MA 108 - Ordinary Differential Equations

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Outline of the lecture

1. Basic Concepts
2. Separable ODEs

Differential equations

Definition

An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a differential equation.

Definition

Let $y(x)$ denote a function in the variable x. An ordinary differential equation (ODE) is an equation containing one or more derivatives of an unknown function y.
In general, a differential equation involving derivative of one or more dependent variables with respect to a single independent variable is called an ODE.

Definition

A differential equation involving partial derivatives of one or more dependent variables with respect to more than one independent variable is called a partial differential equation (PDE).

Motivating Examples

There are many Physical, Biological and Chemical phenomena when expressed in mathematical framework leads to DEs and in particular ODEs.
For example:
Simple harmonic oscillator $y''(x) + y(x) = 0$
Simple pendulum
$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0.$$

Lotka-Volterra model for predator-prey

$$\begin{aligned} x'(t) &= -ax(t) + bx(t)y(t) \\ y'(t) &= cy(t) - dx(t)y(t), \;\; a, b, c, d > 0. \end{aligned}$$

Radioactive decay

$$\frac{dy}{dt} = -ky,$$

where k is the decay constant of the radio active material.

Skydiver

$$m\frac{dv}{dt} = mg - c\,v^2,$$

where m is the mass of the diver and equipment, v is the velocity of the diver, cv^2 is the air resistance.

Note that, the ODE may contain y itself (the 0^{th} derivative), and known functions of x (including constants). In other words, an ODE is a relation between the derivatives y, y' or $\dfrac{dy}{dx}, \ldots, y^{(n)}$ or $\dfrac{d^n y}{dx^n}$ and functions of x:

$$\boxed{F(x, y, y', \ldots, y^{(n)}) = 0.}$$

As we have seen, DE's occur naturally in physics, engineering and so on.

Further classification according to the appearance of the highest derivative appearing in the equation is done now.

Definition

The order of a differential equation is the order of the highest derivative in the equation.

Examples

1. $\dfrac{d^2y}{dx^2} + xy\left(\dfrac{dy}{dx}\right)^2 = 0$ (ODE, 2nd order)
2. $\dfrac{d^4x}{dt^4} + 5\dfrac{d^2x}{dt^2} + 3x = \sin t$ (ODE, 4th order)
3. $\dfrac{\partial v}{\partial t} + \dfrac{\partial v}{\partial s} = v$ (PDE, 1st order)
4. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$ (PDE, 2nd order)
5. $\dfrac{dx}{dt} = f(x,y),\ \dfrac{dy}{dt} = g(x,y),\ \ x = x(t),\ y = y(t).$ (System of ODEs, 1st order)

Linear equations - $F(x, y, y', \ldots, y^{(n)}) = 0$ is linear if F is a linear function of the variables $y, y', \ldots, y^{(n)}$.
Thus, a linear ODE of order n is of the form

$$\boxed{a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_n(x)y = b(x)}$$

where $a_0, a_1, \ldots, a_n, b$ are functions of x and $a_0(x) \neq 0$.

Check list : If the dependent variable is y, derivatives occur upto first degree only, no products of y and/or its derivatives are there.
Examples :

1. $y'' + 5y' + 6y = 0$ - 2nd order, linear
2. $y^{(4)} + x^2y^{(3)} + x^3y' = xe^x$ - 4th order, linear
3. $y'' + 5(y')^3 + 6y = 0$ - 2nd order, non-linear.

Can we solve it?

Given an equation, you would like to solve it. At least, try to solve it.

Questions:

1. What is a solution?
2. Does an equation always have a solution? If so, how many?
3. Can the solutions be expressed in a nice form? If not, how to get a feel for it?
4. How much can we proceed in a systematic manner?

order - first, second, ..., n^{th}, ...
linear or non-linear?

What is a solution?

Consider $F(x, y, y', \ldots, y^{(n)}) = 0$. We assume that it is always possible to solve a differential equation for the highest derivative, obtaining

$$y^{(n)} = f(x, y, y', \cdots, y^{(n-1)})$$

and study equations of this form. This is to avoid the ambiguity which may arise because a single equation $F(x, y, y', \ldots, y^{(n)}) = 0$ may correspond to several equations of the form $y^{(n)} = f(x, y, y', \cdots, y^{(n-1)})$. For example, the equation $y'^2 + xy' + 4y = 0$ leads to the two equations

$$y' = \frac{-x + \sqrt{x^2 - 16y}}{2} \text{ or } y' = \frac{-x - \sqrt{x^2 - 16y}}{2}.$$

Definition

A explicit solution of the ODE $y^{(n)} = f(x, y, y', \cdots, y^{(n-1)})$ on the interval $\alpha < x < \beta$ is a function $\phi(x)$ such that $\phi', \phi'', \cdots, \phi^{(n)}$ exist and satisfy

$$\phi^{(n)}(x) = f(x, \phi, \phi', \cdots, \phi^{(n-1)}),$$

for every x in $\alpha < x < \beta$.

