MA106 Tut4

Q1. Brute Force:

$$A = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \\ v_1 & v_2 & \cdots & v_n \end{bmatrix}$$

$$AA^T = \begin{bmatrix} u_1^2+u_2^2+\cdots & u_1v_1+u_2v_2+\cdots \\ u_1v_1+u_2v_2+\cdots & v_1^2+v_2^2+\cdots \end{bmatrix}$$

$$A^TA = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \\ \vdots & \vdots \\ u_n & v_n \end{bmatrix} \begin{bmatrix} u_1 & u_2 & \cdots & u_n \\ v_1 & v_2 & \cdots & v_n \end{bmatrix}$$

$$= \begin{bmatrix} u_1^2+v_1^2 & u_1u_2+v_1v_2 & \cdots \\ u_1u_2+v_1v_2 & u_2^2+v_2^2 & \\ \vdots & & \ddots \end{bmatrix}$$

Hence the (i,j)th principle minor is

$$\begin{vmatrix} P_{ii} & P_{ij} \\ P_{ji} & P_{jj} \end{vmatrix} = \begin{vmatrix} u_i^2+v_i^2 & u_iv_i+u_jv_j \\ u_iv_i+u_jv_j & u_j^2+v_j^2 \end{vmatrix}$$

$$= (u_iv_j - u_jv_i)^2$$

Then,

$$\sum\sum_{1\le i<j\le n} \begin{vmatrix} P_{ii} & P_{ij} \\ P_{ji} & P_{jj} \end{vmatrix} = \sum\sum_{1\le i<j\le n} (u_iv_j-u_jv_i)^2$$

$$= (u_1^2+u_2^2+\cdots)(v_1^2+v_2^2+\cdots) - (u_1v_1+u_2v_2+\cdots)^2$$

$(u_1^2+u_2^2+\cdots)(v_1^2+v_2^2+\cdots) - (u_1v_1+u_2v_2+\cdots)^2$ $\}$ $= 11$

$= \det(AA^T)$ you can prove this by Brute Force. You have seen this in the JEE proof of Cauchy Schwartz inequality

Nicer method coming up in a week or two. Called the Cauchy-Binet Formula. Prof. Gopal also has a paper on this: http://www.math.iitb.ac.in/~gopal/papers/Cauchy_Binet.pdf

Q2. Define:

expand along this

$$A_n = \begin{vmatrix} c\alpha & 1 & & & 0 \\ 1 & 2c\alpha & 1 & & \\ & 1 & 2c\alpha & 1 & \\ & & & \ddots & 1 \\ 0 & & & 1 & 2c\alpha \end{vmatrix}_{n\times n} \qquad c \to \cos$$

$A_n = \cos n\alpha$, $n \in \mathbb{N}$

$$= 2c\alpha \begin{vmatrix} c\alpha & 1 & & \\ 1 & 2c\alpha & 1 & \\ & 1 & \ddots & \end{vmatrix}_{(n-1)\times(n-1)} - \begin{vmatrix} c\alpha & 1 & & \\ 1 & 2c\alpha & 1 & \\ & & \ddots & \\ & & 1 & 2c\alpha \; 1 \end{vmatrix}_{(n-2)\times(n-2)} \cdot 1$$

$$= 2c\alpha\, A_{n-1} - A_{n-2}$$

Proof By Induction:

Verify that $A_1 = c\alpha$ ($= |c\alpha|$), $A_2 = c(2\alpha)$ ($= \begin{vmatrix} c\alpha & 1 \\ 1 & 2c\alpha \end{vmatrix}$)

Now, use

$$\cos(n\theta) = 2\cos\theta \cos(n-1)\theta - \cos(n-2)\theta$$

$\langle z, z\rangle = \|z\|^2$

Q3. $\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2$

$= \langle x+y, x+y\rangle - \langle x-y, x-y\rangle + i\langle x+iy, x+iy\rangle - i\langle x-iy, x-iy\rangle$

