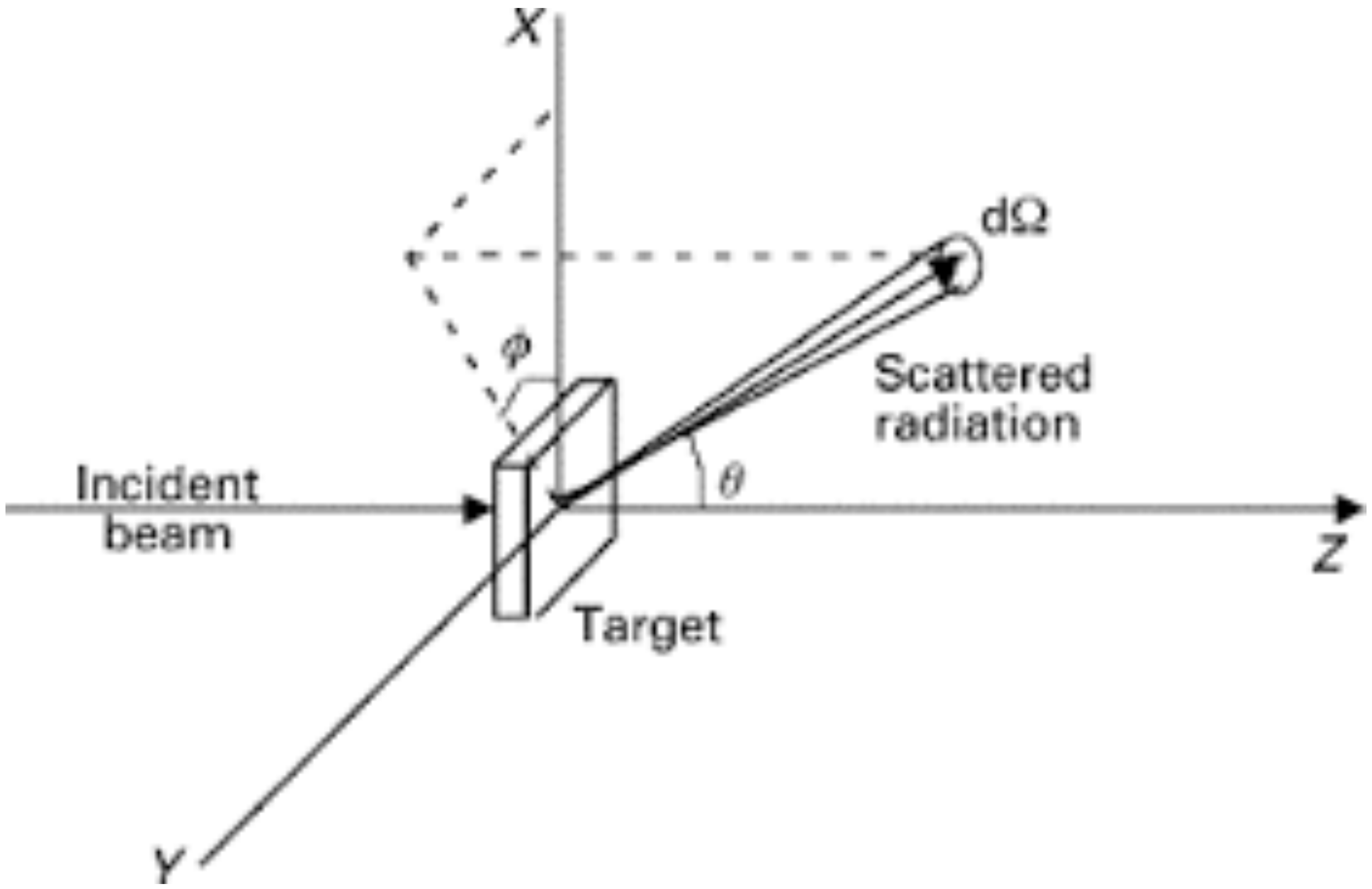
PH-107

Scattering and Step Potential

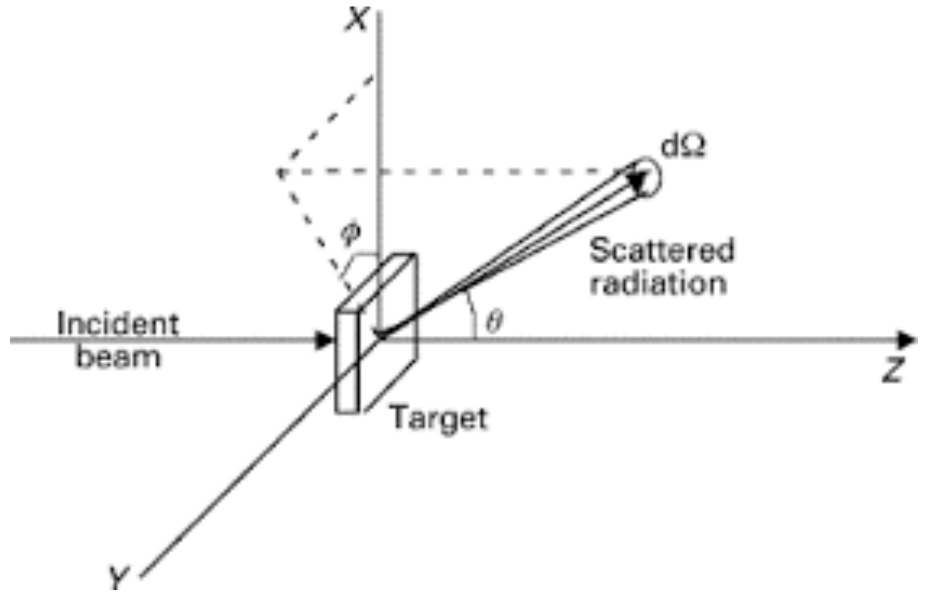
Scattering

In a number of cases, we study the properties of the interaction between two objects by means of scattering.



We shoot projectiles (usually light objects) with a well defined momentum at a target (usually a heavy object at rest). We observe how the projectiles are scattered by the target. That is we measure the momentum of each projectile as it is pushed by the target.

Scattering



Interaction between the projectile and the target is treated in terms of a potential. By observing the pattern of the projectiles scattering off the target, we can figure out the potential.

Some examples of scattering are

- Rutherford Scattering: Shooting α particles off gold nuclei.
- Compton Scattering: Shooting X-rays off electrons in metal.
- Raman Scattering: Shooting light off electrons in molecules.

