its particular equation.



5. Check your answer to Problem 4 graphically. Does your graph agree with the given one?

6. Does the graph in problem 3 have an asymptote? If so, what is it's equation?

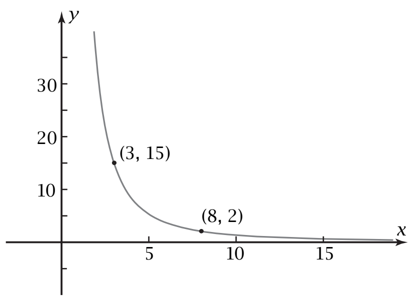
(Over)

Name: ______________________ Group Members: ______________________

Exploration 7-2a: Graphical Patterns in Functions continued

Date: __________

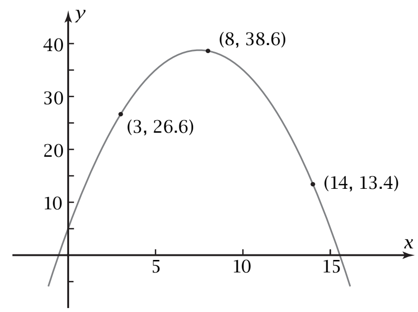
7. What graphical evidence do you have that this function is a power function, not an exponential function? Find its particular equation.



8. Check your answer to Problem 7 graphically. Does your graph agree with the given one?

9. Does the graph in problem 7 have an asymptote? If so, what is it's equation?

10. Identify what kind of function is graphed, and find its particular equation.



11. Check your answer to Problem 10 graphically. Does your graph agree with the given one?

12. Does the graph in problem 10 have an asymptote? If so, what is it's equation?

13. What did you learn as a result of doing this Exploration that you did not know before?

1.5.2 STATS in the TI

Objective

Create appropriate regressions of data in a grapher.

In order to proceed from numerical to algebraic and graphical models of real-world situations, we need to perform **regression analysis**, a thoughtful process of establishing the strength and kind of relationship between two variables. The entirety of Chapter 13 is about regression. For now, we simply seek to make equations from sets of ordered pairs.

Your TI-8* has *some* capacity as a spreadsheet. You may entered tabular data, graph, and analyze the results on this little computer! Most things you will need start by pressing the STAT button. This information is stored in lists, like $\mathtt{L}_1$, $\mathtt{L}_2$, $\mathtt{L}_3$, etc. These names are not letters found with the ALPHA button, but whole characters. They can be accessed via 2ND 1, 2ND 2, etc. The most normal way to proceed is to put x data in $\mathtt{L}_1$ and y's in $\mathtt{L}_2$.

Next, in order to make data you've entered visible, one must turn on a `STAT PLOT`. This is done via 2ND Y. Here you will see three different `STAT PLOT`s you can turn on or off. Press ENTER to select 1, and then again to turn it on. Be sure to turn `STAT PLOT`s off when you're not using them, so as not to obfuscate other graphs.

Windows

One of the trickier things to manage on the TI-8* is the `WINDOW`. The Cartesian plane is a big place, and finding your place in it can be harder stilld. Having a grasp on concepts like domain and range is once again worth it's weight in gold.

First of all, the TI-8* does try to help you out by offering several automatic options, which may or may not be useful. The most commonly used is ZOOM 6: `ZStandard`. This is the common window, $-10 \le x \le 10$ and $-10 \le y \le 10$, with tick-marks every unit. When in doubt, start here.

If, however, the data is given to you, make the `Xmin` the smallest element in the domain, and `Xmax` the biggest (press WINDOW). Similarly, the range should dictate your `Ymin` and `Ymax`. For truly useful graphs, determine the breadth of both the domain and range, by subtracting the smallest element from the biggest. Divide these by 20 and you will get the `Xscl` and `Yscl` respectively (the little tick marks).

Of course, the procedure outlined in the last paragraph will not work for most function definitions, since they will go on forever, often in both domain and range.

This means you just have to know what are the salient features of a given kind of graph, and how to bring them into view. For example, a quadratic function's most important point the vertex, the place where it changes from increasing to decreasing or visa versa. Most of the time, this is involves a process of guess-and-check, combined with every-honing instincts.

Table

2ND GRAPH is the very helpful `TABLE` of data. This is especially useful at comparing multiple functions on the same inputs. The default to start at $x = 1$ and increment by one from there on up. To customize the table, you must press 2ND WIND, which is called `TABLESETUP`. You can either make the table automatically increment through the domain, starting where you wish and skipping by the same amount every time (`AUTO`), or you can manually enter values for the input (`Ask`).

Linear Regression In you are given only two points and told to make a line through them both, then the problem is very easy system of two equations, and can be immediately solved via the RREF of a 2x3 matrix. However, most scientific and real-life data is "fuzzy", and no line can go through all the data points. In such situations, it is best to enter all the data and let the machines find the "line of best fit". In simple cases, it is simply a matter of constructing the line where half the data is below and half is above. But as data multiplies, this becomes harder and harder.

Example 1.4 [Linear Regression] Find the line of best fit for the data in this table.

x	y
1	1
2	2
3	1.3
4	3.75
5	2.25

Solution Using a TI-8*, press STAT, EDIT, and enter the x data in L_1 and the y data in L_2. If there is already data clogging up these lists, move up to their names at the top of the window and press CLEAR, then ENTER. Assuming the appropriate STATPLOT is on, press ZOOM 9 and inspect the graph. It is not a simple linear graph, where any line passing through any two of the points is perfect, so we proceed with the calculator regression

Pressing STAT, moving over to CALC, we scroll down to LINREG(AX+B). Depending upon your version, you may need to type L_1, L_2, Y_1 or select all these setting for a menu. (Y_1 is located under VARS, Y-VARS, FUNCTION...

and should be entered under 'Store EQ'.) Press ENTER or CALCULATE and the answer of $y = .425x + .785$ appears, as well as the coefficient of determination r^2. (If your calculator doesn't show you r^2, you might need to press CATALOG, scroll down to DiagnosticOn and turn it on.). We will learn about r and r^2 in 13.3, but for now it is enough to say it is helpful measure of how closely the equation "hugs" the data.

Finally, press GRAPH to see the equation against the data.

Quadratic Regression When the data rise and falls (or falls and rises), it is possible that a quadratic equation would be a better fit. Visualize the data and then consider running QuadReg and generating an equation of the form $ax^2 + bx + c$. Perfect data can be solved simply with a 3x4 matrix.

Exponential and Logarithmic Regression ExpReg and LnReg are great tools, if you know how to choose between them.

Analyzing Scatter Plots

In Section 2.7 we used linear and quadratic functions to model several data sets. However, in some applications, data can be modeled more closely by using exponential or logarithmic functions. For instance, Figure 4.43 illustrates some scatter plots that can be modeled effectively by exponential and logarithmic functions.

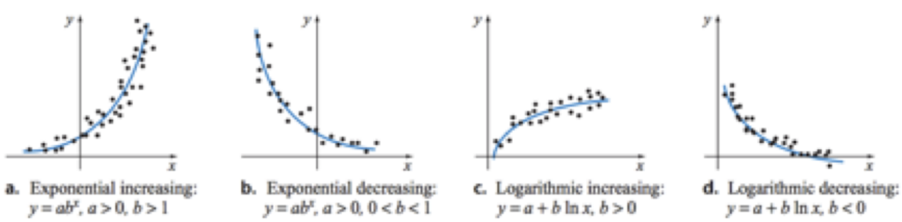


Figure 4.43
Exponential and logarithmic models

Exponential functions have a horizontal asymptote at $y = 0$ if un-translated. Logarithmic functions has a vertical asymptote at $x = 0$ if un-translated. Most practical situations, it is always possible to tell if there is a limiting x or y value. If there are multiple asymptotes, it is probably a power function.

Sinusoidal Regression While we have not formally introduced period functions yet, it is relatively easy to run a SinReg in the calculator and get an equation from data. The only additional questions you must answer are the number of iterations — how many times should the calculator pour over the data (the max is 16, which you should use for all "fuzzy" data) — and the **period** — the distance from peak to peak, or trough to trough. For now, the period will always be given to you.

1.5.3 Exercises

1.6 Review

1.6.1 Chapter Review

This chapter was about functions, and how they model so much of reality. Functions can be described numerically (tables), algebraically (formula), graphically (Cartesian plane curves), or verbally (technical language). Some functions we have seen before or will see in this course are linear, quadratic, exponential, power, rational, logarithmic, and periodic. You need to have memorized the simplest form of each type of equation (Tab. 1.3).

You should review the graphs of each of these. In fact, you should attempt to engage all these forms in all four ways.

Functions are written in function notation, which looks annoyingly like multiplication. Functions can have operations done on them, as numbers do, only they have two place where operators may be applied: inside (before) and outside (afterwards). Outside effects the output (y) and is reasonable. Inside effects x and is contrarian! Adding translates (moves) the graph. Multiplying stretches (dilates) the graph. Negatives 'flip' the graph. Graphs that are the same when flipped left/right are called even, like even powers of x. Graphs that look the same when they are flipped left/right *and* up/down are called odd, like odd powers of x. The calculator can help us turn numerical data into function notation via it's many regression functions.

Some questions you should be prepared to address are: What are functions? What are some of the basic types of functions? What are the four ways we describe functions in this chapter? How do we move between the various descriptions? What happens to $f(x)$ as we vary four constants, like this: $a \cdot f(b(x+c))+d$? What are some of the differences between our models and reality? What have previous classes shown you to be the nature of the mathematic task? What is technical (vs ordinary or poetic) language? What role does technical language play in your life, now and presumedly in the future?

Figure 1.14 is not the final word(s) on relating the four representations, but it is supposed to get you thinking. We often manipulate symbols and change one equation type into another, but we can do the same with graphs. Paper does not allow it, but couldn't we build an object that changes color over time, in order to represent a function? Data could be changed from absolute x and y to "number of steps since last" and "percent change". Try to come up with a simple model of a natural phenomenon and see how many relationships you can fulfill.

