MA 108 - Ordinary Differential Equations

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Outline of the lecture

- Basis of Solutions
- Abel's formula and method of reduction of order
- Second order linear equations with constant coefficients
- Cauchy-Euler equations
- Non-homogeneous equations

General solution

Result: Let y_1 & y_2 be a basis of solutions of the homogeneous second order linear DE $y'' + p(x)y' + q(x)y = 0$ on I, where $p(x)$ and $q(x)$ are continuous on I, an open interval. Then,

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

is a general solution of $y'' + p(x)y' + q(x)y = 0$.
Every solution $y = Y(x)$ of the DE has the form

$$Y(x) = C_1 y_1(x) + C_2 y_2(x),$$

where C_1 and C_2 are arbitrary constants.

Proof

First statement follows, since ODE is linear (How?)
Let $Y(x)$ be a solution of the given ODE.
Aim: To find C_1 and C_2 such that

$$Y(x) = C_1 y_1(x) + C_2 y_2(x).$$

Fix a $x_0 \in I$, and consider the system of linear equations

$$\begin{aligned} Y(x_0) &= C_1 y_1(x_0) + C_2 y_2(x_0) \\ Y'(x_0) &= C_1 y_1'(x_0) + C_2 y_2'(x_0). \end{aligned}$$

i.e.,

$$\begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{pmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} Y(x_0) \\ Y'(x_0) \end{bmatrix}.$$

As y_1 and y_2 form a basis of solutions of the DE,

$$\begin{pmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{pmatrix}$$

is invertible (Justify!), and $W(x_0)$ is not zero.

Proof contd...

Therefore,

$$C_1 = \frac{\begin{vmatrix} Y(x_0) & y_2(x_0) \\ Y'(x_0) & y_2'(x_0) \end{vmatrix}}{W(x_0)},$$

and

$$C_2 = \frac{\begin{vmatrix} y_1(x_0) & Y(x_0) \\ y_1'(x_0) & Y'(x_0) \end{vmatrix}}{W(x_0)}.$$

(Is the representation of Y in terms of C_1 and C_2 unique?)

Now,

$$u(x) = Y(x) - C_1 y_1(x) - C_2 y_2(x)$$

satisfies the given DE, and

$$u(x_0) = 0 = u'(x_0).$$

But the constant function $u(x) \equiv 0$ also satisfies the IVP. Thus,

$$Y(x) = C_1 y_1(x) + C_2 y_2(x) \text{ by the uniqueness theorem.}$$

Abel's formula: Assume that $p(x), q(x)$ are continuous on an open interval I. Then the Wronskian of any two solutions $y_1(x), y_2(x)$ of

$$y'' + p(x)y' + q(x)y = 0, x \in I,$$

satisfies the DE $\boxed{W'(x) = -p(x)W(x)}$ and is given by

$$W(y_1, y_2)(x) = W(y_1, y_2)(x_0)e^{-\int_{x_0}^{x} p(t)dt},$$

for any $x_0 \in I$.

Solution : Set $W(y_1, y_2)(x) = W(x)$. Then,

$$W(x) = (y_1 y_2' - y_1' y_2)(x)$$
$$W'(x) = (y_1 y_2'' - y_1'' y_2)(x).$$

Now,

$$y_1'' = -p(x)y_1' - q(x)y_1$$
$$y_2'' = -p(x)y_2' - q(x)y_2.$$

Thus,

$$\begin{aligned} W'(x) &= -y_1 p y_2' - y_1 q y_2 + y_2 p y_1' + y_2 q y_1 \\ &= -p(y_1 y_2' - y_1' y_2) \\ &= -pW(x). \end{aligned}$$

Hence,

$$W(x) = ce^{-\int_{x_0}^{x} p(t)dt},$$

for a constant c.

For $x = x_0$, we get $W(x_0) = c$. Hence,

$$W(y_1, y_2)(x) = W(y_1, y_2)(x_0)e^{-\int_{x_0}^{x} p(t)dt}.$$

Consider the second order linear homogeneous ODE

$$y'' + p(x)y' + q(x)y = 0.$$

As we remarked earlier, there is no general method to find a basis of solutions. However, if we know one non-zero solution $y_1(x)$ then Abel's formula (also called as Abel-Liouville formula) gives us a method to find $y_2(x)$ such that $y_1(x)$ and $y_2(x)$ are linearly independent.

