MA106 Tut5

Q1. $|B - \lambda I| = 0$

$\longleftrightarrow |P^{-1}AP - \lambda I| = 0$

$\longleftrightarrow |P^{-1}AP - \lambda P^{-1}P| = 0$

$\longleftrightarrow \cancel{|P^{-1}|}\,|A - \lambda I|\,\cancel{|P|} = 0$ $\qquad |AB| = |A||B|$

$\longleftrightarrow |A - \lambda I| = 0$ $\qquad [\,|P| \neq 0\,]$

Q2. $\boxed{\text{M1}}$ Using eigenstuff

Suppose $I - AB$ is not invertible

$\therefore \exists x \neq 0 \quad (I - AB)x = 0$

$\rightarrow ABx = x$

$\therefore 1$ is an eigenvalue of AB

Now,

$B(ABx) = Bx$

$\rightarrow BA(Bx) = (Bx)$

Also, $Bx \neq 0$ as $A(Bx) = x \neq 0$

Let $y = Bx$, $y \neq 0$

$\therefore BAy = y$

$\therefore 1$ is an eigenvalue of BA

$\therefore \exists y \neq 0 \quad (I - BA)y = 0$

$\therefore (I - BA)$ is not invertible

Similary prove the other direction

Lemma: If AB has an eigenvalue λ then, BA also has λ as an eigenvalue

Proof: $\exists x \neq 0 \quad ABx = \lambda x$

If $\lambda \neq 0$:

$BA(Bx) = \lambda(Bx)$

and $Bx \neq 0$ as $ABx \neq 0$

$\therefore \exists y \neq 0 \quad BAy = \lambda y$

A matrix has a 0 eigenvalue iff it is not invertible $\qquad x \neq 0$, $Ax = 0$

If $\lambda = 0$

$\rightarrow |AB| = 0$

$\rightarrow |A| = 0$ or $|B| = 0$

$\rightarrow |BA| = 0$

$\rightarrow$ 0 is an eigenvalue of BA

M2] Explicitely construct Inverse

Suppose the inverse of I-AB is C

$C(I-AB) = I$

$\rightarrow C - CAB = I$

$\rightarrow CA - CABA = A$

$\rightarrow BCA - BCABA = BA$

$\rightarrow BCA(I-BA) = I - (I-BA)$

$\rightarrow (I+BCA)(I-BA) = I$

Q3 Nullity = k, hence rank = n-k

Hence, for all mxm submatrices, where m > n-k; the determinant is 0

Now, consider the characteristic equation:

$$x^n - a_{n-1}x^{n-1} + a_{n-2}x^{n-2} - a_{n-3}x^{n-3} \dots = 0$$

Then,

$$a_{n-i} = \sum \text{det of } (i \times i) \text{ principal minors}$$

Hence, for i > n-k; all the determinants of principle minors would be 0

$$\therefore a_{n-i} = 0 \quad \forall\, i > n-k \rightarrow n-i < k$$

$\therefore$ x^k divides

Another nice solution is by change of basis (taught?)

This also has an important implication:

For any eigenvalue, the algebraic multiplicity is >= the geometric multiplicity

Algebraic multiplicity: Number of occurences of λ in roots of $|xI - A| = 0$

Geometric multiplicity: nullity of $(A - \lambda I)$ or equivalent the number of linearly independent eigenvectors for the eigenvalue λ

Proof: Let nullity of $(A - \lambda I)$ be k

GM = k

Then, the poly:

$$|xI - (A - \lambda I)|$$

is divisible by x^k

Hence, put $y = x+\lambda$

$|yI - A|$

is divisible by $(y-\lambda)^k$

$\therefore$ $|yI - A|$ has atleast k roots as λ

$$\downarrow$$

$AM \geq k$

$\therefore$ $AM \geq GM$

84 manual labour

(b) $$\begin{vmatrix} 4-\lambda & -1 & -2 \\ 2 & 1-\lambda & -2 \\ 1 & -1 & 1-\lambda \end{vmatrix} = 0$$

$$\rightarrow (4-\lambda)(\lambda^2-2\lambda+1) + 2+4 + 2(1-\lambda) - 2(4-\lambda) + 2(1-\lambda) = 0$$

$$\rightarrow 4\lambda^2 - 8\lambda + 4 - \lambda^3 + 2\lambda^2 - \lambda + 6 + 4 - 8 - 6\lambda = 0$$

$$\rightarrow \lambda^3 - 6\lambda^2 + 13\lambda - 6 = 0$$

$$\rightarrow \lambda = 1,2,3$$

Finding eigenvectors:

