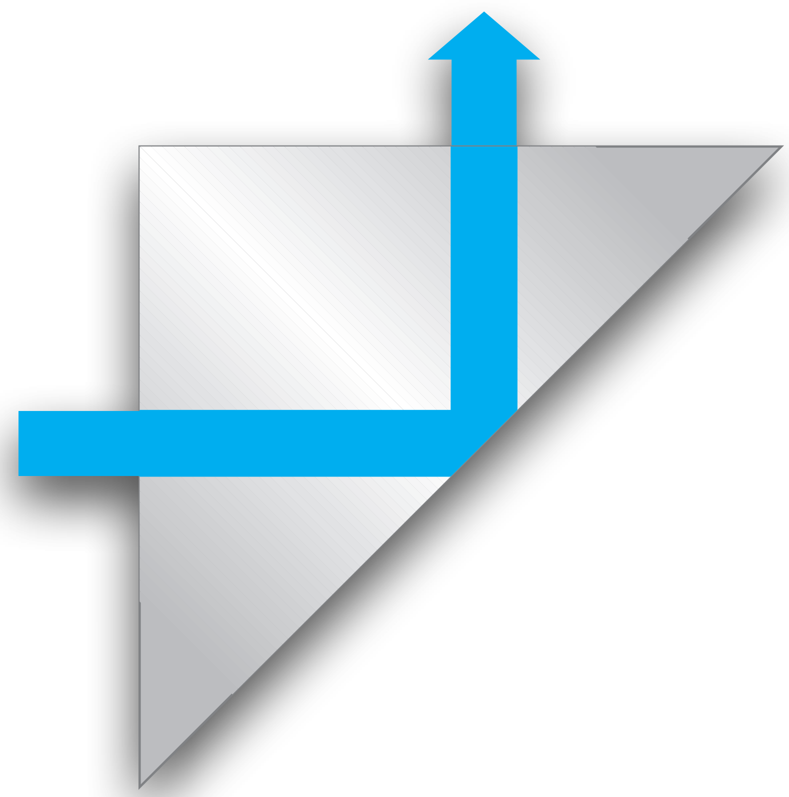
# PH-107

# Explanation of Tunnelling and Examples

# Tunneling for Light waves

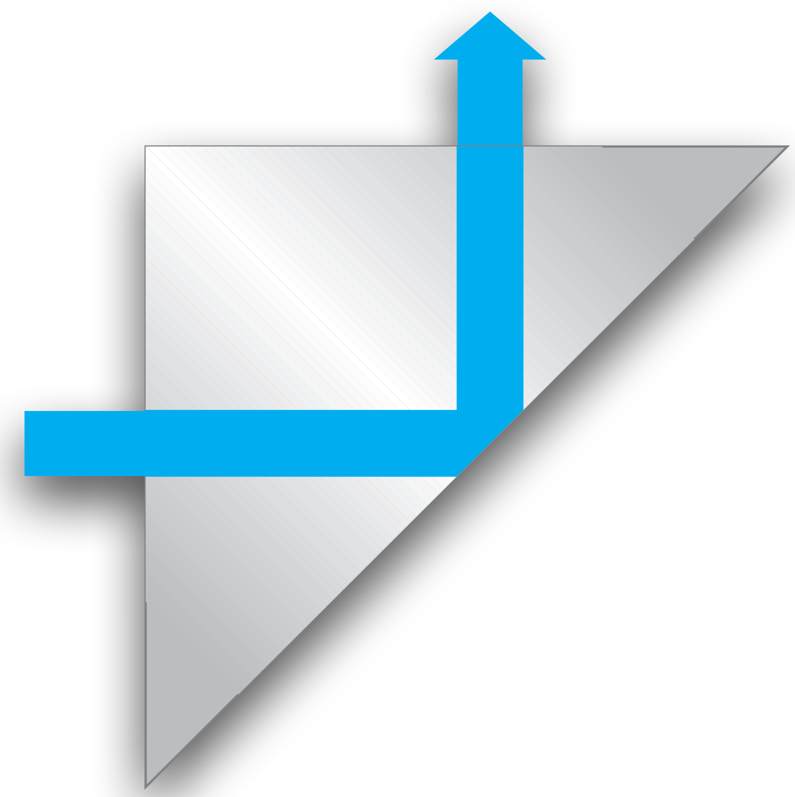
Total internal reflection of light waves at a glass – air boundary.



Light entering a right-angle prism is completely reflected at the hypotenuse face, even though an electro-magnetic wave, the evanescent wave, penetrates into the space beyond the reflecting surface.

# Tunneling for Light waves

Total internal reflection of light waves at a glass – air boundary.



When light is total-internally reflected, Maxwell's equations require that the tangential component of the electric field remains continuous across the boundary of the two media (**Evanescent Wave**).

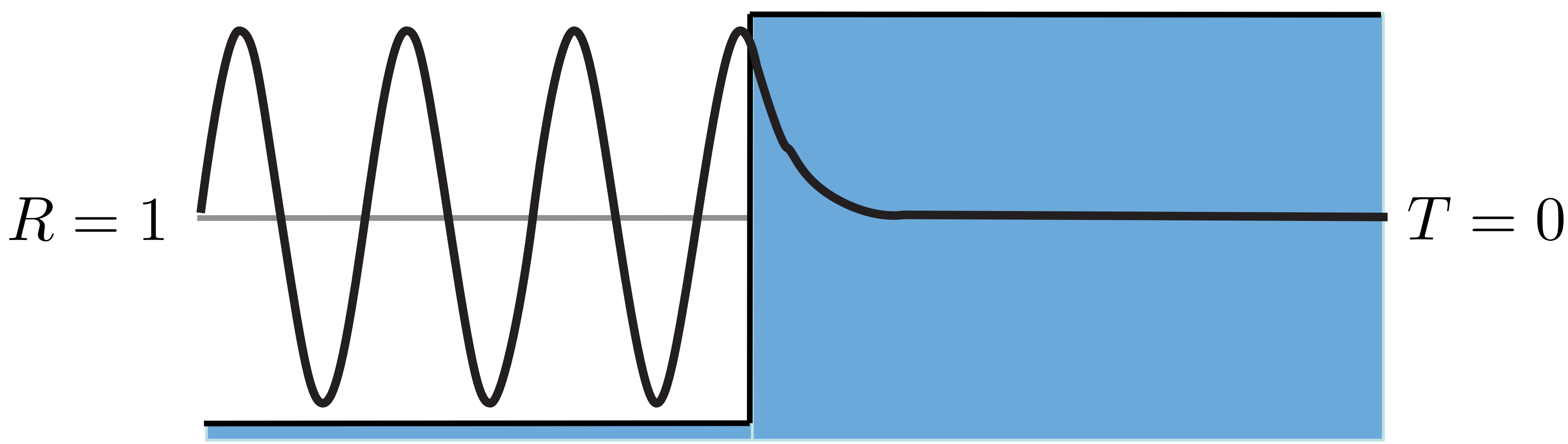
# Tunneling for Light waves

Total internal reflection of light waves at a glass – air boundary.

