Outline

1. Dirac-Delta function: A different way to look at it.
2. Helmholtz Theorem: Finding the vector field using it's divergence and curl.

Objectives

1. To get a much broader view of the Dirac-Delta function.
2. To understand Helmholtz decomposition theorem.
3. To grasp the mathematical techniques used in verifying the Helmholtz theorem.

Recap

- Dirac Delta function:

$$\begin{aligned} \delta(x) &= 0 \quad \forall x \neq 0 \\ &\neq 0 \quad \text{if } x = 0 \\ \int_{-\varepsilon}^{+\varepsilon} \delta(x)dx &= 1 \quad \forall \varepsilon > 0 \end{aligned}$$

- Point charge:

$$\begin{aligned} \rho(\vec{r}) &= 0 \quad \text{if } r \neq 0 \\ &\neq 0 \quad \text{if } r = 0 \\ \int_V \rho(\vec{r})d\tau &= q \end{aligned}$$

- Divergence of Field produced by point source:

$$\nabla \cdot \vec{V} = 4\pi\delta^3(r)$$

Divergence of Field produced by Point Source

- $\vec{V} = \frac{\hat{r}}{r^2}$
- $\oint_S \vec{V} \cdot d\vec{A} = \int_V (\nabla \cdot \vec{V}) d\tau$
- The L.H.S when computed turns out to be 4π.
- $\nabla \cdot \vec{V}$ is zero $\forall r \neq 0$.
- Let $d\tau_0$, represent the infinitesimal volume at $r = 0$.

$$\begin{aligned}
\oint_S \vec{V} \cdot d\vec{A} &= \int_V (\nabla \cdot \vec{V}) d\tau \\
&= \sum_{i=0}^{\infty} (\nabla \cdot \vec{V})_{d\tau_i} d\tau_i \\
&= [(\nabla \cdot \vec{V})_{d\tau_0} d\tau_0] + \sum_{i=1}^{\infty} (\nabla \cdot \vec{V})_{d\tau_i} d\tau_i \\
&= [(\nabla \cdot \vec{V}) d\tau_0]_{r=0} + \int_{V(r\neq 0)} (\nabla \cdot \vec{V}) d\tau \\
4\pi &= [(\nabla \cdot \vec{V}) d\tau_0]_{r=0}
\end{aligned}$$

Divergence of Field produced by Point Source

- $d\tau_0$ is a infinitesimally small volume with $r = 0$, whose product with $(\nabla \cdot \vec{V})_{r=0}$ should equal 4π.

$$[(\nabla \cdot \vec{V})d\tau]_{r=0} = 4\pi$$

- This is possible only if $(\nabla \cdot \vec{V})_{r=0}$ is infinite/divergent.
- But there are different types of infinities. Not all infinities are equivalent.

$$P_1(x) = 2x + 3 \quad (1)$$

$$P_2(x) = 4x + 7 \quad (2)$$

$$P_3(x) = 3x^2 - x + 6 \quad (3)$$

$$P_4(x) = x^2 + 4x + 10 \quad (4)$$

- All the above polynomials $P_1(x), P_2(x), P_3(x), P_4(x)$ diverge to ∞ as $x \to \infty$.

Comparing Infinities

- All polynomials aren't equivalent at ∞. They diverge at different rates.

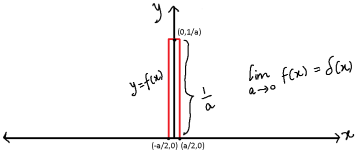
$$\begin{aligned}
\lim_{x\to\infty}\frac{P_1(x)}{P_2(x)} &= \frac{1}{2}\\
\lim_{x\to\infty}\frac{P_2(x)}{P_3(x)} &= 0\\
\lim_{x\to\infty}\frac{P_3(x)}{P_4(x)} &= 3
\end{aligned}$$

