PH 108 : Electricity & Magnetism
Solution Booklet

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Contents

1 Vector Calculus

1.1. Calculate the curl and divergence of the following vector functions. If the curl turns out to be zero, construct a scalar function ϕ of which the vector field is the gradient:

(a) $F_x = x + y$; $F_y = -x + y$; $F_z = -2z$

Solution:

$$\vec{F}(x, y, z) = (x + y, -x + y, -2z) \tag{1.1.1}$$

$$\text{Curl: } \nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + y & -x + y & -2z \end{vmatrix} \tag{1.1.2}$$

$$= 0\hat{\imath} - 0\hat{\jmath} + (-1 - 1)\hat{k} = \boxed{-2\,\hat{k}} \tag{1.1.3}$$

$$\text{Div: } \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x + y) + \frac{\partial}{\partial y}(-x + y) + \frac{\partial}{\partial z}(-2z) \tag{1.1.4}$$

$$= 1 + 1 - 2 = \boxed{0} \tag{1.1.5}$$

(b) $G_x = 2y$; $G_y = 2x + 3z$; $G_z = 3y$

Solution:

$$\vec{G}(x, y, z) = (2y, 2x + 3z, 3y) \tag{1.1.6}$$

$$\text{Curl: } \nabla \times \vec{G} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & 2x + 3z & 3y \end{vmatrix} \tag{1.1.7}$$

$$= (3 - 3)\hat{\imath} - 0\hat{\jmath} + (2 - 2)\hat{k} = \boxed{\vec{0}} \tag{1.1.8}$$

$$\text{Div: } \nabla \cdot \vec{G} = \frac{\partial}{\partial x}(2y) + \frac{\partial}{\partial y}(2x + 3z) + \frac{\partial}{\partial z}(3y) \tag{1.1.9}$$

$$= 0 + 0 + 0 = \boxed{0} \tag{1.1.10}$$

$$\phi\text{: } G_x = \frac{\partial}{\partial x}\phi(x, y, z) = 2y \tag{1.1.11}$$

$$\implies \phi(x, y, z) = 2xy + \psi(y, z) \tag{1.1.12}$$

$$\implies \frac{\partial}{\partial y}\phi(x, y, z) = 2x + \frac{\partial}{\partial y}\psi(y, z) = G_y = 2x + 3z \tag{1.1.13}$$

$$\implies \frac{\partial}{\partial y}\psi(y, z) = 3z \tag{1.1.14}$$

$$\implies \psi(y, z) = 3yz + \omega(z) \tag{1.1.15}$$

$$\implies \phi(x, y, z) = 2xy + 3yz + \omega(z) \tag{1.1.16}$$

$$\implies \frac{\partial}{\partial z}\phi(x, y, z) = 3y + \frac{\partial}{\partial z}\omega(z) \tag{1.1.17}$$

$$\implies \omega(z) = C \tag{1.1.18}$$

$$\implies \phi(x, y, z) = \boxed{2xy + 3yz + C} \tag{1.1.19}$$

(c) $H_x = x^2 - z^2$; $H_y = 2$; $H_z = 2xz$

Solution:

$$\vec{H}(x,y,z) = (x^2 - z^2, 2, 2xz) \tag{1.1.20}$$

$$\text{Curl: } \nabla \times \vec{H} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - z^2 & 2 & 2xz \end{vmatrix} \tag{1.1.21}$$

$$= 0\hat{\imath} - (2z + 2z)\hat{\jmath} + 0\hat{k} = \boxed{-4z\,\hat{\jmath}} \tag{1.1.22}$$

$$\text{Div: } \nabla \cdot \vec{H} = \frac{\partial}{\partial x}(x^2 - y^2) + \frac{\partial}{\partial y}(2) + \frac{\partial}{\partial z}(2xz) \tag{1.1.23}$$

$$= 2x + 0 + 2x = \boxed{4x} \tag{1.1.24}$$

1.2. Calculate the Laplacian of the following functions:

(a) $T_a = x^2 + 2xy + 3z + 4$

Solution:

$$\nabla \cdot \nabla T_a = \frac{\partial^2 T_a}{\partial x^2} + \frac{\partial^2 T_a}{\partial y^2} + \frac{\partial^2 T_a}{\partial z^2} \tag{1.2.1}$$

$$= 2 + 0 + 0 \tag{1.2.2}$$

$$= \boxed{2} \tag{1.2.3}$$

(b) $T_b = \sin(x)\sin(y)\sin(z)$

Solution:

$$\nabla \cdot \nabla T_b = \frac{\partial^2 T_b}{\partial x^2} + \frac{\partial^2 T_b}{\partial y^2} + \frac{\partial^2 T_b}{\partial z^2} \tag{1.2.4}$$

$$= \boxed{-3\sin(x)\sin(y)\sin(z)} \tag{1.2.5}$$

$$\tag{1.2.6}$$

(c) $T_c = e^{-5x}\sin(4y)\cos(3z)$

Solution:

$$\nabla \cdot \nabla T_c = \frac{\partial^2 T_c}{\partial x^2} + \frac{\partial^2 T_c}{\partial y^2} + \frac{\partial^2 T_c}{\partial z^2} \tag{1.2.7}$$

$$= 25e^{-5x}\sin(4y)\cos(3x) - 16e^{-5x}\sin(4y)\cos(3x) \tag{1.2.8}$$

$$- 9e^{-5x}\sin(4y)\cos(3x) \tag{1.2.9}$$

$$= \boxed{0} \tag{1.2.10}$$

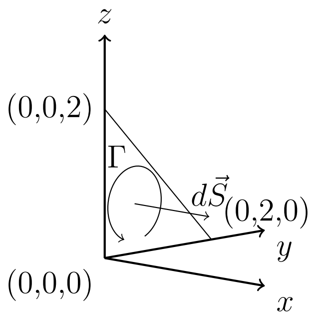
(d) $\vec{v} = x^2\,\hat{\imath} + 3xz^2\,\hat{\jmath} - 2xz\,\hat{k}$

Solution:

$$\nabla^2\vec{v} = \nabla^2 v_x\,\hat{\imath} + \nabla^2 v_y\,\hat{\jmath} + \nabla^2 v_z\,\hat{k} \tag{1.2.11}$$
$$= 2\,\hat{\imath} + 6x\,\hat{\jmath} \tag{1.2.12}$$

1.3. Test the Stokes' theorem for the vector field $\vec{v} = xy\hat{i} + 2yz\hat{j} + 3z\hat{k}$ using a triangular area with vertices $(0,0,0)$, $(0,2,0)$ and $(0,0,2)$.

Solution:



According to Stokes theorem:

$$\oint_\Gamma \vec{v}\cdot d\vec{\Gamma} = \iint_S \nabla\times\vec{v}\cdot d\vec{S} \tag{1.3.1}$$

where

$\vec{v}$ is a vector field (given in our case)

S is the surface (of the triangle) with a given orientation (defined by orientation of Γ)

Γ is the boundary of the surface with a given orientation (orientation specified in figure)

Let us work on the RHS first. For that, we need $d\vec{S}$. We can simply observe that $d\vec{S} = dy\,dz\,\hat{\imath}$. But there might be more complicated surfaces in the future (or in your exams), so we need a more rigorous way of finding $d\vec{S}$. We can use our MA knowledge (existant or not) to parametrize the triangle surface as

$$\vec{r}\,(y,z) = 0\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k};\ \ 0 \le y \le 2,\ 0 \le z \le 2-y \tag{1.3.2}$$

This parametrization is very simple because our surface is really a part of the yz plane. Now $d\vec{S} = \vec{r_y}\times\vec{r_z} = (dy\,\hat{\jmath})\times(dz\,\hat{k}) = dy\,dz\,\hat{\imath}$. Why did we choose $\vec{r_y}\times\vec{r_z}$ and not $\vec{r_z}\times\vec{r_y}$? Because the former leads to the orientation we want. Now since $d\vec{S}$ has only an $\hat{\imath}$ component, we only need the $\hat{\imath}$ component of $\nabla\times\vec{v}$, since it's being dotted with $d\vec{S}$ in the RHS.

$$(\nabla\times\vec{v})_{\hat{\imath}} = \frac{\partial}{\partial y}(3zk) - \frac{\partial}{\partial z}(2yz) = -2y \tag{1.3.3}$$

$$\text{RHS} = \int_{y=0}^{y=2} \int_{z=0}^{z=2-y} -2y \, dz \, dy \tag{1.3.4}$$

$$= -2 \int_{y=0}^{y=2} 2y - y^2 \, dy \tag{1.3.5}$$

$$= -2(4 - \frac{8}{3}) = -\frac{8}{3} \tag{1.3.6}$$

Now we need LHS. Γ can be parametrized as

$$\vec{r_1}(t) = 0\,\hat{\imath} + t\,\hat{\jmath} + (2-t)\,\hat{k};\ 0 \le t \le 2 \implies d\vec{r_1}(t) = dt\,\hat{\jmath} - dt\,\hat{k} \tag{1.3.7}$$

$$\vec{r_2}(t) = 0\,\hat{\imath} + 0\,\hat{\jmath} + t\,\hat{k};\ 0 \le t \le 2 \implies d\vec{r_2}(t) = dt\,\hat{k} \tag{1.3.8}$$

$$\vec{r_3}(t) = 0\,\hat{\imath} + t\,\hat{\jmath} + 0\,\hat{k};\ 0 \le t \le 2 \implies d\vec{r_3}(t) = dt\,\hat{\jmath} \tag{1.3.9}$$

Each of $\vec{r_1}$, $\vec{r_2}$, $\vec{r_3}$ is a parametrization of one side of the triangle.

$$\text{LHS} = \int_2^0 \vec{v} \cdot (\hat{\jmath} - \hat{k})\, dt + \int_2^0 \vec{v} \cdot \hat{k}\, dt + \int_0^2 \vec{v} \cdot \hat{\jmath}\, dt \tag{1.3.10}$$

$$= \int_0^2 v_z - v_y\, dt - \int_0^2 v_z\, dt + \int_0^2 v_y\, dt \tag{1.3.11}$$

$$= \int_0^2 3(2-t)k - 2t(2-t)\, dt \tag{1.3.12}$$

$$- \int_0^2 3tk\, dt \tag{1.3.13}$$

$$+ \int_0^2 2t \cdot 0\, dt \tag{1.3.14}$$

$$= -2 \int_0^2 2t - t^2\, dt \tag{1.3.15}$$

$$= -\frac{8}{3} = \text{RHS} \qquad \blacksquare \tag{1.3.16}$$

1.4. Compute the unit normal vector $\hat{n}$ to the ellipsoidal surfaces defined by constant values of $\Phi(x, y, z) = V\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)$. What is $\hat{n}$ when $a = b = c$?

Solution: The unit normal vector to a surface specified by the equation $\Phi(x, y, z) = C$ is given by the gradient direction $\nabla\Phi$

$$\nabla\Phi = \frac{\partial\Phi}{\partial x}\,\hat{\imath} + \frac{\partial\Phi}{\partial y}\,\hat{\jmath} + \frac{\partial\Phi}{\partial z}\,\hat{k} = V\left(\frac{2x}{a^2}\,\hat{\imath} + \frac{2y}{b^2}\,\hat{\jmath} + \frac{2z}{c^2}\,\hat{k}\right) \tag{1.4.1}$$

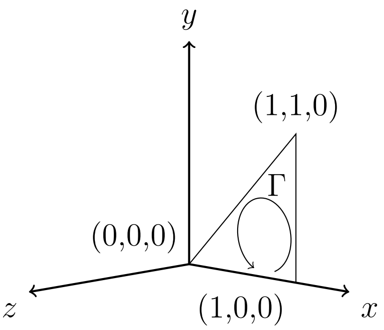
$$\implies \hat{n} = \frac{\nabla\Phi}{|\nabla\Phi|} = \frac{\frac{x}{a^2}\,\hat{\imath} + \frac{y}{b^2}\,\hat{\jmath} + \frac{z}{c^2}\,\hat{k}}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{c^4} + \frac{z^2}{c^4}}} \tag{1.4.2}$$

When $a = b = c$, $\hat{n} = \frac{x\,\hat{\imath}+y\,\hat{\jmath}+z\,\hat{k}}{\sqrt{x^2+y^2+z^2}} = \hat{r}$, which is obvious because the ellipsoids become spheres.

1.5. A force defined by $\vec{F} = A(y^2\hat{i} + 2x^2\hat{j})$ is exerted on a particle which is initially at the origin of the co-ordinate system. A is a positive constant. We transport the particle on a triangular path defined by the points (0,0,0), (1,0,0), (1,1,0) in the counterclockwise direction.

(a) How much work does the force do when the particle travels around the path? Is this a conservative force?

Solution:



We need the work done by the force $= \oint_\Gamma \vec{F} \cdot d\vec{\Gamma}$. Recall MA105 in which a curve was parametrized. We need to parametrize Γ in order to obtain the integral. It might not be absolutely necessary in this example, but starting off this way would make other harder examples easier to solve.

Γ can be parametrized as

$$\vec{r_1}(t) = t\,\hat{\imath} + 0\,\hat{\jmath} + 0\,\hat{k};\ 0 \le t \le 1 \implies d\vec{r_1}(t) = dt\,\hat{\imath} \tag{1.5.1}$$

$$\vec{r_2}(t) = 1\,\hat{\imath} + t\,\hat{\jmath} + 0\,\hat{k};\ 0 \le t \le 1 \implies d\vec{r_2}(t) = dt\,\hat{\jmath} \tag{1.5.2}$$

$$\vec{r_3}(t) = t\,\hat{\imath} + t\,\hat{\jmath} + 0\,\hat{k};\ 0 \le t \le 1 \implies d\vec{r_3}(t) = dt\,\hat{\imath} + dt\,\hat{\jmath} \tag{1.5.3}$$

Each of $\vec{r_1}$, $\vec{r_2}$, $\vec{r_3}$ is a parametrization of one side of the triangle.

$$\mathrm{W} = \int_0^1 \vec{F} \cdot \hat{\imath}\,dt + \int_0^1 \vec{F} \cdot \hat{\jmath}\,dt + \int_1^0 \vec{F} \cdot (\hat{\imath} + \hat{\jmath})\,dt \tag{1.5.4}$$

$$= \int_0^1 F_x\,dt + \int_0^1 F_y\,dt - \int_0^1 F_x + F_y\,dt \tag{1.5.5}$$

$$= A\int_0^1 0^2 + 2 \cdot 1^2 - t^2 - 2t^2\,dt \tag{1.5.6}$$

$$= A\int_0^1 2 - 3t^2\,dt \tag{1.5.7}$$

$$= A \tag{1.5.8}$$

Since the work done in a closed loop is not zero, $\vec{F}$ cannot be conservative. The same can be verified if we calculate $\nabla \times \vec{F}$ which will come out to be non-zero.

(b) The particle is placed at rest right at the origin. Is this a stable situation? Give any argument (mathematical, physical, intuitive) to justify the stability (or instability) of this situation.

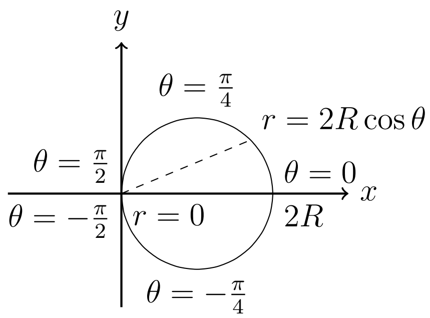
Solution: For the particle to be in a stable equilibrium, the force it experiences upon slight displacement in **any** direction has to be opposite to the displacement. So it can be visualized that if we draw a very small sphere around the origin, the flux of $\vec{F}$ will be negative. So $\nabla \cdot \vec{F}$ should be negative at origin as well.

$$\nabla \cdot \vec{F} = 0 + 0 = 0 \tag{1.5.9}$$

So the particle is **not** in stable equilibrium.

1.6. The area bounded by the curve $r = 2R\cos\theta$ has a surface charge density $\sigma(r,\theta) = \sigma_0 \frac{r}{R}\sin^4\theta$. What is the total amount of charge?

Solution:



The curve is a circle, as we can see, with $-\frac{\pi}{2} \le \theta < \frac{\pi}{2}$

$$\text{Charge } = \iint_{\text{circle}} \sigma\, dA = \sigma_0 \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \int_0^{r=2R\cos\theta} \frac{r}{R}\sin^4\theta\, r\, dr\, d\theta \tag{1.6.1}$$

$$= \frac{8\sigma_0 R^2}{3}\int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \cos^3\theta\, \sin^4\theta\, d\theta \tag{1.6.2}$$

$$= \frac{8\sigma_0 R^2}{3}\int_{\sin\theta=-1}^{\sin\theta=1} \sin^4\theta - \sin^6\theta\, d\sin\theta \tag{1.6.3}$$

$$= \frac{8\sigma_0 R^2}{3}\left(\frac{2}{5} - \frac{2}{7}\right) \tag{1.6.4}$$

$$= \frac{32\sigma_0 R^2}{105} \tag{1.6.5}$$

1.7. Suppose that the height of a certain hill (in feet) is given by

$$h(x, y) = 10(2xy - 3x^2 - 4y^2 + 14x + 10y + 40),$$

where x is the distance (in km) east, y the distance north of the closest town

(a) Where is the top of Sameer Hills located, and how high is it?

Solution: The top of Sameer hills will correspond to a maxima of $h(x, y)$. And we know that $\nabla h = \vec{0}$ at the maxima. So we need to find points satisfying $\nabla h = \vec{0}$.

$$\nabla h = \frac{\partial h}{\partial x}\,\hat{\imath} + \frac{\partial h}{\partial y}\,\hat{\jmath} \tag{1.7.1}$$
$$= 10(2y - 6x + 14)\,\hat{\imath} + 10(2x - 8y + 10)\,\hat{\jmath} \tag{1.7.2}$$
$$\therefore \nabla h = 0 \implies 2y - 6x + 14 = 0;\ 2x - 8y + 10 = 0 \tag{1.7.3}$$
$$\implies (x, y) = (3, 2) \tag{1.7.4}$$
$$h(3, 2) = 710 \tag{1.7.5}$$

So the top of Sameer hills is located 3 km east and 2 km north of the closest town, and it is at a height of 710 feet

(b) How steep is the slope (in feet per km) at a point 1 km north and 1 km east of Hostel 16? In what direction is the slope steepest, at that point?

Solution:

$$\nabla h(1, 1) = 100\,\hat{\imath} + 40\,\hat{\jmath} \tag{1.7.6}$$

So the steepness of the slope at $(1, 1)$ is $|\nabla h(1, 1)| = 20\sqrt{29}$ and the slope is steepest at $\frac{\nabla h(1,1)}{|\nabla h(1,1)|} = \frac{1}{\sqrt{29}}(5\,\hat{\imath} + 2\,\hat{\jmath})$

1.8. The gradient operator ∇ behaves like a vector in " some sense". For example, divergence of a curl ($\nabla.\nabla \times \vec{A} = 0$) for any $\vec{A}$, may suggest that it is just like $\vec{A}.\vec{B} \times \vec{C}$ being zero if any two vectors are equal. Prove that $\nabla \times \nabla \times \vec{F} = \nabla(\nabla.\vec{F}) - \nabla^2\vec{F}$. To what extent does this look like the well known expansion of $\vec{A} \times \vec{B} \times \vec{C}$?

Solution: We will be using Einstein summation convention. According to this convention:

1. The whole term is summed over any index which appears **twice** in that term
2. Any index appearing only once in the term in not summed over and must be present in both LHS and RHS

Don't be confused by this. The convention only is to simply omit the $\sum$ symbol. You can mentally insert the $\sum$ symbol to make sense of what the summation will look like
Now, as may have been done in the lectures:

$$\vec{A} \cdot \vec{B} = A_i B_i \tag{1.8.1}$$
$$\vec{A} \times \vec{B} = \epsilon_{ijk} A_j B_k\,\hat{e}_i \tag{1.8.2}$$

where $\hat{e}_i$ is the unit vector in i direction
When the $\sum$ symbol is reinserted, this simply means $\sum\limits_{i=x,y,z} \sum\limits_{j=x,y,z} \sum\limits_{k=x,y,z} \epsilon_{ijk} A_j B_k \hat{e}_i$

Now we will represent each component of ∇, $\frac{\partial}{\partial i}$ by ∂_i,

$$\nabla \times \left(\nabla \times \vec{F}\right) = \epsilon_{ijk}\partial_j \left(\nabla \times \vec{F}\right)_k \hat{e}_i = \epsilon_{ijk}\epsilon_{klm}\partial_j\partial_l F_m \,\hat{e}_i = \epsilon_{kij}\epsilon_{klm}\partial_j\partial_l F_m \,\hat{e}_i \tag{1.8.3}$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\partial_j\partial_l F_m \,\hat{e}_i \qquad (\because \epsilon_{ijk}\epsilon ilm = \delta_{jl}\delta_{km} - \delta jm\delta kl) \tag{1.8.4}$$

$$= (\partial_m\partial_i F_m - \partial_l\partial_l F_i)\,\hat{e}_i \qquad (\because \delta_{ij}A_j = A_i) \tag{1.8.5}$$

$$= \partial_i(\partial_m F_m)\,\hat{e}_i - \partial_l\partial_l(F_i\,\hat{e}_i) \tag{1.8.6}$$

$$= \nabla(\nabla \cdot \vec{F}) - \nabla^2\vec{F} \qquad \blacksquare \tag{1.8.7}$$

We know that $\vec{A} \times \vec{B} \times \vec{C} = \vec{B}(\vec{A}\cdot\vec{C}) - (\vec{A}\cdot\vec{B})\vec{C}$. If we write $\vec{A} = \nabla$, $\vec{B} = \nabla$, $\vec{C} = \vec{F}$, we get $\nabla \times \nabla \times \vec{F} = \nabla(\nabla\cdot\vec{F}) - (\nabla\cdot\nabla)\vec{F} = \nabla(\nabla\cdot\vec{F}) - \nabla^2\vec{F}$. So this serves as a fake proof of the identity

2 Curvilinear Coordinates

2.1. Consider a planar surface $\mathcal{S}$ parallel to xy plane bounded by a closed curve $\mathcal{C}$. Consider a vector field

$$\vec{F} = 0.5(-y\,\hat{\imath} + x\,\hat{\jmath})$$

Prove that the value of the integral of this vector field along the curve $\mathcal{C}$ is exactly equal to the area of the planar surface $\mathcal{S}$. Using this find the area inside the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$

Solution: We have to prove some line integral to be equal to some area integral. Stokes' theorem immediately springs to mind (this kind of application of Stokes' theorem on a plane is another theorem called Green's theorem).

$$\oint_{\mathcal{C}} \vec{F}\cdot d\vec{l} = \iint_{\mathcal{S}} \nabla \times \vec{F} \cdot d\hat{n} \tag{2.1.1}$$

$$= \iint_{\mathcal{S}} \hat{k}\cdot dx\,dy\,\hat{k} \tag{2.1.2}$$

$$= \iint_{\mathcal{S}} dx\,dy \tag{2.1.3}$$

$$= \text{Area}(S) \tag{2.1.4}$$

$$\tag{2.1.5}$$

Thus we can use this to find the area enclosed inside any curve. Let $\mathcal{C}$ be the curve defined by $x^{\frac{2}{3}} + y^{\frac{2}{3}}$, and $\mathcal{S}$ be the region enclosed by this curve. Let us parametrize $\mathcal{C}$ using the paremeter t

$$\mathcal{C} \coloneqq \{\, (\cos^3 t, \sin^3 t) \mid 0 \le t < 2\pi \,\} \tag{2.1.6}$$

$$d\vec{l} = dx\,\hat{\imath} + dy\,\hat{\jmath} \tag{2.1.7}$$

$$= -3\cos^2 t \sin t\,\hat{\imath} + 3\sin^2 t\cos t\,\hat{\jmath} \tag{2.1.8}$$

$$\begin{aligned}
\text{Area}(S) &= \oint_{C} \vec{F} \cdot d\vec{l} && (2.1.9)\\
&= \frac{1}{2}\int_{t=0}^{2\pi} (-\sin^3 t)(-3\cos^2 t \sin t) + (\cos^3 t)(3\sin^2 t \cos t)\, dt && (2.1.10)\\
&= \frac{3}{2}\int_{t=0}^{2\pi} \sin^2 t \cos^2 t\, dt && (2.1.11)\\
&= \frac{3}{8}\int_{t=0}^{2\pi} \sin^2 2t\, dt && (2.1.12)\\
&= \frac{3}{16}\int_{t=0}^{4\pi} \sin^2 t\, dt && (2.1.13)\\
&= \frac{3}{4}\int_{t=0}^{\pi} \sin^2 t\, dt && (2.1.14)\\
&= \frac{3\pi}{8} && (2.1.15)
\end{aligned}$$

2.2. A ship sails from the southernmost point of India (6.75^oN, 93.84^oE) to the southernmost point of Africa (34.5^oS, 20.00^oE) following the shortest possible path.

(a) Given that the radius of the earth is 6400 km, what is the distance it has covered?

Solution: We cannot directly calculate the distance. We need the angle subtended by the path on Earth's centre. For that we need to convert to Cartesian coordinates

$$\vec{r_1} = R\sin\theta_1\cos\phi_1\,\hat{\imath} + R\sin\theta_1\sin\phi_1\,\hat{\jmath} + R\cos\theta_1\,\hat{k} = R(-0.067\,\hat{\imath} - 0.991\,\hat{\jmath} + 0.118\,\hat{k}) \tag{2.2.1}$$

$$\vec{r_2} = R\sin\theta_2\cos\phi_2\,\hat{\imath} + R\sin\theta_2\sin\phi_2\,\hat{\jmath} + R\cos\theta_2\,\hat{k} = R(0.774\,\hat{\imath} - 0.282\,\hat{\jmath} - 0.566\,\hat{k}) \tag{2.2.2}$$

Where,

$\mathbf{R}$ = 6400 km

$\theta_1 = 90^o - 6.75^o = 83.25^o$

$\theta_2 = 90^o + 34.5^o = 124.5^o$

$\phi_1 = -93.84^o$

$\phi_2 = -20^o$

Let θ' be the angle between $\vec{r_1}$ and $\vec{r_2}$.

$$\begin{aligned}
\cos\theta' &= \frac{\vec{r_1}\cdot\vec{r_2}}{|\vec{r_1}||\vec{r_2}|} && (2.2.3)\\
&= \frac{R^2(-0.067\cdot 0.774 + 0.991\cdot 0.282 - 0.118\cdot 0.566)}{R\cdot R} = 0.161 && (2.2.4)\\
\implies \theta' &= \cos^{-1} 0.161 = 1.409 \text{ rad} && (2.2.5)
\end{aligned}$$

Now we can get the distance by $R\,\theta' = 6400$ km $\cdot\ 1.409 = 9017$ km

(b) If instead of sailing, one had travelled in an aeroplane - by what percentage would the shortest possible distance change?

Solution: The change in distance will only be due to the change in R, as R will be increased by the cruising altitude of an aeroplane $\approx$ 10 km. So percentage change = $\frac{10}{6400} = 0.15\%$

2.3. Compute the divergence of the vector field given by:

$$\vec{v} = r\cos\theta\,\hat{r} \;+\; r\sin\theta\,\hat{\theta} \;+\; r\sin\theta\cos\phi\,\hat{\phi}$$

Check the divergence theorem for this using the volume of an inverted hemisphere of radius R, resting on the xy plane and centered at the origin.

Solution: Let us first calculate the divergence of $\vec{v}$

$$\nabla\cdot\vec{v} = \frac{1}{h_r h_\theta h_\phi}\left(\frac{\partial(v_r h_\theta h_\phi)}{\partial r} + \frac{\partial(h_r v_\theta h_\phi)}{\partial\theta} + \frac{\partial(h_r h_\theta v_\phi)}{\partial\phi}\right) \tag{2.3.1}$$

where

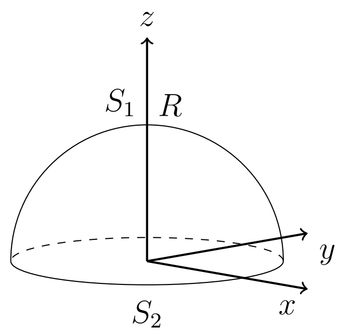
h_r = Scale factor of $r = 1$

h_θ = Scale factor of $\theta = r$

h_ϕ = Scale factor of $\phi = r\sin\theta$

$$\begin{aligned}
\implies \nabla\cdot\vec{v} &= \frac{1}{r^2\sin\theta}\left(\frac{\partial(r^3\sin\theta\cos\theta)}{\partial r} + \frac{\partial(r^2\sin^2\theta)}{\partial\theta} + \frac{\partial(r^2\sin\theta\cos\phi)}{\partial\phi}\right) && (2.3.2)\\
&= \frac{1}{r^2\sin\theta}\left(3r^2\sin\theta\cos\theta + 2r^2\sin\theta\cos\theta - r^2\sin\theta\sin\phi\right) && (2.3.3)\\
&= 5\cos\theta - \sin\phi && (2.3.4)
\end{aligned}$$

Now we need to check the divergence theorem for the hemisphere



According to divergence theorem

$$\iiint_V \nabla \cdot \vec{v}\, dV = \oiint_S \vec{v} \cdot d\vec{S} \tag{2.3.5}$$

Let us first calculate the LHS. For that we need dV. We know that for spherical polar coordinates, $dV = h_r h_\theta h_\phi \, dr\, d\theta\, d\phi = r^2 \sin\theta\, dr\, d\theta\, d\phi$

$$\text{LHS} = \iiint_V \nabla \cdot \vec{v}\, dV = \int_{\phi=0}^{\phi=2\pi} \int_{\phi=0}^{\theta=\frac{\pi}{2}} \int_{r=0}^{r=R} (5\cos\theta - \sin\phi)\, r^2 \sin\theta\, dr\, d\theta\, d\phi \tag{2.3.6}$$

$$= \frac{R^3}{3} \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\frac{\pi}{2}} (5\sin\theta\cos\theta - \sin\theta\sin\phi)\, d\theta\, d\phi \tag{2.3.7}$$

$$= \frac{5\pi R^3}{3} \tag{2.3.8}$$

For the RHS we need $d\vec{S}$. But the surface has 2 parts, the hemispherical cap S_1 and the base S_2. Now for spherical coordinates you know that $d\vec{S_1} = d\vec{S_r} = h_\theta h_\phi \, d\theta\, d\phi\, \hat{r} = r^2 \sin\theta\, d\theta\, d\phi\, \hat{r}$. You can also find this using parametrization. We can see (using visualization or parametrization) that $d\vec{S_2} = r\, dr\, d\phi\, \hat{\theta}$.

$$\vec{v}\Big|_{S_1} = \vec{v}\Big|_{r=R} = R\cos\theta\, \hat{r} + R\sin\theta\, \hat{\theta} + R\sin\theta\cos\phi\, \hat{\phi} \tag{2.3.9}$$

$$\vec{v}\Big|_{S_2} = \vec{v}\Big|_{\theta=\frac{\pi}{2}} = r\, \hat{\theta} + r\cos\phi\, \hat{\phi} \tag{2.3.10}$$

$$\implies \text{RHS} = \oiint_{S_1} \vec{v} \cdot d\vec{S_1} + \oiint_{S_2} \vec{v} \cdot d\vec{S_2} = \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\frac{\pi}{2}} (R\cos\theta)\, R^2 \sin\theta\, d\theta\, d\phi \tag{2.3.11}$$

$$+ \int_{\phi=0}^{\phi=2\pi} \int_{r=0}^{r=R} (r)\, r\, dr\, d\theta \tag{2.3.12}$$

$$= \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\frac{\pi}{2}} R^3 \sin\theta\cos\theta\, d\theta\, d\phi \tag{2.3.13}$$

$$+ \int_{\phi=0}^{\phi=2\pi} \int_{r=0}^{r=R} r^2\, dr\, d\theta \tag{2.3.14}$$

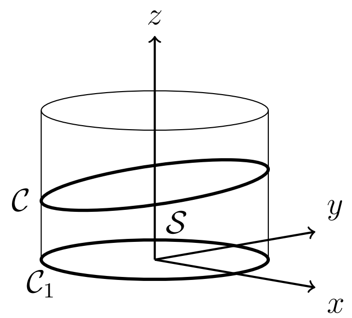
$$= \pi R^3 + \frac{2\pi R^3}{3} = \frac{5\pi R^3}{3} = \text{ LHS} \quad \blacksquare \tag{2.3.15}$$

2.4. Compute the line integral of $\vec{F} = (2xz + y, 2yz + 3x, x^2 + y^2 + 5)$. Use Stokes' theorem to compute

$$\oint_{\mathcal{C}} \vec{F} \cdot \mathrm{d}\vec{r}$$

$\mathcal{C}$ is any positively oriented curve on the surface of the circular cylinder of radius 1 with it's axis along the +ve y axis and with one face as the xy plane with O as centre.

