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MA 108–Ordinary Differential Equations

Spring 2022

Tutorial Sheet No. 1

Q.1. Classify the following equations (order, linear or non-linear):
(i) $\frac{d^3y}{dx^3} + 4(\frac{dy}{dx})^2 = y$ (ii) $\frac{dy}{dx} + 2y = \sin x$ (iii) $y\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + y = 0$
(iv) $\frac{d^4y}{dx^4} + (\sin x)\frac{dy}{dx} + x^2y = 0$. (v)$(1+y^2)\frac{d^2y}{dt^2} + t\frac{d^6y}{dt^6} + y = e^t$.

Q.2. Formulate the differential equations represented by the following functions by eliminating the arbitrary constants a, b and c:

(i) $y = ax^2$ (ii) $y - a^2 = a(x-b)^2$ (iii) $x^2 + y^2 = a^2$ (iv) $(x-a)^2 + (y-b)^2 = a^2$
(v) $y = a\sin x + b\cos x + a$ (vi) $y = a(1-x^2) + bx + cx^3$ (vii) $y = cx + f(c)$.

Also state the order of the equations obtained.

Q.3. Solve the equation $x^3(\sin y)y' = 2$. Find the particular solution such that $y(x) \to \frac{\pi}{2}$ as $x \to +\infty$.

Q.4. Prove that a curve with the property that all its normals pass through a point is a circle.

Q.5. Find the values of m for which

(a) $y = e^{mx}$ is a solution of
(i) $y'' + y' - 6y = 0$ (ii) $y''' - 3y'' + 2y' = 0$.

(b) $y = x^m$ for $x > 0$ is a solution of
(i) $x^2y'' - 4xy' + 4y = 0$ (ii) $x^2y''' - xy'' + y' = 0$.

Q.6. For each of the following linear differential equations verify that the function given in brackets is a solution of the differential equation.

(i) $y'' + 4y = 5e^x + 3\sin x$ $(y = a\sin 2x + b\cos 2x + e^x + \sin x)$

(ii) $y'' - 5y' + 6y = 0$, $(y_1 = e^{3x}, y_2 = e^{2x}, c_1y_1 + c_2y_2)$

(iii) $y''' + 6y'' + 11y' + 6y = e^{-2x}$ $(y = ae^{-x} + be^{-2x} + ce^{-3x} - xe^{-2x})$

(iv) $y''' + 8y = 9e^x + 65\cos x$, $(y = ae^{-2x} + e^x(b\cos\sqrt{3}x + c\sin\sqrt{3}x) + 8\cos x - \sin x + e^x)$

Q.7. Let φ_i be a solution of $y' + ay = b_i(x)$ for $i = 1, 2$.

Show that $\varphi_1 + \varphi_2$ satisfies $y' + ay = b_1(x) + b_2(x)$. Use this result to find the solutions of $y' + y = \sin x + 3\cos 2x$ passing through the origin.

Q.8. Obtain the solution of the following differential equations:

(i) $(x^2+1)dy + (y^2+4)dx = 0;\ \ y(1) = 0$ (ii) $y' = y\cot x;\ \ y(\pi/2) = 1$

(iii) $y' = y(y^2 - 1)$, with $y(0) = 2$ or $y(0) = 1$, or $y(0) = 0$ (iv) $(x+2)y' - xy = 0;\ \ y(0) = 1$

(v) $y' + \dfrac{y-x}{y+x} = 0;\ \ y(1) = 1$ (vi) $y' = (y-x)^2;\ \ y(0) = 2$

(vii) $2(y\sin 2x + \cos 2x)dx = \cos 2x dy;\ \ y(\pi) = 0$. (viii) $y' = \dfrac{1}{(x+1)(x^2+1)}$

Q.9. For each of the following differential equations, find the general solution (by substituting $y = vx$)

(i) $y' = \dfrac{y^2 - xy}{x^2 + xy}$ (ii) $x^2y' = y^2 + xy + x^2$

(iii) $xy^{'} = y + x\cos^2(y/x)$ (iv) $xy' = y(\ln y - \ln x)$

Q.10. Show that the differential equation $\dfrac{dy}{dx} = \dfrac{ax+by+m}{cx+dy+n}$ where a, b, c, d, m and n are constants can be reduced to $\dfrac{dy}{dt} = \dfrac{ax+by}{cx+dy}$ if $ad - bc \neq 0$. Then find the general solution of

(i) $(1 + x - 2y) + y'(4x - 3y - 6) = 0$

(ii) $y' = \frac{y-x+1}{y-x+5}$

(iii) $(x + 2y + 3) + (2x + 4y - 1)y' = 0.$

Q.11. Solve the differential equation $\sqrt{1-y^2}dx + \sqrt{1-x^2}dy = 0$ with the conditions $y(0) = \dfrac{\pm 1}{2}\sqrt{3}$. Sketch the graphs of the solutions and show that they are each arcs of the same ellipse. Also show that after these arcs are removed, the remaining part of the ellipse does not satisfy the differential equation.

