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THE ERGODICITY OF THE COLLATZ PROCESS IN POSITIVE INTEGER FIELD

BOJIN ZHENG, YANGQIAN SU, HONGRUN WU, AND LI KUANG

1. INTRODUCTION

Since 1930s, researchers have deeply investigated the $3x + 1$ problem. Until now, the $3x + 1$ problem has obtained many names, such as the Kakutani's problem, the Syracuse problem and Ulam's problem and so on [11, 9].

The $3x + 1$ problem can be stated from the viewpoint of computer algorithm as follows:

For any given integer x , if x is odd, then let $x := 3x + 1$; if x is even, then let $x := x/2$; if we repeat this process, x will certainly be 1 at some time.

Mathematically, this problem can be presented as the iterations of a function $f(x)$, called the Collatz function shown as equation (1.1), i.e., $\forall x, \exists k, f^k(x) = 1, x \in \mathbb{N}^+$. Here, \mathbb{N}^+ is the set of positive integers.

$$(1.1) \quad f(x) = \begin{cases} 3x + 1, & \text{if } x \text{ is odd} \\ \frac{x}{2}, & \text{if } x \text{ is even} \end{cases} \quad x > 0$$

This problem is very hard to solve because the iteration process is very “random”, although the Collatz function is deterministic.

In spite of the difficulty of this problem, the researchers still attained many fruitful achievements [4]. From the view of probability theory [5], the researchers explored the existence of divergent trajectories; from the view of number theory and diophantine approximations and other mathematical tools, the researchers discussed the existence of the cycles other than $4 \rightarrow 2 \rightarrow 1$ [10, 7, 1, 14, 15, 3, 8]; from the perspective of mathematical logic and theory of algorithms, the researchers studied the solvability of this problem [6, 16]. Moreover, this problem was also tried from the view of the fractal [18, 17], graph theory [2] and computation experiments [13, 11, 4] and so on. Owing to the efforts of Prof. J. C. Lagarias, the related works were collected and commented [12].

In this paper, we treated the Collatz function as a deterministic program (process), and generalized it to the non-deterministic program and set up three models; furthermore, we mapped the programs to the Collatz graphs. By the proposed models and the graph theory, we proved that the Collatz conjecture holds, i.e., all the positive integers can reach 1.

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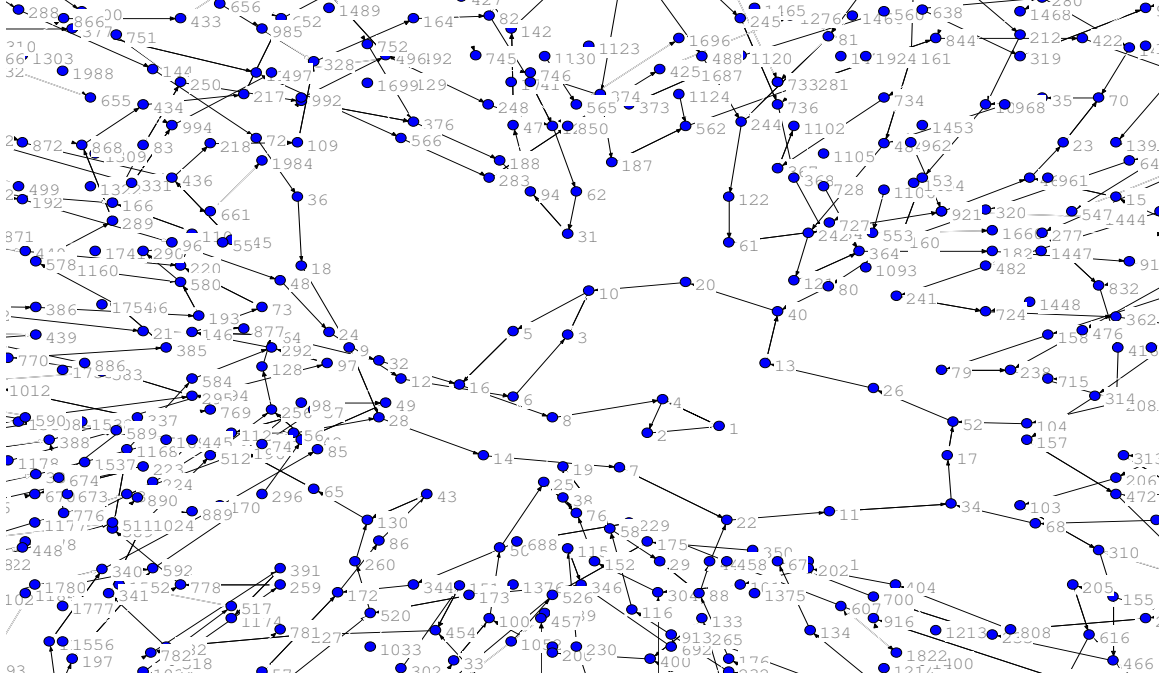


FIGURE 1. The original Collatz graph of model M0

2. MODELS

The Collatz problem, which we called model $M0$, can be mapped to the Collatz graph [9] as shown in Fig. 1.

In Fig. 1, every positive integer is a node. Every node has two directed edges. Every edge responds to an item of Collatz function, which represents a transformation of the value of the variable x .

From the viewpoint of Collatz graph, the Collatz problem can be stated as follows: for any given node, i.e., positive integer A , there exists an $A \Rightarrow 1$ path in Collatz graph.

If there is an $A \Rightarrow 1$ path in Collatz graph, then we say A is reachable. All the reachable positive integers will form a set Ω_0 , thus, the Collatz conjecture can also be stated as $\Omega_0 = \mathbb{N}^+$.

To solve this problem, we generalized the problem to non-deterministic process and set up three models. The first model is named MS , the second is named $M1$ and the third is named $M2$. Recall that the original problem is called $M0$.

