PH108: Basics of Electricity and Magnetism
Mid-Sem Exam (2022)

Maximum Marks: 30 Date: June 1, 2022, Time: 9:30 AM - 11:30 AM

1. Consider a spherical charge distribution given by

$$\rho(r) = \rho_0(\tfrac{r^2}{a^2} - 1) \quad \text{for } r \leq a,$$
$$\rho(r) = 0 \quad \text{for } r > a,$$

where ρ_0 and a are constants. Calculate

(a) Electric field for $r < a$
Soln: For computing $\mathbf{E}_{in}$, the field inside the sphere, consider a spherical Gaussian surface centered at the origin with radius $r \leq a$. Using the spherical symmetry of the problem, the Gauss's Law yields

$$E_{in} 4\pi r^2 = \frac{q_{enc}}{\epsilon_0} = \frac{\int_0^r \rho(r') 4\pi r'^2 dr'}{\epsilon_0},$$

which yields

$$\boxed{\mathbf{E}_{in} = \frac{\rho_0}{\epsilon_0}\left(\frac{r^2}{5a^2} - \frac{1}{3}\right)\mathbf{r}}.$$

(b) Electric potential for $r < a$
Soln: The potential is obtained using

$$V(r) = -\int_a^r \mathbf{E}_{in} \cdot \mathbf{dr} - \int_\infty^a \mathbf{E}_{out} \cdot \mathbf{dr}$$

using $\mathbf{E}_{out}$ computed in part (c), it turns out to be

$$V(r) = \frac{\rho_0}{\epsilon_0}\left[\left(\frac{(a^4 - r^4)}{20a^2}\right) - \frac{(a^2 - r^2)}{6}\right] + \frac{Q}{4\pi\epsilon_0 a}$$

Using the value of Q obtained below

$$\boxed{V(r) = \frac{\rho_0}{\epsilon_0}\left[\frac{r^2}{6} - \frac{r^4}{20a^2} - \frac{a^2}{4}\right]}$$

(c) Electric field for $r > a$.
Soln: Because of the spherical symmetry, the electric field for $r > a$ will be given by

$$\mathbf{E}_{out} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}$$

where Q is the total charge in the distribution given by

$$Q = \int_0^a \rho(r) 4\pi r^2 dr = 4\pi\rho_0 a^3\left(\frac{1}{5} - \frac{1}{3}\right) = -\frac{8\pi\rho_0 a^3}{15}$$

so that

$$\boxed{\mathbf{E}_{out} = -\frac{2\rho_0 a^3}{15\epsilon_0 r^2}\hat{\mathbf{r}}}$$

(d) Electric potential for $r > a$

Soln:

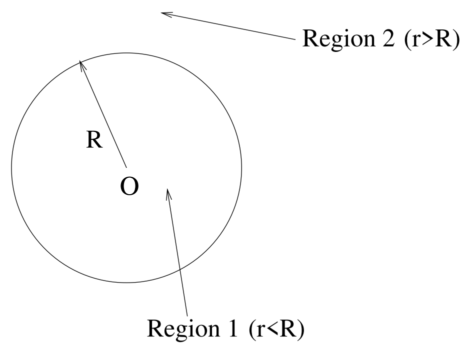
$$\boxed{V_{out}(r) = \frac{Q}{4\pi\epsilon_0 r} = -\frac{2\rho_0 a^3}{15\epsilon_0 r}}$$

Express your answers in terms of r, and the constants of the problem. **[1+1+1+1]**

2. A spherical shell of radius R has a surface charge distribution $\sigma(\theta)$, where θ is a spherical polar coordinate. This charge distribution produces the potential $V(\mathbf{r}) = V_0(2z^2 - x^2 - y^2)$, for $r \leq R$, where V_0 is a constant.

(a) Write $V(\mathbf{r})$ in spherical polar coordinates.

