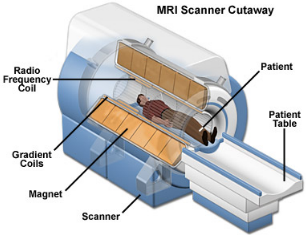
Holy Grail of Communication:
Error Correction Coding

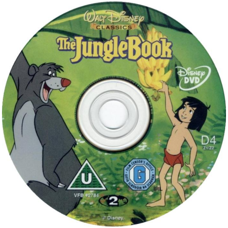
Sibi Raj B Pillai
srbpteach@gmail
Subject:EE113-RollNo

Outline

- Three Systems: Communication, Storage, MRI
- Some Common Ground
- Linear Solvers
- Theory to Practice
- Conclusion

Mission Critical





Courtesy: Google Images

Pick the ODD one out

10×10 gms

10×11 gms

• •

10×10 gms

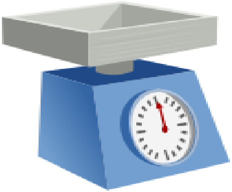
10×10 gms

Pick the ODD one out



• •


10×10 gms 10×11 gms 10×10 gms 10×10 gms



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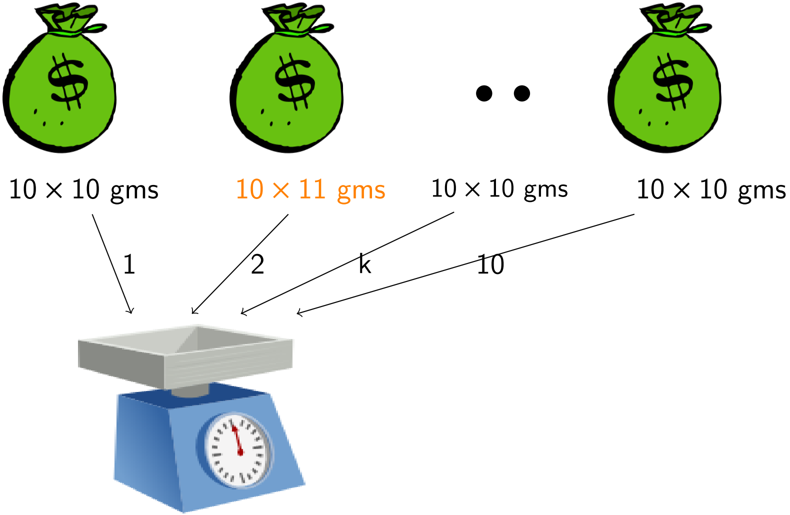

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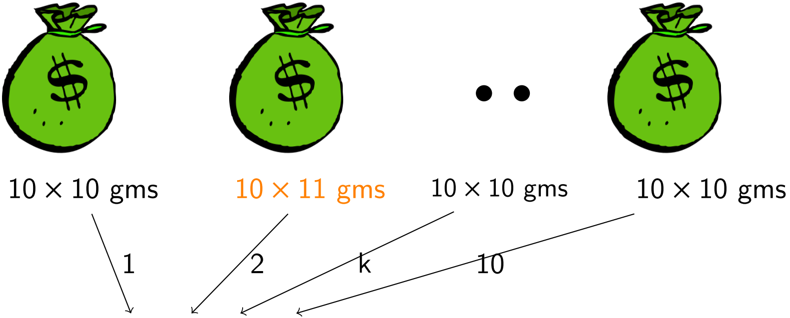
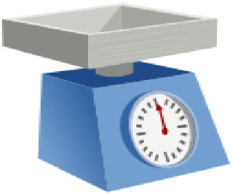
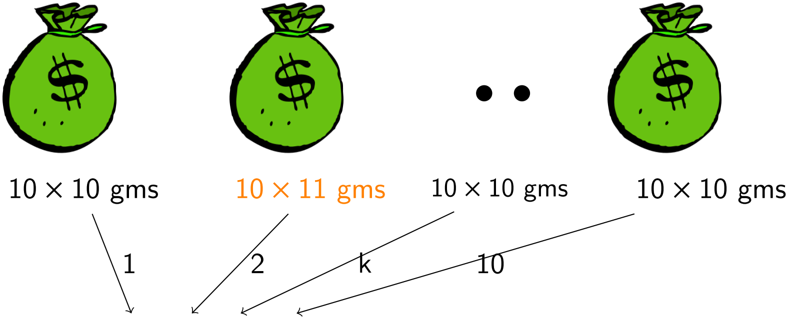