Linear:
$y = m \cdot b$

Inversely Proportional:
$y = \frac{k}{x}$

Power:
$y = a \cdot x^b$

Quadratic:
$y = ax^2 + bx + c$

Exponential:
$y = a \cdot b^x$

Logarithmic:
$y = a + b \ln x$

Periodic:
$y = a \cdot \sin(b(x-c)) + d$

Table 1.3 General equations

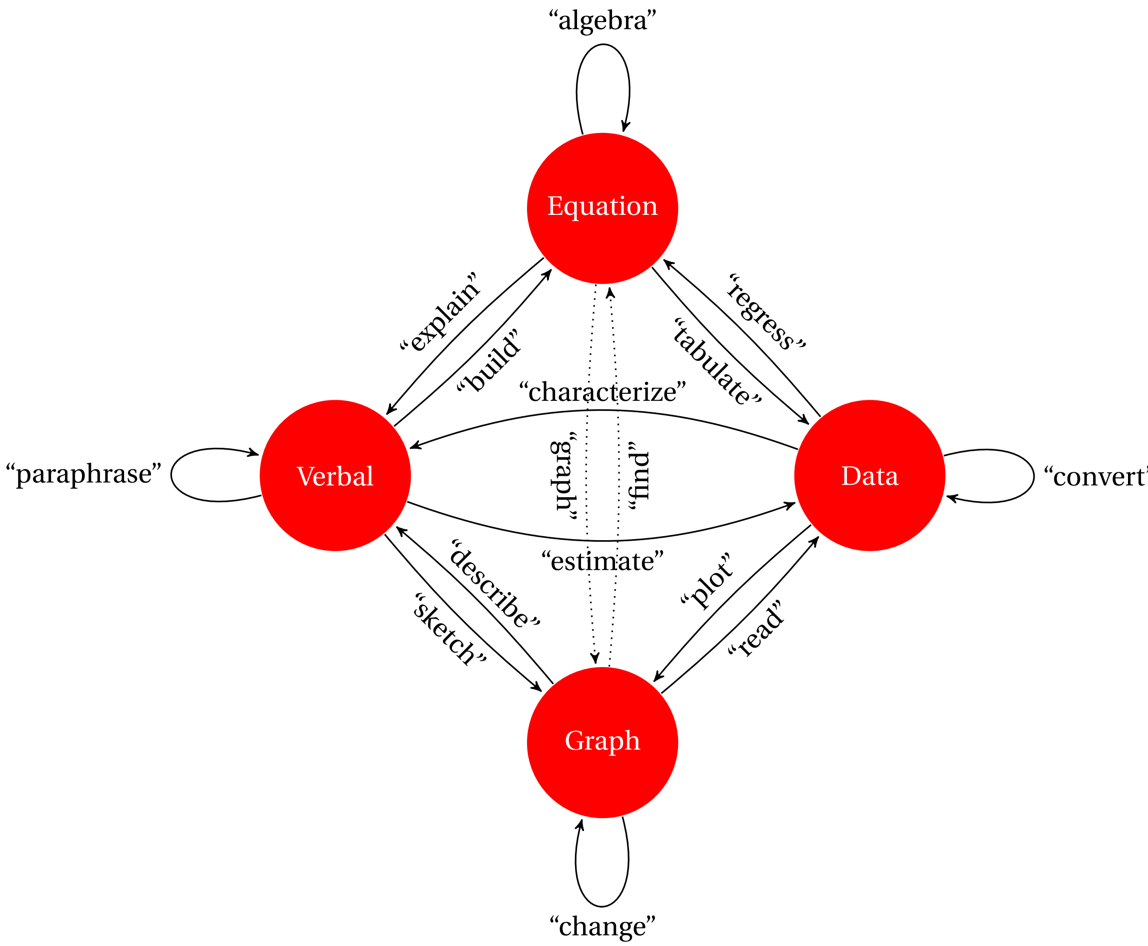


Figure 1.14 Some possible words to describe moving among the four representations

1.6.2 Chapter Test

Name: ________________ Date: ________

Test 1, Sections 1-1 to 1-3 Form

Objective: Identify and transform functions and their graphs.

Part 1: No calculators allowed (1–8)

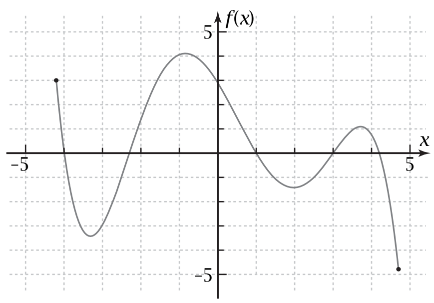
1. The grade you could get on a precalculus test depends on the number of hours you study the night before. If you study too long, however, you might score lower because you are too sleepy on the day of the test. Sketch a reasonable graph.

2. What type of function has a graph like the one you sketched in Problem 1?

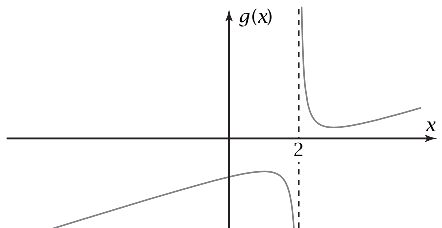
3. If $f(x) = 2 \times x^3$, what type of function is f?

4. If $f(x) = 5 \times 3^x$, what type of function is f?

5. The graph shows a polynomial function. The domain of the function is $-4.2 \le x \le 4.7$. What is the range of the function?

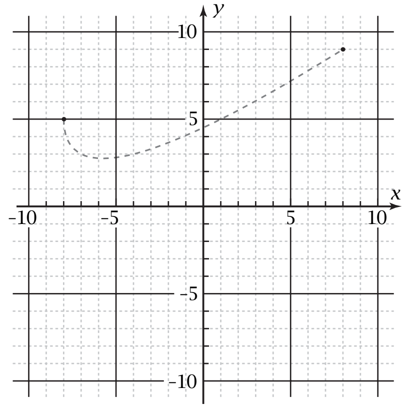


6. What type of function has a graph like this?



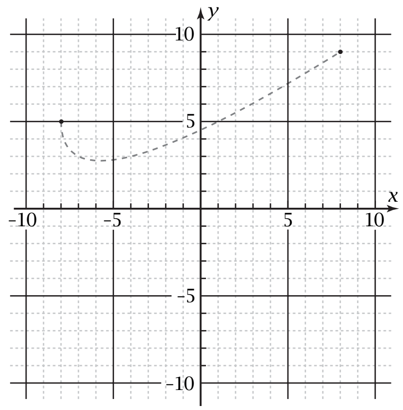
7. The graph of $f(x)$ is shown. Sketch the graph of function g, a horizontal dilation by a factor of $\frac{1}{4}$. Write $g(x)$ in terms of $f(x)$.

Equation: ________



8. If $g(x) = f(x) - 7$, describe the transformation, and sketch the graph of function g.

Verbally: ________

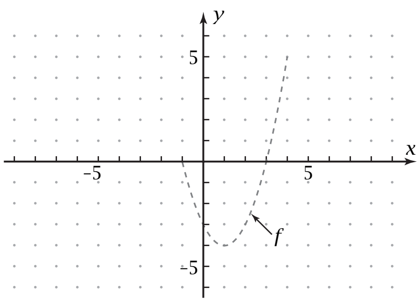


e: ______________________ Date: ______________________

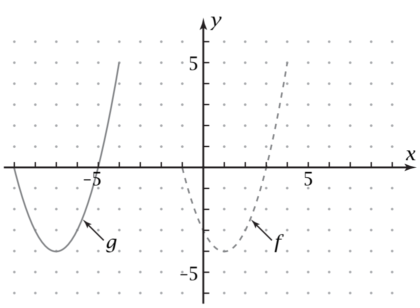
st 1, Sections 1-1 to 1-3 continued Form A

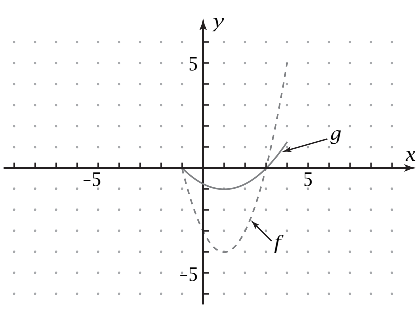
t 2: Graphing calculators allowed (9–21)

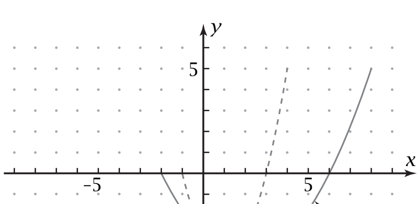
The graph shows $f(x) = x^2 - 2x - 3$ plotted in the domain $-1 \le x \le 4$. Plot this graph using a friendly window. Divide by the Boolean variable $(x \ge -1 \text{ and } x \le 4)$ to get the domain shown. Check your graph with your instructor.



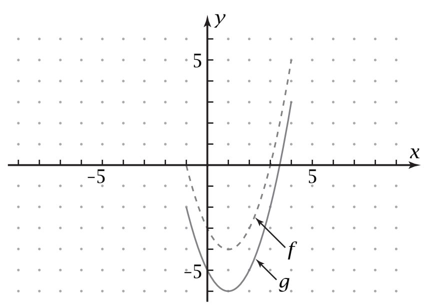
Problems 10–14, identify the transformation of the h of f (dashed) to get the graph of g (solid). Plot the h of g on your grapher and state whether it checks.



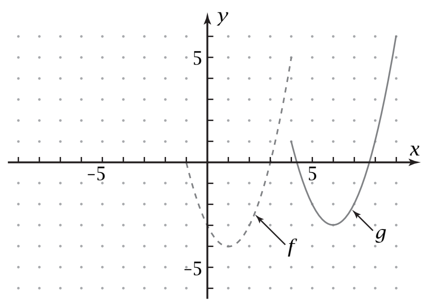




13.



14.



Shopping Cart Problem: For Problems 15–20: The shopping carts at a grocery store are each 52 in. long. A line of 6 carts pushed together has a total length of 109 in.

15. Make a sketch showing what the line of 6 carts would look like.

16. How many inches are added to the line for each cart? Show how you get your answer.

17. Let $f(n)$ be the length in inches for a line of n carts. Write an equation for $f(n)$ in terms of n.

18. What kind of function did you write in Problem 17?

19. Based on your equation in Problem 17, how long would a line of 15 carts be?

20. The store has a space exactly 20 ft long (240 in.) in which to put lines of carts. What is the greatest number of carts they can put in a line without exceeding the 240 in.? Show how you get your answer.

II

Appendices

A. Prerequisites

The purpose of this chapter is to re-present ideas which should already be known coming into this class. This material would make excellent summer assignments, to minimize students' re-entry time once classes begin. However, many of these ideas should have been covered in an Algebra II class.

A.1 Sets

Objective

Manipulate and create sets and intervals using set-builder notation.

In this section, we will review how to describe regions of numbers and sets.

Set

A **set** is a well-defined collection of distinct objects, and is considered as an object in its own right. "Well-defined" means it is readily apparent to anyone if an object is or is not part of the set.

An object in a set is called an "element". If $A = \{1, 2, 5\}$ then the elements of A are 1, 2, and 5[1]. We may also say $5 \in A$, which pronounced aloud as "five is an element of A"[2]. If set B is defined as $B = \{1, 2\}$, then we can say $B \subseteq A$, that is, "B is a subset of A". $\varnothing$ is the symbol for the null set, also called the empty set, i.e. the set with no elements.

