MA-111 Calculus II (D1 & D2)

Lecture 7

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Recap

- Suppose the vector field $\mathbf{F}$ is a continuous conservative field, i.e., $\mathbf{F} = \nabla f$, for some C^1 scalar function f. Then for any smooth path $\mathbf{c}$, we have
$$\int_{\mathbf{c}} \mathbf{F}.\mathbf{ds} = f(\mathbf{c}(b)) - f(\mathbf{c}(a)).$$
- This shows that the value of the line integral of a conservative field depends only on the value of the function at the end points of the curve, not on the curve itself.

Definition

The line integral of a vector field $\mathbf{F}$ is independent of path in a region D if for any $\mathbf{c}_1$ and $\mathbf{c}_2$ paths in D with the same initial and terminal points,

$$\int_{\mathbf{c}_1} \mathbf{F}.\mathbf{ds} = \int_{\mathbf{c}_2} \mathbf{F}.\mathbf{ds}.$$

Equivalently, the line integral of $\mathbf{F}$ is independent of path in D if for any closed curve $\mathbf{c}$ (why?)

$$\int_{\mathbf{c}} \mathbf{F}.\mathbf{ds} = 0.$$

Examples

Example Find the work done by the gravitational field $\mathbf{F}(x, y, z) = -\frac{mMG}{|\mathbf{r}(x,y,z)|^3}\mathbf{r}(x, y.z)$, in moving a particle with mass m and position vector $\mathbf{r}(x, y, z) = (x, y, z)$ from $(3, 4, 12)$ to the point $(2, 2, 0)$ along a piecewise-smooth curve C.

Ans Since the gravitational field is a conservative field and $\mathbf{F}(x, y, z) = \nabla f(x, y, z)$, where

$$f(x, y, z) = \frac{mMG}{|\mathbf{r}(x, y, z)|} = \frac{mMG}{\sqrt{x^2 + y^2 + z^2}}.$$

Using the Fundamental theorem for line integrals, the work done is

$$W = \int_{\mathbf{c}} \mathbf{F}.\mathbf{ds} = f(\mathbf{c}(b) - f(\mathbf{c}(a)) = f(2, 2, 0) - f(3, 4, 12) = mMG(\frac{1}{2\sqrt{2}} - \frac{1}{13}),$$

where $\mathbf{c} : [a, b] \to \mathbb{R}$, a parametrization of curve C with $\mathbf{c}(a) = (3, 4, 12)$ and $\mathbf{c}(b) = (2, 2, 0)$.

Example Evaluate $\int_C y^2 dx + x dy$, where

1. $C = C_1$ is the line segment from $(-5, -3)$ to $(0, 2)$,
2. $C = C_2$ is the part of parabola $x = 4 - y^2$ from $(-5, -3)$ to $(0, 2)$.

Are the line integrals along C_1 and C_2 same?

Ans 1.) Consider parametrization for C_1,
$\mathbf{c}_1(t) = (5t - 5, 5t - 3), \quad t \in [0, 1]$. Thus $\mathbf{c}_1'(t) = (5, 5)$ for all $t \in [0, 1]$. So, $\mathbf{F}(\mathbf{c}_1(t)) = ((5t - 3)^2, 5t - 5)$ and

$$\int_{C_1} y^2 dx + x dy = \int_0^1 \left[(5t - 3)^2 .5 + (5t - 5).5\right] dt = -\frac{5}{6}.$$

2. Consider parametrization for C_2, $\mathbf{c}_2(t) = (4 - t^2, t), \quad t \in [-3, 2]$. Thus $\mathbf{c}_2'(t) = (-2t, 1)$ for all $t \in [-3, 2]$. So, $\mathbf{F}(\mathbf{c}_2(t)) = (t^2, 4 - t^2)$ and

$$\int_{C_2} y^2 dx + x dy = \int_{-3}^2 \left[t^2(-2t) + (4 - t^2)\right] dt = 40\frac{5}{6}.$$

Line integrals along C_1 and C_2 are Not same! Though the endpoints of C_1 and C_2 are same!

Conservative vector fields

• In general, the line integral of a vector field depends on the path.

• Fundemental theorem of calculus for line integrals yields that the line integral of a conservative field is independent of path in D.

• The converse to our previous assertion holds for *path connected* D.

