Solutions To Tutorial Sheet No - 4

Q.1 (i) The given equation is $y'' - 5y' + 6y = 2e^x$. The characteristic equation is:

$$\lambda^2 - 5\lambda + 6 = 0$$

with roots $2, 3$. Therefore, $y_1 = e^{2x}, y_2 = e^{3x}$ are two independent solutions of the homogeneous part. The Wronskian is given by

$$W = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = e^{5x}$$

Therefore a particular solution of the ODE is:

$$\begin{aligned} y_p &= -y_1 \int^x \frac{y_2 r}{W} dt + y_2 \int^x \frac{y_1 r}{W} dt \\ &= -e^{2x} \int^x 2e^{-t} dt + e^{3x} \int^x 2e^{-2t} dt \\ &= 2e^x - e^x = e^x \end{aligned}$$

(ii) $y'' + y = \tan x, \quad 0 < x < \pi/2$.
Here $y_1 = \cos x, y_2 = \sin x, W = 1$ and hence

$$\begin{aligned} y_p &= -\cos x \int^x \sin t \tan t \, dt + \sin x \int^x \cos t \tan t \, dt \\ &= -\cos x(\ln|\sec x + \tan x|) \end{aligned}$$

(iii) $y'' + 4y' + 4y = x^{-2}e^{-2x}, x > 0$.
Here the characteristic equation has a double root $\lambda = -2$. Therefore $y_1 = e^{-2x}, y_2 = xe^{-2x}$ are two independent solutions of the homogeneous part. The Wronskian $W = e^{-4x}$ and

$$\begin{aligned} y_p &= -e^{-2x} \int^x \frac{te^{-2t}t^{-2}e^{-2t}}{e^{-4t}} dt + xe^{-2x} \int^x \frac{e^{-2t}t^{-2}e^{-2t}}{e^{-4t}} dt \\ &= -e^{-2x} \ln x - e^{-2x} \end{aligned}$$

(We can ignore the term e^{-2x} and take $y_p = -e^{-2x} \ln x$.

(iv) $y'' + 4y = 3\text{cosec } 2x, 0 < x < \pi/2$.
$y_1 = \cos 2x, y_2 = \sin 2x, \ W = 2$.

$$y_p = -\frac{3}{2}\cos 2x + \frac{1}{2}\ln \sin 2x.$$

(v) $x^2y'' - 2xy' + 2y = 5x \cos x$. The homogeneous part is a Cauchy-equation whose auxiliary equation is

$$m^2 - 3m + 2 = 0$$

with roots $m = 1, 2$. Therefore, $y_1 = x, y_2 = x^2, W = x^2$.

$$\begin{aligned} y_p &= -x \int^x \frac{t^2(5t\cos t)}{t^2} dt + x^2 \int^x \frac{t(5t\cos t)}{t^2} dt \\ &= -5x \int^x t \cos t dt + 5x^2 \int^x \cos t dt \\ &= -5x \cos x \end{aligned}$$

(vi) $xy'' - y' = x^2e^x$. (2)
Rewrite this as $x^2y'' - xy = x^3e^x$ (3)
so that it is an equation of Cauchy's type. The auxiliary equation is $m^2 - 2m = 0$ and hence $y_1 = 1, y_2 = x^2$ are solutions of the homogeneous part. The Wronskian is $W = 2x$. Before applying the variation of parameters, we rewrite (3) as
$y'' - \frac{y'}{x} = xe^x$ (4)
and now

$$\begin{aligned} y_p &= -\int^x \frac{t^2te^t}{2t}dt + x^2\int^x \frac{te^t}{2t}dt \\ &= \frac{1}{2}[-(x^2 - 2x + 2)e^x + x^2e^x] \\ &= (x-1)e^x. \end{aligned}$$

Q. 2 [CLASSWORK]

$$W' = \begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix} = \begin{vmatrix} y_1 & y_2 \\ -py_1' - qy_1 & -py_2' - qy_2 \end{vmatrix} = -pW.$$

Therefore, $W = k\exp(\int^x p(t)dt)$. If $W(x_0) = 0$ then $k = 0$ and hence $W \sim 0$.

Q.3 [CLASSWORK] Given a solution $y = y_1$ of $y'' + p(x)y' + q(x)y = 0$ we try for another solution of the form

$$y_2 = uy1$$

But then, $y_2' = y_1'u + y_1u'$ and $y_2'' = y_1u'' + 2y_1'u' + y_1''u$. Therefore

$$0 = y_2'' + py_2' + q = y_1u'' + (2y_1' + py_1)u'$$

since y_1 is a solution. Let us set $v = u'$ to reduce the order of this equation:

$$y_1v' + (2y_1' + py_1)v = 0$$

Assuming that y_1 does not vanish in the given interval, we can divide out by y_1, separate the variables, and solve for v :

$$v = \frac{c_1\exp(\int^x p(t)dt)}{y_1^2}$$

Integrating once more to get the required expression: $u = c_2 + c_1\psi$ where

$$\psi(x) = \int^x \frac{\int^t \exp[-p(s)ds]}{y_1(t)^2}\,dt.$$

The point is that the steps are reversible and u obtained in this way satisfies the requirement that $y_2 = uy_1$ is a solution. This can be directly checked also.

Q.4 In each of the following problem, we use the theory, as in Q. 3. Remember to put the given equation in the standard form first. In order to get a second solution y_2 independent of y_1, we can take $c_2 = 0, c_1 = 1$ in the general formula of ψ. Then we simply put $y_2 = y_1\psi$. In what follows, we therefore obtain the expressions for ψ in each case:

(i) $y_1 = \dfrac{\sin x}{\sqrt{x}}$, $p(x) = \dfrac{1}{x}$. Therefore

$$\psi(x) = \int^x \frac{\exp(-\int^t p(u)du)}{y_1(t)^2} dt = \int^x \frac{dt}{\sin^2 t} = -\cot x$$

(ii) $y_1 = e^{x^2}$; and $p(x) = -4x$. Therefore

$$\psi(x) = \int^x \frac{e^{2t^2}}{e^{2t^2}} dt = x.$$

(iii) $y_1 = \dfrac{x}{(x-1)^2}; p = \dfrac{3}{x-1}$. Therefore,

$$\begin{array}{rcl} \psi(x) & = & \int^x \frac{\exp(-3\ln(t-1))}{t^2/(t-1)^4} dt \\ & = & \int^x \frac{(t-1)}{t^2} dt \\ & = & \ln x + \frac{1}{x} \end{array}$$

