### MA106 Tut1

| _  |                           | _       |
|----|---------------------------|---------|
| IO | nics                      | Covered |
| 10 | $\mathbf{p}_{\mathbf{i}}$ |         |

- 1) Basic definitions related to matrices
- 2) Geometric interpretation of matrices as linear transforms
- 3) EROs, REF, RREF
- 4) System of linear equations
  5) Inverse of a matrix
  6) Rank, Nullity, Row space, Column space

- 7) General vector spaces

Consider in general an n x n matrix with r pivotal elements in its 011. reduced REF. This implies it has r non-zero rows (as each non-zero row must have a pivot). Now, we must choose any rout of the n columns to be pivots. The order in which they appear as pivots will automatically get decided. For eg:

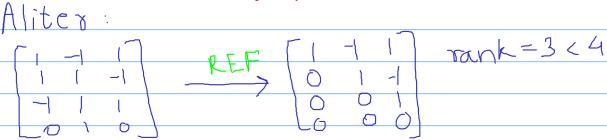
Hence, number of such RREF "types" =

For 
$$n=4$$
  $\gamma=0 \rightarrow 1$  matrix (Null)  
 $\gamma=1 \rightarrow 4$   
 $\gamma=2 \rightarrow 6$   
 $\gamma=3 \rightarrow 4$   
 $\gamma=4 \rightarrow 1$  (Identity)

# MA106 Tut2

92. (j) Observe: 2[010] - [11-1]+[-111] = 0

Hence, linearly dependent



Motivation:

Consider vectors:  $V_1, V_2, V_3, \dots, V_n$ . These will be linearly dependent iff the equation:

 $\chi_1 + \chi_2 + \dots + \chi_n = 0$ has a non-trivial solution. Putting this in matrix form:

$$\begin{bmatrix} \sqrt{1} & \sqrt{2} & \cdots & \sqrt{2} \\ \sqrt{2} & \cdots & \sqrt{2} \\ \cdots & \cdots & \cdots \\ \sqrt{2} &$$

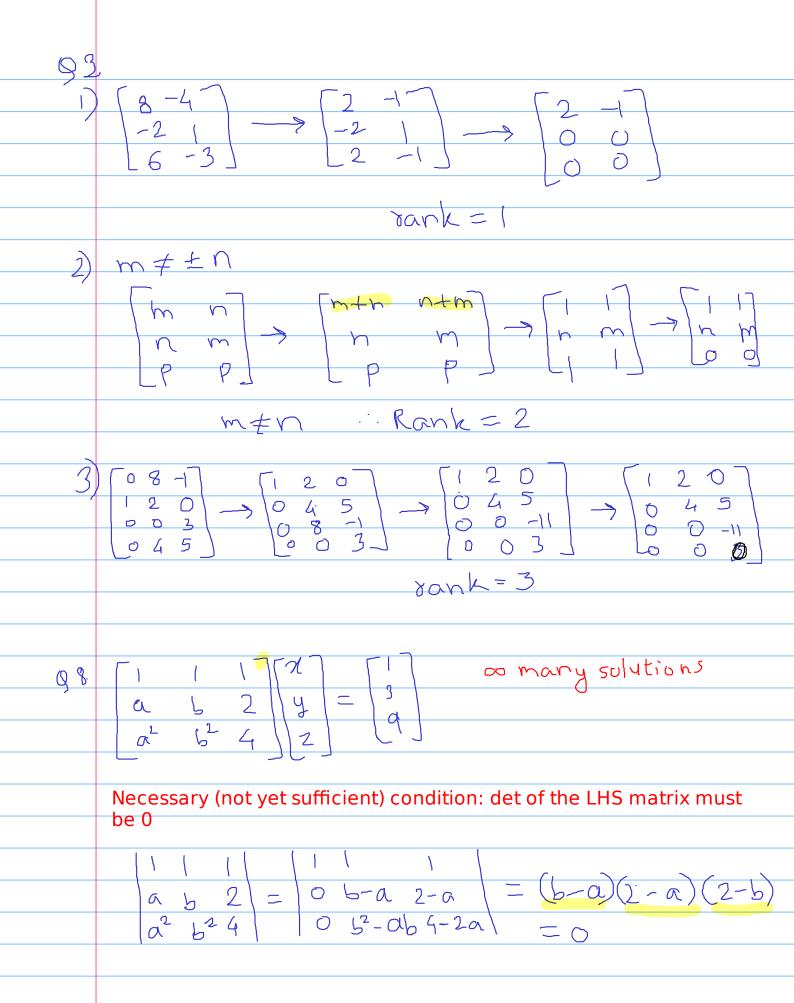
This equation will have multiple solutions iff their is a non-pivotal (free) variable in the REF. Thus, the vectors will be linearly independent iff there is a free variable

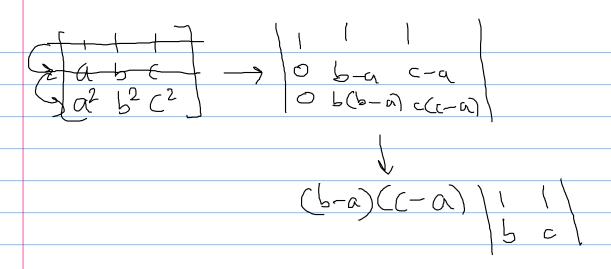
Caveat: In the given question, the vectors are row vectors. We must first convert them to column vectors before applying this method.

