MA-111 Calculus II
(D1 & D2)

Lecture 9

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Green's Theorem

With the preliminaries out of the way, we are now in a position to state the first major theorem of vector calculus, namely Green's Theorem.

Theorem (Green's theorem:)

1. *Let D be a bounded region in $\mathbb{R}^2$ with a positively oriented boundary ∂D consisting of a finite number of non-intersecting simple closed piecewise continuously differentiable curves.*
2. *Let Ω be an open set in $\mathbb{R}^2$ such that $\left(D \cup \partial D\right) \subset \Omega$ and let $F_1 : \Omega \to \mathbb{R}$ and $F_2 : \Omega \to \mathbb{R}$ be $\mathcal{C}^1$ functions. Consider the vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$.*

Then

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \int_{\partial D} F_1 dx + F_2 dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy.$$

The importance of Green's theorem is that it converts a double integral into a line integral. Depending on the situation, one may be easier to evaluate than the other.

Example: Let C be the circle of radius r oriented in the counterclockwise direction, and let $F_1(x, y) = -y$ and $F_2(x, y) = x$. Evaluate

$$\int_C F_1(x, y)dx + F_2(x, y)dy.$$

Solution: Let D denote the disc of radius r. Then $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 2$. Hence, by Green's theorem

$$\int_C F_1(x, y)dx + F_2(x, y)dy = \iint_D 2dxdy = 2\pi r^2.$$

Also by the direct calculation, denoting $\mathbf{F} = (F_1, F_2)$, check $\int_C \mathbf{F}.\mathbf{ds} = \int_C F_1(x, y)dx + F_2(x, y)dy = ?$.

Examples.

Example. Compute the line integral $\int_C ye^{-x}\,dx + (\frac{1}{2}x^2 - e^{-x})\,dy$, where C is the circle in $\mathbb{R}^2$ with origin at $(2,0)$ and radius 1.

Can compute directly using definition of line integral! But is there any better way?

Use Green's theorem: Set $F_1(x,y) = ye^{-x}$ and $F_2(x,y) = (\frac{1}{2}x^2 - e^{-x})$, for all $(x,y) \in D$, where $D = \{(x,y) \in \mathbb{R}^2 \mid (x-2)^2 + y^2 \leq 1\}$. Using Green's theorem,

$$\int_C ye^{-x}\,dx + (\frac{1}{2}x^2 - e^{-x})\,dy = \int\int_D \Big[\frac{\partial F_2}{\partial x}(x,y) - \frac{\partial F_1}{\partial y}(x,y)\Big]\,dxdy.$$

Now see

$$\int\int_D \Big[\frac{\partial F_2}{\partial x}(x,y) - \frac{\partial F_1}{\partial y}(x,y)\Big]\,dxdy = \int\int_D xdxdy,$$

and derive the double integral using polar coordinates: Check!

$$\int\int_D xdxdy = 2\pi.$$

Area of a region

Can the area of a region enclosed be expressed as a line integral?

If C is a positively oriented curve that bounds a region D. Then the area $A(D)$ is given by

$$A(D) = \frac{1}{2}\int_C xdy - ydx = \int_C x\,dy = -\int_C y\,dx.$$

• Put $F_1(x,y) = -\frac{y}{2}$ and $F_2(x,y) = \frac{x}{2}$, for all $(x,y) \in D$. Then $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$, and hence $A(D) := \int\int_D 1\,dxdy = \int\int_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\,dxdy$.
By Green's theorem,

$$\int_D\int_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \int_C F_1\,dx + F_2\,dy = \frac{1}{2}\int_C xdy - ydx,$$

Thus $A(D) = \frac{1}{2}\int_C xdy - ydx$.

• Put $F_1 \equiv 0$ and $F_2(x,y) = x$, on D, $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$. Thus $A(D) = \int_C x\,dy$.

• Put $F_1(x,y) = -y$ and $F_2 \equiv 0$, on D, $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$. Thus $A(D) = -\int_C y\,dx$.

