Outline

1. Divergence in Curvilinear Co-ordinates: Using Divergence theorem(also known as Guass's theorem)
2. Curl in Curvilinear Co-ordinates: Using Stoke's theorem
3. Divergence of field produced by a point charge/mass: Introduction of Delta-Dirac function.

Objectives

1. To derive formulae for Divergence and Curl in Curvilinear Co-ordinates
2. To get a basic idea about Delta-Dirac function and how it represents point charges/masses.

Recap

- Polar, Cylindrical and Spherical Co-ordinates.
- Displacement vector in Generalized Orthogonal Curvilinear Co-ordinates

$$d\vec{l} = \sum_{i=1,2}^{3} \hat{u}_i h_i du_i$$

- Gradient in Curvilinear Co-ordinates.

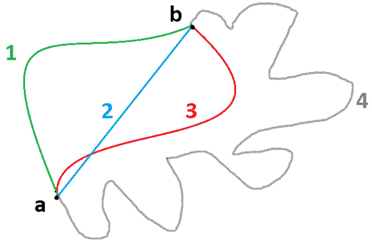
$$\nabla = \sum_{i=1,2}^{3} \hat{u}_i \frac{1}{h_i} \frac{\partial}{\partial u_i}$$

Path Independence of Integral of a Gradient Field

- Let u and $u+du$ be two points in curvilinear co-ordinates with infinitesimal distance between them.
- Let df be the difference between values of a scalar function f at $u+du$ and u.

$$
\begin{aligned}
df &= f(u+du) - f(u) \\
&= f(u) + \sum_{i=1,2}^{3} \frac{\partial f}{\partial u_i} du_i + O(h^2) - f(u) \\
&= \sum_{i=1,2}^{3} \frac{\partial f}{\partial u_i} du_i \\
&= \sum_{i=1,2}^{3} \left(\hat{u}_i \frac{1}{h_i} \frac{\partial f}{\partial u_i} \right) \cdot (\hat{u}_i h_i du_i) \\
&= \nabla f \cdot d\vec{l}
\end{aligned}
$$

Path Independence of Integral of a Gradient Field



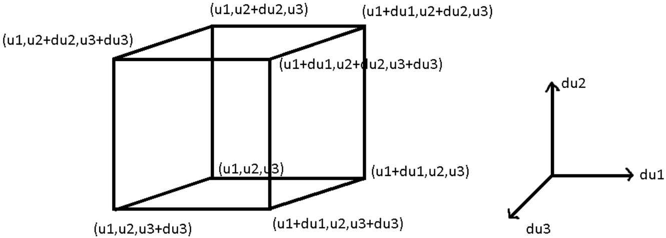
- Let a, b be two points in curvilinear co-ordinates, and $f(a), f(b)$ be the values of function f at a, b respectively.

$$
\begin{aligned}
f(b) - f(a) &= \int_a^b df \\
&= \int_a^b (\nabla f) \cdot d\vec{l}
\end{aligned}
$$

Divergence in Curvilinear Co-ordinates

- The net flux flowing outward from a infinitesimal volume element is calculated and used along with Divergence theorem in order to obtain the formula for Divergence in curvilinear co-ordinates.
- Consider 8-points, $(u_1, u_2, u_3), (u_1 + du_1, u_2, u_3), (u_1, u_2 + du_2, u_3), (u_1 + du_1, u_2 + du_2, u_3), (u_1, u_2, u_3 + du_3), (u_1 + du_1, u_2, u_3 + du_3), (u_1, u_2 + du_2, u_3 + du_3), (u_1 + du_1, u_2 + du_2, u_3 + du_3)$ that are infinitesimally close and approximately form an infinitesimal parallelepiped.

- As only Orthogonal Curvilinear Co-ordinates are being considered, the 8 points will approximately form a cuboid, which is a special case of parallelepiped with right angle between all adjacent sides.

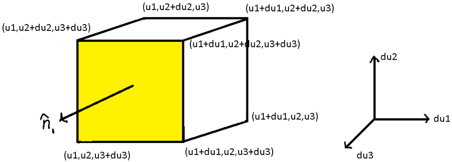


Divergence in Curvilinear Co-ordinates

- Consider a vector field $\vec{V} = V_1\hat{u}_1 + V_2\hat{u}_2 + V_3\hat{u}_3$
- The outward normal to the surface is considered to be the positive direction of flow of flux.
- Flux through all 6 faces should be calculated.
- Later the fluxes should be added to obtain the net flux through the volume element.