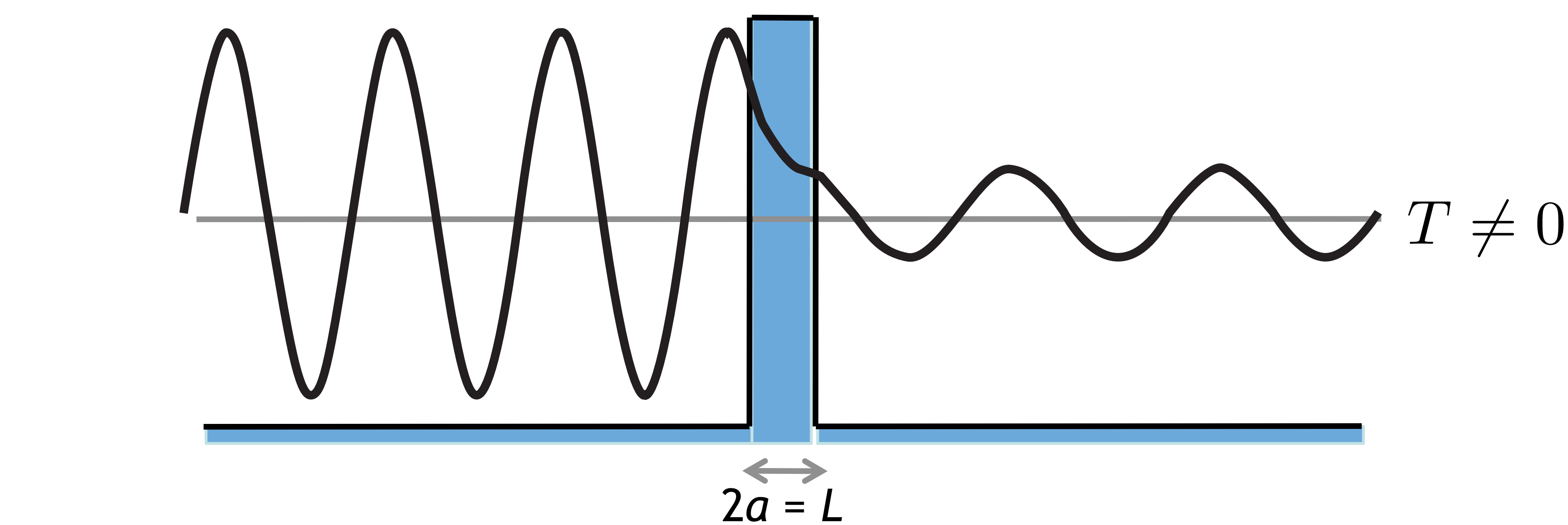
A second prism brought into near contact with the first can “pick up” this evanescent wave, thereby transmitting and redirecting the original beam. The evanescent wave is “picked up” by a neighbouring surface, resulting in transmission across the gap. Notice that the light beam *does not* appear in the gap. This phenomenon, known as **frustrated total internal reflection,** is the optical analog of tunneling: In effect, photons have tunnelled across the gap separating the two prisms.

# Quantum Tunneling Through a Thin Potential Barrier

## Total Reflection at Boundary

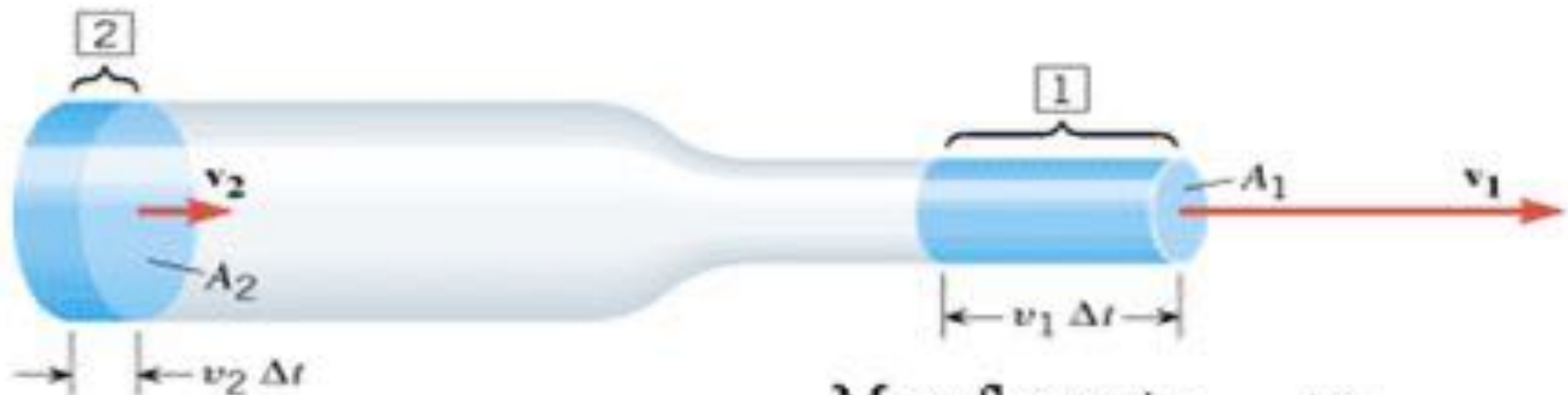



## Frustrated Total Reflection (Tunneling)



# Continuity Equation

## Equation of Continuity




Mass flow rate at position 1 $= \dfrac{\Delta m_1}{\Delta t} = \rho_1 A_1 v_1$

Mass flow rate at position 2 $= \dfrac{\Delta m_2}{\Delta t} = \rho_2 A_2 v_2$

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2$$

# Continuity Equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0$$

**This is know as quantum mechanical version of continuity equation.**

$$\rho = \psi\psi^* \text{ and } j = \frac{\hbar}{2im}\left[\psi^*\frac{\partial \psi}{\partial x} - \psi\frac{\partial \psi^*}{\partial x}\right]$$

**Probability Current Density**

**Probability Density**

The **equation** states that the time derivative of the probability of the particle being measured in V is equal to the rate at which probability flows into V.

Flow of probability in terms of probability per unit time per unit area.

# Continuity Equation

For a plane wave propagating in space: $\Psi(x,t) = A\ e^{i(kx-\omega t)}$

The probability density is constant everywhere;

$$\rho(x,t) = |A|^2 \rightarrow \frac{\partial|\Psi|^2}{\partial t} = 0$$

(that is, plane waves are stationary states) but the probability current is nonzero – the square of the absolute amplitude of the wave times the particle's speed;

$$j(x,t) = |A|^2 \frac{\hbar k}{m} = \rho \frac{|p|}{m} = \rho |v|$$

Illustrating that the particle may be in motion even if its spatial probability density has no explicit time dependence.

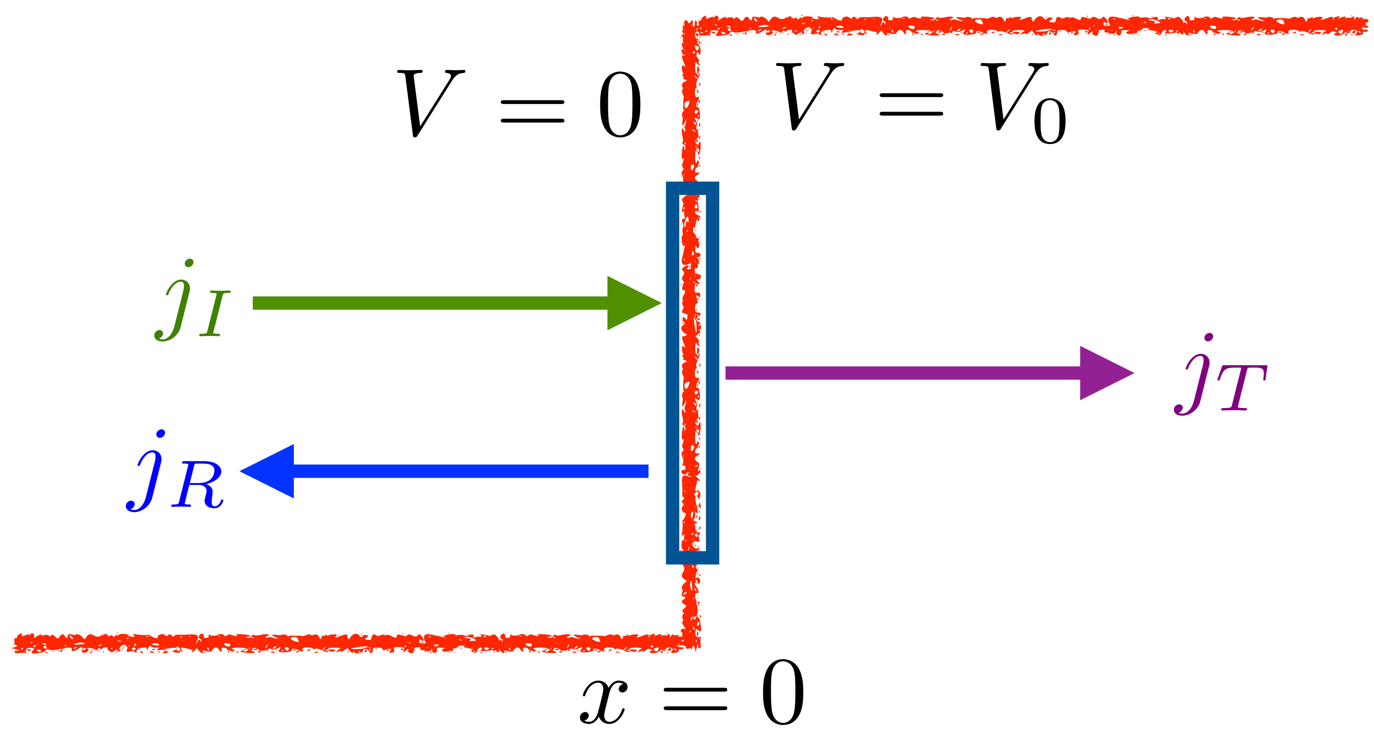
# Continuity Equation

There's no energy transfer, but we can find the particle. How do we understand this ?

