MA 108 - Ordinary Differential Equations

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May 9, 2022

Outline of the lecture

- Separable ODE
- Equations reducible to separable form
- Exact equations

Separable ODE - Example 3

Escape velocity.

A projectile of mass m moves in a direction perpendicular to the surface of the earth. Suppose v_0 is its initial velocity. We want to calculate the height the projectile reaches.

Using Newton's law of gravitation, the equation of motion is

$$m\frac{dv}{dt} = -\frac{mgR^2}{(R+x)^2}; \;\; v(0) = v_0,$$

where x is the height of the projectile from the surface of earth, R is the radius of earth and g is the acceleration due to gravity. By chain rule,

$$\frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = v \cdot \frac{dv}{dx}.$$

Thus,

$$v \cdot \frac{dv}{dx} = -\frac{gR^2}{(R+x)^2}.$$

This ODE is separable. Linear or non-linear? (NL)

Separating the variables and integrating, we get:

$$\frac{v^2}{2} = \frac{gR^2}{R+x} + c.$$

For $x = 0$, we get $\frac{v_0^2}{2} = gR + c$, hence, $c = \frac{v_0^2}{2} - gR$, and,

$$v = \pm\sqrt{v_0^2 - 2gR + \frac{2gR^2}{R+x}}.$$

Suppose the body reaches the maximum height H. Then $v = 0$ at this height.

$$v_0^2 - 2gR + \frac{2gR^2}{(R+H)} = 0.$$

Thus,

$$v_0^2 = 2gR - \frac{2gR^2}{R+H} = 2gR\left(\frac{H}{R+H}\right).$$

The escape velocity is found by taking limit as $H \to \infty$. Thus,

$$v_e = \sqrt{2gR} \sim 11 \text{ km/sec.}$$

Method of separation of variables doesn't yield all solutions!

Solve $y' = 3y^{2/3},\ y(0) = 0$.
$y \equiv 0$ is a solution.
If $y \neq 0$, $\dfrac{dy}{y^{2/3}} = 3dx \Longrightarrow 3y^{1/3} = 3(x + c) \Longrightarrow y = (x + c)^3$.
Initial condition yields $c = 0$.
Hence $\boxed{y = x^3}$ and $\boxed{y = 0}$ are solutions which satisfy the initial conditions.
Consider

$$\phi_k(x) = \begin{cases} 0 & -\infty < x \leq k \\ (x-k)^3 & k < x < \infty \end{cases}$$

Are these functions solutions of the DE? YES.
There are infinitely many functions which are solutions of the IVP.

Homogeneous functions

Definition

A function $f(x_1, \ldots, x_n)$ is called homogeneous of degree d if

$$f(tx_1, \ldots, tx_n) = t^d f(x_1, \ldots, x_n)$$

for all $(x_1, x_2, \cdots, x_n)$, $t \neq 0$.

Examples :

$f(x, y) = x^2 + xy + y^2$ is homogeneous of degree 2.

$f(x, y) = y + x \cos^2\left(\frac{y}{x}\right)$ is homogeneous of degree 1.

Definition

The first order ODE

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

is called homogeneous if M and N are homogeneous of equal degree.

Example :

$$(y^2 - x^2)\frac{dy}{dx} + 2xy = 0$$

is homogeneous.

$$(y^3 - 3x^2y) + 4x^2y^2\frac{dy}{dx} = 0$$

is not homogeneous.

Homogeneous ODE's - Reduction to variable separable form

Let

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

where M and N are homogeneous of degree d. Put

$$\frac{y}{x} = v.$$

Then,

$$\frac{dy}{dx} = x\frac{dv}{dx} + v.$$

Substituting this in the given ODE, we get:

$$M(x, xv) + N(x, xv)\left(x\frac{dv}{dx} + v\right) = 0.$$

Thus,

$$x^d M(1, v) + x^d N(1, v)\left(x\frac{dv}{dx} + v\right) = 0.$$

For $x \neq 0$

$$M(1, v) + N(1, v) \cdot v + N(1, v) \cdot x\frac{dv}{dx} = 0.$$

Thus,

$$\frac{dx}{x} + \frac{N(1, v)}{M(1, v) + N(1, v) \cdot v}dv = 0.$$

This is a separable equation.

