Outline

1. Electrostatic Potential due to Discrete and Continuous Charge Distributions
2. Boundary Conditions on Electric Field and Electrostatic Potential
3. Energy of a Discrete and Continuous Charge Distributions

Objectives

1. To understand how to incorporate boundary conditions.
2. To understand and be able to compute the energy of a charge configuration.

Recap

- Electrostatic Field using Coulomb's Law

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\vec{r}')}{|\vec{R}|^2}\hat{R}d\tau'$$

where $\hat{R}$ is the unit vector along the seperation vector given by, $\vec{R} = \vec{r} - \vec{r}'$.

- Electric Potential $V(\vec{r})$.

$$\nabla \times \vec{E} = 0 \quad \Rightarrow \quad \vec{E}(\vec{r}) = -\nabla V(\vec{r})$$

- Poisson's Equation:

$$\nabla^2 V(\vec{r}) = -\frac{\rho(\vec{r})}{\varepsilon_0}$$

In charge free regions, Poisson's equation turns out to be Laplace's Equation,

$$\nabla^2 V(\vec{r}) = 0$$

Electrostatic Potential

- The external work that should be done to shift a unit test charge with infinitesimally small speed from $\vec{a}$ to $\vec{b}$ is,

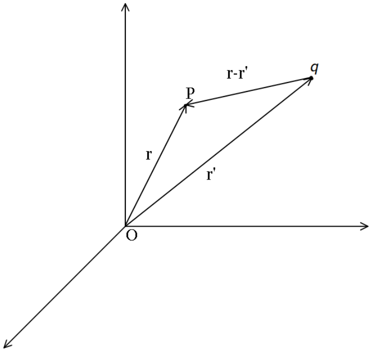
$$\begin{aligned} W_{ext} &= -\int_a^b \vec{E} \cdot d\vec{l} \\ &= \int_a^b (\nabla V) \cdot d\vec{l} \\ &= V(\vec{b}) - V(\vec{a}) \end{aligned}$$

- Potential differences are physically measurable not potentials. Therefore, potentials are defined up to an arbitrary constant, or with respective to a reference potential.
- One of the famous reference voltage is $V \to 0$ as $r \to \infty$. (in general this reference works for all except infinite charge distributions).
- In physical situations, there won't be any infinite distributions. Therefore, $V \to 0$ as $r \to \infty$ fits well into physical situations.

Electrostatic Potential

- Electrostatic potential due to a point charge with charge q poistioned at $\vec{r}'$,

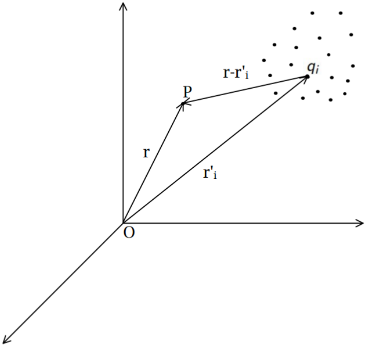
$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{q}{|\vec{r} - \vec{r}'|}$$



Electrostatic Potential

- A collection of point charges with charge q_i located at $\vec{r}_i'$.
- Principle of superposition can be used in this case, as Poisson's equation $\left(\nabla^2 V = \frac{-\rho(\vec{r})}{\varepsilon_0}\right)$ is linear,

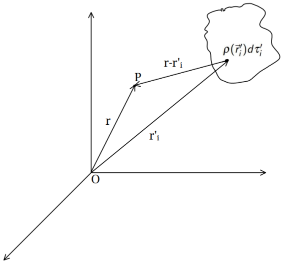
$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{i=0}^{n} \frac{q_i}{|\vec{r} - \vec{r}_i'|}$$



Electrostatic Potential

- A continuous charge distribution can be visualised as a discrete collection of infinite number of point charges with infinitesimally small charges. In simple words, if $n \to \infty$ and $q_i \to \rho\, d\tau_i$,

$$
\begin{aligned}
V(\vec{r}) &= \lim_{n\to\infty} \sum_{i=0}^{n} \frac{1}{4\pi\varepsilon_0} \frac{\rho(\vec{r}_i')d\tau_i'}{|\vec{r} - \vec{r}_i'|} \\
&= \int_V \frac{1}{4\pi\varepsilon_0} \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} d\tau'
\end{aligned}
$$



