MA-111 Calculus II (D1 & D2)

Lecture 1

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Welcome to course MA 111!

- The lectures and interaction session will be held on Zoom. The Zoom link has been shared via Moodle.
  Moodle website:
  `https://moodle.iitb.ac.in/login/index.php`
- The common lectures for both $D1$ and $D2$ in each week are on Mondays and Thursdays, 2-3:30 p.m.
- All lectures will be recorded. The recordings will be uploaded to either Zoom or Google Drive. The link for videos will be shared via Moodle. You can download them according to your convenience.
- Slides of the lectures will be uploaded to Moodle for MA111.

- Interaction session/office hours:

Division	Taken by	Timing
D1 and D2	Instructor	Tuesday: 9:30-10:30 a.m.
D1	TA	Wednesday 10-11 a.m.
D2	TA	Thursday 11:30-12:30 a.m.

- Tutorials: 2-3 p.m on Wednesdays.
  Every week a set of tutorial problems will be assigned and posted on the Moodle class page. Please attend the tutorial section assigned to you. Your TA will discuss some of these problems. You are advised to try problems in advance and use this time to ask questions and doubts.

Evaluation policy

Tentatively: Quiz=40%, Endsem=60%.

Tentatively, the quiz will be on 18th February.

Course content

Syllabus
Double and Triple integration, Jacobians and change of variables formula. Parametrization of curves and surfaces, vector Fields, line and surface integrals. Divergence and curl, Theorems of Green, Gauss, and Stokes.

Texts/References.
1. Hughes-Hallett et al., Calculus - Single and Multivariable (3rd Edition), John-Wiley and Sons (2003).
2. James Stewart, Calculus (5th Edition), Thomson (2003).
3. T. M. Apostol, Calculus, Volumes 1 and 2 (2nd Edition), Wiley Eastern 1980.
4. G. B. Thomas and R. L. Finney, Calculus and Analytic Geometry (9th Edition), ISE Reprint, Addison-Wesley, 1998.

Additional References:
5. J.E Marsden, A. J. Tromba, A. Weinstein. *Basic Multivariable Calculus*, South Asian Edition, Springer (2017).
6. S.R. Ghorpade and B. V. Limaye, *A course in Multivariable Calculus and Analysis*, Springer UTM (2017).

Recall : One variable Integration from MA 109

Let $f : [a, b] \to \mathbb{R}$ be a *bounded function* and $a, b \in \mathbb{R}$.

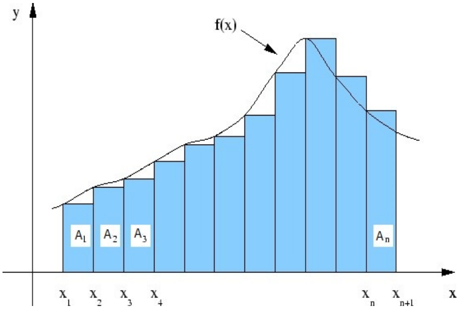
- The area enclosed by the graph of a non-negative function over the region of the interval is $\int_a^b f(t)\ dt$.
- A *partition* of the interval $[a, b]$ is a set of points $P = \{a = x_0 \leq x_1 \leq \ldots x_n = b\}$ for some $n \in \mathbb{N}$. Let M_i and m_j be respectively the supremum and infimum of $f([x_{i-1}, x_i])$, $L(f, P) = \sum_{i=1}^n m_i(x_i - x_{i-1})$, $U(f, P) = \sum_{i=1}^n M_i(x_i - x_{i-1})$.
- The *lower Darboux integral and upper Darboux integral* of f are $L(f) = \sup\{L(f, P)\colon P$ is a partition of $[a, b]\}$, and $U(f) = \inf\{U(f, P)\colon P$ is a partition of $[a, b]\}$, respectively.
- When $L(f) = U(f)$ then f is *Darboux integrable* and

