Indian Institute of Technology Bombay

MA 106 Linear Algebra

Spring 2021 SRG/DP

Solutions and Marking Scheme for Make-up Quiz

Date: April 21, 2021 Max. Marks: 10

Q. 1 Consider the 4×5 matrix

$$A = \begin{bmatrix} 3 & 21 & 0 & b-1 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 2 \\ 6 & 42 & -1 & 13 & a+5 \end{bmatrix},$$

where a and b denote the last two digits of your roll number (e.g., if your roll number is 200010059, then $a = 5$ and $b = 9$).

(i) Determine a row echelon form of A.

(ii) Determine a basis for the column space of A. [3 marks]

Solution:

$$A = \begin{bmatrix} 3 & 21 & 0 & b-1 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 2 \\ 6 & 42 & -1 & 13 & a+5 \end{bmatrix} \xrightarrow{R_1-3R_2, R_3-2R_2, R_4-6R_2} \begin{bmatrix} 0 & 0 & 3 & b+5 & 3 \\ 1 & 7 & -1 & -2 & -1 \\ 0 & 0 & 2 & 10 & 4 \\ 0 & 0 & 5 & 25 & a+11 \end{bmatrix},$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 7 & -1 & -2 & -1 \\ 0 & 0 & 3 & b+5 & 3 \\ 0 & 0 & 2 & 10 & 4 \\ 0 & 0 & 5 & 25 & a+11 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_2} \begin{bmatrix} 1 & 7 & -1 & -2 & -1 \\ 0 & 0 & 2 & 10 & 4 \\ 0 & 0 & 3 & b+5 & 3 \\ 0 & 0 & 5 & 25 & a+11 \end{bmatrix},$$

$$\xrightarrow{R_2/2} \begin{bmatrix} 1 & 7 & -1 & -2 & -1 \\ 0 & 0 & 1 & 5 & 2 \\ 0 & 0 & 3 & b+5 & 3 \\ 0 & 0 & 5 & 25 & a+11 \end{bmatrix} \xrightarrow{R_3-3R_2, R_4-5R_2} \begin{bmatrix} 1 & 7 & -1 & -2 & -1 \\ 0 & 0 & 1 & 5 & 2 \\ 0 & 0 & 0 & b-10 & -3 \\ 0 & 0 & 0 & 0 & a+1 \end{bmatrix}.$$

(i) Thus a **row echelon form** of A is given by:

$$\begin{bmatrix} 1 & 7 & -1 & -2 & -1 \\ 0 & 0 & 1 & 5 & 2 \\ 0 & 0 & 0 & b-10 & -3 \\ 0 & 0 & 0 & 0 & a+1 \end{bmatrix} \qquad [2]$$

[**Note:** *Give 1 mark if the rank is correct (viz. 4), but there are calculation mistakes and a wrong REF is given (e.g., pivots are not in the correct position). Note that different row echelon forms are possible. One way to check them is to confirm that they give the same RCF (which is obtained from the above matrix by making the pivots to equal to 1, and the entries above the pivots to be 0); observe that (1,2)-th entry will not change.*]

(ii) The pivots are in columns 1, 3, 4, 5. Thus the corresponding columns of A form a basis of the column space of A, i.e., a basis is given by

$$\left\{ \begin{bmatrix} 3 \\ 1 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} b-1 \\ -2 \\ 6 \\ 13 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ a+5 \end{bmatrix} \right\} \qquad [1]$$

[**Note:** *Give 1 mark for any correct basis. Although the above basis is preferred, in this particular problem, the column space $\mathcal{C}(A)$ of A is a subset of $\mathbb{K}^4$ and the rank of A is 4, i.e., $\dim \mathcal{C}(A) = 4$. Thus $\mathcal{C}(A) = \mathbb{K}^4$ and so any basis of $\mathbb{K}^4$ is also a correct answer. For instance, this can be $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ or it can consist of the pivotal columns of a REF of A.*]

Q. 2 Let

$$A = \begin{bmatrix} 2a^2 + 3b^2 & ab \\ ab & 3a^2 + 2b^2 \end{bmatrix},$$

where a, b are the last two digits of your roll number if the product of the last two digits of your roll number is not zero, whereas $(a, b) = (2, 3)$ if the product of the last two digits of your roll number is zero. (e.g., if your roll number is 200010059, then $a = 5$ and $b = 9$, whereas if your roll number is 200010050, then $a = 2$ and $b = 3$).

