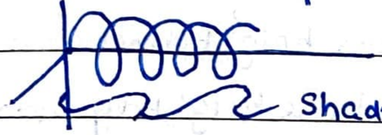
lecture 4

Fourier Analysis

- Consider point P ~~rotating~~ in circular motion.
- x and y components of a rotating pt P ($\theta = \omega t$) Periodic Motion are $x(t)$ and $y(t)$ $\Rightarrow \cos\omega t = \frac{x(t)}{A}$ and $\sin\omega t = \frac{y(t)}{A}$, $\omega = \frac{2\pi}{T}$
- To see the shape make it travel in space:



- General case $x(t) = \underset{\uparrow\,\text{Amplitude}}{A}\cos(\omega t - \underset{\uparrow\,\text{Initial phase}}{\theta_0}) \Rightarrow C\cos\omega t + D\sin\omega t$
  $A^2 = C^2 + D^2$, $\tan\theta = D/C$
- Now consider more points P_n with speed ω_n
  Many waves generated $\omega_n = n\omega_1 \Rightarrow$ ω_1 = Fundamental freq, ω_n = nth harmonic
  $x_n = C_n\cos(n\omega_1 t) + D_n\sin(n\omega_1 t)$
- Add all these. ~~f(t)~~ Now any periodic fn can be represented as a combo of sines and cosines. This is called FOURIER SERIES

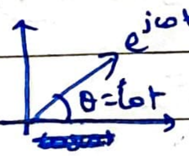
$$f(t) = \underbrace{\frac{C_0}{2}}_{\text{(DC) avg } f(t)} + \sum_{n=1}^{\infty} \underbrace{\left[C_n\cos(\omega n t) + D_n\sin(\omega n t)\right]}_{\text{These have avg}=0} \quad (\star)$$

- C_n and D_n vary as θ_n and Amplitude vary.
  so function can be seen as a set of sines and cosines with varying ω.

$$C_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(\omega n t)\,dt \qquad D_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(\omega n t)\,dt$$

(derivation based on orthogonal property of sines and cosines)

- Fourier series eqn: ($\star$) RHS tells which freqs present in f (freq domain)
  LHS tells how f varies with time (time domain)
- Euler's identity: $e^{j\omega t} = \cos\omega t + j\sin\omega t$ (we want to combine cos & sine)



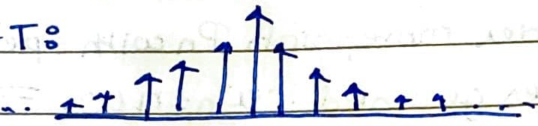
as $t\uparrow$, $\theta\uparrow$ (anti clockwise motion)

$\cos\omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$, $\sin\omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2}$ $\rightarrow$ substitute in ($\star$)

$$\boxed{f(t) = \sum_{-\infty}^{\infty} A_n e^{j\omega n t}}$$

Date:

- We took Σ as $\omega_n = n\omega$ so Σ works for periodic fns.
- For non periodic functions, frequencies not discrete but continuous
- FOURIER TRANSFORM: $F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt$ | INVERSE FOURIER TRANS: $f(t) = \int_{-\infty}^{\infty} F(\omega) e^{j\omega t}\, d\omega$ (Represent fn in terms of its FT)
- F.T of $e^{j\omega_0 t} \Rightarrow 2\pi\, \delta(\omega - \omega_0)$
- $a + b\sin\omega_0 t \Rightarrow$ FT will have δ at 0 whose height depends on a
  FT will have δ at $\pm\omega_0$ whose height depends on b.
- FT is a linear operator.
- Use filters on FT and then take IFT to remove certain frequencies
  eg. square wave FT:



Higher frequencies carry info about sudden transition and lower frequencies carry info about peak values.

- Spacial Fourier transform

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux}\, dx \Rightarrow 1D$$

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)}\, dx\, dy \Rightarrow 2D$$

u similar to f
u and v spacial frequencies.

- So ~~using a~~ obj $\xrightarrow{F.T}$ Δ $\xrightarrow{\text{Low pass Filter}}$ blurred image, soft boundaries sharpness of
  $\xrightarrow{\text{high pass filter}}$ only sharp boundary (outline seen)

