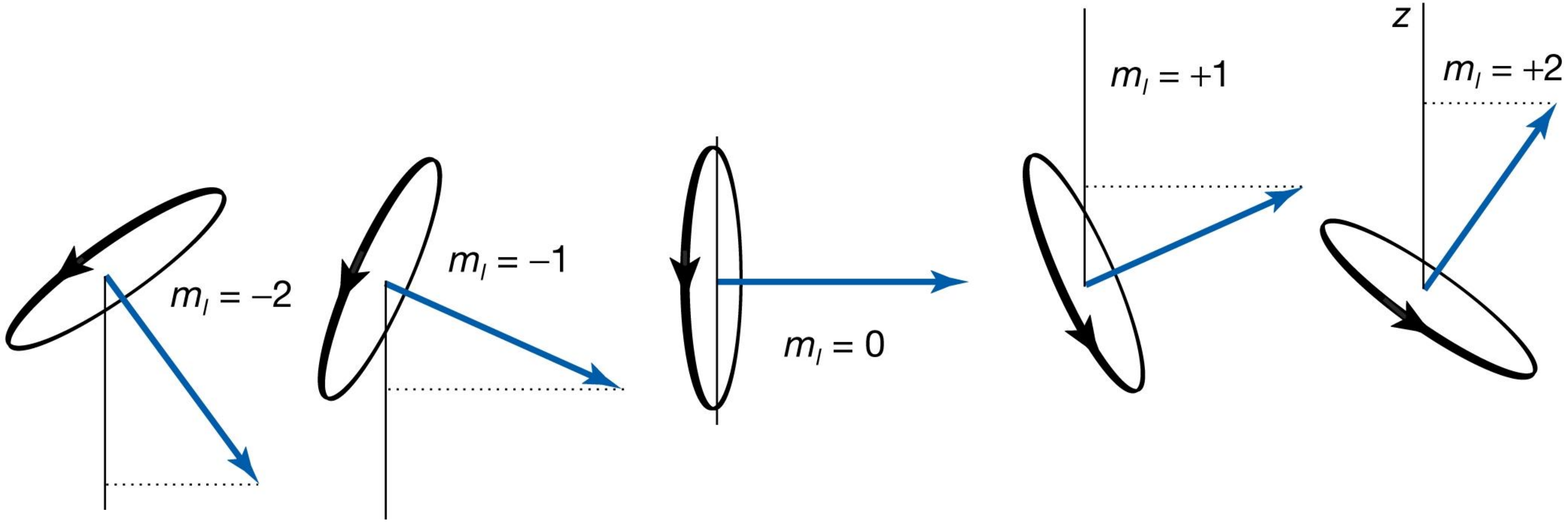
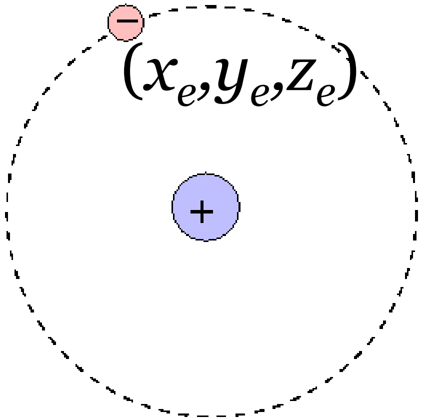
Hydrogen Atom: Magnetic and Azimuthal quantum numbers



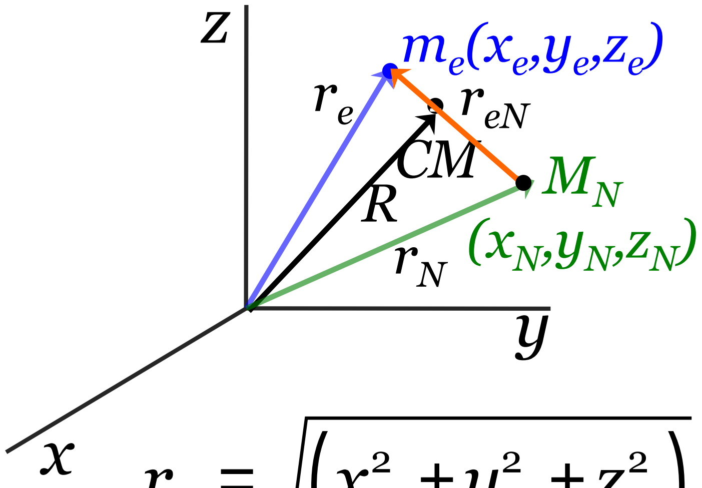
Atkins' Physical Chemistry

Hydrogen Atom: Relative Frame of Reference



$$\left(-\frac{\hbar^2}{2m_N}\nabla_N^2-\frac{\hbar^2}{2m_e}\nabla_e^2-\frac{QZe^2}{r_{eN}}\right)\Psi_{Total}=E_{Total}\cdot\Psi_{Total}$$