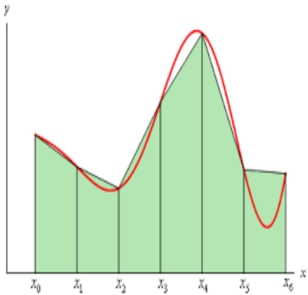
$$\int_a^b f := L(f) = U(f).$$

- A *tagged partition* is a partition P with a set of points $t = \{t_1, \dots t_n\}$ where $t_j \in [x_{j-1}, x_j]$ for all $j = 1, \dots, n$.
- Define $S(f, P, t) = \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1})$ and define the *norm* of a partition P as $\|P\| = \max_j\{|x_j - x_{j-1}|\}, \quad 1 \le j \le n$.
- A function $f : [a, b] \to \mathbb{R}$ is said to be *Riemann integrable* if for some $S \in \mathbb{R}$ and every $\epsilon > 0$ there exists $\delta > 0$ such that $|S(f, P, t) - S| < \epsilon$, whenever $\|P\| < \delta$. The Riemann integral of f is then S.
- The Riemann integral exists if and only if the Darboux integral exists. Further, the two integrals are equal.
- Unlike the Darboux integral, Reimann integral can be computed as a limit. This is clearly advantageous in computations.
- Let $f : [a, b] \to \mathbb{R}$ be a function that is *bounded*, and continuous at all but finitely many points of $[a, b]$. Then f is *Riemann integrable* on $[a, b]$.
- For computing integrals, we use the *Fundamental theorem of calculus*. If $f : [a, b] \to \mathbb{R}$ is continuous and $f = g'$ for some continuous function $g : [a, b] \to \mathbb{R}$ which is differentiable on (a, b), then $\displaystyle\int_a^b f = g(b) - g(a)$.

Think about it...

For the Riemann integral, we take the sum of areas of these rectangles, and then take the limit as the norm $||P|| \to 0$. What happened if we took the sum of areas of the *trapezoids* instead of *rectangles*? Do we get the same answer?





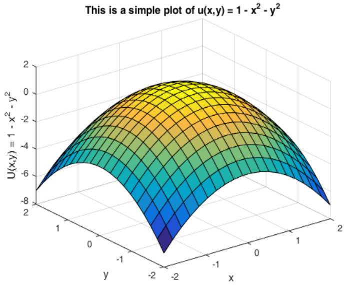
Functions in two variables

A *closed, bounded rectangle* R in $\mathbb{R}^2$ is a subset of the form

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b, \; c \leq y \leq d.\}$$

This is also called the *Cartesian product* of the two closed intervals $[a, b]$ and $[c, d]$, where $a, b, c, d \in \mathbb{R}$.

The graph of a function $f : R \subset \mathbb{R}^2 \to \mathbb{R}$ is the subset $\{(x, y, f(x, y)) \in \mathbb{R}^3 \mid (x, y) \in R\}$ in $\mathbb{R}^3$.



Partitions for rectangles

A partition P of a rectangle $R = [a, b] \times [c, d]$ is the Cartesian product of a partition P_1 of $[a, b]$ and a partition P_2 of $[c, d]$. Let

$$P_1 = \{x_0, x_1, \cdots x_m\}, \quad \text{with} \quad a = x_0 < x_1 < x_2 < \cdots < x_m = b,$$

$$P_2 = \{y_0, y_1, \cdots y_n\}, \quad \text{with} \quad c = y_0 < y_1 < y_2 < \cdots < y_n = d,$$

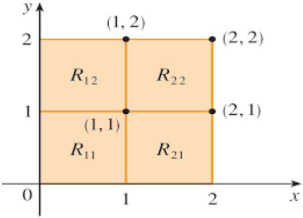
and $P = P_1 \times P_2$ be defined by

$$P = \{(x_i, y_j) \mid i \in \{0, 1, \cdots m\}, \quad j \in \{0, 1, \cdots, n\}\}.$$

The points of P divide the rectangle R into *nm non-overlapping sub-rectangles* denoted by

$$R_{ij} := [x_i, x_{i+1}] \times [y_j, y_{j+1}], \quad \forall\, i = 0, \cdots m-1, \quad j = 1, \cdots, n-1.$$

Note $R = \cup_{i,j} R_{ij}$.



The area of each R_{ij}: $\Delta_{ij} := (x_{i+1} - x_i) \times (y_{j+1} - y_j)$, for all $i = 0, \cdots, m-1$, $j = 0, \cdots, n-1$. The area of R is the sum of the areas of all R_{ij}.

Norm of the partition P:
$\|P\| := \max\{(x_{i+1} - x_i), (y_{j+1} - y_j) \mid i = 0, \cdots, m-1, \quad j = 0, \cdots, n-1\}$.

Question. Why do we not define the norm by
$\max\{(x_{i+1} - x_i) \times (y_{j+1} - y_j) \mid i = 0, \cdots, m-1, \quad j = 0, \cdots, n-1\}$?

Darboux integral

Let $f : R \to \mathbb{R}$ be a bounded function where R is a rectangle . Let $m(f) = \inf\{f(x,y) \mid (x,y) \in R\}$, $M(f) = \sup\{f(x,y) \mid (x,y) \in R\}$. For all $i = 0, 1, \cdots, m-1, \quad j = 0, 1, \cdots, n-1$, let,
$m_{ij}(f) := \inf\{f(x,y) \mid (x,y) \in R_{ij}\}$, and
$M_{ij}(f) := \sup\{f(x,y) \mid (x,y) \in R_{ij}\}$.