If y_2 is any other solution of the ODE, the Abel's formula tells us that

$$W(y_1, y_2)(x) \;=\; W(y_1, y_2)(x_0)e^{-\int_{x_0}^{x} p(t)dt}, \;\; x \in I.$$

For $W(y_1, y_2)(x_0) \neq 0$, y_2 satisfies the first order ODE

$$y_2' - \frac{y_1'(x)}{y_1(x)}y_2 = W(y_1, y_2)(x_0)\frac{1}{y_1(x)}e^{-\int_{x_0}^{x} p(t)dt}, \;\; x \in I.$$

Method of reduction of order

Abel formula:Method of reduction of order

Integrating factor is

$$e^{\int -\frac{y_1'(x)}{y_1(x)}dx} = \frac{1}{y_1(x)}.$$

Hence

$$y_2(x) = y_1(x) \int \frac{e^{-\int p(x)dx}}{y_1(x)^2}.$$

Example

Given that $y = x$ is a solution, find a l.i. solution of

$$(x^2+1)y'' - 2xy' + 2y = 0$$

.

$y_2 = vy_1 = vx$, where

$$v(x) = \int \frac{e^{-\int p dx}}{y_1^2} dx = \int \frac{e^{-\int \frac{-2x}{x^2+1} dx}}{x^2} dx = \int \frac{x^2+1}{x^2} dx = \int (1+\frac{1}{x^2}) dx$$

Hence, $v(x) = x - \dfrac{1}{x}$ and $y_2 = x\left(x - \dfrac{1}{x}\right) = x^2 - 1.$

Are y_1 & y_2 l.i.? What is the general solution?

Second Order Linear ODE's with constant coefficients

We have developed enough theory to now find all solutions of

$$y'' + py' + qy = 0,$$

where p and q are in $\mathbb{R}$; that is, a second order homogeneous linear ODE with constant coefficients.
Suppose e^{mx} is a solution of this equation. On substituting in the DE we get

$$m^2 e^{mx} + pme^{mx} + qe^{mx} = 0,$$

and this implies

$$m^2 + pm + q = 0.$$

This is called the characteristic equation or auxiliary equation of the linear homogeneous ODE with constant coefficients. The roots of this equation are

$$\boxed{m_1, m_2 = \frac{-p \pm \sqrt{p^2 - 4q}}{2}.}$$

$\boxed{\text{Case I: Real \& unequal roots}}$ $m_1, m_2 \in \mathbb{R}, m_1 \neq m_2$.

When $p^2 - 4q > 0$, m_1 and m_2 are distinct real numbers. Moreover,

$$\frac{e^{m_1 x}}{e^{m_2 x}} = e^{(m_1 - m_2)x}$$

is not a constant function. Hence, $e^{m_1 x}$ and $e^{m_2 x}$ are linearly independent. So the general solution of

$$y'' + py' - qy = 0$$

is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x},$$

where $c_1, c_2 \in \mathbb{R}$.

$\boxed{\text{Case II: Equal roots}}$ $m_1 = m_2 \in \mathbb{R}$.

$$m_1 = m_2 \iff p^2 - 4q = 0,$$

and in this case $m = -\dfrac{p}{2}$. Hence $e^{-\frac{px}{2}}$ is one solution. To find the other solution, let

$$g(x) = v(x)e^{-\frac{px}{2}}.$$

Then,

$$\begin{aligned} v(x) &= \int \frac{e^{-\int p dx}}{e^{-px}} dx \\ &= ax + b, \end{aligned}$$

for some $a, b \in \mathbb{R}$. Choose $v(x) = x$. Then, $g(x) = xe^{-\frac{px}{2}}$. Hence the general solution is

$$y = c_1 e^{-\frac{px}{2}} + c_2 xe^{-\frac{px}{2}},$$

with $c_1, c_2 \in \mathbb{R}$.

$\boxed{\text{Case III : Complex roots}}$ $m_1 \neq m_2 \in \mathbb{C} \setminus \mathbb{R}$.