(iv) $y_1 = \cos x^2; p = \dfrac{1}{x}$. Therefore

$$\psi(x) = \int^x \frac{tdt}{\cos^2 t^2} = \frac{1}{2}\tan x^2.$$

(v) $y_1 = \sqrt{\dfrac{1-x^2}{x}}$; $p = -\dfrac{x}{1-x^2}$. Therefore,

$$\begin{array}{rcl} \psi(x) & = & \displaystyle\int^x \frac{(1-t^2)^{-1/2}t}{1-t^2} dt \\ & = & \displaystyle\int^x \frac{t}{(1-t^2)^{3/2}} dt = \frac{1}{(1-x^2)^{1/2}}. \end{array}$$

(vi) $y_1 = 1 + x^2$, $p = \dfrac{2}{x(1+3x^2)} = \dfrac{2}{x} - \dfrac{6x}{1+3x^2}$. Therefore

$$\begin{array}{rcl} \psi(x) & = & \int^x \frac{3x^2+1}{x^2(1+x^2)^2} \\ & = & \int^x \left(\frac{1-t^2}{(1+t^2)^2} + \frac{1}{t^2}\right) dt \\ & = & \frac{x}{1+x^2} - \frac{1}{x} = -\frac{1}{x(1+x^2)} \end{array}$$

Q.5 Putting $y_i = e^{r_i x}$, we have,$y^j = r_i^j y_i$ for all i, j. Therefore $W(y_1, \ldots, y_n) = \exp((\sum_i r_i)x)V$ where V is the Vandermond determinant

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_n \\ r_1^2 & r_2^2 & \cdots & r_n^2 \\ \cdots & \cdots & \cdots & \cdots \\ r_1^{n-1} & r_2^{n-1} & \cdots & r_n^{n-1} \end{vmatrix}$$

which is equal to $\prod_{i>j}(r_i - r_j)$. Since r_i are distinct, it follows that W never vanishes. Therefore y_i which are solutions of an n^{th} order linear differential equation are independent.

Aliter Suppose there exist real numbers α_i such that $\sum_i \alpha_i y_i \cong 0$. By taking derivatives up to $(n-1)$-th order, this yields

$$\widehat{W}(\alpha_1, \dots, \alpha_n)^t = (0, 0, \dots, 0)^t$$

where $\widehat{W}$ is the Wronskian matrix (not the determinant). Since the Wronskian itself is not zero, this matrix is invertible. Therefore

$$(\alpha_1, \dots, \alpha_n)^t = (0, 0 \dots, 0)^t.$$

Q.6 [CLASSWORK] We set $y = \sum_i c_i y_i$ where c_i are some smooth functions whose derivatives satisfy

$$\sum_i c_i' y_i^{(j)} = 0 \text{ for all } j = 0, 1, \dots, n-2. \tag{1}$$

Under this condition, we check that y is a solution of the given equation iff

$$\sum_i c_i' y_i^{(n-1)} = r(x) \tag{2}$$

If $\widehat{W}$ is the Wronskian matrix, then condition (1) and (2) are together equivalent to say

$$\widehat{W}(c_1', \dots, c_n')^t = (0, \dots, 0, r(x))^t$$

By Cramer's rule this is the same as saying that

$$c_i' = \frac{D_i}{W}$$

Upon integration, this is the same as

$$c_i(x) = \int^x \frac{D_i(s)}{W(s)} ds.$$

Q.7 We first observe that if y_1, y_2 are two solutions of a non homogeneous linear equation, then $y_1 - y_2$ is a solution of the homogeneous part. Applying this to the present situation we get $y_1 - y_2, y_2 - y_3$ are solutions of the homogeneous part. We then check that these two solutions are independent (by computing the Wronskian, for instance or otherwise). Therefore, the general solution of the given equation may be written as

$$c_1(y_1 - y_2) + c_2(y_2 - y_3) + y_1.$$

Q.8 Similar to Q.4, we first write down the expression for ψ in each case. The second solution of the homogeneous equation (independent of the first one) is given by $y_2 = y_1\psi$. We then apply Q. 6 to obtain a particular solution Y_p of the inhomogeneous equation and then the general solution of the given equation can be written; $y = c_1 y_1 + c_2 y_2 + y_p$.

(i) $y_1 = x;\ p = \dfrac{2x}{1+x^2};\ r = x.$

$$\psi(x) = \int^x \frac{1+t^2}{t^2} dt = x - \frac{1}{x}.$$

$$y_2 = x^2 - 1, \quad W = x^2 + 1.$$

$$\begin{array}{rcl} y_p & = & -y_1 \int^x \frac{y_2 r}{W} dt + y_2 \int^x \frac{y_1 r}{W} dt \\ & = & -x \int^x \frac{(t^2-1)t}{t^2+1}\, dt + (x^2-1) \int^x \frac{t^2}{t^2+1}\, dt \\ & = & x \ln(x^2+1) - (x^2-1)\tan^{-1} x + \frac{1}{2} x(x^2-2). \end{array}$$

(ii) $y_1 = e^x;\quad p = -\dfrac{1}{x}; r = x.$

$$\psi(x) = \int^x \frac{t}{e^{2t}} dt = -\frac{e^{-2x}(2x+1)}{4}.$$

$$y_2 = e^{-x}(2x+1); \quad W = 4x.$$

$$\begin{array}{rcl} y_p & = & -\frac{1}{4}\left(-e^x \int^x (2t+1)e^{-t}\, dt + (2x+1)e^{-x} \int^x e^t\, dt\right) \\ & = & -(x+1). \end{array}$$