Example: Let us use the formula above to find the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution: We parametrise the curve C by $\mathbf{c}(t) = (a\cos t, b\sin t)$, $0 \leq t \leq 2\pi$. By the formula above, we get

$$
\begin{aligned}
\text{Area} &= \frac{1}{2}\int_C xdy - ydx \\
&= \frac{1}{2}\int_0^{2\pi} (a\cos t)(b\cos t)dt - (b\sin t)(-a\sin t)dt \\
&= \frac{1}{2}\int_0^{2\pi} abdt = \pi ab.
\end{aligned}
$$

Polar coordinates

Suppose we are given a simple positively oriented closed curve $C : (r(t), \theta(t))$ in polar coordinates. Thus for $t \in [a, b]$ $x(t) = r(t)\cos(\theta(t))$ and $y(t) = r(t)\sin(\theta(t))$ and using chain rule formula:

$$\frac{dx}{dt}(t) = \cos(\theta(t))\frac{dr}{dt}(t) - r(t)\sin\theta(t)\frac{d\theta}{dt}(t),$$
$$\frac{dy}{dt}(t) = \sin(\theta(t))\frac{dr}{dt}(t) + r(t)\cos\theta(t)\frac{d\theta}{dt}(t).$$

Then, by the area formula above, we know that the area enclosed by C is given by

$$\begin{aligned}
&\frac{1}{2}\int_C xdy - ydx := \frac{1}{2}\int_C \left(x(t)\frac{dy}{dt}(t) - y(t)\frac{dx}{dt}(t)\right) dt \\
&= \frac{1}{2}\int_a^b r(t)\cos\theta(t)\sin\theta(t)\frac{dr}{dt}dt + \frac{1}{2}\int_a^b r^2(t)\cos^2\theta(t)\frac{d\theta}{dt}dt \\
&-\frac{1}{2}\int_a^b r(t)\sin\theta(t)\cos\theta(t)\frac{dr}{dt}dt + \frac{1}{2}\int_a^b r(t)^2\sin^2\theta(t)\frac{d\theta}{dt}dt \\
&= \frac{1}{2}\int_C r^2 d\theta.
\end{aligned}$$

Exercise: Find the area of the cardioid $r = a(1 - \cos\theta)$, $0 \leq \theta \leq 2\pi$.

Solution: Using the formula we have just derived, the desired area is simply

$$\frac{1}{2}\int_0^{2\pi} a^2(1-\cos\theta)^2 d\theta = a^2\int_0^{2\pi} -2\cos\theta + \frac{\cos 2\theta}{2} + \frac{3}{2} d\theta$$
$$= \frac{3a^2\pi}{2}.$$

A proof of Green's theorem for regions of special type

We give a proof of Green's theorem when the region D is both of type 1 and type 2 .

Examples: Rectangles, Discs are examples of such region.

Assume that D is of Type 1

$$D = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, \quad \phi_1(x) \leq y \leq \phi_2(x)\},$$

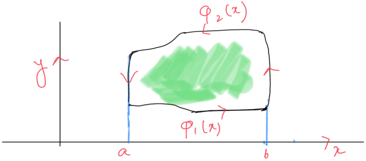
for some continuous functions ϕ_1 and ϕ_2.

Also assume there exist two continuous functions ψ_1 and ψ_2 such that D can be written as Type 2:

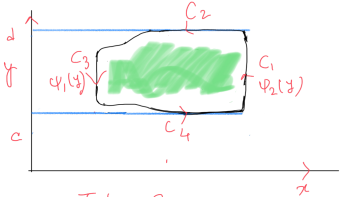
$$D = \{(x, y) \in \mathbb{R}^2 \mid c \leq y \leq d, \quad \psi_1(y) \leq x \leq \psi_2(y)\}.$$

The proof follows two main steps:

- Double integrals can be reduced to iterated integrals.
- Then the fundamental theorem of calculus can be applied to the resulting one-variable integrals.



Type 1.



Type 2.

The proof of Green's theorem, contd.

To prove

$$\int_{\partial D} F_1 dx + F_2 dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy,$$

we show

Step 1 Using the fact that D is a region of Type 2,

$$\iint_D \frac{\partial F_2}{\partial x} = \int_{\partial D} F_2 dy.$$

Step 2 Using the fact that D is a region of Type 1,

$$-\iint_D \frac{\partial F_1}{\partial y} = \int_{\partial D} F_1 dx.$$

Then combining the both equalities, we get our result.
Since D is a region of Type 2, it gives

$$\iint_D \frac{\partial F_2}{\partial x} dxdy = \int_c^d \int_{x=\psi_1(y)}^{\psi_2(y)} \frac{\partial F_2}{\partial x}(x,y) dxdy.$$

Using the Fundamental Theorem of Calculus we get

$$\int_c^d \int_{x=\psi_1(y)}^{\psi_2(y)} \frac{\partial F_2}{\partial x}(x,y) dxdy = \int_c^d F_2(\psi_2(y), y) - F_2(\psi_1(y), y) dy$$

The proof of Green's theorem contd.