Divergence in Curvilinear Co-ordinates

- Let's start with face-1 formed by the co-ordinates $(u_1, u_2, u_3+du_3), (u_1+du_1, u_2, u_3+du_3), (u_1, u_2+du_2, u_3+du_3), (u_1+du_1, u_2+du_2, u_3+du_3)$.

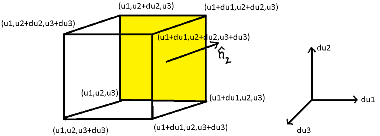


- Flux of a vector $\vec{P}$ through an area $|d\vec{A}|$ and outward normal $\hat{n}$is $|d\vec{A}|\vec{P}\cdot\hat{n}$. In case of face-1 the normal vector $\hat{n}$ is along $\hat{u}_3$.
- Flux of $\vec{V}$ through face-1 is, [1]

$$
\begin{aligned}
&= |h_1 du_1 \hat{u}_1 \times h_2 du_2 \hat{u}_2|\vec{V}\cdot\hat{n}_1 \\
&= (h_1 du_1 h_2 du_2 \hat{u_3})\vec{V}\cdot\hat{u}_3 \\
&= (V_3 h_1 h_2)_{u_3+du_3} du_1 du_2
\end{aligned}
$$

[1] Remember that h_1, h_2, h_3 are also dependent on u_1, u_2, u_3.

- Let's now calculate the flux of $\vec{V}$ through face-2(the one opposite to the face-1) with vertices $(u_1, u_2, u_3), (u_1 + du_1, u_2, u_3), (u_1, u_2 + du_2, u_3), (u_1 + du_1, u_2 + du_2, u_3)$,



- Flux of a vector $\vec{P}$ through an area $|d\vec{A}|$ and outward normal $\hat{n}$ is $|d\vec{A}|\vec{P}\cdot\hat{n}$. Incase of face-2 the normal vector $\hat{n}$ is along $-\hat{u}_3$
- Flux of $\vec{V}$ through face-2 is,

$$\begin{aligned} &= |h_1 du_1 \hat{u}_1 \times h_2 du_2 \hat{u}_2| \vec{V} \cdot \hat{n}_1 \\ &= (h_1 du_1 h_2 du_2 \hat{u}_3) \vec{V} \cdot -\hat{u}_3 \\ &= -(V_3 h_1 h_2)_{u_3} du_1 du_2 \end{aligned}$$

Divergence in Curvilinear Co-ordinates

- Net flux through faces 1,2 is,

$$
\begin{aligned}
&= [(V_3 h_1 h_2)_{u_3+du_3} - (V_3 h_1 h_2)_{u_3}] du_1 du_2 \\
&= \frac{\partial (V_3 h_1 h_2)}{\partial u_3} du_1 du_2 du_3 \\
&= \frac{1}{h_1 h_2 h_3} \cdot \frac{\partial (V_3 h_1 h_2)}{\partial u_3} \cdot d\tau
\end{aligned}
$$

where $d\tau$ is the volume of the cuboid formed by the 8 points.

- Similarly, net flux through faces 3,4 and 5,6 can also be obtained.
- Adding all the flux passing through faces 1,2,3,4,5,6 with their signs will result in,

$$
= \frac{d\tau}{h_1 h_2 h_3} \left[\frac{\partial (V_3 h_1 h_2)}{\partial u_3} + \frac{\partial (V_2 h_3 h_1)}{\partial u_2} + \frac{\partial (V_1 h_2 h_3)}{\partial u_1} \right]
$$

Divergence in Curvilinear Co-ordinates

- Using Divergence/Guass's theorem,

$$\oint_S \vec{V} \cdot d\vec{a} = \int_{d\tau} (\nabla \cdot \vec{V}) dV$$

- The L.H.S of the above equation gives the net outward flux through the volume element, which we have already calculated.
- $\nabla \cdot \vec{V}$ can be considered to be constant for an infinitesimal volume element $d\tau$. $(\int_{d\tau} (\nabla \cdot \vec{V}) dV = (\nabla \cdot \vec{V}) d\tau)$

$$\begin{aligned}
(\nabla \cdot \vec{V}) d\tau &= \frac{d\tau}{h_1 h_2 h_3} \left[\frac{\partial (V_3 h_1 h_2)}{\partial u_3} + \frac{\partial (V_2 h_3 h_1)}{\partial u_2} + \frac{\partial (V_1 h_2 h_3)}{\partial u_1} \right] \\
\nabla \cdot \vec{V} &= \frac{1}{h_1 h_2 h_3} \left[\frac{\partial (V_3 h_1 h_2)}{\partial u_3} + \frac{\partial (V_2 h_3 h_1)}{\partial u_2} + \frac{\partial (V_1 h_2 h_3)}{\partial u_1} \right] \\
&= \frac{1}{h_1 h_2 h_3} \sum_{i=1}^{3} \left[\frac{\partial (V_i h_j h_k)}{\partial u_i} \right]
\end{aligned}$$

