Indian Institute of Technology Bombay

MA 106 Linear Algebra

Spring 2021 SRG/DP

Solutions and Marking Scheme for Quiz 2 (held on 9.4.2021)

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1. (i) Define when a square matrix over $\mathbb{C}$ is diagonalizable.

   (ii) Is it true that any square matrix with entries in $\mathbb{C}$ is diagonalizable? Justify your answer. [3 marks]

   Solution: (i) A square matrix $\mathbf{A}$ over $\mathbb{C}$ is said to be **diagonalizable** if it is similar to a diagonal matrix, that is, $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a diagonal matrix for some invertible matrix $\mathbf{P}$ over $\mathbb{C}$. [1]

   (ii) It is **not** true that any square matrix with entries in $\mathbb{C}$ is diagonalizable. [1]

   Justification: For example, the matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has entries over $\mathbb{C}$ and if we had $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$ for some invertible matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$, then both the diagonal entries of $\mathbf{D}$ have to be zero (because similar matrices have the same eigenvalues, and clearly, the only eigenvalue of $\mathbf{A}$ is 0). But then $\mathbf{D}$ is the zero matrix, and hence so is $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$, which is a contradiction. Thus $\mathbf{A}$ is not diagonalizable. [1]

   [*Note that other examples of non-diagonalizable matrices are possible. Give 1 mark for any valid counterexample.*]

2. Consider the linear map $T : \mathbb{R}^3 \to \mathbb{R}^4$ defined by $T\mathbf{x} = \mathbf{A}\mathbf{x}$, for $\mathbf{x} \in \mathbb{R}^3 = \mathbb{R}^{3\times 1}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 3 & -2 \\ 0 & 1 & 3 \\ 1 & -1 & 1 \end{bmatrix}.$$

   Let $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ and $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \mathbf{f}_4)$ denote the standard ordered bases of $\mathbb{R}^3$ and $\mathbb{R}^4$ respectively. Also, let a and b denote the last two digits of your roll number (for example, if your roll number is 200010059, then $a = 5$ and $b = 9$). Find the matrix of T with respect to the ordered basis $(\mathbf{e}_3,\ a\,\mathbf{e}_1 + 2\,\mathbf{e}_2 + b\,\mathbf{e}_3,\ \mathbf{e}_1)$ of $\mathbb{R}^3$ and the ordered basis $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \mathbf{f}_4)$ of $\mathbb{R}^4$. [3 marks]

   Solution: We note that

$$\begin{aligned} T(\mathbf{e}_3) &= 3\mathbf{f}_1 - 2\mathbf{f}_2 + 3\mathbf{f}_3 + \mathbf{f}_4 \\ T(a\mathbf{e}_1 + 2\mathbf{e}_2 + b\mathbf{e}_3) &= (a - 4 + 3b)\mathbf{f}_1 + (-2a + 6 - 2b)\mathbf{f}_2 + (2 + 3b)\mathbf{f}_3 + (a - 2 + b)\mathbf{f}_4 \\ T(\mathbf{e}_1) &= \mathbf{f}_1 - 2\mathbf{f}_2 + 0 \cdot \mathbf{f}_3 + \mathbf{f}_4 \end{aligned}$$

[1]

Hence the matrix of T with respect to the ordered basis $(\mathbf{e}_3,\ a\,\mathbf{e}_1{+}2\,\mathbf{e}_2{+}b\,\mathbf{e}_3,\ \mathbf{e}_1)$ of $\mathbb{R}^3$ and the ordered basis $(\mathbf{f}_1,\ \mathbf{f}_2,\ \mathbf{f}_3,\ \mathbf{f}_4)$ of $\mathbb{R}^4$ is given by

$$\begin{bmatrix} 3 & a-4+3b & 1 \\ -2 & -2a+6-2b & -2 \\ 3 & 2+3b & 0 \\ 1 & a-2+b & 1 \end{bmatrix}. \qquad [2]$$

[*For the first step, give 1 mark for any reasonable attempt of expressing values of the action of T on the elemenets of the given ordered basis of $\mathbb{R}^3$ into the given ordered basis of $\mathbb{R}^4$ (even if there are numerical errors). For the next step, give 2 marks for the correct matrix of T. Note that the second column will depend on the last two digits of the roll number and will have to be checked carefully. Give partial credit of 1 mark if the columns are permuted or if the second column is correct, but there is a mistake in the first or the third column. Also give* 1 *mark if $T(a\mathbf{e}_1 + 2\mathbf{e}_2 + b\mathbf{e}_3)$ is expressed correctly in terms of $\mathbf{f}_1,\ \mathbf{f}_2,\ \mathbf{f}_3,\ \mathbf{f}_4$, but there is an error when writing down the matrix. Give 0 marks if the values of a and b differ from the last two digits of the roll number.*]

