PH-107

Quantum Physics and Applications

Elements of Statistical Physics-IV

Black-Body Radiation

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Radiation:

Two questions can be asked…

- ✳ Why should a (hot) body emit radiation?
- ✳ How does the intensity of radiation vary with wavelength (at different temperatures)?

Radiation:

Any heated solid emits radiation in a continuous spectrum. Some empirical observations:

✳The hotter the body the higher the frequency of radiation. (First the body becomes red hot and then becomes white hot).

✳The frequency of radiation is independent of the object being heated. It depends only on the temperature.

A Universal character of all heated objects

Black-body Radiation (Recap)

Recap from XI class

CHAPTER ELEVEN

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THERMAL PROPERTIES OF MATTER

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11.9.3 Radiation

11.9.4 Blackbody Radiation

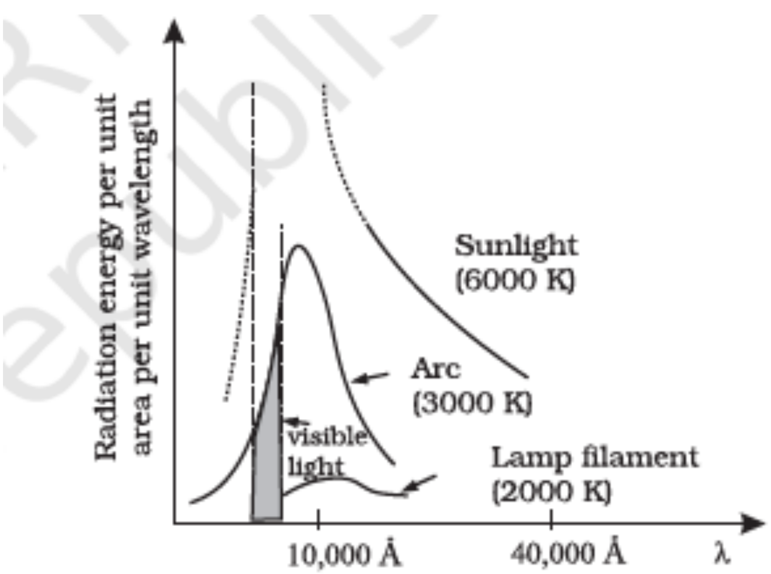


Fig. 11.18: *Energy emitted versus wavelength for a blackbody at different temperatures*

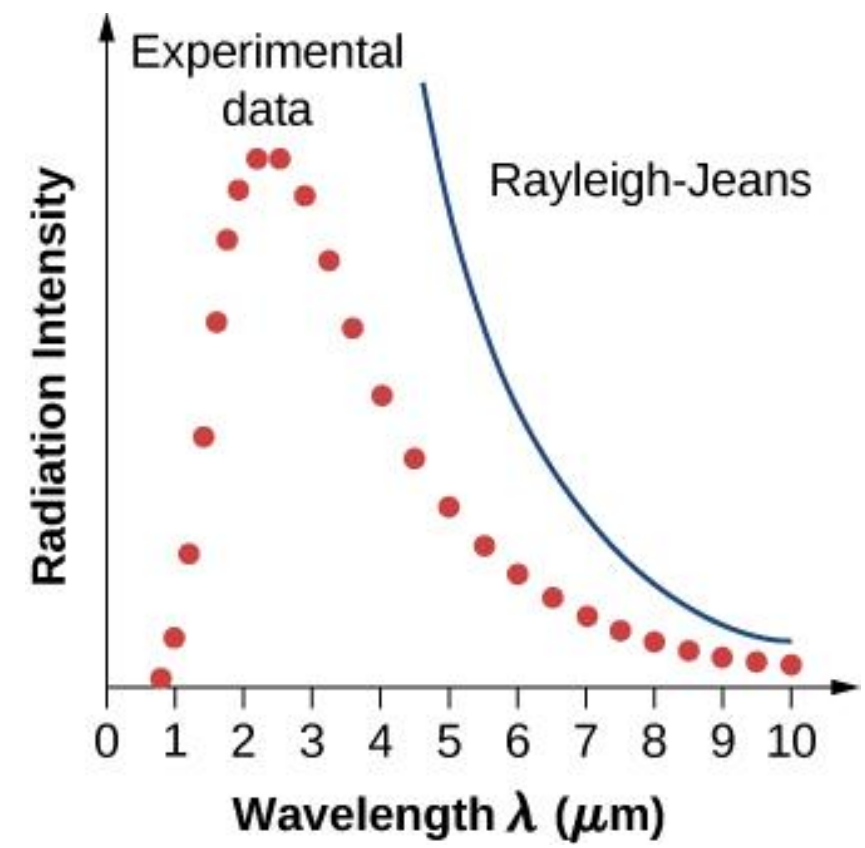
Gustav Kirchhoff (1824-1887)

