MA 108 - Ordinary Differential Equations

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Outline of the lecture

- Picard's iteration
- Second order linear equations

Let R be a rectangle containing (x_0, y_0) in the domain D,

- $f(x, y)$ be continuous at all points (x, y) in $R : |x - x_0| < a,\ |y - y_0| < b$ and
- bounded in R, that is, $|f(x, y)| \leq K\ \forall (x, y) \in R$.

Then, the IVP $\boxed{y' = f(x, y),\ y(x_0) = y_0}$ has at least one solution $y(x)$ defined for all x in the interval $|x - x_0| < \alpha$, where

$$\alpha = \min\left\{a, \frac{b}{K}\right\}.$$

In addition to the above conditions, if f satisfies the Lipschitz condition with respect to y in R, that is,

$$|f(x, y_1) - f(x, y_2)| \leq M|y_1 - y_2|\ \ \forall (x, y_1), (x, y_2) \text{ in } R,$$

then, the solution $y(x)$ defined at least for all x in the interval $|x - x_0| < \alpha$, with α defined above is unique [1].

[1] Existence - Peano, Existence & uniqueness -Picard

Recall : Picard's iteration method

[2] AIM : To solve

$$y' = f(x,y), \; y(x_0) = y_0 \tag{1}$$

METHOD

1. Integrate both sides of (1) to obtain

$$y(x) - y(x_0) = \int_{x_0}^{x} f(t, y(t))\, dt$$

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt \tag{2}$$

Note that any solution of (1) is a solution of (2) and vice-versa.

[2] Picard used this in his existence-uniqueness proof

Picard's method

2. Solve (2) by iteration:

$$\begin{aligned} y_1(x) &= y_0 + \int_{x_0}^{x} f(t, y_0)\, dt \\ y_2(x) &= y_0 + \int_{x_0}^{x} f(t, y_1(t))\, dt \\ &\vdots \\ y_n(x) &= y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\, dt \end{aligned}$$

3. Under the assumptions of existence-uniqueness theorem, the sequence of approximations converges to the solution $y(x)$ of (1). That is,

$$y(x) = \lim_{n\to\infty} y_n(x).$$

Uniqueness of solution

Suppose ϕ_1 and ϕ_2 are both solutions of $\boxed{y' = f(x, y),\ y(x_0) = y_0}$.
Thus, both these satisfy the integral equation

$$\phi_i(x) = y_0 + \int_{x_0}^{x} f(t, \phi_i(t))\, dt \qquad i = 1, 2.$$

For $x \geq x_0$,

$$\phi_1(x) - \phi_2(x) = \int_{x_0}^{x} (f(t, \phi_1(t)) - f(t, \phi_2(t)))\, dt.$$

Thus,

$$|\phi_1(x) - \phi_2(x)| \leq \int_{x_0}^{x} |f(t, \phi_1(t)) - f(t, \phi_2(t))|\, dt.$$

Since f satisfies Lipschitz condition w.r.t. the second variable, we have

$$|f(t, \phi_1(t)) - f(t, \phi_2(t))| \leq M|\phi_1(t) - \phi_2(t)|.$$

Uniqueness - proof contd..

That is,

$$
\begin{aligned}
|\phi_1(x) - \phi_2(x)| &\leq \int_{x_0}^{x} |f(t, \phi_1(t) - f(t, \phi_2(t)|\, dt \\
&\leq \int_{x_0}^{x} M|\phi_1(t) - \phi_2(t)|\, dt \qquad (3)
\end{aligned}
$$

Set $U(x) = \int_{x_0}^{x} |\phi_1(t) - \phi_2(t)|\, dt$. Then,

$$U(x_0) = 0, \; U(x) \geq 0, \; \forall x \geq x_0.$$

Further, $U(x)$ is differentiable and

$$U'(x) = |\phi_1(x) - \phi_2(x)|.$$

Hence, (3) yields

$$U'(x) - MU(x) \leq 0.$$

Multiplying the above equation by e^{-Mx} gives

$$e^{-Mx}U(x)' - Me^{-Mx}U(x) \leq 0 \text{ for } x \geq x_0.$$

$\implies (e^{-Mx}U(x))' \leq 0$ for $x \geq x_0$. Integrating this from x_0 to x we get $e^{-Mx}U(x) \leq 0$ for $x \geq x_0$, i.e., $U(x) \leq 0$ for $x \geq x_0$. So

$$U(x) = 0 \; \forall x \geq x_0 \implies U'(x) \equiv 0 \implies \phi_1(x) \equiv \phi_2(x)$$

which contradicts the initial hypothesis.
Use a similar argument to show for $x \leq x_0$.
Thus, $\boxed{\phi_1(x) \equiv \phi_2(x).}$

Examples

1. If $p(x)$, $q(x)$ are continuous functions in $\overline{I}$ where I is a finite open interval, prove that $y'(x) + p(x)y(x) = q(x), \ y(x_0) = y_0$ for $x_0 \in I$ has a unique solution .