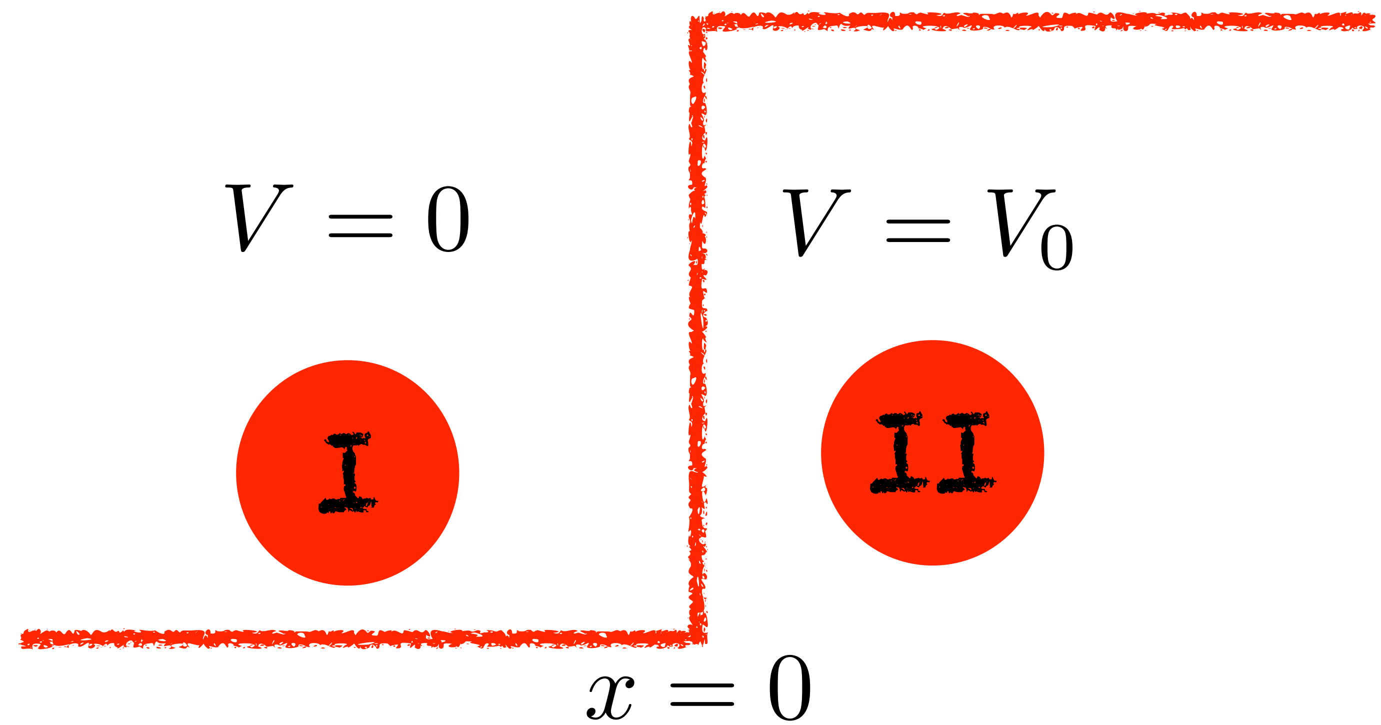
Here we consider only the projectile. We will not worry about what the target is. We assume that the target gives rise to a potential V(x) and the projectile is affected by this potential.

