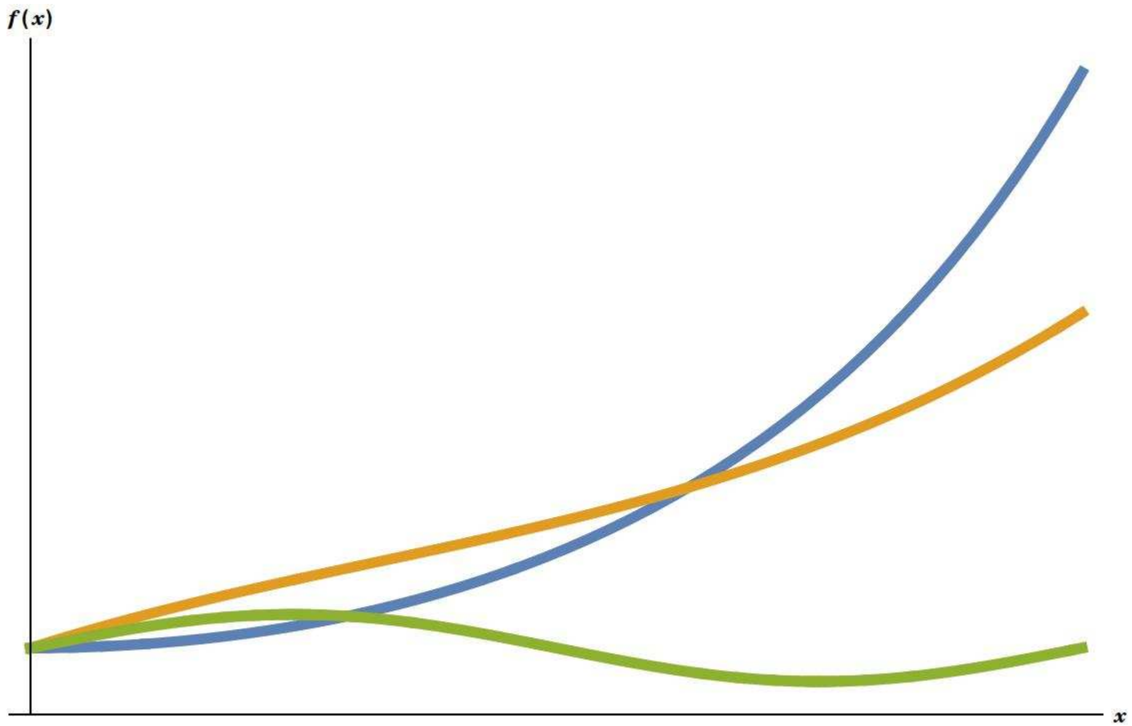
MATHEMATICAL PRELIMINARIES

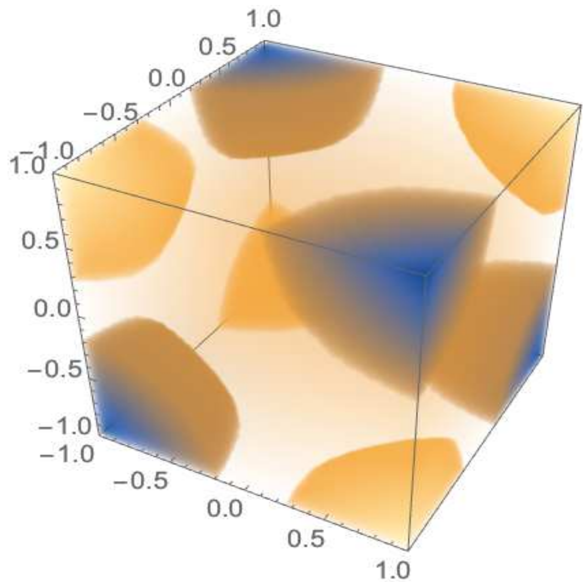
DERIVATIVE OF A FUNCTION



$$\frac{df(x)}{dx}$$

The derivative of a function measures how the function f(x) changes as we change x.

GRADIENT OF A SCALAR FIELD



Temperature T(x,y,z)

How fast does the temperature vary?

