①

$$\overset{M_1}{\begin{pmatrix} I_n & -iI_n \\ 0 & I_n \end{pmatrix}} \begin{pmatrix} A & B \\ -B & A \end{pmatrix} = \begin{pmatrix} A+iB & (-i)(A+iB) \\ -B & A \end{pmatrix}$$

$$\boxed{\text{let } X = (A+iB)^{-1}}$$

$$\begin{pmatrix} A+iB & (-i)(A+iB) \\ -B & A \end{pmatrix} \overset{M_2}{\begin{pmatrix} I_n & iI_n \\ 0 & I_n \end{pmatrix}} = \begin{pmatrix} A+iB & 0 \\ -B & A-iB \end{pmatrix}$$

$$\underset{M_3}{\begin{pmatrix} I_n & 0 \\ BX & I_n \end{pmatrix}} \begin{pmatrix} A+iB & 0 \\ -B & A-iB \end{pmatrix} = \begin{pmatrix} A+iB & 0 \\ 0 & A-iB \end{pmatrix}$$

$$\therefore \quad M_3\left(\left(M_1 \begin{pmatrix} A & B \\ -B & A \end{pmatrix}\right) M_2\right) = \begin{pmatrix} A+iB & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & A-iB \end{pmatrix}$$

as $\det(M_i) = 1$ for $i = 1, 2, 3$

$$\det\begin{pmatrix} A & B \\ -B & A \end{pmatrix} = \det(A+iB)\det(A-iB)$$
$$= \det(A+iB)\,\overline{\det(A+iB)}$$

(As det is a polynomial function in its entries)

as $\det(A+iB) \neq 0$

$$\therefore \det\begin{pmatrix} A & B \\ -B & A \end{pmatrix} > 0$$

Q2)

$$AB = \underbrace{\begin{bmatrix} 2 & 0 & 6 & 0 & 4 \\ 5 & 3 & 2 & 2 & 7 \\ 2 & 5 & 7 & 5 & 5 \\ 2 & 0 & 9 & 2 & 7 \\ 7 & 8 & 4 & 2 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 10^4 \\ 0 & 1 & 0 & 0 & 10^3 \\ 0 & 0 & 1 & 0 & 10^2 \\ 0 & 0 & 0 & 1 & 10^1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{B}$$

$$= \underbrace{\begin{bmatrix} 2 & 0 & 6 & 0 & 20604 \\ 5 & 3 & 2 & 2 & 53227 \\ 2 & 5 & 7 & 5 & 25755 \\ 2 & 0 & 9 & 2 & 20927 \\ 7 & 8 & 4 & 2 & 78421 \end{bmatrix}}_{AB}$$

as $\det(B) = 1$, and each entry of the last column of AB is divisible by 17. $\therefore$ 17 divides $\det(AB) = \det(A)$

Q3)

let α be a common root of

$$\left.\begin{aligned} x^2 + ax + b = 0 \\ x^2 + px + q = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} \alpha^2 + a\alpha + b = 0 \\ \alpha^2 + p\alpha + q = 0 \end{aligned}$$

$$\underset{A}{\begin{bmatrix} 1 & a & b & 0 \\ 0 & 1 & a & b \\ 1 & p & q & 0 \\ 0 & 1 & p & q \end{bmatrix}} \underset{B}{\begin{bmatrix} 1 & 0 & 0 & \alpha^3 \\ 0 & 1 & 0 & \alpha^2 \\ 0 & 0 & 1 & \alpha \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & a & b & \alpha(\alpha^2 + a\alpha + b) \\ 0 & 1 & a & \alpha^2 + a\alpha + b \\ 1 & p & q & \alpha(\alpha^2 + p\alpha + q) \\ 0 & 1 & p & \alpha^2 + p\alpha + q \end{bmatrix} = \begin{bmatrix} 1 & a & b & 0 \\ 0 & 1 & a & 0 \\ 1 & p & q & 0 \\ 0 & 1 & p & 0 \end{bmatrix}$$

$\therefore \det(AB) = 0$ as $\det(B) = 1$

$\therefore \det(A) = 0$

Q2)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 6 & \beta \end{bmatrix}$$

coefficient matrix.

$$\det(A) = (3\beta - 6) + 2(1-\beta) + 3(3)$$
$$= \beta + 5$$

for $\beta \neq -5$, det of coefficient matrix $\neq 0$

$\therefore$ Cramer's Rule is applicable.

For $\beta = -5$, Augmented matrix

$$[A:b] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 20 \\ 1 & 3 & 1 & 13 \\ 1 & 6 & -5 & -5 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 20 \\ 0 & 1 & -2 & -7 \\ 0 & 4 & -8 & -25 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 20 \\ 0 & \boxed{1} & -2 & -7 \\ 0 & 0 & 0 & \boxed{3} \end{array}\right]$$

$\therefore$ Rank of $A = 2$

Rank of $[A:b] = 3$ $\Rightarrow$ System is not solvable for $\beta = -5$

Q3)

Vectors are linearly independent iff $\det A \neq 0$

where $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$

$$\det(A) = acb - a^3 + bac - b^3 + cab - c^3$$
$$= 3abc - a^3 - b^3 - c^3$$
$$= (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$
$$= (a+b+c) \times ((a-b)^2 + (b-c)^2 + (a-c)^2) \times \frac{1}{2}$$

$\det(A) = 0$ iff $a+b+c=0$
(or)
$a=b=c$

$\therefore$ vectors are linearly dependent iff $a+b+c=0$ or $a=b=c$

Q9)

$$H = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$$

Cofactor matrix of H = C

$$C = \begin{bmatrix} (+1)\left(\frac{1}{15\times16}\right) & (-1)\left(\frac{2}{12\times10}\right) & (+1)\left(\frac{1}{8\times9}\right) \\ (-1)\left(\frac{1}{12\times10}\right) & (+1)\left(\frac{4}{5\times9}\right) & (-1)\left(\frac{2}{4\times6}\right) \\ (+1)\left(\frac{1}{8\times9}\right) & (-1)\left(\frac{2}{4\times6}\right) & (+1)\left(\frac{1}{3\times4}\right) \end{bmatrix}$$

Adjugate (H) = $C^t = C$ (as C is symmetric)

$$\det(H) = \frac{1}{15\times16} - \left(\frac{1}{2}\times\frac{2}{12\times10}\right) + \frac{1}{3\times8\times9}$$

$$= \frac{1}{240} - \frac{1}{120} + \frac{1}{216} = \frac{1}{216} - \frac{1}{6\times40} = \left(\frac{1}{6}\right)\left(\frac{4}{36\times40}\right)$$

$(\cancel{6\times36})$

$$= \frac{1}{2160}$$

$\therefore$ $H^{-1} = (\det(H))^{-1}\, C^t = (2160)\, C$

$$= \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix}$$

11)

$c_1 f_1 + \cdots + c_n f_n = 0$ on (a,b)

$\Rightarrow$ $c_1 f_1^{(i)} + \cdots + c_n f_n^{(i)} = 0$ on (a,b)

for $i = 1, 2, \ldots, n-1$

we set

$$\begin{bmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & & & \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \quad x \in (a,b)$$

If $W_{f_1, \ldots, f_n}(x_0) \neq 0$ for some $x_0 \in (a,b)$

Then $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$