MA-111 Calculus II
(D1 & D2)

Lecture 3

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Subtleties in Fubini's theorem - II

We will give examples of two functions f, g with the following properties:
(1) The function g may be double integrable but the one dimensional integrals $\int_c^d g(x, y)\, dy$ may not exist.
(2) Both the iterated integrals of f exist and are equal, but f is not double integrable.

Example

The examples depend on dyadic rationals, so we itroduce them.
Dyadic rationals: A rational number p/q in its reduced form (i.e. p and q are coprime) is called a *dyadic rational* if q is a power of 2. For example, $1/2, 3/2, 3/8$ are dyadic rationals but $1/3$ is not. The dyadic rationals are dense in $\mathbb{R}$. The complement of the set of dyadic rationals is also dense in $\mathbb{R}$.

Example continued...

Consider the functions f and g on the rectangle $R = [0,1] \times [0,1]$ defined by:

(1) For any integer $i \geq 0$, for any odd integer $j \in [0, 2^i]$, for any integer $k \geq 0$, for any odd integer $\ell \in [0, 2^k]$, define $f(j/2^i, \ell/2^k) = \delta_{ik}$ (here, δ_{ik} is the Kronecker delta, equal to one if $i = k$ and 0 if not) and $g(j/2^i, \ell/2^k) = 1/2^i$; and

(2) $\forall x, y \in [0,1]$, if either x or y is not a dyadic rational, define $f(x,y) = 0$ and $g(x,y) = 0$.

- g is Riemann integrable on R but, for any dyadic rational $x = i/2^j \in [0,1]$, the Riemann integral $\int_0^1 g(x,y)\,dy$ does not exist.
- For the function f, both the iterated integrals $\int_0^1 \int_0^1 f(x,y)\,dx\,dy$ and $\int_0^1 \int_0^1 f(x,y)\,dy\,dx$ exist and are equal to zero. However, f is not Riemann integrable over R.

Proposition.

Let R be a rectangle in $\mathbb{R}^2$ and let $f : R \to \mathbb{R}$ be a continuous function. Then f is integrable on R. Moreover, both iterated integrals of f exist and are equal to the double integral of f over R.

Examples:

Example : Find the integral of $f(x, y) = x^2 + y^2$ on the rectangle $[0, 1] \times [0, 1]$ if it exists.

Solution: Since f is continuous, it is integrable. By Fubini's theorem, the double integral is equal to any of the iterated integrals. Let us now compute the integral using iterated integrals.

$$\begin{aligned}
\int\int_{[0,1]\times[0,1]} x^2 + y^2 \; dxdy &\overset{\text{Fubini}}{=} \int_0^1 \int_0^1 x^2 + y^2 \; dxdy \\
&= \int_0^1 [\frac{x^3}{3} + xy^2]_0^1 dy \\
&= \int_0^1 (\frac{1}{3} + y^2) dy \\
&= [\frac{y}{3} + \frac{y^3}{3}]_0^1 = \frac{2}{3}
\end{aligned}$$

Example (Marsden, Tromba and Weinstein page 288): Compute $\int\int_R \sin(x+y)dxdy$, where $R = [0,\pi] \times [0,2\pi]$.

Solution:

$$\begin{aligned}
\int\int_R \sin(x+y)dxdy &= \int_0^{2\pi} \left[\int_0^{\pi} \sin(x+y)dx\right] dy \\
&= \int_0^{2\pi} [-\cos(x+y)|_{x=0}^{\pi}]dy \\
&= \int_0^{2\pi} [\cos y - \cos(y+\pi)]dy \\
&= [\sin y - \sin(y+\pi)]|_{y=0}^{2\pi} = 0
\end{aligned}$$

Example (Marsden, Tromba and Weinstein, page 289): If D is a plate defined by $1 \leq x \leq 2, 0 \leq y \leq 1$ (measured in centimeters), and the mass density $\rho(x,y) = ye^{xy}$ grams per square centimeter. Find the mass of the plate.
Solution: The total mass of the plate is got by integrating over the rectangular region covered by D:

$$\begin{aligned} \int\int_D \rho(x,y)dxdy &= \int_0^1 \int_1^2 ye^{xy}dxdy = \int_0^1 (e^{xy}|_{x=1}^2)dy \\ &= \int_0^1 (e^{2y} - e^y)dy = \tfrac{e^2}{2} - e + \tfrac{1}{2} \end{aligned}$$

Special case Let $\phi : [a,b] \to \mathbb{R}$ and $\psi : [c,d] \to \mathbb{R}$ be Riemann integrable. Define $f(x,y) := \phi(x)\psi(y)$, for all $(x,y) \in R = [a,b] \times [c,d]$. Then f is integrable on R and

$$\int\int_R f(x,y)\,dx\,dy = \Big(\int_a^b \phi(x)\,dx\Big)\Big(\int_c^d \psi(y)\,dy\Big).$$

Example Let $0 < a < b$ and $0 < c < d$ and $r \geq 0$ and $s \geq 0$. Denote $R = [a,b] \times [c,d]$. Compute $\int\int_R x^r y^s\,dx\,dy$.