with (i, j, k) in cyclic order, and $i \neq j \neq k$

Divergence in various Curvilinear Co-ordinates

- Spherical co-ordinates:

$$\nabla\cdot\vec{V} = \frac{1}{r^2\sin\theta}\left[\frac{\partial(V_r r^2\sin\theta)}{\partial r}+\frac{\partial(V_\theta r\sin\theta)}{\partial\theta}+\frac{\partial(V_\phi r)}{\partial\phi}\right]$$

$$= \frac{1}{r^2}\frac{\partial(r^2 V_r)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial(V_\theta\sin\theta)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial(V_\phi)}{\partial\phi}$$

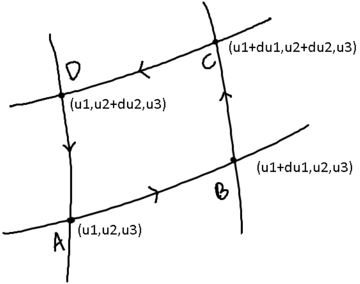
- Cylindrical co-ordinates:

$$\nabla\cdot\vec{V} = \frac{1}{\rho}\left[\frac{\partial(V_\rho\rho)}{\partial\rho}+\frac{\partial(V_\theta)}{\partial\theta}+\frac{\partial(V_z\rho)}{\partial z}\right]$$

$$= \frac{1}{\rho}\frac{\partial(\rho V_\rho)}{\partial\rho}+\frac{1}{\rho}\frac{\partial V_\theta}{\partial\theta}+\frac{\partial V_z}{\partial z}$$

- Stoke's Theorem is used to obtain curl in curvilinear co-ordinates.

$$\oint \vec{F} \cdot d\vec{l} \quad = \quad \int \int_S (\nabla \times \vec{F}) \cdot d\vec{A}$$



- $\oint \vec{F} \cdot d\vec{l}$ can be calculated by adding the path integrals of $\vec{F}$ along AB, BC, CD, DA.

Curl in Curvilinear Co-ordinates

- $A \to B : \int_A^B \vec{F} \cdot d\vec{l} = (F_1 h_1)_{u_2} du_1$
- $B \to C : \int_B^C \vec{F} \cdot d\vec{l} = (F_2 h_2)_{u_1 + du_1} du_2$
- $C \to D : \int_C^D \vec{F} \cdot d\vec{l} = -(F_1 h_1)_{u_2 + du_2} du_1$
- $D \to A : \int_A^B \vec{F} \cdot d\vec{l} = -(F_2 h_2)_{u_1} du_2$

$$
\begin{aligned}
\oint \vec{F} \cdot d\vec{l} &= du_1 [(F_1 h_1)_{u_2} - (F_1 h_1)_{u_2 + du_2}] \\
&\quad + du_2 [(F_2 h_2)_{u_1 + du_1} - (F_2 h_2)_{u_!}] \\
&= du_1 du_2 \left(\frac{\partial (F_2 h_2)}{\partial u_1} - \frac{\partial (F_1 h_1)}{\partial u_2} \right) \\
&= \frac{1}{h_1 h_2} \left(\frac{\partial (F_2 h_2)}{\partial u_1} - \frac{\partial (F_1 h_1)}{\partial u_2} \right) \hat{u}_3 \cdot d\vec{A}_{12} \\
&= \frac{1}{h_1 h_2} \int \int_{dA_{12}} \left(\frac{\partial (F_2 h_2)}{\partial u_1} - \frac{\partial (F_1 h_1)}{\partial u_2} \right) \hat{u}_3 \cdot d\vec{S}
\end{aligned}
$$

Curl in Curvilinear Co-ordinates

$$\oint \vec{F} \cdot d\vec{l} = \int\int_{dA_{12}} \frac{1}{h_1 h_2} \left(\frac{\partial (F_2 h_2)}{\partial u_1} - \frac{\partial (F_1 h_1)}{\partial u_2} \right) \hat{u}_3 \cdot d\vec{S}$$

- Comparing this with Stoke's theorem,

$$(\nabla \times \vec{F})_{u_3} = \frac{1}{h_1 h_2} \left(\frac{\partial (F_2 h_2)}{\partial u_1} - \frac{\partial (F_1 h_1)}{\partial u_2} \right)$$

