Uncertainty Relations

December 29, 2021

1 Distributions

In statistics, we consider the distributions. For example, you measure the heights of a group of people. Classify the heights into bins of one cm. Plot a histogram of "height" on x-axis and the number of people of such height on y-axis. This gives you the "height distribution". You can calculate the "mean" and the "standard deviation" of this distribution. We pay particular emphasis to these two numbers because they describe the **essence** of the distribution.

Suppose you have done the height distribution of three different groups. Suppose they are characterised by

- Group-1: Mean is 170 cm and standard deviation is 10 cm
- Group-2: Mean is 175 cm and standard deviation is 10 cm
- Group-3: Mean is 175 cm and standard deviation is 15 cm

Comparing groups 1 and 2, we can say that members of group-2 have access to better nutrition. Comparing groups-2 and 3, we say that there is a significant fraction of group-3, which does NOT have access to the kind of nutrition that members of group-2 have access to.

How do we get the second conclusion? That is because the standard deviation is a **measure of the spread of the distribution**. In a typical distribution, the number of "data points" in the range (mean - s.d) to (mean + s.d.) is about two thirds of the total number of data points. Even if your range is $(0, \infty)$ nominally, a very large fraction of the data points fall within the above range. That is why a distribution is typically characterised by its mean and its standard deviation.

2 Wave Packet

We have described wave packet as a **localized wave**. Which means that it has peak some where and will fall off to zero on both sides of the peak. That is, **it looks like a distribution!** So we can use the same language to describe a wave packet what we use to describe a distribution. As discussed in the Fourier analysis note, we can explicitly calculate the mean and standard deviation for a wave packet. Or, we can take a short cut method by labelling the **maximum position as the mean** and taking the **full width at half maximum** to be twice the standard deviation. In a large number of cases, the short cut gives quite good results.

Given a wave packet in $f(x)$, we can take its Fourier transform

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx.$$

From various examples in Fourier analysis, we saw that $A(k)$ is a **wave packet in the variable** k. There is a spread $2\Delta x$ in $f(x)$ and there is a spread $2\Delta k$ in $A(k)$. From the examples we saw in Fourier analysis, we know that the product of these two spreads is a number of order 1,

$$\Delta x \Delta k \simeq 1.$$

It can be mathematically proved that the product should be $\geq 1/2$.

3 Heisenberg Uncertainty Relations

de Broglie said that we should describe a massive particle such as an electron by a wave packet. That itself sounds strange!!! Does it lead to some other strange results? It leads to **Heisenberg Uncertainty Relations**.

From de Broglie hypothesis, we have $p = h/\lambda = (h/(2\pi))(2\pi/\lambda)$ leading to $p = \hbar k$, where $\hbar = h/(2\pi)$. When we have a wave packet with spread $2\Delta x$, the momentum associated with this wave packet has a spread $2\Delta p = 2\hbar \Delta k$. And the two **uncertainties** are related to each other by

$$\Delta x \Delta p \geq \frac{\hbar}{2}.$$

This is the famous Heisenberg Uncertainty Relation. We note that it arises from

1. describing a particle by a localised wave packet AND
2. assuming that the wavelength is related to momentum by de Broglie hypothesis.

If the motion is purely along x direction, the spread in the wave packet is along x direction, its momentum is along x direction. To emphasize this, the above relation is written as

$$\Delta x \Delta p_x \geq \frac{\hbar}{2}.$$

Similarly, we can write

$$\Delta y \Delta p_y \geq \frac{\hbar}{2}, \quad \Delta z \Delta p_z \geq \frac{\hbar}{2}.$$

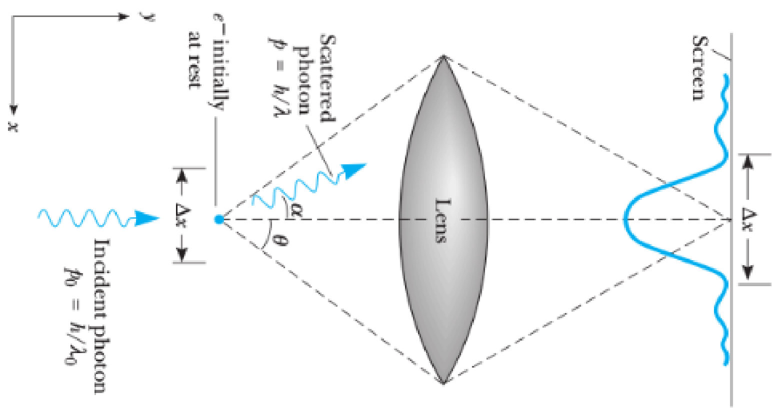
For a general three dimensional motion, we sometimes talk of spherical coordinates, which involve the r (radial distance travelled which is the magnitude of the displacement) and p_r momentum in the radial direction (which is also the magnitude of the momentum). In such a case, we will have

$$\Delta r \Delta p_r \geq \frac{\hbar}{2}.$$

In Fourier analysis, I concentrated on the relation between x and k. It is straight forward to see that a similar relation exists between t and ω. If we consider a time pulse of a finite duration $2\Delta t$, its Fourier transform will have a spread of frequencies $2\Delta\omega$. From that it can be shown that

$$\Delta t \Delta \omega \geq \frac{1}{2} \quad \text{and} \quad \Delta t \Delta E \geq \frac{\hbar}{2}$$