Q.12. The differential equation $y = xy' + f(y')$ is called a Clairaut equation (or Clairaut's equation). Show that the general solution of this equation is the family of straight lines $y = cx + f(c)$. In addition to these show that it has a special solution given by $f'(p) = -x$ where $p = y'$. This special solution which does not (in general) represent one of the straight lines $y = cx + f(c)$, is called a singular solution. Hint: Differentiate the differential equation.

Q.13. Determine the general solutions as well as the singular solutions of the following Clairaut equations. In each of the two examples, sketch the graphs of these solutions.

(i) $y = xy' + 1/y'$. (ii) $y = xy' - y'/\sqrt{1+y'^2}$

Q.14. For the parabola $y = x^2$ find the equation of its tangent at (c, c^2) and find the ordinary differential equation for this one parameter family of tangents. Identify this as a Clairaut equation. More generally take your favourite curve and determine the ODE for the one parameter family of its tangents and verify that it is a Clairut's equation. N.B: Exercise 13 shows that the converse is true.

Q.15. In the preceeding exercises, show that in each case, the envelope of the family of straight lines is also a solution of the Clairaut equation.

Q.16. Show that the differential equation $y' - y^3 = 2x^{-3/2}$ has three distinct solutions of the form $A/\sqrt{x}$ but that only one of these is real valued.

Tutorial Sheet No. 2

Q.1. State the conditions under which the following equations are exact.

(i) $[f(x) + g(y)]dx + [h(x) + k(y)]dy = 0$

(ii) $(x^3 + xy^2)dx + (ax^2y + bxy^2)dy = 0$

(iii) $(ax^2 + 2bxy + cy^2)dx + (bx^2 + 2cxy + gy^2)dy = 0$

Q.2. Solve the following exact equations

(i) $3x(xy - 2)dx + (x^3 + 2y)dy = 0$ (ii) $(\cos x \cos y - \cot x)dx - \sin x \sin y dy = 0.$

(iii) $e^x y(x + y)dx + e^x(x + 2y - 1)dy = 0$

Q.3. Determine (by inspection suitable) Integrating Factors (IF's) so that the following equations are exact.

(i) $ydx + xdy = 0$ (ii) $d(e^x \sin y) = 0$

(iii) $dx + (\frac{y}{x})^2 dy = 0$ (iv) $ye^{x/y}dx + (y - xe^{x/y})dy = 0$

(v) $(2x + e^y)dx + xe^y dy = 0$, (vi) $(x^2 + y^2)dx + xydy = 0$

Q.4. Verify that the equation $Mdx + Ndy = 0 \ldots$ (1) can be expressed in the form

$$\frac{1}{2}(Mx + Ny)d(\ln xy) + \frac{1}{2}(Mx - Ny)d\ln(\frac{x}{y}) = 0.$$

Hence, show that (i) if $Mx + Ny = 0$, then $\dfrac{1}{Mx - Ny}$ is an IF of (1) and

(ii) if $Mx - Ny = 0$, then $\dfrac{1}{Mx + Ny}$ is an IF of (1).

Also show that (iii) if M and N are homogeneous of the same degree then $\dfrac{1}{Mx + Ny}$ is an IF of (1).

Q.5. If $\mu(x, y)$ is an IF of $Mdx + Ndy = 0$ then prove that

$$M_y - N_x = N\frac{\partial}{\partial x}\ln|\mu| - M\frac{\partial}{\partial y}\ln|\mu|.$$

Use the relation to prove that if $\dfrac{1}{N}(M_y - N_x) = f(x)$ then there exists an IF $\mu(x)$ given by $\exp(\int_a^x f(t)dt)$ and if $\dfrac{1}{M}(M_y - N_x) = g(y)$, then there exists an IF $\mu(y)$ given by $\exp(-\int_a^y g(t)dt)$. Further if $M_y - N_x = f(x)N - g(y)M$ then $\mu(x, y) = \exp(\int_a^x f(x')dx' + \int_a^y g(y')dy')$ is an IF, where a is any constant.

