MA-111 Calculus II
(D1 & D2)

Lecture 8

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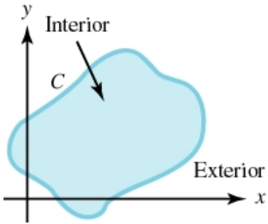
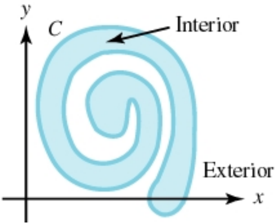
Jordan curve theorem

This is a celebrated theorem in topology:

Theorem

If $\mathbf{c} : [a, b] \to \mathbb{R}^2$ *is a simple closed path then* $\mathbb{R}^2 - \mathbf{c}\big([a, b]\big)$ *is divided in two connected parts, 'interior' and 'exterior', such that any path from one of them to the other would have to intersect* $\mathbf{c}\big([a, b]\big)$.





The bounded part is called the **interior** of the curve and the unbounded part is called the **exterior** of the curve.

Orientation of the boundary of an enclosed region in plane

By Jordan's theorem, a simply closed curve C encloses a bounded region D in the plane. There is a natural notion of positive orientation of the region D - clearly it is given by the vector field $\mathbf{k}$ - the unit normal vector pointing in the direction of the positive z axis.

A curve C can be obtained as the boundary of a region D in the plane, but C may now consists of several components or pieces and D may have "holes".

Then how to define the orientation of the boundary curve C?

• The goal is now to obtain a two-dimensional analog of the Fundamental theorem of calculus to express a double integral over a 'plane region D' as a line integral along the closed curve which is the boundary of D. This is the content of Green's theorem.

• **Positive orientation of a simple closed curve**
By convention, *the positive orientation* of a simple closed curve on a plane corresponds to the anti-clockwise direction and the *negative orientation* of a simple closed curve on a plane corresponds to the clock-wise direction.

• **Positive orientation of the boundary of a region**
The boundary curve C of a bounded region D in $\mathbb{R}^2$ is *positively oriented* if the region D always lies to the left of an observer walking along the curve in the chosen direction. otherwise, we say that the curve is negatively oriented.

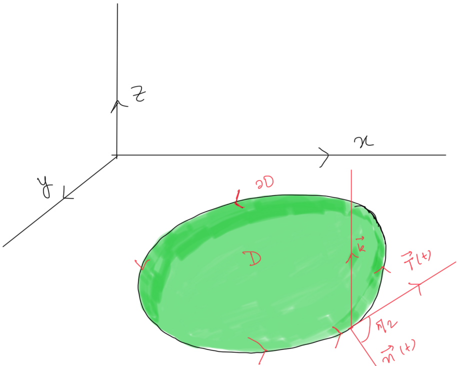
Orienting the boundary curve

Definition: The positive orientation of a curve C in $\mathbb{R}^2$ is given by the vector field

$$\mathbf{k} \times \mathbf{n}_{\text{out}},$$

where $\mathbf{n}_{\text{out}}$ is the unit normal vector field pointing outward along the curve.

Though the planar curve C lies in $\mathbb{R}^2$, here the path is parametrized as $\mathbf{c}(t) = (x(t), y(t), 0)$ for $t \in [a, b]$ with range in $\mathbb{R}^3$, taking 0 in the 3rd component.



Orienting the boundary curve contd.

Physically, this means that if one walks along C in the direction of the positive orientation, the region D is always on one's left.

As we shall see later, if C is a closed curve in space bounding an oriented surface S, the orientation of S naturally induces an orientation on the boundary C. The above example is a special case of this.

Example: Simple closed curve

Positively oriented curve

Ex. $\gamma(\theta) = (\cos\theta, \sin\theta, 0)$, $\theta \in [0, 2\pi]$. Then $\gamma'(\theta) = (-\sin\theta, \cos\theta, 0)$ and $\mathbf{n}_{\text{out}}(\theta) = (\cos\theta, \sin\theta, 0)$.
Note

$$\mathbf{k} \times \mathbf{n}_{\text{out}}(\theta) = (-\sin\theta, \cos\theta, 0) = \gamma'(\theta),$$

and so the curve is positively oriented.

Negatively Oriented curve

Ex. $\gamma_1(\theta) = (\cos\theta, -\sin\theta, 0)$, $\theta \in [0, 2\pi]$. Then $\gamma_1'(\theta) = (-\sin\theta, -\cos\theta, 0)$ and $\mathbf{n}_{\text{out}}(\theta) = (\cos\theta, -\sin\theta, 0)$.
Note

$$\mathbf{k} \times \mathbf{n}_{\text{out}}(\theta) = (\sin\theta, \cos\theta, 0) = -\gamma_1'(\theta),$$

and so the curve is negatively oriented.

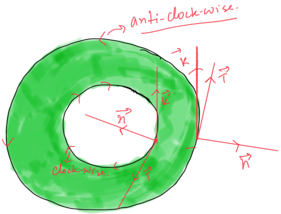
Example: Orientation of boundary of a region with hole

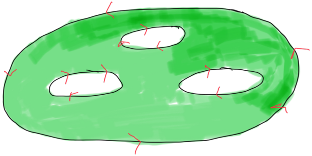
Annulus: For $D = \{(x, y) \in \mathbb{R}^2 \mid a^2 \leq x^2 + y^2 \leq b^2\}$, the boundary $\partial D = C_1 \cup C_2$, where C_1 is the outer boundary $C_1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = b^2\}$ and C_2 is the inner boundary $C_2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = a^2\}$.
what is the positive orientation of ∂D, boundary of D?

C_1 oriented anti-clockwise: $\mathbf{c}_1(\theta) = (b\cos\theta, b\sin\theta, 0)$ for $\theta \in [0, 2\pi]$ and $\mathbf{n}_{\text{out}}(\theta) = (b\cos\theta, b\sin\theta, 0)$ at C_1 and $\mathbf{c}_1'(\theta) = \mathbf{k} \times \mathbf{n}_{\text{out}}(\theta)$.

C_2 oriented clockwise $\mathbf{c}_2(\theta) = (a\cos\theta, -a\sin\theta, 0)$ for $\theta \in [0, 2\pi]$ and $\mathbf{n}_{\text{out}}(\theta) = (-a\cos\theta, a\sin\theta, 0)$ at C_2 and $\mathbf{c}_2'(\theta) = \mathbf{k} \times \mathbf{n}_{\text{out}}(\theta)$.

Then the outer boundary curve is given the anti-clockwise orientation, while the inner boundary curves are oriented in the clockwise direction.





Positive-orientation:

Outer boundary anti-clock-wise.
Inner boundary clock-wise.

Green's Theorem

With the preliminaries out of the way, we are now in a position to state the first major theorem of vector calculus, namely Green's Theorem.

Theorem (Green's theorem:)

1. *Let D be a bounded region in $\mathbb{R}^2$ with a positively oriented boundary ∂D consisting of a finite number of non-intersecting simple closed piecewise continuously differentiable curves.*
2. *Let Ω be an open set in $\mathbb{R}^2$ such that $\left(D \cup \partial D\right) \subset \Omega$ and let $F_1 : \Omega \to \mathbb{R}$ and $F_2 : \Omega \to \mathbb{R}$ be $\mathcal{C}^1$ functions. Consider the vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j}$.*

Then

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \int_{\partial D} F_1 dx + F_2 dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy.$$

The importance of Green's theorem is that it converts a double integral into a line integral. Depending on the situation, one may be easier to evaluate than the other.