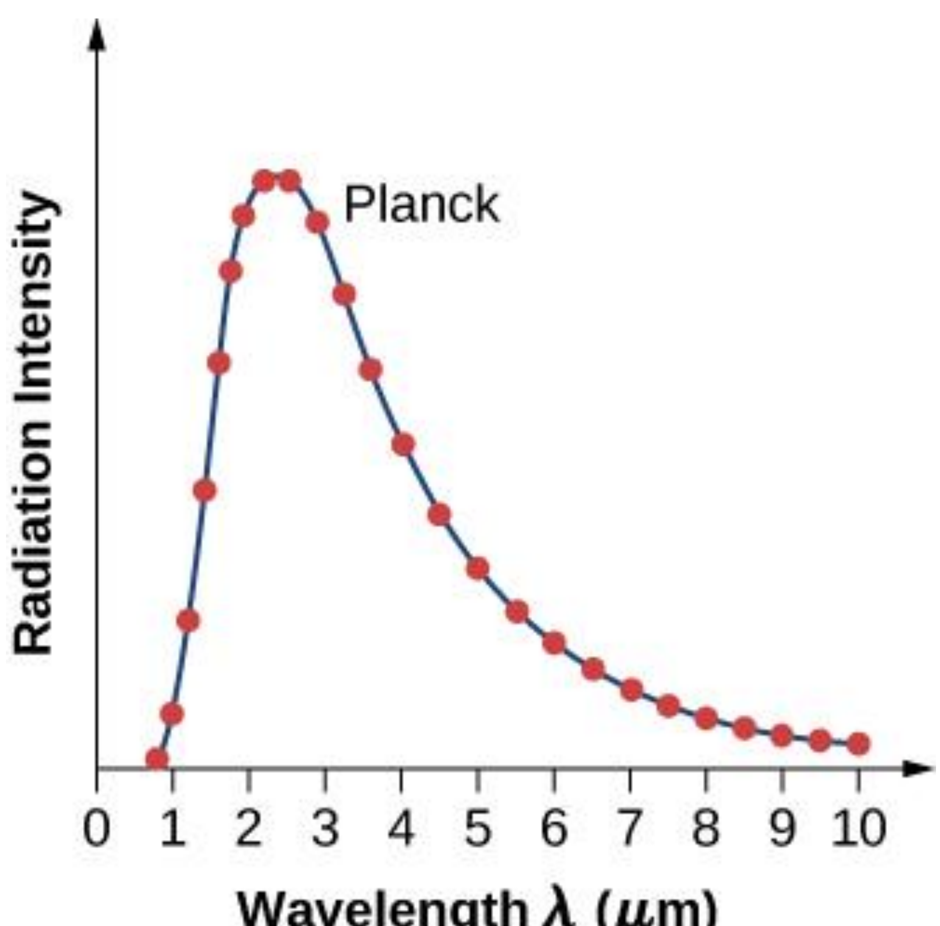
temperature. The relation between λ_m and T is given by what is known as **Wien's Displacement Law:**

$$\lambda_m T = \text{constant} \qquad (11.15)$$

Black-body Radiation (Recap)

Rayleigh–Jeans (1900): "Sources of radiation are atoms in a state of oscillation (classical oscillators)"





Max-Planck (1901): "The elementary oscillators could emit and absorb EM radiation ONLY in discrete packets"

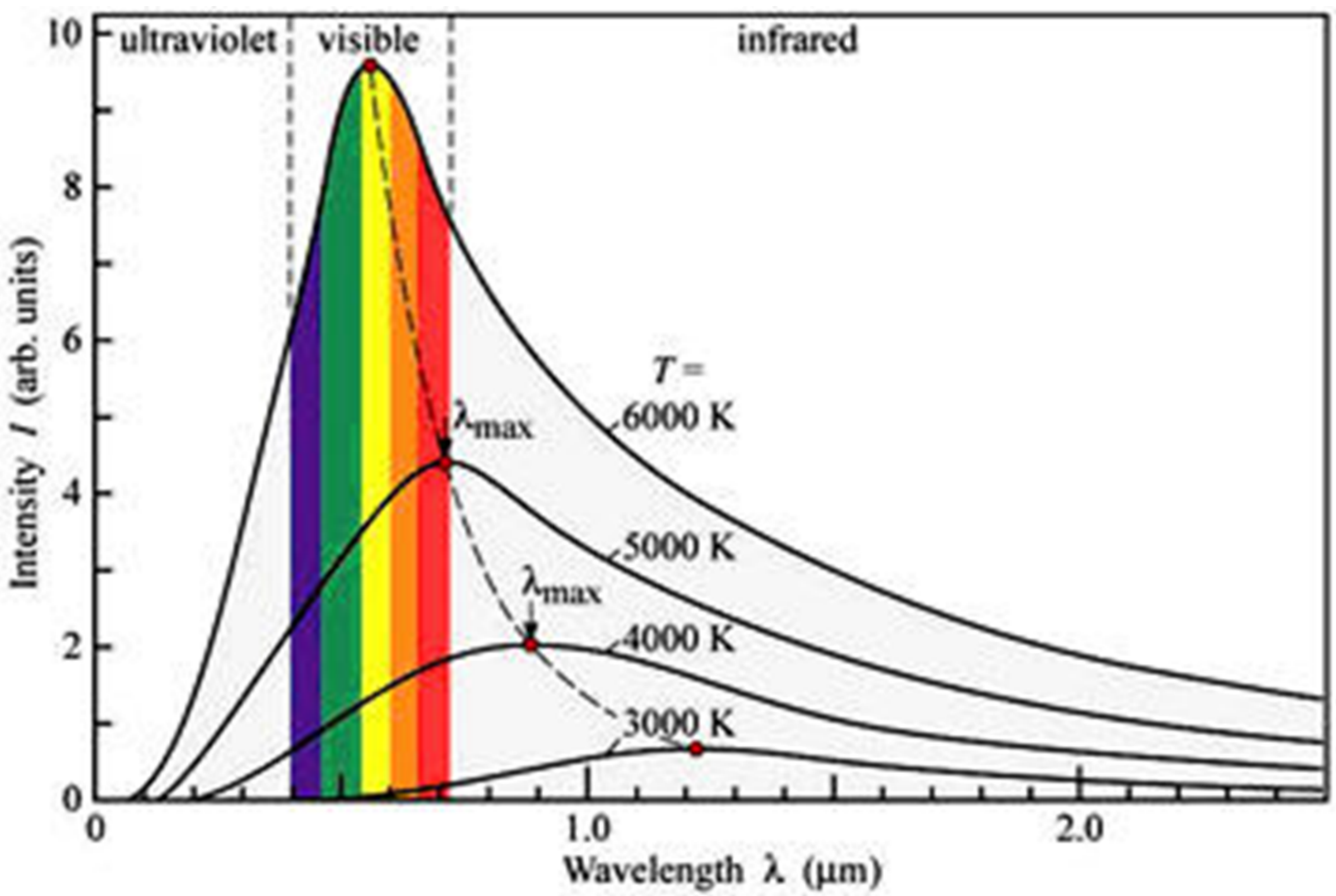
$$E = nh\nu \qquad h \sim 6.6 \times 10^{-34} \text{ SI units}$$

Birth of Quantum Physics !!