Determine an IF for the following differential equations:

(i) $y(8x - 9y)dx + 2x(x - 3y)dy = 0.$

(ii) $3(x^2 + y^2)dx + (x^3 + 3xy^2 + 6xy)dy = 0$

(iii) $4xy + 3y^2 - x)dx + x(x + 2y)dy = 0$

Q.6. Find the general solution of the following differential equations.

(i) $(y - xy') + a(y^2 + y') = 0$ (ii) $[y + xf(x^2 + y^2)]dx + [yf(x^2 + y^2) - x]dy = 0$

(iii) $(x^3 + y^2\sqrt{x^2 + y^2})dx - xy\sqrt{x^2 + y^2}dy = 0$ (iv) $(x + y)^2 y' = 1$

(v) $y' - x^{-1}y = x^{-1}y^2$ (vi) $x^2 y' + 2xy = \sinh 3x$

(viii) $y' + y\tan x = \cos^2 x$ (ix) $(3y - 7x + 7)dx + (7y - 3x + 3)dy = 0.$

Q.7. Solve the following homogeneous equations.

(i) $(x^3 + y^3)dx - 3xy^2 dy = 0$ (ii) $(x^2 + 6y^2)dx + 4xydy = 0$

(iii) $xy' = y(\ln y - \ln x)$ (iv) $xy' = y + x\cos^2 \frac{y}{x}$

Q.8. Solve the following first order linear equations.

(i) $xy' - 2y = x^4$ (iii) $y' = 1 + 3y\tan x$

(ii) $y' + 2y = e^{-2x}$ (iv) $y' = \text{cosec}\ x + y\cot x.$

(v) $y' = \text{cosec}\ x - y\cot x.$ (vi) $y' - my = c_1 e^{mx}$

Q.9. A differential equation of the form $y' + f(x)y = g(x)y^\alpha$ is called a Bernoulli equation. Note that if $\alpha = 0$ or 1 it is linear and for other values it is nonlinear. Show that the transformation $y^{1-\alpha} = u$ converts it into a linear equation. Use this to solve the following equations.

(i) $e^y y' - e^y = 2x - x^2$

(ii) $2(y+1)y' - \frac{2}{x}(y+1)^2 = x^4$

(iii) $xy' = 1 - y - xy$

(iv) $(xy + x^3y^3)\frac{dy}{dx} = 1.$

(v) $\frac{dy}{dx} = xy + x^3y^3$

(vi) $xy' + y = 2x^6y^4$

(vii) $6y^2dx - x(2x^3 + y)dy = 0$ (Bernoulli in x).

Q.10. (i) Solve $(x^2 + 6y^2)dx - 4xydy = 0$ as a Bernoulli equation.

(ii) Consider the initial value problem $y' = y(1-y), y(0) = 0$. Can this be solved by the meethod of separation of variables? As a Bernoulli equation?

Put $y = 1 - u$, $u(0) = 1$ and solve the resulting equation as a Bernoulli equation.

(iii) Solve $2ydx + x(x^2 \ln y - 1)dy = 0$. Hint: The equation is Bernoulli in x.

(iv) Solve $\cos y \sin 2x dx + (\cos^2 y - \cos^2 x)dy = 0$

(**Hint:** Put $z = -\cos^2 x$; resulting ODE is Bernoulli in z.)

Q.11. Find the orthogonal trajectories of the following families of curves.

(i) $x^2 - y^2 = c^2$ (ii) $y = ce^{-x^2}$ (iii) $e^x \cos y = c$ (iv) $x^2 + y^2 = c^2$

(v) $y^2 = 4(x+h)$ (vi) $y^2 = 4x^2(1 - cx)$ (vii) $y^2 = x^3/(a-x)$

(viii) $y = c(\sec x + \tan x)$. (ix) $xy = c(x+y)$

(x) $x^2 + (y-c)^2 = 1 + c^2$

Q.12. Find the ODE for the family of curves $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1, \quad (0 < b < a)$ and find the ODE for the orthogonal trajectories.

Q.13. A differential equation of the form $y' = P(x) + Q(x)y + R(x)y^2$ is called Riccati's equation. In general, the equation cannot be solved by elementary methods. But if a particular solution $y = y_1(x)$ is known, then the general solution is given by $y(x) = y_1(x) + u(x)$ where u satisfies the Bernoulli equation

$$\frac{du}{dx} - (Q + 2Ry_1)u = Ru^2.$$

(i) Use the method to solve $y' + x^3y - x^2y^2 = 1$, given $y_1 = x$.

(ii) Use the method to solve $y' = x^3(y-x)^2 + x^{-1}y$ given $y_1 = x$.

Q.14. Determine by Picard's method, successive approximations to the solutions of the following initial value problems. Compare your results with the exact solutions.

