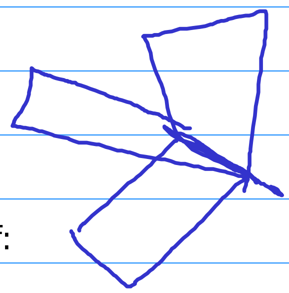
MA106: Tut0

Q1. $\hat{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$ $\hat{b} = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$ $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}}_{M} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cdot & - & \cdot \\ \cdot & - & \cdot \end{vmatrix}$$



$$\det(M) = 0$$

Recall Cramer's Rule: A solution in the case D=0, exists iff:

$$D_x = D_y = D_z = 0$$

$$D_x = \begin{vmatrix} b_x & -a_z & a_y \\ b_y & 0 & -a_x \\ b_z & a_x & 0 \end{vmatrix} = 0$$

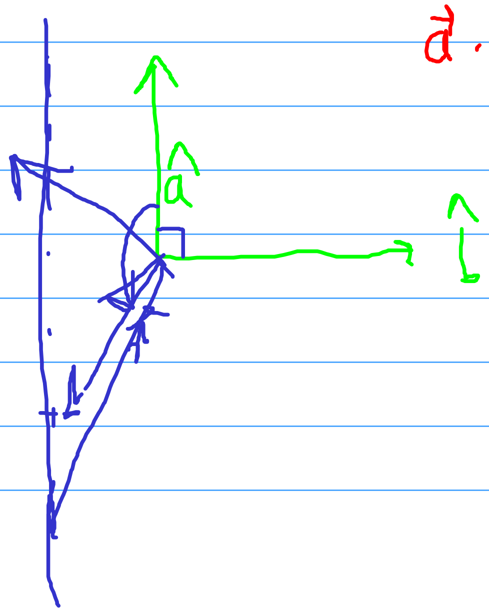
$$\Rightarrow \quad a_x(a_x b_x + a_y b_y + a_z b_z) = 0$$

$$a_y(\quad \cdots \quad) = 0$$

$$a_z(\quad \cdots \quad) = 0$$

$$(a_x b_x + a_y b_y + a_z b_z) = 0$$

$$\vec{a} \cdot \vec{b} = 0$$



$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ x & y & z \end{vmatrix}$$

$$= \hat{i}(a_y z - a_z y)$$
$$+ \hat{j}(a_z x - a_x z)$$
$$+ \hat{k}(a_x y - a_y x)$$
$$= b_x \hat{i} + b_y \hat{j} + b$$

Q2.

$$A(\theta) = \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix}$$

short forms
c = cos, s = sin

Notice:

$$A(\theta_1) \cdot A(\theta_2) = A(\theta_1 + \theta_2)$$

Proof using cos (x+y) = cosx cosy - sinx siny
sin(x+y) = sinx cosy + siny cosx

Also det(A) = 1, so A is invertible

Suppose {p, Ap, A^2p, A^3p ...} is finite: then there must be repetition of elements

If: for i < j :

$$A^i p = A^j p$$

$$\text{Then,} \quad A^{i+k} p = A^{j+k} p$$

$$A^i, A^{i+1}, A^{i+2}, \ldots, A^j, A^{j+1}, A^{j+2}$$

So, set becomes:

$$\{p, Ap, A^2p, \ldots, A^{j-1}p\}$$

$$A^{j-i} P = P \qquad A^n(\theta) = A(n\theta)$$

$$(j-i)\theta = \phi$$

$$A^{j-i}(\theta) = A((j-i)\theta) = A(\phi)$$

$$\begin{bmatrix} c\phi & -s\phi \\ s\phi & c\phi \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{cases} (1-c)x = sy \\ -sx = (1-c)y \end{cases}$$

If $c = 1, s = 0 \rightarrow$ Good

Else, $\dfrac{x}{y} = \dfrac{s}{1-c} = \dfrac{1-c}{-s}$

$$-s^2 = 1 + c^2 - 2c$$

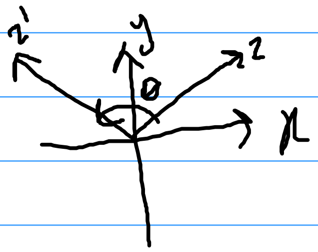
$$\rightarrow c = 1 \quad \times$$

Hence,

$$n = j-i$$

$$\phi = 2k\pi$$

$$\rightarrow \theta = \frac{2k\pi}{n} \quad \text{for some } k \in \mathbb{Z},\ n \in \mathbb{N}$$



Q3.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$[x \; y \; z] A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 3$$

