MA 108 - Ordinary Differential Equations

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Outline of the lecture

- n^{th} order
- Method of variation of parameters

n^{th} ORDER Linear ODE

n^{th} order Linear ODE

Consider an n-th order linear ODE :

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x)y = g(x).$$

Assume that the functions $a_0(x), a_1(x), \ldots, a_n(x), g(x)$ are continuous on an open interval I. Also assume that $a_0(x) \neq 0$ for every $x \in I$.

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = r(x).$$

is called a n-th order linear ODE in normal form / standard form.
If $r(x) \equiv 0$ that is,

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = 0$$

then the ODE is said to be homogeneous. Otherwise it is called non-homogeneous.

Initial Value Problem- Existence/Uniqueness

We consider IVP for n^{th} order of the form

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = 0,$$
$$y(x_0) = k_0, y^{(1)}(x_0) = k_1, \ldots, y^{(n-1)}(x_0) = k_{n-1}$$

with $x_0 \in I$.

Existence - Uniqueness theorem : If $p_i(x)$ are continuous in an open interval I containing x_0, then the IVP has a unique solution on I.

Note that both existence and uniqueness are guaranteed on the same I where continuity of the coefficients is given.

Wronskian

The Wronskian of n differentiable functions $y_1(x), y_2(x), \cdots, y_n(x)$ is defined by

$$W(y_1, \cdots, y_n)(x) := \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix}.$$

Result 1: Suppose that

$$\boxed{Ly := y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = 0}$$

has continuous coefficients on an open interval I and $y_1, y_2, \cdots, y_n$ be solutions of $Ly = 0$. Then y_1, $y_2, \cdots, y_n$ are linearly dependent on I iff their Wronskian is 0 at some $x_0 \in I$.

Proof for n th order - $\Longrightarrow$

Let $y_1, \cdots, y_n$, be linearly dependent in I. That is, $\exists\ k_1, \cdots, k_n$ with $k_i \neq 0$ for some i such that

$$\begin{aligned}
&k_1 y_1(x) + \cdots + k_n y_n(x) = 0\\
&k_1 y_1'(x) + \cdots + k_n y_n'(x) = 0\\
&\vdots\\
&k_1 y_1^{(n-1)}(x) + \cdots + k_n y_n^{(n-1)}(x) = 0
\end{aligned}$$

For $x_0 \in I$, in particular,

$$\begin{pmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \cdots & y_n'(x_0) \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

has a non-trivial solution $\Longrightarrow W(y_1, \cdots, y_n)(x_0) = 0$.

Conversely, let $W(y_1, \cdots, y_n)(x_0) = 0$ for some $x_0 \in I$.
Consider the linear system of equations :

$$
\begin{aligned}
&k_1 y_1(x_0) + \cdots + k_n y_n(x_0) = 0 \\
&k_1 y_1'(x_0) + \cdots + k_n y_n'(x_0) = 0 \\
&\vdots \\
&k_1 y_1^{(n-1)}(x_0) + \cdots + k_n y_n^{(n-1)}(x_0) = 0
\end{aligned}
$$

$W(y_1, \cdots, y_n)(x_0) = 0 \Longrightarrow \exists$ non-trivial $k_1, \cdots, k_n$ solving the above linear system.

Let

$$y(x) = k_1 y_1(x) + k_2 y_2(x) + \cdots + k_n y_n(x).$$

We have, $y(x_0) = y'(x_0) = \cdots = y^{(n-1)}(x_0) = 0$.
By existence-uniqueness theorem, $y(x) \equiv 0$ is the unique solution of the IVP

$$Ly \ = 0, y(x_0) = \cdots = y^{n-1}(x_0) = 0$$

$$\implies k_1 y_1(x) + k_2 y_2(x) + \cdots + k_n y_n(x) = 0$$

with $k_1, k_2, \cdots, k_n$ not all identically zero.
Hence, $y_1, y_2, \cdots, y_n$ are l.d.