(i) $y' = 2\sqrt{y}; \quad y(1) = 0$

(ii) $y' - xy = 1; \quad y(0) = 1$

(iii) $y' = x - y^2; \quad y(0) = 1.$

Q.16. Show that the function $f(x,y) = |\sin y| + x$ satisfies the Lipschitz's condition

$$|f(x,y_2) - f(x,y_1)| \le M|y_2 - y_1|$$

with $M = 1$, on the whole xy plane, but f_y does not exist at $y = 0$.

Q.17. Examine whether the following functions satisfy the Lipschitz condition on the xy plane. Does $\frac{\partial f}{\partial y}$ exist ? Compute the Lipschitz constant wherever possible.

(i) $f = |x| + |y|$

(ii) $f = 2\sqrt{y}$ in $\Re : |x| \le 1, 0 \le y \le 1$ or in $\Re : |x| \le 1, \frac{1}{2} < y < 1$

(iii) $f = x^2|y|$ in $\Re : |x| \le 1, |y| \le 1$

(iv) $f = x^2 \cos^2 y + y \sin^2 x, |x| \le 1, |y| < \infty$

Tutorial Sheet No. 3

Q.1. Find the curve $y(x)$ through the origin for which $y'' = y'$ and the tangent at the origin is $y = x$.

Q.2. Find the general solutions of the following differential equations.
(i) $y'' - y' - 2y = 0$ (ii) $y'' - 2y' + 5y = 0$

Q.3. Find the differential equation of the form $y'' + ay' + by = 0$, where a and b are constants for which the following functions are solutions:
(i) $e^{-2x}, 1$ (ii) $e^{-(\alpha+i\beta)x}, e^{-(\alpha-i\beta)x}$.

Q.4. Are the following statements true or false. If the statement is true, prove it, if it is false, give a counter example showing it is false. Here Ly denotes $y'' + P(x)y' + Q(x)y$.

(i) If $y_1(x)$ and $y_2(x)$ are linearly independent on an interval I, then they are linearly independent on any interval containing I.

(ii) If $y_1(x)$ and $y_2(x)$ are linearly dependent on an interval I, then they are linearly dependent on any subinterval of I.

(iii) If $y_1(x)$ and $y_2(x)$ are linearly independent solution of $L(y) = 0$ on an interval I, they are linearly independent solution of $L(y) = 0$ on any interval I contained in I.

(iv) If $y_1(x)$ and $y_2(x)$ are linearly dependent solutions of $L(y) = 0$ on an interval I, they are linearly dependent on any interval J contained in I.

Q.5. Are the following pairs of functions linearly independent on the given interval?
(i) $\sin 2x, \cos(2x + \frac{\pi}{2}); x > 0$ (ii) $x^3, x^2|x|; -1 < x < 1$
(iii) $x|x|, x^2; 0 \le x \le 1$ (iv) $\log x, \log x^2; x > 0$ (v) $x, x^2, \sin x; x \in \mathbb{R}$

Q.6. Solve the following:
(i) $y'' - 4y' + 3y = 0,\ y(0) = 1, y'(0) = -5;$ (ii) $y'' - 2y' = 0,\ y(0) = -1, y(\frac{1}{2}) = e - 2.$

Q.7. Solve the following initial value problems.

(i) $(D^2+5D+6)y=0, \quad y(0)=2, y'(0)=-3$ (ii) $(D+1)^2y=0, \quad y(0)=1, y'(0)=2$

(iii) $(D^2+2D+2)y=0, \quad y(0)=1, y'(0)=-1$

Q.8. Solve the following initial value problems.

(i) $(x^2D^2-4xD+4)y=0, y(1)=4, y'(1)=1$

(ii) $(4x^2D^2+4xD-1)y=0, y(4)=2, y'(4)=-1/4$

(iii) $(x^2D^2-5xD+8)y=0, y(1)=5, y'(1)=18$

Q.9. Using the Method of Undetermined Coefficients, determine a particular solution of the following equations. Also find the general solutions of these equations.

(i) $y''+2y'+3y=27x$ (ii) $y''+y'-2y=3e^x$

(iii) $y''+4y'+4y=18\cosh x$ (iv) $y''''+y=6\sin x$

(v) $y''+4y'+3y=\sin x+2\cos x$ (vi) $y''-2y'+2y=2e^x\cos x$

(vii) $y''+y=x\cos x+\sin x$ (viii) $2y''''+3y''+y=x^2+3\sin x$

(ix) $y'''-y'=2x^2e^x$ (x) $y'''-5y''+8y'-4y=2e^x\cos x$

Q.10. Solve the following initial value problems.

