Conductors

Conductors

Outline

I. Conductors and their electrical properties
II. Examples using (i) spherical shell and (ii) a cavity in a spherical shell
III. Force on a conductor with example
IV. Applications

Conductors

Learning Objectives

I. To learn about conductors.

II. To learn to show the electrical properties of conductors.

III. To learn about the force on a charged conductor.

Conductors

Learning Outcomes

I. To be able to analytically show the electrical properties of conductors.

II. To be able to calculate the electric field and potential in the presence of conductors

III. To be able to calculate force on a charged conductor.

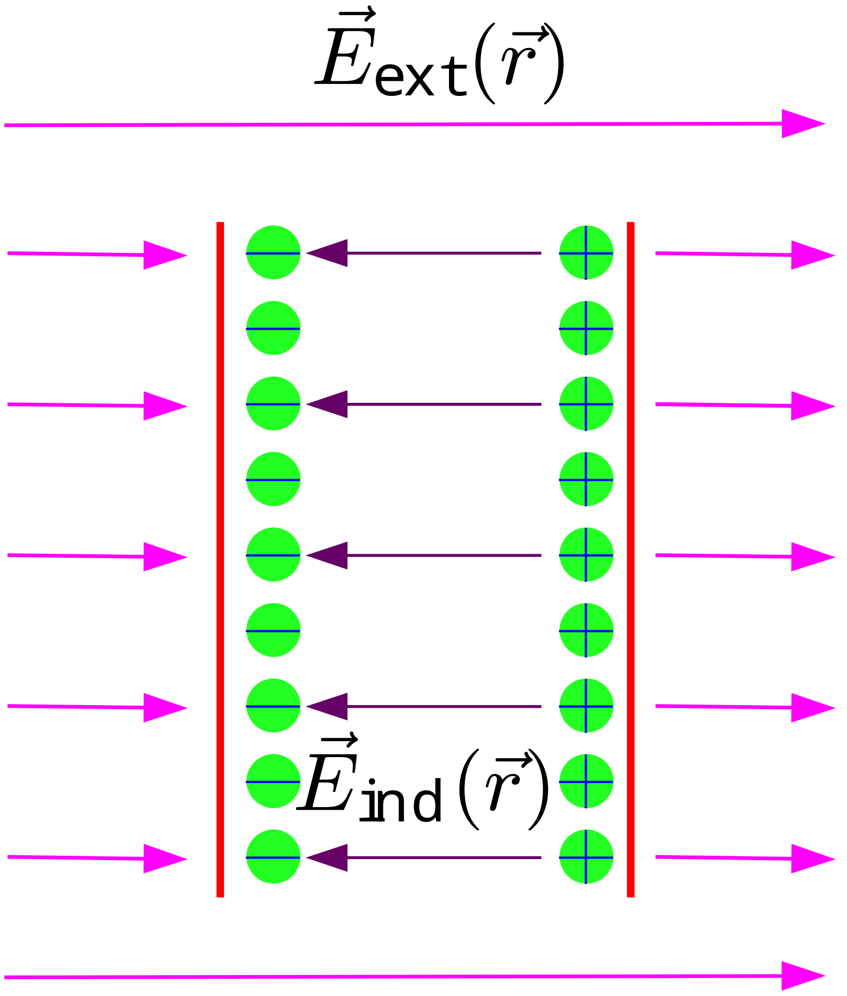
Conductors

The kinds of material based on their conducting properties are:

Material	Resistivity (Ω-m)
Silver	1.6×10^{-8}
Copper	1.7×10^{-8}
Gold	2.4×10^{-8}
Iron	1.0×10^{-7}
Sea water	0.2
Polyethylene	2.0×10^{11}
Glass	$\sim10^{12}$
Fused quartz	7.5×10^{17}

Conductors

Consider a rectangular block of conductor in the presence of an external electric field as shown in the figure.



Typical free electron response time ~10^{-15} s to 10^{-12} s.

Conductors

Some of the properties of a conductor are:

1. $\vec{E}_{\mathtt{net}}(\vec{r}) \equiv 0$ inside a conductor.

2. The volume free charge density, $\rho^{\mathtt{free}}_{\mathtt{inside}}(\vec{r}) = 0$ inside a conductor.

3. Any induced charges on a conductor can only reside on surface or surfaces of the conductor – as surface charge distribution, $\sigma_{\mathtt{free}}$.

4. The entire volume and the surface of a conductor is an equipotential.

5. Just outside of the surface of a conductor, $\vec{E}_{\mathtt{outside}}(\vec{r})$ is perpendicular to the surface.

Conductors

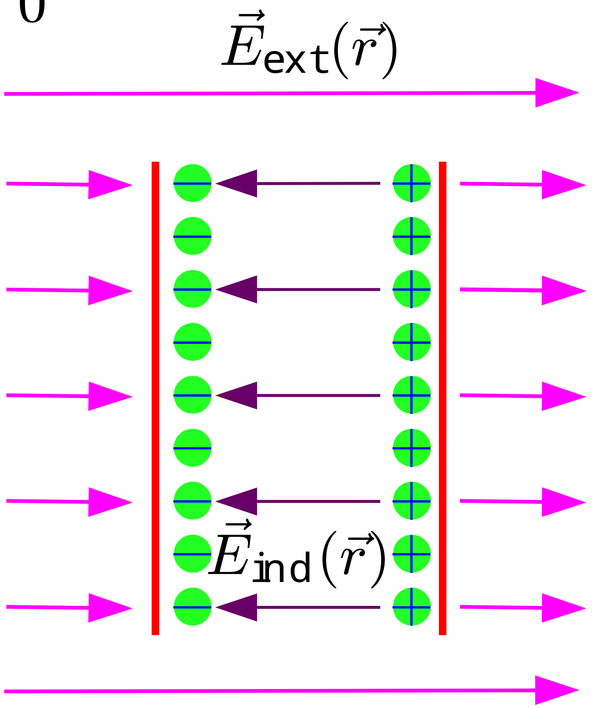
1. $\vec{E}_{\text{net}}(\vec{r}) \equiv 0$ inside a conductor:

Consider a rectangular block of conductor in the presence of an external electric field $\vec{E}_{\text{ext}}(\vec{r})$.

In response to the external electric field, the free electrons redistribute themselves such that the net electric field inside the conductor is zero,

$$\vec{E}_{\substack{\text{net}\\\text{inside}}}(\vec{r}) = \vec{E}_{\text{ext}}(\vec{r}) + \vec{E}_{\substack{\text{induced}\\\text{inside}}}(\vec{r}) = 0$$

$$\Rightarrow \vec{E}_{\substack{\text{induced}\\\text{inside}}}(\vec{r}) = -\vec{E}_{\text{ext}}(\vec{r})$$



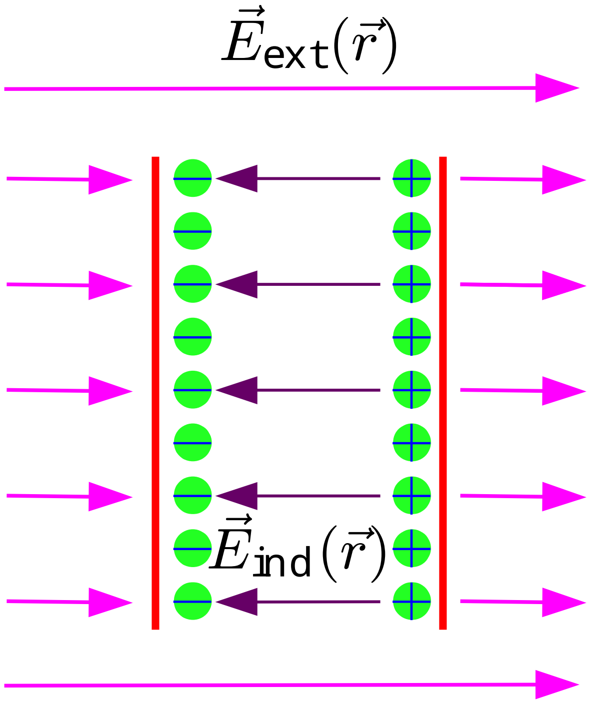
So that $\vec{E}_{\text{net}}(\vec{r}) \equiv 0$ inside a conductor.

