MA-111 Calculus II (D1 & D2)

Lecture 10

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Recap: Surfaces

Definition

Let E be a path connected subset in $\mathbb{R}^2$ with non-zero area. A parametrised surface is a continuous function $\mathbf{\Phi} : E \to \mathbb{R}^3$.

This definition is the analogue of what we called paths in one dimension and what are often called parametrized curves.

Examples:

- Graphs of real valued functions of two independent variables.
- A cylinder, A sphere, A cone.
- Surface of revolution.

Note that for a given $(u, v) \in E$, $\mathbf{\Phi}(u, v)$ can be written as

$$\mathbf{\Phi}(u, v) = (x(u, v), y(u, v), z(u, v)),$$

where x, y and z are scalar functions on E.

The parametrized surface $\mathbf{\Phi}$ is said to be a smooth parametrized surface if the functions x, y, z have continuous partial derivatives in a open subset of $\mathbb{R}^2$ containing E.

Tangent vectors for a parametrised surface

Let $\mathbf{\Phi}(u, v)$ be a smooth parametrised surface. If we fix the variable v, say $v = v_0$, we obtain a curve $\mathbf{c}(u, v_0)$ that lies on the surface. Thus

$$\mathbf{c}(u) = x(u, v_0)\mathbf{i} + y(u, v_0)\mathbf{j} + z(u, v_0)\mathbf{k}.$$

Since this curve is C^1 we can talk about its tangent vector at the point u_0. This is given by

$$\mathbf{c}'(u_0) = \frac{\partial x}{\partial u}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\mathbf{k}.$$

We can *define* the partial derivative of a vector valued function as

$$\mathbf{\Phi}_u(u_0, v_0) = \frac{\partial \mathbf{\Phi}}{\partial u}(u_0, v_0) := \mathbf{c}'(u_0).$$

Similarly, by fixing u and varying v we obtain a curve $\mathbf{l}(u_0, v)$ and we can set

$$\mathbf{\Phi}_v(u_0, v_0) = \frac{\partial \mathbf{\Phi}}{\partial v}(u_0, v_0) := \frac{\partial x}{\partial v}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\mathbf{k}.$$

The tangent plane

Let for any given point on the surface, $P_0 = (x_0, y_0, z_0) := \mathbf{\Phi}(u_0, v_0)$ for some $(u_0, v_0) \in D$.
The two tangent vectors $\mathbf{\Phi}_u(u_0, v_0)$ and $\mathbf{\Phi}_v(u_0, v_0)$ at P_0 define a plane.
We call this plane as the tangent plane to the surface at P_0.

The normal to this plane at P_0, $\mathbf{n}(u_0, v_0) = \mathbf{\Phi}_u(u_0, v_0) \times \mathbf{\Phi}_v(u_0, v_0)$.

Thus for a given point $(x_0, y_0, z_0) = \mathbf{\Phi}(u_0, v_0)$ in $\mathbb{R}^3$ the equation of the tangent plane is given by

$$\mathbf{n}(u_0, v_0) \cdot (x - x_0, y - y_0, z - z_0) = 0,$$

provided $\mathbf{n} \neq 0$.

In particular, if $\mathbf{\Phi}_u(u_0, v_0) \times \mathbf{\Phi}_v(u_0, v_0) = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$, then the equation of the tangent plane at P_0 is given by

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

Let us find the equation of the tangent plane at points on the various parametrised surfaces we have already looked at.

Example 1: Let D be a path-connected subset of $\mathbb{R}^2$ and $f : D \to \mathbb{R}$ be a C^1 function. The surface given by the graph of the function $z = f(x, y)$ is parametrized by $\mathbf{\Phi}(x, y) = (x, y, f(x, y))$. In this case, at $P_0 = \mathbf{\Phi}(x_0, y_0)$ for $(x_0, y_0) \in D$,

$$\mathbf{\Phi}_x(x_0, y_0) = \mathbf{i} + \frac{\partial f}{\partial x}(x_0, y_0)\mathbf{k} \quad \text{and} \quad \mathbf{\Phi}_y(x_0, y_0) = \mathbf{j} + \frac{\partial f}{\partial y}(x_0, y_0)\mathbf{k}.$$

