Outline

1. Coordinate systems in 2-dimensions: Cartesian and plane polar coordinate systems and their relationship. Length and area elements
2. Coordinate systems in 3-dimensions: Cylindrical and Spherical Polar Coordinate systems, line, surface and volume elements

Objectives

1. To learn to use symmetry adapted coordinate systems
2. To understand as to how to construct line, surface, and volume elements for various coordinate systems

Using Symmetries in Physics

- Using a coordinate system which is consistent with the symmetry of the physical system simplifies calculations
- If a planar system has circular symmetry, use of plane-polar coordinate system will simplify calculations
- For systems with cylindrical symmetry, use of cylindrical polar coordinates is advised
- Likewise for spherical systems, use of spherical polar coordinate system will be beneficial

Coordinate Systems in Two Dimensions: Cartesian Coordinates

- Location of a point in a flat plane is given by coordinates (x, y).
- Differential line element $\overrightarrow{dl}$ is given by $\overrightarrow{dl} = dx\hat{i} + dy\hat{j}$
- A general vector is given by $\vec{A} = A_x\hat{i} + A_y\hat{j}$.
- Infinitesimal area element $\overrightarrow{dA}_{12}$ in a plane described by orthogonal coordinates 1 and 2 can be computed for any coordinate system as

$$\overrightarrow{dA}_{12} = \overrightarrow{dl}_1 \times \overrightarrow{dl}_2 \tag{1}$$

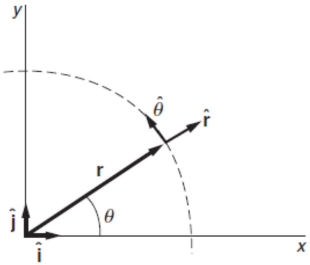
- For Cartesian coordinates it yields

$$\overrightarrow{dA} = dx\hat{i} \times dy\hat{j} = dxdy\,\hat{k}$$

$$\text{or } dA = dxdy$$

2D Coordinates: Plane Polar Coordinates

Unit vectors denoted as $\hat{r}$ and $\hat{\theta}$ are shown below



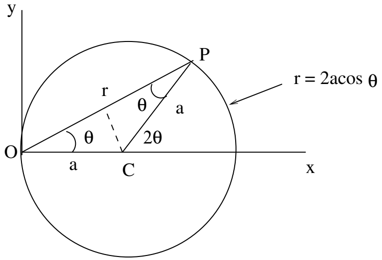
- Location of a point in a flat plane is given by coordinates (r, θ).
- Differential line element $\overrightarrow{dl}$ is given by $\overrightarrow{dl} = dr\hat{r} + rd\theta\hat{\theta}$
- Infinitesimal surface area is $\overrightarrow{dA} = dr\hat{r} \times rd\theta\hat{\theta} = rdrd\theta\hat{k}$, or $dA = rdrd\theta$

Relationship between Cartesian and Plane Polar Coordinates

- $x = r\cos\theta$, $y = r\sin\theta$
- $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, where $-\infty \leq x, y \leq \infty$; $0 \leq r \leq \infty$, $0 \leq \theta \leq 2\pi$.
- And unit vectors are related as $\hat{r} = \cos\theta\hat{i} + \sin\theta\hat{j}$, and $\hat{\theta} = -\sin\theta\hat{i} + \cos\theta\hat{j}$
- $\hat{i} = \cos\theta\hat{r} - \sin\theta\hat{\theta}$, $\hat{j} = \sin\theta\hat{r} + \cos\theta\hat{\theta}$
- Using these relations, one can easily transform vectors expressed in one coordinate system, into the other one.
- Area of a circle of radius a

$$A = \int\int dA = \int_0^a \int_0^{2\pi} r dr d\theta = \int_0^a r dr \int_0^{2\pi} d\theta = \pi a^2$$

- Consider a circle of radius a as shown in the figure below



- The origin of the (r, θ) coordinate system O is assumed to be on the circumference
- Clearly, the distance of any point P on the circle, from the origin is $r = 2a\cos\theta$
- Thus, the equation of the circle in this coordinate system will be $r = 2a\cos\theta$,

Area of a circle...