Solution:



Let $\mathcal{C}_1$ be the unit circle on the $z = 0$ plane with the same orientation as $\mathcal{C}$ and $\mathcal{S}$ is the surface on the cylinder enclosed between $\mathcal{C}$ and $\mathcal{C}_1$, with the normal direction such that it matches the orientation of $\mathcal{C}$

Now, by Stokes theorem

$$-\oint_{\mathcal{C}_1} \vec{F} \cdot d\vec{l} + \oint_{\mathcal{C}} \vec{F} \cdot d\vec{l} = \iint_{\mathcal{S}} \nabla \times \vec{F} \cdot d\hat{n} \tag{2.4.1}$$

But,

$$\nabla \times \vec{F} = 2\,\hat{k} \tag{2.4.2}$$

$$\implies \iint_{\mathcal{S}} \nabla \times \vec{F} \cdot d\hat{n} = 0 \tag{2.4.3}$$

Since $d\hat{n} \cdot \hat{k} = 0$

$$\implies \oint_{\mathcal{C}} \vec{F} \cdot d\vec{l} = \oint_{\mathcal{C}_1} \vec{F} \cdot d\vec{l} \tag{2.4.4}$$

Thus the line integral of any closed curve on the cylinder (which goes around the cylinder) is equal to the line integral of the unit circle at $z = 0$ plane centered at origin. Let us find this line integral

Paremetrizing $\mathcal{C}_1$ using the variable t

$$\mathcal{C}_1 := \{\,(\cos t, \sin t, 0) \mid 0 \le t < 2\pi\,\} \tag{2.4.5}$$

$$\implies d\vec{l} = dx\,\hat{\imath} + dy\,\hat{\jmath} \tag{2.4.6}$$

$$= -\sin t\,\hat{\imath} + \cos t\,\hat{\jmath} \tag{2.4.7}$$

$$\implies \oint_{\mathcal{C}_1} \vec{F} \cdot d\vec{l} = \int_{t=0}^{2\pi} (\sin t)(-\sin t) + (3\cos t)(\cos t)\,dt \tag{2.4.8}$$

$$= \boxed{2\pi} \tag{2.4.9}$$

2.5. If $\vec{a}$ and $\vec{b}$ are constant vectors, $\phi(\vec{r}) = (\vec{a} \times \vec{r}) \cdot (\vec{b} \times \vec{r})$ is the potential over all space, then find the electric field and charge density in spherical co-ordinates.
[Hint: $\vec{E} = -\nabla\phi$, $\rho = \epsilon_0 \nabla \cdot \vec{E}$]

Solution: First let us simplify $\phi(\vec{r})$

$$\begin{aligned}
\phi(\vec{r}) &= (\vec{a} \times \vec{r}) \cdot (\vec{b} \times \vec{r}) && (2.5.1)\\
&= \vec{a} \cdot (\vec{r} \times \vec{b} \times \vec{r}) && \because \vec{a} \times \vec{b} \cdot \vec{c} = \vec{a} \cdot \vec{b} \times \vec{c} \quad (2.5.2)\\
&= \vec{a} \cdot ((\vec{r} \cdot \vec{r})\vec{b} - (\vec{b} \cdot \vec{r})\vec{r}) && \text{The } \textit{bac-cab} \text{ rule} \quad (2.5.3)\\
&= r^2(\vec{a} \cdot \vec{b}) - (\vec{a} \cdot \vec{r})(\vec{b} \cdot \vec{r}) && (2.5.4)
\end{aligned}$$

Now to find $\vec{E}$

$$\begin{aligned}
\vec{E} &= -\nabla\phi && (2.5.5)\\
&= -(\vec{a} \cdot \vec{b})\nabla r^2 + \nabla(\vec{a} \cdot \vec{r})(\vec{b} \cdot \vec{r}) && \because (\vec{a} \cdot \vec{b}) \text{ is const, we pull it out} \quad (2.5.6)\\
&= -(\vec{a} \cdot \vec{b})\frac{1}{h_r}\frac{\partial r^2}{\partial r}\,\hat{r} + \nabla(\vec{a} \cdot \vec{r})(\vec{b} \cdot \vec{r}) && \text{Evaluate } \nabla \text{ in spherical} \quad (2.5.7)\\
&= -2r\,\hat{r}(\vec{a} \cdot \vec{b}) + (\vec{b} \cdot \vec{r})\nabla(\vec{a} \cdot \vec{r}) + (\vec{a} \cdot \vec{r})\nabla(\vec{b} \cdot \vec{r}) && \because \text{scalar product rule of gradient} \quad (2.5.8)\\
&= \boxed{-2\vec{r}(\vec{a} \cdot \vec{b}) + (\vec{b} \cdot \vec{r})\vec{a} + (\vec{a} \cdot \vec{r})\vec{b}} && \because \vec{a}, \vec{b} \text{ const} \quad (2.5.9)
\end{aligned}$$

Now to find ρ

$$\begin{aligned}
\rho &= \epsilon_0 \nabla \cdot \vec{E} && (2.5.10)\\
&= -2\epsilon_0(\vec{a} \cdot \vec{b})\nabla \cdot \vec{r} + \epsilon_0 \nabla \cdot ((\vec{b} \cdot \vec{r})\vec{a} + (\vec{a} \cdot \vec{r})\vec{b}) && (2.5.11)\\
&= -2\epsilon_0(\vec{a} \cdot \vec{b})\frac{1}{h_r h_\theta h_\phi}\frac{\partial r h_\theta h\phi}{\partial r} + \epsilon_0 \nabla \cdot ((\vec{b} \cdot \vec{r})\vec{a} + (\vec{a} \cdot \vec{r})\vec{b}) && \text{Evaluate } \nabla\cdot \text{ in spherical} \quad (2.5.12)\\
&= -6\epsilon_0(\vec{a} \cdot \vec{b}) + \epsilon_0((\vec{b} \cdot \vec{r})\nabla \cdot \vec{a} + \vec{a} \cdot \nabla(\vec{b} \cdot \vec{r})) && (2.5.13)\\
&\quad + \epsilon_0((\vec{a} \cdot \vec{r})\nabla \cdot \vec{b} + \vec{b} \cdot \nabla(\vec{a} \cdot \vec{r})) && \because \text{product rule of div} \quad (2.5.14)\\
&= -6\epsilon_0(\vec{a} \cdot \vec{b}) + \epsilon_0(\vec{a} \cdot \nabla(\vec{b} \cdot \vec{r}) + \vec{b} \cdot \nabla(\vec{a} \cdot \vec{r})) && (2.5.15)\\
&= -6\epsilon_0(\vec{a} \cdot \vec{b}) + \epsilon_0(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a}) && \because \text{scalar product rule of gradient} \quad (2.5.16)\\
&= \boxed{-4\epsilon_0(\vec{a} \cdot \vec{b})} && (2.5.17)
\end{aligned}$$

2.6. A vector field is given by

$$\vec{v} = ay\,\hat{i} + bx\,\hat{j}$$

where a, b are constants.

(a) Find the line integral of this field over a circular path of radius R, lying in the xy plane and centered at the origin using

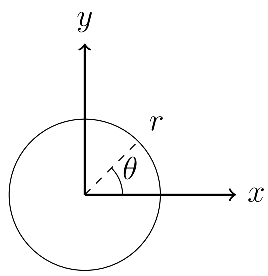
i. Plane polar

Solution: Let us take the anti-clockwise line integral. We first need to write $\vec{v}$ in terms of polar coordinates.

$$\begin{pmatrix} \hat{\imath} \\ \hat{\jmath} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{r} \\ \hat{\theta} \end{pmatrix} \tag{2.6.1}$$

$$\vec{v} = ay\,\hat{\imath} + bx\,\hat{\jmath} = (ay\cos\theta + bx\sin\theta)\,\hat{r} + (-ay\sin\theta + bx\cos\theta)\,\hat{\theta} \tag{2.6.2}$$

$$= r\cos\theta\sin\theta(a+b)\,\hat{r} + r(-a\sin^2\theta + b\cos^2\theta)\,\hat{\theta} \tag{2.6.3}$$



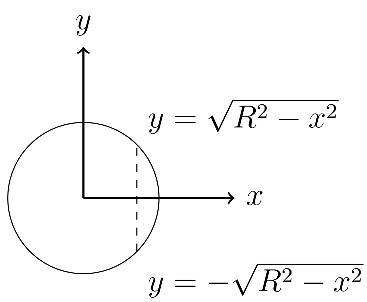
For a circle, $r = R$

$$\implies d\vec{\Gamma} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} = \left(r\,\hat{\theta} + \frac{dr}{d\theta}\,\hat{r}\right)\,d\theta = r\,d\theta\,\hat{\theta} \tag{2.6.4}$$

$$\oint_\Gamma \vec{v}\cdot d\vec{\Gamma} = \int_{\theta=0}^{\theta=2\pi} R^2(-a\sin^2\theta + b\cos^2\theta)\,d\theta = (b-a)\pi R^2 \tag{2.6.5}$$

ii. Cartesian

Solution:



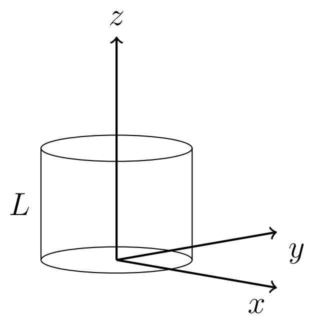
Let us divide the circle into two parts $y \geq 0$ & $y < 0$. For a circle, $y = \pm\sqrt{R^2 - x^2}$

$$\implies d\vec{\Gamma} = dx\,\hat{\imath} + dy\,\hat{\jmath} = \left(\hat{\imath} + \frac{dy}{dx}\,\hat{\jmath}\right)\,dx = \left(\hat{\imath} \mp \frac{x}{\sqrt{R^2 - x^2}}\,\hat{\jmath}\right)\,dx \tag{2.6.6}$$

$$\oint_\Gamma \vec{v}\cdot d\vec{\Gamma} = \int_{x=R}^{x=-R} (a\sqrt{R^2-x^2}\,\hat{\imath} + bx\,\hat{\jmath})\cdot(\hat{\imath} - \frac{x}{\sqrt{R^2-x^2}}\,\hat{\jmath})\,dx \tag{2.6.7}$$
$$+ \int_{x=-R}^{x=R} (-a\sqrt{R^2-x^2}\,\hat{\imath} + bx\,\hat{\jmath})\cdot(\hat{\imath} + \frac{x}{\sqrt{R^2-x^2}}\,\hat{\jmath})\,dx \tag{2.6.8}$$
$$= \int_{x=-R}^{x=R} -2a\sqrt{R^2-x^2} + \frac{2bx^2}{\sqrt{R^2-x^2}}\,dx \tag{2.6.9}$$
$$= (b-a)\pi R^2 \tag{2.6.10}$$

(b) Imagine a right circular cylinder of length L with its axis parallel to the z axis standing on the circle. Use cylindrical co-ordinate system to show that the Stokes theorem is valid over its surface.

Solution:



According to Stokes theorem,

$$\iint_S \nabla\times\vec{v}\cdot\, d\vec{S} = \oint_\Gamma \vec{v}\cdot d\vec{\Gamma} \tag{2.6.11}$$

Let us find $\nabla\times\vec{v}$

$$\nabla\times\vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ay & bx & 0 \end{vmatrix} = (b-a)\,\hat{k} = (b-a)\,\hat{z} \tag{2.6.12}$$

We need to find $d\vec{S_1}$ of the curved part of the cylinder. We can find it out using parametrization (Ugh! Not MA 105 again!):

$$\vec{r} = R\cos\phi\,\hat{\imath} + R\sin\phi\,\hat{\jmath} + z\,\hat{k} \tag{2.6.13}$$
$$\vec{r}_\phi = (R\sin\phi\,\hat{\imath} + R\cos\phi\,\hat{\jmath})\,d\phi \tag{2.6.14}$$
$$\vec{r}_z = (z\,\hat{k})\,dz \tag{2.6.15}$$
$$d\vec{S} = \frac{\vec{r}_\phi\times\vec{r}_z}{|\vec{r}_\phi\times\vec{r}_z|} = (\cos\phi\,\hat{\imath} + \sin\phi\hat{\jmath})\,d\phi\,dz = d\phi\,dz\,\hat{\rho} \tag{2.6.16}$$

We also need $d\vec{S_2}$ of the top part of the cylinder. It can similarly be found to be

$d\vec{S}_2 = \rho\, d\rho\, d\phi\, \hat{z}$

$$\text{LHS} = \iint_S \nabla \times \vec{v} \cdot d\vec{S} = \int_{\phi=0}^{\phi=2\pi} \int_{z=\frac{L}{2}}^{z=\frac{L}{2}} \cancelto{0}{(b-a)\,\hat{z}\cdot\hat{\rho}\, d\phi\, dz} \tag{2.6.17}$$

$$+ \int_{\phi=0}^{\phi=2\pi} \int_{r=0}^{r=R} (b-a)\,\hat{z}\cdot\hat{z}\,\rho\, d\rho\, d\phi \tag{2.6.18}$$

$$= (b-a)\pi R^2 = \text{ RHS} \qquad \blacksquare \tag{2.6.19}$$

2.7. Although the gradient, divergence and curl theorems are the fundamental integral theorems of vector calculus, it is possible to derive a number of corollaries from them. Show that:

(a) $\int_V (\nabla T)\, dV = \oint_S T\, d\vec{S}$ [Hint: Let $\vec{v} = \vec{c}T$ where $\vec{c}$ is a constant, in the divergence theorem]

Solution:

$$\iiint_V \nabla \cdot \vec{v}\, dV = \oiint_S \vec{v} \cdot d\vec{S} \tag{2.7.1}$$

$$\implies \iiint_V \nabla \cdot (\vec{c}T)\, dV = \oiint_S (\vec{c}T) \cdot d\vec{S} \tag{2.7.2}$$

$$\implies \cancelto{0}{\iiint_V T\,(\nabla \cdot \vec{c})}\, dV + \vec{c} \cdot \iiint_V (\nabla T)\, dV = \vec{c} \cdot \oiint_S T\, d\vec{S} \tag{2.7.3}$$

$$\implies \iiint_V (\nabla T)\, dV = \oiint_S T\, d\vec{S} \qquad \blacksquare \tag{2.7.4}$$

(b) $\int_V (\nabla \times \vec{v})\, dV = -\oint_S \vec{v} \times d\vec{S}$ [Hint: Replace $\vec{v}$ by $\vec{v} \times \vec{c}$, where $\vec{c}$ is a constant in the divergence theorem]

Solution:

$$\iiint_V \nabla \cdot (\vec{v} \times \vec{c})\, dV = \oiint_S (\vec{v} \times \vec{c}) \cdot d\vec{S} \tag{2.7.5}$$

$$\implies \vec{c} \cdot \iiint_V T\,(\nabla \times \vec{v})\, dV - \cancelto{0}{\iiint_V \vec{v} \cdot (\nabla \times \vec{c})}\, dV = -\vec{c} \cdot \oiint_S \vec{v} \times d\vec{S} \tag{2.7.6}$$

$$\implies \iiint_V (\nabla \times \vec{v})\, dV = -\oiint_S \vec{v} \times d\vec{S} \qquad \blacksquare \tag{2.7.7}$$

3 Helmholtz Theorem, Delta Functions

3.1. Consider a vector field $\vec{F}(\vec{r})$ which dies faster than $\frac{1}{r}$ as $r \to \infty$, show the following results

(a) Using Helmholtz theorem as discussed in Lecture 5, show that $\vec{F}(\vec{r})$ may be written as

$$\vec{F}(\vec{r}) = -\nabla \frac{1}{4\pi} \iiint_{\mathcal{V}} \frac{\nabla' \cdot \vec{F}(\vec{r}')}{|\vec{r} - \vec{r}'|}\, d\tau' + \nabla \times \frac{1}{4\pi} \iiint_{\mathcal{V}} \frac{\nabla' \times \vec{F}(\vec{r}')}{|\vec{r} - \vec{r}'|}\, d\tau'$$

Solution: We have to directly apply Helmholtz Theorem

$$\vec{F}(\vec{r}) = -\nabla U(\vec{r}) + \nabla \times \vec{A}(\vec{r}) \tag{3.1.1}$$

Where

$$U(\vec{r}) = \frac{1}{4\pi} \iiint_{\mathcal{V}} \frac{D(\vec{r}')}{|\vec{r} - \vec{r}'|}\, d\tau' \tag{3.1.2}$$

$$\vec{A}(\vec{r}) = \frac{1}{4\pi} \iiint_{\mathcal{V}} \frac{\vec{C}(\vec{r}')}{|\vec{r} - \vec{r}'|}\, d\tau' \tag{3.1.3}$$

$$D(\vec{r}) = \nabla \cdot \vec{F}(\vec{r}) \tag{3.1.4}$$

$$\vec{C}(\vec{r}) = \nabla \times \vec{F}(\vec{r}) \tag{3.1.5}$$

Simply putting in $U(\vec{r})$ and $\vec{A}(\vec{r})$ into 3.1.1 gives us exactly what we want

(b) Derive the same expression for $\vec{F}(\vec{r})$ using

$$\vec{F}(\vec{r}) = \iiint_{\mathcal{V}} d\tau' \, \vec{F}(\vec{r}')\delta^3(\vec{r} - \vec{r}')$$

boundary of the integral is to be understood at ∞

Hint: Use the following

(i) $-4\pi\delta^3(\vec{r} - \vec{r}') = \nabla^2 \frac{1}{|\vec{r}-\vec{r}'|}$

(ii) $\nabla \times \nabla \times \vec{v} = \nabla\nabla \cdot \vec{f} - \nabla^2 \vec{f}$

(iii) $\nabla \frac{1}{|\vec{r}-\vec{r}'|} = -\nabla' \frac{1}{|\vec{r}-\vec{r}'|}$

(iv) $\nabla \times \frac{\vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|} = -\vec{F}(\vec{r}) \times \nabla \left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$

(v) Q2.7b

Solution:

$$\begin{aligned}
\vec{F}(\vec{r}) &= \iiint_{\mathcal{V}} d\tau' \, \vec{F}(\vec{r}')\delta^3(\vec{r} - \vec{r}') && && (3.1.6)\\
&= -\frac{1}{4\pi} \iiint_{\mathcal{V}} d\tau' \, \vec{F}(\vec{r}')\nabla^2 \left(\frac{1}{|\vec{r} - \vec{r}'|}\right) && \text{Using (i)} && (3.1.7)\\
&= -\frac{1}{4\pi}\nabla^2 \iiint_{\mathcal{V}} \frac{\vec{F}(\vec{r}')}{|\vec{r} - \vec{r}'|} \, d\tau' && \because \vec{r}' \text{ is constant wrt } \nabla && (3.1.8)\\
&= -\frac{1}{4\pi}\nabla \left(\nabla \cdot \left(\iiint_{\mathcal{V}} \frac{\vec{F}(\vec{r}')}{|\vec{r} - \vec{r}'|} \, d\tau'\right)\right) && && (3.1.9)\\
&+ \frac{1}{4\pi}\nabla \times \left(\nabla \times \left(\iiint_{\mathcal{V}} \frac{\vec{F}(\vec{r}')}{|\vec{r} - \vec{r}'|} \, d\tau'\right)\right) && \text{Using (ii)} && (3.1.10)\\
&= -\frac{1}{4\pi}\nabla \left(\iiint_{\mathcal{V}} \vec{F}(\vec{r}') \cdot \nabla \left(\frac{1}{|\vec{r} - \vec{r}'|}\right) \, d\tau'\right) && && (3.1.11)\\
&+ \frac{1}{4\pi}\nabla \times \left(\iiint_{\mathcal{V}} \vec{F}(\vec{r}') \times \nabla \left(\frac{1}{|\vec{r} - \vec{r}'|}\right) \, d\tau'\right) && \because \text{product rules and (iv)} && (3.1.12)\\
&= \frac{1}{4\pi}\nabla \left(\iiint_{\mathcal{V}} \vec{F}(\vec{r}') \cdot \nabla' \left(\frac{1}{|\vec{r} - \vec{r}'|}\right) \, d\tau'\right) && && (3.1.13)\\
&- \frac{1}{4\pi}\nabla \times \left(\iiint_{\mathcal{V}} \vec{F}(\vec{r}') \times \nabla' \left(\frac{1}{|\vec{r} - \vec{r}'|}\right) \, d\tau'\right) && \text{Using (iii)} && (3.1.14)
\end{aligned}$$

$$= \frac{1}{4\pi}\nabla\left(\iiint_{\mathcal{V}} \nabla'\cdot\left(\frac{\vec{F}(\vec{r'})}{|\vec{r}-\vec{r'}|}\right)d\tau'\right) \tag{3.1.15}$$

$$- \frac{1}{4\pi}\nabla\left(\iiint_{\mathcal{V}} \frac{\nabla'\cdot\vec{F}(\vec{r'})}{|\vec{r}-\vec{r'}|}\,d\tau'\right) \tag{3.1.16}$$

$$- \frac{1}{4\pi}\nabla\times\left(\iiint_{\mathcal{V}} \nabla\times\left(\frac{\vec{F}(\vec{r'})}{|\vec{r}-\vec{r'}|}\right)d\tau'\right) \tag{3.1.17}$$

$$+ \frac{1}{4\pi}\nabla\times\left(\iiint_{\mathcal{V}} \frac{\nabla\times\vec{F}(\vec{r'})}{|\vec{r}-\vec{r'}|}\,d\tau'\right) \qquad \because \text{product rules of div and curl} \tag{3.1.18}$$

$$= \frac{1}{4\pi}\nabla\left(\cancelto{0}{\oiint_{\mathcal{S}} \frac{\vec{F}(\vec{r'})}{|\vec{r}-\vec{r'}|}\cdot d\vec{S'}}\right) \tag{3.1.19}$$

$$- \frac{1}{4\pi}\nabla\left(\iiint_{\mathcal{V}} \frac{\nabla'\cdot\vec{F}(\vec{r'})}{|\vec{r}-\vec{r'}|}\,d\tau'\right) \tag{3.1.20}$$

$$+ \frac{1}{4\pi}\nabla\times\left(\cancelto{0}{\oiint_{\mathcal{S}} \left(\frac{\vec{F}(\vec{r'})}{|\vec{r}-\vec{r'}|}\right)\times d\vec{S'}}\right) \tag{3.1.21}$$

$$+ \frac{1}{4\pi}\nabla\times\left(\iiint_{\mathcal{V}} \frac{\nabla\times\vec{F}(\vec{r'})}{|\vec{r}-\vec{r'}|}\,d\tau'\right) \qquad \text{Using Div theorem and 2.7b} \tag{3.1.22}$$

$$= -\frac{1}{4\pi}\nabla\left(\iiint_{\mathcal{V}} \frac{\nabla'\cdot\vec{F}(\vec{r'})}{|\vec{r}-\vec{r'}|}\,d\tau'\right) + \frac{1}{4\pi}\nabla\times\left(\iiint_{\mathcal{V}} \frac{\nabla\times\vec{F}(\vec{r'})}{|\vec{r}-\vec{r'}|}\,d\tau'\right) \qquad \blacksquare \tag{3.1.23}$$

3.2. (a) Using the identity $\delta(ax) = \frac{\delta(x)}{|a|}$, $a \neq 0$, prove that:

$$\delta(g(x)) = \sum_{m \text{ s.t } g(x_m)=0,\ g'(x_m)\neq 0} \frac{\delta(x-x_m)}{|g'(x_m)|}$$

Solution: Whenever we want to prove equality of two expressions involving delta functions, we need to prove them under the integral sign with an arbitrary test function

$$\int_{x=-\infty}^{\infty} f(x)\delta(g(x))\,dx \tag{3.2.1}$$

$$= \sum_{m \text{ s.t } g(x_m)=0} \int_{x=x_m-\epsilon_0}^{x_m+\epsilon_0} f(x)\delta(g(x))\,dx \tag{3.2.2}$$

$$= \sum_{m \text{ s.t } g(x_m)=0} \int_{\epsilon=-\epsilon_0}^{\epsilon_0} f(x_m+\epsilon)\delta(g(x_m)+\epsilon g'(x_m)+\mathcal{O}(\epsilon^2))\,d\epsilon \quad \text{Taylor Expansion} \tag{3.2.3}$$

$$= \sum_{m \text{ s.t } g(x_m)=0} \int_{\epsilon=-\epsilon_0}^{\epsilon_0} f(x_m+\epsilon)\delta(\epsilon g'(x_m)+\mathcal{O}(\epsilon^2))\,d\epsilon \quad g(x_m)=0 \tag{3.2.4}$$

$$= \sum_{m \text{ s.t } g(x_m)=0,\ g'(x_m)\neq 0} \int_{\epsilon=-\epsilon_0}^{\epsilon_0} f(x_m+\epsilon)\frac{\delta(\epsilon)}{|g'(x_m)|}\,d\epsilon \quad \text{Using given identity} \tag{3.2.5}$$

$$= \sum_{m \text{ s.t } g(x_m)=0,\ g'(x_m)\neq 0} \int_{x=x_m-\epsilon_0}^{x_m+\epsilon_0} f(x)\frac{\delta(x-x_m)}{|g'(x_m)|}\,dx \tag{3.2.6}$$

$$= \int_{-\infty}^{\infty} f(x) \sum_{m \text{ s.t } g(x_m)=0,\ g'(x_m)\neq 0} \frac{\delta(x-x_m)}{|g'(x_m)|}\,dx \tag{3.2.7}$$

Since this works for any arbitrary ordinary function $f(x)$, the expressions $\delta(g(x))$ and $\sum_{m \text{ s.t } g(x_m)=0,\ g'(x_m)\neq 0} \frac{\delta(x-x_m)}{|g'(x_m)|}$ are equal

(b) Confirm that $I = \int_{x=0}^{\infty} \delta(\cos x)e^{-x}\,dx = \frac{1}{2\sinh\left(\frac{\pi}{2}\right)}$

Solution: We need x_m s.t. $\cos(x_m) = 0, -\sin(x_m) \neq 0$
Thus $x_m = (2n+1)\frac{\pi}{2}$, $n \in \mathbb{I}$
We have $|g'(x_m)| = 1\ \forall\ n \in \mathbb{I}$
Thus,

$$I = \int_{x=0}^{\infty} \delta(\cos x)e^{-x}\,dx = \int_{x=0}^{\infty} \sum_{n\in\mathbb{N}} \delta\left(x-(2n+1)\frac{\pi}{2}\right)e^{-x}\,dx \tag{3.2.8}$$

$$= \sum_{n\in\mathbb{N}} e^{-(2n+1)\frac{\pi}{2}} \tag{3.2.9}$$

$$= e^{-\frac{\pi}{2}}\frac{1}{1-e^{-\pi}} \tag{3.2.10}$$

$$= \frac{1}{e^{\frac{\pi}{2}}-e^{-\frac{\pi}{2}}} \tag{3.2.11}$$

$$= \frac{1}{2\sinh\left(\frac{\pi}{2}\right)} \tag{3.2.12}$$

(c) Show that,

$$\lim_{m\to\infty} \frac{\sin \text{ mx}}{\pi x}$$

is a representation of $\delta(x)$ by showing that $\int_{-\infty}^{\infty} dx f(x) D(x) = f(0)$

Solution:

$$\int_{\mathbb{R}} f(x)D(x)\,dx = \int_{x=-\infty}^{\infty} f(x) \lim_{m\to\infty} \frac{\sin(mx)}{\pi x}\,dx \tag{3.2.13}$$

$$= \int_{y=-\infty}^{\infty} \lim_{m\to\infty} f\left(\frac{y}{m}\right) \frac{\sin y}{\pi y}\,dy \tag{3.2.14}$$

$$= \int_{y=-\infty}^{\infty} f(0)\frac{\sin y}{\pi y}\,dy \tag{3.2.15}$$

$$= \frac{f(0)}{\pi} \int_{y=-\infty}^{\infty} \frac{\sin y}{y}\,dy \tag{3.2.16}$$

$$= f(0) \tag{3.2.17}$$

Hence $D(x)$ is equivalent to $\delta(x)$

3.3. Show that D($\vec{r}$,ϵ) demonstrates the peak-character & goes to $\delta^3(\vec{r})$ as $\epsilon \to 0$

$$D(\vec{r},\epsilon) = -\frac{1}{4\pi}\nabla^2 \frac{1}{\sqrt{r^2+\epsilon^2}}$$

Hint:

(i) Show that $D(\vec{r},\epsilon) = \frac{3\epsilon^2}{4\pi((r^2+\epsilon^2)^{5/2})}$

(ii) Check that $D(0,\epsilon) \to \infty$ as $\epsilon \to 0$

(iii) Check that $D(\vec{r},\epsilon) \to 0$ as $\epsilon \to 0$, for all r $\neq 0$

(iv) Check that the integral of $D(\vec{r},\epsilon)$ over all the space is 1

Solution: First, we will evaluate $D(r,\epsilon)$ in terms of r and ϵ

$$D(r,\epsilon) = -\frac{1}{4\pi}\nabla^2 \frac{1}{\sqrt{r^2+\epsilon^2}} \tag{3.3.1}$$

$$\implies D(r,\epsilon) = -\frac{1}{4\pi}\nabla\cdot\nabla\left(\frac{1}{\sqrt{r^2+\epsilon^2}}\right) \tag{3.3.2}$$

$$\implies D(r,\epsilon) = \frac{1}{4\pi}\nabla\cdot\frac{r\,\hat{r}}{(r^2+\epsilon^2)^{\frac{3}{2}}} \tag{3.3.3}$$

$$\implies D(r,\epsilon) = \frac{1}{4\pi r^2}\frac{d}{dr}\left(\frac{r^3}{(r^2+\epsilon^2)^{\frac{3}{2}}}\right) \tag{3.3.4}$$

$$\implies D(r,\epsilon) = \frac{3\epsilon^2}{4\pi(r^2+\epsilon^2)^{\frac{5}{2}}} \tag{3.3.5}$$

Now, we need to show that as $\epsilon \to 0$, $D(r, \epsilon)$ behaves like $\delta^3(r)$. For an ordinary function $f(\vec{r})$

$$lim_{\epsilon \to 0} \iiint_{\text{all space}} f(\vec{r}) D(r, \epsilon) d\tau' \tag{3.3.6}$$

$$= \lim_{\epsilon \to 0} \iiint_{\text{all space}} \frac{3\epsilon^2 f(\vec{r})}{4\pi (r^2 + \epsilon^2)^{\frac{5}{2}}} d\tau' \tag{3.3.7}$$

$$= \lim_{\epsilon \to 0} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{\infty} \frac{3\epsilon^2 f(\vec{r})}{4\pi (r^2 + \epsilon^2)^{\frac{5}{2}}} \cdot r^2 \, dr \sin\theta \, d\theta \, d\phi \tag{3.3.8}$$

$$= \lim_{\epsilon \to 0} \int_{r=0}^{\infty} \frac{3\epsilon^2 r^2 f(\vec{r})}{(r^2 + \epsilon^2)^{\frac{5}{2}}} dr \tag{3.3.9}$$

$$= \lim_{\epsilon \to 0} \left(f(\vec{r}) \cdot \int \frac{3\epsilon^2 r^2}{(r^2 + \epsilon^2)^{\frac{5}{2}}} dr \bigg|_{r=0}^{\infty} - \int_{r=0}^{\infty} \left(\int \frac{3\epsilon^2 r^2}{(r^2 + \epsilon^2)^{\frac{5}{2}}} \right) \cdot f'(\vec{r}) \right) \tag{3.3.10}$$

$$= \lim_{\epsilon \to 0} \left(\frac{f(\vec{r}) r^3}{(r^2 + \epsilon^2)^{\frac{3}{2}}} \bigg| r =_0^{\infty} \right) - \int_{r=0}^{\infty} \lim_{\epsilon \to 0} \left(\frac{f'(\vec{r}) r^3}{(r^2 + \epsilon^2)^{\frac{3}{2}}} \right) \tag{3.3.11}$$

$$= f(0) = \iiint_{\text{all space}} f(\vec{r}) \delta^3(r) d\tau' \tag{3.3.12}$$

Hence $\lim_{\epsilon \to 0} D(\vec{r}, \epsilon)$ is equivalent to $\delta^3(\vec{r})$

3.4. Evaluate the following integral

$$\iiint_{\mathcal{V}} \vec{r} \cdot (\vec{d} - \vec{r}) \delta^3(\vec{e} - \vec{r}) \, d\tau$$

where $\vec{d} = (5, 5, 5)$, $\vec{e} = (15, 19, 17)$, and $\mathcal{V}$ is a sphere of radius 7 centered at $(10, 15, 19)$.

