PH107 D1T5: Tutorial 9

19-02-2022

SCATTERING PROBLEMS

Question 1

A potential barrier is defined by $V = 0$ for $x < 0$ and $V = V_0$ for $x > 0$. Particles with energy $E(< V_0)$ approaches the barrier from left.

(a) Find the value of $x = x_0 (x_0 > 0)$, for which the probability density is $1/e$ times the probability density at $x = 0$.

From TISE:

$$\frac{1}{\psi(x)}\frac{d^2\psi(x)}{dx^2} = \begin{cases} -k_1^2 & x < 0 \\ k_2^2 & x \geq 0 \end{cases}$$

$$\psi(x) = \begin{cases} Ae^{ik_1x} + Be^{-ik_1x} & x < 0 \\ Ce^{k_2x} + De^{-k_2x} & x \geq 0 \end{cases}$$

Where

$$k_1^2 = \frac{2mE}{\hbar^2}$$
$$k_2^2 = \frac{2m(V_0 - E)}{\hbar^2}$$

We must have $C = 0$ such that $\psi(x)$ does not blow up at infinity.
We know that $ProbabilityDensity(x \geq 0) = \psi(x)^*\psi(x) = |C|^2 e^{-2k_2x}$

$$\frac{PD(x_0)}{PD(0)} = \frac{1}{e}$$
$$\implies x_0 = \frac{1}{2k_2} = \frac{\hbar}{\sqrt{8m(V_0 - E)}}$$

(b) Take the maximum allowed uncertainty Δx for the particle to be localized in the classically forbidden region as x_0. Find the uncertainty this would cause in the energy of the particle. Can then one be sure that its energy E is less than V_0.

We want to analyse what happens to the wavefunction, and hence the particle if it was originally in the energy eigenstate with eigenvalue $E < V_0$, and a position observation leads us to find the position result as $x > 0$
Of course our position detector will not be perfect, and won't perform a perfect position measurement.
Hence after the measurement the wavefunction will collapse to a state close to the position eigenstate (a delta function), i.e. it will collapse to a gaussian packet instead, one with $\sigma_x = \Delta x = x_0$. Thus is an assumption of the physical imperfection of our detector.
This collapsed wavefunction is no longer an energy eigenstate, hence it does not have any

energy eigenvalue, and the expectation of energy is certainly not $E < V_0$. Instead this collapsed wavefunction will be composed of a spread in energies, with

$$\begin{aligned}\Delta E &= \sqrt{\langle E^2\rangle - \langle E\rangle^2} \\ &\approx \frac{\Delta p^2}{2m} = \frac{\hbar^2}{8m^2\Delta x^2} \\ &= V_0 - E\end{aligned}$$

Therefore the energy has enough spread such that it takes up the expectation of energy well above V_0.

Question 2

Consider a potential

$$V(x) = \begin{cases} 0 & x < 0 \\ -V_0 & x > 0 \end{cases}$$

Consider a beam of non-relativistic particles of energy $E > 0$ coming from $x \to -\infty$ and being incident on the potential. Calculate the reflection and transmission coefficients.

Reflection coefficient in case of $E > V_0$ case is :

$$R = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2$$

Transmission coefficient in case of $E > V_0$ case is :

$$T = \frac{4k_1k_2}{(k_1 + k_2)^2}$$

where

$$\begin{aligned}k_1^2 &= \frac{2mE}{\hbar^2} \\ k_2^2 &= \frac{2m(E + V_0)}{\hbar^2}\end{aligned}$$

Question 3

A potential barrier is defined by $V = 0eV$ for $x < 0$ and $V = 7eV$ for $x > 0$. A beam of electrons with energy $3eV$ collides with this barrier from left. Find the value of x for which the probability of detecting the electron will be half the probability of detecting it at $x = 0$.

Same as Q1 part(a)

$$\frac{PD(x_0)}{PD(0)} = \frac{1}{2}$$
$$\implies x_0 = \frac{\log 2}{2k_2} = \frac{\hbar \log 2}{\sqrt{8m(V_0 - E)}}$$
$$= 0.33\text{Å}$$

Question 4

A beam of particles of energy E and de Broglie wavelength λ, traveling along the positive x-axis in a potential free region, encounters a one-dimensional potential barrier of height $V = E$ and width L.

(a) Obtain an expression for the transmission coefficient.