Hence,

$$\mathbf{n}(x_0, y_0) = \mathbf{\Phi}_x(x_0, y_0) \times \mathbf{\Phi}_y(x_0, y_0) = \left(-\frac{\partial f}{\partial x}(x_0, y_0), -\frac{\partial f}{\partial y}(x_0, y_0), 1\right).$$

Thus the equation of the tangent plane is

$$(x - x_0, y - y_0, z - z_0) \cdot \left(-\frac{\partial f}{\partial x}(x_0, y_0), -\frac{\partial f}{\partial y}(x_0, y_0), 1\right) = 0;$$

which yields,

$$z - z_0 = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0).$$

Tangent Plane: Examples

Example 2: Let us consider a cylinder parametrized as

$$\mathbf{\Phi}(u, v) = (a\cos u, a\sin u, v), \quad \forall (u, v) \in [0, 2\pi] \times [0, h],$$

where $a > 0$. Then

$$\mathbf{\Phi}_u(u, v) \times \mathbf{\Phi}_v(u, v) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin u & a\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = (a\cos u, a\sin u, 0).$$

Since this is non-zero on $[0, 2\pi] \times [0, h]$ for any $h > 0$, we can define the tangent plane to $\mathbf{\Phi}$ at any point $P_0 = (x_0, y_0, z_0) = \mathbf{\Phi}(u_0, v_0)$ as

$$(a\cos u_0, a\sin u_0, 0).(x - x_0, y - y_0, z - z_0) = 0.$$

Now using $(x_0, y_0, z_0) = \mathbf{\Phi}(u_0, v_0) = (a\cos u_0, a\sin u_0, v_0)$, we get the equation for the tangent plane to $\mathbf{\Phi}$ at P_0 is

$$(\cos u_0)x + (\sin u_0)y = a.$$

Example 3: The sphere: $x^2 + y^2 + z^2 = a^2$, for some $a > 0$. Let us consider the parametrization

$$\mathbf{\Phi}(u, v) = (a \cos u \sin v, a \sin u \sin v, a \cos v), \quad \forall\,(u, v) \in [0, 2\pi] \times [0, \pi].$$

Check $\mathbf{\Phi}_u(u, v) \times \mathbf{\Phi}_u(u, v) = (a \sin v)\mathbf{\Phi}(u, v)$, for all $(u, v) \in [0, 2\pi] \times [0, \pi]$.

Note for $(u_0, v_0) \in [0, 2\pi] \times (0, \pi)$, $\mathbf{\Phi}_u(u_0, v_0) \times \mathbf{\Phi}_u(u_0, v_0) \neq (0, 0, 0)$ and the tangent plane at $P_0 = \mathbf{\Phi}(u_0, v_0)$ is

$$(\sin v_0 \cos u_0)x + (\sin v_0 \sin u_0)y + (\cos v_0)z = a.$$

Example 4: This was the example of the right circular cone. The parametric surface was given by

$$\mathbf{\Phi}(u, v) = (u\cos v, u\sin v, u), \quad (u, v) \in [0, \infty) \times [0, 2\pi].$$

In this case we get

$$\mathbf{\Phi}_u(u, v) = \cos v\mathbf{i} + \sin v\mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{\Phi}_v(u, v) = -u\sin v\mathbf{i} + u\cos v\mathbf{j},$$

where $\mathbf{n}(u, v) = \mathbf{\Phi}_u(u, v) \times \mathbf{\Phi}_v(u, v) = (-u\cos v, -u\sin v, u)$.

For any $(u_0, v_0) \in (0, \infty) \times [0, 2\pi]$, $\mathbf{n}(u_0, v_0) \neq (0, 0, 0)$ and the tangent plane check

$$(\cos v_0)x + (\sin v_0)y = z.$$

Note that if $(u, v) = (0, 0)$, then $\mathbf{n}(0, 0) = 0$, so the tangent plane is not defined at the origin. However, it is defined at any other point.

