Outline

1. Maxwell's Equations and Coulomb's Law
2. Electrostatic Field: Electric field due to a time invariant charge distribution
3. Gauss's Law: Finding Electric field of symmetric distributions
4. Electrostatic Potential: Poisson's and Laplace Equation

Objectives

1. To Understand how Maxwell's equations and Coulomb's Law are equivalent.
2. To use symmetry of the charge system effectively along with Guass's Law.

Recap

- If the divergence and curl of a vector field $\vec{F}(\vec{r})$ are $D(\vec{r})$, and $\vec{C}(\vec{r})$ respectively, and they both go to zero faster than $1/r^2$ as $r \to \infty$, then the vector field is given by,

$$\vec{F}(\vec{r}) = -\nabla U(\vec{r}) + \nabla \times \vec{W}(\vec{r})$$

where, $U(\vec{r}) = \frac{1}{4\pi} \int \frac{D(\vec{r}')}{|\vec{r}-\vec{r}'|} d\tau'$ and $\vec{W}(\vec{r}) = \frac{1}{4\pi} \int \frac{\vec{C}(\vec{r}')}{|\vec{r}-\vec{r}'|} d\tau'$.

- $\vec{F}(\vec{r}) = -\nabla U(\vec{r}) + \nabla \times \vec{W}(\vec{r})$ is a unique solution for the above problem, if $\vec{F}(\vec{r})$ goes to zero as $r \to \infty$.

Maxwell's Equations

Theorem (Maxwell's Equations)

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \tag{1}$$

$$\nabla \cdot \vec{B} = 0 \tag{2}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{3}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{4}$$

- These equations give the divergence and curl of the electric and magnetic fields, which can be used, along with the Helmholtz theorem, to obtain $\vec{E}$, and $\vec{B}$.

Electrostatic Field

- Let's start with a simpler set of equations, with no time varying fields. $\left(\frac{\partial \vec{B}}{\partial t} = 0, \frac{\partial \vec{E}}{\partial t} = 0\right)$

$$\begin{aligned} \nabla \cdot \vec{E} &= \frac{\rho}{\varepsilon_0} \\ \nabla \times \vec{E} &= 0 \end{aligned}$$

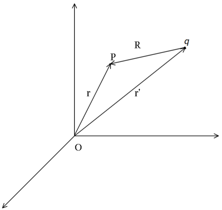
- Using Helmholtz theorem,

$$\begin{aligned} \vec{E} &= -\nabla U + \nabla \times \vec{W} \\ &= -\nabla_{(r)} \left(\frac{1}{4\pi} \int \frac{\rho(\vec{r}')}{\varepsilon_0 |\vec{r} - \vec{r}'|} d\tau' \right) + 0 \\ &= -\frac{1}{4\pi\varepsilon_0} \int \rho(\vec{r}') \nabla_{(r)} \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) d\tau' \\ &= -\frac{1}{4\pi\varepsilon_0} \int \rho(\vec{r}') \left(\frac{-(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \right) d\tau' \end{aligned}$$

Electrostatic Field

- Let, $\vec{R}$ be the separation vector between $\vec{r}$ and $\vec{r}'$, which is given by $\vec{R} = \vec{r} - \vec{r}'$.

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \rho(\vec{r}') \frac{\hat{R}}{|\vec{R}|^2} d\tau'$$



- Helmholtz theorem along with the Maxwell's equations resulted in Coulomb's law!
- One can also start with the Coulomb's law, and then compute the divergence and curl of the electric field to obtain the Maxwell's equations.

Electrostatic Field

- Coulomb's law can be easily modified for various types of charge distributions.
- Point Charges:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{R_i^2}\hat{R}_i$$

- Line Charges:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda(\vec{r}')}{|\vec{R}|^2}\hat{R}\,dl'$$

- Surface Charges:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma(\vec{r}')}{|\vec{R}|^2}\hat{R}\,da'$$

where $\vec{R}$ is the separation vector between $\vec{r}$ and $\vec{r}'$.