In the model MS , we extended the Collatz function to three items, i.e., adding a non-deterministic item that is inverse to the $3x + 1$ item without the constraint of parity, shown in equation (2.1).

$$(2.1) \quad f_s(x) = \begin{cases} 3x + 1, & \text{if } x \bmod 2 \equiv 1 \\ \frac{x}{2}, & \text{if } x \bmod 2 \equiv 0 \\ \frac{x-1}{3}, & \text{if } x \bmod 3 \equiv 1 \end{cases} \quad x > 0$$

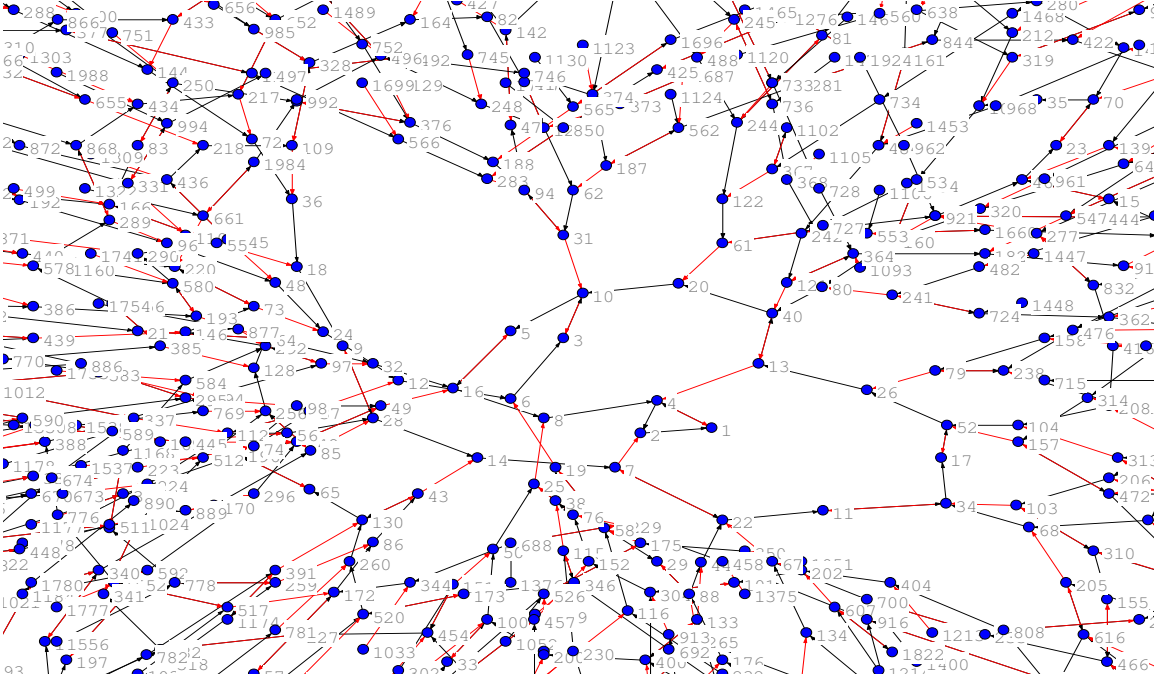


FIGURE 2. The Collatz graph of model MS

By using a similar method to that of the Collatz graph, we can draw a graph to reflect equation (2.1) as Fig. 2. In honor of L. Collatz, we called the generalized graphs as the Collatz graphs.

From Fig. 2, we can see that every positive integer, i.e., every node, has a few options to connect to the other nodes. For examples, node 7 can connect to 22 by $3x + 1$ or connect to 2 by $\frac{x-1}{3}$; node 10 can connect to 3 by $\frac{x-1}{3}$ and 5 by $\frac{x}{2}$. Therefore, the nodes in the Collatz graphs can be categorized into a few classes. Or say, the Collatz graphs have fruitful patterns.

In Fig. 2, the red edges are different edges in contrast to Fig. 1 of model $M0$.

Like $M0$, if there exists an $A \Rightarrow 1$ path, we say A is reachable. We denote the reachable set of MS as Ω_s .

In the second model $M1$, we removed more constraints. The function of $M1$ is presented as equation (2.2).

$$(2.2) \quad f_1(x) = \begin{cases} 3x + 1 \\ \frac{x}{2}, & \text{if } x \bmod 2 \equiv 0 \\ 2x \\ \frac{x-1}{3}, & \text{if } x \bmod 3 \equiv 1 \end{cases} \quad x > 0$$

Compared with the mode MS , model $M1$ adds a new item, i.e., $2x$, which is inverse to the item $x/2$.

With respect to equation (2.2), the Collatz graph can be shown as in Fig. 3.

The structure of Fig. 3 of model $M1$ is the same as Fig. 2 of model MS except that: some edges in Fig. 2 are one-way, and all the edges in Fig. 3 are two-way.

