Laplace's Equation

Laplace's Equation

Outline

I. General properties of Laplace's eq. in 1-, 2- and 3-D, boundary conditions and uniqueness theorem

II. Methods for solving Laplace's eq.: (i) Separation of variables and (ii) Method of images with examples

III. Electric dipole as a prelude to the dielectric materials

IV. Applications

Laplace's Equation

Learning Objectives

I. To learn about the properties of Laplace's eq, associated boundary conditions and uniqueness theorem.

II. To learn how to solve Laplace's eq. using (i) separation of variables and (ii) method of images.

III. To refresh our knowledge of electric dipoles.

Laplace's Equation

Learning Outcomes

I. To be able to solve Laplace's eq. with azimuthal symmetry using separation of variables in spherical polar coordinates.

II. To be able to solve Laplace's eq. using method of images.

III. To be able to calculate the electric potential and the electric field due to an electric dipole.

Poisson's Equation and Laplace's Equation

We have characterized the electric field by calculating its divergence and curl,

$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\epsilon_0} \quad \text{and} \quad \vec{\nabla} \times \vec{E}(\vec{r}) = 0$$

We want to see how these equations change with the introduction of potential. The curl equation becomes

$$\vec{\nabla} \times \vec{\nabla} V(\vec{r}) = 0$$

while the divergence equation reduces to

$$\vec{\nabla} \cdot (-\vec{\nabla} V(\vec{r})) = -\nabla^2 V(\vec{r}) = \frac{\rho(\vec{r})}{\epsilon_0}$$

or, $\boxed{\nabla^2 V(\vec{r}) = -\frac{\rho(\vec{r})}{\epsilon_0}}$ $\Leftarrow$ Poisson's equation

which is known as the Poisson's equation

Poisson's Equation and Laplace's Equation

If the volume of interest doesn't contain any charge density so that $\rho(\vec{r}) = 0$ then Poisson's equation becomes

$$\boxed{\nabla^2 V(\vec{r}) = 0} \Leftarrow \boxed{\text{Laplace's equation}}$$

which is known as the Laplace's equation. The Laplacian, ∇^2 , in different coordinate systems is given by

$$Cartesian\,Coordinates: \quad \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$Cylindrical\,Coordinates: \quad \nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

$Spherical\,Coordinates:$

$$\nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

Laplace's Equation: General Properties

First, we will consider the general properties of Laplace's equation,

$$\boxed{\nabla^2 V(\vec{r}) = 0} \Leftarrow \boxed{\text{Laplace's equation}}$$

in one (1-D), two (2-D) and three (3-D) dimensions, respectively. In 1-D, the Laplace's eqaution reduces to

$$\frac{d^2V}{dx^2} = 0$$

which is a second order differential equation. Its general solution can be obtained in closed form by integrating it twice.

Laplace's Equation: General Properties

Integrating it once,

$$\int \frac{d^2V(x)}{dx^2}dx = \int \frac{d}{dx}\left(\frac{dV(x)}{dx}\right)dx = \int d\left(\frac{dV(x)}{dx}\right) = \frac{dV(x)}{dx} = m$$

where *m* is the first constant of integration. Again,

$$\int \frac{dV(x)}{dx}dx = \int m dx = m\int dx$$

or, $\displaystyle\int dV(x) = V(x) = mx + b$

Hence, the general solution of Laplace's equation in 1-D is given by

$$V(x) = b + mx$$

A unique solution is obtained by specifying two boundary conditions determining *m* and *b*.

Laplace's Equation (1-D): General Properties

In 1-D, the solution of Laplace's equation exhibits the following properties.

1. The potential V(x) is the average of V(x+a) and V(x-a) for any a.

$$V(x) = \frac{1}{2}\left[V(x+a) + V(x-a)\right]$$

2. There are no local maxima or minima in the potential.

To prove 1, we consider,

$$\begin{aligned}\frac{1}{2}\left[V(x+a) + V(x-a)\right] &= \frac{1}{2}\left[(mx+ma+b) + (mx-ma+b)\right] \\ &= \frac{1}{2}\left[2(mx+b)\right] \\ &= V(x)\end{aligned}$$

Laplace's Equation (2-D): General Properties

In 2-D, the Laplace's equation becomes,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

which is a second-order partial differential equation. There are no general solutions in closed form.

1. The potential $V(x,y)$ at a point is an average of potentials at nearby points. In particular, if one draws a circle of radius R centered at point (x,y) then

$$V(x,y) = \frac{1}{2\pi R} \oint_{\text{circle}} V\,dl$$

2. There are no local maxima or minima in the potential $V(x,y)$. The extreme values occur at the boundary.

Laplace's Equation (3-D): General Properties

In 3-D, the Laplace's equation becomes,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

which is a second-order partial differential equation. There are no general solutions in closed form.

1. The potential $V(x,y,z)$ at a point (x,y,z) is average of the potential over a sphere of radius R and centered at (x,y,z).

$$V(x,y.z) = \frac{1}{4\pi R^2} \oint_{\text{sphere}} V da$$

2. There are no local maxima or minima in the potential $V(x,y,z)$. The extreme values occur at the bounding surface.

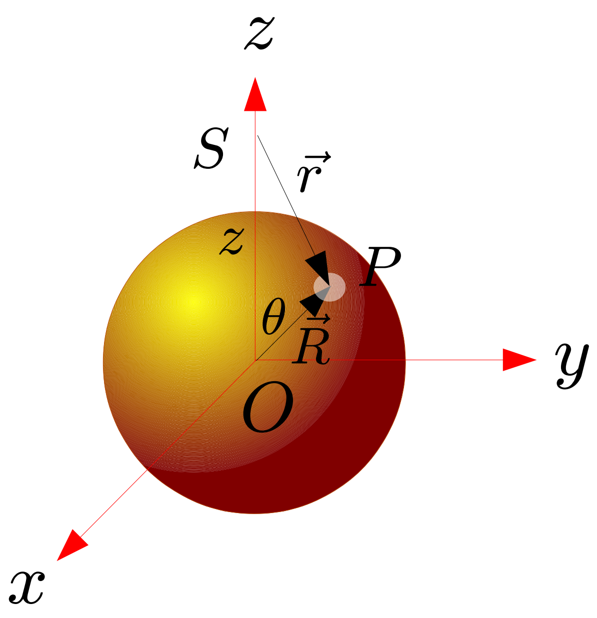
Laplace's Equation (3-D): General Properties

In order to prove

$$V_{\mathrm{o}}(x, y.z) = \frac{1}{4\pi R^2} \oint_{\text{sphere}} V\, da$$

we consider a sphere of radius *R* centered at the origin, and a charge *q* at a distance *z* along the z-axis as shown in the figure. The potential at a point P on the surface of the sphere can be written as

$$V_{\mathrm{P}} = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$



where $r^2 = R^2 + z^2 - 2zR\cos\theta$

Hence, $V_{\mathrm{P}} = \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{\sqrt{R^2 + z^2 - 2zR\cos\theta}}$

