MA106 Tut1

Topics Covered:
1) Basic definitions related to matrices
2) Geometric interpretation of matrices as linear transforms
3) EROs, REF, RREF
4) System of linear equations
5) Inverse of a matrix
6) Rank, Nullity, Row space, Column space
7) General vector spaces

$Ax = 0 \rightarrow x = 0$

Q1. Consider in general an n x n matrix with r pivotal elements in its reduced REF. This implies it has r non-zero rows (as each non-zero row must have a pivot). Now, we must choose any r out of the n columns to be pivots. The order in which they appear as pivots will automatically get decided. For eg:

$n = 4$ $r = 3$. Suppose I pick columns $\{1, 2, 4\}$

RREF: $\begin{bmatrix} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Hence, number of such RREF "types" = ${}^{n}C_{r}$

For $n = 4$
$r = 0 \rightarrow 1$ matrix (Null)
$r = 1 \rightarrow 4$
$r = 2 \rightarrow 6$
$r = 3 \rightarrow 4$
$r = 4 \rightarrow 1$ (Identity)

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Q2. (i) Observe: $2[0\ 1\ 0] - [1\ 1\ -1] + [-1\ 1\ 1] = 0$

Hence, linearly dependent

Aliter:

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{\text{REF}} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

rank $= 3 < 4$

Motivation:

Consider vectors : $v_1, v_2, v_3 \cdots, v_n$. These will be linearly dependent iff the equation:

Column Vector

$$x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = 0$$

has a non-trivial solution. Putting this in matrix form:

$$[v_1\ v_2 \cdots v_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0$$

This equation will have multiple solutions iff their is a non-pivotal (free) variable in the REF. Thus, the vectors will be linearly independent iff there is a free variable

Caveat : In the given question, the vectors are row vectors. We must first convert them to column vectors before applying this method.

(ii) Linearly independent

Let us convert to column vectors and then use the method above

$$\begin{bmatrix} 1 & 2 & 2 \\ 9 & 0 & 0 \\ 9 & 0 & 0 \\ 8 & 3 & 8 \end{bmatrix} \xrightarrow{\text{REF}} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

Q3

1) $\begin{bmatrix} 8 & -4 \\ -2 & 1 \\ 6 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & -1 \\ -2 & 1 \\ 2 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

rank = 1

2) $m \neq \pm n$

$\begin{bmatrix} m & n \\ n & m \\ p & p \end{bmatrix} \rightarrow \begin{bmatrix} m+n & n+m \\ n & m \\ p & p \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ n & m \\ 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ n & m \\ 0 & 0 \end{bmatrix}$

$m \neq n \quad \therefore$ Rank = 2

3) $\begin{bmatrix} 0 & 8 & -1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 4 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 5 \\ 0 & 8 & -1 \\ 0 & 0 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 5 \\ 0 & 0 & -11 \\ 0 & 0 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 4 & 5 \\ 0 & 0 & -11 \\ 0 & 0 & 0 \end{bmatrix}$

rank = 3

Q8 $\begin{bmatrix} 1 & 1 & 1 \\ a & b & 2 \\ a^2 & b^2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$ ∞ many solutions

Necessary (not yet sufficient) condition: det of the LHS matrix must be 0

$\begin{vmatrix} 1 & 1 & 1 \\ a & b & 2 \\ a^2 & b^2 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & 2-a \\ 0 & b^2-ab & 4-2a \end{vmatrix} = (b-a)(2-a)(2-b) = 0$

$$\begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix} \longrightarrow \begin{vmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b(b-a) & c(c-a) \end{vmatrix}$$

$$\downarrow$$

$$(b-a)(c-a)\begin{vmatrix} 1 & 1 \\ b & c \end{vmatrix}$$

Exercise: Show that for n numbers x1, x2, x3, ..., x_(n-1), find the value of the following determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_1 & x_n \\ x_1^2 & x_1^2 & x_n^2 \\ \vdots & \vdots & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_n^{n-1} \end{vmatrix}$$