$= \cancel{\langle x, x\rangle} + \langle x, y\rangle + \langle y, x\rangle + \cancel{\langle y, y\rangle}$
$- \cancel{\langle x, x\rangle} + \langle x, y\rangle + \langle y, x\rangle - \cancel{\langle y, y\rangle}$
$+ \cancel{i\langle x, x\rangle} + i\langle x, iy\rangle + i\langle iy, x\rangle + \cancel{i\langle iy, iy\rangle}$
$- \cancel{i\langle x, x\rangle} - i\langle x, -iy\rangle - i\langle -iy, x\rangle - \cancel{i\langle -iy, -iy\rangle}$

$= 2\langle x, y\rangle + 2\langle y, x\rangle + 2\langle x, y\rangle - 2\langle y, x\rangle$

$= 4\langle x, y\rangle$

$$\cos(n+1)\theta + \cos(n-1)\theta = 2\cos(n\theta)\cos\theta$$

$$\langle a+c, b\rangle = \langle a,b\rangle + \langle c,b\rangle$$

$$\langle a, b+c\rangle = \langle a,b\rangle + \langle a,c\rangle$$

$$\langle x, \lambda y\rangle = \bar{\lambda}\langle x,y\rangle$$

$$\langle \lambda x, y\rangle = \lambda\langle x,y\rangle$$

$$A = \begin{bmatrix} \gamma_1^T \\ \gamma_2^T \\ \vdots \\ \gamma_n^T \end{bmatrix} \qquad \langle \gamma_i, \gamma_j\rangle = 0,\ i \neq j \qquad \langle \gamma_i, \gamma_i\rangle = 1$$

$$\begin{bmatrix} \langle \gamma_1, \gamma_1\rangle & \langle \gamma_1, \gamma_2\rangle & \cdots \\ \langle \gamma_2, \gamma_1\rangle & \langle \gamma_2, \gamma_2\rangle & \\ & \vdots & \end{bmatrix} = I$$

$$\langle x, y\rangle = x^T \bar{y}$$

$$= \begin{bmatrix} \gamma_1^T \bar{\gamma}_1 & \gamma_1^T \bar{\gamma}_2 \\ \gamma_2^T \bar{\gamma}_1 & \ddots \end{bmatrix} = \underbrace{\begin{bmatrix} \gamma_1^T \\ \gamma_2^T \\ \vdots \end{bmatrix}}_{A} \begin{bmatrix} \bar{\gamma}_1 & \bar{\gamma}_2 & \cdots \end{bmatrix}$$

$$\bar{A}^T = A^*$$

$AA^* = I$ equivalent to rows of A are orthonormal

$$|A| \neq 0$$

$$A^* = A^{-1} \rightarrow (A^*)A^{**} = I$$

$$A = A^{**}$$

Rows of A* are orthonormal

Rows of A* are columns of A conjugate -> columns of A are orthonormal

Q5. If rows of A are orthonormal, then:

$$AA^* = I$$

as $A = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \end{bmatrix} \rightarrow AA^* = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \end{bmatrix} \begin{bmatrix} r_1^* & r_2^* & \cdots \end{bmatrix}$

$$= \begin{bmatrix} \langle r_1, r_1 \rangle & \langle r_1, r_2 \rangle & \cdots \\ \langle r_2, r_1 \rangle & \langle r_2, r_2 \rangle & \\ \vdots & & \end{bmatrix}$$

$$= I$$

Hence,

$A^{-1} A A^* = A^{-1}$ [why is A inv.]

$\rightarrow A^* = A^{-1}$

$\rightarrow A^* A = I$

$\rightarrow$ Columns are orthonormal

Q6. M1] Take a random vector, hope it does not lie in the plane spanned by v and w; then use Gram-Schmidt to orthonormalize it