Soln:



Using $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$, we obtain

$$\begin{aligned}(2z^2 - x^2 - y^2) &= 2r^2\cos^2\theta - r^2\sin^2\theta\cos^2\phi - r^2\sin^2\theta\sin^2\phi \\ &= 2r^2\cos^2\theta - r^2\sin^2\theta \\ &= r^2(3\cos^2\theta - 1) \\ &= 2r^2 P_2(\cos\theta)\end{aligned}$$

so the potential in region 1 $V_1(r,\theta)$ (valid for $r \leq R$) can be written as

$$\boxed{V_1(r,\theta) = 2V_0 r^2 P_2(\cos\theta).}$$

Expression $V_1(r,\theta) = V_0 r^2(3\cos^2\theta - 1)$ will also get full marks.

(b) There are no charges other than $\sigma(\theta)$ anywhere is space. Evaluate $\sigma(\theta)$.

Soln: Because the charge distribution is localized, so the potential must vanish at infinity, i.e, $V_2(r \to \infty, \theta) = 0$. So the only possible form for $V_2(r,\theta)$

$$V_2(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta).$$

Potential must be continuous at $r = R$, i.e, $V_1(R,\theta) = V_2(R,\theta)$

$$2V_0 R^2 P_2(\cos\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta),$$

leading to

$$\begin{aligned}B_2 &= 2V_0 R^5 \\ B_l &= 0, \quad \text{for } l \neq 2,\end{aligned}$$

so that

$$V_2(r,\theta) = \frac{2V_0R^5}{r^3}P_2(\cos\theta).$$

$\sigma(\theta)$ can be obtained from the discontinuity condition on the normal components of the electric field across a surface charge density

$$E_{n2} - E_{n1} = \frac{\sigma}{\epsilon_0} \tag{1}$$

Here the discontinuity is on the surface of the sphere of radius R, so there $E_n = -\partial V/\partial r$. So Eq. 1 leads to

$$\frac{\partial V_1}{\partial r}|_{r=R} - \frac{\partial V_2}{\partial r}|_{r=R} = \frac{\sigma}{\epsilon_0}$$

or

$$\boxed{\sigma(\theta) = \epsilon_0\{4V_0R + 6V_0R\}P_2(\cos\theta) = 10\epsilon_0V_0RP_2(\cos\theta).}$$

[1+2]

3. An electric field is expressed in the cylindrical polar coordinates (s,ϕ,z) as

$$\mathbf{E} = C(2sz\sin\phi\hat{e}_s + sz\cos\phi\hat{e}_\phi + s^2\sin\phi\hat{e}_z),$$

where C is a constant and $\hat{e}_s$, $\hat{e}_\phi$, and $\hat{e}_z$ are the unit vectors of the coordinate system.

(a) What is the electrostatic potential corresponding to this $\mathbf{E}$?
Soln: We have

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial s}\hat{e}_s - \frac{1}{s}\frac{\partial V}{\partial \phi}\hat{e}_\phi - \frac{\partial V}{\partial z}\hat{e}_z$$

comparison of LHS and RHS for the given $\mathbf{E}$ yields the three partial differential equations

$$\begin{aligned} -\frac{\partial V}{\partial s} &= 2Csz\sin\phi \\ -\frac{1}{s}\frac{\partial V}{\partial \theta} &= Csz\cos\phi \\ -\frac{\partial V}{\partial z} &= Cs^2\sin\phi \end{aligned}$$

Integration of the first equation yields $V(s,\phi,z) = -Cs^2z\sin\phi + f(\phi,z)$. Substituting this in the second equation above yields $-\frac{1}{s}\frac{\partial f}{\partial \phi} = 0$ which upon integration yields $f = f(z) + c$, where c is some constant. So we have now $V(s,\phi,z) = -Cs^2z\sin\phi + f(z) + c$. Substituting this in the third equation above yields $-\frac{df}{dz} = 0$, leading to the final result

$$\boxed{V(s,\phi,z) = -Cs^2z\sin\phi + c,}$$

where c is a constant.