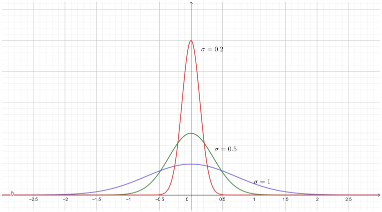
- Even though all the polynomials diverge at infinty, we can still define some sort of degree of divergence by comparing them to one another.
- Similarily, we want to associate a measure for $(\nabla\cdot\vec{V})_{r=0}$ in terms of the Dirac Delta function.

$$\begin{aligned}
\nabla\cdot\vec{V} &= 4\pi\delta^3(r)\\
&= 4\pi\delta(x)\delta(y)\delta(z)
\end{aligned}$$

Properties of Dirac-Delta Function

- $\int_b^c \delta(x-a)dx = 1 \quad \text{if } b < a < c$
- $\int_b^c f(x)\delta(x-a)dx = f(a) \quad \text{if } b < a < c$
- The Dirac Delta function is extracting the value of the function at a specific point in space. This is extremely helpful in solving differential equations.





Helmholtz Theorem

Theorem (Helmholtz Decomposition)

If divergence and curl of a vector field are given, then the vector field can be determined. If $\nabla \cdot \vec{F} = D(\vec{r})$ *and* $\nabla \times \vec{F} = \vec{C}(\vec{r})$, *then*

$$\vec{F}(\vec{r}) = -\nabla U(\vec{r}) + \nabla \times \vec{W}(\vec{r})$$

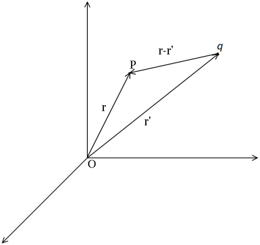
$$U(\vec{r}) = \frac{1}{4\pi} \int \frac{D(\vec{r}')}{|\vec{r} - \vec{r}'|} d\tau'$$

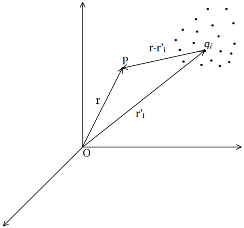
$$\vec{W}(\vec{r}) = \frac{1}{4\pi} \int \frac{\vec{C}(\vec{r}')}{|\vec{r} - \vec{r}'|} d\tau'$$

Electrostatic Potential

- What is the electrostatic potential due to a single point charge with charge q, placed at a point $\vec{r}'$ in space?

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{r}-\vec{r}'|}$$





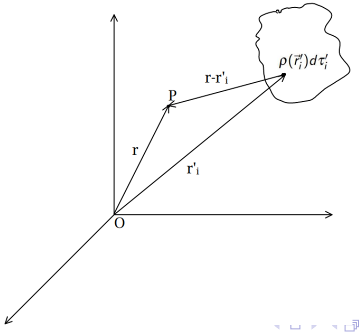
- What is the electrostatic potential due to several point charges with charge $q_1, q_2, q_3 \ldots.$,placed at points $\vec{r}_1', \vec{r}_2', \vec{r}_3' \ldots..$ respectively in space?

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{|\vec{r}-\vec{r}_i'|}$$

Electrostatic Potential

- What is the electrostatic potential due to a continuous charge distribution in 3D, with charge density $\rho(\vec{r})$?

$$
\begin{aligned}
V(\vec{r}) &= \frac{1}{4\pi\varepsilon_0}\sum_i \frac{\rho(\vec{r}_i')d\tau_i'}{|\vec{r}-\vec{r}_i'|} \\
&= \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}d\tau'
\end{aligned}
$$



Similarity between Helmholtz and Electrostatic Potential

- Notice the similarity between the $V(\vec{r})$ and $U(\vec{r}), \vec{W}(\vec{r})$.