Laplace's Equation (3-D): General Properties

The average potential on the surface of the sphere is

$$V_{\text{avg}} = \frac{1}{4\pi R^2} \int_{\text{sphere}} V_{\text{P}}\, R^2 \sin\theta d\theta d\phi$$

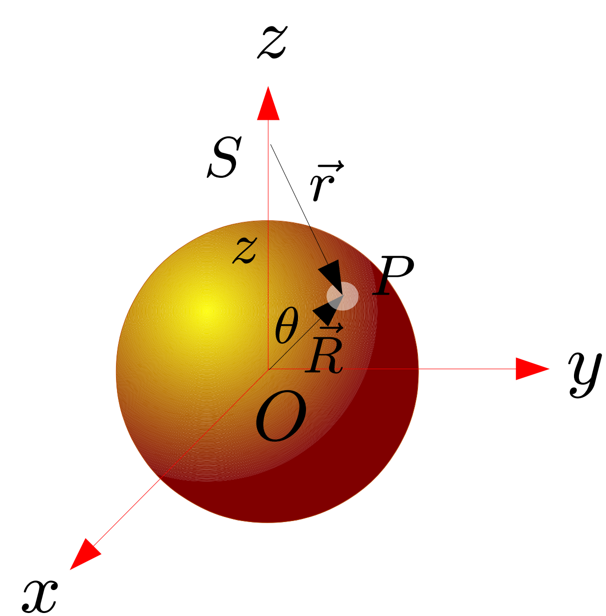
or, $$V_{\text{avg}} = \frac{1}{4\pi} \frac{1}{4\pi\epsilon_0} \int \frac{q}{\sqrt{R^2 + z^2 - 2zR\cos\theta}} 2\pi \sin\theta d\theta$$

$$= \frac{q}{8\pi\epsilon_0} \int_{-1}^{1} \frac{d(\cos\theta)}{\sqrt{R^2 + z^2 - 2zR\cos\theta}}$$

$$= \frac{q}{8\pi\epsilon_0} \left. \frac{\sqrt{R^2 + z^2 - 2zR\cos\theta}}{zR} \right|_0^{\pi}$$

$$= \frac{q}{8\pi\epsilon_0} \left(\frac{z+R}{zR} - \frac{z-R}{zR} \right)$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q}{z} \quad = V_{\text{o}}(x, y, z)$$



Boundary Conditions and Uniqueness Theorems

The solution of Poisson's or Laplace's equation requires a clear specification of boundary conditions. To show that a solution of the Poisson's equation or Laplace's equation satisfying the boundary conditions is unique, we first prove the Uniqueness theorems.

Consider Poisson's equation in a region of volume *V* bounded by a surface *S*. Let us assume that V_1 and V_2 are solutions of Poisson's equation inside *V*, and they satisfy the boundary condition on *S*. That is,

$$\nabla^2 V_1 = -\rho/\epsilon_0 \quad \text{and} \quad \nabla^2 V_2 = -\rho/\epsilon_0$$

Assume that the two solutions, V_1 and V_2, are different inside the volume, and then consider a function *U* given by

$$U \equiv V_2 - V_1$$

Boundary Conditions and Uniqueness Theorems

For U, we have

$$\nabla^2 U = \nabla^2 V_2 - \nabla^2 V_1 = -\rho/\epsilon_0 - (-\rho/\epsilon_0) = 0$$

inside V. To proceed further, we make use of Green's theorem

$$\int_V [\phi \vec{\nabla}^2 \psi + (\vec{\nabla}\phi) \cdot (\vec{\nabla}\psi)] d\tau = \oint_S (\phi \vec{\nabla}\psi) \cdot d\vec{a}$$

with $\phi = \psi = U$

$$\int_V (U\nabla^2 U + (\nabla U)^2) d\tau = \oint_S U \frac{\partial U}{\partial n} da$$

We consider two ways in which the boundary conditions on S can be specified. We can either specify the value of the potential on the surface, known as the *Dirichlet boundary condition*, or specify its normal derivative, known as the *Neumann boundary condition*. In either case the RHS of the above equation vanishes.

Boundary Conditions and Uniqueness Theorems

We have

$$\int_V |\nabla U|^2 \, d\tau = 0$$

$$\Rightarrow \;\; \nabla U = 0$$

$$\Rightarrow \;\; U \;\; \text{is a constant}$$

Since $U = V_2 - V_1 = 0 \;\; \text{on} \;\; \text{S}$

$$\Rightarrow \;\; U = 0 \quad \text{inside V}$$

From which we get the two solutions, V_1 and V_2, to be identical inside the volume,

$$V_1 = V_2 \quad \text{inside V}$$

That is, the potential inside the volume is uniquely determined if the potential on the boundary is specified.

Boundary Conditions and Uniqueness Theorems

In case of the normal derivative of the potential being specified on the boundary, we get

$$\frac{\partial U}{\partial n} = \frac{\partial V_2}{\partial n} - \frac{\partial V_1}{\partial n} = 0$$

$$\Rightarrow\ U \ \ \text{is} \ \ \text{a} \ \ \text{constant} \ \ \text{on} \ \ \text{S}$$

Therefore,

$$V_2 = V_1 + \text{some} \ \ \text{constant}$$

That is, the potential inside the volume is known within a constant term. Note that the electric field is unique. This proves the uniqueness theorem.

Green's Theorem

For two scalar functions $\phi \text{ and } \psi$, we want to show that

$$\int_V [\phi\vec{\nabla}^2\psi + (\vec{\nabla}\phi)\cdot(\vec{\nabla}\psi)]d\tau = \oint_S (\phi\vec{\nabla}\psi)\cdot d\vec{a}$$

This is known as the Green's theorem.

Consider the divergence theorem for a vector function $\vec{A}$,

$$\int_V \vec{\nabla}\cdot\vec{A}d\tau = \oint_S \vec{A}\cdot d\vec{a}$$

Let $\vec{A} = \phi\vec{\nabla}\psi$, then the divergence theorem becomes

$$\int_V \vec{\nabla}\cdot(\phi\vec{\nabla}\psi)d\tau = \oint_S (\phi\vec{\nabla}\psi)\cdot d\vec{a}$$

Using $\quad \vec{\nabla}\cdot(\phi\vec{\nabla}\psi) = \phi\vec{\nabla}\cdot(\vec{\nabla}\psi) + (\vec{\nabla}\phi)\cdot(\vec{\nabla}\psi)$

Green's Theorem

or, $\vec{\nabla} \cdot (\phi \vec{\nabla} \psi) = \phi \vec{\nabla}^2 \psi + (\vec{\nabla} \phi) \cdot (\vec{\nabla} \psi)$

We have, $\int_V [\phi \vec{\nabla}^2 \psi + (\vec{\nabla} \phi) \cdot (\vec{\nabla} \psi)] d\tau = \oint_S (\phi \vec{\nabla} \psi) \cdot d\vec{a}$

which is the Green's theorem.

