MA 108 - Ordinary Differential Equations

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Lerch's theorem - Statement

In this lecture, we discuss properties of Laplace transforms. We begin with the following result.

Theorem : Suppose $f, g : [0, \infty) \to \mathbb{R}$ are piecewise continuous functions such that $L(f)(s) = L(g)(s)$, $s > \alpha$. Then $f(t) = g(t)$ at all points of continuity of f and g.
If $L(f)(s) = F(s), s > \alpha, \alpha \in \mathbb{R}$, then we define the inverse Laplace trasform as $L^{-1}(F)(t) = f(t), t \geq 0$ which are points of continuous of f.

Let's now see how to solve IVP's using Laplace transforms.
Solve

$$y'' - y' - 2y = 0, y(0) = 1, y'(0) = 0.$$

Apply Laplace transform through out to obtain :

$$L(y'') - L(y') - 2L(y) = 0.$$

Look for a solution y such that
y, y' are of exponential order.
Thus,

$$(s^2 L(y) - sy(0) - y'(0)) - (sL(y) - y(0)) - 2L(y) = 0$$

$$\implies (s^2 - s - 2)L(y) - s + 1 = 0.$$

So,

$$L(y)(s) = \frac{s-1}{(s-2)(s+1)} = \frac{1/3}{s-2} + \frac{2/3}{s+1}.$$

The rhs is

$$\frac{1}{3}L(e^{2t}) + \frac{2}{3}L(e^{-t}).$$

Thus,

$$y = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}.$$

Remark: If you had done this via characteristic equation, first you would have got

$$y(t) = c_1 e^{2t} + c_2 e^{-t},$$

and you would evaluate c_1, c_2 from the constraints.

Example

Solve $y'' - 2y' + 5y = 8\sin t - 4\cos t,\; y(0) = 1,\; y'(0) = 3.$
Taking Laplace transforms,

$$L(y'') - 2L(y') + 5L(y) = \frac{8}{s^2+1} - \frac{4s}{s^2+1}$$

$$(s^2L(y) - sy(0) - y'(0)) - 2(sL(y) - y(0)) + 5L(y) = \frac{8-4s}{s^2+1}.$$

Using the initial conditions,

$$L(y)(s^2-2s+5)-s-3+2 = \frac{8-4s}{s^2+1} \implies L(y)(s^2-2s+5) = \frac{4(2-s)}{s^2+1}+s+1$$

This yields

$$L(y) = \frac{s^3+s^2-3s+9}{(s^2-2s+5)(s^2+1)}.$$

That is, $y = L^{-1}\left(\frac{s^3+s^2-3s+9}{(s^2-2s+5)(s^2+1)}\right).$

Example contd..

$$\frac{s^3+s^2-3s+9}{(s^2-2s+5)(s^2+1)} = \frac{As+B}{s^2+1} + \frac{C(s-1)+D}{(s-1)^2+2^2}$$

That is,

$$(As+B)(s^2-2s+5) + (C(s-1)+D)(s^2+1) = s^3+s^2-3s+9.$$

This yields, $A = 0,\ B = 2,\ C = 1,\ D = 0.$
Hence,

$$y = L^{-1}\left(\frac{2}{s^2+1}\right) + L^{-1}\left(\frac{s-1}{(s-1)^2+2^2}\right) = 2\sin t + e^t \cos 2t.$$

Property 5: Integration

Let f be piecewise continuous and suppose there exist $K,\ t_0 \geq 0$ and $\alpha \geq 0$ such that

$$|f(t)| \leq Ke^{\alpha t},$$

for $t \geq t_0$. Also, let $L(f)(s) = F(s), s > \alpha$. Then,

$$L\left(\int_0^t f(\tau)\, d\tau\right)(s) = \frac{F(s)}{s}, \qquad \text{for } s > \alpha.$$

Proof: We need to show that

$$L\left(\int_0^t f(\tau)\, d\tau\right)(s) = \frac{1}{s} L(f)(s),$$

for $s > \alpha$. Set

$$g(t) = \int_0^t f(\tau) d\tau.$$

Then, $g'(t) = f(t)$, except at the points of discontinuities of $f(t)$.