- We know that in plane polar coordinates area element is

$$dA = r dr d\theta,$$

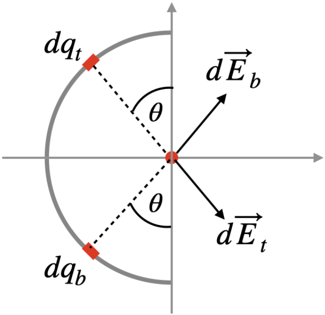
- therefore, the total area will be the double integral (pay attention to the limits)

$$\begin{aligned} A &= \int_{\theta=-\pi/2}^{\pi/2} d\theta \int_{r=0}^{2a\cos\theta} r dr \\ &= 2a^2 \int_{-\pi/2}^{\pi/2} \cos^2\theta d\theta \\ &= a^2 \int_{-\pi/2}^{\pi/2} (1+\cos 2\theta) d\theta \\ &= \pi a^2 \end{aligned}$$

- Note the same result as before

Electric Field Due to a Uniformly Charged Semicircular Wire

- Consider a uniformly charged semicircular wire shown below



- Clearly due to symmetry, y-components of the $\vec{E}$ field will cancel
- and the net field will be in the x direction

Uniformly charged Semicircular Wire

- Let the linear charge density be λ, and the radius of the semicircle be a
- If the length elements subtend angles $d\theta$ at the origin, then clearly $dq_t = dq_b = \lambda dl = \lambda a d\theta$
- Electric field at the origin due to dq_t is

$$\overrightarrow{dE}_t = \frac{dq_t}{4\pi\varepsilon_0 a^2}\left(\sin\theta\hat{i} - \cos\theta\hat{j}\right)$$
$$= \frac{\lambda d\theta}{4\pi\varepsilon_0 a}\left(\sin\theta\hat{i} - \cos\theta\hat{j}\right)$$
$$\implies \vec{E}(r=0) = \int \overrightarrow{dE}_t = \frac{\lambda}{4\pi\varepsilon_0 a}\int_0^{\pi}\left(\sin\theta\hat{i} - \cos\theta\hat{j}\right)d\theta$$
$$= \frac{\lambda}{2\pi\varepsilon_0 a}\hat{i} + 0\hat{j}$$

Kinematics in plane polar coordinates

- Computing quantities such as velocity and acceleration is a bit more complicated in plane polar coordinates
- The reason: $\hat{\mathrm{r}}$ and $\hat{\theta}$ are direction dependent
- Let us compute the velocity

$$\mathrm{v} = \frac{d\mathrm{r}}{dt} = \frac{d}{dt}\left(r\hat{\mathrm{r}}\right).$$

- Using the chain rule, we have

$$\mathrm{v} = \frac{dr}{dt}\hat{\mathrm{r}} + r\frac{d\hat{\mathrm{r}}}{dt}.$$

- Note that as the particle moves, vector $\hat{\mathrm{r}}$ also changes so that $\frac{d\hat{\mathrm{r}}}{dt} \neq 0$.
- But, how to compute $\frac{d\hat{\mathrm{r}}}{dt}$?
- A geometric calculation is possible, but let us take a different approach.

Velocity in plane polar coordinates

- Let us use Cartesian coordinates for the purpose

$$\hat{\mathrm{r}} = \cos\theta\hat{\mathrm{i}} + \sin\theta\hat{\mathrm{j}}$$

- Because Cartesian basis vectors $\hat{\mathrm{i}}$ and $\hat{\mathrm{j}}$ have fixed directions in space, so they don't change with time
- Therefore

$$\frac{d\hat{\mathrm{r}}}{dt} = \frac{d\cos\theta}{dt}\hat{\mathrm{i}} + \frac{d\sin\theta}{dt}\hat{\mathrm{j}}$$

- Now

$$\frac{d\cos\theta}{dt} = -\sin\theta\frac{d\theta}{dt} = -\sin\theta\dot{\theta}$$
$$\frac{d\sin\theta}{dt} = \cos\theta\frac{d\theta}{dt} = \cos\theta\dot{\theta}$$

- So that

$$\frac{d\hat{\mathrm{r}}}{dt} = -\sin\theta\dot{\theta}\hat{\mathrm{i}} + \cos\theta\dot{\theta}\hat{\mathrm{j}} = \dot{\theta}\left(-\sin\theta\hat{\mathrm{i}} + \cos\theta\hat{\mathrm{j}}\right) = \dot{\theta}\hat{\theta}$$

Velocity in plane polar coordinates contd.