Planck's Contribution

The total energy $u(\nu)d\nu$ per unit volume in the frequency interval ν and $\nu + d\nu$

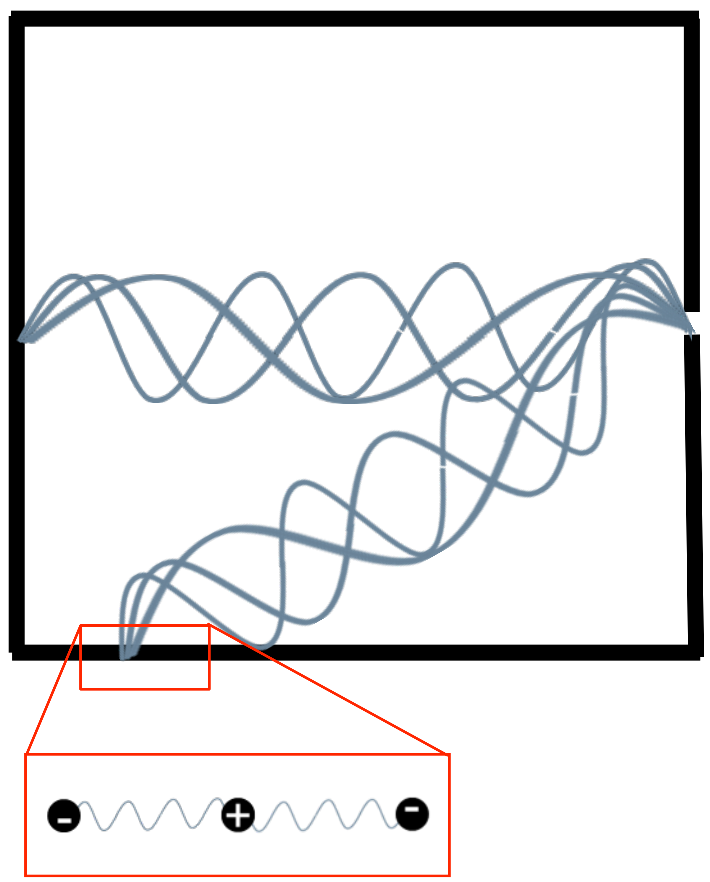
$$u(\nu)d\nu = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/k_B T} - 1}d\nu$$



Black-body Radiation

Why should a (hot) body emit radiation?

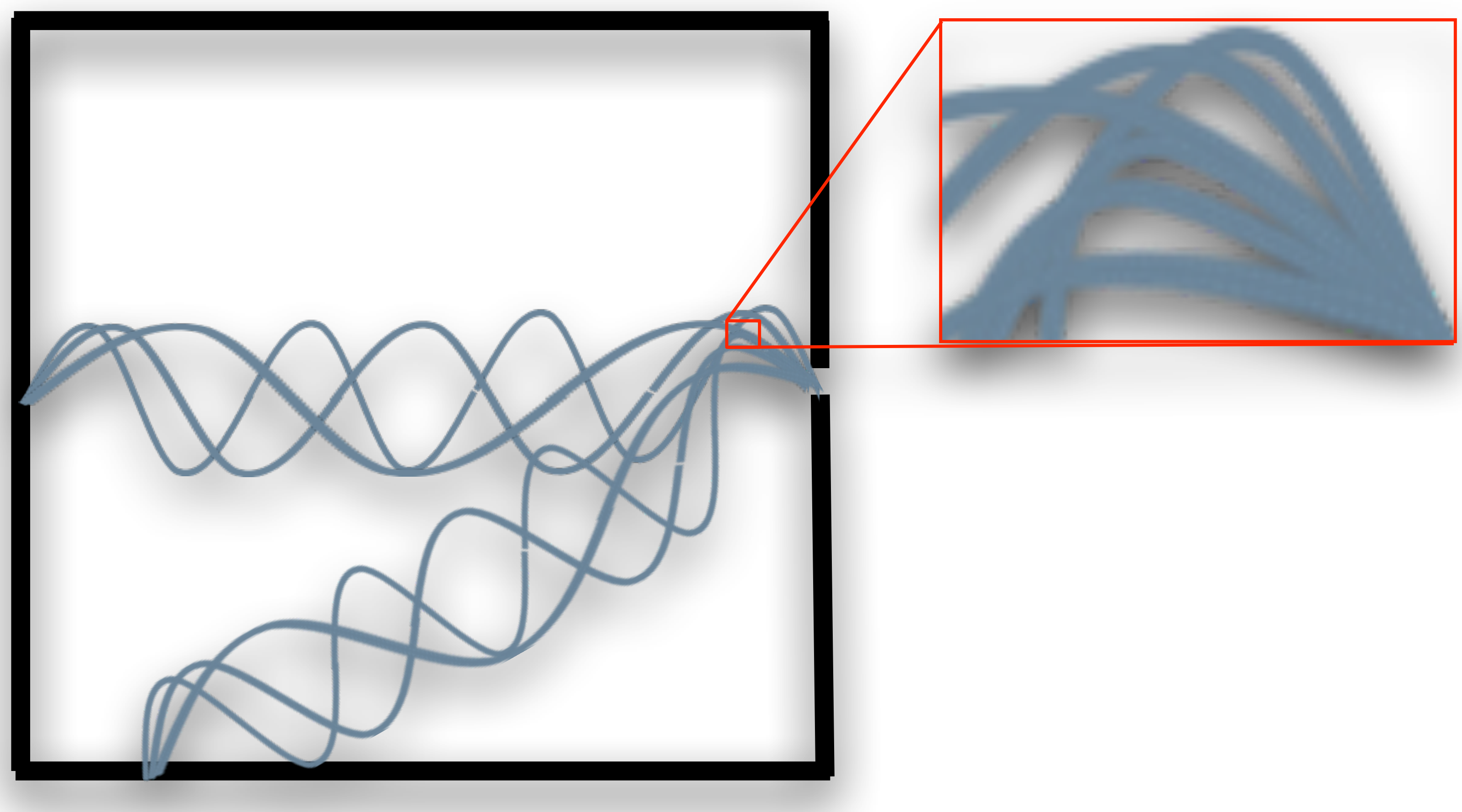
EM waves are generated by periodically oscillating charges **(Maxwell, 1864).**