(b) What is the charge density $\rho(s,\phi,z)$ giving rise to this $\mathbf{E}$?
Soln: From the first Maxwell equation we have

$$\rho = \epsilon_0\nabla\cdot\mathbf{E}$$

which yields

$$\begin{aligned}\rho(r,\theta,z) &= C\epsilon_0\left\{\frac{1}{s}\frac{\partial(s(2sz\sin\phi))}{\partial s}+\frac{1}{s}\frac{\partial(sz\cos\phi)}{\partial\phi}+\frac{\partial(s^2\sin\phi)}{\partial z}\right\}\\ &= C\epsilon_0\{4z\sin\phi - z\sin\phi\} = 3C\epsilon_0 z\sin\phi\end{aligned}$$

Thus

$$\boxed{\rho(r,\theta,z) = 3C\epsilon_0 z\sin\phi}$$

(c) Calculate the total charge enclosed in a cylinder of length L, and radius R, centered at the origin, with its axis along the z axis. **[2+1+1]**
Soln: Total charge Q in the given volume

$$\boxed{Q = \int_{-L/2}^{L/2} dz \int_0^R s ds \int_{\phi=0}^{2\pi} \rho(s,\phi,z)d\phi = 0,}$$

because of the ϕ integration for $\rho(s,\phi,z)$ obtained in part (b).

4. Consider an infinitely long hollow cylinder of radius R. The potential on the surface of the cylinder is given by $V(\phi) = V_0\cos\phi$, where V_0 is a constant. Using the method of separation of variables, the potential anywhere in the space can be written as $V(s,\phi,z) = f(s)g(\phi)h(z)$, where s, ϕ, and z are cylindrical coordinates. Assume $f(s) = As^l + Bs^{-l}$, for $l > 0$, with A and B as constants.

(a) The general expression for $h(z)$ in this case is –
Soln. The expression for $h(z)$ for this case is constant which we choose to be 1.

(b) The corresponding expression for $g(\phi)$ is –
Soln. Due to the symmetry in the z-direction, the potential is independent of z. Thus, the potential is $V(s,\phi,z) = f(s)g(\phi)$. The Laplace's equation then becomes

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial\phi^2} = 0$$

Substituting $V(s,\phi)$ above and then dividing by V, we can write down the differential equation as

$$\frac{s}{f}\frac{d}{ds}\left(s\frac{df}{ds}\right) + \frac{1}{g}\frac{d^2 g}{d\phi^2} = 0$$

We assume,

$$\frac{s}{f}\frac{d}{ds}\left(s\frac{df}{ds}\right) = l^2$$

and the solution of the same is $f(s) = As^l + Bs^{-l}$ for $l > 0$.
The ϕ-equation is

$$\frac{d^2 g}{d\phi^2} = -l^2 g.$$

We identify the solution of this to be

$$\boxed{g(\phi) = C\sin l\phi + D\cos l\phi\,.}$$

(c) For the case of part (b), calculate the potential outside the cylinder by fixing all the necessary constants.
Soln. For this case, after fixing all the necessary constants, the expression for the potential outside the cylinder is – The general solution of $V(s,\phi,z)$ is

$$V(s,\phi,z) = \sum_{l=1}^{\infty} s^l (A_l \sin l\phi + B_l \cos l\phi) + s^{-l}(C_l \sin l\phi + D_l \cos l\phi) \qquad l > 0$$