Abel's formula

Theorem

Let $p_1, p_2, \ldots, p_n$ are continuous on I and $x_0 \in I$. If $y_1, y_2, \ldots, y_n$ are solutions of

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = 0,$$

then,

$$W(y_1, \ldots, y_n)(x) = W(y_1, \ldots, y_n)(x_0)e^{-\int_{x_0}^{x} p_1(t)dt}, x \in I.$$

Proof

Proceeding as in the proof for second order case, we need to show

$$W' = -p_1(x)W.$$

Notice that the derivative of

$$\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

is

$$\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix}$$

Proof

In this, substituting $y_i''' = -p_1(x)y_i'' - p_2(x)y_i' - p_3(x)y_i$, we get

$$-p_1(x)\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} - p_2(x)\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1' & y_2' & y_3' \end{vmatrix} - p_3(x)\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1 & y_2 & y_3 \end{vmatrix}.$$

$$= -p_1(x)W(x).$$

Thus the claim is thus proved for $n = 3$. The proof for any $n \geq 4$ is similar.

Basis of solutions & General solution (n^{th} order)

Result 2: If $p_1(x), \ldots, p_n(x)$ are continuous on an open interval I, then

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = 0$$

has n linearly independent solutions $y_1, \ldots, y_n$ on I (basis of solutions).

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x)$$

is the general solution of the DE.
Every solution $y = Y(x)$ of the DE has the form

$$Y(x) = C_1 y_1(x) + \cdots + C_n y_n(x),$$

where $C_1, \cdots, C_n$ are arbitrary constants. (Prove this !)

Consider the non-homogeneous DE

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = r(x)$$

where $p_1(x), \cdots, p_n(x), r(x)$ are continuous functions on an interval I.
Let $y_p(x)$ be any solution of

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = r(x)$$

and $y_1(x), \cdots, y_n(x)$ be a basis of the solution space of the corresponding homogeneous DE.
Then the set of solutions of the non-homogeneous DE is

$$\{c_1y_1(x) + \cdots + c_ny_n(x) + y_p(x) \mid c_1, \cdots, c_n \in \mathbb{R}\}.$$

Summary:

In order to find the general solution of a non-homogeneous DE, we need to

- get the general solution of the corresponding homogeneous DE.
- get one particular solution of the non-homogeneous DE

General Solution of Homogeneous Equations with Constant Coefficients

Consider

$$Ly := y^{(n)} + p_1 y^{(n-1)} + \ldots + p_n y = 0$$

where $p_1, \cdots, p_n$ are in $\mathbb{R}$; that is, an n$^{\text{th}}$ order homogeneous linear ODE with constant coefficients.
Suppose e^{mx} is a solution of this equation. Then,

$$m^n e^{mx} + p_1 m^{n-1} e^{mx} + \cdots + p_n e^{mx} = 0,$$

and this implies

$$P(m) := m^n + p_1 m^{n-1} + \cdots + p_n = 0.$$

This is called the characteristic equation or auxiliary equation of the linear homogeneous ODE with constant coefficients.

The polynomial $P(m)$ of degree n has n zeros say $m_1, \cdots, m_n$, some of which may be equal and hence the characteristic polynomial can be written in the form

$$(m-m_1)(m-m_2)\cdots(m-m_n).$$

Also we write

$$Ly = (D-m_1)(D-m_2)\ldots(D-m_n)y.$$

Depending on the the nature of these zeros (real & unequal, real & equal, complex), we write down the basis and general solution of the homogeneous DE.

Example 1

Find a basis of solutions and general solution of the DE:

$$y''' - 7y' + 6y = 0.$$

1. The characteristic equation is

$$m^3 - 7m + 6 = 0.$$

2. The roots are $1, 2, -3$.
3. Hence a basis for solutions is $\{e^x, e^{2x}, e^{-3x}\}$ Why?
   $W(0) = 2 \neq 0$.
4. Thus, the general solution (also called complementary function) is of the form

$$c_1 e^x + c_2 e^{2x} + c_3 e^{-3x},$$

where $c_1, c_2, c_3 \in \mathbb{R}$.