Cavity walls composed of billions of miniscule charged oscillators.

Black-body Radiation

Radiation inside the cavity at absolute temperature *T* are a series of standing EM waves.



How many standing waves in the frequency interval *v* and *v*+*dv*?

Black-body Radiation

We model the enclosure heated to temperature T as a gas of photons (Bosons).

So, the number of photons within the energy range E and $E+dE$ is given by

$$dN(E) = N(E)dE = g(E)\ f_{\mathrm{BE}}(E)dE$$

and the energy of the particles by

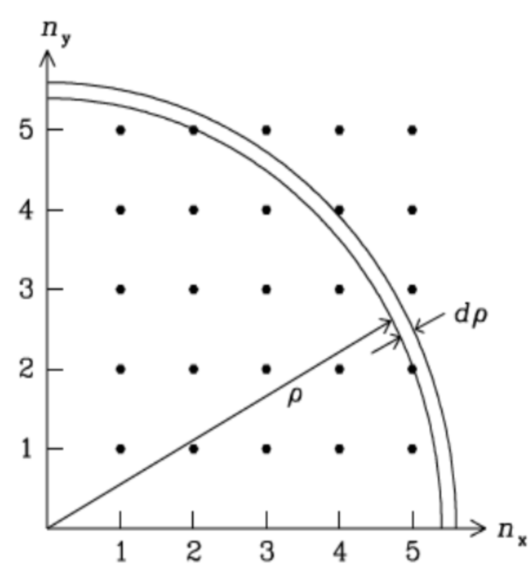
$$dU = EN(E)dE = g(E)\ Ef_{\mathrm{BE}}(E)dE$$

All we need to find is $g(E)$.

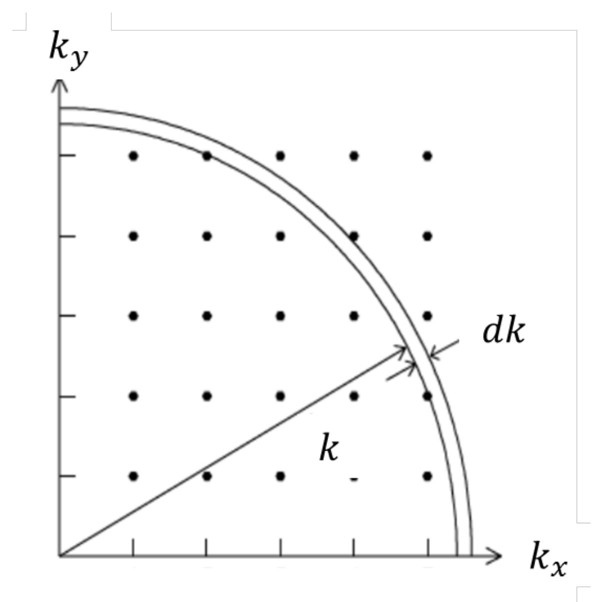
Density of State (Recap)

Let us recap how we estimated density of state *g(E)* in previous lecture.

We have written the energy as $E = \frac{\pi^2 \hbar^2}{2mL^2}(n_x^2 + n_y^2 + n_z^2)$



$$n_{x,y,z} = \frac{k_{x,y,z}\ L}{\pi},$$



What is the area (volume) of the 2D (3D) k-space occupied by each point ?

Density of states in the k space $F(k)\ d\mathbf{k} = \frac{V}{\pi^3}\left(\frac{1}{8}\right)\ 4\pi k^2\ dk$

$$f(k)d\mathbf{k} = \frac{1}{\pi^3}\left(\frac{1}{8}\right)4\pi k^2 dk = g(E)dE$$

