MA-111 Calculus II
(D1 & D2)

Lecture 6

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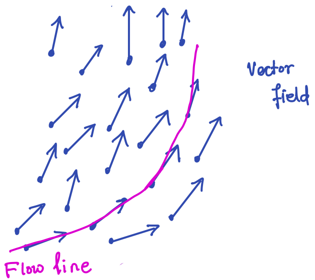
Flow lines for vector field

Vector fields also arise as the tangent vectors to the fluid flow.
Or conversely, given a vector field we can talk about its flow lines.

Definition If $\mathbf{F}$ is a vector field defined from $D \subset \mathbb{R}^n$ to $\mathbb{R}^n$, a flow line or integral curve is a path i.e., a map $\mathbf{c} : [a, b] \to D$ such that

$$\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t)), \quad \forall t \in [a, b].$$

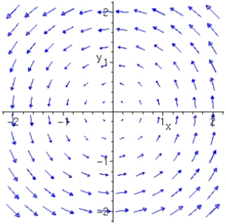
In particular, $\mathbf{F}$ yields the velocity field of the path $\mathbf{c}$.



Example: Show that $\mathbf{c}(t) = (\cos t, \sin t)$ for $t \in [0, 2\pi]$, is a flow line for the vector field $\mathbf{F}(x, y) = (-y, x)$.

Ans $\mathbf{c}'(t) = (-\sin t, \cos t)$ and $\mathbf{F}(\mathbf{c}(t)) = (-\sin t, \cos t)$.

Does it have other flow lines? Can you guess by looking at the vector field?



Ans Yes! $\mathbf{c}(t) = (a\cos t, \sin t)$, $t \in [0, 2\pi]$ and any $a > 0$.

Flow line: System of ODEs

Finding the flow line for a given vector field involves solving a system of differential equations, if $\mathbf{c}(t) = (x(t), y(t), z(t))$ then

$$\begin{aligned} x'(t) &= P(x(t), y(t), z(t)) \\ y'(t) &= Q(x(t), y(t), z(t)) \\ z'(t) &= R(x(t), y(t), z(t)), \end{aligned}$$

where the vector field is given by $\mathbf{F} = (P, Q, R)$.

Such questions are dealt with in MA108.

Curve and path

Recall a path in $\mathbb{R}^n$ is a continuous map $\mathbf{c} : [a, b] \to \mathbb{R}^n$.

A curve in $\mathbb{R}^n$ is the image of a path $\mathbf{c}$ in $\mathbb{R}^n$.

Both the curve and path are denoted by the same symbol $\mathbf{c}$.

• Let $n = 3$ and $\mathbf{c}(t) = (x(t), y(t), z(t))$, for all $t \in [a, b]$. The path $\mathbf{c}$ is continuous iff each component x, y, z is continuous. Similarly, $\mathbf{c}$ is a C^1 path, i.e., continuously differentiable if and only if each component is C^1.

• A path $\mathbf{c}$ is called closed if $\mathbf{c}(a) = \mathbf{c}(b)$.

• A path $\mathbf{c}$ is called simple if $\mathbf{c}(t_1) \neq \mathbf{c}(t_2)$ for any $t_1 \neq t_2$ in $[a, b]$ other than $t_1 = a$ and $t_2 = b$ endpoints.

• If we write $\mathbf{c}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ in vector notation, the tangent vector to $\mathbf{c}(t)$ is $\mathbf{c}'(t)$, i.e.,

$$\mathbf{c}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}.$$

• If a C^1 curve $\mathbf{c}$ is such that $\mathbf{c}'(t) \neq 0$ for all $t \in [a, b]$, the curve is called a regular or non-singular parametrised curve.

Examples of curves

• Let $\mathbf{c}(t) = (\cos 2\pi t, \sin 2\pi t)$ where $0 \leq t \leq 1$. This is a simple closed C^1 (actually smooth) curve.

• Let $\mathbf{c}(t) = (t, t^2)$ where $-1 \leq t \leq 5$ is a simple curve but not closed.

• Let $\mathbf{c}(t) = (\sin(2t), \sin t)$ where $-\pi \leq t \leq \pi$. It traces out a figure 8. It is not a simple but a closed C^1 curve.