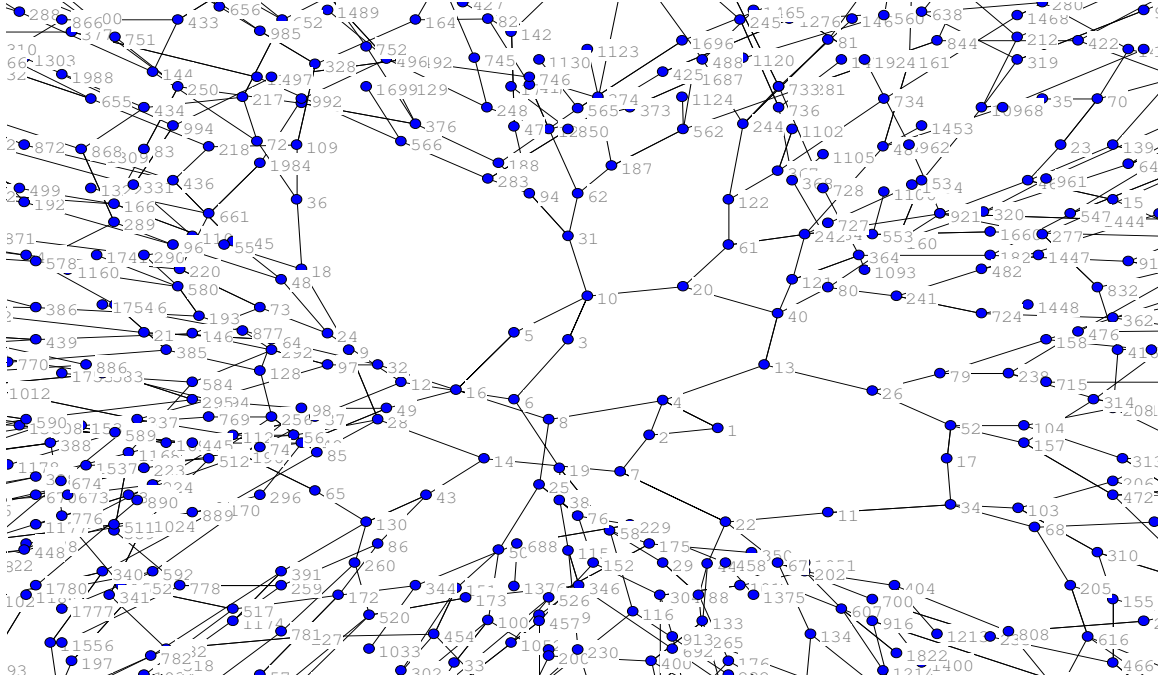


FIGURE 3. The Collatz graph of model M1

Similarly, we denote the reachable set of $M1$ as Ω_1 .

According to equation (2.2), $f_1(x)$ has four items, which we call “actions”.

Definition 2.1 (Action). An action is an optional transformation of functions, i.e., an item of functions.

We use T for $3x + 1$; B for $x/2$; F for $(x - 1)/3$ and D for $2x$, respectively.

According to the functions, action T is inverse to F , and B is inverse to D .

We further generalize $M1$ to the third model $M2$, which can be formulated as equation (2.3).

$$(2.3) \quad f_2(x) = \begin{cases} 3x + 1 \\ \frac{x}{2} \\ 2x \\ \frac{x-1}{3} \end{cases} \quad x > 0$$

Obviously, in model $M2$, the functional values can be the rational numbers, i.e., $M2$ has many nodes with the rational values.

Similarly, We denoted the reachable set of $M2$ as Ω_2 .

After the introduction of the proposed models, we further explored the properties of the proposed models. From model $M2$, we got some valuable clues for $M1$, and further we got the structural features of $M1$ and MS . Finally, we proved that $\Omega_0 = \mathbb{N}^+$.

3. THE MODEL M_2

Theorem 3.1 (The succession theorem). *The action sequence 'TDDFFBBT' is the succession function for any given positive integer, i.e., equation (3.1) always holds for any given positive integer.*

$$(3.1) \quad 'TDDFFBBT'(x) = TBBFFDDT(x) = x + 1$$

Proof. The calculations are listed as follows,

$$\begin{aligned} 'TDDFFBBT'(x) &= TBBFFDDT(x) \\ &= \frac{(x*3+1)*2*2-1}{3} * \frac{1}{2} * \frac{1}{2} * 3 + 1 \\ &= \frac{4x}{3} * \frac{1}{4} * 3 + 1 \\ &= x + 1 \end{aligned}$$

□

Lemma 3.2. *Every positive integer is reachable in M_2 , i.e., $\Omega_2 = \mathbb{N}^+$.*

According to the succession theorem, M_2 demonstrates a spiral structure. The positive integers are the central pillars.

Theorem 3.3 (The 2 successions theorem). *The action sequence 'DFFBTT' is the succession of succession function for any positive integer, i.e., equation (3.2) always holds for any given positive integer.*

$$(3.2) \quad 'DFFBTT'(x) = TTBBFFD(x) = x + 2$$

Proof. The calculations are listed as follows,

$$\begin{aligned} 'DFFBTT'(x) &= TTBBFFD(x) \\ &= \left(\frac{\frac{x*2-1}{3}-1}{3} * \frac{1}{2} * 3 + 1 \right) * 3 + 1 \\ &= \left(\frac{2x-4}{9} * \frac{3}{2} + 1 \right) * 3 + 1 \\ &= x + 2 \end{aligned}$$

□

According to the 2 successions theorem, there often exist shorter action sequences to transform a number to another number in M_2 .

Theorem 3.4 (The 3 successions theorem). *The action sequence 'DDFFBBTT' is the succession of succession of succession function for any given positive integer, i.e., equation (3.3) always holds for any given positive integer.*

$$(3.3) \quad 'DDFFBBTT'(x) = TTBBFFDD(x) = x + 3$$

Proof. The calculations are listed as follows,

$$\begin{aligned} 'DDFFBBTT'(x) &= TTBBFFDD(x) \\ &= \left(\frac{\frac{x*2*2-1}{3}-1}{3} * \frac{1}{2} * \frac{1}{2} * 3 + 1 \right) * 3 + 1 \\ &= \left(\frac{4x-4}{9} * \frac{3}{4} + 1 \right) * 3 + 1 \\ &= x + 3 \end{aligned}$$

□

Theorem 3.5 (The 4 successions theorem). *The action sequence 'TDDFDDFF-BBBBTT' is the succession of succession of succession of succession function for any positive integer, i.e., equation (3.4) always holds for any given positive integer.*

$$(3.4) \quad 'TDDFDDFFBBBBTT'(x) = TTBBBBFFDDFDDT(x) = x + 4$$

Proof. The calculations are listed as follows,

$$\begin{aligned} & 'TDDFDDFFBBBBTT'(x) = TTBBBBFFDDFDDT(x) \\ & = \left(\frac{\frac{(3x+1)*2*2-1}{3} - 1}{3} * \frac{1}{2} * \frac{1}{2} * \frac{1}{2} * \frac{1}{2} * 3 + 1 \right) * 3 + 1 \\ & = \left(\frac{\frac{(4x+1)*4-1}{3} - 1}{16} + 1 \right) * 3 + 1 \\ & = x + 4 \end{aligned}$$

□

Because all the edges are two-way, there exists the precursor theorems corresponding to the succession theorems.

