Analyzing whether Mathematical induction method could be used in Collatz conjecture

Preprint · April 2018

DOI: 10.13140/RG.2.2.17650.81602/1

Mercedes Orús-Lacort

Analyzing Collatz conjecture with mathematical complete induction method

Mercedes Orús–Lacort
C/ Escorial 29, 08024 Barcelona, Spain. Email: mercedes.orus@gmail.com

June 12, 2018

Abstract

In this paper we show a demonstration of the Collatz conjecture using the mathematical complete induction method. We show that this conjecture is satisfied for the first values of natural numbers, and analyzing the sequence generated by the odd numbers, we can deduce a formula for the general term of Collatz sequence, for any n odd natural number after several iterations. This formula is used in one case that we analyze in the process of demonstration by mathematical complete induction method.

Key Words: Number theory, Collatz conjecture.

AMS Classification: 11–02

1 Introduction

Collatz conjecture is one of the best-known unsolved problems in sequences and series of number theory. It states:

"If we start with any positive integer n, then each sequence term is obtained from the previous term as follows: if the previous term is even, the next term is one half the previous term. Otherwise, the next term is 3 times the previous term plus 1. And we will always reach 1 (and therefore cycle 4, 2, 1) for any number n that we start with".

The conjecture is named after Lothar Collatz introduced the idea in 1937, two years after receiving his doctorate [1].

It is also known as the $3n+1$ conjecture, the Ulam conjecture (after Stanisław Ulam), Kakutani's problem (after Shizuo Kakutani), the Thwaites conjecture (after Sir Bryan Thwaites), Hasse's algorithm (after Helmut Hasse), or the Syracuse problem [2] [3]; the sequence of numbers involved is referred to as the hailstone sequence or hailstone numbers (because the values are usually subject to multiple descents and ascents like hailstones in a cloud) [4] [5], or as wondrous numbers [6].

Although the conjecture has not been proven, many mathematicians have studied it, and have achieved important results, see Refs. from [7] to [26]. Most mathematicians who have looked into the problem think the conjecture is true, because experimental evidence and heuristic arguments support it [16] [18].

In this paper we show a demonstration of the Collatz conjecture using the mathematical complete induction method.

We show that this conjecture is satisfied for the first values of natural numbers. From this analysis we can deduce a formula for the general term of Collatz sequence for any n odd natural number after several iterations, and this formula is used in one case that we analyze in the process of demonstration by mathematical complete induction method.

The paper is organized as follows: in Sec.2 we show that the conjecture is satisfied for the first values of natural numbers, and we deduce a formula for the general term of Collatz sequence for any n odd natural number, and after several iterations. In Sec.3 we show a demonstration of the Collatz conjecture using the mathematical complete induction method. Finally, in Sec.4 we wrap up our conclusions.

2 Sequences for the first natural numbers and formula for the general term of Collatz sequence for any n odd natural number

Formally, each term of the sequence of numbers is equivalent to applying the following function to n, and to each term of the sequence:

$$f(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ 3n+1, & \text{if } n \text{ is odd} \end{cases}$$

Therefore, given any natural number, we can consider its orbit, that is, the successive images when iterating the function, in the following way.

For example, if $n = 13$:

$$\begin{aligned} x_1 &= f(13) = 3 \cdot 13 + 1 = 40, \\ x_2 &= f(40) = \frac{40}{2} = 20, \\ x_3 &= f(20) = \frac{20}{2} = 10; \text{ etc.} \end{aligned} \tag{1}$$

The conjecture says that we will always reach 1 (and therefore cycle 4, 2, 1) for any natural number that we start with.

Let us see what happens with the first natural numbers.