Solving Poisson's or Laplace's Equation

There are various methods for solving Poisson's or Laplace's equation. Once we find a solution of Poisson's or Laplace's equation by any method, in light of uniqueness theorem, we know that we have found *the* solution. In the following, we will consider two methods for solving the electrostatics boundary value problems satisfying either the Dirichlet or Neumann boundary condition.

1. Method of separation of variables,

2. Method of images

Separation of Variables in Spherical Polar Coordinates

1. Method of separation of variables: We consider separation of variables of Laplace's equation in spherical polar coordinates for the potentials having azimuthal symmetry.

In spherical polar coordinates, the Laplace's equation

$$\nabla^2 V(r,\theta) = 0$$

becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0$$

Due to azimuthal symmetry, the last term in LHS vanishes,

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0$$

Separation of Variables in Spherical Polar Coordinates

Assume a solution of the form

$$V(r,\theta) = R(r)\Theta(\theta)$$

Substituting and then dividing by $R(r)\Theta(\theta)$, we get

$$\frac{\Theta}{R\Theta}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{R}{R\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) = 0$$

$$\text{or,}\quad \frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = 0$$

The first term depends only on r, while the second term depends only on θ, they must be constant.

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1)$$

and $$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -l(l+1)$$

Separation of Variables in Spherical Polar Coordinates

Consider the radial part,

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1)R$$

The solution is given by (one can check by substitution)

$$R(r) = Ar^l + Br^{-(l+1)}$$

where A and B are constants. Next, we consider the angular part

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -l(l+1)\Theta\sin\theta$$

The solution is given by

$$\boxed{\Theta(\theta) = P_1(\cos\theta)}$$

The other solution $\Theta(\theta) = \ln\left(\tan\frac{\theta}{2}\right)$, is not acceptable.

Legendre Polynomials

The solution $P_l(\cos\theta)$, are called Legendre polynomials. A convenient way of defining Legendre polynomials is through the Rodrigues formula

$$P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l$$

They are orthogonal over the interval -1 to 1,

$$\int_{-1}^{1} P_l(x)P_{l'}(x)dx = \int_0^\pi P_l(\cos\theta)P_{l'}(\cos\theta)\sin\theta d\theta$$

$$= \begin{cases} 0, & if\, l' \neq l \\ \frac{2}{2l+1}, & if\, l' = l \end{cases}$$

Combining, we can write the orthogonality property as

$$\text{or,}\ \int_{-1}^{1} P_l(x)P_{l'}(x)dx = \frac{2}{2l+1}\delta_{l'l}$$

Legendre Polynomials

The Legendre polynomials also form a complete set of basis functions for expanding any function *f*(*x*) over the interval -1 to 1,

$$f(x) = \sum_{l=0}^{\infty} A_l P_l(x)$$

where the expansion coefficients A_1 are determined with the help of the orthogonality relation,

$$A_l = \frac{2l+1}{2} \int_{-1}^{1} f(x) P_l(x) dx$$

For evaluating the above integral, most of the time, it is useful to express *f*(*x*) in terms of $P_l(x)$, and then use the orthogonality relations of Legendre polynomials.

Legendre Polynomials

Some of the Legendre polynomials are given belew.

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = (3x^2 - 1)/2$$

$$P_3(x) = (5x^3 - 3x)/2$$

$$P_4(x) = (35x^4 - 30x^2 + 3)/8$$

$$P_5(x) = (63x^5 - 70x^3 + 15x)/8$$

Solutions in Spherical Polar Coordinates

Coming back to the Laplace equation, the most general, separable solution can be written as

$$V(r,\theta) = R(r)\Theta(\theta) = \left(A_l r^l + B_l r^{-(l+1)}\right) P_l(\cos\theta)$$

The separation of variables gives one solution for each *l*. Therefore, the most general solution is

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-(l+1)}\right) P_l(\cos\theta)$$

Laplace's Equation: Examples with Azimuthal Symmetry

Example 3.6: The potential $V_0(\theta)$ is specified on the surface of a hollow sphere of radius *R*. Find the potential inside the sphere.

From the general solution

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-(l+1)} \right) P_l(\cos\theta)$$

We find that all B_1's must vanish because $r = 0$ is included in the volume of interest. Therefore, the general solution reduces to

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

Applying the boundary condition at $r = R,$

$$V(R,\theta) = V_0(\theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta)$$

Laplace's Equation: Examples with Azimuthal Symmetry

Multiplying both sides by $P_{l'}(\cos\theta)\sin\theta$ and integrating from 0 to π, we get

$$\int_0^\pi V_0(\theta)P_{l'}(\cos\theta)\sin\theta d\theta = \sum_{l=0}^{\infty} A_l R^l \int_0^\pi P_l(\cos\theta)P_{l'}(\cos\theta)\sin\theta d\theta$$

$$\text{or,}\ \int_0^\pi V_0(\theta)P_{l'}(\cos\theta)\sin\theta d\theta = \sum_{l=0}^{\infty} A_l R^l \frac{2}{2l+1}\delta_{l'l}$$

$$\text{or,}\ \int_0^\pi V_0(\theta)P_{l'}(\cos\theta)\sin\theta d\theta = A_{l'} R^{l'} \frac{2}{2l'+1}$$

Finally, the expansion coefficients are given by

$$A_l = \frac{2l+1}{2R^l}\int_0^\pi V_0(\theta)P_l(\cos\theta)\sin\theta d\theta$$

Laplace's Equation: Examples with Azimuthal Symmetry

$$\text{Let, } V_0(\theta) = k\sin^2(\theta/2) = \frac{k}{2}(1-\cos\theta)$$

Rewriting the boundary condition in terms of Legendre polynomials,

$$V_0(\theta) = \frac{k}{2}[P_0(\cos\theta) - P_1(\cos\theta)]$$

Substituting in the expression for the expansion coefficient

$$A_l = \frac{2l+1}{2R^l}\int_0^{\pi}\frac{k}{2}[P_0(\cos\theta) - P_1(\cos\theta)]P_l(\cos\theta)\sin\theta d\theta$$

Only *l*=0 and *l*=1 terms will survive

$$A_0 = \frac{1}{2}\frac{k}{2}2 = \frac{k}{2} \quad \text{and} \quad A_1 = \frac{3}{2R}\frac{k}{2}[-\frac{2}{3}] = -\frac{k}{2R}$$