Electrostatic Field

- We can find electric field for all sort of charge distributions using the above formulas.
- Once we have electric field we can find the force on a test charge q_o:

$$\vec{F}(\vec{r}) \quad = \quad q_o \vec{E}(\vec{r})$$

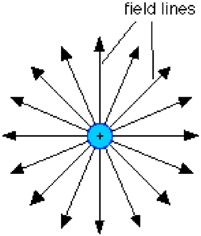
- To find electric fields for symmetric objects, Coulomb's law or Gauss's law can be used directly.
- But calculation of integrals for arbitrary charge distributions requires the use of numerical methods.

Electric Field Lines

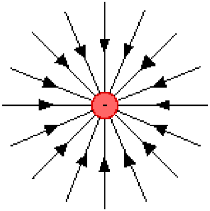
- Electric field lines always originate from a positive charge or can be coming from infinity.
- They terminate at negative charge or extend up to infinity.
- The strength of the electric field at a point is indicated by the density of electric field lines at that point.
- Field lines never cross each other.

Electric Field Lines

- Isolated point charge:
  - If positive point charge:

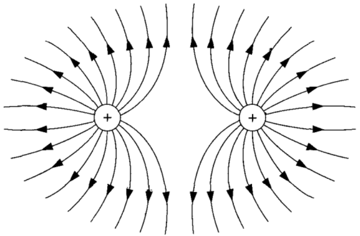


  - If negative point charge:

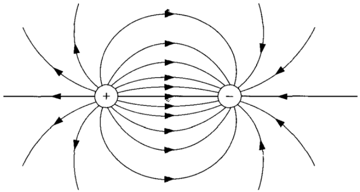


Electric Field Lines

- Two point charges at a fixed distance:
  - If both are positive:



  - If one is positive and the other one is negative charges:



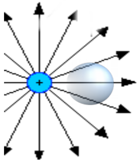
Guass's Law

Theorem (Gauss's Law)

The net electric flux through any closed surface equals $\frac{1}{\varepsilon_0}$ times the net electric charge in the closed surface.

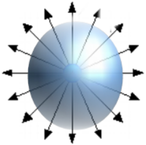
$$\Phi_E = \oint_S \vec{E} \cdot d\vec{S} = \frac{Q_{enc}}{\varepsilon_0}$$

- Flux is proportional to the number of field lines passing through the surface.
- Electric flux due to a charge lying outside the surface is zero.



Electric Flux

- Gauss's law can be easily verified for a point charged particle.

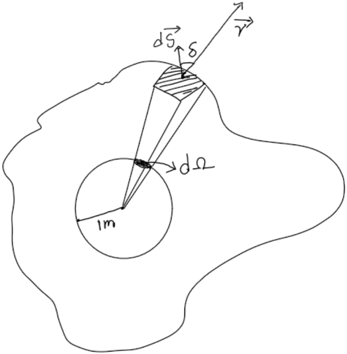


$$
\begin{aligned}
\oint_S \vec{E} \cdot d\vec{S} &= \int_0^{2\pi} \int_0^{\pi} \frac{q}{4\pi\varepsilon_0 r^2} \hat{r} \cdot d\vec{S} \\
&= \int_0^{2\pi} \int_0^{\pi} \frac{q}{4\pi\varepsilon_0 r^2} \left(r^2 \sin\theta \, d\theta \, d\phi \right) \\
&= \frac{q}{\varepsilon_0}
\end{aligned}
$$

- What if the surface isn't a sphere but an arbitrary one?

Electric Flux

- The solid angle subtended by a surface S can be defined as the surface area of a unit sphere covered by the projection of surface S onto the unit sphere.



$$d\Omega = \frac{d\vec{S}\cdot\hat{r}}{r^2}$$

Electric Flux

- $d\vec{S} \cdot \hat{r}$ gives the area projected onto a sphere with radius r, and dividing with r^2 scales down the projected area onto a unit sphere.

$$d\Omega = \frac{|d\vec{S}|\cos\delta}{r^2}$$

- One can use $d\vec{S} = (r^2 \sin\theta d\theta d\phi)\hat{r} + (r\sin\theta dr d\phi)\hat{\phi} + (r dr d\theta)\hat{\theta}$,

$$\begin{aligned} d\Omega &= \frac{d\vec{S} \cdot \hat{r}}{r^2} \\ &= \sin\theta d\theta d\phi \end{aligned}$$

  where r, θ, ϕ are polar co-ordinates.