Of course, there exists more than one action sequence to perform the arbitrary successions and precursors.

Besides, these theorems above will be used in the model $M1$ to demonstrate the structures of $M1$ and MS .

4. THE MODEL $M1$

Compared with $M2$, model $M1$ only eliminates all non-integers from $M2$. Here, we also need to prove that all positive integers are reachable in $M1$, i.e., $\Omega_1 = \mathbb{N}^+$.

From now, we use a new notational method to represent a positive integer to facilitate the calculation. Basically, we use the 3-based numeral system.

For any given positive integer A represented in the 3-based numeral system, if A accepts a ' T ' action, the value would be $A1$. Therefore, we use $A1$ to represent the number $3A + 1$. Formally, we use $A11$ to represent a number like $(9(A)_3 + 4)_{10}$.

To represent the carry in the 3-based numeral system, we use $(A + 1)11$ to represent a number like $((A)_3 * 9 + 9 + 4)_{10}$.

We also use A^D to represent the double value of A , i.e., the value after action " D ", and A^D11 to represent a number like $(2 * (A)_3 * 9 + 4)_{10}$.

We also use A_1 to represent $\lfloor A/2 \rfloor$, A_2 to $\lfloor \lfloor A/2 \rfloor / 2 \rfloor$, and A_3 to $\lfloor \lfloor \lfloor A/2 \rfloor / 2 \rfloor / 2 \rfloor$. Obviously, A_1 is the value of A after a ' B ' action with consideration of the carry.

As to the ' F ' action, we only need to erase the last ' 1 ' symbol of ' $A1$ '.

Moreover, for positive integers A and C , if there is an action sequence that can transform A to C , we denote it as $A \Rightarrow C$.

Definition 4.1 (9-cluster). For any given positive integer A , the set $\{A00, A01, A02, A10, A11, A12, A20, A21, A22\}$, i.e., in decimal, $\{9k + 0, 9k + 1, 9k + 2, 9k + 3, 9k + 4, 9k + 5, 9k + 6, 9k + 7, 9k + 8\}$ is called a 9-cluster. Here, $(A)_3 = (k)_{10}$.

Definition 4.2 (5-cluster). For any given positive integer A , the set $\{A00, A01, A02, A10, A11\}$, i.e., in decimal, $\{9k + 0, 9k + 1, 9k + 2, 9k + 3, 9k + 4\}$ is called a 5-cluster. Here, $(A)_3 = (k)_{10}$.

Definition 4.3 (3-cluster). For any given positive integer A , the set $\{A12, A20, A21\}$, i.e., $\{9k + 5, 9k + 6, 9k + 7\}$ is called a 3-cluster. Here, $(A)_3 = (k)_{10}$.

The proof procedure can be organized as follows:

- (1) first we prove that every 5-cluster can form an internally connected subgraph from Lemma 4.4 to Theorem 4.12.
- (2) next we prove that every 3-cluster can form an internally connected subgraph from Lemma 4.14 to 4.24 and Theorem 4.41 .
- (3) then we prove that every 3-cluster can connect to its corresponding 5-cluster.
- (4) also we prove that every 9-cluster can form an internally connected subgraph.
- (5) at last we show that all 9-clusters can connect to 1.

Actually, there exist simpler proofs on $\Omega_1 = \mathbb{N}^+$. However, to illustrate the structure of the Collatz graphs, we used the proof method stated above.

Here, we firstly prove that every 5-cluster can form an internally connected subgraph.

Lemma 4.4. *The action sequence 'TDDFFBBT' can transform A10 to A11.*

The proof of $A10 \Rightarrow A11$. The calculations are listed as follows,

$$\begin{aligned}
 T(A10) &= A101 \\
 D(A101) &= A^D 202 \\
 D(A^D 202) &= (A^{DD} + 1)111 \\
 F((A^{DD} + 1)111) &= (A^{DD} + 1)11 \\
 F((A^{DD} + 1)11) &= (A^{DD} + 1)1 \\
 B((A^{DD} + 1)1) &= A^D 2 \\
 B(A^D 2) &= A1 \\
 T(A1) &= A11
 \end{aligned}$$

□

Lemma 4.5. *The action sequence $(\text{'TDDFFBBT'})^{-1} = \text{'FDDTTBBF'}$ can transform A11 to A10.*

Because the procedure is inverse to that in Lemma 4.4 and all edges in $M1$ is two-way, we omit the detailed proof procedure. Moreover, we will omit the proof procedure on all inverse lemmas.