(iii) $y_1 = e^{2x};\ p = -4\dfrac{x+1}{2x+1};\ r = \dfrac{e^{2x}}{2x+1}.$

$$\psi(x) = \int^x \frac{e^{2t}(2t+1)}{e^{4t}}\, dt = -e^{-2x}(x+1).$$

Therefore $y_2 = x + 1;\quad W = -e^{2x}(2x+1).$

$$\begin{array}{rcl} y_p & = & e^{2x} \int^x \frac{e^{2t}(t+1)}{(2t+1)e^{2t}(2t+1)}\, dt - (x+1)\int^x \frac{e^{2t}e^{2t}}{(2t+1)(2t+1)e^{2t}}\, dt \\ & = & \frac{1}{4}\ln(2x+1) - \frac{1}{4(2x+1)} - (x+1)\int^x \frac{e^{2t}}{(2t+1)^2}\, dt \end{array}$$

(iv) $y_1 = x^2;\quad p = -\dfrac{x^2+2x-2}{x^2-x} = -\left(1 + \dfrac{2}{x} + \dfrac{1}{x-1}\right);\quad r(x) = \dfrac{(x-1)e^x}{x}.$

$$\psi(x) = \int^x \frac{e^t t^2(t-1)}{t^4}\, dt = \frac{e^x}{x}.$$

Therefore $y_2 = xe^x;\ W = x^2(x-1)e^x$; and

$$\begin{array}{rcl} y_p & = & -x^2 \int^x \frac{te^t(t-1)e^t}{t^2(t-1)e^t}\, dt + xe^x \int^x \frac{t^2(t-1)e^t}{t^2(t-1)e^t}\, dt \\ & = & x^2 e^x - x^2 \int^x \frac{e^t}{t^2}\, dt \end{array}.$$

Q.9 In each case, substitute $y = ux$ in the given ODE and observe that the u-term vanishes. Now put $u' = v$ to get an ODE of lower order:

$$y = ux;\ y' = u'x + u;\ y'' = u''x + 2u';\ y''' = u'''x + 3u''.$$

(i) $x^4 v'' + (3x^3 - 3x^2)v' - x^4 v = 0.$

(ii) $xv'' + (x^3 + x + 3)v' + (2 + 2x^2 - 2x^3)v = 0.$

Q.10 We shall solve this exercise using M.U.C.

(i) $(D^2+1)^2 y = \sin x$. The auxiliary equation is $(\lambda^2+1)^2 = 0$ with roots $\pm i$ repeated twice. Therefore the complementary function (i.e., the general solution of the homogeneous part) is:

$$y_c = c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x.$$

Now we find a linear operator L such that $L(\sin x) = 0$. The simplest one is $L = D^2 + 1$. It follows that if y_p is a solution of (i), then

$$(D^2+1)^3(y_p) = 0.$$

The fundamental system

$$\{\cos x, \sin x, x \cos x, x \sin x\}$$

for the operator $(D^2+1)^2$ is extended to a fundamental system

$$\{\cos x, \sin x, x \cos x, x \sin x, x^2 \cos x, x^2 \sin x\}$$

for $(D^2+1)^3$. Therefore, y_p can be chosen to be of the form

$$y_p = Ax^2 \cos x + Bx^2 \sin x.$$

We now compute $(D^2+1)^2(y_p)$ and determine the values of A, B.

$$\begin{array}{rcl} D(x \sin x) & = & x \cos x + \sin x. \\ D(x \cos x) & = & -x \sin x + \cos x \\ (D^2+1)(x \sin x) & = & 2 \cos x \\ (D^2+1)(x \cos x) & = & -2 \sin x \\ (D^2+1)(x^2 \sin x) & = & 4x \cos x + 2 \sin x \\ (D^2+1)^2(x^2 \sin x) & = & -8 \sin x. \\ (D^2+1)(x^2 \cos x) & = & 2 \cos x - 4x \sin x \\ (D^2+1)^2(x^2 \cos x) & = & -8 \cos x \end{array}$$

Therefore $\sin x = -8(A \cos x + B \sin x)$ from which we get on comparing coefficients $A = 0, B = -\dfrac{1}{8}$ and hence $y_p = -\dfrac{1}{8}x^2 \sin x$.

[**Alternate Method**(for computing A, B):

By operator method, a particular solution is found by

$$y_p = \frac{1}{F(D)} r(x) = \frac{1}{(D^2+1)^2} \sin x$$

Since $\imath$ is a repeated root of the auxiliary equation, we directly try with $y_p = cx^2 \sin x$; (the computation above justifies this) and see that $c = -1/8$.]

(ii) Here we have $(D-1)^3(D+2)y = xe^x + 3e^{-2x}$. So, the complementary function is:

$$y_c = c_1e^x + c_2xe^x + c_3x^2e^x + c_4e^{-2x}.$$

To compute y_p we use the following formula:

$$\begin{array}{lcl} (D-a)^j(x^ke^{ax}) & = & k(k-1)\cdots(k-j+1)x^{k-j}e^{ax}\ ; k \geq j \geq 1 \\ (D-a)^j(x^ke^{ax}) & = & 0;\ 0 \leq k < j. \\ (D-a)^j(e^{bx}) & = & (b-a)^je^{bx};\ a \neq b. \\ (D-a)x^ke^{bx} & = & [kx^{k-1} + (b-a)]e^{bx}\ ; a \neq b, k \geq 1. \end{array}$$

So, we try $y_p = \alpha x^4e^x + \beta e^x - \gamma xe^{-2x}$. Direct computations yield: $\alpha = 1/72, \beta = -1/9$ and $\gamma = -1/9$.

Q.11 In the homogeneous part, we put $y = x^\lambda$, get the auxiliary equation and solve it for λ to obtain the complementary solutions. We then try $y_p = cx^3$ and determine c.
(i) $F(D)y := (x^2D^2 + 2xD + 1)y = x^3$.
The auxiliary equation is
$m^2 + m + 1 = 0$ with $\omega, \bar{\omega}$ as roots.
Since $D^4x^3 = 0$, the MUC suggests that $y_p = ax^3 + bx^2 + cx + d$. However, the homogeneous nature of the Cauchy operator $F(D)$ shows that $F(D)(x^k) = \alpha_kx^k$ for all k. Hence we may directly take $y_p = ax^3$. Now $F(D)(ax^3) = 13x^3$. Therefore $y_p = x^3/13$.