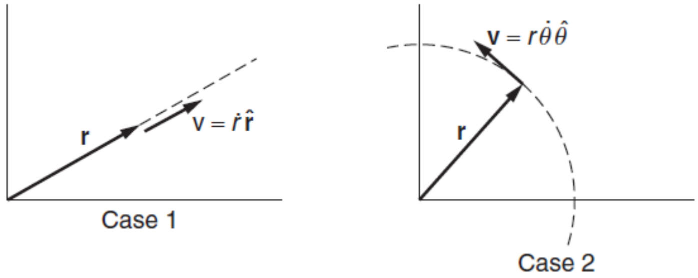
- Therefore, finally we have

$$\mathbf{v} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$$
$$= v_r\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}}$$

- Thus, we have obtained an expression for velocity in terms of its radial and angular (also called tangential) components
- What is the physical significance of v_r and v_θ?

Velocity in polar coordinates: Physical Significance

- Consider the figure



- **Case 1:** This case corresponds to motion along the radial direction, with θ held fixed ($\dot{\theta} = 0$), so that $\mathrm{v} = \dot{r}\hat{\mathrm{r}}$.
- **Case 2:** Here there is no radial motion ($\dot{r} = 0$), so velocity will be along the arc of a circle with $\mathrm{v} = r\dot{\theta}\hat{\theta}$

Acceleration in polar coordinates

- Acceleration can be computed as

$$
\begin{aligned}
\mathrm{a} &= \frac{d\mathrm{v}}{dt} \\
&= \frac{d}{dt}\left(\dot{r}\hat{\mathrm{r}} + r\dot{\theta}\hat{\theta}\right) \\
&= \frac{d\dot{r}}{dt}\hat{\mathrm{r}} + \dot{r}\frac{d\hat{\mathrm{r}}}{dt} + \frac{d(r\dot{\theta})}{dt}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt} \\
&= \ddot{r}\hat{\mathrm{r}} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt}
\end{aligned}
$$

- We compute $\frac{d\hat{\theta}}{dt}$, by expressing $\hat{\theta}$ in Cartesian coordinates

$$
\begin{aligned}
\frac{d\hat{\theta}}{dt} &= \frac{d}{dt}\left(-\sin\theta\hat{\mathrm{i}} + \cos\theta\hat{\mathrm{j}}\right) \\
&= -\cos\theta\dot{\theta}\hat{\mathrm{i}} - \sin\theta\dot{\theta}\hat{\mathrm{j}} \\
&= -\dot{\theta}\hat{\mathrm{r}}
\end{aligned}
$$

Acceleration in polar coordinates....

- On substituting the expression of $\frac{d\hat{\theta}}{dt}$, we obtain

$$\mathrm{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathrm{r}} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\theta}$$

- Different terms have the following interpretations
  - $\ddot{r}$ due to change of radial speed, points in radial direction
  - $-r\dot{\theta}^2$ centripetal acceleration, pointing radially inwards
  - $2\dot{r}\dot{\theta}$ Coriolis acceleration, present whenever both radial and angular velocities are zero, points in tangential direction
  - $r\ddot{\theta}$ tangential angular acceleration, due to changing angular velocity, points in tangential direction

Derivation of Kepler's Second Law

- Kepler's second law states that the areal velocity of each planet with respect to Sun is constant
- The force applied by Sun on planets is gravitational and of the form $\mathbf{f}(\mathbf{r}) = -\frac{C}{r^2}\hat{\mathbf{r}}$
- This force depends only on the distance r between Sun and the planet, and is along the line joining them
- Such forces are called central forces, which have the general form

$$\mathbf{f}(\mathbf{r}) = f(r)\hat{\mathbf{r}}$$

- We know that in plane polar coordinates

$$\mathbf{a} = \ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$$

Kepler's second law...

- Therefore, the equation of motion for a planet (or a particle) of mass m $m\ddot{\mathrm{r}} = f(r)\hat{\mathrm{r}}$, becomes

$$m(\ddot{r} - r\dot{\theta}^2)\hat{\mathrm{r}} + m(2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta} = f(r)\hat{\mathrm{r}}$$

- On comparing both sides, we obtain following two equations

$$m(\ddot{r} - r\dot{\theta}^2) = f(r)$$
$$m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0$$

- By multiplying second equation on both sides by r, we obtain

$$\frac{d}{dt}(mr^2\dot{\theta}) = 0$$

Equations of motion...

- This equation yields

$$mr^2\dot{\theta} = L\,(\text{constant}),$$

we called this constant L because it is nothing but the angular momentum of the particle about the origin. Note that $L = I\omega$, with $I = mr^2$.