• Let $\mathbf{c}(t) = (t^3, t)$ where $-1 \leq t \leq 1$ for some real numbers a, b is a part of the graph of the function $y = x^{1/3}$. This is simple but not a closed curve. Though the function $y = x^{\frac{1}{3}}$ is not a smooth function at origin, but this parametrization is regular!

Line integrals of vector fields

Assume that the vector field $\mathbf{F} : D \subset \mathbb{R}^n \to \mathbb{R}^n$, for $n = 1, 2$, is continuous and the curve $\mathbf{c} : [a, b] \to D$ is C^1.

Then we define the line integral of $\mathbf{F}$ over $\mathbf{c}$ as:

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} := \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt.$$

If $\mathbf{F} = (F_1, F_2, F_3)$ and $\mathbf{c}(t) = (x(t), y(t), z(t))$, we see that

$$\begin{aligned}
&\int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt \\
&= \int_a^b \left(F_1(\mathbf{c}(t)) \frac{dx(t)}{dt} + F_2(\mathbf{c}(t)) \frac{dy(t)}{dt} + F_3(\mathbf{c}(t)) \frac{dz(t)}{dt} \right) dt.
\end{aligned}$$

Because of the form of the right hand side the line integral is sometimes written as

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b F_1 dx + F_2 dy + F_3 dz.$$

The expression on the right hand side is just alternate notation for the line integral. It does not have any independent meaning.

Physical interpretation

The work done by a *constant force* $\mathbf{F}$ on a particle that moves a displacement $\mathbf{s}$ is given by $W = \mathbf{F} \cdot \mathbf{s}$.

The work done by a *force field* $\mathbf{F}$ in moving a particle along a curve $\mathbf{c}$ is given by the line integral

$$W = \int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}.$$

An example

Example 1: Evaluate

$$\int_{\mathbf{c}} x^2 dx + xydy + dz,$$

where $\mathbf{c} : [0, 1] \to \mathbb{R}^3$ is given by $\mathbf{c}(t) = (t, t^2, 1)$.

Solution: Let $\mathbf{c}(t) = (t, t^2, 1)$.
Let $\mathbf{F}(x, y, z) = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)) = (x^2, xy, 1)$.

Thus $F_1(t, t^2, 1) = t^2$, $F_2(t, t^2, 1) = t^3$ and $F_3(t, t^2, 1) = 1$.

We have $\mathbf{c}'(t) = (1, 2t, 0)$, hence

$$(F_1(t, t^2, 1), F_2(t, t^2, 1), F_3(t, t^2, 1)) \cdot \mathbf{c}'(t) = t^2 + 2t^4 + 0.$$

$$\int_{\mathbf{c}} x^2 dx + xydy + dz = \int_0^1 (t^2 + 2t^4) dt = 11/15.$$

Example 2 (Marsden, Tromba, Weinstein): Find the work done by the force field $\mathbf{F} = (x^2 + y^2)(\mathbf{i} + \mathbf{j})$ around the loop $\mathbf{c}(t) = (\cos t, \sin t)$, $0 \leq t \leq 2\pi$.

Solution: The work done is given by

$$\begin{aligned} W &= \int_{\mathbf{c}} \mathbf{F} \cdot ds \\ &= \int_0^{2\pi} \mathbf{F}(\mathbf{c}(t)).\mathbf{c}'(t)\ dt \\ &= \int_0^{2\pi} (\cos t + \sin t) dt \\ &= (\sin t - \cos t)|_0^{2\pi} = 0 \end{aligned}$$

Integrating along successive paths

It is easy to see that if $\mathbf{c}_1$ is a path joining two points P_0 and P_1 and $\mathbf{c}_2$ is a path joining P_1 and P_2 and $\mathbf{c}$ is the union of these paths (that is, it is a path from P_0 to P_2 passing through P_1), which is C^1 then

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}_1} \mathbf{F} \cdot d\mathbf{s} + \int_{\mathbf{c}_2} \mathbf{F} \cdot d\mathbf{s}.$$

This property follows directly from the corresponding property for Riemann integrals:

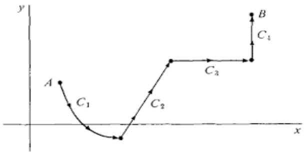
$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt,$$

where c is a point between a and b.