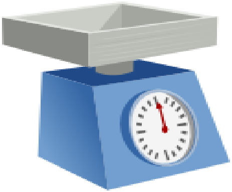


$$\text{Bag Number} = W - 550$$

$$W = \sum a_i x_i, \text{ where } a_i = i, x_i \in \{10, 11\}$$

Pick the ODD one out





$$\text{Bag Number} = W - 550$$

$$W = \sum a_i x_i, \text{ where } a_i = i, x_i \in \{10, 11\}$$

1) Linear combination to pick odd ones

2) Measurements should cover all bags

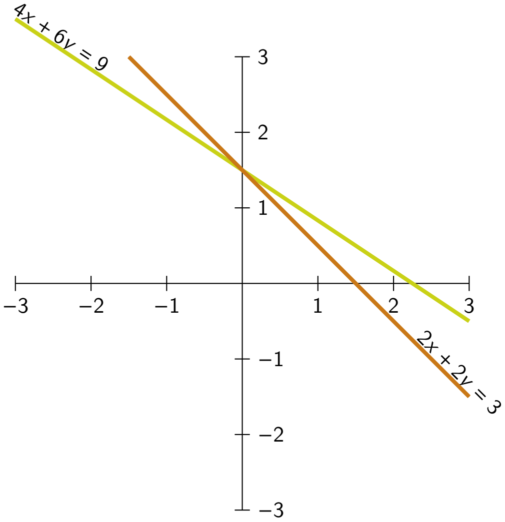
Solving Equations

$$4x_1 + 6x_2 = 9$$
$$2x_1 + 2x_2 = 3$$

$$\begin{bmatrix} 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

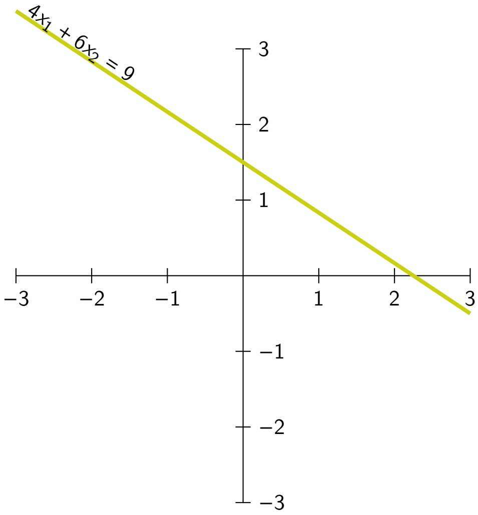
$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\bar{x} = A^{-1}\ \bar{c}$$



More Unknowns

$$4x_1 + 6x_2 = 9$$



- Many Solutions in general!
- Which one(s) do we need?

Linear Solvers

- We have M linear equations and N unknowns.

$$y_j = \sum_i a_{ji} x_i \quad , \; 0 \leq i \leq M - 1.$$

$$\underset{M \times 1}{\bar{y}} = \underset{M \times N}{A} \; \underset{N \times 1}{\bar{x}}$$

- An **under-determined** set of equations, $M < N$.

- However, assume x to be **sparse** (a few *odd* values).

- **Sparsity** s represents the number of non-zero entries of $\bar{x}$.

System and Objectives

$$\begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ y_M \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1N} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2N} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{M1} & a_{M2} & \cdot & \cdot & a_{MN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_N \end{bmatrix}$$

$$y_i = < \vec{a}_i, \bar{x} >$$

$$\bar{y} = \sum_j x_j \bar{a}_j$$

System and Objectives

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- Our aim is to find the sparse signal(s) $\hat{x}$ satisfying the above.

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- Our aim is to find the sparse signal(s) $\hat{x}$ satisfying the above.
- Need to design **matrix** A, as well as a **recovery algorithm**.

Design Example 1

- $N = 8,\ M = 1,\ s = 1,\ x_i \in \{0, 1\}$:

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = y_1$$

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- Solution:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{bmatrix}$$

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If $y_1 = j$, declare $x_j = 1$ and all others are zero.

Design Example 2

- $N = 8,\ M = 2,\ s = 1,\ x_i \in \{0, 1, 2 \cdot \cdot\}$:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

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- Solution:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{bmatrix}$$

Set $j = \frac{y_2}{y_1}$ and declare $x_j = y_1$; all others are zero.