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}_i'|}d\tau'$$

- $\rho(\vec{r})$ is the source for electrostatic potential $V(\vec{r})$.

$$U(\vec{r}) = \frac{1}{4\pi}\int \frac{D(\vec{r}')}{|\vec{r}-\vec{r}_i'|}d\tau'$$

- Thus, divergence of $\vec{F}(\vec{r})$, i.e., $D(\vec{r})$ ($\nabla\cdot\vec{F} = D(\vec{r})$) is the source for $U(\vec{r})$. And

$$\vec{W}(\vec{r}) = \frac{1}{4\pi}\int \frac{\vec{C}(\vec{r}')}{|\vec{r}-\vec{r}_i'|}d\tau'$$

- similarly, curl of $\vec{F}(\vec{r})$, i.e., $\vec{C}(\vec{r})$ ($\nabla\times\vec{F} = \vec{C}(\vec{r})$) is the source for $\vec{W}(\vec{r})$.

Verifying Helmholtz Theorem

- Proof of Helmholtz theorem is tedious and unnecessarily complicated. However, we can check if the proposition made in Helmholtz theorem is correct.
- The solution should satisfy the following two equations, $\nabla \cdot \vec{F} = D(\vec{r})$ and $\nabla \times \vec{F} = \vec{C}(\vec{r})$

$$
\begin{aligned}
\nabla \cdot \vec{F} &= D(\vec{r}) \\
\nabla \cdot \vec{F} &= \nabla \cdot \left(-\nabla U(\vec{r}) + \nabla \times \vec{W}(\vec{r})\right) \\
&= -\nabla^2 U(\vec{r}) + \nabla \cdot (\nabla \times \vec{W}(\vec{r})) \\
&= -\nabla^2_{(r)} \left(\frac{1}{4\pi} \int \frac{D(\vec{r}')}{|\vec{r} - \vec{r}'|} d\tau'\right) \\
&= \frac{-1}{4\pi} \int D(\vec{r}') \nabla^2_{(r)} \left(\frac{1}{|\vec{r} - \vec{r}'|}\right) d\tau'
\end{aligned}
$$

- Above we used the fact that divergence of a curl is zero, i.e., $\nabla \cdot (\nabla \times \vec{W}(\vec{r})) = 0$

Verifying Helmholtz Theorem

$$
\begin{aligned}
\nabla\cdot\vec{F} &= \frac{-1}{4\pi}\int D(\vec{r}\,')\nabla_{(r)}\cdot\left[\nabla_{(r)}\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right)\right]d\tau' \\
&= \frac{-1}{4\pi}\int D(\vec{r}\,')\nabla_{(r)}\cdot\left[\frac{-\hat{R}}{R^2}\right]d\tau' \qquad (\text{ here } \vec{R}=\vec{r}-\vec{r}\,') \\
&= \frac{-1}{4\pi}\int D(\vec{r}\,')(-4\pi\delta^3(\vec{R})d\tau' \\
&= \int D(\vec{r}\,')\delta^3(\vec{r}-\vec{r}\,')d\tau' = D(\vec{r})
\end{aligned}
$$

- The divergence of the solution stated in Helmholtz theorem has come out to be correct.

Verifying Helmholtz Theorem

- The curl of the solution stated in the Helmholtz theorem should come out to be $\vec{C}(\vec{r})$.

$$\begin{aligned}
\nabla \times \vec{F} &= \vec{C}(\vec{r}) \\
\nabla \times \vec{F} &= \nabla \times (\nabla \times \vec{W}) \\
&= -\nabla^2 \vec{W} + \nabla(\nabla \cdot \vec{W})
\end{aligned}$$