- As the particle moves along the trajectory so that the angle θ changes by an infinitesimal amount $d\theta$, the area swept with respect to the origin is

$$\begin{aligned} dA &= \frac{1}{2}r^2 d\theta \\ \Longrightarrow \quad \frac{dA}{dt} &= \frac{1}{2}r^2\dot{\theta} = \frac{L}{2m} = \text{constant}, \end{aligned}$$

because L is constant, thus, proving Kepler's second law

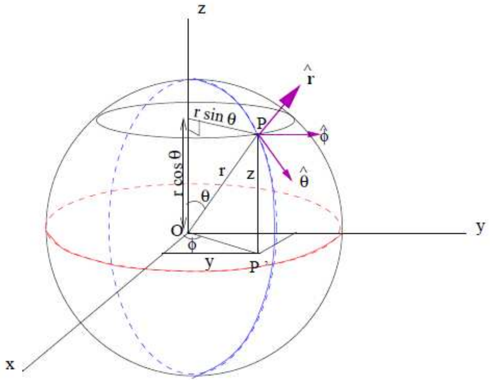
- Note, constancy of areal velocity is a property of all central forces, not just the gravitational forces.
- And it holds due to the conservation of angular momentum

Coordinate Systems in 3D: Cartesian Coordinates

- Location of a point is is given by coordinates (x, y, z).
- Differential line element $\overrightarrow{dl}$ is given by $\overrightarrow{dl} = dx\hat{i} + dy\hat{j} + dz\hat{k}$
- Infinitesimal area element depends upon the plane. For xy plane it will be
$$\overrightarrow{dA}_{xy} = dxdy\hat{k}$$
- Infinitesimal volume element for any orthogonal 3D coordinate system is given by
$$dV = dl_1 dl_2 dl_3$$
for this case $dV = dxdydz$

3D Coordinates: Spherical Polar Coordinates

- Location of a point is specified by three coordinates (r, θ, ϕ), as shown below



- What is the range of r, θ, and ϕ?

3D Coordinates...

- Clearly $0 \leq r \leq \infty$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$
- Relationship with Cartesian coordinates $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$
- Inverse relationship

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right)$$

$$\phi = \tan^{-1}\frac{y}{x}$$

- Differential line element $\overrightarrow{dl}$ is given by $\overrightarrow{dl} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta d\phi\hat{\phi}$
- Cross products given by $\hat{\theta} \times \hat{\phi} = \hat{r}$, $\hat{\phi} \times \hat{r} = \hat{\theta}$, and $\hat{r} \times \hat{\theta} = \hat{\phi}$
- Note that ϕ of this coordinate system is like θ of plane polar system

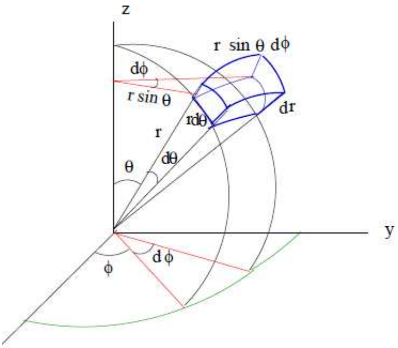
Spherical Polar Coordinates...

- Relationship between Cartesian and Spherical unit vectors

$$\hat{r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$
$$\hat{\theta} = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$$
$$\hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

- Area element on the surface of a sphere or radius R,
$\overrightarrow{dA}_{\theta\phi} = \overrightarrow{dl}_{\theta} \times \overrightarrow{dl}_{\phi} = Rd\theta\hat{\theta} \times R\sin\theta d\phi\hat{\phi} = R^2\sin\theta d\theta d\phi\hat{r}$
- Similarly, one can calculate $\overrightarrow{dA}_{r\theta}$ and $\overrightarrow{dA}_{\phi r}$
- Area of the surface of a sphere
$A = R^2\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}d\phi = 4\pi R^2$

Spherical Polar Coordinates...