Separation of $\hat{H}$ into Center of Mass and Internal co-ordinates



$$r_N=\sqrt{\left(x_N^2+y_N^2+z_N^2\right)}$$

$$r_e=\sqrt{\left(x_e^2+y_e^2+z_e^2\right)}$$

$$x=x_e-x_N$$

$$y=y_e-y_N$$

$$z=z_e-z_N$$

$$r=r_{eN}=r_e-r_N$$

$$=\sqrt{\left(x^2+y^2+z^2\right)}$$

$$X=\frac{m_e x_e+m_N x_n}{m_e+m_N}$$

$$Y=\frac{m_e y_e+m_N y_n}{m_e+m_N}$$

$$Z=\frac{m_e z_e+m_N z_n}{m_e+m_N}$$

$$R=\frac{m_e r_e+m_N r_N}{m_e+m_N}$$

Hydrogen Atom: Separation of CM motion

$$\left(-\frac{\hbar^2}{2M}\nabla_R^2 - \frac{\hbar^2}{2\mu}\nabla_r^2 - \frac{QZe^2}{r}\right)\Psi_{Total} = E_{Total}\cdot\Psi_{Total}$$

$$\hat{H} = \hat{H}_N + \hat{H}_e \qquad \Psi_{Total} = \chi_N \times \psi_e \qquad E_{Total} = E_N + E_e$$

$$\hat{H}_N\chi_N = \left(-\frac{\hbar^2}{2M}\nabla_R^2\right)\chi_N = E_N\chi_N$$

Free particle!
Kinetic energy of the atom

$$E_N = \frac{\hbar^2 k^2}{2M}$$

Hydrogen Atom: Electronic Hamiltonian

$$\widehat{H}_e \cdot \psi_e = \left(-\frac{\hbar^2}{2\mu} \nabla_r^2 - \frac{QZe^2}{r} \right) \psi_e = E_e \cdot \psi_e$$

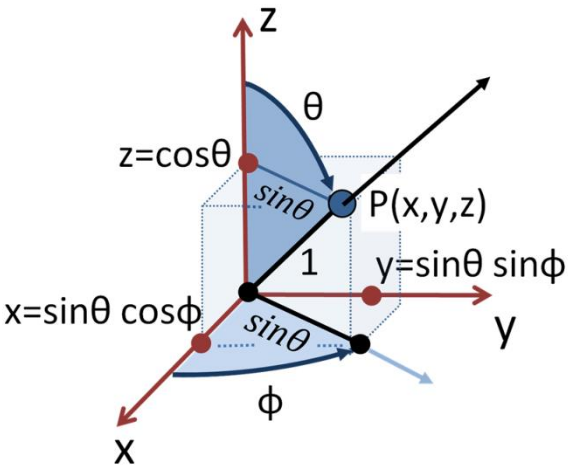
$$y_e \text{ Þ } y_e(x,y,z)$$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\frac{QZe^2}{\sqrt{\left(x^2 + y^2 + z^2\right)}} \psi_e(x,y,z)$$

Not possible to separate out into three different co-ordinates.
Need a new co-ordinate system

Spherical Polar Co-ordinates

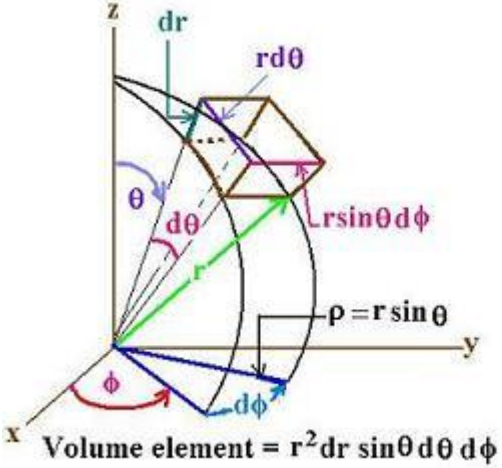


$$z = r\cos q$$
$$x = r\sin q\cos f$$
$$y = r\sin q\sin f$$

r: 0 to ∞

θ: 0 to π

ϕ: 0 to 2π



$$r = \sqrt{\left(x^2 + y^2 + z^2\right)}$$
$$\theta = \cos^{-1}\left(\frac{z}{r}\right)$$
$$\phi = \tan{-1}\left(\frac{y}{x}\right)$$

$$dt = r^2 \times dr \times \sin q \times dq \times df$$

Separation of variables

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{2\mu r Q Z e^2}{\hbar^2}+\frac{2\mu r^2}{\hbar^2}E_e=\beta$$

Radial equation

$$\frac{\sin\theta}{\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\beta\sin^2\theta=m^2$$

Angular equation

Separation of variables

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{2\mu rQZe^2}{\hbar^2}+\frac{2\mu r^2}{\hbar^2}E_e=\beta$$

$$\frac{\sin\theta}{\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\beta\sin^2\theta=m^2$$

$$\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2}=-m^2$$

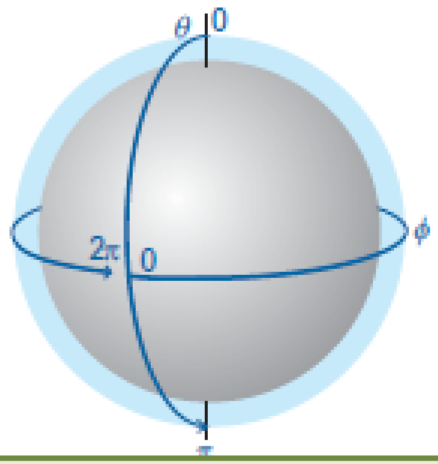
The three variables r, θ and ϕ are separated

Solution to ϕ part

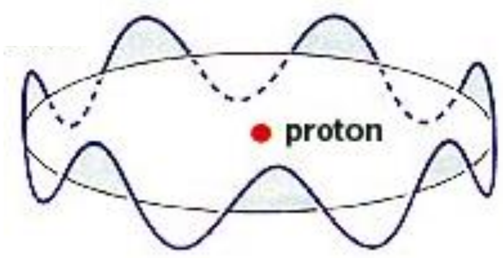
$$\frac{1}{\Phi(\phi)}\frac{\partial^2\Phi(\phi)}{\partial\phi^2}+m^2=0 \quad\Rightarrow\quad \frac{\partial^2\Phi(\phi)}{\partial\phi^2}=-m^2\Phi(\phi)$$