With these tools for sets, we may define variables using **set-builder notation**. For example, we might say:

$$\{x | x > 0, x \in \{-3, -2, -1, 0, 1, 2, 3\}\} \tag{A.1}$$

You should read A.1 if it were the prose paragraph, "There is a variable x such that x is greater than 0, and x is in the set from -3 to 3." Can you figure out what x is actually equal to? All that was a very long way to write that $x = \{1, 2, 3\}$! Why would anyone write this way? The power of this notation becomes obvious later, when we bring in infinite sets. To begin with, we will use "..." (called an "ellipsis") to make infinite sets that keep going as before.

Set-Builder Notation

A subject and a predicate, separated by a pipe and enclosed in curly-brackets.

[1] Notice that the convention is to choose capital letters close to the start of the alphabet for set names, and then delimit the entries in a comma-separated list, inside curly brackets.

[2] You might remember this symbol by noting how much it looks like a capital E, which is the first letter of "element", as in "is an element of".

Example A.1

Use set-builder notation to describe all the odd numbers greater than 3.

Solution: Odd numbers are one unit away from even numbers, which are themselves two times *any* number.

$$\{x|x \in \{5,7,9,...\}\}$$
$$\{2x+1|x \in \{2,3,4,...\}\}$$
$$\{2x+1|x > 2, x \in \{1,2,3,...\}\}$$

A.1.1 Union and Intersection

Lastly, sets may be joined, looking only for the overlap, or united, look for anything in either. The overlap is called "intersection" and is symbolized with a $\cap$, while "anything in either" is called "union" and is symbolized with a $\cup$. This can be most easily visualized via Venn diagrams:

☞ $\cup$ is an easy symbol to memorize, since its name is **u**nion, and it looks like a U. You might consider intersection as it opposite.

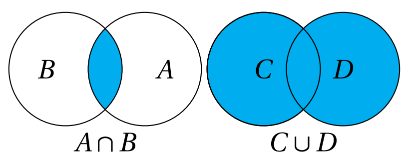


Figure A.1 Union and Intersection

Example A.2

Given that that $A = \{1,2,3,4,5\}$, $B = \{-3,-1,1,3,5\}$, and $C = \{-1,2,3,5,6\}$, find $(A \cap C) \cup (B \cap C)$.

Solution: $A \cap C$ means "What do A and C have in common?", which is 1, 3, and 5. $B \cap C$ means "What do B and C have in common?", which is -1, 3, 5. $\{1,3,5\} \cup \{-1,3,5\}$ means, "What is in either one?", which is $\{-1,1,3,5\}$.

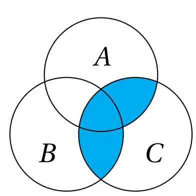


Figure A.2 $(A \cap C) \cup (B \cap C)$

Exercises

Set builder notation, both mathematical and verbal

PA.1 FileNameWrite each given set in the Set-Builder Form:

(a) 2, 4, 6, 8, 10

(b) 2, 3, 5, 7, 11

(c) January, June, July

(d) a, e, i, o, u

(e) Tuesday, Thursday

(f) 1, 4, 9, 16, 25

(g) 5, 10, 15, 20, 25, 30

PA.2 FileName Write the following sets in Set-Builder Form or Rule form:

(i) A = 1, 3 5, 7, 9

(ii) B = 16, 25, 36, 49, 64

(iii) C = a, e, i, o, u

(iv) D = violet, indigo, blue, green, yellow, orange, red

(v) E = January, March, May, July, August, October, December

Answers for the worksheet on sets in Set-builder form are given below to check the exact answers of the above questions on sets in Set-builder form or Rule form.

Answers:

1. (i) x : x is an even natural numbers less than 12

(ii) x : x is a prime numbers less than 12

(iii) x : x is a month whose name starts with letter J

(iv) x : x is a vowel in English alphabets

(v) x : x is a day of the week whose name starts with letter T

(vi) x : x is a perfect square natural number upto 25

(vii) x : x is a natural number upto 30 and divisible by 5

2. (i) A = x | x is an odd number less than 10.

(ii) B = x | x is a perfect square natural number between 15 and 65

(iii) C = x | x is a vowel in English small alphabet.

(iv) D = x | x is a color in rainbow.

(v) E = x | x is a month having 31 days.

A.2 Kinds of Numbers

Objective

Explain the Reals and how to build them.

Numbers are abstract entities which represent quantities, measurements, or arrangements. There are various kinds of numbers, and thinking about the sizes and definitions of these kinds is a surprisingly vast and complicated field. Perhaps this enterprise is as old as mathematics, beginning when someone thought, "There is no number so large that one cannot be added to it".

It is assumed that all human cultures with counting began with the **natural numbers**, that is $\{1, 2, 3, \ldots\}$. The symbols for "natural numbers" is $\mathbb{N}$. For many areas of life, analyses need proceed no further. But, various cultures at various times have needed to distinguish zero from an actual lack of an answer. More formally, $0 \neq \varnothing$. , and it plays an important role in mathematics as a number. We call 0, 1, 2, 3, ... the whole numbers, or $\mathbb{W}$.

Example A.3

If $A \in \mathbb{N}$ and $B \in \mathbb{W}$, does $(A \cap \{0\}) \cap B = \varnothing$?

Solution: $A \cap \{0\}$ is asking that the natural numbers and zero have in common, which is nothing. B and nothing have nothing in common. Therefore, yes, $(A \cap \{0\}) \cap B = \varnothing$

Many mathematics textbooks discuss negative numbers next, but we will save that discussion for the next section. Suffice it to say, whole numbers and their negative counterparts make up the **integers** ($\mathbb{Z}$).

Most cultures have a need to portion out large things, and these portions are typically thought of as proportional. That is, we imagine a ratio of possessed pieces to overall divisions. The most common fraction is $\frac{1}{2}$, which means division into two pieces and existence of only one of those pieces. Quite plainly, we can call these numbers **rational numbers** because they can be expressed as ratios. The mathematical symbol is $\mathbb{Q}$, because the Latinate word for a ratio is 'quotient'.

Positive Rational Number

A positive rational number may be written as $\frac{a}{b}$, where $\{a|a \in \mathbb{W}\}$ and $\{b|b \in \mathbb{N}\}$. That is, the denominator may not be zero.

Decimal to Fraction

Any repeating or terminating decimal may be written as a rational number and visa versa.

Example A.4

Convert $1.\overline{126}$ into a ratio.

Solution: Let $x = 1.\overline{126}$. Because there are three repeating digits, we multiply by 10^3.

$$\begin{array}{rr} 1000x = & 1126.\overline{126} \\ -x = & -1.\overline{126} \\ \hline 999x = & 1125.000 \end{array}$$

$$\therefore x = \frac{1125}{999} = \frac{125}{111}$$

☞ You calculator can help save time by reducing fractions for you. MATH (►FRAC) is a powerful tool.

A.2.1 Irrationals

Some numbers may not be expressed as fractions. They are called **irrational numbers** and their symbol is $\mathbb{I}$. Some irrational numbers are nevertheless solutions to algebra problems, such as $\sqrt{2}$ or $10^{2.3}$. These are called the **algebraic** numbers and are symbolized with a $\mathbb{A}$. Others may be described, but are not the result of arithmetic, numbers such as π or e. Such values are called **transcendental** numbers, and they are proven to be the vast majority of numbers.

Exercises

Convert repeating decimals to fractions

PA.3 Convert 0.11111... into a fraction

Answer: $\frac{1}{9}$

PA.4 Convert 1.212121212... into a fraction

PA.5 Convert 0.123123123123... into a fraction

PA.6 Convert 0.123456789 into a fraction

PA.7 Convert 0.12345678901234567890123... into a fraction

PA.8 Put in order without a calculator $\sqrt{10}, \pi, \frac{22}{7}, 3.14$

A.3 Number Line

Objective

Solve absolute value linear equation and inequalities.

A.3.1 Negative Numbers

As mentioned in the previous section, negative number arose as an extension of the whole numbers which accounted for all cases of subtraction. What is the difference between someone who has no money, and someone who owes \$3? You can't tell until they are paid \$3: one will have \$3 and the other will be broke!

The imagery of debt brings in a helpful cognitive metaphor: quantities can have a direction, and not simply be a **scalar**. Once direction is involved, it is necessary to speak of opposites, which are easily framed as a positive and a negative frame of reference. The discovery of negative numbers recast what has simply been "numbers" as *positive* numbers. In our system, positive is generally rightward, while negative is leftward.

Example A.5

What is five plus negative three? What is five minus negative three? Why?

Solution: Suppose we are discussing money. Someone having \$5 has added a debt of \$3 to his or her ledger. The net worth of such a person has decreased to \$2. However, if someone worth \$5 has a \$3 debt *removed* from his or her account, the net worth of such a person has increased (do to removed debt), and is now \$8.

A.3.2 Absolute Value

After the introduction of directionality to numbers, it is often necessary to disregard such information. Rather than caring about how far forward or backwards something has travelled, we only need to know the *distance*. The operation of ignoring the sign of a quantity is called the *absolute value*. Without other arguments, it answers the question, "How far is a given quantity from the center, from zero?" For example, $|x-0| = 4$ is an algebraic way of asking for the *two* numbers four away from zero. To include the subtraction operator, we should really ask for the "the two numbers who's difference with zero is four."

Example A.6

What are the two numbers who's difference with negative seven is six? How would one represent this problem in algebra?

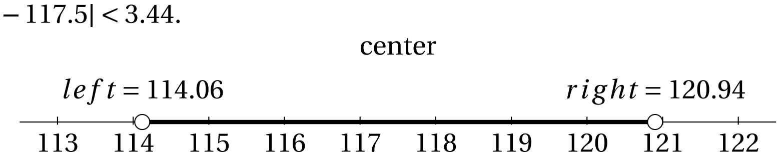
Solution: If we travel six units to the right from negative seven, we arrive at negative one. If we travel six units to the left, negative thirteen. This can be written as $|x-(-7)| = 6$ or even better $|x+7| = 6 \therefore x = \{-13, -1\}$.

Looking only at the algebra, it can be beneficial to think of an absolute value sign in terms of the two cases it presents: a leftward possibility, and a rightward one. Once an absolute value expression is reduced down to a simple equality, either the contents of the absolute value is positive or negative. For exampe,

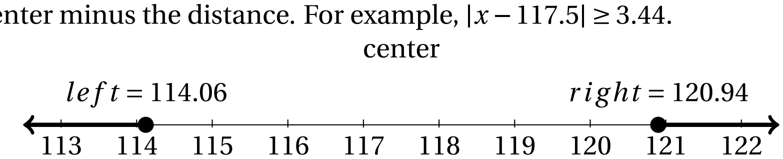
$$
\begin{aligned}
2|3x+1|+4 &= 12 \\
|3x+1| &= 4 \\
3x+1 &= 4 \\
&\text{or} \\
3x+1 &= -4 \\
x &= \{1, -\frac{5}{3}\}
\end{aligned}
$$

A.3.3 Inequalities

Absolute value inequalities require logical reasoning, because if the distance to a given center is *less* than some number, the solution must be less than the center plus the distance *AND* greater than the center minus the distance. For example, $|x-117.5| < 3.44$.