Non-singular surfaces

In analogy with the situation for curves, we will call $\mathbf{\Phi}$ a regular or non-singular parametrised surface if $\mathbf{\Phi}$ is C^1 and $\mathbf{\Phi}_u \times \mathbf{\Phi}_v \neq 0$ at all points.

As we just saw, the right circular cone is not a regular parametrised surface.

For a regular surface parametrized by $\mathbf{\Phi} : D \to \mathbb{R}^3$, the unit normal $\hat{\mathbf{n}}$ to the surface at any point $P_0 = \mathbf{\Phi}(u_0, v_0)$ is defined by

$$\hat{\mathbf{n}}(u_0, v_0) := \frac{\mathbf{\Phi}_u(u_0, v_0) \times \mathbf{\Phi}_v(u_0, v_0)}{\|\mathbf{\Phi}_u(u_0, v_0) \times \mathbf{\Phi}_v(u_0, v_0)\|}.$$

Surface Area

Let $\mathbf{\Phi} : E \to \mathbb{R}^3$ be a smooth parametrized surface, where E is a path-connected, bounded subset of $\mathbb{R}^2$ having a non-zero area. Also assume ∂E, the boundary of E, is of content zero.

Let $(u, v) \in E$. For $h, k \in \mathbb{R}$ with $|h|, |k|$ small, assuming $\mathbf{\Phi}$ is C^1 we can get the following approximations;

$$P := \mathbf{\Phi}(u, v), \quad P_1 := \mathbf{\Phi}(u + h, v) \approx \mathbf{\Phi}(u, v) + h\, \mathbf{\Phi}_u(u, v),$$

$$P_2 := \mathbf{\Phi}(u, v + k) \approx \mathbf{\Phi}(u, v) + k\, \mathbf{\Phi}_v(u, v), \quad Q := \mathbf{\Phi}(u + h, v + k).$$

Area of the parallelogram with sides PP_1 and PP_2

$$= \|(P_1 - P) \times (P_2 - P)\| \approx \|\mathbf{\Phi}_u(u, v) \times \mathbf{\Phi}_v(u, v)\|\, |h||k|.$$

In view of this approximation, we define

$$\text{Area}(\mathbf{\Phi}) := \iint_E \|(\mathbf{\Phi}_u \times \mathbf{\Phi}_v)(u,v)\|\, du\, dv.$$

Since the subset E of $\mathbb{R}^2$ is bounded with boundary ∂E which is of content zero and the function $\|\mathbf{\Phi}_u \times \mathbf{\Phi}_v\|$ is continuous on E, the integral in the definition of $\text{Area}(\mathbf{\Phi})$ is well-defined.
In analogy with the differential notation $ds = \|\gamma'(t)\| dt$, we introduce the following differential notation:

$$dS = \|\mathbf{\Phi}_u \times \mathbf{\Phi}_v\|\, dudv.$$

Thus $\text{Area}(\mathbf{\Phi}) := \iint_E dS$.

Examples
• Graph of a function: Given a subset E of $\mathbb{R}^2$ have an area, $f : E \to \mathbb{R}$ be a smooth function, and $\mathbf{\Phi}(u,v) = (u, v, f(u,v))$ for $(u,v) \in E$. Then

$$\begin{aligned} \text{Area}(\mathbf{\Phi}) &= \iint_E \|(-f_u, -f_v, 1)\| dudv \\ &= \iint_E \sqrt{1 + f_u^2 + f_v^2}\, dudv \end{aligned}$$

Example: Let $E := [0, 2\pi] \times [0, h]$, $\mathbf{\Phi}(\theta, z) := (a\cos\theta, a\sin\theta, z)$, and $\mathbf{\Psi}(\theta, z) := (a\cos 2\theta, a\sin 2\theta, z)$ for $(\theta, z) \in E$. Then

$$\begin{aligned} \text{Area}(\mathbf{\Phi}) &= \iint_E \|\mathbf{\Phi}_\theta \times \mathbf{\Phi}_z\| d\theta dz = \iint_E a\, d\theta dz = 2\pi a\, h, \\ \text{Area}(\mathbf{\Psi}) &= \iint_E \|\mathbf{\Psi}_\theta \times \mathbf{\Psi}_z\| d\theta dz = \iint_E 2a\, d\theta dz = 4\pi a\, h. \end{aligned}$$

We note that $\mathbf{\Psi}(E) = \mathbf{\Phi}(E)$, but $\text{Area}(\mathbf{\Psi}) = 2\,\text{Area}(\mathbf{\Phi})$.

