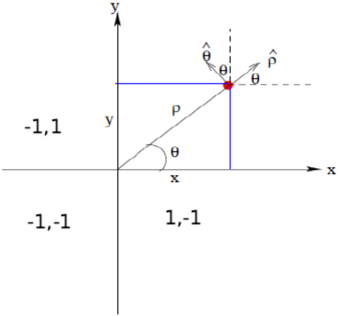
Outline

1. Coordinate systems in 2-dimensions: Cartesian and plane polar coordinate systems and their relationship. Length and area elements
2. Coordinate systems in 3-dimensions: Cylindrical and Spherical Polar Coordinate systems, line, surface and volume elements

Objectives

1. To learn to use symmetry adapted coordinate systems
2. To understand as to how to construct line, surface, and volume elements for various coordinate systems

- Using a coordinate system which is consistent with the symmetry of the physical system simplifies calculations
- If a planar system has circular symmetry, use of plane-polar coordinate system will simplify calculations
- For systems with cylindrical symmetry, use of cylindrical polar coordinates is advised
- Likewise for spherical systems, use of spherical polar coordinate system will be beneficial



Cartesian Coordinates:

- Location of a point in a flat plane is given by coordinates (x, y).
- Differential line element $\overrightarrow{dl}$ is given by $\overrightarrow{dl} = dx\hat{i} + dy\hat{j}$

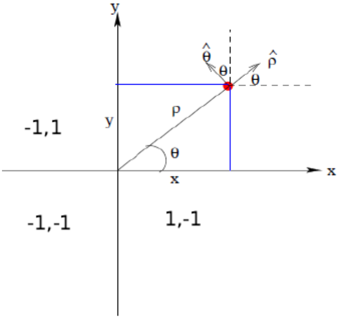
2D Coordinates Continued

- A general vector is given by $\vec{A} = A_x \hat{i} + A_y \hat{j}$.
- Infinitesimal area element $\overrightarrow{dS}_{12}$ in a plane described by orthogonal coordinates 1 and 2 can be computed for any coordinate system as

$$\overrightarrow{dS}_{12} = \overrightarrow{dl}_1 \times \overrightarrow{dl}_2 \tag{1}$$

- For Cartesian coordinates it yields

$$\overrightarrow{dS} = dx\hat{i} \times dy\hat{j} = dxdy\,\hat{k}$$
$$\text{or } dS = dxdy$$



Plane Polar Coordinates:

- Location of a point in a flat plane is given by coordinates (ρ, θ).
- Differential line element $\overrightarrow{dl}$ is given by $\overrightarrow{dl} = d\rho\hat{\rho} + \rho d\theta\hat{\theta}$
- Infinitesimal surface area is $\overrightarrow{dS} = d\rho\hat{\rho} \times \rho d\theta\hat{\theta} = \rho d\rho d\theta\hat{k}$, or $dS = \rho d\rho d\theta$

Relationship between Cartesian and Plane Polar Coordinates

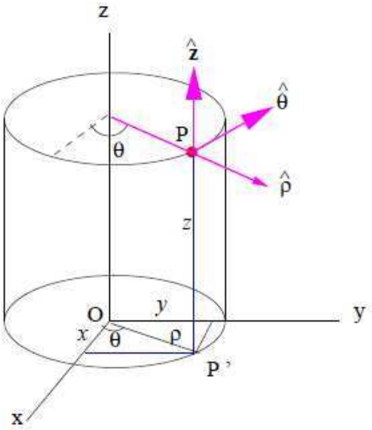
- $x = \rho \cos\theta, \; y = \rho \sin\theta$
- $\rho = \sqrt{x^2 + y^2}, \; \theta = \tan^{-1}\left(\frac{y}{x}\right)$, where $-\infty \leq x, y \leq \infty$; $0 \leq \rho \leq \infty, \; 0 \leq \theta \leq 2\pi$.
- And unit vectors are related as $\hat{\rho} = \cos\theta \hat{i} + \sin\theta \hat{j}$, and $\hat{\theta} = -\sin\theta \hat{i} + \cos\theta \hat{j}$
- $\hat{i} = \cos\theta \hat{\rho} - \sin\theta \hat{\theta}, \; \hat{j} = \sin\theta \hat{\rho} + \cos\theta \hat{\theta}$
- Using these relations, one can easily transform vectors expressed in one coordinate system, into the other one.
- Area of a circle of radius R, $A = \int_0^R \rho \, d\rho \int_0^{2\pi} d\theta = \pi R^2$

Cartesian Coordinates:

