Outline

1. Scalar and vector fields
2. Line, surface, and volume integrals involving scalar and vector fields, and some examples

Scalar and Vector Fields

- A scalar field Φ is a function of position $\mathbf{r}$ without any direction
- That is

$$\Phi = \Phi(\mathbf{r})$$

- Some examples of scalar fields are $r = \sqrt{x^2 + y^2 + z^2}$, $(x^2y + y^2z + z^2x)$, $r^2 + a^2\cos^2\theta + b^2\sin^2\phi$, $\rho^2 + z + a\cos\theta$ etc.
- A vector field $\mathbf{V}$, as the name suggests is a vector function with a direction, which is in general position dependent, i.e.,

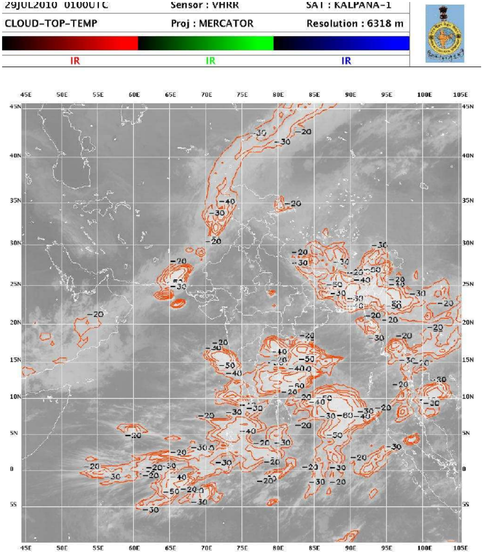
$$\mathbf{V} = \mathbf{V}(\mathbf{r})$$

- Some examples of vector fields are $\mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $x^2y\hat{i} + y^2z\hat{j} + z^2x\hat{k}$, $r^2\hat{r} + b^2\sin\phi\,\hat{\theta} + c^2\cos^2\theta\,\hat{\phi}$, $\rho^2\hat{\rho} + \rho\sin\theta\,\hat{\theta} + z^2\hat{z}$

Temperature of a rain cloud at different locations: Example of a Scalar Field

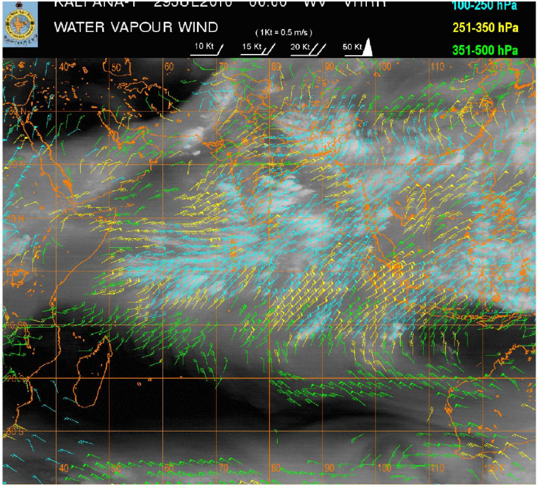
Scalar and Vector fields

Scalar field: Cloud-Top Temperature



Velocity of a cloud at different locations: Example of a Vector field

Vector field: Water Vapour Wind



Line Integrals: These are the integrals along a given path, and are of the following type

- For a scalar field $\Phi(\mathbf{r})$

$$\mathbf{I} = \int \Phi(\mathbf{r})d\mathbf{l}$$

- For a vector field $\mathbf{V}(\mathbf{r})$, following two types are possible
  - $$I = \int \mathbf{V} \cdot d\mathbf{l}$$
  - $$\mathbf{I} = \int \mathbf{V} \times d\mathbf{l}$$

Surface Integrals: These are the integrals along a given surface, and are of the following type

- For a scalar field $\Phi(\mathbf{r})$

$$\mathbf{I} = \int \Phi(\mathbf{r}) d\mathbf{S}$$

- For a vector field $\mathbf{V}(\mathbf{r})$, following two types are possible
  - $$I = \int \mathbf{V} \cdot d\mathbf{S}$$
  - $$\mathbf{I} = \int \mathbf{V} \times d\mathbf{S}$$

Volume Integrals: These are the integrals along a given volume. Because volume is a scalar quantity, they are of the following type

- For a scalar field $\Phi(\mathbf{r})$

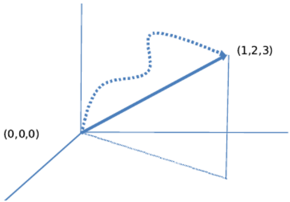
$$I = \int \Phi(\mathbf{r}) dV$$

- For a vector field $\mathbf{A}(\mathbf{r})$, following two types are possible

$$\mathbf{I} = \int \mathbf{A} dV$$

Line Integral of a Vector Field:

Example 1: A force $\mathbf{F} = zy\hat{i} + x\hat{j} + z^2x\hat{k}$ acts on a particle which travels from the origin to the point $(1,2,3)$ along a straight line as shown. Calculate the work done.



