Outline

1. We introduce vector differential operations of gradient, divergence, and curl
2. Gauss's theorem and Stokes theorem are discussed

Learning Objectives

1. To understand the physical meaning of vector differential operators gradient, divergence and curl
2. To learn to evaluate the gradient, divergence, and curl of given functions
3. To understand theorems involving these operators, particularly Gauss's and Stokes theorems

Gradient of a Scalar Function

- Consider a scalar function $T(x, y, z)$
- We want to compute the change in T, as we move from initial coordinates $\mathbf{r} \equiv (x, y, z)$ infinitesimally to the new position $\mathbf{r} + \mathbf{dr} \equiv (x + dx, y + dy, z + dz)$
- Using Taylor expansion (for multi-variables), and retaining terms up to first order

$$T(\mathbf{r} + \mathbf{dr}) = T(\mathbf{r}) + dx\frac{\partial T}{\partial x} + dy\frac{\partial T}{\partial y} + dz\frac{\partial T}{\partial z} + \text{higher order terms}$$

- Or, to the first order terms,

$$T(\mathbf{r} + \mathbf{dr}) = T(\mathbf{r}) + \mathbf{dr} \cdot \nabla T$$

- Where

$$\mathbf{dr} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

$$\nabla T = \frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k}$$

Gradient contd...

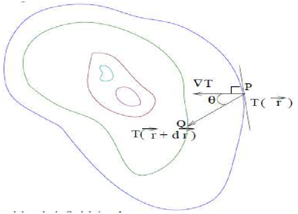
- Defining $T(\mathbf{r}+\mathbf{dr}) = T(\mathbf{r}) + dT$, we conclude

$$dT = \mathbf{dr} \cdot \nabla T,$$

  where the vector ∇T defined above is called the gradient of scalar field T.
- Thus ∇T defines the rate of change of the scalar field with respect to the spatial coordinates, and is itself a vector quantity
- Let us examine ∇T a bit more

Physical Meaning of Gradient

- Let us plot the constant surfaces of a given scalar field T



- As per the figure, we can write the change in the scalar field dT as

$$dT = \mathbf{dr} \cdot \nabla T = |\mathbf{dr}||\nabla T| \cos\theta$$

- Let us consider two possibilities:
  - **dr** is along a constant T surface
  - **dr** is in an arbitrary direction

Gradient, physical meaning...

- If **dr** is along a constant T surface then $dT = 0$. This means

$$|\mathbf{dr}||\nabla T|\cos\theta = 0$$
$$\implies \cos\theta = 0$$

- Thus the direction of ∇T at a given point **r** is always perpendicular to the constant T surface passing through that point
- Let us consider **dr** to be in an arbitrary direction
- Then from $dT = |\mathbf{dr}||\nabla T|\cos\theta$, it is obvious that the magnitude of the maximum possible change in T is

$$dT_{max} = |\mathbf{dr}||\nabla T|,$$

  i.e., when $\cos\theta = 1$.
- Thus the direction of ∇T is also the direction of maximum change in the scalar function T.

- Thus, at a given point **r**, if one moves in the direction of ∇T, maximum change in T will take place
- This property of gradient is used in optimization problems involving location of maxima/minima of scalar functions

Examples:

1. Let us consider a scalar function

$$T = r^2 = x^2 + y^2 + z^2$$

It is easy to see that

$$\nabla T == \frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k} = 2x\hat{i} + 2y\hat{j} + 2z\hat{k} = 2\mathbf{r}$$

2. Consider $\Phi(x,y,z) = x^2y + y^2z + z^2x + 2xyz$

Gradient calculation

Clearly

$$\nabla\Phi = \frac{\partial\Phi}{\partial x}\hat{i} + \frac{\partial\Phi}{\partial y}\hat{j} + \frac{\partial\Phi}{\partial z}\hat{k}$$
$$= (2xy + 2yz + z^2)\hat{i} + (2yz + 2xz + x^2)\hat{j} + (2zx + 2xy + y^2)\hat{k}$$