• More generally, let the curve $\mathbf{c}$ be a union of curves $\mathbf{c}_1, \ldots, \mathbf{c}_n$. We often write this as $\mathbf{c} = \mathbf{c}_1 + \mathbf{c}_2 + \ldots \mathbf{c}_n$, where end point of $\mathbf{c}_i$ is the starting point of $\mathbf{c}_{i+1}$ for all $i = 1, \ldots, n-1$.

Then we can define

$$\int_{\mathbf{c}} \mathbf{F} \cdot ds := \int_{\mathbf{c}_1} \mathbf{F} \cdot ds + \ldots + \int_{\mathbf{c}_n} \mathbf{F} \cdot ds.$$



• Let $\mathbf{c}$ be a curve on $[a, b]$ and $\widetilde{\mathbf{c}}(t) = \mathbf{c}(a + b - t)$, that is the curve $\widetilde{\mathbf{c}}$ traversed in the reverse direction and is denoted by $-\mathbf{c}$.
$\int_c \mathbf{F} \cdot ds = -\int_{-c} \mathbf{F} \cdot ds$ (use change of variables formula).

Different parametrizations of the same path

Example 1: Let $\mathbf{c}_1(t) = (\cos t, \sin t)$ for $0 \leq t \leq 2\pi$. Then $c_2(t) = (\cos 2t, \sin 2t)$ for $0 \leq t \leq \pi$, the paths are different as a function but the curves traversed are the same.

Example: 2: Here is an example of a simple path with three different parametrisations with the same domain.
Take the straight line segment between $(0, 0, 0)$ and $(1, 0, 0)$.

Here are three different ways of parametrising it:

$$\{t, 0, 0)\}, \quad \{(t^2, 0, 0)\} \quad \text{and} \quad \{(t^3, 0, 0)\},$$

where $0 \leq t \leq 1$.

Reparametrisation preserving the orientation

Let $\mathbf{c} : [a, b] \to \mathbb{R}^n$ be a path which is non-singular, that is, $\mathbf{c}'(t) \neq 0$ for all $t \in [a, b]$.

- Suppose we now make change of variables $t = h(u)$, where h is $\mathcal{C}^1$ diffeomorphism (this means that h is bijective, $\mathcal{C}^1$ and so is its inverse) from $[\alpha, \beta]$ to $[a, b]$. We let $\gamma(u) = \mathbf{c}(h(u))$.
- We will assume that $h(\alpha) = a$ and $h(\beta) = b$.
- Then γ is called a reparametrisation of $\mathbf{c}$.
- Because h is a C^1 diffeomorphism, γ is also a C^1curve.

The line integral of a vector field $\mathbf{F}$ along γ is given by

$$\int_\gamma \mathbf{F} \cdot d\mathbf{s} = \int_\alpha^\beta \mathbf{F}(\gamma(u)) \cdot \gamma'(u) du = \int_\alpha^\beta \mathbf{F}(c(h(u)) \cdot \mathbf{c}'(h(u)) h'(u) du,$$

where the last equality follows from the chain rule. Using the fact that $h'(u) du = dt$, we can change variables from u to t to get

$$\int_\gamma \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt = \int_\mathbf{c} \mathbf{F} \cdot d\mathbf{s}.$$

Reparametrization reversing the orientation

For given two points P and Q on $\mathbb{R}^n$ for $n = 2, 3$, and a path connecting them, we can ask whether the path is traversed from P to Q or from Q to P?

Since a path from P to Q is a mapping $\mathbf{c} : [a, b] \to \mathbb{R}^n$ with $\mathbf{c}(a) = P$ and $\mathbf{c}(b) = Q$, (or vice-versa), it allows us to determine the direction in which the path is traversed. This direction of the path is called its Orientation.