$$W = \int \mathbf{F}\cdot\mathbf{dl} = \int_0^1 zydx + \int_0^2 xdy + \int_0^3 z^2xdz$$

Equation of the straight line

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} = m$$

$$x = m; \quad y = 2m; \quad z = 3m$$

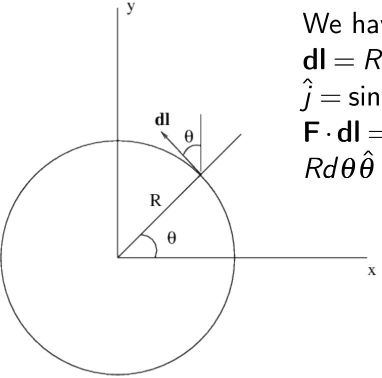
$$\implies dx = dm; \quad dy = 2dm; \quad dz = 3dm$$

Line integral contd.

So that

$$W = \int_0^1 6m^2 dm + \int_0^1 2m dm + \int_0^1 27m^3 dm = 2 + 1 + \frac{27}{4} = 9.75$$

Example 2: Calculate the line integral of $\mathbf{F} = y\hat{i} + a\hat{j}$ over a circle of radius R, in the anti-clockwise direction.



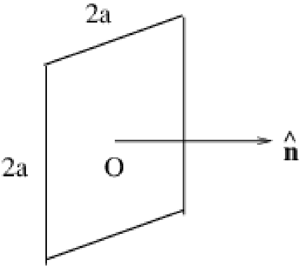
We have to evaluate $\int \mathbf{F} \cdot \mathbf{dl}$. In plane polar coordinates $\mathbf{dl} = Rd\theta\hat{\theta}$, $y = R\sin\theta$, $\hat{i} = \cos\theta\hat{\rho} - \sin\theta\hat{\theta}$, and $\hat{j} = \sin\theta\hat{\rho} + \cos\theta\hat{\theta}$. Therefore, $\mathbf{F} \cdot \mathbf{dl} = R\sin\theta(\cos\theta\hat{\rho} - \sin\theta\hat{\theta}) \cdot Rd\theta\hat{\theta} + a(\sin\theta\hat{\rho} + \cos$ $Rd\theta\hat{\theta} = -R^2\sin^2\theta d\theta + aR\cos\theta d\theta$. So that

$$\int \mathbf{F} \cdot \mathbf{dl} = -R^2 \int_0^{2\pi} \sin^2\theta d\theta + aR \int_0^{2\pi} \cos\theta d\theta$$
$$= -\pi R^2$$

Surface Integral of a Vector Field:

Consider a vector field $\mathbf{V}$. Its integral over a given surface $\int \mathbf{V}\cdot\mathbf{dS}$ is called the flux of $\mathbf{V}$, passing through the surface. Next, we will consider some examples of calculation of flux of a given vector field.

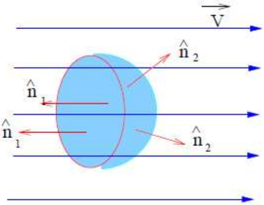
Example 1: Calculate the flux of the vector field $\mathbf{V} = z^2\hat{i} + 2x^2\hat{j} + (x^2+2y^2)\hat{k}$ passing through a square of sides $2a$ located on the xy-plane ($z=0$), and centered at the origin, as shown in the figure.



Clearly, here $\hat{n} = \hat{k}$, so that $\mathbf{dS} = dxdy\hat{k}$. Thus $\mathbf{V}\cdot\mathbf{dS} = (x^2+2y^2)dxdy$. Therefore,

$$\int \mathbf{V}\cdot\mathbf{dS} = \int_{-a}^{a}\int_{-a}^{a}(x^2+2y^2)dxdy$$
$$= (2a)(2\frac{a^3}{3}) + 2(2a)(2\frac{a^3}{3})$$
$$= 4a^3$$

Example 2: Consider a hemispherical bowl of radius R such that the circular base is located on the xy-plane centered at the origin. If a vector field $\mathbf{V} = V\hat{k}$ (V is a constant) is passing through the space as shown, calculate its flux passing through both the flat and the curved surface of the hemisphere.



For the flat surface, $\hat{n}_1 = -\hat{k}$, while for the curved surface $\hat{n}_2 = \hat{r}$. So, for the flat surface

$$\int \mathbf{V}\cdot\mathbf{dS} = -V\int dS = -V\pi R^2$$

For the curved surface $\mathbf{dS} = R^2 \sin\theta\, d\theta\, d\phi\, \hat{r}$. So that

$$\mathbf{V}\cdot\mathbf{dS} = VR^2 \sin\theta\, d\theta\, d\phi (\hat{k}\cdot\hat{r})$$

But $\hat{k}\cdot\hat{r} = \cos\theta$.

Flux calculation....

Therefore

$$\int \mathbf{V} \cdot \mathbf{dS} = VR^2 \int_{\theta=0}^{\pi/2} \cos\theta \sin\theta \, d\theta \int_{\phi=0}^{2\pi} d\phi$$
$$= V\pi R^2.$$

Thus flux through the two surfaces is equal and opposite. Is it surprising?