Indian Institute of Technology Bombay

MA 106 Linear Algebra

Spring 2021 SRG/DP

Solutions and Marking Scheme for Common Quiz 1

Date: March 24, 2021 Max. Marks: 10
Time: 8.30 AM – 9.15 AM

Q1: Consider the 4×5 matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 & a \\ 1 & 1 & 5 & 2 & b \\ 1 & 2 & 8 & 4 & 12 \\ 3 & 4 & 18 & 8 & 27 \end{bmatrix},$$

where a and b denote the last two digits of your roll number (e.g., if your roll number is 200010059, then $a = 5$ and $b = 9$). Determine:

(i) The row canonical form of A.

(ii) The nullity of A. [3 marks]

(i)

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 & a \\ 1 & 1 & 5 & 2 & b \\ 1 & 2 & 8 & 4 & 12 \\ 3 & 4 & 18 & 8 & 27 \end{bmatrix} \xrightarrow{R_2-R_1, R_3-R_1, R_4-3R_1} \begin{bmatrix} 1 & 0 & 2 & 1 & a \\ 0 & 1 & 3 & 1 & b-a \\ 0 & 2 & 6 & 3 & 12-a \\ 0 & 4 & 12 & 5 & 27-3a \end{bmatrix},$$

$$\xrightarrow{R_3-2R_2, R_4-4R_2} \begin{bmatrix} 1 & 0 & 2 & 1 & a \\ 0 & 1 & 3 & 1 & b-a \\ 0 & 0 & 0 & 1 & 12+a-2b \\ 0 & 0 & 0 & 1 & 27+a-4b \end{bmatrix} \xrightarrow{R_4-R_3} \begin{bmatrix} 1 & 0 & 2 & 1 & a \\ 0 & 1 & 3 & 1 & b-a \\ 0 & 0 & 0 & 1 & 12+a-2b \\ 0 & 0 & 0 & 0 & 15-2b \end{bmatrix},$$

$$\xrightarrow{R_4/(15-2b)} \begin{bmatrix} 1 & 0 & 2 & 1 & a \\ 0 & 1 & 3 & 1 & b-a \\ 0 & 0 & 0 & 1 & 12+a-2b \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1-aR_4, R_2-(b-a)R_4, R_3-(12+a-2b)R_4} \begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1-R_3, R_2-R_3} \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus the **Row Canonical form (RCF)** of A is given by $\begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

[Marking scheme: *Correct RCF: 2 marks. Incorrect RCF but correct rank: 1 mark. All other cases: 0 marks. In case the values of a, b in the matrix A do not correspond to the last two digits of the roll number, give* 0 *marks even if the RCF is fully correct.*]

(ii) **Nullity**$(A) = 5 - \text{rank}(A) = 1$.

[Marking scheme: *1 mark for the value of nullity that is consistent with the rank of A as indicated by the RCF. In case the value of the nullity is given to be* 1*, but this does not match with the rank indicated by the RCF, give* 0 *marks.*]

Q2: Let $r_1, \ldots, r_6$ denote the last six digits of your roll number (so that r_6 is the last digit, r_5 the second last digit, and so on). Consider the matrices

$$\mathbf{a} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{bmatrix}, \quad \mathbf{b} = [r_6\ r_5\ r_4\ r_3\ r_2\ r_1] \quad \text{and} \quad \mathbf{A} = \mathbf{ab}.$$

of sizes 6×1, 1×6, and 6×6, respectively. Compute the rank of $\mathbf{A}$ and write down a basis for the column space of $\mathbf{A}$. [3 marks]

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \end{bmatrix} [r_6\ r_5\ r_4\ r_3\ r_2\ r_1] = \begin{bmatrix} r_1r_6 & r_1r_5 & r_1r_4 & r_1r_3 & r_1r_2 & r_1r_1 \\ r_2r_6 & r_2r_5 & r_2r_4 & r_2r_3 & r_2r_2 & r_2r_1 \\ r_3r_6 & r_3r_5 & r_3r_4 & r_3r_3 & r_3r_2 & r_3r_1 \\ r_4r_6 & r_4r_5 & r_4r_4 & r_4r_3 & r_4r_2 & r_4r_1 \\ r_5r_6 & r_5r_5 & r_5r_4 & r_5r_3 & r_5r_2 & r_5r_1 \\ r_6r_6 & r_6r_5 & r_6r_4 & r_6r_3 & r_6r_2 & r_6r_1 \end{bmatrix}.$$