Proof:

$$y' = q(x) - p(x)y.$$

Since $p(x), q(x)$ are continuous on $\overline{I}$, there exists $M > 0$ such that

$$|p(x)| \leq M, \ |q(x)| \leq M, \ x \in I.$$

So, $f(x, y) = q(x) - p(x)y$ satisfies the following. For $x \in I$

$$|f(x, y_2) - f(x, y_1)| = |p(x)||y_2 - y_1| \leq M|y_2 - y_1|$$

for some $M \geq 0$. i.e., $f(x, y)$ is Lipschitz continuous w.r.to. y on $D = I \times \mathbb{R}$.

Examples

Using method of integrating factors, we can see that the ODE has a solution

$$y(x) = e^{-\int_{x_0}^{x} p(t)dt}\left(y_0 + \int_{x_0}^{x} e^{\int_{x_0}^{t} p(s)ds} q(t)dt\right), \;\; x \in I.$$

Hence, by using the above uniqueness result, the IVP has a unique solution on I.

Summary - First Order Equations

- Linear Equations - Solution
  - Reducible to linear - Bernoulli
- Non-linear equations
  - Variable separable
  - Reducible to variable separable
  - Exact equations - Integrating factors
  - Reducible to Exact
- Existence & Uniqueness results for IVP :

  $$\boxed{y' = f(x, y), \;\; y(x_0) = y_0}$$

  - Peano's existence theorem
  - Picard's existence-uniqueness theorem
- Picard's iteration method

Recall that a general second order linear ODE is of the form

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

An ODE of the form

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)$$

is called a second order linear ODE in standard form.
Though there is no formula to find all the solutions of such an ODE, we study the existence, uniqueness and number of solutions of such ODE's.

Homogeneous Linear Second Order DE

If $r(x) \equiv 0$ in the equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x),$$

that is,

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0,$$

then the ODE is said to be homogeneous.
Otherwise it is called non-homogeneous.

An initial value problem of a second order homogeneous linear ODE is of the form:

$$y'' + p(x)y' + q(x)y = 0,\ \ y(x_0) = a, y'(x_0) = b,$$

where $p(x)$ and $q(x)$ are assumed to be continuous on an open interval I with $x_0 \in I$, has a unique solution $y(x)$ in the interval I.

Linearly independent functions & Wronskian

Definition

The functions $\phi_1(x)$ and $\phi_2(x)$ are said to be linearly independent on an open interval I if

$$C_1\phi_1(x) + C_2\phi_2(x) = 0 \ \ \forall x \in I \Longrightarrow C_1 = C_2 = 0.$$

Definition

Wronskian determinant

The Wronskian $W(y_1, y_2)$ of two differentiable functions $y_1(x)$ and $y_2(x)$ is defined by

$$W(y_1, y_2)(x) := \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}.$$

Test for l.i.

Suppose that

$$\boxed{y'' + p(x)y' + q(x)y = 0}$$

has continuous coefficients on an open interval I. Then

1. two solutions y_1 and y_2 of the DE on I are linearly dependent iff their Wronskian is 0 at some $x_0 \in I$.
2. Wronskian $=0$ for some $x = x_0 \implies W \equiv 0$ on I.
3. if there exists an $x_1 \in I$ at which $W \neq 0$, then y_1 and y_2 are linearly independent on I.

Proof

1. two solutions y_1 and y_2 of the DE on I are linearly dependent iff their Wronskian is 0 at some $x_0 \in I$.

Let y_1, y_2 be linearly dependent. Then, $y_1(x) = ky_2(x)$, for some constant k. Then,

$$W(y_1, y_2) = W(ky_2, y_2) = \begin{vmatrix} ky_2 & y_2 \\ (ky_2)' & y_2' \end{vmatrix} \equiv 0.$$

Proof

Conversely, let $W(y_1, y_2) = 0$ for some $x_0 \in I$. That is,

$$W(y_1, y_2)(x_0) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = 0.$$

Consider the linear system of equations :

$$\begin{aligned} k_1 y_1(x_0) + k_2 y_2(x_0) &= 0 \\ k_1 y_1'(x_0) + k_2 y_2'(x_0) &= 0 \end{aligned}$$

$W(y_1, y_2)(x_0) = 0 \implies \exists$ non-trivial k_1 & k_2 solving the above linear system.
Let $y(x) = k_1 y_1(x) + k_2 y_2(x)$ for this choice of k_1 and k_2. Then $y(x)$ solves the DE. Now, $y(x_0) = y'(x_0) = 0$.
By existence-uniqueness theorem, $y(x) \equiv 0$ is the unique solution of $y'' + p(x)y' + q(x)y = 0;\ y(x_0) = 0, y'(x_0) = 0$. Thus $k_1 y_1(x) + k_2 y_2(x) = 0$ with k_1, k_2 both not identically zero. Hence, y_1, y_2 are linearly dependent.

