FourierAnalysis

December 2021

1 Fourier Series

Consider a periodic function, which has the property

$$f(x+L) = f(x),$$

where L is a constant. We also define a wave-number constant $k_L = 2\pi/L$. A French mathematician called Joseph Fourier proved an important theorem in which such periodic functions are expressed in terms of an infinite series of sine and cosine terms. The explicit form of the series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nk_L x) + b_n \sin(nk_L x) \right].$$

Here a_n are called n^{th} cosine Fourier coefficient and b_n are called n^{th} sine Fourier coefficient. The constant a_0 is given by

$$a_0 = \frac{1}{L} \int_0^L f(x) dx$$

The Fourier coefficients are given by

$$\begin{aligned} a_n &= \frac{2}{L} \int_0^L f(x) \cos(nk_L x) dx \\ b_n &= \frac{2}{L} \int_0^L f(x) \sin(nk_L x) dx \end{aligned}$$

We can write the sine and cosine functions in terms of exponential functions as

$$\cos(nk_L x) = \frac{e^{ink_L x} + e^{-ink_L x}}{2} \quad \text{and} \quad \sin(nk_L x) = \frac{e^{ink_L x} - e^{-ink_L x}}{2i}$$

Substituting them in the Fourier series expansion, we get

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[A_n e^{ink_L x} + A_n^* e^{-ink_L x} \right],$$

where $A_n = (a_n - ib_n)/2$. From the definitions of a_n and b_n, we can show that

$$A_n = \frac{2}{L} \int_0^L f(x) e^{(-ink_L x)}.$$

What is the point of this Fourier series expansion? There are a number of cases where a very close approximation of the function $f(x)$ is obtained by taking only the first few terms in the series expansion. How many terms do we need to include depends on two factors:

- How closely do we want the series to approximate the original function? To ten percent level, to one percent level or to an even more accurate level?

- How the Fourier Coefficients a_n and b_n vary with n? In the worst case scenario, $a_n/b_n \sim 1/n$ and we needs to take a large number of terms (of the order of 100) to get a one percent approximation. If, one the other hand, $a_n/b_n \sim (1/n^2$ or even faster, then only the first few terms (less than ten) are enough to give a very good approximation of the function.

Fourier series expansions are used extensively in **signal processing**.

2 Fourier Integral

In the previous section, we have done a super-position, which is characterised by n, **a variable which takes integer values**. Now, consider a super-position where the variable is a **continuous variable**. In Fourier series, we had a summation over terms with k_L, $2k_L$, $3k_L$ etc, where k_L is a fixed quantity. Our wave-numbers are labelled nk_L with n taking integer values. Suppose we consider a very different case, where the wave-numbers are labelled by a **continuous variable** k, which, in principle, can vary from $-\infty$ to ∞. In such a situation, we need to replace the summation over n with an integral over k.

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} A(k) e^{ikx} dk.$$

The factor $1/\sqrt{2\pi}$ in front is put in for reasons technical reasons, which will not concern us. The right-hand side of the above equation is called **Fourier Integral** and the quantity $A(k)$ is called **Fourier Amplitude** of the wave with *wave-number* k. It can be shown that the Fourier Amplitude is given by

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ikx} dx.$$

In the definition of Fourier coefficients, we had factors $2/L$ in front. In Fourier integrals, two **normalizations** are used in literature.

- In the expansion of $f(x)$, the factor in front is 1 and in the definition of $A(k)$ the factor in front is $1/(2\pi)$. This definition matches the definition of Fourier coefficients. But, it leads to a confusion of which definition has the $1/(2\pi)$ factor in front.
- To avoid the above confusion, a factor of $1/\sqrt{2\pi}$ is put in front, in the definitions of both $f(x)$ and $A(k)$ to make the expressions look symmetric.

In doing any calculation, one should fix one of the above definitions and stick to it. For the issues that we will discuss in this course, do not depend which definition we use.