There is no transfer of probability density $\rho(x) = |\varphi(x)|^2$ either. This is equivalent to saying that no particle crosses the potential step.

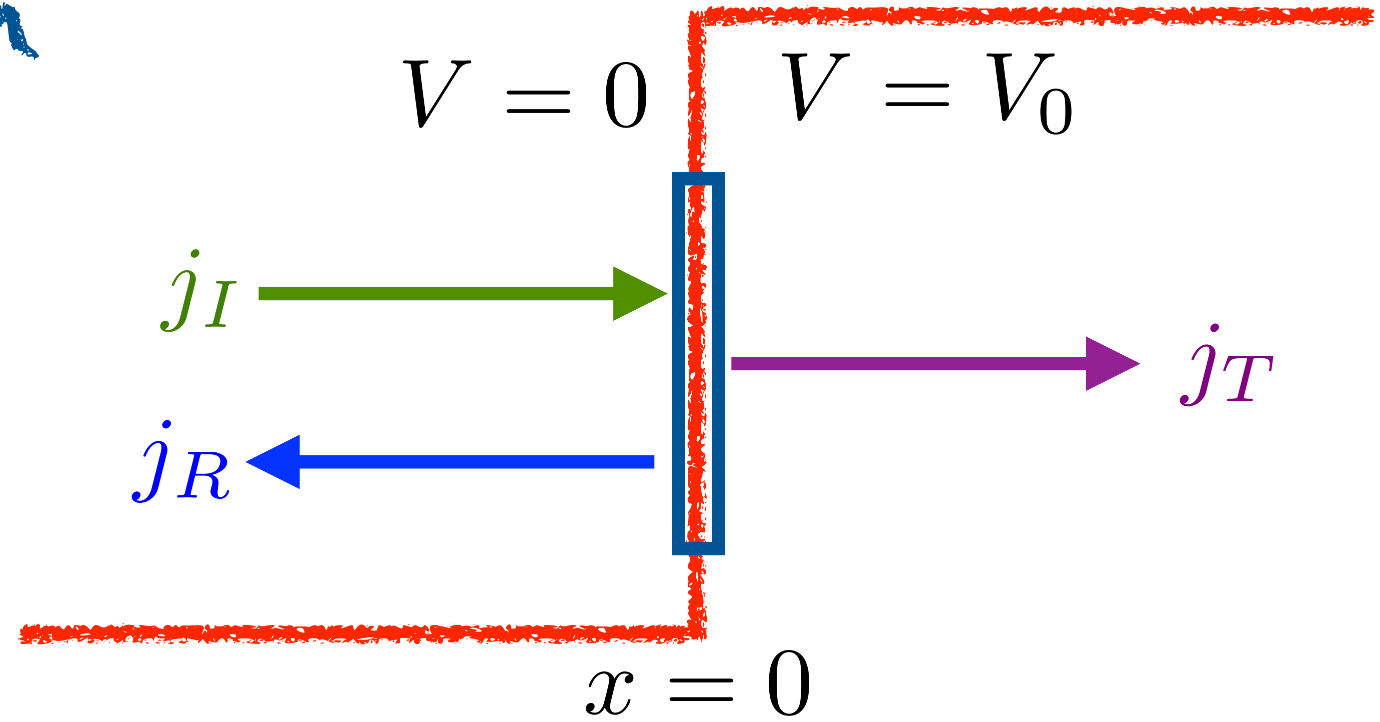
The Reflection and the Transmission coefficients are given as

$$R = \frac{j_R}{j_I} \quad \text{and} \quad T = \frac{j_T}{j_I}$$



# Continuity Equation

$$R = \frac{j_R}{j_I} \quad \text{and} \quad T = \frac{j_T}{j_I}$$



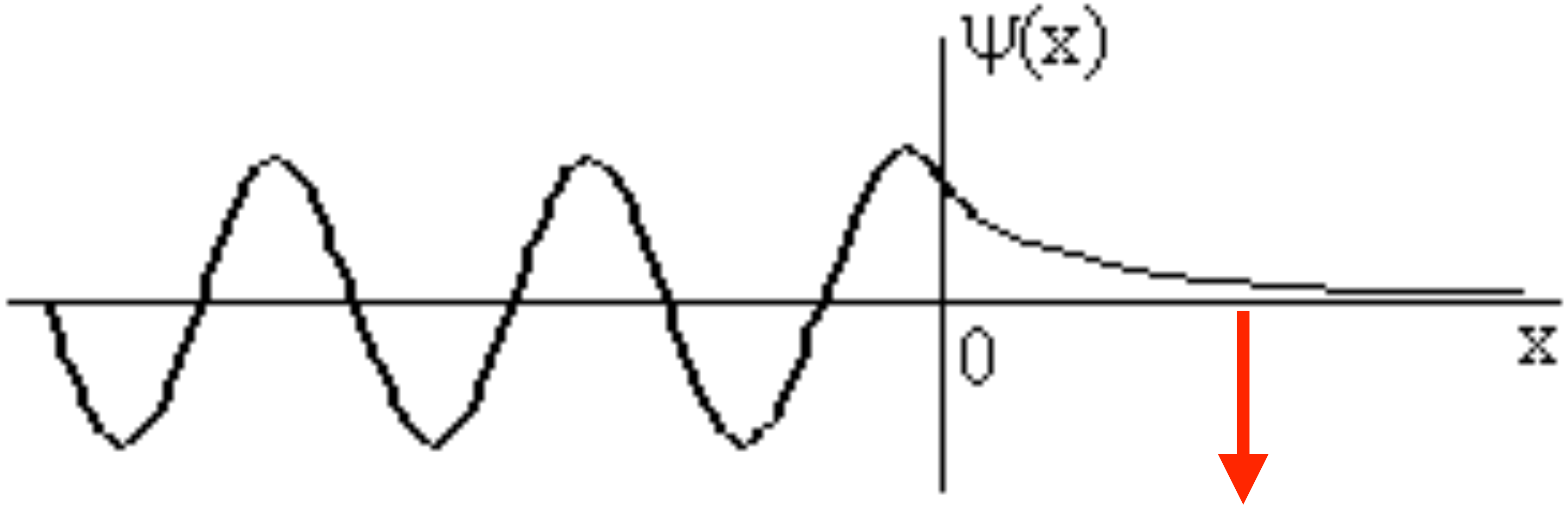

We know

$$j = \frac{\hbar}{2im}\left[\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right]$$

In region II $j_{II} = \frac{\hbar}{2im}\left[\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right] = 0$

As $\psi(x) = \psi^*(x) = C\ e^{-\alpha x}$

# Energy in Region II




$$\psi(x) = C\ e^{-\alpha x}$$

Wave function rapidly approaches zero beyond $x = 1/\alpha$. Therefore the probability density is appreciable only near $x = 0$, in the range

$$\Delta x = \frac{1}{\alpha} = \frac{\hbar}{\sqrt{2m(V_0 - E)}}$$

So one may say that the particle is predominantly localized within the length $\Delta x$.