Solution: First, we verify if the vector $\vec{e} \in V$.
$\vec{e} = (15, 19, 17)$ and centre of sphere is $\vec{r_1} = (10, 15, 19)$
Since $\|\vec{e} - \vec{r_1}\| = \sqrt{45} < 7 \implies \vec{e} \in \mathcal{V}$
Using property,

$$\iiint_{\mathcal{V}} f(r') \delta^3(r' - r_0) d\tau' = f(r_0), \qquad r_0 \in \mathcal{V} \tag{3.4.1}$$

the required integral

$$\iiint_{\mathcal{V}} r \cdot (d - r) \delta^3(e - r) d\tau' = e \cdot (d - e) \tag{3.4.2}$$

$$= (15, 19, 17) \cdot (-10, -14, 12) \tag{3.4.3}$$

$$= \boxed{-620} \tag{3.4.4}$$

3.5. Let $\vec{F}$ be a vector field whose divergence and curl are given as

$$\nabla \cdot \vec{F} = \delta(x)\delta(y) \quad \text{and} \quad \nabla \times \vec{F} = \vec{0}$$

Using the Helmholtz theorem, determine $\vec{F}(x, y, z)$.

Solution: We are given that $D(\vec{r'}) = \delta(x')\delta(y')$, $\vec{C}(\vec{r'}) = \vec{0}$. Let us find $U(\vec{r})$ and $\vec{A}(\vec{r})$ of Helmholtz theorem

$$\vec{A}(\vec{r}) = \iiint \frac{\vec{C}(\vec{r'})}{|\vec{r} - \vec{r'}|} \, d\tau' = 0 \tag{3.5.1}$$

$$U(\vec{r}) = \iiint \frac{D(\vec{r'})}{|\vec{r} - \vec{r'}|} \, d\tau' \tag{3.5.2}$$

$$= \int_{x'=-\infty}^{\infty} \int_{y'=-\infty}^{\infty} \int_{z'=-\infty}^{\infty} \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \, dz' \delta(y') \, dy' \delta(x') \, dx' \tag{3.5.3}$$

$$= \int_{z'=-\infty}^{\infty} \frac{1}{\sqrt{x^2 + y^2 + (z-z')^2}} \, dz' \tag{3.5.4}$$

Now, this integral is divergent. (You can guess so, since $\delta(x)\delta(y)$ represents the charge distribution of a charged wire along z axis, whose potential we know does indeed diverge when reference is taken as infinity). Thus we directly attempt to find $\vec{F}$ by calculating the grad of this potential.

$$\implies \vec{F}(\vec{r}) = -\nabla U \tag{3.5.5}$$

$$= -\int_{z'=-\infty}^{\infty} \nabla \frac{1}{\sqrt{x^2 + y^2 + (z-z')^2}} \, dz' \tag{3.5.6}$$

$$= \int_{z'=-\infty}^{\infty} \frac{x}{\sqrt{x^2 + y^2 + (z-z')^2}^3} \, dz' \, \hat{\imath} \tag{3.5.7}$$

$$+ \int_{z'=-\infty}^{\infty} \frac{y}{\sqrt{x^2 + y^2 + (z-z')^2}^3} \, dz' \, \hat{\jmath} \tag{3.5.8}$$

$$+ \int_{z'=-\infty}^{\infty} \frac{z-z'}{\sqrt{x^2 + y^2 + (z-z')^2}^3} \, dz' \, \hat{k} \tag{3.5.9}$$

$$= \left. \frac{x(z-z')}{(x^2+y^2)\sqrt{x^2 + y^2 + (z-z')^2}} \right|_{z'=-\infty}^{\infty} \hat{\imath} \tag{3.5.10}$$

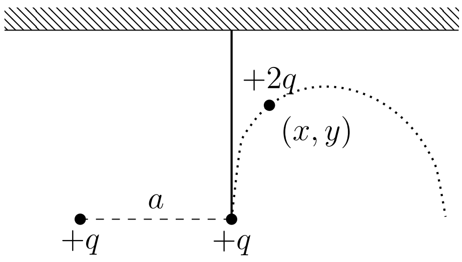
$$+ \left. \frac{y(z-z')}{(x^2+y^2)\sqrt{x^2 + y^2 + (z-z')^2}} \right|_{z'=-\infty}^{\infty} \hat{\jmath} \tag{3.5.11}$$

$$- \left. \frac{1}{\sqrt{x^2 + y^2 + (z-z')^2}} \right|_{z'=-\infty}^{\infty} \hat{k} \tag{3.5.12}$$

$$= \frac{2x}{x^2+y^2} \, \hat{\imath} + \frac{2y}{x^2+y^2} \, \hat{\jmath} \tag{3.5.13}$$

3.6. A small ball with a positive charge $+q$ hangs by an insulating thread.Holding this ball vertical, a second ball having charge $+q$ is kept at a distance a along the horizontal direction. There are an infinite number of points where a third ball with charge $+2q$ may be positioned so that the first ball continues to remain vertical when released. Find the equation of the curve describing these points.

Solution:



We basically want to balance all the forces on the hanging $+q$.
Now,

$$\vec{F}_{\text{thread}} = f\,\hat{\jmath} \tag{3.6.1}$$

$$\vec{F}_{+q} = q\vec{E}_{+q} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{a^2}\,\hat{\imath} \tag{3.6.2}$$

$$\vec{F}_{+2q} = q\vec{E}_{+2q} = -\frac{1}{4\pi\epsilon_0}\frac{2q^2x}{\sqrt{x^2+y^2}^3}\,\hat{\imath} - \frac{1}{4\pi\epsilon_0}\frac{2q^2y}{\sqrt{x^2+y^2}^3}\,\hat{\jmath} \tag{3.6.3}$$

Now we know that the thread can only apply a pulling force, so $f > 0$. So to balance the forces in the y direction, $y > 0$.
Next we balance forces in the x direction

$$\frac{1}{4\pi\epsilon_0}\frac{q^2}{a^2} = \frac{1}{4\pi\epsilon_0}\frac{2q^2x}{\sqrt{x^2+y^2}^3} \tag{3.6.4}$$

$$\implies \left(x^2+y^2\right)^3 = (2a^2x)^2 \tag{3.6.5}$$

$$\implies y^2 = (2a^2x)^{\frac{2}{3}} - x^2 \tag{3.6.6}$$

$$\implies y = +\sqrt{(2a^2x)^{\frac{2}{3}} - x^2} \tag{3.6.7}$$

3.7. After an extremely precise measurement, it was revealed that the actual force between two point charges is given by

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{r^2}\left(1+\frac{r}{\lambda}\right)e^{-\frac{r}{\lambda}}\,\hat{r}$$

Where λ is a constant with dimensions of length, such that $\lambda \gg 1$, hence the correction factor is negligible, but non zero
Does this electric field results from a scalar potential? Justify.
And if yes, find the potential due to a point charge q placed at the origin using infinity as your reference.

Solution: Yes. The field of a point charge at the origin is radial and symmetric, so $\nabla \times \vec{E}(\vec{r}) = \vec{0}$

$$U(\vec{r}) = -\int_{r'=\infty}^{r} \vec{E}(\vec{r'}) \cdot d\vec{l'} = -\frac{1}{4\pi\epsilon_0} q \int_{r'=\infty}^{r} \frac{1}{r'^2}\left(1+\frac{r'}{\lambda}\right) e^{-\frac{r'}{\lambda}}\, dr' \tag{3.7.1}$$

$$= \frac{1}{4\pi\epsilon_0} q \int_{r'=r}^{\infty} \frac{1}{r'^2}\left(1+\frac{r'}{\lambda}\right) e^{-\frac{r'}{\lambda}}\, dr' = \frac{q}{4\pi\epsilon_0}\left(\int_{r'=r}^{\infty} \frac{1}{r'^2} e^{-\frac{r'}{\lambda}}\, dr' + \frac{1}{\lambda}\int_{r'=r}^{\infty} \frac{1}{r'} e^{-\frac{r'}{\lambda}}\, dr'\right) \tag{3.7.2}$$

Now $\int \frac{1}{r'^2} e^{-\frac{r'}{\lambda}}\, dr' = -\frac{e^{-\frac{r'}{\lambda}}}{r'} - \frac{1}{\lambda}\int \frac{e^{-\frac{r'}{\lambda}}}{r'}\, dr' \longleftarrow$ exactly right to kill the last term. Therefore

$$V(\vec{r}) = \frac{q}{4\pi\epsilon_0}\left(-\left.\frac{e^{-\frac{r'}{\lambda}}}{r'}\right|_{r'=r}^{\infty}\right) = \boxed{\frac{q}{4\pi\epsilon_0}\frac{e^{-\frac{r}{\lambda}}}{r}} \tag{3.7.3}$$

3.8. Which one of the following is possible expression for an electrostatic field? For the right expression, find a potential which determines this field with the origin as the reference.

(a) $\mathbf{E} = A\left(x^2yz\,\hat{\imath} + 2xz\,\hat{\jmath} - 3yz\,\hat{k}\right)$

Solution: We know that the curl of the electric field has to be $\mathbf{0}$, so if the curl of this purported electric field is nonzero, then it cannot be the expression of an electrostatic field.
In this case,

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x^2yz & 2xz & -3yz \end{vmatrix} \tag{3.8.1}$$

$$= \left(-3z - 2x\right)\hat{\imath} + x^2y\,\hat{\jmath} + \left(2z - x^2z\right)\hat{k} \tag{3.8.2}$$

$$\neq \mathbf{0} \tag{3.8.3}$$

(∂_x denotes the partial derivative w.r.t. x.)

(b) $\mathbf{E} = A\left(\left[3xz^2 + y^2\right]\hat{\imath} + 2xy\,\hat{\jmath} + 3x^2z\,\hat{k}\right)$

Solution: The curl of this $\mathbf{E}$ is $\mathbf{0}$ (verify), so therefore, this is a possible expression for an electrostatic field.
To find the potential $\Phi(x, y, z)$, we need to solve the system of three partial differential equations:

$$\frac{\partial \Phi}{\partial x} = 3xz^2 + y^2 \tag{3.8.4}$$

$$\frac{\partial \Phi}{\partial y} = 2xy \tag{3.8.5}$$

$$\frac{\partial \Phi}{\partial z} = 3x^2 z \tag{3.8.6}$$

Solving them, we get that

$$\Phi = \frac{3}{2}x^2z^2 + xy^2 + g(y,z) \tag{3.8.7}$$

$$\Phi = xy^2 + h(x,z) \tag{3.8.8}$$

$$\Phi = \frac{3}{2}x^2z^2 + f(x,y) \tag{3.8.9}$$

$$\tag{3.8.10}$$

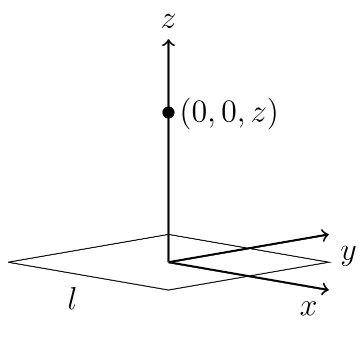
where $f(x,y)$ is a function purely of x and y, $g(y,z)$ is a function purely of y and z, and $h(x,z)$ is a function purely of x and z.

Putting it all together, we get that

$$\Phi(x,y,z) = \frac{3}{2}x^2z^2 + xy^2 \tag{3.8.11}$$

3.9. Find the electric field a distance z above center of a square loop of side l carrying uniform line charge density λ.

Solution:



By symmetry, we know that only the z component of the electric field contributes to the total electric field.

Hence, to find the total electric field, we simply integrate over the z-component of the electric field due to one side of the square loop and then multiply by 4:

$$E_z = \frac{4}{4\pi\varepsilon_0}\int \frac{\mathrm{d}q\sin\theta}{r^2} = \frac{4}{4\pi\varepsilon_0}\int \frac{z\,\mathrm{d}q}{r^3} \tag{3.9.1}$$

$$E_z = \frac{1}{\pi\varepsilon_0}\int_{-l/2}^{l/2} \frac{z\lambda\,\mathrm{d}y}{\left(\left(\frac{l}{2}\right)^2 + z^2 + y^2\right)^{3/2}} \tag{3.9.2}$$

$$E_z = \frac{2z\lambda}{\pi\varepsilon_0}\int_{0}^{l/2} \frac{\mathrm{d}y}{\left(\left(\frac{l}{2}\right)^2 + z^2 + y^2\right)^{3/2}} \tag{3.9.3}$$

Now, to solve the integral, we use the trig. substitution $y = a\tan\theta$, where $a = \left(\frac{l}{2}\right)^2 + z^2$:

$$E_z = \frac{2z\lambda}{\pi\varepsilon_0}\int_{y=0}^{y=l/2} \frac{a\sec^2\theta\,\mathrm{d}\theta}{a^3\sec^3\theta} \tag{3.9.4}$$

$$E_z = \frac{2z\lambda}{\pi\varepsilon_0}\int_{y=0}^{y=l/2} \frac{\cos\theta\,\mathrm{d}\theta}{a^2} \tag{3.9.5}$$

$$E_z = \frac{2z\lambda}{\pi\varepsilon_0}\cdot\left.\frac{\sin\theta}{a^2}\right|_{y=0}^{y=l/2} \tag{3.9.6}$$

$$E_z = \frac{2z\lambda}{\pi\varepsilon_0}\cdot\frac{1}{\left(\frac{l}{2}\right)^2 + z^2}\cdot\left.\frac{y}{\sqrt{y^2 + \frac{l^2}{4} + z^2}}\right|_{y=0}^{y=l/2} \tag{3.9.7}$$

$$E_z = \frac{zl\lambda}{\pi\varepsilon_0\left(\frac{l^2}{4} + z^2\right)\sqrt{\frac{l^2}{2} + z^2}} \tag{3.9.8}$$

Therefore,

$$\vec{F} = \frac{zl\lambda}{\pi\varepsilon_0\left(\frac{l^2}{4} + z^2\right)\sqrt{\frac{l^2}{2} + z^2}}\,\hat{k} \tag{3.9.9}$$

4 Electrostatic Fields, Potentials, Energy

4.1. Consider a conducting sphere A which is initially uncharged. Another conducting sphere B is given a charge $+Q$, brought into contact with A and then moved far away. The charge on B is then increased to its original value $+Q$ and again brought into contact with A. Show that if this process is repeated many times, the charge on A will tend to the limit $\dfrac{Qq}{Q-q}$, where q is the charge acquired by A after its first contact with B.

Solution: It is obvious that after each transfer of charge from B to A, A has k fraction of total charge of A and B
Initially after one transfer $q_A = q = kQ$
After another transfer $q_A = (k^2 + k)Q$
After 2 transfers $q_A = (k^3 + k^2 + k)Q$
So $q_A \to (\ldots + k^3 + k^2 + k)Q = \frac{Qk}{1-k} = \frac{Qq}{Q-q}$ ∎

4.2. A hemisphere of radius R has $z = 0$ as its equatorial plane and lies entirely in the region $z \geq 0$. The hemisphere has a uniform volume charge density ρ. Determine the field at the center of the hemisphere.

Solution: We know that the field at $\vec{r}$ due to a 3D charge distribution is given by

$$\vec{E}(r) = \frac{1}{4\pi\epsilon_0} \iiint_{\mathcal{V}} \frac{\rho(\vec{r'})}{|\vec{\mathscr{r}}\,|^2}\, \hat{\mathscr{r}}\ dV' = \frac{1}{4\pi\epsilon_0} \iiint_{\mathcal{V}} \frac{\rho(\vec{r'})}{|\vec{\mathscr{r}}\,|^3}\, \vec{\mathscr{r}}\ dV' \tag{4.2.1}$$

We will use spherical coordinates for convenience
Here $\vec{r} = \vec{0}$ (origin), so $\vec{\mathscr{r}} = -r'\,\hat{r} = -r'\left(\sin\theta\cos\phi\,\hat{\imath} + \sin\theta\sin\phi\,\hat{\jmath} + \cos\theta\,\hat{k}\right)$

$$\implies \vec{E}(\vec{0}) = \frac{1}{4\pi\epsilon_0} \iiint_{\mathcal{V}} \frac{\rho(\vec{r'})}{|\vec{\mathscr{r}}\,|^3}\, \vec{\mathscr{r}} = \frac{1}{4\pi\epsilon_0} \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r'=0}^{r'=R} \frac{\rho}{r'^3}(-r'\,\hat{r})\, r'^2 \sin\theta\, dr\, d\theta\, d\phi \tag{4.2.2}$$

$$= -\frac{\rho}{4\pi\epsilon_0} \cancelto{0}{\int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r'=0}^{r'=R} \sin^2\theta\cos\phi\, dr\, d\theta\, d\phi}\ \hat{\imath} \tag{4.2.3}$$

$$- \frac{\rho}{4\pi\epsilon_0} \cancelto{0}{\int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r'=0}^{r'=R} \sin^2\theta\sin\phi\, dr\, d\theta\, d\phi}\ \hat{\jmath} \tag{4.2.4}$$

$$\tag{4.2.5}$$

$$- \frac{\rho}{4\pi\epsilon_0} \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\frac{\pi}{2}} \int_{r'=0}^{r'=R} \sin\theta\cos\theta\, dr\, d\theta\, d\phi\, \hat{k} \tag{4.2.6}$$

$$= -\frac{\rho R}{4\epsilon_0}\, \hat{k} \tag{4.2.7}$$

4.3. The potential takes the constant value ϕ_0 on the closed surface S which bounds a volume V. The total charge inside V is Q. There is no charge anywhere else. Show that the electrostatic energy contained in the space outside of S is $U_E(\text{out}) = \frac{Q\phi_0}{2}$

Solution: We know, for a volume charge density ρ, $U = \frac{1}{2}\iiint_{\mathcal{V}} \rho V \, d\tau$

$$\rho = \epsilon_0 \nabla \cdot \vec{E} \tag{4.3.1}$$

$$\implies U = \frac{\epsilon_0}{2} \iiint_{\mathcal{V}} (\nabla \cdot \vec{E}) V \, d\tau \tag{4.3.2}$$

Use integration by parts to transfer the derivative from $\vec{E}$ to V

$$\implies U = \frac{\epsilon_0}{2} \left(- \iiint_{\mathcal{V}} \vec{E} \cdot (\nabla V) \, d\tau + \oiint_{\mathcal{S}} V \vec{E} \cdot d\vec{S} \right) \tag{4.3.3}$$

$$\implies U = \frac{\epsilon_0}{2} \left(\iiint_{\mathcal{V}} \vec{E} \cdot \vec{E} \, d\tau + \oiint_{\mathcal{S}} V \vec{E} \cdot d\vec{S} \right) \tag{4.3.4}$$

$$\implies U = U_E(\text{in}) + \frac{\epsilon_0}{2} \oiint_{\mathcal{S}} V \vec{E} \cdot d\vec{S} \tag{4.3.5}$$

$$U - U_E(\text{in}) = U_E(\text{out}) = \frac{\epsilon_0}{2} \oiint_{\mathcal{S}} V \vec{E} \cdot d\vec{S} = \frac{\phi_0 \epsilon_0}{2} \oiint_{\mathcal{S}} \vec{E} \cdot d\vec{S} \tag{4.3.6}$$

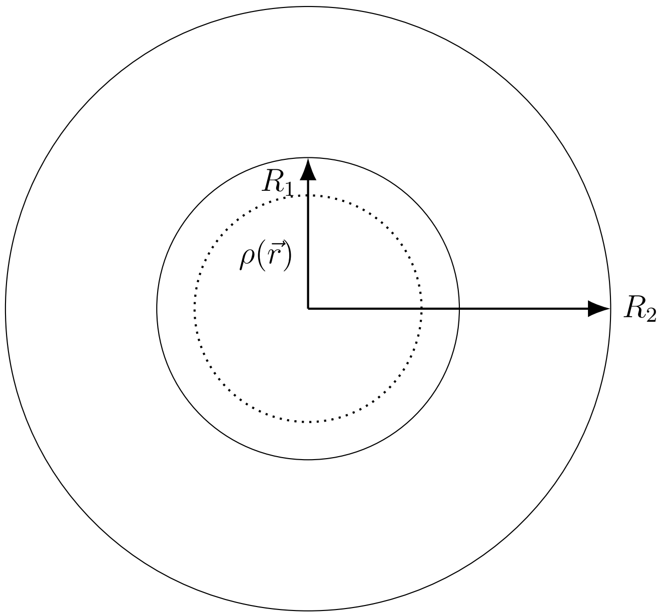
$$= \frac{Q\phi_0}{2} \qquad \blacksquare \tag{4.3.7}$$

Notice that the last step is due to Gauss's law

4.4. The inside of a grounded spherical metallic shell (inner radius R_1 and outer radius R_2) is filled with space charge of charge density $\rho(\vec{r}) = a + br$.

(a) Find the potential at the center.

Solution:



We know $\rho(\vec{r}) = a + br$.
Hence, by symmetrical arguments, the potential and magnitude of electric field will be independent of θ and ϕ coordinates.

Additionally, the charge in the bulk of the conductor and outside the conductor is 0. So is the potential and the electric field. Consider a Gaussian surface in the form of a sphere of radius r. Using Gauss Law,

$$\iiint_{\mathcal{V}} \rho(\vec{r})d\tau = \epsilon_0 \oiint \vec{E}(r) \cdot d\vec{A}$$
$$\implies \int_0^r 4\pi r^2(a+br)dr = 4\pi\epsilon_0 r^2 \vec{E}(r)$$
$$\implies \vec{E}(r) = \frac{1}{\epsilon_0}\left(\frac{ar}{3} + \frac{br^2}{4}\right)$$

Taking a line integral

$$\int_{\vec{a}}^{\vec{b}} \vec{E}(r) = V(\vec{a}) - V(\vec{b})$$
$$\int_r^{R_1} \vec{E}(r)dr = V(r) - V(R_1)$$
$$\implies \frac{1}{\epsilon_0}\int_r^{R_1} \frac{ar}{3} + \frac{br^2}{4}dr = V(r) - V(R_1)$$
$$\implies V(r) = \frac{1}{\epsilon_0}\left(\frac{a}{6}(R_1^2 - r^2) + \frac{b}{12}(R_1^3 - r^3)\right) \tag{4.4.1}$$
$$\implies V(0) = \frac{1}{\epsilon_0}\left(\frac{aR_1^2}{6} + \frac{bR_1^3}{12}\right) \tag{4.4.2}$$

(b) Calculate the electrostatic energy of the system

Solution:

Now to find energy

$$\begin{aligned} Energy &= \frac{1}{2}\iiint_{\mathcal{V}} \rho(\vec{r})V(\vec{r})d\tau \\ &= \int_0^{R_1} \frac{(a+br)4\pi r^2}{\epsilon_0}\left(\frac{a}{6}(R_1^2 - r^2) + \frac{b}{12}(R_1^3 - r^3)\right) \\ &= \frac{2\pi R_1^5}{\epsilon_0}\left(\frac{a^2}{45} + \frac{abR_1}{36} + \frac{b^2R_1^2}{112}\right) \end{aligned}$$

(c) Show that the system attains minimum energy configuration when $b = \frac{-14a}{9R_1}$

Solution:

For minimum energy, we differentiate wrt b

$$\frac{\partial E}{\partial b} = \frac{2\pi R_1^5}{\epsilon_0}\left(\frac{aR_1}{36} + \frac{bR_1^2}{56}\right) = 0$$
$$\implies b = \frac{-14a}{9R_1} \tag{4.4.3}$$

4.5. Show that the field is uniquely determined when the charge density ρ is specified within a bounded region and either the potential ϕ or its normal derivative $\frac{\partial \phi}{\partial n}$ is specified on each boundary

Solution: We let there be 2 solutions to the $\vec{E}$ field, $\vec{E_1}$ and $\vec{E_2}$, with potentials V_1 and V_2 such that $\vec{E_1} = -\nabla V_1$, $\vec{E_2} = -\nabla V_2$
Let $\vec{E_3} = \vec{E_2} - \vec{E_1}$, $V_3 = V_2 - V_1$

$$\nabla \cdot (\vec{E_3}V_3) = V_3\nabla \cdot \vec{E_3} + \vec{E_3} \cdot \nabla V_3 \tag{4.5.1}$$
$$= V_3(\nabla \cdot \vec{E_2} - \nabla \cdot \vec{E_1}) - |\vec{E_3}|^2 \tag{4.5.2}$$
$$= \cancelto{0}{V_3(\rho - \rho)} - |\vec{E_3}|^2 \tag{4.5.3}$$
$$\implies \iiint \nabla \cdot (\vec{E_3}V_3)\, d\tau = -\iiint |\vec{E_3}|^2\, d\tau \tag{4.5.4}$$
$$\implies \oiint_{\text{boundary}} V_3\vec{E_3} \cdot \hat{n}|dS| = -\iiint |\vec{E_3}|^2\, d\tau \tag{4.5.5}$$

Now if ϕ is specified at boundary, then $V_3 = V_2 - V_1 = \phi - \phi = 0$ at boundary. If $\frac{\partial \phi}{\partial n}$ is specified at boundary, then $\vec{E_3} \cdot \hat{n} = \frac{\partial \phi}{\partial n} - \frac{\partial \phi}{\partial n} = 0$. Either way $V_3\vec{E_3} \cdot \hat{n} = 0$

$$\implies \iiint |\vec{E_3}|^2\, d\tau = 0 \tag{4.5.6}$$
$$\implies \vec{E_3} = 0 \tag{4.5.7}$$
$$\implies \vec{E_1} = \vec{E_2} \tag{4.5.8}$$

Hence no two distinct solutions can exist

4.6. A ring of radius R has a total charge $+Q$ uniformly distributed on it.

(a) Calculate the electric field and potential at the center of the ring.

Solution: Potential at centre =

$$\frac{Q}{4\pi\epsilon_0 R}$$

Field at centre = 0

(b) Calculate field at a height z above the center. Compare this result with Coulomb's law in large z limit.

> **Solution:** At height z above centre, field is
>
> $$\frac{Qz\hat{z}}{4\pi\epsilon_0(R^2+z^2)^{\frac{3}{2}}}$$
>
> As $z \to \infty, E \to \frac{Q\hat{z}}{4\pi\epsilon_0 z^2}$
> which resembles the field from a point charge.

(c) Consider a charge $-Q$ constrained to slide along the axis of the ring. Show that the charge will execute simple harmonic motion for small displacements perpendicular to the plane of the ring. Calculate time period.

> **Solution:** We know the equation for simple harmonic motion is
>
> $$m\frac{d^2x}{dt^2} = -kx$$
>
> Hence,Force acting on the charged particle
>
> $$F = -Q\vec{E}(z)\hat{z} = -\frac{Q^2 z}{4\pi\epsilon_0(R^2+z^2)^{\frac{3}{2}}}$$
>
> Thus,
>
> $$k = -F'(0) = \frac{Q^2}{4\pi\epsilon_0(R^2+z^2)^{\frac{3}{2}}}$$
>
> Time period is
>
> $$2\pi\sqrt{\frac{k}{m}} = \frac{2\pi R}{Q}\sqrt{4\pi\epsilon_0 Rm}$$

4.7. Assume that an electron is a small sphere of radius R in which the charge $-|e|$ is distributed uniformly over its volume. Calculate the total energy of the system as a function of R. Now suppose you equate this energy to m_0c^2 where m_0 is the rest mass of the electron. What value of R do you get? How does it compare with the size of a hydrogen atom?

> **Solution:** We can first calculate the energy of a spherical charge distribution with constant volume charge density.
> This energy equals
>
> $$\frac{3q^2}{20\pi\epsilon_0 R}$$
>
> where R is the radius of the distribution and q is total charge.

Equating this to m_0c^2, we get

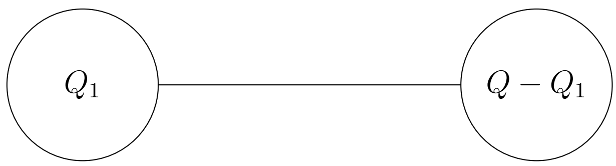
$$
\begin{aligned}
R &= \frac{3e^2}{20\pi\epsilon_0 m_0 c^2} && (4.7.1)\\
\implies R &\approx 10^{-7}m \approx 1000r_a && (4.7.2)
\end{aligned}
$$

where r_a is Bohr radius of Hydrogen atom

5 Conductors & Image Charges

5.1. A metal sphere with radius R_1 has charge Q. A second metal sphere with radius R_2 has zero charge. Now connect the spheres together using a fine conducting wire. Assume that the spheres are separated by a distance R which is large enough that the charge distribution on each ball remains uniform. Derive an expression for the final charge on the sphere with radius R_1.

Solution:



We need to equate the potentials at each end of the wire, because that is when the whole conductor system will become an equipotential

$$
\begin{aligned}
V_{\text{left end}} &= V_{\text{right end}} && (5.1.1)\\
\frac{Q_1}{R_1} + \frac{Q-Q_1}{R} &= \frac{Q-Q_1}{R_2} + \frac{Q_1}{R} && (5.1.2)\\
\implies Q_1 &= \frac{(R-R_2)R_1}{RR_1 + RR_2 - 2R_1R_2} && (5.1.3)
\end{aligned}
$$

$R \gg R_1, R_2$

$$
\begin{aligned}
\implies Q_1 &= Q\frac{\left(1-\frac{R_2}{R}\right)}{\left(1-2\frac{R_2}{R\left(1+\frac{R_2}{R_1}\right)}\right)\left(1+\frac{R_2}{R_1}\right)} && (5.1.4)\\
&\approx Q\frac{R_1}{R_1+R_2}\left(1-\frac{R_2}{R}\right)\left(1+2\frac{R_2}{R\left(1+\frac{R_2}{R_1}\right)}\right) && (5.1.5)\\
&\approx Q\frac{R_1}{R_1+R_2}\left(1+\frac{R_2(R_1-R_2)}{R(R_1+R_2)}\right) && (5.1.6)
\end{aligned}
$$

5.2. (a) Show that the capacitance, C, of a conducting sphere of radius a is given by $C = 4\pi\epsilon_0 a$

Solution: Assume a conducting sphere of radius a carrying a total charge Q, which will be distributed uniformly. Potential on surface of the sphere is

$$\frac{Q}{4\pi\epsilon_0 a}$$

Since our reference for potential measurement is with respect to infinity, potential at infinity is 0 Hence, Capacitance is

$$C = \frac{Q}{V} = 4\pi\epsilon_0 a \tag{5.2.1}$$

(b) Two isolated conducting spheres, both of radius a, initially carry charges of q_1 and q_2 and are held far apart. The spheres are connected together by a conducting wire until equilibrium is reached, whereupon the wire is removed. Show that the total electrostatic energy stored in the spheres decreases by an amount U , given by

$$\Delta U = \frac{1}{16\pi\epsilon_0 a}(q_1 - q_2)^2$$

What happens to this energy?

