Tut 7

Q1 (a) Suppose I-A is not invertible. Then $\exists x \neq 0$ such that

$$(I-A)x = 0$$
$$\rightarrow \quad x = Ax$$
$$\rightarrow \quad Ax = A^2 x$$
$$\vdots$$
$$A^{k-1}x = A^k x = 0$$
$$\therefore x = 0 \qquad \text{Contradiction}$$

Explicitely constructing inverse:

$$(I-A)(\underbrace{I + A + A^2 + \dots + A^{k-1}}) = I$$
$$= I - A^k$$

(b) If λ is an eigenvalue, $\exists x \neq 0$

$$Ax = \lambda x$$
$$\rightarrow A^2 x = \lambda A x = \lambda^2 x$$
$$\vdots$$
$$A^k x = \lambda^k x = 0$$
$$\hookrightarrow \lambda = 0$$

$\therefore$ Char eqn: $x^n = 0$

(c) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow$ rank $= 3 \rightarrow$ nullity $= 1$

(d) If commuting: $A^{k_1} = 0 \quad B^{k_2} = 0, \quad k = \max(k_1, k_2)$

$$(AB)^k = A^k B^k = 0$$

$(AB)^2 = ABAB = A^2B^2$

If non-commuting:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ Non commuting

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ Nilpotent

Product is not nilpotent :

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Aside: How do check if $A_{n\times n}$ is nilpotent i.e. atmost how many k's do you need to check

Lemma: If a matrix $A_{n\times n}$ is nilpotent then, $\boxed{A^n = 0}$

Proof: Recall Schur's theorem: every matrix is unitarily upper triangularizable

$$\therefore \; A = \underbrace{U}_{\text{}} \, V \, U^T$$

unitary

upper $\triangle$

Moreover, the diagonal entries of the upper triangular matrix obtained are the eigenvalues of A with their geometric multiplities

(Proof: it has the same eigenvalues of A. Now look at its characteristic equation)

$$\therefore \quad V = \begin{bmatrix} 0 & & & * \\ & 0 & & \\ & & 0 & \\ 0 & & & \ddots \\ & & & & 0 \end{bmatrix}_{n\times n}$$

$$A^2 = U V^2 U^T$$

Now notice that V^2 has the diagonal besides the principle diagonal also as all zeroes. In general, we can easily prove (by induction), that V^k will have zeroes till the $(k-1)^{th}$ beside the principle

$$V^2 = \begin{bmatrix} 0 & 0 & & * \\ & 0 & 0 & \\ & & 0 & \ddots \\ 0 & & & 0 \\ & & & & 0 \end{bmatrix} \qquad V^k = \begin{bmatrix} 0 & \cdots & 0 & * \\ & 0 & & 0 \\ & & 0 & \ddots \\ 0 & & & 0 \end{bmatrix} \Big\} k$$

Hence,

$$\underline{V^n = 0}$$

$UV(U^T U)V(U^T) \cdots$

$$\therefore \; \underline{A^n} = U V^n U^T = \underline{0}$$

Q2 (a) $P^2 = P$

$$(I-P)^2 = I^2 - 2P + P^2 = I - 2P + P = I - P$$

(b) $\exists x \neq 0, \lambda \in \mathbb{C}$:

$$Px = \lambda x$$
$$\to P^2 x = \lambda^2 x = Px = \lambda x$$
$$\to \lambda(\lambda - 1) x = 0$$
$$\to \lambda = 0, 1$$

(c) $P^2 = P$

If invertible:

$$P^{-1} P^2 = P^{-1} P \to P = I$$

(d) Suppose :

$$\exists\, \alpha_{k+1}, \alpha_{k+2}, \dots, \alpha_n \text{ s.t.}$$

$$\alpha_{k+1} P v_{k+1} + \alpha_{k+2} P v_{k+2} + \dots = 0$$

$$\to P(\alpha_{k+1} v_{k+1} + \alpha_{k+2} v_{k+2} + \dots) = 0$$

Hence $(\alpha_k v_k + \alpha_{k+1} v_{k+1} + \dots)$ must belong to the null space