Now let us calculate $\int_{\partial D} F_2 dy$. Note that ∂D can be written as union of four curves C_1, C_2, C_3 and C_4 such that
On C_1: $C_1 = \{(\psi_2(y), y) \in \mathbb{R}^2 \mid c \leq y \leq d\}$ with direction upwards. So,

$$\int_{C_1} F_2\, dy = \int_c^d F_2(\psi_2(y), y)\, dy.$$

On C_3: $C_3 = \{(\psi_1(y), y) \in \mathbb{R}^2 \mid c \leq y \leq d\}$ with direction downwards. So,

$$\int_{C_3} F_2\, dy = -\int_{-C_3} F_2\, dy = -\int_c^d F_2(\psi_1(y), y)\, dy.$$

On C_2 and C_4: $C_2 = \{(x, d) \mid \psi_1(d) \leq x \leq \psi_2(d)\}$ going from right to left and $C_4 = \{(x, c) \mid \psi_1(c) \leq x \leq \psi_2(c)\}$ going from left to right. In particular, they are vertical lines and y is constant along these lines. Thus, for any parametirzation of C_2 and C_4, $\frac{dy}{dt} = 0$, and

$$\int_{C_2} F_2\, dy = 0 = \int_{C_4} F_2\, dy.$$

The proof of Green's theorem contd.

Noting that

$$\int_{\partial D} F_2 dy = \int_{C_1} F_2\, dy + \int_{C_2} F_2\, dy + \int_{C_3} F_2\, dy + \int_{C_4} F_2\, dy,$$

and using previous results, we obtain

$$\int_{\partial D} F_2 dy = \int_c^d F_2(\psi_2(y), y)\, dy - \int_c^d F_2(\psi_1(y), y)\, dy,$$

and thus

$$\iint_D \frac{\partial F_2}{\partial x} dxdy = \int_{\partial D} F_2 dy.$$

Similarly, using the fact that D can be written as a region of Type 1, we get

$$\iint_D \frac{\partial F_1}{\partial y} dxdy = -\int_{\partial D} F_1 dx.$$

Where does the minus sign come from?

From the fact that $y = \phi_2(x)$ is oriented in the direction of decreasing x.

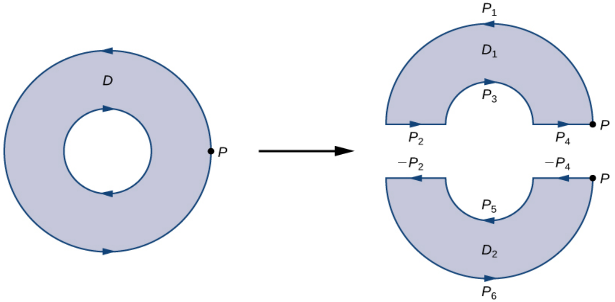
Subtracting the two equations above, we get

$$\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy = \int_{\partial D} F_1 dx + \int_{\partial D} F_2 dy.$$

A more general case

How does one proceed in general, that is for more general regions which may not be of both type 1 and type 2. We can try proceeding as follows:

- Break up D into smaller regions each of which is of both type 1 and type 2 but so that any two pieces meet only along the boundary.
- Apply Green's theorem to each piece.
- Observe that the line integrals along the interior boundaries cancel, leaving only the line integral around the boundary of D.



Del operator on vector fields

The del operator operates on vector fields as in two different ways. For a vector field $\mathbf{F} = (F_1, F_2, F_3)$ we define the curl of $\mathbf{F}$:

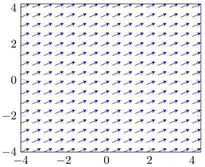
$$\operatorname{curl}\mathbf{F} := \nabla \times \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.$$

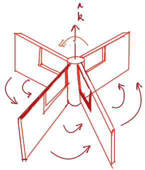
It is often written as a determinant;

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}.$$

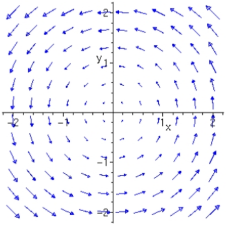
Curl as a measure of rotation

Curl of a vector field is measuring the extent to which the field rotate a particle. For instance ,





Imagine putting a small paddle wheel as shown in the above figure at any point in the plane with the vector field acting on it and visualize how it will rotate. Clearly in this example it will not rotate.