Solution: Initial Charges on the spheres are q_1 and q_2. When the spheres are connected by the wire, they become equipotential. Assume the final charges are q_1' and q_2'

$$\begin{aligned} q_1 + q_2 &= q_1' + q_2' \\ \frac{q_1'}{4\pi\epsilon_0 a} &= \frac{q_2'}{4\pi\epsilon_0 a} \\ \implies q_1' = q_2' &= \frac{q_1 + q_2}{2} \end{aligned}$$

Initial Energy is

$$E_1 = \frac{q_1^2 + q_2^2}{2C}$$

Final Energy is

$$E_2 = \frac{q_1'^2 + q_2'^2}{2C}$$

Substituting

$$\Delta U = \frac{1}{16\pi\epsilon_0 a}(q_1 - q_2)^2 \tag{5.2.2}$$

As the charges move from one sphere to another under the influence of an electric field(higher to lower potential), they gain kinetic energy. When the reach the second sphere, this energy gets dissipated as heat. In other words, this energy gets dissipated due to resistive heating of the wire

5.3. A metal sphere of radius R carries a total charge Q. What is the force of repulsion between the "northern" hemisphere and the "southern" hemisphere?

Solution: Since the sphere is made of metal(conductor), all the charge will be uniformly distributed on the surface(with $\sigma = \frac{Q}{4\pi R^2}$). Let $\mathcal{S}$ represent the surface of the "northern" hemisphere. Electrostatic Pressure $P = \frac{\sigma^2}{2\epsilon_0} = \frac{Q^2}{32\pi^2\epsilon_0 R^4}$

$$F = \oiint_{\mathcal{S}} P(r)\, d\vec{A} \tag{5.3.1}$$

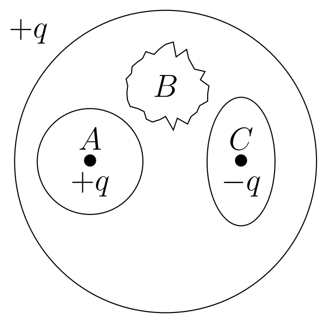
By symmetry, we know that force along z-axis only will survive. Hence we take the component of each elemental contribution to force along with z-axis

$$F = \oiint_{\mathcal{S}} \frac{Q^2}{32\pi^2\epsilon_0 R^4} \cos\theta\, d\vec{A} \tag{5.3.2}$$

$$= \frac{Q^2}{32\pi^2\epsilon_0 R^4} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\frac{\pi}{2}} R^2 \cos\theta \sin\theta\, d\theta\, d\phi \tag{5.3.3}$$

$$\implies F = \frac{Q^2}{32\pi\epsilon_0 R^2} \tag{5.3.4}$$

5.4. A solid spherical conductor encloses 3 cavities, a cross-section of which are as shown in the figure. A net charge $+q$ resides on the outer surface of the conductor. Cavities A and C contain point charges $+q$ and $-q$, respectively. What are the net charges on the surfaces of the cavities?



Solution: Consider a Gaussian surface just outside the cavity A inside the interior of the bulk of the conductor. Since electric field is 0 everywhere inside the bulk of a conductor, flux through the Gaussian surface is 0. By Gauss' Law, the net charge inside the Gaussian surface is 0. Hence,

$$Q_{inside\ cavity} + Q_{on\ surface} = 0$$

Hence charge on surface of cavity A $= -q$
Similarly, charge on surface of cavity C $= +q$

5.5. A metal sphere of radius R, carrying charge q, is surrounded by a thick concentric metal shell (inner radius a, outer radius b). The shell carries no net charge.

(a) Find the surface charge density σ at R, at a, and at b.

Solution: $\sigma_R = \frac{q}{4\pi R^2}$, as any leftover charge on conductor will distribute itself uniformly on the surface
This $+q$ in the cavity of the shell induces a total charge of $-q$ on the inner surface of shell. Hence $\sigma_a = \frac{-q}{4\pi a^2}$
The leftover charge on shell distributes itself uniformly on the outer surface. Hence $\sigma_b = \frac{q}{4\pi b^2}$

(b) Find the potential at the center, using infinity as the reference point.

Solution: $V(b) = \frac{q}{4\pi\epsilon_0 b}$ (The interior of shell is shielded)
$V(a) = V(b)$
$V(R) = V(a) + \frac{q}{4\pi\epsilon_0}\left(\frac{1}{R} - \frac{1}{a}\right)$
$V(a) = V(R) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{b} - \frac{1}{a} + \frac{1}{R}\right)$

(c) Now the outer surface is touched to a grounding wire, which drains off charge and lowers its potential to zero (same as at infinity). How do your answers to (a) and (b) change?

Solution: The only difference is there will be no leftover charge $+q$ on the outer surface of shell. So $\sigma_b = 0$
Also, $V(a) = V(b) = 0$
So $V(0) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{R} - \frac{1}{a}\right)$

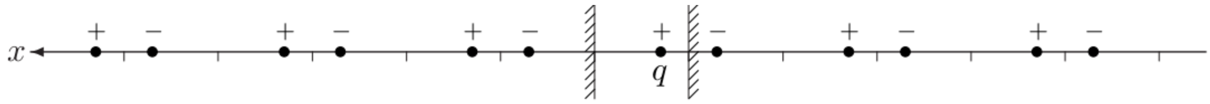
5.6. A point charge q of mass m is released from rest at a distance d from an infinite grounded conducting plane. How long will it take for the charge to hit the plane?

Solution: We will first find the velocity of the charge after falls to a distance of x from the plane, by energy conservation

$$\begin{aligned}
\frac{1}{2}mv^2 &= \frac{q^2}{16\pi\epsilon_0}\left(\frac{1}{x} - \frac{1}{d}\right) \\
\implies v &= \sqrt{\frac{q^2}{8\pi\epsilon_0 m}\left(\frac{1}{x} - \frac{1}{d}\right)} \\
\implies T &= \sqrt{\frac{8\pi\epsilon_0 m}{q^2}} \int_{x=d}^{x=0} \left(\frac{1}{x} - \frac{1}{d}\right)^{-\frac{1}{2}} \\
&= \frac{\pi d}{q}(2\pi\epsilon_0 m d)^{\frac{1}{2}}
\end{aligned}$$

5.7. Two infinite parallel grounded conducting plates are held a distance a apart. A point charge q is placed between them, at a distance x from one plate. Find the force on q. Check that your answer is correct for the special cases $a \to \infty$ and $x = \frac{a}{2}$.

Solution:



The image configuration is shown in the figure, the positive image charges cancel in pairs. The net force of the negative image charges is:

$$F = \frac{1}{4\pi\epsilon_0}q^2\left\{\frac{1}{[2(a-x)]^2} + \frac{1}{[2a+2(a-x)]^2} + \frac{1}{[4a+2(a-x)]^2} + \dots\right. \tag{5.7.1}$$

$$\left. - \frac{1}{(2x)^2} - \frac{1}{(2a+2x)^2} - \frac{1}{(4a+2x)^2} - \dots\right\} \tag{5.7.2}$$

$$= -\frac{q^2}{16\pi\epsilon_0}\left\{\sum_{n=0}^{\infty}\frac{1}{(na+x)^2} - \sum_{n=1}^{\infty}\frac{1}{(na-x)^2}\right\} \tag{5.7.3}$$

$$= -\frac{q^2}{16\pi\epsilon_0}\left\{\frac{1}{x^2} - 4ax\sum_{n=1}^{\infty}\frac{n}{(n^2a^2-x^2)^2}\right\} \tag{5.7.4}$$

Now when $a \to \infty$, the sum term tends to 0, and we are left with

$$F = \frac{q^2}{4\pi\epsilon_0}\frac{1}{(2x)^2} \tag{5.7.5}$$

which is the same force as the case for a single plane.
If $x = \frac{a}{2}$ then we expect no force, and:

$$F = \frac{q^2}{16\pi\epsilon_0}\left\{\frac{4}{a^2} - \frac{2}{a^2}\sum_{n=1}^{\infty}\frac{n}{\left(n^2-\frac{1}{4}\right)^2}\right\} = 0 \tag{5.7.6}$$

5.8. A conducting sphere (or a shell) of radius R has a charge Q.

(a) Find the force of repulsion between the two hemispheres

Solution: Already calculated in 5.3

(b) Now suppose one has a solid sphere of radius R with charge Q distributed uniformly over its volume.

Solution: Let $\mathcal{V}$ represent the volume of the sphere.

$$\vec{F} = \iiint_{\mathcal{V}} \rho\vec{E}(r)d\tau \tag{5.8.1}$$

$$= \iiint_{\mathcal{V}} \frac{3Q}{4\pi\epsilon_0 R^3}\frac{Qr\hat{r}}{4\pi\epsilon_0 R^3}d\tau \tag{5.8.2}$$

Since we know field is in z-direction by symmetry

$$= \iiint_{\mathcal{V}} \frac{3Q}{4\pi R^3} \frac{Qr}{4\pi\epsilon_0 R^3} \cos\theta d\tau \tag{5.8.3}$$

$$= \frac{3Q^2}{16\pi^2\epsilon_0 R^6} \int_{r=0}^{R} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\frac{\pi}{2}} r^3 \cos\theta \sin\theta \, dr \, d\theta \, d\phi \tag{5.8.4}$$

$$\implies F = \frac{3Q^2}{64\pi\epsilon_0 R^2} \tag{5.8.5}$$

(c) Which case (a) vs (b) has the larger force of repulsion?

Solution: Clearly, the force of repulsion is greater in (b) than (a) as in the case of the conductor, the charges redistribute to minimise repulsion.

5.9. (a) Find the average potential over a spherical surface of radius R due to a point charge q located inside. Show that, in general,

$$V_{\text{ave}} = V_{\text{center}} + \frac{Q_{enc}}{4\pi\epsilon_0 R}$$

where V_{center} is the potential at the center due to all the external charges and Q_{enc} is the total enclosed charge.

Solution: The average value of V at a point r over a spherical surface of radius R is given as (assuming $\mathcal{S}$ represents the sphere surface)

$$V_{\text{ave}}(r) = \frac{1}{4\pi R^2} \oint\!\!\!\oint_{\mathcal{S}} V \, dS \tag{5.9.1}$$

In this case since the point charge is located inside so we can break $V_{\text{ave}} = V_{\text{int}} + V_{\text{ext}}$ where V_{int} is the average due to the internally located charges and V_{ext} is the due to the externally located charges.
The potential V_{int} due to the point charge q located inside the sphere at $\vec{r'} = (0, 0, z)$ can be calculated as follows

$$V_{\text{int}} = \frac{1}{4\pi R^2} \oint\!\!\!\oint_{\mathcal{S}} \left(\frac{1}{4\pi\epsilon_0} \frac{q}{|\vec{r} - \vec{r'}|} \right) dS \tag{5.9.2}$$

$$= \frac{q}{16\pi^2\epsilon_0 R^2} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{1}{\sqrt{R^2 + z^2 - 2zR\cos\theta}} R^2 \sin\theta \, d\theta \, d\phi \tag{5.9.3}$$

$$= \frac{q}{8\pi\epsilon_0} \int_{\theta=0}^{\pi} \frac{\sin\theta}{\sqrt{R^2 + z^2 - 2zR\cos\theta}} \, d\theta \tag{5.9.4}$$

$$= \frac{q}{4\pi\epsilon_0 R} \tag{5.9.5}$$

Thus from superposition principle, $V_{\text{int}} = \frac{Q_{\text{enc}}}{4\pi\epsilon_0 R}$
For V_{ext}, the potential is same as given at the center of the sphere V_{center}.

So finally, we have

$$V_{\text{ave}} = V_{\text{center}} + \frac{Q_{enc}}{4\pi\epsilon_0 R}$$

(b) Find the general solution to Laplace's equation in spherical coordinates for the case where V depends only on r. Do the same for cylindrical coordinates assuming V depends only on s.

Solution: In case of the spherical coordinates we will have

$$\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = 0$$

for which the solution will be

$$V = \frac{-c}{r} + k$$

where c, k is some constant.
In case of the cylindrical coordinates we will have

$$\nabla^2 V = \frac{1}{s}\frac{d}{ds}\left(s\frac{dV}{ds}\right) = 0$$

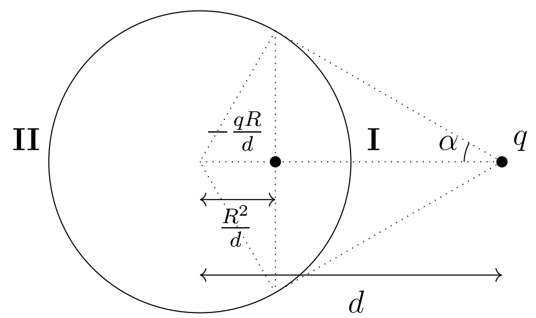
and the solution for the above case is

$$V = c\ln s + k$$

where c, k is some constant.

5.10. A point charge $+q$ is placed at a distance d from the centre of a conducting sphere of radius R $(d > R)$. Show that if the sphere is grounded, the ratio of the charge on the part of the sphere visible from $+q$ to that on the rest is $\sqrt{\frac{d+R}{d-R}}$.

Solution:



We need the ratio of charge on **I** and **II**

$$\frac{\iint_{\mathbf{I}} \sigma\, dS}{\iint_{\mathbf{II}} \sigma\, dS} \tag{5.10.1}$$

But remember, $\sigma = \epsilon_0 \vec{E}\hat{n}$, because this is a conductor. Replacing this in the above ratio, we get

$$= \frac{\iint_{\mathbf{I}} \vec{E}\cdot d\vec{S}}{\iint_{\mathbf{II}} \vec{E}\cdot d\vec{S}} = \frac{\text{Flux}_{\mathbf{I}}}{\text{Flux}_{\mathbf{II}}} \tag{5.10.2}$$

Here we are talking about flux of the electric field. Now, as we can see contribution of q towards flux through I and II is the same (except for the sign), and is $\frac{q\Omega}{4\pi\epsilon_0}$, where Ω is the solid angle subtended by the sphere on q.

But $\Omega = 2\pi(1-\cos\alpha) = 2\pi\left(1-\frac{\sqrt{d^2-R^2}}{d}\right)$. So the flux due to q is $\frac{q\left(1-\frac{\sqrt{d^2-R^2}}{d}\right)}{2\epsilon_0}$ through **II** and $-\frac{q\left(1-\frac{\sqrt{d^2-R^2}}{d}\right)}{2\epsilon_0}$ through **I**

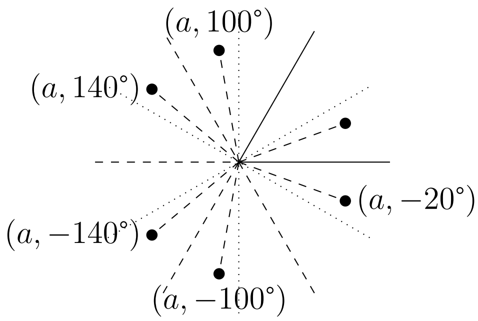
Now, the flux due to the image charge $-\frac{qR}{d}$ is $-\frac{qR}{2\epsilon_0 d}$ through both **I** and **II**

So ultimately,

$$\frac{-\frac{q\left(1-\frac{\sqrt{d^2-R^2}}{d}\right)}{2\epsilon_0} - \frac{qR}{2\epsilon_0 d}}{\frac{q\left(1-\frac{\sqrt{d^2-R^2}}{d}\right)}{2\epsilon_0} - \frac{qR}{2\epsilon_0 d}} = \frac{R+d-\sqrt{d^2-R^2}}{R-d+\sqrt{d^2-R^2}} = \sqrt{\frac{d+R}{d-R}} \quad \blacksquare \tag{5.10.3}$$

5.11. Two infinite conducting plates (both grounded and perpendicular to the $x-y$ plane) meet at an angle of 60°. A point charge $+q$ in the xy plane has plane polar coordinates $(a, 20°)$. Find all the image charges and their positions in polar coordinates.

Solution:



5.12. A rectangular pipe running parallel to the z-axis (from $-\infty$ to $+\infty$) has three grounded metal sides at $y = 0, y = a$ and $x = 0$. The fourth side at $x = b$ is maintained at a specified potential $V_0(y)$.

(a) Develop a general formula for the potential within the pipe.

Solution: We need to solve

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

with the boundary conditions given as

$$\begin{aligned} V(x,0) &= 0 \\ V(x,a) &= 0 \\ V(0,y) &= 0 \\ V(b,y) &= V_0(y) \end{aligned}$$

Using separation of variables, the solution to the above equation can be written as

$$V(x,y) = \left(Ae^{kx} + Be^{-kx}\right)(C\sin ky + D\cos ky)$$

Using the boundary conditions we can deduce that

$$\begin{aligned} D &= 0 \\ B &= -A \\ k &= \frac{n\pi}{a} \end{aligned}$$

Using the above we can write the most general form of the solution as

$$V(x,y) = \sum_{n=1}^{\infty} C_n \sinh(n\pi x/a)\sin(n\pi y/a)$$

wherein C_n can be determined using the boundary condition $V(b,y) = V_0(y)$ and is given as

$$C_n = \frac{2}{a\sinh(n\pi b/a)}\int_0^a V_0(y)\sin(n\pi y/a)dy$$

(b) Find the potential explicitly, for the case $V_0(y) = V_0$ (a constant).

Solution: In this case C_n turns out to be

$$C_n = \frac{2}{a\sinh(n\pi b/a)}V_0\int_0^a \sin(n\pi y/a)dy$$

The integral in the above eq. is 0 when n is even and $\frac{2a}{n\pi}$ when n is odd. So, finally we will get

$$V(x,y) = \frac{4V_0}{\pi}\sum_{n=1,3,5}\frac{\sinh(n\pi x/a)\sin(n\pi y/a)}{n\sinh(n\pi b/a)}$$

6 Solutions to Laplace Equation

6.1. Two infinitely long grounded metal plates, at $y = 0$ and $y = a$, are connected at $x = \pm b$ by metal strips maintained at a constant potential V_0, (a thin layer of insulation at each corner prevents them from shorting out). Find the potential inside the resulting rectangular pipe.

Solution: We need to solve the equations

$$\frac{\partial^2 V(x,y)}{\partial x^2} + \frac{\partial^2 V(x,y)}{\partial y^2} = 0 \qquad \text{Laplace's Equation} \tag{6.1.1}$$

$$V(-b,y) = V_0 \qquad \text{Boundary Conditions} \tag{6.1.2}$$

$$V(b,y) = V_0 \tag{6.1.3}$$

$$V(x,0) = 0 \tag{6.1.4}$$

$$V(x,a) = 0 \tag{6.1.5}$$

in the region $|x| \leq b$, $0 \leq y \leq a$

Using separation of variables, we know that the general solution can be written as

$$V(x,y) = \sum_k (A_k e^{kx} + B_k e^{-kx})(C_k \cos ky + D_k \sin ky) \tag{6.1.6}$$

Using (6.1.4) and (6.1.5)

$$0 = \sum_k C_k (A_k e^{kx} + B_k e^{-kx}) \tag{6.1.7}$$

$$0 = \sum_k (A_k e^{kx} + B_k e^{-kx})(C_k \cos ka + D_k \sin ka) \tag{6.1.8}$$

The trivial way to satisfy (6.1.7) is $C_k = 0 \; \forall \; k$

Putting this back into (6.1.8) we can see that $\sin ka = 0$ will satisfy it, hence only those D_k terms are non zero, which satisfy $k = \frac{n\pi}{a}$

We can thus index our coefficients by n instead of k

We get the general solution as

$$V(x,y) = \sum_n (D_n A_n e^{n\pi \frac{x}{a}} + D_n B_n e^{-n\pi \frac{x}{a}}) \sin\left(n\pi \frac{y}{a}\right) \tag{6.1.9}$$

Using (6.1.2) and (6.1.3) we can see that $V(b,y) = V(-b,y)$

$$\implies \sum_n A_n e^{n\pi \frac{b}{a}} + B_n e^{-n\pi \frac{b}{a}} = \sum_n A_n e^{-n\pi \frac{b}{a}} + B_n e^{n\pi \frac{b}{a}} \tag{6.1.10}$$

$$\implies \sum_n (A_n - B_n)(e^{n\pi \frac{b}{a}} + e^{-n\pi \frac{b}{a}}) = 0 \tag{6.1.11}$$

$$\implies A_n = B_n \;\; \forall \, n \tag{6.1.12}$$

We can finally simplify our general solution as

$$V(x,y) = \sum_{n=1}^{\infty} 2 D_n A_n \cosh\left(n\pi \frac{x}{a}\right) \sin\left(n\pi \frac{y}{a}\right) \tag{6.1.13}$$

In order to solve this, we use (6.1.3) (and write our coefficient as $K_n = 2D_n A_n$)

$$V_0 = \sum_{n=1}^{\infty} K_n \cosh\left(n\pi \frac{b}{a}\right) \sin\left(n\pi \frac{y}{a}\right) \tag{6.1.14}$$

Using Fourier's trick

$$\int_{y=0}^{a} V_0 \sin\left(n'\pi\frac{y}{a}\right) dy = \sum_{n=1}^{\infty} K_n \cosh\left(n\pi\frac{b}{a}\right) \int_{y=0}^{a} \sin\left(n\pi\frac{y}{a}\right) \sin\left(n'\pi\frac{y}{a}\right) dy \tag{6.1.15}$$

$$\implies \sum_{n=1}^{\infty} \frac{aK_n}{2} \cosh\left(n\pi\frac{b}{a}\right)\delta_{nn'} = \begin{cases} 0 & n' \text{ even} \\ \frac{2aV_0}{n'\pi} & n' \text{ odd} \end{cases} \tag{6.1.16}$$

$$\implies K_n = \begin{cases} 0 & n \text{ even} \\ \frac{4V_0}{n\pi}\frac{1}{\cosh\left(n\pi\frac{b}{a}\right)} & n \text{ odd} \end{cases} \tag{6.1.17}$$

Putting this back into (6.1.13),

$$V(x,y) = \boxed{\frac{4V_0}{\pi} \sum_{n=1,3,5...} \frac{\cosh\left(n\pi\frac{x}{a}\right)}{n\cosh\left(n\pi\frac{b}{a}\right)} \sin\left(n\pi\frac{y}{a}\right)} \tag{6.1.18}$$

We can indeed verify that this does satisfy the boundary conditions and the Laplace's equation, hence by Uniqueness theorem it must be the only solution

6.2. A cubical box of side length a consists of five metal plates welded together and grounded. The sixth plate at the top is insulated from the rest and maintained at V_0.

(a) Argue that the potential at the centre should be $\frac{V_0}{6}$

Solution: Let us consider 6 seperate cases, where in the i^{th} case the i^{th} face is at potential V_i and all others are at 0 potential (the charges on the faces arrange in such a way to facilitate this scenario in each case)
It is not very difficult to see that each case is identical upto a rotation and scaling of the charges on each face. Thus the potential at the centre must be proportional to V_i in the i^{th} case. Hence if we take a superposition of all cases, we can easily see that the resultant scenario in which i^{th} face is at V_i must have the potential at centre $\propto \sum_i V_i$.
If we take the case of all $V_i = V_0$, we can see that the proportionality constant is $\frac{1}{6}$
Hence the potential at centre is $\frac{V_0}{6}$ when one face is held at V_0 and all others at 0

(b) Find the potential inside the box.

Solution: We need to solve the equations

$$\frac{\partial^2 V(x,y,z)}{\partial x^2} + \frac{\partial^2 V(x,y,z)}{\partial y^2} + \frac{\partial^2 V(x,y,z)}{\partial z^2} = 0 \quad \text{Laplace's Equation} \tag{6.2.1}$$

$$V(x,y,a) = V_0 \quad \text{Boundary Conditions} \tag{6.2.2}$$

$$V(x,y,0) = 0 \tag{6.2.3}$$

$$V(0,y,z) = 0 \tag{6.2.4}$$

$$V(a,y,z) = 0 \tag{6.2.5}$$

$$V(x,0,z) = 0 \tag{6.2.6}$$

$$V(x,a,z) = 0 \tag{6.2.7}$$

For the region $0 \le x, y, z \le a$
Using separation of variables we know that the general solution can be written as

$$V(x,y,z) = \sum_{k,l} (A_k \cos kx + B_k \sin kx)(C_l \cos ly + D_l \sin ly)(E_{kl} e^{\sqrt{k^2+l^2}z} + K_{kl} e^{-\sqrt{k^2+l^2}z}) \tag{6.2.8}$$

Applying (6.2.4), (6.2.5), (6.2.6), (6.2.7), we can see that

$$A_k = 0 \ \forall \ k \tag{6.2.9}$$

$$B_k = 0 \ \forall \ k \neq \frac{n\pi}{a} \tag{6.2.10}$$

$$C_l = 0 \ \forall \ l \tag{6.2.11}$$

$$D_l = 0 \ \forall \ l \neq \frac{m\pi}{a} \tag{6.2.12}$$

Applying (6.2.3),

$$E_{kl} + K_{kl} = 0 \ \forall \ k, l \tag{6.2.13}$$

Thus we can write the general solution as (indexing the coefficients with m, n instead of k, l)

$$V(x,y,z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} 2B_n D_m E_{nm} \sin\left(n\pi\frac{x}{a}\right) \sin\left(m\pi\frac{y}{a}\right) \sinh\left(\sqrt{n^2+m^2}\pi\frac{z}{a}\right) \tag{6.2.14}$$

In order to solve this, we can use (6.2.2) (and write the coefficient $K_n m = 2B_n D_m E_{nm}$)

$$V_0 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} K_{nm} \sin\left(n\pi\frac{x}{a}\right) \sin\left(m\pi\frac{y}{a}\right) \sinh\left(\sqrt{n^2+m^2}\pi\right) \tag{6.2.15}$$

Applying Fourier's trick,

$$\int_{x=0}^{a} \int_{y=0}^{a} V_0 \sin\left(n'\pi\frac{x}{a}\right) \sin\left(m'\pi\frac{y}{a}\right) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} K_{nm} \sinh\left(\sqrt{n^2+m^2}\pi\right) \tag{6.2.16}$$

$$\times \int_{x=0}^{a} \int_{y=0}^{a} \sin\left(n\pi\frac{x}{a}\right) \sin\left(n'\pi\frac{x}{a}\right) \sin\left(m\pi\frac{y}{a}\right) \sin\left(m'\pi\frac{y}{a}\right) dx\, dy \tag{6.2.17}$$

$$\implies \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a^2 K_{nm}}{4} \sinh\Big(\sqrt{n^2+m^2}\pi\Big)\delta_{nn'}\delta_{mm'} = \begin{cases} 0 & n' \text{ even or } m' \text{ even} \\ \frac{4a^2V_0}{\pi^2 n'm'} & n' \text{ odd and } m' \text{ odd} \end{cases} \tag{6.2.18}$$

$$\implies K_{nm} = \begin{cases} 0 & n \text{ even or } m \text{ even} \\ \frac{16V_0}{\pi^2 nm}\frac{1}{\sinh\left(\sqrt{n^2+m^2}\pi\right)} & n \text{ odd and } m \text{ odd} \end{cases} \tag{6.2.19}$$

Thus we can write the solution as

$$V(x,y,z) = \frac{16V_0}{\pi^2} \sum_{n=1,3,5\ldots}\sum_{m=1,3,5\ldots} \frac{\sinh\left(\sqrt{n^2+m^2}\pi\frac{z}{a}\right)}{nm\sinh\left(\sqrt{n^2+m^2}\pi\right)} \sin\Big(n\pi\frac{x}{a}\Big)\sin\Big(m\pi\frac{y}{a}\Big) \tag{6.2.20}$$

Summing it up numerically upto 49 terms gives us $0.166666666676V_0 \approx \frac{V_0}{6}$, confirming our "guess"

6.3. A rectangular pipe, running parallel to the z-axis (from $-\infty$ to $+\infty$), has three grounded metal sides, at $y = 0$, $y = a$, and $x = 0$. The fourth side, at $x = b$, is maintained at a specified potential $V_0(y)$. Develop a general formula for the potential inside the pipe.

Solution: We need to solve the equations

$$\frac{\partial^2 V(x,y)}{\partial x^2} + \frac{\partial^2 V(x,y)}{\partial y^2} = 0 \qquad \text{Laplace's Equation} \tag{6.3.1}$$

$$V(b,y) = V_0(y) \qquad \text{Boundary Conditions} \tag{6.3.2}$$

$$V(0,y) = 0 \tag{6.3.3}$$

$$V(x,0) = 0 \tag{6.3.4}$$

$$V(x,a) = 0 \tag{6.3.5}$$

We can use separation of variables to write the general solution

$$V(x,y) = \sum_k (A_k e^{kx} + B_k e^{-kx})(C_k \cos ky + D_k \sin ky) \tag{6.3.6}$$

We can apply (6.3.3), (6.3.4), (6.3.5) to get

$$A_k + B_k = 0\,\forall\, k \tag{6.3.7}$$

$$C_k = 0\,\forall\, k \tag{6.3.8}$$

$$D_k = 0\,\forall\, k \neq \frac{n\pi}{a} \tag{6.3.9}$$

Thus we can write the general solution as

$$V(x,y) = \sum_{n=1}^{\infty} 2A_k D_k \sinh\Big(n\pi\frac{x}{a}\Big)\sin\Big(n\pi\frac{y}{a}\Big) \tag{6.3.10}$$

We use (6.3.2) (Writing the coefficient as $K_n = 2A_kD_k$)

$$V_0(y) = \sum_{n=1}^{\infty} K_n \sinh\left(n\pi\frac{b}{a}\right)\sin\left(n\pi\frac{y}{a}\right) \tag{6.3.11}$$

Applying Fourier's trick,

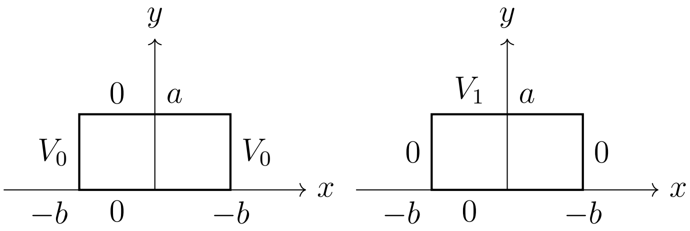
$$K_n = \frac{2}{a}\frac{1}{\sinh\left(n\pi\frac{b}{a}\right)}\int_{y=0}^{a} V_0(y)\sin\left(n\pi\frac{y}{a}\right)dy \tag{6.3.12}$$

We can finally write the solution as

$$V(x,y) = \frac{2}{a}\sum_{n=1}^{\infty}\frac{\int_{y=0}^{a} V_0(y)\sin\left(n\pi\frac{y}{a}\right)dy}{\sinh\left(n\pi\frac{b}{a}\right)}\sinh\left(n\pi\frac{x}{a}\right)\sin\left(n\pi\frac{y}{a}\right) \tag{6.3.13}$$

6.4. Two infinitely long metal plates at $y = 0$ and $y = a$ are connected at $x = \pm b$ by metal strips maintained at a constant potential V_0. The potential on the bottom ($y = 0$) is zero, however the potential on the top ($y = a$) is a nonzero constant V_1. A thin layer of insulation at each corner prevents the plates from shorting out. Find the potential inside the resulting rectangular pipe.