Design Example 3

- $N = 8,\ M = 4,\ s = 2,\ x_i \in \{0, 1, \cdot\cdot 9\}$:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

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- $N = 8,\ M = 4,\ s = 2,\ x_i \in \{0, 1, \cdot \cdot 9\}$:

$$\begin{bmatrix} 1 & 2^0 & 3^0 & 4^0 & 5^0 & 6^0 & 7^0 & 8^0 \\ 1 & 2^1 & 3^1 & 4^1 & 5^1 & 6^1 & 7^1 & 8^1 \\ 1 & 2^2 & 3^2 & 4^2 & 5^2 & 6^2 & 7^2 & 8^2 \\ 1 & 2^3 & 3^3 & 4^3 & 5^3 & 6^3 & 7^3 & 8^3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

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- Challenge:

Design an integer-valued A with $0 \leq a_{ij} \leq 10$ for this problem.

Quick Recap

- A set of equations

$$\underset{M\times 1}{\bar{y}} = \underset{M\times N}{A} \; \underset{N\times 1}{\bar{x}}$$

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- Under-determined in general, little hope of recovering x.
- However, we wish to recover a sparse input x.
- **Goal:** Design the matrix A and a recovery strategy.

CD Writing

Discrete Fourier Transform

Recall our Vander Monde matrix (with $N' = N - 1$)

$$F = \begin{bmatrix} \alpha_0^0 & \alpha_1^0 & \alpha_2^0 & \cdots & \alpha_{N'}^0 \\ \alpha_0^1 & \alpha_1^1 & \alpha_2^1 & \cdots & \alpha_{N'}^1 \\ \alpha_0^2 & \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_{N'}^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_0^{N'} & \alpha_1^{N'} & \alpha_2^{N'} & \cdots & \alpha_{N'}^{N'} \end{bmatrix} \tag{1}$$

Erasure Coding

$$\begin{array}{ccccccc}
\boxed{x_{11}} & \boxed{x_{12}} & \boxed{x_{13}} & \boxed{x_{14}} & \boxed{x_{15}} & \cdots & \boxed{x_{1n}} \\
\boxed{x_{21}} & \boxed{x_{22}} & \boxed{x_{23}} & \boxed{x_{24}} & \boxed{x_{25}} & \cdots & \boxed{x_{2n}} \\
\boxed{x_{31}} & \boxed{x_{32}} & \boxed{x_{33}} & \boxed{x_{34}} & \boxed{x_{35}} & \cdots & \boxed{x_{3n}} \\
\boxed{x_{41}} & \boxed{x_{42}} & \boxed{x_{43}} & \boxed{x_{44}} & \boxed{x_{45}} & \cdots & \boxed{x_{4n}} \\
\boxed{x_{51}} & \boxed{x_{52}} & \boxed{x_{53}} & \boxed{x_{54}} & \boxed{x_{55}} & \cdots & \boxed{x_{5n}} \\
\boxed{x_{61}} & \boxed{x_{62}} & \boxed{x_{63}} & \boxed{x_{64}} & \boxed{x_{65}} & \cdots & \boxed{x_{6n}} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\boxed{x_{L1}} & \boxed{x_{L2}} & \boxed{x_{L3}} & \boxed{x_{L4}} & \boxed{x_{L5}} & \cdots & \boxed{x_{Ln}}
\end{array}$$

- Data symbols (byte)

$$\bar{d} = [d_1, \cdots, d_k]^T$$

- Stored symbols (int).

$$\bar{x} = [x_1, \cdots, x_n]^T$$

- Read values (int/?).

$$\bar{y} = [y_1, \cdots, y_n]^T$$

- Upto m erasures/row.

Erasure Coding

x_{11}	x_{12}	x_{13}	?	x_{15}	$\cdots$	x_{1n}
x_{21}	x_{22}	x_{23}	x_{24}	x_{25}	$\cdots$	x_{2n}
?	?	x_{33}	x_{34}	?	$\cdots$	x_{3n}
x_{41}	x_{42}	x_{43}	x_{44}	?	$\cdots$	x_{4n}
x_{51}	?	x_{53}	x_{54}	x_{55}	$\cdots$	?
x_{61}	x_{62}	x_{63}	?	x_{65}	$\cdots$	?
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{L1}	x_{L2}	x_{L3}	?	x_{L5}	$\cdots$	x_{Ln}

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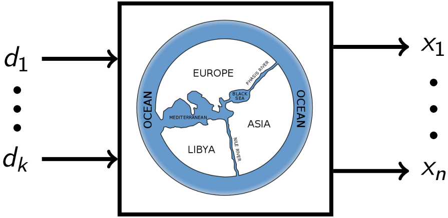
Encoding Strategy