Conductors

2. The volume free charge density, $\rho^{\text{free}}_{\text{inside}}(\vec{r}) = 0$ inside a conductor.

Since $E_{\text{inside}}(\vec{r})$ is zero inside the conductor, we have from Gauss' law,

$$\vec{\nabla} \cdot \vec{E}_{\text{inside}}(\vec{r}) = \frac{\rho^{\text{free}}_{\text{inside}}(\vec{r})}{\epsilon_0}$$



Since,

$$\vec{E}_{\substack{\text{net}\\\text{inside}}}(\vec{r}) = \vec{E}_{\text{ext}}(\vec{r}) + \vec{E}_{\substack{\text{induced}\\\text{inside}}}(\vec{r}) = 0$$

So that

$$\vec{\nabla} \cdot \vec{E}_{\text{inside}}(\vec{r}) = 0$$

$$\Rightarrow \rho^{\text{free}}_{\text{inside}}(\vec{r}) = 0$$

The volume free charge density inside a conductor is zero.

Conductors

3. Any induced charges on a conductor can only reside on surface or surfaces of the conductor – as surface charge distribution, σ_{free}.

We have said that in the presence of an external electric field,

$$\rho_{\text{inside}}^{\text{free}}(\vec{r}) = 0$$

Therefore, any induced free charge must reside on the surface of the conductor as $\sigma_{\text{free}}(\vec{r})$.

Conductors

4. The entire volume and the surface of a conductor is an equipotential.

Consider two points *a* and *b* on the surface of a conductor. The potential difference between the two points is given by

$$\triangle V_{\mathsf{ab}} = V(b) - V(a) = -\int_{\mathsf{a}}^{\mathsf{b}} \vec{E}(\vec{r}) \cdot \vec{dl} = 0$$

$$\Rightarrow V_{\mathsf{a}} = V_{\mathsf{b}}$$

If the integral is not zero then the charges would move. Therefore, the entire volume and the surface of a conductor is an equipotential.

Conductors

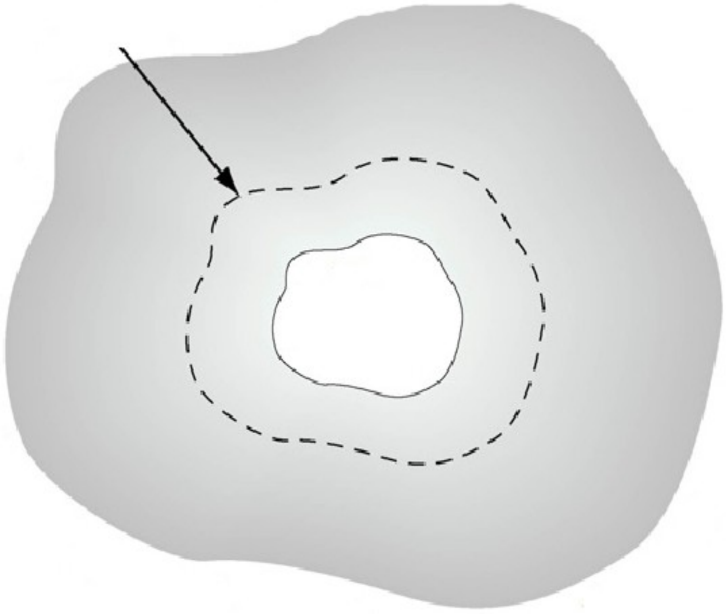
5. Just outside of the surface of a conductor, $\vec{E}_{\mathrm{outside}}(\vec{r})$ is perpendicular to the surface.

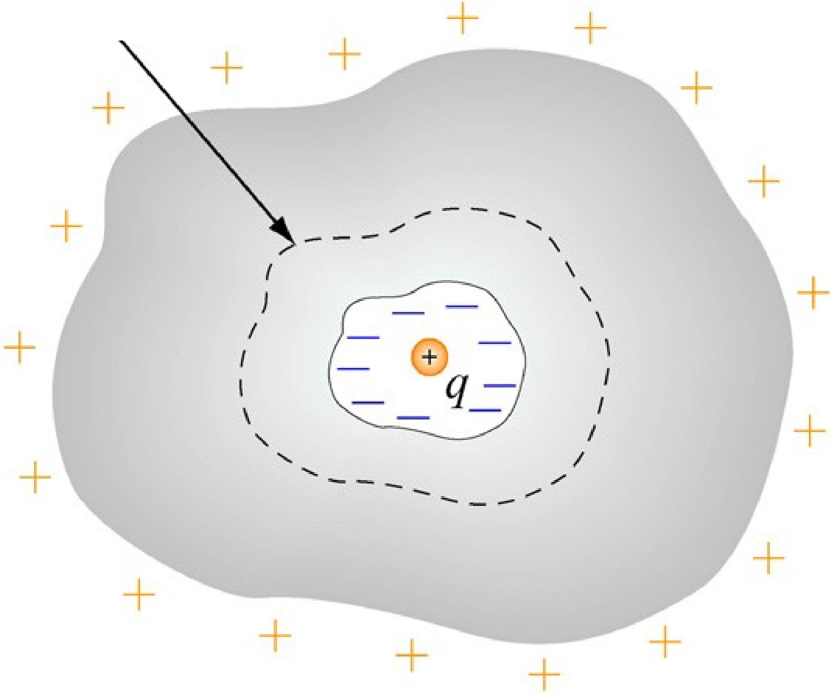
If $\vec{E}_{\mathrm{tangential}}(\vec{r}) \neq 0,$ then charges would move, which is inconsistent with electrostatics. Hence, just outside of the surface of a conductor, the electric field is perpendicular to the surface.

Conductors

(i) The potential $V(\vec{r})$ inside the cavity of a conductor is constant if there is no charge inside the cavity. The corresponding electric field $\vec{E}(\vec{r})$ inside the cavity is zero.

(ii) A charge q placed inside the cavity induces a surface charge on the inner wall of the cavity and $q_{\text{ind}} = -q$.

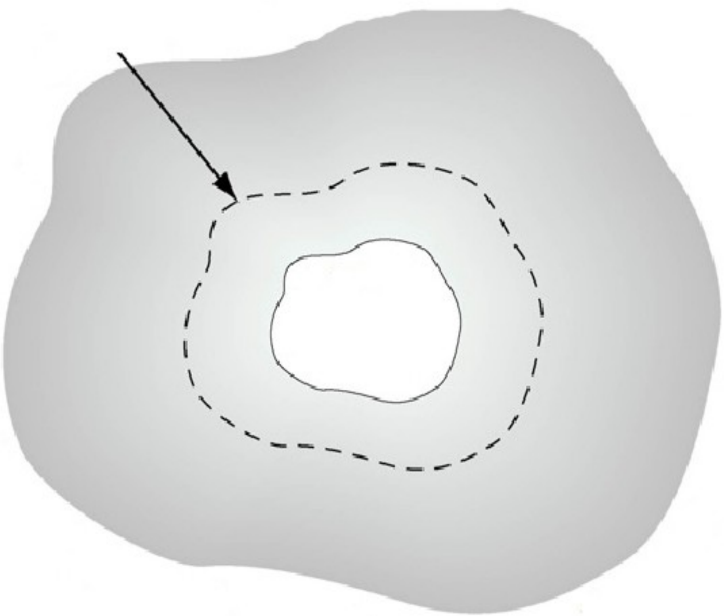


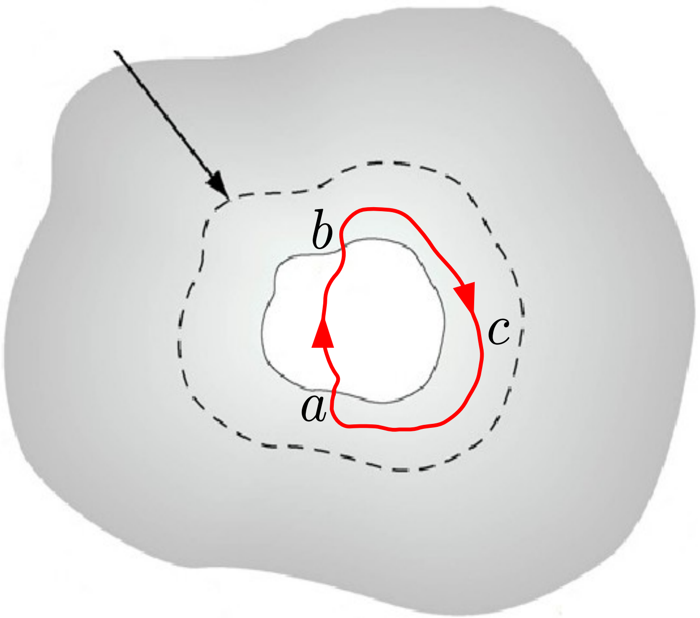


Conductors

Consider a conductor with a cavity. Find $\vec{E}(\vec{r})$ and $V(\vec{r})$ inside the cavity.