Depends on the direction we look at!

$$dT = \left(\frac{\partial T}{\partial x}\right) dx + \left(\frac{\partial T}{\partial y}\right) dy + \left(\frac{\partial T}{\partial z}\right) dz$$

$$dT = \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right).\left(dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}\right) = (\nabla T).(d\vec{l})$$

$$\boxed{\nabla T = \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right)}$$

Gradient of a scalar field is a vector.

In fact $\nabla T(x,y,z)$ is a vector field!

The gradient points in the direction of maximum increase of the field.

THE DEL OPERATOR

$$\nabla T = \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right)$$

$$\nabla \equiv \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)$$

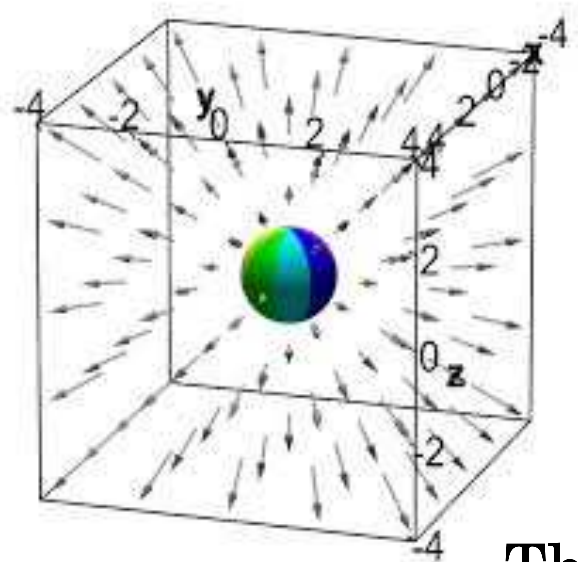
This is a ***vector* operator**. It needs to acts upon a quantity to have any meaning.

- Multiplication by a scalar ➔ Gradient of a scalar field ∇T
- Dot product with vector ➔ Divergence of a vector field $\nabla . \vec{v}$
- Cross product with vector ➔ Curl of a vector field $\nabla \times \vec{v}$

DIVERGENCE OF A VECTOR FIELD

$$\nabla \cdot \vec{v} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot \left(v_x \hat{x} + v_y \hat{y} + v_z \hat{z}\right)$$

$$\nabla \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

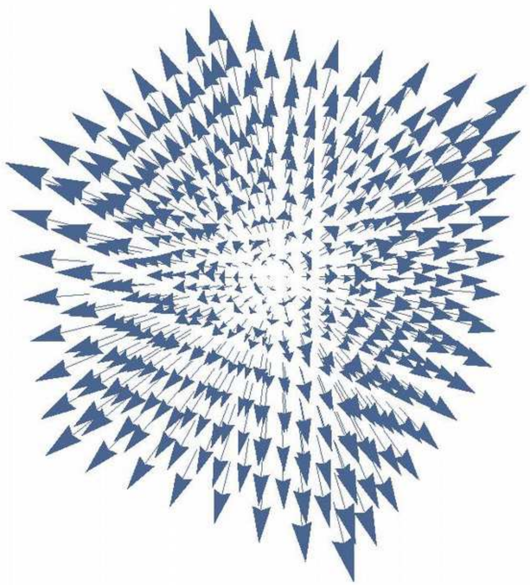


The divergence is a measure of how much the vector spreads out from the point in question.

The divergence of a vector is a scalar quantity.

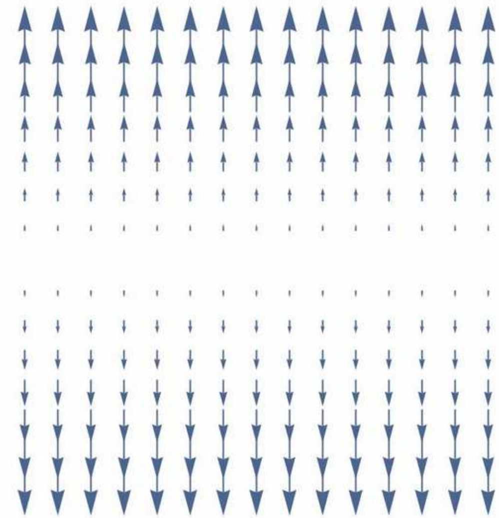
A point of positive divergence is a source, a point of negative divergence is a sink.