For $n = 1$

$$\begin{aligned} x_1 &= f(1) = 3 \cdot 1 + 1 = 4, \\ x_2 &= f(4) = \frac{4}{2} = 2, \\ x_3 &= f(2) = \frac{2}{2} = 1. \end{aligned} \tag{2}$$

For $n = 2$

$$x_1 = f(2) = \frac{2}{2} = 1. \tag{3}$$

For $n = 3$

$$\begin{aligned}
x_1 &= f(3) = 3 \cdot 3 + 1 = 10, \\
x_2 &= f(10) = \frac{10}{2} = 5, \\
x_3 &= f(5) = 3 \cdot 5 + 1 = 16, \\
x_4 &= f(16) = \frac{16}{2} = 8, \\
x_5 &= f(8) = \frac{8}{2} = 4, \\
x_6 &= f(4) = \frac{4}{2} = 2, \\
x_7 &= f(2) = \frac{2}{2} = 1.
\end{aligned} \tag{4}$$

Therefore, for the first values of n, we observe that the conjecture is satisfied. And it is also satisfied for many others greater than 3, and for numbers power of 2 we always reach 1 too, dividing successively by 2.

Given that, at a certain point in the demonstration process by mathematical complete induction method we will need a formula that would represent the general term of Collatz sequence for any n odd natural number after several iterations, we will see bellow how this formula would be.

If n is an odd natural number, using the definition of $f(n)$, the first iteration would be:

$$x_1 = 3n + 1.$$

Since $3n + 1$ is always an even number, the next iteration would be:

$$x_2 = \frac{3n + 1}{2}.$$

The next iteration will depend on whether the result obtained previously is even or odd, and since in the following or next iterations we can obtain an even number, let us assume that an even number is obtained in the following $r_1 - 1$ iterations, until we obtain again an odd number ($r_1 \geq 1$). Therefore, the result obtained after $r_1 - 1$ iterations would be:

$$x_{1+r_1} = \frac{3n + 1}{2^{r_1}}.$$

Now, if $\frac{3n + 1}{2^{r_1}}$ is an odd number, the next iteration would be:

$$x_{r_1+2} = 3 \cdot \frac{3n+1}{2^{r_1}} + 1$$
$$x_{r_1+2} = \frac{3^2 n + 3 + 2^{r_1}}{2^{r_1}}.$$

Since $\dfrac{3^2 n + 3 + 2^{r_1}}{2^{r_1}}$ is an even number, the next iteration would be:

$$x_{r_1+3} = \frac{3^2 n + 3 + 2^{r_1}}{2^{r_1+1}}.$$

And for the same reason as before, we will assume that in the next $r_2 - 1$ iterations we obtain an even number, until we obtain again an odd number ($r_2 \geq 1$). Therefore the result would be:

$$x_{r_1+r_2+2} = \frac{3^2 n + 3 + 2^{r_1}}{2^{r_1+r_2}}.$$

And so on, following the same reasoning, and after several iterations, as for example $r_1 + r_2 + ... + r_k + k$ iterations, we would obtain that the term $x_{r_1+r_2+...+r_k+k}$ of the sequence would be:

$$x_{r_1+r_2+...+r_k+k} = \frac{3^k n + 3^{k-1} + 3^{k-2} 2^{r_1} + 3^{k-3} 2^{r_1+r_2} + ... + 2^{r_1+r_2+...+r_{k-1}}}{2^{r_1+r_2+...+r_k}}. \tag{5}$$

This formula will be used in the next section, when we analyze a specific case using the mathematical complete induction method.

3 Proof of Collatz conjecture using mathematical complete induction

We need to prove that for all $n \in \mathbb{N}$, the sequence obtained reaches 1.

In the previous section, we have seen that it is true for values of n from 1 to 3. And it is proven to be true also for values higher than $n = 3$, notice that it is also true for example for all $n = 2^s$, $s \in \mathbb{N}$, because we always reach 1 dividing successively by 2.

To apply the method of mathematical complete induction, we will assume that for a certain $m \in \mathbb{N}$, big enough, and for any other natural number less than m we can reach with successive iterations the number 1.

Hence, if we can prove that it is true for $m + 1$, we can conclude that it is true for all $n \in \mathbb{N}$.

Therefore, in our induction hypothesis we assume that for a large enough $m \in \mathbb{N}$, and for any other natural number less than m it is true.