Hence $g'(t)$ is piecewise continuous. Hence,

$$L(f)(s) = L(g')(s) = sL(g) - g(0) = sL(g),$$

for $s > \alpha$. Thus,

$$L\left(\int_0^t f(\tau)\, d\tau\right)(s) = \frac{1}{s} L(f)(s).$$

Example

If $L(f) = \dfrac{1}{s(s^2+\omega^2)}$, find $f(t)$.

Solution:

We know,

$$L\left(\frac{\sin \omega t}{\omega}\right)(s) = \frac{1}{s^2+\omega^2}, s > 0.$$

$$g(t) = \frac{\sin \omega t}{\omega}, t \geq 0$$

is continuous and is with exponential growth.

Hence using property 5,

$$L\Big(\int_0^t \frac{\sin \omega \tau}{\omega} d\tau\Big)(s) = \frac{1}{s(s^2+\omega^2)}, s > 0.$$

So

$$f(t) \quad = \quad \int_0^t \frac{\sin \omega \tau}{\omega}\, d\tau = -\frac{\cos \omega \tau}{\omega^2}\Big|_0^t = \frac{1}{\omega^2}(1 - \cos \omega t).$$

Example

$$\mathcal{L}^{-1}\left(\frac{1}{s^2(s^2+\omega^2)}\right) = \mathcal{L}^{-1}\left(\frac{1}{s}\frac{1}{s(s^2+\omega^2)}\right)$$
$$= \int_0^t \frac{1-\cos\omega\tau}{\omega^2}\,d\tau = \frac{1}{\omega^2}\left(\tau - \frac{\sin\omega\tau}{\omega}\right)_0^t = \frac{1}{\omega^2}\left(t - \frac{\sin\omega t}{\omega}\right).$$

Property 6 : Differentiation of Laplace transforms

Suppose $f\colon [0,\infty) \to \mathbb{R}$ is piecewise continuous and of exponential order and let $L(f(t))(s) = F(s), s > \alpha$. Then,

$$F'(s) = -L(tf(t))(s), s > \alpha.$$

Also,

$$L(t^n f(t))(s) = (-1)^n F^{(n)}(s), s > \alpha.$$

Proof.

$$F(s) = \int_0^\infty e^{-st} f(t)dt.$$

Then,

$$\begin{aligned}
\frac{dF(s)}{ds} &= \int_0^\infty \frac{\partial}{\partial s}(e^{-st})f(t)dt \\
&= \int_0^\infty -te^{-st}f(t)dt \\
&= -L(tf(t))(s).
\end{aligned}$$

Exercise : Prove for nth derivative.

Examples

We know that

$$L(\cos\beta t) = \frac{s}{s^2+\beta^2},\ \ L(\sin\beta t) = \frac{\beta}{s^2+\beta^2}, s > 0.$$

Therefore, using

$$F'(s) = -L(tf(t)),$$

$$L(t\cos\beta t)(s) = -\frac{d}{ds}\left(\frac{s}{s^2+\beta^2}\right) = \frac{s^2-\beta^2}{(s^2+\beta^2)^2},$$

and

$$L(t\sin\beta t)(s) = -\frac{d}{ds}\left(\frac{\beta}{s^2+\beta^2}\right) = \frac{2s\beta}{(s^2+\beta^2)^2}, s > 0.$$

Examples

$$L(t\sin\beta t)(s) = -\frac{d}{ds}\left(\frac{\beta}{s^2+\beta^2}\right) = \frac{2s\beta}{(s^2+\beta^2)^2}, s > 0.$$

Thus,

$$\frac{s}{(s^2+\beta^2)^2} = L\Big(\frac{t\sin\beta t}{2\beta}\Big)(s).$$

Thus, from Property 5 $\left(L\left(\int_0^t f(\tau)\,d\tau\right) = \frac{F(s)}{s},\ \text{for } s > \alpha\right)$

$$\frac{1}{(s^2+\beta^2)^2} = L\Big(\int_0^t \frac{\tau\sin\beta\tau}{2\beta}d\tau\Big)(s), s > 0$$

and from Property 4 $\left(L(f') = sL(f) - f(0)\right)$,

$$L\Big(\frac{d}{dt}\left(\frac{t\sin\beta t}{2\beta}\right)\Big)(s) = \frac{s^2}{(s^2+\beta^2)^2}, s > 0.$$