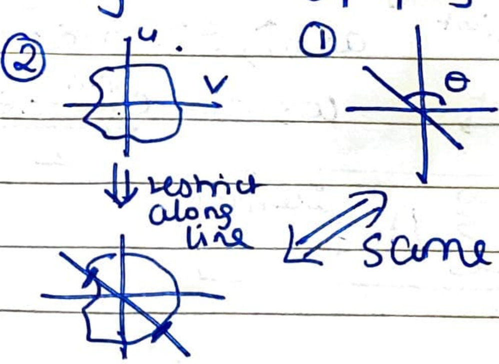
$\Rightarrow$ Use in CT Scan

measurement

$$S_\theta(\omega) = \int_{-\infty}^{\infty} P_\theta(t) e^{-j\omega t}\, dt \Rightarrow \text{1D FT of measurement} \rightarrow (1)$$

$$\tilde{\mu}(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(x,y) e^{-j2\pi(ux+vy)}\, dx\, dy \Rightarrow \text{2D FT of object property} \rightarrow (2)$$

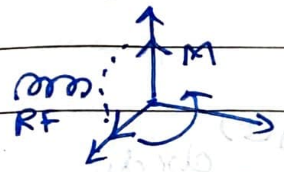
Restrict (2) along a line that gives (1).
Thus, Rotate (1) and take various lines, this will reconstruct $\mu(u,v)$



Now take IFT to get $\mu(x,y)$

Less blurred than previous approach.

MRI

- H_2O dipole due to spinning proton.
- Net (avg) ~~field ind~~ moment zero as randomly alligned.
- Apply B_0 (const external field) $\Rightarrow$ dipoles alligned in B_0 direction and cause induced B (magnetic field).
- B_0 also determines processional Frequency, ω_0 (how fast proton spins)
- Magnetisation M = density of magnetic moment.
- 
  Initially ~~protons~~ dipoles alligned along B_0.
  We give radio freq at $90°$ to B_0 with same freq ω_0
  This causes dipoles to flip by $90°$ and they start rotating along xy plane.
  This induces current signal in the RF coil placed in the xy plane ~~$\Rightarrow$ this sign~~
  This signal measured as $s(t)$
- We can write M as $m(r,t) = M_0(r)\, e^{-j\omega t}$ (rotates)
- $t\, \omega_0 = \gamma B_0 t$ $\Rightarrow$ this works for const B_0
  $\uparrow$ gyromagnetic ratio.
- If B changes spacially,
  $B(r,t) = B_0 + \Delta B(r,t)$
  then phase term $\omega_0 t + \gamma \int^{t} \Delta B(r,t')\,dt'$
  so $m(r,t) = m_0(r) e^{-i\omega_0 t} \times e^{-i\gamma \int_0^t \Delta B(r,t')dt'}$
- We want to include points at all locations
  so $s(t) = \iint m(x,y)\, e^{-i\gamma \left[\int_0^t \Delta B(r,t')dt'\right]} dx\,dy \rightarrow$ (a)
  ω_0 is filtered out.
- Define $\Delta B(r,t)$ as $\Delta B(x,y,t) \Rightarrow G_x(t)\, x + G_y(t)\, y \rightarrow$ (b)
  (as a linear fn in x and y where G varies with time)
- This shows that (on substituting (b) in (a))
  The 2d FT of conc of H atoms of human body (m) which is a tissue property (RHS)
  is equal to the value measured $s(t)$ (LHS)
- So simply take ~~IFT~~ 2D IFT of signal to get the body tissue image

- We used 3 fields.

  (1) B_0 → main field to allign magnets.

  (2) R.F → flips M by 90° rotating in xy plane and induces $S(t)$

  (3) $S(t)$ is from all over the body, we want to localise souse Gradient field (ΔB) which tells us which slice of the body the ~~waves~~ signals are coming from.

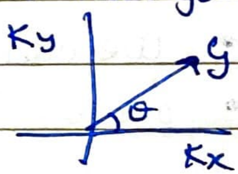
- Define $K_x(t) = \frac{\gamma}{2\pi} \int_0^t G_x(a)\,da$

$$K_y(t) = \frac{\gamma}{2\pi} \int_0^t G_y(a)\,da$$

$$\text{So } S(t) = \iint m(x,y)\, e^{-2\pi(x \underbrace{K_x(t)}_{u} + y \underbrace{K_y(t)}_{v})}\,dx\,dy$$

Measurement
- If G_x & G_y const for a short time,
  K_x and K_y linear ( )
  Change magnitudes of G_x and G_y to get different $K_x K_y$ slopes

$$\left.\begin{aligned} K_x &= G_x t \\ K_y &= G_y t \end{aligned}\right\} G = G_x^2 + G_y^2, \quad \theta = \tan^{-1}\frac{G_y}{G_x}$$



  The $K_x K_y$ plot is the U V plot
  and so it is the 2D FT of $m(x,y)$
  Like we did for CT scan, plot at diff θ to get complete 2D FT
  Then take 2D IFT to get tissue Image.

- MRI can be in many directions, no need for radial so benefit