# MA 108 - Ordinary Differential Equations

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# Outline of the lecture

- Laplace tranform of periodic functions
- Gamma function
- Partial Fractions

# Property 10. Laplace transform of periodic functions

Let $f$ be a piecewise continuous periodic function with period $p$ whose Laplace transform exists. Then,

$$L(f) = \frac{1}{1-e^{-sp}} \int_0^p e^{-st} f(t)dt.$$

$$\begin{aligned}
L(f)(s) &= \int_0^\infty e^{-st} f(t)dt \\
&= \int_0^p e^{-st} f(t)dt + \int_p^{2p} e^{-st} f(t)dt + \int_{2p}^{3p} e^{-st} f(t)dt + \dots \\
& \quad \text{setting } t = u - (n-1)p \text{ in the } n^{th} \text{ integral} \\
&= \int_0^p e^{-st} f(t)dt + \int_0^p e^{-s(u+p)} f(t)dt + \int_0^p e^{-s(u+2p)} f(t)dt \\
& \quad + \dots \\
&= \frac{1}{1-e^{-sp}} \int_0^p e^{-st} f(t)dt.
\end{aligned}$$

Tut 5, Q 6

# Example

Solve

$$2y_1' - y_2' - y_3' = 0, y_1' + y_2' = 4t + 2, y_2' + y_3 = t^2 + 2;$$
$$y_1(0) = 0, y_2(0) = 0, y_3(0) = 0.$$

Taking Laplace transforms and denoting $L(y_i)(s) = Y_i(s), i = 1, 2, 3,$ we have

$$\begin{aligned} 2sY_1 - sY_2 - sY_3 &= 0 \\ sY_1 + sY_2 &= \frac{4}{s^2} + \frac{2}{s} \\ sY_2 + Y_3 &= \frac{2}{s^3} + \frac{2}{s}. \end{aligned}$$

Solving:

$$Y_1 = \frac{2}{s^3}, Y_2 = \frac{2}{s^3} + \frac{2}{s^2}, Y_3 = \frac{2}{s^3} - \frac{2}{s^2}.$$

Thus,

$$y_1(t) = t^2, y_2(t) = t^2 + 2t, y_3(t) = t^2 - 2t.$$

# Solution of a system of DE using LT

Solve $x' = x + y, y' = 4x + y$.
Denoting $X(s)$ and $Y(s)$ as the LT's of $x$ and $y$ respectively.
Taking Laplace transforms,

$$\begin{aligned} sX - x(0) &= X + Y \\ sY - y(0) &= 4X + Y. \end{aligned}$$

Solving:

$$X(s) = \frac{(s-1)x(0) + y(0)}{s^2 - 2s - 3}, \; Y(s) = \frac{4x(0) + (s-1)y(0)}{s^2 - 2s - 3}.$$

Take $L^{-1}$ to get $x(t)$ and $y(t)$.
Tut. Sheet 5, Q. 4, 17

# Properties

| | | |
|---|---|---|
| 1. | Linearity | $L(af(t)+bg(t)) = aL(f(t)) + bL(g(t))$ |
| 2. | I Shifting theorem | $L(e^{at}f(t)) = F(s-a)$ |
| 3. | Scaling | $L(f(ct)) = \frac{1}{c}F\left(\frac{s}{c}\right), \; c > 0$ |
| 4. | Laplace transform of derivative | $L(f') = sL(f) - f(0)$<br>$L(f'') = s^2L(f) - sf(0) - f'(0)$ |
| 5. | L.T. of integral | $L\left(\int_0^t f(\tau)\,d\tau\right) = \frac{F(s)}{s},$ for . |
| 6. | Dervative of L.T. | $F'(s) = -L(tf(t))$<br>$L(t^n f(t)) = (-1)^n F^{(n)}(s)$ |
| 7. | Integral of L.T. | $\int_s^\infty F(\tilde{s})d\tilde{s} = L\left(\frac{f(t)}{t}\right), \quad s > \alpha.$ |
| 8. | II shifting theorem | $L(u_c(t)f(t-c)) = e^{-cs}F(s)$ |
| 9. | Convolution & L.T. | $(f * g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$<br>$L(f * g) = L(f)\cdot L(g)$ |
| 10. | L.T. of Periodic function | $L(f) = \frac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt$ |