On the other hand, if the distance asked for this *greater* than some number, the solution must be more than the center plus the distance *OR* less than the center minus the distance. For example, $|x-117.5| \geq 3.44$.



Notice the convention is to indicate values which are attainable ("or equal to") with solid dots, while values not possible are marked as boundaries with hollow circles.

Exercises

Graph on a number line

PA.9 x < 2

PA.10 x <= 3

PA.11 x > 5 bowl x <=10

PA.12 |x| = 3

PA.13 |x| = -2

PA.14 |x-4| = 0

PA.15 |x - 4| = 2

PA.16 |x + 4| = 2

PA.17 |x| > 3

PA.18 |x-1| > 2

PA.19 4|x+2| < 8

A.4 Analytic Geometry

Objective

Calculate distances and midpoints on the Cartesian coordinate plane for order pairs from functions and relations.

A.4.1 Cartesian Plane

Descartes did many strange things in philosophy, but he was a real boon for mathematics. The idea of graphing values of one variable (x) to the left and right, while simultaneously graphing another variable (y) on the up and down is credited to him. This is called the **Cartesian plane** or "Rectangular Coordinates".

Quadrants

It is customary to split the coordinate system into four quadrants, numbered with Roman numerals. Quadrant I is where x and y are both positive, to the upper right. Quadrant II is where x is negative but y is positive, to the upper left. Quadrant III is where both are negative, to the lower left. Quadrant IV is where x is positive and y is negative, to the lower right.

A.4.2 Triangles

Just as we might find the middle between two numbers on the number line, so too we can find a point midway between two other points, simply by taking the average of their xs and the average of their ys.

Midpoint

Given two points (x_1, y_1) and (x_2, y_2) the arithmetic mean of the x's is x of the midpoint, and the same for y.

Midpoint $(x, y) = (\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$

If we draw a right-triangle with its hypotenuse being the shortest path between two points, and its legs going strictly left-to-right and up-to-down, then calculating their distance is simply the Pythagorean Theorem.

Pythagorean Theorem

The distance from (x_1, y_1) to (x_2, y_2) is the hypotenuse of a right triangle with legs $|x_2 - x_1|$ and $|y_2 - y_1|$.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{A.2}$$

A.4.3 Functions and Relations

Chapter 1 proper of this textbook is about functions, but even more basic are relations. For number, it will suffice to say that any equation with xs and ys in it is a relation[3].

like

Relation

A relation between two sets is a collection of ordered pairs containing one object from each set. If the object x is from the first set and the object y is from the second set, then the objects are said to be related if the ordered pair (x, y) is in the relation.

[3]Note that these need not be "nice". For example, $x^2 y^2 = 4$ produces an infinite, four-pointed star, but must be put into the TI-8* as `Y1=sqrt(4/x2̂)` and `Y2=-sqrt(4/x2̂)`.

Exercises

PA.20 Graph the relation $x = |y|$. How could we put this into our TI-8*?

PA.21 Use the Pythagorean theorem to find 12 lattice points 5 units from the origin.

PA.22 Find the relation describing all the ordered pairs 5 units from the origin.

PA.23 What are the points 5 units away from 4,2 with a y value of 10?

PA.24 What are the points 6 units away from -2,-1 with a x value of -10

PA.25 What are the points root 10 units away from 0,2 and root 10 units away from 2,0?

A.5 Properties and Operators

A.5.1 Closure

Suppose you attempted to help a young child with some mathematic problems. Suppose further this child knew next to nothing about fractions yet. Perhaps they understand the notion of debt — negative numbers — but no greater sets of numbers. What can you talk to them about? What must you avoid? Consider the following list of procedures you know:

- Addition
- Subtraction
- Multiplication
- Division
- Distribution of Multiplication over Addition
- Squaring
- Square roots

The question before us is one of **closure**. A set is closed under an operator if doing the operation on any member of the set produces another member of the set. For example, the integers are closed under addition, subtraction, and multiplication, but not division.

Example A.7

Are the Real numbers closed under square rooting?

Solution: No. Disproof by counterexample: $\sqrt{-4} = 4i$

A.5.2 Identity and Invertibility

Over a given set, most operators have an **identity element** that leaves them unchanged. For example, the identity of addition is 0, because adding zero does not change a number. Similarly, multiplication has the identity element of 1, because multiplying by one leaves a number unchanged.

Example A.8

What is the identity of set union?

Solution: $\varnothing$ when joined to any set will not change the set in any way.

It is possible to have more than one identity for a given operator over a particular set. One effect of having an identity element is that it becomes possible to define "undoing" a particular operator. For instance, multiplicative inverse of every number except 0 is one, which shows that multiplication by m can be undo by multiplying by $\frac{1}{m}$. The additive inverse of some number x is $-x$, because together they add to the additive identity 0. An operation may be invertible without being commutative (see below)

A.5.3 Associativity

Suppose we have one operator and three operands. In what order do we proceed? Many times, it does not matter. Addition and multiplication are associative over most numbers, so $(3+2)+1$ is the same as $3+(2+1)$ and $(2\cdot 3)\cdot 4$ is the same as $2\cdot(3\cdot 4)$. Division is not associative, and neither are powers.

☞ A cute way to remember this property is to think of friends associating together. If three girls are friend — call them Jill, Sandra, and Bree — then it doesn't matter which we we draw the association. Jill and Sandra are friends, and together they're friends with Bree. That is the same as Sandra and Bree being friends, and together being friends with Jill. $(J\cup S)\cup B = J\cup(S\cup B)$.

Example A.9

Is rock-paper-scissors[4] associative?

Solution: No. Disproof by counterexample:
(rock vs. paper) vs. scissors → paper vs. scissors → scissors
rock vs. (paper vs. scissors) → rock vs scissors → rock

A.5.4 Commutativity

Many operators do not care in which order the operands come. Addition and multiplication are the same forwards and backwards! But subtraction and division yield very different results when done in the reverse. This property is called commutativity.

A.5.5 Distributivity

A very complicated property, yet a very well known one, is the distributive property. This property requires *two* operators to be performed on a set. For example, multiplication *is* distributive over addition. e.g. $2(3+4) = 2{\cdot}7 = 2{\cdot}3+2{\cdot}4 = 6+8 = 14$.

Example A.10

Is division distributive over addition? Are powers distributive over multiplication?

Solution: No and yes.
No. $2\div(3+4) \neq 2\div 3+2\div 4$. Yes. $(3\cdot 4)^2 = 12^2 = 3^2\cdot 4^2 = 9\cdot 16 = 144$

Exercise: clock-math closure

[4] Or even better, rock-paper-scissors-lizard-Spock!

A.6 Factoring and Primes

Humans seeks to understand the fundamental nature of numbers. We have learned many things, but the underlying, deep structure is not fully known. We believe will know a great deal more if we can understand the distribution of Prime Numbers.

Prime Number A natural number greater than 1 that has no positive divisors other than 1 and itself.

Fundamental Theorem of Arithmetic Every integer greater than 1 either is prime itself or is the unique product of prime numbers.

Factorizing numbers show the algebraic nature of all natural numbers. As we seek to simplify algebraic expression through factorization, we find many helpful patterns which are important to know both forwards (distribution of multiplication over addition) and backwards.

A.6.1 Patterns of Distribution

The square of a binomial always follows a pattern, which can be seen algebraically or via algebra block. A binomial multiplied by its conjugate produces the difference of squares. When the first element of two binomial is the same, the result will have like terms that combine as a sum.

$$
\begin{aligned}
&(a+b)^2 = a^2 + 2ab + b^2 \\
&(a+b)(a-b) = a^2 - b^2 \\
&(x+a)(x+b) = x^2 + (a+b)x + ab
\end{aligned}
\tag{A.3}
$$

A.6.2 Factoring Quadratics

When factoring, the first check should always be for a common factor to take out. Then, the above techniques seen in distribution work in reverse for factoring. If a quadratic expression is of the form $ax^2 + bx + c$, there are two very different techniques, depending upon whether $a = 1$ of not.

Simple Squared Term

After checking for a GCF,
Perfect Square Binomial: $a^2 + 2ab + b^2 = (a+b)^2$
Difference of Squares: $a^2 - b^2 = (a+b)(a-b)$
“Adds to, Multiplies to”: $x^2 + (a+b)x + ab = (x+a)(x+b)$

Split the middle When $a \neq 1$, more elaborate techniques are called for. Typically, working with $a < 0$ is much more difficult, so factoring out a negative is helpful. Next, splitting the linear term in two may yield results. Looking at $ax^2 + bx + c$, seek to find factors of $a \cdot c$ which add to b. Factor the terms, two at a time (called "factoring by grouping". In large polynomials, groups of arbitrary size are possible.) You will know you have succeed when you have two different terms multiplied agains the same term in parentheses.

Example A.11

Factor $15x^2 - 11x - 14$.

Solution: Begin by multiplying $15 \cdot (-14) = -210$. We are looking for factors of -210 which are 11 apart. Only 10 and and -21 fit the bill.

$15x^2 - 11x - 14$
$15x^2 + 10x - 21x - 14$
$5x(3x+2) - 7(3x+2)$
$(5x-7)(3x+2)$

Sometimes, none of the above techniques yields any results. There are two equally valid moves at that point, but one of which is made from the other.

Complete the Square and the Quadratic Formula First, if $a \neq 1$, factor it out of the terms with an unknown. Second, use the definition of a Perfect Square Binomial to discover what constant would be needed to "complete the square". Insert this term and its additive inverse (which is like adding zero) and factor to a square.

Example A.12

Rewrite by completing the square: $3x^2 + 12x + 7$

Solution: $3(x^2 + 4x) + 7$
$3(x^2 + 4x + 4 - 4) + 7$
$3[(x+2)^2 - 4] + 7$
$3(x+2)^2 - 5$

Quadratic Formula For any quadratic $y = ax^2 + bx + c$, the values of x where $y = 0$ may be are the two solutions $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Exercises

PA.26 sum and difference of cubes

PA.27 prove q.f. by completing the square

A.7 Matrices

Matrix A **matrix** (plural matrices) is a rectangular array of numbers, symbols, or expressions, arranged in meaningful rows and columns. The individual items in a matrix are called its elements or entries.

Size, Entries A matrix's size is described by the number of rows, by the number of columns. If a matrix is given a name, an entry may be referred to by a subscript of row and column on that letter. For example, on matrix **[A]**, one might refer to the entry in the second row and third column as $\mathbf{A}_{2,3}$

Example A.13

Given that A is 1234 what is $A_{1,4}$?