- Electric flux:

$$\begin{aligned} \Phi_E &= \oint \vec{E} \cdot d\vec{S} \\ &= \frac{q}{4\pi\varepsilon_0} \oint \frac{\hat{r} \cdot d\vec{S}}{r^2} \end{aligned}$$

Electric Flux

$$\Phi_E = \frac{q}{4\pi\varepsilon_0}\oint \sin\theta\, d\theta\, d\phi$$
$$= \frac{q}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^{\pi} \sin\theta\, d\theta\, d\phi$$
$$= \frac{q}{\varepsilon_0}$$

- What if there are multiple charges inside the arbitrary surface?
  - Principle of superposition.(as Maxwell's equations are linear)

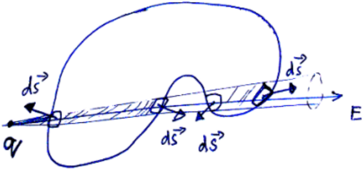
$$E = \sum_{i=1}^{n} E_i$$
$$\oint \vec{E}\cdot d\vec{S} = \sum_{i=1}^{n}\left(\oint E_i \cdot d\vec{S}\right)$$
$$= \sum_{i=0}^{n}\left(\frac{q_i}{\varepsilon_0}\right)$$
$$= \frac{Q_{enc}}{\varepsilon_0}$$

Electric Flux

- Solid angle subtended by a closed surface at a point inside the closed surface,

$$\begin{aligned} d\Omega &= \oint \sin\theta \, d\theta \, d\phi \\ &= \int_0^{2\pi} \int_0^{\pi} \sin\theta \, d\theta \, d\phi \\ &= 4\pi \end{aligned}$$

- Electric flux through a closed surface due to a point charge located outside the surface is zero. The contributions exactly cancel.



Gauss's Law- Differential form

- Gauss's Law:

$$\oint \vec{E} \cdot d\vec{S} = \frac{Q_{enc}}{\varepsilon_0}$$

- Divergence Theorem:

$$\begin{aligned}
\int_V (\nabla \cdot \vec{E}) dV &= \oint \vec{E} \cdot d\vec{S} \\
&= \frac{Q_{enc}}{\varepsilon_0} \\
&= \int_V \frac{\rho(r)}{\varepsilon_0} dV \\
\nabla \cdot \vec{E} &= \frac{\rho(r)}{\varepsilon_0}
\end{aligned}$$

Gauss's Law- Differential form

- Maxwell's equations along with Helmholtz theorem lead to Coulomb's law.
- Coulomb's law along with the Gauss's law result in Maxwell's equations.
- Maxwell's equations or Coulomb's law can only be derived only if the other is assumed to be true. Therefore, both Coulomb's law and Maxwell's Equations are equivalent
- Both are based on empirical, i.e., experimental observations

Helmholtz Theorem

$$\nabla . \boldsymbol{E} = \frac{\rho(\boldsymbol{r})}{\epsilon_0} \qquad \boldsymbol{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\boldsymbol{r}}$$

Valid for moving charges!

Only for static charges

Gauss's Law

Curl of the Electric Field

$$
\begin{aligned}
\vec{E} &= \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} \\
\nabla \times \vec{E} &= 0
\end{aligned}
$$

- Line integral of Electric Field:

$$
\begin{aligned}
\int_a^b \vec{E}\cdot d\vec{l} &= \int_a^b \left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}\right)\cdot(\hat{r}\,dr + \hat{\theta}\,r\,d\theta + \hat{\phi}\,r\sin\theta\,d\phi) \\
&= \int_a^b \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}dr \\
&= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_a}-\frac{1}{r_b}\right) \\
\oint \vec{E}\cdot d\vec{l} &= 0
\end{aligned}
$$

Curl of the Electric Field

- Stoke's law:

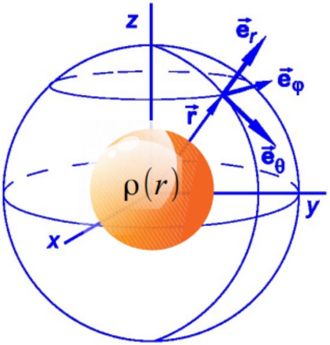
$$
\begin{aligned}
\int_S (\nabla \times \vec{E}) \cdot d\vec{S} &= \oint \vec{E} \cdot d\vec{l} \\
&= 0 \\
\nabla \times \vec{E} &= 0
\end{aligned}
$$

- This can be proved to be true for any number of charges using superposition.
- Remember that this is valid only for static charges.