(i) $y''+y'-2y=14+2x-2x^2, y(0), y'(0)=0.$

(ii) $y''+y'-2y=-6\sin 2x-18\cos 2x;\ y(0)=2, y'(0)=2.$

(iii) $y''-4y'+3y=4e^{3x}, y(0)=-1, y'(0)=3.$

Q.11. For each of the following equations, write down the form of the particular solution. Do not go further and compute the Undetermined Coefficients.

(i) $y''+y=x^3\sin x$ (ii) $y''+2y'+y=2x^2e^{-x}+x^3e^{2x}$ (iii) $y'+4y=x^3e^{-4x}$ (iv) $y^{(4)}+y=xe^{x/\sqrt{2}}\sin(x/\sqrt{2})$.

Q.12. Solve the Cauchy-Euler equations: (i) $x^2y''-2y=0$ (ii) $x^2y''+2xy'-6y=0$. (iii) $x^2y''+2xy'+y/4=1/\sqrt{x}$

Q.13. Find the solution of $x^2y''-xy'-3y=0$ satisfying $y(1)=1$ and $y(x)\longrightarrow 0$ as $x\longrightarrow\infty$.

Q.14. Show that every solution of the constant coefficient equation $y''+\alpha y'+\beta y=0$ tends to zero as $x\to\infty$ if and only if the real parts of the roots of the characteristic polynomial are negative.

Tutorial Sheet No. 4

Q.1. Using the Method of Variation of Parameters, determine a particular solution for each of the following.

(i) $y''-5y'+6y=2e^x$ (ii) $y''+y=\tan x, 0<x<\dfrac{\pi}{2}$

(iii) $y''+4y'+4y=x^{-2}e^{-2x}, x>0$ (iv) $y''+4y=3\ \text{cosec}\ 2x, 0<x<\dfrac{\pi}{2}$

(v) $x^2y''-2xy'+2y=5x^3\cos x$ (vi) $xy''-y'=(3+x)x^3e^x$

Q.2. Let $y_1(x)$ and $y_2(x)$ be two solutions of the homogeneous equation $y'' + p(x)y' + q(x)y = 0$, $a < x < b$, and let $W(x)$ be the Wronskian of these two solutions. Prove that $W'(x) = -p(x)W(x)$. If $W(x_0) = 0$ for some x_0 with $a < x_0 < b$, then prove that $W(x) = 0$ for each x with $a < x < b$.

Q.3. Let $y = y_1(x)$ be a solution of $y'' + p(x)y' + q(x)y = 0$. Let I be an interval where $y_1(x)$ does not vanish, and $a \in I$ be any element. Prove that the general solution is given by

$$y = y_1(x)[c_2 + c_1\psi(x)] \text{ where } \psi(x) = \int_a^x \frac{\exp[-\int_a^t p(u)du]}{y_1^2(t)} dt.$$

Q.4. For each of the following ODEs, you are given one solution. Find a second solution.

(i) $4x^2y'' + 4xy' + (4x^2 - 1)y = 0; y_1(x) = \sin x/\sqrt{x}$

(ii) $y'' - 4xy' + 4(x^2 - 2)y = 0; y_1 = e^{x^2}$

(iii) $x(x-1)y'' + 3xy' + y = 0; y_1 = x/(x-1)^2$;

(iv) $xy'' - y' + 4x^3y = 0, y_1 = \cos x^2$

(v) $x^2(1-x^2)y'' - x^3y' - \left(\frac{3-x^2}{4}\right)y = 0,\ y_1 = \sqrt{\frac{1-x^2}{x}}$.

(vi) $x(1+3x^2)y'' + 2y' - 6xy = 0, y_1 = 1 + x^2$

(vii) $(\sin x - x\cos x)y'' - (x\sin x)y' + (\sin x)y = 0, y_1 = x$.

Q.5. Computing the Wronskian or otherwise, prove that the the functions $e^{r_1x}, e^{r_2x}, \ldots, e^{r_nx}$, where $r_1, r_2, \ldots, r_n$ are distinct real numbers, are linearly independent.