Example: Let $E := [0, \pi] \times [0, 2\pi]$, and $\mathbf{\Phi}(\varphi, \theta) = (a\sin\varphi\cos\theta, a\sin\varphi\sin\theta, a\cos\varphi)$ for $(\varphi, \theta) \in E$. Then

$$\begin{aligned} \text{Area}(\mathbf{\Phi}) &= \iint_E \|\mathbf{\Phi}_\varphi \times \mathbf{\Phi}_\theta\| d\varphi d\theta = \iint_E a^2 \sin\varphi\, d\varphi d\theta \\ &= \int_0^{2\pi} \left(\int_0^\pi a^2 \sin\varphi\, d\varphi \right) d\theta = 4\pi a^2. \end{aligned}$$

Let C be a smooth curve in $\mathbb{R}^2 \times \{0\}$ given by $\boldsymbol{\gamma}(t) := (x(t), y(t))$, $t \in [\alpha, \beta]$. If C lies on or above the x-axis, and C is revolved about the x-axis, then it generates a surface parametrized by

$$\mathbf{\Phi}(t, \theta) := (x(t),\ y(t)\cos\theta,\ y(t)\sin\theta) \quad \text{for } (t, \theta) \in E,$$

where $E := [\alpha, \beta] \times [0, 2\pi]$. For all $(t, \theta) \in E$,

$$\begin{aligned}
(\mathbf{\Phi}_t \times \mathbf{\Phi}_\theta)(t, \theta) &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x'(t) & y'(t)\cos\theta & y'(t)\sin\theta \\ 0 & -y(t)\sin\theta & y(t)\cos\theta \end{vmatrix} \\
&= (y(t)y'(t), -x'(t)y(t)\cos\theta, -x'(t)y(t)\sin\theta).
\end{aligned}$$

By the Fubini theorem, we obtain

$$\begin{aligned}
\text{Area}(\mathbf{\Phi}) &= \iint_E \sqrt{y(t)^2 y'(t)^2 + x'(t)^2 y(t)^2}\, d(t, \theta) \\
&= 2\pi \int_\alpha^\beta y(t)\sqrt{x'(t)^2 + y'(t)^2}\, dt,
\end{aligned}$$

Note: $\mathbf{\Phi}$ is non-singular $\iff$ $\boldsymbol{\gamma}$ is non-singular and $y(t) \neq 0$ for $t \in [\alpha, \beta]$.

The area vector of an infinitesimal surface element

We see that $\mathbf{\Phi}$ takes the small rectangle R to the parallelogram given by the vectors $\mathbf{\Phi}_u \Delta u$ and $\mathbf{\Phi}_v \Delta v$.

It follows that the 'area vector' $\Delta \mathbf{S}$ of this parallelogram is

$$\Delta \mathbf{S} = (\mathbf{\Phi}_u \times \mathbf{\Phi}_v) \Delta u \Delta v.$$

Thus the surface 'area vector' is to be thought of as a vector pointing in the direction of the normal to the surface and in differential notation:

$$d\mathbf{S} = (\mathbf{\Phi}_u \times \mathbf{\Phi}_v)\, du\, dv.$$

The magnitude of the surface 'area vector' is given by

$$dS = \|d\mathbf{S}\| = \|\mathbf{\Phi}_u \times \mathbf{\Phi}_v\|\, du\, dv.$$

If the parametric surface $\mathbf{\Phi}$ is non-singular, we can write

$$d\mathbf{S} = \hat{\mathbf{n}} dS,$$

where $\hat{\mathbf{n}}$ is the unit vector normal to the surface.