Let us see below if it is true also for $m + 1$. Notice that $m + 1$ can be odd or even number, depending on if m is even or odd number. So, we will analyze both cases:

Case 1: $m + 1$ is an even number

If $m + 1$ is an even number it is because m is an odd number, that is, $m = 2t + 1$, for $t \in \mathbb{N}$. So, $m + 1 = 2t + 2$ and the first iteration applying the definition of $f(n)$ would be:

$$x_1 = \frac{2t+2}{2} = t+1.$$

Notice that $t+1 < 2t+1 = m$, and as we were assuming that for all natural numbers less or equal than m we reach number 1, then for $t+1$ we reach number 1.

Therefore, if $m+1$ is an even number, the sequence obtained reaches 1.

Case 2: $m+1$ is an odd number

If $m+1$ is an odd number it is because m is an even number, that is, $m = 2t$, for $t \in \mathbb{N}$, and $t > 1$. So, $m+1 = 2t+1$ and the first iteration applying the definition of $f(n)$ would be:

$$x_1 = 3 \cdot (2t+1) + 1 = 6t+4.$$

And given that $6t+4$ is an even number, the next iteration would be:

$$x_2 = \frac{6t+4}{2} = 3t+2.$$

At this point, note that the next iteration depends on whether t is an even or odd number; if t is an even number then $3t+2$ will be an even number, however if t is an odd number then $3t+2$ will be an odd number.

Next, we will analyze both options:

Option 1: t is an even number:

If t is an even number, then $3t+2$ is an even number, and the next iteration applying the definition of $f(n)$ would be:

$$x_3 = \frac{3t+2}{2}.$$

And notice that $\frac{3t+2}{2} \leq 2t = m$, because $t \geq 2$, and as we were assuming that for all natural numbers less or equal than m we reach number 1, then for $\frac{3t+2}{2}$ we can reach number 1, and so we can reach number 1 for $m+1$.

Therefore, if $m+1$ is an odd number and t an even number, the sequence obtained reaches 1.

Option 2: t is an odd number:

If t is an odd number, then $3t+2$ is an odd number, and now using the formula Eq.5 that represents the general term of Collatz sequence for any odd natural number after several iterations, we have that after $r_1 + r_2 + ... + r_k + k$ iterations, we would obtain:

$$x_{r_1+r_2+...+r_k+k} = \frac{3^k(3t+2) + 3^{k-1} + 3^{k-2}2^{r_1} + 3^{k-3}2^{r_1+r_2} + ... + 2^{r_1+r_2+...+r_{k-1}}}{2^{r_1+r_2+...+r_k}}. \tag{6}$$

Rewriting the above formula we have:

$$\frac{3^k t + 3^{k-1} + 3^{k-2}2^{r_1} + 3^{k-3}2^{r_1+r_2} + ... + 2^{r_1+r_2+...+r_{k-1}}}{2^{r_1+r_2+...+r_k}} + \frac{3^k(2t+2)}{2^{r_1+r_2+...+r_k}}. \quad (7)$$

At this point, notice that for $m = 2t$, t an odd natural number, we were assuming that the sequence reaches number 1, and calculating the first terms for the sequence of $m = 2t$ we have:

$$\begin{aligned} x_1 &= \frac{2t}{2} = t, \\ x_2 &= 3t + 1, \\ x_3 &= \frac{3t+1}{2}. \end{aligned} \quad (8)$$

The next iteration will depend on whether the result obtained previously is even or odd, and since in the following or next iterations we can obtain an even number, using the formula Eq.5 and after the next $s_1 + s_2 + ... + s_u + u - 3$ iterations we would obtain:

$$x_{s_1+s_2+...+s_u+u} = \frac{3^u t + 3^{u-1} + 3^{u-2}2^{s_1} + 3^{u-3}2^{s_1+s_2} + ... + 2^{s_1+s_2+...+s_{u-1}}}{2^{s_1+s_2+...+s_u}}. \quad (9)$$

If we call $S = \sum_{i=1}^{u} s_i$, and since for $m = 2t$ we were assuming that the sequence reaches number 1, it means that the limit of x_{S+u} when S and u tend to infinity is equal to 1.