To compute the potential outside the cylinder, $V \to 0$ for $s \to \infty$. Thus, $A_l = B_l = 0$. Applying the boundary condition, at the surface of the cylinder $V = V_0 \cos\phi$, we get

$$V_0 \cos\phi = \sum_{l=1}^{\infty} s^{-l}(C_l \sin l\phi + D_l \cos l\phi)$$

From the orthonormal property of sin and cos, we conclude that $C_l = 0$ and $D_l = 0$ for$l \neq 1$. This gives, $V_0 \cos\phi = D_1 \cos\phi/R \Rightarrow D_1 = V_0 R$. Therefore, the potential outside is

$$\boxed{V(s,\phi,z) = V_0 \frac{R}{s} \cos\phi}$$

[1+1+2]

5. A static charge distribution produces a radial electric field $\mathbf{E} = k\frac{\exp(-br)}{r^2}\hat{\mathbf{r}}$ where k, b are constants and r is the distance from the origin. The charge density at an arbitrary point in space is –
Soln.
From differential form of Gauss's law, the volume charge density is

$$\begin{aligned} \rho(r) &= \epsilon_0 \nabla \cdot \mathbf{E} \\ &= k\epsilon_0 \left[\exp(-br)\left(\nabla \cdot \frac{\hat{r}}{r^2}\right) + \frac{\hat{r}}{r^2} \cdot \nabla \exp(-br)\right] \\ \rho(r) &= k\epsilon_0 \left[\exp(-br) 4\pi\delta^3(r) - b\frac{\exp(-br)}{r^2}\right] \end{aligned}$$

$$\boxed{\rho(r) = k\epsilon_0 \left[\exp(-br) 4\pi\delta^3(r) - b\frac{\exp(-br)}{r^2}\right]}$$

[2]

6. An isolated conducting sphere of radius R carrying a charge $+Q$ is surrounded by a concentric conducting spherical shell of inner radius a and outer radius b. The spherical shell is electrically neutral.

(a) What are the charges on the inner surface and the outer surface of the spherical shell?
Soln: $-Q$ on inner surface and $+Q$ on outer surface.

(b) What is the potential at the centre of the charged sphere?

$$\begin{aligned}
\mathbf{E}(r < R) &= \mathbf{E}_1 = 0 \\
\mathbf{E}(R < r < a) &= \mathbf{E}_2 = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}} \\
\mathbf{E}(a < r < b) &= \mathbf{E}_3 = 0 \\
\mathbf{E}(r > b) &= \mathbf{E}_4 = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}
\end{aligned}$$

The potential at the centre,

$$\begin{aligned}
V &= -\left(\int_\infty^b \mathbf{E}_4.d\mathbf{l} + \int_b^a \mathbf{E}_3.d\mathbf{l} + \int_a^R \mathbf{E}_2.d\mathbf{l} + \int_R^0 \mathbf{E}_1.d\mathbf{l}\right) \\
&= \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{b} + \frac{1}{R} - \frac{1}{a}\right)
\end{aligned}$$

$$\boxed{V = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{b} + \frac{1}{R} - \frac{1}{a}\right)}$$

[1+1]

7. An isolated conducting sphere of radius R carries a charge +Q. A point charge $+q$ is placed at a distance $2R$ from the centre of the sphere. At this position the net force on this point charge is zero. Determine the ratio Q/q. **[2]**
Soln. The total number of image charges will be 2.
Image charge 1 : $-\frac{q}{2}$ at $r = \frac{R}{2}$
Image charge 2 : $Q + \frac{q}{2}$ at $r = 0$
The net force due at a point $r = 2R$ is,

$$\mathbf{F} = \frac{q}{4\pi\epsilon_0}\left(-\frac{q/2}{(3R/2)^2} + \frac{Q}{4R^2} + \frac{q/2}{4R^2}\right)\hat{\mathbf{r}} = 0$$

therefore,

$$\frac{-2q}{9} + \frac{Q}{4} + \frac{q}{8} = 0$$

and

$$\boxed{Q/q = \frac{7}{18}}$$

8. A disk of radius R has a uniform surface charge density σ. The potential at any point on the circumference of the disk is – **[2]**
Soln: We place the origin of the plane polar coordinate system at point a point C on the rim, and consider a small elemental area on the disk $dA = rdrd\theta$, which is located at (r, θ). The charge dq on it is

