INDIAN INSTITUTE OF TECHNOLOGY BOMBAY
MA205 Complex Analysis Autumn 2012

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- Consider $s_n = sup\{b_n, b_{n+1}, \ldots\}$. Then s_n is a monotonically decreasing sequence. So, the following makes sense.
- **Definition:** $Limsup_n\{b_n\} := \lim\limits_{n\to\infty} s_n$.

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- Two important properties of the Limsup are:
- (**Limsup-I**) If $\alpha > \limsup_n\{b_n\}$ then there exists n_0 such that for all $n \geq n_0$ we have, $b_n < \alpha$.
- (**Limsup-II**) If $\beta < \limsup_n\{b_n\}$ then there exists infinitely many n_j such that $b_{n_j} > \beta$.

Recall Limsup

- Indeed, $\limsup$ can be characterized by these two properties.

Recall Limsup

- It is important to note that $\limsup$ always exists and can be any value in $[-\infty, \infty]$. When a sequence is convergent (including $\pm\infty$) the $\limsup_n$ of the sequence will be equal to the limit.

Cauchy-Hadamard Theorem

Definition

A formal power series $P(t) = \sum_n a_n t^n$ is said to be *convergent* if there exists a **non zero number** z (real or complex) such that the series of complex numbers $\sum_n a_n z^n$ is convergent.

Cauchy-Hadamard Theorem

Theorem

(**Cauchy-Hadamard**) *Let* $P = \sum_{n \geq 0} a_n t^n$ *be a power series over* $\mathbb{C}$. *Then*

(a) *there exists* $0 \leq R \leq \infty$ *such that for all* $0 < r < R$, *the series* $P(z)$ *is absolutely and uniformly convergent in* $|z| \leq r$ *and for all* $|z| > R$ *the series is divergent.*

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Convention:

$$\frac{1}{0} = \infty; \ \frac{1}{\infty} = 0.$$

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- **Proof:** Indeed, all that we have to prove is to take R as given by (b) and show that it satisfies (a).

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- So, let $0 < r < R$. Choose $r < s < R$. Then $1/s > 1/R$ and hence by property (Limsup-I), we must have n_0 such that for all $n \geq n_0$, $\sqrt[n]{|a_n|} < 1/s$.

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- So, let $0 < r < R$. Choose $r < s < R$. Then $1/s > 1/R$ and hence by property (Limsup-I), we must have n_0 such that for all $n \geq n_0$, $\sqrt[n]{|a_n|} < 1/s$.
- Therefore, for all $|z| \leq r$, $|a_n z^n| < (r/s)^n, \quad n \geq n_0$.

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- Thus we have proved that inside the disc $|z| < R$ the series is convergent.

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- On the other hand, suppose $|z| > R$. We fix s such that $|z| > s > R$.
- Then $1/s < 1/R$, and hence by (Limsup-II) there exist infinitely many n_j, for which $\sqrt[n_j]{|a_{n_j}|} > 1/s$.
- This means that $|a_{n_j} z^{n_j}| > (R/s)^{n_j}$. It follows that the sequence $\{a_n z^n\}$ is unbounded and hence the series $\sum_n a_n z^n$ is divergent. ♠

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- The second part of the theorem gives you the formula for it. This is called the Cauchy-Hadamard formula.
- Thus the collection of all points at which a given power series converges consists of an open disc centered at the origin and perhaps some points on the boundary of the disc.

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- Observe that if $P(z)$ is convergent for some z, then from the last part of (a), the radius of convergence of P is at least $|z|$.
- Also observe that the theorem does not say anything about the convergence of the series at points on the boundary $|z| = R$.

Radius of Convergence:

Theorem

Any given power series, its derived series and its integrated series all have the same radius of convergence.

Newton's Binomial Series

Example

Consider the principal branch $f(z)$ of $(1+z)^\alpha$ in the open disc $|z| < 1$. The MacLaurin's series $f(z) = \sum_{n=0}^\infty a_n z^n$ is called the Newton's binomial series.

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$$a_n = \binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}.$$

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If $r > 1$ then $(1+z)^\alpha$ is complex differentiable as many times as you like at $z = -1$.
which leads to a contradiction.

