Tut 6

Q1

Equation	Surface	Eigenvalues of A
$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	ellipsoid	all three positive
$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z}{c}=0$	elliptic paraboloid	two positive, one zero
$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$	elliptic cone	two positive, one negative
$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	1-sheeted hyperboloid	two positive, one negative
$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	2-sheeted hyperboloid	one positive, two negative
$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z}{c}=0$	hyperbolic paraboloid	one positive, one negative, one zero.

(a) $\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 1$

$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ $\quad |A - \lambda I| = 0$

$\rightarrow \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0 = -\lambda^3 + 2 + 3\lambda$

$\rightarrow \lambda^3 - 3\lambda - 2 = 0$

$\rightarrow \lambda(\lambda-1)(\lambda+1) - 2(\lambda+1) = 0$

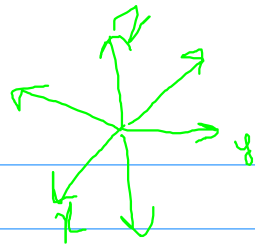
$\rightarrow (\lambda-2)(\lambda+1)^2 = 0 \rightarrow \underbrace{2, -1, -1}$

2-sheeted hyperboloid

(b) 1-sheeted hyperboloid

(c) 1-sheeted hyperboloid

Q2. $\displaystyle\int_{R^3} e^{-(2x^2+5y^2+2z^2-4xy-2xz+4yz)}\,dx\,dy\,dz$



$(2x^2+5y^2+2z^2-4xy-2xz+4yz)$

$$= \begin{bmatrix} x & y & z \end{bmatrix} \underbrace{\begin{bmatrix} 2 & -2 & -1 \\ -2 & 5 & 2 \\ -1 & 2 & 2 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$|A-\lambda I| = 0$

$$\rightarrow \begin{vmatrix} 2-\lambda & -2 & -1 \\ -2 & 5-\lambda & 2 \\ -1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$\rightarrow (\lambda^2-4\lambda+4)(5-\lambda) + 4 + 4 + (\lambda-5) + 8(\lambda-2) = 0$

$\rightarrow 5\lambda^2-20\lambda+20-\lambda^3+4\lambda^2-4\lambda+8+9\lambda-21 = 0$

$\rightarrow -\lambda^3+9\lambda^2-15\lambda+7 = 0$

$\rightarrow \lambda = 1, 1, 7$

Finding eigenvectors

$\lambda = 1$ $\quad \begin{bmatrix} 1 & -2 & -1 \\ -2 & 4 & 2 \\ -1 & 2 & 1 \end{bmatrix} X = 0$

$\rightarrow \begin{bmatrix} 1 & -2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} X = 0 \rightarrow$ Basis of N(A):

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$\rightarrow$ Orthonormal basis (Gram-Schmidt)

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$\lambda = 7$

$$\begin{bmatrix} -5 & -2 & -1 \\ -2 & -2 & 2 \\ -1 & 2 & -5 \end{bmatrix} X = 0$$

$$\rightarrow \begin{bmatrix} -1 & 2 & -5 \\ 0 & -6 & 12 \\ 0 & -12 & 24 \end{bmatrix} X = 0 \quad \rightarrow \begin{bmatrix} -1 & 2 & -5 \\ 0 & -6 & 12 \\ 0 & 0 & 0 \end{bmatrix} X = 0$$

$$\rightarrow X = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

Hence, orthogonally diagonalizing A:

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$= UDU^T$$

$$x^T A x = x^T U D U^T x = (U^T x)^T D (U^T x)$$

$$\int_{R^3} e^{-(U^T v)^T A (U^T v)} \, dx\,dy\,dz \qquad v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Perform change of variables for integration

$$w = U^T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \leftarrow \text{New vars}$$

Jacobian $= U^T$

$\rightarrow |\det(J)| = 1$

$$= \int_{\mathbb{R}^3} e^{-w^T D w}\, d\alpha\, d\beta\, d\gamma$$

$$w^T D w = \underbrace{\alpha^2 + \beta^2 + 7\gamma^2}$$

$$= \int_{\mathbb{R}^3} e^{-(\alpha^2+\beta^2+7\gamma^2)}\, d\alpha\, d\beta\, d\gamma$$

$$= \int_{-\infty}^{\infty} e^{-\alpha^2} d\alpha \cdot \int_{-\infty}^{\infty} e^{-\beta^2} d\beta \cdot \int_{-\infty}^{\infty} e^{-7\gamma^2} d\gamma$$

$$= \sqrt{\pi} \cdot \sqrt{\pi} \cdot \sqrt{\frac{\pi}{7}} = \frac{\pi^{3/2}}{\sqrt{7}}$$

Q3 $\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Specific case: when f=g=h=0

$$ax^2 + by^2 + cz^2$$

remember the JEE way of doing this?