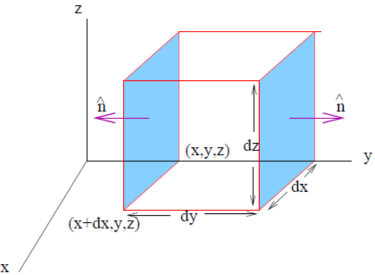
- Thus, in Cartesian coordinates, the gradient operator can be denoted as
$$\nabla \equiv \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$
- In curvilinear coordinates the gradient operator has more complicated forms

$$\text{Cylindrical } \nabla \equiv \frac{\partial}{\partial \rho}\hat{\rho} + \frac{\partial}{\rho\partial\theta}\hat{\theta} + \frac{\partial}{\partial z}\hat{k}$$
$$\text{Spherical } \nabla \equiv \frac{\partial}{\partial r}\hat{r} + \frac{\partial}{r\partial\theta}\hat{\theta} + \frac{\partial}{r\sin\theta\partial\phi}\hat{\phi}$$

Some Properties of Gradient

1. $$\nabla(U+V)=\nabla U+\nabla V$$
2. $$\nabla(UV)=U\nabla V+V\nabla U$$
3. $$\nabla(V^n)=nV^{n-1}\nabla V$$

Let us consider a vector field $\mathbf{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$, and evaluate its flux $\Phi = \int \mathbf{F}\cdot\mathbf{dS}$ over the six surfaces of a cuboid shown below



Clearly, outward flux through the shaded planes

$$\Phi_y = [F_y(x, y+dy, z) - F_y(x, y, z)]\, dxdz$$

Using first order Taylor expansion

$$F_y(x, y+dy, z) = F_y(x, y, z) + dy\frac{\partial F_y}{\partial y}.$$

So that

$$\Phi_y = \frac{\partial F_y}{\partial y} dxdydz$$

Divergence contd.

- If we similarly calculate the flux through all the remaining faces of the cuboid and add it to obtain the total flux $d\Phi$, we obtain

$$d\Phi = \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) dV,$$

  where $dV = dxdydz$ is the volume of the cuboid.

- Thus we get for the entire cuboid

$$\mathbf{F} \cdot \mathbf{dS} = (\nabla \cdot \mathbf{F}) dV,$$

  where

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

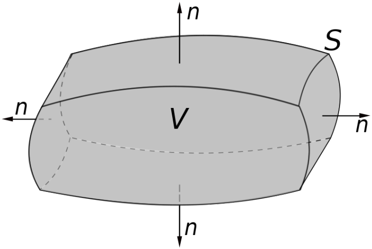
  is called the divergence of the vector field $\mathbf{F}$.

- Note that divergence can be seen as a dot product of the operator $\nabla \equiv \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$, and the vector field $\mathbf{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$.

- Divergence Theorem (Gauss's Theorem): Flux of a vector field **F** calculated over the surface of an arbitrarily shaped volume satisfies the following result, which is called Gauss's Theorem or Divergence Theorem

$$\oint_S \mathbf{F} \cdot \mathbf{dS} = \int_V \nabla \cdot \mathbf{F} dV, \tag{1}$$

that is the flux of a vector field over the surface enclosing a given volume is equal to the volume integral of its divergence over the given volume.



Proof of Divergence Theorem

- Divergence theorem is easy to prove using the result we proved for an infinitesimal cuboid

$$\mathbf{F} \cdot \mathbf{dS} = (\nabla \cdot \mathbf{F}) dV \tag{2}$$

- Divide the given volume into a large number of such infinitesimal cuboids.
- Perform the sum of Eq. 2 for all cuboids.
- L.H.S. of the sum will only have the contribution from the external surface S of the volume V, because contribution from those surfaces of the cuboids which are inside the volume will cancel due to opposite orientations for the adjacent cuboids. This will lead to the LHS of Eq. 1.
- R.H.S. of the sum will simply be the volume integral of $\nabla \cdot \mathbf{F}$ over the entire volume V, i.e., RHS of Eq. 1
- Therefore QED.