Electrostatic Potential

- Electrostatic potential for a volume charge distribution can be modulated to surface charge distribution and linear charge distribution.

$$\rho d\tau = \quad \sigma da \quad = \lambda dl = dq$$

- Electrostatic potential due to a surface charge distribution,

$$V(\vec{r}) \quad = \quad \frac{1}{4\pi\varepsilon_0} \int_S \frac{\sigma(\vec{r}')}{|\vec{r} - \vec{r}'|} da'$$

- Electrostatic potential due to a linear charge distribution,

$$V(\vec{r}) \quad = \quad \frac{1}{4\pi\varepsilon_0} \int_L \frac{\lambda(\vec{r}')}{|\vec{r} - \vec{r}'|} dl'$$

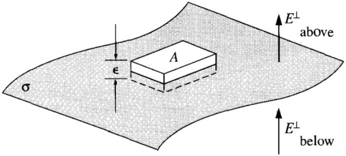
Electrostatic Potential

- In general, electrostatic potential is easier to deal with, as compared to the electric field.
- Potential is a scalar with only one component and with no direction associated.
- Most of the measurements that are done in laboratory are energy related, and energy can be easily related to potential.

Boundary Conditions on Electric Field

- How does the electric field behave very close to the surface?
  - Is it discontinuous or continuous?Is it always discontinuous or only in the presence of charge?
  - Is it discontinuous along all directions, or just along a specific direction?
- Consider a cuboid with a small top/bottom rectangular area(A) locally parallel to the surface and with a much smaller height(ε). Applying Gauss law on this cuboid,

$$\int_S \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$$



Boundary Conditions on Electric Field

- Length of the cuboid ε can be made arbitrarily small, such that the flux passing through the lateral surface of the cuboid is negligible when compared to the total flux through the cuboid.

$$\begin{aligned}
(E^{\perp}_{above}\hat{n})\cdot(A\hat{n}) + (E^{\perp}_{below}\hat{n})\cdot(-A\hat{n}) &= \frac{Q_{enc}}{\varepsilon_0} \\
(E^{\perp}_{above} - E^{\perp}_{below})A &= \frac{Q_{enc}}{\varepsilon_0} \\
E^{\perp}_{above} - E^{\perp}_{below} &= \frac{\sigma}{\varepsilon_0}
\end{aligned}$$

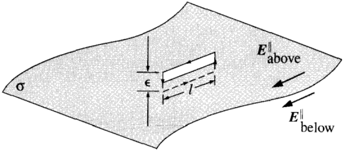
- This indicates discontinuity in $E^{\perp}$ at a charged surface.
- If the surface is charge free, then $E^{\perp}$ is continuous.
- Note that $E^{\perp}$ in the above formula are just above and just below the surface, not far away.

Boundary Conditions on Electric Field

- Consider a rectangular loop with a small length(l) locally parallel to the surface and much smaller width(w) perpendicular to the surface.

$$\oint \vec{E} \cdot d\vec{l} \quad = \quad 0$$

- Width(w) can be made arbitrarily small, such that the line integral along width(w) is negligible compared to the line integral along length(l).



Boundary Conditions on Electric Field

$$\begin{aligned}(E^{\parallel}_{above} - E^{\parallel}_{below})l &= 0 \\ E^{\parallel}_{above} &= E^{\parallel}_{below}\end{aligned}$$

- This indicates that the $E^{\parallel}$component is continuous across a charged surface.
- Also note that the perpendicular to the surface direction is unique, whereas the parallel direction isn't unique.
- However one can prove that $E^{\parallel}_{above} = E^{\parallel}_{below}$ along all parallel directions in the exact same way.
- Note that $E^{\parallel}$ in the above formula are just above and just below the surface, not far away.

Boundary Conditions on Electric Field

- We have obtained boundary conditions for both perpendicular and parallel components of Electric field.
- Now, summing up these boundary conditions into a single equation,

$$\vec{E}_{above} - \vec{E}_{below} \quad = \quad \frac{\sigma}{\varepsilon_0}\hat{n}$$

  where $\hat{n}$ is the unit normal to the surface from below to above.
- Note that $\vec{E}$ in the above formula are just above and just below the surface, not far away.