$$E = \frac{\hbar^2 k^2}{2m}$$

$$g(E) = \frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}}\sqrt{E}$$

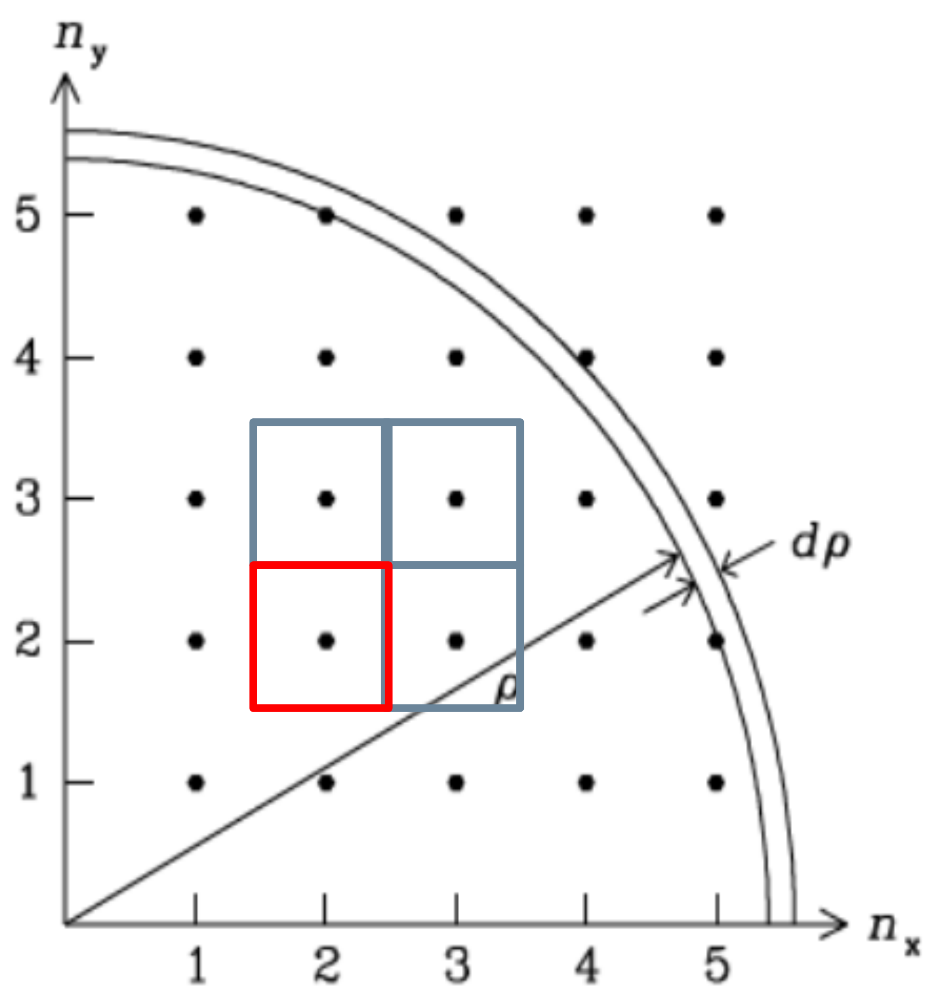
Density of State

Integer space

In integer space, let us calculate the density of states X(ρ)

$\mathbf{X}(\rho)d\rho$ could be equated to $G(E)dE$ as

$$\mathbf{X}(\rho)d\rho = G(E)dE$$

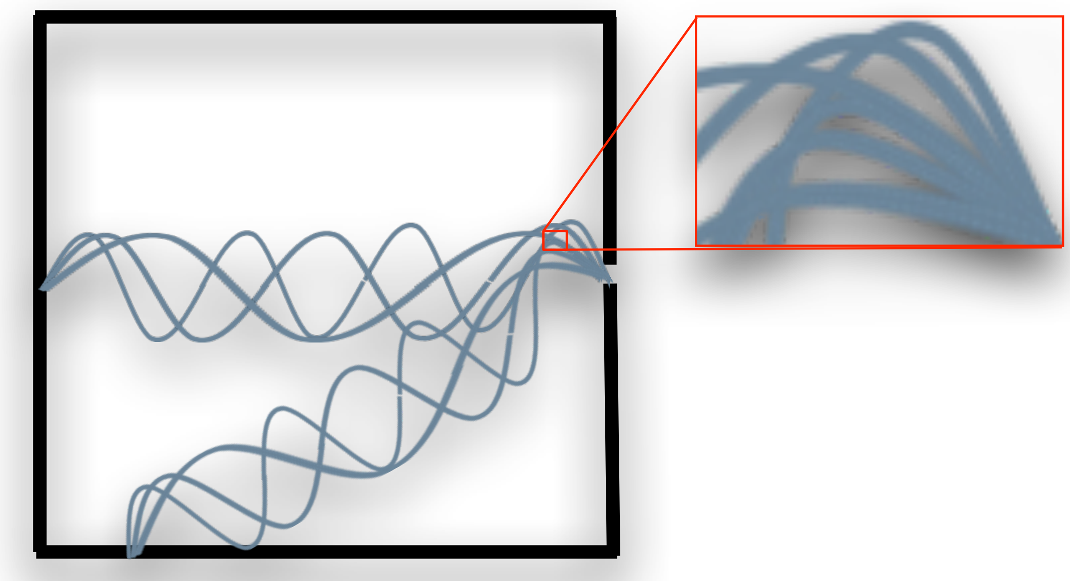


How do we get X(ρ)?

We know the size of the box enclosing each point is 1×1 (No unit, it's a number space!)