Singularity

- Definition

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  A point $z \in U$ is called an **isolated singularity** *of f if f is defined and holomorphic in a neighborhood of z except perhaps at z.*

Examples

- (i) If $p(z)$ is a polynomial, then $1/p(z)$ has all its singularities isolated and these are nothing but the zeros of $p(z)$.

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Examples

- (i) If $p(z)$ is a polynomial, then $1/p(z)$ has all its singularities isolated and these are nothing but the zeros of $p(z)$.
- (ii) Since for any holomorphic function f, the zeros of f are isolated, it follows that all the singularities of $1/f$ are isolated.
- (iii) Natural examples of holomorphic functions which have non isolated singularities are branches of logarithmic function and inverse-trigonometric functions. For instance, $\mathrm{Ln}(z)$ has singularities along the negative real

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- Later on we weakened this hypothesis to include those functions f which are holomorphic on $U_1 = U \setminus A$, where A is a finite subset and
- satisfying the weaker condition that f is continuous at $a \in A$.
- (Or just bounded in nbd of a.)

Removable Singularities

- Thus

$$f(w) = \frac{1}{2\pi\imath}\int_C \frac{f(z)dz}{z-w} \qquad (1)$$

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- Therefore, it follows that f is holomorphic even at points of A.

Removable Singularities

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Definition

An isolated singularity z_0 of a holomorphic function is called a **removable singularity** if $f(z_0)$ can be defined in such a way that f becomes complex differentiable at z_0, i.e., there exists a holomorphic function $g : U \to \mathbb{C}$ such that for all $z \in U \setminus \{a\}$ we have $f(z) = g(z)$.

Removable Singularities

Theorem

Let U be a domain, $a \in U$ be any point. Suppose f is holomorphic in $U \setminus \{a\}$. A necessary and sufficient condition that there exists a holomorphic function $g : U \to \mathbb{C}$ such that $g(z) = f(z), z \in U \setminus \{a\}$ is that f is **continuous at** *a.*

Removable Singularity

- **Proof:** If such a g exists as stated in the definition, then

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- Conversely, suppose the above limit exists. Take a circular region D around a contained in U and so that f is holomorphic in $D \setminus \{a\}$. It is enough to find a holomorphic function g on D such that $f(z) = g(z)$ for all $z \in D \setminus \{a\}$.

Removable Singularity

- But then C.I.F. says that

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- On the other hand, if we take $g(z)$ as RHS, then we know that g is holomorphic function throughout D. (by differentiating under the integral sign). This completes the proof. ♠

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- Obviously $z = 0$ is an isolated singularity. We easily see that $\lim\limits_{z\to 0} f(z)$ exists. Hence, $z = 0$ is a removable singularity. Also we see that $\lim\limits_{z\to 0} f(z) = 1$. So we can define $f(0) = 1$ and make f holomorphic at $z = 0$ also.

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- Other similar example are $\frac{e^z - 1}{z}$, $z \cot z$ etc. for which $z = 0$ is a removable singularity.

Removable Singularities

Remark

One easy way a removable singularity z_0 can arise is by taking a genuine holomorphic function f around this point and then brutally redefining the value of f to be something else only at z_0 or merely pretending as if f is not defined at z_0.

Poles

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- Theorem

  Let $z = a$ be a pole of $f(z)$ defined and holomorphic in $U \setminus \{a\}$. Then there exists a positive integer k such that in a disc around a, $\lim_{z \to a}(z-a)^k f(z)$ *exists; (equivalently* $\lim_{z \to a}(z-a)^{k+1} f(z) = 0$*).*

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- Consider $g(z) = 1/f(z)$ on $B_\delta(a) \setminus \{a\} =: U_1$. Then $g(z)$ is holomorphic on U_1. Moreover, $\lim_{z\to a} g(z) = 0$.

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- Consider $g(z) = 1/f(z)$ on $B_\delta(a) \setminus \{a\} =: U_1$. Then $g(z)$ is holomorphic on U_1. Moreover, $\lim_{z\to a} g(z) = 0$.
- Hence $z = a$ is a removable singularity of $g(z)$. We are forced to define $g(a) = 0$ in order to obtain a holomorphic function on $B_\delta(a)$.