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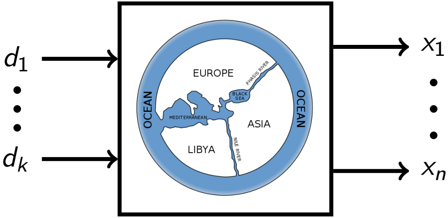

Using our *Vandermonde* matrix F.

$$\bar{x} = F\bar{d}, \text{ where } \bar{d}^T = [d_1, \cdots, d_k, 0, \cdots, 0].$$

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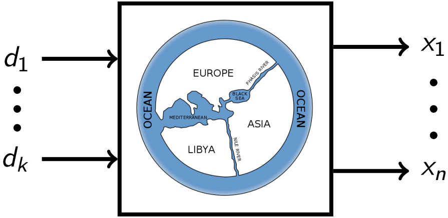

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Decoding possible if $k \leq n - m$.

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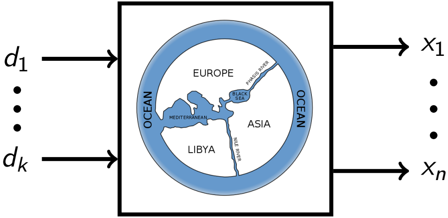

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Decoding possible if $k \leq n - m$.

Efficiency: $\boxed{\frac{k}{n} = 1 - \frac{m}{n}}$

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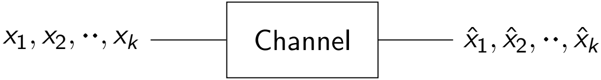
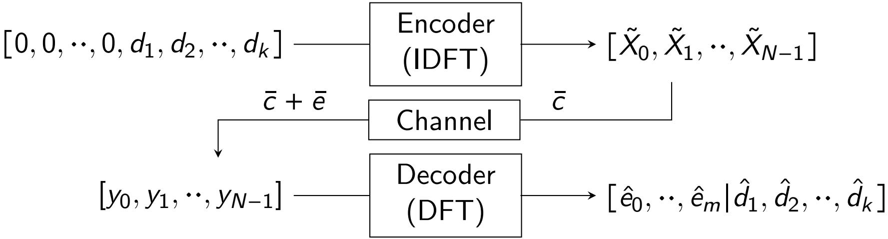
Correcting Errors

- Communication Channel



- Channel Coding



- If there are no errors, $\hat{e}_i = 0$ and $\hat{d}_i = d_i$, $\forall i$.

How it works

- For s-errors, consider the first $m = 2s$ rows of Fourier matrix.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \alpha_1^1 & \alpha_2^1 & \alpha_3^1 & \alpha_4^1 & \alpha_5^1 & \alpha_6^1 & \alpha_7^1 \\ 1 & \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \alpha_4^2 & \alpha_5^2 & \alpha_6^2 & \alpha_7^2 \\ 1 & \alpha_1^3 & \alpha_2^3 & \alpha_3^3 & \alpha_4^3 & \alpha_5^3 & \alpha_6^3 & \alpha_7^3 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_8 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_m \end{bmatrix}$$

- Pick **any** $2s$ columns from this restricted matrix.

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \alpha_k^1 & \alpha_l^1 & \alpha_m^1 & \alpha_n^1 \\ \alpha_k^2 & \alpha_l^2 & \alpha_m^2 & \alpha_n^2 \\ \alpha_k^3 & \alpha_l^3 & \alpha_m^3 & \alpha_n^3 \end{bmatrix}$$

- $det(B) \neq 0$ if $\alpha_i \neq \alpha_j \implies$ cols. linearly independent.

Why so many Pixels

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Lungern, Switzerland, Oct 2006

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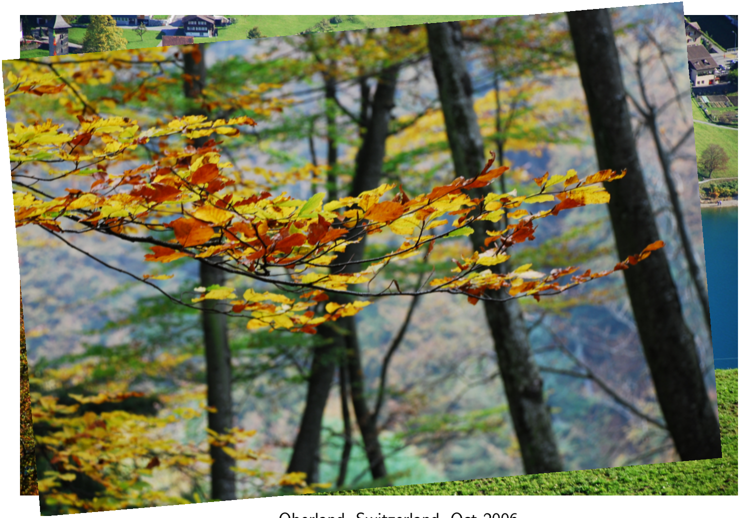