Lemma 4.6. *The action sequence 'DFFBTT' can transform A02 to A11.*

The proof of $A02 \Rightarrow A11$. The calculations are listed as follows,

$$\begin{aligned}
 D(A02) &= A^D 11 \\
 F(A^D 11) &= A^D 1 \\
 F(A^D 1) &= A^D \\
 B(A^D) &= A \\
 T(A) &= A1 \\
 T(A1) &= A11
 \end{aligned}$$

□

Lemma 4.7. *The action sequence $(\text{'DFFBTT'})^{-1} = \text{'FFDTTB'}$ can transform A11 to A02.*

Lemma 4.8. *The action sequence 'DDFFBBTT' can transform A01 to A11.*

The proof of $A01 \Rightarrow A11$. The calculations are listed as follows,

$$D(A01) = A^D 02$$

$$\begin{aligned}
D(A^D 02) &= (A^{DD})11 \\
F(A^{DD} 11) &= A^{DD} 1 \\
F(A^{DD} 1) &= A^{DD} \\
B(A^{DD}) &= A^D \\
B(A^D) &= A \\
T(A) &= A1 \\
T(A1) &= A11
\end{aligned}$$

□

Lemma 4.9. *The action sequence $(\text{'DDFFBBT'})^{-1} = \text{'FFDDTTBB'}$ can transform $A11$ to $A01$.*

Lemma 4.10. *The action sequence 'TDDFDDFFBBBBTT' can transform $A00$ to $A11$.*

The proof of $A00 \Rightarrow A11$. The calculations are listed as follows,

$$\begin{aligned}
T(A00) &= A001 \\
D(A001) &= A^D 002 \\
D(A^D 002) &= A^{DD} 011 \\
F(A^{DD} 011) &= A^{DD} 01 \\
D(A^{DD} 01) &= A^{DDD} 02 \\
D(A^{DDD} 02) &= A^{DDDD} 11 \\
F(A^{DDDD} 11) &= A^{DDDD} 1 \\
F(A^{DDDD} 1) &= A^{DDDD} \\
B(A^{DDDD}) &= A^{DD} \\
B(A^{DD}) &= A^D \\
B(A^D) &= A \\
T(A) &= A1 \\
T(A1) &= A11
\end{aligned}$$

□

Lemma 4.11. *The action sequence $(\text{'TDDFDDFFBBBBTT'})^{-1} = \text{'FFDDDDTTTBBTBBF'}$ can transform $A11$ to $A00$.*

Theorem 4.12 (The 5-cluster connection theorem). *For any given positive integer A , nodes $A00$, $A01$, $A02$, $A10$ and $A11$ are internally connected.*

This theorem follows from Lemmas 4.4, 4.5, 4.6, 4.7, 4.8, 4.9, 4.10 and 4.11.

Theorem 4.13 (The 5-cluster attaching theorem). *For any given positive integer A , there is at least a path from A to a 5-cluster.*

Proof. $T(A) = A1$

$$T(A1) = A11$$

□

From the 5-cluster attaching theorem, the action sequence 'TT' can assure that arbitrary positive integer is connected to at least one 5-cluster.

Here, we prove that the 3-clusters are internally connected.

Lemma 4.14. *The action sequence 'TDDFFBBT' can transform $A20$ to $A21$.*

The proof of $A20 \Rightarrow A21$. The calculations are listed as follows,

$$\begin{aligned}
T(A20) &= A201 \\
D(A201) &= (A^D + 1)102
\end{aligned}$$

$$\begin{aligned}
D((A^D + 1)102) &= ((A^D + 1)^D)211 \\
F(((A^D + 1)^D)211) &= (A^D + 1)^D 21 \\
F((A^D + 1)21) &= (A^D + 1)^D 2 \\
B((A^D + 1)^D 2) &= (A^D + 1)1 \\
B((A^D + 1)1) &= A2 \\
T(A2) &= A21
\end{aligned}$$

□

Lemma 4.15. *The action sequence $(\text{'TDDFFBBT'})^{-1} = \text{'FDDTTBBF'}$ can transform $A21$ to $A20$.*

Lemma 4.16. *The action sequence 'DDDDFFBBTBT' can transform $A12$ to $A21$.*

The proof of $A12 \Rightarrow A21$. The calculations are listed as follows,

$$\begin{aligned}
D(A12) &= (A^D + 1)01 \\
D((A^D + 1)01) &= (A^D + 1)^D 02 \\
D((A^D + 1)^D 02) &= (A^D + 1)^{DD} 11 \\
F((A^D + 1)^{DD} 11) &= (A^D + 1)^{DD} 1 \\
F((A^D + 1)^{DD} 1) &= (A^D + 1)^{DD} \\
B((A^D + 1)^{DD}) &= (A^D + 1)^D \\
B((A^D + 1)^D) &= (A^D + 1) \\
T((A^D + 1)) &= (A^D + 1)1 \\
B((A^D + 1)1) &= A2 \\
T(A2) &= A21
\end{aligned}$$

□

Lemma 4.17. *The action sequence $(\text{'DDDDFFBBTBT'})^{-1} = \text{'FDFDDTTBBB'}$ can transform $A21$ to $A12$.*

Theorem 4.18 (The 3-cluster connection theorem). *For any given positive integer A , nodes $A12$, $A20$ and $A21$ are internally connected.*

According to Lemma 4.14, 4.15, 4.16 and 4.17.

From now on, we will prove that every 3-cluster can connect to its corresponding 5-cluster.