# Energy in Region II

Uncertainty principle then requires that

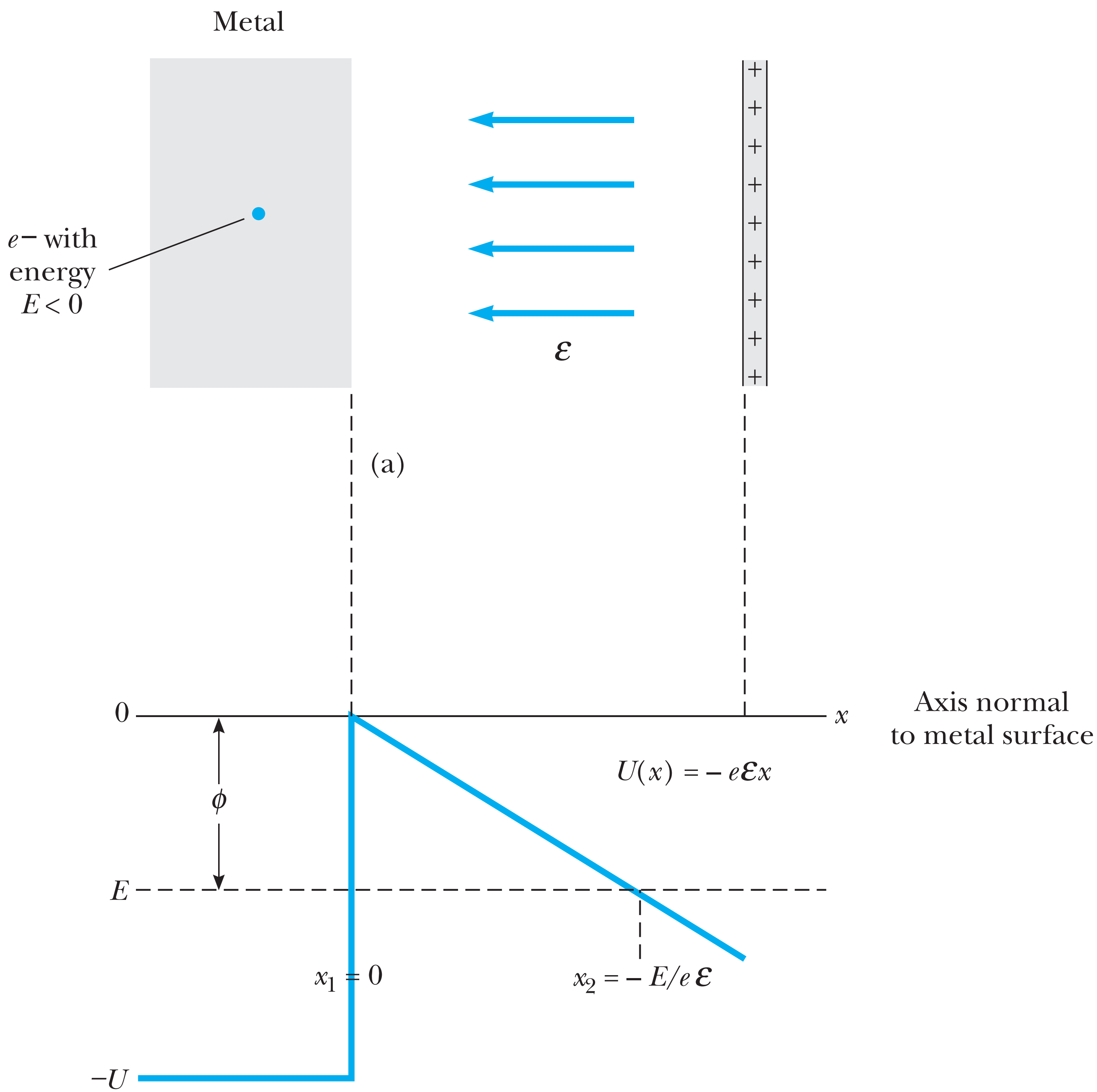
$$\Delta p \simeq \frac{\hbar}{(2\Delta x)} = \sqrt{2m(V_0 - E)}$$

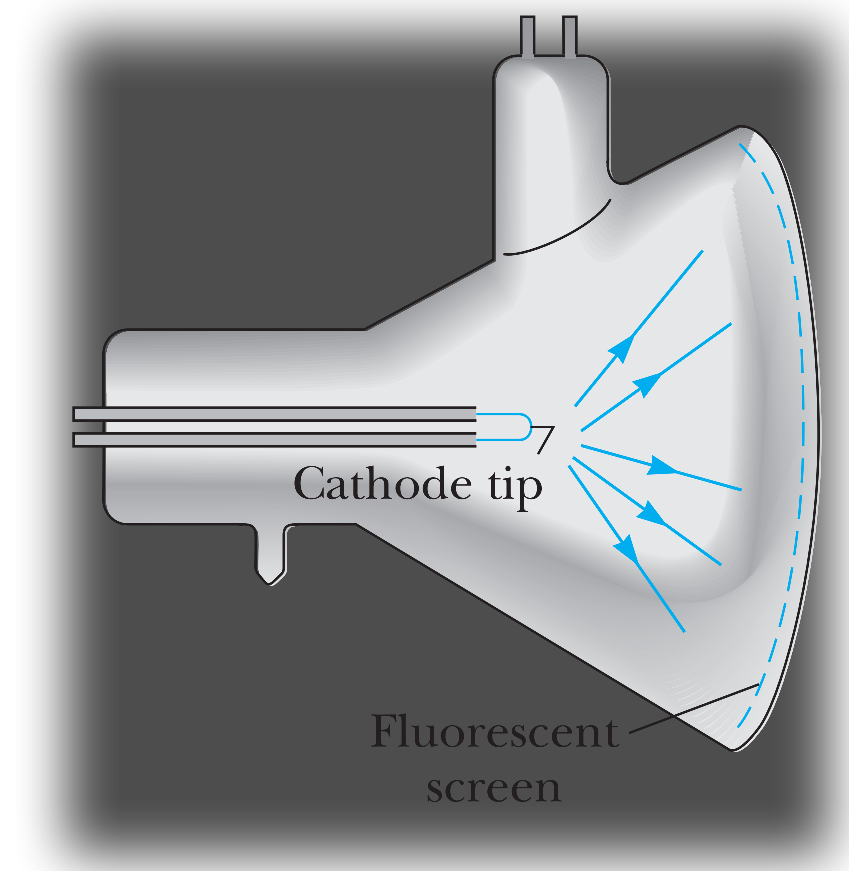
Uncertainty in the energy of the particle

$$\Delta E = \frac{(\Delta p)^2}{2m} \simeq (V_0 - E)$$

So, it is impossible to determine whether the energy of the particle is less than or greater than the barrier.

