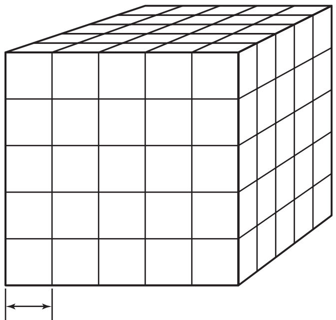
Name: ____________________ Group Members: ____________________

Exploration 11-6a: Hausdorff's Definition of (Fractal) Dimension

Date: __________

Objective: Learn a definition of the concept of *dimension*, and apply it to self-similar figures generated iteratively.

Here is a (three-dimensional) self-similar cube of edge 1 unit. It is cut into smaller cubes of edge $r = \frac{1}{5}$ unit. Note that the smaller cubes are similar to the original cube.



1. How many small cubes will there be? __________
2. Write the answer to Problem 1 as a power of 5.

   ____

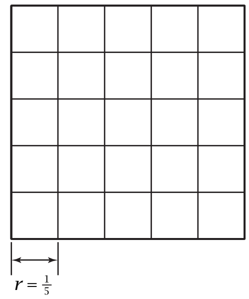
3. What physical quantity does the exponent in Problem 2 represent?
4. Suppose that the cube in Problem 1 is divided into N smaller cubes, each of edge r units. Write N in terms of r.
5. Transform the equation $\left(\frac{1}{r}\right)^3 = N$ so that 3 is by itself on the left and N and r are on the right. (Recall the properties of logarithms!)

The results of Problem 5 are the basis for the definition of dimension, D, of a self-similar object. It was proposed by Felix Hausdorff, who lived from 1868 to 1942.

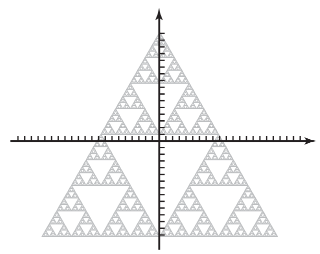
$$D = \frac{\log N}{\log \left(\frac{1}{r}\right)}$$

where N is the number of self-similar pieces into which an object can be divided and r is the length of a given piece as a fraction of the length of the original pre-image. In order for this definition to apply, you must be able to subdivide the object infinitely.

6. Apply Hausdorff's definition to the square in the next column and thereby show that a square is (as you would expect) two-dimensional.



7. In a previous exercise, you generated Sierpiński's gasket, shown here.



On the drawing, show that the gasket is self-similar by dividing it into three congruent pieces, each of which is similar to the whole figure. Then show how one of these pieces can be divided into three other pieces, each of which is also similar to the whole figure.

8. Because the subdividing into self-similar pieces can go on infinitely, Hausdorff's definition of dimension applies. Calculate the dimension of the Sierpiński gasket. Surprising?
9. The original pre-image in Problem 7 is a triangle of base 20 and altitude 30. What is its area? What is the total area of the first iteration? the second iteration? What is the total area of the tenth iteration? the 100th iteration? What limit do the total areas of the iterations approach as the number of iterations approaches infinity? Surprising?
10. What did you learn as a result of doing this Exploration that you did not know before?