DIVERGENCE OF A VECTOR FIELD



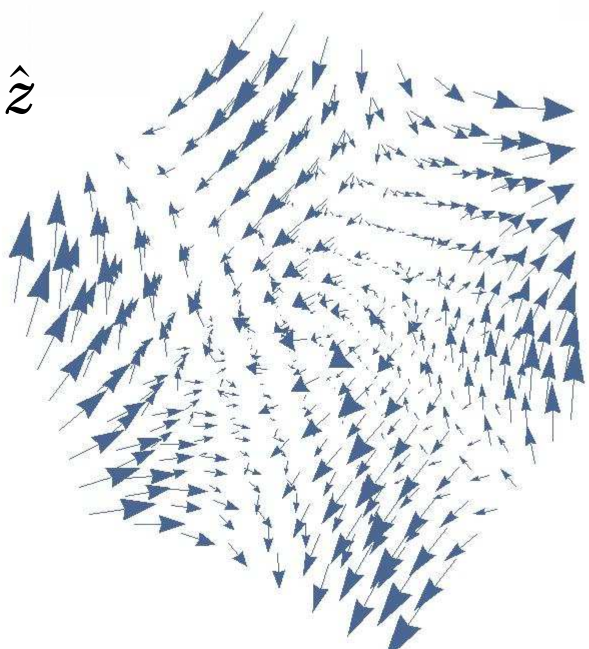
$\vec{v} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$