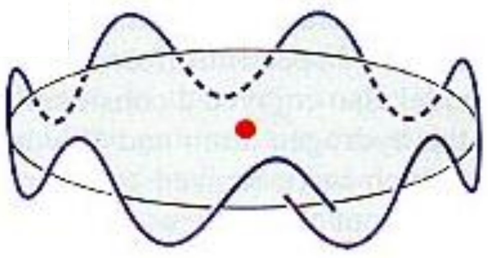
Trial solution: $\Phi(\phi)=Ae^{\pm im\phi}$

$$\frac{\partial\Phi}{\partial\phi}=\pm im\Phi$$



‘ϕ’ ranges from 0 to 2π





Wavefunction has to be single-valued

$$\Rightarrow \Phi(\phi+2\pi)=\Phi(\phi)$$

Periodic Boundary Condition

z-component of angular momentum

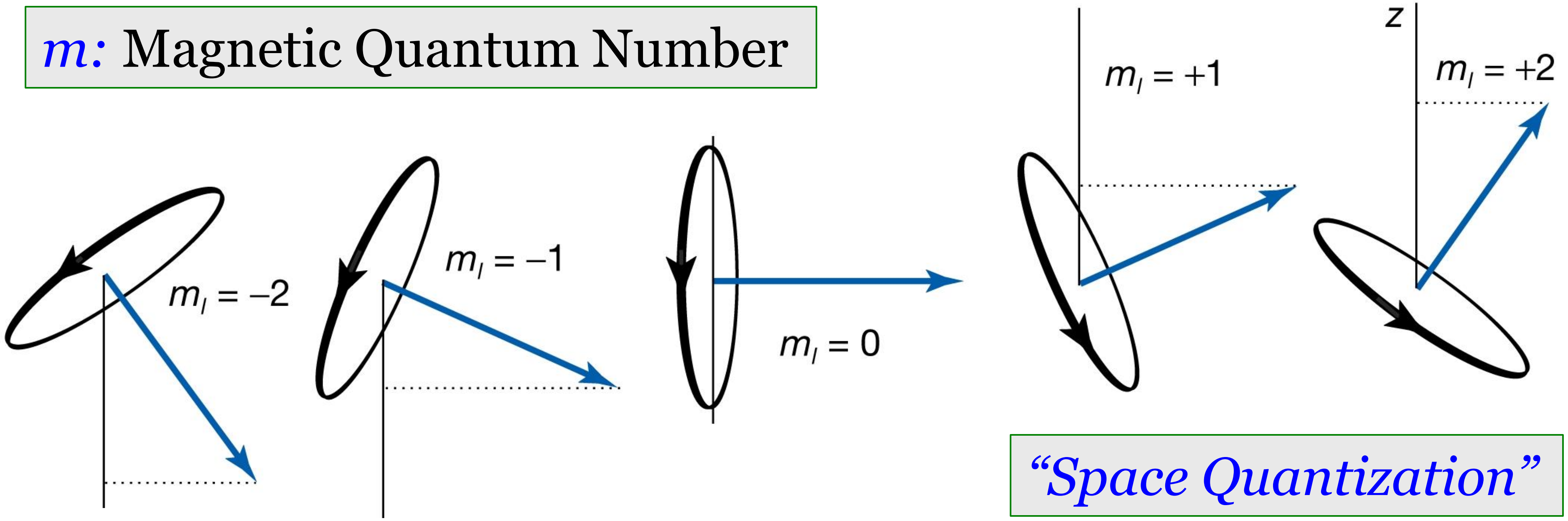
$$\widehat{L_z} = \frac{\hbar}{i}\frac{\partial}{\partial \phi}$$

$$\Phi(\phi) = Ae^{\pm im\phi}$$

$$\widehat{L_z}\Phi = \frac{\hbar}{i}\frac{\partial}{\partial \phi}\Phi = \frac{\hbar}{i}\, im\, \Phi = \boxed{m\hbar}\,\Phi$$

z-component of angular momentum

m: Magnetic Quantum Number



"Space Quantization"

Solution to ϕ part: Magnetic quantum number

- m=0, ±1, ±2, ±3, ±4,....

- m is the **magnetic quantum number**

- m is restricted by another quantum number (orbital Angular momentum), l, such that $|m|<l$

$$\frac{\sin\theta}{\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\beta\sin^2\theta=m^2$$

$$\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2}=-m^2$$

The Θ part

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{\partial\theta}\right) - \frac{m^2}{\sin^2\theta}\Theta(\theta) + \beta\Theta(\theta) = 0$$

$$P_l^m(\cos\theta) = \frac{(-1)^m}{2^l l!}(1-\cos^2\theta)^{m/2}\frac{d^{l+m}}{dx^{l+m}}(\cos^2\theta - 1)^l$$

$\Theta(\theta) =$

$$P_l^{-m}(\cos\theta) = (-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(\cos\theta)$$

$$\boxed{\beta = l(l+1)}$$

$P_l^m(\cos\theta)$: Associated Legendre Polynomials

Azimuthal quantum number l = *0,1,2,3…,*

$m \le l$

The angular (Θ·Φ) part

The angular part of the solution

$Y_l^m(\theta,\phi) \Rightarrow \Theta(\theta)\cdot\Phi(\phi)$ are called spherical harmonics

$$Y_l^m(q,f) = \sqrt{\frac{(2l+1)}{4p}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos q)e^{imf}$$

l=0,1,2,3…

m=0, ±1, ±2, ±3… and $|m| \leq l$

The angular ($\Theta\cdot\Phi$) part:

$$\vec{L} = \left(yp_z - zp_y\right)\vec{i} + \left(xp_z - zp_{yx}\right)\vec{j} + \left(xp_y - yp_x\right)k$$

$$\widehat{L^2} = \widehat{L_x}.\widehat{L_x} + \widehat{L_y}.\widehat{L_y} + \widehat{L_z}.\widehat{L_z}$$

$$= -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$ in spherical polar co-ords.