Step Potential

Potential

$$V = 0 \quad \forall x \leq 0$$

$$= V_0 \quad \forall x > 0$$

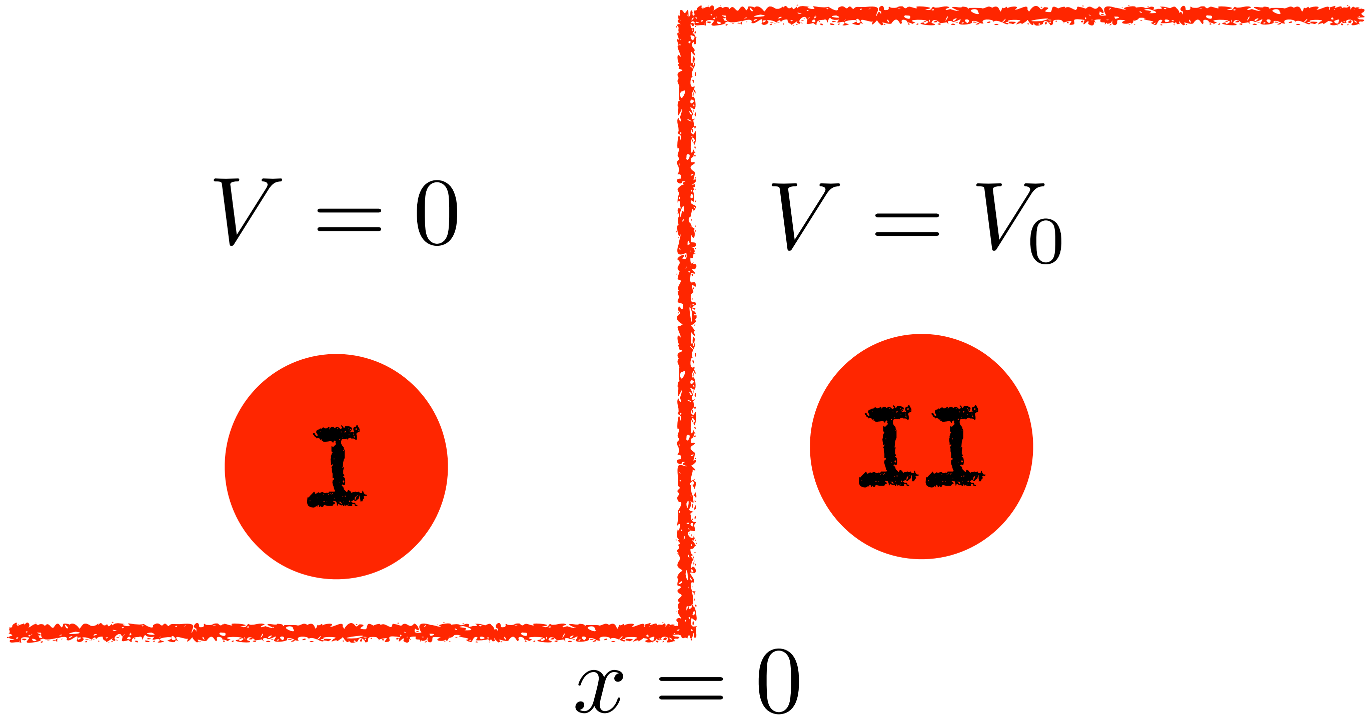


For $E > V_0$

I $$\phi_I(x) = Ae^{ik_1 x} + Be^{-ik_1 x}; \;\; k_1^2 = \frac{2mE}{\hbar^2}$$

II $$\phi_{II}(x) = Ce^{ik_2 x} + De^{-ik_2 x}; \;\; k_2^2 = \frac{2m(E - V_0)}{\hbar^2}$$

Step Potential



As there is no incidence from the right side, so

$$\phi_{II}(x) = Ce^{ik_2 x} + De^{-ik_2 x} \implies D = 0$$

Boundary conditions:

$$\phi_I(0) = \phi_{II}(0) \implies A + B = C$$

$$\phi'_I(0) = \phi'_{II}(0) \implies ik_1(A - B) = ik_2 C$$

Step Potential

$$A + B = C$$

$$A - B = \left(\frac{k_2}{k_1}\right) C$$

$$A = \left(1 + \frac{k_2}{k_1}\right) \frac{C}{2}$$

and

$$B = \left(1 - \frac{k_2}{k_1}\right) \frac{C}{2}$$

$$\frac{C}{A} = \frac{2k_1}{k_1 + k_2}$$

and

$$\frac{B}{A} = \frac{k_1 - k_2}{k_1 + k_2}$$

Step Potential

So we can write the wave functions as

$$\phi_I(x) = A\left(e^{ik_1x} + \left(\frac{k_1 - k_2}{k_1 + k_2}\right)e^{-ik_1x}\right)$$