3. Let a and b denote the last two digits of your roll number (for example, if your roll number is 200010059, then $a = 5$ and $b = 9$). Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3 & 4 & a & b \end{bmatrix}.$$

Calculate the characteristic polynomial of $\mathbf{A}$. [2 marks]

Solution: The characteristic polynomial of $\mathbf{A}$ is given by

$$p_{\mathbf{A}}(t) = \det(\mathbf{A} - t\mathbf{I}) = \det \begin{bmatrix} -t & 1 & 0 & 0 \\ 0 & -t & 1 & 0 \\ 0 & 0 & -t & 1 \\ 3 & 4 & a & b-t \end{bmatrix} \qquad [1]$$

Expanding by the first row, we find

$$\begin{aligned} p_{\mathbf{A}}(t) &= -t \begin{vmatrix} -t & 1 & 0 \\ 0 & -t & 1 \\ 4 & a & b-t \end{vmatrix} - \begin{vmatrix} 0 & 1 & 0 \\ 0 & -t & 1 \\ 3 & a & b-t \end{vmatrix} \\ &= \left[t^2\left(-t(b-t) - a\right) + t(-4)\right] - 3 \\ &= t^4 - bt^3 - at^2 - 4t - 3. \qquad [1] \end{aligned}$$

[**Remark:** It may be noted that the last row of $\mathbf{A}$ is reflected in the coefficients of the characteristic polynomial of $\mathbf{A}$. This is a special case of a general fact. Namely if $\mathbf{C}$ is an $n \times n$ matrix with 1 on the superdiagonal and 0 elsewhere in the first $n-1$ rows, while the last row is given by $[c_0\ c_1\ \ldots\ c_{n-1}]$, then the characteristic polynomial of $\mathbf{C}$ is given by $p(t) = t^n - c_{n-1}t^{n-1} - \cdots - c_1 t - c_0$. The matrix C is called the **companion matrix** of the polynomial $p(t)$. In particular, we see that every monic polynomial of degree n with coefficients in $\mathbb{K}$ is the characteristic polynomial of some $n \times n$ matrix over $\mathbb{K}$.]

4. Let a and b denote the last two digits of your roll number (for example, if your roll number is 200010059, then $a = 5$ and $b = 9$). Consider the vectors

$$\mathbf{v}_1 = [2\ 0\ 0]^\mathsf{T}, \quad \mathbf{v}_2 = [a\ 3\ 0]^\mathsf{T}, \quad \mathbf{v}_3 = [b\ 2\ 1]^\mathsf{T}.$$

in $\mathbf{x} \in \mathbb{R}^3 = \mathbb{R}^{3\times 1}$ and let V be the vector subspace of $\mathbb{R}^3$ spanned by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Use the Gram-Schmidt orthonormalization process to find an orthonormal basis of V. [2 marks]

Solution: Applying Gram-Schmidt orthogonalization process to $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, we obtain

$$\begin{aligned}
\mathbf{y}_1 &= \mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \\
\mathbf{y}_2 &= \mathbf{v}_2 - P_{\mathbf{y}_1}(\mathbf{v}_2) = \begin{bmatrix} a \\ 3 \\ 0 \end{bmatrix} - \frac{\langle \mathbf{y}_1, \mathbf{v}_2 \rangle}{\langle \mathbf{y}_1, \mathbf{y}_1 \rangle} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ 3 \\ 0 \end{bmatrix} - \frac{2a}{4} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \\
\mathbf{y}_3 &= \mathbf{v}_3 - P_{\mathbf{y}_1}(\mathbf{v}_3) - P_{\mathbf{y}_2}(\mathbf{v}_3) = \begin{bmatrix} b \\ 2 \\ 1 \end{bmatrix} - \frac{2b}{4} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} - \frac{6}{9} \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.
\end{aligned}$$

This gives an orthogonal basis $(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3)$ of V. To obtain an orthonormal basis, we consider $\frac{\mathbf{y}_j}{\|\mathbf{y}_j\|}$, which is clearly $\mathbf{e}_j$ for $j = 1, 2, 3$. Thus an orthonormal basis of V is given by the standard unit vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$.

[**Marking Scheme:** *1 mark for using the Gram-Schmidt process correctly and 1 mark for the correct orthonormal basis. Also give 1 mark (out of 2) if the answer is not fully correct, but one among* $\mathbf{y}_2, \mathbf{y}_3$ *is correct and one among* $\mathbf{e}_2, \mathbf{e}_3$ *is correctly obtained.*]