#### (11) Linearly independent

Let us convert to column vectors and then use the method above







Exercise: Show that for n numbers x1, x2, x3, ...,  $x_{n-1}$ , find the value of the following determinant

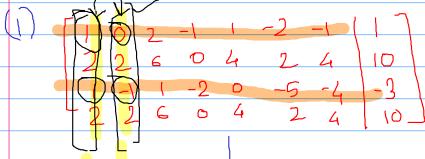
$$\begin{array}{c} \alpha_{1}C_{1}+\alpha_{2}C_{1}\cdots+\alpha_{r}C_{1}=0 \quad \Rightarrow \quad \alpha_{1}=0 \quad \forall i \\ \beta_{1}EC_{1}+\beta_{2}EC_{1}+\cdots=0 \\ \\ \Rightarrow \quad \beta_{1}C_{1}+\beta_{2}C_{1}+\cdots=0 \end{array}$$

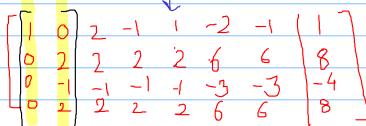
Exercise: Suppose we have A(nxn) and B(nxn) with rank(A)=m and rank(B)=r. Then show that:

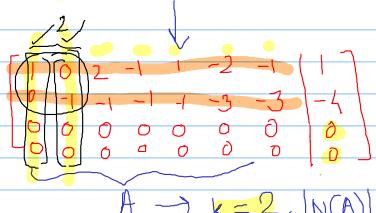
- 1) If m=n, then rank(AB)=r
- 2) In general rank(AB) $\leq$  min(m,r)

| (2) Dimension of column space is unchanged by elementary row operations  |
|--|
| Suppose ERO is represented by the matrix $E$ and the columns are $C_1, C_2, \cdots C_n$ . Then, new columns are $E_1, E_2, \cdots, E_n$  |
| If out of these Ci Cia Cia are linearly independent,   |
| are $C_1, C_2, \cdots, C_n$ . Then, new columns are $C_1, C_2, \cdots, C_n$ If out of these $C_1, C_{1_2}, C_{1_3}, \cdots, C_{1_n}$ are linearly independent,  then, so are $C_1, C_2, \cdots, C_n$ as $C_1, C_2, \cdots, C_n$ as $C_1, C_2, \cdots, C_n$ |
| So, the dimensions of the column space remain unchanged  |
| Note: The column space itself may get altered by EROs  |
| Eg: 7  |
|  |
| Proves row-rank = col. Yank  |
| 10 y c 3   |
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### Handwritten Problem





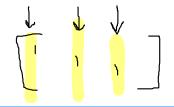


$$\tilde{A} \rightarrow k=2$$
,  $N(A) = 5$ 

- Yes, it has infinitely many solutions. Rank( $\mathbb{A}^{\dagger}$ ) = Rank( $\mathbb{A}$ )
- We have a clever way to do this, rather than checking every 2x2 submatrix. Will discuss at the end
- (4)Null space will have 5 basis vectors. Construct these vectors by putting each of the free variables to 1 and the others to 0 turn-by-turn. Then

|      |   | _  |     |    |
|------|---|----|-----|----|
| [-2] |   | [- | [2] |    |
| -1   | - | -1 | -3  | -3 |
| O    | • | 0  | 0   | 0  |
| 0    | 0 | 1  | 0   | 0  |
| 0    | 0 | O  |     | 0  |
|      |   |    |     |    |

for each of these vectors obtain other elements using solutions of Ax = 0



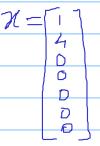
IMPORTANT: The row space remains unchanged by EROs. The column space changes, only its dimension remains the same.

To obtain the basis vectors for the column space, take the indices of the pivotal elements of the REF and the columns at those indices in the original matrix will form a basis of the column space

Proof: Consider the matrix (call it B) whose columns are only those columns of the original matrix which are at pivotal indices. Then, if on B, we apply all the EROs that were applied to the original matrix to get its REF; we get the REF of B. And this REF will have all columns as pivotal i.e. it will be full rank. This shows that the columns we considered are linearly independent and so form the basis of the column space

We have already obtained the basis of the null-space of A. We just need to obtain one particular solution of Ax=b and then the full set of solutions can be obtained by adding the particular solution and the null-space

Particular solution: Put all free variables to 0

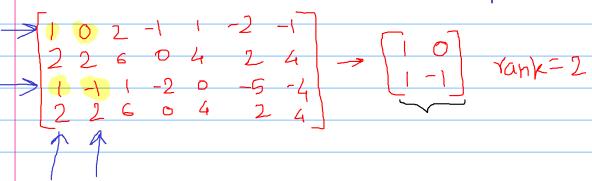


basis vectors of null space

Full set of solutions: 74+

# Obvious

We will take those columns which form the basis of the column space and those rows which make the basis of the row space



Reason: Recall proof of result:

A matrix has rank atleast k iff there exists a kxk submatrix with rank k

Let  $C_1, C_2, C_3, \cdots C_k$  be k linearly independent columns

Then, if we take the submatrices of these with indices same as the rows that contribute to the row space and call them

 $C_1', C_2', C_3', \cdots C_k'$ , then, these must also be linearly independent.

Proof: Assume that there exist coefficients  $\langle \langle , \langle \langle \rangle \rangle, \cdots, \langle \langle \rangle \rangle$ , such that

$$\propto C' + \propto 2C'_2 + \dots + \propto NC'_k = 0$$

then

For, those rows that are already in the row space, the summing up to 0 is evident. For other rows, they can be expressed as a linear combination of the rows in the row space, so even they sum to 0

But,  $C_1$ ,  $C_2$ ,  $C_3$ , ...,  $C_k$  are linearly independent, so  $C_1$ ,  $C_2$ , ...,  $C_k$  are all  $C_1$ 

So,  $C_1'$ ,  $C_2'$ ,  $C_3'$ , ...  $C_k'$  are linearly independent. Hence, the rank of the k x k matrix formed by them as columns is k