# Application I: Field Emission



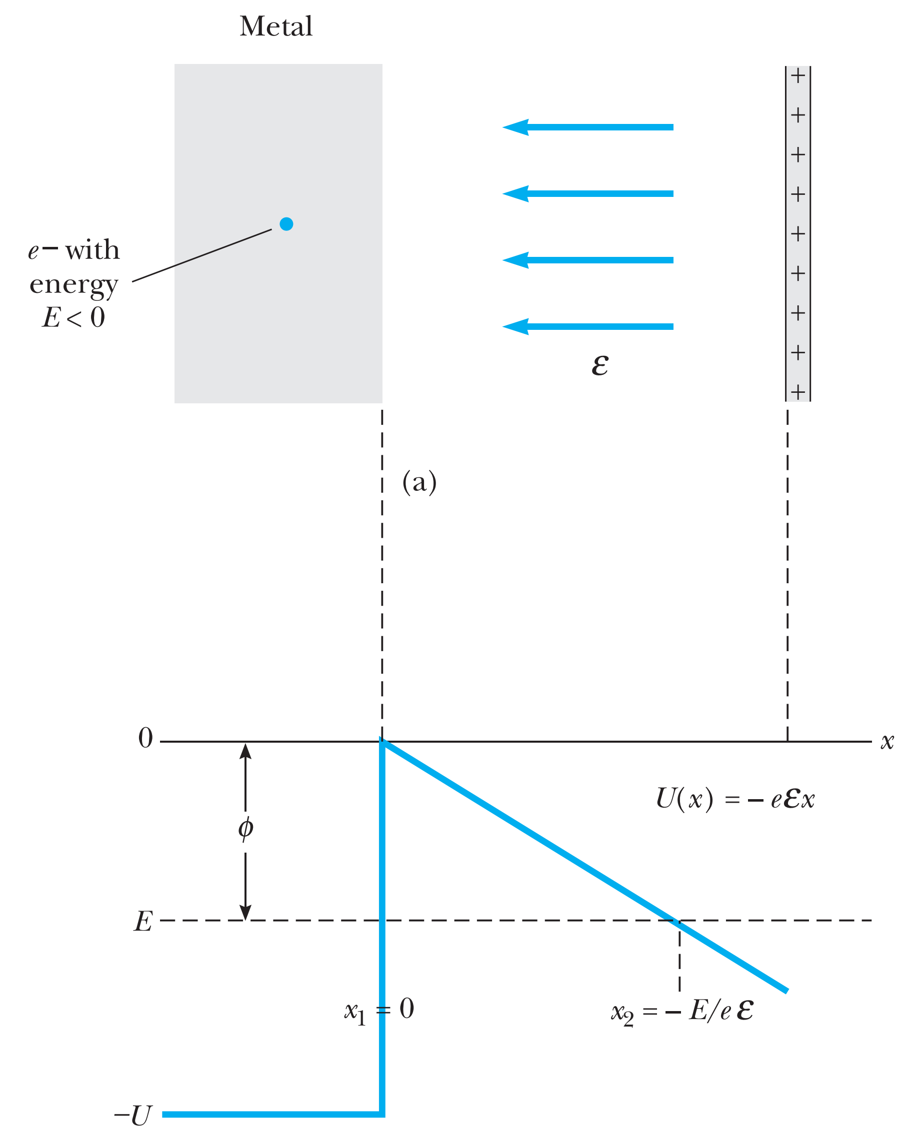

Electrons are emitted from a metal under the action of a strong electric field.

# Application I: Field Emission

$$T(E) \simeq \exp\left(-\frac{2}{\hbar}\sqrt{2m}\int\sqrt{U(x)-E}\,dx\right)$$

$$T(E)_{\mathrm{FE}} \simeq \exp\left(\left\{-\frac{4}{3e\hbar}\sqrt{2m}|E|^{3/2}\right\}\frac{1}{\epsilon}\right)$$

$$\simeq \exp\left(-\frac{\epsilon_c}{\epsilon}\right)$$




Some numbers: $|E| = \phi = 4.0\ eV$. $\varepsilon_C = 5.5\times 10^{10}\ V/m$
Emission rate ($10^9$ e$^-$/s) = collision frequency x $T(E)_{FE}$ can be realized for $\varepsilon = 10^9\ V/m$

Example 7.1 from Serway:

Two conducting copper wires are separated by an insulating layer of copper-oxide. We model the oxide layer as a rectangular barrier of height 10 eV. Calculate the transmission coefficient for penetration by 7 eV electrons, if the layer thickness is (a) 5 nm and (b) 1 nm.

$$\kappa = \frac{\sqrt{2m(V_0 - E)}}{\hbar} == \sqrt{2 \times 511000 \times 3}1973 = 0.9(\mathrm{Angstrom})^{-1}.$$

For $L \gg 1$, $\kappa L \approx 45 \implies \sinh(\kappa L) \approx \exp(\kappa L)/2$.

Thus we $T \approx 4\exp(-2\kappa L)$ leading to

$$\frac{T(L=50)}{T(L=10)} = 4\exp(-2 \times 0.9 \times 40) \approx 10^{-31}.$$

Because of the exponential factor, small changes of the barrier height or width lead to large changes in the tunnelling probability.