The magnitude of the area vector

It remains to compute the magnitude dS. To do this we must find $\|\mathbf{\Phi}_u \times \mathbf{\Phi}_v\|$. Writing this out in terms of x, y and z, we see that

$$d\mathbf{S} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} dudv.$$

Hence,

$$dS = \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2} dudv,$$

where $\frac{\partial(y,z)}{\partial(u,v)}$, $\frac{\partial(x,z)}{\partial(u,v)}$, $\frac{\partial(x,y)}{\partial(u,v)}$ are the determinant of corresponding Jacobian matrix. For example

$$\frac{\partial(y,z)}{\partial(u,v)} = \frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v},$$

$$\frac{\partial(x,z)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial z}{\partial u}, \quad \frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u},$$

The surface area integral

Because of the calculations we have just made, the surface area is given by the double integral

$$\iint_S dS = \iint_E \sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2} dudv.$$

The area is nothing but the integral of the constant function 1 on the surface S. We integrate any bounded scalar function $f : S \to \mathbb{R}$:

$$\iint_S fdS = \iint_E f(x,y,z)\sqrt{\left[\frac{\partial(y,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,z)}{\partial(u,v)}\right]^2 + \left[\frac{\partial(x,y)}{\partial(u,v)}\right]^2} dudv,$$

provided the R.H.S double integral exists. If Σ is a union of parametrised surfaces S_i that intersect only along their boundary curves, then we can define

$$\iint_\Sigma fdS = \sum_i \iint_{S_i} fdS.$$

The surface integral of a vector field

Let $\mathbf{F}$ be a bounded vector field (on $\mathbb{R}^3$) such that the domain of $\mathbf{F}$ contains the non-singular parametrised surface $\mathbf{\Phi} : E \to \mathbb{R}^3$. Then the surface integral of $\mathbf{F}$ over S is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} := \iint_E \mathbf{F}(\mathbf{\Phi}(u, v)) \cdot (\mathbf{\Phi}_u \times \mathbf{\Phi}_v) du dv,$$

provided the R.H.S double integral exists. This can also be written more compactly as

$$\iint_S \mathbf{F} \cdot d\mathbf{S} := \iint_S \mathbf{F} \cdot \hat{\mathbf{n}} dS,$$

which is the surface integral of the scalar function given by the normal component of $\mathbf{F}$ over S.

Let a subset E of $\mathbb{R}^2$ have an area, and let $f : E \to \mathbb{R}$ be a smooth function. Let the smooth parametrized surface $\mathit{\Phi} : E \to \mathbb{R}^3$ represent the graph of f, and let $\boldsymbol{F} : \boldsymbol{\Phi}(E) \to \mathbb{R}^3$ be a continuous vector field. If $\boldsymbol{F} := (P, Q, R)$, then

$$\iint_{\boldsymbol{\Phi}} \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_E (-P\, f_x - Q\, f_y + R) d(x, y)$$

since $d\boldsymbol{S} = (\boldsymbol{\Phi}_x \times \boldsymbol{\Phi}_y) dxdy = (-f_x, -f_y, 1) dxdy$.

Examples

(i) Using above result, let $E := [0, 1] \times [0, 1]$, $f(x, y) := x + y + 1$ for $(x, y) \in E$. If $\boldsymbol{F}(x, y, z) := (x^2, y^2, z)$ for $(x, y, z) \in \mathbb{R}^3$, then

$$\begin{aligned} \iint_{\mathit{\Phi}} \boldsymbol{F} \cdot d\boldsymbol{S} &= \iint_E \big(- x^2 - y^2 + (x + y + 1)\big) d(x, y) \\ &= \int_0^1 \left(\int_0^1 (x + y + 1 - x^2 - y^2) dy \right) dx \\ &= = \frac{1}{2} + \frac{1}{2} + 1 - \frac{1}{3} - \frac{1}{3} = \frac{4}{3}. \end{aligned}$$

Examples Contd.