From TISE:

$$\frac{1}{\psi(x)}\frac{d^2\psi(x)}{dx^2} = \begin{cases} -k^2 & x < 0 \\ 0 & 0 \leq x \leq L \\ -k^2 & x \geq L \end{cases}$$

$$\psi(x) = \begin{cases} Ae^{-ikx} + Be^{ikx} & x < 0 \\ Cx + D & 0 \leq x < L \\ Ee^{-ikx} + Fe^{ikx} & x \geq L \end{cases}$$

Where

$$k^2 = \frac{2mE}{\hbar^2}$$

We want to analyse the case where the particle is coming in from $x < 0$ and getting reflected and transmitted. Hence $E = 0$.
Continuity and Differentiability at $x = 0$

$$A + B = D$$
$$ik(-A + B) = C$$

Continuity and Differentiability at $x = L$

$$CL + D = Fe^{ikL}$$
$$C = ikFe^{ikL}$$
$$\implies D = Fe^{ikL}(1 - ikL)$$

Putting values of C and D in terms of F

$$A + B = Fe^{ikL}(1 - ikL)$$
$$-A + B = Fe^{ikL}$$
$$\implies B = F\left(1 - \frac{ikL}{2}\right)$$
$$T = \frac{|G|^2}{|B|^2}$$
$$\implies T = \frac{1}{1 + \frac{k^2L^2}{4}}$$
$$= \frac{1}{1 + \frac{mEL^2}{2\hbar^2}}$$

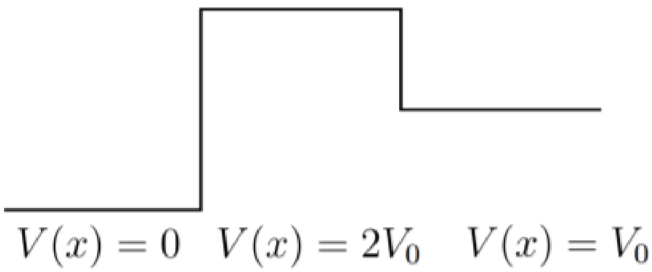
(b) Find the value of L (in terms of λ) for which the reflection coefficient will be half.

We know that if $R = \frac{1}{2} \implies T = \frac{1}{2}$

$$\implies \frac{1}{1 + \frac{k^2L^2}{4}} = \frac{1}{2}$$
$$\implies kL = 2$$
$$\implies L = \frac{\lambda}{\pi}$$

Question 5

A beam of particles of energy $E < V_0$ is incident on a barrier (see figure below) of height $V = 2V_0$. It is claimed that the solution is $\psi_1 = A\exp(-k_1 x)$ for region I($0 < x < L$) and $\psi_{II} = B\exp(-k_2 x)$ for region II($x > L$), where $k_1 = \sqrt{\frac{2m(2V_0 - E)}{\hbar^2}}$ and $k_2 = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$. Is this claim correct? Justify your answer.



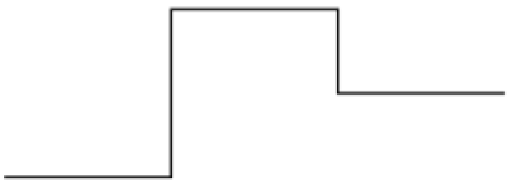
Continuity and Differentiability at $x = L$

$$\begin{aligned} Ae^{-k_1 L} &= Be^{-k_2 L} \\ Ak_1 e^{-k_1 L} &= Bk_2 e^{-k_2 L} \\ \implies k_1 &= k_2 \\ \implies V_0 &= 0 \end{aligned}$$

Thus the only possibility is if both the regions have same potential height, which is not what happens. Hence the claim is incorrect.

Question 6

A beam of particles of mass m and energy $9V_0$ (V_0 is a positive constant with the dimension of energy) is incident from left on a barrier, as shown in figure below. $V = 0$ for $x < 0, V = 5V_0$ for $x \le d$ and $V = nV_0$ for $x > d$. Here n is a number, positive or negative and $d = \frac{\pi\hbar}{\sqrt{8mV_0}}$. It is found that the transmission coefficient from $x < 0$ region to $x > d$ region is 0.75.



$V(x) = 0 \quad V(x) = 5V_0 \quad V(x) = nV_0$

(a) Find n. Are there more than one possible values for n?