How about in this example?

Angular velocity

Consider a solid body B rotating around the z-axis on the x-y-plane.

Let $\mathbf{v}$ denote the velocity vector, $\mathbf{w}$ the angular velocity vector at a point $\mathbf{r}$ in B. Note $\mathbf{w} = \omega\mathbf{k}$, where ω is the angular speed. Further,

$$\mathbf{v} = \mathbf{w} \times \mathbf{r} = -\omega y\mathbf{i} + \omega x\mathbf{j}.$$

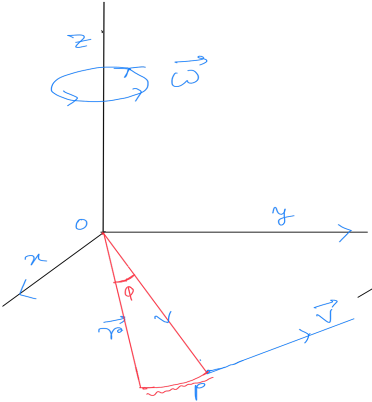
(Hint: put $\mathbf{r} = a(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$, so $\mathbf{v} = d\mathbf{r}/dt = a(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j})\frac{d\theta}{dt}$.)
Now,

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = 2\omega\mathbf{k} = 2\mathbf{w}.$$

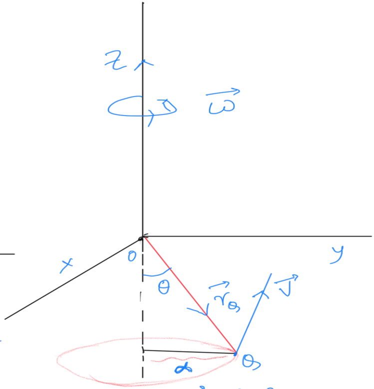
Thus, the curl of velocity is twice the angular velocity.

If a vector field $\mathbf{F}$ represents the flow of a fluid, then the value of $\nabla \times \mathbf{F}$ at a point is twice the rotation vector of a rigid body that rotates as the fluid does near that point. In particular, $\nabla \times \mathbf{F} = 0$ at a point P means that the fluid is free from the rigid rotations at P.
The curl free vector field is called irrotational field.

Angular velocity



$$\vec{V} = \vec{\omega} \times \vec{r}.$$



$$\alpha = |\vec{r}_\theta| \sin\theta.$$

The curl of a gradient

Suppose that $\mathbf{F} = \nabla f$ for some scalar function f and f is $\mathcal{C}^2$. Then

$$\nabla \times \nabla f = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\mathbf{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right)\mathbf{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\mathbf{k}.$$

Clearly,

$$\nabla \times \mathbf{F} = 0.$$

In particular, this gives a criterion for deciding whether a vector field arises as the gradient of a function. This gives that curl$\mathbf{F} = 0$ is a necessary condition for any smooth vector field $\mathbf{F}$ to be the gradient field.

Theorem (Green's theorem using curl)

Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ be a C^1 vector field on an open connected region D with ∂D be positively oriented. Then

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \iint_D (curl\,\mathbf{F} \cdot \mathbf{k})\ dxdy.$$

Is the condition $\nabla \times \mathbf{F} = 0$ sufficient for $\mathbf{F}$ to be a gradient field?

Example Consider the vector field

$$\mathbf{F} = \frac{y}{x^2+y^2} \cdot \mathbf{i} + \frac{-x}{x^2+y^2} \cdot \mathbf{j},$$

Check that $\nabla \times \mathbf{F} = 0$.
$\mathbf{F}$ is not conservative because the line integral of $\mathbf{F}$ over the unit circle is non-zero.

Conservative field and its curl in $\mathbb{R}^2$

Theorem

1. *Let Ω be an open, simply connected region in $\mathbb{R}^2$.*
2. *if $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$ is such that F_1 and F_2 have continuous first order partial derivatives on Ω.*

Then $\mathbf{F}$ is a conservative field in Ω if and only if

$$\mathit{curl}\,\mathbf{F} = 0, \quad \text{in} \quad \Omega.$$

Outline of the proof: Let the assumptions on Ω and $\mathbf{F}$ in the statement hold.

