Lecture 3

X ray tomography (toy model)

- $E = h\nu$ so energy ↑ as frequency (so visible light safer than
- Imp concepts are penetration depth, scattering, absorption
- X rays don't scatter easily.
- Make a toy model using visible light.
  medium: water: little scattering (you can see beam and beam is a straight line); ~~some~~ little absorption
  ink: little scattering (you can see beam and beam is a straight line); more absorption
  milk: more scattering (beam deviates); more absorption.
- Take sample, let beam interact, detect change.
- Use feedback to reduce difference b/w guess and measurement.
- Intensity is $f(x)$ as the beam is a straight line along x axis.
- If we considered scattering it would be $f(x,y,z)$

Computed tomography (CT scan)

- Assumptions: 1 wavelength
  Straight line path (no refraction / diffraction) (everything $f(x)$)
  Follows Beer's law.
- Beer's law: • The change in intensity $dI = I(x+dx) - I(x) \propto dx$
  - Attenuation cumulative, i.e. dI is more if starting value of I is more, for the same dx.
    i.e. $dI \propto I$ too.
  - Proportionality const $\mu(x) \rightarrow$ absorption coeff (How ~~much do~~ fast decay?)
  - $dI = \underset{\text{-ve as decay.}}{-\mu(x) I(x) dx}$

$$\frac{dI}{I} = -\mu(x)dx \xRightarrow{\text{Integ}} \log\left[\frac{I(x)}{I_0}\right] = -\int_0^x \mu(x')dx'$$

$$\Rightarrow I(x) = I_0 e^{-\int_0^x \mu(x')dx'}$$

- Many media:

$I_0 \leftarrow l_1 \rightarrow I_1 \leftarrow l_2 \rightarrow I_2 \leftarrow l_3 \rightarrow I_4$

$I_1 = I_0 e^{-\mu_1 l_1}$, $I_2 = I_1 e^{-\mu_2 l_2}$, $I_3 = I_2 e^{-\mu_3 l_3}$

$I = I_0 e^{\underbrace{-(\mu_1 l_1 + \mu_2 l_2 + \mu_3 l_3)}_{\Sigma \text{ (like } \int)}}$ If l tiny, similar to $I_0 e^{-\int_0^x \mu dx'}$

Here, • Forward model : find $I(x)$ given I_0, μ_0.

• Inverse model : find μ (obj property) given Beer's law (eqn), I_0, I

Linear model

- Easy to compute
- Properties: Additive : If $x_1 \Rightarrow y_1$ and $x_2 \Rightarrow y_2$ Then $x_1 + x_2 \Rightarrow y_1 + y_2$
  Homogeniety : If $x_1 \Rightarrow y_1$ Then $\alpha x_1 \Rightarrow \alpha y_1$
  $\Big\}\ \alpha x_1 + \beta x_2 \Rightarrow \alpha y_1 + \beta y_2$
- To linearise, define $P = \log\left(\frac{I_0}{I}\right)$ new measurement
  ↖ take $\frac{I_0}{I}$ to remove − sign

  $= \int_0^x \mu(x')\,dx' \Rightarrow$ Integral and $\frac{d}{dx}$ are linear operators

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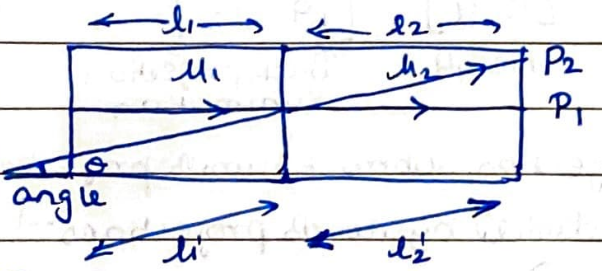
CT scan in reality

- 2 unknowns, one eqn so to get 2 eqns, measure at different angles.

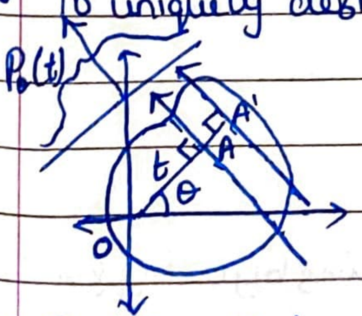


$P_1 = \mu_1 l_1 + \mu_2 l_2$

$P_2 = \mu_1 l_1' + \mu_2 l_2'$

(rotate source)

- Dont give a single ray as it may miss tumour Give a set of ∥ rays.
  $\theta \in (0 \text{ to } 180°)$
- To uniquely define one X-ray, $OA = t$, $\angle \theta = \theta \Rightarrow$ one θ & t uniquely define a x ray.



eqn of line $x\cos\theta + y\sin\theta = t$

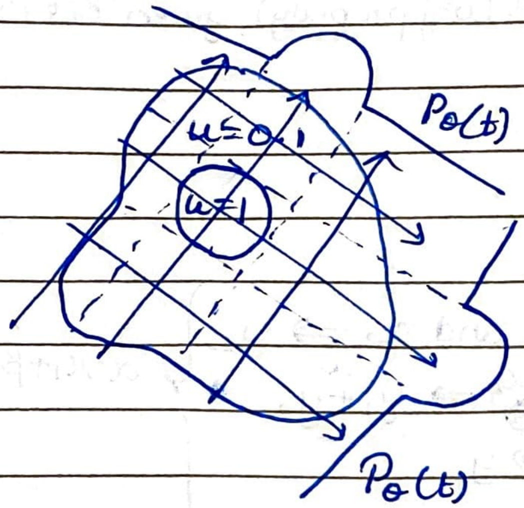
$P_\theta(t)$ is integral of absorption coeff along a particular x ray.

For a given angle, you can plot $P_\theta(t)$ vs t

- $P_\theta(t)$ called radon transform of μ

Plot of $P_\theta t$ as a fn of θ and t called a sinogram.

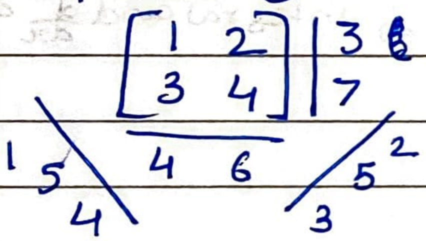
Reconstruction Methods



- measure Po(t), plot it.
- Project measurement to obj, superimpose (add) and get reconstruction

Matrix problem

- Represent μ as a matrix in 2D.



(Summation of absorption coeffs along different views of xrays)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

↑ unknowns find M's inverse prob.

$$\begin{bmatrix} 3 & 3 \\ 7 & 7 \end{bmatrix} + \begin{bmatrix} 5 & 2 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} 4 & 6 \\ 4 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 5 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} 13 & 16 \\ 19 & 22 \end{bmatrix}$$

view1 view2 view3 view4 Back projection summation

Trick: from each entry of back projection, subtract sum of projections and divide by no. of projections − 1 (4−1=3)

view 1 3+7=10
2 3+5+2=10
3 4+6=10
4 1+5+4=10

$$\frac{1}{3}\begin{bmatrix} 13-10 & 16-10 \\ 19-10 & 22-10 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

- Simple back projection is blurred, get rid of blurring by using Fourier slice theorem. Blurred = original convolved with $f(r) = 1/r$.