Q.6. Let $y_1(x), y_2(x) \ldots, y_n(x)$ be n linearly independent solutions of the nth order homogeneous linear differential equation $y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_{n-1}(x)y + p_n(x)y = 0$. Prove that $y(x) = c_1(x)y_1(x) + c_2(x)y_2(x) + \ldots + c_n(x)y_n(x)$ is a solution of the nonhomogeneous equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_{n-1}(x)y = r(x),$$

where $c_1(x), c_2(x), \ldots, c_n(x)$ are given by $c_i(x) = \int \frac{D_i(x)}{W(x)} dx$, where $D_i(x)$ is the determinant of the matrix obtained from the matrix defining the Wronskian $W(x)$ by replacing its ith column by $[0\ 0\ 0\ \ldots\ r(x)\ ]^T$

Q.7. Three solutions of a certain second order non-homogeneous linear differential equation are

$$y_1(x) = 1 + e^{x^2} \quad y_2(x) = 1 + xe^{x^2}, \quad y_3(x) = (1+x)e^{x^2} - 1.$$

Find the general solution of the equation.

Q.8. For the following nonhomogeneous equations, a solution y_1 of the corresponding homogeneous equation is given. Find a second solution y_2 of the corresponding homogeneous equation and the general solution of the nonhomogeneous equation using the Method of Variation of Parameters.

(i) $(1+x^2)y'' - 2xy' + 2y = x^3 + x, \quad y_1 = x$ (ii) $xy'' - y' + (1-x)y = x^2, \quad y_1 = e^x$

(iii) $(2x+1)y'' - 4(x+1)y' + 4y = e^{2x}, \quad y_1 = e^{2x}$

(iv) $(x^3 - x^2)y'' - (x^3 + 2x^2 - 2x)y' + (2x^2 + 2x - 2)y = (x^3 - 2x^2 + x)e^x, \quad y_1 = x^2$

Q.9. Reduce the order of the following equations given that $y_1 = x$ is a solution.

(i) $x^3y''' - 3x^2y'' + (6-x^2)xy' - (6-x^2)y = 0$ (ii) $y''' + (x^2+1)y'' - 2x^2y' + 2xy = 0$

Q.10. Find the complementary function and particular integral for the following differential equations

(i) $y^{(4)} + 2y^{(2)} + y = \sin x$ (ii) $y^{(4)} - y^{(3)} - 3y^{(2)} + 5y' - 2y = xe^x + 3e^{-2x}$

Q.11. Solve the following Cauchy-Euler equations

(i) $x^2y'' + 2xy' + y = x^3$ (ii) $x^4y^{(4)} + 8x^3y^{(3)} + 16x^2y^{(2)} + 8xy' + y = x^3$

(iii) $x^2y'' + 2xy' + \frac{y}{4} = \frac{1}{\sqrt{x}}$

Q.12. Find a particular solution of the following inhomogeneous Cauchy-Euler equations.

(i) $x^2y'' - 6y = \ln x$ (ii) $x^2y'' + 2xy' - 6y = 10x^2$.

Q. 13. Find a second solution of

(i) $(x^2 - x)y'' + (x+1)y' - y = 0$ given that $(1+x)$ is a solution.

(ii) $(2x+1)y'' - 4(x+1)y' + 4y = 0$ given that e^{2x} is a solution.

Q. 14. Find a homogeneous linear differential equation on $(0, \infty)$ whose general solution is $c_1x^2e^x + c_2x^3e^x$. Does there exist a homogeneous differential equation with constant coefficients with general solution $c_1x^2e^x + c_2x^3e^x$?

Tutorial Sheet No. 5

Q.1. Find the Laplace Transform of the following functions.

(i) $t\cos wt$ (ii) $t\sin wt$ (iii) $e^{-t}\sin^2 t$ (iv) t^2e^{-at} (v) $(1+te^{-t})^3$ (vi) $(5e^{2t} - 3)^2$

(vii) $te^{-2t}\sin wt$ (viii) t^ne^{at} (ix) $t^2e^{-at}\sin bt$ (xi) $\cosh at \cos at$

Q.2. Find the inverse Laplace transforms of the following functions.

(i) $\dfrac{s^2 - w^2}{(s^2+w^2)^2}$ (ii) $\dfrac{2as}{(s^2-a^2)^2}$ (iii) $\dfrac{1}{(s^2+w^2)^2}$ (iv) $\dfrac{s^3}{(s^4+4a^4)}$ (v) $\dfrac{s-2}{s^2(s+4)^2}$ (vi) $\dfrac{1}{s^4 - 2s^3}$ (vii) $\dfrac{1}{s^4(s^2+\pi^2)}$

(viii) $\dfrac{s^2+a^2}{(s^2-a^2)^2}$ (ix) $\dfrac{s^3+3s^2-s-3}{(s^2+2s+5)^2}$ (x) $\dfrac{s^3-7s^2+14s-9}{(s-1)^2(s-2)^2}$

Q.3. Solve the following intial value problems using Laplace transforms and convolutions.