Laplace's Equation: Examples with Azimuthal Symmetry

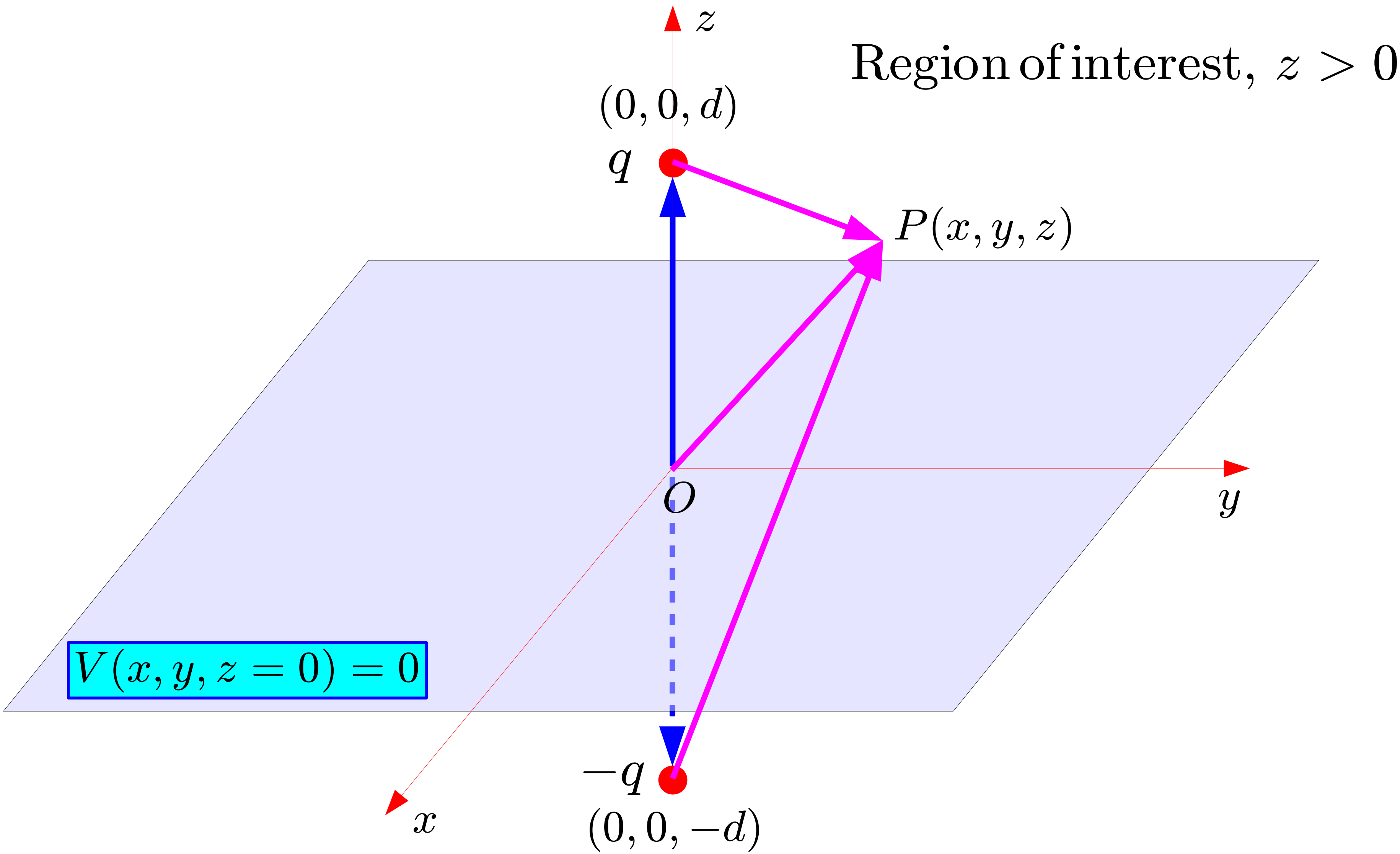
The solution inside the sphere becomes

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) = A_0 r^0 P_0(\cos\theta) + A_1 r^1 P_1(\cos\theta)$$

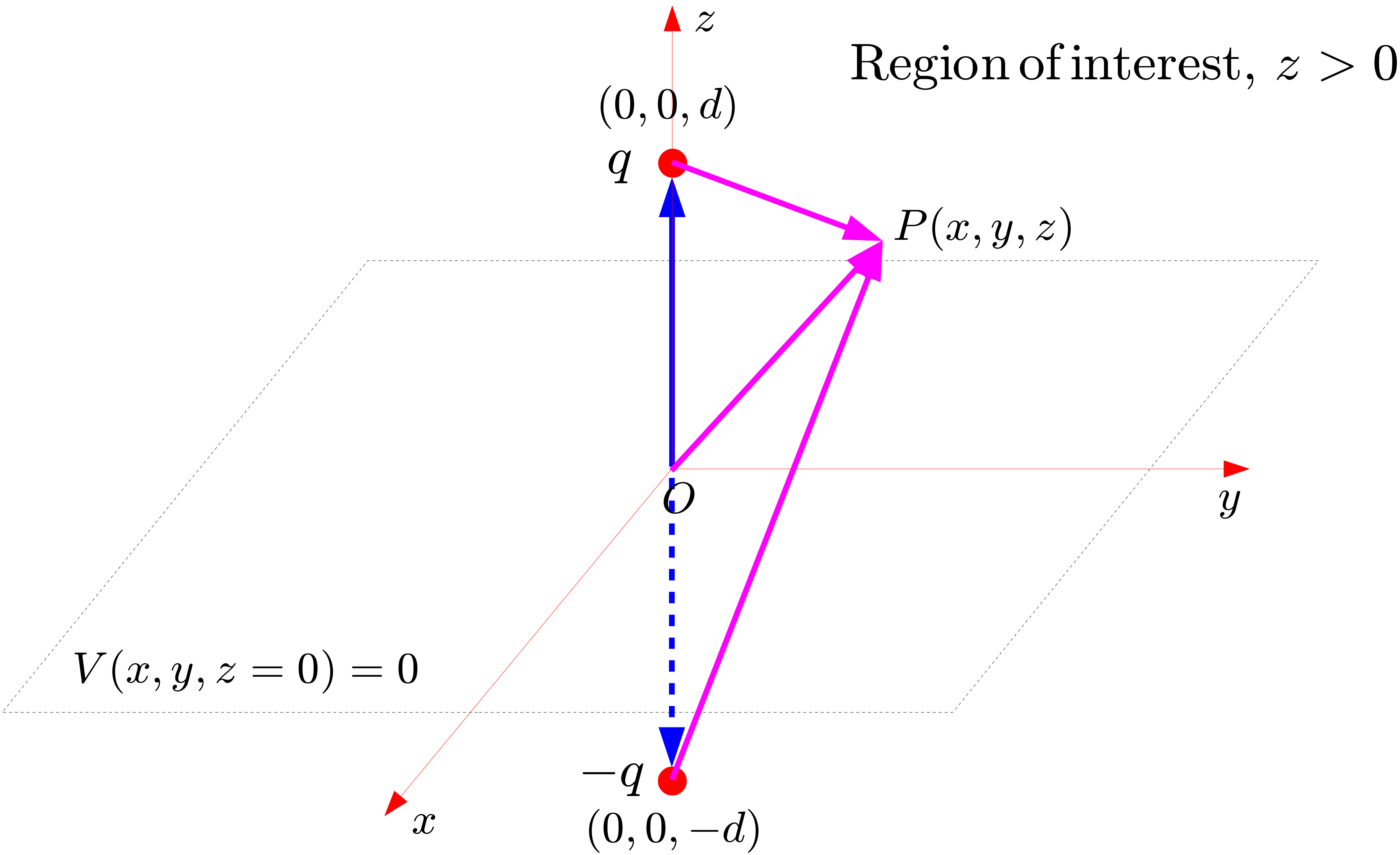
Substituting the values of the non-zero expansion coefficients,