Boundary Conditions on Electrostatic Potential

- How does the electrostatic potential behave very close to the surface? Does the presence of charge affect the continuity of Potential?

$$
\begin{aligned}
V_{above} - V_{below} &= -\int_{below}^{above} \vec{E}\cdot d\vec{l} \\
&= -\lim_{\varepsilon\to 0}\int_{z_0-\varepsilon}^{z_0+\varepsilon} \vec{E}\cdot d\vec{l}
\end{aligned}
$$

- As $\vec{E}$ is finite for all sort of surface charge distributions, the above integral is zero in the limit $\varepsilon \to 0$.

$$V_{above} = V_{below}$$

- Potential is continuous across a charged surface.
- Note that V in the above formula are just above and just below the surface, not far away.

Boundary Conditions on Electrostatic Potential

- Boundary conditions on Electric field can be converted to Boundary conditions on Electrostatic Potential.

$$
\begin{aligned}
E^{\perp}_{above} - E^{\perp}_{below} &= \frac{\sigma}{\varepsilon_0} \\
\frac{\sigma}{\varepsilon_0} &= \vec{E}_{above} \cdot \hat{n} - \vec{E}_{below} \cdot \hat{n} \\
&= (\nabla V_{below}) \cdot \hat{n} - (\nabla V_{above}) \cdot \hat{n} \\
&= \left.\frac{\partial V}{\partial n}\right|_{below} - \left.\frac{\partial V}{\partial n}\right|_{above}
\end{aligned}
$$

where $\frac{\partial V}{\partial n}$ represents the directional derivative along $\hat{n}$, $\frac{\partial V}{\partial n} = (\nabla V) \cdot \hat{n}$.

$$
\left.\frac{\partial V}{\partial n}\right|_{above} - \left.\frac{\partial V}{\partial n}\right|_{below} = -\frac{\sigma}{\varepsilon_0}
$$

Energy of a Discrete Charge Distribution

- What is the external work that should be done to arrange three point charges q_1, q_2, q_3 at positions $\vec{r}_1, \vec{r}_2, \vec{r}_3$?
- As electrostatic force is conservative, energy is a state function and is independent of the path one follows.
- Let's assume that initially q_1 is brought from infinity to $\vec{r}_1$ with infinitesimally small speed. As there is no other charge to affect, the work done in the process is zero.
- External work done that should be done to bring charge q_2 from infinity to $\vec{r}_2$ can be found using work-energy theorem,

$$\begin{aligned} W_{ext_1} &= q_2 V_1(\vec{r}_2) - q_2 V_1(\infty) + \Delta K.E \\ &= q_2 \frac{1}{4\pi\varepsilon_0} \frac{q_1}{|R_{21}|} - 0 + 0 \\ &= \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{|R_{21}|} \end{aligned}$$

where $\vec{R}_{12}$ is the seperation vector given by, $\vec{R}_{21} = \vec{r}_2 - \vec{r}_1$.

Energy of a Discrete Charge Distribution

- External work done that should be done to bring charge q_3 from infinity to $\vec{r}_3$ can be found using work-energy theorem,

$$\begin{aligned} W_{ext_2} &= q_3 V_1(\vec{r}_3) - q_3 V_1(\infty) + q_3 V_2(\vec{r}_3) - q_3 V_2(\vec{r}_3) + \Delta K.E \\ &= q_3 \frac{1}{4\pi\varepsilon_0}\frac{q_1}{|R_{31}|} - 0 + q_3 \frac{1}{4\pi\varepsilon_0}\frac{q_2}{|R_{32}|} - 0 + 0 \\ &= \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1 q_3}{|R_{31}|} + \frac{q_2 q_3}{|R_{32}|}\right] \end{aligned}$$

- Total work done in arranging these three charges q_1, q_2, q_3 at positions $\vec{r}_1, \vec{r}_2, \vec{r}_3$ is,

$$\begin{aligned} W &= \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1 q_2}{|R_{21}|} + \frac{q_1 q_3}{|R_{31}|} + \frac{q_2 q_3}{|R_{32}|}\right] \\ &= W_{12} + W_{23} + W_{31} \end{aligned}$$