(i) $y'' + y = \sin 3t;\ y(0) = y'(0) = 0$ (ii) $y'' + 3y' + 2y = e^{-t};\ y(0) = y'(0) = 0$

(iii) $y'' + 2y' - 8y = 0;\ y(0) = 1; y'(0) = 8$ (iv) $y'' + 2y' + y = 2\cos t;\ y(0) = 3, y'(0) = 0$

(v) $y'' - 2y' + 5y = 8\sin t - 4\cos t;\quad y(0) = 1; y'(0) = 3$

(vi) $y'' - 2y' - 3y = 10\sin h2t;\quad y(0) = 0; y'(0) = 4$

Q.4. Solve the following systems of differential equations using Laplace transforms.

(i) $x' = x + y,\ y' = 4x + y$ (ii) $x' = 3x + 2y,\ y' = -5x + y$

(iii) $x'' - x + y' = y = 1,\ y'' + y + x' - x = 0$ (iv) $x' = 5x + 8y + 1,\ y' = -6x - 9y + t,\ x(0) = 4,\ y(0) = -3$

(v) $y_1' + y_2 = 2\cos t;\ y_1 + y_2' = 0; y_1(0) = 0; y_2(0) = 1$

(vi) $y_1'' + y_2 = -5\cos 2t;\ y_2'' + y_1 = 5\cos 2t;\ \ y_1(0) = 1, y_1'(0) = 1, y_2(0) = -1, y_2'(0) = 1$

(vii) $2y_1' - y_2' - y_3' = 0; y_1' + y_2' = 4t + 2;\ \ y_2' + y_3 = t^2 + 2, y_1(0) = y_2(0) = y_3(0) = 0$

(viii) $y_1'' = y_1 + 3y_2; y_2'' = 4y_1 - 4e^t;\ \ y_1(0) = 2; y_1'(0) = 3, y_2(0) = 1, y_2'(0) = 2$

Q.5. Assuming that for a Power series in $\frac{1}{s}$ with no constant term the Laplace transform can be obtained term-by-term, i.e., assuming that $\mathcal{L}^{-1}[\sum_0^\infty \frac{A_k}{s^{k+1}}] = \sum_0^\infty A_k \frac{t^k}{k!}$, where $A_0, A_1 \ldots A_k \ldots$ are real numbers, prove that

(i) $\mathcal{L}^{-1}(\frac{1}{s-1}) = e^t$ (ii) $\mathcal{L}^{1}(\frac{1}{s^2+1}) = \sin t$

(iii) $\mathcal{L}^{1}(\frac{1}{s}e^{-b/s}) = J_0(2\sqrt{bt})\ \ (b > 0)$ (iv) $\mathcal{L}^{-1}(\frac{1}{\sqrt{s^2+a^2}}) = J_0(at)\ \ (a > 0)$

(v) $\mathcal{L}^{-1}(\frac{e^{-b/s}}{\sqrt{s}}) = \frac{1}{\sqrt{\pi t}}\cos(2\sqrt{bt})\ \ (b > 0)$ (vi) $\mathcal{L}^{-1}(\tan^{-1}\frac{1}{s}) = \frac{\sin t}{t}$

Q.6. Find the Laplace transform of the following periodic functions.

(i) $f(t), f(t+p) = f(t)$ for all $t > 0$ and $f(t)$ piecewise continuous

(ii) $f(t) = |\sin wt|$

(iii) $f(t) = 1(0 < t < \pi); f(t) = -1(\pi < t < 2\pi);\ f(t+2\pi) = f(t)$

(iv) $f(t) = t(0 \le t \le 1), f(t) = 2 - t(1 \le t \le 2); f(t+2) = f(t)$

(v) $f(t) = \sin t(0 \le t \le \pi), f(t) = 0(\pi \le t \le 2\pi); f(t+2\pi) = f(t)$

Q.7. Find the Laplace Transform of $f(t)$ where $f(t) = n,\ n-1 \le t \le n, n = 1, 2, 3, \ldots$

Q.8. Find $f(t)$ given $\mathcal{L}[f(t)] = (e^{-s} - e^{-2s} - e^{-3s} + e^{-4s})/s^2$

Q.9. Find the Laplace Transform of (i) $f(t) = u_\pi(t)\sin t$ (ii) $f(t) = u_1(t)e^{-2t}$ where $u_\pi(u_1)$ is the Heaviside step function.