Lemma 4.19. *When A is even, there exists at least one action sequence to transform $A21$ to $A11$, i.e., $A21 \Rightarrow A11$.*

The proof of $A21 \Rightarrow A11$. The calculations are listed as follows,

$$\begin{aligned}
T(A21) &= A211 \\
B(A211) &= A_1 102 \\
\text{'DFFBTT'}(A_1 102) &= A_1 111 \text{ (Lemma 4.6)} \\
FFF(A_1 111) &= A_1 \\
\text{'DTT'}(A_1) &= A11
\end{aligned}$$

□

Lemma 4.20. *When A is even, there exists at least one sequence to transform $A11$ to $A21$, i.e., $A11 \Rightarrow A21$.*

Lemma 4.21. *When A is odd, and $A = R0$, i.e., the last symbol of A is '0', there exists at least one action sequence to transform $A21$ to $A11$, i.e., $R021 \Rightarrow R011$.*

The proof of $A21 \Rightarrow A11$. The calculations are listed as follows,

$$\begin{aligned}
& R021 \Rightarrow R^D112 \text{ (D)} \\
& \Rightarrow R^D11211 \text{ (TT)} \\
& \Rightarrow R02102 \text{ (B)} \\
& \Rightarrow R02111 \text{ (Lemma 4.6)} \\
& \Rightarrow R02 \text{ (FFF)} \\
& \Rightarrow R11 \text{ (Lemma 4.6)} \\
& \Rightarrow R01 \text{ (Lemma 4.9)} \\
& \Rightarrow R011 \text{ (T)}
\end{aligned}$$

□

Lemma 4.22. When A is odd, and $A = R0$, there exists at least one action sequence to transform $A11$ to $A21$, i.e., $R011 \Rightarrow R021$.

Lemma 4.23. When A is odd, and $A = R1$, there exists at least one action sequence to transform $A21$ to $A11$, i.e., $R121 \Rightarrow R111$.

The proof of $A21 \Rightarrow A11$. The calculations are listed as follows,

$$\begin{aligned}
& R121 \Rightarrow R112 \text{ (Lemma 4.17)} \\
& \Rightarrow R11211 \text{ (TT)} \\
& \Rightarrow R_102102 \text{ (B, here } R \text{ is even because } A \text{ is odd and } A = R1) \\
& \Rightarrow R_102111 \text{ (FFF)} \\
& \Rightarrow R_102 \text{ (D)} \\
& \Rightarrow R11 \text{ (T)} \\
& \Rightarrow R111 \text{ (T)}
\end{aligned}$$

□

Lemma 4.24. When A is odd, and $A = R1$, there exists at least one action sequence to transform $A11$ to $A21$, i.e., $R111 \Rightarrow R121$.

Here, we firstly discuss the relationship between $A22$ and $A11$ and then come back to discuss the circumstance when A is an odd and $A = R2$.

Lemma 4.25. When A is even, there exists at least one action sequence to transform $A22$ to $A11$, i.e., $A22 \Rightarrow A11$.

The proof of $A22 \Rightarrow A11$. The calculations are listed as follows,

$$\begin{aligned}
& B(A22) = A_111 \\
& 'FFD TT'(A_111) = A11
\end{aligned}$$

□

Lemma 4.26. When A is even, there exists at least one sequence to transform $A11$ to $A22$, i.e., $A11 \Rightarrow A22$.

Lemma 4.27. When A is odd, and $A = R0$, there exists at least one action sequence to transform $A22$ to $A11$, i.e., $R022 \Rightarrow R011$.

The proof of $R022 \Rightarrow R011$. The calculations are listed as follows,

$$\begin{aligned}
& R022 \Rightarrow R^D121 \text{ (D)} \\
& \Rightarrow R^D112 \text{ (Lemma 4.17)} \\
& \Rightarrow R021 \text{ (B)} \\
& \Rightarrow R02 \text{ (F)} \\
& \Rightarrow R11 \text{ (Lemma 4.6)} \\
& \Rightarrow R01 \text{ (Lemma 4.9)} \\
& \Rightarrow R011 \text{ (T)}
\end{aligned}$$

□

Lemma 4.28. *When A is odd, and $A = R0$, there exists at least one action sequence to transform $A11$ to $A22$, i.e., $R011 \Rightarrow R022$.*

Lemma 4.29. *When A is odd, and $A = R1$, there exists at least one action sequence to transform $A22$ to $A11$, i.e., $R122 \Rightarrow R111$.*

The proof of $R122 \Rightarrow R111$. The calculations are listed as follows,

$$\begin{aligned} R122 &\Rightarrow R1011 \\ &\Rightarrow R101 \\ &\Rightarrow R02 \\ &\Rightarrow R11 \\ &\Rightarrow R111 \end{aligned}$$

□

Lemma 4.30. *When A is odd, and $A = R1$, there exists at least one action sequence to transform $A11$ to $A22$, i.e., $R111 \Rightarrow R122$.*

Lemma 4.31. *There exists at least one action sequence to transform $R0 \overbrace{2 \cdots 2}^n$ to $R0 \overbrace{2 \cdots 2}^{n+1}$, i.e., $R0 \overbrace{2 \cdots 2}^n \Rightarrow R0 \overbrace{2 \cdots 2}^{n+1}$.*

Proof. The calculations are listed as follows,

$$\begin{aligned} R0 \overbrace{2 \cdots 2}^n &\Rightarrow R0 \overbrace{2 \cdots 21}^n \\ &\Rightarrow R^D \overbrace{12 \cdots 212}^{n-1} \\ &\Rightarrow R^D \overbrace{12 \cdots 221}^{n-1} \\ &\Rightarrow R0 \overbrace{2 \cdots 222}^{n-1} \Rightarrow R0 \overbrace{2 \cdots 2}^{n+1} \end{aligned}$$

□

Lemma 4.32. *There exists at least one action sequence to transform $R0 \overbrace{2 \cdots 2}^{n+1}$ to $R0 \overbrace{2 \cdots 2}^n$, i.e., $R0 \overbrace{2 \cdots 2}^{n+1} \Rightarrow R0 \overbrace{2 \cdots 2}^n$.*

Lemma 4.33. *There exists at least one action sequence to transform $R1 \overbrace{2 \cdots 2}^n$ to $R1 \overbrace{2 \cdots 2}^{n+1}$, i.e., $R1 \overbrace{2 \cdots 2}^n \Rightarrow R1 \overbrace{2 \cdots 2}^{n+1}$.*