(ii) Here we have

$$F(D) = x^4D^4 + 8x^3D^3 + 16x^2D^2 + 8xD + 1.$$

The auxiliary equation is:

$$(m^2 + m + 1)^2 = 0$$

with $\omega, \bar{\omega}$ as repeated roots. So, $y_h = k_1e^{\omega x} + k_2e^{\bar{\omega}x} + k_3xe^{\omega x} + k_42e^{\bar{\omega}x}$. As in the previous case, we try $y_p = cx^3$. Since $F(D)(cx^3) = 121x^3$ we take $y_p = x^3/121$.

Q.12 (i) If $y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x)$, where $y_1(x)$ and $y_2(x)$ are solutions of the corresponding homogeneous equation then we have $y_1 = x^3$, $y_2 = x^{-2}$, $c_1 = \frac{-1}{5}(x^{-1} + 1)\ln x$ and $c_2 = \frac{-1}{5}\left(\frac{x^4}{4}\ln x - \frac{x^3}{3}\right)$.
(ii) With notation as in (i), $y_1 = x^{-3}$, $y_2 = x^2$, $c_1 = 12x^5$, $c_2 = -x^2$.

Q.13 Routine.

Q.14 The Wronskian W of $y_1 = x^2e^x$ and $y_2 = x^3e^x$ is $W = x^4e^x$. From (4.2) we know that $W = e^{-\int p dx}$, whence it follows that $p(x) = -2 - 4/x$. Substituting y_1 in the homogeneous equation given in (4.2) we see that $q(x) = 1 + 6/x^2$.
Any constant coefficient homogeneous differential equation with x^2e^x as a solution must also have e^x and xe^x as solutions. Thus its general solution cannot have the form $c_1x^2e^x + c_2x^3e^x$.

Solutions To Tut-Sheet 5

Q.1 (i) and (ii) A direct integration by parts would be tedious. Instead, differentiate with respect to ω and invert the order of $\mathcal{L}$ and $\dfrac{d}{d\omega}$ in

$$\mathcal{L}(\cos \omega t) = \frac{s}{s^2+\omega^2}, \quad \mathcal{L}(\sin \omega t) = \frac{\omega}{s^2+\omega^2}.$$

to get the formulae

$$\mathcal{L}(t\cos \omega t) = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}, \quad \mathcal{L}(t\sin \omega t) = \frac{2s\omega}{(s^2+\omega^2)^2}.$$

(iii) Work in the half plane $s > 0$.

$$F(s) = \mathcal{L}(e^{-t}\sin^2 t) = \int_0^\infty e^{-t(s+1)}\sin^2 t dt$$

Then we get

$$2F(s-1) = \int_0^\infty e^{-ts}(1-\cos 2t)dt = \frac{1}{s} - \mathcal{L}(\cos 2t) = \frac{1}{s} - \frac{s}{s^2+4}$$

From which follows

$$\mathcal{L}(e^{-t}\sin^2 t) = \frac{1}{2(s+1)} - \frac{s+1}{2(s^2+2s+5)}$$

(iv) Assume a to be real[1]. Then putting $t(s+a) = u$ in the defining integral,

$$\mathcal{L}(t^2e^{-at}) = \int_0^\infty t^2e^{-t(s+a)}dt = \frac{1}{(s+a)^3}\int_0^\infty u^2e^{-u}du = \frac{\Gamma(3)}{(s+a)^3} = \frac{2!}{(s+a)^3}$$

More generally for complex a and b the following holds:

$$\mathcal{L}(t^ae^{-bt}) = \frac{\Gamma(a+1)}{(s+b)^{a+1}}, \qquad (*)$$

(v)-(viii) Use the result $\mathcal{L}(t^ae^{-bt}) = \dfrac{\Gamma(a+1)}{(s+b)^{a+1}}$, by writing the trigonometric and hyperbolic functions in terms of exponentials.

(ix) $\mathcal{L}(t^2e^{-at}\sin bt) = \displaystyle\int_0^\infty t^2 \sin bte^{-(a+s)t}dtF(a+s)$ where $F(s)$ is the Laplace transform of $t^2\sin bt$. But then (see note below) $F(s) = \dfrac{2b(3s^2-b^2)}{(s^2+b^2)^2}$ so that

$$\mathcal{L}(t^2e^{-at}\sin bt) = \frac{2b(3(s+a)^2-b^2)}{(s+a)^2+b^2}$$

[1]The result is true for complex a but the proof given does not go through. One must use Cauchy's integral formula to deform the path of the integral into the complex domain. The result however may be used even for complex a

Note: To obtain $F(s)$ differentiate twice with respect to the parameter b in the formula for $\mathcal{L}(\sin bt)$.

(xi) Starting with the defining formula for hyperbolic cosine,

$$\mathcal{L}(\cosh at \cos at) = \frac{1}{2}\Big(\mathcal{L}(e^{at}\cos at) + \mathcal{L}(e^{at}\sin at)\Big)$$

If $F(s) = \mathcal{L}(\cos at)$ then $F(s-a) = \mathcal{L}(e^{at}\cos at)$, so that $\mathcal{L}(e^{at}\cos at) = \dfrac{s-a}{(s-a)^2+a^2}$. Similarly compute $\mathcal{L}(e^{at}\sin at)$

Q.2 (i) Partial Fraction decomposition gives

$$\frac{s^2-\omega^2}{(s^2+\omega^2)^2} = \frac{A_1}{s+i\omega} + \frac{A_2}{(s+i\omega)^2} + \frac{B_1}{S-i\omega} + \frac{B_2}{(s-i\omega)^2}$$

A calculation shows that $A_1 = B_1 = 0$ and $A_2 = B_2 = 1/2$. Hence

$$\mathcal{L}^{-1}\Big(\frac{s^2-\omega^2}{(s^2+\omega^2)^2}\Big) = \frac{1}{2}\Big[\mathcal{L}^{-1}\Big(\frac{1}{(s+i\omega)^2}\Big) + \mathcal{L}^{-1}\Big(\frac{1}{(s-i\omega)^2}\Big)\Big] = \frac{1}{2}(xe^{-i\omega x} + xe^{i\omega x}) = x\cos\omega x$$