Given: $a < b$

1) Suppose $a = 2$

Augmented matrix:

$$A^+ = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & b & 2 & 3 \\ 4 & b^2 & 4 & 9 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & b-2 & 0 & 1 \\ 0 & b^2-4 & 0 & 5 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & b-2 & 0 & 1 \\ 0 & 0 & 0 & 3-b \end{array}\right]$$

For ∞ sol^n: $b = 3$

$[\text{rank}(A) = \text{rank}(A)]$

2) Try $b = 2 \longrightarrow a = 3$ ⊗ $a < b$

Q9. (1) Row space is unchanged by row operations

(a) Exchange: Set of vectors remains same; so space spanned is same

(b) Multiplication by non-zero number: Suppose rows are $r_1, r_2, \dots r_n$ and number is $\alpha \neq 0$

$$x = x_1 r_1 + x_2 r_2 + \dots + x_i r_i + \dots + x_n r_n$$

$$\updownarrow$$

$$x = x_1 r_1 + x_2 r_2 + \dots + \frac{x_i}{\alpha}(\alpha r_i) + \dots + x_n r_n$$

(c) Multiplication of another row by a number and addition to row

Suppose rows are $r_1, r_2, \dots r_n$ and α

$$x = x_1 r_1 + x_2 r_2 + \dots + x_i r_i + \dots + x_j r_j + \dots$$

$$\updownarrow$$

$$x = x_1 r_1 + \dots + x_i(r_i + \alpha r_j) + \dots + (x_j - \alpha x_i) r_j + \dots$$

$$\alpha_1 C_{i_1} + \alpha_2 C_{i_2} \cdots + \alpha_r C_{i_r} = 0 \rightarrow \alpha_i = 0 \ \forall i$$

$$\beta_1 E C_{i_1} + \beta_2 E C_{i_2} + \cdots = 0$$

$$\times E^{-1}$$

$$\beta_1 C_{i_1} + \beta_2 C_{i_2} + \cdots = 0$$

Exercise: Suppose we have A(nxn) and B(nxn) with rank(A)=m and rank(B) = r. Then show that:

1) If m=n, then rank(AB)=r
2) In general rank(AB)<= min(m,r)

(2) Dimension of column space is unchanged by elementary row operations

$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ $\quad \lambda \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad \lambda \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Suppose ERO is represented by the matrix E and the columns are $C_1, C_2, \dots C_n$. Then, new columns are $EC_1, EC_2, \dots, EC_n$

If out of these $\overbrace{C_{i_1}, C_{i_2}, C_{i_3}, \dots, C_{i_r}}^{r}$ are linearly independent, then, so are $(EC_{i_1}, EC_{i_2}, \dots, EC_{i_r})$ as E is invertible

$\therefore$ So, the dimensions of the column space remain unchanged

Note: The column space itself may get altered by EROs

Eg: ?

Proves row-rank = col. rank

Handwritten Problem

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 2 & -1 & 1 & -2 & -1 & 1 \\ 2 & 2 & 6 & 0 & 4 & 2 & 4 & 10 \\ 1 & -1 & 1 & -2 & 0 & -5 & -4 & -3 \\ 2 & 2 & 6 & 0 & 4 & 2 & 4 & 10 \end{array}\right]$$

(1)

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 2 & -1 & 1 & -2 & -1 & 1 \\ 2 & 2 & 6 & 0 & 4 & 2 & 4 & 10 \\ 1 & -1 & 1 & -2 & 0 & -5 & -4 & -3 \\ 2 & 2 & 6 & 0 & 4 & 2 & 4 & 10 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 2 & -1 & 1 & -2 & -1 & 1 \\ 0 & 2 & 2 & 2 & 2 & 6 & 6 & 8 \\ 0 & -1 & -1 & -1 & -1 & -3 & -3 & -4 \\ 0 & 2 & 2 & 2 & 2 & 6 & 6 & 8 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 2 & -1 & 1 & -2 & -1 & 1 \\ 0 & -1 & -1 & -1 & -1 & -3 & -3 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