$\nabla . \vec{v} = 3$



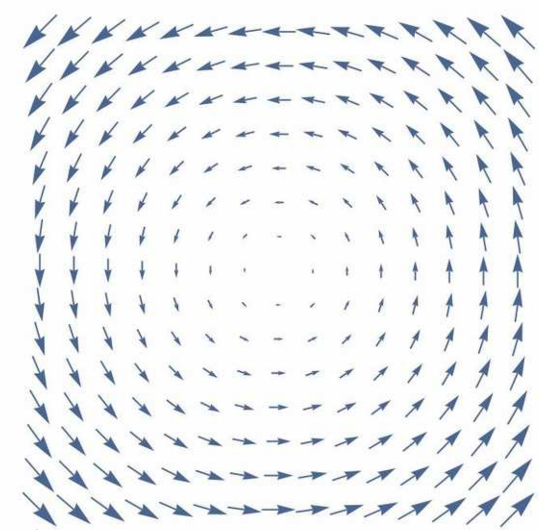
$\vec{v} = z\,\hat{z}$

$\nabla . \vec{v} = 1$



$\vec{v} = xy\,\hat{x} + 2\,yz\,\hat{y} + 3\,zx\,\hat{z}$

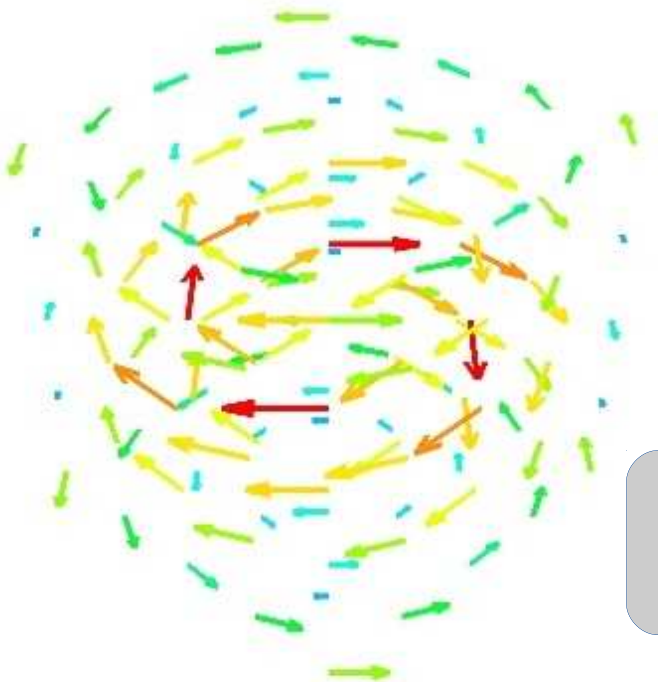
$\nabla . \vec{v} = y + 2\,z + 3\,x$



$\vec{v} = -y\,\hat{x} + x\,\hat{y}$

$\nabla . \vec{v} = 0$

CURL OF A VECTOR FIELD



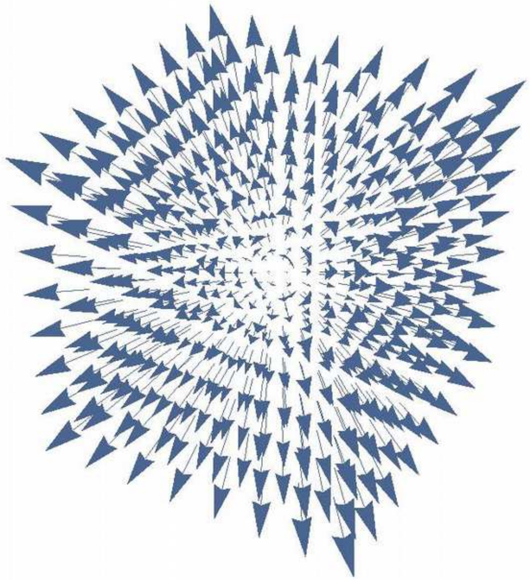
$$\nabla \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

$$\nabla \times \vec{v} = \hat{x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

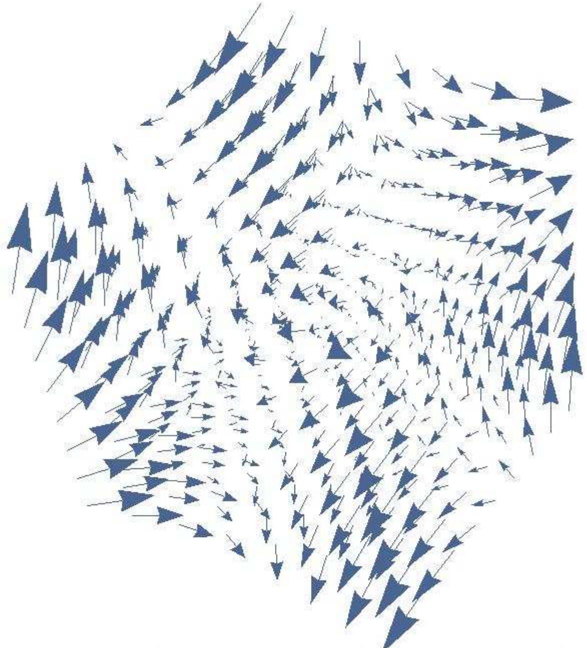
The curl of a vector is a measure of how much the vector curls around the point in question.

The curl of a vector is a vector itself.