$m^2 + px + m = 0$ has distinct complex roots if and only if $p^2 - 4q < 0$. In this case, let

$$m_1 = a + \imath b, m_2 = a - \imath b.$$

Thus,

$$e^{m_1 x} = e^{(a+\imath b)x} = e^{ax}(\cos bx + \imath \sin bx),$$

and

$$e^{m_2 x} = e^{(a-\imath b)x} = e^{ax}(\cos bx - \imath \sin bx).$$

As our aim is to find solutions which are real valued functions, we take

$$f(x) = \frac{e^{m_1 x} + e^{m_2 x}}{2} = e^{ax} \cos bx,$$

and

$$g(x) = \frac{e^{m_1 x} - e^{m_2 x}}{2\imath} = e^{ax} \sin bx.$$

Now, $\dfrac{g(x)}{f(x)} = \tan bx$ is not a constant function. Thus the general solution is of the form

$$y = e^{ax}(c_1 \cos bx + c_2 \sin bx),$$

with $c_1, c_2 \in \mathbb{R}$.

Solve $4y'' - 8y' + 3y = 0,\ y(0) = 2,\ y'(0) = \frac{1}{2}$.

The characteristic equation is $4m^2 - 8m + 3 = 0 \implies m = \frac{3}{2}, \frac{1}{2}$.

The general solution is

$$y = c_1 e^{\frac{3}{2}x} + c_2 e^{\frac{1}{2}x}.$$

Now,

$$y' = \frac{3}{2} c_1 e^{\frac{3}{2}x} + \frac{1}{2} c_2 e^{\frac{1}{2}x}$$

$y(0) = 2 \implies c_1 + c_2 = 2$

$y'(0) = \frac{1}{2} \implies \frac{3}{2} c_1 + \frac{1}{2} c_2 = \frac{1}{2}$

Solving, $\boxed{c_1 = -\frac{1}{2}}$, $\boxed{c_2 = \frac{5}{2}}$.

Therefore,

$$\boxed{y = -\frac{1}{2} e^{\frac{3}{2}x} + \frac{5}{2} e^{\frac{1}{2}x}.}$$

Example 2

Solve $y'' - 4y' + 4y = 0,\ y(0) = 3,\ y'(0) = 1.$
The characteristic equation is $(m-2)^2 = 0 \Longrightarrow m = 2$
The general solution is

$$y = c_1 e^{2x} + c_2 x e^{2x} = (c_1 + c_2 x)e^{2x}.$$

Now,

$$y' = 2(c_1 + c_2 x)e^{2x} + c_2 e^{2x}$$

$$y(0) = 3 \Longrightarrow c_1 = 3,$$

$$y'(0) = 1 \Longrightarrow 2c_1 + c_2 = 1.$$

Hence, $c_2 = -5$.
Therefore,

$$\boxed{y = (3 - 5x)e^{2x}.}$$

Example 3

Solve $y'' - 6y' + 25y = 0, \; y(0) = -3, \; y'(0) = -1.$

The characteristic equation is $m^2 - 6m + 25 = 0$

$$\Longrightarrow m_1 = 3 + 4\imath, \; m_2 = 3 - 4\imath.$$

The general solution is

$$y = e^{3x}(c_1 \cos 4x + c_2 \sin 4x).$$

Now,

$$y' = 3e^{3x}(c_1 \cos 4x + c_2 \sin 4x) + e^{3x}(-4c_1 \sin 4x + 4c_2 \cos 4x)$$

$$y(0) = -3 \Longrightarrow c_1 = -3$$

$$y'(0) = -1 = 3c_1 + 4c_2 \Longrightarrow c_2 = 2.$$

Therefore,

$$\boxed{y = e^{3x}(-3 \cos 4x + 2 \sin 4x).}$$

The equation

$$x^2y'' + axy' + by = 0$$

where $a, b \in \mathbb{R}$ is called a $\boxed{\text{Cauchy-Euler equation}}$. Assume $x > 0$.
Suppose $y = x^m$ is a solution to this DE. Then,

$$x^2 m(m-1)x^{m-2} + axmx^{m-1} + bx^m = 0.$$

We get:

$$\boxed{m(m-1) + am + b = 0.}$$

that is,

$$m^2 + (a-1)m + b = 0.$$

This is called the auxiliary equation of the given Cauchy-Euler equation.
The roots are

$$m_1, m_2 = \frac{(1-a) \pm \sqrt{(a-1)^2 - 4b}}{2}.$$

$\boxed{\text{Case I: Distinct real roots.}}$
Are x^{m_1} and x^{m_2} linearly independent? Yes. Hence the general solution is given by

$$y = c_1 x^{m_1} + c_2 x^{m_2},$$

for $c_1, c_2 \in \mathbb{R}$.