# Application II: Alpha-Decay

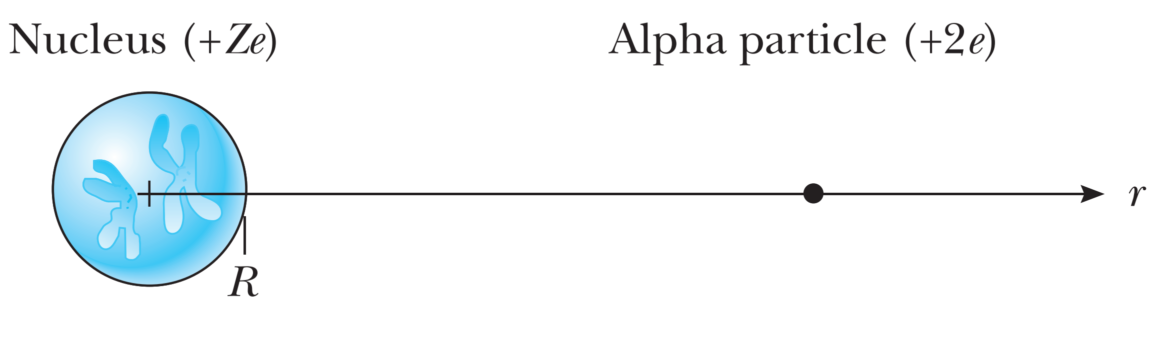


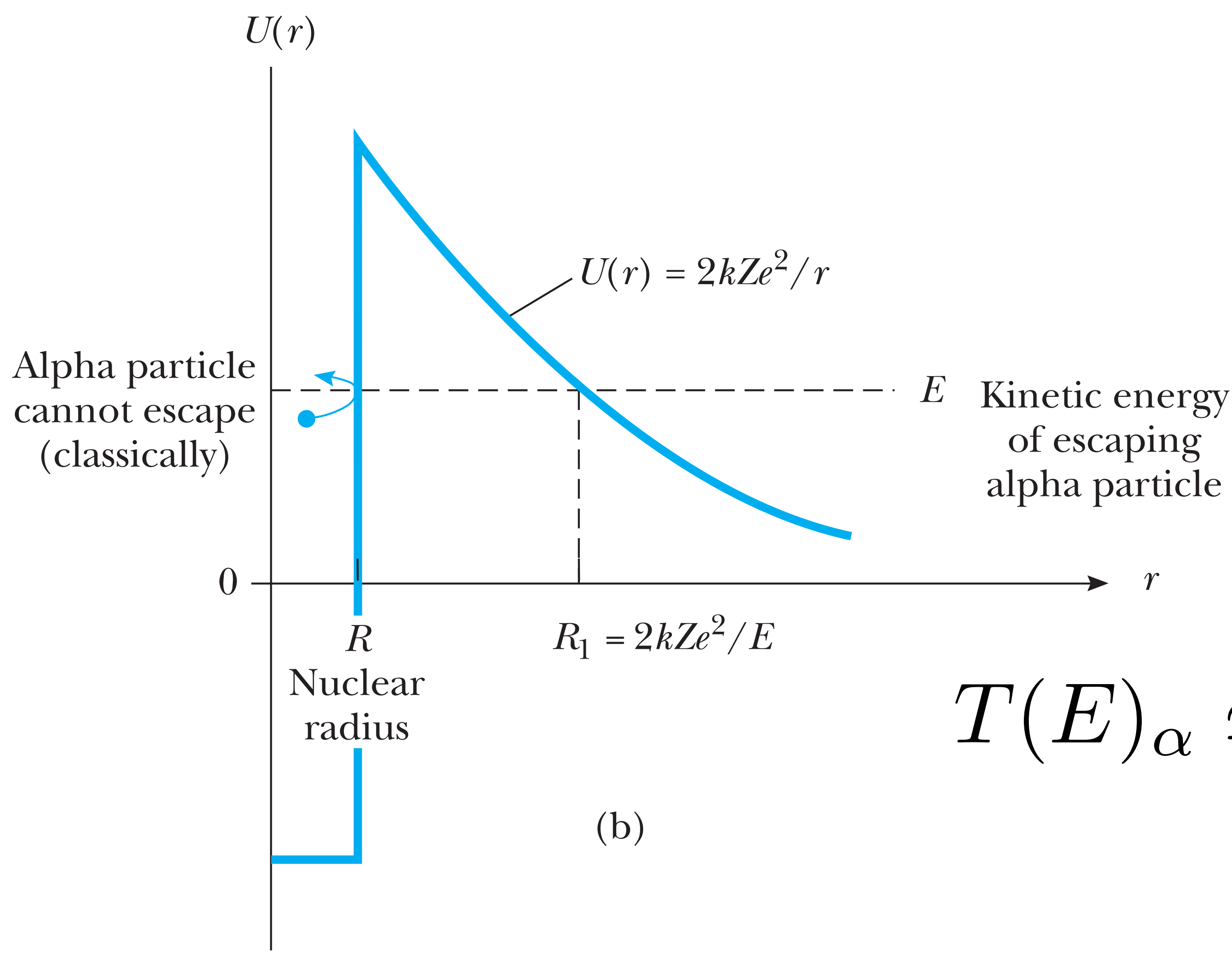

**Table 7.1 Characteristics of Some Common $\alpha$ Emitters**

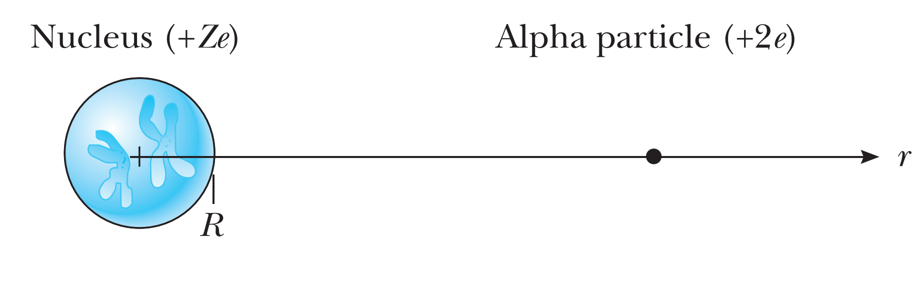
| Element | $\alpha$ Energy | Half-Life* |
|---|---|---|
| $^{212}_{84}$Po | 8.95 MeV | $2.98 \times 10^{-7}$ s |
| $^{240}_{96}$Cm | 6.40 MeV | 27 days |
| $^{226}_{88}$Ra | 4.90 MeV | $1.60 \times 10^{3}$ yr |
| $^{232}_{90}$Th | 4.05 MeV | $1.41 \times 10^{10}$ yr |

*Note that half-lives range over 24 orders of magnitude when $\alpha$ energy changes by a factor of 2.

Decay of radioactive elements with emission of $\alpha$-particles (helium nuclei) was puzzling until 1928 (Gamow and Gurney).

# Application II: Alpha-Decay




$$T(E)_\alpha \simeq \exp\left(-4\pi Z\sqrt{\frac{E}{E_0}} + 8\sqrt{\frac{ZR}{r_0}}\right)$$

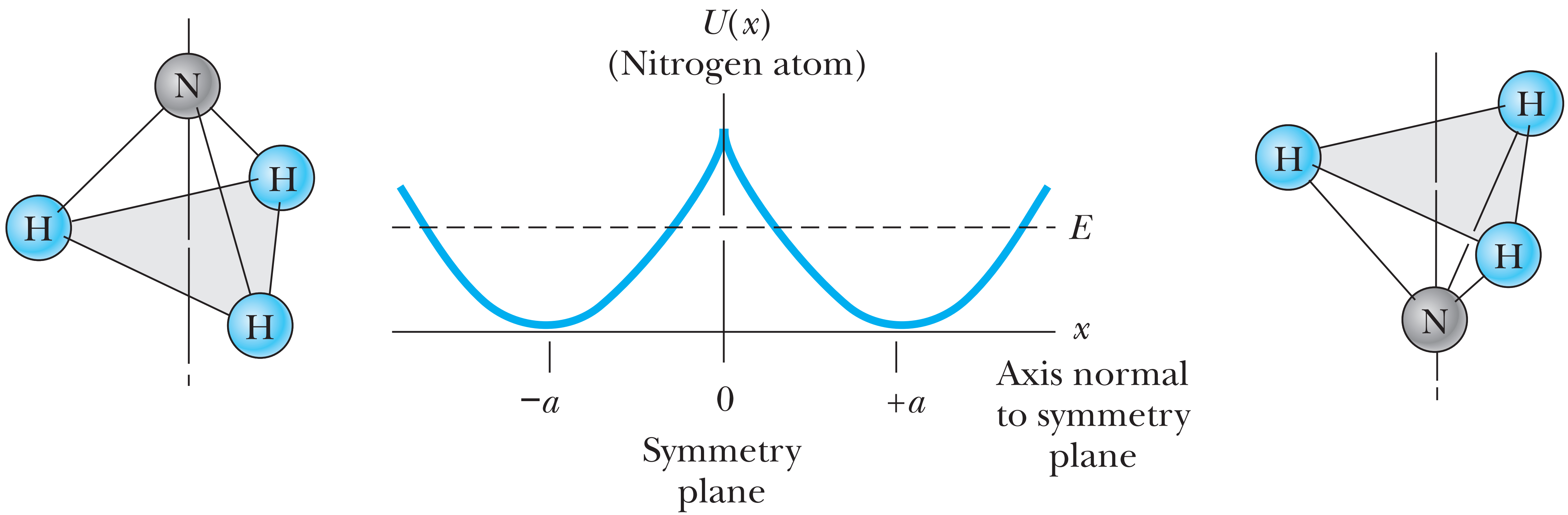
$r_0$ = 7.25 **fm** is like the "Bohr radius" of the **α**-particle.

$E_0$ = 0.0993 MeV is analogous to the Rydberg constant.