Definition: A subset D of $\mathbb{R}^n$ is called connected if it cannot be written as a disjoint union of two non-empty subsets $D_1 \cup D_2$, with $D_1 = D \cap U_1$ and $D_2 = D \cap U_2$, where U_1 and U_2 are open sets.

Definition: A subset of D of $\mathbb{R}^n$ is said to be path connected if any two points in the subset can be joined by a path (that is the image of a continuous curve) inside D.

• In general, path connected sets are connected.

• In $\mathbb{R}^n$ we can show that an open subset is connected if and only if it is path connected.

Examples

Example. $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < a^2\}$ is path-connected.

Ans. If $P = (x_0, y_0)$ and $Q = (x_1, y_1)$ are in D, then which path lying in D can be defined connecting P and Q?

Example. $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\} \cup \{(2, 2)\}$ is connected in $\mathbb{R}^2$?
Ans No. (Why?)

Example. $D = \{(x, \sin(\frac{1}{x}) \in \mathbb{R}^2 \mid x \in (0, 1]\} \cup \{(0, 0)\}$ is connected in $\mathbb{R}^2$ but not path-connected.

Theorem: Let $\mathbf{F} : D \to \mathbb{R}^3$ be a continuous vector field on a connected open region D in $\mathbb{R}^3$. If the line integral of $\mathbf{F}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field in D.

Proof: Let the line integral of $\mathbf{F}$ be path-independent in D, where D is an open, connected set of $\mathbb{R}^n$, for $n = 3$.
Goal: Find a differentiable function $V : D \to \mathbb{R}$ such that

$$\mathbf{F}(x, y, z) = \nabla V(x, y, z), \quad \text{for all} \quad (x, y, z) \in D.$$

We construct such V in the following way.
Step 1 Let $P_0 = (x_0, y_0, z_0)$ be a fixed point in D. Let $P = (x, y, z)$ be an arbitrary point in D. We define

$$V(x, y, z) = \int_{\mathbf{c}_P} \mathbf{F} \cdot d\mathbf{s}, \quad \text{for all} \quad (x, y, z) \in D,$$

where $\mathbf{c}_P : [a, b] \to D$ is any path from P_0 to P.
Since D is path connected, there always exists a path from P_0 to any point $P \in D$. Hence V is defined on the whole of D.

Since the line integral of $\mathbf{F}$ is path-independent in D, $V(x, y, z)$ does not depend on which path we took from P_0 to P and hence is well-defined.

The proof of theorem contd.

Step 2 It remains to show that $\mathbf{F} = \nabla V$.

Let $\mathbf{F} = (F_1, F_2, F_3)$. Then we have to show

$$\frac{\partial V}{\partial x} = F_1, \quad \frac{\partial V}{\partial y} = F_2, \quad \frac{\partial V}{\partial z} = F_3, \quad \text{on} \quad D.$$

Evaluate $\frac{\partial V}{\partial x}(x, y, z) = \lim_{h \to 0} \frac{V(x+h,y,z) - V(x,y,z)}{h}$ for all $(x, y, z) \in D$.

From definition of V,

$$V(x + h, y, z) = \int_{\mathbf{c}_{P_h}} \mathbf{F}.\mathbf{ds},$$

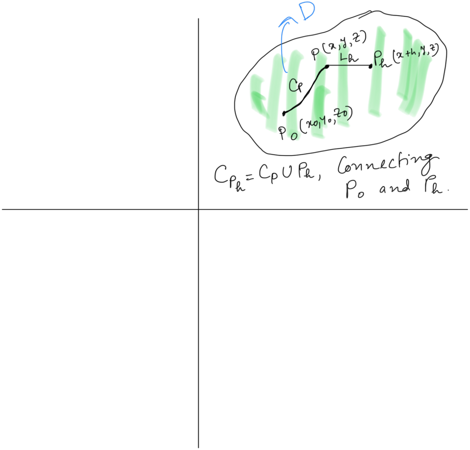
where $P_h = (x + h, y, z)$ and $\mathbf{c}_{P_h}$ is any path joining P_0 and P_h in D.

Choose $\mathbf{c}_{Ph}$ conveniently: Since D is open, for a given $P = (x, y, z) \in D$, there exists a disk contained in D with center P containing points $P_h = (x + h, y, z)$ for all h such that h is small enough. Thus for all h with $|h|$ suitable small, the straight line $\mathbf{L}_h$ joining P and P_h lies in D, where

$$\mathbf{L}_h(t) = (x + th, y, z) \quad \forall\, 0 \le t \le 1.$$

The proof of theorem contd.