Let us consider a few examples of $f(x)$ and compute $A(k)$ for them. For the first example, consider the function

$$\begin{aligned} f(x) &= \cos(\pi x/2a) = \cos(k_1 x) \text{ for } -a \leq x \leq a \\ &= 0 \text{ for} |x| > a, \end{aligned} \tag{1}$$

where $k_1 = \pi/(2a)$. We calculate $A(k)$ to be

$$\begin{aligned} A(k) &= \frac{1}{\sqrt{2\pi}} \frac{1}{2} \int_{-a}^{a} \left(e^{i(k_1-k)x} + e^{-i(k_1+k)x} \right) dx \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{2} \left(\frac{e^{i(k_1-k)x}}{i(k_1-k)} + \frac{e^{-i(k_1+k)x}}{-i(k_1+k)} \right)_{-a}^{a} \end{aligned}$$

In the above equation, $k_1 a = \pi/2$. We have $e^{i\pi/2} = i$ and $e^{-i\pi/2} = -i$. Substituting this, we get

$$\begin{aligned} A(k) &= \frac{1}{\sqrt{2\pi}} \frac{1}{2} \left(\frac{e^{ika} + e^{-ika}}{k_1 - k)} + \frac{e^{-ika} + e^{ika}}{(k_1+k)} \right) \\ &= \frac{1}{\sqrt{2\pi}} \frac{2k_1 \cos(ka)}{k_1^2 - k^2} \end{aligned} \tag{2}$$

The plot of the function is given in Figure 1. It has a peak at $k = 0$.

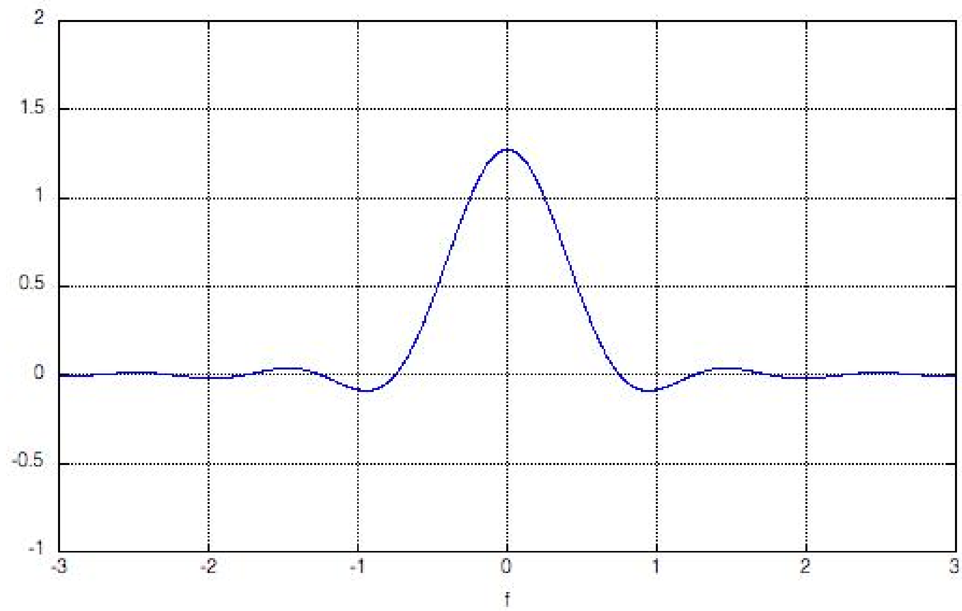

Figure 1: Fourier Integral of a truncated Cosine

At $k = \pm k_1 a$, it **does not** vanish because of the $(k_1^2 - k^2)$ factor in the denominator. If we set $k = k_1 - \delta k$ in the above function and take the limit $\delta k \to 0$, we find that $A(k)$ is finite at $k = \pm k_1$. The first set of zeros for $A(k)$ occur at $k = 3k_1 = 3\pi/(2a)$. We can say that the **width** of $A(k)$ is $3\pi/a$. Multiplying this by the width of $f(x)$, which is $2a$, we get $6\pi \approx 19$.

We find that this will be the case for all wave packets. The spread Δx in x of the wave packet $\psi(x)$ and the spread Δk in k of its Fourier transform $A(k)$ will always be related by

$$\Delta x \Delta k \sim \mathcal{O}(1)$$

that is, the product is a number of order 1. Let us consider a wave packet which extends from $-\infty$ to $+\infty$ to check this statement. We consider the function

$$f(x) = e^{(-|x|/a)} = e^{-\alpha|x|}$$

If we want wave packets to be localized, we must choose them to be of finite extent or that they should vanish as $|x| \to \pm\infty$. The Fourier components of the above function are

$$
\begin{aligned}
A(k) &= \int_{-\infty}^{0} e^{(\alpha - ik)x} dx + \int_{0}^{\infty} e^{-(\alpha \ + ik)x} dx \\
&= \frac{1}{\alpha - ik} + \frac{1}{\alpha + ik} = \frac{2\alpha}{\alpha^2 \ + k^2}.
\end{aligned} \tag{3}
$$