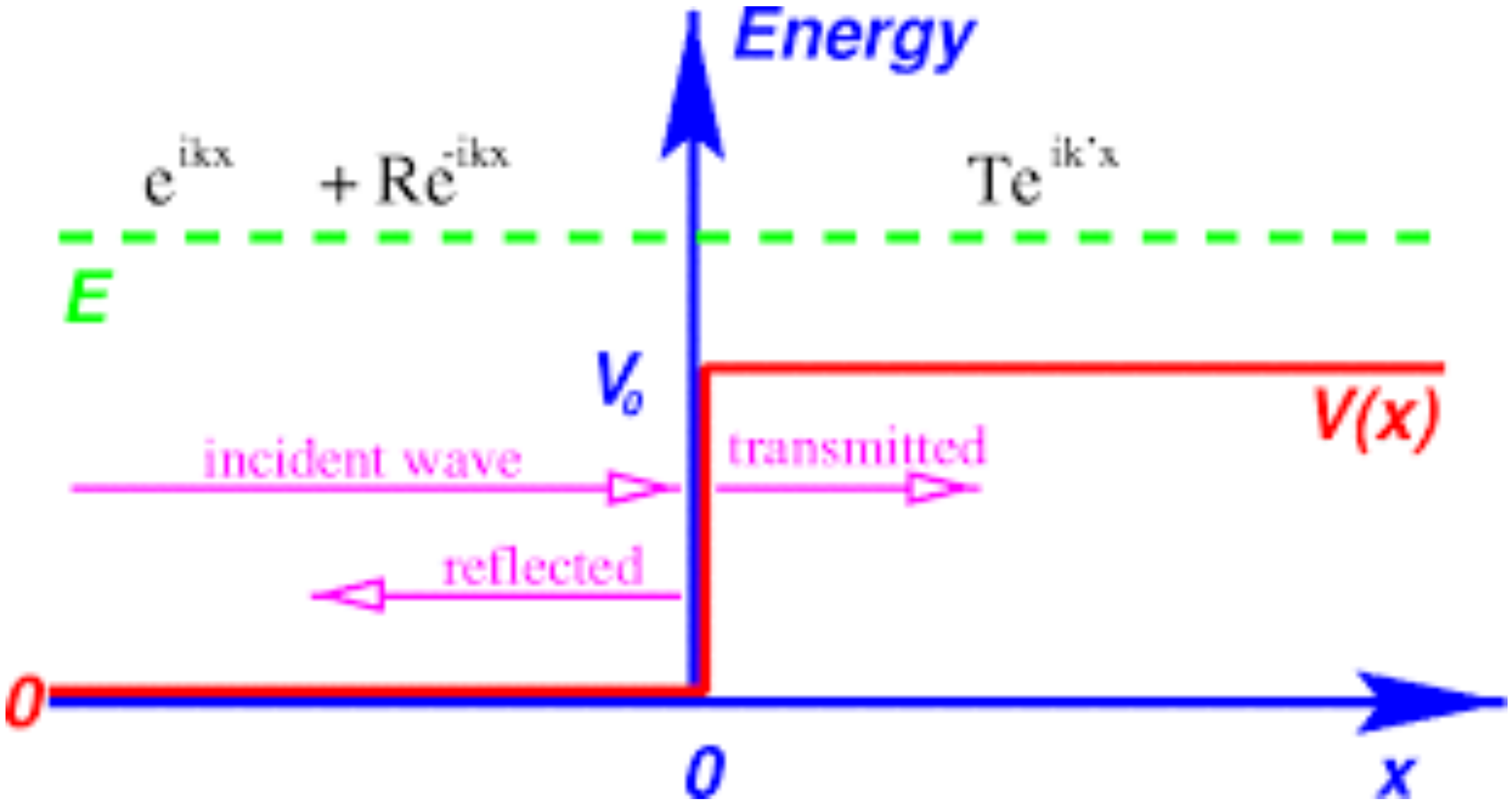
and

$$\phi_{II}(x) = A\left(\frac{2k_1}{k_1 + k_2}\right)e^{ik_2x}$$

This implies that the probability of the particle being **reflected** is non-zero.

Classically, this is forbidden

Step Potential



Step Potential

Reflection coefficient

$$R = \left|\frac{B}{A}\right|^2 = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2$$

Transmission coefficient

$$T = \frac{v_2}{v_1}\left|\frac{C}{A}\right|^2 = \frac{k_2}{k_1}\left|\frac{C}{A}\right|^2 = \frac{4k_1k_2}{(k_1 + k_2)^2}$$

The rate at which the incident particles approach the barrier is $(\hbar k_1/m)|A|^2$. The rate at which they are reflected is $(\hbar k_1/m)|B|^2$ and the rate at which they move forward is $(\hbar k_2/m)|C|^2$. This accounts for the factor k_2/k_1 in the definition of T.