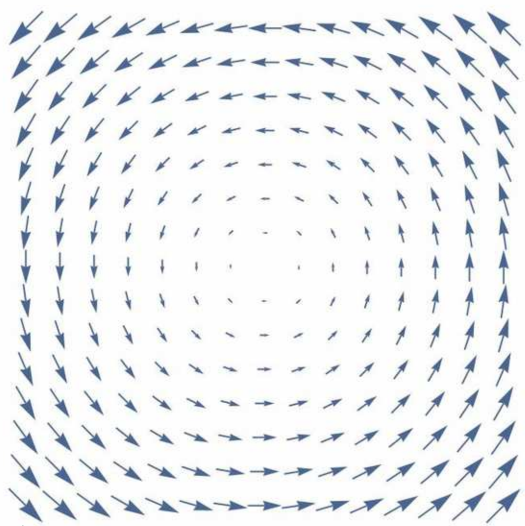
CURL OF A VECTOR FIELD



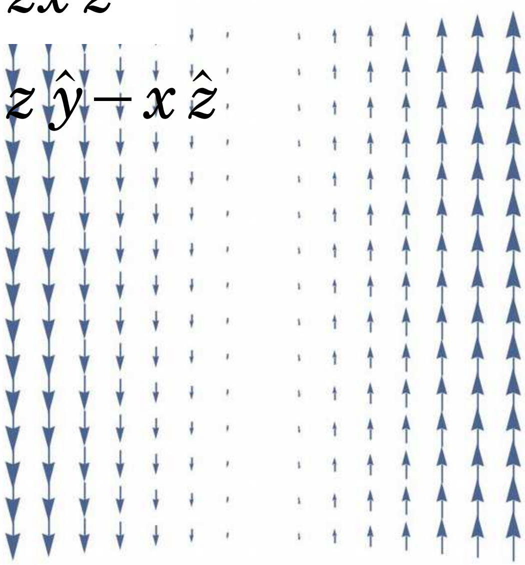
$$\vec{v} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$$
$$\nabla \times \vec{v} = 0$$



$$\vec{v} = xy\,\hat{x} + 2\,yz\,\hat{y} + 3\,zx\,\hat{z}$$
$$\nabla \times \vec{v} = -2\,y\,\hat{x} - 3\,z\,\hat{y} - x\,\hat{z}$$



$$\vec{v} = -y\,\hat{x} + x\,\hat{y}$$
$$\nabla \times \vec{v} = 2\,\hat{z}$$



$$\vec{v} = x\,\hat{y}$$
$$\nabla \times \vec{v} = \hat{z}$$

SUM RULES

$$\frac{d}{dx}(f+g)=\frac{df}{dx}+\frac{dg}{dx}$$

$$\nabla(f+g)=\nabla f+\nabla g$$

$$\nabla.(\vec{A}+\vec{B})=(\nabla.\vec{A})+(\nabla.\vec{B})$$

$$\nabla\times(\vec{A}+\vec{B})=(\nabla\times\vec{A})+(\nabla\times\vec{B})$$

MULTIPLICATION BY A CONSTANT

$$\frac{d}{dx}(kf)=k\frac{df}{dx}$$

$$\nabla(kf)=k\nabla f$$

$$\nabla.(k\vec{A})=k(\nabla.\vec{A})$$

$$\nabla\times(k\vec{A})=k(\nabla\times\vec{A})$$

PRODUCT RULES

$$\frac{d}{dx}(fg)=f\frac{dg}{dx}+g\frac{df}{dx}$$

$$\nabla(fg)=f\nabla g+g\nabla f$$

$$\nabla(\vec{A}.\vec{B})=\vec{A}\times(\nabla\times\vec{B})+\vec{B}\times(\nabla\times\vec{A})$$
$$+(\vec{A}.\nabla)\vec{B}+(\vec{B}.\nabla)\vec{A}$$

$$\nabla.(f\vec{A})=f(\nabla.\vec{A})+\vec{A}.(\nabla f)$$

$$\nabla.(\vec{A}\times\vec{B})=\vec{B}.(\nabla\times\vec{A})-\vec{A}.(\nabla\times B)$$

$$\nabla\times(f\vec{A})=f(\nabla\times\vec{A})-\vec{A}\times(\nabla f)$$

$$\nabla\times(\vec{A}\times\vec{B})=(\vec{B}.\nabla)\vec{A}-(\vec{A}.\nabla)\vec{B}$$
$$+\vec{A}(\nabla.\vec{B})-\vec{B}(\nabla.\vec{A})$$

GRADIENT OF DOT PRODUCT OF TWO VECTORS - PROOF

$$\nabla(\vec{A}.\vec{B}) = \vec{A}\times(\nabla\times\vec{B}) + \vec{B}\times(\nabla\times\vec{A}) + (\vec{A}.\nabla)\vec{B} + (\vec{B}.\nabla)\vec{A}$$

$$\vec{A}\times(\nabla\times\vec{B}) = \hat{x}\left\{A_y\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) - A_z\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right)\right\} + \hat{y}\{\ldots\} + \hat{z}\{\ldots\}$$

$$\vec{B}\times(\nabla\times\vec{A}) = \hat{x}\left\{B_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - B_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right\} + \hat{y}\{\ldots\} + \hat{z}\{\ldots\}$$

