The method of separation of variables applied to Laplace Equation

Outline

Laplace Equation

- We know that the Laplace equation is

$$\nabla^2 V = 0, \tag{1}$$

where, in 3D Cartesian coordinates

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

with

$$V(\mathrm{r}) \equiv V(x, y, z)$$

specified on various boundaries. Boundary conditions may be of the Dirichlet type (V specfied) or of the Neumann kind ($\frac{\partial V}{\partial n}$ specified).

- Our aim is to develop a method based on the concept of the separation of variables to solve the Laplace equation, consistent with the boundary conditions.

- The method of separation of variables is based upon the conjecture (guess, tukka,...) that the solution can be written in the form

$$V(x,y,z) = X(x)Y(y)Z(z), \tag{2}$$

where $X(x)$, $Y(y)$, and $Z(z)$ are, respectively, functions of the variables, x, y, and z, only.

- "Separation of Variables" implies the product form of $V(x,y,z)$.
- Substituting Eq. 2, the Laplace equation, we get

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)X(x)Y(y)Z(z) = 0 \tag{3}$$

$\Longrightarrow$

$$Y(y)Z(z)\frac{d^2X}{dx^2} + X(x)Z(z)\frac{d^2Y}{dy^2} + X(x)Y(y)\frac{d^2Z}{dz^2} = 0 \tag{4}$$

3D Separation of Variables, contd.

- Note that partial derivatives have been replaced by total derivatives, why?
- This is because functions $X(x)$, $Y(y)$, and $Z(z)$ are functions of one variable only.
- On dividing Eq. 4 by $V(x,y,z) = X(x)Y(y)Z(z)$, we obtain

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0$$

$\Longrightarrow$

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} - \frac{1}{Z}\frac{d^2Z}{dz^2} \tag{5}$$

- Note that LHS of this equation depends only on x, while RHS on y and z. What does it mean?

3D separation of variables, contd.

- Eq. 5 can be satisfied only if both sides are equal to the same constant, say, $-l^2$
- $\Longrightarrow$

$$\frac{1}{X}\frac{d^2X}{dx^2} = -l^2$$

and

$$-\frac{1}{Y}\frac{d^2Y}{dy^2} - \frac{1}{Z}\frac{d^2Z}{dz^2} = -l^2$$

- X equation becomes

$$\frac{d^2X}{dx^2} + l^2X = 0$$

and Y and Z equation can be rewritten as

$$-\frac{1}{Y}\frac{d^2Y}{dy^2} = \frac{1}{Z}\frac{d^2Z}{dz^2} - l^2$$

3D separation of variables, contd.

- Separation of variable argument leads to

$$-\frac{1}{Y}\frac{d^2Y}{dy^2} = m^2$$

and

$$\frac{1}{Z}\frac{d^2Z}{dz^2} - l^2 = m^2,$$

where m^2 is another constant. So that

$$\frac{d^2Y}{dy^2} + m^2Y = 0$$

and

$$\frac{d^2Z}{dz^2} - (l^2 + m^2)Z = 0$$

3D separation of variables, contd.

- Defining $n^2 = -(l^2 + m^2)$ we obtain

$$\frac{d^2 Z}{dz^2} + n^2 Z = 0$$

- Finally, we get three ordinary differential equations in X, Y, and Z, in place of a partial differential equation (PDE)

$$\begin{aligned}
\frac{d^2 X}{dx^2} + l^2 X &= 0 \\
\frac{d^2 Y}{dy^2} + m^2 Y &= 0 \\
\frac{d^2 Z}{dz^2} + n^2 Z &= 0
\end{aligned}$$

where $l^2 + m^2 + n^2 = 0$.

These three ordinary differential equations (ODEs) can be solved, in conjunction with the boundary conditions.

An Example in 2D

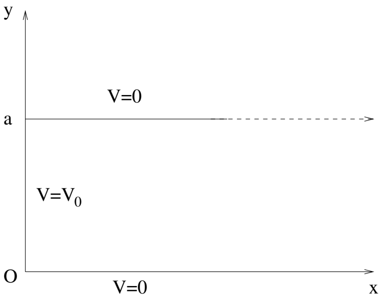

Figure: 2D Boundary conditions

- We aim to solve for $V = V(x, y)$ satisfying the 2D Laplace equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0,$$

subject to boundary conditions above.

2D Laplace eqn.

- Use separation of variables conjecture $V(x,y) = X(x)Y(y)$ in 2D Laplace equation to obtain

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = k^2 \,(\text{say}),$$

  where k^2 is a constant.

- $\Longrightarrow$

$$\frac{d^2X}{dx^2} - k^2 X = 0 \quad (6)$$

$$\frac{d^2Y}{dy^2} + k^2 Y = 0 \quad (7)$$

- Eqs. 6 and 7 subject to boundary conditions

$$\begin{aligned} X(x=0) &= V_0, \qquad X(x\to\infty)=0 \\ Y(y=0) &= \qquad Y(y=a)=0 \end{aligned}$$

- Eqs. 6 and 7 have solutions

$$\begin{aligned} X(x) &= Ae^{-kx}+Be^{kx} \\ Y(y) &= C\sin(ky)+D\cos(ky) \end{aligned}$$

- $X(x\to\infty)=0 \implies B=0$
- $Y(y=0)=0 \implies D=0$
- $Y(y=a)=0 \implies \sin(ka)=0 \implies ka=n\pi$
- $k\equiv k_n = n\pi/a$

- So most general solution satisfying the given BCs

$$V(x,y) = X(x)Y(y) = \sum_{n=1}^{\infty} A_n e^{-n\pi x/a} \sin(n\pi y/a),$$

  where A_n are constants to be determined. Use the BC $V(x=0,y) = V_0$, for $0 \leq y \leq a$.

- $\Longrightarrow$

$$\sum_{n=1}^{\infty} A_n \sin(n\pi y/a) = V_0$$

- Multilply both sides by $\sin(m\pi y/a)$ (m is an integer) and integrated for $0 \leq y \leq a$

$$\sum_{n=1}^{\infty} A_n \int_0^a \sin(m\pi y/a)\sin(n\pi y/a)dy = V_0 \int_0^a \sin(m\pi y/a)dy$$

- Using $\int_0^a \sin(m\pi y/a)\sin(n\pi y/a)dy = (a/2)\delta_{m,n}$, and $\int_0^a \sin(m\pi y/a)dy = (1-\cos(m\pi))(a/m\pi)$

- Where $\delta_{m,n}$ is called Kronecker delta and defined as

$$\begin{aligned} \delta_{m,n} &= 1, \quad \text{for } m = n \\ &= 0, \quad \text{for } m \neq n \end{aligned}$$

- Using this, we get

$$A_n = \frac{2V_0}{n\pi}(1 - \cos n\pi)$$

- So that

$$\begin{aligned} A_n &= 0 \quad \text{for even values of } n \\ A_n &= \tfrac{4V_0}{n\pi} \quad \text{for odd } n \end{aligned}$$

- Thus, the final solution is

$$V(x,y) = \frac{4V_0}{\pi} \sum_{n=1,3,5,\dots} \frac{1}{n} e^{-n\pi x/a} \sin(n\pi y/a)$$