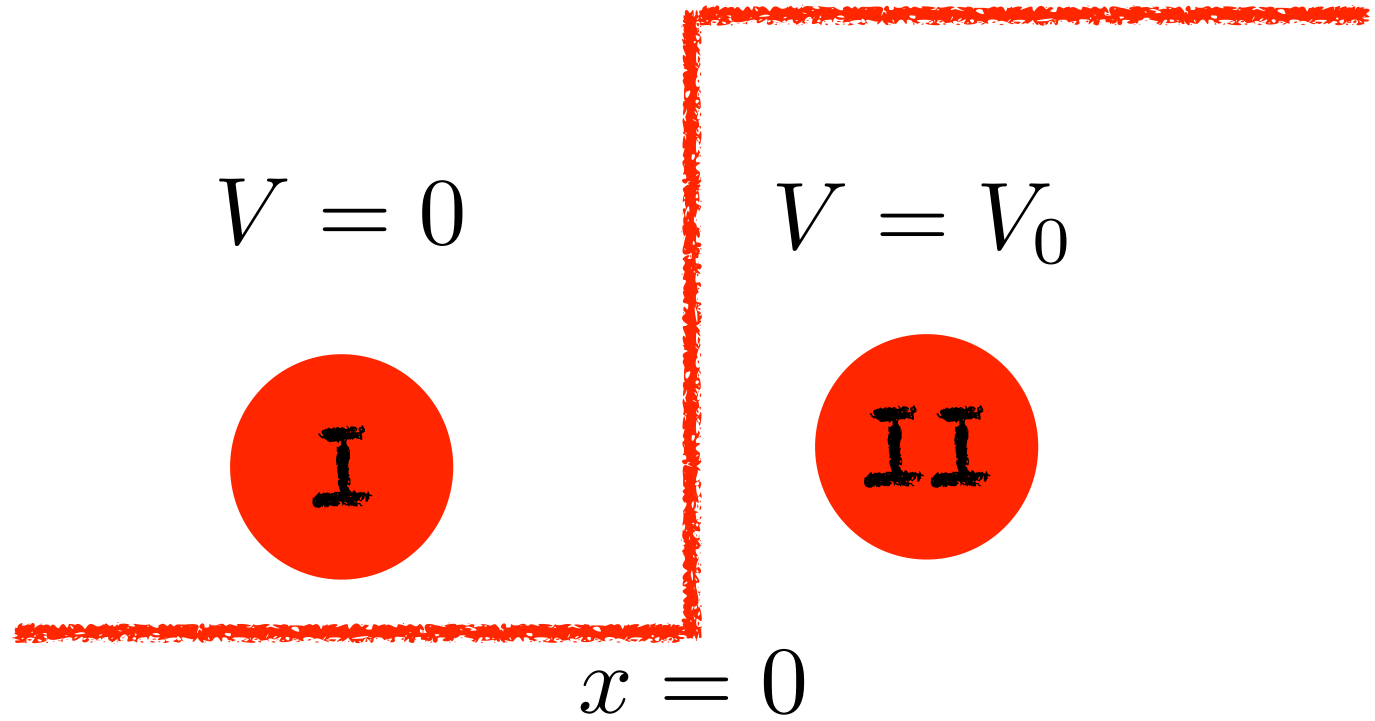
You can verify that $R + T = 1$

Step Potential

Potential

$$V = 0 \quad \forall x \leq 0$$

$$= V_0 \quad \forall x > 0$$

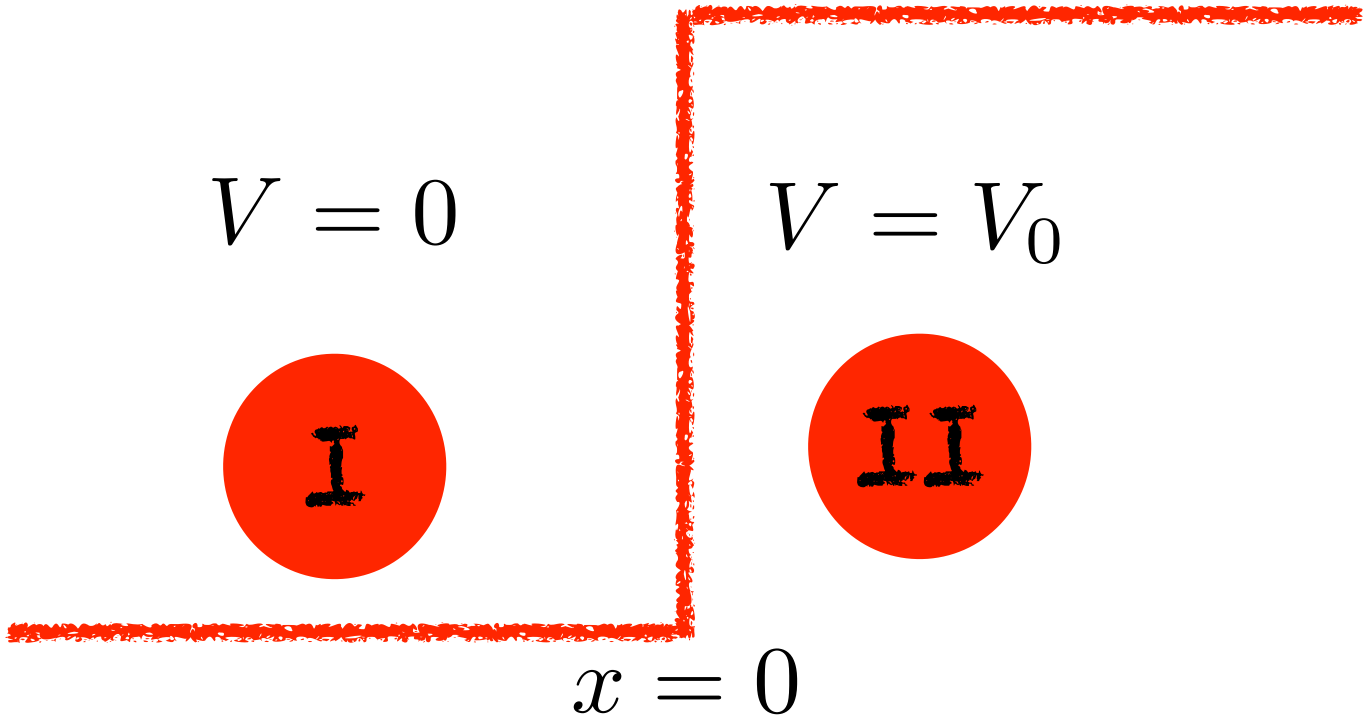


For $E < V_0$

I $$\phi_I(x) = Ae^{ikx} + Be^{-ikx};\; k^2 = \frac{2mE}{\hbar^2}$$

II $$\phi_{II}(x) = Ce^{-\alpha x} + De^{\alpha x};\; \alpha^2 = \frac{2m(V_0 - E)}{\hbar^2}$$

Step Potential



As there is no incidence from the right side, so

$$\phi_{II}(x) = Ce^{-\alpha x} + De^{\alpha x} \implies D = 0$$

Boundary conditions:

$$\phi_I(0) = \phi_{II}(0) \implies A + B = C$$

$$\phi_I'(0) = \phi_{II}'(0) \implies ik(A - B) = -\alpha C$$

Step Potential

Finding the coefficients:

Trick: Put k_1 = k and k_2 = $i\alpha$

$$\frac{C}{A} = \frac{2k_1}{k_1 + k_2}$$

and

$$\frac{B}{A} = \frac{k_1 - k_2}{k_1 + k_2}$$

For $E > V_0$

$$\Longrightarrow$$

$$\frac{C}{A} = \frac{2k}{k + i\alpha}$$

and

$$\frac{B}{A} = \frac{k - i\alpha}{k + i\alpha}$$

For $E < V_0$

Step Potential

Reflection coefficient

$$R = \left|\frac{B}{A}\right|^2 = \left(\frac{k - i\alpha}{k + i\alpha}\right)\left(\frac{k + i\alpha}{k - i\alpha}\right) = 1$$

The Reflection Coefficient = 1, implying that the probability of reflection is 100%. However, $C \neq 0$ means that the particle penetrates into region II.