Suppose $f\colon [0,\infty) \to \mathbb{R}$ is piecewise continuous of exponential order. Suppose further that $\lim_{t\to 0^+} \frac{f(t)}{t}$ exists. Then,

$$L\left(\frac{f(t)}{t}\right)(s) = \int_s^\infty F(\tilde{s})\, d\tilde{s}, \quad s > \alpha.$$

Proof:

$$\begin{aligned}
\int_s^\infty F(\tilde{s})\, d\tilde{s} &= \int_s^\infty \left(\int_0^\infty e^{-\tilde{s}t} f(t)dt\right) d\tilde{s} \\
&= \int_0^\infty \int_s^\infty e^{-\tilde{s}t} f(t) d\tilde{s} dt \\
&= \int_0^\infty f(t) \left[\frac{e^{-\tilde{s}t}}{-t}\right]_s^\infty dt \\
&= \int_0^\infty e^{-st} \frac{f(t)}{t} dt = L\left(\frac{f(t)}{t}\right)(s).
\end{aligned}$$

Example

Find L^{-1} of $\ln(1+\frac{\omega^2}{s^2})$.

We have:

$$\ln\left(1+\frac{\omega^2}{s^2}\right) = -\int_s^\infty \frac{d}{d\tilde{s}}\left(\ln\left(1+\frac{\omega^2}{\tilde{s}^2}\right)\right)\,d\tilde{s}.$$

$$\begin{aligned} -\frac{d}{ds}\left(\ln(1+\frac{\omega^2}{s^2})\right) &= \frac{2\omega^2}{s(s^2+\omega^2)} \\ &= \frac{2}{s} - \frac{2s}{s^2+\omega^2} \\ &:= F(s), s>0. \end{aligned}$$

Now

$$f(t) = L^{-1}(F(s))(t) = 2 - 2\cos\omega t, t \geq 0.$$

$$\lim_{t \to 0+} \frac{f(t)}{t} = 0.$$

From Property 7,

$$\begin{aligned} L^{-1}\left(\ln(1+\frac{\omega^2}{s^2})\right) &= L^{-1}\left(\int_s^\infty F(\tilde{s})\, d\tilde{s}\right) \\ &= \frac{f(t)}{t}, \quad s > \alpha \\ &= \frac{2 - 2\cos\omega t}{t}. \end{aligned}$$

Exercises

(1) $L^{-1}\left(ln\,(1-\frac{a^2}{s^2})\right)$

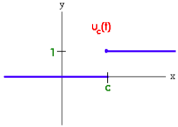
(2) $L^{-1}(\tan^{-1}(\frac{1}{s}))$.

Heaviside function

For $c \geq 0$, the function

$$u_c(t) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t \geq c \end{cases}$$

is called the Heaviside function.



$$L(u_c(t))(s) = \frac{e^{-cs}}{s}.$$

$$\begin{aligned} L(u_c(t))(s) &= \int_0^{\infty} e^{-st} u_c(t)\,dt \\ &= \int_c^{\infty} e^{-st}\,dt \\ &= \frac{e^{-cs}}{s}, \end{aligned}$$

for $s > 0$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Consider the new function

$$g(t) = \begin{cases} 0 & \text{if } t < c \\ f(t-c) & \text{if } t \geq c. \end{cases}$$

Note that

$$g(t) = u_c(t) f(t-c).$$

Property 8 : II Shifting theorem

Suppose $L(f(t)) = F(s)$ for $s > a \geq 0$. If $c > 0$, then for $s > a$,

$$L(u_c(t)f(t-c)) = e^{-cs}F(s).$$

Proof:

$$\begin{aligned}
L(u_c(t)f(t-c)) &= \int_0^\infty e^{-st}u_c(t)f(t-c)dt \\
&= \int_c^\infty e^{-st}f(t-c)dt \\
&= \int_0^\infty e^{-s(u+c)}f(u)du \\
&= e^{-cs}F(s).
\end{aligned}$$

Example

Find the Laplace transform of

$$f(t) = \begin{cases} \sin t & 0 \leq t < \frac{\pi}{4} \\ \sin t + \cos(t - \frac{\pi}{4}) & t \geq \frac{\pi}{4}. \end{cases}$$

Write

$$f(t) = \sin t + u_{\frac{\pi}{4}}(t) \cdot \cos(t - \frac{\pi}{4}).$$

Hence,

$$\begin{aligned} L(f(t)) &= L(\sin t) + L(u_{\frac{\pi}{4}}(t) \cdot \cos(t - \frac{\pi}{4})) \\ &= \frac{1}{s^2+1} + e^{-\frac{\pi}{4}s} \cdot \frac{s}{s^2+1} \\ &= \frac{1 + e^{-\frac{\pi}{4}s}s}{s^2+1}. \end{aligned}$$

(use $L(u_c(t)f(t-c)) = e^{-cs}F(s)$).