Poles

- Suppose a is a zero of g of order k. Then $g(z) = (z-a)^k h(z)$ for a holomorphic function h on U_1 and $h(a) \neq 0$. Therefore, $\lim_{z\to a}(z-a)^k f(z) = \lim_{z\to a} 1/h(z)$ exists. This implies $\lim_{z\to a}(z-a)^{k+1} f(z) = 0$.

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- A partial converse is also true.
- For, it follows that the function $(z-a)^{k+1} f(z)$ is holomorphic around a and vanishes at a.
- If the order of zero at a is m then we have $(z-a)^{k+1} f(z) = (z-a)^m \alpha(z)$ for a holomorphic function α with $\alpha(a) \neq 0$.

Poles

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- If $m < k+1$ then

$$|f(z)| = \left|\frac{\alpha(z)}{(z-a)^{k-m+1}}\right| \to \infty$$

as $z \to a$.

Poles

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- Definition
  If the order is 1 then the pole is called a **simple pole;** if the order is bigger than 1, then the pole is called a **multiple pole.**

Poles

- Indeed we have just proved that $f(z) = (z-a)^{-k}h(z)$, for all z in a neighborhood of a, where $h(z)$ is holomorphic and $h(a) \neq 0$, where k is the order of the pole. The number k with this property is unique.

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- Definition
A function which has all its singularities, if any, as poles, is called a *meromorphic* function in U. [Observe that the poles of a meromorphic function are required to be isolated.

Poles: Examples

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- (i) The simplest example of a function with a pole at $z = 0$ of order k is $1/z^k$.
- (ii) More generally, functions of the form $\dfrac{P(z)}{Q(z)}$ where P, Q are polynomials are meromorphic functions.
- (iii) Sums, products and scalar multiples of meromorphic functions are meromorphic.

Poles: Examples

- (iv) If f and g are non zero meromorphic functions then so is f/g. Further, the zeros of g become poles of f/g, in general. However, if $z = a$ is a common zero of f and g, it becomes a removable singularity of f/g provided the order of the zero of f at a is bigger than or equal to that of g.

Poles: Examples

- (iv) If f and g are non zero meromorphic functions then so is f/g. Further, the zeros of g become poles of f/g, in general. However, if $z = a$ is a common zero of f and g, it becomes a removable singularity of f/g provided the order of the zero of f at a is bigger than or equal to that of g.
- (v) A typical example of this type is $(\sin z)/z$, which is indeed a holomorphic function.

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- (v) A typical example of this type is $(\sin z)/z$, which is indeed a holomorphic function.
- (vi) Amongst trigonometric functions we have $\tan z$ and $\cot z$ which have infinitely many poles

Poles: Singular part

- Let f have a pole of order k at $z = a$ and consider $h(z) = (z-a)^k f(z)$, and apply the Taylor's expansion to $h(z) =$
$$b'_0 + b'_1(z-a) + \cdots + b'_{k-1}(z-a)^{k-1} + \phi(z)(z-a)^k$$
where ϕ_k is holomorphic at $z = a$.

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  $$b'_0 + b'_1(z-a) + \cdots + b'_{k-1}(z-a)^{k-1} + \phi(z)(z-a)^k$$
  where ϕ_k is holomorphic at $z = a$.
- For $z \neq a$, we can divide this expression by $(z-a)^k$ and write $b_1 = b'_{k-1},\ b_2 = b'_{k-2}, \ldots, b_k = b'_0$, to obtain
  $$f(z) = \frac{b_k}{(z-a)^k} + \frac{b_{k-1}}{(z-a)^{k-1}} + \cdots + \frac{b_1}{(z-a)} + \phi(z)$$

Poles: Singular part

- The sum of terms which involve b_i is called the *principal part* of $f(z)$ at $z = a$. Observe that f minus its principal part is a holomorphic function.

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Poles: Singular part

- The sum of terms which involve b_i is called the *principal part* of $f(z)$ at $z = a$. Observe that f minus its principal part is a holomorphic function.
- Further, if we write Taylor's expansion for $\phi(z)$ on the rhs above we get *Laurent*[1] *expansion* for $f(z)$.

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