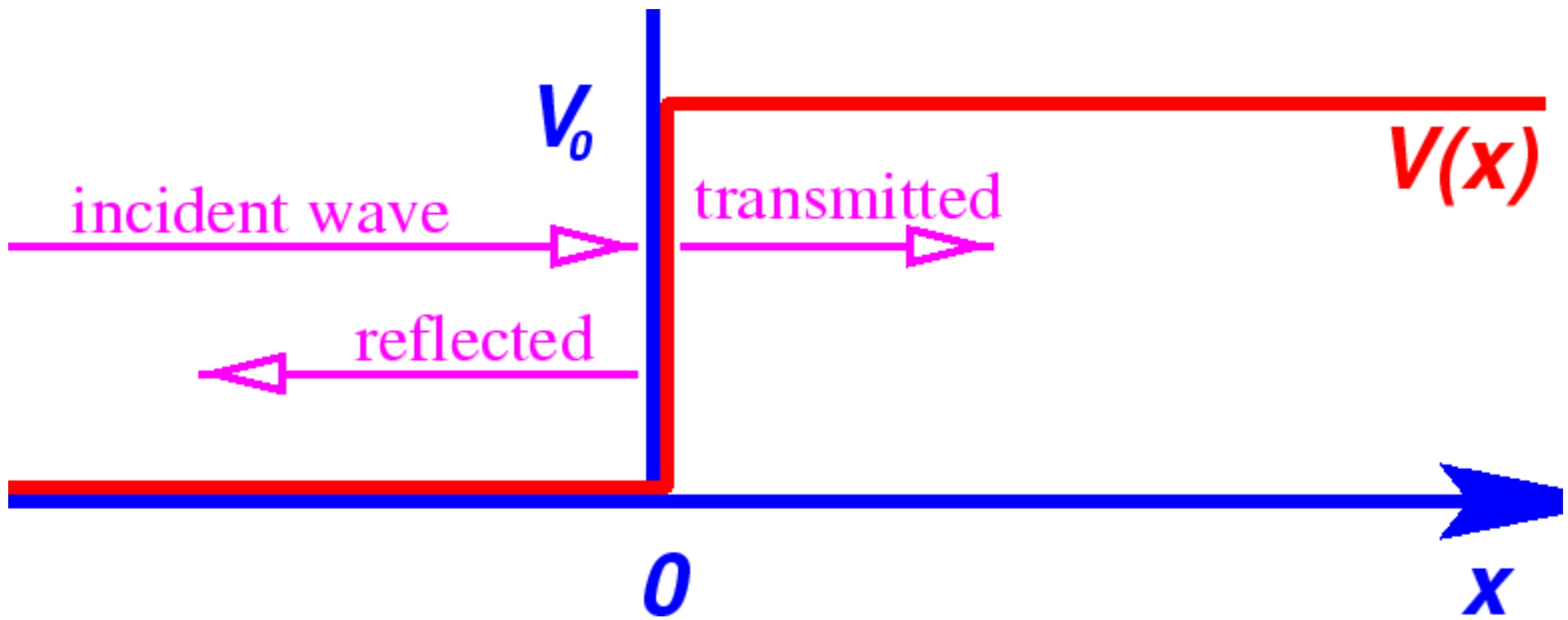
The particle actually penetrates the potential barrier upto a depth of about $1/\alpha = \hbar/\sqrt{2m(V_0 - E)}$

Classically, this is forbidden

Step Potential

Reflection coefficient

$$R = \left|\frac{B}{A}\right|^2 = \left(\frac{k - i\alpha}{k + i\alpha}\right)\left(\frac{k + i\alpha}{k - i\alpha}\right) = 1$$



No power transmission or energy transfer across the potential step. Yet the probability of finding the particle is non-zero in region II