Solution: 4

A.7.1 Addition and Scalars

Matrices may be added if and only if they are the exact same size. A matrix maybe multiplied by a number (called a **scalar**), which is simply multiplied against every element in the matrix. Two matrices are added just by adding the for corresponding entries, i.e. $A_{i,j} + B_{i,j}$ produces the new entry at i, j.

Example A.14

If A is 1234, what is 2A?

Solution: 2468?

Example A.15

If A is 1234 and B is 0102 what is A+B?

Solution: 1336

A.7.2 Matrix Multiplication

The product of two matrices is the coming together of rows of the first, with columns of the second. For example, to compute the top left entry in the product of two matrices, one multiplies each entry in the first row of the first matrix, against the corresponding entry in the first column of the second matrix. (See below for a helpful visual.) Naturally, this means that the rows and columns must match up.

A.7.3 Square Matrices

identity, inverses, determinants

A.7.4 Gaussian Elimination

augmented matrices

B. Matrices

Numbers in boxes

B.1 Transformations

Name: ______________________ Group Members: ______________________________________

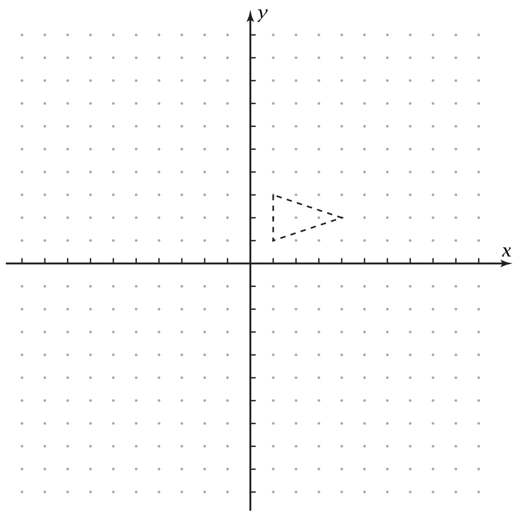
Exploration 11-3a: Matrix Images and Transformations

Date: ____________

Objective: Transform an image by multiplying by a matrix.

1. Matrix $[M]$ describes the triangle shown here. What do the columns represent?

$$[M] = \begin{bmatrix} 1 & 4 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$



2. Matrix $[A]$ is a **transformation matrix.** Multiply $[A][M]$ and write the answer here.

$$[A] = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

3. Explain why you can't multiply $[M][A]$.

4. The figure corresponding to the answer in Problem 2 is called the **image** of $[M]$ for the transformation $[A]$. Plot the image on the given figure.

5. How would you describe the transformation defined by matrix $[A]$?

6. The original triangle whose matrix is $[M]$ is called the **pre-image.** Why is this its name?

7. Matrix $[B]$ is another transformation matrix.

$$[B] = \begin{bmatrix} \cos 90° & \cos 180° \\ \sin 90° & \sin 180° \end{bmatrix}$$

Apply the transformation by calculating $[B][M]$. Plot the result on the given figure.

8. Matrix $[B]$ in Problem 7 **rotates** the pre-image by 90° counterclockwise. Write a matrix $[C]$ that you think will rotate the pre-image by 40° counterclockwise. To do this, you should realize that the angle in the second column is 90° more than the angle in the first column. Apply the transformation by calculating $[C][M]$. Write the answer with elements rounded to one decimal place.

9. Plot the image from Problem 8 on the given figure. Extend one side of the pre-image and the corresponding side of the image and measure the angle with a protractor. Did the transformation really rotate the pre-image by 40°?

10. Write a transformation matrix $[D]$ that both **dilates** by a factor of 2 and rotates by 240°. Calculate $[D][M]$. Write the image matrix with elements rounded to one decimal place, and plot the image on the given figure.

B.1.1 Shifts

B.1.2 Reflections

B.1.3 Rotations

B.1.4 Derivatives

B.1.5 Exercises

B.2 Walks

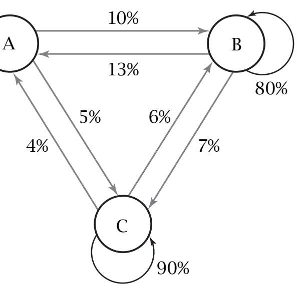
e: ____________ Group Members: ____________

ploration 11-4d: Markov Chain Problem

Date: ________

ective: Use iterative multiplication of matrices in a nongeometrical real-world problem.

vision Network Loyalty Problem: In a poll of television ers, a research company found the percentages of ers who change from the evening news on one vork to that on another network one month later.



nstance, for network A, 85% stayed with A, 10% ched to network B, and 5% switched to network C. The pany uses these numbers as probabilities of what will pen in subsequent months. They arrange the numbers **ansition matrix** $[T]$.

$$\begin{array}{cc} & \begin{array}{ccc} & & \end{array} \\ T] = & \begin{bmatrix} 0.85 & 0.10 & 0.05 \\ 0.13 & 0.80 & 0.07 \\ 0.04 & 0.06 & 0.90 \end{bmatrix} \begin{array}{c} A \\ B \\ C \end{array} \\ & \begin{array}{ccc} \quad A & \quad\; B & \quad\; C \end{array} \end{array}$$

element represents the probability that a viewer who hes the network in the *row* one month will be hing the network in the *column* the next month.

At the beginning of January, A has 35 million viewers, B has 20 million, and C has 43 million. These numbers are recorded in the "viewers" matrix, $[V_0]$.

$$\begin{array}{c} [V_0] = [35 \quad 20 \quad 43] \\ \qquad\quad A \quad\; B \quad\; C \end{array}$$

Explain why the number of viewers at the beginning of February is given by

$$[V_1] = [V_0][T]$$

Show that the number of viewers for A in $[V_1]$ is equal to the number who stayed with A plus the number that transferred from B and from C to A.

3. Show that the viewers matrix $[V_2]$ at the beginning of March can be found either as

$$[V_2] = [V_1][T] \text{ or } [V_2] = [V_0][T]^2$$

4. In the most time-efficient way, find the number of viewers predicted for each network the following January, one full year later. Assume that the probabilities remain constant.

5. Assuming that the probabilities remain constant, the number of viewers for each network approach a fixed limit as the number of months becomes very large. Find approximations for these limits numerically.

6. The matrices $[V_0],[V_1],[V_2], \ldots$ form a **Markov chain.** On the Internet or in some other reference source, find out who Markov is or was. Write a paragraph summarizing your findings.

7. What did you learn as a result of doing this Exploration that you did not know before?

B.2.1 Adjacency Maps

B.2.2 Unidirectional Lines

B.2.3 Exercises

B.3 Vector-Spaces

B.3.1 Null Space

B.3.2 Dependent Systems

B.3.3 Exercises

B.4 Determinants

B.4.1 Eigenvalues

B.4.2 Eigenvectors

B.4.3 Exercises

B.5 Fractals

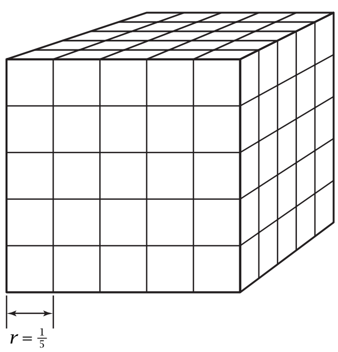
Name: ______________ Group Members: ______________

Exploration 11-6a: Hausdorff's Definition of (Fractal) Dimension

Date: ______

Objective: Learn a definition of the concept of *dimension,* and apply it to self-similar figures generated iteratively.

Here is a (three-dimensional) self-similar cube of edge 1 unit. It is cut into smaller cubes of edge $r = \frac{1}{5}$ unit. Note that the smaller cubes are similar to the original cube.



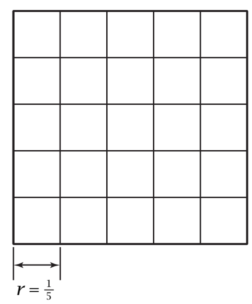
1. How many small cubes will there be? ______
2. Write the answer to Problem 1 as a power of 5.

   ______
3. What physical quantity does the exponent in Problem 2 represent?
4. Suppose that the cube in Problem 1 is divided into N smaller cubes, each of edge r units. Write N in terms of r.
5. Transform the equation $\left(\frac{1}{r}\right)^3 = N$ so that 3 is by itself on the left and N and r are on the right. (Recall the properties of logarithms!)

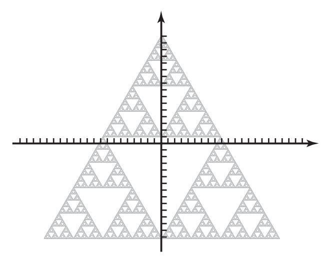
The results of Problem 5 are the basis for the definition of dimension, D, of a self-similar object. It was proposed by Felix Hausdorff, who lived from 1868 to 1942.

$$D = \frac{\log N}{\log\left(\frac{1}{r}\right)}$$

where N is the number of self-similar pieces into which an object can be divided and r is the length of a given piece as a fraction of the length of the original pre-image. In order for this definition to apply, you must be able to subdivide the object infinitely.



7. In a previous exercise, you generated Sierpiński's gasket, shown here.



   On the drawing, show that the gasket is self-simil by dividing it into three congruent pieces, each o which is similar to the whole figure. Then show h one of these pieces can be divided into three othe pieces, each of which is also similar to the whole figure.
8. Because the subdividing into self-similar pieces c go on infinitely, Hausdorff's definition of dimens applies. Calculate the dimension of the Sierpińsk gasket. Surprising?
9. The original pre-image in Problem 7 is a triangle base 20 and altitude 30. What is its area? What is total area of the first iteration? the second iterati What is the total area of the tenth iteration? the 100th iteration? What limit do the total areas of t iterations approach as the number of iterations approaches infinity? Surprising?

B.5.1 Text

Nature is fractals, not smooth (locally linear)

B.5.2 Exercises

e: ______________________ Group Members: ______________________

ploration 11-6c: Fractal Dimension of a River

Date: __________

ective: Apply Hausdorff's definition of *dimension* to the Guadalupe River.

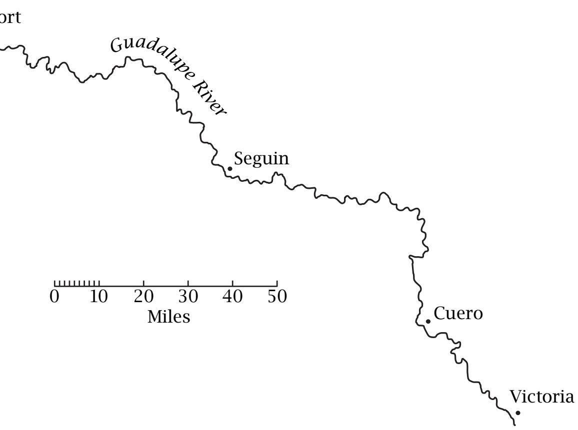


figure shows the Guadalupe River between the towns omfort and Victoria. As the crow flies, the two towns 150 miles apart. But the river is longer because it nders. In this Exploration, you will find out something rising about the actual length of the river.