- One can guess that $-\nabla^2 \vec{W} = \vec{C}$, after seeing $-\nabla^2 U = D$. Following the same procedure

$$\begin{aligned}
-\nabla^2 \vec{W} &= -\nabla^2_{(r)} \left(\frac{1}{4\pi} \int \frac{\vec{C}(\vec{r}')}{|\vec{r} - \vec{r}'|} d\tau' \right) \\
&= \frac{-1}{4\pi} \int \vec{C}(\vec{r}') \nabla^2_{(r)} \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) d\tau' \\
&= \frac{-1}{4\pi} \int \vec{C}(\vec{r}')(-4\pi \delta^3(\vec{r} - \vec{r}')) d\tau' \\
&= \vec{C}(\vec{r})
\end{aligned}$$

Verifying Helmholtz Theorem

- This means that $\nabla(\nabla \cdot \vec{W})$ should be zero.
- Let's first evaluate $\nabla \cdot \vec{W}$, and then we can compute the gradient of it.

$$\nabla \cdot \vec{W} \quad = \quad \nabla \cdot \left[\frac{1}{4\pi} \int \frac{\vec{C}(\vec{r}')}{|\vec{r} - \vec{r}_i'|} d\tau' \right]$$

- $\nabla_{(r)} \cdot \left(\frac{\vec{C}(\vec{r}')}{|\vec{r}-\vec{r}_i'|} \right) = \vec{C}(\vec{r}') \cdot \nabla_{(r)} \left(\frac{1}{|\vec{r}-\vec{r}'|} \right) + \frac{1}{|\vec{r}-\vec{r}'|} (\nabla_{(r)} \cdot \vec{C}(\vec{r}'))$
- $\nabla_{(r)} \cdot \vec{C}(\vec{r}') = 0$ as $\vec{C}$ is independent of $\vec{r}$, leading to

$$\nabla \cdot \vec{W} \quad = \quad \frac{1}{4\pi} \int \vec{C}(\vec{r}') \cdot \left[\nabla_{(r)} \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) \right] d\tau'$$

Verifying Helmholtz Theorem

- Now, $\nabla_{(r)}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = -\nabla_{(r')}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$, so that

$$\nabla \cdot \vec{W} = -\frac{1}{4\pi}\int \vec{C}(\vec{r}') \cdot \left[\nabla_{(r')}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)\right] d\tau'$$

  Because,
  $\nabla_{(r')} \cdot \left[\frac{\vec{C}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right] = \vec{C}(\vec{r}') \cdot \left[\nabla_{(r')}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)\right] + \frac{1}{|\vec{r}-\vec{r}'|}\left(\nabla_{(r')} \cdot \vec{C}(\vec{r}')\right)$,
  we have

$$\vec{C}(\vec{r}') \cdot \left[\nabla_{(r')}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)\right] = \nabla_{(r')} \cdot \left[\frac{\vec{C}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right]$$

- Because, $\nabla_{(r')} \cdot \vec{C}(\vec{r}') = 0$ as divergence of a curl is zero. Thus,

$$\nabla \cdot \vec{W} = -\frac{1}{4\pi}\int \nabla_{(r')} \cdot \left[\frac{\vec{C}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right] d\tau'$$

Verifying Helmholtz Theorem

- Using Gauss's theorem $\left\{\int_V (\nabla \cdot \vec{P}) d\tau = \oint_S \vec{P} \cdot d\vec{S}\right\}$ above, we obtain

$$\nabla \cdot \vec{W} = -\frac{1}{4\pi} \oint_S \frac{\vec{C}(\vec{r}')}{|\vec{r} - \vec{r}'|} \cdot d\vec{S}'$$

- As the volume integral is all over space, S will be a closed surface at infinity.