Case II: Equal real roots.
that is,

$$m_1 = m_2 = \frac{1-a}{2}.$$

Hence $y = f(x) = x^{\frac{1-a}{2}}$ is a solution. The DE in standard form is

$$y'' + \frac{a}{x}y' + \frac{b}{x^2}y = 0.$$

To get a solution $g(x)$ linearly independent from $f(x)$, set $g(x) = v(x)f(x)$.

$$v(x) = \int \frac{e^{-\int \frac{a}{x}dx}}{x^{1-a}}dx = \int \frac{dx}{x} = \ln x$$

Hence,

$$g(x) = (\ln x)x^{\frac{1-a}{2}}.$$

Thus the general solution is given by

$$y = c_1x^{\frac{1-a}{2}} + c_2x^{\frac{1-a}{2}}\ln x,$$

$c_1, c_2 \in \mathbb{R}$.

$\boxed{\text{Case III : Complex roots}}$

Roots are $m_1 = \mu + \imath\nu, \ m_2 = \mu - \imath\nu$.

$$x^{m_1} = x^{\mu} e^{\imath\nu \ln x} = x^{\mu}(\cos(\nu \ln x) + \imath \sin(\nu \ln x)),$$

$$x^{m_2} = x^{\mu} e^{-\imath\nu \ln x} = x^{\mu}(\cos(\nu \ln x) - \imath \sin(\nu \ln x)),$$

General solution is given by

$$y = x^{\mu}(c_1 \cos(\nu \ln x) + c_2 \sin(\nu \ln x)),$$

$c_1, c_2 \in \mathbb{R}$.

Examples

Solve:

1. $2x^2y'' + 3xy' - y = 0, \ x > 0.$
2. $x^2y'' + 5xy' + 4y = 0, \ x > 0.$
3. $x^2y'' + xy' + y = 0, \ x > 0.$

Solutions :

1. $y = c_1\sqrt{x} + c_2/x$
2. $y = x^{-2}(c_1 + c_2 \ \ln x).$
3. $y = c_1 \cos(\ln x) + c_2 \sin(\ln x).$

Non-homogeneous Second Order Linear ODE's

Consider the non-homogeneous DE

$$y'' + p(x)y' + q(x)y = r(x)$$

where $p(x), q(x), r(x)$ are continuous functions on an interval I.
The associated homogeneous DE is

$$y'' + p(x)y' + q(x)y = 0.$$

Can we relate the solutions of the above two DE's?

Non-homogeneous Second Order Linear ODE's

Theorem

Let $y_p(x)$ be any solution of

$$y'' + p(x)y' + q(x)y = r(x)$$

and $y_1(x), y_2(x)$ be a basis of the solution space of the corresponding homogeneous DE.
Then the set of solutions of the non-homogeneous DE is

$$\{c_1 y_1(x) + c_2 y_2(x) + y_p(x) \mid c_1, c_2 \in \mathbb{R}\}.$$

Proof: Let

$$L(y) = y'' + p(x)y' + q(x)y$$

and $\phi(x)$ be any solution of $L(y) = r(x)$. Then,

$$L(\phi(x) - y_p(x)) = L(\phi(x)) - L(y_p(x)) = r(x) - r(x) = 0.$$

Hence, $\phi(x) - y_p(x)$ is a solution of the homogeneous DE. Thus,

$$\phi(x) - y_p(x) = c_1 y_1(x) + c_2 y_2(x),$$

for $c_1, c_2 \in \mathbb{R}$. Hence,

$$\phi(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x).$$

Summary: In order to find the general solution of a non-homogeneous DE, we need to

- get the general solution of the corresponding homogeneous DE.
- get one particular solution of the non-homogeneous DE.