Divergence in Curvilinear Coordinates

- For vector $\mathbf{V} = v_\rho \hat{\rho} + v_\theta \hat{\theta} + v_z \hat{z}$, in cylindrical polar coordinates, the divergence is given by

$$\nabla \cdot \mathbf{V} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho v_\rho) + \frac{1}{\rho} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}$$

- For vector $\mathbf{V} = v_r \hat{r} + v_\theta \hat{\theta} + v_\phi \hat{\phi}$, in spherical polar coordinates, the divergence is given by

$$\nabla \cdot \mathbf{V} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (\sin\theta \, v_\theta) + \frac{1}{r \sin\theta} \frac{\partial v_\phi}{\partial \phi}$$

- Note the different forms of divergence operator in different coordinate systems.

Examples of Divergence Calculation

- Calculate the divergence of vector $\mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k}$
    - In Cartesian coordinates

$$\nabla \cdot \mathbf{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

- In spherical polar coordinates $\mathbf{r} = r\hat{r}$, so

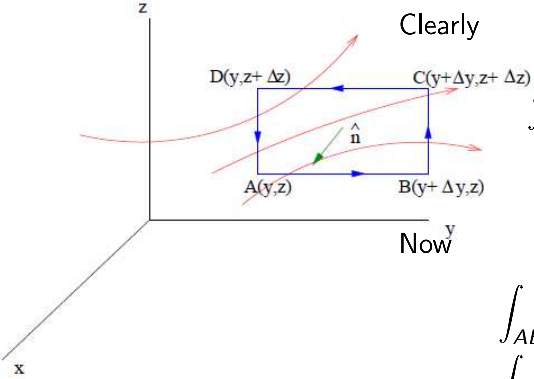
$$\nabla \cdot \mathbf{r} = \frac{1}{r^2}\frac{\partial (r^2 r)}{\partial r} = 3$$

- In cylindrical polar coordinates $\mathbf{r} = \rho\hat{\rho} + z\hat{z}$, so

$$\nabla \cdot \mathbf{r} = \frac{1}{\rho}\frac{\partial (\rho\rho)}{\partial \rho} + \frac{\partial z}{\partial z} = 2 + 1 = 3$$

Curl of a Vector Field

Let us consider a vector field $\mathbf{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$, and evaluate its line integral along a infinitesimal rectangular path shown below



Clearly

$$\oint \mathbf{F.dl} = \int_{AB} \mathbf{F.dl} + \int_{BC} \mathbf{F.dl} + \int_{CD} \mathbf{F.dl} + \int_{DA} \mathbf{F.dl}$$

Now

$$\int_{AB} \mathbf{F.dl} = \int \mathbf{F}.dy\hat{j} \approx F_y(y,z)\Delta y$$

$$\int_{BC} \mathbf{F.dl} = \int \mathbf{F}.dz\hat{k} \approx F_z(y+\Delta y, z)\Delta z$$

Using first order Taylor expansion

$$F_z(y+\Delta y, z) = F_z(y,z) + \frac{\partial F_z}{\partial y}\Delta y$$

So that

$$\int_{AB} \mathbf{F.dl} + \int_{BC} \mathbf{F.dl} = \left(F_y \Delta y + F_z \Delta z + \frac{\partial F_z}{\partial y} \Delta z \Delta y \right)$$

Similarly one can show (by integrating in AD and DC directions)

$$\int_{CD} \mathbf{F.dl} + \int_{DA} \mathbf{F.dl} = -\left(F_y \Delta y + F_z \Delta z + \frac{\partial F_y}{\partial z} \Delta z \Delta y \right)$$

By adding all the contributions we obtain

$$\oint \mathbf{F.dl} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \Delta S_x \qquad (3)$$

Curl contd....