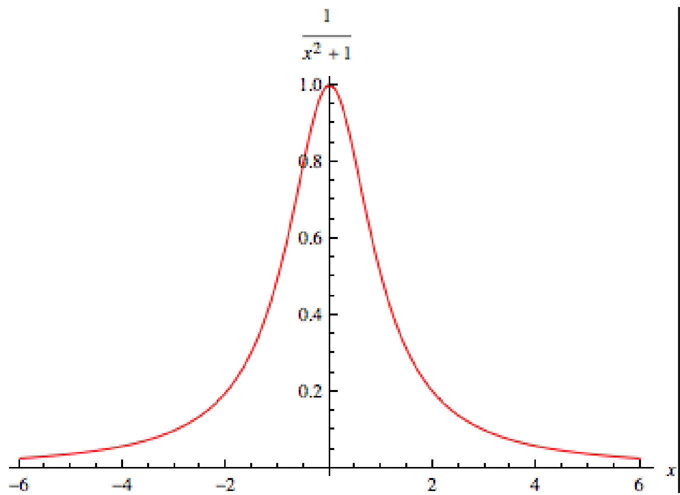

Figure 2: Fourier Integral of a two-sided decaying exponential

$A(k)$ is plotted in Figure 2. We note that both $f(x)$ and $A(k)$ extend from $-\infty$ to $+\infty$. How do we define ranges for such functions? One way is to use the definition of standard deviation. A short cut is to define a quantity called **Full Width at Half Maximum** (FWHM). If P is the peak value of a function, then find the range for which the value of function is greater than $P/2$. This is called FWHM. Consider a histogram of mean μ and standard deviation σ. In the range $(\mu \pm \sigma)$, we find that the values of the histogram will be greater than $P/2$, where P is the peak value of the histogram. Thus, FWHM represents 2σ.

Given that $e^{-0.693} = 0.5$, FWHM for the exponentially decaying function is $2 \times 0.693/\alpha$. From the definition of $A(k)$, it is easy to see that its FWHM is 2α (from $-\alpha$ to $+\alpha$). If we take these FWHM to represent $2\Delta x$ and $2\Delta k$, we find that the product

$$\Delta x \Delta k = 0.693,$$

again a number of order 1.

Exercise: Compute FWHM for $f(x)$ and $A(k)$ for the truncated cosine of the first example and find the product of Δx and Δk.

In principle, we can repeat this for any wave packet and we will get the same answer: $\Delta k \sim 1/\Delta x$. This is a feature which is intrinsic to the construction of the wave packet. If we want to construct a narrow wave packet (Δx small), then we should use a broad range of wavenumbers because $f(x)$ should change quite a bit within a narrow range of x. If, on the other hand, the wave packet is broad, then we need to use only a much smaller range of wavenumbers, for the same reason as above. If we ask the question, for which wave packet is the product the smallest, the answer is **Gaussian wave packet** shown in Figure 3. This can be proved using Calculus of variations. It turns out $A(k)$ for Gaussian is also a Gaussian. By calculating

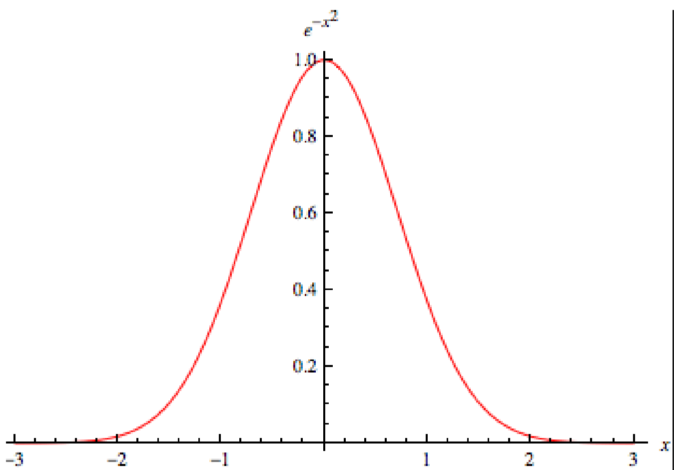

Figure 3: A Gaussian Wave Packet

the FWHM (which is equal to twice the standard deviation for Gaussian), we can show that $\Delta x \Delta k = 1/2$ for Gaussian.