From TISE:

$$\frac{1}{\psi(x)}\frac{d^2\psi(x)}{dx^2} = \begin{cases} -k_1^2 & x < 0 \\ -k_2^2 & 0 \le x \le d \\ -k_3^2 & x > d \end{cases}$$

$$\psi(x) = \begin{cases} Ae^{-ik_1 x} + Be^{ik_1 x} & x < 0 \\ Ce^{-ik_2 x} + De^{ik_2 x} & 0 \le x \le d \\ Ee^{-ik_3 x} + Fe^{ik_3 x} & x > d \end{cases}$$

Where

$$\begin{aligned} k_1^2 &= \frac{18mV_0}{\hbar^2} \\ k_2^2 &= \frac{8mV_0}{\hbar^2} \\ k_3^2 &= \frac{2mV_0(9-n)}{\hbar^2} \end{aligned}$$

$F = 0, k_2 d = \pi \implies e^{ik_2 d} = -1$

Continuity and Differentiability at $x = 0$

$$A + B = C + D$$
$$k_1(-A + B) = k_2(-C + D)$$

Continuity and Differentiability at $x = d$

$$-C - D = Fe^{ik_3d}$$
$$-k_2(-C + D) = k_3Fe^{ik_3d}$$

$$\implies A + B = -Fe^{ik_3d}$$
$$\implies -A + B = -\frac{k_3}{k_1}Fe^{ik_3d}$$
$$\implies B = -\frac{k_1 + k_3}{2k_1}Fe^{ik_3d}$$
$$T = \frac{k_3|G|^2}{k_1|B|^2} = 0.75$$
$$\implies T = \frac{4k_3k_1}{(k_3 + k_1)^2} = 0.75$$
$$\implies \frac{k_3}{k_1} = 3\text{or}\frac{k_3}{k_1} = \frac{1}{3}$$

Substituting values of k_1 and k_3

$$\implies n = 8\text{or}n = -72$$

(b) Find the un-normalized wave function in all the regions in terms of the amplitude of the incident wave for each possible value of n.

Solving for A, B, C, D using the boundary condition equations,

$$A = -\frac{k_1 - k_3}{2k_1}Fe^{ik_3d}$$
$$B = -\frac{k_1 + k_3}{2k_1}Fe^{ik_3d}$$
$$C = -\frac{k_2 - k_3}{2k_2}Fe^{ik_3d}$$
$$D = -\frac{k_2 + k_3}{2k_2}Fe^{ik_3d}$$

For $n = 8, k_3d = \frac{\pi}{2} \implies e^{ik_3d} = i$
For $n = -72, k_3d = \frac{9\pi}{2} \implies e^{ik_3d} = i$
Since we cannot normalize our wavefunction, and it can be scaled by any constant,we can set $F = i$ WLOG

$$A = \frac{k_1 - k_3}{2k_1}$$
$$B = \frac{k_1 + k_3}{2k_1}$$

$$C = \frac{k_2 - k_3}{2k_2}$$
$$D = \frac{k_2 + k_3}{2k_2}$$
$$\psi(x) = \begin{cases} \frac{k_1-k_3}{2k_1} e^{-ik_1x} + \frac{k_1+k_3}{2k_1} e^{ik_1x} & x < 0 \\ \frac{k_2-k_3}{2k_2} e^{-ik_2x} + \frac{k_2+k_3}{2k_2} e^{ik_2x} & 0 \leq x \leq d \\ e^{ik_3x} & x > d \end{cases}$$

(c) Is there a phase change between the incident and the reflected beam at $x = 0$? If yes, determine the phase change for each possible value of n. Give your answers by explaining all the steps and clearly writing the boundary conditions used.

As we can see, for $n = 8, k_1 > k_3$, hence A has the same sign as B. Therefore no phase change.
If $n = -72, k_1 < k_3$, hence A has the opposite sign as B, therefore there is a phase change of π rad.

Question 7

A scanning tunneling microscope (STM) can be approximated as an electron tunneling into a step potential [$V(x) = 0$ for $x \leq 0, V(x) = V_0$ for $x > 0$]. The tunneling current (or probability) in an STM reduces exponentially as a function of the distance from the sample. Considering only a single electron-electron interaction, an applied voltage of 5 V and the sample work function of 7 eV, calculate the amplification in the tunneling current if the separation is reduced from 2 atoms to 1 atom thickness (take approximate size of an atom to be 3 A).

From TISE:
$$\frac{1}{\psi(x)} \frac{d^2\psi(x)}{dx^2} = \begin{cases} -k_1^2 & x < 0 \\ k_2^2 & x \geq 0 \end{cases}$$
$$\psi(x) = \begin{cases} Ae^{ik_1x} + Be^{-ik_1x} & x < 0 \\ Ce^{k_2x} + De^{-k_2x} & x \geq 0 \end{cases}$$
Where
$$k_1^2 = \frac{2mE}{\hbar^2}$$
$$k_2^2 = \frac{2m(V_0 - E)}{\hbar^2}$$
Because the wavefunction cannot blow up at infinity $C = 0$
If the seperation between the start of the step potential and the microscope is d , then the tunneling current is proportional to $|\psi(d)|^2$
$$\implies \frac{Current_{1atom}}{Current_{2atom}} = e^{2k_2d} = 78.25\text{Å}$$