Verifying Helmholtz

- If $\vec{C}(\vec{r})$ vanishes faster than $1/r$, as $r \to \infty$, then the integral will be zero.

$$
\begin{aligned}
\nabla \cdot \vec{W} &= -\lim_{r' \to \infty} \frac{1}{4\pi} \oint_S \frac{\vec{C}(\vec{r}')}{|\vec{r} - \vec{r}'|} \cdot (r'^2 \sin\theta' d\theta' d\phi' \hat{r}' + r' \sin\theta' dr' d\phi' \hat{\theta} \\
&\quad + r' dr' d\theta' \hat{\phi}) \\
\nabla \cdot \vec{W} &= -\lim_{r' \to \infty} \frac{1}{4\pi} \oint_S \frac{C_r(\vec{r}')}{|\vec{r} - \vec{r}'|} r'^2 \sin\theta' d\theta' d\phi' \\
&\quad -\lim_{r' \to \infty} \frac{1}{4\pi} \oint_S \frac{C_\theta(\vec{r}')}{|\vec{r} - \vec{r}'|} r' \sin\theta' dr' d\phi' \\
&\quad -\lim_{r' \to \infty} \frac{1}{4\pi} \oint_S \frac{C_\phi(\vec{r}')}{|\vec{r} - \vec{r}'|} r' dr' d\theta'
\end{aligned}
$$

Verifying Helmholtz

- As $r' \to \infty$, $|\vec{r} - \vec{r}'| \sim |\vec{r}'|$.

$$\begin{aligned}
\nabla \cdot \vec{W} &= -\frac{1}{4\pi} \oint_S \lim_{r' \to \infty} (r' C_r(\vec{r}')) sin\theta' d\theta' d\phi' \\
&\quad -\frac{1}{4\pi} \lim_{r' \to \infty} \oint_S C_\theta(\vec{r}') sin\theta' dr' d\phi' \\
&\quad -\frac{1}{4\pi} \lim_{r' \to \infty} \oint_S C_\phi(\vec{r}') dr' d\theta'
\end{aligned}$$

- Above three integrals become zero $\forall \vec{C}(\vec{r})$ converging to 0 faster than $1/r$, as $r \to \infty$.
- If $\nabla \cdot \vec{W}$ is zero, then $\nabla(\nabla \cdot \vec{W})$ is zero. Therefore, the curl of the solution stated in Helmholtz theorem is also correct.

Verifying Helmholtz

- But is it guaranteed that $\vec{C}(\vec{r})$ converges to 0 faster than $1/r$, as $r \to \infty$?
- The very existence of functions $U(\vec{r}), \vec{W}(\vec{r})$ enforces convergence faster than $1/r^2$ on $D(\vec{r}), \vec{C}(\vec{r})$, as $r \to \infty$.

$$
\begin{aligned}
U &\sim \int_0^{2\pi} d\phi' \int_0^{\pi} sin\theta' d\theta' \int_0^{\infty} \frac{D(\vec{r}')}{r'} r'^2 dr' \\
&\sim 4\pi \int_0^{\infty} [r' D(\vec{r}')] dr'
\end{aligned}
$$

- For this integral to be finite, $D(\vec{r})$ must converge to 0 faster than $1/r^2$, as $r \to \infty$.
- One can use to the same argument to conclude that $\vec{W}(\vec{r})$ is finite only if $\vec{C}(r)$ converges to 0 faster than $1/r^2$, as $r \to \infty$.

Loophole in Helmholtz?

- If there is a function $\vec{G}(\vec{r})$, whose divergence and curl are zero, then $\vec{F}(\vec{r}) = -\nabla U(\vec{r}) + \nabla \times \vec{W}(\vec{r}) + \vec{G}(\vec{r})$ also gives $\nabla \cdot \vec{F} = D(\vec{r}), \nabla \times \vec{F} = \vec{C}(\vec{r})$.
- Typically, all the fields we deal with, either electric or magnetic, will vanish as $r \to \infty$, therefore we require $\vec{F}(\vec{r})$ to vanish as $r \to \infty$.
- This boundary condition at $r \to \infty$ allows us to uniquely determine $\vec{F}(\vec{r})$, as there is no such $G(\vec{r})$, that has zero curl and zero divergence everywhere and goes to zero as $r \to \infty$.
- Therefore, if we know the curl and divergence of a vector field along with the boundary conditions, the vector field can be uniquely determined.