Remark : What is important for the above method to work is that the ODE can be put into the form

$$y' = f(\frac{y}{x}).$$

Solve the ODE:

$$(y^2 - x^2)\frac{dy}{dx} + 2xy = 0.$$

Put $y = vx$. Thus, $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$.
Substituting this in the given ODE, we get:

$$(v^2x^2 - x^2)\left(v + x\frac{dv}{dx}\right) + 2x^2v = 0.$$

Thus, for $x \neq 0$,

$$(v^2 - 1)v + x(v^2 - 1)\frac{dv}{dx} + 2v = 0;$$

i.e.,

$$(v^3 + v) + x(v^2 - 1)\frac{dv}{dx} = 0.$$

Thus, we have a separable ODE:

$$\frac{v^2-1}{v(v^2+1)}dv + \frac{dx}{x} = 0.$$

Integrating, we get:

$$\ln|x| + \int \left(\frac{2v}{v^2+1} - \frac{1}{v} \right) dv = c_1.$$

Thus,

$$\ln|x| + \ln(v^2+1) - \ln|v| = c_1.$$

Hence,

$$\frac{x(v^2+1)}{v} = 2c,$$

or

$$y^2 + x^2 = 2cy,$$

which is

$$x^2 + (y-c)^2 = c^2.$$

1. Solve $(4x + 2y + 5)y' + (2x + y - 1) = 0$.
   Hint :
   Substitute $v = 2x + y$. Reduces to separable form.
2. Solve $y' = \dfrac{x + y - 3}{x - y - 1}$.
   Hint :
   - Substitute $x = x_1 + h,\ y = y_1 + k$ for some $h,\ k$ which will be determined.
   - $\dfrac{dy_1}{dx_1} = \dfrac{x_1 + y_1 + h + k - 3}{x_1 - y_1 + h - k - 1}$.
   - Choose $h,\ k$ such that $h + k - 3 = 0,\ h - k - 1 = 0$. This choice makes the equation homogeneous.
   - Formal solution : $e^{\tan^{-1}(\frac{y-1}{x-2})} = C\sqrt{(x-2)^2 + (y-1)^2}$.

1. The DE $e^x y' + 3y = x^2 y$ is linear & separable. TRUE OR FALSE?
2. The DE $yy' + 3x = 0$ is linear & separable. TRUE OR FALSE?
3. Is the DE $\dfrac{dx}{dt} = \dfrac{x + 2xt + \cos t}{1 + t^2}$ linear/non-linear & separable/not separable?
4. For the linear differential equation $\dfrac{dy}{dx} + \dfrac{x}{1+x} y = 1 + x$, the integrating factor is ——?
(Integrating factor $= e^{\int P(x)dx}$ for $y' + P(x)y = Q(x)$.)

Exact ODE's

Definition

A first order ODE

$$M(x, y) + N(x, y)y' = 0$$

is called exact, if there is a function $u(x, y)$ such that

$$\frac{\partial u}{\partial x} = M \;\&\; \frac{\partial u}{\partial y} = N.$$

Example : Is

$$(2x + y^2) + 2xy\frac{dy}{dx} = 0$$

exact? Consider the function $u(x, y) = x^2 + xy^2$.

Exact ODE's

$\boxed{\text{Recall from calculus}}$ Given a function $u(x, y)$ with continuous first partial derivatives, its differential is

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$$

If the ODE $M(x, y) + N(x, y)y' = 0$ is exact, then there exist such $u(x, y)$ with $\dfrac{\partial u}{\partial x} = M$ & $\dfrac{\partial u}{\partial y} = N$, and hence

$$0 = M(x, y)dx + N(x, y)dy = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = du.$$

Integrating $du = 0$, we get $\boxed{u(x, y) = c}$ as an implicit/formal solution to the given ODE.