$$V(r,\theta) = \frac{k}{2}[1 - \frac{r}{R} P_1(\cos\theta)]$$

Method of Images: A point Charge in front of a Grounded, Conducting Infinite Plane at *z*=0



Method of Images: A point Charge in front of a Grounded, Conducting Infinite Plane at *z*=0



Method of Images: A point Charge in front of a Grounded, Conducting Infinite Plane at *z=0*

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Method of Images: A point Charge in front of a Grounded, Conducting Infinite Plane at *z*=0

Consider a point charge q at (0,0,d) in front of a grounded, conducting, infinite plane at z=0, so that the charge density can be written as

$$\rho(\vec{r}) = q\delta(x)\delta(y)\delta(z-d)$$

In the region z>0, we want to solve the Poisson equation

$$\nabla^2 V(\vec{r}) = -\frac{\rho(r)}{\epsilon_0}$$

with the boundary conditions

$$1.\, V(x,y,z=0) = 0, \text{ and}$$

$$2.\, V(x,y,z) \to 0 \text{ for } (x^2+y^2+z^2) \gg d^2$$

From symmetry, we realize that the image charge is -q at (0,0,-d) but let us put an image charge q_i at z_i and impose the boundary condition $V(x,y,z=0)=0$.

Method of Images: A point Charge in front of a Grounded, Conducting Infinite Plane at *z*=0

The potential at a point P(*x*,*y*,*z*) can be written as

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{1}{4\pi\epsilon_0}\frac{q_{\mathrm{i}}}{\sqrt{x^2+y^2+(z+z_{\mathrm{i}})^2}}$$

For *V*(*x*,*y*,*z*=0)=0, we have

$$q_{\mathrm{i}} = -q, \quad \text{and} \quad z_{\mathrm{i}} = d$$

So that

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{x^2+y^2+(z+d)^2}}$$

Next, we must show that it satisfies the Poisson equation in the region of interest z>0.

$$\nabla^2 V(\vec{r}) = \nabla^2\left[\frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

Method of Images: A point Charge in front of a Grounded, Conducting Infinite Plane at *z*=0

Using $\nabla^2 \frac{1}{|\vec{r} - \vec{r}'|} = -4\pi\delta(\vec{r} - \vec{r}')$, we have

$$\nabla^2 V(\vec{r}) = \frac{q}{4\pi\epsilon_0}[-4\pi\delta(x)\delta(y)\delta(z-d) - (-)4\pi\delta(x)\delta(y)\delta(z+d)]$$

For z>0, only the first Dirac-delta function survives,

$$\nabla^2 V(\vec{r}) = \frac{q}{4\pi\epsilon_0}[-4\pi\delta(x)\delta(y)\delta(z-d)] = -\frac{\rho(r)}{\epsilon_0}$$

which shows that the potential satisfies the Poisson equation in the region of interest z>0.

Now we can calculate the following properties,
(i) *Electric field*, (ii) *Induced surface charge density*,
(iii) *Induced total charge*, (iv) *Force*, and (v) *Work done.*

Method of Images: A point Charge in front of a Grounded, Conducting Infinite Plane at *z*=0

Electric field: The electric field is given by

$$\vec{E} = -\vec{\nabla} V$$

which will be correct in the region z>0. Consider the z-component,

$$\begin{aligned}
(\vec{E})_z &= -\frac{\partial V}{\partial z} \\
&= -\frac{q}{4\pi\epsilon_0}\frac{\partial}{\partial z}\left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right] \\
&= -\frac{q}{4\pi\epsilon_0}\left[\frac{(-\frac{1}{2})2(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}} - \frac{(-\frac{1}{2})2(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}}\right] \\
&= \frac{q}{4\pi\epsilon_0}\left[\frac{(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}} - \frac{(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}}\right]
\end{aligned}$$

Method of Images: A point Charge in front of a Grounded, Conducting Infinite Plane at *z*=0

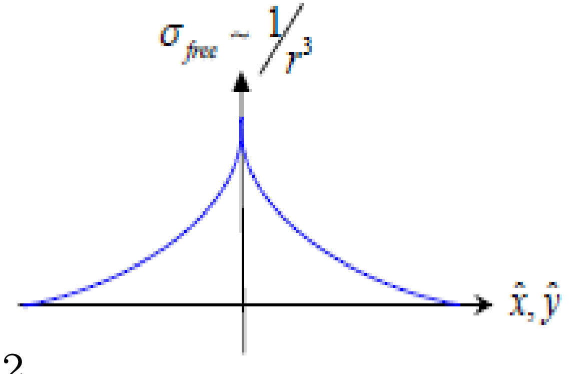
Induced surface charge density σ(x,y): Now the induced surface charge density can be calculated from

$$\sigma = -\epsilon_0 \left.\frac{\partial V}{\partial z}\right|_{z=0}$$

$$= \epsilon_0 \frac{q}{4\pi\epsilon_0}\left[\frac{-d}{[x^2+y^2+d^2]^{3/2}} - \frac{d}{[x^2+y^2+d^2]^{3/2}}\right]$$

$$= -\frac{q}{2\pi}\left[\frac{d}{[x^2+y^2+d^2]^{3/2}}\right]$$

$$= -\frac{q}{2\pi}\left[\frac{d}{[r_\perp^2+d^2]^{3/2}}\right]$$



For z=0 plane, we have written $x^2+y^2=r_\perp^2$.

Method of Images: A point Charge in front of a Grounded, Conducting Infinite Plane at *z*=0

Induced total charge *Q*: The total charge induced on the *z*=0 plane can be obtained by carrying out the integration in plane polar coordinates,

$$Q = \int \sigma(x,y)dxdy = 2\pi \int_0^\infty \sigma(r_\perp)r_\perp dr_\perp$$

$$= -qd \int_0^\infty \frac{1}{[r_\perp^2 + d^2]^{3/2}} r_\perp dr_\perp$$

$$= -qd\frac{1}{2} \left. \frac{[r_\perp^2 + d^2]^{-3/2+1}}{-\frac{3}{2}+1} \right|_0^\infty$$

$$= qd[0 - \frac{1}{d}] = -q$$

Thus, the total charge induced is equal to -*q*.