Solution: To solve this we need to break this up into 2 situations



Case I Case II

We already know the solution to Case I (from Q6.1),

$$V_{\mathrm{I}}(x,y) = \frac{4V_0}{\pi}\sum_{n=1,3,5\dots}\frac{\cosh\left(n\pi\frac{x}{a}\right)}{n\cosh\left(n\pi\frac{b}{a}\right)}\sin\left(n\pi\frac{y}{a}\right) \tag{6.4.1}$$

Let us solve for Case II:
We need to solve the equations

$$\frac{\partial^2 V_{\mathrm{II}}(x,y)}{\partial x^2} + \frac{\partial^2 V_{\mathrm{II}}(x,y)}{\partial y^2} = 0 \qquad \text{Laplace's Equation} \tag{6.4.2}$$

$$V_{\mathrm{II}}(x,a) = V_1 \qquad \text{Boundary Conditions} \tag{6.4.3}$$

$$V_{\mathrm{II}}(x,0) = 0 \tag{6.4.4}$$

$$V_{\mathrm{II}}(y,-b) = 0 \tag{6.4.5}$$

$$V_{\mathrm{II}}(y,b) = 0 \tag{6.4.6}$$

in the region $|x| \leq b$, $0 \leq y \leq a$
Using separation of variables we can write the general solution as

$$V_{\text{II}}(x, y) = \sum_k (A_k \cos kx + B_k \sin kx)(C_k e^{ky} + D_k e^{-ky}) \tag{6.4.7}$$

Using (6.4.4) we get

$$C_k + D_k = 0 \tag{6.4.8}$$

Thus our general solution simplifies to

$$V_{\text{II}}(x, y) = \sum_k 2C_k (A_k \cos kx + B_k \sin kx) \sinh(ky) \tag{6.4.9}$$

Putting (6.4.5) and (6.4.6) we get

$$A_k \cos kb + B_k \sin kb = 0 \tag{6.4.10}$$
$$A_k \cos kb - B_k \sin kb = 0 \tag{6.4.11}$$
$$\implies A_k \cos kb = B_k \sin kb = 0 \tag{6.4.12}$$
$$\implies A_k = 0 \,\forall\, k = \frac{2n\pi}{2b} \tag{6.4.13}$$
$$B_k = 0 \,\forall\, k = \frac{(2n+1)\pi}{2b} \tag{6.4.14}$$
$$A_k = B_k = 0 \,\forall\, k \neq \frac{n\pi}{2b} \tag{6.4.15}$$

Hence we can write out our general solution as (indexing the coefficients accordingly)

$$V_{\text{II}}(x, y) = 2C_1 A_1 \cos\left(\pi \frac{x}{2b}\right) \sinh\left(\pi \frac{y}{2b}\right) + 2C_2 B_2 \sin\left(2n\pi \frac{x}{2b}\right) \sinh\left(2\pi \frac{y}{2b}\right) \tag{6.4.16}$$
$$+ 2C_3 A_3 \cos\left(3n\pi \frac{x}{2b}\right) \sinh\left(3\pi \frac{y}{2b}\right) \dots \tag{6.4.17}$$

$$\implies V_{\text{II}}(x, y) = 2C_1 A_1 \sin\left(\pi \frac{x+b}{2b}\right) \sinh\left(\pi \frac{y}{2b}\right) - 2C_2 A_2 \sin\left(2\pi \frac{x+b}{2b}\right) \sinh\left(2\pi \frac{y}{2b}\right) \tag{6.4.18}$$
$$+ - 2C_3 A_3 \sin\left(3\pi \frac{x+b}{2b}\right) \sinh\left(3\pi \frac{y}{2b}\right) \dots \tag{6.4.19}$$

Writing $\pm C_n A_n = K_n$,

$$V_{\text{II}}(x, y) = \sum_{n=1}^{\infty} K_n \sin\left(n\pi \frac{x+b}{2b}\right) \sinh\left(n\pi \frac{y}{2b}\right) \tag{6.4.20}$$

Using (6.1.2)

$$V_1 = \sum_{n=1}^{\infty} K_n \sin\left(n\pi \frac{x+b}{2b}\right) \sinh\left(n\pi \frac{a}{2b}\right) \tag{6.4.21}$$

Applying Fourier's trick,

$$K_n = \begin{cases} 0 & n \text{ even} \\ \frac{4V_1}{n\pi}\frac{1}{\sinh\left(n\pi\frac{a}{2b}\right)} & n \text{ odd} \end{cases} \tag{6.4.22}$$

Thus

$$V_{\text{II}}(x,y) = \frac{4V_1}{\pi}\sum_{n=1,3,5\ldots}\frac{\sinh\left(n\pi\frac{y}{2b}\right)}{n\sinh\left(n\pi\frac{a}{2b}\right)}\sin\left(n\pi\frac{x+b}{2b}\right) \tag{6.4.23}$$

We finally have

$$\begin{aligned} V(x,y) &= V_{\text{I}} + V_{\text{II}} && (6.4.24)\\ &= \frac{4V_0}{\pi}\sum_{n=1,3,5\ldots}\frac{\cosh\left(n\pi\frac{x}{a}\right)}{n\cosh\left(n\pi\frac{b}{a}\right)}\sin\left(n\pi\frac{y}{a}\right) && (6.4.25)\\ &+ \frac{4V_1}{\pi}\sum_{n=1,3,5\ldots}\frac{\sinh\left(n\pi\frac{y}{2b}\right)}{n\sinh\left(n\pi\frac{a}{2b}\right)}\sin\left(n\pi\frac{x+b}{2b}\right) && (6.4.26) \end{aligned}$$

6.5. Consider a spherical surface of a large radius R, the potential on the spherical surface is given below. Using the separation of variables find the potential $V(r,\theta)$ for $r \geq R$ upto order $\mathcal{O}(\frac{1}{r^6})$

$$V(R,\theta) = \begin{cases} +V & 0 \leq \theta < \frac{\pi}{2} \\ -V & \frac{\pi}{2} \leq \theta \leq \pi \end{cases}$$

Solution: We need to solve the equations (remembering that we have azimuthal symmetry)

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V(r,\theta)}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V(r,\theta)}{\partial\theta}\right) = 0 \qquad \text{Laplace's Equation} \tag{6.5.1}$$

$$V(R,\theta) = \begin{cases} +V & 0 \leq \theta < \frac{\pi}{2} \\ -V & \frac{\pi}{2} \leq \theta \leq \pi \end{cases} \qquad \text{Boundary Conditions} \tag{6.5.2}$$

$$\lim_{r\to\infty} V(r,\theta) = 0 \tag{6.5.3}$$

in the region $r \geq R$

Using separation of variables we can write the general solution as

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta) \tag{6.5.4}$$

Using (6.5.3), $A_l = 0 \; \forall \; l$
We can solve for the coefficients using (6.5.3)

$$V(R,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta) \tag{6.5.5}$$

Applying Fourier's trick,

$$\int_{\theta=0}^{\pi} V(R,\theta) P_{l'}(\cos\theta) \sin\theta \, d\theta = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} \int_{\theta=0}^{\pi} P_l(\cos\theta) P_{l'}(\cos\theta) \sin\theta \, d\theta \tag{6.5.6}$$

$$\implies V \int_{\theta=0}^{\frac{\pi}{2}} P_{l'}(\cos\theta) \sin\theta \, d\theta - V \int_{\theta=\frac{\pi}{2}}^{\pi} P_{l'}(\cos\theta) \sin\theta \, d\theta \tag{6.5.7}$$

$$= \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} \int_{\theta=0}^{\pi} P_l(\cos\theta) P_{l'}(\cos\theta) \sin\theta \, d\theta \tag{6.5.8}$$

$$\implies B_l = \begin{cases} 0 & l \text{ even} \\ (2l+1) V R^{l+1} \int_{x=0}^{1} P_l(x) \, dx & l \text{ odd} \end{cases} \tag{6.5.9}$$

Thus our solution is

$$V(r,\theta) = V \sum_{l=1,3,5\ldots} (2l+1) \frac{R^{l+1}}{r^{l+1}} P_l(\cos\theta) \int_{x=0}^{1} P_l(x) \, dx \tag{6.5.10}$$

We only need to calculate upto $l = 5$ ($\mathcal{O}(\frac{1}{r^6})$)

$$\int_{x=0}^{1} P_1(x) \, dx = \frac{1}{2} \tag{6.5.11}$$

$$\int_{x=0}^{1} P_3(x) \, dx = -\frac{1}{8} \tag{6.5.12}$$

$$\int_{x=0}^{1} P_5(x) \, dx = \frac{1}{16} \tag{6.5.13}$$

Thus,

$$V(r,\theta) = V \left(\frac{3R^2}{2r^2} P_1(\cos\theta) - \frac{7R^4}{8r^4} P_3(\cos\theta) + \frac{11R^6}{16r^6} P_5(\cos\theta) \right) + \mathcal{O}(\frac{1}{r^8}) \tag{6.5.14}$$

6.6. Let a sphere of radius R have potential $V(R,\theta,\phi) = V_0\cos^2\theta$. Find the potential everywhere inside and outside the sphere.

Solution: Let us begin by calculating potential outside. As always, we have to solve

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V_{\text{out}}(r,\theta)}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V_{\text{out}}(r,\theta)}{\partial\theta}\right) = 0 \quad \text{Laplace's Equation} \tag{6.6.1}$$

$$V_{\text{out}}(R,\theta) = V_0\cos^2\theta \quad \text{Boundary Conditions} \tag{6.6.2}$$

$$\lim_{r\to\infty} V_{\text{out}}(r,\theta) = 0 \tag{6.6.3}$$

in the region $r \geq R$ We can write the general solution using separation of variables

$$V_{\text{out}}(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta) \tag{6.6.4}$$

Using (6.6.3), $A_l = 0$
We can solve for the coefficients using (6.6.2)

$$V_0\cos^2\theta = \sum_{l=0}^{\infty}\frac{B_l}{R^{l+1}}P_l(\cos\theta) \tag{6.6.5}$$

We can see that $B_l = 0\ \forall\ l > 2$ by observing the degree of $\cos\theta$ in the LHS

$$\implies V_0\cos^2\theta = \frac{B_0}{R} + \frac{B_1}{R^2}\cos\theta + \frac{B_2}{2R^3}\left(3\cos^2\theta - 1\right) \tag{6.6.6}$$

Comparing the coefficients of like degrees of $\cos\theta$,

$$B_2 = \frac{2V_0R^3}{3} \tag{6.6.7}$$

$$B_1 = 0 \tag{6.6.8}$$

$$B_0 = \frac{V_0R}{3} \tag{6.6.9}$$

$$\implies V_{\text{out}}(r,\theta) = \frac{V_0R}{3r}\left(1 + \frac{R^2}{r^2}\left(3\cos^2\theta - 1\right)\right) \tag{6.6.10}$$

We can similarly find $V_{\text{in}}(r,\theta)$,

$$V_{\text{in}}(r,\theta) = \frac{V_0}{3}\left(1 + \frac{r^2}{R^2}\left(3\cos^2\theta - 1\right)\right) \tag{6.6.11}$$

6.7. An uncharged, conducting sphere of radius R is placed in a region where the electric field is uniform i.e. $\vec{E} = \vec{E}_0$. Can you guess whether l value will be odd or even in potential outside the sphere? Find the electric field in the region after the sphere is put in place.

Solution: We can begin by observing that charges will distribute themeselves such that the charge density function will be odd in $\cos\theta$, and hence l will also be odd
WLOG, Let the external electric field be along $\hat{k}$, thus $\vec{E_0} = E_0\hat{k}$. In spherical polar coordinates, $\vec{E_0} = E_0 \cos\theta\,\hat{r} - E_0 \sin\theta\,\hat{\theta}$
This time we must be careful in setting the reference of potential, since we cannot set it to its usual at infinity (since everything near the sphere will have an infinite potential difference wrt infinity due to the constant $\vec{E_0}$ field).
Remember that since our potential reference is now **not** at infinity, all of our intuitions regarding absolute potentials go out the window. Any potentials we must specify must also be accompanied by what they are in respect to.
We can simply choose the reference to be the sphere surface itself. We can see that then the whole xy plane will also be at 0 potential wrt the sphere ($\because$ due to symmetry, $\vec{E}\cdot\hat{r} = 0$ on the xy plane).
Thus we observe that far away from sphere (where charges on sphere won't have any effect), $V = -E_0 z = -E_0 r\cos\theta$ Thus we need to solve the equations

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V(r,\theta)}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V(r,\theta)}{\partial\theta}\right) = 0 \qquad \text{Laplace's Equation} \tag{6.7.1}$$

$$V(R,\theta) = 0 \qquad \text{Boundary Conditions} \tag{6.7.2}$$

$$\lim_{r\to\infty} V(r,\theta) = -E_0 r\cos\theta \tag{6.7.3}$$

in the region $r \geq R$ As always, we can write the general solution using separation of variables

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta) \tag{6.7.4}$$

Applying (6.7.2),

$$B_l = -A_l R^{2l+1}\,\forall\, l \tag{6.7.5}$$

Applying (6.7.3), we can see that $A_l = 0\ \forall\ l > 1$. Comparing coefficients,

$$-E_0 r\cos\theta = A_0 + A_1 r\cos\theta \tag{6.7.6}$$

$$\implies A_0 = 0 \tag{6.7.7}$$

$$A_1 = -E_0 \tag{6.7.8}$$

Putting it all together,

$$V(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta \tag{6.7.9}$$

Of course, remember that this potential is **in reference to the sphere**
To find electric field,

$$\vec{E} = -\nabla V \tag{6.7.10}$$

$$= E_0\left(1 + \frac{2R^3}{r^3}\right)\cos\theta\,\hat{r} - E_0\frac{1}{r}\left(r - \frac{R^3}{r^2}\right)\sin\theta\,\hat{\theta} \tag{6.7.11}$$

$$= \left(E_0\cos\theta\,\hat{r} - E_0\sin\theta\,\hat{\theta}\right) + E_0\frac{R^3}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right) \tag{6.7.12}$$

We can see that in addition to $\vec{E_0}$, a dipole field is induced by the sphere

6.8. Suppose the potential $V_0(\theta)$ at the surface of the sphere is specified, and there is no charge outside or inside the sphere. Show that charge density on the surface of the sphere is given by

$$\sigma(\theta) = \frac{\epsilon_0}{2R}\sum_{l=0}^{\infty}(2l+1)^2 C_l P_l(\cos\theta)$$

where C_l is

$$C_l = \int_{\theta=0}^{\pi} V_0(\theta) P_l(\cos\theta)\sin\theta\, d\theta$$

Solution: We know that (after a simple application of separation of variables followed by Fourier's trick)

$$V_{\text{out}}(r,\theta) = \sum_{l=0}^{\infty} \frac{\int_{\theta=0}^{\pi} V_0(\theta) P_l(\cos\theta)\sin\theta\, d\theta}{r^{l+1}} \frac{(2l+1)R^{l+1}}{2} P_l(\cos\theta) \tag{6.8.1}$$

$$= \sum_{l=0}^{\infty} \frac{C_l}{r^{l+1}} \frac{(2l+1)R^{l+1}}{2} P_l(\cos\theta) \tag{6.8.2}$$

$$V_{\text{in}}(r,\theta) = \sum_{l=0}^{\infty} \frac{\int_{\theta=0}^{\pi} V_0(\theta) P_l(\cos\theta)\sin\theta\, d\theta}{R^{l}} \frac{(2l+1)r^{l}}{2} P_l(\cos\theta) \tag{6.8.3}$$

$$= \sum_{l=0}^{\infty} \frac{C_l}{R^{l}} \frac{(2l+1)r^{l}}{2} P_l(\cos\theta) \tag{6.8.4}$$

We also know that

$$\frac{\sigma(\theta)}{\epsilon_0} = -\left.\frac{\partial V_{\text{out}}}{\partial r}\right|_{r=R} + \left.\frac{\partial V_{\text{in}}}{\partial r}\right|_{r=R} \tag{6.8.5}$$

$$= \sum_{l=0}^{\infty} \frac{(2l+1)C_l P_l(\cos\theta)}{2}\left(\frac{l}{R} + \frac{l+1}{R}\right) \tag{6.8.6}$$

$$= \frac{1}{2R}\sum_{l=0}^{\infty}(2l+1)^2 C_l P_l(\cos\theta) \tag{6.8.7}$$

$$\implies \sigma(\theta) = \frac{\epsilon_0}{2R}\sum_{l=0}^{\infty}(2l+1)^2 C_l P_l(\cos\theta) \tag{6.8.8}$$

7 Dipoles & Multipole expansion

7.1. Consider a thin spherical shell (thickness $\to 0$) of radius R with a surface charge density

$$\sigma(\theta) = \sigma_0(\cos\theta + \cos^2\theta)$$

Using solutions of Laplace's equation, find the potential $V(r,\theta)$ everywhere, both for $r > R$ and $r < R$.

Solution: As before, we know that,

$$V_{\text{out}}(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) \tag{7.1.1}$$

$$V_{\text{in}}(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \tag{7.1.2}$$

We can apply $V_{\text{in}}(R,\theta) = V_{\text{out}}(R,\theta)$ to get $B_l = A_l R^{2l+1}$
We also know that

$$\sigma(\theta) = \epsilon_0 \left.\frac{\partial V_{\text{in}}(r,\theta)}{\partial r}\right|_{r=R} - \epsilon_0 \left.\frac{\partial V_{\text{out}}(r,\theta)}{\partial r}\right|_{r=R} \tag{7.1.3}$$

$$= \epsilon_0 \sum_{l=0}^{\infty} (2l+1) A_l R^{l-1} P_l(\cos\theta) \tag{7.1.4}$$

We can immediately see that $A_l = 0 \;\forall\; l > 2$. Comparing coefficients,

$$\sigma_0 \cos\theta + \sigma_0 \cos^2\theta = \frac{A_0\epsilon_0}{R} + 3A_1\epsilon_0\cos\theta + \frac{5A_2R\epsilon_0}{2}\left(3\cos^2\theta - 1\right) \tag{7.1.5}$$

$$\implies A_2 = \frac{2\sigma_0}{15R\epsilon_0} \tag{7.1.6}$$

$$A_1 = \frac{\sigma_0}{3\epsilon_0} \tag{7.1.7}$$

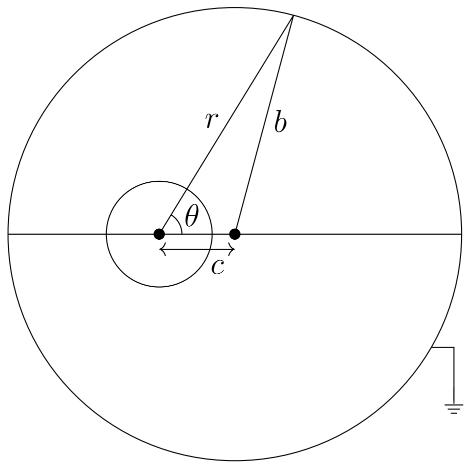
$$A_0 = \frac{\sigma_0 R}{3\epsilon_0} \tag{7.1.8}$$

Thus we can write the potentials

$$V_{\text{out}} = \frac{\sigma_0 R^2}{3\epsilon_0 r} + \frac{\sigma_0 R^3}{3\epsilon_0 r^2}\cos\theta + \frac{\sigma_0 R^4}{15\epsilon_0 r^3}\left(3\cos^2\theta - 1\right) \tag{7.1.9}$$

$$V_{\text{in}} = \frac{\sigma_0 R}{3\epsilon_0} + \frac{\sigma_0 r}{3\epsilon_0}\cos\theta + \frac{\sigma_0 r^2}{15\epsilon_0 R}\left(3\cos^2\theta - 1\right) \tag{7.1.10}$$

7.2. In the following system (see figure), the inner conducting sphere of radius a carries charge Q and the outer sphere of radius b is grounded. The distance between the centres is c which is a small quantity.



(a) Show that to the first order in c, the equation describing the outer sphere, using the centre of inner sphere as origin, is $r(\theta) = b + c\cos\theta$.

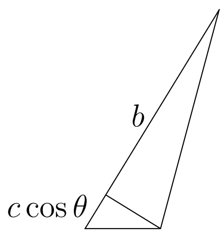
Solution: We can use trigonometry and observe

$$b^2 = c^2 + r^2 - 2cr\cos\theta \tag{7.2.1}$$

Solving using the Quadratic Formula (and taking the positive root)

$$\begin{aligned} r &= c\cos\theta + \sqrt{b^2 - c^2\sin^2\theta} && (7.2.2)\\ &= c\cos\theta + b\sqrt{1 - \frac{c^2\sin^2\theta}{b^2}} && (7.2.3)\\ &= c\cos\theta + b\left(1 - \frac{c^2\sin^2\theta}{2b^2}\right) + \mathcal{O}(c^4) && (7.2.4)\\ &= b + c\cos\theta + \mathcal{O}(c^2) && (7.2.5) \end{aligned}$$

We can also see this geometrically



(b) If the potential between two spheres contains only $l = 0$ and $l = 1$ angular components, determine it to first order in c.

Solution: As before, we have

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta) \tag{7.2.6}$$

$$\frac{\partial V(a,\theta)}{\partial\theta} = 0 \qquad \text{Inner sphere equipotential} \tag{7.2.7}$$

$$-\epsilon_0 \int_{\text{inner sphere}} \left.\frac{\partial V(r,\theta)}{\partial r}\right|_{r=a} = Q \qquad \text{Total charge on inner sphere} \tag{7.2.8}$$

$$V(b + c\cos\theta + \mathcal{O}(c^2), \theta) = 0 \qquad \text{Outer sphere grounded} \tag{7.2.9}$$

Using (7.2.7),

$$B_l = -A_l a^{2l+1} \,\forall\, l > 0 \tag{7.2.10}$$

Using, (7.2.8)

$$-2\pi\epsilon_0 a^2 \int_{\theta=0}^{\pi} \left(-\frac{B_0}{a^2} + \sum_{l=1} (2l+1) A_l a^{l-1} P_l(\cos\theta) \right) \sin\theta \, d\theta = Q \tag{7.2.11}$$

$$\implies 4\pi\epsilon_0 B_0 + \sum_{l=1}^{\infty} a^{l+1}(2l+1)A_l \int_{x=-1}^{1} P_l(x)\, dx = Q \tag{7.2.12}$$

$$\implies 4\pi\epsilon_0 B_0 + \sum_{l=1}^{\infty} a^{l+1}(2l+1)A_l \int_{x=-1}^{1} P_l(x)P_0(x)\, dx = Q \tag{7.2.13}$$

$$\implies B_0 = \frac{Q}{4\pi\epsilon_0} \tag{7.2.14}$$

Applying (7.2.9)

$$0 = A_0 + B_0 \left(b + c\cos\theta + \mathcal{O}(c^2)\right)^{-1} \tag{7.2.15}$$

$$+ \left(A_1 \left(b + c\cos\theta + \mathcal{O}(c^2)\right) + B_1 \left(b + c\cos\theta + \mathcal{O}(c^2)\right)^{-2} \right) \cos\theta + \mathcal{O}(\cos^2\theta) \tag{7.2.16}$$

$$\implies A_0 + \frac{B_0}{b} + \left(A_1 b + \frac{B_1}{b^2} - \frac{B_0 c}{b^2} \right) \cos\theta + \mathcal{O}(c^2) + \mathcal{O}(\cos^2\theta) = 0 \tag{7.2.17}$$

Solving upto first order in c and equating coefficients of 1 and $\cos\theta$,

$$A_0 = -\frac{Q}{4\pi\epsilon_0 b} \tag{7.2.18}$$

$$A_1 = -\frac{Q}{4\pi\epsilon_0(b^3 - a^3)} \tag{7.2.19}$$

Putting it all together,

$$V(r,\theta) = \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{r} - \frac{1}{b} - \frac{c}{b^3 - a^3} \left(r - \frac{a^3}{r^2} \right) \cos\theta \right) + \mathcal{O}(c^2) + \mathcal{O}(\cos^2\theta) \tag{7.2.20}$$

7.3. Static charges are distributed along the x axis (one-dimensional) in the interval $-a \le x' \le a$.

The charge density is :

$$\begin{cases} \rho(x') & -a \le x' \le a \\ 0 & |x'| > a \end{cases}$$

(a) Write down the multipole expansion for the electrostatic potential $\phi(x)$ at a point x on the axis in terms of $\rho(x')$, valid for $x > a$

Solution:

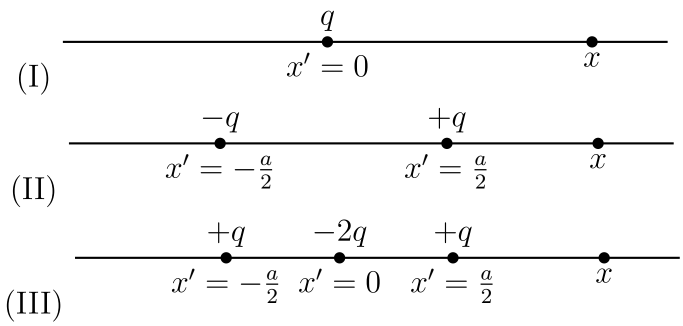
$$\phi(x) = \frac{1}{4\pi\epsilon_0} \int_{x=-a}^{a} \frac{\rho(x')}{|x - x'|} dx' \tag{7.3.1}$$

$$\phi(x) = \frac{1}{4\pi\epsilon_0} \int_{x=-a}^{a} \frac{\rho(x')}{x|1 - \frac{x'}{x}|} dx' \tag{7.3.2}$$

$$= \frac{1}{4\pi\epsilon_0} \int_{x=-a}^{a} \frac{\rho(x')}{x} dx' + \frac{1}{4\pi\epsilon_0} \int_{x=-a}^{a} \frac{\rho(x')x'}{x^2} dx' + \frac{1}{4\pi\epsilon_0} \int_{x=-a}^{a} \frac{\rho(x')x'^2}{x^3} dx' \dots \tag{7.3.3}$$

$$= \sum_{l=0}^{\infty} \frac{1}{4\pi\epsilon_0} \int_{x=-a}^{a} \frac{\rho(x')x'^l}{x^{l+1}} dx' \tag{7.3.4}$$

(b) For each charge configuration given in below figure, find (a) total charge $Q = \int \rho(x')\, dx'$, (b) dipole moment $P = \int x'\rho(x')\, dx'$



Solution: (I) $Q = q$, $P = 0$
(II) $Q = 0$, $P = qa$
(III) $Q = 0$, $P = 0$

7.4. A circular disc of radius R lies in the $z = 0$ plane, centred at the origin. It has the following charge density frozen on

$$\sigma(r', \phi) = \sigma_0 r' \cos(\phi)$$

(a) What is the monopole moment of the configuration?

Solution:

$$Q = \int_{\text{all space}} \rho \, d\tau \tag{7.4.1}$$
$$= \sigma_0 \int_{r'=0}^{R} \int_{\phi=0}^{2\pi} r'^2 \cos\phi \, dr' \, d\phi \tag{7.4.2}$$
$$= 0 \tag{7.4.3}$$

(b) Calculate the dipole contribution to the potential due to the configuration at $(0, 0, z)$ using the expression in polar form

Solution: We know that $\theta' = \frac{\pi}{2}$, so $\cos\theta' = 0$, therefore dipole contribution is also 0

(c) Now calculate the the cartesian components of the dipole moment of the con guration. Use this to calculate the dipole contribution at $(0, 0, z)$. Verify your answer with the expression obtained in (b)

Solution:

$$\vec{P} = \int_{\text{disc}} \vec{r}\rho(r') \, d\tau' \tag{7.4.4}$$
$$= \sigma_0 \int_{r'=0}^{R} \int_{\phi=0}^{2\pi} r'^3 (\cos\phi \, \hat{\imath} + \sin\phi \, \hat{\jmath}) \cos\phi \, dr' \, d\phi \tag{7.4.5}$$
$$= \frac{\pi\sigma_0 R^4}{4} \hat{\imath} \tag{7.4.6}$$

$V(0, 0, z) = \frac{1}{4\pi\epsilon_0 z^2} \frac{\pi\sigma_0 R^4}{4} \hat{\imath} \cdot \hat{k} = 0$

7.5. If the total amount of charge (monopole) contained in a distribution is zero, show that the dipole moment is independent of the choice of the origin.

Solution:

$$\vec{P} = \int \vec{r'}\rho(\vec{r'}) \, d\tau' \tag{7.5.1}$$

$\vec{r'} \to \vec{r'} - \vec{a}$

$$\vec{P_1} = \int \vec{r'}\rho(\vec{r'}) \, d\tau' - \vec{a} \int \rho(\vec{r'}) \, d\tau' \tag{7.5.2}$$
$$= \int \vec{r'}\rho(\vec{r'}) \, d\tau' = \vec{P} \tag{7.5.3}$$

7.6. Find the dipole moment of:

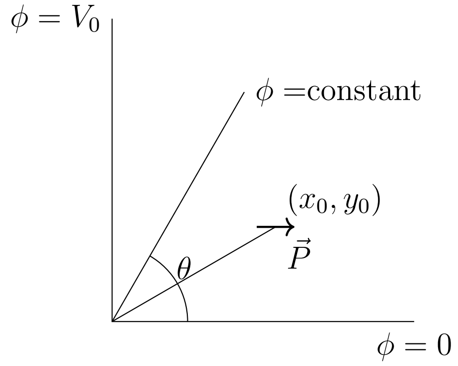
(a) A ring with charge per unit length $\lambda = \lambda_0 \cos\phi$ where ϕ is the angular variable in cylindrical coordinates.

> **Solution:**
>
> $$\vec{P} = \lambda_0 R^2 \int_{\phi=0}^{2\pi} (\cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath}) \cos\phi\, d\phi \tag{7.6.1}$$
> $$= \lambda_0 \pi R^2\,\hat{\imath} \tag{7.6.2}$$

(b) A sphere with charge per unit area $\sigma = \sigma_0 \cos\theta$ where θ is the polar angle measured from the positive z axis.

> **Solution:**
>
> $$\vec{P} = \sigma_0 R^3 \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} (\sin\theta\cos\phi\,\hat{\imath} + \sin\theta\sin\phi\,\hat{\jmath} + \cos\theta\hat{k}) \cos\theta \sin\theta\, d\theta\, d\phi \tag{7.6.3}$$
> $$= \frac{4\pi}{3}\sigma_0 R^3\,\hat{k} \tag{7.6.4}$$

7.7. An electric dipole of moment $\vec{P} = (P_x, 0, 0)$ is located at the point $(x_0, y_0, 0)$ where $x_0 > 0$ and $y_0 > 0$. The planes $x = 0$ and $y = 0$ are conducting plates with a tiny gap at the origin. The potential of the plate at $x = 0$ is maintained at V_0 and the plate at $y = 0$ is grounded. The dipole is sufficiently weak so that you can ignore the charges induced on the plates.



(a) Based on the figure, deduce a simple expression for the electrostatic potential $\phi(x, y)$.