(iii) The same procedure gives

$$\frac{1}{(s^2+\omega^2)^2} = \frac{A_1}{s+i\omega} + \frac{A_2}{(s+i\omega)^2} + \frac{B_1}{S-i\omega} + \frac{B_2}{(s-i\omega)^2}$$

with $B_2 = A_2 = \dfrac{-1}{4\omega}$ and $-B_1 = A_1 = \dfrac{i}{4\omega^3}$ whereby

$$\mathcal{L}^{-1}\Big(\frac{1}{(s^2+\omega^2)^2}\Big) = \frac{i}{4\omega^3}(e^{-i\omega x} - e^{i\omega x}) - \frac{x}{4\omega^2}(2\cos\omega x) = \frac{\sin\omega x}{2\omega^3} - \frac{x\cos\omega x}{2\omega^2}$$

(iv) Proceeding as before,

$$\frac{s^3}{s^4+4a^4} = \frac{1}{4}\Big[\frac{1}{s+(a-ia)} + \frac{1}{s+(a+ia)} + \frac{1}{s-(a-ia)} + \frac{1}{s-(a+ia)}\Big]$$

whereby

$$\mathcal{L}^{-1}\Big(\frac{s^3}{s^4+4a^4}\Big) = \frac{1}{4}\Big[e^{-ax+iax} + e^{-ax-iax} + e^{ax-iax} + e^{ax+iax}\Big] = \frac{\cos ax}{2}(e^{-ax}+e^{ax}) = \cos ax\cosh ax$$

(v) In this case the decomposition is

$$\frac{s-2}{s^2(s+4)^2} = \frac{1}{8}\Big[\frac{1}{s} - \frac{1}{s^2} - \frac{1}{s+4} - \frac{3}{(s+4)^2}\Big]$$

and the inverse Laplace transform is given by

$$\mathcal{L}^{-1}\Big(\frac{s-2}{s^2(s+4)^2}\Big) = \frac{1}{8}\Big[1 - x - e^{-4x} - 3xe^{-4x}\Big]$$

(vi) In this case the decomposition is

$$\frac{1}{s^4-2s^3} = \frac{-1}{8s} - \frac{1}{s^2} - \frac{1}{4s^2} - \frac{1}{2s^3} + \frac{1}{8(s-2)}$$

and the inverse Laplace transform is

$$\mathcal{L}^{-1}\Big(\frac{1}{s^4-2s^3}\Big) = \frac{-1}{8} - \frac{x}{4} - \frac{x^2}{4} + \frac{e^{2x}}{8}$$

(vii) Ans: $\dfrac{x^3}{6\pi^2} - \dfrac{x}{\pi^2} + \dfrac{\sin \pi x}{\pi^5}$

(viii) The partial fraction decomposition is

$$\frac{s^2+a^2}{(s^2-a^2)^2} = \frac{1}{2}\Big[\frac{1}{(s-a)^2} + \frac{1}{(s+a)^2}\Big]$$

so that

$$\mathcal{L}^{-1}\Big(\frac{s^2+a^2}{(s^2-a^2)^2}\Big) = \frac{x}{2}(e^{ax}+e^{-ax}) = x\cosh ax.$$

(ix) The partial fraction decomposition is

$$\frac{s^3+3s^2-s-3}{(s^2+2s+5)^2} = \frac{1}{2}\Big[\frac{1}{s+1-2i} + \frac{1}{s+1+2i} + \frac{i}{(s+1-2i)^2} - \frac{i}{(s+1+2i)^2}\Big]$$

and so

$$\begin{aligned}\mathcal{L}^{-1}\Big(\frac{s^3+3s^2-s-3}{(s^2+2s+5)^2}\Big) &= \frac{1}{2}\Big[e^{(-1+2i)x} + e^{(-1-2i)x} + 2ixe^{(-1+2i)x} - 2ixe^{(-1-2i)x}\Big] \\ &= e^x(\cos 2x - 2x\sin 2x)\end{aligned}$$

(x) Ans: $e^x(1-x-xe^{-x})$

Comments: The Laplace transform method is superior to other methods inasmuch as the initial conditions are incorporated at the outset and the problem of determining the constants of integration is absent. This is particularly advantageous in the case of systems of differential equations with constant coefficients. Moreover the convolution theorem gives an integral representation of the solution.

Q.3 (i) Laplace tranforming the DE and using the initial conditions we get

$$(s^2+1)\mathcal{L}y = \frac{3}{s^2+9}$$

and so

$$\mathcal{L}y = \frac{-3i}{16}\Big[\frac{1}{s-i} - \frac{1}{s+i}\Big] + \frac{i}{16}\Big[\frac{1}{s-3i} - \frac{1}{s+3i}\Big]$$

On taking the inverse Laplace transforms,

$$y = \frac{3}{8}\sin x - \frac{1}{8}\sin 3x.$$

The procedure is similar for the remaining bits. We merely record the answers.

(ii) Ans: $y = te^{-t} - e^{-t} + e^{-2t}$

(iii) Ans: $y = 2e^{2t} - e^{-4t}$

(iv) Ans: $y(x) = 3e^{-x} + 3xe^{-x} + \frac{1}{2}\sin x - \frac{1}{2}xe^{-x}$

(v) Ans: $y(x) = e^{x}\cos 2x + 2\sin x$

(vi) $y(x) = -\frac{1}{2}e^{3x} - \frac{1}{6}e^{-x} - \frac{5}{3}e^{2x} + e^{-2x}$

Q.4 The procedure is routine. Question 17 is similar and is to be done by the student entirely by himself. Lower case letters refer to the unkowns and the corresponding upper case letter refers to the Laplca transform. This notation is standard and used extensively in books. Thus $Y(s)$ is the Laplace transform of $y(t)$. First Laplace transform the given system. Solve the resulting algebraic system of linear equations and proceed to compute the inverse Laplace transform. The initial conditions are incorporated in the solution procedure.

(i) $sX - x(0) = X + Y$, $sY - y(0) = 4X + Y$. WE take $x(0) = a$ and $y(0) = b$ and solve for X and Y:

$$X = \frac{a(s-1)+b}{(s-3)(s+1)}, \qquad Y = \frac{4a+(s-1)b}{(s-3)(s+1)}$$

A partial fraction decomposition gives

$$X = \Big(\frac{2a+b}{4}\Big)\frac{1}{s-3} + \Big(\frac{2a-b}{4}\Big)\frac{1}{s+1}$$

Taking the inverse Laplace transform gives

$$x(t) = \Big(\frac{2a+b}{4}\Big)e^{3t} + \Big(\frac{2a-b}{4}\Big)e^{-t}$$

A simple computation shows

$$y(t) = \Big(\frac{2a+b}{2}\Big)e^{3t} + \Big(\frac{b-2a}{2}\Big)e^{-t}$$

(ii), (iii) and (iv) are similar.

