MA 106 TUT3

Handwritten :

Q1 **Evaluating determinants of Block Matrices:**

Suppose the matrix given is block upper triangular i.e. looks like:

$$\begin{bmatrix} A_{11} & A_{12} & \cdots \\ 0 & A_{22} & \cdots \\ 0 & 0 & A_{nn} \end{bmatrix}$$

k

Then determinant is just equal to $= |A_{11}||A_{22}| \cdots |A_{nn}|$

Proof: To calculate the determinant, we will have to pick one element from each row and column, multiply them all and put a suitable sign; then add over all such cases

For the last rows, it makes sense to only pick elements out of Ann as otherwise the element would be 0. Now, since in the last rows, we have exhausted all the last columnss so, in the second last row, we would have to pick only from A(n-1)(n-1). And so on...

Hence, we try to convert every block matrix to block upper triangular to find its determinant

To do this, we can multiply with the following "Block ERO matrices", each of which have determinant +-1 and so don't alter the magnitude of determinant (keep track of sign)

$(\det AB = \det A \times \det B)$

Row exchange:

$$\begin{bmatrix} I & 0 & 0 & \cdots \\ 0 & 0 & I & \cdots \\ 0 & I & 0 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

Multiply by matrix and add:

$$\begin{bmatrix} I & 0 & 0 & \cdots \\ P & I & 0 & \cdots \\ 0 & 0 & I & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

similar operations can also be done on columns

Back to Q1

$$\det(A+iB) \neq 0$$

$$\det\begin{bmatrix} A & B \\ -B & A \end{bmatrix} = 1\times\det\begin{bmatrix} A & B \\ -B & A \end{bmatrix} = \det\begin{bmatrix} A & B \\ -B & A \end{bmatrix}\det\begin{bmatrix} I & 0 \\ Ii & I \end{bmatrix}$$

$$= \det\begin{bmatrix} A+iB & B \\ -B+iA & A \end{bmatrix}$$

$$= \det\begin{bmatrix} A+iB & B \\ i(A+iB) & A \end{bmatrix}$$

$$\begin{bmatrix} c_1 & & & \\ & c_2 & & \\ & & \ddots & \\ & & & c_n \end{bmatrix}$$

$$= \det\begin{bmatrix} I & 0 \\ -Ii & I \end{bmatrix}\det\begin{bmatrix} A+iB & B \\ i(A+iB) & A \end{bmatrix}$$

$$= \det\begin{bmatrix} A+iB & B \\ 0 & A-iB \end{bmatrix} = \det(A+iB)\cdot\det(A-iB)$$

$$= \det(A+iB)\cdot\overline{\det(A+iB)} > 0$$

Aside : What is the inverse of this matrix?

Suppose $A+iB$ has inverse $C+iD$

$$(A+iB)(C+iD) = I$$

$$\rightarrow AC-BD = I, \quad AD+BC = 0$$

$$\begin{bmatrix} C & D \\ -D & C \end{bmatrix}\begin{bmatrix} A & B \\ -B & A \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

EXTRA

Application to geometry: Complex vector spaces of n dimensions expressed as real vector spaces of 2n dimensions have the same orientation irrespective of choice of basis

Q2 $\det\begin{bmatrix} 2 & 0 & 6 & 0 & 4 \\ 5 & 3 & 2 & 2 & 7 \\ 2 & 5 & 7 & 5 & 5 \\ 2 & 0 & 9 & 2 & 7 \\ 7 & 8 & 4 & 2 & 1 \end{bmatrix} = \det\begin{bmatrix} 2 & 0 & 6 & 0 & 20604 \\ 5 & 3 & 2 & 2 & 53227 \\ 2 & 5 & 7 & 5 & 25755 \\ 2 & 0 & 9 & 2 & 20927 \\ 7 & 8 & 4 & 2 & 78421 \end{bmatrix}$

Q3 Note: "Necessary" Condition. Suppose c is common root

Consider :

$$c^2 + ac + b = 0 \quad -(1)$$

$$\to \quad c^3 + ac^2 + bc = 0 \quad -(2)$$

$$c^2 + pc + q = 0 \quad -(3)$$

$$\to \quad c^3 + pc^2 + qc = 0 \quad -(4)$$

$\begin{bmatrix} c^3 \\ c^2 \\ c \\ 1 \end{bmatrix}$ $Ax = 0$

Sufficient condition?

$$|A| = \begin{vmatrix} 0 & 1 & a & b \\ 1 & a & b & 0 \\ 0 & 1 & p & q \\ 1 & p & q & 0 \end{vmatrix} = 0$$

1,2,3,4 : linear in 4 vars $1, c, c^2, c^3$

Has non-trivial sol^n

$$\therefore \begin{vmatrix} 1 & a & b & 0 \\ 0 & 1 & a & b \\ 1 & p & q & 0 \\ 0 & 1 & p & q \end{vmatrix} = 0$$

Tut sheet 3

Q2 Cramer Rule : $AX = b$

$$[C_1 \ C_2 \ C_3 \ \cdots] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \end{bmatrix}$$

if $|A| \neq 0$

then

$$x_i = \frac{\left|[C_1 \ C_2 \cdots C_{i-1} \ b \ C_{i+1} \cdots]\right|}{|A|}$$ (Proof ?)