Energy of a Discrete Charge Distribution

- What is the external work that should be done to arrange n point charges $q_1, q_2 ..., q_n$ at positions $\vec{r}_1, \vec{r}_2, ..., \vec{r}_n$?

$$\begin{aligned} W &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} W_{ij} \\ &= \frac{1}{2}\cdot\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n}\frac{q_i q_j}{|R_{ij}|} \end{aligned}$$

- Can this be expressed in terms of potentials?

$$\begin{aligned} W &= \frac{1}{2}\cdot\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n} q_i\left[\sum_{j=1, j\neq i}^{n}\frac{q_j}{|R_{ij}|}\right] \\ &= \frac{1}{2}\sum_{i=1}^{n} q_i V(\vec{r}_i') \end{aligned}$$

where $V(\vec{r}_i')$ is the potential due to all charges except q_i at $\vec{r}_i$.

Energy of a Continuous Charge Distribution

- We can use the expression derived for discrete charge distribuion here in the limit $n \to \infty$.

$$\begin{aligned} W &= \frac{1}{2} \lim_{n\to\infty} \sum_{i=1}^{n} q_i V(\vec{r}_i') \\ &= \frac{1}{2} \lim_{n\to\infty} \sum_{i=1}^{n} \rho(\vec{r}_i') V(\vec{r}_i') d\tau_i' \\ &= \frac{1}{2} \int V(\vec{r}') \rho(\vec{r}') d\tau' \end{aligned}$$

Using $\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$,

$$W = \frac{\varepsilon_0}{2} \int V(\nabla \cdot \vec{E}) d\tau'$$

Energy of a Continuous Charge Distribution

- Using $\nabla \cdot (V\vec{E}) = V(\nabla \cdot \vec{E}) + \vec{E} \cdot (\nabla V)$,

$$W = \frac{\varepsilon_0}{2}\left[\int \nabla \cdot (V\vec{E}) d\tau - \int \vec{E} \cdot (\nabla V) d\tau\right]$$

Using Divergence theorem on the first integral,

$$W = \frac{\varepsilon_0}{2}\left[\oint_S (V\vec{E}) \cdot d\vec{a} - \int \vec{E} \cdot (\nabla V) d\tau\right]$$

- If we consider the whole space,

$$\begin{aligned}
\oint_S (V\vec{E}) \cdot d\vec{a} &\sim \lim_{r\to\infty} \int_0^{2\pi} \int_0^{\pi} \left(\frac{1}{r} \cdot \frac{1}{r^2}\right) r^2 \sin\theta \, d\theta \, d\phi \\
&\sim \lim_{r\to\infty} \frac{4\pi}{r} \\
&\sim 0
\end{aligned}$$

One can consider an arbitrary surface also, but will end up with the same result in the limit $r \to \infty$.

Energy of a Continuous Charge Distribution

- Using the above result $\oint_S (V\vec{E}) \cdot d\vec{a} = 0$,

$$W = -\frac{\varepsilon_0}{2} \int \vec{E} \cdot (\nabla V) d\tau$$

Using $\vec{E} = -\nabla V$,

$$\begin{aligned} W &= \frac{\varepsilon_0}{2} \int \vec{E} \cdot \vec{E} d\tau \\ &= \int \left(\frac{1}{2} \varepsilon_0 E^2 \right) d\tau \end{aligned}$$

- Therefore the energy of a continuous charge distribution is,

$$W = \frac{\varepsilon_0}{2} \int E^2 d\tau$$

Energy of a Point Charge

- What is the energy needed to create a point charge?

$$
\begin{aligned}
W &= \frac{\varepsilon_0}{2}\int_{r=0}^{\infty}\frac{q^2}{(4\pi\varepsilon_0)^2 r^4} r^2 \sin\theta \, dr\, d\theta\, d\phi \\
&\sim \int_0^{\infty}\frac{1}{r^2}dr
\end{aligned}
$$

- Energy needed to create a point charge diverges according to classical electrodynamics.