Density of points in integer space X(ρ) = 1

Density of State



For standing waves, nodes at boundary implies

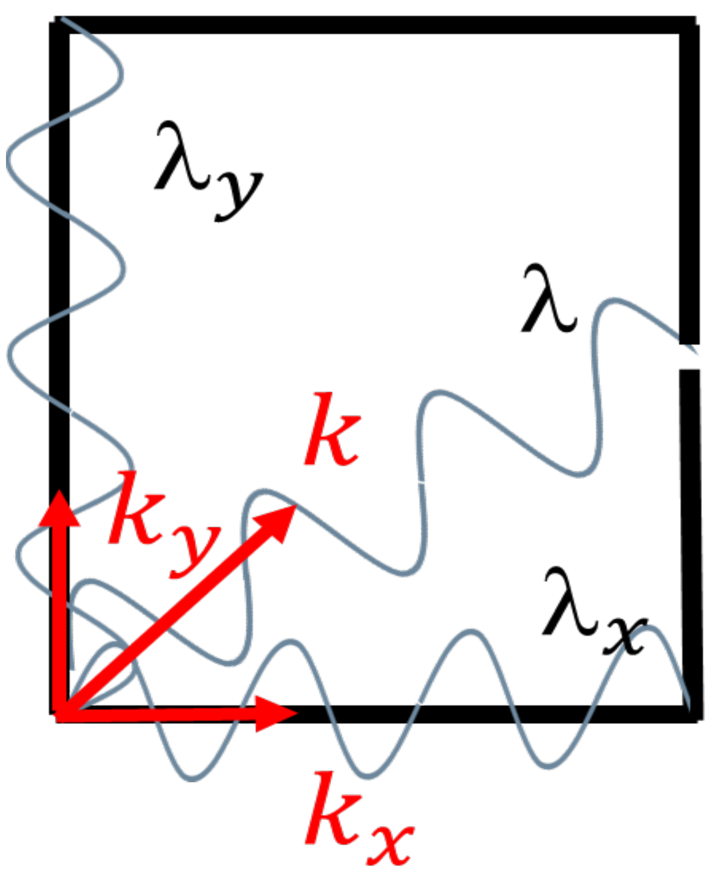
$$\lambda = 2L, \quad L, \quad \frac{2L}{3} \cdots \text{ or } n = \frac{2L}{\lambda}, \quad n = 1, 2, 3 \ldots$$

For wave in *x*- and *y*-directions

$$n_x \lambda_x = 2L \text{ and } n_y \lambda_y = 2L$$

The condition above can be written as

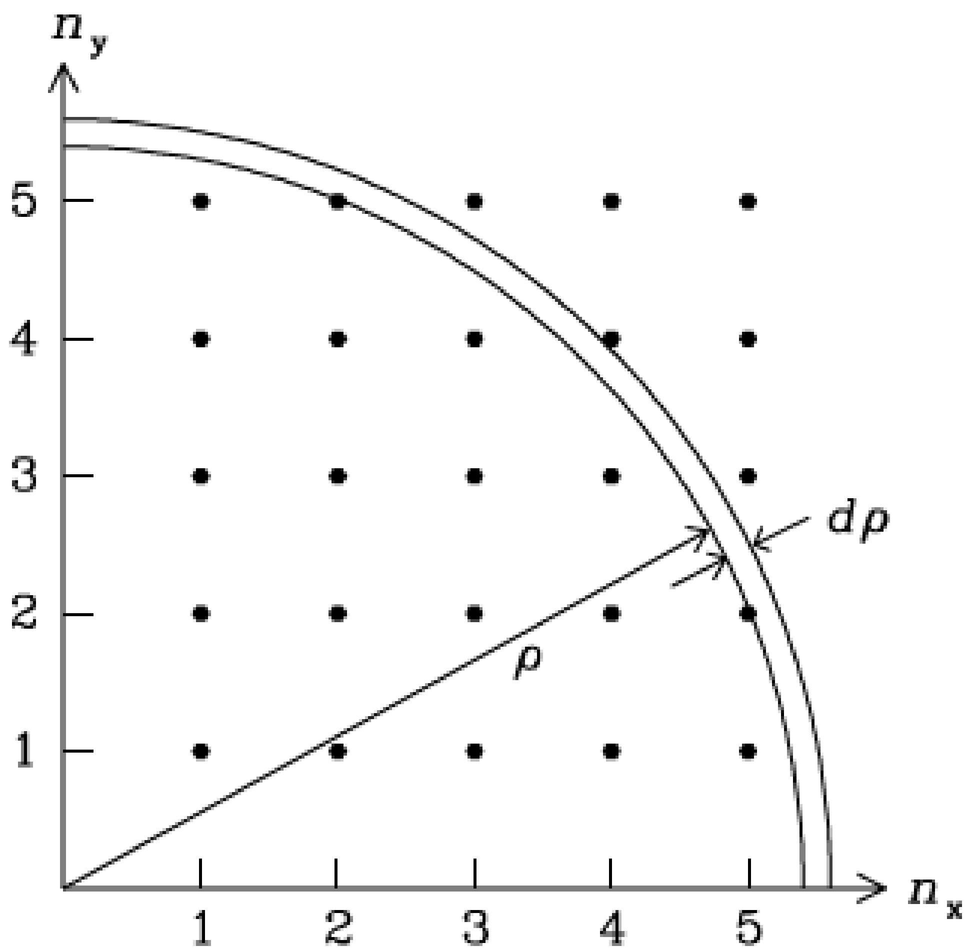
$$k_x = \frac{\pi n_x}{L} \text{ and } k_y = \frac{\pi n_y}{L}$$



$$k^2 = \left(\frac{\pi n_x}{L}\right)^2 + \left(\frac{\pi n_y}{L}\right)^2 \text{ or } n_x^2 + n_y^2 = \left(\frac{2L}{\lambda}\right)^2$$

$$n_x^2 + n_y^2 + n_z^2 = \left(\frac{2L}{\lambda}\right)^2$$

Density of State



Density of states is 1 state per unit volume of the number space

$$d\boldsymbol{\rho} = \left(\frac{1}{8}\right) 4\pi\rho^2 d\rho$$

So, number of states between $\boldsymbol{\rho}$ and $\boldsymbol{\rho} + d\boldsymbol{\rho}$

$$\mathbf{X}(\rho)d\boldsymbol{\rho} = \left(\frac{1}{8}\right) 4\pi\rho^2 d\rho$$

But, $\mathbf{X}(\rho)d\boldsymbol{\rho} = F(k)d\boldsymbol{k} = G(E)dE = \theta(\nu)d\nu$