# Variable coefficients - an example

Compute the Laplace transform of a solution of

$$ty'' + y' + ty = 0,\ t > 0,\ \ y(0) = k,\ L(y)(1) = 1/\sqrt{2}.$$

$$
\begin{aligned}
& L(ty'' + y' + ty) = 0 \\
& -\frac{d}{ds}L(y'') + (sL(y) - y(0)) - \frac{d}{ds}(L(y)) = 0 \\
& -\frac{d}{ds}(s^2L(y) - sy(0) - y'(0)) + (sL(y) - y(0)) - \frac{d}{ds}(L(y)) = 0 \\
\Rightarrow\ \ & -\frac{d}{ds}(s^2Y(s) - sy(0) - y'(0)) + sY(s) - y(0) - Y'(s) = 0 \\
\Rightarrow\ \ & (s^2 + 1)Y'(s) + sY(s) = 0 \Longrightarrow Y(s) = \frac{C}{\sqrt{s^2 + 1}} \\
& Y(1) = 1/\sqrt{2} \Longrightarrow Y(s) = \frac{1}{\sqrt{s^2 + 1}}.
\end{aligned}
$$

Let us now introduce the gamma function.
$\Gamma : (0, \infty) \to \mathbb{R}$ is defined by

$$\Gamma(a) = \int_0^\infty e^{-x} x^{a-1} dx, a > 0$$

Now we show that the right hand side integral converges.
Write it as

$$\int_0^1 e^{-x} x^{a-1} dx + \int_1^\infty e^{-x} x^{a-1} dx,$$

and we need to check that both these integrals do converge.
Why do these integrals converge?
For the first integral, note that $e^{-x} \leq 1$ for $x \geq 0$ and

$$0 \leq x^{a-1} e^{-x} \leq x^{a-1} \qquad a > 0.$$

Hence,

$$0 \leq \int_0^1 x^{a-1} e^{-x}\, dx \leq \int_0^1 x^{a-1}\, dx = \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 x^{a-1}\, dx = \lim_{\varepsilon \to 0^+} \left( \frac{x^a}{a} \right)_\varepsilon^1 .$$

If $a > 0$, $\varepsilon^a \to 0$ as $\varepsilon \to 0^+$, so the integral converges to $1/a$ in the first case.

For the second integral, note that $\lim_{x\to\infty} e^{-x/2}x^r = 0,\ r \in \mathbb{R}$.
That is, given $a > 0$, $\exists N$ such that,

$$0 \le e^{-x/2}x^{a-1} \le 1, \qquad x \ge N.$$

$$\int_1^\infty e^{-x}x^{a-1}dx = \int_1^N e^{-x}x^{a-1}dx + \int_N^\infty e^{-x}x^{a-1}dx.$$

The first integral above is finite. For the second, note that

$$e^{-x}x^{a-1} = (e^{-\frac{x}{2}}x^{a-1})e^{-\frac{x}{2}} \le e^{-x/2}.$$

$$\int_N^\infty e^{-x}x^{a-1}dx \le \int_N^\infty e^{-x/2}dx = \lim_{\varepsilon\to\infty}\left(-2e^{-x/2}\right)_N^\varepsilon = -2e^{-N/2}.$$

The second term is convergent and hence the second integral is also convergent.

# Gamma Function

The gamma function satisfies a nice functional equation:

$$\Gamma(a+1) = a\Gamma(a).$$

Proof: Let $0 < s < t$. Use integration by parts to see:

$$\begin{aligned}
\int_s^t e^{-x} x^a \, dx &= [-x^a e^{-x}]_s^t + a \int_s^t e^{-x} x^{a-1} \, dx \\
&= s^a e^{-s} - t^a e^{-t} + a \int_s^t e^{-x} x^{a-1} \, dx.
\end{aligned}$$

Take limit as $t \to \infty$ and $s \to 0^+$ to get the functional equation.
In particular,

$$\Gamma(n+1) = n!.$$

Thus, the gamma function interpolates the factorial function.