Find an Orthogonal matrix C, and a diagonal matrix D such that $C^T AC = D$, a diagonal matrix. [4 marks]

Solution: By looking at the characteristic polynoimal of the matrix, we readily find that the two eigenvalues of A are $2(a^2 + b^2)$ and $3(a^2 + b^2)$. [1]

To find an eigenvector of A corresponding to the eigenvalue $2(a^2 + b^2)$, we need to find solutions of $(A - 2(a^2 + b^2)I)\mathbf{x} = \mathbf{0}$. Now

$$A - 2(a^2 + b^2)I = \begin{bmatrix} b^2 & ab \\ ab & a^2 \end{bmatrix} \xrightarrow{\frac{1}{b}R_1,\ \frac{1}{a}R_2} \begin{bmatrix} b & a \\ b & a \end{bmatrix}$$

Therefore, it is clear that $\mathbf{u} = [a \ \ -b]^\mathsf{T}$ is a solution of $(A - 2(a^2 + b^2)I)\mathbf{x} = \mathbf{0}$. Similarly,

$$A - 3(a^2 + b^2)I = \begin{bmatrix} -a^2 & ab \\ ab & -b^2 \end{bmatrix} \xrightarrow{\frac{-1}{a}R_1,\ \frac{1}{b}R_2} \begin{bmatrix} a & -b \\ a & -b \end{bmatrix}$$

Therefore, it is clear that $\mathbf{v} = [b \ \ a]^\mathsf{T}$ is a solution of $(A - 3(a^2 + b^2)I)\mathbf{x} = \mathbf{0}$. [1]

Now $\{\mathbf{u}, \mathbf{v}\}$ forms a basis of $\mathbb{K}^2$ consisting of eigenvectors of A corresponding to the eigenvalues $2(a^2+b^2)$ and $3(a^2+b^2)$. An orthonormal basis of eigenvectors of A is obtained simply by dividing $\mathbf{u}$ and $\mathbf{v}$ by their norm, viz., $\sqrt{a^2 + b^2}$. It follows that $C^T AC = D$, for

$$C = \frac{1}{\sqrt{a^2 + b^2}} \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 2(a^2 + b^2) & 0 \\ 0 & 3(a^2 + b^2) \end{bmatrix}. \qquad [2]$$

[**Note:** *Give at most 2 marks if the eigenvalues and the matrix D are obtained correctly, but C is not computed or is completely wrong. In case the columns of C and the diagonal entries of D are both switched, this is still a correct answer. Deduct 1 mark if only one columns of C is correct, or if the columns of C are taken as* $\mathbf{u}$ *and* $\mathbf{v}$*, i.e., they are not divided by* $\sqrt{a^2 + b^2}$*. Give* 0 *marks if wrong values of a and b are used.*]

Q. 3 Let A be an $n \times n$ normal matrix with entries in $\mathbb{C}$ and let $\lambda \in \mathbb{C}$.

(i) Show that $A - \lambda I$ is a normal matrix.

(ii) Show that if $A\mathbf{x} = \lambda\mathbf{x}$ for some $\mathbf{x} \in \mathbb{C}^n$, then $A^*\mathbf{x} = \bar{\lambda}\mathbf{x}$. [3 marks]

Solution: (i) We have $AA^* = A^*A$ and $(A - \lambda I)^* = A^* - \bar{\lambda}I$. Thus,

$$\begin{aligned}(A - \lambda I)(A^* - \bar{\lambda}I) &= (AA^*) - (\lambda A^* + \bar{\lambda}A) + \lambda\bar{\lambda}I \\ &= (A^*A) - (\lambda A^* + \bar{\lambda}A) + \lambda\bar{\lambda}I \\ &= (A^* - \bar{\lambda}I)(A - \lambda I).\end{aligned}$$

This shows that $A - \lambda I$ is a normal matrix. [1]

(ii) Suppose $\mathbf{x} \in \mathbb{C}^n$ is such that $A\mathbf{x} = \lambda\mathbf{x}$. Consider $B = A - \lambda I$. Then $B\mathbf{x} = \mathbf{0}$ and by (i) above, B is a normal matrix. [1]

Now $B\mathbf{x} = \mathbf{0}$ implies that $\|B\mathbf{x}\|^2 = 0$, i.e., $\langle B\mathbf{x},\ B\mathbf{x}\rangle = 0$, and hence $\langle \mathbf{x},\ B^*B\mathbf{x}\rangle = 0$. Since $B^*B = BB^*$, we obtain

$$0 = \langle \mathbf{x},\ B^*B\mathbf{x}\rangle = \langle \mathbf{x},\ BB^*\mathbf{x}\rangle = \langle B^*\mathbf{x},\ B^*\mathbf{x}\rangle = \|B^*\mathbf{x}\|^2$$

This shows that $B^*\mathbf{x} = \mathbf{0}$, i.e., $A^*\mathbf{x} = \bar{\lambda}\mathbf{x}$. [1]

[**Note:** *If part* (ii) *is deduced from quoting the result (proved in the class) that if B is normal, then $\|B\mathbf{x}\| = \|B^*\mathbf{x}\|$, then this may be permitted.*]