> **Solution:** We need to solve the equations
>
> $$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial\phi(\rho,\theta)}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\phi(\theta)}{\partial\theta^2} = 0 \qquad \text{Laplace's Equation} \tag{7.7.1}$$
> $$\phi(\rho, 0) = 0 \qquad \text{Boundary Conditions} \tag{7.7.2}$$
> $$\phi(\rho, \frac{\pi}{2}) = V_0 \tag{7.7.3}$$
>
> in the region $0 \leq \theta \leq \frac{\pi}{2}$
> We can write the general solution using separation of variables (the linear solution in

θ no longer needs to be periodic since our domain is no longer the full range of θ)

$$\phi(\rho,\theta) = (A_0 + B_0 \ln\rho)(C_0 + D_0\theta) + \sum_{m=1}^{\infty}(A_m\rho^m + B_m\rho^{-m})(C_m\cos m\theta + D_m \sin m\theta) \tag{7.7.4}$$

Applying (7.7.2) and (7.7.3)

$$B_0 = 0 \tag{7.7.5}$$
$$C_m = 0\,\forall\, m \tag{7.7.6}$$
$$D_m = 0\,\forall\, m > 0 \tag{7.7.7}$$
$$\tag{7.7.8}$$

We are left with

$$\phi(\rho,\theta) = A_0D_0\theta \tag{7.7.9}$$

Applying (7.7.3) gives us $A_0D_0 = \frac{2V_0}{\pi}$

$$\implies \phi(\theta) = \frac{2V_0}{\pi}\theta \tag{7.7.10}$$
$$\implies \phi(x,y) = \frac{2V_0}{\pi}\tan^{-1}\left(\frac{y}{x}\right) \tag{7.7.11}$$

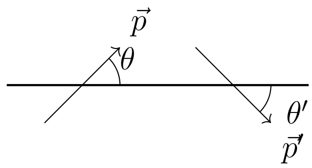
(b) Calculate the force on the dipole

Solution:

$$\vec{E} = -\nabla\phi \tag{7.7.12}$$
$$= -\frac{1}{\rho}\frac{2V_0}{\pi}\,\hat{\theta} \tag{7.7.13}$$
$$= \frac{2V_0 y}{(x^2+y^2)\,\pi}\,\hat{\imath} - \frac{2V_0 x}{(x^2+y^2)\,\pi}\,\hat{\jmath} \tag{7.7.14}$$
$$\vec{F} = (\vec{P}\cdot\nabla)\vec{E} \tag{7.7.15}$$
$$= -\frac{4V_0x_0y_0P_x}{\left(x_0^2+y_0^2\right)^2\pi}\,\hat{\imath} + \left(-\frac{2V_0P_x}{(x_0^2+y_0^2)\pi} + \frac{4V_0x_0^2P_x}{(x_0^2+y_0^2)^2\pi}\right)\hat{\jmath} \tag{7.7.16}$$
$$= \frac{2V_0P_x}{(x_0^2+y_0^2)^2\pi}\left(2x_0y_0\,\hat{\imath} + (x_0^2-y_0^2)\,\hat{\jmath}\right) \tag{7.7.17}$$

8 Biot-Savart Law

8.1. Two co-planar dipoles are oriented as shown in the figure. Find equilibrium value of θ' if θ is fixed.



Solution: In equilibrium, the energy of the system will be minimized (or maximized).

The positions of $\vec{p}$ and $\vec{p}'$ are fixed, so the only degree of freedom the energy depends on is the orientation of $\vec{p}'$. The energy of $\vec{p}'$ in the field of $\vec{p}$ will be minimized (or maximized) when $\vec{p}'$ is along (or opposite to) $\vec{E}_{\vec{p}}$.

We can easily find the direction of $\vec{E}_{\vec{p}}$ at $\vec{p}'$. Let the distance between the dipoles be $\vec{r}$, and let the axis joining their centres be the x axis, and the dipoles be oriented in the xy plane

$$\vec{E}_{\vec{p}} = \frac{1}{4\pi\epsilon_0}\frac{2|\vec{p}|\cos\theta}{r^2}\,\hat{\imath} - \frac{1}{4\pi\epsilon_0}\frac{|\vec{p}|\sin\theta}{r^3}\,\hat{\jmath} \tag{8.1.1}$$

$$\implies \tan(\theta') = \frac{-\vec{E}_{\vec{p}}\cdot\hat{\jmath}}{\vec{E}_{\vec{p}}\cdot\hat{\imath}} \tag{8.1.2}$$

$$= \frac{\sin\theta}{2\cos\theta} \tag{8.1.3}$$

$$= \frac{\tan(\theta)}{2} \tag{8.1.4}$$

$$\implies \theta' = \tan^{-1}\left(\frac{\tan(\theta)}{2}\right) \tag{8.1.5}$$

Thus the dipole will align itself along (or opposite to) θ'

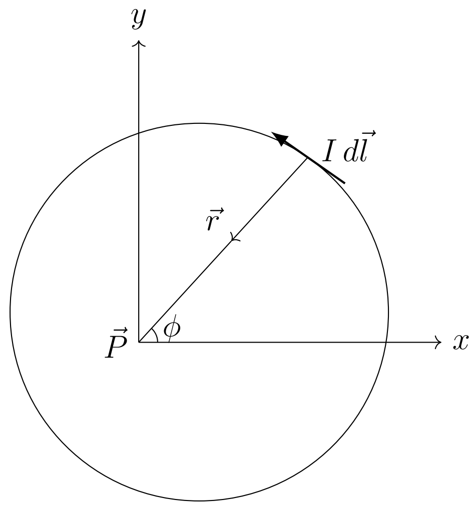
8.2. Let I be the current carried by a wire bent into a planar loop. Place the origin of coordinates at an observation point $\vec{P}$ in the plane of the loop.

(a) Show that the magnitude of the magnetic field at the point $\vec{P}$ is

$$|\vec{B}(\vec{P})| = \frac{\mu_0 I}{4\pi}\int_0^{2\pi}\frac{d\phi}{r(\phi)}$$

where $r(\phi)$ is the distance from the origin of coordinates at $\vec{P}$ to the point on the loop located at an angle ϕ from the positive x axis.

Solution: To obtain the limits of the integral mentioned in the question, we have to assume the point lies *inside* the loop.



$$\vec{B}(\vec{P}) = \frac{\mu_0 I}{4\pi}\oint \frac{d\vec{l}\times\hat{r}}{|\vec{r}|^2} \tag{8.2.1}$$

$$= \frac{\mu_0 I}{4\pi}\oint \frac{\left(dr\,\hat{r} + |\vec{r}|\,d\phi\,\hat{\phi}\right)\times\hat{r}}{|\vec{r}|^2} \tag{8.2.2}$$

$$= \frac{\mu_0 I}{4\pi}\oint \frac{d\phi}{|\vec{r}|}\,d\hat{k} \tag{8.2.3}$$

$$\implies |\vec{B}(\vec{P})| = \frac{\mu_0 I}{4\pi}\int_{\phi=0}^{2\pi}\frac{d\phi}{|\vec{r}|} \tag{8.2.4}$$

(b) Show that the magnetic field at the center of a current-carrying wire bent into an ellipse with major and minor axes $2a$ and $2b$ is proportional to the integral of the form (with constant k),

$$\int_0^{\frac{\pi}{2}} d\phi\sqrt{1-k^2\sin^2\phi}$$

What are the magnetic fields when $a = b$ and when $a \to \infty$ with b fixed?

Solution: Applying the result obtained in the previous part,

$$r(\phi) = \frac{ab}{\sqrt{a^2\sin^2\phi + b^2\cos^2\phi}} \tag{8.2.5}$$

$$\implies |\vec{B}(\vec{P})| = \frac{\mu_0 I}{4\pi}\int_{\phi=0}^{2\pi}\frac{\sqrt{a^2\sin^2\phi + b^2\cos^2\phi}}{ab}\,d\phi \tag{8.2.6}$$

$$= \boxed{\frac{\mu_0 I}{\pi a}\int_{\phi=0}^{\frac{\pi}{2}}\sqrt{1-\left(1-\frac{a^2}{b^2}\sin^2\phi\right)}\,d\phi} \tag{8.2.7}$$

When $a = b$

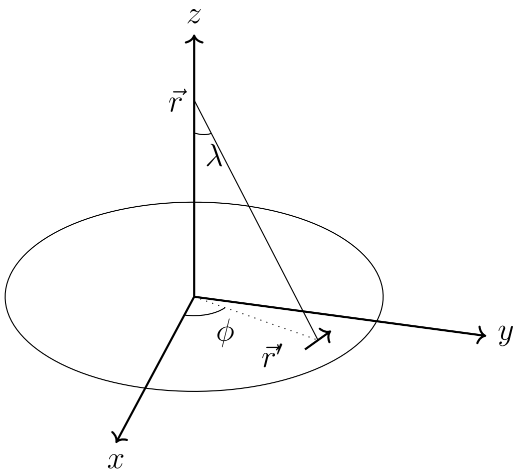
$$|\vec{B}(\vec{P})| = \frac{\mu_0 I}{\pi a}\int_{\phi=0}^{\frac{\pi}{2}} d\phi = \frac{\mu_0 I}{2a} \tag{8.2.8}$$

When $a \to \infty$

$$|\vec{B}(\vec{P})| = \frac{\mu_0 I}{\pi b} \int_{\phi=0}^{\frac{\pi}{2}} \sin\phi \, d\phi = \frac{\mu_0 I}{\pi b} \tag{8.2.9}$$

(c) An infinitesimally thin wire is wound in the form of a planar coil which can be modeled using an effective surface current density $\vec{K} = K\,\hat{\phi}$. Find the magnetic field at a point $\vec{P}$ on the symmetry axis of the coil. Express your answer in terms of the angle α subtended by the coil at P.

Solution:



$$\vec{\mathscr{r}} = \vec{r} - \vec{r'} = z\,\hat{k} - r'\,\hat{r} \tag{8.2.10}$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \iint \frac{\vec{K}(\vec{r'}) \times \hat{\mathscr{r}}}{\mathscr{r}^{\,2}}\, dS \tag{8.2.11}$$

$$= \frac{\mu_0 K}{4\pi} \iint \frac{\hat{\phi} \times \vec{\mathscr{r}}}{(z^2 + r'^2)^{\frac{3}{2}}}\, dS \tag{8.2.12}$$

$$= \frac{\mu_0 K}{4\pi} \iint \frac{(-\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}) \times (z\,\hat{k} - r'\cos\phi\,\hat{\imath} - r'\sin\phi\,\hat{\jmath})}{(z^2 + r'^2)^{\frac{3}{2}}}\, dS \tag{8.2.13}$$

$$= \frac{\mu_0 K}{4\pi} \left(\int_{r'=0}^{R} \cancelto{0}{\int_{\phi=0}^{2\pi} \frac{z\cos\phi}{(z^2 + r'^2)^{\frac{3}{2}}} r'\, d\phi}\, dr'\,\hat{\imath} + \int_{r'=0}^{R} \cancelto{0}{\int_{\phi=0}^{2\pi} \frac{z\sin\phi}{(z^2 + r'^2)^{\frac{3}{2}}} r'\, d\phi}\, dr'\,\hat{\jmath} \right.$$

$$+ \int_{r'=0}^{R} \int_{\phi=0}^{2\pi} \frac{r'^2}{(z^2 + r'^2)^{\frac{3}{2}}} \, d\phi \, dr' \, \hat{k} \Bigg) \tag{8.2.14}$$

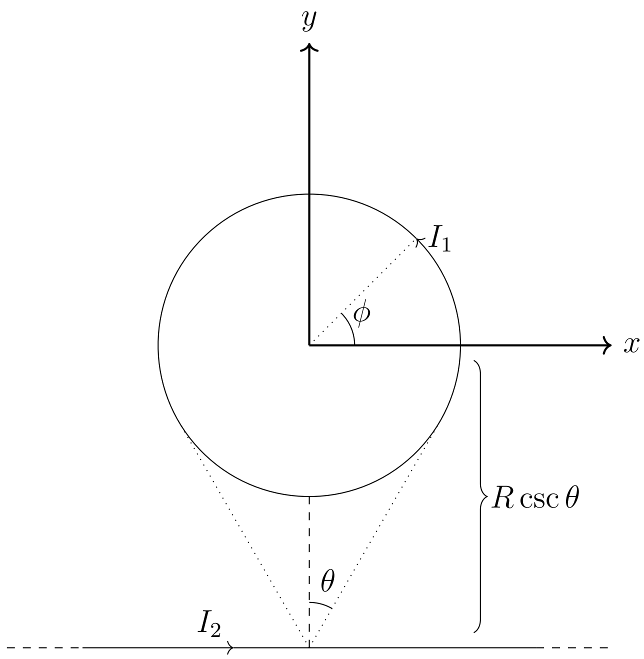
$$= \frac{\mu_0 K}{2} \int_{r'=0}^{R} \frac{r'^2}{(z^2 + r'^2)^{\frac{3}{2}}} \, dr' \, \hat{k} \tag{8.2.15}$$

$$= \frac{\mu_0 K}{2} \int_{\lambda=0}^{\alpha} (\sec\lambda - \cos\lambda) \, d\lambda \, \hat{k} \tag{8.2.16}$$

$$= \frac{\mu_0 K}{2} \left(\ln(\sec\alpha + \tan\alpha) - \sin\alpha\right) \tag{8.2.17}$$

8.3. A circular loop of wire carries a current I_1. A long straight wire in the plane of the loop carries a current I_2. The loop subtends an angle 2θ at a point on the wire which is nearest to it. Show that the force between the wire and the loop has a magnitude $\mu_0 I_1 I_2 (\sec\theta - 1)$

Solution:



Let us find force on I_1 due to field of I_2

$$\vec{B}_{I_2}(\phi) = \frac{\mu_0 I_2}{2\pi(R\sin\phi + R\csc\theta)}\,\hat{k} \tag{8.3.1}$$

$$\implies \vec{F} = \oint I_1\, d\vec{l} \times \vec{B}_{I_2}(\phi) \tag{8.3.2}$$

$$= \frac{\mu_0 I_1 I_2}{2\pi}\int_{\phi=0}^{2\pi} \frac{d\phi}{\sin\phi + \csc\theta}\,\hat{\phi}\times\hat{k} \tag{8.3.3}$$

$$= \frac{\mu_0 I_1 I_2}{2\pi}\int_{\phi=0}^{2\pi} \frac{d\phi}{\sin\phi + \csc\theta}\,(-\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath})\times\hat{k} \tag{8.3.4}$$

$$= \frac{\mu_0 I_1 I_2}{2\pi}\left(\cancelto{0}{\int_{\phi=0}^{2\pi} \frac{\cos\phi\, d\phi}{\sin\phi + \csc\theta}}\,\hat{\imath}\right. \tag{8.3.5}$$

$$\left. + \int_{\phi=0}^{2\pi} \frac{\sin\phi\, d\phi}{\sin\phi + \csc\theta}\,\hat{\jmath}\right) \tag{8.3.6}$$

$$= \frac{\mu_0 I_1 I_2}{2\pi}\left(2\pi - \csc\theta\int_{\phi=0}^{2\pi} \frac{1}{\sin\phi + \csc\theta}\,d\phi\right)\hat{\jmath} \tag{8.3.7}$$

$$= -\mu_0 I_1 I_2(\sec\theta - 1)\,\hat{\jmath} \tag{8.3.8}$$

8.4. A long cylindrical conductor of radius R carries current through it, with current density $J = kr$. Find the expression for magnetic field B at a distance r ($r < R$).

Solution: Using Ampere's Law,

$$\iint \vec{J}\cdot d\vec{a} = \oint B\cdot d\vec{l} \tag{8.4.1}$$

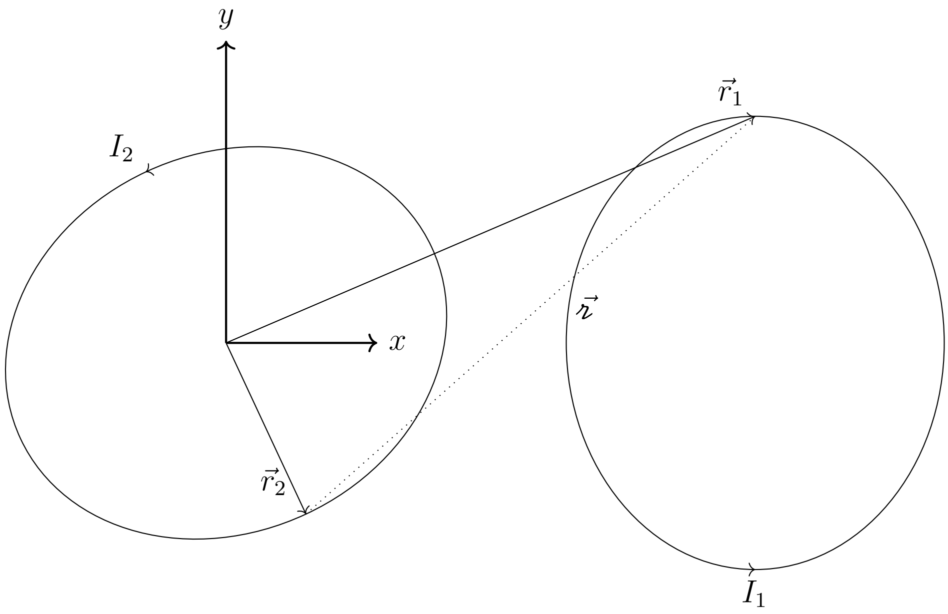
Using symmetry, magnetic field is independent of θ and using the fact that its divergence is 0, the radial component of B is also 0

$$\implies \int_0^r kr\cdot 2\pi r dr = B\cdot 2\pi r \tag{8.4.2}$$

$$\implies B = \frac{kr^2}{3} \tag{8.4.3}$$

8.5. Calculate the force between two closed circuits carrying steady currents.

Solution:



$$\vec{B}(\vec{r}_1) = \frac{\mu_0 I_2}{4\pi} \oint_2 \frac{d\vec{l}_2 \times \vec{\mathscr{r}}}{|\vec{\mathscr{r}}|^3} \tag{8.5.1}$$

$$= \frac{\mu_0 I_2}{4\pi} \oint_2 \frac{d\vec{l}_2 \times (\vec{r}_1 - \vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|^3} \tag{8.5.2}$$

$$\implies \vec{F}_{12} = I_1 \oint_1 d\vec{l}_1 \times \vec{B}(\vec{r}_1) \tag{8.5.3}$$

$$= \frac{\mu_0 I_1 I_2}{4\pi} \oint_1 \oint_2 \frac{d\vec{l}_1 \times (d\vec{l}_2 \times (\vec{r}_1 - \vec{r}_2))}{|(\vec{r}_1 - \vec{r}_2)^3|} \tag{8.5.4}$$

$$= \frac{\mu_0 I_1 I_2}{4\pi} \left(\oint_2 d\vec{l}_2 \oint_1 \frac{d\vec{l}_1 \cdot (\vec{r}_1 - \vec{r}_2)}{|(\vec{r}_1 - \vec{r}_2)^3|} \right.$$

$$\left. - \oint_2 \oint_1 (\vec{r}_1 - \vec{r}_2) \frac{d\vec{l}_1 \cdot d\vec{l}_2}{|\vec{r}_1 - \vec{r}_2|^3} \right) \qquad \because \text{BAC-CAB rule} \tag{8.5.5}$$

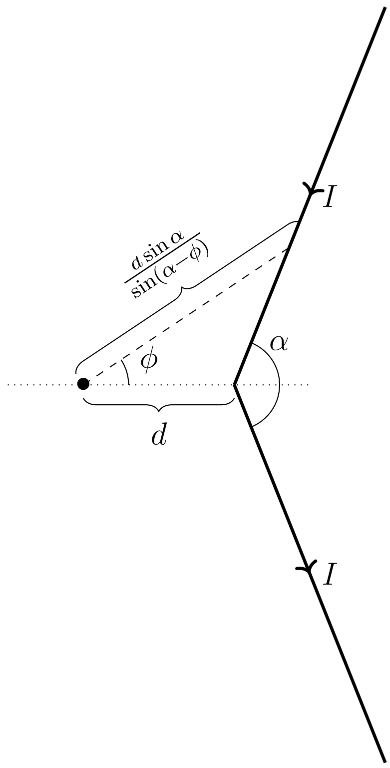
$$= -\frac{\mu_0 I_1 I_2}{4\pi} \left(\oint_2 d\vec{l}_2 \cancelto{0}{\oint_1 \nabla \left(\frac{1}{|\vec{r}_1 - \vec{r}_2|} \right) \cdot d\vec{l}_1} \right.$$

$$\left. - \oint_2 \oint_1 (\vec{r}_1 - \vec{r}_2) \frac{d\vec{l}_1 \cdot d\vec{l}_2}{|\vec{r}_1 - \vec{r}_2|^3} \right) \tag{8.5.6}$$

$$= \frac{\mu_0 I_1 I_2}{4\pi} \oiint_{1,2} \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_1 - \vec{r}_2|^3} \, d\vec{l}_1 \cdot d\vec{l}_2 \tag{8.5.7}$$

8.6. Using Biot and Savart law, find $\vec{B}(\vec{r})$ in the plane of the wire at a distance d from the bend along the axis of symmetry.

Solution:



We can directly use the results of Q 8.2 a

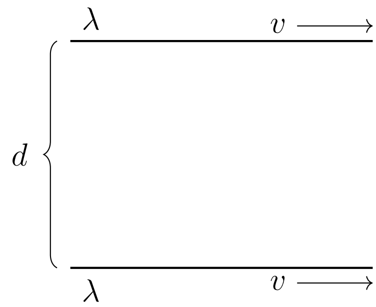
$$\vec{B} = \frac{\mu_0 I}{4\pi} \int_{\text{wire}} \frac{d\phi}{r}\,\hat{k} \tag{8.6.1}$$

$$= \frac{\mu_0 I \csc\alpha}{2\pi d} \int_{\phi=0}^{\phi=\alpha} \sin(\alpha - \phi)\, d\phi\, \hat{k} \tag{8.6.2}$$

$$= \frac{\mu_0 I \csc\alpha(1 - \cos\alpha)}{2\pi d} \tag{8.6.3}$$

$$= \frac{\mu_0 I}{2\pi d} \tan\frac{\alpha}{2} \tag{8.6.4}$$

8.7. Suppose you have two infinite straight-line charges λ, a distance d apart, moving along at a constant v (see Figure below). How fast would v have to be in order for the magnetic attraction to balance the electrical repulsion?



Solution: Assume velocity is v. Hence current is $\frac{dQ}{dt} = \lambda v$

Electrostatic force per unit length between 2 charged wires is

$$F_E = \frac{\lambda^2}{2\pi\epsilon_0 r} \tag{8.7.1}$$

Magnetic Force per unit length between 2 current carrying wires is

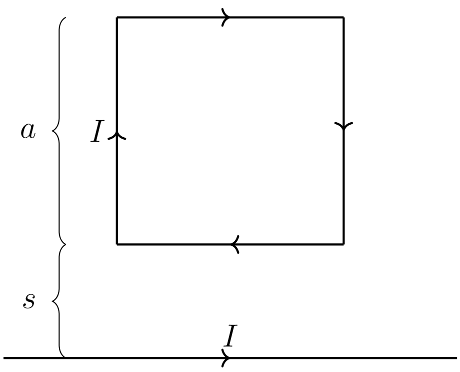
$$\begin{aligned} F_B &= \frac{\mu_0 I^2}{2\pi r} \\ &= \frac{\mu_0 \lambda^2 v^2}{2\pi r} \end{aligned} \tag{8.7.2}$$

Equating the 2 forces, we get

$$v = \frac{1}{\sqrt{\mu_o \epsilon_o}} = c \tag{8.7.3}$$

Where c is speed of light

8.8. (a) Find the force on a square loop placed as shown in the figure , near an infinite straight wire. Both the loop and the wire carry a steady current I.

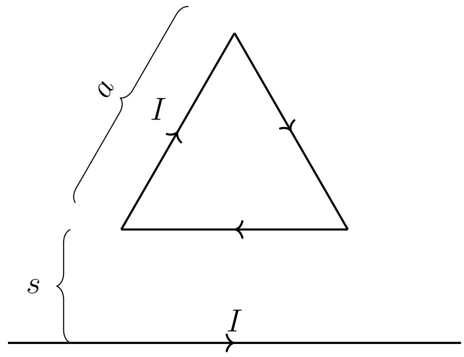


Solution: There is no force on the sides perpendicular to straight wire
We can calculate the difference of forces on upper and lower side

$$\vec{F} = \frac{\mu_0 I^2 a}{2\pi}\left(\frac{1}{s} - \frac{1}{s+a}\right)\hat{\jmath} \tag{8.8.1}$$

$$= \frac{\mu_0 I^2 a^2}{2\pi s(s+a)}\hat{\jmath} \tag{8.8.2}$$

(b) Find the force on the triangular loop in figure

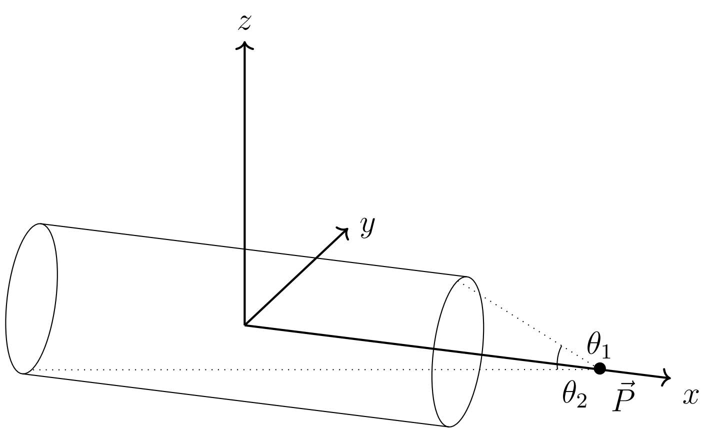


Solution: We again calculate the force on each side. We focus only on the $\hat{\jmath}$ component as the $\hat{\imath}$ component is 0 by symmetry

$$\vec{F} = \frac{\mu_0 I^2}{2\pi}\left(\frac{a}{s} - 2\cdot\frac{1}{2}\int_{l=0}^{a}\frac{dl}{s+\frac{l\sqrt{3}}{2}}\right)\hat{\jmath} \tag{8.8.3}$$

$$= \frac{\mu_0 I^2}{2\pi}\left(\frac{a}{s} - \frac{2}{\sqrt{3}}\ln\left(\frac{2s+\sqrt{3}a}{2s}\right)\right)\hat{\jmath} \tag{8.8.4}$$

8.9. Find the magnetic field at point $\vec{P}$ on the axis of a tightly wound solenoid (helical coil) consisting of n turns per unit length wrapped around a cylindrical tube of radius a and carrying current I. Express your answer in terms of θ_1 and θ_2. Consider the turns to be essentially circular. What is the field on the axis of an infinite solenoid (infinite in both directions)?



Solution:

$$\vec{B} = \frac{\mu_0 n I}{2} \int \frac{R^2}{(R^2+z^2)^{\frac{3}{2}}}\, dz\, \hat{\imath} \tag{8.9.1}$$

$$= \frac{\mu_0 n I}{2} \int_{\theta=\theta_1}^{\theta_2} \sin\theta\, d\theta\, \hat{\imath} \tag{8.9.2}$$

$$= \frac{\mu_0 n I}{2} (\cos\theta_2 - \cos\theta_1)\hat{\imath} \tag{8.9.3}$$

For infinite solenoid, $\theta_1 = \pi$, $\theta_2 = 0$

$$\implies \vec{B} = \mu_0 n I\, \hat{\imath} \tag{8.9.4}$$

8.10. The current I flowing along the edges of one face of a cube (see Figure(a)) produces a magnetic field in the center of the cube of magnitude B_0. Consider another cube where the current I flows along a path shown in Figure (b). What magnetic field will now exist at the center of the cube?

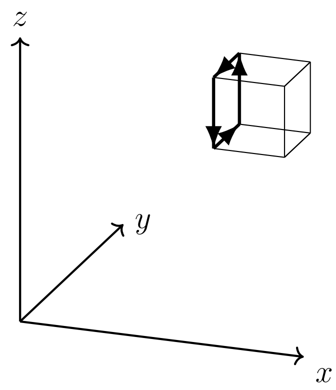


Figure (a)

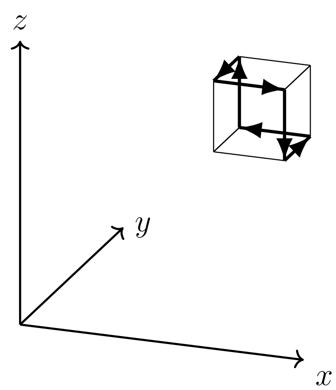


Figure (b)

Solution: We can replace a branch carrying no current as a superposition of 2 branches carrying equal and opposite current. Hence, Figure (b) can be expressed as a superposition of 3 loops of Figure (a). Hence the magnetic field is a superposition of the magnetic field due to the 3 loops

9 Dielectrics

9.1. Calculate the magnetic force of attraction between the northern and southern hemispheres of a spinning charged magnetic shell (with radius R, angular speed ω and surface charge density σ)

Solution: Let us orient the z axis along $\vec{\omega}$
We already know the vector potential

$$\vec{A}(\vec{r}) = \begin{cases} \frac{\mu_0\omega\sigma}{3} r\sin\theta\,\hat{\phi} & r \le R \\ \frac{\mu_0\omega R^4\sigma}{3}\frac{\sin\theta}{r^2}\,\hat{\phi} & r \ge R \end{cases} \tag{9.1.1}$$

We need the $\vec{B}_{\text{ave}} = \frac{\vec{B}_{\text{in}}(R)+\vec{B}_{\text{out}}(R)}{2}$ at the surface of the shell

$$\vec{B}_{\text{in}}(R) = \nabla\times\vec{A}_{\text{in}}\Big|_{r=R} = \frac{2}{3}\mu_0 R\omega\sigma\,\hat{z} \tag{9.1.2}$$

$$\vec{B}_{\text{out}}(R) = \nabla\times\vec{A}_{\text{out}}\Big|_{r=R} = \frac{\mu_0 R\omega\sigma}{3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}) \tag{9.1.3}$$

$$\implies \vec{B}_{\text{ave}} = \frac{\mu_0 R\omega\sigma}{6}(4\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}) \tag{9.1.4}$$

Now to find the force due to this $\vec{B}$ on the surface current $\vec{K}$
We know that $\vec{K} = \sigma\vec{v} = \omega R\sin\theta\,\hat{\phi}$

$$\vec{K}\times\vec{B}_{\text{ave}} = \frac{\mu_0}{6}\sin\theta(\sigma\omega R)^2(4\cos\theta\,\hat{\theta} + \sin\theta\,\hat{r}) \tag{9.1.5}$$

Finally,

$$\vec{F} = \int_{\mathcal{S}} \vec{K}\times\vec{B}_{\text{ave}}\,dA \tag{9.1.6}$$

Only the z component will survive the integral

$$\vec{F} = -\frac{\mu_0}{2}(\sigma\omega R^2)R^2\,\hat{z}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\frac{\pi}{2}} \sin^3\theta\cos\theta\,d\theta\,d\phi = -\frac{\mu_0\pi}{4}(\sigma\omega R^2)^2\,\hat{z} \tag{9.1.7}$$

9.2. What current density would produce the vector potential, A = k$\hat{\phi}$ (where k is a constant), in cylindrical coordinates?

Solution: Given $A_\phi = k$, so using $B = \nabla \times A$, in cylindrical coordinates, we get $B = \frac{k}{s}\hat{z}$.
Next, by using $J = \frac{\nabla \times B}{\mu_0}$, in cylindrical coordinates, we get $J = \frac{k}{\mu_0 s^2}\hat{\phi}$.

9.3. A sphere of radius R carries a polarization $\mathrm{P(r)} = k\mathrm{r}$ where k is a constant and r is the vector from the center.

(a) Calculate the bound charges σ_b and ρ_b

Solution: $\sigma_b = P.\hat{n} = kR$, $\rho_b = -\nabla.P = -3k$

(b) Find the electric field inside and outside the sphere.

Solution: Inside the sphere, $E = \frac{\rho r}{3\epsilon_0}\hat{r}$, so using ρ, we get $E = \frac{-k}{\epsilon_0}r$
For calculating E outside the sphere, we can use the fact that the net charge at the center is 0, so effectively E=0.