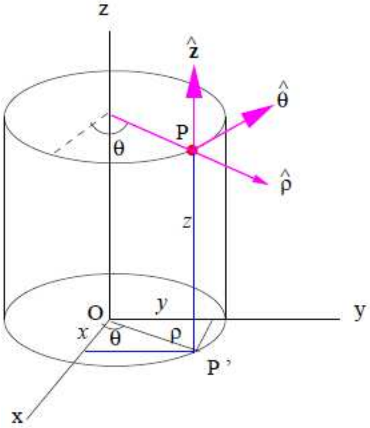


- Elementary volume element
  $dV = dl_r dl_\theta dl_\phi = dr r d\theta r \sin\theta d\phi = r^2 \sin\theta dr d\theta d\phi$
- Volume of a sphere of radius R

$$V = \int_0^R r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi = \frac{4}{3}\pi R^3$$

3D Coordinate Systems: Cylindrical Coordinates

- Location of a point specified by three coordinates (ρ, θ, z)



- Relationship with Cartesian coordinates $x = \rho\cos\theta$, $y = \rho\sin\theta$, $z = z$
- Inverse relationship $\rho = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, $z = z$
- Differential line element $\overrightarrow{dl}$ is given by $\overrightarrow{dl} = d\rho\hat{\rho} + \rho d\theta\hat{\theta} + dz\hat{k}$
- Area element in different planes can be obtained using the relation $\overrightarrow{dA_{ij}} = \overrightarrow{dl}_i \times \overrightarrow{dl}_j$
- For $\rho - \theta$ plane it will be $\overrightarrow{dA}_{\rho\theta} = \overrightarrow{dl}_\rho \times \overrightarrow{dl}_\theta = d\rho\hat{\rho} \times \rho d\theta\hat{\theta} = \rho d\rho d\theta\hat{k}$
- Volume element $dV = dl_1 dl_2 dl_3 = \rho d\rho d\theta dz$
- Volume of a cylinder of height L, and radius R $V = \int_{\rho=0}^{R} \rho d\rho \int_{z=0}^{L} dz \int_{\theta=0}^{2\pi} d\theta = \pi R^2 L$

- Let us consider a general curvilinear coordinate system in 3D
- The coordinates are defined as (u_1, u_2, u_3),which may or may not have dimensions of distance
- Corresponding unit vectors are $(\hat{u}_1, \hat{u}_2, \hat{u}_3)$
- Orthonormality condition of the unit vectors is $\hat{u}_i \cdot \hat{u}_j = \delta_{ij}$, for $i, j = 1, 2, 3$
- Symbol δ_{ij} is called Kronecker delta and it is $\delta_{ii} = 1$, and $\delta_{ij} = 0$, for $i \neq j$
- The displacement vector $\overrightarrow{dl}$ between points (u_1, u_2, u_3) and $(u_1 + du_1, u_2 + du_2, u_3 + du_3)$ is given by

$$\overrightarrow{dl} = h_1 du_1 \hat{u}_1 + h_2 du_2 \hat{u}_2 + h_3 du_3 \hat{u}_3 = \sum_{i=1}^{3} h_i du_i \hat{u}_i$$

- What are h_i's?

Curvilinear Coordinates...

- Because u_i's don't necessarily have dimensions of length, therefore, we need h_i's
- They are defined such that $h_i du_i$ have dimensions of length
- By comparing with the Cartesian, Spherical Polar, and the Cylindrical coordinates, we have

$$
\begin{array}{llll}
h_x = 1, & h_y = 1, & h_z = 1 & \text{for Cartesian} \\
h_r = 1, & h_\theta = r, & h_\phi = r\sin\theta & \text{for spherical} \\
h_\rho = 1, & h_\theta = \rho & h_z = 1 & \text{for cylindrical}
\end{array}
$$

Gradient in Curvilinear Coordinates

- Let us consider a scalar field f which is a function of three curvilinear coordinates $f(u_1, u_2, u_3) \equiv f(u_i)$
- Using multi-variable Taylor expansion we have

$$\begin{aligned} f(u_i + du_i) &= f(u_i) + \sum_{i=1}^{3} \frac{\partial f}{\partial u_i} du_i + O(h_i^2) + \cdots \\ &= f(u_i) + \sum_{i=1}^{3} \frac{1}{h_i} \frac{\partial f}{\partial u_i} h_i du_i + \cdots \\ &= f(u_i) + \nabla f \cdot \overrightarrow{dl} + \cdots \end{aligned}$$

- Leading to

$$\nabla f = \sum_{i=1}^{3} \frac{1}{h_i} \frac{\partial f}{\partial u_i} \hat{u}_i$$

Gradient...

- From which we deduce the expression for the gradient operator in a general orthogonal 3D curvilinear coordinate system

$$\nabla \equiv \sum_{i=1}^{3} \hat{u}_i \frac{1}{h_i} \frac{\partial}{\partial u_i}$$

using which, and the values of h_i, one can derive the expression for gradient in any curvilinear coordinate system