Convolution of functions

The convolution of $f(t)$ and $g(t)$ is defined as:

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)\, d\tau.$$

Check:

1. $f * g = g * f$ (Put $y = t - \tau$.)
2. $f * (g_1 + g_2) = f * g_1 + f * g_2$
3. $(f * g) * h = f * (g * h)$
4. $f * 0 = 0 * f = 0$.

Remark: $f * 1$ need not be f.
Check that $\sin t * 1 = 1 - \cos t$.

Suppose $L(f)$ and $L(g)$ exist for all $s > a \geq 0$. Then,

$$L(f * g) = L(f) \cdot L(g),$$

for $s > a$.

Proof: Let $L(f) = F(s)$ and $L(g) = G(s)$. Fix $\tau \geq 0$.

$$\begin{aligned} e^{-s\tau} G(s) &= L\left(u_\tau(t) g(t-\tau)\right)(s) \text{ using II shifting theorem} \\ &= \int_0^\infty e^{-st} u_\tau(t) g(t-\tau)\, dt \\ &= \int_\tau^\infty e^{-st} g(t-\tau)\, dt. \end{aligned}$$

$$\begin{aligned} L(f)(s)\, L(g)(s) &= F(s)\, G(s) = \left(\int_0^\infty e^{-s\tau} f(\tau)\, d\tau\right) G(s) \\ &= \int_0^\infty e^{-s\tau} G(s) f(\tau)\, d\tau \\ &= \int_0^\infty f(\tau) \left(\int_\tau^\infty e^{-st} g(t-\tau) dt\right) d\tau \end{aligned}$$

That is,

$$\begin{aligned}
L(f)(s)\,L(g)(s) &= \int_0^\infty f(\tau)\left(\int_\tau^\infty e^{-st}g(t-\tau)dt\right)\,d\tau \\
&= \int_0^\infty e^{-st}\left(\int_0^t f(\tau)g(t-\tau)d\tau\right)\,dt \\
&= \int_0^\infty e^{-st}(f*g)(t)dt \\
&= L(f*g)(s).
\end{aligned}$$

Example

Find L^{-1} of

$$F(s) = \frac{a}{s^2(s^2 + a^2)}.$$

Recall

$$L(t) = \frac{1}{s^2},$$

and

$$L(\sin at) = \frac{a}{s^2 + a^2}.$$

Thus,

$$L(t * \sin at) = F(s).$$

Now,

$$t * \sin at = \int_0^t (t - \tau) \sin a\tau \; d\tau = \frac{at - \sin at}{a^2}.$$

Example

Solve the IVP:

$$y'' + 4y = g(t), \;\; y(0) = 3, y'(0) = -1.$$

Taking Laplace transforms:

$$L(y'') + 4L(y) = L(g) = G(s).$$

Thus,

$$s^2 L(y) - sy(0) - y'(0) + 4L(y) = G(s).$$

Therefore,

$$\begin{aligned} L(y) &= \frac{3s-1}{s^2+4} + \frac{G(s)}{s^2+4} \\ &= 3 \cdot \frac{s}{s^2+4} - \frac{1}{2} \cdot \frac{2}{s^2+4} + \frac{1}{2} \cdot \frac{2}{s^2+4} \cdot G(s) \\ &= 3L(\cos 2t) - \frac{1}{2} L(\sin 2t) + \frac{1}{2} L(\sin 2t) \cdot L(g) \\ &= 3L(\cos 2t) - \frac{1}{2} L(\sin 2t) + \frac{1}{2} L(\sin 2t * g). \end{aligned}$$

Hence,

$$y = 3\cos 2t - \frac{1}{2}\sin 2t + \frac{1}{2}\int_0^t \sin 2(t-x) g(x)\, dx.$$