Proof. The calculations are listed as follows,

$$\begin{aligned} \text{When } R \text{ is even,} \\ R1 \overbrace{2 \cdots 2}^n &\Rightarrow R1 \overbrace{2 \cdots 21}^n \\ &\Rightarrow R1 \overbrace{02 \cdots 22}^n \\ &\Rightarrow R1 \overbrace{02 \cdots 221}^n \\ &\Rightarrow R1 \overbrace{02 \cdots 212}^n \end{aligned}$$

$$\Rightarrow R \overbrace{12 \cdots 2}^n 21$$

$$\Rightarrow R \overbrace{12 \cdots 2}^{n+1}$$

When R is odd, let $R = (P + 1)$,

$$(P + 1) \overbrace{12 \cdots 2}^n$$

$$\Rightarrow P_1 \overbrace{21 \cdots 1}^n$$

$$\Rightarrow P_1 \overbrace{21 \cdots 1}^{n+1}$$

$$\Rightarrow (P + 1) \overbrace{12 \cdots 2}^{n+1}$$

Therefore, for any given R , $R \overbrace{12 \cdots 2}^n \Rightarrow R \overbrace{12 \cdots 2}^{n+1}$

□

Lemma 4.34. *There exists at least one action sequence to transform $R \overbrace{12 \cdots 2}^{n+1}$ to $R \overbrace{12 \cdots 2}^n$, i.e., $R \overbrace{12 \cdots 2}^{n+1} \Rightarrow R \overbrace{12 \cdots 2}^n$.*

Theorem 4.35 (The 2 appending theorem). *There exists at least one action sequence to transform $R \overbrace{2 \cdots 2}^n$ to $R \overbrace{2 \cdots 2}^{n+1}$, i.e., $R \overbrace{2 \cdots 2}^n \Rightarrow R \overbrace{2 \cdots 2}^{n+1}$.*

According to Lemma 4.31 to 4.33, this theorem is obvious.

Theorem 4.36 (The 2 backspace theorem). *There exists at least one action sequence to transform $R \overbrace{2 \cdots 2}^{n+1}$ to $R \overbrace{2 \cdots 2}^n$, i.e., $R \overbrace{2 \cdots 2}^{n+1} \Rightarrow R \overbrace{2 \cdots 2}^n$.*

According to Theorem 4.35, this theorem is obvious.

Lemma 4.37. *When A is odd, and $A = R2$, there exists at least one action sequence to transform $A22$ to $A11$, i.e., $R222 \Rightarrow R211$.*

$R222 \Rightarrow R211$. '0R' should include at least one '1' or '0' before a series of '2'.

Therefore, $R222 \Rightarrow R2 \Rightarrow R211$, or say, $A22 \Rightarrow A \Rightarrow A11$.

□

From Lemma 4.25 to 4.37, we can obtain a conclusion as Theorem 4.38.

Theorem 4.38. *For any given positive integer A , there exists at least an action sequence to transform $A22$ to $A11$, i.e., $A22 \Rightarrow A11$.*

This proposition follows from Lemma 4.25 to 4.37.

Lemma 4.39. *For any given positive integer A , there exists at least an action sequence to transform $A11$ to $A22$.*

Now, we can come back to discuss $A21$ when A is odd.

Lemma 4.40. *When A is odd, and $A = R2$, there exists at least one action sequence to transform $A21$ to $A11$, i.e., $R221 \Rightarrow R211$.*

Proof of $R221 \Rightarrow R211$. The calculations are listed as follows,

$$\begin{aligned} R221 &\Rightarrow R22 \\ &\Rightarrow R2 \\ &\Rightarrow R211 \end{aligned}$$

□

Theorem 4.41. *For any given positive integer A , there exists at least one action sequence to transform $A21$ to $A11$.*

Lemma 4.42. *For any given positive integer A , there exists at least one action sequence to transform $A11$ to $A21$.*

According to Theorems 4.38, 4.41 and 4.12, we can obtain Theorem 4.43.

Theorem 4.43. *For any given positive integer A , the corresponding 9-cluster is internally connected.*

So we can prove that every positive integer can reach $(11)_3$, i.e., $(4)_{10}$; Of course, can also connect to 1.

Theorem 4.44. *For any given positive integer A , there exists at least one action sequence to transform A to 11.*

Proof. If the numbers of symbols A is odd, then let $A = 0A$, i.e., add an additional 0 to the head of A , to make the numbers of symbols is even.

$A ** \Rightarrow A11 \Rightarrow A$, here $*$ is arbitrary one of '0', '1' and '2'.

By repeating this process, we obtain $A \Rightarrow 11$.

□

According to the proofs above, we can obtain more conclusions.

Lemma 4.45. *For any given positive integer A , there exists at least one action sequence H in $M1$ such that $H(A) < A$.*

Lemma 4.46. *For any given positive integer A , there exists at least one action sequence to transform A to 1, i.e., $\Omega_1 = \mathbb{N}^+$.*

Because $M1$ can be mapped into Fig. 3, Theorem 4.44 and Lemma 4.46 indicate that there is at least one path from any given positive integer to 4 or 1.

Theorem 4.47 (The node loop existence theorem). *For any given positive integer A , A is in a loop.*

Proof. 1) If A is odd, $A \Rightarrow A111 \Rightarrow A1202 \Rightarrow A1211 \Rightarrow A12 \Rightarrow A1$. Because $A_1 < A$ and $A_1 \Rightarrow 1$, therefore, A is in a loop.

2) If A is even, $A \Rightarrow A11 \Rightarrow A102 \Rightarrow A111 \Rightarrow A1$. Because $A_1 < A$ and $A_1 \Rightarrow 1$, therefore, A is in a loop.

□

5. THE MODEL MS

Compared with model $M1$, all nodes in $M1$ are still in MS . However, some edges of model MS become directed. That is, the Collatz graph in Fig. 2 is a weakly connected graph and all the positive integers are weakly connected in it.