Method of Images: A point Charge in front of a Grounded, Conducting Infinite Plane at *z*=0

Force: The force on q is due to the electric field produced by the induced charge in $z>0$. Since the electric field is correctly described by the image charge, the force on q is

$$\vec{F} = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{(2d)^2}\hat{k}$$

Work done: The work done in bringing the charge q from infinity to d is given by

$$W = -\int_{\infty}^{d} \vec{F}\cdot\vec{dl}$$

which, in the present case, reduces to

$$W = -\int_{\infty}^{d} F_z dz$$

Method of Images: A point Charge in front of a Grounded, Conducting Infinite Plane at *z*=0

Substituting for the force, we have

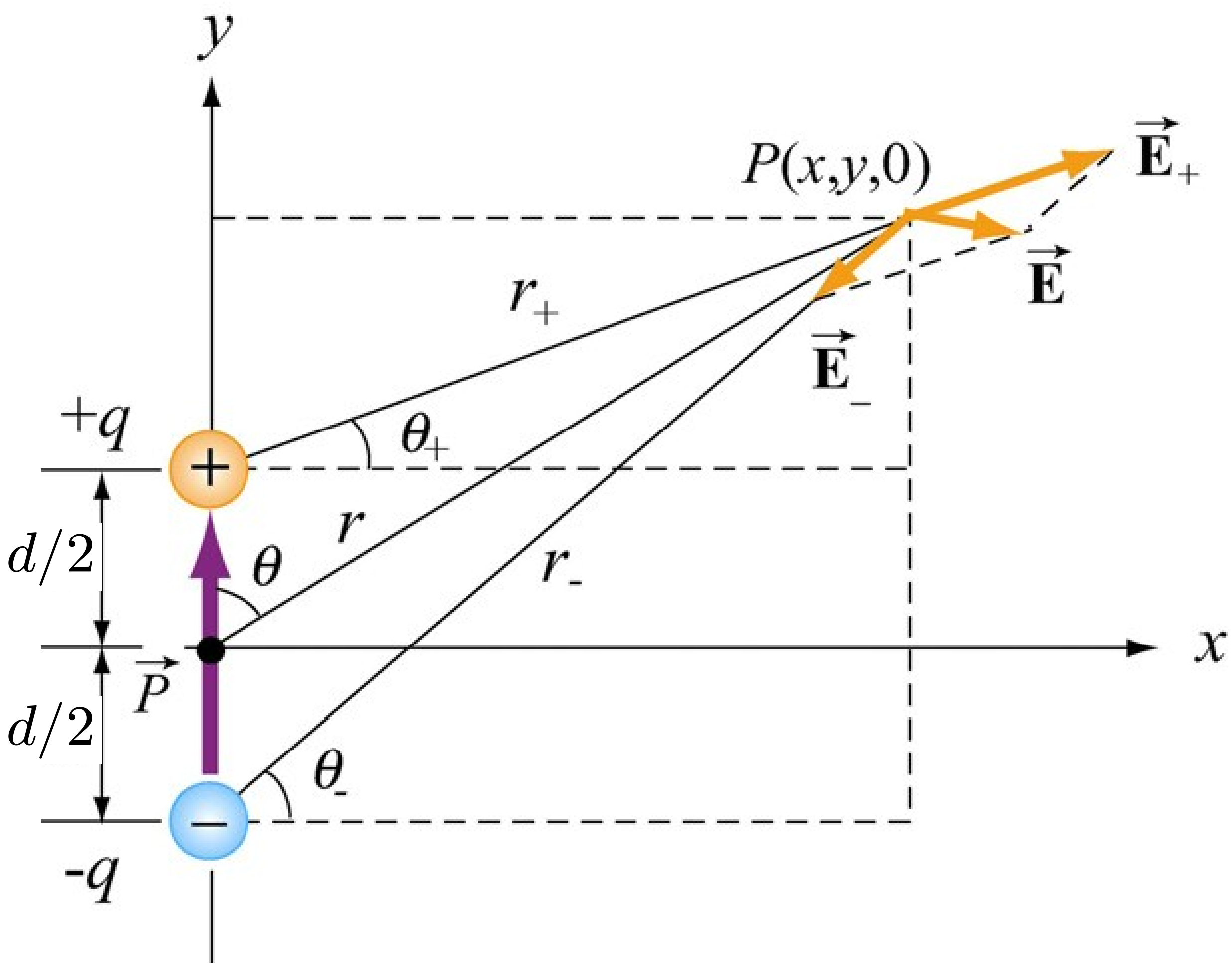
$$W = \int_{\infty}^{d} \frac{1}{4\pi\epsilon_0} \frac{q^2}{(2z)^2} dz = \frac{q^2}{16\pi\epsilon_0} \; \frac{-z^{-2+1}}{-2+1} \Bigg|_{\infty}^{d}$$

$$= \frac{q^2}{16\pi\epsilon_0} \left[-\frac{1}{d} + \frac{1}{\infty} \right]$$

$$\text{or, } W = -\frac{q^2}{16\pi\epsilon_0 d}$$

The minus sign indicates that the energy is released. Note that this is not the work done in moving charge q in the presence of the image charge q_i.

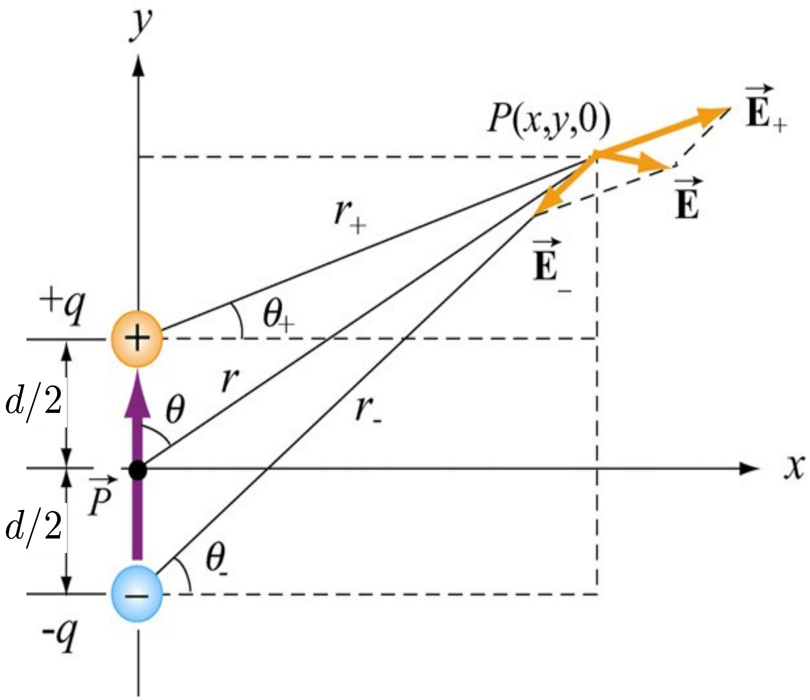
Multipoles: Electric Dipole



Electric Dipole: Electric Potential

The electric potential at a point P due to +q and -q charges,

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_+} - \frac{q}{r_-}\right)$$