Q.10. Find (i) $\mathcal{L}^{-1}\left[\ln\frac{s^2+4s+5}{s^2+2s+5}\right]$

Q.11. If $\mathcal{L}[f(t)] = F(s), \mathcal{L}[g(t)] = G(s)$ prove that $\mathcal{L}^{-1}[F(s)G(s)] = \int_0^t f(u)g(t-u)du$. Also show that

$$\mathcal{L}^{-1}[\frac{F(s)}{(s+a)^2+a^2}] = \frac{1}{a}e^{-at}\int_0^t f(u)e^{au}\sin a(t-u)du.$$

Q.12. Compute the Laplace transform of a solution of $ty'' + y' + ty = 0,\ t > 0$, satisfying $y(0) = k$, $Y(1) = 1/\sqrt{2}$, where k is a real constant and Y denotes the Laplace transform of y.

Q.13. Compute the convolution of $t^{a-1}u(t)$ and $t^{b-1}u(t)$ and use the convolution theorem to prove

$$\Gamma(a)\Gamma(b) = \Gamma(a+b)B(a,b)$$

where $B(a,b)$ denotes the Beta function and $\Gamma(a)$ the Gamma function. Use this to find the value of $\Gamma(1/2)$ and hence of $\int_{-\infty}^{\infty}\exp(-x^2)dx$.

Q.14. Suppose $f(x)$ is a function of exponential type and $\mathcal{L}f = 1/\sqrt{s^2+1}$. Determine $f * f$.

Q.15. Evaluate the following integrals by computing their Laplace transforms.

(i) $f(t) = \displaystyle\int_0^\infty \frac{\sin(tx)}{x}dx$ (ii) $f(t) = \displaystyle\int_0^\infty \frac{\cos tx}{x^2+a^2}dx$ (iii) $f(t) = \displaystyle\int_0^\infty \sin(tx^a)dx, \quad a > 1$

(iv) $\displaystyle\int_0^\infty \frac{1}{x^2}(1-\cos tx)dx$ (v) $\displaystyle\int_0^\infty \frac{\sin^4 tx}{x^3}dx$ (vi) $\displaystyle\int_0^\infty \Big(\frac{x^2-b^2}{x^2+b^2}\Big)\frac{\sin tx}{x}dx$

Q.16. Solve the following integral/integro-differential equations

(i) $y(t) = 1 - \sinh t + \displaystyle\int_0^t (1+x)y(t-x)dx$ (ii) $A = \displaystyle\int_0^t \frac{y(x)dx}{\sqrt{t-x}}$, where A is a constant.

(iii) $\dfrac{dy}{dt} = 1 - \displaystyle\int_0^t y(t-\tau)d\tau$, $y(0) = 1$.

Q.17. Find a real general solution of the following nonhomogeneous linear systems.

(i) $y_1' = y_2 + e^{3t}, \quad y_2' = y_1 - 3e^{3t}$.

(ii) $y_1' = 3y_1 + y_2 - 3\sin 3t, \quad y_2' = 7y_1 - 3y_2 + 9\cos 3t - 16\sin 3t$.

(iii) $y_1' = y_2 + 6e^{2t}, \quad y_2' = y_1 - 3e^{2t}, \quad y_1(0) = 11, \quad y_2(0) = 0$.

(iv) $y_1' = 5y_2 + 23, \quad y_2' = -5y_1 + 15t, \quad y_1(0) = 1, \quad y_2(0) = -2$.

(v) $y_1' = y_2 - 5\sin t, \quad y_2' = -4y_1 + 17\cos t, \quad y_1(0) = 5, \quad y_2(0) = 2$.

(vi) $y_1' = 5y_1 + 4y_2 - 5t^2 + 6t + 25, \quad y_2' = y_1 + 2y_2 - t^2 + 2t + 4, \quad y_1(0) = 0, \quad y_2(0) = 0$.

Q.18 Prove that the Laplace transform of $(1-e^{-t})^\nu$ is $B(s, \nu+1)$ where $B(a,b)$ is the beta function.

Q.19 Show that if $f(t) = 1/(1+t^2)$ then its Laplace transform $F(s)$ satisfies the differential equation $F'' + F = 1/s$. Deduce that $F(s) = \displaystyle\int_0^\infty \frac{\sin\lambda d\lambda}{(\lambda+s)}$.

Q.20 Show that the Laplace transform of $\log t$ is $-s^{-1}\log s - Cs^{-1}$. Identify the constant C in terms of the gamma function.

Q.21 Evaluate the integral $\displaystyle\int_0^\infty \exp\Big\{-\Big(at+\frac{b}{t}\Big)\Big\}\frac{dt}{\sqrt{t}}$ where a and b are positive. Use this result to compute the Laplace transform of $\dfrac{1}{\sqrt{t}}\exp\Big(\dfrac{-b}{t}\Big)$., $b > 0$.