Spherically Symmetric Distribution

- Rotation of co-ordinate system about z-axis and x-axis by some arbitrary angles doesn't change the charge distribution.
- Charge density function $\rho(r)$ is independent of θ, ϕ. Therefore, electric field is also independent of θ, ϕ.

$$\vec{E}(r, \theta + \theta_o, \phi + \phi_o) = \vec{E}(r, \theta, \phi)$$



Spherically Symmetric Distribution

- Line integral about a closed circular loop around z-axis,

$$\begin{aligned}
\oint \vec{E}\cdot d\vec{l} &= \oint \vec{E}\cdot(\hat{\phi} r\sin\theta\, d\phi) \\
0 &= r\sin\theta \int_0^{2\pi} E_\phi\, d\phi \\
2\pi r\sin\theta E_\phi &= 0 \\
E_\phi &= 0
\end{aligned}$$

- Line integral about a closed circular loop around x-axis,

$$\begin{aligned}
\oint \vec{E}\cdot d\vec{l} &= \oint \vec{E}\cdot(\hat{\theta} r d\theta) \\
0 &= r \int_0^{2\pi} E_\theta\, d\theta \\
2\pi r E_\theta &= 0 \\
E_\theta &= 0
\end{aligned}$$

- Note that above we have extended the θ limits to 2π from π, to make the loop a closed one

Spherically Symmetric Distribution

- Guass's law: $\oint_S \vec{E} \cdot d\vec{S} = \frac{Q_{enc}}{\varepsilon_0}$ along with $E_\theta = E_\phi = 0$ leads to,

$$
\begin{aligned}
\oint E_r r^2 \sin\theta d\theta d\phi &= \frac{\int_V \rho(r) r^2 \sin\theta dr d\theta d\phi}{\varepsilon_0} \\
E_r &= \frac{\int_0^r \rho(r') r'^2 dr'}{\varepsilon_0 r^2}
\end{aligned}
$$

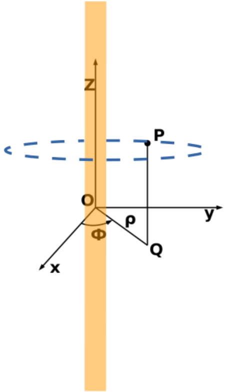
- In simple terms,

$$
E_r = \frac{Q_{enc}}{4\pi\varepsilon_0 r^2}
$$

Cylindrically Symmetric Distributions

- Rotation of co-ordinate system about the z-axis and translation of origin along z axis doesn't change the charge distribution .
- Charge density function $\rho(r)$ is independent of θ, z. Therefore, electric field is also independent of θ, z.

$$\vec{E}(r, \theta + \theta_0, z + z_o) = \vec{E}(r, \theta, z)$$



Cylindrically Symmetric Distribution

- Note, instead of ρ, ϕ, we are using the symbols r, θ. Line integral about a closed circular loop around z-axis,

$$
\begin{aligned}
\oint \vec{E} \cdot d\vec{l} &= \oint (\vec{E} \cdot \hat{\theta}) r d\theta \\
0 &= r \int_0^{2\pi} E_\theta d\theta \\
2\pi r E_\theta &= 0 \\
E_\theta &= 0
\end{aligned}
$$

- Flip the charge distribution about x-y plane. If there was a E_z component earlier, it should be $-E_z$ now.
- However, the charge distribution still remains the same as earlier.

$$
\begin{aligned}
E_z &= -E_z \\
E_z &= 0
\end{aligned}
$$

Cylindrically Symmetric Distribution

- Gauss's law: $\oint_S \vec{E}\cdot d\vec{S} = \frac{Q_{enc}}{\varepsilon_0}$ by considering a cylinder of finite length as Gaussian surface.
- $E_z = E_\theta = 0$ and $\vec{E}$ is along $\hat{r}$ which is perpendicular to the area of two circular discs in the top and bottom of the cylinder.

$$\begin{aligned}
\int (E_r\hat{r})\cdot(\hat{r}rd\theta dz) &= \frac{\int_V \rho(r)rdrd\theta dz}{\varepsilon_0} \\
2\pi\rho L E_r &= \frac{2\pi L\int_0^r \rho(r')r'dr'}{\varepsilon_0} \\
E_r &= \frac{\int_0^r \rho(r')r'dr'}{\varepsilon_0 r}
\end{aligned}$$