From cosine rule,

$$r_\pm^2 = r^2 + (d/2)^2 \mp rd\cos\theta$$

$$= r^2\left(1 \mp \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right)$$

For, $r \gg d$

$$r_\pm = r\left(1 \mp \frac{d}{r}\cos\theta\right)^{1/2}$$

So that, $\dfrac{1}{r_\pm} = \dfrac{1}{r}\left(1 \mp \dfrac{d}{r}\cos\theta\right)^{-1/2} \cong \dfrac{1}{r}\left(1 \pm \dfrac{d}{2r}\cos\theta\right)$

Electric Dipole: Electric Potential

or, $\frac{1}{r_\pm} \cong \left(\frac{1}{r} \pm \frac{d}{2r^2}\cos\theta\right)$

Thus, $\frac{1}{r_+} - \frac{1}{r_-} \cong \left(\frac{1}{r} + \frac{d}{2r^2}\cos\theta\right) - \left(\frac{1}{r} - \frac{d}{2r^2}\cos\theta\right)$

or, $\frac{1}{r_+} - \frac{1}{r_-} \cong \frac{d}{r^2}\cos\theta$

Substituting it in the expression for the potential,

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{qd\cos\theta}{r^2}$$

The potential at large distances due to a dipole of dipole moment $\vec{p} = q\vec{d}$ becomes

$$V_{dip}(r,\theta) = \frac{1}{4\pi\epsilon_0}\frac{\hat{r}\cdot\vec{p}}{r^2} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$

Electric Dipole: Electric Field

The electric field at large distances due to a dipole can now be calculated from the potential

$$V_{dip}(r,\theta) = \frac{1}{4\pi\epsilon_0}\frac{\hat{r}\cdot\vec{p}}{r^2} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$

as $\vec{E} = -\vec{\nabla}V$

$$E_r = -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3}$$

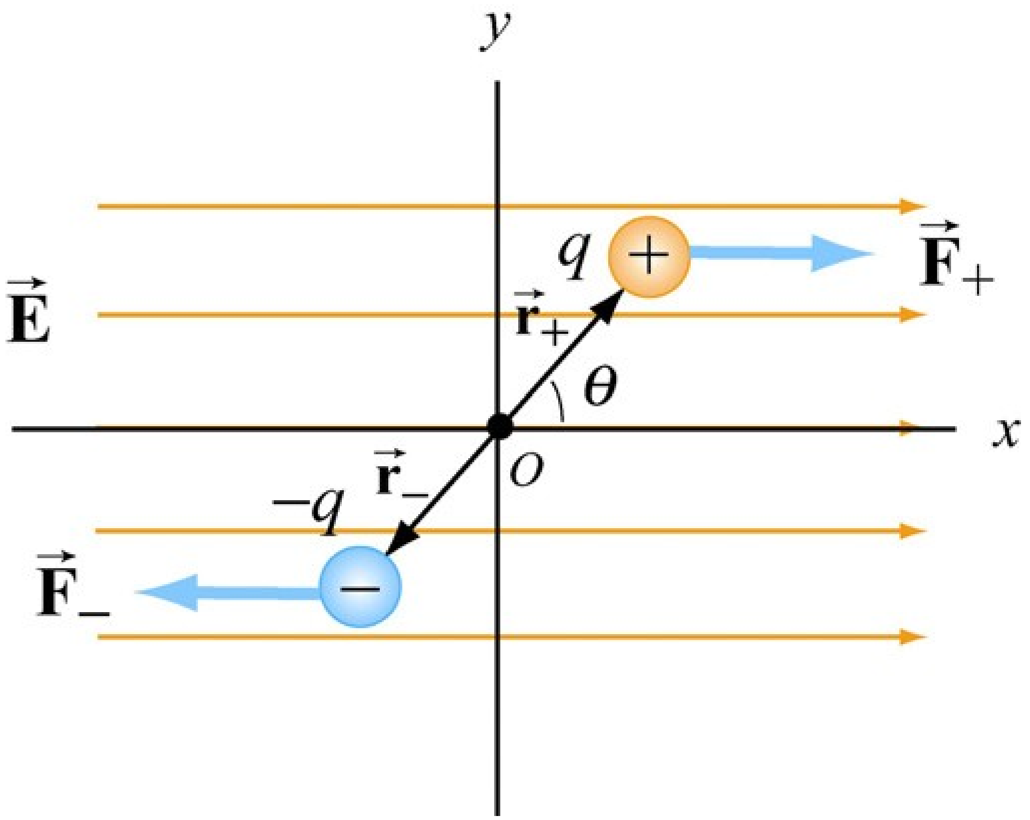
$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{p\sin\theta}{4\pi\epsilon_0 r^3}$$

$$E_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} = 0$$

Thus, the electric field due to a dipole is

$$E_{dip}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta})$$

Electric Dipole in a Uniform Electric Field



Electric Dipole in a Uniform Electric Field

Consider a dipole in a uniform electric field $\vec{E} = E\hat{i}$. The net force on the dipole is zero but the torque $\vec{\tau}$ is non-zero,

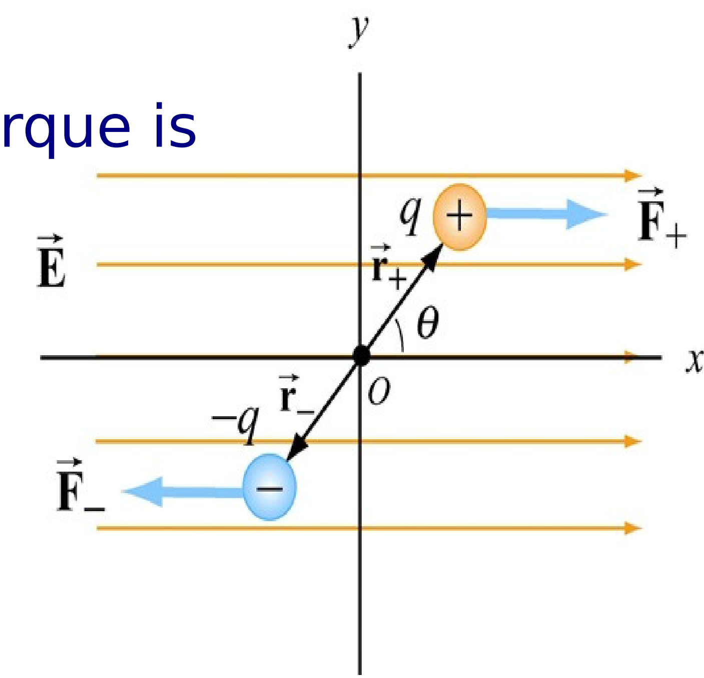
$$\begin{aligned}\vec{\tau} &= \vec{r}_+ \times \vec{F}_+ + \vec{r}_- \times \vec{F}_- \\ &= (d/2)(\cos\theta\hat{i} + \sin\theta\hat{j})(F_+\hat{i}) + (d/2)(-\cos\theta\hat{i} - \sin\theta\hat{j})(-F_-\hat{i}) \\ &= (d/2)(\sin\theta F_+(-\hat{k})) + (d/2)(\sin\theta F_-(-\hat{k})) \\ &= dF\sin\theta(-\hat{k})\end{aligned}$$