Existence of integrals over rectangles -I

Theorem

If f is bounded and monotonic in each of two variables, then f is integrable on R.

Again the proof follows by using Riemann condition.

Example: Let $f(x, y) := [x + y]$, for all $(x, y) \in R$, where $[u]$ means the greatest integer less than equal to u, for any $u \in \mathbb{R}$. Since f is monotonic in each of two variables, f is integrable on R.

Existence of integrals on rectangles - II

Sets of measure zero Let $A \subset \mathbb{R}^n$. We say A has *measure zero* in $\mathbb{R}^n$ if for every $\epsilon > 0$, there is a covering $Q_1, Q_2, \ldots$ of A by countably many rectangles such that

$$\sum_{i=1}^{\infty} \mathit{volume}(Q_i) < \epsilon.$$

If this inequality holds, we often say that the total volume of the rectangles $Q_1, Q_2, \ldots$ is less than ϵ.

Remark If A is closed, bounded and has measure zero, then the collection $\{Q_j\}_j$ could be chosen to be finite. We say A has *content zero*. Not all measure zero sets have content zero, for example, $[0, 1] \cap \mathbb{Q}$ has measure zero but not content zero.

- If $B \subset A$ and A has measure zero in $\mathbb{R}^n$, then so does B.
- Let A be the union of countable collection of sets $A_1, A_2, \ldots$. If each A_i has measure zero in $\mathbb{R}^n$. In particular, every countable set has measure zero.
- If Q is a rectangle in $\mathbb{R}^n$, then the boundary of Q has measure zero in $\mathbb{R}^n$. (Q has measure zero if and only if the area of Q is zero.)
- Let $f : [a, b] \to \mathbb{R}$. The graph of f is the subset

  $$\Gamma_f = \{(x, y) : y = f(x)\}$$

  of $\mathbb{R}^2$. If f is continuous, then Γ_f has measure zero in $\mathbb{R}^2$.

Continued...

Think about it... Let $D \subset \mathbb{R}^2$ be a measure zero set. Suppose $0 \leq \theta < 2\pi$ and let R_θ be the rotation by angle θ i.e. $R_\theta(x, y) = (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)$. Is $R_\theta(D)$ a set of measure zero?

Theorem Let Q be a rectangle in $\mathbb{R}^2$, let $f : Q \to \mathbb{R}$ be a bounded function. Let D be the set of points of Q at which f fails to be continuous. Then f is integrable over Q if and only if D has measure zero in $\mathbb{R}^2$.

Theorem Let Q be a rectangle in $\mathbb{R}^2$, let $f : Q \to \mathbb{R}$ be integrable. Let E be the set of points of Q where f is nonzero. If E has measure zero, then $\int\int_Q f = 0$.

- **Example.** Let $R := [-1,1] \times [-1,1]$,

$$f(x,y) = \begin{cases} \frac{xy}{(x^2+y^2)}, & (x,y) \in R, \quad (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

  Since $f \leq 1/2$ and continuous on the complement of the origin, it is integrable over R.

- **Example:** Let $R = [0,1] \times [0,1]$ and

$$f(x,y) = \begin{cases} 1, & 0 \leq x < y, \quad y \in [0,1], \\ 0, & y \leq x \leq 1, \quad y \in [0,1]. \end{cases}$$

  The set of discontinuities of f is on the diagonal line, which is of measure zero as it is the graph of the function $y = x$. Therefore f is integrable over R.

- **Bivariate Thomae function:** $f : [0,1] \times [0,1] \to \mathbb{R}$ is defined by

$$f(x,y) = \begin{cases} 1, & \text{if} \quad x = 0, \quad y \in \mathbb{Q} \cap [0,1], \\ \frac{1}{q}, & x, y \in \mathbb{Q} \cap [0,1] \quad \text{and} \quad x = \frac{p}{q}, \\ & \qquad\qquad p, q \in \mathbb{N} \quad \text{are relatively prime}, \\ 0, & \text{otherwise}. \end{cases}$$

  Then f is continuous at (x,y) if x is irrational, so the points of discontinuities is contained in the measure zero set $\mathbb{Q} \times \mathbb{R}$. This function is integrable on $R = [0,1] \times [0,1]$, and $\int\int_R f = 0$.