Angular equation:

$$-\left[\frac{1}{\Theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + \frac{1}{\Phi}\frac{1}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}\right] = \beta$$

Multiply by

$$Y(\theta,\phi) = \Theta\Phi$$

The angular (Θ·Φ) part:

$$\vec{L} = \left(yp_z - zp_y\right)\vec{\imath} + \left(xp_z - zp_{yx}\right)\vec{\jmath} + \left(xp_y - yp_x\right)k$$

$$\widehat{L^2} = \widehat{L_x}.\widehat{L_x} + \widehat{L_y}.\widehat{L_y} + \widehat{L_z}.\widehat{L_z}$$

$$= -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$ in spherical polar co-ords.

Angular equation:

$$-\left[\frac{1}{\Theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + \frac{1}{\Phi}\frac{1}{\sin^2\theta}\frac{\partial^2\Phi}{\partial f^2}\right] = b$$

Multiply by

$$Y(\theta,\phi) = \Theta\Phi$$

$$-\left[\frac{\Phi}{\sin\theta}\frac{\partial\Theta}{\partial\theta}\sin\theta\frac{\partial\Theta}{\partial\theta} + \frac{\Theta}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}\right] = \beta\,\Phi\Theta$$

The angular (Θ·Φ) part:

$$\vec{L} = \left(yp_z - zp_y\right)\vec{\imath} + \left(xp_z - zp_{yx}\right)\vec{\jmath} + \left(xp_y - yp_x\right)k$$

$$\widehat{L^2} = \widehat{L_x}.\widehat{L_x} + \widehat{L_y}.\widehat{L_y} + \widehat{L_z}.\widehat{L_z}$$

$$= -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$ in spherical polar co-ords.

Angular equation:

$$-\left[\frac{1}{\Theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + \frac{1}{\Phi}\frac{1}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}\right] = \beta$$

Multiply by

$Y(\theta,\phi) = \Theta\Phi$

$$-\hbar^2\left[\frac{\Phi}{\sin\theta}\frac{\partial\Theta}{\partial\theta}\sin\theta\frac{\partial\Theta}{\partial\theta} + \frac{\Theta}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}\right] = \hbar^2\beta\,\Phi\Theta$$

The angular (Θ·Φ) part:

$$\vec{L} = \left(yp_z - zp_y\right)\vec{\imath} + \left(xp_z - zp_{yx}\right)\vec{\jmath} + \left(xp_y - yp_x\right)k$$

$$\widehat{L^2} = \widehat{L_x}.\widehat{L_x} + \widehat{L_y}.\widehat{L_y} + \widehat{L_z}.\widehat{L_z}$$

$$= -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$ in spherical polar co-ords.

Angular equation:

$$-\left[\frac{1}{\Theta}\frac{1}{\sin q}\frac{\partial}{\partial q}\left(\sin q\frac{\partial\Theta}{\partial q}\right) + \frac{1}{\Phi}\frac{1}{\sin^2 q}\frac{\partial^2\Phi}{\partial f^2}\right] = b$$

Multiply by

$Y(\theta,\phi) = \Theta\Phi$

$$-\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\Phi\Theta = \hbar^2\beta\,\Phi\Theta$$

The angular (Θ·Φ) part:Total angular momentum

$$\vec{L} = (yp_z - zp_y)\vec{i} + (xp_z - zp_{yx})\vec{j} + (xp_y - yp_x)k$$

$$\widehat{L^2} = \widehat{L_x}.\widehat{L_x} + \widehat{L_y}.\widehat{L_y} + \widehat{L_z}.\widehat{L_z}$$

$$= -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

in spherical polar co-ords.

Angular equation:

$$-\left[\frac{1}{\Theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + \frac{1}{\Phi}\frac{1}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}\right] = \beta$$

Multiply by

$$Y(\theta,\phi) = \Theta\Phi$$

$$-\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]Y(\theta,\phi) = \hbar^2\beta Y(\theta,\phi)$$

