Method of images: A point charge in front of a grounded conducting sphere

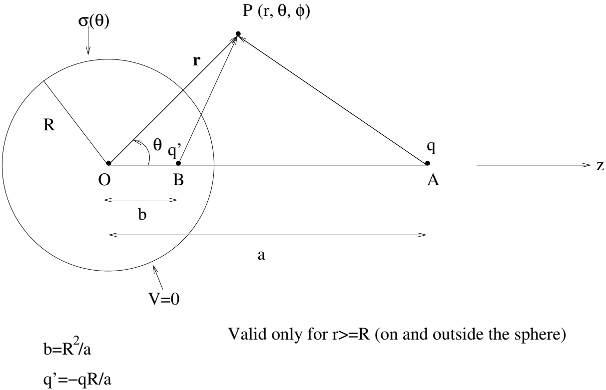

Figure: A point charge in front of a grounded sphere: Location and value of the image charge

- Using the location and the value of the image charge given in the previous figure, the potential in the region of interest $(r > R)$ can be written as

$$V(\mathrm{r}) = \frac{q}{4\pi\varepsilon_0}\left\{\frac{1}{|\mathrm{r} - a\hat{k}|} - \frac{R}{a|\mathrm{r} - b\hat{k}|}\right\}, \tag{1}$$

  where $b = R^2/a$.

- or

$$V(r,\theta) = \frac{q}{4\pi\varepsilon_0}\left\{\frac{1}{\sqrt{r^2 + a^2 - 2ar\cos\theta}} - \frac{R}{a\sqrt{r^2 + \frac{R^4}{a^2} - 2\frac{R^2}{a}r\cos\theta}}\right\}. \tag{2}$$

- By setting $r = R$ in Eq. (2), one can easily verify that $V(r = R, \theta) = 0$, i.e., the given potential satisfies the correct boundary conditions.

- Using the result

$$\nabla^2 \left(\frac{1}{|\mathrm{r} - \mathrm{r}'|}) \right) = -4\pi\delta(\mathrm{r} - \mathrm{r}')$$

- We obtain from Eq. (1) that in the region of interest $(r > R)$

$$\nabla^2 V = -\frac{\rho(\mathrm{r})}{\varepsilon_0}, \tag{3}$$

where $\rho(\mathrm{r}) = q\delta(\mathrm{r} - a\hat{k})$. Note that Eq. 3 is Laplace equation at all points except where the charge is located.

- So the potential of Eq. (1) satisfies the correct boundary condition, as well as the Laplace/Poisson equation in the region of interest. Thus, by virtue of the uniqueness theorem, this is the correct solution for the potential.

- Using Eq. (2) and the expression for the gradient in spherical polar coordinates, one can easily calculate the electric field in the region of interest ($r > R$).
- Induced charge density on the surface of the sphere is calculated as

$$\sigma_{ind} = \varepsilon_0 E_{surface} = -\varepsilon_0 \left. \frac{\partial V}{\partial r} \right|_{r=R}$$

- Using Eq. (2), we readily get

$$\sigma_{ind}(\theta) = -\frac{q}{4\pi R^2}\left(\frac{R}{a}\right)\frac{1-\frac{R^2}{a^2}}{(1+\frac{R^2}{a^2}-2\frac{R}{a}\cos\theta)^{3/2}}$$

- If we integrate σ_{ind} along the surface of the sphere, we will get the image charge, i.e.,

$$2\pi R^2 \int_0^{\pi} \sigma_{ind}(\theta) \sin\theta \, d\theta = -\frac{qR}{a}.$$

- The force F_q experienced by the charge q due to the induced charge is easily calculated as the force between the charge q and the image charge q'

$$\mathrm{F}_q = -\frac{Rq^2}{4\pi\varepsilon_0 a(a - \frac{R^2}{a})^2}\hat{k}$$

- Question 1: What if the charge q was inside the sphere, i.e., $a < R$? And we have to compute the potential inside the spherical shell, i.e. for $r \leq R$.
- Answer: Nothing changes really, the same formulas apply. It is obvious that the image charge will now be placed outside the sphere.
- Question 2: What if the sphere is not grounded, but is at a constant potential V_0?
- Answer: In this case there will be an additional image charge located at the center of the sphere, required to raise its potential to V_0. Obviously, this charge will be $q_0 = 4\pi R V_0$.