Inside the conductor, $\vec{E}(\vec{r}) = 0$ and $V(\vec{r}) = \Phi$, a constant





In order to show that $\vec{E}(\vec{r}) = 0$ and $V(\vec{r}) = \Phi$ inside the cavity, assume $\vec{E}(\vec{r})$ to be non-zero and consider the line integral along the contour $a \to b \to c \to a$ such that the line element along $a \to b$ follows $\vec{E}(\vec{r})$.

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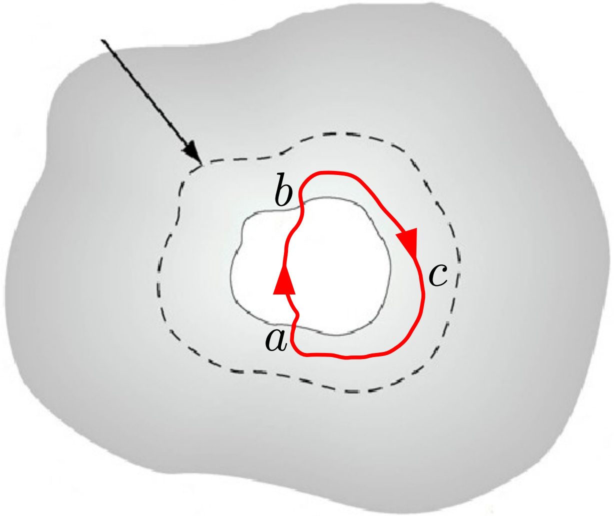
That is

$$\oint_C \vec{E}(\vec{r}) \cdot d\vec{l} = \int_{a\to b} \vec{E}(\vec{r}) \cdot d\vec{l} + \int_{b\to c\to a} \vec{E}(\vec{r}) \cdot d\vec{l}$$



Since $\vec{E}(\vec{r})$ is zero inside the conductor,

$$\int_{b\to c\to a} \vec{E}(\vec{r}) \cdot d\vec{l} = 0$$

and as $\vec{E}(\vec{r})$ is along $d\vec{l}$, the integral

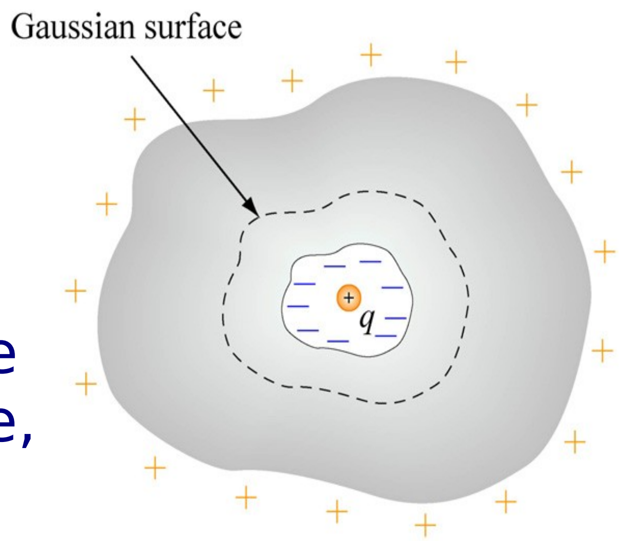
$$\int_{a\to b} \vec{E}(\vec{r}) \cdot d\vec{l} = \int_{a\to b} E dl$$

can be zero only if $\vec{E}(\vec{r})$ is zero inside the cavity. In addition, since the inner wall of the cavity is at the potential $V(\vec{r}) = \Phi$, we must have the same potential inside the cavity.

Conductors

A charge *q* placed inside the cavity induces a surface charge on the inner wall of the cavity and $q_{\text{ind}} = -q$.

Consider a Gaussian surface inside the conductor as shown in the figure. Using Gauss' law and the fact that $\vec{E}(\vec{r}) = 0$ inside the conductor, we have



$$\oint_{S} \vec{E} \cdot d\vec{S} = \frac{1}{\epsilon_0} \int_{V} \rho dV = 0$$

So that the total charge enclosed inside the volume must be zero, and therefore,

$$q_{\text{ind}} = -q$$

on the inner wall of the cavity. Consequently, the outer surface of the conductor gets a positive charge *q*.

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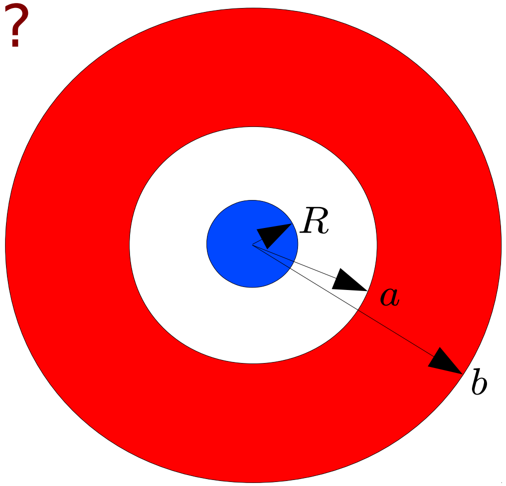
Example 2.35: A metal sphere of radius *R*, carrying charge *q*, is surrounded by a thick concentric metal shell (inner radius *a*, outer radius *b*). The shell carries no net charge.

(i) Find the surface charge density σ at *R*, at *a*, and at *b*.

(ii) Find the potential at the center of the sphere, using infinity as reference.

(iii) Now the outer surface is touched to a grounding wire which lowers its potential to zero. How do your answers to (i) and (ii) change?



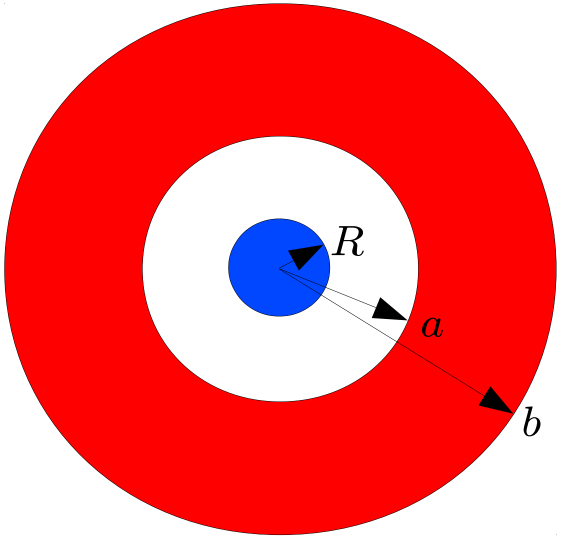
Conductors

(i) The surface charge densities at the various surfaces are:

On the surface of the sphere of radius *R*, $\sigma_{\mathrm{R}} = \frac{q}{4\pi R^2}$

On the inner surface of the spherical shell $\sigma_{\mathrm{a}} = -\frac{q}{4\pi a^2}$