Since $v_1, v_2, \dots v_n$ is a basis, this is possible iff $\alpha_{k+1} \dots = \alpha_n = 0$

Everything till here was quite general, noting specific to P being idempotent. Now here's the trick:

$$P^2 v_{k+1} = P v_{k+1} \quad \text{as} \quad P^2 = P$$
$P \cdot P$ $\quad$ and $P v_{k+1} \neq 0$

Hence, $P v_{k+1}, \dots, P v_n$ are linearly independent eigenvectors of P for the eigenvalue 1

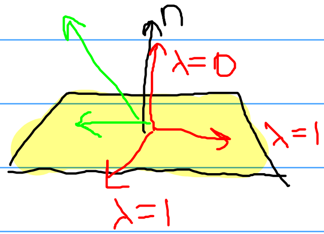
So, P has n linearly independent eigenvectors; the k basis vectors of the null space for eigenvalue 0 and $P v_{k+1}, \dots, P v_n$ for eigenvalue 1. So it is diagonalizable.

Q3. $H_0 = I - nn^T$

$$H_0 x = \lambda x$$
$$\rightarrow x - nn^T x = \lambda x$$
$$\rightarrow n(n^T x) = x(1-\lambda)$$

Hence, $n^T x = 0$ or $x \parallel n$

$\downarrow$ $\lambda = 1$ $\downarrow$ $\lambda = 0$



diagonalizable

Also from prev. Q

It is also idempotent

$$H_0^2 = (I - nn^T)(I - nn^T)$$
$$= I - 2nn^T + n(n^T n)n^T$$
$$= I - nn^T = H_0$$

Geometrically, this is because, H_0 projects any vector into the plane, and then acts on any vector in a plane trivially

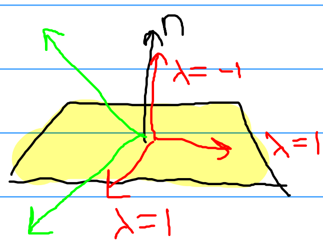
$H = I - 2nn^T$

diagonalizable

$$2n(n^T x) = x(1-\lambda)$$

Hence $n^T x = 0$ or $x \parallel n$

$\downarrow$ $\lambda = -1$ $\downarrow$ $\lambda = 0$



$H = 2H_0 - I$

In general, if X is diagonalizable so is $\alpha X + \beta I$

$$P^{-1} X P$$
$$P^{-1}(\alpha X + \beta I) P$$
$$= \alpha \underbrace{P^{-1} X P} + \underbrace{\beta I}$$

Not idempotent:

$$H^2 = (I - 2nn^T)(I - 2nn^T)$$
$$= I - 4nn^T + 4n(n^T n)n^T = I \neq H$$

Q4. $A = \begin{bmatrix} 3 & 0 & 0 \\ e & 2 & 0 \\ 10^8 & 10^2 & 1 \end{bmatrix}$ $(x-3)(x-2)(x-1)=0$ $B = \begin{bmatrix} 2 & 10^5 & 10^9 \\ 0 & 1 & \pi \\ 0 & 0 & 3 \end{bmatrix}$ $\begin{bmatrix} 1 & & \\ & 2 & \\ & & 3 \end{bmatrix} = D$

These two are clearly diagonalizable matrices as they have 3 distinct eigenvalues.
They will be similar iff they are similar to there is a diagonal matrix that they are both similar to.