If the reparametrization $\gamma(\cdot) = \mathbf{c}(h(\cdot))$ preserves the orientation of $\mathbf{c}$, then

$$\int_\gamma \mathbf{F}.\mathbf{ds} = \int_{\mathbf{c}} \mathbf{F}.\mathbf{ds}.$$

If the reparamtrization reverses the orientation, then

$$\int_\gamma \mathbf{F}.\mathbf{ds} = -\int_{\mathbf{c}} \mathbf{F}.\mathbf{ds}.$$

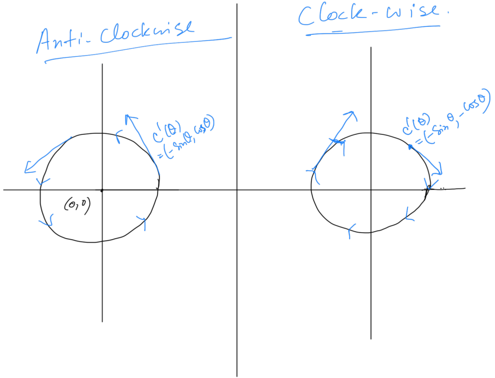
Orientation of closed curves on plane

Let us consider the paths lying in $\mathbb{R}^2$, namely, Planar curves.

For a simple closed planar curve, we get a choice of direction- clockwise or anti-clockwise.

Ex. $\gamma(\theta) = (\cos(\theta), \sin(\theta))$, $\theta \in [0, 2\pi]$. This is a circle with direction anti-clockwise.

Set $\gamma_1(\theta) = (\cos(\theta), -\sin(\theta))$, $\theta \in [0, 2\pi]$. It is circle with clockwise direction.



The argument just made shows that the line integral is independent of the choice of parametrisation - it depends only on the image of the non-singular parametrised path and the orientation.

- A geometric curve C is a set of points in the plane or in the space that can be traversed by a parametrized path in the given direction. Often the line integral of a vector field $\mathbf{F}$ along a 'geometric curve' C is represented by $\int_C \mathbf{F}.\mathbf{ds}$ or by $\int_C F_1 dx + F_2 dy + F_3 dz$.
- To evaluate $\int_C \mathbf{F}.\mathbf{ds}$, choose a convenient parametrization $\mathbf{c}$ of C traversing C in the given direction and then

$$\int_C \mathbf{F}.\mathbf{ds} := \int_{\mathbf{c}} \mathbf{F}.\mathbf{ds}.$$

- '$\oint_C$' means the line integral over a closed curve C.

The arc length parametrisation

There is a natural choice of parametrisation we can make which is useful in many situations. This is the parametrisation by arc length.

Recall the length of a curve $\mathbf{c}$ for a path $\mathbf{c} : [a, b] \to \mathbb{R}^3$, called its arc length ,is given by

$$\ell(\mathbf{c}) = \int_a^b \|\mathbf{c}'(t)\| \; dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \; dt.$$

We now set

$$s(t) = \int_a^t \|\mathbf{c}'(u)\| du,$$

where $\mathbf{c} : [a, b] \to \mathbb{R}^3$ is a non-singular curve, from which it follows that $s'(t) = \|\mathbf{c}'(t)\|$. Why? Fundamental theorem of Calculus.

It is easy to see that s is a strictly increasing differentiable function. Let $h : [0, \ell(\mathbf{c})] \to [a, b]$ be its inverse. Then it is differentiable and its derivative is not vanishing. Define $\tilde{\mathbf{c}}(u) := \mathbf{c}(h(u))$ for $u \in [0, \ell(\mathbf{c})]$. This is called the arc length parametrization.

Let $h(u) = t \in [a, b]$ or $s(t) = u$.
Note that

$$\begin{aligned} \frac{d\tilde{\mathbf{c}}(u)}{du} &= \mathbf{c}'(h(u))h'(u) \\ &= \mathbf{c}'(h(u))\frac{1}{s'(h(u))} \\ &= \mathbf{c}'(t)\frac{1}{\|\mathbf{c}'(t)\|} \end{aligned}$$

Using the reparmetrization theorem we get that

$$\int_{\mathbf{c}} \mathbf{F}.\mathbf{ds} = \int_{\tilde{\mathbf{c}}} \mathbf{F}.\mathbf{ds}.$$

Note,

$$\begin{aligned}
\int_{\tilde{\mathbf{c}}} \mathbf{F} \cdot d\mathbf{s} &= \int_0^{\ell(\mathbf{c})} \mathbf{F}(\tilde{\mathbf{c}}(u)) \cdot \tilde{\mathbf{c}}'(u) du \\
&= \int_0^{\ell(\mathbf{c})} \mathbf{F}(\mathbf{c}(h(u))) \cdot \frac{\mathbf{c}'(h(u))}{\|\mathbf{c}'(h(u))\|} du \\
&= \int_0^{\ell(\mathbf{c})} \mathbf{F}(\mathbf{c}(h(u)) \cdot \mathbf{T}(h(u))\ du
\end{aligned}$$

where $\mathbf{T}(t) = \frac{\mathbf{c}'(t)}{\|\mathbf{c}'(t\|}$ is the unit tangent vector along the curve.