On the outer surface of the spherical shell $\sigma_{\mathrm{b}} = \frac{q}{4\pi b^2}$



Conductors

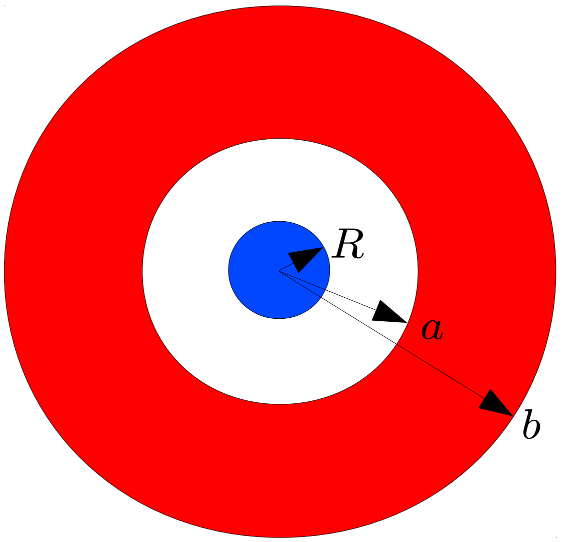
(ii) The potential at the center of the sphere is given by

$$V_0 = -\int_{\infty}^{0} \vec{E}(\vec{r}) \cdot \vec{dl} = -\frac{q}{4\pi\epsilon_0}\int_{\infty}^{b} \frac{1}{r^2} dr - \frac{q}{4\pi\epsilon_0}\int_{a}^{R} \frac{1}{r^2} dr$$

or, $V_0 = \dfrac{q}{4\pi\epsilon_0}\left[\dfrac{1}{b} - \dfrac{1}{a} + \dfrac{1}{R}\right]$

(iii) When the outer surface of the shell is grounded then the charge density on the outside shell becomes zero. Now, the potential at the center of the sphere becomes

$$V_0 = -\int_{b}^{0} \vec{E}(\vec{r}) \cdot \vec{dl} = -\frac{q}{4\pi\epsilon_0}\int_{a}^{R} \frac{1}{r^2} dr$$



or, $V_0 = \dfrac{q}{4\pi\epsilon_0}\left[\dfrac{1}{R} - \dfrac{1}{a}\right]$

Conductors

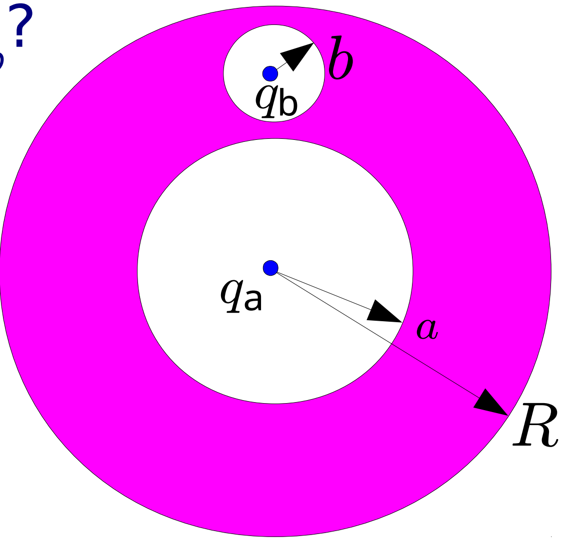
Example 2.36: Two spherical cavities, of radii *a* and *b* are hollowed out from the interior of a (neutral) conducting sphere of radius *R*. At the centre of each cavity a point charge is placed – call these charges q_a and q_b.

(i) Find the surface charge density σ at *R*, at *a*, and at *b*.

(ii) What is the field outside the conductor?

(iii) What is the field within each cavity?

(iv) What is the force on q_a and q_b?



Conductors

(i) The surface charge densities at the various surfaces are:

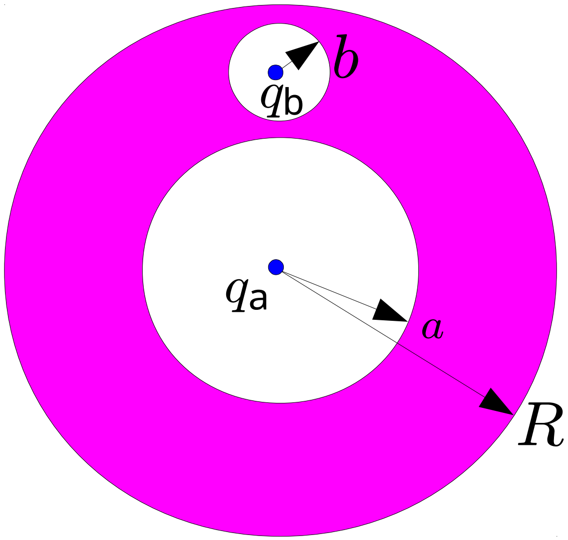
The inner surface of the cavity of radius *a*, $\sigma_{\mathrm{a}} = -\frac{q_{\mathrm{a}}}{4\pi a^2}$

The inner surface of the cavity of radius *b*, $\sigma_{\mathrm{b}} = -\frac{q_{\mathrm{b}}}{4\pi b^2}$

The surface of the sphere of radius *R*, $\sigma_{\mathrm{R}} = \frac{q_{\mathrm{a}} + q_{\mathrm{b}}}{4\pi R^2}$

(ii) The electric field outside the conductor is:

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q_{\mathrm{a}} + q_{\mathrm{b}}}{r^2} \hat{r}, \quad r > R$$



Conductors

(iii) The electric field inside cavity *a*,

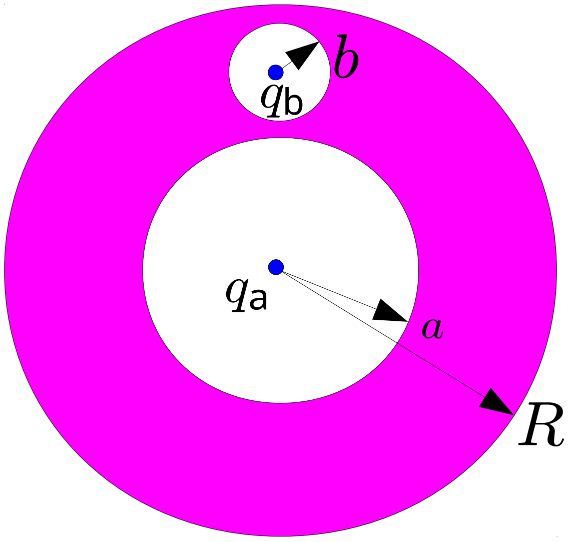
$$\vec{E}_{\mathsf{a}} = \frac{1}{4\pi\epsilon_0}\frac{q_{\mathsf{a}}}{r_{\mathsf{a}}^2}\hat{r}_{\mathsf{a}}, \;\; r_{\mathsf{a}} < a$$

The electric field inside cavity *b*,

$$\vec{E}_{\mathsf{b}} = \frac{1}{4\pi\epsilon_0}\frac{q_{\mathsf{b}}}{r_{\mathsf{b}}^2}\hat{r}_{\mathsf{b}}, \;\; r_{\mathsf{b}} < b$$