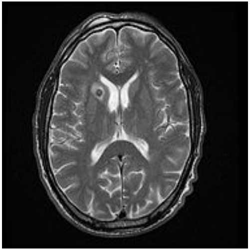
Oberland, Switzerland, Oct 2006

Why so many Pixels

EPFL, Switzerland, May 2007

Magnetic Resonance Imaging

MRI image, source:wikipedia



Magnetic Resonance Imaging

Shepp-Logan Phantom 256×256

Candes Romberg Tao' 2006

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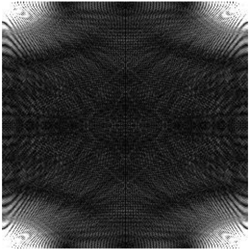
Shepp-Logan Phantom 256×256



Frequency grid

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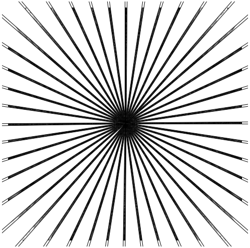
Magnetic Resonance Imaging

Shepp-Logan Phantom 256×256



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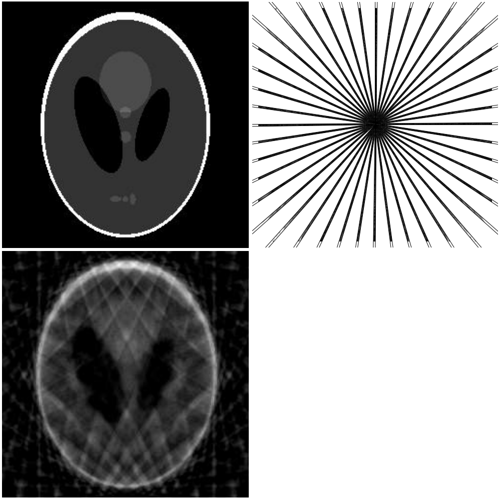
Magnetic Resonance Imaging



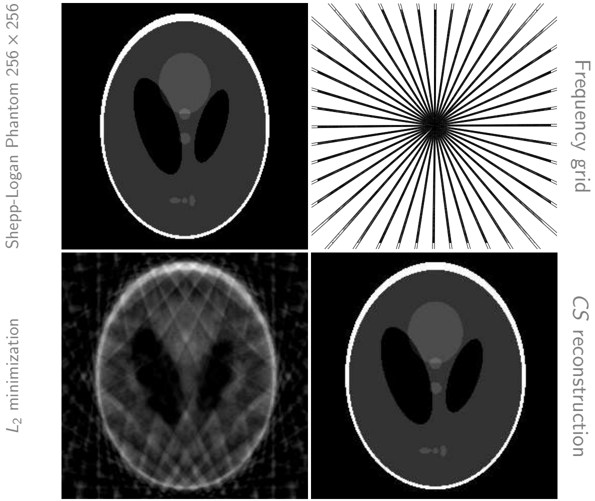
Shepp-Logan Phantom 256×256

Frequency grid

L_2 minimization

Candes Romberg Tao' 2006

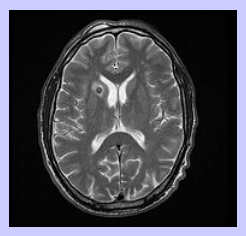
Magnetic Resonance Imaging

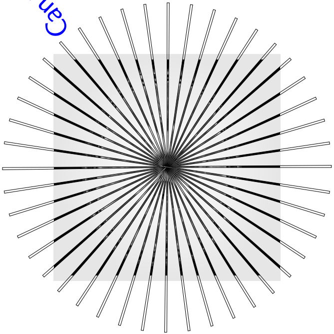


Candes Romberg Tao' 2006

CS Magic?

Can we get back the original





- Surely you are joking Mr. Xxxxman.
- Linear equations seem to do wonders here.

Conclusion

- We discussed three problems
  1. Space communication.
  2. CD Information storage and retrieval.
  3. MRI Imaging
- Somewhat cute that linear solvers are the key to all three.
- Comments and queries to bsraj@ee.iitb.ac.in