# Exercise : Gamma Function

1. Prove that

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

Hint: Let $I =$ lhs. Compute $I^2$ as a double integral by changing to polar coordinates.

2. Find $\Gamma(\frac{1}{2}), \Gamma(\frac{3}{2}), \ldots$.

$$\Gamma(\frac{1}{2}) = \int_0^\infty e^{-x}x^{-\frac{1}{2}}\,dx.$$

Put $x = t^2$. Thus,

$$\Gamma(\frac{1}{2}) = 2\int_0^\infty e^{-t^2}\,dt = 2\frac{\sqrt{\pi}}{2} = \sqrt{\pi}.$$

Now,

$$\Gamma(\frac{3}{2}) = \frac{1}{2}\cdot\Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}.$$

# Laplace transform of $t^p$, $p > -1$

Determine $L(t^p), p > -1$.

$$L(t^p) = \int_0^\infty e^{-st} t^p dt.$$

Put $x = st$. Thus, $dt = \dfrac{dx}{s}$. Thus,

$$\begin{aligned} L(t^p)(s) &= \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^p \cdot \frac{dx}{s} \\ &= \frac{1}{s^{p+1}} \int_0^\infty e^{-x} x^p dx \\ &= \frac{\Gamma(p+1)}{s^{p+1}}, \end{aligned}$$

where $s > 0$. Hence $L(t^n) = \dfrac{n!}{s^{n+1}}$, $n = 0, 1, \ldots$. For $p = \dfrac{1}{2}$, we get, for $s > 0$,

$$L(t^{-1/2}) = \frac{\Gamma(1/2)}{s^{1/2}} = \sqrt{\frac{\pi}{s}}, \; L(t^{1/2}) = \frac{\Gamma(3/2)}{s^{3/2}} = \frac{\sqrt{\pi}}{2s^{3/2}}.$$

Result: Let $f : [0, \ \infty) \to \mathbb{R}$ be piecewise continuous and of exponential order. Then

$$\lim_{s\to\infty} L(f)(s) = 0.$$

Proof: Note that there exists $t_0 > 0, K > 0$ and $a \in \mathbb{R}$ such that

$$|f(t)| \leq Ke^{at}, t \geq t_0.$$

Take $M =$l.u.b $\{|f(t)| : 0 \leq t \leq t_0\}$. For $s > a$,

$$\begin{aligned} |L(f)(s)| &\leq \int_0^{t_0} e^{-st}|f(t)|dt + \int_{t_0}^{\infty} e^{-st}|f(t)|dt \\ &\leq \frac{M}{s}(1 - e^{-st_0}) + \frac{K}{s-a}e^{-(s-a)t_0}. \end{aligned}$$

In particular, it follows that

$$L(f)(s) \to 0,$$

as $s \to \infty$.

Remark: This limiting behaviour is true for any $f$ such that $L(f)$ exists; i.e., even without assuming exponential order etc. Proof is tough!

Remark: Thus, $\frac{s-1}{s+1}, \frac{e^s}{s}, s^2, \frac{s}{\ln s}$ etc are not the Laplace transform of any function!

# Example

Solve $y'' + ty' - 2y = 4,\ y(0) = -1,\ y'(0) = 0.$

$$L(y'') + L(ty') - 2L(y) = L(4)$$

$$(s^2L(y) - sy(0) - y'(0)) - (sL(y) - y(0))' - 2L(y) = \frac{4}{s}$$

Denoting $Y(s) = L(y)$, by simplifying the above expression and using the initial conditions, we obtain

$$Y'(s) + (\frac{3}{s} - s)Y(s) = 1 - \frac{4}{s^2}.$$

Solving this DE, we obtain :

$$Y(s) = \frac{2}{s^3} - \frac{1}{s} + \frac{c}{s^3}e^{s^2/2},$$

where $c$ is a constant.

