PH108 : Electricity & Magnetism : Problem Set 3

Only * problems are to be solved in the tut session

1. * Consider a vector field $\vec{F}(r)$, where $r = \vec{r}$ and $\vec{F}(r)$ dies faster than $\frac{1}{r}$ as $r \to \infty$, show the following results

   (a) Using Helmholtz theorem as discussed in Lecture 5, Show that $\vec{F}(r)$ may be written as

   $$\vec{F}(r) = -\nabla \cdot \frac{1}{4\pi} \int_V \frac{\nabla' \cdot \vec{F}(r')}{|r - r'|} d\tau' + \nabla \times \frac{1}{4\pi} \int_V \frac{\nabla' \times \vec{F}(r')}{|r - r'|} d\tau'$$

   (b) Derive the same expression for $\vec{F}(r)$ using

   $$\vec{F}(r) = \int_V d\tau' \vec{F}(r') \delta^3(r - r')$$

   boundary of the integral is to be understood at ∞.

   Hint: Use the following

   (i) $-4\pi\delta^3(r - r') = \nabla^2 \frac{1}{|r-r'|}$

   (ii) $\nabla \times \nabla \times = \nabla\nabla \cdot -\nabla^2$

   (iii) $\nabla \frac{1}{|r-r'|} = -\nabla' \frac{1}{|r-r'|}$

   (iv) $\nabla \times \frac{\vec{F}(r')}{|r-r'|} = -\vec{F}(r') \times \nabla \left(\frac{1}{|r-r'|} \right)$ and $7(b)$ from Problem Set 2.

2. *

   (a) Using the identity: $\delta(ax) = \frac{\delta(x)}{|a|}$, a $\neq$0
   Prove that: $\delta[g(x)] = \sum_m \frac{1}{|g'(x_m)|} \delta(x - x_m)$, where, $g(x_m) = 0$ and $g'(x_m) \neq 0$

   (b) Confirm that: I=$\int_0^\infty dx \delta(\text{cosx})\exp(-x) = \frac{1}{2\text{sinh}(\frac{\pi}{2})}$

   (c) Show that,

   $$D(x) = \lim_{m \to \infty} \frac{\text{sin mx}}{\pi x}$$

   is a representation of $\delta(x)$ by showing that $\int_{-\infty}^{\infty} dx f(x) D(x) = f(0)$

3. Show that D(r,ϵ) demonstrates the peak-character & goes to $\delta^3(r)$ as $\epsilon \to 0$

   $$D(r, \epsilon) = -\frac{1}{4\pi} \nabla^2 \frac{1}{\sqrt{r^2 + \epsilon^2}}$$

   Hint:

   (a) Show that $D(r, \epsilon) = \frac{3\epsilon^2}{4\pi((r^2+\epsilon^2)^{5/2})}$

   (b) Check that $D(0, \epsilon) \to \infty$ as $\epsilon \to 0$

   (c) Check that $D(r, \epsilon) \to 0$ as $\epsilon \to 0$, for all r $\neq$ 0

   (d) Check that the integral of $D(r, \epsilon)$ over all the space is 1

4. Evaluate the following integral

$$\int_V \mathbf{r} \cdot (\mathbf{d} - \mathbf{r}) \delta^3 (\mathbf{e} - \mathbf{r}) d\tau$$

where $\mathbf{d} = (5, 5, 5)$, $\mathbf{e} = (15, 19, 17)$, and V is a sphere of radius 7 centered at (10, 15, 19).

5. * Let $\mathbf{F}$ be a vector field whose divergence and curl are given as

$$\nabla . \mathbf{F} = \delta(x)\delta(y) \quad \text{and} \quad \nabla \times \mathbf{F} = 0$$

Using the Helmholtz theorem, determine $\mathbf{F}(x, y, z)$.

6. * A small ball with a positive charge +q hangs by an insulating thread. Holding this ball vertical, a second ball having charge +q is kept at a distance a along the horizontal direction. There are an infinite number of points where a third ball with charge +2q may be positioned so that the first ball continues to remain vertical when released. Find the equation of the curve describing these points.

7. * After an extremely precise measurement, it was revealed that the actual force between two point charges is given by -

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \left(1 + \frac{r}{\lambda}\right) e^{-r/\lambda} \hat{\boldsymbol{r}}$$

Where λ is a constant with dimensions of length, and it is a huge number which is why the correction is tiny and difficult to notice.
Does this electric field results from a scalar potential? Justify.
And if yes, find the potential due to a point charge q placed at the origin using infinity as your reference.

8. Which one of the following is possible expression for an electrostatic field? For the right expression, find a potential which determines this field with the origin as the reference.

   (a) $\mathbf{E} = A(x^2yz\hat{i} + 2xz\hat{j} - 3yz\hat{k})$
   (b) $\mathbf{E} = A([3xz^2 + y^2]\hat{i} + 2xy\hat{j} + 3x^2z\hat{k})$

9. Find the electric field a distance z above center of a square loop of side l carrying uniform line charge density λ.