- If $\mathbf{F}$ is C^1 and a conservative field, i.e., $\mathbf{F} = \nabla f$, for some f is C^2. Then a direct calculation gives $\text{curl} F = 0$.
- Now conversely, if $\mathbf{F}$ is C^1 and $\text{curl}\,\mathbf{F} = 0$ on Ω. Then by Green's theorem we can show that the line integral of F over any simple closed curve is 0. That is, the line integral of F in Ω is path independent. Hence the result follows.

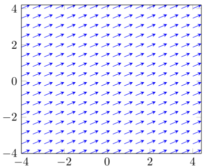
The divergence of a vector field

The del operator can be made to operate on vector fields to give a scalar function as follows.

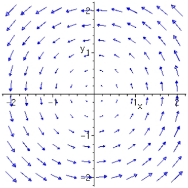
Definition: Let $\mathbf{F} = (F_1, F_2, F_3)$ be a vector field. The divergence of $\mathbf{F}$ is the scalar function defined by

$$\operatorname{div} \mathbf{F} := \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

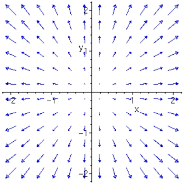
One way to interpret divergence of a velocity vector field at a point P as the amount of fluid flowing in versus the amount of fluid flowing out.



If $\mathbf{F}$ is a constant vector field then at any point what is flowing in is flowing out and the divergence is 0.



Is the divergence for this vector field 0?



This should have non-zero divergence. But what is it measuring?

Physical interpretation

If $\mathbf{F}$ is the velocity field of a fluid, the divergence of $\mathbf{F}$ gives the rate of expansion of the volume of the fluid per unit volume as the volume moves with the flow. In the case of planar vector fields we get the corresponding rate of expansion of area.

Example : $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$. The flow lines of this vector field point radially outward from the origin, so it is clear that the fluid is expanding as it flows. This is reflected in the fact that

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) = 2 > 0.$$

Example : If we look at the vector field $\mathbf{F} = -x\mathbf{i} - y\mathbf{j}$, we see that $\nabla \cdot \mathbf{F} = -2$. This is consistent with the fact that the flow lines of the vector field all point towards the origin, and the fluid is getting compressed.

Example : $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$. In this case the fluid is moving countercolockwise around the origin - so it is neither being compressed, nor is it expanding. One checks easily that $\nabla \cdot \mathbf{F} = 0$.

The change in area in a flow

Let us assume the vector field $\mathbf{v} = (u, v)$ represents the velocity field of a fluid in $\mathbb{R}^2$.

Let us compute the rate of change of unit area of the fluid as it flows along the curve.

We assume that we start at time $t = 0$ at a point $P = (x, y)$ in $\mathbb{R}^2$. Let the point evolve under the velocity field $\mathbf{v}$ to a point (X, Y) at time t. In particular,

$$X = X(x, y, t), Y = Y(x, y, t).$$

The change of variables formula tells us how an elementary area changes. Computing the Jacobian determinant for mapping $h(x, y) = (X, Y)$

$$J(x, y, t) = \begin{vmatrix} \frac{\partial X}{\partial x}(x, y, t) & \frac{\partial X}{\partial y}(x, y, t) \\ \frac{\partial Y}{\partial x}(x, y, t) & \frac{\partial Y}{\partial y}(x, y, t) \end{vmatrix}.$$

Now computing $\frac{\partial J(x,y,t)}{\partial t}$,

$$\begin{aligned} \frac{\partial J}{\partial t} &= \frac{\partial^2 X}{\partial x \partial t}\frac{\partial Y}{\partial y} + \frac{\partial X}{\partial x}\frac{\partial^2 Y}{\partial y \partial t} - \left(\frac{\partial^2 X}{\partial y \partial t}\frac{\partial Y}{\partial x} + \frac{\partial X}{\partial y}\frac{\partial^2 Y}{\partial x \partial t}\right) \\ &= (\nabla \cdot \mathbf{v})J. \end{aligned}$$