Lower double sum: $L(f,P) := \displaystyle\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} m_{ij}(f)\Delta_{ij}$, and

Upper double sum: $U(f,P) := \displaystyle\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} M_{ij}(f)\Delta_{ij}$,

Note that for any partition P of R

$$m(f)(b-a)(d-c) \leq L(f,P) \leq U(f,P) \leq M(f)(b-a)(d-c).$$

Lower Darboux integral: $L(f) := \sup\{L(f,P) \mid P$ is any partition of $R\}$.

Upper Darboux integral:$U(f) := \inf\{U(f,P) \mid P$ is any partition of $R\}$.

Note $L(f) \leq U(f)$.

Darboux integral contd.

Definition (Darboux integral)

A bounded function $f : R \to \mathbb{R}$ is said to be *Darboux integrable* if $L(f) = U(f)$. The Double integral of f is the common value $U(f) = L(f)$ and is denoted by

$$\int\int_R f, \quad \text{or} \quad \int\int_R f(x,y)dA, \quad \text{or} \quad \int\int_R f(x,y)dxdy.$$

Theorem (Riemann condition)

Let $f : R \to \mathbb{R}$ be a bounded function. Then f is integrable if and only if for every $\epsilon > 0$ there is a partition P_ϵ of R such that

$$|U(f,P_\epsilon) - L(f,P_\epsilon)| < \epsilon.$$

Examples

(1) The constant function $f(x, y) = \alpha$ is always integrable over any rectangle R. Indeed, for any partition P,

$$L(f, P) = U(f, P) = \alpha \, \mathrm{area}(R).$$

(2) The bivariate Dirichlet function over $[0, 1] \times [0, 1]$ is defined as follows.

$$f(x, y) := \begin{cases} 1 & \text{if both } x \text{ and } y \text{ are rational numbers,} \\ 0 & \text{otherwise.} \end{cases}$$

This is not integrable over $[0, 1] \times [0, 1]$. Indeed, given any partition P, $L(f, P) = 0$ and $U(f, P) = 1$.

Riemann Integral

Riemann integral: Let P be any partition of a rectangle $R = [a, b] \times [c, d]$. We define a *tagged partition* (P, t) where

$$t = \{t_{ij} \mid t_{ij} \in R_{ij}, \quad i = 0, 1, \dots m-1, \quad j = 0, 1, \dots n-1\}.$$

Let $t_{ij} = (x_{ij}^*, y_{ij}^*)$.
The *Riemann sum* of f associate to (P, t) is defined by

$$S(f, P, t) = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} f(t_{ij}) \Delta_{ij} \text{ where, } \Delta_{ij} = (x_{i+1} - x_i)(y_{j+1} - y_j)$$

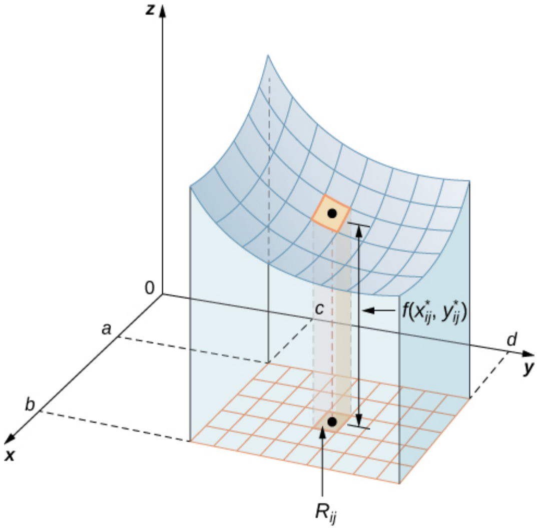
Definition (Riemann integral)

A bounded function $f : R \to \mathbb{R}$ is said to be *Riemann integrable* if there exists a real number S such that for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|S(f, P, t) - S| < \epsilon,$$

for every tagged partition (P, t) satisfying $\|P\| < \delta$ and S is the value of Riemann integral of f.

Geometrically, the Riemann integral represents the (signed) volume of the solid bounded below by the rectangle R and above by the surface $z = f(x, y)$.



Riemann Integral contd.

- **Theorem.** Let $R \subseteq \mathbb{R}^2$ be any rectangle and let $f : R \to \mathbb{R}$ be bounded. Then f is Darboux integrable over R if and only if it is Riemann integrable over R. In this case, we simply say that $f : R \to \mathbb{R}$ is *integrable* on R.
- **Theorem** Every continuous function $f : R \to \mathbb{R}$ is integrable.
- **Think about it...**
  For the Riemann integral, one takes the sum of the volumes of the rectangular prisms and then takes the limit as $||P|| \to 0$. What happened if one took the sum of the volumes of the "trapezoidal prisms" instead of rectangular prisms?