Where $\Delta S_x = \Delta y \Delta z$, is the area of the infinitesimal loop, directed along the x axis. Let us define a quantity called curl, denoted as $\nabla\times$

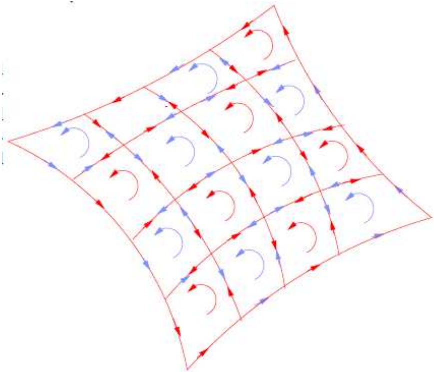
$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}.$$

Using this we can cast Eq. 3 as

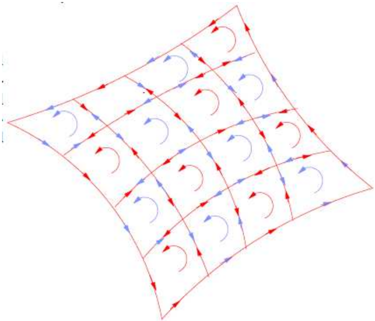
$$\oint \mathbf{F}.\mathbf{dl} = (\nabla \times \mathbf{F})_x \Delta S_x = (\nabla \times \mathbf{F}) \cdot \Delta \mathbf{S} \tag{4}$$

Stokes' Theorem: If a vector field **F** is integrated along a closed loop of an arbitrary shape, then the line integral is equal to the surface integral of the curl of **F**, evaluated over the area enclosed by the loop

$$\oint \mathbf{F}.\mathbf{dl} = \int_S (\nabla \times \mathbf{F}) \cdot \mathbf{dS}$$



Outline of the proof:



We can split the area enclosed by the loop into a large number of infinitesimal loops as shown, for each one of which Eq. 4 will hold. Upon adding the contribution of all such loops, we get the desired result

$$\oint \mathbf{F}.\mathbf{dl} = \int_S (\nabla \times \mathbf{F}) \cdot \mathbf{dS}.$$

Note that in the line integral, the contribution only from the boundary of the loop will survive because the contribution from the internal lines gets canceled from adjacent loops.

Curl in Cylindrical Coordinates: For a vector field $\mathbf{A} = A_\rho \hat{\rho} + A_\theta \hat{\theta} + A_z \hat{\mathbf{z}}$

$$\nabla \times \mathbf{A} = \left(\frac{1}{\rho} \frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z} \right) \hat{\rho} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right) \hat{\theta} + \frac{1}{\rho} \left(\frac{\partial (\rho A_\theta)}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi} \right) \hat{\mathbf{z}}$$

Curl in Spherical Polar Coordinate System

Curl in Spherical Polar Coordinates: For a vector field $\mathbf{A} = A_r \hat{\boldsymbol{r}} + A_\theta \hat{\theta} + A_\phi \hat{\phi}$

$$\begin{aligned}
\nabla \times \mathbf{A} &= \frac{1}{r \sin\theta} \left(\frac{\partial}{\partial \theta} \left(A_\phi \sin\theta \right) - \frac{\partial A_\theta}{\partial \phi} \right) \hat{\boldsymbol{r}} \\
&+ \frac{1}{r} \left(\frac{1}{\sin\theta} \frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r} \left(r A_\phi \right) \right) \hat{\theta} \\
&+ \frac{1}{r} \left(\frac{\partial}{\partial r} \left(r A_\theta \right) - \frac{\partial A_r}{\partial \theta} \right) \hat{\phi}
\end{aligned}$$

Examples of Calculation of Curl

1. Calculate the curl of the vector field $\mathbf{F} = -y\hat{i} + z\hat{j} + x^2\hat{k}$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & z & x^2 \end{vmatrix}$$
$$= -\hat{i} + 2x\hat{j} + \hat{k}$$

2. Easy to verify that $\nabla \times \mathbf{r} = 0$, where $\mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k}$.
3. Verify the result of 2, using spherical coordinates ($\mathbf{r} = r\hat{r}$) and cylindrical coordinates ($\mathbf{r} = \rho\hat{\rho} + z\hat{z}$)