PH-107: Introduction to Quantum Mechanics
Tutorial Sheet 5

* marked problems will be solved in the Wednesday tutorial class

Operators and Wave function

1. Which of the operators A_i defined in the following are linear operators? Which of these are hermitian? All the functions $\psi(x)$ are well behaved functions vanishing at $\pm\infty$.
   (a) $\hat{A}_1\psi(x) = \psi(x)^2$
   (b) $\hat{A}_2\psi(x) = \frac{\partial\psi(x)}{\partial x}$
   (c) $\hat{A}_3\psi(x) = \int_a^x \psi(x')\, dx'$
   (d) $\hat{A}_4\psi(x) = 1/\psi(x)$
   (e) $\hat{A}_5\psi(x) = -\psi(x+a)$
   (f) $\hat{A}_6\psi(x) = \sin(\psi(x))$
   (g) $\hat{A}_7\psi(x) = \frac{\partial^2\psi(x)}{\partial x^2}$

2. (a) If $\hat{A}$ and $\hat{B}$ are Hermitian and $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} = i\hat{C}$, prove that $\hat{C}$ is Hermitian
   (b) An operator is said to be anti-Hermitian if $\hat{O}^\dagger = -\hat{O}$. Prove that $[\hat{A}, \hat{B}]$ is anti-Hermitian.

3. * Prove that if $\hat{K}$ is a Hermitian operator, $\exp\left(i\hat{K}\right)$ is an unitary operator, and if $\hat{U}$ is an Unitary operator, then there is an operator K such that $\hat{U} = \exp\left(i\,\hat{K}\right)$, and this $\hat{K}$ is Hermitian.

4. If $\hat{A}$ and $\hat{B}$ are operators, prove
   (a) that $\left(\hat{A}^\dagger\right)^\dagger = \hat{A}$
   (b) that $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger$
   (c) that $\hat{A} + \hat{A}^\dagger, i\left(\hat{A} - \hat{A}^\dagger\right)$, and that $\hat{A}\hat{A}^\dagger$ are Hermitian operators.

5. An operator is given by
$$\hat{G} = i\hbar\frac{\partial}{\partial x} + Bx$$
   where B is a constant. Find the eigen function $\phi(x)$. If this eigen function is subjected to a boundary condition $\phi(a) = \phi(-a)$ find out the eigen values.

6. $\Psi_1(x)$ and $\Psi_2(x)$ are the normalized eigen functions of an operator $\hat{P}$, with eigen values P_1 and P_2 respectively. If the wave function of a particle is $0.25\Psi_1(x) + 0.75\Psi_2(x)$ at $t = 0$, find the probability of observing P_1.

7. * Consider a large number (N) of identical experimental set-ups. In each of these, a single particle is described by a wave function $\Phi(x) = A\exp(-x^2/b^2)$ at $t = 0$, where A is the normalization constant and b is another constant with the dimension of length. If a measurement of the position of the particle is carried out at time $t = 0$ in all these set-ups, it is found that in 100 of these, the particle is found within an infinitesimal interval of $x = 2b$ to $2b + dx$. Find out, in how many of the measurements, the particle would have been found in the infinitesimal interval of $x = b$ to $b + dx$.

8. * An observable A is represented by the operator $\hat{A}$. Two of its normalized eigen functions are given as $\Phi_1(x)$ and $\Phi_2(x)$, corresponding to distinct eigenvalues a_1 and a_2, respectively. Another observable B is represented by an operator $\hat{B}$. Two normalized eigen functions of this operator are given as $u_1(x)$ and $u_2(x)$ with distinct eigenvalues b_1 and b_2, respectively. The eigen functions $\Phi_1(x)$ and $\Phi_2(x)$ are related to $u_1(x)$ and $u_2(x)$ as, $\Phi_1 = D(3u_1+4u_2)$; $\Phi_2 = F(4u_1 - Pu_2)$ At time t = 0, a particle is in a state given by $\frac{2}{3}\Phi_1 + \frac{1}{3}\Phi_2$.

(a) Find the values of D, F and P.
(b) If a measurement of A is carried out at t = 0, what are the possible results and what are their probabilities ?
(c) Assume that the measurement of A mentioned above yielded a value a1. If a measurement of B is carried out immediately after this, what would be the possible outcomes and what would be their probabilities ?
(d) If instead of following the above path, a measurement of B was carried out initially at t = 0, what would be the possible outcomes and what would be their probabilities ?
(e) Assume that after performing the measurements described in (c), the outcome was b_2. What would be the possible outcomes, if A were measured immediately after this and what would be the probabilities ?