Example 2

Find the general solution of the DE:

$$Ly = (D^3 - D^2 - 8D + 12)(y) = 0.$$

1. The characteristic equation is

$$m^3 - m^2 - 8m + 12 = (m-2)^2(m+3) = 0.$$

2. The roots are $2, 2, -3$.
3. Hence a basis for solutions is $\{e^{2x}, xe^{2x}, e^{-3x}\}$. Why? repeated root $m = 2$ gives solutions e^{2x}, xe^{2x} (justify this) and $W(0) \neq 0$.
4. Thus, the general solution is of the form

$$c_1 e^{2x} + c_2 xe^{2x} + c_3 e^{-3x}, c_1, c_2, c_3 \in \mathbb{R}.$$

Find the general solution of the DE:

$$Ly = (D^6 + 2D^5 - 2D^3 - D^2)(y) = 0.$$

The characteristic equation is $m^2(m-1)(m+1)^3 = 0$. The general solution is (justify)

$$c_1 + c_2 x + c_3 e^x + c_4 e^{-x} + c_5 x e^{-x} + c_6 x^2 e^{-x}, c_i \in \mathbb{R}.$$

Find the general solution of

$$y^{(5)} - 9y^{(4)} + 34y^{(3)} - 66y^{(2)} + 65y' - 25y = 0.$$

The characteristic polynomial is

$$(m-1)(m^2 - 4m + 5)^2.$$

The zeros are

$$1, 2 \pm \imath, 2 \pm \imath.$$

The general solution is (justify)

$$y = c_1 e^x + c_2 e^{2x} \cos x + c_3 x e^{2x} \cos x + c_4 e^{2x} \sin x + c_5 x e^{2x} \sin x, c_i \in \mathbb{R}.$$

Variation of Parameters

We consider the non-homogeneous linear ODE

$$Ly := y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y = r(x).$$

We proceed exactly the same way as $n = 2$.
Let the general solution of the associated homogeneous DE be

$$y = c_1y_1 + c_2y_2 + \ldots + c_ny_n.$$

Method of variation of parameters suggest particular solution of the form

$$y_p = v_1y_1 + v_2y_2 + \ldots + v_ny_n \; (*)$$

We use the following ansatz ($(n-1)$th order contact conditions).

$$\begin{aligned}
v_1'y_1 + \ldots + v_n'y_n &= 0, \\
v_1'y_1' + \ldots + v_n'y_n' &= 0, \\
\ldots &. \\
v_1'y_1^{(n-2)} + \ldots + v_n'y_n^{(n-2)} &= 0.
\end{aligned}$$

Differentiate (*), we get

$$y_p' = v_1 y_1' + \ldots + v_n y_n' + v_1' y_1 + \ldots + v_n' y_n$$

Using the Ansatz, we get

$$y_p' = v_1 y_1' + \ldots + v_n y_n'.$$

Differentiate again and use the anstaz, we get

$$y_p'' = v_1 y_1'' + \ldots + v_n y_n''.$$

repeat this, we get

$$\ldots\ldots$$

$$y_p^{(n-2)} = v_1 y_1^{(n-2)} + \ldots + v_n y_n^{(n-2)}.$$

$$y_p^{(n-1)} = v_1 y_1^{(n-1)} + \ldots + v_n y_n^{(n-1)}.$$

Differentiate the above, we get

$$y_p^{(n)} = v_1 y_1^{(n)} + \ldots + v_n y_n^{(n)} + v_1' y_1^{(n-1)} + \ldots + v_n' y_n^{(n-1)}.$$