By using the remark in the previous slide, we have $c = 0$!

# Partial Fractions

Suppose $f(x)$ takes the form

$$f(x) = \sum_i \left( \frac{a_{i1}}{x - x_i} + \frac{a_{i2}}{(x - x_i)^2} + \cdots + \frac{a_{ik_i}}{(x - x_i)^{k_i}} \right).$$

Then,

$$a_{ij} = \frac{1}{(k_i - j)!} \lim_{x \to x_i} \frac{d^{k_i - j}}{dx^{k_i - j}} \left( (x - x_i)^{k_i} f(x) \right).$$

When $f(x)$ above is given in the form, $f(x) = \dfrac{P(x)}{Q(x)}$ and $x = x_i$ is a simple root of $Q(x)$, then

$$a_{i1} = \frac{P(x_i)}{Q'(x_i)},$$

Remark: Note that any $f(x)$ can be put into this form over $\mathbb{C}$. So we can do this over $\mathbb{C}$, and then club conjugate terms to get partial fractions over $\mathbb{R}$.

Example:

$$f(x) = \frac{x^2-5}{(x^2-1)(x^2+1)} = \frac{x^2-5}{(x+1)(x-1)(x+i)(x-i)}.$$

This can be decomposed into rational functions whose denominators are $x+1, x-1, x+\imath, x-\imath$. Note that each term is of power one. Let $x_i = -1, 1, -\imath, \imath$. Note that

$$\frac{P(x_i)}{Q'(x_i)} = \frac{x_i^2-5}{4x_i^3},$$

and we get $1, -1, \frac{3\imath}{2}, -\frac{3\imath}{2}$ respectively. Thus,

$$\begin{aligned} f(x) &= \frac{1}{x+1} - \frac{1}{x-1} + \frac{3\imath}{2}\frac{1}{x+\imath} - \frac{3\imath}{2}\frac{1}{x-\imath} \\ &= \frac{1}{x+1} - \frac{1}{x-1} + \frac{3}{x^2+1}. \end{aligned}$$

Example: Solve the IVP:

$$y'' - 3y' + 2y = 4t, \;\; y(0) = 1, y'(0) = -1.$$

Apply Laplace transform:

$$s^2 L(y) - sy(0) - y'(0) - 3sL(y) + 3y(0) + 2L(y) = \frac{4}{s^2}.$$

Thus,

$$L(y) = \frac{s^3 - 4s^2 + 4}{s^2(s-1)(s-2)}.$$

Need to write
$$\frac{s^3-4s^2+4}{s^2(s-1)(s-2)}$$
in partial fractions:

Coefficient of $\dfrac{1}{s-1}$ is $\dfrac{s^3-4s^2+4}{4s^3-9s^2+4s}(1)=-1.$

Coefficient of $\dfrac{1}{s-2}$ is $\dfrac{s^3-4s^2+4}{4s^3-9s^2+4s}(2)=-1.$

Coefficient of $\dfrac{1}{s}$ is $\dfrac{1}{1!}\lim_{s\to 0}\dfrac{d}{ds}\dfrac{s^3-4s^2+4}{(s-1)(s-2)}=3.$

Coefficient of $\dfrac{1}{s^2}$ is $\dfrac{1}{0!}\lim_{s\to 0}\dfrac{s^3-4s^2+4}{(s-1)(s-2)}=2.$

Thus,
$$L(y)=\frac{3}{s}+\frac{2}{s^2}-\frac{1}{s-1}-\frac{1}{s-2}.$$
So,
$$y=3+2t-e^t-e^{2t}.$$

# Additional Examples

Example:

$$\begin{aligned} L(e^{-t}\sin^2 t) &= L(\frac{1}{2}(e^{-t}(1-\cos 2t)) \\ &= \frac{1}{2}\left[\frac{1}{s+1} - \frac{s+1}{(s+1)^2+4}\right]. \end{aligned}$$