(v) $sY_1 + Y_2 - y_1(0) = \dfrac{2s}{s^2+1}$ and $Y_1 + sY_2 - y_2(0) = 0$. Putting in the given initial conditions and solving $Y_2 = \dfrac{s}{s^2+1}$ whence $y_2(t) = \cos t$ and $y_1 = -y_2' = \sin t$.

(vi) Laplace transforming the equations we get the pair:

$$\begin{aligned} s^2Y_1 + Y_2 - sy_1(0) - y_1'(0) &= \frac{-5s}{s^2+4}, \quad s^2Y_1 + Y_2 = \frac{s^3+s^2-s+4}{s^2+4} \\ s^2Y_2 + Y_1 - sy_2(0) - y_2' &= \frac{5s}{s^2+4}, \quad Y_1 + s^2Y_2 = \frac{-s^3+s^2+s+4}{s^2+4} \end{aligned}$$

Either solve directly for Y_1 and Y_2 or else in this case it is easier to add and subtract to get

$$\begin{aligned} Y_1 + Y_2 &= \frac{2}{s^2+1}, \quad y_1(t) + y_2(t) = 2\sin t \\ Y_1 - Y_2 &= \frac{2s}{s^2+4}, \quad y_1(t) - y_2(t) = 2\cos 2t \end{aligned}$$

and we get from these the solutions

$$y_1(t) = \sin t + \cos 2t, \quad y_2(t) = \sin t - \cos 2t.$$

(vii) Laplace Transforming the system of ODEs we get

$$\begin{aligned} 2Y_1 - Y - 2 - Y_3 = 0, \\ Y_1 + Y_2 = \frac{4}{s^3} + \frac{2}{s^2}, \\ sY_1 + Y_3 = \frac{2}{s^3} + \frac{2}{s} \end{aligned}$$

On solving,

$$Y_1 = \frac{1}{s+3}\Big[\frac{6}{s^3} + \frac{2}{s^2} + \frac{2}{s}\Big], \quad y_1(t) = \frac{2}{3} + t^2 - \frac{2}{3}e^{-3t}$$

Now $y_2' = 4t + 2 - y_1' = 2t + 2 - 2e^{-3t}$ and finally $y_3(t) = -y_2' + t^2 + 2$ from which $y_3(t)$ may be computed.

(viii) Ans: $y_1(t) = e^{2t} + e^t, \; y_2(t) = e^{2t}$.

Q.5 (i) $\mathcal{L}^{-1}\Big(\frac{1}{s-1}\Big) = \mathcal{L}^{-1}\Big(\frac{1}{s}\Big(\frac{1}{1-\frac{1}{s}}\Big)\Big) = \mathcal{L}^{-1}\Big(\frac{1}{s}\sum_{j=0}^{\infty}\frac{1}{s^j}\Big) = \mathcal{L}^{-1}\Big(\sum_{j=0}^{\infty}\frac{1}{s^{j+1}}\Big) = \sum_{j=0}^{\infty}\frac{t^j}{j!} = e^t$

(ii) $\mathcal{L}^{-1}\Big(\frac{1}{s^2+1}\Big) = \mathcal{L}^{-1}\Big(\sum_{j=0}^{\infty}\frac{(-1)^j}{s^{2j+2}}\Big) = \sum_{j=0}^{\infty}\frac{(-1)^j t^{2j+1}}{(2j+1)!} = \sin t$

(iii) $\mathcal{L}^{-1}\Big(\frac{1}{s}e^{-b/s}\Big) = \mathcal{L}^{-1}\Big(\sum_{j=0}^{\infty}\frac{(-1)^j b^j}{j!s^{j+1}}\Big) = \sum_{j=0}^{\infty}\frac{(-1)^j b^j t^j}{(j!)^2} = J_0(2\sqrt{bt})$

(iv) We use the binomial theorem to expand $\sqrt{s^2+a^2}$ into an infinite series.

$$\begin{aligned}
\mathcal{L}^{-1}\Big(\frac{1}{\sqrt{s^2+a^2}}\Big) &= \mathcal{L}^{-1}\Big(\frac{1}{s}\sum_{j=0}^{\infty}\frac{(-1)^j(1\cdot 3\cdot 5\cdots(2j-1))a^{2j}}{2^j j! s^{2j}}\Big)\\
&= \mathcal{L}^{-1}\Big(\frac{1}{s}\sum_{j=0}^{\infty}\frac{(-1)^j(2j)!a^{2j}}{4^j(j!)^2 s^{2j}}\Big)\\
&= \sum_{j=0}^{\infty}\frac{(-1)^j(ax)^{2j}}{4^j j!^2} = J_0(at)
\end{aligned}$$

(v) $\mathcal{L}^{-1}\Big(\frac{e^{-b/s}}{\sqrt{s}}\Big) = \sum_{j=0}^{\infty}\frac{(-1)^j b^j}{j!}\mathcal{L}^{-1}\Big(\frac{1}{s^{j+\frac{1}{2}}}\Big)$. Now using the formula $\mathcal{L}(x^p) = \Gamma(p+1)/s^{p+1}$ we get

$$\mathcal{L}^{-1}\Big(\frac{1}{s^{j+\frac{1}{2}}}\Big) = \frac{x^{j-\frac{1}{2}}}{\Gamma(j+\frac{1}{2})} = \frac{2^j x^{j-\frac{1}{2}}}{(2j-1)(2j-3)\dots 3\cdot 1\sqrt{\pi}} = \frac{4^j j! x^{j-\frac{1}{2}}}{(2j)!\sqrt{\pi}}$$

Hence

$$\mathcal{L}^{-1}\Big(\frac{e^{-b/s}}{\sqrt{s}}\Big) = \frac{1}{\sqrt{\pi x}}\cos(2\sqrt{bx})$$