Example : by inspection

Solve the DE:

$$(2x + y^2) + 2xy\frac{dy}{dx} = 0.$$

Consider the function $u(x, y) = x^2 + xy^2$. Note that

$$\frac{\partial u}{\partial x} = 2x + y^2, \ \frac{\partial u}{\partial y} = 2xy.$$

Hence $x^2 + xy^2 = c$ is the solution of the given ODE.

Working Rule

Given an exact ODE $M(x, y) + N(x, y)y' = 0$, the function $u(x, y)$ can be found either by inspection or by the following method:

1. Integrate $\dfrac{\partial u}{\partial x} = M(x, y)$ with respect to x to obtain

$$u(x, y) = \int M(x, y)dx + k(y),$$

   where $k(y)$ is a constant of integration. (y is treated as a constant during integration).

2. To determine $k(y)$, differentiate the above equation with respect to y, to obtain

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\int M(x, y)dx\right) + k'(y).$$

3. As the given ODE is exact, we get

$$N(x, y) = k'(y) + \frac{\partial}{\partial y}\left(\int M(x, y)dx\right).$$

   We use this to determine $k(y)$ and hence u.

Test for exactness

Theorem

Let M, N and their first order partial derivatives exist and be continuous in a region $D \subseteq \mathbb{R}^2$. We have:

1. *If $M(x,y)dx + N(x,y)dy = 0$ is an exact differential equation, then $M_y = N_x$.*
2. *If D is convex, then $M_y = N_x \Longrightarrow M(x,y)dx + N(x,y)dy = 0$ is exact.*

Proof: Let the ODE be exact. So there is a u such that $M = \dfrac{\partial u}{\partial x}$ and $N = \dfrac{\partial u}{\partial y}$. Then,

$$M_y = \frac{\partial^2 u}{\partial y \partial x} \;\&\; N_x = \frac{\partial^2 u}{\partial x \partial y}.$$

By the theorem on mixed partials, $M_y = N_x$.

Conversely, let D be convex, and $M_y = N_x$. Consider the vector field

$$H(x, y) = (M(x, y),\ N(x, y)).$$

By our assumptions, H is continuously differentiable throughout D. The curl of H is given by

$$\nabla \times H = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = (N_x - M_y)\hat{k} = \mathbf{0}.$$

As D is convex, "curl free is grad"; i.e., there is a function $\phi(x, y)$ such that

$$H = \nabla\phi = (\phi_x, \phi_y).$$

Hence $\phi_x = M$, $\phi_y = N$ and thus $Mdx + Ndy = 0$ is exact.

Example

Solve the DE:

$$(y\cos x + 2xe^y) + (\sin x + x^2e^y - 1)y' = 0.$$

Let $M = y\cos x + 2xe^y$ and $N = \sin x + x^2e^y - 1$.

Do we have an exact DE?

How to find $u(x,y)$ such that $u_x = M$ and $u_y = N$?

1. $$u(x,y) = \int (y\cos x + 2xe^y)dx + k(y) = y\sin x + x^2e^y + k(y).$$

2. $$u_y = \sin x + x^2e^y + k'(y) = \sin x + x^2e^y - 1.$$

3. Thus, $k'(y) = -1$.
4. Choosing $k(y) = -y$, we obain :

$$u(x,y) = y\sin x + x^2e^y - y = c$$

as an implicit solution (Why implicit?) to the given DE.

Remarks

1. We have seen that solutions are given in form $u(x, y) = c$. To see whether the formal solution is implicit or not. We can use the following **implicit function theorem**.

Let $u(x, y)$ and its partial derivatives u_x, u_y are continuous in a region $D \subseteq \mathbb{R}^2$. Let $(x_0, y_0) \in D$ be such that

$$u(x_0, y_0) = c, \ \frac{\partial u}{\partial y}(x_0, y_0) \neq 0.$$

Then there exists a differentiable function φ defined on an interval I containing x_0 such that

$$y_0 = \varphi(x_0), \ \ u(x, \varphi(x)) = c \text{ on } I.$$

2. The method fails if attempt to solve non-exact equations. Consider $(3x + y^2) + (x^2 + xy)y' = 0$. Is the equation exact ? Does the method work?

Can we use integrating factors!?