M2] Solve the equations: $u \cdot v = 0$, $u \cdot w = 0$

Then, normalize u

$\downarrow$

$\langle u, v \rangle = 0 \quad \langle u, w \rangle = 0$

Tut sheet

Q4. $V_1 = [1\ 1\ 0\ 0]$

$$V_2 = [1\ 0\ 1\ 0] - \frac{[1\ 0\ 1\ 0]\begin{bmatrix}1\\1\\0\\0\end{bmatrix}}{[1\ 1\ 0\ 0]\begin{bmatrix}1\\1\\0\\0\end{bmatrix}} [1\ 1\ 0\ 0]$$

$$= [1\ 0\ 1\ 0] - \frac{1}{2}[1\ 1\ 0\ 0] = \left[\frac{1}{2}\ \ \frac{-1}{2}\ \ 1\ \ 0\right]$$

$$V_3 = [1\ 0\ 0\ 1] - \frac{[1\ 0\ 0\ 1]\begin{bmatrix}1/2\\-1/2\\1\\0\end{bmatrix}}{[1/2\ -1/2\ 1\ 0]\begin{bmatrix}1/2\\-1/2\\1\\0\end{bmatrix}} [1/2\ -1/2\ 1\ 0] - \frac{[1\ 0\ 0\ 1]\begin{bmatrix}1\\1\\0\\0\end{bmatrix}}{[1\ 1\ 0\ 0]\begin{bmatrix}1\\1\\0\\0\end{bmatrix}} [1\ 1\ 0\ 0]$$

$$= \frac{1}{3}[1\ -1\ 1\ 3]$$

Do remaining vectors similarly using Gram Schmidt:

$$V_4 = \frac{1}{2}[-1\ 1\ 1\ 1] \qquad V_5 = V_6 = 0$$

We have a basis for the space $\mathbb{R}^4$, so Bessel's inequality must yeild equality. Verify:

$$\|u\|^2 = \sum_{i=1}^{6} \left(u \cdot \frac{V_i}{\|V_i\|}\right)^2$$

Q8. Again GS procedure. Take care of order in inner product

$$w_1 = [1 \; i \; 0 \; 0 \; 0]$$

$$w_2 = \frac{1}{2}[i \; 1 \; 2i \; 0 \; 0]$$

$$w_3 = \frac{1}{3}[-1 \; i \; 1 \; 3i \; 0]$$

$$w_4 = \frac{1}{4}[-i, -1, i, 1, 4i]$$

So, the orthonormal basis is:

$$\frac{1}{\sqrt{2}}[1 \; i \; 0 \; 0 \; 0] \qquad \frac{1}{\sqrt{6}}[i \; 1 \; 2i \; 0 \; 0]$$

$$\frac{1}{2\sqrt{3}}[-1 \; i \; 1 \; 3i \; 0] \qquad \frac{1}{2\sqrt{5}}[-i, -1, i, 1, 4i]$$

Check in Bessel's inequality:

$$\|[1 \; i \; 1 \; i \; 1]\|^2 = 5$$

$$\sum |\langle v \cdot u_i \rangle|^2 = \frac{24}{5}$$

$$\frac{24}{5} < 5 \rightarrow \text{Not in span}$$

Suppose that you have a vector space V equiped with an inner product $\langle \, , \, \rangle$

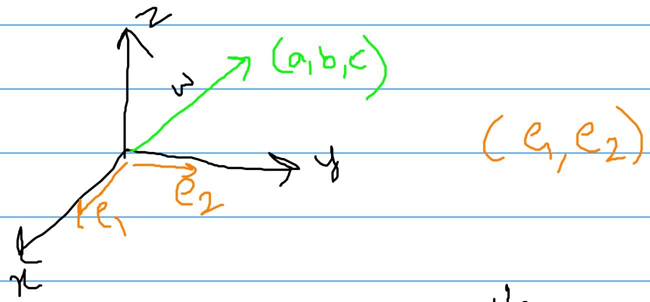
$(e_1, e_2, \dots e_k) \rightarrow$ Orthonormal set of vectors

$w \rightarrow$ Any vector in V

$$\|w\|^2 \geq \sum_{i=1}^{k} |\langle w, e_i \rangle|^2$$

$\langle w, e_i \rangle$

equality holds iff w lies in the space spanned by e_i's



$$a^2 + b^2 + c^2 = \|w\|^2$$
$$\geq |\langle w, e_1 \rangle|^2 + |\langle w, e_2 \rangle|^2$$
$$= a^2 + b^2$$