Regular partitions

However, the current definition isn't truly helpful in making computations. We define *Regular* partitions.

The regular partition of R of order any $n \in \mathbb{N}$ is defined by $x_0 = a$ and $y_0 = c$, and for $i = 0, 1, \cdots, n-1, \quad j = 0, 1, \cdots, n-1$,

$$x_{i+1} = x_i + \frac{b-a}{n}, \quad y_{j+1} = y_j + \frac{d-c}{n}.$$

We take $t = \{t_{ij} \in R_{ij} \mid i, j \in \{0, 1, \cdots, n-1\}\}$ any arbitrary tag. To check the integrability of a function f, it is enough to consider a sequence of regular partitions P_n of R.

Theorem

A bounded function $f : R \to \mathbb{R}$ is Riemann integrable if and only if the Riemann sum

$$S(f, P_n, t) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} f(t_{ij}) \Delta_{ij},$$

tends to the same limit $S \in \mathbb{R}$ as $n \to \infty$, for any choice of tag t.

Example

The function $f(x,y) = x^2 + y^2$ is continuous on $R = [0,1] \times [0,1]$, hence integrable over R. We want to compute the double integral over R using regular partitions.
Choose regular partition with subrectangles $[i/n, (i+1)/n] \times [j/n, (j+1)/n]$ and tag $t = \{(\frac{i}{n}, \frac{j}{n}) \mid i = 0, \dots n-1, j = 0, \dots, n-1\}$, so that the Riemann sum

$$S(f, P_n, t) = (\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (\frac{i}{n})^2 + (\frac{j}{n})^2) \frac{1}{n^2}.$$

The right side is $\frac{1}{n^4} \sum_j \sum_i (i^2 + j^2) = \frac{1}{n^4} \times n \times 2 \sum_{i=1}^{n-1} i^2$. If you substitute the formula

$$\sum_{i=1}^{n-1} i^2 = \frac{(n-1)n(2n-1)}{6},$$

and take $n \to \infty$, you get the answer $= 2/3$.

A similar example

$f(x, y) = xy$ over the same rectangle $R = [0, 1] \times [0, 1]$, with the same regular partition and the same tags as before. The Riemann sum is

$$S(f, P_n, t) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (\frac{ij}{n^2}) \frac{1}{n^2}.$$

The right hand side is as before

$$\frac{1}{n^4} \sum_{j=1}^{n-1} j \sum_{i=1}^{n-1} i = \frac{1}{n^4} \times (\frac{n(n-1)}{2})^2.$$

As $n \to \infty$, this tends to $1/4$.

Conventions

Based on our definition, we make the following convention: Let $a, b, c, d \in \mathbb{R}$

- If $a = b$ or $c = d$, then $\int \int_{[a,b]\times[c,d]} f(x,y)dxdy := 0$.
- If $a < b$ and $c < d$:
  $$\int \int_{[b,a]\times[c,d]} f(x,y)dxdy := -\int \int_{[a,b]\times[c,d]} f(x,y)dxdy,$$
  $$\int \int_{[a,b]\times[d,c]} f(x,y)dxdy := -\int \int_{[a,b]\times[c,d]} f(x,y)dxdy,$$
  $$\int \int_{[b,a]\times[d,c]} f(x,y)dxdy := \int \int_{[a,b]\times[c,d]} f(x,y)dxdy.$$

Properties of integrals over rectangles

(Domain Additivity Property:) Let R be a rectangle and $f : R \to \mathbb{R}$ be a bounded function. Partition R into finitely many non-overlapping sub-rectangles. Then f is integrable on R if and only if it is integrable on each sub-rectangle. When it exists, the integral of f on R is the sum of the integrals of f on the sub-rectangles.

Algebraic properties :

Let $R := [a, b] \times [c, d]$. Let f and g are integrable on R.

- If f is defined as $f(x, y) = \alpha \in \mathbb{R}$ for all $(x, y) \in \mathbb{R}^2$ then $\int\int_R f = \alpha A(R)$ where A is the area of R.
- The function $f + g$ is integrable, and $\int\int_R f + g = \int\int_R f + \int\int_R g$.
- For all $\alpha \in \mathbb{R}$, αf is integrable and $\int\int_R \alpha f = \alpha \int\int_R f$.
- If $f(x, y) \le g(x, y)$ for all $(x, y) \in R$, then $\int\int_R f \le \int\int_R g$.
- $|f|$ is integrable and $|\int\int_R f| \le \int\int_R |f|$.
- The function $f.g$ is integrable.
- If $\frac{1}{f}$ is well defined and bounded on R, then $\frac{1}{f}$ is integrable on R.

All these follow by applying the definition and properties of limits. An immediate consequence is that all polynomial functions are integrable.