Divergence free is area preserving

Putting $\frac{\partial X}{\partial t}(x, y, t) = u(X(x, y, t), Y(x, y, t))$ and $\frac{\partial Y}{\partial t}(x, y, t) = v(X(x, y, t), Y(x, y, t))$, $\frac{\partial J}{\partial t}$ is equal to

$$\frac{\partial J}{\partial t} = (\nabla \cdot \mathbf{v})J.$$

Thus, $\nabla \cdot \mathbf{v} = 0$ if and only if J is independent of t. Since at $t = 0$, $J(x, y, 0) = 1$, $J(x, y, t) = J(x, y, 0) = 1$, for all t. There is no change of coordinates and hence Jacobian is trivial.
Clearly, $J = 1$ means that there is no change in the area,

$$Area(D) = \iint_D dXdY = \iint_{D'} |J(x, y)| dxdy = \iint_{D'} dxdy = Area(D').$$

The divergence free vector field is called incompressible field.

The divergence of any curl is zero. In other words, if $\mathbf{G}$ is a $\mathcal{C}^2$ vector field,

$$\operatorname{div}(\operatorname{curl}\mathbf{G}) = \nabla \cdot (\nabla \times \mathbf{G}) = 0.$$

Qn : If $\nabla \cdot \mathbf{F} = 0$, does it imply that $\mathbf{F} = \nabla \times \mathbf{G}$ for some vector field $\mathbf{G}$?

This question is related to the topological properties to of the domain of the vector field as in the case of when a curl free vector field is a gradient vector field. We will be able to show that this is the case when the domain is $\mathbb{R}^n$ for $n = 2, 3$. We postpone it for later.

Next, we mention the Divergence theorem in $\mathbb{R}^2$:
Let ∂D be a non-singular, positively oriented curve in $\mathbb{R}^2$, parametrized by $\mathbf{c} : [a, b] \to \mathbb{R}^3$ such that $\mathbf{c}(t) = (x(t), y(t), 0)$. Then the unit tangent to the curve $\mathbf{c}$ and the unit outward normal to the curve are denoted by

$$\mathbf{T}(t) = \frac{\mathbf{c}'(t)}{\|\mathbf{c}'(t)\|}, \quad \mathbf{n}(t) = \mathbf{T}(t) \times \mathbf{k}, \quad \forall\, t \in [a, b].$$

Divergence form of Green's theorem

Divergence form or Normal form:

$$\int_{\partial D} \mathbf{F}.\mathbf{n} ds = \int\int_D \text{div}\mathbf{F} dxdy.$$

Gauss's divergence theorem is a 3-dimensional analogue of the above result.

Outline of its proof: Since $\mathbf{n}(t) = \mathbf{T}(t) \times \mathbf{k}$, for all $t \in [a, b]$, using the definition of $\mathbf{c}(t)$, we get $\mathbf{n}(t) = \left(\frac{y'(t)}{\|\mathbf{c}'(t)\|}, \frac{-x'(t)}{\|\mathbf{c}'(t)\|}, 0\right)$. Thus, for $\mathbf{F} = (F_1, F_2, 0)$, using $ds = \|\mathbf{c}'(t)\| dt$

$$\int_{\partial D} \mathbf{F}.\mathbf{n} ds = \int_{\partial D} \left[F_1(\mathbf{c}(t)) \frac{y'(t)}{\|\mathbf{c}'(t)\|} - F_2(\mathbf{c}(t)) \frac{x'(t)}{\|\mathbf{c}'(t)\|}\right] ds$$
$$= \int_{\partial D} [F_1(\mathbf{c}(t))y'(t) - F_2(\mathbf{c}(t))x'(t)] dt = \int_{\partial D} F_1\, dy - F_2\, dx.$$

Now by Green's theorem, we get

$$\int_{\partial D} F_1\, dy - F_2\, dx = \iint_D \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) dxdy = \iint_D \text{div}\mathbf{F} dxdy.$$

Physical interpretation of Divergence theorem

We can interpret the above theorem in the context of fluid flow. If $\mathbf{F}$ represents the flow of a fluid, then the left hand side of the divergence theorem represents the net flux of the fluid across the boundary ∂D. On the other hand, the right hand side represents the integral over D of the rate $\nabla \cdot \mathbf{F}$ at which fluid area is being created. In particular if the fluid is incompressible (or, more generally, if the fluid is being neither compressed nor expanded) the net flow across ∂D is zero.

We can talk about volume analogously in the three dimensional case after proving Stokes theorem.

Surfaces : Definition

A curve is a "one-dimensional" object. Intuitively, this means that if we want to describe a curve, it should be possible to do so using just one variable or parameter.