Thus, if we call $R = \sum_{i=1}^{k} r_i$ in equation Eq.7, and we calculate the limit when R and k tend to infinity, we have:

$$\lim_{(R,k)\to(\infty,\infty)} \left(\frac{3^k t + 3^{k-1} + 3^{k-2}2^{r_1} + 3^{k-3}2^{r_1+r_2} + ... + 2^{r_1+r_2+...+r_{k-1}}}{2^{r_1+r_2+...+r_k}} + \frac{3^k(2t+2)}{2^{r_1+r_2+...+r_k}}\right) =$$

$$\lim_{(R,k)\to(\infty,\infty)} \frac{3^k t + 3^{k-1} + 3^{k-2}2^{r_1} + 3^{k-3}2^{r_1+r_2} + ... + 2^{r_1+r_2+...+r_{k-1}}}{2^{r_1+r_2+...+r_k}} + \lim_{(R,k)\to(\infty,\infty)} \frac{3^k(2t+2)}{2^{r_1+r_2+...+r_k}} =$$

$$\lim_{(S,u)\to(\infty,\infty)} \frac{3^u t + 3^{u-1} + 3^{u-2}2^{s_1} + 3^{u-3}2^{s_1+s_2} + ... + 2^{s_1+s_2+...+s_{u-1}}}{2^{s_1+s_2+...+s_u}} + \lim_{(R,k)\to(\infty,\infty)} \frac{3^k(2t+2)}{2^{r_1+r_2+...+r_k}} =$$

$$= 1 + \lim_{(R,k)\to(\infty,\infty)} \frac{3^k(2t+2)}{2^R}.$$

Now, notice that $R \geq k$, then $R = k + v$, for $v \in \mathbb{N}$, and since we are assuming that m is a large enough number, we can assume without losing any generality that $v \geq k$. So we have, $v = k + w$ for $w \in \mathbb{N}$, $R = 2k + w$, and the above limit when R and k tend to infinity would be calculated as:

$$1 + \lim_{(R,k)\to(\infty,\infty)} \frac{3^k(2t+2)}{2^R} = 1 + \lim_{k\to\infty} \frac{3^k(2t+2)}{2^{2k+w}} = 1 + \lim_{k\to\infty} (\frac{3}{4})^k \frac{2t+2}{2^w} = 1 + 0 = 1.$$

Thus, for $m + 1$ an odd number, and t an odd number, the sequence obtained also reaches 1.

Therefore, for $m + 1$ even or odd number, the sequence reaches 1, and so it means that for all n natural number the Collatz sequence always reaches 1, as we want to prove.

4 Conclusions

We have shown how to use mathematical complete induction method to prove Collatz conjecture. We show that this conjecture is satisfied for the first values of natural numbers. From this analysis we can deduce a formula for the general term of Collatz sequence, for any n odd natural number, after several iterations. This formula is used in one case that we analyze during the demonstration by mathematical complete induction. And using mathematical complete induction method, we have shown that Collatz conjecture is true.

Acknowledgements: we acknowledge Jean-Claude Evard and Roman Orus for proposing the dissemination of this result.

References

[1] O'Connor, J.J.; Robertson, E.F. (2006). "Lothar Collatz". St Andrews University School of Mathematics and Statistics, Scotland.

[2] Maddux, Cleborne D.; Johnson, D. Lamont (1997). Logo: A Retrospective. New York: Haworth Press. p. 160. ISBN 0-7890-0374-0. The problem is also known by several other names, including: Ulam's conjecture, the Hailstone problem, the Syracuse problem, Kakutani's problem, Hasse's algorithm, and the Collatz problem".

[3] According to Lagarias (1985),[3] p. 4, the name "Syracuse problem" was proposed by Hasse in the 1950s, during a visit to Syracuse University.

[4] Pickover, Clifford A. (2001). Wonders of Numbers. Oxford: Oxford University Press. pp. 116–118. ISBN 0-19-513342-0.