Energy of a Uniformly Charged Spherical Shell

- What is the energy needed to create a uniformly charged spherical shell of radius R and charge Q?

$$
\begin{aligned}
W &= \frac{\varepsilon_0}{2}\int_{r=0}^{\infty} E^2 d\tau \\
&= \frac{\varepsilon_0}{2}\left[\int_{r=0}^{R} E^2 d\tau + \int_{r=R}^{\infty} E^2 d\tau\right] \\
&= \frac{\varepsilon_0}{2}\left[0 + \int_{r=R}^{\infty} \frac{Q^2}{(4\pi\varepsilon_0)^2 r^4} r^2 \sin\theta\, dr d\theta d\phi\right] \\
&= \frac{Q^2}{8\pi\varepsilon_0 R}
\end{aligned}
$$

- Will we obtain the same result if we use the initial expression $\frac{1}{2}\int V(\vec{r}')\rho(\vec{r}')d\tau'$?

Energy of a Uniformly Charged Spherical Shell

$$
\begin{aligned}
W &= \frac{1}{2}\oint_S V(\vec{r}\,')\sigma(\vec{r}\,')da' \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^{\pi} \frac{Q}{4\pi\varepsilon_0 R}\cdot\frac{Q}{4\pi R^2} R^2 \sin\theta \, d\theta \, d\phi \\
&= \frac{Q^2}{8\pi\varepsilon_0 R}
\end{aligned}
$$

- Yes, we have obtained the same result through both the methods.

Energy of a Uniformly Charged Solid Sphere

- Energy of a Solid Sphere can be obtained in multiple ways.
  - Using the energy formula in terms of Electric field.

$$W = \frac{1}{2}\varepsilon_0 \int E^2 d\tau$$

  - Using the energy formula in terms of Electrostatic Potential.

$$W = \frac{1}{2}\int V(\vec{r}')\rho(\vec{r}')d\tau'$$

  - A charge dq is brought from infinity and smeared over a sphere of charge q and radius r, to obtain a sphere of charge $q + dq$ and radius $r + dr$. Energy required in the process is,

$$\begin{aligned} dW &= V(r)dq = \frac{q(r)}{4\pi\varepsilon_0 r}dq \\ &= \frac{1}{4\pi\varepsilon_0 r}\cdot Q\frac{r^3}{R^3}\cdot Q\frac{3r^2}{R^3}dr \end{aligned}$$

- All the above methods lead to the same answer, $W = \frac{3Q^2}{20\pi\varepsilon_0 R}$.

Some Comments

- Energy of a charge distribution doesn't follow the principle of superposition because of its non-linear (quadratic) dependence on $\vec{E}$, i.e., $W_{1+2} \neq W_1 + W_2$, for two charge distributions ρ_1 and ρ_2, giving rise to electric fields $\vec{E}_1$ and $\vec{E}_2$

$$W_{1+2} = \frac{1}{2}\varepsilon_0 \int |\vec{E}_1 + \vec{E}_2|^2 d\tau \quad \neq \quad \frac{1}{2}\varepsilon_0 \int \left[E_1^2 + E_2^2\right] d\tau = W_1 + W_2$$

- What does the each term mean in W_{1+2}?

$$W_{1+2} \quad = \quad \frac{1}{2}\varepsilon_0 \int \left[E_1^2 + E_2^2 + 2\vec{E}_1 \cdot \vec{E}_2\right] d\tau$$

  - The first and second terms represent the electrostatic self energy of charge distributions 1 and 2, respectively.
  - The third term represents the interaction energy between those charge distributions.

- Where is the energy stored?
    - All over the space in form of electrostatic field?

$$W = \frac{1}{2}\varepsilon_0 \int E^2 d\tau$$

    - All over the charge distribution as potential energy of charge?

$$W = \frac{1}{2}\int V(\vec{r}')\rho(\vec{r}')d\tau'$$

    - Both the views work as far as electrostatics is concerned.

Some Comments

- There is a slight difference between energy of discrete and continuous charge distribution.
  - In the discrete charge distribution, we assume that the point charges exist initially and we calculate just the energy required to assemble the configuration.
  - In the continuous charge distribution, we calaculate the total energy stored in the charge distribution.
  - In the formula for discrete case, $V(\vec{r}_i')$ is the potential due to all charges except q_i.
  - Whereas in continuous case, $V(\vec{r}')$ is the potential due to the whole distribution.
  - Energy required to assemble the configuration(Interaction energy) might be negative.
  - Total energy(Self energy+Interaction energy) in the charge distribution is always positive.