$R$ ~ 10 **fm**; $U(r)$ ~ 30 MeV

**Decay rate** $\lambda \propto T(E)_\alpha = t_{1/2}^{-1}$

# Application III: Ammonia Inversion

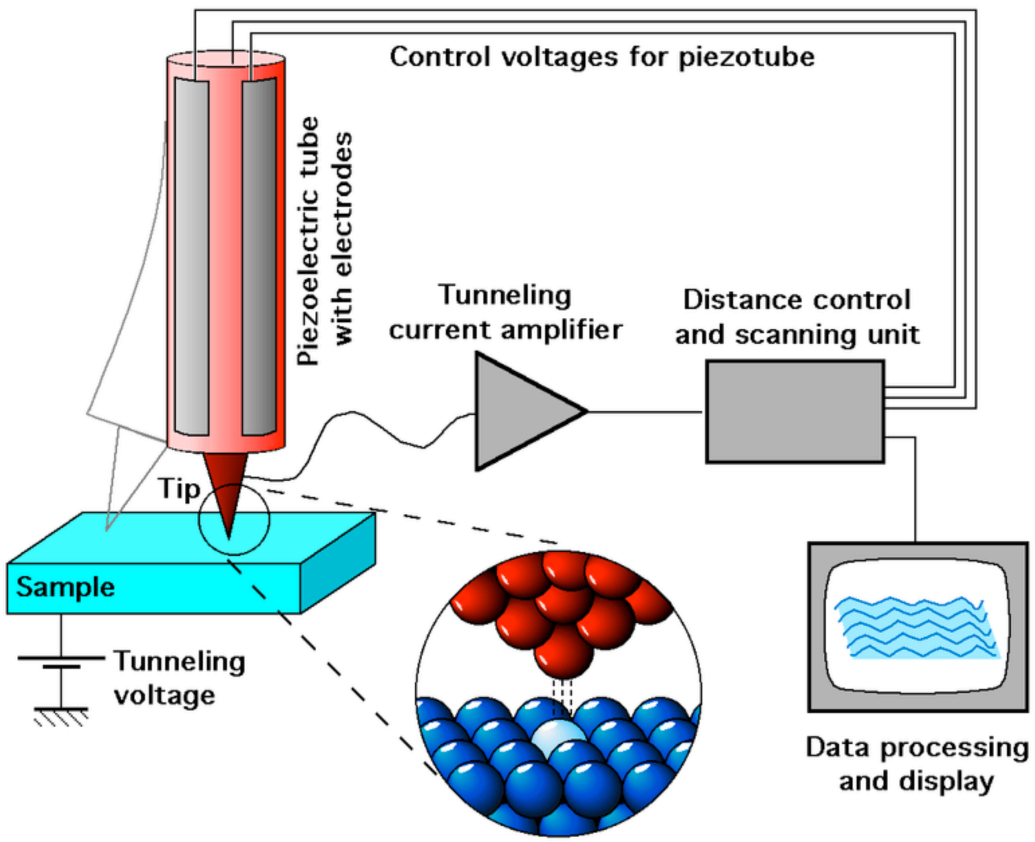


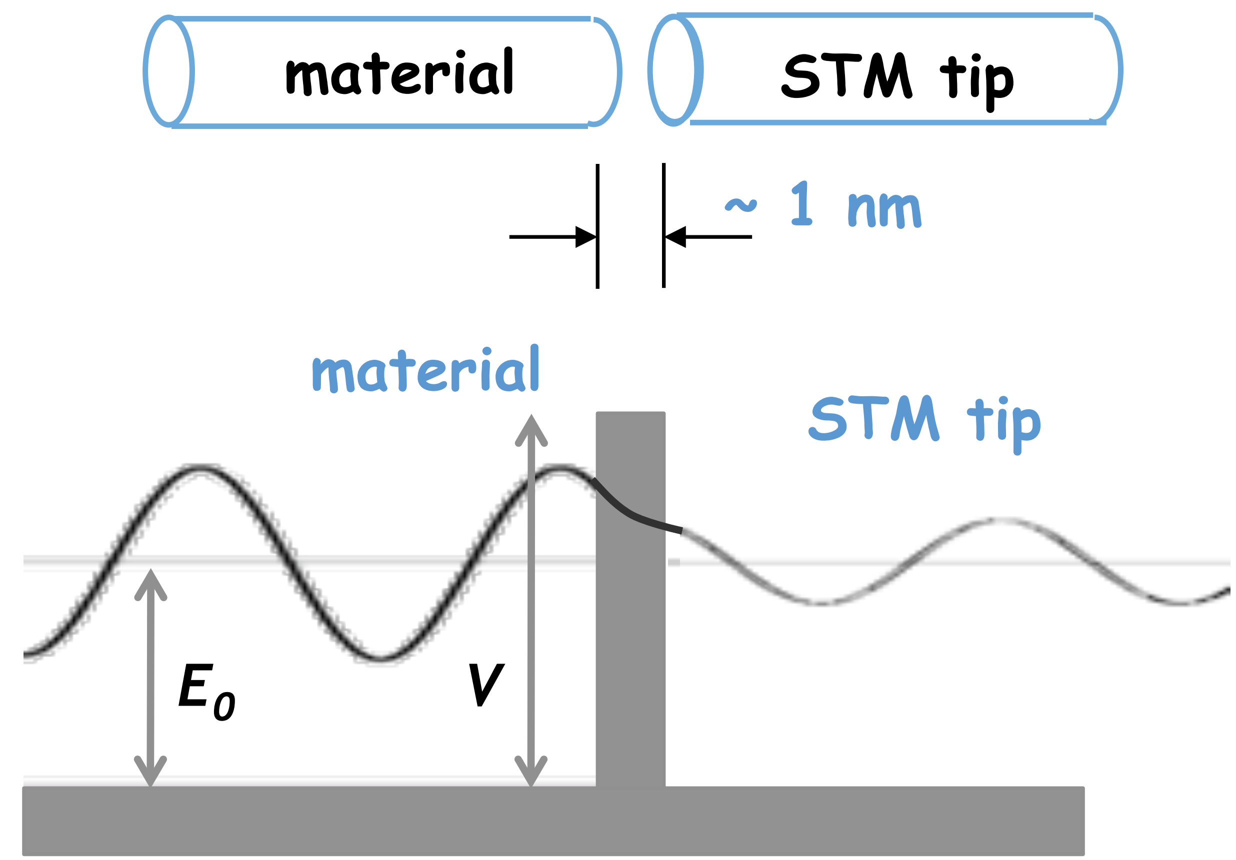

$$U(x) = \frac{1}{2} M\omega^2 (|x| - a)^2$$

$$\frac{4}{A^2}\int_0^{a-A} \sqrt{(x-a)^2 - A^2}\, dx = \sinh(2y_0) - 2y_0$$

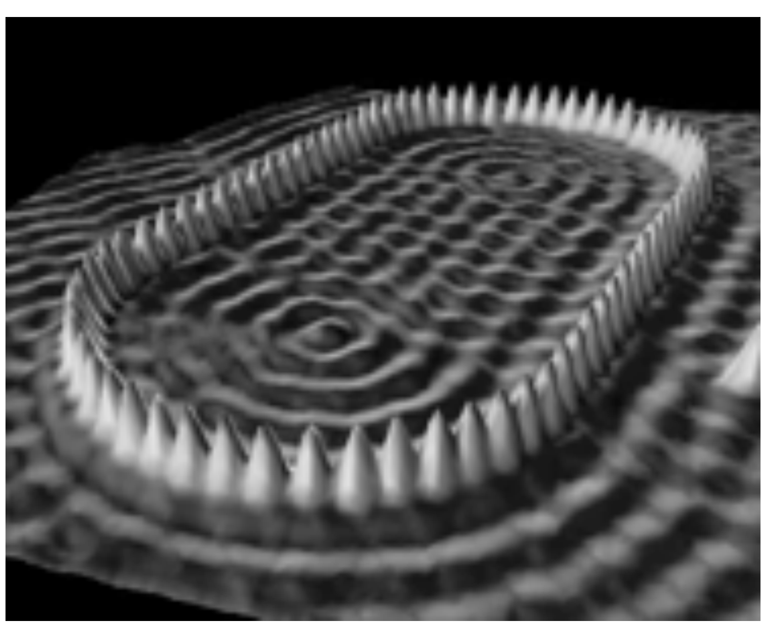
$$T = e^{-[\sinh(2y_0) - 2y_0]}$$

# Application IV: STM




**Sodium atoms on metal:**



# *Flash