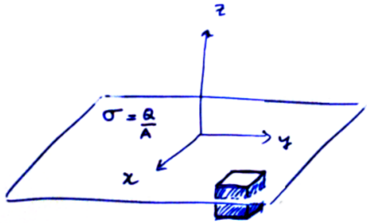
- In simple terms,

$$E_r = \frac{Q_{enc}}{2\pi r L\varepsilon_0} = \frac{\lambda}{2\pi\varepsilon_0 r}$$

Uniformly Charged Infinite Plane

- Translation of the origin in the $x - y$ plane doesn't change the charge distribution.
- Charge density function σ is constant and independent of x, y. Therefore, electric field is also independent of x, y.

$$E(x + x_o, y + y_o, z) = E(x, y, z)$$



Uniformly Charged Infinite Plane

- Flip the charge distribution about x-z plane, If there was a E_y component earlier, it should be $-E_y$ now.
- However, the charge distribution still remains the same as earlier.

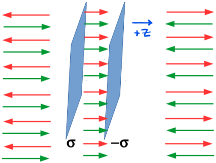
$$\begin{aligned} E_y &= -E_y \\ E_y &= 0 \end{aligned}$$

- Similarly, flipping the charge distribution about y-z plane results in $E_x = 0$.
- Gauss's law: $\oint_S \vec{E} \cdot d\vec{S} = \frac{Q_{enc}}{\varepsilon_0}$ by considering a cube as the Gaussian surface.

$$\begin{aligned} \int E_z dxdy &= \frac{\int \sigma dxdy}{\varepsilon_0} \\ 2A|E_z| &= \frac{\sigma A}{\varepsilon_0} \\ \vec{E} &= \frac{\sigma}{2\varepsilon_0}\hat{n} \end{aligned}$$

Two Uniformly and Oppositely Charged Parallel Plates

- Principle of superposition can be used.



- Electric Field Between the plates:

$$\vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{z} + \frac{-\sigma}{2\varepsilon_0}(-\hat{z}) = \frac{\sigma}{\varepsilon_0}\hat{z}$$

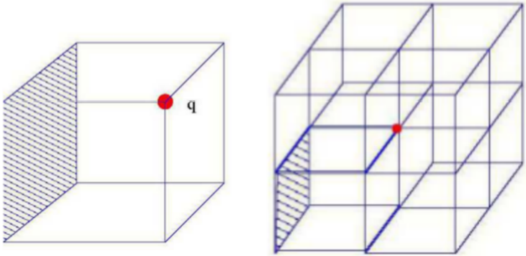
- Electric Field at the left side of both plates:

$$\vec{E} = \frac{\sigma}{2\varepsilon_0}(-\hat{z}) + \frac{-\sigma}{2\varepsilon_0}(-\hat{z}) = 0$$

- Electric Field at the right side of both plates:

$$\vec{E} = \frac{\sigma}{2\varepsilon_0}(\hat{z}) + \frac{-\sigma}{2\varepsilon_0}(\hat{z}) = 0$$

Flux using Symmetry



- The large cube is made of eight small cubes, which means that the flux through each small cube is 1/8 times the flux through large cube.
- Each small cube has 6 faces, out of which 3 have zero flux passing through them.
- Therefore, the flux through the shaded area is 1/24 times the flux through large cube.

$$\Phi_E = \frac{q}{24\varepsilon_0}$$

Electrostatic Potential

- Gauss's law is extremely powerful in cases with symmetry. It may not be helpful in general cases.
- If Gauss's law isn't helpful, then what can be done?
  - One can move to the potential picture from the field picture.

    $$\nabla \times \vec{E} = 0$$

    Curl of gradient is zero. Therefore, we assume a function V, such that

    $$\vec{E} = -\nabla V$$

- Maxwell equations can be written in terms of electrostatic potential $V(\vec{r})$.

  $$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \quad \Rightarrow \quad \nabla^2 V = -\frac{\rho}{\varepsilon_0} \text{ Poisson's equation}$$

  If regions of no charge,

  $$\nabla^2 V = 0 \quad \text{Laplace's equation}$$