Density of State

Note that

$$\rho = \sqrt{n_x^2 + n_y^2 + n_z^2} = \frac{2L}{\lambda} = \frac{2\pi}{\lambda}\frac{L}{\pi} = \frac{kL}{\pi}$$

$$\mathbf{X}(\rho)d\boldsymbol{\rho} = \left(\frac{1}{8}\right)4\pi\rho^2 d\rho = \left(\frac{L^3}{2\pi^2}\right)k^2 dk = F(k)d\boldsymbol{k}$$

Note that $L^3 = V$ (the volume of the enclosure) and $k = \frac{2\pi\nu}{c}$,

we can write

$$\mathbf{X}(\rho)d\boldsymbol{\rho} = \left(\frac{1}{2\pi^2}\right)k^2 dk = \frac{4\pi\nu^2}{c^3}d\nu = \theta(\nu)d\nu$$

Density of State

BBR spectral Density

Finally, we have to account for the fact that each *k*-state is 2-fold degenerate due to the two possible ***polarizations*** of the E-field for each mode, So we get

$$\theta(\nu)d\nu = \frac{8\pi\nu^2}{c^3}d\nu$$

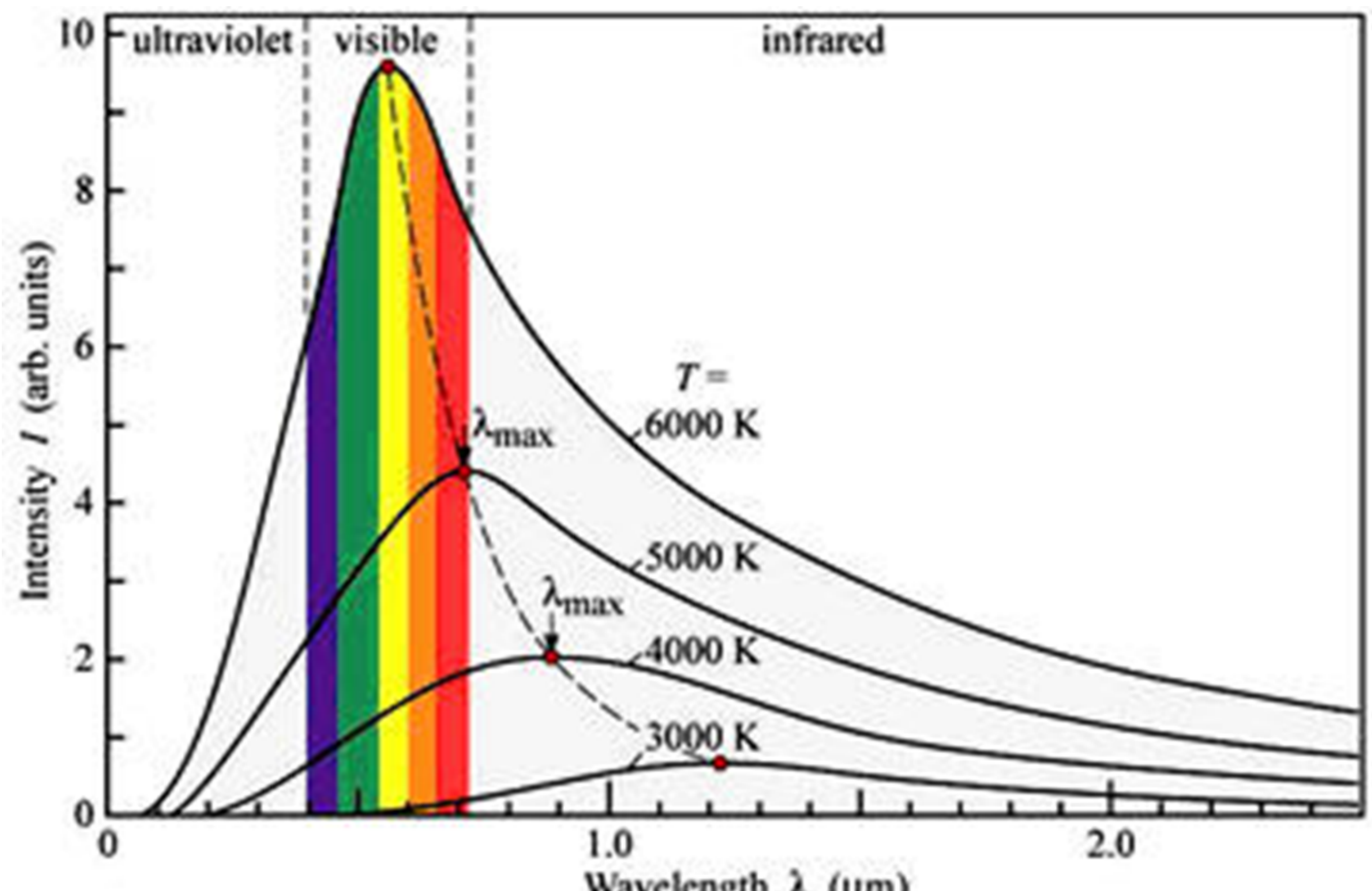
Now you can verify

$$u(\nu)d\nu = \frac{U(\nu)}{V}d\nu = \theta(\nu)\ f_{\mathrm{BE}}(\nu)\ E(\nu)d\nu = \frac{8\pi\nu^2}{c^3}\ \frac{1}{e^{\frac{h\nu}{k_BT}}-1}\ h\nu\ d\nu$$