Draw the bee-line distance from Comfort to Victoria. Do you agree that this distance is 150 miles?

____

If you use a ruler 150 miles long, the river would measure $N = 1$ ruler length. If you use a ruler only 50 miles long, then measuring the distance between points on the river, you will get *more* than three ruler lengths. Starting at Comfort, draw a line segment to a point on the river a direct distance of 50 miles from Comfort. Draw other 50-mile segments connecting points on the river until you reach Victoria. (The last segment will be a fraction of 50 miles.) How long does the river appear to be, using a 50-mile ruler? To two decimal places, how many ruler lengths is this?

This table shows the number of ruler lengths, N, as a function of the length of the ruler. Do you agree with N for a 20-mile ruler? __________

Ruler (mi)	*N* Pieces	Ratio, *r*	$\frac{1}{r}$
150	1.0		
50	3.16		
20	8.2		
10	17.2		

4. In the table, fill in the columns for the ratio

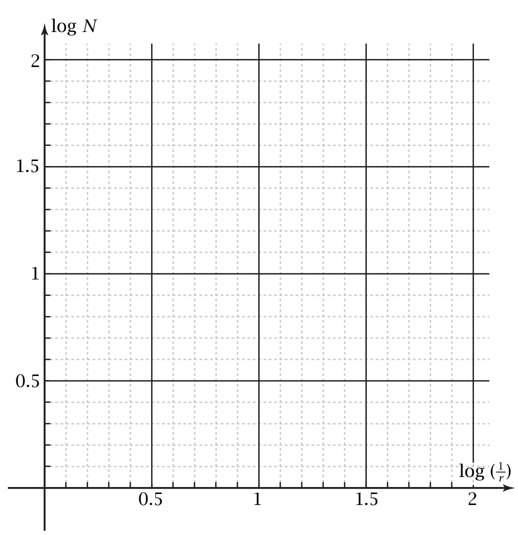
$$r = \frac{\text{ruler length}}{150}$$

and for the reciprocal, $\frac{1}{r}$.

5. Enter lists for $\frac{1}{r}$ and N on your grapher. Use these to calculate other lists containing values of $\log \frac{1}{r}$ and $\log N$. Write the values here, rounded to two decimal places.

$\log \frac{1}{r}$	$\log N$

6. Plot $\log N$ versus $\log \frac{1}{r}$ on this graph paper. If any points do not lie on a straight line, go back and check your work.



7. Perform linear regression to find the best-fitting linear function for $\log N$ as a function of $\log \frac{1}{r}$. Write the particular equation here.

B.6 Review

B.6.1 Chapter Review

B.6.2 Chapter Test

C. Vectors

C.1 Addition and Scalars

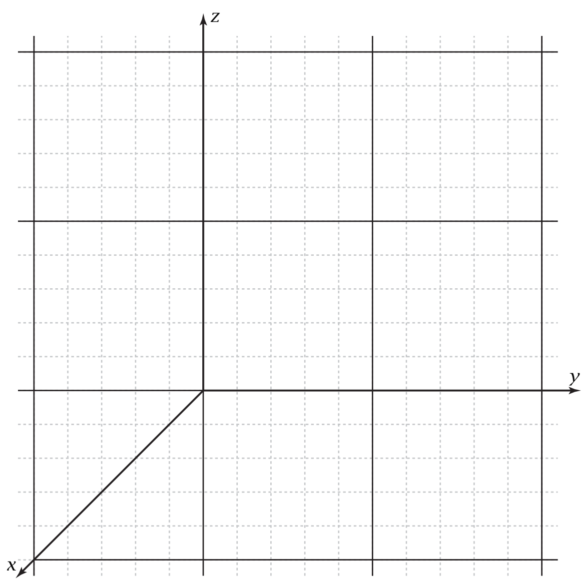
Name: ______________________ Group Members: ______________________

Exploration 10-3a: Introduction to Three-Dimensional Vectors

Date: __________

Objective: Plot and do operations with three-dimensional vectors by using their components.

1. Draw vector $\vec{v} = 5\vec{i} + 7\vec{j} + 10\vec{k}$ as a position vector in three-dimensional coordinates. Do this by plotting the three components head to tail, starting with $5\vec{i}$. Show the "box" surrounding the vector that makes the vector look three-dimensional.



2. Show that you know what the three unit vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ mean by sketching them starting from the origin in Problem 1.

3. Find the length of $\vec{v}$ in Problem 1 using the three-dimensional Pythagorean theorem.

4. Write the vector $3\vec{v}$ as a sum of components.

5. Show by means of the three-dimensional Pythagorean theorem that $3\vec{v}$ is really three times as

6. Write as a sum of components a *unit* vector in th direction of $\vec{v}$.

7. Let $\vec{a} = 4\vec{i} - 3\vec{j} + 8\vec{k}$. Quickly find $\vec{v} + \vec{a}$, where $\vec{v} = 5\vec{i} + 7\vec{j} + 10\vec{k}$, as in Problem 1.

8. Find displacement vector $\vec{b}$ from the point (1, 5, the point (7, 3, 11).

9. Find the position vector of the point 0.3 of the w from the point (1, 5, 2) to the point (7, 3, 11).

10. What did you learn as a result of doing this Exploration that you did not know before?

C.1.1 Unit Vectors

C.1.2 Magnitude

3d Pythagorus

C.1.3 Octants

C.2 Dot Products

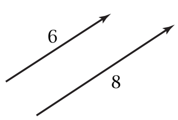
e: ______________________ Group Members: ______________________________________

ploration 10-4a: Introduction to the alar Product of Two Vectors

Date: __________

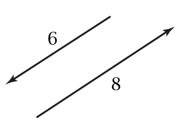
ective: With guidance from your instructor or text, find out the meaning of the inner uct of two vectors.

The figure shows two vectors pointing in the same direction, $\vec{a}$ 8 units long and $\vec{b}$ 6 units long. What is the **dot product,** $\vec{a} \cdot \vec{b}$ of these two vectors?



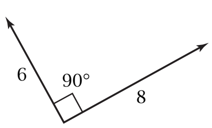
$\vec{a} \cdot \vec{b}$ = ____________________

The figure shows two vectors as in Problem 1 but with $\vec{b}$ pointing in the direction *opposite* $\vec{a}$. Find the dot product.



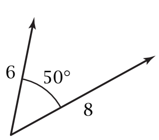
$\vec{a} \cdot \vec{b}$ = ____________________

The figure shows two vectors, $\vec{a}$ and $\vec{b}$, at right angles to each other. Find the dot product.



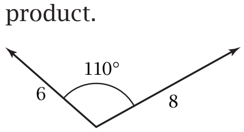
$\vec{a} \cdot \vec{b}$ = ____________________

The figure shows two vectors, $\vec{a}$ and $\vec{b}$, placed tail-to-tail, with an angle of 50° between them. Find the dot product.



$\vec{a} \cdot \vec{b}$ = ____________________

The figure shows two vectors with an angle of 110° between them when placed tail-to-tail. Find the dot product.



$\vec{a} \cdot \vec{b}$ = ____________________

6. State the definition of *dot product*.

7. Find:

   $\vec{i} \cdot \vec{i}$ = ________, $\vec{j} \cdot \vec{j}$ = ________, $\vec{k} \cdot \vec{k}$ = ________

   $\vec{i} \cdot \vec{j}$ = ________, $\vec{j} \cdot \vec{k}$ = ________, $\vec{i} \cdot \vec{k}$ = ________

8. Let $\vec{a} = 2\vec{i} + 5\vec{j} + 7\vec{k}$.

   Let $\vec{b} = 9\vec{i} + 3\vec{j} + 4\vec{k}$.

   Find $\vec{a} \cdot \vec{b}$.

9. Give two other names for *dot product*.

10. From what you found in Problem 8, there is an easy way to find the dot product of two vectors. Show that you have discovered this way by finding quickly the dot product $\vec{c} \cdot \vec{d}$.

    $\vec{c} = 4\vec{i} - 6\vec{j} + 9\vec{k}$

    $\vec{d} = 2\vec{i} + 5\vec{j} - 3\vec{k}$

C.2.1 angles

C.2.2 projection

C.2.3 Components

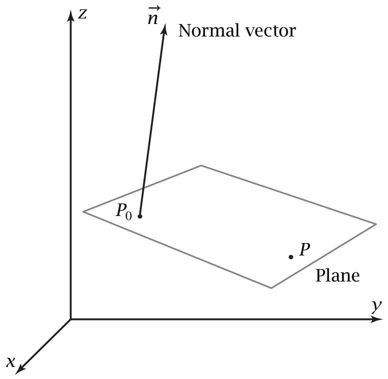
C.3 Plane, Lines, Parametric

Name: ______________________ Group Members: ______________________

Exploration 10-5a: Equation of a Plane Normal to a Vector

Date: __________

Objective: Given a point on a plane and a vector normal to the plane, find a Cartesian equation of the plane.



The figure shows, in general, a plane and its normal vector (vector perpendicular to the plane). Assume that the normal vector is $\vec{n} = 3\vec{i} + 7\vec{j} + 10\vec{k}$ and that it starts at point $P_0(8, 9, 4)$ on the plane.

1. Let $P(x, y, z)$ be a (variable) point on the plane. Draw displacement vector $\overrightarrow{P_0P}$.
2. Explain why the dot product $\vec{n} \cdot \overrightarrow{P_0P}$ equals 0.
3. Write $\overrightarrow{P_0P}$ in terms of the coordinates $P_0(8, 9, 4)$ and $P(x, y, z)$.
4. Substitute the answer to Problem 3 and the components of $\vec{n}$ into the equation $\vec{n} \cdot \overrightarrow{P_0P} = 0$. Simplify the result.
5. The equation in Problem 4 is a Cartesian equatio[n of] the plane shown. By observing the pattern in the answer, show how you can get the equation *quick*[ly.]
6. Use the equation of the plane to find the z-coordinate of the point for which $x = 5$ and $y =$
7. Find the x-intercept of the plane, the value of x where the plane crosses the x-axis.
8. Explain what is meant by a *variable* point in the plane.
9. What did you learn as a result of doing this Exploration that you did not know before?

C.3.1 Normal vectors

C.3.2 plane equations

C.4 Cross Products

e: ______________________ Group Members: ______________________

ploration 10-6a: Introduction to e Cross Product

Date: __________

ective: Discover the meaning of and way of computing the cross product of two vectors.