Example:

$$\begin{aligned} L(t^2 e^{-at}) &= \frac{d^2}{ds^2}\left(\frac{1}{s+a}\right) \\ &= \frac{2}{(s+a)^3}. \end{aligned}$$

Example:

$$L(t^a e^{-bt}) = \frac{\Gamma(a+1)}{(s+b)^{a+1}}.$$

# Additional Example

Solve $y'' + y = 1 - u_{\pi/2}(t),\ y(0) = 0,\ y'(0) = 1.$

$$
\begin{aligned}
L(y'') + L(y) &= L(1 - u_{\pi/2}(t)) \\
s^2 L(y) - sy(0) - y'(0) + L(y) &= \frac{1}{s} - \frac{e^{-\frac{\pi}{2}s}}{s} \\
L(y)(s^2+1) &= \frac{1}{s} - \frac{e^{-\frac{\pi}{2}s}}{s} + 1
\end{aligned}
$$

$$
\begin{aligned}
L(y) = L^{-1}\left(\frac{1}{s(s^2+1)}\right) - L^{-1}\left(\frac{e^{-\frac{\pi}{2}s}}{s(s^2+1)}\right) + \frac{1}{s^2+1} \\
y = 1 - \cos t - u_{\pi/2}(t)(1 - \cos(t - \pi/2)) + \sin t.
\end{aligned}
$$

That is, $y(t) = 1 - \cos t + \sin t$, if $t < \pi/2$ and
$y(t) = \cos t + 2\sin t$, if $t > \pi/2$.

Example: Solve the IVP:

$$2y'' + y' + 2y = u_5(t) - u_{20}(t), \;\; y(0) = 0, y'(0) = 0.$$

Take Laplace transforms:

$$2(s^2 L(y) - sy(0) - y'(0)) + (sL(y) - y(0)) + 2L(y) = L(u_5(t) - u_{20}(t));$$

i.e.,

$$(2s^2 + s + 2)L(y) = \frac{e^{-5s} - e^{-20s}}{s}.$$

Put

$$H(s) = \frac{1}{s(2s^2 + s + 2)},$$

and

$$L(h(t)) = H(s).$$

Then,

$$y(t) = u_5(t)h(t-5) - u_{20}(t)h(t-20).$$

To find $h(t)$, write

$$\frac{1}{s(2s^2+s+2)} = \frac{a}{s} + \frac{bs+c}{2s^2+s+2}.$$

Check:

$$a = \frac{1}{2}, b = -1, c = -\frac{1}{2}.$$

Thus,

$$\begin{aligned} H(s) &= \frac{1/2}{s} + \frac{(-s-\frac{1}{2})}{2s^2+s+2} \\ &= \frac{1/2}{s} - \frac{1}{2} \cdot \frac{(s+\frac{1}{4}) + \frac{1}{4}}{(s+\frac{1}{4})^2 + \frac{15}{16}}. \end{aligned}$$

Thus,

$$H(s) = \frac{1/2}{s} - \frac{1}{2} \cdot \frac{(s+\frac{1}{4})}{(s+\frac{1}{4})^2 + \frac{15}{16}} - \frac{1}{8}\frac{4}{\sqrt{15}} \cdot \frac{\sqrt{15}/4}{(s+\frac{1}{4})^2 + \frac{15}{16}}.$$

Therefore,

$$h(t) = \frac{1}{2} - \frac{1}{2}e^{-\frac{t}{4}} \cos\frac{\sqrt{15}t}{4} - \frac{1}{2\sqrt{15}}e^{-\frac{t}{4}} \sin\frac{\sqrt{15}t}{4}.$$

Note that the function $y(t)$ is defined everywhere and $y'(t)$ exists everywhere, but $y''(t)$ does not exist at $t = 5, 20$. Thus, $y(t)$ is a solution only in the intervals $(0, 5), (5, 20), (20, \infty)$, and not throughout. You could have done these three cases directly by earlier methods as well.

WISH YOU ALL THE VERY BEST