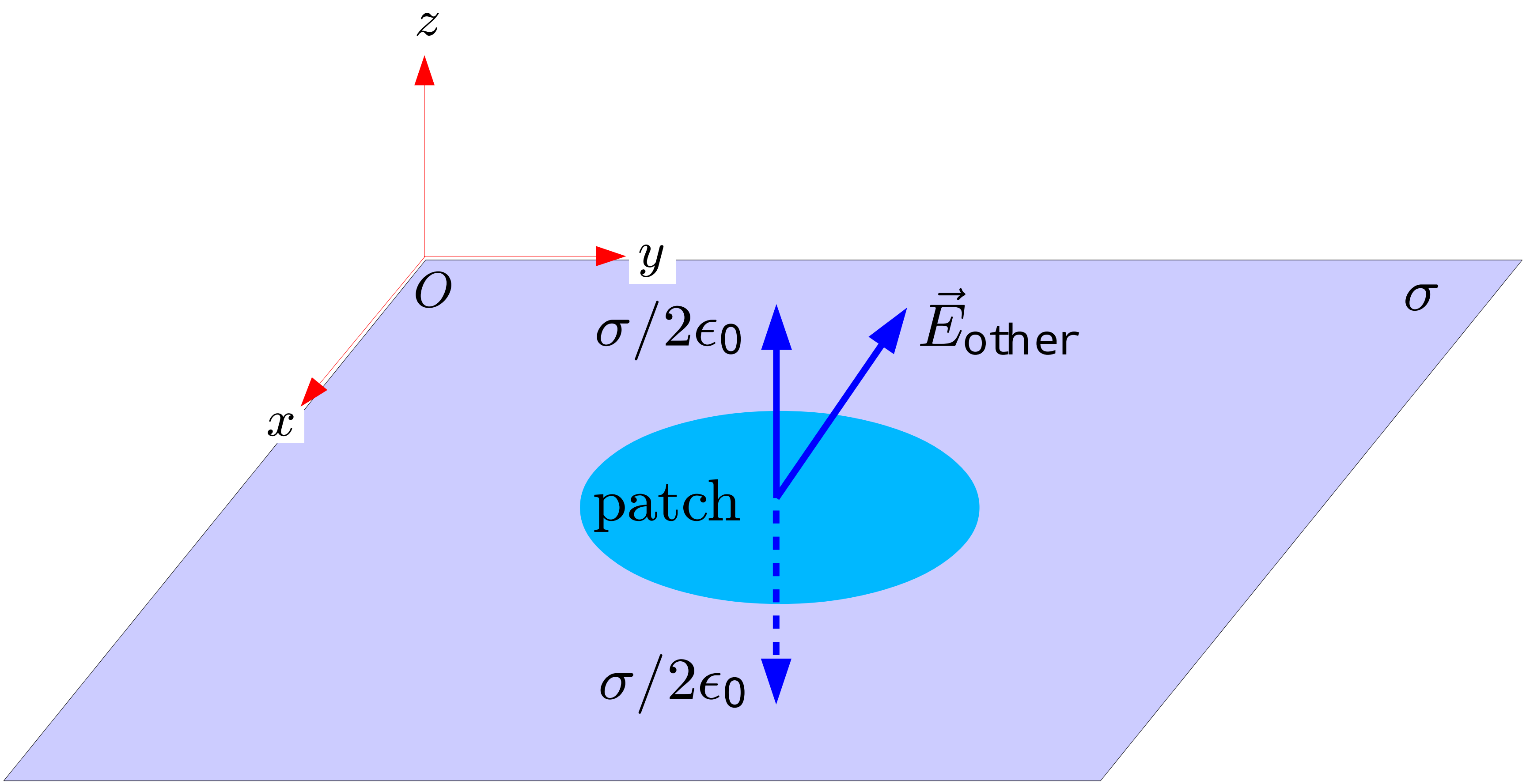
The vectors $\vec{r}_{\mathsf{a}}$ and $\vec{r}_{\mathsf{b}}$ are measured from the respective centers of the cavities.

(iv) The force on q_{a} and q_{b} is zero.



Conductors : Force on a Conductor

Consider a small patch of area A on the conducting surface with a surface charge density σ. What is the electrostatic force acting on this patch?



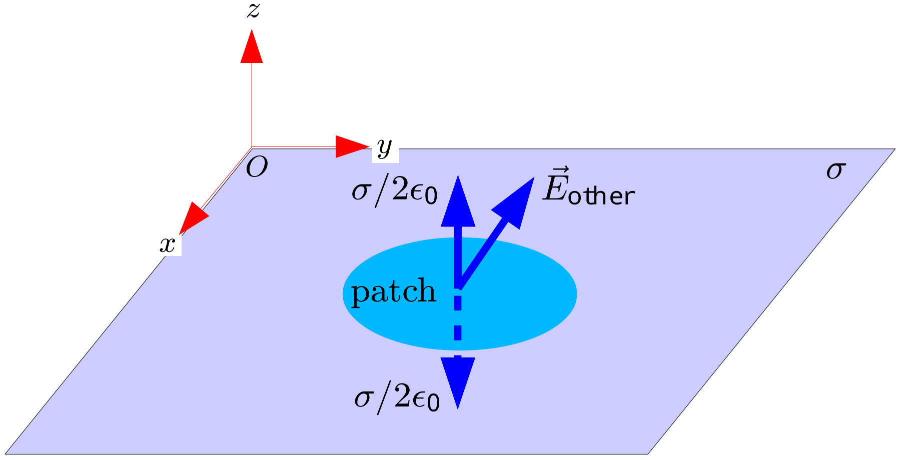
Conductors : Force on a Conductor

The electric fields, above and below the surface, due to the patch are

$$\vec{E}^{\text{patch}}_{\text{above}} = \frac{\sigma}{2\epsilon_0}\hat{n} \quad \text{and} \quad \vec{E}^{\text{patch}}_{\text{below}} = -\frac{\sigma}{2\epsilon_0}\hat{n}$$

Let the electric field at the patch due to the rest of the surface be $\vec{E}_{\text{other}}$. Then,

$$\vec{E}_{\text{above}} = \vec{E}_{\text{other}} + \vec{E}^{\text{patch}}_{\text{above}}$$



So that,

$$\vec{E}_{\text{above}} = \vec{E}_{\text{other}} + \frac{\sigma}{2\epsilon_0}\hat{n}$$

and

$$\vec{E}_{\text{below}} = \vec{E}_{\text{other}} - \frac{\sigma}{2\epsilon_0}\hat{n}$$

Hence, $\vec{E}_{\text{other}} = \frac{1}{2}\left(\vec{E}_{\text{above}} + \vec{E}_{\text{below}}\right)$

That is, the electric field at the patch due to the rest of the charges is the average of $\vec{E}_{\text{above}}$ and $\vec{E}_{\text{below}}$.

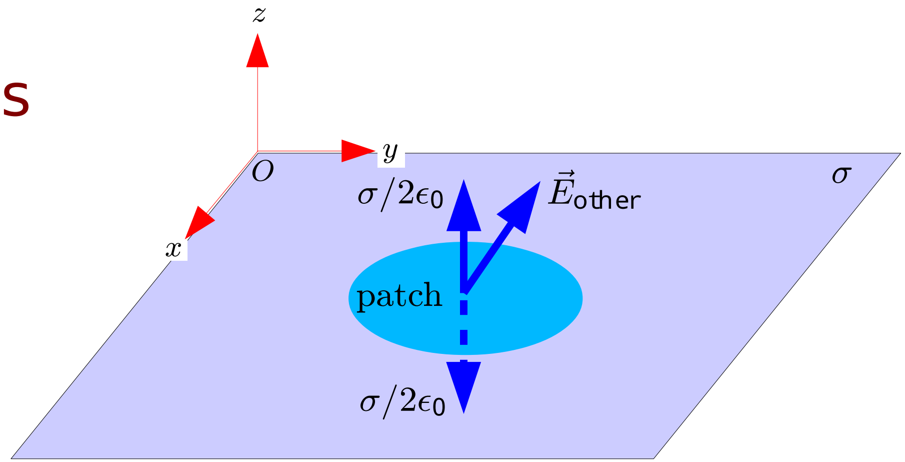
Conductors : Force on a Conductor

To apply the previous result, we note that for the conducting surface with charge density σ, we have

$$\vec{E}_{\text{above}} = \frac{\sigma}{\epsilon_0}\hat{n}, \quad \vec{E}_{\text{below}} = 0$$

Therefore, the average field is

$$\vec{E}_{other} = \frac{1}{2}\left(\frac{\sigma}{\epsilon_0}\hat{n} + \vec{0}\right)$$