Integrals over any bounded region in $\mathbb{R}^2$

Let D be any bounded subset (not necessarily rectangle) of $\mathbb{R}^2$. Then there exists a rectangle R in $\mathbb{R}^2$ containing D, i.e., $D \subset R$.

Proof: Indeed, since D is a bounded subset of R^2, there exists $a > 0$ such that any $(x, y) \in D$ satisfies $x^2 + y^2 < a^2$, i.e, $D \subset B_a = \{(x, y) \mid x^2 + y^2 \leq a^2\}$. But $B_a \subset [-a, a] \times [-a, a]$, so the rectangle $R := [-a, a] \times [-a, a]$ contains D. $\square$

Let $f : D \to \mathbb{R}$ be a function. Extend f from D to R by defining

$$f^*(x, y) := \begin{cases} f(x, y), & (x, y) \in D, \\ 0, & (x, y) \notin D. \end{cases}.$$

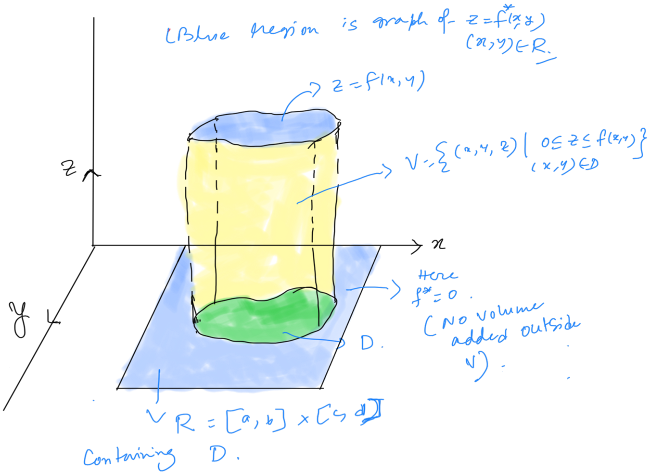
Definition

The function $f : D \to \mathbb{R}$ is said to be integrable on bounded $D \subset \mathbb{R}^2$, if f^* *is integrable on* R and the integral of f on D is defined by

$$\int\int_D f(x, y)\, dx\, dy := \int\int_R f^*(x, y)\, dx\, dy.$$

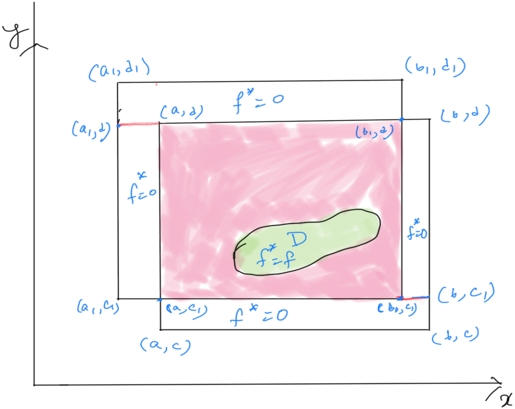
- If $f \geq 0$ on $D \subset \mathbb{R}^2$ and f is integrable on D, then the double integral of f on D is the volume of the solid that lies above D in the x-y plane and below the graph of the surface $z = f(x, y)$ for all $(x, y) \in D$.

$\int \int_D f$ = volume of V



Independent of choice of rectangle

- The choice of rectangle R containing D is not unique.
- But the value of the integral of f on D does not depend on the choice of the rectangle R containing D.
- Use the additivity property of integrals on rectangle and note that only 'zero' is getting added outside D.



Properties of Integrals over bounded sets in $\mathbb{R}^2$

Let D be a bounded subset of $\mathbb{R}^2$. Let $f : D \to \mathbb{R}$ be an integrable function.

- The algebraic properties for integrals on any bounded set D in $\mathbb{R}^2$ hold similarly to those of the case of integrals on rectangle.

Domain additivity property: Let $D \subseteq \mathbb{R}^2$ be a bounded set. Let $D_1, D_2 \subseteq D$ such that $D = D_1 \cup D_2$. Let $f : D \to \mathbb{R}^2$ be a bounded function. If f is integrable over D_1 and D_2 and $D_1 \cap D_2$ has content zero then f is integrable on D and

$$\int\int_D f = \int\int_{D_1} f + \int\int_{D_2} f.$$