Planck's Lucky Guess

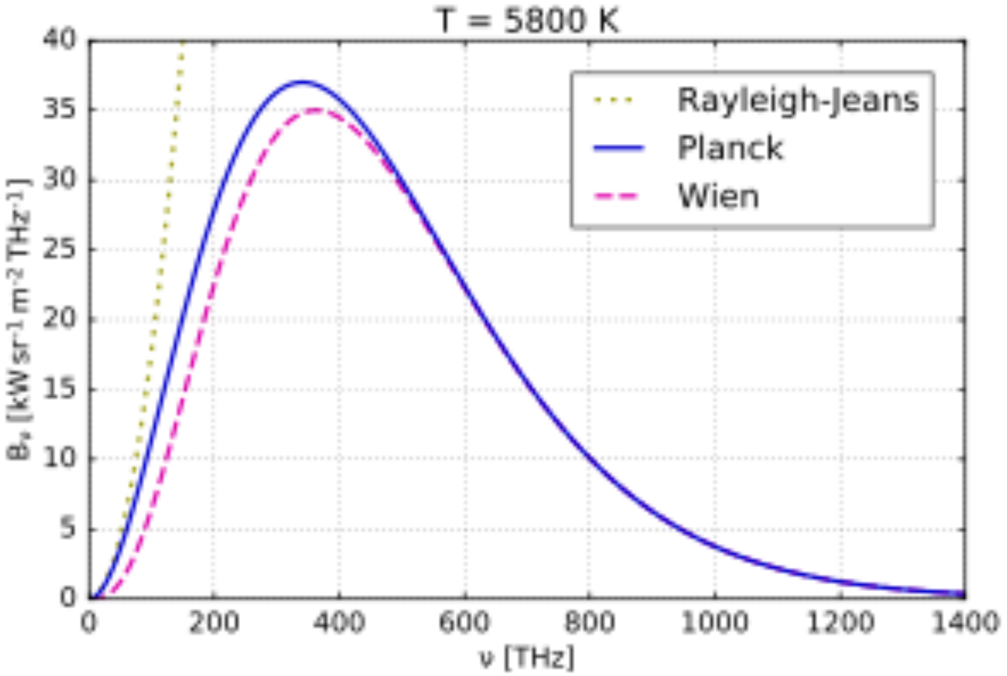
The total energy $u(\nu)d\nu$ per unit volume in the frequency interval ν and $\nu + d\nu$

$$u(\nu)d\nu = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/k_B T} - 1}d\nu$$



The total energy $\mathrm{u}(\nu)\mathrm{d}\nu$ per unit volume in the frequency interval ν and $\nu + d\nu$

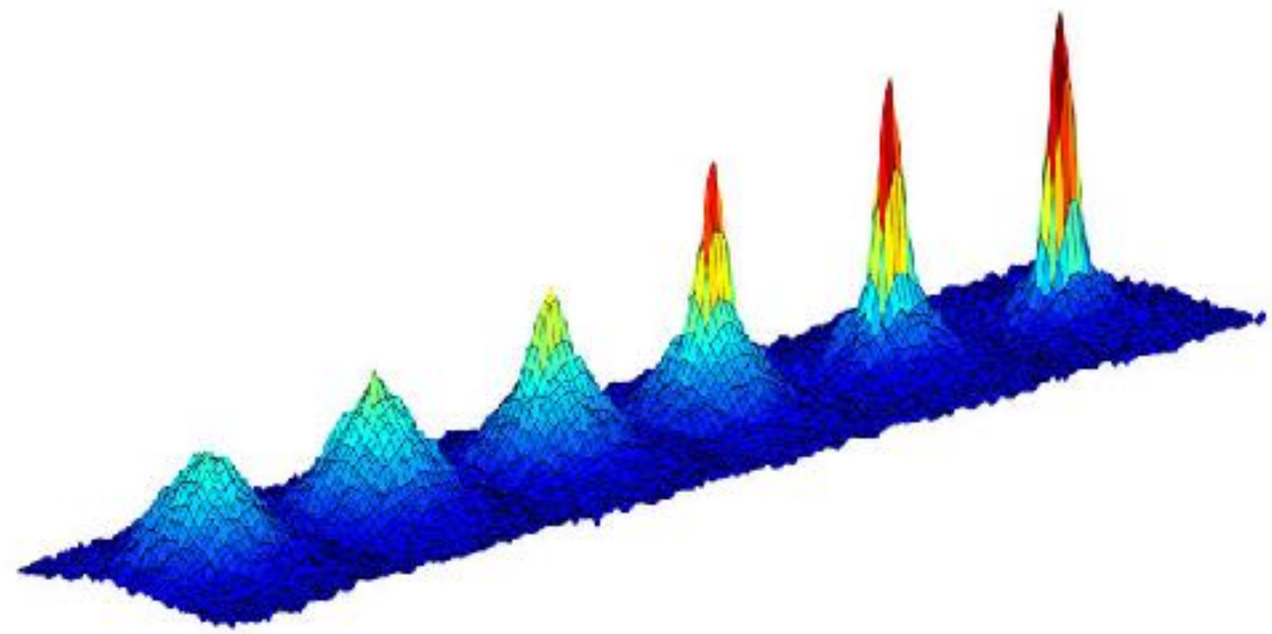
$$\mathrm{u}(\nu)\mathrm{d}\nu = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/k_B T} - 1}d\nu$$



Deduce the results of Rayleigh-Jeans Law and Wien's Law from Planck's results.

Distribution Function

$$f_{\mathrm{BE}}(E) = \frac{1}{e^{(E/k_B T)} - 1}$$



Case I:

$$E/k_B T \to 0, \quad f_{\mathrm{BE}}(E) \to \infty$$

Case II:

$$E/k_B T \to \infty, \quad f_{\mathrm{BE}}(E) \to e^{-E/k_B T}$$

$$f_{\mathrm{BE}}(E) \to f_{\mathrm{MB}}(E)$$

Occupation probability of lowest energy states increases exponentially, at sufficiently low temperature, all particles drop down to the ground energy state: Bose-Einstein Condensation

Recommended Readings

Statistical Physics, Chapter 10, Section 10.4