- The above equation is obtained by changing u_1, u_2 and keeping u_3 as constant.
- Similarly, one can keep u_1 constant by changing u_2, u_3 to obtain the component of curl along $\hat{u}_1$.

$$(\nabla \times \vec{F})_{u_1} = \frac{1}{h_2 h_3} \left(\frac{\partial (F_3 h_3)}{\partial u_2} - \frac{\partial (F_2 h_2)}{\partial u_3} \right)$$

- To obtain component of curl along $\hat{u}_2$ by changing u_1, u_3.

$$(\nabla \times \vec{F})_{u_2} = \frac{1}{h_1 h_3} \left(\frac{\partial (F_1 h_1)}{\partial u_3} - \frac{\partial (F_3 h_3)}{\partial u_2} \right)$$

- Curl in curvilinear co-ordinates as a whole can be written as

$$
\begin{aligned}
\nabla \times \vec{F} &= \frac{1}{h_2 h_3}\left(\frac{\partial(F_3 h_3)}{\partial u_2} - \frac{\partial(F_2 h_2)}{\partial u_3}\right)\hat{u}_1 \\
&+ \frac{1}{h_1 h_3}\left(\frac{\partial(F_1 h_1)}{\partial u_3} - \frac{\partial(F_3 h_3)}{\partial u_2}\right)\hat{u}_2 \\
&+ \frac{1}{h_1 h_2}\left(\frac{\partial(F_2 h_2)}{\partial u_1} - \frac{\partial(F_1 h_1)}{\partial u_2}\right)\hat{u}_3 \\
\nabla \times \vec{F} &= \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 \hat{u}_1 & h_2 \hat{u}_2 & h_3 \hat{u}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ F_1 h_1 & F_2 h_2 & F_3 h_3 \end{vmatrix}
\end{aligned}
$$

Curl in Various Curvilinear Co-ordinates

- Spherical co-ordinates:

$$\nabla \times \vec{V} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix}$$

- Cylindrical co-ordinates:

$$\nabla \times \vec{V} = \frac{1}{\rho} \begin{vmatrix} \hat{\rho} & \rho\hat{\theta} & \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ F_\rho & \rho F_\theta & F_z \end{vmatrix}$$

Divergence of field produced by a point source

- Gravitational fields and Electrostatic fields both obey the inverse square law.
- That is, these fields can be written in the following form,

$$\vec{V} = V_0 \frac{\hat{r}}{r^2}$$

- For the sake of simplicity, let's consider $V_0 = 1$.
- What is Divergence of the following field?

$$\nabla \cdot \vec{V} = \frac{1}{r^2}\frac{\partial(r^2 V_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(V_\theta \sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial(V_\phi)}{\partial\phi}$$

Divergence of field produced by a point source

- As the field is radial, $V_\theta = 0, V_\phi = 0$.

$$\begin{aligned}\nabla \cdot \vec{V} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot \frac{1}{r^2}\right) \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}(1) \\ &= 0\end{aligned}$$

- But is the divergence really zero? Can this be checked using Divergence/Guass's theorem?

$$\begin{aligned}\int\int\int(\nabla \cdot \vec{V})d\tau &= \oint_S \vec{V} \cdot d\vec{S} \\ &= \oint_S \frac{\hat{r}}{r^2} \cdot (r^2 \sin\theta d\theta d\phi)\hat{r} \\ &= \int_0^{2\pi}\int_0^{\pi} \sin\theta d\theta d\phi \\ &= 4\pi\end{aligned}$$

Domains are important

- How can the volume integral of a zero function be non-zero? What's going wrong?
- While calculating divergence, we have considered $\left(r^2 \cdot \frac{1}{r^2}\right)$ to be 1, which is true for all $r \in \mathbb{R} - \{0\}$

$$r^2 \cdot \frac{1}{r^2} \quad = \quad 1 \qquad \text{if r} \neq 0$$

- Divergence of this field is zero everywhere except at origin.

$$\begin{aligned} \nabla \cdot \vec{V} \quad &= \quad 0 \quad \forall r \neq 0 \\ &\neq \quad 0 \quad \text{if } r = 0 \end{aligned}$$

- The total flux is coming from a single point, so the flux density(divergence) at origin is not finite(in other words infinite).