In Q2,

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 6 & \beta \end{vmatrix} \neq 0 \rightarrow 3\beta + 2 + 18 - 9 - 6 - 2\beta \neq 0$$

$$\rightarrow \boxed{\beta \neq -5}$$

For $\beta = -5$,

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 20 \\ 1 & 3 & 1 & 13 \\ 1 & 6 & -5 & -5 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 20 \\ 0 & 1 & -2 & -7 \\ 0 & 4 & -8 & -25 \end{array}\right]$$

Digression: Grammian Matrix

$$G = \left[v_i^T v_j \right]_{n \times n}$$

$$= \begin{bmatrix} v_1^T v_1 & v_2^T v_1 & \cdots \\ v_1^T v_2 & & \\ \vdots & & \end{bmatrix}_{n \times n}$$

$$= \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \checkmark$$

$$|G| = \left| \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \right|^2 \geq 0$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 20 \\ 0 & 1 & -2 & -7 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

Hence, no solution

Q3

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix}$$

$$= (a+b+c)\left(bc+ab+ac-a^2-b^2-c^2 \right)$$

$$= -\frac{1}{2}(a+b+c)\left((a-b)^2+(b-c)^2+(c-a)^2\right)$$

Linearly dependent iff = 0

$\therefore$ $a+b+c=0$ or $a=b=c$

Q9. Lot of Pain

Find adjoint, then use $A^{-1} = \frac{adj(A)}{|A|}$

Q11. Given:

On (a,b) : $c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0$

Since it is an open interval and assume all the f's are sufficiently differentiable:

$$c_1 f_1^r + c_2 f_2^r + \cdots + c_n f_n^r = 0$$

$$\forall r \in \{0, 1, 2 \cdots, n-1\}$$

where:

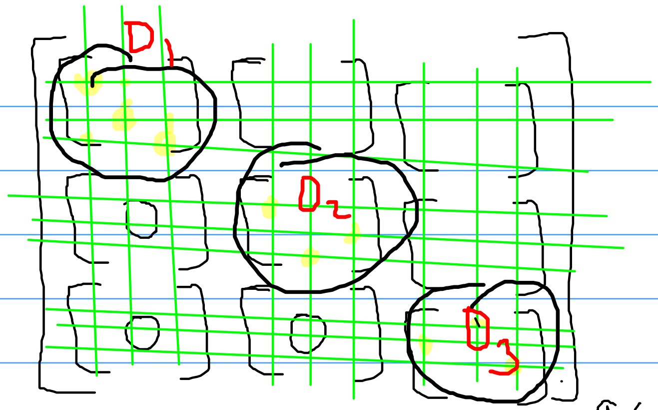
f_i^r = r^{th} derivative of f_i

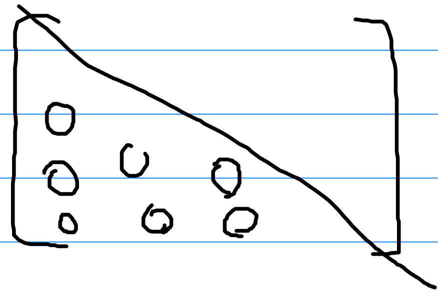
Hence,

$$\underbrace{\begin{bmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ & \vdots & & \\ f_1^{n-1} & f_2^{n-1} & \cdots & f_n^{n-1} \end{bmatrix}}_{W} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = 0$$

If $\det(W) \neq 0 \longrightarrow c_i = 0 \ \forall i$

Q1 Better Explained

1) 



9x9

$$(\;|D_1|\;)(\;|D_2|\;)(\;|D_3|\;)$$

2) $\begin{bmatrix} A & B \\ -B & A \end{bmatrix} \longrightarrow \begin{bmatrix} A+iB & B \\ -B+iA & A \end{bmatrix} \longrightarrow [$

$$\begin{bmatrix} I & \\ 0 & I \end{bmatrix}$$

Proof Of $\det(AB) = (\det A)(\det B)$

$A_{n \times n} \quad B_{n \times n}$

1) A is not invertible $\rightarrow$ $|AB| = |A||B|$

$|A| = 0$

$|AB| = 0$

2) A is invertible

A can be converted to diagonal matrix via EROs

$$(E_1 \cdots E_r)A = \begin{bmatrix} t_1 & & & \\ & t_2 & & \\ & & t_3 & \\ & & & \ddots \end{bmatrix} = T$$

$$(E_1 \quad E_r)AB = \begin{bmatrix} t_1 & & & \\ & t_2 & & \\ & & t_3 & \\ & & & t_n \end{bmatrix} B$$

$$B = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \end{bmatrix}$$

$$|A||B| = |T||B|$$

$$|AB| = |TB|$$

$$TB = \begin{bmatrix} t_1\gamma_1 \\ t_2\gamma_2 \\ \vdots \end{bmatrix}$$

$$|TB| = \underbrace{t_1 t_2 \cdots}_{|T|} |B|$$