9.4. A thick spherical shell having inner radius a and outer radius b is made of dielectric material with a "frozen-in" polarization $\mathbf{P(r)} = \frac{k}{r}\hat{r}$ where k is a constant and r is the distance from the center and no free charge is present, then find D and E for all 3 regions. Refer Figure 1.

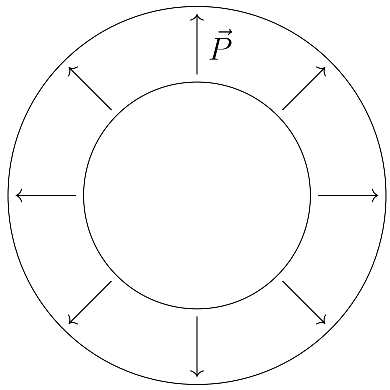


Figure 1: Shell with $\vec{P}(\vec{r})$

Solution: Since no free charge is present so D=0 everywhere.
By using $D = \epsilon_0 E + P$, we get $E = -\frac{1}{\epsilon_0}P$, So $E \neq 0$ only for $a < r < b$ in which case we get $E = \frac{-k}{\epsilon_0 r}\hat{r}$. For $r < a, r > b$ the net charge is 0, hence E=0.

9.5. In the lecture, we have calculated the total field inside the sphere when a sphere made of homogeneous linear isotropic dielectric (with dielectric constant ϵ_r) material is placed in an otherwise uniform electric field $\vec{E_0}$ by solving Laplace's equation:

$$\vec{E} = \frac{3}{\epsilon_r + 2}E_0$$

Attempt an alternate (and possibly more intuitive) approach to this, as follows. First find the polarization $\vec{P}_0$ due to $\vec{E}_0$. This polarization generates a field of its own, $\vec{E}_1$ which in turns causes an additional polarization $\vec{P}_1$, which further generates an additional field $\vec{E}_2$ and so on. Show that resultant field inside the sphere $\vec{E} = \vec{E}_0 + \vec{E}_1 + \vec{E}_2 + \dots$ matches that what we get by solving Laplace's Equation.

Solution: We know that

$$\vec{P}_i = \chi_e \epsilon_0 \vec{E}_i = (\epsilon_r - 1)\epsilon_0 \vec{E}_i \tag{9.5.1}$$

We also have

$$\vec{P}_{i+1} = -\frac{1}{3\epsilon_0}\vec{E}_i \tag{9.5.2}$$

Thus we end up with

$$\vec{E}_i = \left(-\frac{\epsilon_r - 1}{3}\right)^i \vec{E}_0 \tag{9.5.3}$$

$$\implies \vec{E} = \sum_{i=0}^{\infty} \vec{E}_i = \frac{3}{\epsilon_r + 2}\vec{E}_0 \tag{9.5.4}$$

9.6. Find the electric potential inside and outside a homogeneous linear isotropic dielectric sphere (with dielectric constant ϵ_r and radius R), at the centre of which a pure dipole $\vec{p}$ is imbedded.

Solution: We will find solution's to Laplace's equation in the region $r < R$ (excluding a small region containing the dipole), and $r > R$ separately.
Using separation of variables,

$$V_{\text{in}}(r,\theta) = \sum_{l=0}^{\infty} \left(A_l^{(\text{in})} r^l + \frac{B_l^{(\text{in})}}{r^{l+1}} \right) P_l(\cos\theta) \tag{9.6.1}$$

$$V_{\text{out}}(r,\theta) = \sum_{l=0}^{\infty} \left(A_l^{(\text{out})} r^l + \frac{B_l^{(\text{out})}}{r^{l+1}} \right) P_l(\cos\theta) \tag{9.6.2}$$

Subject to boundary conditions

$$\lim_{r\to\infty} V_{\text{out}}(r,\theta) = 0 \tag{9.6.3}$$

$$\lim_{r\to 0} V_{\text{in}}(r,\theta) = \frac{1}{4\pi\epsilon_0\epsilon_r}\frac{p\cos\theta}{r^2} \tag{9.6.4}$$

$$V_{\text{in}}(R,\theta) = V_{\text{out}}(R,\theta) \qquad \because \text{ continuity of } V \tag{9.6.5}$$

$$\epsilon_0\epsilon_r \left.\frac{\partial V_{\text{in}}}{\partial r}\right|_{r=R} = \epsilon_0 \left.\frac{\partial V_{\text{out}}}{\partial r}\right|_{r=R} \qquad \because D^{\perp}_{\text{out}} - D^{\perp}_{\text{in}} = \sigma_f = 0 \tag{9.6.6}$$

Applying (9.6.3) and (9.6.4) we can immediately see that $A_l^{(\text{out})} = 0 \;\forall\; l$; $B_l^{(\text{in})} = 0 \;\forall\; l \neq 1$; $B_1^{(\text{in})} = \frac{p}{4\pi\epsilon_0\epsilon_r}$
Applying (9.6.5) and (9.6.6),

$$A_l^{(\text{in})} = 0 \;\forall\; l \neq 1 \tag{9.6.7}$$

$$B_l^{(\text{out})} = 0 \;\forall\; l \neq 1 \tag{9.6.8}$$

$$A_1^{(\text{in})} R + \frac{p}{4\pi\epsilon_0\epsilon_r R^2} = \frac{B_1^{(\text{out})}}{R^2} \tag{9.6.9}$$

$$\epsilon_r A_1^{(\text{in})} - \frac{2p}{4\pi\epsilon_0 R^3} = \frac{-2B_1^{(\text{out})}}{R^3} \tag{9.6.10}$$

Solving,

$$B^{(\text{out})} = \frac{p}{4\pi\epsilon_0}\frac{3}{\epsilon_r + 2} \tag{9.6.11}$$

$$A^{(\text{in})} = \frac{p}{4\pi\epsilon_0 R^3}\frac{2(\epsilon_r - 1)}{\epsilon_r(\epsilon_r + 2)} \tag{9.6.12}$$

Thus we finally have,

$$V_{\text{in}}(r,\theta) = \frac{p\cos\theta}{4\pi\epsilon_0\epsilon_r r^2}\left(1 + \frac{r^3}{R^3}\frac{2(\epsilon_r - 1)}{\epsilon_r + 2}\right) \tag{9.6.13}$$

$$V_{\text{out}}(r,\theta) = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}\frac{3}{\epsilon_r + 2} \tag{9.6.14}$$

9.7. A cylinder of radius R and height L is positioned such that the origin is at the center and the z-axis is along the axis of the cylinder. The cylinder carries a frozen polarisation $\vec{P} = P_o\hat{z}$. Calculate $\vec{E}$ and $\vec{D}$ at all points on the z-axis. Are the quantities $\vec{D}$ and $\vec{E}$ proportional to each other inside the material?

Solution: Since $\vec{P}$ is uniform, the bound volume charge density is $-\nabla\cdot\vec{P} = 0$. There exists only a uniform bound surface charge density at the ends of the bar electret, $\sigma = \vec{P}\cdot\hat{n} = \pm P_0$
Therefore the only $\vec{E}$ field is due to the two discs of radius $\vec{R}$, parallel to xy plane and centered at $(0, 0, \pm\frac{L}{2})$, and having uniform surface charge density P_0 and $-P_0$

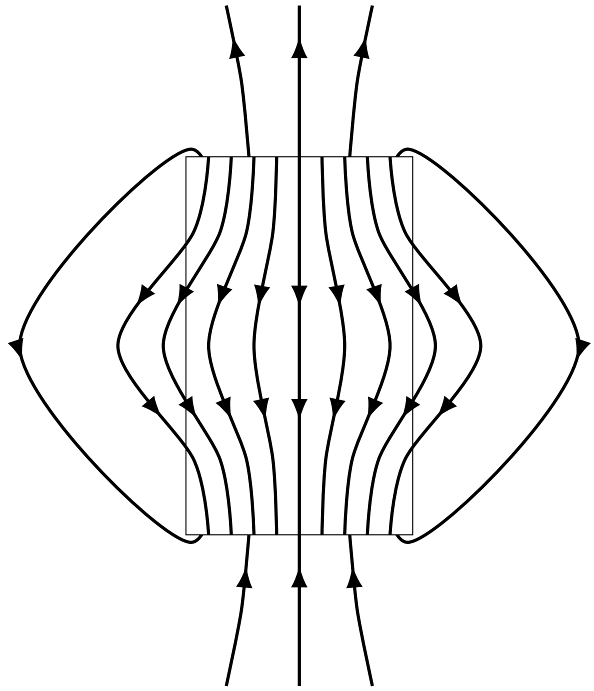


Figure 2: $\vec{E}$ field for the bar electret

This $\vec{E}$ field is obviously curl-less and suffers from discontinuities at the electret ends owing to the surface (bound) charges

We can find the $\vec{E}$ field on the z axis quantitatively since we know the field on the axis of a disc

$$\vec{E} = \begin{cases} -\frac{P_0}{2\epsilon_0}\left(2 - \frac{\frac{L}{2}-z}{\sqrt{(\frac{L}{2}-z)^2+R^2}} - \frac{\frac{L}{2}+z}{\sqrt{(\frac{L}{2}+z)^2+R^2}}\right)\hat{k} & -\frac{L}{2} < z < \frac{L}{2} \\ \frac{P_0}{2\epsilon_0}\left(\frac{\frac{L}{2}+z}{\sqrt{(\frac{L}{2}+z)^2+R^2}} - \frac{z-\frac{L}{2}}{\sqrt{(z-\frac{L}{2})^2+R^2}}\right)\hat{k} & z > \frac{L}{2} \\ \frac{P_0}{2\epsilon_0}\left(\frac{\frac{L}{2}-z}{\sqrt{(\frac{L}{2}-z)^2+R^2}} - \frac{-z-\frac{L}{2}}{\sqrt{(\frac{L}{2}+z)^2+R^2}}\right)\hat{k} & z < -\frac{L}{2} \end{cases} \tag{9.7.1}$$

We can thus find $\vec{D}$

$$\vec{D} = \epsilon_0\vec{E} + \vec{P} = \begin{cases} \frac{P_0}{2}\left(\frac{\frac{L}{2}-z}{\sqrt{(\frac{L}{2}-z)^2+R^2}} + \frac{\frac{L}{2}+z}{\sqrt{(\frac{L}{2}+z)^2+R^2}}\right)\hat{k} & -\frac{L}{2} < z < \frac{L}{2} \\ \frac{P_0}{2}\left(\frac{\frac{L}{2}+z}{\sqrt{(\frac{L}{2}+z)^2+R^2}} - \frac{z-\frac{L}{2}}{\sqrt{(z-\frac{L}{2})^2+R^2}}\right)\hat{k} & z > \frac{L}{2} \\ \frac{P_0}{2}\left(\frac{\frac{L}{2}-z}{\sqrt{(\frac{L}{2}-z)^2+R^2}} - \frac{-z-\frac{L}{2}}{\sqrt{(\frac{L}{2}+z)^2+R^2}}\right)\hat{k} & z < -\frac{L}{2} \end{cases} \tag{9.7.2}$$

$$= \frac{P_0}{2}\left(\frac{\frac{L}{2}-z}{\sqrt{(\frac{L}{2}-z)^2+R^2}} + \frac{\frac{L}{2}+z}{\sqrt{(\frac{L}{2}+z)^2+R^2}}\right)\hat{k} \tag{9.7.3}$$

$\vec{D}$ is of course only $\propto \vec{E}$ outside the electret

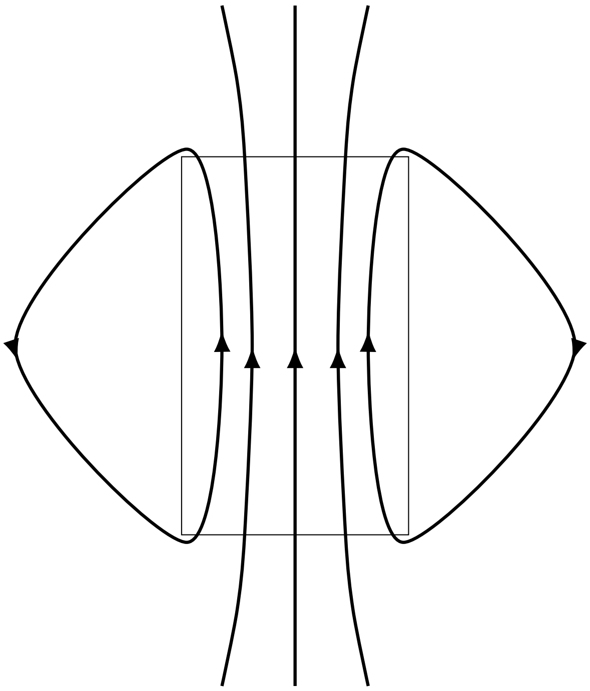


Figure 3: $\vec{D}$ field for the bar electret

This $\vec{D}$ field has no discontinuities, but is not curl-less (it's curl is non-zero at the curved surface of electret and is equal to the curl of $\vec{P}$)

This is a clear demonstration that there is no analog of Coulomb's law for $\vec{D}$ and ρ_f, as $\rho_f = 0$ everywhere, yet $\vec{D} \neq 0$

9.8. A conducting sphere of radius R is half submerged in a linear, homogeneous, semi-infinite liquid dielectric medium of dielectric constant κ. The sphere is at a potential V_0. Assuming there is no bound charge at the liquid-air interface, calculate

(a) the potential at a point outside the sphere

Solution: We begin by noticing that the problem has azimuthal symmetry. We also know that the potential must be continous across the surface of the dielectric medium. We can intuitively begin by trying a radially symmetric solution. If our potential has no dependance on θ, then it's dependence on r must be $\propto \frac{1}{r}$ by separation of variables technique
$\therefore V(\vec{r}) = \frac{V_0 R}{r}$
We can immediately see that this satisfies our boundary conditions, and also Laplace's equations in the regions of interest, therefore by appealing to Uniqueness theorem, this must be the solution
Alternate Solution:
We can try a more rigorous approach to this situation, by dividing up space into 2 regions, $z \geq 0$ and $z \leq 0$ (both cases $r \geq R$), and finding solutions of Laplace's

equation in each region. Thus,

$$V(r,\theta) = \begin{cases} V_{z^+} & z \geq 0 \\ V_{z^-} & z \leq 0 \end{cases} \tag{9.8.1}$$

$$V_{z^+}(r,\theta) = \sum_{l=0}^{\infty} \left(A_l^+ r^l + \frac{B_l^+}{r^{l+1}} \right) P_l(\cos\theta) \tag{9.8.2}$$

$$V_{z^-}(r,\theta) = \sum_{l=0}^{\infty} \left(A_l^- r^l + \frac{B_l^-}{r^{l+1}} \right) P_l(\cos\theta) \tag{9.8.3}$$

$$\lim_{r\to\infty} V_z^+(r,\theta) = \lim_{r\to\infty} V_z^-(r,\theta) = 0 \tag{9.8.4}$$

$$V_{z^-}(r,\frac{\pi}{2}) = V_{z^+}(r,\frac{\pi}{2}) \tag{9.8.5}$$

$$\kappa \left.\frac{\partial V_{z^-}(r,\theta)}{\partial\theta}\right|_{\theta=\frac{\pi}{2}} = \left.\frac{\partial V_{z^+}(r,\theta)}{\partial\theta}\right|_{\theta=\frac{\pi}{2}} \tag{9.8.6}$$

$$V_{z^+}(R,\theta) = V_{z^-}(R,\theta) \tag{9.8.7}$$

Applying (9.8.4), we can see that $A_l^+ = A_l^- = 0 \; \forall \; l$
Applying (9.8.5), we can see that $B_l^+ = B_l^- \; \forall$ even l, and from (9.8.6), we get that $B_l^+ = \kappa B_l^- \; \forall$ odd l
We can also enforce from (9.8.7) the fact that potential at R is a constant to get $B^l + = B^l - = 0 \; \forall \; l \neq 0$
Finally from (9.8.7), $V(r,\theta) = \frac{V_0 R}{r}$

(b) the electric field, the electric displacement, and the surface and volume bound charge density in the dielectric

Solution: Let $z < 0$ be the region of dielectric, with the sphere centered at origin.

$$\vec{E} = -\nabla V = \boxed{V_0 \frac{R}{r^2}\,\hat{r}} \tag{9.8.8}$$

$$\implies \vec{D} = \epsilon\vec{E} = \boxed{\begin{cases} \epsilon_0 V_0 \frac{R}{r^2}\,\hat{r} & z \geq 0 \\ \epsilon_0 \kappa V_0 \frac{R}{r^2}\,\hat{r} & z \leq 0 \end{cases}} \tag{9.8.9}$$

$$\implies \vec{P} = \chi_e \epsilon_0 \vec{E} = \begin{cases} 0 & z \geq 0 \\ (\kappa - 1)\epsilon_0 V_0 \frac{R}{r^2}\,\hat{r} & z \leq 0 \end{cases} \tag{9.8.10}$$

$$\sigma_b(r = R) = -\vec{P}\cdot\hat{r} = \boxed{\begin{cases} 0 & z \geq 0 \\ (1-\kappa)\epsilon_0 V_0 \frac{1}{R} & z \leq 0 \end{cases}} \tag{9.8.11}$$

$$\sigma_b(z = 0) = \vec{P}\cdot\hat{z} = \boxed{0} \tag{9.8.12}$$

$$\rho_b = -\frac{\chi_e}{1+\chi_e}\rho_f = \boxed{0} \tag{9.8.13}$$

(c) the total free charge on the conductor.

Solution: We can use Gauss law on a spherical Gaussian surface with $r > R$

$$\int_{\mathcal{S}} \vec{D} \cdot \hat{r} = Q_f \tag{9.8.14}$$

$$\implies Q_f = 2\pi\epsilon_0 V_0 (1+\kappa) R \tag{9.8.15}$$

10 Magnetodynamics

10.1. A cylinder of radius a and length $2L$ is placed with its axis along $\hat{z}$ and its center at the origin. It has an uniform frozen electric polarisation $\vec{P} = P_0\hat{z}$. It is set rotating with angular velocity ω about the direction of polarisation. Calculate the magnetic field $\vec{B}$ at a point on the z axis.

Solution: We know that the only charge present is the two discs of radius a with surface (bound) charge density $\sigma = \pm P_0$ at $z = \pm L$

We need field at $\vec{r} = z\,\hat{k}$.

The surface current at cylindrical coordinates $\vec{r'} = (r', \phi, \pm L)$ is $\vec{K} = \sigma\vec{v} = \pm P_0\omega r'\hat{\phi} = \pm P_0\omega r'(-\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath})$

Writing $\vec{r'}$ in cartesian basis, $\vec{r'} = r'\cos\phi\,\hat{\imath} + r'\sin\phi\,\hat{\jmath} \pm L\,\hat{k}$

Putting it all together,

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\iint \frac{\vec{K}\times\vec{\mathscr{r}}}{|\vec{\mathscr{r}}\,|^3} dS \tag{10.1.1}$$

$$= \frac{\mu_0 P_0\omega}{4\pi}\int_{r'=0}^{a} dr' \left(\cancelto{0}{\int_{\phi=0}^{2\pi} \frac{r'^2(z-L)\cos\phi\,\hat{\imath}}{((z-L)^2+r'^2)^{\frac{3}{2}}}\,d\phi} + \cancelto{0}{\int_{\phi=0}^{2\pi} \frac{r'^2(z-L)\sin\phi\,\hat{\jmath}}{((z-L)^2+r'^2)^{\frac{3}{2}}}\,d\phi} \right. \tag{10.1.2}$$

$$\left. + \int_{\phi=0}^{2\pi} \frac{r'^3\hat{k}}{((z-L)^2+r'^2)^{\frac{3}{2}}}\,d\phi \right) \tag{10.1.3}$$

$$- \frac{\mu_0 P_0\omega}{4\pi}\int_{r'=0}^{a} dr' \left(\cancelto{0}{\int_{\phi=0}^{2\pi} \frac{r'^2(z+L)\cos\phi\,\hat{\imath}}{((z+L)^2+r'^2)^{\frac{3}{2}}}\,d\phi} + \cancelto{0}{\int_{\phi=0}^{2\pi} \frac{r'^2(z+L)\sin\phi\,\hat{\jmath}}{((z+L)^2+r'^2)^{\frac{3}{2}}}\,d\phi} \right. \tag{10.1.4}$$

$$\left. + \int_{\phi=0}^{2\pi} \frac{r'^3\hat{k}}{((z+L)^2+r'^2)^{\frac{3}{2}}}\,d\phi \right) \tag{10.1.5}$$

$$= \frac{\mu_0 P_0\omega}{2}\int_{r'=0}^{a} \frac{r'^3}{((z-L)^2+r'^2)^{\frac{3}{2}}} + \frac{r'^3}{((z+L)^2+r'^2)^{\frac{3}{2}}}\,dr'\,\hat{k} \tag{10.1.6}$$

$$= \frac{\mu_0 P_0\omega}{2}\left(\frac{2(z-L)^2+a^2}{\sqrt{(z-L)^2+a^2}} - 2\sqrt{(z-L)^2} \right. \tag{10.1.7}$$

$$\left. + \frac{2(z+L)^2+a^2}{\sqrt{(z+L)^2+a^2}} - 2\sqrt{(z+L)^2} \right)\hat{k} \tag{10.1.8}$$

10.2. A sphere of linear dielectric material has embedded in it a uniform free charge density ρ. Find the potential at the center of the sphere (relative to infinity), if its radius is R and the dielectric constant is ϵ_r.

Solution: For $r < R$

$$\oint_S d\vec{s} \cdot \vec{D} = Q_{f_{enc}} \quad \Rightarrow \vec{D} = \frac{1}{3}\rho\vec{r} \quad \Rightarrow \vec{E} = \frac{\rho\vec{r}}{3\epsilon_0\epsilon_r}$$

For $r > R$

$$\oint_S d\vec{s} \cdot \vec{D} = Q_{f_{enc}} \quad \Rightarrow \vec{D} = \frac{\rho R^3}{3r^2}\hat{r} \quad \Rightarrow \vec{E} = \frac{\rho R^3}{3\epsilon_0 r^2}\hat{r}$$

Then the potential is

$$V(0) = -\int_\infty^0 d\vec{l} \cdot \vec{E} = \frac{\rho R^2}{3\epsilon_0}\left(1 + \frac{1}{2\epsilon_r}\right)$$

10.3. An uncharged conducting sphere of radius a is coated with a thick insulating shell (dielectric constant ϵ_r) out to radius b . This object is now placed in an otherwise uniform electric field E_0. Find the electric field in the insulator.

Solution: Let us assume the external field is along z, thus $\vec{E} = E_0\,\hat{k}$

We again have the situation where we cannot use infinity as our potential reference. So we set the conductor surface as $V = 0$, and keep in mind that every potential we specify is with respect to the conductor surface.

With that, we start out with the general solution to Laplace's equation in spherical coordinates, and apply boundary conditions

$$V_{\text{out}}(r,\theta) = \left(A_l^{\text{out}} r^l + \tfrac{B_l^{\text{out}}}{r^{l+1}}\right) P_l(\cos\theta) \quad r \geq b \tag{10.3.1}$$

$$V_{\text{in}}(r,\theta) = \left(A_l^{\text{in}} r^l + \tfrac{B_l^{\text{in}}}{r^{l+1}}\right) P_l(\cos\theta) \quad a \leq r \leq b \tag{10.3.2}$$

$$V_{\text{in}}(b,\theta) = V_{\text{out}}(b,\theta) \quad \text{Continuity of V} \tag{10.3.3}$$

$$V_{\text{in}}(a,\theta) = 0 \quad \text{Conductor} \tag{10.3.4}$$

$$\lim_{r\to\infty} V_{\text{out}}(r,\theta) = -E_0 r\cos\theta \quad \text{External field} \tag{10.3.5}$$

$$\left.\frac{\partial V_{\text{out}}(r,\theta)}{\partial r}\right|_{r=b} - \epsilon_r \left.\frac{\partial V_{\text{in}}(r,\theta)}{\partial r}\right|_{r=b} = 0 \quad \text{Continuity of } \vec{D} \tag{10.3.6}$$

$$\oiint \frac{\partial V_{\text{in}}(r,\theta)}{\partial r} dS = 0 \quad \text{Uncharged conductor} \tag{10.3.7}$$

Applying (10.3.4), (10.3.5), (10.3.6) we get

$$A_l^{\text{out}} = 0 \;\forall\; l \;\neq 1 \tag{10.3.8}$$

$$A_1^{\text{out}} = -E_0 \tag{10.3.9}$$

$$B_l^{\text{in}} = -A_l^{\text{in}} a^{2l+1} \;\forall\; l \tag{10.3.10}$$

Thus we write the potentials as

$$V_{\text{out}}(r,\theta) = -E_0 r\cos\theta + \frac{B_l^{\text{out}}}{r^{l+1}} P_l(\cos\theta) \tag{10.3.11}$$

$$V_{\text{in}}(r,\theta) = A_l^{\text{in}}\left(r^l - \frac{a^{2l+1}}{r^{l+1}}\right) P_l(\cos\theta) \tag{10.3.12}$$

Applying (10.3.3) and (10.3.6)

$$A_1^{\text{in}} = \frac{-3E_0}{2\left(1-\frac{a^3}{b^3}\right) + \epsilon_r\left(1+\frac{2a^3}{b^3}\right)} \tag{10.3.13}$$

$$B_1^{\text{out}} = \frac{E_0(b^3+2a^3)(\epsilon_r - 1)}{2\left(1-\frac{a^3}{b^3}\right) + \epsilon_r\left(1+\frac{2a^3}{b^3}\right)} \tag{10.3.14}$$

$$A_l^{\text{in}} = 0 \;\forall\; l \neq 1 \tag{10.3.15}$$

$$B_l^{\text{out}} = 0 \;\forall\; l \neq 1 \tag{10.3.16}$$

Thus we can find potential in dielectric

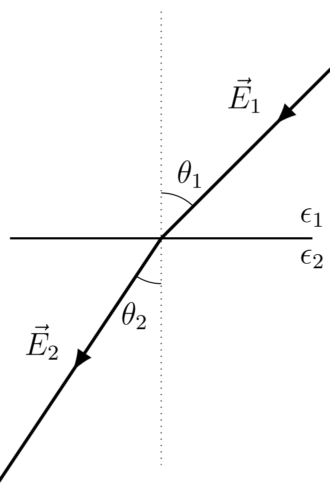
$$V_{\text{in}}(r,\theta) = \frac{-3E_0}{2\left(1-\frac{a^3}{b^3}\right) + \epsilon_r\left(1+\frac{2a^3}{b^3}\right)}\left(r - \frac{a^3}{r^2}\right)\cos\theta \tag{10.3.17}$$

And thus the electric field

$$\vec{E}_{\text{in}} = -\nabla V_{\text{in}} = \frac{3}{2\left(1-\frac{a^3}{b^3}\right) + \epsilon_r\left(1+\frac{2a^3}{b^3}\right)}\left(E_0\,\hat{k} + \frac{2a^3\cos\theta}{r^3}\,\hat{r} + \frac{a^3\sin\theta}{r^3}\,\hat{\theta}\right) \tag{10.3.18}$$

Thus we have the original external field (albeit scaled), along with a dipole field

10.4. Two dielectrics having permittivity ϵ_1 and ϵ_2 have an interface which has no free charges. The electric field in medium 1 makes an angle θ_1 with perpendicular of interface while, the field in medium 2 makes an angle θ_2. Find the Relationship between two angles.



Solution: We know that $\vec{E}$ is curl-less at the boundary, so component along the boundary is going to be continuous

$$E_1 \sin\theta_1 = E_2 \sin\theta_2 \tag{10.4.1}$$

We also know that the component of $\vec{D}$ along the normal is going to be continuous, since no free charges exist at the interface

$$\epsilon_1 E_1 \cos\theta_1 = \epsilon_2 E_2 \cos\theta_2 \tag{10.4.2}$$

Combining the two,

$$\tan\theta_1 = \frac{\epsilon_1}{\epsilon_2} \tan\theta_2 \tag{10.4.3}$$

10.5. Consider an infinite cylindrical region with its axis along $\hat{z}$ and radius R. Consider another parallel cylinder running along $\hat{z}$ of radius a with axis at a distance b from the larger cylinder. Assume that b is small enough that the smaller cylinder is a 'cavity' in the larger. Suppose that a magnetic field $\vec{B}(t) = B_0 t\,\hat{z}$ exists everywhere inside the larger cylinder except for the cavity. Find the Electric field induced inside the cavity.

Solution: Consider the scenario as a superposition of a cylindrical flux through the larger cylinder and a negative flux through the smaller one. On the edge of a circle with uniform changing magnetic field inside, symmetry argument along with the integral form of Ampere's law produces the vector form of electric field as

$$\mathbf{E} = \frac{B_0}{2}\hat{z} \times \overline{r}$$

when origin is the centre of the circle. A straightforward vector addition for our scenario yields

$$\mathbf{E}_{cav} = \frac{B_0}{2}\hat{z} \times \overline{b}$$

10.6. A charge Q is distributed uniformly on a non-conducting ring of radius R and mass M. The ring is dropped from rest from a height h and falls to the ground through a non-uniform magnetic field $\vec{B}(r)$. The plane of the ring remains horizontal during its fall.

(a) Explain qualitatively why the ring rotates as it falls.

Solution: As the ring falls, since B_z is varying, the flux through the ring is changing. Hence a motional emf arises which makes the ring rotate in response (to produce an appropriate current)

We might be tempted to think that there is an induced electric field which produces this torque on the ring. This is true in the frame of the ring itself, but from the ground frame there is **no** electric field, induced or otherwise. Why? Remember that though the flux through the ring is changing, this is because the area itself is **moving**. But at no fixed point is the $\vec{B}$ field time varying. So there is no $\vec{E}$ field, and it is purely the $\vec{B}$ field exerting all forces.

(*Technically* the rotating ring also produces time varying fields of its own, and hence will face some extra forces trying to impede it's rotation (analogous to self inductance). We will ignore that effect for this part, and assume that the time varying fields are negligible)

(b) Use Faraday's flux rule to show that the velocity of the center of mass of the ring when it hits the ground is

$$v_{\text{CM}} = \sqrt{2gh - \frac{Q^2R^2}{4M}(B_z(0) - B_z(h))^2}$$

Solution:

Applying Faraday's law, we know that the (motional) emf (clockwise from top) on the ring can be given by

$$\mathcal{E}(z) = -\frac{d\phi}{dt} = -\pi R^2 \frac{dB_z(z(t))}{dt} = -\pi R^2 \frac{dB_z(z)}{dz} v(z) \tag{10.6.1}$$

Where $B_z(z)$ is the z component of magnetic field averaged over ring area when the ring has fallen to height z

But,

$$\mathcal{E} = \oint \vec{f} \cdot d\vec{l} = \oint f_\phi dl = \frac{1}{\lambda R} \oint \tau_z \, dl \tag{10.6.2}$$

Where $\vec{f}$ is the force per unit charge and $\vec{\tau}$ is the torque per unit length

$$\implies MR^2 \frac{d\omega_z(z(t))}{dt} = MR^2 v(z) \frac{d\omega_z(z)}{dz} = \text{Torque}_z(z) = \lambda R \mathcal{E}(z) = -\frac{QR^2 v(z)}{2} \frac{dB_z(z)}{dz} \tag{10.6.3}$$

$$\implies \int_{z=h}^{0} d\omega_z(z) = -\frac{Q}{2M} \int_{z=h}^{0} dB_z(z) \tag{10.6.4}$$

$$\implies \omega_z(0) = \frac{Q}{2M}(B_z(h) - B_z(0)) \tag{10.6.5}$$

We also know that all the forces are due to $\vec{B}$ fields which do no work, so the energy must be conserved.

We know that the only dynamic fields are generated by the ring itself, and those dynamic fields are going to have azimuthal symmetry, as the ring rotation itself has azimuthal symmetry. Hence the fields themselves carry no momentum in the xy plane. Thus the ring itself is not imparted any momentum in the xy plane

We also know that the ring plane stays horizontal. Thus the ring is not imparted any angular momentum in the xy plane.