$$\widehat{L^2}Y(\theta,\phi) = \hbar^2 l(l+1)Y(\theta,\phi)$$

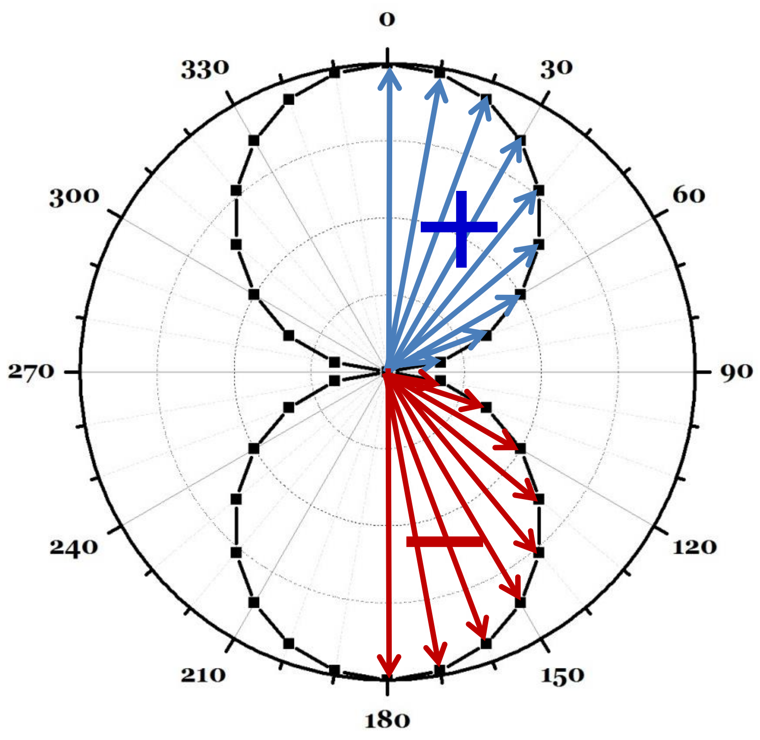
Angular Distribution Functions

p-Orbitals

$$\psi_{210} = \psi_{2p_z} = N\rho e^{-\rho/2}\cos\theta$$

$$Y_{210} = Y_{2p_z} = C\cos q \quad (m = 0)$$

Angular part: Polar plot of $2p_z$ --- $\cos\theta$



θ	cosθ
0	1.000
10	0.985
20	0.940
30	0.866
40	0.766
50	0.643
60	0.500
70	0.342
80	0.174
90	0.000
120	-0.500
150	-0.866
180	-1.000

Hydrogen Atom: Principal quantum number

$n = 4$ $l = 0, 1, 2, 3$

$n = 3$ $l = 0, 1, 2$

$m = 0, \pm 1, \pm 2, ...$

$n = 2$ $l = 0, 1$

$n = 1$ $l = 0$

Separation of variables for Schrodinger equation

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{2\mu rQZe^2}{\hbar^2}+\frac{2\mu r^2}{\hbar^2}E_e=\beta$$

$$\frac{\sin\theta}{\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\beta\sin^2\theta=m^2$$

$$\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2}=-m^2$$

ϕ-part: Magnetic quantum number

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{2\mu r QZe^2}{\hbar^2}+\frac{2\mu r^2}{\hbar^2}E_e=\beta$$

$$\frac{\sin\theta}{\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\beta\sin^2\theta=m^2$$

$$\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2}=-m^2$$

$$\Phi(\phi)=Ae^{\pm im\phi}$$

$$\widehat{L_z}\Phi=\frac{\hbar}{i}\frac{\partial}{\partial\phi}\Phi=\frac{\hbar}{i}\,im\,\Phi=m\hbar\,\Phi$$

z-component of angular momentum

m: Magnetic Quantum Number

θ-part: Azimuthal quantum number

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{2\mu rQZe^2}{\hbar^2}+\frac{2\mu r^2}{\hbar^2}E_e=\beta$$

$$\frac{\sin\theta}{\Theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\beta\sin^2\theta=m^2$$

$$\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2}=-m^2$$

$\Theta(\theta)=$

$$P_l^m(\cos\theta)=\frac{(-1)^m}{2^l l!}(1-\cos^2\theta)^{m/2}\frac{d^{l+m}}{dx^{l+m}}(\cos^2\theta-1)^l$$

$$P_l^{-m}(\cos\theta)=(-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(\cos\theta)$$

Associated Legendre Polynomials

Azimuthal quantum number

$l = 0,1,2,3\ldots,$

$m\leq l$

$$\beta=l(l+1)$$

$$\widehat{L^2}Y(\theta,\phi)=\boxed{\hbar^2 l(l+1)}Y(\theta,\phi)$$

r- part

$$\boxed{\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{2\mu r QZe^2}{\hbar^2}+\frac{2\mu r^2}{\hbar^2}E_e=\beta}$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R(r)}{\partial r}\right)+\frac{2\mu r^2}{\hbar^2}\left(\frac{QZe^2}{r}+E_e\right)R(r)-\beta R(r)=0$$

Solution to R(r) are

$$a=\frac{\hbar^2}{Q\mu e^2}=\frac{4\pi\varepsilon_o\hbar^2}{\mu e^2}$$

$$R_{nl}(r)=-\left[\frac{(n-l-1)!}{2n\left[(n+l)!\right]^3}\right]^{1/2}\left(\frac{2Z}{na}\right)^{l+3/2} r^l e^{-Zr/na_o} L_{n+l}^{2l+1}\left(\frac{2Zr}{na}\right)$$

Restriction on $l<n$

Where $L_{n+l}^{2l+1}\left(\frac{2Zr}{na_o}\right)$ are called associated *Laguerre* functions

The new quantum number is 'n' called principal quantum number

Energy of the Hydrogen Atom

$$E_n = -\frac{2Q^2 Z^2 \mu e^4}{\hbar^2 n^2} = -\frac{Z^2 \mu e^4}{8\varepsilon_o^2 h^2 n^2} = -\frac{Z^2 e^4}{8\pi\varepsilon_o a_o n^2} \left(\mu \approx m_e\right)$$

$$E_n = \frac{-13.6eV}{n^2}$$

Energy is dependent only on ‘*n*’

Energy obtained by full quantum mechanical treatment is equal to Bohr energy

Potential energy term is only dependent on the ***Radial*** part and has no contribution from the ***Angular*** parts

Quantum Numbers of Hydrogen Atom

n	**Principal Quantum number** Specifies the energy of the electron
l	**Orbital Angular Momentum Quantum number** Specifies the magnitude of the electron's orbital angular momentum
m	**Z-component of Angular Momentum Quantum number** Specifies the orientation of the electron's orbital angular momentum
S	**Spin Angular Momentum Quantum number** Specifies the orientation of the electron's spin angular momentum

