MA-111 Calculus II (D1 & D2)

Lecture 2

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Evaluating Integrals

Recall in 1 dimension, for a integrable function f on $[a, b]$, we use *Fundamental theorem of Calculus:* Find a function F on $[a, b]$ such that $F'(x) = f(x)$ for all $x \in (a, b)$ and then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Can we compute double integrals using this?

Cavalieri's principle Suppose two regions in three-space (solids) are included between two parallel planes. If every plane parallel to these two planes intersects both regions in cross-sections of equal area, then the two regions have equal volumes.

Picture from wikipedia. Two stacks of coins with the same volume, illustrating Cavalieri's principle in three dimensions.

Geometrically, if f is non-negative then the double integral is the volume of the region D between the rectangle and under the surface $z = f(x, y)$. Then first compute area of each slice $A(x) = \int_c^d f(x, y)\ dy$ of the cross section of D perpendicular to the x-axis (or alternately the area $B(y) = \int_a^b f(x, y)\ dx$ of the cross section perpendicular to the y-axis)

Then the volume of $D = \int_a^b A(x)\ dx = \int_c^d B(y)\ dy$.

Fubini theorem and Iterated integrals

Theorem

Let $R := [a, b] \times [c, d]$ and $f : R \to \mathbb{R}$ be integrable. Let I denote the integral of f on R.

1. *If for each $x \in [a, b]$, the Riemann integral $\int_c^d f(x, y)\, dy$ exists, then the iterated integral $\int_a^b (\int_c^d f(x, y)\, dy)\, dx$ exists and is equal to I.*
2. *If for each $y \in [c, d]$, the Riemann integral $\int_a^b f(x, y)\, dx$ exists, then the iterated integral $\int_c^d (\int_a^b f(x, y)\, dx)\, dy$ exists and is equal to I.*

As a consequence, if f is integrable on R and if both iterated integrals exist in 1. and 2. in above theorem, then

$$\int_a^b \Big(\int_c^d f(x, y)\, dy \Big)\, dx = I = \int_c^d \Big(\int_a^b f(x, y)\, dx \Big)\, dy.$$

Sketch of the proof

The proof is using Riemann condition.

- Since f is double integrable over R, for any given $\epsilon > 0$, there exists a partition $P_\epsilon = \{(x_i, y_j) \mid i = 0, 1, \cdots k-1, \quad j = 0, \cdots n-1\}$ of R such that
$$U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon.$$
- Assume for each fixed $x \in [a, b]$, the Riemann integral $\int_c^d f(x, y)\, dy$ exists. Define
$$A(x) := \int_c^d f(x, y)\, dy, \quad \forall\, x \in [a, b].$$
- Claim: The function A is integrable over $[a, b]$. Note that $m(f)(d-c) \le A(x) \le M(f)(d-c)$ for all $x \in [a, b]$ and hence A is bounded. Also by domain additivity, $A(x) = \sum_{j=0}^{n-1} \int_{y_j}^{y_{j+1}} f(x, y)\, dy$, for all $x \in [a, b]$.
- Thus for each fixed $i \in \{0, \cdots k-1\}$, for $x \in [x_i, x_{i+1}]$, we obtain
$$\sum_{j=0}^{n-1} m_{ij}(f)(y_{j+1} - y_j) \le A(x) \le \sum_{j=0}^{n-1} M_{ij}(f)(y_{j+1} - y_j).$$

Sketch of the proof contd.

- Denoting $m_i(A) := \inf\{A(x) \mid x \in [x_i, x_{i+1}]\}$ and $M_i(A) := \sup\{A(x) \mid x \in [x_i, x_{i+1}]\}$, we have

$$\sum_{j=0}^{n-1} m_{ij}(f)(y_{j+1} - y_j) \leq m_i(A) \leq M_i(A) \leq \sum_{j=0}^{n-1} M_{ij}(f)(y_{j+1} - y_j).$$

  Multiplying by $(x_{i+1} - x_i)$ and summing over $i = 0, \dots, k-1$, we obtain

$$L(f, P_\epsilon) \leq \sum_{i=0}^{k-1} m_i(A)(x_{i+1} - x_i) \leq \sum_{i=0}^{k-1} M_i(A)(x_{i+1} - x_i) \leq U(f, P_\epsilon).$$

  and it yields that there exists a partition $P_1 := \{x_0, \cdots, x_{k-1}\}$ of $[a, b]$ such that

$$U(A, P_1) - L(A, P_1) < \epsilon.$$

- Thus the function of A is integrable and

$$\int\int_R f\, dx\, dy = \int_a^b A(x)\, dx = \int_a^b \Big(\int_c^d f(x, y)\, dy \Big)\, dx.$$

Subtleties in Fubini's theorem - I

An example where both iterated integrals exist but are not equal, the function f is not double integrable.

Example

$R := [0,1] \times [0,1]$, $f(x,y) = \begin{cases} \frac{(x^2-y^2)}{(x^2+y^2)^2}, & x \neq 0 \neq y, \\ 0, & x = 0 \text{ or } y = 0. \end{cases}$

We have

$$\int_0^1 \int_0^1 f(x,y)dxdy = \int_0^1 \frac{-1}{1+y^2} dy = \frac{-\pi}{4}$$

and

$$\int_0^1 \int_0^1 f(x,y)dydx = \int_0^1 \frac{1}{1+x^2} dx = \frac{\pi}{4}$$

A similar example

$R := [0,1] \times [0,1]$, $f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{(x^2+y^2)^3}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$

Compute both the iterated integrals and compare them.