Thus the energy from rotation must be drawn from the energy of translation in z direction

Thus applying conservation of energy

$$Mgh = \frac{1}{2}Mv(0)^2 + \frac{1}{2}MR^2\omega_z(0)^2 \tag{10.6.6}$$

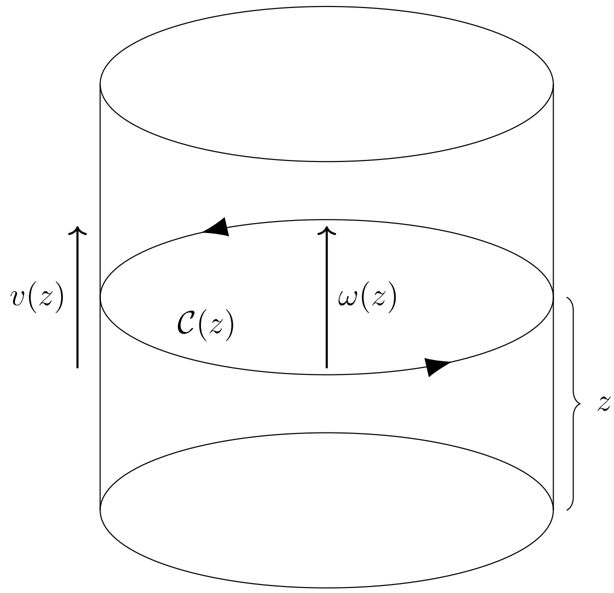
$$\implies v(0) = \boxed{-\sqrt{2gh - \frac{Q^2R^2}{4M^2}(B_z(h) - B_z(0))^2}} \tag{10.6.7}$$

Alternate Solution:

Wait a minute! We have been saying that $\vec{B}$ exerts all forces, but how can B_z apply any force in the vertical direction to slow the ring down? A closer inspection will reveal that it is B_r that is capable of exerting force in both tangential (virtue of ring's downward motion) and upward (virtue of ring's tangential motion) directions.

But wait. We have no information of B_r, only about B_z. Ah! Here we need to recall the fact that $\nabla \cdot B = 0$, so the inhomogenous nature of B_z automatically fixes B_r to preserve $\nabla \cdot B = 0$

Can we get the same result with this analysis?



We know that the velocity of each charge element is $\vec{v} = R\omega(z)\,\hat{\phi} + v(z)\,\hat{k}$

Thus the magnetic force on each charge element is

$$d\vec{F} = dq\,\vec{v} \times \vec{B} \tag{10.6.8}$$

$$= \frac{Q}{2\pi R}(B_r\,\hat{r} + B_\phi\,\hat{\phi} + B_z\,\hat{k}) \times (R\omega(z)\,\hat{\phi} + v(z)\,\hat{k})\,dl \tag{10.6.9}$$

$$= \frac{Q}{2\pi R}\Big((R\omega(z)B_z - v(z)B_\phi)\,\hat{r} + v(z)B_r\,\hat{\phi} - R\omega(z)B_r\,\hat{k}\Big)\,dl \tag{10.6.10}$$

$$MR^2\frac{d\omega(z(t))}{dt} = \tau_z = R\oint_{\mathcal{C}(z)} d\vec{F}\cdot\hat{\phi} \tag{10.6.11}$$

$$\implies MR\cancel{v(z)}\frac{d\omega(z)}{dz} = \frac{Q}{2\pi R}\cancel{v(z)}\oint_{\mathcal{C}(z)} B_r\,dl \tag{10.6.12}$$

$$\implies \frac{2M\pi R^2}{Q}\int_{z'=h}^{z} d\omega(z') = \int_{z'=h}^{z}\oint_{\mathcal{C}(z')} B_r(z')\,dl\,dz' \tag{10.6.13}$$

We can simplify the RHS. Let us imagine a cylinder with top surface $\mathcal{C}(h)$, and bottom surface $\mathcal{C}(z)$. We can calculate $\nabla\cdot\vec{B}$ over this volume

$$\iiint \nabla\cdot\vec{B} = 0 \tag{10.6.14}$$

$$\implies \oiint \vec{B}\cdot d\vec{S} = 0 \tag{10.6.15}$$

$$\implies \iint_{\mathcal{C}(h)} B_z(h)\,dS - \iint_{\mathcal{C}(z)} B_z(z)\,dS + \int_{z'=z}^{h} dz'\oint_{\mathcal{C}(z')} B_r(z')\,dl = 0 \tag{10.6.16}$$

$$\implies \int_{z'=h}^{z}\oint_{\mathcal{C}(z')} B_r(z')\,dl\,dz' = \pi R^2\left(\overline{B_z(h)} - \overline{B_z(z)}\right) \tag{10.6.17}$$

Where $\overline{B_z(h)}$ and $\overline{B_z(z)}$ are the averaged versions of $B_z(h)$ and $B_z(z)$ over the corresponding ring areas
Putting this back into (10.6.13)

$$\frac{2M\pi R^2}{Q}\omega(z) = \pi R^2\left(\overline{B_z(h)} - \overline{B_z(z)}\right) \tag{10.6.18}$$

$$\implies \omega(z) = \frac{Q}{2M}\left(\overline{B_z(h)} - \overline{B_z(z)}\right) \tag{10.6.19}$$

Now to find $v(z)$, we apply Newton's Second Law in the z direction

$$Ma = Mv(z)\frac{dv(z)}{dz} = -Mg + \oint_{\mathcal{C}(z)} d\vec{F}\cdot\hat{k} \tag{10.6.20}$$

$$\implies M\int_{z'=h}^{z} v(z')\,dv(z) = Mg\int_{z'=z}^{h} dz' - \frac{Q}{2\pi}\int_{z'=h}^{z}\omega(z')\oint_{\mathcal{C}(z')} B_r(z')\,dl\,dz' \tag{10.6.21}$$

We can subsitute $\oint_{\mathcal{C}(z')} B_r(z')\,dl$ from (10.6.12)

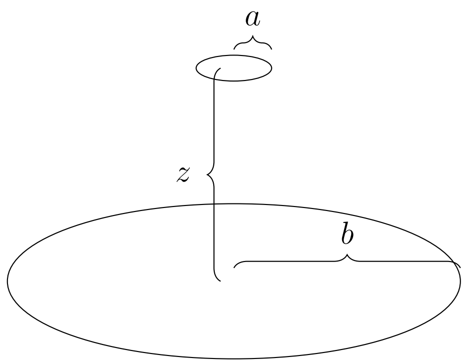
$$\implies M\int_{z'=h}^{z} v(z')\,dv(z) = Mg\int_{z'=z}^{h} dz' - MR^2\int_{z'=h}^{z}\omega(z')\frac{d\omega(z')}{dz'}\,dz' \tag{10.6.22}$$

$$\implies \frac{v(z)^2}{2} = g(h-z) - R^2\frac{\omega(z)^2}{2} \tag{10.6.23}$$

$$\implies v(z) = \sqrt{2g(h-z) - \frac{Q^2R^2}{4M^2}\left(\overline{B_z(h)} - \overline{B_z(z)}\right)^2} \tag{10.6.24}$$

This gives us right what we had before! Thus we can conclusively say that it is only B_r that is exerting all forces, and it takes translational energy from the ring and pumps it in the form of rotational energy, doing no net work in the process

10.7. Two circular loops of wire share the same axis but are displaced vertically by a distance z. The wire of radius a is considerably smaller than the wire of radius b.



(a) The larger loop (of radius b) carries a current I. What is the magnetic flux through the smaller loop due to the larger? (Hint: The field of the large loop may be considered constant in the region of the smaller loop.)

Solution: The field on the axis of the larger loop is given by $\mu_0 I \frac{b^2}{2(z^2+b^2)^{\frac{3}{2}}}$

Denoting the smaller loop as 1 and larger loop as 2, $\phi_{12} = \frac{\mu_0 I \pi a^2 b^2}{2(z^2+b^2)^{\frac{3}{2}}}$

(b) If the same current I now flows in the smaller loop, then what is the magnetic flux through the larger loop? (Hint: The field of the smaller loop may be treated as a dipole.)

Solution: Taking the center of 1 as origin, and z axis from 1 to 2

$$\vec{B}_1(r,\theta) = \frac{\mu_0 I \pi a^2}{4\pi}\left(\frac{2\cos\theta}{r^3}\hat{r} + \frac{\sin\theta}{r^3}\hat{\theta}\right) \tag{10.7.1}$$

$$\implies \vec{B}_1(r,\theta)\cdot\hat{k} = \frac{\mu_0 I a^2}{4}\left(\frac{2\cos^2\theta}{r^3} - \frac{\sin^2\theta}{r^3}\right) \tag{10.7.2}$$

$$\implies \phi_{21} = 2\pi\int_{\rho=0}^{b}\vec{B}_1\left(\sqrt{\rho^2+z^2}, \tan^{-1}\left(\frac{\rho}{z}\right)\right)\cdot\hat{k}\,\rho\,d\rho \tag{10.7.3}$$

$$= \frac{\mu_0 I \pi a^2}{2}\int_{\rho=0}^{b}\left(\frac{2\frac{z^2}{(\rho^2+z^2)} - \frac{\rho^2}{\rho^2+z^2}}{(\rho^2+z^2)^{\frac{3}{2}}}\rho\,d\rho\right) \tag{10.7.4}$$

$$= \frac{\mu_0 I \pi a^2}{2}\int_{u=z^2}^{z^2+b^2}\frac{3z^2-u}{u^{\frac{5}{2}}}\,du \tag{10.7.5}$$

$$= \frac{\mu_0 I \pi a^2 b^2}{2(z^2+b^2)^{\frac{3}{2}}} \tag{10.7.6}$$

(c) What is the mutual inductance of this system? Show that $M_{12} = M_{21}$

Solution: We can clearly see that $M_{12} = M_{21} = \frac{\phi_{12}}{I} = \frac{\phi_{21}}{I} = \frac{\mu_0 \pi a^2 b^2}{2(z^2+b^2)^{\frac{3}{2}}}$

10.8. A rectangular capacitor with side lengths a and b has separation s, with s much smaller than a and b. It is partially filled with a dielectric with dielectric constant κ. The overlap distance is x. The capacitor is isolated and has constant charge Q

(a) What is the energy stored in the system? (Treat the capacitor like two capacitors in parallel)

Solution: We know that the dielectric will raise the capacitance by κ. Hence we have 2 capacitors in parallel, with capacitance $\frac{\epsilon_0 a(b-x)}{s}$ and $\frac{\kappa\epsilon_0 ax}{s}$
$\implies C = \frac{\epsilon_0 a}{s}\left(b + (\kappa - 1)x\right)$
$\implies U = \frac{Q^2}{2C} = \frac{Q^2 s}{2\epsilon_0 a(b+(\kappa-1)x)}$

(b) What is the force on the dielectric? Does this force pull the dielectric into the capacitor or push it out?

Solution: $F = -\frac{dU}{dx} = \frac{Q^2 s(\kappa-1)}{2\epsilon_0 a(b+(\kappa-1)x)^2}$
This force is in direction of increasing x. That means the dielectric is pulled inwards

11 EM Waves

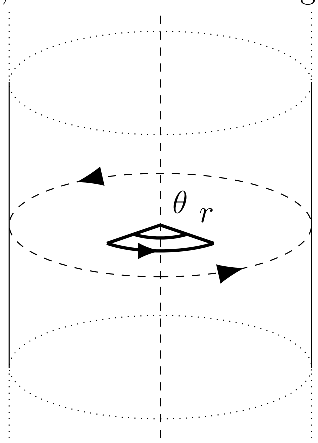
11.1. A very long solenoid of n turns per unit length carries a current which increases uniformly with time, $i = Kt$.

(a) Calculate the electric field and magnetic field inside the solenoid at time t (neglect retardation).

Solution:
Assume that the solenoid axis is along $\hat{k}$
We already know that magnetic field inside solenoid due to solenoidal current is $\vec{B} = \mu_0 ni\,\hat{k} = \mu_0 nKt\,\hat{k}$

$$\implies \vec{B}(\vec{r}) = \begin{cases} \mu_0 nKt\,\hat{k} & r < R \\ 0 & r > R \end{cases} \tag{11.1.1}$$

To calculate the magnetic field, consider the following amperian loop



We know that $\vec{E}$ field will have azimuthal as well as z translational symmetry
This rules out any z component of $\vec{E}$ field. We can also rule out the radial component, since a non zero (and azimuthally symmetric) radial component will have non zero divergence which cannot be because $\rho = 0$ everywhere
Thus only an azimuthally symmetric azimuthal component remains. Applying Ampere's law on the aforementioned loop

$$\oint \vec{E} \cdot dl = -\frac{d\phi}{dt} = -\mu_0 nK \frac{r^2\theta}{2} \tag{11.1.2}$$

$$\implies \vec{E} = -\frac{\mu_0 nKr}{2}\hat{\theta} \qquad r < R \tag{11.1.3}$$

(b) Consider a cylinder of length l and radius equal to that of the solenoid, and coaxial with the solenoid. Find the rate at which energy flows into the volume enclosed by this cylinder and show that it is equal to $\frac{d(\frac{1}{2}lLi^2)}{dt}$, where L is the self-inductance per unit length of the solenoid.

Solution: Let us find the rate of inflow of energy (mechanical and electromagnetic) at the surface of the cylinder. This will be given by the influx of Poynting vector just *inside* the solenoid surface (just outside the surface $\vec{B}$ and thus the Poynting vector are both zero)

$$\frac{dU_{\mathcal{V}}}{dt} = -\oint_{\mathcal{S}} \vec{S} \cdot d\vec{S} \tag{11.1.4}$$

$$= -\frac{R}{\mu_0}\int_{z=0}^{l}\int_{\theta=0}^{2\pi} \vec{E}(R) \times \vec{B}(R) \cdot d\theta\, dz \tag{11.1.5}$$

$$= \mu_0 n^2 K^2 \pi R^2 lt \tag{11.1.6}$$

As we have said before, the Poynting vector just outside the surface is 0. So this points to the fact that this energy is flowing from what is **at** the surface, into the volume. But what *is* at the surface? The current!
What happens is that as the current increases, due to self inductance it faces some resistance to the increase of current. Thus extra energy has to be pumped in (by the battery or whatever) to keep the current increasing at the stated rate.
This energy flows from the current into the fields (specifically, the magnetic field, as the electric field and thus its energy is static)
Therefore this is what the magnetic energy $\frac{1}{2}lLi^2$ is. Let us calculate that

$$L = \mu_0 n^2 \pi R^2 \tag{11.1.7}$$

$$\implies \frac{1}{2}lLi^2 = \frac{\mu_0 n^2 \pi R^2 K^2 t^2 l}{2} \tag{11.1.8}$$

$$\implies \frac{d(\frac{1}{2}lLi^2)}{dt} = \mu_0 n^2 K^2 \pi R^2 lt = \frac{dU_{\mathcal{V}}}{dt} \tag{11.1.9}$$

11.2. It has been proposed that a spacecraft may be propelled by harnessing the pressure of sunlight. Assume that a spacecraft is sufficiently away from earth and is under the sun's gravitational

eld alone. A very large and fully reflecting sail is oriented at right angles to the sun's rays and attached to the craft. How large must the sail be so that the craft can start sailing away from the sun? The sun radiates 10^{26} W and has a mass of 10^{30} kg, the total mass of the spacecraft and sail is 1500 kg.

Solution: Let sail area be A, and it be a distance of r from the sun

Thus the power hitting the sail is $P_{\text{tot}}\frac{A}{4\pi r^2}$

Thus the force on the sail is $\frac{2P_{\text{tot}}A}{4\pi r^2 c}$

Thus must be equal to the gravitational force

$$\frac{P_{\text{tot}}A}{2\pi \cancel{r^2} c} = \frac{GMm}{\cancel{r^2}} \tag{11.2.1}$$

$$\implies A = \frac{2\pi c GMm}{P_{\text{tot}}} \approx 1.9 \times 10^6 \text{ m}^2 \tag{11.2.2}$$

11.3. An infinite wire carries a current up the rotational symmetry axis of a toroidal solenoid with N tightly wound turns and a circular cross section. The inner radius of the toroid is a and the outer radius is b. Find the mutual inductance M between the wire and the solenoid

Solution: Let us calculate flux of field due to straight wire through a single loop, and multiply that by N to get the total flux through the toroidal solenoid

Let the radius of revolution be $R_1 = \frac{b+a}{2}$ and the radius of single loop be $R_2 = \frac{b-a}{2}$

$$\phi = N \int_{\text{loop}} \frac{\mu_0 I}{2\pi r}\, dA \tag{11.3.1}$$

$$= \frac{\mu_0 N I}{2\pi} \int_{x=a}^{b} \int_{y=-\sqrt{R_2^2-(x-R_1)^2}}^{\sqrt{R_2^2-(x-R_1)^2}} \frac{1}{x}\, dx\, dy \tag{11.3.2}$$

$$= \frac{\mu_0 N I}{\pi} \int_{x=a}^{b} \frac{\sqrt{R_2^2-(x-R_1)^2}}{x}\, dx \tag{11.3.3}$$

$$= \frac{\mu_0 N I}{\pi} \int_{x=a}^{b} \frac{\sqrt{(b-x)(x-a)}}{x}\, dx \tag{11.3.4}$$

$$= \frac{\mu_0 N I(\sqrt{b}-\sqrt{a})^2}{2} \tag{11.3.5}$$

$$\implies M = \frac{\mu_0 N(\sqrt{b}-\sqrt{a})^2}{2} \tag{11.3.6}$$

11.4. A rectangular coil has length $2L$ and width $2w$. The coil is in the xz plane, centred at the origin and rotates about the z axis with uniform angular velocity ω. A uniform magnetic field of B_0 is applied in the y direction. Determine the emf induced in the coil using motional EMF. Then check your answer using rate of change of flux in a coil

Solution: When the coil is at an angle of ϕ with the xz plane, the velocity of the vertical parts of the coil can be written as $\vec{v} = \pm w\omega\,\hat{\phi} = \omega(-y\,\hat{\imath} + x\,\hat{\jmath})$. The force per unit charge can be written as $\vec{f} = \vec{v} \times B_0\,\hat{\jmath} = -\omega B_0 y\,\hat{k}$

Similarly the force per unit charge on the horizontal parts are also $-\omega B_0 y\,\hat{k}$, but they don't contribute to emf as $d\vec{l}$ is perpendicular to $\hat{k}$

Putting $y = \omega \sin\phi$, the net emf is $4w\omega B_0 L \sin\phi$

Let us find the negative rate of change of flux

$$-\frac{d\phi}{dt} = -B_0(2L)(2w)\frac{d\cos\phi}{dt} = 4w\omega B_0 L \sin\phi \tag{11.4.1}$$

11.5. Write down the (real) electric and magnetic fields for a monochromatic plane wave of amplitude E_0 and frequency ω and phase angle zero, that is travelling

(a) in the negative x direction and polarized in z direction

Solution:

$$\vec{k} = -\frac{\omega}{c}\,\hat{\imath} \tag{11.5.1}$$

$$\vec{E} = E_0 \cos\left(\vec{k}\cdot\vec{r} - \omega t\right)\hat{k} \tag{11.5.2}$$

$$= E_0 \cos\left(\frac{\omega x}{c} + \omega t\right)\hat{k} \tag{11.5.3}$$

$$\vec{B} = \frac{\vec{k}}{\omega} \times \vec{E} \tag{11.5.4}$$

$$= \frac{E_0}{c}\cos\left(\frac{\omega x}{c} + \omega t\right)\hat{\jmath} \tag{11.5.5}$$

(b) traveling in the direction from the origin to the point $(1, 1, 1)$ with polarization parallel to the xz plane.

Solution:

$$\vec{k} = \frac{\omega}{c}\frac{\hat{\imath} + \hat{\jmath} + \hat{k}}{\sqrt{3}} \tag{11.5.6}$$

The direction of polarization is perpendicular to both $\vec{k}$ and $\hat{\jmath}$

$$\vec{E} = \pm E_0 \cos\left(\vec{k}\cdot\vec{r} - \omega t\right) \frac{\vec{k}\times\hat{\jmath}}{|\vec{k}\times\hat{\jmath}|} \tag{11.5.7}$$

$$= \pm E_0 \cos\left(\frac{\omega(x+y+z)}{c\sqrt{3}} - \omega t\right) \frac{\hat{\imath}-\hat{k}}{\sqrt{2}} \tag{11.5.8}$$

$$\vec{B} = \frac{\vec{k}}{\omega}\times\vec{E} \tag{11.5.9}$$

$$= \pm\frac{E_0}{c} \cos\left(\frac{\omega(x+y+z)}{c\sqrt{3}} - \omega t\right) \frac{-\hat{\imath}+2\hat{\jmath}-\hat{k}}{\sqrt{6}} \tag{11.5.10}$$

11.6. Consider a propagating wave in free space given by

$$\vec{E} = E_0 \frac{\sin\theta}{r}\left(\cos(kr-\omega t) - \frac{\sin(kr-\omega t)}{kr}\right)\hat{\phi}$$

(a) Calculate the magnetic field $\vec{B}$ and the Poynting vector $\vec{S}$. You would need to use the expansions of $\nabla\times$ in spherical co-ordinates.

Solution: Since this is not a planar wave, we cannot apply $\vec{B} = \frac{\vec{k}}{\omega}\times\vec{E}$ here.
Let us apply the Maxwell's Equation $\nabla\times E = -\frac{\partial B}{\partial t}$
Writing the complex form of the field

$$\vec{E} = E_0 \frac{\sin\theta}{r} e^{i(kr-\omega t)}\left(1 + \frac{e^{i\frac{\pi}{2}}}{kr}\right)\hat{\phi} \tag{11.6.1}$$

$$\nabla\times E = \frac{2E_0\cos\theta}{r^2} e^{i(kr-\omega t)}\left(1 + \frac{e^{i\frac{\pi}{2}}}{kr}\right)\hat{r} \tag{11.6.2}$$

$$-\frac{E_0\sin\theta}{r} e^{i(kr-\omega t+\frac{\pi}{2})}\left(k + \frac{e^{i\frac{\pi}{2}}}{r} - \frac{1}{kr^2}\right)\hat{\theta} \tag{11.6.3}$$

$$\implies \vec{B} = -\int \nabla\times\vec{E}\,dt \tag{11.6.4}$$

Ignoring the integration constant (as we are not interested in constant terms)

$$\implies \vec{B} = \frac{2E_0\cos\theta}{\omega r^2} e^{i(kr-\omega t-\frac{\pi}{2})}\left(1 + \frac{e^{i\frac{\pi}{2}}}{kr}\right)\hat{r} \tag{11.6.5}$$

$$-\frac{E_0\sin\theta}{\omega r} e^{i(kr-\omega t)}\left(k + \frac{e^{i\frac{\pi}{2}}}{r} - \frac{1}{kr^2}\right)\hat{\theta} \tag{11.6.6}$$

Interestingly, this field resembles the field of an oscillating *magnetic* monopole. You can verify that the divergence of $\vec{B}$ has a delta term leftover, while the divergence of $\vec{E}$ is zero.
Before calculating the Poynting vector we must be careful to switch back to the real representation

$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0} \tag{11.6.7}$$

$$= \frac{E_0^2 \sin^2\theta}{\mu_0 \omega r^2} \left(k\cos^2(kr - \omega t) - \sin(2(kr - \omega t)) \left(\frac{1}{r} - \frac{1}{2k^2 r^3} \right) - \frac{\cos(2(kr - \omega t))}{kr^2} \right) \hat{r} \tag{11.6.8}$$

$$+ \frac{E_0 \sin 2\theta}{\mu_0 \omega r^3} \left(\frac{\sin(2(kr - \omega t))}{2} \left(1 - \frac{1}{k^2 r^2} \right) + \frac{\cos(2(kr - \omega t))}{kr} \right) \hat{\theta} \tag{11.6.9}$$

Wow that's a mouthful! Fortunately on a time averaged scale most terms will cancel out

(b) What is the total average power radiated by the source?

Solution: We want the average power radiated, which is going to be the outflux of the time averaged Poynting vector. We know that $\langle \cos(u(t)) \rangle = \langle \sin(u(t)) \rangle = 0$ and $\langle \cos^2(u(t)) \rangle = \langle \sin^2(u(t)) \rangle = \frac{1}{2}$

$$\implies \langle \vec{S} \rangle = \frac{E_0^2 \sin^2\theta}{2\mu_0 c r^2} \hat{r} \tag{11.6.10}$$

$$\implies \langle P \rangle = \oiint \langle \vec{S} \rangle \cdot d\vec{s} = \iint \frac{E_0^2 \sin^2\theta}{2\mu_0 c} \sin\theta \, d\theta \, d\phi \tag{11.6.11}$$

$$= \frac{4\pi E_0^2}{3\mu_0 c} \tag{11.6.12}$$

11.7. A cylinder of radius R and infinite length is made of permanently polarized dielectric. The polarization vector $\vec{P}$ is proportional to radial vector $\vec{r}$ everywhere, $\vec{P} = a\vec{r}$ where a is positive constant. The cylinder rotates around its axis with an angular speed ω. This is a non-relativistic problem.

(a) Calculate electric field $\vec{E}$ at a radius r both inside and outside the cylinder.

Solution: We know that there is a volume charge $-\nabla \cdot \vec{P} = -2a$ inside cylinder and a surface charge $\vec{P} \cdot \hat{n} = aR$ on the cylinder

We can use a gaussian cylindrical surface to compute $\vec{E}$. Since the net charge is 0, the field outside is naturally zero

$$\implies \vec{E} = \begin{cases} -\frac{ar}{\epsilon_0} \hat{r} & r < R \\ 0 & r > R \end{cases} \tag{11.7.1}$$

(b) Calculate magnetic field $\vec{B}$ at a radius r both inside and outside the cylinder.

Solution: There is a volume current inside $\vec{J} = \rho\vec{v} = -2a\omega r\hat{\phi}$. There is also a surface current of $\vec{K} = \sigma\vec{v} = a\omega R^2\,\hat{\phi}$

Since there are no currents outside, the $\vec{B}$ field outside must be 0. To find the $\vec{B}$ field inside, we can use a rectangular amperian, whose one arm is at a radius of r inside the cylinder, and the other is outside

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{\text{enc}} = \mu_0\left(\iint \vec{J}\cdot d\vec{S} + \int \vec{K}\cdot\hat{n}dl\right) \tag{11.7.2}$$

$$\implies \vec{B} = \begin{cases}\mu_0 a\omega r^2 & r < R\\ 0 & r > R\end{cases} \tag{11.7.3}$$

(c) What is the total electromagnetic energy stored per unit length of the cylinder before it started spinning and while it is spinning? Where did the extra energy come from?

Solution: The energy before spinning is impossible to find. Since there is a frozen polarization (hence non linear dielectric) involved, the work done into making this arrangement depends on the history of the dielectric. However we could technically say that the pure electromagnetic energy is the work done in assembling the bound charges (disregarding the work done against the atomic springs). That way the energy per unit length before spinning is

$$U_E = \int_{\phi=0}^{2\pi}\int_{r=0}^{R}\frac{\epsilon_0\vec{E}\cdot\vec{E}}{2}\,r\,dr\,d\phi = \frac{\pi R^4 a^2}{4\epsilon_0} \tag{11.7.4}$$

After the spinning is set up, the additional energy per unit length is

$$U_M = \int_{\phi=0}^{2\pi}\int_{r=0}^{R}\frac{\vec{B}\cdot\vec{B}}{2\mu_0}r\,dr\,d\phi = \frac{\pi\mu_0 a^2\omega^2 R^6}{6} \tag{11.7.5}$$

When we start to spin the cylinder, it faces some impedance to the rotational acceleration. This is due to self inductance. Thus extra energy has to be pumped (by whatever agent is making the cylinder rotate) into the cylinder to keep it rotating and get it to the required angular speed. This energy makes its way into the magnetic field

11.8. Suppose,

$$\vec{E}(\vec{r},t) = \frac{-1}{4\pi\epsilon_0}\frac{q}{r^2}\Theta(vt-r)\,\hat{r} \qquad\qquad \vec{B}(\vec{r}) = 0 \tag{11.8.1}$$

(a) Show that they satisfy Maxwell's equations

Solution: $\nabla\cdot\vec{B}$

$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = 0$ both are easily seen to be satisfied

$$\nabla \cdot \vec{E} = -\frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\frac{\partial\,\Theta(vt-r)}{\partial r} - \frac{q}{4\pi\epsilon_0}\Theta(vt-r)\nabla\cdot\frac{\hat{r}}{r^2} \tag{11.8.2}$$

$$= \frac{q\,\delta(vt-r)}{4\pi\epsilon_0 r^2} - \frac{q}{\epsilon_0}\,\delta^3(\vec{r}) \tag{11.8.3}$$

$$\implies \rho = \frac{q\,\delta(vt-r)}{4\pi r^2} - q\,\delta^3(\vec{r}) \tag{11.8.4}$$

$$\nabla\times\vec{B} - \epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} = \frac{\mu_0 q}{4\pi r^2}\frac{\partial\,\Theta(vt-r)}{\partial t}\,\hat{r} \tag{11.8.5}$$

$$\implies \vec{J} = \frac{qv}{4\pi r^2}\delta(vt-r)\,\hat{r} \tag{11.8.6}$$

The final test for consistency, the continuity equation

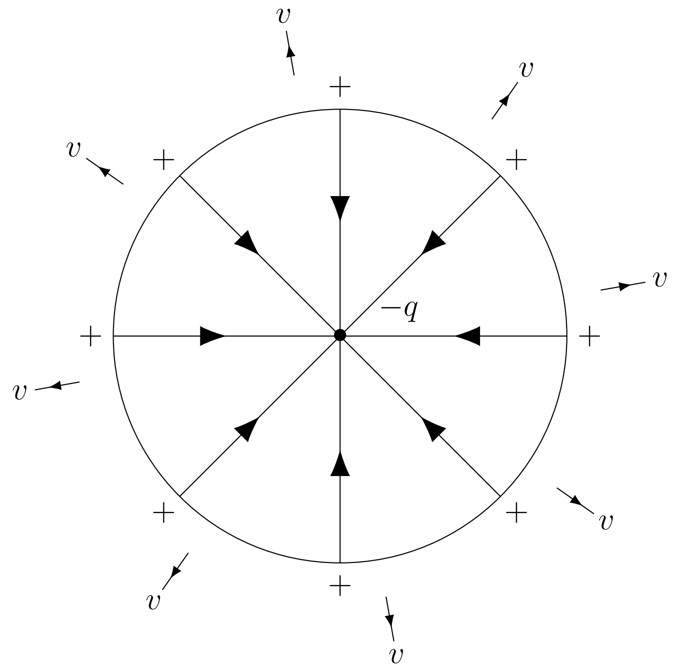
$$-\nabla\cdot\vec{J} = -\frac{qv}{4\pi r^2}\frac{\partial\,\delta(vt-r)}{\partial r} - \frac{qv}{4\pi}\delta(vt-r)\nabla\cdot\frac{\hat{r}}{r^2} \tag{11.8.7}$$

$$= \frac{qv}{4\pi r^2}\frac{\partial\,\delta(vt-r)}{\partial r} \tag{11.8.8}$$

$$\frac{\partial\rho}{\partial t} = \frac{q}{4\pi r^2}\frac{\partial\,\delta(vt-r)}{\partial t} = -\frac{qv}{4\pi r^2}\frac{\partial\,\delta(vt-r)}{\partial r} = -\nabla\cdot\vec{J} \tag{11.8.9}$$

Thus Maxwell's equations are consistently satisfied.

Interestingly, the physical situation is a negative point charge sitting at the origin with a positively charged spherical shell expanding out with speed v



(b) Determine ρ and $\vec{J}$

Solution: Solved in the above part