$$(\vec{A}.\nabla)\vec{B} = \hat{x}\left\{\left(A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z}\right)B_x\right\} + \hat{y}\{\ldots\} + \hat{z}\{\ldots\}$$

$$(\vec{B}.\nabla)\vec{A} = \hat{x}\left\{\left(B_x\frac{\partial}{\partial x} + B_y\frac{\partial}{\partial y} + B_z\frac{\partial}{\partial z}\right)A_x\right\} + \hat{y}\{\ldots\} + \hat{z}\{\ldots\}$$

$$\Rightarrow \mathrm{RHS} = \hat{x}\Big\{A_y\frac{\partial B_y}{\partial x} - A_y\cancel{\frac{\partial B_x}{\partial y}} - A_z\cancel{\frac{\partial B_x}{\partial z}} + A_z\frac{\partial B_z}{\partial x} + B_y\frac{\partial A_y}{\partial x} - B_y\cancel{\frac{\partial A_x}{\partial y}} - B_z\cancel{\frac{\partial A_x}{\partial z}} + B_z\frac{\partial A_z}{\partial x}$$

$$+ A_x\frac{\partial B_x}{\partial x} + A_y\cancel{\frac{\partial B_x}{\partial y}} + A_z\cancel{\frac{\partial B_x}{\partial z}} + B_x\frac{\partial A_x}{\partial x} + B_y\cancel{\frac{\partial A_x}{\partial y}} + B_z\cancel{\frac{\partial A_x}{\partial z}}\Big\} + \hat{y}\{\ldots\} + \hat{z}\{\ldots\}$$

$$= \hat{x}\left\{\frac{\partial}{\partial x}\left(A_x B_x + A_y B_y + A_z B_z\right)\right\} + \hat{y}\{\ldots\} + \hat{z}\{\ldots\} = \nabla(\vec{A}.\vec{B})$$

SECOND DERIVATIVES

Divergence of a gradient $\nabla . (\nabla T)$

Curl of a gradient $\nabla \times (\nabla T)$

Gradient of a divergence $\nabla (\nabla . \vec{v})$

Divergence of curl $\nabla . (\nabla \times \vec{v})$

Curl of curl $\nabla \times (\nabla \times \vec{v})$

SECOND DERIVATIVES

Divergence of a gradient $\nabla . (\nabla T)$

$$\nabla . (\nabla T) = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \right) . \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z} \right)$$

$$= \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

Laplacian: $\nabla^2 T \equiv \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$

Laplacian of a vector $\nabla^2 \vec{v} = (\nabla^2 v_x)\hat{x} + (\nabla^2 v_y)\hat{y} + (\nabla^2 v_z)\hat{z}$

SECOND DERIVATIVES

Gradient of a divergence $\nabla(\nabla.\vec{v})$

$$\nabla(\nabla.\vec{v})=\left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right).\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)$$

$$=\hat{x}\left(\frac{\partial}{\partial x}\left[\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right]\right)$$

$$+\hat{y}\left(\frac{\partial}{\partial y}\left[\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right]\right)$$

$$+\hat{z}\left(\frac{\partial}{\partial z}\left[\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right]\right)$$

$$\nabla^2\vec{v}\neq\nabla(\nabla.\vec{v})$$

The Laplacian of a vector is not the same as gradient of the divergence.

SECOND DERIVATIVES

Curl of curl $\nabla\times(\nabla\times\vec{v})$

$$\nabla\times(\nabla\times\vec{v})=\nabla\times\left(\hat{x}\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)+\hat{y}\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)+\hat{z}\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)\right)$$

$$=\hat{x}\left\{\frac{\partial}{\partial y}\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)-\frac{\partial}{\partial z}\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)\right\}+\hat{y}\{\ldots\}+\hat{z}\{\ldots\}$$

$$=\hat{x}\left\{\frac{\partial}{\partial x}\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}\right)-\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right)\right\}+\ldots$$

$$\nabla\times(\nabla\times\vec{v})=\nabla(\nabla.\vec{v})-\nabla^2\vec{v}$$