If we can prove that, for any given positive integer A , there exists an action sequence H such that $H(A) < A$, then there should exist an path from A to 1, i.e., $\Omega_s = \mathbb{N}^+$, because 1 is the smallest value.

Lemma 5.1. *For any given A , there exists an action sequence H , such that $H(A2) < A2$.*

Proof. If A is even, then $B(A2) = A_11$, $F(A_11) = A_1$. Because $A_1 < A2$, this proposition holds.

If A is odd, then $T(A2) = A21$, $B(A21) = A_122$. For any given $k \geq 1$, if A_k is even, then $B \overbrace{BT \cdots BT}^k(A2) = A_{k+1} \overbrace{1 \cdots 1}^{k+1}$. Further, $\overbrace{F \cdots F}^{k+1}(A_{k+1} \overbrace{1 \cdots 1}^{k+1}) = A_{k+1}$. Because $A_{k+1} < A2$, this proposition holds.

Because there exists a k such that $A_k = 0$, this proposition holds. \square

Lemma 5.2. *For any given A , there exists an action sequence H , such that $H(A1) < A1$.*

Proof. $F(A1) = A < A1$ \square

Lemma 5.3. *For any given A , there exists an action sequence H , such that $H(A0) < A0$.*

Proof. If A is even, $B(A0) = A_10 < A0$, this proposition holds. If A is odd, $T(A0) = A01$, $B(A01) = A_112$. For A_112 , if A_1 is odd, $BF(A_112) = A_22 < A0$, this proposition holds; if A_1 is even, $TB(A_112) = A_2022$. According to Lemma 5.1, this proposition also holds.

Therefore, this proposition always holds. \square

Theorem 5.4 (The descending theorem). *For any given positive integer A , there exists an action sequence H in MS such that $H(A) < A$.*

Proof. According to Lemmas 5.3, 5.2 and 5.1, this theorem holds. \square

Theorem 5.5 (The edge loop existence theorem). *For any given positive even integer A , the edge $A \leftarrow A1$ is in a loop.*

Proof. Because A is even, $A1 \Rightarrow A11 \Rightarrow A102$.

- (1) If A_1 is even, $A102 \Rightarrow A_201 \Rightarrow A_20$. Because $A_20 < A$ and $A_1 \Rightarrow 1$, the edge $A \leftarrow A1$ is in a loop.
- (2) If A_1 is odd, $A102 \Rightarrow A_1021 \Rightarrow A_2122$.
- (3) If A_2 is odd, $A_2122 \Rightarrow A_3211 \Rightarrow A_32$. Because $A_32 < A$, this theorem holds.
- (4) If A_2 is even, $A_2122 \Rightarrow A_3222$. By repeating this analysis process, we obtain $A_3222 \Rightarrow A_{k+1}$ because the result should finally be even in some time k . Since $A_{k+1} < A$, this theorem holds.

Therefore, the edge $A \leftarrow A1$ is in a loop. \square

Lemma 5.6. *For any given positive integer A , there exists a path from A to 1.*

Lemma 5.7. *In model MS , $\Omega_s = \mathbb{N}^+$.*

Definition 5.8 (The removable edge). In Fig. 2 (model MS), when an edge e is removed, Ω_s does not change, then the edge e is said removable.

Theorem 5.9 (The de-looping theorem). *For any given edge e , if e belongs to the edge set E_1 indicated by $\frac{x-1}{3}$ if $x \bmod 6 \equiv 1$, then e is removable.*

Proof. According to Theorem 5.5, and notice that the edge e has a different direction to the other path is the loop, therefore, e is removable. \square

6. THE MODEL M_0

The graph of M_0 is a directed graph. Compared with the model MS , model M_0 eliminates all the edges indicated by the action F .

Lemma 6.1 (The Collatz conjecture). *For any given positive integer A , A is reachable in M_0 .*

Proof. According to Lemma 5.7, all positive integers which exist in Fig. 2 (model MS) still exist in Fig. 1 (model M_0).

According to the de-looping theorem in model MS , all these edges in E_1 can be removed one by one starting from $7 \rightarrow 2$.

After the removals of the edge set E_1 , all the edges belonging to the edge set E_4 , which is indicated by $\frac{x-1}{3}$ if $x \bmod 6 \equiv 4$, would be abundant (have no successions nodes) in MS , and hence are removable.

After the removals of the edge sets E_1 and E_4 , MS becomes M_0 . Therefore, the Collatz conjecture holds. \square

Theorem 6.2. *There is only one circle $4 \rightarrow 2 \rightarrow 1 \rightarrow 4$ in M_0 .*

Proof. Because every node in M_0 has only an out-link, according to graph theory and Fig. 1 (model M_0) is a connected graph, only the positive integer 1 has an extra out-link, and it out-links to the positive integer 4, so only one circle $4 \rightarrow 2 \rightarrow 1 \rightarrow 4$ exists. \square

7. CONCLUSIONS

This paper proves the $3x + 1$ problem. The result shows that all the positive integers can be transformed to 1 by the iteration of f . Equivalently, there are no other cycles other than $4 \rightarrow 2 \rightarrow 1 \rightarrow 4$ cycle and there is no divergent trajectories.

The result in this paper would be useful to the research of chaos [18, 17], computer science [6], complex systems and so on.

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ABSTRACT. The $3x + 1$ problem, also called the Collatz conjecture, is a very interesting unsolved mathematical problem related to computer science. This paper generalized this problem by relaxing the constraints, i.e., generalizing this deterministic process to non-deterministic process, and set up three models. This paper analyzed the ergodicity of these models and proved that the ergodicity of the Collatz process in positive integer field holds, i.e., all the positive integers can be transformed to 1 by the iterations of the Collatz function.

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