Radial Functions of Hydrogen Atom

$$R_{nl}(r) = -\left[\frac{(n-l-1)!}{2n[(n+l)!]^3}\right]^{1/2}\left(\frac{2}{na_o}\right)^{l+3/2} r^l e^{-r/na_o} L_{n+l}^{2l+1}\left(\frac{2r}{na_o}\right)$$

$n=1; l=0 \quad 2\left(\frac{1}{a_o}\right)^{3/2} e^{-r/a_o}$

$n=2; l=0 \quad \frac{1}{8^{1/2}}\left(\frac{1}{a_o}\right)^{3/2}\left(2-\frac{r}{a_o}\right)e^{-r/2a_o}$

$n=2; l=1 \quad \frac{1}{24^{1/2}}\left(\frac{1}{a_o}\right)^{3/2}\left(\frac{r}{a_o}\right)e^{-r/2a_o}$

$n=3; l=0 \quad 2\left(\frac{1}{3a_o}\right)^{3/2}\left(1-\frac{2}{3}\left[\frac{r}{a_o}\right]-\frac{2}{27}\left[\frac{r}{a_o}\right]^2\right)e^{-r/3a_o}$

$n=3; l=1 \quad \frac{1}{486^{1/2}}\left(\frac{1}{a_o}\right)^{3/2}\left(4-\frac{2r}{3a_o}\right)e^{-r/3a_o}$

$n=3; l=2 \quad \frac{1}{2430^{1/2}}\left(\frac{1}{a_o}\right)^{3/2}\left(\frac{2r}{3a_o}\right)^2 e^{-r/3a_o}$

$$\rho = \frac{2Zr}{na}$$

$$a = \frac{4\pi\varepsilon_o\hbar^2}{\mu e^2}$$

$$a = a_o \quad (\text{for } \mu = m_e)$$

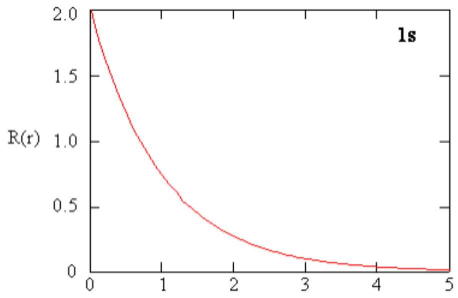
Number of radial nodes =

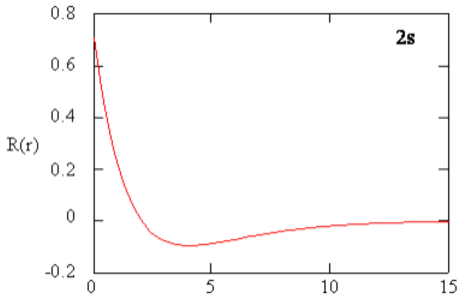
n-l-1

1s and 2s Orbitals

$$\psi_{1,0,0} = \psi_{1s} = \frac{1}{\sqrt{\pi}}\left(\frac{1}{a_o}\right)^{3/2} e^{-r/a_o}$$

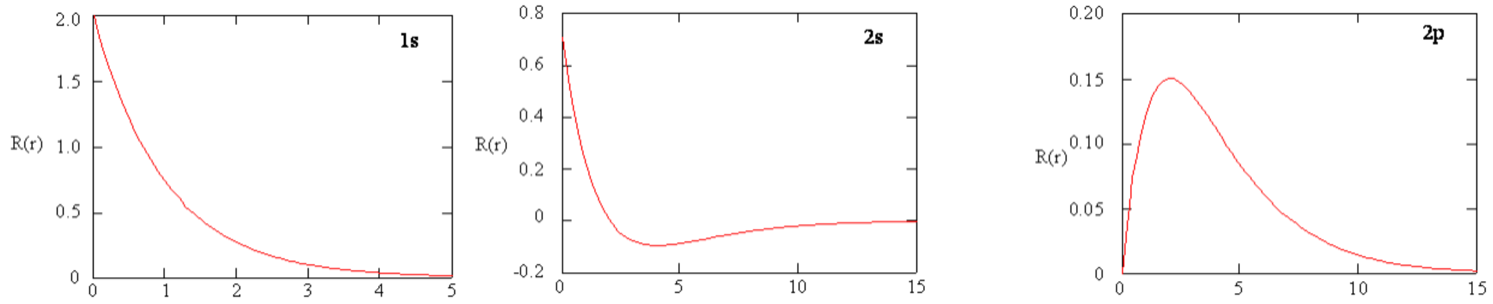
$$\psi_{2,0,0} = \psi_{2s} = \frac{1}{4\sqrt{2\pi}}\left(\frac{1}{a_o}\right)^{3/2}\left(2-\frac{r}{a_o}\right)e^{-r/2a_o}$$





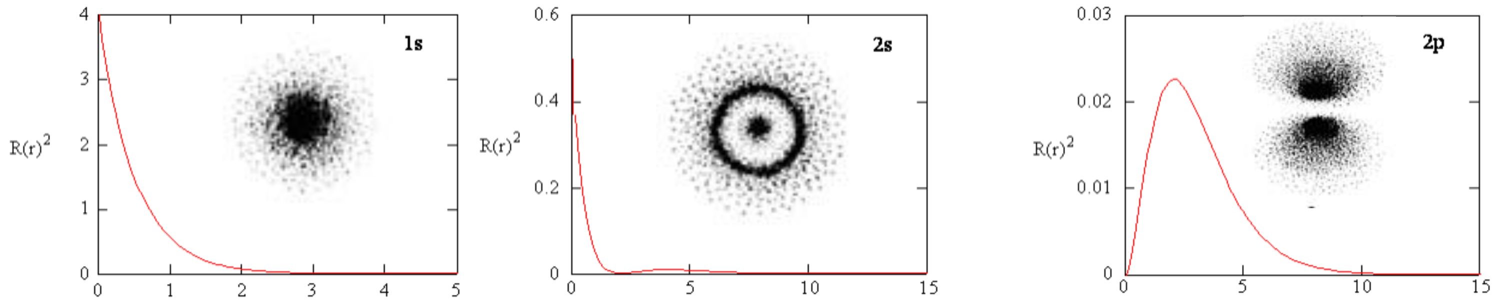
Functions of only 'r'