Memory*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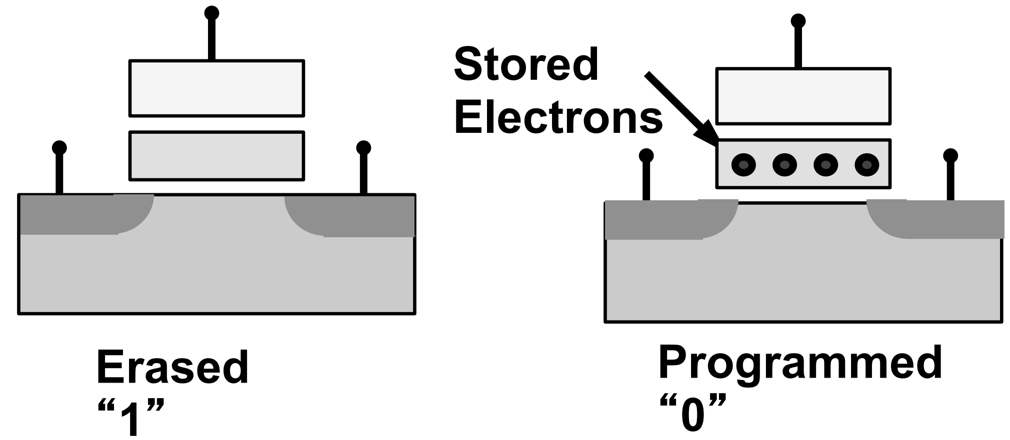


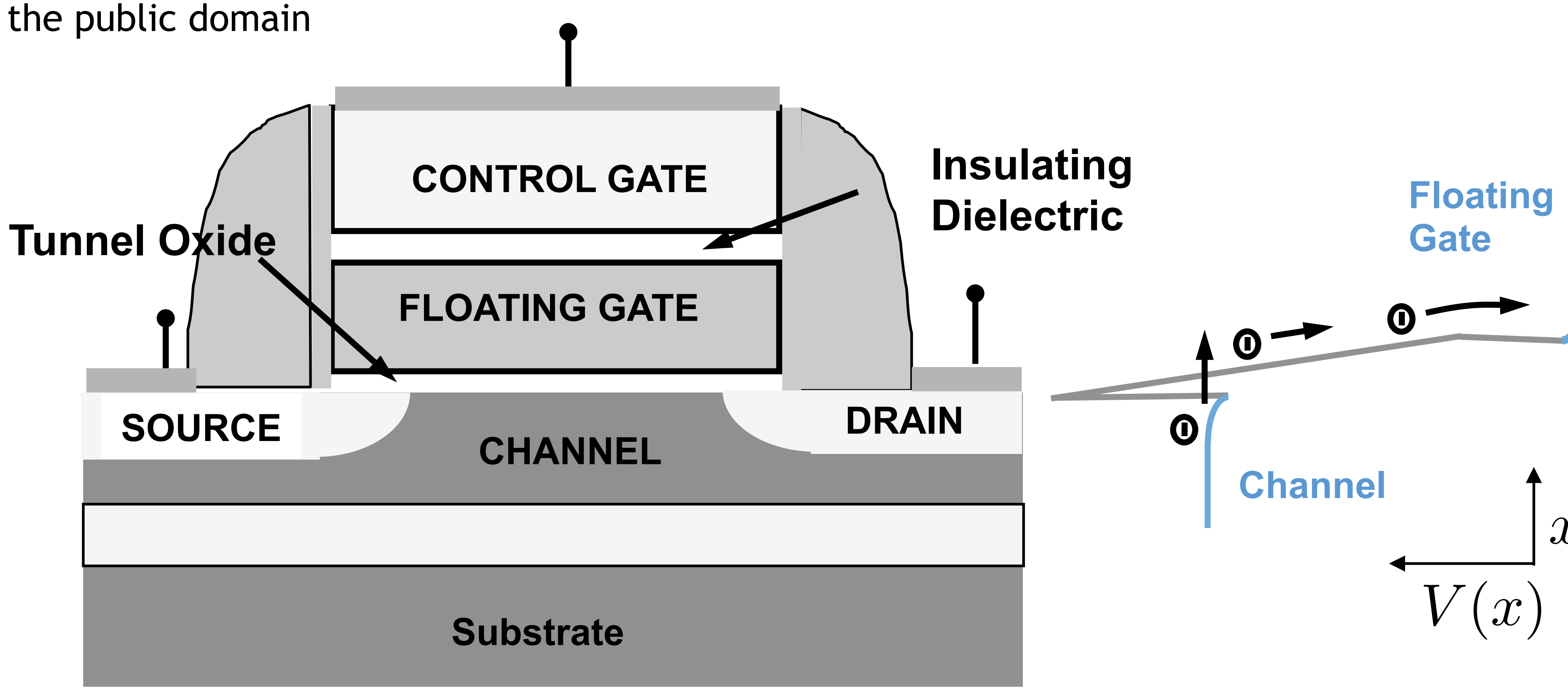


Electrons tunnel preferentially when a voltage is applied

# *MOSFET: Transistor in a Nutshell*

Conduction electron flow

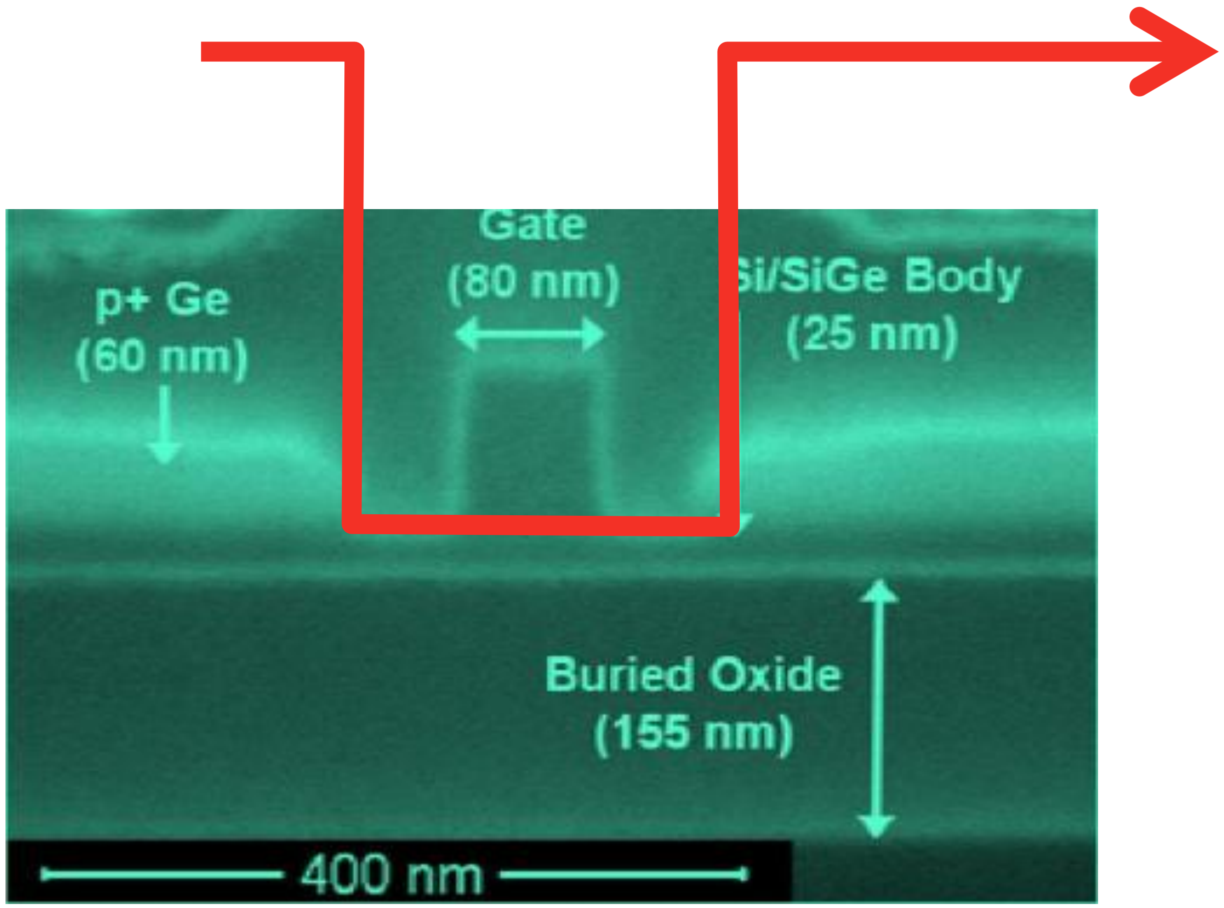


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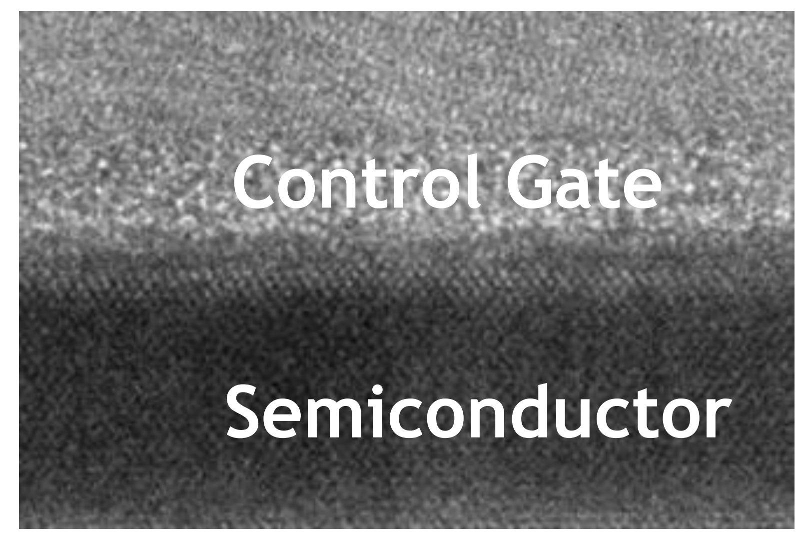
Photo by L. Gomez

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Tunneling causes thin insulating layers to become leaky !