Definition

A relation $g(x, y) = 0$ is called an implicit solution of $y^{(n)} = f(x, y, y', \cdots, y^{(n-1)})$ if this relation defines at least one function $\phi(x)$ on an interval $\alpha < x < \beta$, such that, this function is an explicit solution of $y^{(n)} = f(x, y, y', \cdots, y^{(n-1)})$ in this interval.

Examples :

1. $x^2 + y^2 - 25 = 0$ is an implicit solution of $x + yy' = 0$ in $-5 < x < 5$, because it defines two functions

$$\phi_1(x) = \sqrt{25 - x^2},\ \phi_2(x) = -\sqrt{25 - x^2}$$

which are solutions of the DE in the given interval. Verify!

2. Consider $x^2 + y^2 + 25 = 0 \Longrightarrow x + yy' = 0 \Longrightarrow y' = -\dfrac{x}{y}$. We say $x^2 + y^2 + 25 = 0$ formally satisfies $x + yy' = 0$. But it is NOT an implicit solution of DE as this relation doesn't yield ϕ which is an explicit solution of the DE on any real interval I.

First order ODE & Initial Value Problem for first order ODE

Consider a linear first order ODE of the form $\boxed{y' + a(x)y = b(x)}$.
If $b(x) = 0$, then we say that the equation is homogeneous.

Note that the set of all solutions of the above homogeneous differential equation form a vector space under usual addition and scalar multiplication

Definition

Initial value problem (IVP) : A DE along with an initial condition is an IVP. For example,

$$y' = f(x, y), \; y(x_0) = y_0$$

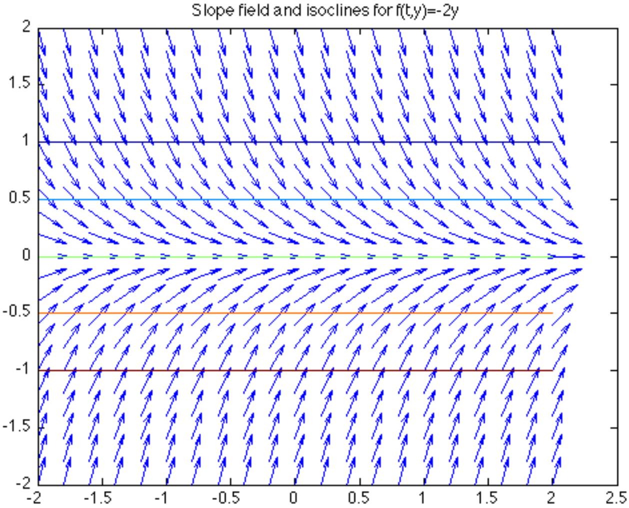
is a first order IVP.

Given an amount of a radioactive substance, say 1 gm, find the amount present at any later time.
The relevant ODE is
$$\frac{dy}{dt} = -k\,y.$$
Initial amount given is 1 gm at time $t = 0$. i.e.,
$$y(0) = 1.$$
An IVP . By inspection, $y = ce^{-kt}$, for an arbitrary constant c, is a solution of the above ODE. The initial condition determines $c = 1$. Hence
$$y = e^{-kt}$$
is a particular solution to the above ODE with the given initial condition.

Geometrical meaning : $\dfrac{dy}{dt} = -2\,y$

1. Suppose that y has certain value. From the RHS of the DE, we obtain $\dfrac{dy}{dt}$. For instance, if $y = 1.5$, $\dfrac{dy}{dt} = -3$. This means that the slope of a solution $y = y(t)$ has the value -3 at any point where $y = 1.5$.
2. Display this information graphically in ty-plane by drawing short line segments or arrows (direction fields) at several points on $y = 1.5$.
3. Similarly proceed for other values of y.
4. The figures given in the next slide and the slide after two slides are examples of direction fields or slope fields.
5. An isocline (a series of lines with the same slope) is often used to supplement the slope field. In an equation of the form $\dfrac{dy}{dt} = f(t, y)$, the isocline is a line in the ty-plane obtained by setting $f(t, y)$ equal to a constant.