(vi) Recalling from the chapter on power series, the series development for the inverse tangent function, we get

$$\mathcal{L}^{-1}\Big(\tan^{-1}(1/s)\Big) = \mathcal{L}^{-1}\Big(\sum_{j=0}^{\infty}\frac{(-1)^j}{2j+1}\frac{1}{s^{2j+1}}\Big) = \sum_{j=0}^{\infty}\frac{(-1)^j x^{2j}}{(2j+1)!} = \frac{\sin x}{x}$$

Q.6 (i) Either an L-type or an I-type problem. The purpose is to derive the formula

$$\mathcal{L}(f) = \frac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)dt \tag{Π}$$

for a periodic function of period p. The importance of this is not so much the computation of Laplace transforms but to note that, in conjunction with Fourier series, it provides the classical partial fraction expansions (Mittag-Leffler formulae) for the functions such as $\coth x$. For instance, compute the Fourier series for the square wave defined by $f(x) = \mathrm{sgn}(x)$ on $[-\pi,\pi]$ and Laplace transform the resulting expansion. The following Mittag-Leffler identity pops out (!):

$$\frac{1}{s}\tanh\frac{\pi s}{2} = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2+s^2}$$

Now replace s by $2is$ and we get (!!)

$$\pi\tan\pi s = \sum_{n=0}^{\infty}\frac{8s}{(2n+1)^2-4s^2}$$

Term by term integration in turn yields the factorization (!!!)

$$\cos\pi s = \prod_{n=0}^{\infty}\Big(1-\frac{4s^2}{(2n+1)^2}\Big)$$

We stop here to keep the iterated factorials down to three.

(ii) Period for $|\sin \omega t|$ equals $p = \pi/\omega$.

$$\int_0^p f(t)e^{-st}dt = \int_0^{\pi/\omega} e^{-st}\sin\omega t dt = \frac{\omega}{\omega^2+s^2}(1+e^{-\pi s/\omega})$$

Now substitute in the formula (Π).

(iii) This is the example of the square wave discussed in (i) above. Period is 2π and

$$\mathcal{L}(f) = \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} f(t)e^{-st}dt = \frac{1}{s}\frac{(1-e^{-\pi s})^2}{1-e^{-2\pi s}} = \frac{1}{s}\Big(\frac{1-e^{-\pi s}}{1+e^{-\pi s}}\Big) = \frac{1}{s}\tanh\frac{\pi s}{2}.$$

(iv) Period $p = 2$ and

$$\mathcal{L}(f) = \frac{1}{1-e^{-2s}}\int_0^2 f(t)e^{-st}dt = \frac{1}{1-e^{-2s}}\frac{(1-e^{-s})^2}{s^2} = \frac{1}{s^2}\coth\frac{s}{2}$$

(v) Period $p = 2\pi$.

$$\mathcal{L}(f) = \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} f(t)e^{-st}dt = \frac{1}{s^2+1}\frac{1+e^{-\pi s}}{1-e^{-2\pi s}} = \frac{1}{(s^2+1)(1-e^{-\pi s})}$$

Q.7 The defining integral has to be broken up into pieces:

$$\mathcal{L}(f) = \sum_{n=1}^{\infty}\int_{n-1}^{n} ne^{-st}dt = \frac{e^s-1}{s}\sum_{n=1}^{\infty} ne^{-ns} = \frac{e^s(e^s-1)}{s(e^s-1)^2} = \frac{e^s}{s(e^s-1)}$$

Q.8 Define $\mathbf{u}(t-a) = 0$ if $t \le a$ and $\mathbf{u}(t-a) = 1$ if $t > a$. Then $\mathcal{L}(\mathbf{u}(t-a)) = e^{-as}/s$. Assume $a > 0$ and let us solve

$$\frac{dv}{dt} = \mathbf{u}(t-a), \quad v(0) = 0.$$

The solution is given by $\mathbf{v}_a(t) = 0$ if $t < a$ and $\mathbf{v}_a(t) = t-a$ if $t \ge a$. Now

$$\begin{aligned}\mathcal{L}\Big(\frac{d\mathbf{v}_a}{dt}\Big) &= \mathcal{L}(\mathbf{u}(t-a)) = e^{-as}/s.\\ s\mathcal{L}(\mathbf{v}_a) &= e^{-as}/s\end{aligned}$$

which gives us the requisite formula $\mathcal{L}(\mathbf{v}_a) = \dfrac{e^{-as}}{s^2}$. Now,

$$f(t) = \mathcal{L}^{-1}\Big(\frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2} - \frac{e^{-3s}}{s^2} + \frac{e^{-4s}}{s^2}\Big) = \mathbf{v}_{-1}(t) - \mathbf{v}_{-2}(t) - \mathbf{v}_{-3}(t) + \mathbf{v}_{-4}(t)$$

Unraveling the right hand side gets us the answer:

$$\begin{aligned} f(t) &= 0 \quad \text{if } t < 1\\ &= t-1 \quad \text{if } 1 \le t < 2\\ &= 1 \quad \text{if } 2 \le t < 3\\ &= 4-t \quad \text{if } 3 \le t < 4\\ &= 0 \quad \text{if } t \ge 4\end{aligned}$$

Q.9 (i) We have to compute the Laplce transform of $\mathbf{u}(t-\pi)\sin t$. Write $\sin t = -\sin(t-\pi)$ and use the shifting theorem.

$$\mathcal{L}(\mathbf{u}(t-\pi)\sin t) = -e^{-\pi s}\mathcal{L}(\sin t) = -e^{-\pi s}/(s^2+1)$$

(ii) Writing e^{-2t} as $e^2 e^{-2(t-1)}$ prepares us to apply the shifting theorem.

$$\mathcal{L}(\mathbf{u}(t-1)) = e^2\mathcal{L}(\mathbf{u}(t-1)e^{-2(t-1)}) = e^{2-s}\mathcal{L}(e^{-2t}) = e^{2-s}/(s+2).$$

Q.10 First observe that

$$\ln\big(\frac{s+a}{s+b}\big) = \ln\Big(1+\frac{a-b}{s+b}\Big) = \sum_{j=0}^{\infty}\frac{(-1)^j}{j+1}\Big(\frac{a-b}{s+b}\Big)^{j+1}$$

The series converges for $s > |a-b|+|b|$ and so

$$\ln\Big(\frac{s+a}{s+b}\Big) = \sum_{j=0}^{\infty}\frac{(-1)^j(a-b)^{j+1}}{(j+1)!}\mathcal{L}(e^{-bx}x^j)$$

Hence

$$\mathcal{L}^{-1}\Big[\ln\Big(\frac{s+a}{s+b}\Big)\Big] = -\frac{e^{-bx}}{x}\sum_{j=0}^{\infty}\frac{(x(b-a))^{j+1}}{(j+1)!} = \frac{e^{-bx}-e^{-ax}}{x}$$

Likewise

$$\mathcal{L}^{-1}\Big[\ln\Big(\frac{s+a}{s+b}\Big)\Big(\frac{s+c}{s+d}\Big)\Big] = \frac{1}{x}(e^{-bx}-e^{-ax}+e^{-dx}-e^{-cx})$$

In this example $a=-2+i$, $c=-2-i$, $b=-1+2i$ and $d=-1-2i$.