$P^{-1}AP = D$
$= Q^{-1}BQ$
$\rightarrow A = PQ^{-1}BQP^{-1}$
$= (QP^{-1})^{-1} B (QP^{-1})$

Now, both of these are similar to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, hence they are similar to each other

Q5 Suppose λ is a real eigenvalue of the skew symmetric matrix A

$\exists x \neq 0 \quad Ax = \lambda x$
$\rightarrow x^T A^T = \lambda x^T$
$\rightarrow x^T A^T x = \lambda x^T x$

Also,
$x^T A x = \lambda x^T x$
$\therefore x^T (A + A^T) x = 2\lambda x^T x = 0$
$\rightarrow \lambda = 0$

So, for odd order skew symmetric real matrix, atleast one real eigenval exists, so it must be so det is 0

Hence, a skew symmetric matrix can only have 0 as a real eigenvalue. Complex eigenvalues are in general are purely imaginary:

$\exists x \neq 0 \quad Ax = \lambda x \rightarrow x^* A^* = \bar{\lambda} x^*$
$\hookrightarrow x^* A x = \lambda x^* x \qquad \hookrightarrow x^* A^* x = \bar{\lambda} x^* x$
Since $A^* = A^T = -A \rightarrow (\lambda + \bar{\lambda}) x^* x = 0$ $\quad \|x\|^2 > 0$
$\rightarrow \lambda + \bar{\lambda} = 0$

Skew symmetric matrices are not diagonalizable using real eigenvectors. However a skew symmetric matrix with real entries allowing complex eigenvectors is normal, hence unitarily diagonalizable.

$AA^* = AA^T = -A^2 = A^T A$
$= A^* A$

Q6 $f : R^3 \to R^3$, $f(0) = 0$, $\|f(x) - f(y)\| = \|x - y\|$

Put $y=0$

$$\|f(x)\| = \|x\| \quad \forall x$$

Hence,

$$\|f(x) - f(y)\|^2 = \|x - y\|^2$$

$$\to \cancel{\|f(x)\|^2} + \cancel{\|f(y)\|^2} - 2f(x)^T f(y) = \cancel{\|x\|^2} + \cancel{\|y\|^2} - 2x^T y$$

$$\to f(x)^T f(y) = x^T y$$

Consider e_1, e_2, e_3 as the orthonormal basis of R^3

$\hat{i}, \hat{j}, \hat{k}$

$f(e_1)^T f(e_2) = 0$ and so on

Also, $\|f(e_1)\| = \|e_1\| = 1 \cdots$

Let: $g_1 = f(e_1)$, $g_2 = f(e_2)$, $g_3 = f(e_3)$

these form an orthonormal basis for R^3

For $x = a_1 e_1 + a_2 e_2 + a_3 e_3$

$$f(x)^T g_1 = a_1 \cdots = x^T e_1$$

$$g_2 = a_2$$

$\vdots$

Hence:

$$f(x) = a_1 g_1 + a_2 g_2 + a_3 g_3$$

Clearly:

$$f(x+y) = f(x) + f(y)$$

$$f(\alpha x) = \alpha f(x)$$

Hence, f(x) is linear, so f(x)=Ax for some A

Given that: $f(x) = Ax$

$\rightarrow \|A(x-y)\| = \|x-y\|$

put $z = x-y$

$\rightarrow \|Az\| = \|z\|$ $\quad \|z\|^2 = z^T z$

square both sides:

$\rightarrow z^T A^T A z = z^T z$

$\rightarrow z^T(A^T A - I)z = 0 \quad \forall z$

Now, $(A^T A - I)$ is a real symmetric matrix, hence, it is orthogonally diagonalizable. If λ is some eigenvalue and z is the corresponding eigenvector, then:

$$(A^T A - I)z = \lambda z$$

$$\therefore \lambda z^T z = 0$$

$$\therefore \lambda = 0$$

Hence, all eigenvalues are 0. So $(A^T A - I)$ is similar to the diagonal matrix O, so

$$(A^T A - I) = 0 \qquad P^{-1} O P = 0$$

$$\rightarrow A^T A = I \checkmark$$

Hence, A is orthogonal ✓