Proof contd..

$\boxed{\text{2. Wronskian } =0 \text{ for some } x = x_0 \implies W \equiv 0 \text{ on } I.}$

If Wronskian $=0$ for some $x = x_0$, then by the first part of the result, y_1 & y_2 are linearly dependent
$\implies$ for some constant k, $y_1 = ky_2 \implies W(y_1, y_2)(x) = 0 \;\; \forall x \in I$.

$\boxed{\text{3. if there exists an } x_1 \in I \text{ at which } W \neq 0\text{, then } y_1 \text{ and } y_2 \text{ are l.i. on } I.}$

$W(y_1, y_2)(x_1) \neq 0 \implies W(y_1, y_2)(x) \neq 0, x \in I$ (Using 2.)
$\implies y_1$ & y_2 can't be linearly dependent
$\implies y_1$ & y_2 are l.i.

Uniqueness Proof

The uniqueness of the IVP played a crucial role in establishing 1. in the previous result. So will see a proof of the uniqueness part of the existence uniqueness solution of the IVP.

Uniqueness: Let $p(x), q(x)$ be continuous on an open interval I and $x_0 \in I$. Then $y(x) = 0, x \in I$ is the only solution of

$$y'' + p(x)y' + q(x)y = 0, \;\; y(x_0) = 0 = y'(x_0).$$

Proof : Let $y(x)$ is a solution of the IVP. Consider the function

$$E(x) = y(x)^2 + y'(x)^2, x \in I.$$

E is called an energy function.

Uniqueness Proof

For $x > x_0, x \in I$, let

$$M = \text{l.u.b.}\{|p(t)| + |q(t)|| x_0 \leq t \leq x\}.$$

Then for $x_0 \leq t \leq x$

$$\begin{aligned}
E'(t) &= 2yy' + 2y'y'' \\
&= 2yy' - 2y'(p(t)y' + q(t)y) \\
&= 2yy'(1 - q(t)) - 2p(t)(y')^2 \\
&\leq 2|yy'|(1 + M) + 2(y')^2 M \\
&\leq (y^2 + (y')^2)(1 + M) + 2(y^2 + (y')^2)M \\
&= (1 + 3M)E(t).
\end{aligned}$$

i.e.

$$E'(t) - (1 + 3M)E(t) \leq 0, \;\; x \geq t \geq x_0.$$

Uniqueness Proof

Multiply the above differential inequality by $e^{-(1+3M)t}$ we get

$$\left(e^{-(1+3M)t}E(t)\right)' \leq 0$$

Integrate the above from x_0 to x, we get

$$e^{-(1+3M)x}E(x) \leq e^{-(1+3M)x_0}E(x_0) = 0 \implies E(x) = 0.$$

A similar argument implies $E(x) = 0$ for $x < x_0, x \in I$.
Hence $y(x) = 0$ for all $x \in I$.

Examples

1. Consider the DE

$$x^2y'' - 4xy' + 6y = 0.$$

Then, x^2 and x^3 are linearly independent solutions, but
$W(x^2, x^3) = x^4$ and so $W(x^2, x^3)(0) = 0$.
Does this contradicts the result 2 in the previous slide ?
Useful observations: Note that $p(x) = -\frac{4}{x}$, $q(x) = \frac{6}{x^2}, x \neq 0$ and $p(0), q(0)$ are not defined.
The IVP with initial condition $y(0) = y'(0) = 0$ has many solutions

2. Consider $y_1(x) = x^2$ and

$$y_2(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0, \end{cases}$$

Then, $W(y_1, y_2)(x) = 0$ for all $x \in \mathbb{R}$, but y_1 and y_2 are linearly independent.
Does it contradict the result in the previous slide? No.

Basis of solutions

Definition

A basis or fundamental set of solutions of $y'' + p(x)y' + q(x)y = 0$ on an interval I is a pair $y_1,\ y_2$ of linearly independent solutions of $y'' + p(x)y' + q(x)y = 0$ on I.

Result : If $p(x)$ and $q(x)$ are continuous on an open interval I, then $y'' + p(x)y' + q(x)y = 0$ has a basis of solutions on I.

Proof : Consider the IVP's

$$y'' + p(x)y' + q(x)y = 0, \;\; y(x_0) = 1, \; y'(x_0) = 0$$
$$y'' + p(x)y' + q(x)y = 0, \;\; y(x_0) = 0, \; y'(x_0) = 1$$

By existence-uniqueness theorem of IVP, the above problems have unique solutions $y_1(x)$ and $y_2(x)$ respectively on I.

Now, $W(y_1, y_2)(x_0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0 \Longrightarrow y_1 \,\& y_2$ are l.i. Why?

Hence, they form a basis of solutions of $y'' + p(x)y' + q(x)y = 0$.