Q.11 The first part is the convolution theoerm for which the text of Kreyszig may be referred to. For the second part, let

$$G(s) = \frac{1}{(s+a)^2+a^2} = \frac{1}{a}\mathcal{L}(e^{-at}\sin at)$$

so that by the convolution theorem,

$$\mathcal{L}^{-1}(F(s)G(s)) = f(t)*g(t) = \frac{1}{a}\int_0^t e^{-a(t-u)}\sin a(t-u)f(u)du$$

Q.12 In this problem we see an application to linear differential equations with variable coefficients. We compute the Fourier transform of the Bessel's function. Use formula (1) on p275 of Kreyszig (Eighth edition) and we get

$$-\frac{d}{ds}(s^2Y - sy(0) - y'(0)) + sY - k - Y' = 0$$

which gives the first order ODE for $Y(s)$:

$$(s^2+1)Y' + sY = 0$$

which integrates to

$$Y = C/\sqrt{s^2+1}$$

The value of the constant is seen to be one and $Y = 1/\sqrt{s^2+1}$.

Q.13 Note that the convolution is given by

$$G(\lambda) = \int_{-\infty}^{\infty} t^{a-1}\mathbf{u}(t)(\lambda - t)^{b-1}\mathbf{u}(\lambda - t)dt$$

The presence of the factor $\mathbf{u}(t)$ reduces the integral to

$$G(\lambda) = \int_{0}^{\infty} t^{a-1}(\lambda - t)^{b-1}\mathbf{u}(\lambda - t)dt$$

It is clear the convolution is zero if $\lambda < 0$ so that $G(\lambda) = \mathbf{u}(\lambda)G(\lambda)$. For $t > \lambda$ the integrand in the last integral vanishes and

$$G(\lambda) = \mathbf{u}(\lambda)\int_{0}^{t} t^{a-1}(\lambda - t)^{b-1}dt = \mathbf{u}(\lambda)\lambda^{a+b-1}\int_{0}^{1} t^{a-1}(1 - t)^{b-1}dt = \lambda^{a+b-1}\mathbf{u}(\lambda)B(a, b).$$

In other words,

$$\mathbf{u}(t)t^{a-1} * \mathbf{u}(t)t^{b-1} = \mathbf{u}(t)t^{a+b-1}B(a, b) \qquad (III)$$

Let us compute the Laplace transform of $\mathbf{u}(t)t^{c-1}$:

$$\mathcal{L}(\mathbf{u}(t)t^{c-1}) = \int_{0}^{\infty} t^{c-1}e^{-st}dt = \frac{\Gamma(c)}{s^c}$$

Applying the convolution theorem to (III) we see that

$$\frac{\Gamma(a)}{s^a}\frac{\Gamma(b)}{s^b} = \frac{\Gamma(a+b)}{s^{a+b}}B(a, b)$$

from which we get the fundamental relation

$$\Gamma(a)\Gamma(b) = \Gamma(a + b)B(a, b)$$

Q.14 Denoting the Laplace transform of f by $F(s)$ we see that

$$\mathcal{L}(f) \cdot \mathcal{L}(f) = (F(s))^2 = \frac{1}{s^2 + 1}$$

By convolution theorem,

$$\mathcal{L}(f * f) = (F(s))^2 = \frac{1}{s^2 + 1}$$

so that $f * f = \sin x$.

Q.15 Define $f(t) = 0$ for $t < 0$ and taking the Laplace transform of $f(t)$ we get

$$F(s) = \mathcal{L}\Big(\int_{0}^{\infty} \frac{\sin tx}{x}dx\Big) = \int_{0}^{\infty} \frac{1}{x}\mathcal{L}(\sin tx)dx = \int_{0}^{\infty} \frac{dx}{s^2 + x^2} = \frac{1}{s}\frac{\pi}{2}$$

so that $f(t) = \frac{\pi}{2}$ giving the formula

$$\int_{0}^{\infty} \frac{\sin tx}{x}dx = \frac{\pi}{2}, \quad t > 0.$$

The case $t < 0$ can be reduced to the above case.

(ii) The value of the integral is independent of the sign of t and define $f(t) = 0$ for $t < 0$ and for $t > 0$ take $f(t)$ to be the integral to be computed. Then as before,

$$F(s) = \int_0^\infty \frac{dx}{x^2 + a^2} \mathcal{L}(\cos tx) = \int_0^\infty \frac{sdx}{(x^2 + a^2)(s^2 + x^2)} = \frac{\pi s}{2(s^2 - a^2)} \left\{ \frac{1}{a} - \frac{1}{s} \right\} = \frac{\pi}{2a(s + a)}$$

Taking the inverse Laplace transform we get

$$\int_0^\infty \frac{\cos xtdx}{x^2 + a^2} = \frac{\pi}{2a} e^{-at}$$

Q.16 Solving Abel's integral equation:

Note that the integral is the convolution of $y(t)\mathbf{u}(t)$ and $(1/\sqrt{t})\mathbf{u}(t)$. By the convolution theorem,

$$\frac{A}{s} = Y(s)\sqrt{\pi/s}$$

which gives

$$Y(s) = \frac{A}{\sqrt{\pi s}}, \quad y(t) = \frac{A}{\pi \sqrt{t}}$$

Observe that this is really a special case of Q.13.

Q.17 See the comments in Q.4.

Q.18-21 Cancelled.