We choose the path $\mathbf{c}_{P_h}$ from P_0 to P_h as the union of the two paths $\mathbf{c}_P$ from P_0 to P and the straith line $\mathbf{L}_h$ from P to P_h.



The proof of theorem contd.

From the property of line integrals we mentioned earlier

$$\int_{\mathbf{c}_{P_h}} \mathbf{F} \cdot \mathbf{d}s = \int_{\mathbf{c}_P} \mathbf{F} \cdot d\mathbf{s} + \int_{\mathbf{L}_h} \mathbf{F} \cdot \mathbf{d}s.$$

Hence it yields

$$V(x+h,y,z) = V(x,y,z)$$

$$+ \int_0^1 (F_1(x+th,y,z), F_2(x+th,y,z), F_3(x+th,y,z)) \cdot (h,0,0)dt$$

Thus

$$\frac{\partial V}{\partial x}(x,y,z) = \lim_{h\to 0} \frac{V(x+h,y,z) - V(x,y,z)}{h} = \lim_{h\to 0} \int_0^1 F_1(x+th,y,z)dt.$$

Due to the continuity of F_1,

$$\lim_{h\to 0} \int_0^1 F_1(x+th,y,z)dt = F_1(x,y,z).$$

The proof of theorem contd.

Hence we get

$$\frac{\partial V}{\partial x}(x, y, z) = F_1(x, y, z), \quad \forall\,(x, y, z) \in D.$$

We can similarly show that

$$\frac{\partial V}{\partial y} = F_2 \quad \text{and} \quad \frac{\partial V}{\partial z} = F_3, \quad \text{on} \quad D.$$

This proves our theorem.

In summary, for a given continuous vector field $\mathbf{F}$ in $\mathbb{R}^n$ defined on D, an open, path connected subset of $\mathbb{R}^n$, the vector field $\mathbf{F}$ is a conservative field if and only if the line integral of $\mathbf{F}$ in D is independent of path in D.

Examples Contd.

Example Determine whether or not the vector field

$$\mathbf{F}(x, y) = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}, \quad \text{in} \quad \mathbb{R}^2 \backslash \{(0, 0)\},$$

is conservative.
Ans Check for the closed curve $\mathbf{c} = (\cos t, \sin t)$, $t \in [0, 2\pi]$, the line integral of $\mathbf{F}$ along $\mathbf{c}$?

$$\int_{\mathbf{c}} \mathbf{F}.\mathbf{ds} = \int_0^{2\pi} (\sin t)(\sin t) + (\cos t)(\cos t)\, dt = 2\pi.$$

so, $\int_{\mathbf{c}} \mathbf{F}.\mathbf{ds} \neq 0$, though $\mathbf{c}$ is a closed curve, and hence $\mathbf{F}$ cannot be conservative field.

However, the equivalent formulation of conservative field and the path independency of the line integral of the vector field may not be always useful to determine if a vector field conservative.

Necessary condition for conservative fields

Theorem

- *For $n = 2$, if $\mathbf{F}(x, y) = F_1(x, y)\mathbf{i} + F_2(x, y)\mathbf{j}$ is a conservative vector field, where F_1 and F_2 have continuous first-order partial derivatives on an open region D in $\mathbb{R}^2$, then*

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \text{on} \quad D.$$

- *For $n = 3$, if $\mathbf{F}(x, y, z) = F_1(x, y, z)\mathbf{i} + F_2(x, y, z)\mathbf{j} + F_3(x, y, z)\mathbf{k}$ is a conservative vector field, where F_1, F_2, F_3 have continuous first-order partial derivatives on an open region D in $\mathbb{R}^3$, then*

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y} \quad \text{on} \quad D.$$

The theorem follows from a direct calculation using the fact that $\mathbf{F} = \nabla V$ and using the properties of the mixed partial derivatives of V.

Example Determine whether or not the vector field

$$\mathbf{F}(x, y) = (x - y)\mathbf{i} + (x - 2)\mathbf{j}, \quad \text{in} \quad \mathbb{R}^2$$

is conservative.
Ans Here $F_1(x, y) = x - y$ and $F_2(x, y) = x - 2$. Then

$$\frac{\partial F_1}{\partial y} = -1, \quad \text{and} \quad \frac{\partial F_2}{\partial x} = 1.$$

So by previous theorem, **F** cannot be a conservative field.