To do line integration, we further required some extra properties of the curve - that it should be $\mathcal{C}^1$ and non-singular.

We will now discuss the two dimensional analog, namely, surfaces. In order to describe a surface, which is a two-dimensional object, we clearly need two parameters.

Definition

Let D be a path connected subset in $\mathbb{R}^2$. A parametrised surface is a continuous function $\mathbf{\Phi} : D \to \mathbb{R}^3$.

This definition is the analogue of what we called paths in one dimension and what are often called parametrized curves.

Geometric parametrised surfaces

As with curves and paths, we will distinguish between the surface $\mathbf{\Phi}$ and its image. Similarly, the image $S = \mathbf{\Phi}(D)$ will be called the geometric surface corresponding to $\mathbf{\Phi}$.

Note that for a given $(u, v) \in D$, $\mathbf{\Phi}(u, v)$ is a vector in $\mathbb{R}^3$. Each of the coordinates of the vector depends on u and v. Hence we write

$$\mathbf{\Phi}(u, v) = (x(u, v), y(u, v), z(u, v)),$$

where x, y and z are scalar functions on D.

The parametrized surface $\mathbf{\Phi}$ is said to be a smooth parametrized surface if the functions x, y, z have continuous partial derivatives in a open subset of $\mathbb{R}^2$ containing D.

Examples

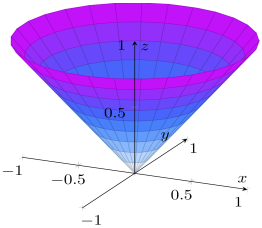
Example 1: Graphs of real valued functions of two independent variables are parametrised surfaces.
Let $f(x, y)$ be a scalar function and let $z = f(x, y)$, for all $(x, y) \in D$, where D is a path connected region in $\mathbb{R}^2$. We can define the parametrised surface $\mathbf{\Phi}$ by

$$\mathbf{\Phi}(u, v) = (u, v, f(u, v)), \quad \forall\, (u, v) \in D.$$

More specifically, we have $x(u, v) = u$, $y(u, v) = v$ and $z(u, v) = f(u, v)$.

Example 2: Consider the cylinder, $x^2 + y^2 = a^2$. Then this is parametrized surface defined by $\mathbf{\Phi} : [0, 2\pi] \times \mathbb{R} \to \mathbb{R}^3$ defined as $\mathbf{\Phi}(u, v) = (a \cos u, a \sin u, v)$.

Example 3: Consider the sphere of radius a, $S = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$. Is it a parametrized surface? Recall using spherical coordinates we can represent it using the following parametrization, $\mathbf{\Phi} : [0, 2\pi] \times [0, \pi] \to \mathbb{R}^3$ defined as $\mathbf{\Phi}(u, v) = (a \cos u \sin v, a \sin u \sin v, a \cos v)$.



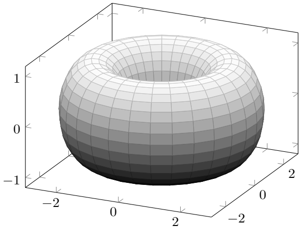
Example 4: The graph of $z = \sqrt{x^2 + y^2}$ can also be parametrized. We use the idea that at each value of z we get a circle of radius z. We can describe the cone as the parametrized surface $\mathbf{\Phi} : [0, \infty) \times [0, 2\pi] \to \mathbb{R}^3$ as $\mathbf{\Phi}(u, v) = (u \cos v, u \sin v, u)$.

Example 5: If we have parametrized curve on the z-y-plane $(0, y(u), z(u))$ which we rotate around z-axis, we can parametrise it as follows:

$$x = y(u)\cos v, \quad y = y(u)\sin v, \quad \text{and} \quad z = z(u).$$

Here $a \le u \le b$ if $[a, b]$ is the domain of the curve, and $0 \le v \le 2\pi$.

Surfaces of revolution around the z-axis



For instance we can parametrize a torus by taking a circle in the y-z plane with center $(0, a, 0)$ of radius b. This is given by the curve $(0, a + b\cos u, b\sin u)$.

Then the parametrization of the torus is then $\mathbf{\Phi}(u, v) = ((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u)$ where $0 \le u \le 2\pi$ and $0 \le v \le 2\pi$.

Parametrised surfaces are more general than graphs of functions.