Every column of $\mathbf{A}$ is a scalar multiple of $\mathbf{a}$. Also, $\mathbf{a}$ can not be a zero vector. Thus $\text{rank}(A) = 1$, and $\{\mathbf{a}\}$ is a basis of the column space of $\mathbf{A}$ [In fact, any nonzero column of $\mathbf{A}$ (or any of its nonzero scalar multiple) gives a basis of the column space of $\mathbf{A}$.]

[Marking scheme: *Correct value of Rank: 1 mark. Correct basis for the column space: 2 marks. Give 0 marks if numbers other than the last 6 digits of the roll number are used (even if the answer is correct).*]

Q3. Let V denote the subspace of $\mathbb{R}^{1\times 4}$ spanned by $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{a}_3$, where

$$\mathbf{a}_1 = [-1\ \ 0\ \ 1\ \ 2]\,, \quad \mathbf{a}_2 = [3\ \ 4\ \ -2\ \ 5]\,, \quad \text{and} \quad \mathbf{a}_3 = [1\ \ 4\ \ 0\ \ 9]\,.$$

Find the dimension of V. Further let $\mathbf{A}$ be the 3×4 matrix whose row vectors given by $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{a}_3$, and let $\mathbf{c}$ be the 3×1 column vector

$$\mathbf{c} = \begin{bmatrix} 0 \\ a \\ b \end{bmatrix},$$

where a and b denote the last two digits of your roll number (e.g., if your roll number is 200010059, then $a = 5$ and $b = 9$). Then determine if the linear system $\mathbf{A}\mathbf{x} = \mathbf{c}$ has (i) no solution, (ii) unique solution, or (iii) infinitely many solutions. [4 marks]

Solution Since $2\mathbf{a}_1 + \mathbf{a}_2 = \mathbf{a}_3$, and since $\mathbf{a}_1$ and $\mathbf{a}_2$ are not multiple of each other, we see that $\{\mathbf{a}_1, \mathbf{a}_2\}$is a basis of V (being a subset that is linearly independent and which spans V). Hence $\dim V = 2$.

[Marking scheme: *Correct value of dimension: 1 mark. Correct justification: 1 mark (note that there are also other ways of justifying that* $\dim V = 2$*. As long as a mathematically valid justification is given, 1 mark may be given.) For just writing the dimension of* V *correctly, but without any justification, give only 1 mark.*]

Next, for the system $\mathbf{A}\mathbf{x} = \mathbf{c}$, we can consider the augmented matrix and apply Elementary Row Operations so as to transform the coefficient matrix $\mathbf{A}$ to a REF:

$$[\mathbf{A}|\mathbf{c}] = \left[\begin{array}{cccc|c} -1 & 0 & 1 & 2 & 0 \\ 3 & 4 & -2 & 5 & a \\ 1 & 4 & 0 & 9 & b \end{array}\right]$$

$$\xrightarrow{R_2+3R_1, R_3+R_1} \left[\begin{array}{cccc|c} -1 & 0 & 1 & 2 & 0 \\ 0 & 4 & 1 & 11 & a \\ 0 & 4 & 1 & 11 & b \end{array}\right] \xrightarrow{R_3-R_2} \left[\begin{array}{cccc|c} -1 & 0 & 1 & 2 & 0 \\ 0 & 4 & 1 & 11 & a \\ 0 & 0 & 0 & 0 & b-a \end{array}\right]$$

Conclusion:

(i) No solution if $a \neq b$. (ii) Infinitely many solutions if $a = b$. (iii) Unique solution never happens.

[Marking scheme: *Correct conclusion about the number of solutions and correct last row of the transformed matrix (including the* $(3,5)$*-th entry, which is* $b-a$*): 2 marks. Give 1 mark if correct values of* a, b *are used, but there is a calculation error somewhere, or if the entries of* $\mathbf{A}$ *are copied incorrectly, as long as the conclusion is consistent with the transformed matrix. Give 0 marks if wrong values of the last two digits of the roll number (a and b) are used even if the final answer is correct.*]