$\lambda = 1$

$$\begin{bmatrix} 4 & -1 & -2 \\ 2 & 1 & -2 \\ 1 & -1 & 1 \end{bmatrix} X = X$$

$$\rightarrow \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 1 & -1 & 0 \end{bmatrix} X = 0$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & -2 \\ 0 & 2 & -2 \end{bmatrix} X = 0 \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} X = 0$$

$$X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ is one sol}^n$$

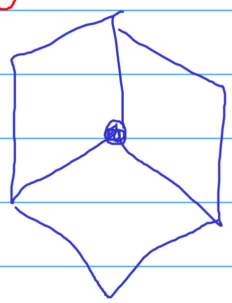
Do the rest :)

1) 5

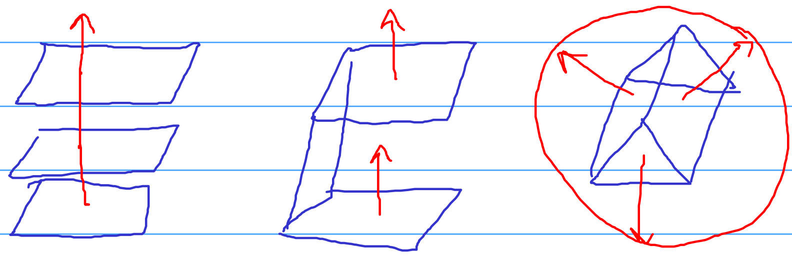
2) AB

$Ax=b$

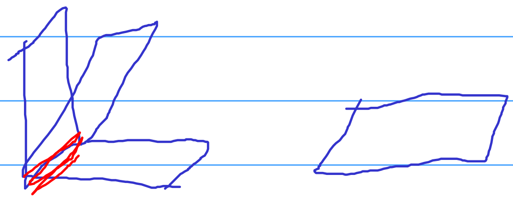
① $|A| \neq 0$



② $|A| = 0$ & No sol^n



③ $|A| = 0$ ∞ sol^n



3) BC

f

Av

① $f(u+v) = f(u)+f(v)$

② $f(\alpha v) = \alpha f(u)$

$[1,0,0]$ $[-2,0,0]$

$f(u+v) = [1,0,0]$

$f(u)+f(v) = [3,0,0]$

4) 21

$3,1,4,2,6,5$

$i = 4$

$\epsilon = (-1)^i = 1$

5) d

6) a, d

u, v, z

$z e^{i\varphi}$

set S is a vector space over F:

1) $u, v \in S \rightarrow u+v \in S$

2) $u \in S,\ \alpha \in F \rightarrow \alpha u \in S$

Consider the set of all skew-hermitian matrices

$$A + A^* = 0$$

is this a vector space? If yes, what is F?

Yes $\cancel{\mathbb{C}}$ $\mathbb{R}$

$$A = \begin{bmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad A^* = \begin{bmatrix} 0 & -i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\downarrow$$

$$A + A^* = 0$$

A, B

$(A+B)^* = A^* + B^*$

$(\alpha A)^* = \alpha A^*$

$\alpha \in \mathbb{R}$

$$B = iA = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad B^* = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B + B^* \neq 0$$

$$(\alpha A) + (\alpha A)^*$$

$$= \alpha (A + A^*) = 0$$

Previous tut Problem 1

$\underbrace{A_{k\times n}}_{\leq k}$ $k<n$

rank(AB) $\leq$ min(rank A, rank B)

Lemma: the non-zero eigenvalues of AA^T are also of A^TA and vice-versa

$$AA^Tv = \lambda v \qquad \lambda \neq 0$$

$$A^TA(A^Tv) = \lambda(A^Tv)$$

Non zero

$$A^TAy = \lambda y \qquad y \neq 0$$

infact, these eigenvalues have the same multiplicities

$A^TA \rightarrow n\times n$ matrix $\rightarrow$ rank$(A^TA) \leq k$

$AA^T \rightarrow k\times k$ matrix $\rightarrow$

$$|xI - A^TA| = 0$$

atleast $(n-k)$ roots are 0

$$\sum k\times k \text{ PM} = \text{coef. of } x^{n-k}(-1)^{n-k}$$

$=$ $\underbrace{\text{sum of roots taken k at a time}}$

atmost one non-zero term

this is simply the product of eigenvalues of AA^T

$\det(AA^T)$