Derivation of the Poynting Theorem

Poynting Theorem

- Poynting theorem, which leads to Poynting vector, is all about the flow of electromagnetic energy through space
- Let us consider a configuration of charges and currents in free space which produces electric and magnetic fields. Under the influence of these fields, charges may move. The question is how much work does the EM field do in the process?
- Let us consider a point charge q, exposed to fields $\mathbf{E}$ and $\mathbf{B}$, moving with a velocity $\mathbf{v}$. If it moves a distance $\mathbf{dl}$, in time dt, the work done dW can be computed using the expression for the Lorentz force

$$dW = \mathbf{F} \cdot \mathbf{dl} = q(\mathbf{E} + (\mathbf{v} \times \mathbf{B})) \cdot \mathbf{dl}$$

- Using the fact that $\mathbf{v} = \frac{d\mathbf{l}}{dt}$

$$dW = q(\mathbf{E} + (\mathbf{v} \times \mathbf{B})) \cdot \mathbf{v}dt = q\mathbf{E} \cdot \mathbf{v}dt$$

$$\text{or } \frac{dW}{dt} = q\mathbf{E} \cdot \mathbf{v}$$

- For a charge distribution, this expression can be generalized to

$$\frac{dW}{dt} = \int_V \rho \mathbf{E} \cdot \mathbf{v} d\tau,$$

where the integral is confined to the volume containing the charge distribution.
- Using the expression $\mathbf{J} = \rho \mathbf{v}$, we obtain

$$\frac{dW}{dt} = \int_V \mathbf{E} \cdot \mathbf{J} d\tau$$

- We use the Maxwell equation $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$ to obtain

$$\frac{dW}{dt} = \frac{1}{\mu_0} \int_V \mathbf{E} \cdot (\nabla \times \mathbf{B} - \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}) d\tau$$

- Using the identity

$$\nabla \cdot (\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{B}),$$

we obtain

$$\frac{dW}{dt} = \frac{1}{\mu_0} \int_V (\mathbf{B} \cdot (\nabla \times \mathbf{E}) - \nabla \cdot (\mathbf{E} \times \mathbf{B}) - \mu_0 \varepsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t}) d\tau$$

- Using $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$, and the divergence theorem, we obtain

$$\frac{dW}{dt} = -\int_V \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0}B^2\right) d\tau - \frac{1}{\mu_0}\oint_S (\mathbf{E}\times\mathbf{B})\cdot d\mathbf{a}, \tag{1}$$

  above S denotes the closed surface bounding the volume V. We also used the fact that $\frac{1}{2}\frac{\partial E^2}{\partial t^2} = \frac{1}{2}\frac{\partial}{\partial t}(\mathbf{E}\cdot\mathbf{E}) = \mathbf{E}\cdot\frac{\partial \mathbf{E}}{\partial t}$, valid for any vector. Eq. (1) is called Poynting's Theorem, which is the work-energy theorem of electrodynamics.
- We define the Poynting vector as

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) \tag{2}$$

- In light of Eq. 1 it is obvious that the Poynting vector has the dimensions of energy per unit area per unit time.

Interpretation of Poynting Vector

- Eq. 1 can be rewritten as

$$\frac{dW}{dt} = -\frac{dU_{em}}{dt} - \oint_S \mathbf{S} \cdot d\mathbf{a}, \qquad (3)$$

where $U_{em} = \int_V u_{em} d\tau = \int_V \left(\frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2 \right) d\tau$ is the total energy stored in the EM field contained in the given volume V.

- This equation simply states the mechanical work done by EM field is consistent with the decrease in the energy of the EM field, and any energy flux which flows out of the region.
- If we define a mechanical energy density u_{mech} as

$$\frac{dW}{dt} = \frac{d}{dt} \int_V u_{mech} d\tau,$$

then using the Divergence theorem again we can write Eq. (3) as

$$\frac{d}{dt} \int_V (u_{em} + u_{mech}) d\tau = -\oint_S \mathbf{S} \cdot d\mathbf{a} = -\int_V \nabla \cdot \mathbf{S} d\tau$$

Interpretation...

- From which we deduce

$$\frac{\partial u_{tot}}{\partial t} = \frac{\partial (u_{mech} + u_{em})}{\partial t} = -\nabla \cdot \mathbf{S}$$

- or

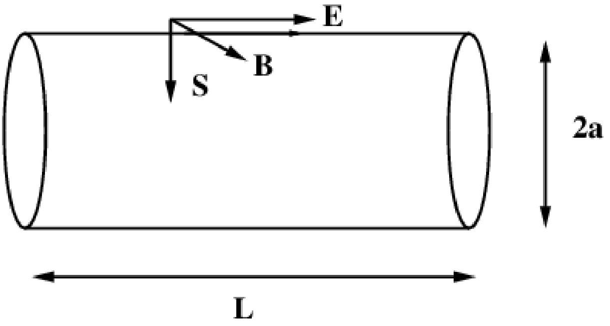
$$\frac{\partial u_{tot}}{\partial t} + \nabla \cdot \mathbf{S} = 0 \tag{4}$$

  which is similar to the charge continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0.$$

- Just like the charge continuity equation implies the conservation of electric charge, similarly Eq. (4) implies conservation of energy, with the Poynting vector **S** playing the role of the energy current.

An example: Joule heating of a current carrying wire



- Let us consider a cylindrical conductor of length $L(\to \infty)$as shown in the figure throgh which a current I is flowing from left to right.
- Directions of **E** and **B**, and hence that of **S** at the curved surface, are as shown.

- If the potential across the conductor is V, and the current is I, then at its surface

$$E = \frac{V}{L}$$
$$B = \frac{\mu_0 I}{2\pi a}$$

- Because **E** and **B** are mutually perpendicular at the curved surface, the Poynting vectors magnitude will be

$$S = \frac{EB}{\mu_0} = \frac{\mu_0 IV}{\mu_0 2\pi aL} = \frac{VI}{2\pi aL}$$

- So the total power input at the curved surface is

$$P = S(\text{area of the surface}) = \left(\frac{VI}{2\pi aL}\right)(2\pi aL) = VI$$

which is nothing but the power corresponding to the Joule heating.