In other words, the line integral is nothing but the (Riemann) integral of the tangential component of $\mathbf{F}$ with respect to arc length.

Note for this reparametrization we need to assume $\mathbf{c}$ is a non singular curve.

Integrals of scalar functions along path

To this end, as before we set

$$s(t) = \int_a^t \|\mathbf{c}'(u)\| du,$$

where $\mathbf{c} : [a, b] \to \mathbb{R}^3$ is a non-singular curve, from which it follows that $ds = \|\mathbf{c}'(t)\| dt$.

Integrals of scalar functions along path: Let $f : D \to \mathbb{R}$ be a continuous scalar function and $\mathbf{c} : [a, b] \longrightarrow D$ be a non-singular path. Then the path integral of f along $\mathbf{c}$ is defined by

$$\int_{\mathbf{c}} f \, ds := \int_a^b f(\mathbf{c}(t)) \|\mathbf{c}'(t)\| \, dt.$$

Example. Find the circumference of the circle in $\mathbb{R}^2$ whose center is at origin and radius is r, for some $r > 0$.

Ans. Check $\int_{\mathbf{c}} f \, ds$ for $f = 1$ and $\mathbf{c}(t) = (r \cos t, r \sin t)$, for $t \in [0, 2\pi]$.

Quiz announcement

- The quiz on Friday, 18th February 2022 from 8:30-9:30am will include everything from Lecture 1 to the previous slide.
- The quiz will be conducted on SAFE. Please make sure when you login with IITB id into SAFE, MA1112022 is visible.
- The question format is *likely* to be objective.
- Further instructions will follow on Moodle.
- Fill the Google form circulated by Tutors (essential for you to take the quiz) before 10pm Friday 11th February 2022.

Characterization of gradient fields

The main observation about line integrals of a gradient field is the following. This is a form of Fundamental theorem of calculus.

Theorem

Let $n = 2, 3$ and let $D \subset \mathbb{R}^n$.

1. *Let $\mathbf{c} : [a, b] \to D \subset \mathbb{R}^n$ be a smooth path.*
2. *Let $f : D \to \mathbb{R}$ be a differentiable function and let ∇f be continuous on $\mathbf{c}$.*

Then $\int_{\mathbf{c}} \nabla f.\mathbf{ds} = f(\mathbf{c}(b)) - f(\mathbf{c}(a))$.

Proof. From definition, it follows that

$$\int_{\mathbf{c}} \nabla f.\mathbf{ds} = \int_a^b \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t)dt$$

Now the integrand on the right hand side is nothing but the directional derivative of f in the direction of $\mathbf{c}(t)$. Hence, we obtain

$$\int_a^b \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t)dt = \int_a^b \frac{d}{dt} f(\mathbf{c}(t))dt = f(\mathbf{c}(b)) - f(\mathbf{c}(a)).$$

- Suppose the vector field $\mathbf{F}$ is a continuous conservative field, i.e., $\mathbf{F} = \nabla f$, for some C^1 scalar function f. Then for any smooth path $\mathbf{c}$, we have
$$\int_{\mathbf{c}} \mathbf{F}.\mathbf{ds} = f(\mathbf{c}(b)) - f(\mathbf{c}(a)).$$
- This shows that the value of the line integral of a conservative field depends only on the value of the function at the end points of the curve, not on the curve itself.

Definition

The line integral of a vector field $\mathbf{F}$ is independent of path in a domain if for any $\mathbf{c}_1$ and $\mathbf{c}_2$ paths in D with the same initial and terminal points,

$$\int_{\mathbf{c}_1} \mathbf{F}.\mathbf{ds} = \int_{\mathbf{c}_2} \mathbf{F}.\mathbf{ds}.$$

Equivalently, the line integral of $\mathbf{F}$ is independent of path in D if for any closed curve $\mathbf{c}$ (why?)
$$\int_{\mathbf{c}} \mathbf{F}.\mathbf{ds} = 0.$$