[5] "Hailstone Number". MathWorld. Wolfram Research.

[6] Hofstadter, Douglas R. (1979). Godel, Escher, Bach. New York: Basic Books. pp. 400–2. ISBN 0-465-02685-0.

[7] Guy, Richard K. (2004). "E17: Permutation Sequences". Unsolved problems in number theory (3rd ed.). Springer–Verlag. pp. 336–7. ISBN 0-387-20860-7. Zbl 1058.11001.

[8] Guy, R. K. (1983). "Don't try to solve these problems". Amer. Math. Monthly. 90: 35–41. doi:10.2307/2975688. JSTOR 2975688.

[9] Lagarias, Jeffrey C., ed. (2010). The ultimate challenge: the $3x + 1$ problem. Providence, R.I.: American Mathematical Society. p. 4. ISBN 0821849409.

[10] Leavens, Gary T.; Vermeulen, Mike (December 1992). "$3x + 1$ Search Programs". Computers and Mathematics with Applications. 24 (11): 79–99. doi:10.1016/0898-1221(92)90034-F.

[11] Roosendaal, Eric. "$3x + 1$ Delay Records". Retrieved 27 November 2011. (Note: "Delay records" are total stopping time records).

[12] Simons, J.; de Weger, B. (2003). "Theoretical and computational bounds for m–cycles of the $3n + 1$ problem" (PDF). Acta Arithmetica. 117 (1): 51–70. doi:10.4064/aa117-1-3.

[13] Steiner, R. P. (1977). "A theorem on the syracuse problem". Proceedings of the 7th Manitoba Conference on Numerical Mathematics. pp. 553–9.

[14] Simons, John L. (2005). "On the nonexistence of 2–cycles for the 3x+1 problem". Math. Comp. 74: 1565–72. doi:10.1090/s0025-5718-04-01728-4. MR 2137019.

[15] Roosendaal, Eric. "On the $3x + 1$ Problem". Retrieved 23 September 2017.

[16] Garner, Lynn E. "On The Collatz 3n + 1 Algorithm" (PDF). Retrieved 27 March 2015.

[17] Lagarias (1985),[3] section "A heuristic argument".

[18] Krasikov, Ilia, Lagarias, Jeffrey C. (2003). "Bounds for the 3x+1 problem using difference inequalities". Acta Arithmetica. 109 (3): 237–258. arXiv:math/0205002 Freely accessible. doi:10.4064/aa109-3-4. MR 1980260.

[19] Colussi, Livio (9 September 2011). "The convergence classes of Collatz function". Theoretical Computer Science. 412 (39): 5409–5419. doi:10.1016/j.tcs.2011.05.056.

[20] Hew, Patrick Chisan (7 March 2016). "Working in binary protects the repetends of 1/3h: Comment on Colussi's The convergence classes of Collatz function". Theoretical Computer Science. 618: 135–141. doi:10.1016/j.tcs.2015.12.033.

[21] Terras, Riho (1976), "A stopping time problem on the positive integers" (PDF), Polska Akademia Nauk, 30 (3): 241–252, MR 0568274.

[22] Chamberland, Marc (1996). "A continuous extension of the 3x + 1 problem to the real line". Dynam. Contin. Discrete Impuls Systems. 2 (4): 495–509.

[23] Letherman, Simon; Schleicher, Dierk; Wood, Reg (1999). "The $(3n + 1)$–Problem and Holomorphic Dynamics". Experimental Mathematics. 8 (3): 241–252. doi:10.1080/10586458.1999.10504402.

[24] Scollo, Giuseppe (2007), "Looking for Class Records in the 3x+1 Problem by means of the COMETA Grid Infrastructure", Grid Open Days at the University of Palermo (PDF).

[25] Lagarias (1985),[3] Theorem D.

[26] Conway, John H. (1972), "Unpredictable Iterations", Proc. 1972 Number Theory Conf., Univ. Colorado, Boulder, pp. 49–52.