The force per unit area $\vec{f}$ is

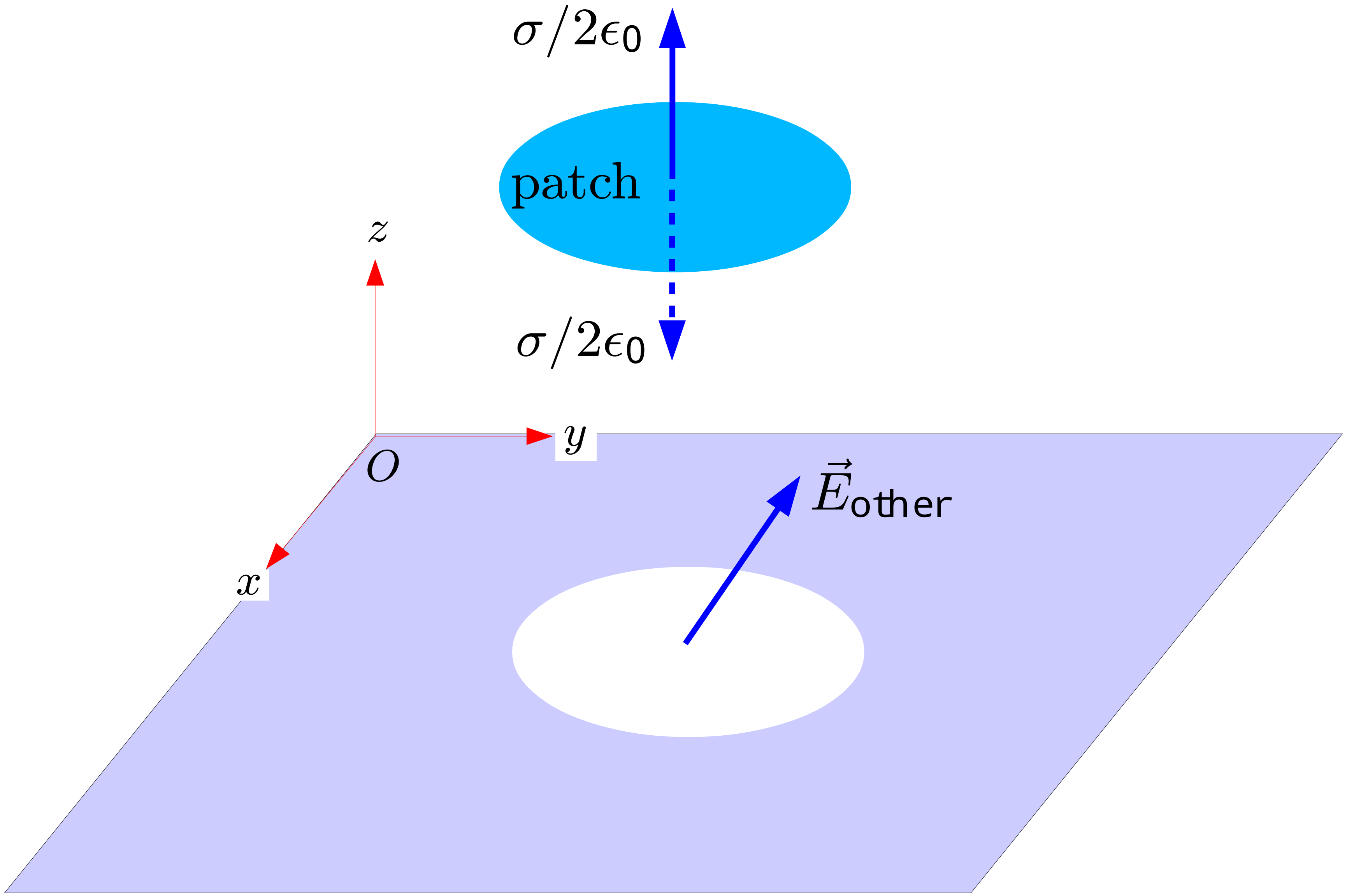
$$\vec{f} = \sigma\vec{E}_{\text{other}} = \frac{\sigma^2}{2\epsilon_0}\hat{n}$$

This is known as the electrostatic pressure. In terms of the electric field just above the conductor, the electrostatic pressure P is,

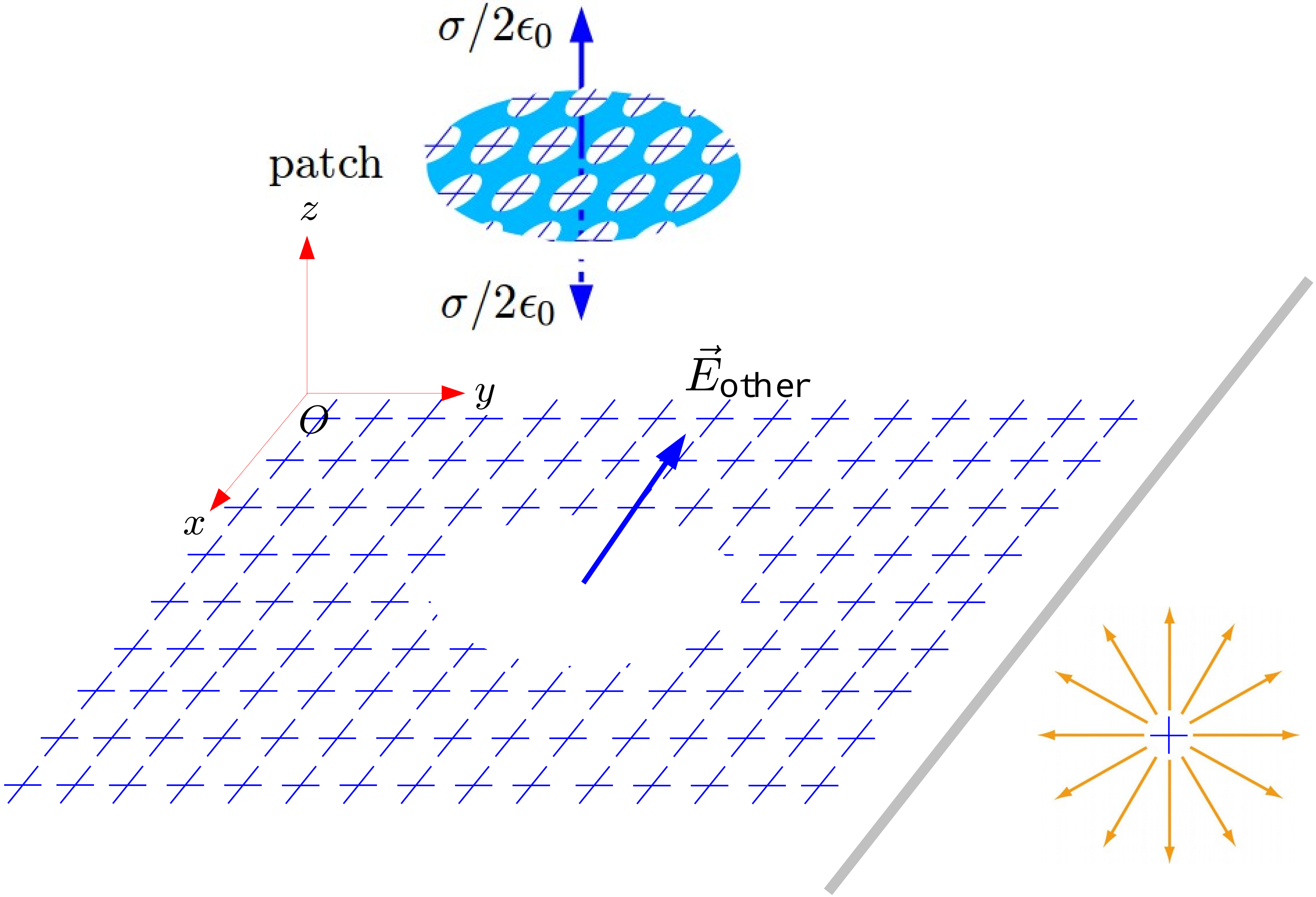
$$\boxed{P = \frac{1}{2}\epsilon_0 E^2}$$