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + (\underbrace{a_{12} + a_{21}}_{})^{2a_{12}} xy$$

$$+ (a_{23} + a_{32}) yz + (a_{13} + a_{31}) = 3$$

For symmetric case, just compare coefficients:

$$a_{11} = 1 \quad a_{22} = 1 \quad a_{33} = -1$$

$$a_{12} = a_{21} = \frac{Z}{2}, \quad a_{23} = a_{32} = -\frac{3}{2}$$

$$a_{13} = a_{31} = 3$$

In not necessarily symmetric case, not unique

Q4. $A = I - uu^T$

Class 12: $|A| = 0 \leftarrow$ Check

Naive & Painful

[Need to use $\|u\|^2 = 1$]

Better way: Use fundamentals: A is invertible if there is a B such that:

$$BA = AB = I \qquad AC = BA = I$$

Notice:

$$Au = u - u(u^T u) = (I - uu^T)u$$
$$= u - u \quad [|u|^2 = u^T u = 1]$$
$$= 0$$

$$\text{so, } BAu = B0 = 0$$
$$BA = I$$
$$\rightarrow u = 0 \quad \times \text{ Contradiction}$$

$$(I - 2uu^T)x$$

Hence, A is not invertible

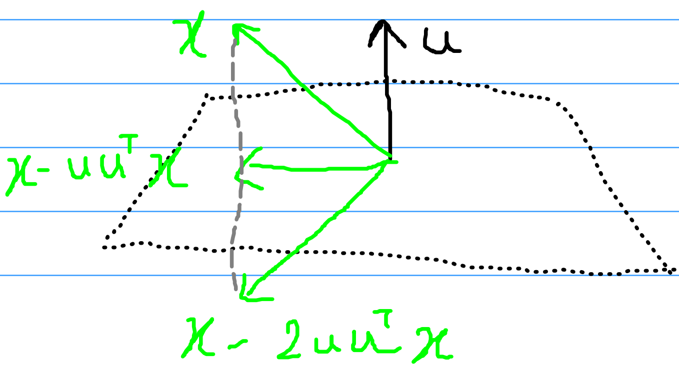
Geometry:

$u^T x$ = Dot product of u and x : the projection of x along u

$u(u^T x)$ = Component of x along u

$x - uu^T x$ = Component of x perpendicular to u

$x - 2uu^T x$ = Mirror image of x on plane perpendicular to u



$$u^T u = \begin{bmatrix} u_x & u_y & u_z \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$$

$$= u_x^2 + u_y^2 + u_z^2 = |u|^2$$

$$= 1$$

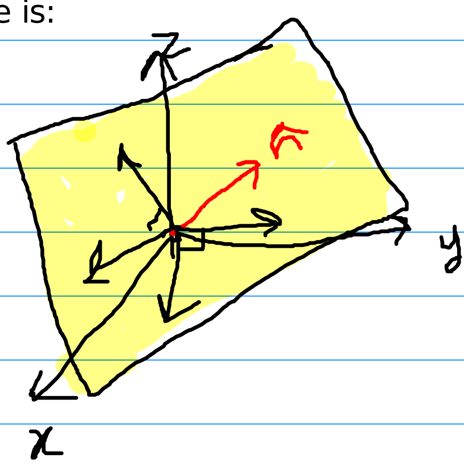
$u^T x =$

$x - 2u(u^T x)$

$\hat{n} \cdot \vec{r} = d$

Q5. Plane : x+y+z = 0, so Normal vector to plane is:

$$\hat{n} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$



WLOG: take u to have x component = 0

$$\hat{u} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Solve u.n = 0

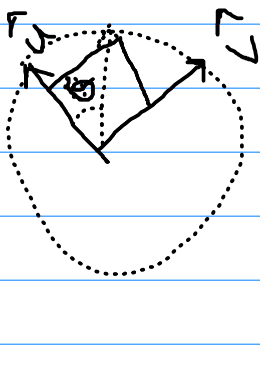
To find v, we can take it to be along the cross product of u and n, that gives:

$$\hat{v} = \frac{1}{\sqrt{6}} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

The circle will be formed by the intersection of the given unit sphere with the plane. Since the plane passes through the center of the circle, hence the circle will have radius and center equal to that of the sphere and will lie in the plane

Param : $\theta \in [0, 2\pi)$

$$c(\theta) = \hat{u}\cos\theta + \hat{v}\sin\theta$$



$x, y \in R^3 \quad x \perp y \qquad z = x \times y$

$x, y \in R^n \qquad\qquad z \quad n-2$ dimensional

$n > 3$

The cross product is defined only for n=3 and n=7

$$z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} \qquad x \cdot z = 0 \quad y \cdot z = 0 \qquad z = 0 \quad z = kx + y$$

$z \quad 1D$

$x + y + z = d.$

$x^2 + y^2 + z^2 = r \qquad \frac{d}{\sqrt{3}} < r$