(ii) Let $E := [0, 2\pi] \times [0, h]$, and $\mathbf{\Phi}(u, v) := (a\cos u, a\sin u, v)$ for $(u, v) \in E$. If $\boldsymbol{F}(x, y, z) := (y, z, x)$ for $(x, y, z) \in \mathbb{R}^3$, then

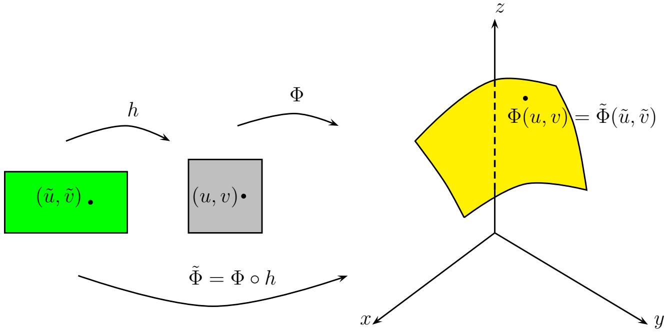
$$\iint_{\mathbf{\Phi}} \mathbf{F} \cdot d\boldsymbol{S} = \iint_{E} (a^2 \cos u \sin u + v\, a \sin u + 0) du dv = 0,$$

since $d\boldsymbol{S} = (\mathbf{\Phi}_u \times \mathbf{\Phi}_v) du dv = (a\cos u, a\sin u, 0) du dv$.

Reparametrization of a Surface

Let E be a path-connected subset of $\mathbb{R}^2$ having an area, and let $\mathbf{\Phi} : E \to \mathbb{R}^3$ be a smooth parametrized surface.
Let $\widetilde{E}$ be a path-connected subset of $\mathbb{R}^2$ having an area, and let $h : \widetilde{E} \to \mathbb{R}^2$ be a continuously differentiable and one-one function such that $h(\widetilde{E}) = E$ and its Jacobian $J(h)$ does not vanish on $\widetilde{E}$. Then the smooth surface $\tilde{\mathbf{\Phi}} := \mathbf{\Phi} \circ h$ is called a **reparametrization** of $\mathbf{\Phi}$. We have

$$\left(\tilde{\mathbf{\Phi}}_{\tilde{u}} \times \tilde{\mathbf{\Phi}}_{\tilde{v}}\right)(\tilde{u}, \tilde{v}) = \left(\mathbf{\Phi}_u \times \mathbf{\Phi}_v\right)(h(\tilde{u}, \tilde{v}))J(h)(h(\tilde{u}, \tilde{v})).$$



Examples: Let $E := (0,\pi) \times [-\pi,\pi]$, and define
$\mathbf{\Phi}(\varphi,\theta) := (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$ for $(\varphi,\theta) \in E$.

If $\widetilde{E} := [-\pi,\pi] \times (0,\pi)$, and we define
$\widetilde{\mathbf{\Phi}}(\theta,\varphi) := (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$ for $(\theta,\varphi) \in \widetilde{E}$,
then $\widetilde{\mathbf{\Phi}}$ is a reparametrization of $\mathbf{\Phi}$ since $\widetilde{\mathbf{\Phi}}(\theta,\varphi) = \mathbf{\Phi}(h(\theta,\varphi))$, where
$h : \widetilde{E} \to E$ is given by $h(\theta,\varphi) := (\varphi,\theta)$ with $J(h) = -1$.

Similarly, if $\widetilde{E} := (0,\pi/2) \times [-\pi/2,\pi/2]$, and we define
$\widetilde{\mathbf{\Phi}}(\varphi,\theta) := (\sin 2\varphi\cos 2\theta, \sin 2\varphi\sin 2\theta, \cos 2\varphi)$ for $(\varphi,\theta) \in \widetilde{E}$,
then $\widetilde{\mathbf{\Phi}}$ is a reparametrization of $\mathbf{\Phi}$ since $\widetilde{\mathit{\Phi}}(\varphi,\theta) = \mathit{\Phi}(h(\varphi,\theta))$, where
$h : \widetilde{E} \to E$ is given by $h(\varphi,\theta) := (2\varphi, 2\theta)$ with $J(h) = 4$.