Dirac-Delta Function

- There is a standard mathematical function with similar properties: Dirac-Delta Function;

$$\begin{aligned} \delta(x) &= 0 \quad \forall x \neq 0 \\ &\neq 0 \quad \text{if } x = 0 \end{aligned}$$

- In fact the Dirac-Delta function has an additional property,

$$\begin{aligned} \int_{-\varepsilon}^{+\varepsilon} \delta(x) dx &= 1 \quad \forall \varepsilon > 0 \\ \int_{-\varepsilon}^{+\varepsilon} f(x)\delta(x) dx &= f(0) \quad \forall \varepsilon > 0 \\ \int_a^b f(x)\delta(x - x_0) dx &= f(x_0) \quad \text{if } a < x_0 < b \\ &= 0 \quad else \end{aligned}$$

Divergence of field produced by a point source

- Assume the divergence to be the following,

$$\begin{aligned}\nabla \cdot \vec{V} &= 4\pi\delta^3(\vec{r})\\ &= 4\pi\delta(x)\delta(y)\delta(z)\end{aligned}$$

- Checking if the assumption is correct or wrong,

$$\begin{aligned}\int\int\int(\nabla \cdot \vec{V})d\tau &= \int\int\int 4\pi\delta(x)\delta(y)\delta(z)dxdydz\\ &= 4\pi\left(\int\delta(x)dx\right)\left(\int\delta(y)dy\right)\left(\int\delta(z)dz\right)\\ &= 4\pi\end{aligned}$$

- This result is in agreement with the result obtained from Divergence/Gauss's theorem.

Density of Point Mass/Charge:

- Density of a point mass/charge is infinite as a finite amount of mass/charge is contained at a single point(volume = 0).

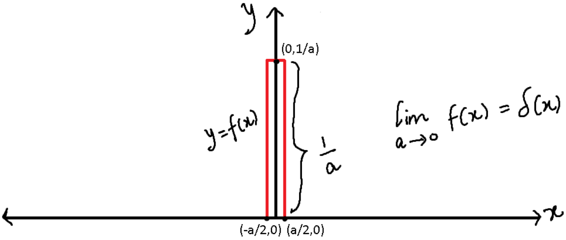
$$\begin{aligned}
\rho(\vec{r}) &= 0 \quad \text{if } r \neq 0 \\
&\neq 0 \quad \text{if } r = 0 \\
\int_V \rho(\vec{r}) d\tau &= q
\end{aligned}$$

- These are the exact properties of the Dirac-Delta function.

Dirac-Delta Visualised

- How can one think of a point source?
  - A sphere containing charge q with radius tending to zero,
  - A cylinder containing charge q with height and radius tending to zero,etc.
- How can one think of Dirac Delta?
  - Rectangular function of unit area with width tending to zero. Mathematically,

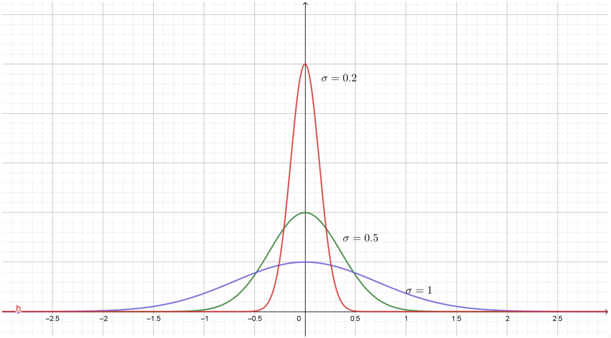
$$
\begin{aligned}
\delta(x) &= \lim_{a\to 0}\left(\frac{1}{a}\right) \quad \text{if } x < \frac{|a|}{2} \\
&= 0 \quad \text{otherwise.}
\end{aligned}
$$



Delta-Dirac Visualised

- How can one think of Dirac Delta?
  - Guassian/Normal distribution with variance tending to zero. Inother words guassian function with $\sigma \to 0$. Mathematically,

$$\delta(x) \quad = \quad \lim_{\sigma \to 0} \frac{1}{\sigma\sqrt{\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$



Some Interesting Properties

- $\delta(-x) = \delta(x)$
- $\delta(ax) = \frac{1}{|a|}\delta(x)$
- $\delta(g(x)) = \sum_i \frac{\delta(x-x_i)}{|g'(x_i)|}$; where $x_i \in$ zeroes of g(x)
- Let, $\Theta(x) = \int_{-\infty}^{x} \delta(t)dt$, then $\delta(x) = \frac{d\Theta(x)}{dx}$.

$$\begin{aligned} \Theta(x) &= 0 \quad \text{if } x < 0, \\ &= 1 \quad \text{if } x > 0. \end{aligned}$$

  $\Theta(x)$ is called the Heaviside step function.
- $\int\int\int_V \delta(\vec{r}-\vec{a})f(\vec{r})d\tau = f(\vec{a})$, where $\vec{a}$ is inside the volume V.