Now substituting the above equations for $y_p, y_p', \cdots, y_p^{(n)}$ below, we get

$$\begin{aligned} r(x) &= y_p^{(n)} + p_1(x) y_p^{(n-1)} + \ldots + p_n(x) y_p \\ &= v_1 L(y_1) + v_2 L(y_2) + \ldots + v_n L(y_n) \\ &\quad + v_1' y_1^{(n-1)} + \ldots + v_n' y_n^{(n-1)} \\ &= v_1' y_1^{(n-1)} + \ldots + v_n' y_n^{(n-1)}. \end{aligned}$$

Variation of Parameters

Thus we have,

$$\begin{bmatrix} y_1 & y_2 & \cdot & y_n \\ y_1' & y_2' & \cdot & y_n' \\ \cdot & \cdot & \cdot & \cdot \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdot & y_n^{(n-1)} \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \\ \cdot \\ v_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ r(x) \end{bmatrix}.$$

Use Cramer's rule to solve for

$$v_1', v_2', \ldots, v_n',$$

and integrate to get the formulae for

$$v_1, v_2, \ldots, v_n,$$

and a particular solution

$$y_p = v_1 y_1 + v_2 y_2 + \ldots + v_n y_n.$$

Geometric interpretation of the Ansatz

As in the second order case, let $Y(x, \xi)$ denote the solution of the modified non homogeneous equation by 'killing' $r(x)$ from ξ onwards.
Then Ansatz is given by the $(n-1)$th order contact conditions of the curves y_p and $Y(\cdot, \xi)$ at the point $(\xi, y_p(\xi))$. The $(n-1)$th order contact conditions are: at $x = \xi$

$$
\begin{aligned}
Y(x, \xi) &= y_p(x) \\
Y'(x, \xi) &= y_p'(x) \\
\cdots &= \cdots \\
Y^{(n-1)}(x, \xi) &= y_p^{(n-1)}(x).
\end{aligned}
$$

Now by substituting

$$Y(x, \xi) = v_1(\xi) y_1(x) + \ldots + v_n(\xi) y_n(x)$$

into the contact conditions and proceed as in the case of $n = 2$ we get the ansatz.

Example

Solve

$$y''' - y'' - y' + y = r(x).$$

Characteristic polynomial for the homogeneous equation is

$$m^3 - m^2 - m + 1 = (m-1)^2(m+1).$$

Hence, a basis of solutions is

$$\{e^x, xe^x, e^{-x}\}.$$

(justify this). We need to calculate $W(x)$. Use Abel's formula:

$$W(x) = W(0)e^{-\int_0^x p_1(t)dt} = W(0) \cdot e^x.$$

Now,

$$W(x) = \begin{vmatrix} e^x & xe^x & e^{-x} \\ e^x & e^x + xe^x & -e^{-x} \\ e^x & 2e^x + xe^x & e^{-x} \end{vmatrix}.$$

Thus,

$$W(0) = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 4.$$

Hence,

$$W(x) = 4e^x.$$

Variation of Parameters

Set

$$W_1(x) = \begin{vmatrix} 0 & xe^x & e^{-x} \\ 0 & e^x + xe^x & -e^{-x} \\ r(x) & 2e^x + xe^x & e^{-x} \end{vmatrix} = -r(x)(2x+1).$$

Obtained by replacing first column of $W(x)$ by $(0, 0, r(x))^T$.
Similarly,

$$W_2(x) = 2r(x), \;\; W_3(x) = r(x)e^{2x}.$$

Note that

$$\begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \\ v_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ r(x) \end{bmatrix}.$$

Variation of Parameters

Hence

$$v_i' = \frac{W_i(x)}{W(x)}, \;\; i = 1, 2, 3.$$

Therefore,

$$y(x) = e^x \int_0^x \frac{-r(t)(2t+1)}{4e^t}\, dt + xe^x \int_0^x \frac{2r(t)}{4e^t}\, dt + e^{-x} \int_0^x \frac{r(t)e^{2t}}{4e^t}\, dt.$$