Slope field


Slope field and isoclines for f(t,y)=-2y
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5

Find the curve through the point $(1, 1)$ in the xy-plane having at each of its points, the slope $-\frac{y}{x}$.
The relevant ODE is

$$y' = -\frac{y}{x}.$$

By inspection,

$$y = \frac{c}{x}$$

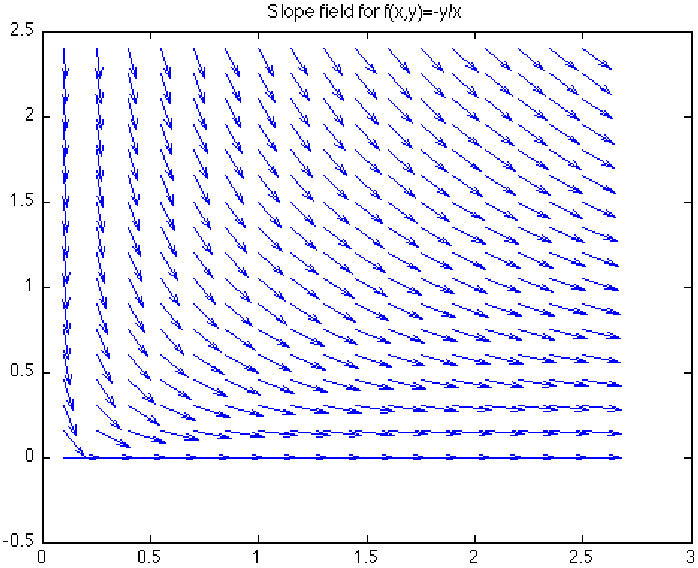
is its general solution for an arbitrary constant c; that is, a family of hyperbolas.
The initial condition given is

$$y(1) = 1,$$

which implies $c = 1$. Hence the particular solution for the above problem is

$$y = \frac{1}{x}.$$

Slope field



Examples

A first order IVP can have

1. NO solution : $|y'| + |y| = 0,\ y(0) = 3.$
2. Precisely one solution : $y' = x,\ y(0) = 1.$ What is the solution?
3. Infinitely many solutions: $xy' = y - 1,\ y(0) = 1$ The solutions are $y = 1 + cx$.

Motivation to study conditions under which the solution would exist and the conditions under which it will be unique!

We first start with a few methods for finding out the solution of first order ODEs, discuss the geometric meaning of solutions and then proceed to study existence-uniqueness results.

Separable ODE's

An ODE of the form

$$M(x) + N(y)y' = 0$$

is called a separable ODE.
ODE can be written as

$$M(x) + N(y(x))y'(x) = 0$$

Integrate the above equation with respect to x, we get

$$\int M(x)dx + \int N(y(x))y'(x)dx = c,$$

where c is a constant.
Using change of variable formula for integration, we get

$$\int M(x)dx + \int N(y)dy = c.$$

i.e.,

$$H_1(x) + H_2(y) = c,$$

where

$$H_1(x) = \int M(x)dx, \;\; H_2(y) = \int N(y)dy.$$

Note: This method many times gives us an implicit solution and not necessarily an explicit one!

Separable ODE - Example 1

Solve the DE :

$$y' = -2xy.$$

Separating the variables, we get :

$$\frac{dy}{y} = -2xdx.$$

Integrating both sides, we obtain :

$$\ln|y| = -x^2 + c_1.$$

Thus, the solutions are

$$y = ce^{-x^2}.$$

How do they look?

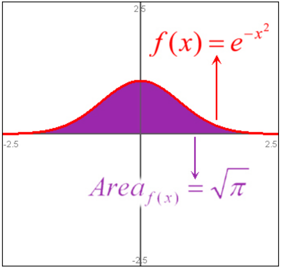
If we are given an initial condition

$$y(x_0) = y_0,$$

then we get:

$$c = y_0 e^{x_0^2}$$

and $\boxed{y = y_0 e^{x_0^2 - x^2}.}$



$$(y_0 = e^{-x_0^2})$$

Find the solution to the initial value problem:

$$\frac{dy}{dx} = \frac{y\cos x}{1+2y^2}; \; y(0) = 1.$$

Assume $y \neq 0$. Then,

$$\frac{1+2y^2}{y}dy = \cos x \, dx.$$

Integrating,

$$\ln|y| + y^2 = \sin x + c.$$

As $y(0) = 1$, we get $c = 1$. Hence a particular solution to the IVP is

$$\ln|y| + y^2 = \sin x + 1.$$

Note: $y \equiv 0$ is a solution to the DE but it is not a solution to the given IVP.