- Location of a point is is given by coordinates (x, y, z).
- Differential line element $\overrightarrow{dl}$ is given by $\overrightarrow{dl} = dx\hat{i} + dy\hat{j} + dz\hat{k}$
- Infinitesimal area element depends upon the plane. For xy plane it will be
$$\overrightarrow{dS}_{xy} = dxdy\hat{k}$$
- Infinitesimal volume element for any orthogonal 3D coordinate system is given by
$$dV = dl_1 dl_2 dl_3$$
for this case $dV = dxdydz$

Cylindrical Coordinates:

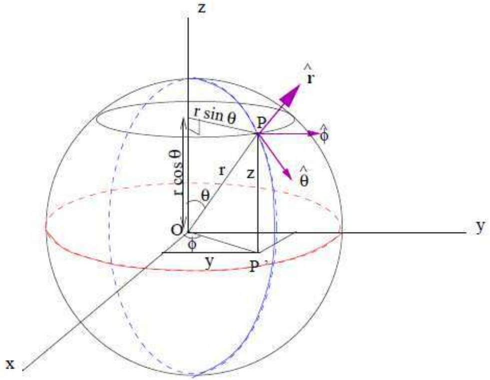
- Location of a point specified by three coordinates (ρ, θ, z)



- Relationship with Cartesian coordinates $x = \rho\cos\theta$, $y = \rho\sin\theta$, $z = z$
- Inverse relationship $\rho = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, $z = z$
- Differential line element $\overrightarrow{dl}$ is given by $\overrightarrow{dl} = d\rho\hat{\rho} + \rho d\theta\hat{\theta} + dz\hat{k}$
- Area element in different planes can be obtained using the relation $\overrightarrow{dS}_{ij} = \overrightarrow{dl}_i \times \overrightarrow{dl}_j$
- For $\rho - \theta$ plane it will be $\overrightarrow{dS}_{\rho\theta} = \overrightarrow{dl}_\rho \times \overrightarrow{dl}_\theta = d\rho\hat{\rho} \times \rho d\theta\hat{\theta} = \rho d\rho d\theta\hat{k}$
- Volume element $dV = dl_1 dl_2 dl_3 = \rho d\rho d\theta dz$
- Volume of a cylinder of height L, and radius R $V = \int_{\rho=0}^{R}\rho d\rho \int_{z=0}^{L} dz \int_{\theta=0}^{2\pi} d\theta = \pi R^2 L$

Spherical Polar Coordinates:

- Location of a point is specified by three coordinates (r, θ, ϕ), as shown below



- What is the range of r, θ, and ϕ?

3D Coordinates...

- Clearly $0 \leq r \leq \infty$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$
- Relationship with Cartesian coordinates $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r \cos\theta$
- Inverse relationship

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right)$$

$$\phi = \tan^{-1}\frac{y}{x}$$

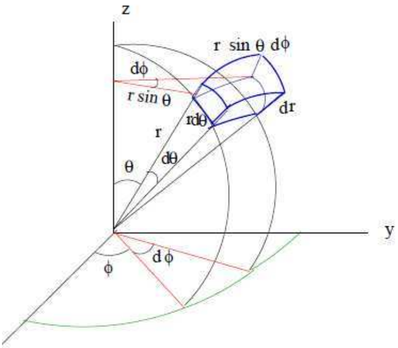
- Differential line element $\overrightarrow{dl}$ is given by $\overrightarrow{dl} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi}$
- Cross products given by $\hat{\theta} \times \hat{\phi} = \hat{r}$, $\hat{\phi} \times \hat{r} = \hat{\theta}$, and $\hat{r} \times \hat{\theta} = \hat{\phi}$

Spherical Polar Coordinates...

- Relationship between Cartesian and Spherical unit vectors

$$\hat{r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$
$$\hat{\theta} = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$$
$$\hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

- Area element on the surface of a sphere or radius R,
$\overrightarrow{dS}_{\theta\phi} = \overrightarrow{dl}_{\theta} \times \overrightarrow{dl}_{\phi} = Rd\theta\hat{\theta} \times R\sin\theta d\phi\hat{\phi} = R^2\sin\theta d\theta d\phi\hat{r}$
- Area of the surface of a sphere
$A = R^2\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}d\phi = 4\pi R^2$



- Elementary volume element
  $dV = dl_r dl_\theta dl_\phi = dr r d\theta r \sin\theta d\phi = r^2 \sin\theta dr d\theta d\phi$
- Volume of a sphere of radius R

$$V = \int_0^R r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi = \frac{4}{3}\pi R^3$$