$$dq = \sigma r dr d\theta.$$

Thus, the potential $d\phi(C)$ due to it, at C is

$$d\phi(C) = \frac{dq}{4\pi\epsilon_0 r} = \frac{\sigma r dr d\theta}{4\pi\epsilon_0 r},$$

from which we compute the total potential by integrating over the entire area of the disk

$$\phi(C) = \frac{\sigma}{4\pi\epsilon_0} \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int\limits_{0}^{2Rcos\theta} dr$$

$$\implies \boxed{\phi(C) = \frac{\sigma R}{\pi\epsilon_0}}$$

9. The following vector field

$$\mathbf{E}(x, y, z) = C\left[f(x, y)\,\hat{\imath} + (-x + y)\,\hat{\jmath} - 2z\,\hat{k}\right]$$

is to represent an electrostatic field in a charge-free region. The function $f(x, y)$ is – **[2]**
Soln: Electrostatic field $\implies \vec{\nabla} \times \vec{E} = 0$

$$f(x, y) = -y + g(x)$$

Charge free region $\implies \vec{\nabla} \cdot \vec{E} = 0$

$$f(x, y) = x + g(y)$$

Hence

$$\boxed{f(x, y) = x - y + c,}$$

where c is a constant.

10. The electrostatic potential in the space is given in terms of cylindrical polar coordinates (s, ϕ, z) to be

$$V(s, \phi, z) = \frac{f(\phi)}{s^2}.$$

(a) Expression for the electric field in cylindrical coordinate is – (You can leave the result in terms of the unknown function f.)
Soln: $\mathbf{E}$ is given by

$$\mathbf{E} = -\nabla V.$$

Using the cylindrical polar coordinate expression for ∇, we have

$$\begin{aligned} \mathbf{E} &= -\frac{\partial V}{\partial s}\hat{s} - \frac{\partial V}{s\partial\phi}\hat{\phi} - \frac{\partial V}{\partial z}\hat{z} \\ &= \frac{2f}{s^3}\hat{s} - \frac{f'}{s^3}\hat{\phi} - 0. \end{aligned}$$

Final expression

$$\boxed{\mathbf{E} = \frac{2f}{s^3}\hat{\rho} - \frac{f'}{s^3}\hat{\phi}}$$

Above $f' = \frac{df}{d\phi}$.

(b) What property must $f(\phi)$ obey for V to be a single valued function?
Soln: For V and $\mathbf{E}$ to be single valued, f must satisfy

$$\boxed{f(\phi + 2\pi) = f(\phi)}$$

[1+1]

11. The most general solution $V(x)$ for a one dimensional Laplace's equation has the form – **[1]**
Soln: $V(x) = ax + b$, where a and b are constants.

12. Three charges q, q_1, and q_2 are located at points $(0,0,0)$, $(1,0,0)$ and $(0,1,0)$, respectively. The charge $q \neq 0$, but q_1, q_2 may be positive, negative, or zero. It is observed that that at the point $(0,y,0)$ (where $y \gg 1$), the monopole and dipole terms in the multipole expansion vanish. What are the values of q_1 and q_2 in terms of q and y? **[2]**
Soln: If monopole contribution is zero that means that the total charge vanishes

$$q + q_1 + q_2 = 0$$

Dipole moment of the system with respect to the origin is

$$\mathbf{p} = q_1\hat{\imath} + q_2\hat{\jmath}$$

Potential due to dipole term is

$$V_{dip} = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2},$$

above $\hat{\mathbf{r}} = \hat{\jmath}$. If this contribution vanishes then

$$\mathbf{p}\cdot\hat{\mathbf{r}} = q_2 = 0$$

$$\boxed{q_2 = 0}$$

Using $q + q_1 + q_2 = 0$

$$\boxed{q_1 = -q}$$