4 Heisenberg Microscope



<u>**Note**</u>: In this section and in the next, I will discuss things using "order of magnitude estimates". Which means that the expressions and the numbers I use are not exact but are correct upto factors of 2 or 3. Such estimates are often used in Physics to illustrate important points in a simple way.

Heisenberg microscope is an imaginary experiment which shows you **physically** why Heisenberg's uncertainty relations should hold. Key point to realize is that measurement process disturbs the object that is being observed. The state of the object changes as a result of the measurement. In the Heisenberg microscope, let us imagine there is an electron at rest at some point (in contradiction to the uncertainty relation). But how are we to know about it? We have to make a measurement. We do that by scattering a photon off it. If this photon is to be collected by the lens, it has to be scattered within at an angle within the range $(-\theta, \theta)$ as shown in the figure.

The scattering imparts a momentum to the electron which is in the range $(-h \sin\theta/\lambda, +h \sin\theta/\lambda)$ where λ is the wavelength of the scattered photon. The photon, after being collected within the lens aperture, is diffracted to the screen far away. The photon hits (strictly speaking, "most likely hits") the screen within the diffraction central maximum, whose spread is given by $\lambda/2\sin\theta$.

This spread represents the uncertainty in the measurement of the position of the electron Δx. The act of measurement created an uncertainty $\Delta p_x = 2h\sin\theta/\lambda$ in the momentum of the electron. The product of the two uncertainties is

$$\Delta x \Delta p_x = \frac{\lambda}{2\sin\theta}\frac{2h\sin\theta}{\lambda} = h.$$

satisfying the Heisenberg's uncertainty principle.

5 Examples of Heisenberg Uncertainty Relation

- All of you have seen on TV, "Hawky-eye" tracking a cricket ball moving at speed of about 40 m/s or so. Does the cricket ball satisfy Heisenberg uncertainty principle? Tracking is done using radar, which uses microwaves. To detect an object of size d, must use a probe whose sensitivity is better than d. The sensitivity of an electromagnetic wave is its wavelength. Lowest energy microwaves have frequency of 100 MHz $= 10^8$ Hz implying $\lambda \sim 3$ m. Objects of similar or larger size, such as enemy planes coming to attack, could be detected by them. Radar was invented during World War II, for that purpose. The size of a cricket ball is a few cm. One needs microwaves with wavelengths of similar size. Assume "Hawk-eye" uses a frequency of 100 GHz, which means the wavelength is 3 mm. This will be our Δx. Then Heisenberg Uncertainty relations predict that the momentum of the cricket ball will be uncertain by

  $$\Delta p > \frac{\hbar}{\Delta x} = \frac{10^{-34} J - s}{3 \times 10^{-3} m}.$$

  This leads to an uncertainty in the speed of the cricket ball of 10^{-31} m/s.

- Rutherford scattering experiment established that all the mass and the all the positive charge of the atom is contained in a nucleus. The same experiment allows us estimate the size of the nucleus to be 10,000 times smaller than that of the atom. An early model of the nucleus was proposed in terms of the two known fundamental particles: proton and electron. A nucleus of mass number A and atomic number Z contains A protons and (A-Z) electrons (as opposed to the present picture of Z protons and (A-Z) neutrons). In beta decay, electrons are emitted by a nucleus. This provided a further"proof" of the above picture. Can it be true?

This above picture can refuted using Heisenberg Uncertainty relation. Size of the nucleus is 10^{-15} m. If we take this to be Δx then $\Delta p = 10^{-34}/10^{-15} = 10^{-19}$ (kg-m/s). The momentum of the electron has to be at least this much. Calculating $pc = 10^{-19} \times 10^{8} = 10^{-11}$ J, which is about 10^{8} eV, or 100 MeV. This is much larger than the rest mass energy 0.5 MeV, of the electron. The electrons emitted in the beta decay have energies of a few MeV. There is nothing in the above picture of nucleus, which can reduce the energy of the electron from about 100 MeV to a few MeV. So the above picture can be ruled out.

Within the present picture of nucleus, we understand beta decay as the decay of the neutron in the nucleus. And the energy of the emitted electron of the order of the mass difference between the mother nucleus and daughter nucleus.