Radial functions



$$\rho = \frac{r}{a_o}$$

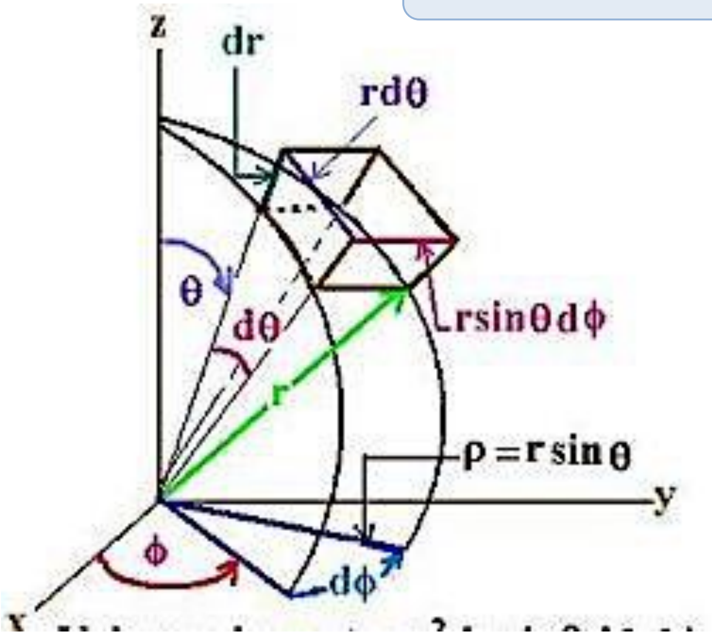
$$\psi_{1s}^{100} = N' e^{-\rho} \qquad \psi_{2s}^{200} = N''(2-\rho)e^{-\rho/2} \qquad \psi_{2p_z}^{210} = N''' \rho e^{-\rho/2}$$



For s-Orbitals the maximum probability denisty of finding the electron is on the nucleus

Probability: Volume element in spherical polar coordinates

$$Volume\ element = r^2 dr\ sin\theta d\theta\ d\phi$$



For a *s* orbital

$$P = \int \psi^* \psi d\tau$$

2

2π

$$= \int_0^\infty R^2 r^2 dr \int_0^\pi \Theta^2 sin\ \theta d\theta \int_0^{2\pi} \Phi^* \Phi d\phi$$

Radial Probability distribution function

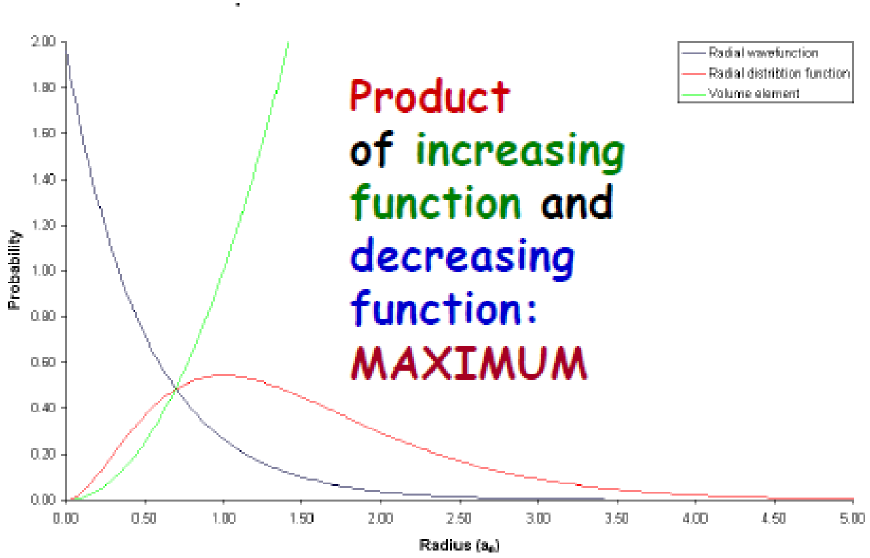
Radial Distribution Functions

Probability of finding the electron in a shell of thickness dr at radius r =

$4pr^2R_{nl}^2(r)dr$ (for s)

$r^2 \rightarrow$ increasing function

$4pr^2R_{nl}^2(r)dr \rightarrow 0$ as $4pr^2dr \rightarrow 0$



For s-Orbitals :

- Maximum probability denisty of finding the electron is on the nucleus
- Probability of finding the electron on the nucleus zero