THEOREM: The surface integral of a continuous scalar field over a smooth surface is invariant under reparametrization upto a sign. In particular, the area of a smooth surface is invariant under reparametrization.

Oriented surfaces

Intuitively, an *oriented* surface S is one that has two different sides, so that one side can be specified as the outside (or positive side) and the other side as the inside (or negative side).

Note Not all surfaces have two different sides, as we will see.

Recall that for us a vector field on a surface S is a vector field is a function $\mathbf{F} : U \to \mathbb{R}^3$ defined on a open set containing $S \subseteq U$. We say $\mathbf{F}$ is continuous (or C^1) if it is continuous on U.

Definition: A surface S is said to be orientable if there exists a continuous vector field $\mathbf{F} : S \to \mathbb{R}^3$ such that for each point P in S, $\mathbf{F}(P)$ is a unit vector normal to the surface S at P.

At each point of S there are two possible directions for the normal vector to S. The question is whether the normal vector field be can be chosen so that the resulting vector field is continuous.

Note This definition is independent of the choice of parametrization and only dependent on the geometric surface.

Examples of orientable surfaces

Example: For the unit sphere in $\mathbb{R}^3$ we can choose an orientation by selecting the unit vector $\hat{\mathbf{n}}(x, y, z) = \hat{\mathbf{r}}$, where $\mathbf{r}$ points outwards from the surface of the sphere.

More explicitly, we define

$$\mathbf{F}(x, y, z) = (x, y, z).$$

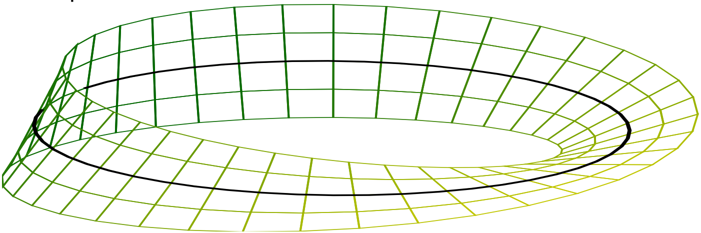
This obviously defines a continuous vector field on S. Hence, we see that the unit sphere in $\mathbb{R}^3$ is orientable.

Notice, that we can also define a vector field $\mathbf{G}(x, y, z) = -(x, y, z)$. The vector field $\mathbf{G} = -\mathbf{F}$ is also obviously continuous. There are two possible choices of orientation.

Non-orientable surfaces

Definition: A surface on which there exists no continuous vector field consisting of unit normal vectors is called non-orientable.

Exercise 1: Make a Möbius strip out of a piece of paper. Starting at the top draw a series of stick figures, head to toe, and label their left and right hands. When the stick figure comes back to the top (on the underside) compare the left and right hands of the two stick figures at the top.



Choosing an orientation

As we have just seen in the preceding example, if S is an orientable surface and $\mathbf{F}$ is a continuous vector field of unit normal vectors, so is $-\mathbf{F}$.

An orientable surface together with a specific choice of continuous vector field $\mathbf{F}$ of unit normal vectors is called an oriented surface. The choice of vector field is called an orientation.

Once one has chosen a particular vector field of normal vectors it makes sense to talk about the "outside" or "positive side" of the surface: usually, it is the side given by the direction of the unit normal vector. The other side is then called the "inside" or "negative side". However, which side one calls "positive" or "negative" is a matter of choice.

The orientation of parametrised surfaces

Let us suppose that we are given an oriented geometric surface S that is described as a $\mathcal{C}^1$ non-singular parametrised surface $\mathbf{\Phi}(u, v)$.

Notice that a oriented parametrised surface $\mathbf{\Phi}$ comes equipped with a natural vector field of unit normal vectors:

$$\hat{\mathbf{n}} = \frac{\mathbf{\Phi}_u \times \mathbf{\Phi}_v}{\|\mathbf{\Phi}_u \times \mathbf{\Phi}_v\|}.$$

Definition: If the unit normal vector $\hat{\mathbf{n}}$ agrees with the given orientation of S we say that the parametrisation $\mathbf{\Phi}$ is orientation preserving. Otherwise we say that $\mathbf{\Phi}$ is orientation reversing.
Note this gives a orientation only when we already know that the surface is oriented.