Using $F = qE$, the magnitude of the torque is

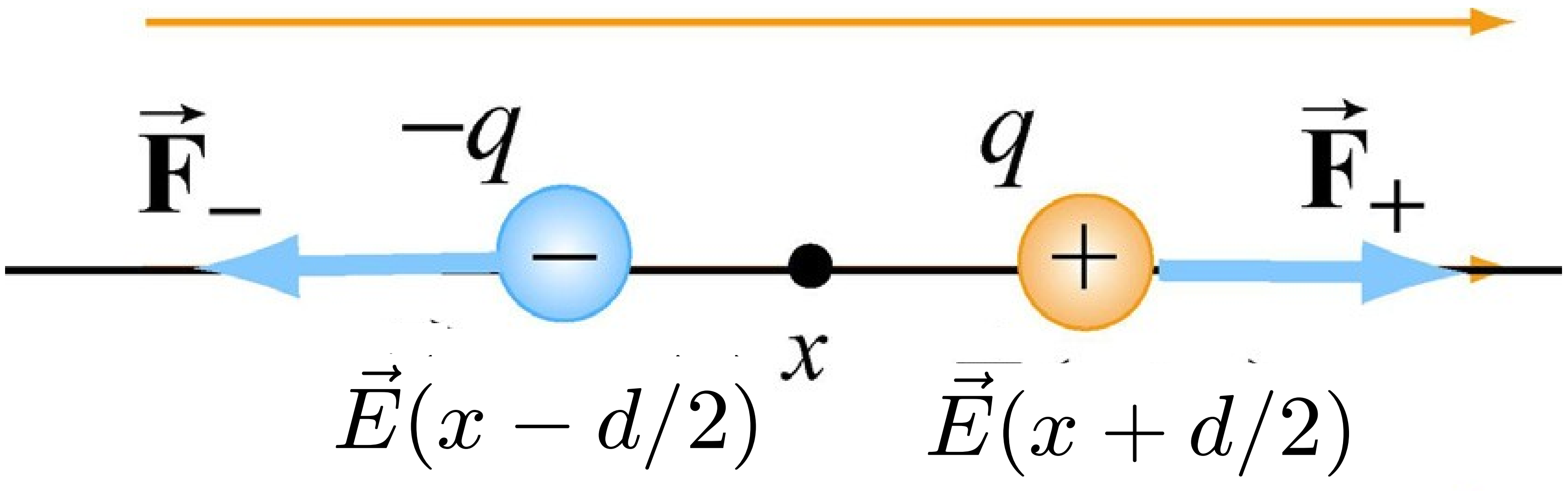
$$\tau = dF\sin\theta = dqE\sin\theta = pE\sin\theta$$

Thus, the torque can be written as

$$\boxed{\vec{\tau} = \vec{p} \times \vec{E}}$$



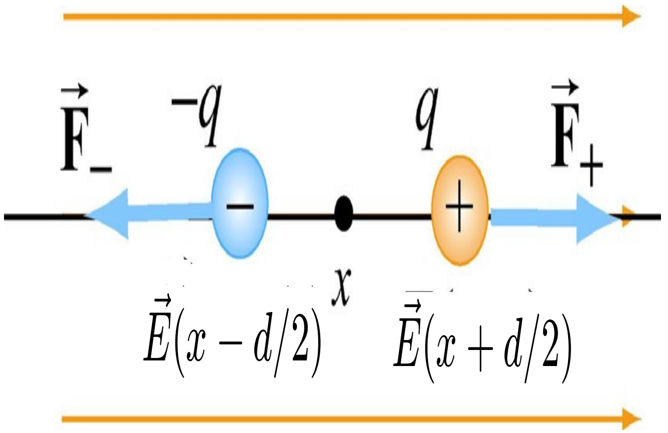
Electric Dipole in a Non-uniform Electric Field



Electric Dipole in a Non-uniform Electric Field

If the applied electric field is non-uniform then the forces on +*q* and -*q* charges can be obtained by approximating the field at those points as,

$$E_+(x+d/2) \approx E(x) + (d/2)\left(\frac{dE}{dx}\right)$$



$$\text{and, } E_-(x-d/2) \approx E(x) - (d/2)\left(\frac{dE}{dx}\right)$$

Thus, the force on the dipole is

$$\vec{F} = q(\vec{F}_+ - \vec{F}_-) = q\left[E(x) + (d/2)\left(\frac{dE}{dx}\right) - E(x) + (d/2)\left(\frac{dE}{dx}\right)\right]\hat{i}$$

or, $\vec{F} = qd\left(\frac{dE}{dx}\right)\hat{i} = p\left(\frac{dE}{dx}\right)\hat{i}$

In general, $\vec{F} = (\vec{p}\cdot\vec{\nabla})\vec{E}$

Laplace's Equation

Applications

I. Numerical solution of Laplace's equation

WolframCDFPlayer RelaxationMethodForElectrostaticProblem.cdf

Laplace's Equation

Summary

I. The potential *V*(*x*,*y*,*z*) at a point (*x*,*y*,*z*) is average of the potential over a sphere of radius *R* and centered at (*x*,*y*,*z*).

$$V(x, y.z) = \frac{1}{4\pi R^2} \oint_{\text{sphere}} V da$$

II. There are no local maxima or minima in the potential *V*(*x*,*y*,*z*). The extreme values occur at the bounding surface.

III. Uniqueness theorem: For the *Dirichlet boundary condition,* the potential is uniquely obtained inside the bounding surface. For the *Neumann boundary condition,* the potential is obtained within a constant inside the bounding surface.

Laplace's Equation

Contd....

IV. The most general solution of Laplace's eq. with azimuthal symmetry is

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-(l+1)} \right) P_l(\cos\theta)$$

V. The electric dipole: (a) The electric potential at large distances is

$$V_{dip}(r,\theta) = \frac{1}{4\pi\epsilon_0} \frac{\hat{r}\cdot\vec{p}}{r^2} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$

(b) The electric field is

$$E_{dip}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta})$$

(c)The force on the dipole is

$$\vec{F} = (\vec{p}\cdot\vec{\nabla})\vec{E}$$