2

$$A \rightarrow k = 2, \quad |N(A)| = 5$$

(2) Yes, it has infinitely many solutions. Rank(A^+) = Rank(A)

(3) We have a clever way to do this, rather than checking every 2x2 submatrix. Will discuss at the end

(4) Null space will have 5 basis vectors. Construct these vectors by putting each of the free variables to 1 and the others to 0 turn-by-turn. Then for each of these vectors obtain other elements using solutions of Ax = 0

$$\begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 2 \\ -3 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

(5) IMPORTANT: The row space remains unchanged by EROs. The column space changes, only its dimension remains the same.
To obtain the basis vectors for the column space, take the indices of the pivotal elements of the REF and the columns at those indices in the original matrix will form a basis of the column space

Proof: Consider the matrix (call it B) whose columns are only those columns of the original matrix which are at pivotal indices. Then, if on B, we apply all the EROs that were applied to the original matrix to get its REF; we get the REF of B. And this REF will have all columns as pivotal i.e. it will be full rank. This shows that the columns we considered are linearly independent and so form the basis of the column space

(6) We have already obtained the basis of the null-space of A. We just need to obtain one particular solution of Ax=b and then the full set of solutions can be obtained by adding the particular solution and the null-space

Particular solution: Put all free variables to 0

$$x = \begin{bmatrix} 1 \\ 4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

basis vectors of null space

Full set of solutions: $x + \sum_{i=1}^{5} \alpha_i x_i$

(7) Obvious

(8) We will take those columns which form the basis of the column space and those rows which make the basis of the row space

$$\begin{bmatrix} 1 & 0 & 2 & -1 & 1 & -2 & -1 \\ 2 & 2 & 6 & 0 & 4 & 2 & 4 \\ 1 & -1 & 1 & -2 & 0 & -5 & -4 \\ 2 & 2 & 6 & 0 & 4 & 2 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \quad \text{rank} = 2$$

Reason: Recall proof of result :

A matrix has rank atleast k iff there exists a kxk submatrix with rank k

Let $C_1, C_2, C_3, \cdots C_k$ be k linearly independent columns

Then, if we take the submatrices of these with indices same as the rows that contribute to the row space and call them

$C_1', C_2', C_3', \cdots C_k'$, then, these must also be linearly independent.

Proof: Assume that there exist coefficients $\alpha_1, \alpha_2, \cdots, \alpha_k$,such that

$$\alpha_1 C_1' + \alpha_2 C_2' + \cdots + \alpha_k C_k' = 0$$

then

$$\alpha_1 C_1 + \alpha_2 C_2 + \cdots + \alpha_k C_k = 0$$

For, those rows that are already in the row space, the summing up to 0 is evident. For other rows, they can be expressed as a linear combination of the rows in the row space, so even they sum to 0

But, $C_1, C_2, C_3, \cdots C_k$ are linearly independent, so $\alpha_1, \alpha_2, \cdots, \alpha_k$ are all 0

So, $C_1', C_2', C_3', \cdots C_k'$ are linearly independent. Hence, the rank of the k x k matrix formed by them as columns is k

$$\overset{2}{\begin{array}{ccc} 1 & 2 & \\ \left[\begin{matrix} 1 \\ 2 \\ 1 \end{matrix}\right. & \begin{matrix} 2 \\ 4 \\ 7 \end{matrix} & \left.\begin{matrix} 3 \\ 6 \\ 9 \end{matrix}\right] \end{array}} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 5 & 6 \end{bmatrix} \longrightarrow \overset{2}{\begin{array}{ccc} 1 & 2 & \\ \left[\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}\right. & \begin{matrix} 2 \\ 5 \\ 0 \end{matrix} & \left.\begin{matrix} 3 \\ 6 \\ 0 \end{matrix}\right] \end{array}}$$

$$\underset{2}{\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & 7 \end{bmatrix}} \longrightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 5 \end{bmatrix} \longrightarrow \underset{2}{\begin{bmatrix} 1 & 2 \\ 0 & 5 \\ 0 & 0 \end{bmatrix}}$$