Examples

(i) Let $E \subset \mathbb{R}^2$ have an area, and $f : E \to \mathbb{R}$ be a smooth scalar field. Consider the graph $S := \{(x, y, f(x, y)) : (x, y) \in E\}$ of f.
For $P := (x, y, z) \in S$, define

$$\hat{\mathbf{n}}(P) := (-f_x(P), -f_y(P), 1)/\|(-f_x(P), -f_y(P), 1)\|.$$

This continuous assignment of upward unit normal vectors gives an orientation of S. Hence S is orientable.

Clearly, the parametrization of S given by $\mathbf{\Phi}(x, y) := (x, y, f(x, y))$ for $(x, y) \in E$, is orientation-preserving.

(ii) Let $S := \{(x, y, z) :\in \mathbb{R}^3 : x^2 + y^2 = a^2 \text{ and } 0 \leq z \leq h\}$. For $P := (x, y, z) \in S$, define $\hat{\mathbf{n}}\,(P) := (x/a, y/a, 0)$.
This continuous assignment of outward unit normal vectors gives an orientation of S. Hence the cylinder S is orientable.

Let $E := [0, 2\pi] \times [0, h]$ and $\mathbf{\Phi}(u, v) := (a \cos u, a \sin u, v)$ for $(u, v) \in E$.

Examples contd.

If $P := \mathbf{\Phi}(u,v) = (x,y,z) \in S$, then

$$\frac{\mathbf{\Phi}_u \times \mathbf{\Phi}_v}{\|\mathbf{\Phi}_u \times \mathbf{\Phi}_v\|}(u,v) = \frac{(a\cos u, a\sin u, 0)}{a} = \left(\frac{x}{a}, \frac{y}{a}, 0\right) = \hat{\mathbf{n}}(P).$$

Hence $\mathbf{\Phi}$ is an orientation-preserving parametrization.

(iii) Let $S := \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = a^2\}$. For $P := (x,y,z) \in S$, define $\hat{\mathbf{n}}(P) := (x/a, y/a, z/a)$.
This continuous assignment of outward unit normal vectors gives an orientation of the sphere S. Hence S is orientable.
Let $E := [0, 2\pi] \times (0, \pi)$ and
$\mathbf{\Phi}(u,v) := (a\cos u \sin v, , a\sin u \sin v, a\cos v)$ for $(u,v) \in E$.
If $P := \mathbf{\Phi}(u,v) = (x,y,z) \in S$, then

$$\frac{\mathbf{\Phi}_u \times \mathbf{\Phi}_v}{\|\mathbf{\Phi}_u \times \mathbf{\Phi}_v\|}(u,v) = -\frac{(a\sin v)\mathbf{\Phi}(u,v)}{a^2 \sin v} = -\left(\frac{x}{a}, \frac{y}{a}, \frac{z}{a}\right) = -\hat{\mathbf{n}}(P).$$

Hence $\mathbf{\Phi}$ is an orientation-reversing parametrization of $S \setminus \{(0,0,\pm a)\}$.

Independence of parametrisation

Let S be an oriented surface. Let $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ be two $\mathcal{C}^1$ non-singular parametrisations of S and let $\mathbf{F}$ be a continuous vector field on S.

- If $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2$ are orientation preserving, then

$$\iint_{\mathbf{\Phi}_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathbf{\Phi}_2} \mathbf{F} \cdot d\mathbf{S}.$$

- If $\mathbf{\Phi}_1$ is orientation preserving and $\mathbf{\Phi}_2$ is orientation reversing, then

$$\iint_{\mathbf{\Phi}_1} \mathbf{F} \cdot d\mathbf{S} = -\iint_{\mathbf{\Phi}_2} \mathbf{F} \cdot d\mathbf{S}.$$

For an oriented surface, the notation

$$\iint_{S} \mathbf{F} \cdot d\mathbf{S} = \iint_{S} \mathbf{F} \cdot \hat{\mathbf{n}} \, dS,$$

is unambiguous.