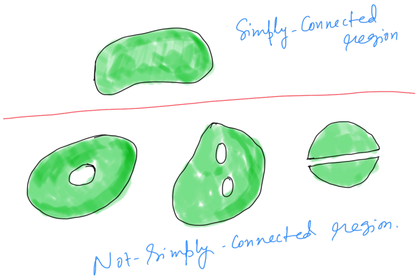
What about the converse of the theorem?
The converse is partially true under some additional hypothesis on D.
However, it is often a convenient method verifying if a vector field is conservative.

Simply connected domain

Definition

A subset D of $\mathbb{R}^n$ for $n = 2, 3$, is simply connected, if D is a connected region such that any simple closed curve lying in D encloses a region that is in D.

Basically, a simply-connected region contains no hole and cannot consist of two separate pieces.



Sufficient condition for conservative field

Theorem

Let $n = 2, 3$ and let D be an open, simply connected region in $\mathbb{R}^n$.

1. *For $n = 2$, if $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ is such that F_1 and F_2 have continuous first order partial derivatives on D satisfying*

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \text{on} \quad D,$$

   Then $\mathbf{F}$ is a conservative field.

2. *For $n = 3$, if $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is such that F_1, F_2 and F_3 have continuous first order partial derivatives on D satisfying*

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y} \quad \text{on} \quad D,$$

   Then $\mathbf{F}$ is a conservative field.

We postpone the proof of the theorem for later as it can be derived using Green's theorem.

Examples

Example. Determine whether or not the vector field

$$\mathbf{F}(x, y) = (3 + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}, \quad \text{in} \quad \mathbb{R}^2$$

is conservative.

Ans Note that the region $\mathbb{R}^2$ is open and simply-connected and $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ is continuously differentiable.
Let $F_1(x, y) = (3 + 2xy)$ and $F_2(x, y) = x^2 - 3y^2$. Then

$$\frac{\partial F_1}{\partial y}(x, y) = 2x = \frac{\partial F_2}{\partial x}.$$

Thus using the previous theorem, we conclude that $\mathbf{F}$ is a conservative field.
How to find a potential function f such that $\mathbf{F} = \nabla f$, for above example?

Example contd.

Let $\mathbf{F} = \nabla f$, then $\frac{\partial f}{\partial x}(x,y) = F_1(x,y)$ and $\frac{\partial f}{\partial y}(x,y) = F_2(x,y)$.
Step 1 Fixing y, solve the ODE with respect to x-variable:

$$\frac{\partial f}{\partial x}(x,y) = F_1(x,y).$$

Integrating with respect to x in both side, we get

$$f(x,y) = \int_0^x F_1(s,y)\,dx + c(y) = 3x + x^2 y + c(y).$$

Step 2 Determine the $c(y)$ using $\frac{\partial f}{\partial y}(x,y) = F_2(x,y)$. Differentiating $f(x,y)$ with respect to y,

$$\frac{\partial f}{\partial y}(x,y) = x^2 + c'(y),$$

and it has to be equal to $F_2(x,y)$.
so, $x^2 + c'(y) = x^2 - 3y^2$ and thus $c'(y) = -3y^2$. Now solving this ODE with respect to y variable:

$$c(y) = -y^3 + K,$$

for some constant K.
Thus $f(x,y) = 3x + x^2 y - y^3 + K$ such that $\mathbf{F} = \nabla f$.

In summary, for a given vector field $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$ for $n = 2, 3$:

1. If $\mathbf{F}$ is a continuous, conservative vector field, i.e., $\mathbf{F} = \nabla f$, for some C^1 scalar function, then the line integral of $\mathbf{F}$ along any path C from P to Q in D given by

$$\int_C \mathbf{F}.\mathbf{ds} = f(Q) - f(P),$$

   and it only depends on the value of f, the potential function, at the initial and terminal points of the path.

2. Let $\mathbf{F}$ be a continuous field and let D be an open connected set in $\mathbb{R}^n$. Then $\mathbf{F}$ is a conservative field if and only if the line integral of $\mathbf{F}$ is path-independent in D.
3. If $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ is a C^1 conservative vector field on an open region D, then $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$ on D. Similar result holds in $\mathbb{R}^3$.
4. Let D be an open, simply connected region in $\mathbb{R}^2$ and let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ be C^1 on D. Then $\mathbf{F}$ is conservative in D if and only if $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$ on D. Similar result holds in $\mathbb{R}^3$.