Conductors : Force on a Conductor

More About Electrostatic Pressure



Conductors : Force on a Conductor



Conductors : Force on a Conductor

It can be shown that $\vec{E}_{\text{other}}$ is equal to

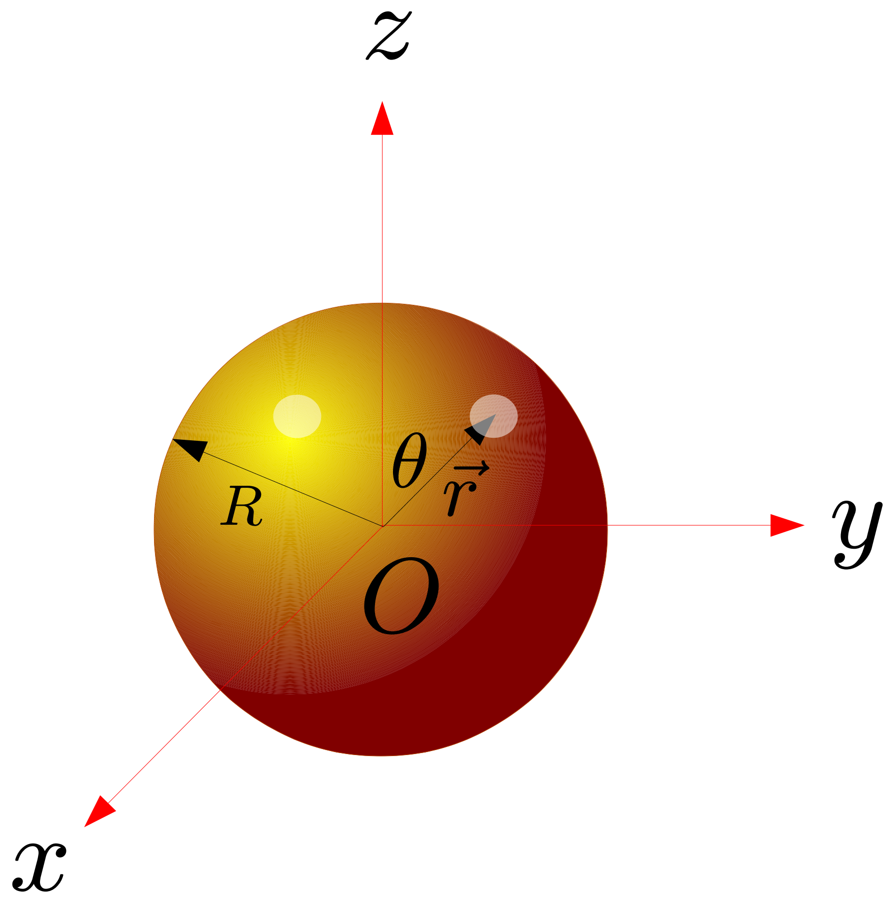
$$\boxed{\vec{E}_{\text{other}} = \frac{\sigma}{2\epsilon_0}\hat{n}}$$

From which we recover the previous result for the electrostatic pressure,

$$\boxed{P = \frac{1}{2}\epsilon_0 E^2}$$

Conductors : Force on a Conductor

Example 2.38: A metal sphere of radius R carries a total charge Q. What is the force of repulsion between the "northern" hemisphere and the "southern" hemisphere?



Conductors : Force on a Conductor

Example 2.38: A metal sphere of radius *R* carries a total charge Q. What is the force of repulsion between the "northern" hemisphere and the "southern" hemisphere?

The surface charge density on the sphere is

$$\sigma = \frac{Q}{4\pi R^2}$$

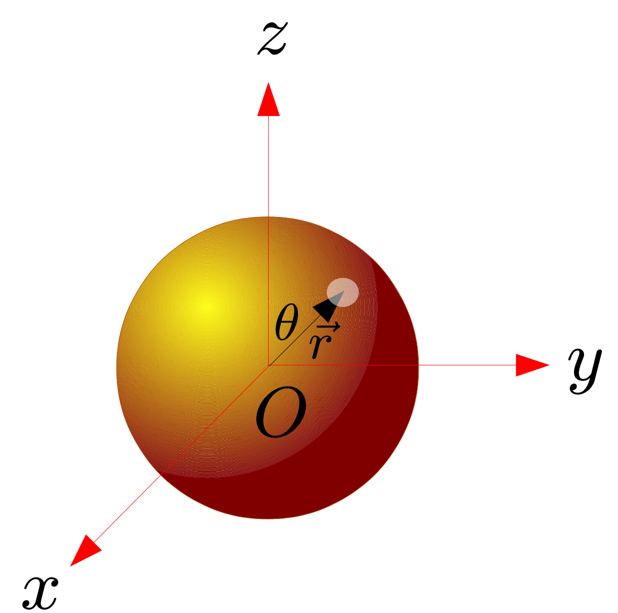
Consider an elemental patch of area *da* containing charge *dq*=σ*da*, as shown in the figure. The electric fields, outside and inside the sphere are

$$\vec{E}_{\text{inside}} = \vec{0}, \quad r < R$$

and $$\vec{E}_{\text{outside}} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}, \quad r > R$$

Therefore,

$$\vec{E}_{\text{avg}} = \vec{E}_{\text{other}} = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}\hat{r},$$



Conductors : Force on a Conductor

The z-component of the force per unit area on *da* is

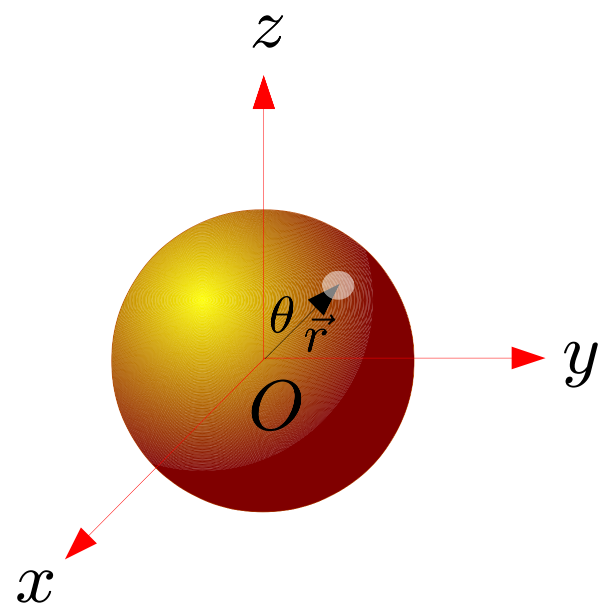
$$f_z = \sigma(\vec{E}_{\text{avg}})_z = \frac{Q}{4\pi R^2}\left(\frac{1}{2}\frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}\hat{r}\cdot\hat{k}\right)$$

Thus, the total force on the "northern" hemisphere

$$F_z = \int f_z da = \frac{Q}{4\pi R^2}\frac{1}{2}\frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}\int_0^{\pi/2}(\cos\theta)R^2\sin\theta d\theta d\phi$$

$$= \left(\frac{Q}{4R}\right)^2\frac{1}{\pi\epsilon_0}\int_0^{\pi/2}\cos\theta\sin\theta d\theta$$

$$= \left(\frac{Q}{4R}\right)^2\frac{1}{\pi\epsilon_0}\left(\frac{1}{2}\sin^2\theta\right)\Big|_0^{\pi/2}$$



The total force F_z is given by

$$F_z = \frac{Q^2}{32\pi\epsilon_0 R^2}$$

Conductors : Force on a Conductor

The surface charge density on the sphere is

$$\sigma = \frac{Q}{4\pi R^2}$$

Consider an elemental patch of area *da* containing charge *dq*=σ*da*. The electric field at the patch due to the rest of the charges is

$$\vec{E}_{\mathrm{other}} = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r},$$