look at it as a quadratic in x

$$= a\left(x^2 + \frac{b}{a} y^2 + \frac{c}{a} z^2\right) \qquad \text{if } a \neq 0$$

$$= a(x - \alpha_1)(x - \alpha_2)$$

$$\alpha_1, \alpha_2 = \frac{\pm\sqrt{-4(by^2+cz^2)}}{2a}$$

seperable as linear factors if α_1, α_2 are linear in y,z
hence, (by^2+cz^2) must be a perfect square of a
complex linear in y,z

$\rightarrow$ $b=0$ or $c=0$

Symmetrically, $a=0$ or $b=0$ or $c=0$

Hence, in the case when f=g=h=0, the quadratic is factorizable into linear terms iff abc=0

In general,

$$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$ is orthogonally diagonalizable

$$\therefore \quad A = UDU^T, \quad D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$\therefore \quad x^T U D U^T x = \underbrace{(U^T x)^T}_{w^T} D \underbrace{(U^T x)}_{w} = w^T D w$$

w is linear in x. Hence $x^T A x$ is factorizable into linear factors iff $w^T D w$ is factorizable into linear factors

Hence, atleast one of the eigenvalues must be 0

$$\therefore \quad \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

$$\begin{bmatrix} \cos\theta & s\theta \\ -s\theta & c\theta \end{bmatrix} \rightarrow e^{\pm i\theta}$$

Q4. Consider the 3x3 orthogonal matrix U: It must have a real eigenvalue as char eqn is odd degree

$$U^T U = I$$

$\therefore$ For $x \neq 0$

$$Ux = \lambda x \quad \text{Real}$$

$$\rightarrow \quad x^T U^T = \lambda x^T$$

$$\rightarrow \quad x^T \underbrace{U^T U}_{I} x = \lambda^2 x^T x \quad \rightarrow (\lambda^2 - 1)\underbrace{(x^T x)}_{\text{Non zero as } x \in \mathbb{R}^3} = 0$$

$$\rightarrow \lambda = \pm 1$$

$|A| =$ Product of eigenvalues taken with geometric multiplicities

If no eigenvalue is 1, then $|A|$ would be -1 if all roots real

If two eigenvalues are complex, they will be conjugates of each other, hence product is positive. So real eigenvalue is 1

$$\therefore |A| = 1 \rightarrow \lambda_1 = 1 \text{ exists}$$

Q5 Example: $\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad \lambda = 1, e^{i\theta}, e^{-i\theta}$

$$A(\rho + i\sigma) = (\alpha + i\beta)(\rho + i\sigma)$$

$$\rightarrow A\rho + iA\sigma = (\alpha\rho - \beta\sigma) + i(\alpha\sigma + \beta\rho)$$

compare real and imaginary parts:

$$A\rho = \alpha\rho - \beta\sigma$$
$$A\sigma = \alpha\sigma + \beta\rho$$

Notice that A also quallifies as a normal matrix as $AA^* = A^*A \; (= I)$

Hence, under the complex inner product, $v, \rho + i\sigma, \rho - i\sigma$ are orthogonal

$$\therefore v^T(\rho - i\sigma) = 0 \rightarrow v^T\rho = 0, \; v^T\sigma = 0$$
$$v^T(\rho + i\sigma) = 0$$

$$(\rho + i\sigma)^T(\rho + i\sigma) = 0$$

$$\rightarrow (\rho^T\rho - \sigma^T\sigma) + i(\rho^T\sigma + \sigma^T\rho) = 0$$

$$\rho^T\rho = \sigma^T\sigma \qquad \rho^T\sigma$$
$$\rho^T\sigma = 0$$

Hence, v, ρ, σ are orthogonal under the real inner product

Now, we know:

$$O^TAO = \begin{bmatrix} v^T \\ \rho^T \\ \sigma^T \end{bmatrix} A \begin{bmatrix} v & \rho & \sigma \end{bmatrix} = \begin{bmatrix} v^T \\ \rho^T \\ \sigma^T \end{bmatrix} \begin{bmatrix} v & \alpha\rho - \beta\sigma & \alpha\sigma + \beta\rho \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -\beta & \alpha \end{bmatrix}$$

Fibonacci Series:

$$f_0 = 0 \quad f_1 = 1$$

$$f_{n+1} = f_n + f_{n-1} \qquad n \geq 1$$

In general what is a closed form expression for f_n

$$\begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f_n \\ f_{n-1} \end{bmatrix}$$

$$\begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{A}{}^{n} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A = P \underbrace{D}_{} P^{-1}$$

$$A^n = P D^n P^{-1}$$

$$|A - \lambda I| = 0 \rightarrow \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\rightarrow \quad \lambda^2 - \lambda - 1 = 0$$

$$\rightarrow \quad \lambda = \frac{1 \pm \sqrt{5}}{2} = \alpha, \beta$$

If the corresponding eigenvectors are v,w

$$A = \begin{bmatrix} v & w \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} v & w \end{bmatrix}^{-1}$$

$$A^n = \begin{bmatrix} v & w \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} v & w \end{bmatrix}^{-1}$$

$$\begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix} = \begin{bmatrix} v & w \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} v & w \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$