$\vec{a} = 3\vec{i} + 5\vec{j} + 7\vec{k}$.

$\vec{b} = 11\vec{i} + 2\vec{j} + 13\vec{k}$.

vector $\vec{c} = 51\vec{i} + 38\vec{j} - 49\vec{k}$ is the **cross product** of $\vec{a}$ $\vec{b}$, written $\vec{a} \times \vec{b}$. In this Exploration, it is your ctive to find out the meaning of *cross product* and it is calculated.

Find $|\vec{a}|$, $|\vec{b}|$, and $|\vec{a} \times \vec{b}|$. Does the length of the cross product vector equal the product of the lengths of the two factors?

Find the angle θ between $\vec{a}$ and $\vec{b}$ when they are placed tail-to-tail. Show that

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$$

Find $(\vec{a} \times \vec{b}) \cdot \vec{a}$ and $(\vec{a} \times \vec{b}) \cdot \vec{b}$. From the answers, what do you conclude about the direction of the cross product with respect to the direction of the two vectors being cross multiplied?

4. Look up the **right-hand rule** in Section 10-6. When all members of your group understand it, have one representative of your group demonstrate it to your instructor. ______________________

5. State the formal definition of *cross product*.

6. Explain why $\vec{i} \times \vec{i}$, $\vec{j} \times \vec{j}$, and $\vec{k} \times \vec{k}$ all equal zero.

7. Explain why $\vec{i} \times \vec{j} = \vec{k}$, $\vec{j} \times \vec{k} = \vec{i}$, and $\vec{k} \times \vec{i} = \vec{j}$.

8. Explain why $\vec{j} \times \vec{i}$ is the *opposite* of $\vec{i} \times \vec{j}$.

C.4.1 Right Hand Rule

C.5 Quaternions

C.5.1 Non-Commutative

C.6 Review

C.6.1 Chapter Review

C.6.2 Chapter Test

D. Conics

Rotated, Eccentricity

D.1 Introduction to Conics

Name: ______________________ Group Members: ______________________

Exploration 12-2b: Cartesian Equations of Conic Sections

Date: ________

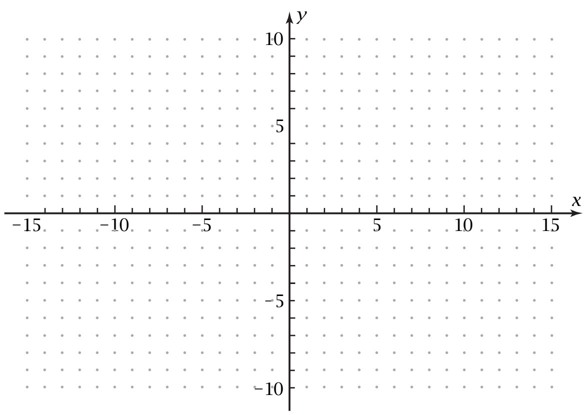
Objective: Sketch graphs of dilated and translated conic sections, and confirm by plotting parametrically.

1. For the equation

$$\left(\frac{x-7}{2}\right)^2 + \left(\frac{y+4}{5}\right)^2 = 1$$

which conic section will it be? ____________

2. Sketch the graph of the conic section in Problem 1.



3. Write parametric equations for the conic section in Problem 1.

4. Put your grapher in parametric and radian modes. Set the t-range from 0 to 2π, and use the window shown in Problem 2. Plot the graph. Does it agree with your sketch in Problem 2? ____________

5. Transform the equation in Problem 1 to the form

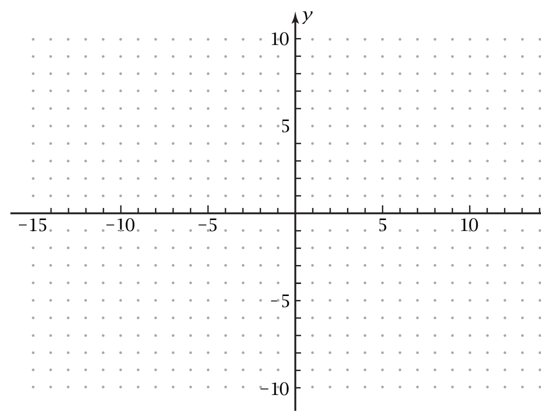
$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

7. For the equation

$$-\left(\frac{x+6}{4}\right)^2 + \left(\frac{y-1}{3}\right)^2 = 1$$

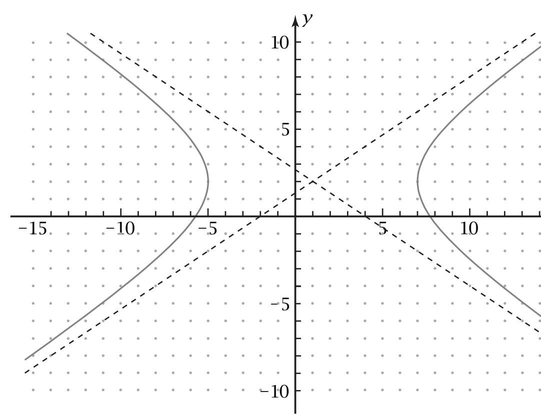
which conic section will it be? ____________

8. Sketch the graph of the conic section in Problem



9. Write parametric equations for the conic section Problem 6. Plot on your grapher. Does the graph agree with your sketch in Problem 8? ____________

10. Write parametric equations for the hyperbola graphed here. Do the parametric equations give t graph? ____________



11. Write a Cartesian equation for the hyperbola

D.1.1 Graphing form

of circles, parabolas, ellipses, and hyperbola

D.1.2 Algebra General Form

D.1.3 Using a Calculator

D.2 Algebra Manipulations

e: ______________________ Group Members: ______________________________________

ploration 12-2a: Parametric Equations Conic Sections

Date: __________

ective: Plot graphs of conic sections in parametric form, and relate the result to the esian equation.

Put your grapher in parametric mode and radian mode. Set a t-range of 0–2π and a window, with equal scales on the two axes, that has an x-range of [−10, 10]. Plot these **parametric equations** and sketch the result.

$x = \cos t$

$y = \sin t$

Square both sides of both equations in Problem 1. Then add the two equations, left side to left side and right side to right side. Use the Pythagorean property of circular functions to show that the result is equivalent to the **unit circle** $x^2 + y^2 = 1$.

Plot these parametric equations.

$x = 5 \cos t$

$y = 3 \sin t$

Describe verbally how the resulting **ellipse** is related to the unit circle in Problem 1.

Plot these parametric equations.

$x = 2 + 5 \cos t$

$y = -1 + 3 \sin t$

5. Plot the **unit hyperbola** with these parametric equations and sketch the result.

$x = \sec t$

$y = \tan t$

6. Square both sides of both equations in Problem 5, and then combine the two equations in a way that shows that $x^2 - y^2 = 1$.

7. Plot this unit hyperbola and explain how it is related to the one in Problem 5.

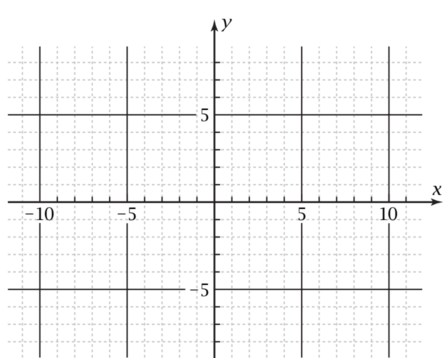
$x = \tan t$

$y = \sec t$

8. Plot this hyperbola. Sketch on the given axes.

$x = -4 + 3 \sec t$

$y = 1 + 2 \tan t$



D.2.1 Discriminant

D.2.2 Completing the Square

D.2.3 Cartesian Forms

D.2.4 Parametric Forms

D.2.5 Polar Forms

D.3 Rotated Conics

D.3.1 Rotated Polar Conics

D.3.2 Cartesian Rotation Equations

D.3.3 Parametric Rotation by Matrix

D.4 Eccentricity

D.4.1 Range of Eccentricity

D.4.2 Foci

D.4.3 Directrices

D.4.4 Distance Formulae

D.5 3D Conics

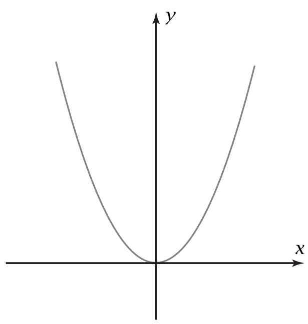
Name: ____________________ Group Members: ____________________

Exploration 12-3a: Quadric Surfaces

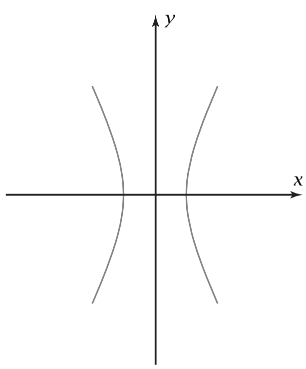
Date: ________

Objective: Sketch a figure formed by rotating a conic section about one of its axes.

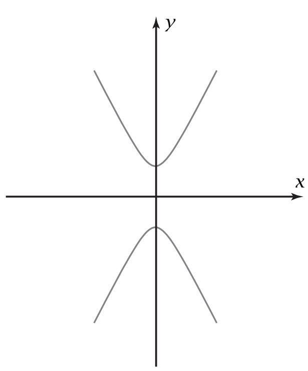
1. The graph shows the parabola $y = x^2$. Sketch the paraboloid formed by rotating this parabola about the y-axis.



2. The graph shows the hyperbola $4x^2 - y^2 = 4$. Sketch the hyperboloid formed by rotating this hyperbola about the y-axis.

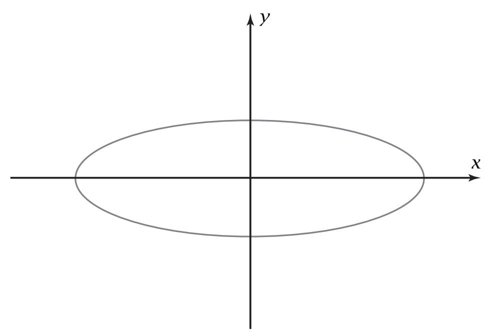


3. The graph shows the hyperbola $-4x^2 + y^2 = 1$. Sketch the hyperboloid formed by rotating this hyperbola about the y-axis.

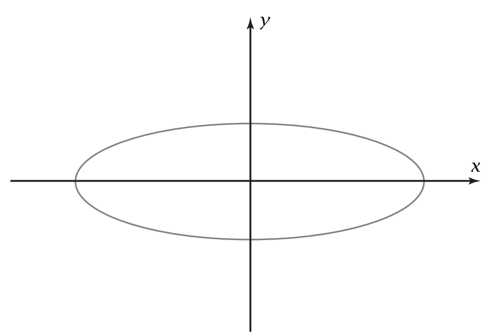


4. Why do you think the surface in Problem 2 is called a **hyperboloid of one sheet** and the surface in Problem 3 a **hyperboloid of two sheets**?