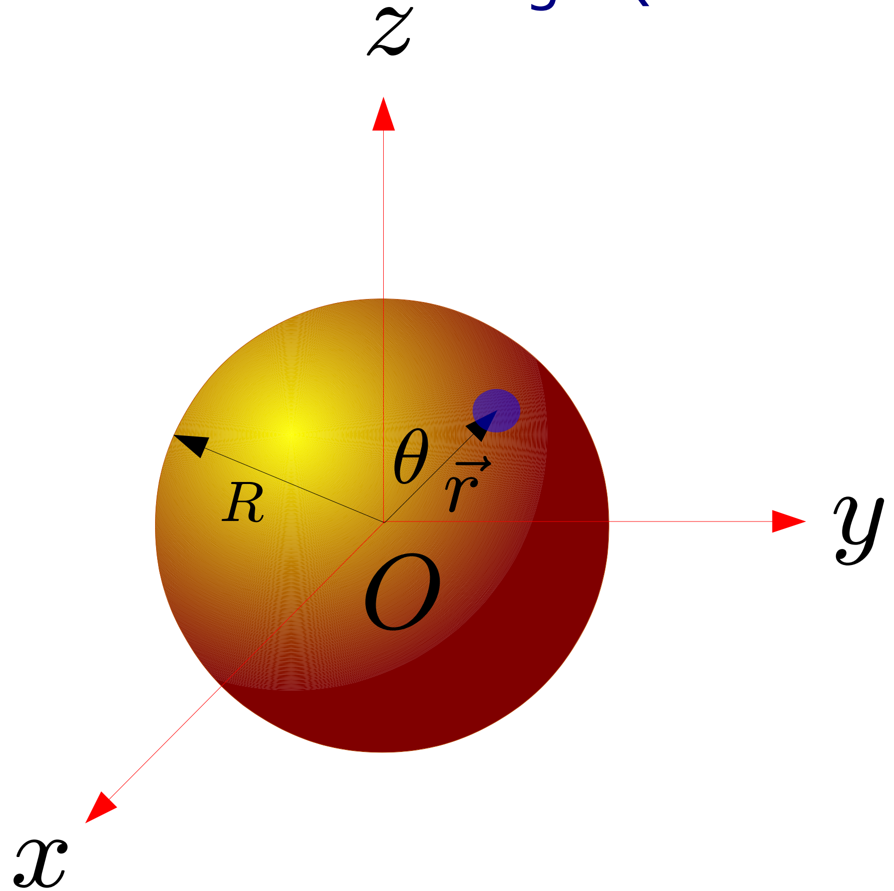
The z-component of the force per unit area on *da* is

$$f_{\mathrm{z}} = \sigma(\vec{E}_{\mathrm{other}})_{\mathrm{z}} = \frac{Q}{4\pi R^2}\left(\frac{1}{2}\frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}\hat{r}\cdot\hat{k}\right)$$

The rest of the calculation is similar to the previous approach.

Conductors : Force on a Conductor

Example 2.43: Find the net force that the southern hemisphere of a uniformly charged sphere exerts on the northern hemisphere. Express your answer in terms of the radius R and the total charge Q?



Conductors : Force on a Conductor

Since the sphere contains a volume charge density, the electric field doesn't suffer any discontinuity, and therefore, the force (per unit volume) on a volume element dτ at $\vec{r}$ containing charge $dq = \rho d\tau$ is given by

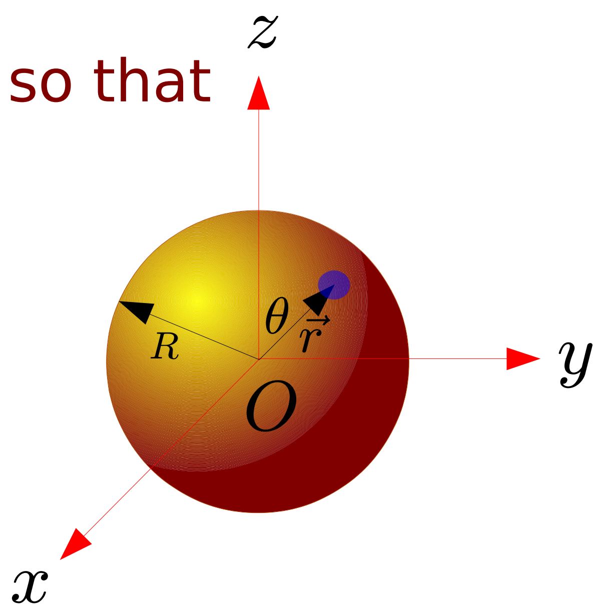
$$\vec{f} = \rho \vec{E}$$

The electric field inside the sphere is

$$\vec{E} = \frac{Qr}{4\pi\epsilon_0 R^3}\hat{r}, \quad r < R$$

Only z-component of the force survives, so that

$$f_z == \frac{3Q}{4\pi R^3}\left(\frac{Qr}{4\pi\epsilon_0 R^3}\hat{r} \cdot \hat{k}\right)$$



Conductors : Force on a Conductor

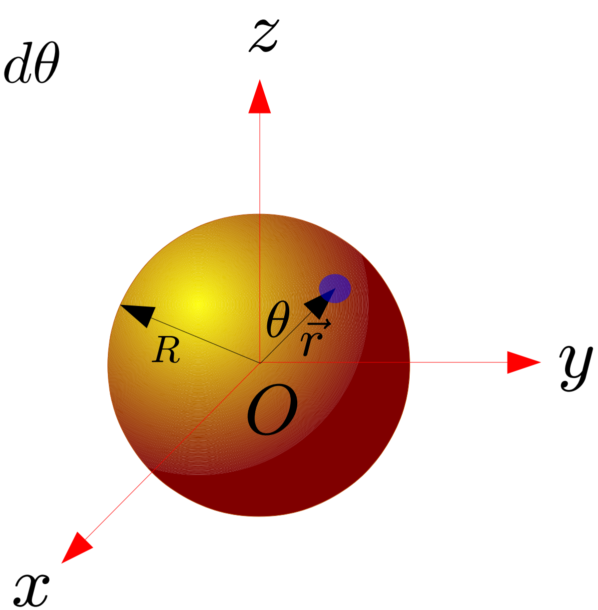
Thus, the total force on the "northern" hemisphere

$$F_z = \int f_z d\tau = \frac{3Q^2}{\epsilon_0 (4\pi R^3)^2} \int_0^R r dr \int_0^{\pi/2} r^2 \cos\theta \sin\theta d\theta \int_0^{2\pi} d\phi$$

$$= \frac{3Q^2}{\epsilon_0 (4\pi R^3)^2} 2\pi \int_0^R r^3 dr \int_0^{\pi/2} \cos\theta \sin\theta d\theta$$

$$= \frac{3Q^2}{\epsilon_0 (4\pi R^3)^2} 2\pi \frac{R^4}{4} \left(\frac{1}{2} \sin^2\theta)\right|_0^{\pi/2}$$

$$= \frac{3Q^2}{\epsilon_0 (4\pi R^3)^2} 2\pi \frac{R^4}{4} \frac{1}{2}$$



The total force F_z is given by

$$F_z = \frac{3Q^2}{64\pi\epsilon_0 R^2}$$

Conductors

Applications

I. Use of copper in making electrical wires

Ordinary cables

Windmill generator

Superconducting cables

Conductors

Summary

1. $\vec{E}_{\text{net}}(\vec{r}) \equiv 0$ inside a conductor.

2. The volume free charge density, $\rho^{\text{free}}_{\text{inside}}(\vec{r}) = 0$ inside a conductor.

3. Any induced charges on a conductor can only reside on surface or surfaces of the conductor – as surface charge distribution, σ_{free}.

4. The entire volume and the surface of a conductor is an equipotential.

5. Just outside of the surface of a conductor, $\vec{E}_{\text{outside}}(\vec{r})$ is perpendicular to the surface.

6. The force per unit area is $\vec{f} = \sigma \vec{E}_{\text{other}} = \dfrac{\sigma^2}{2\epsilon_0}\hat{n}$

7. The electrostatic pressure is $P = \dfrac{1}{2}\epsilon_0 E^2$