Radial Distribution Functions

$$4\pi r^2 R_{nl}^2(r)$$

$n=1; l=0 \quad 2\left(\frac{1}{a_o}\right)^{3/2} e^{-r/a_o}$

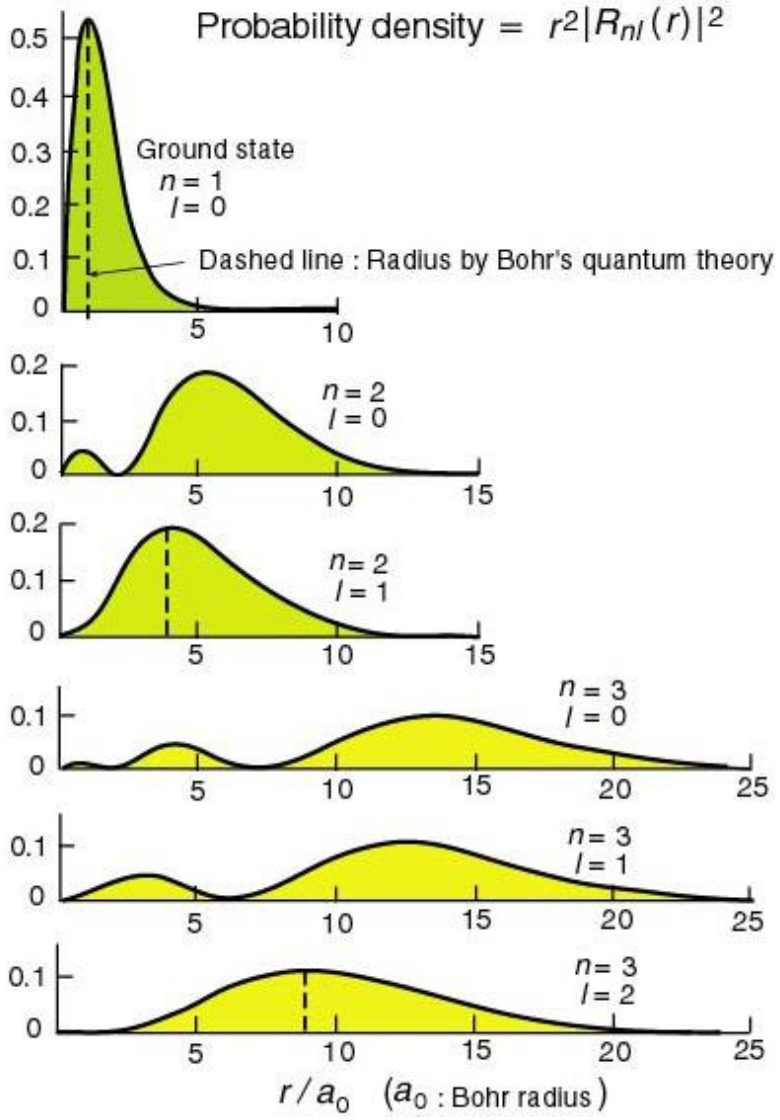
$n=2; l=0 \quad \frac{1}{8^{1/2}}\left(\frac{1}{a_o}\right)^{3/2}\left(2-\frac{r}{a_o}\right)e^{-r/2a_o}$

$n=2; l=1 \quad \frac{1}{24^{1/2}}\left(\frac{1}{a_o}\right)^{3/2}\left(\frac{r}{a_o}\right)e^{-r/2a_o}$

$n=3; l=0 \quad 2\left(\frac{1}{3a_o}\right)^{3/2}\left(1-\frac{2}{3}\left[\frac{r}{a_o}\right]-\frac{2}{27}\left[\frac{r}{a_o}\right]^2\right)e^{-r/3a_o}$

$n=3; l=1 \quad \frac{1}{486^{1/2}}\left(\frac{1}{a_o}\right)^{3/2}\left(4-\frac{2r}{3a_o}\right)e^{-r/3a_o}$

$n=3; l=2 \quad \frac{1}{2430^{1/2}}\left(\frac{1}{a_o}\right)^{3/2}\left(\frac{2r}{3a_o}\right)^2 e^{-r/3a_o}$



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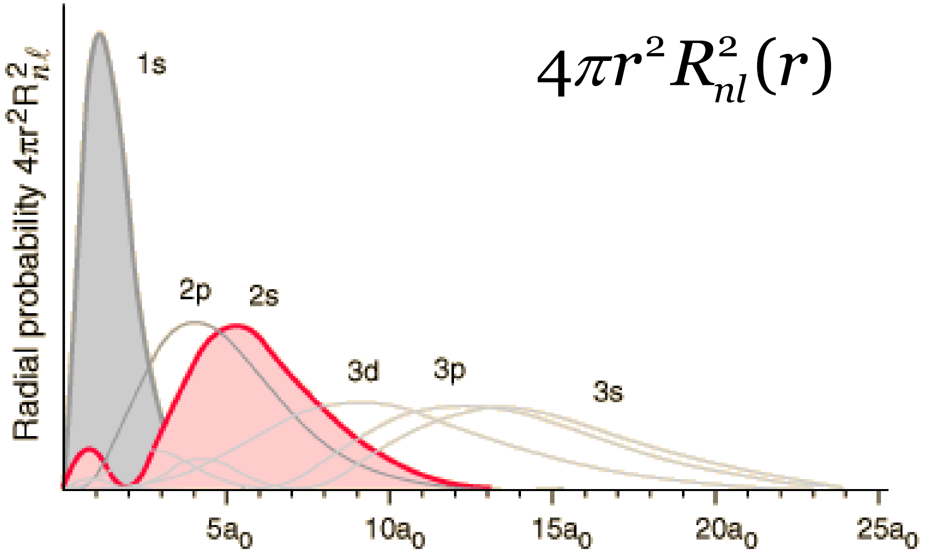

$4\pi r^2 R_{nl}^2(r)$

Average value of radius:

$$\langle r\rangle = \langle \Psi_{ns}|r|\Psi_{ns}\rangle$$

Most probable value of radius:

$$\frac{dP_r}{dr} = 0$$

Coming next: Put them all together