5. The graph shows the ellipse $x^2 + 9y^2 = 9$. Sketch the ellipsoid formed by rotating this ellipse about the x-axis. Do something to make it look three-dimensional.



6. On this copy of the ellipse in Problem 5, sketch the ellipsoid formed by rotating the ellipse about the y-axis.



7. What did you learn as a result of doing this Exploration that you did not know before?

D.5.1 2.5D

D.5.2 x, y, z

D.5.3 r, θ, z

D.5.4 ρ, ϕ, θ

D.6 Chapter Review

E. Solutions

E.1 Unit 1

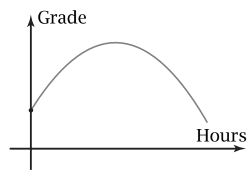
E.1.1 Chapter 1

§1.5

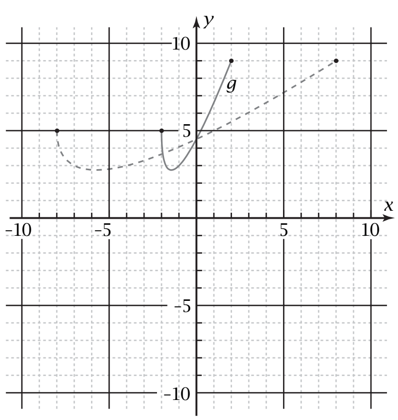
Solutions to Tests

Test 1 Form A

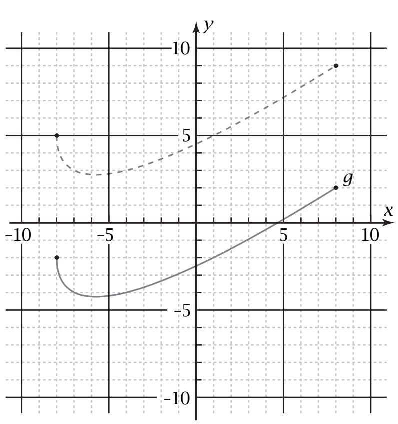
1.



2. Quadratic function
3. Power function (Accept cubic or polynomial.)
4. Exponential function
5. Range: $-4.9 \le y \le 4.1$
6. Rational algebraic function
7. Equation: $g(x) = f(4x)$



8. Vertical translation by −7

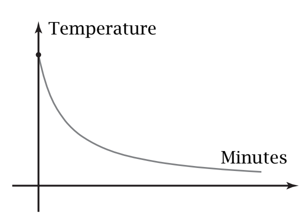


9. Enter $y_1 = x^2 - 2x - 3/(x \ge -1 \text{ and } x \le 4)$.
   The graph checks.
10. Horizontal translation by −8
    Equation: $g(x) = f(x + 8)$
    Check? Yes
11. Vertical dilation by a factor of $\frac{1}{4}$
    Equation: $g(x) = \frac{1}{4}f(x)$ (Accept $g(x) = \frac{1}{5}f(x)$.)
    Check? Yes
12. Horizontal dilation by a factor of 2
    Equation: $g(x) = f\left(\frac{1}{2}x\right)$
    Check? Yes
13. Vertical translation by −2
    Equation: $g(x) = f(x) - 2$
    Check? Yes
14. Horizontal translation by 5
    Vertical translation by 1
    Equation: $g(x) = f(x - 5) + 1$
    Check? Yes
15.

16. Adding five carts (6 − 1) increases the length by 57 inche
    (109 − 52). So each cart adds $\frac{57}{5} = 11.4$ inches.
17. $f(n) = 52 + 11.4(n - 1)$ or $f(n) = 40.6 + 11.4n$
18. Linear function
19. $f(15) = 211.6$ in.
20. $f(n) = 240$
    $40.6 + 11.4n = 240$
    $11.4n = 199.4$
    $n = 17.4912\ldots$
    17 carts (round downward)
21. Answers will vary.

Test 1 Form B

1.



E.1.2 Chapter 2

§2.1

E.1.3 Chapter 3

§3.1

E.2 Unit 2

E.2.1 Chapter 5

§5.1

E.2.2 Chapter 7

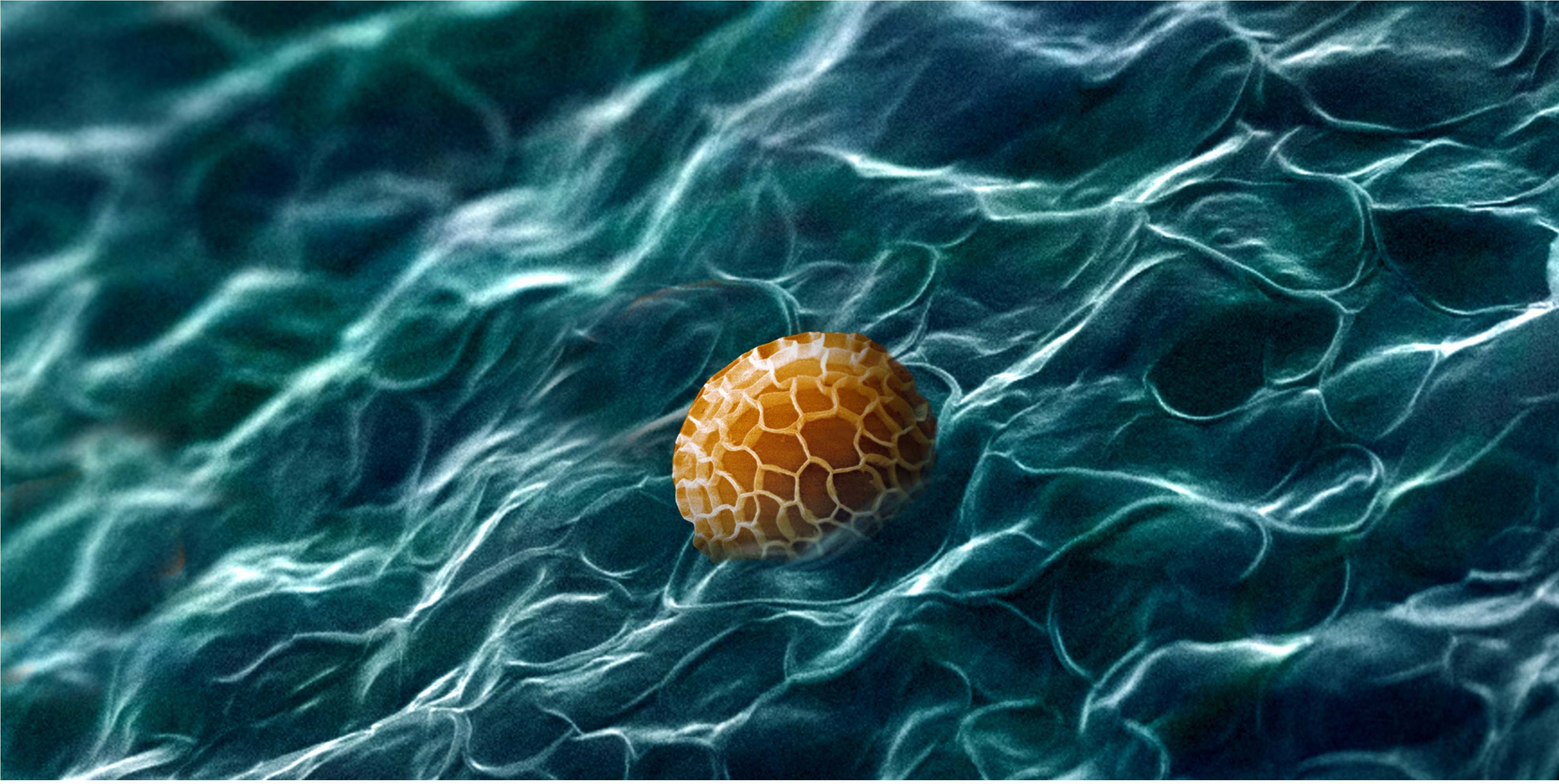
§7.1

§7.3

E.2.3 Chapter 8

§8.2

Addenda



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Glossary

A

algebraic Involving only the operations addition, subtraction, multiplication, division, powers, and roots a finite number of times. 7

C

Cartesian plane A coordinate system, uniquely identifying every point thereupon by a pair of numerical coordinate, which are the signed distances from two, perpendicular axes. 5

D

dependent variable The output of a function, equation, or formula. 5, 6

discrete Not continuous. That is, dealing with countable sets, and therefore excluding calculus and analysis. 6

domain The set of values that the independent variable of a function can have. For Reals, this is typically only limited to exclude even roots of negative numbers and division by zero. 5, 6

E

exponential A transcendental function, where the independent variable appears as an exponent. An untranslated, general equation form is $a \cdot b^x$, where a and b are constants. 18

extrapolation The act of estimating values outside of given data points via a function. 5

F

function A relationship between two variable quantities for which there is exactly one value of the dependent variable for each value of the independent variable in the domain. 5, 6

function notation A way of writing functions invented by Leonhard Euler in 1734, where the function is named first, followed by in the input variables in parentheses, separated by commas. This is equated then to the function. In this notation, inverse functions are written $f^{-1}(x)$. 5

G

graph In two dimensions, the representation of the collection of all ordered pairs $(x, f(x))$, in the form of a curve on a Cartesian plane, together with the axes, and other labels. 6, 17

I

independent variable The input of a function, equation, or formula. 5, 6

indeterminate form A quasi-numeric state of a limit that does not yield enough information to constitute an answer. Typical forms are $0/0, \infty/\infty, 0 \cdot \infty, \infty - \infty, 0^0, 1^\infty, \infty^0$. 47

interpolation The act of estimating values between given data points via a function. 5

L

linear An algebraic function, equivalent to a polynomial of degree 1. The general equation is $ax + b$, where a and b are constants. The graph is a straight line. The numerical data displays an add-add pattern. Verbally, one variable is directly proportional to another, but with a potentially different starting point. 17

logarithm The inverse of an exponential function. $y = \log_a x$ is equivalent to $a^y = x$. 18

M

mathematical model A function which is used to describe and/or predict a real-world phenomenon, expressing relationships between quantities. 5

P

π The ratio of the circumference of circle to its diameter. 7

Q

quadratic An algebraic function, equivalent to a polynomial of degree 2. The general equation is $ax^2 + bx + c$, where a, b, c are constants and $a \neq 0$. The graphical shape is called a parabola. The numerical data displays an add-second difference pattern. Verbally, one variable is directly proportional to the square of another. 17

R

range The set of all values of the dependent variable that correspond to values of the independent variable in the domain. 5, 6

S

set A well-defined collection of distinct objects, as well as an object in its own right. 6