MA 108 - Ordinary Differential Equations

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Exponetial order funcions

Recall:

- For a given f, $L(f)$ may or may not exist.
- Sufficient conditions under which convergence is guaranteed for the integral in the definition of the Laplace transform is that f is piecewise continuous and is of exponential order.
- Piecewise continuity - The function is continuous except possibly for finitely many jump discontinuities.
- A function f is said to be of exponential order if there exists $a \in \mathbb{R}$ and positive constants t_0 and K such that

$$|f(t)| \leq Ke^{at},$$

for all $t \geq t_0 > 0$.

Exponetial order funcions-Examples

- If f is a bounded function defined on $[0, \ \infty)$, then it is of exponetial order with $a = 0$. In particular, $f(t) = \sin bt, \cos bt, t \geq 0$ are of exponetial order.
- $e^{\alpha t} \sin bt, t \geq 0$ is of exponetial order with $a = \alpha$.
- For $\alpha > 0, \ f(t) = t^{\alpha}, t \geq 0$ is of exponetial order, since for any $a > 0$,
$$\lim_{t \to \infty} t^{\alpha} e^{-at} = 0.$$
- If the functions f_1, f_2 defined on $[0, \ \infty)$ are of exponential order, so are $f_1 \pm f_2, f_1 f_2$.

Laplace transform-existence

Theorem: Let f be a piecewise continuous on $[0, \infty)$ and is of exponential order. Then $L(f)$ exists on a domain which contain (α, ∞) where α is such that

$$|f(t)| \leq Ke^{\alpha t}, \ t \geq t_0, K > 0, t_0 \geq 0.$$

Proof: Consider

$$\begin{aligned} \int_0^\infty e^{-st}|f(t)|dt &= \int_0^{t_0} e^{-st}|f(t)|dt + \int_{t_0}^\infty e^{-st}|f(t)|dt \\ &\leq \int_0^{t_0} e^{-st}|f(t)|dt + K\int_{t_0}^\infty e^{-(s-\alpha)t}dt. \end{aligned}$$

First integral exists, since it is a definite integral and f is piecewise continuous. Second integral exists for all $s > a$. This we have already seen. Now comparison test for integrals implies that $L(f)(s)$ exists for all all $s > a$.

The conditions given in the previous theorem are sufficient but not necessary.

Let $f(t) = \dfrac{1}{\sqrt{t}}, t > 0$ and $f(0) = 0$. Then f is not continuous at 0 and hence not piecewise continuous on $[0, \beta]$, for any $\beta > 0$.

Now

$$\int_0^\infty e^{-st}\frac{1}{\sqrt{t}}dt = \int_0^1 e^{-st}\frac{1}{\sqrt{t}}dt + \int_1^\infty e^{-st}\frac{1}{\sqrt{t}}dt$$

Both the integrals exists using comparison test for integrals. Hence $L(f)$ exists on $(0, \ \infty)$.

Property 1 : Linearity

Let $f, g : [0, \infty) \to \mathbb{R}$ be functions such that $L(f)$ and $L(g)$ exist on (α, ∞). Let $a, b \in \mathbb{R}$. Then,

$$L(af + bg)(s) = aL(f)(s) + bL(g)(s), s > \alpha.$$

Proof: For $s > \alpha$,

$$\begin{aligned} L(af + bg)(s) &= \int_0^\infty e^{-st}(af(t) + bg(t))dt \\ &= \int_0^\infty e^{-st} af(t)dt + \int_0^\infty e^{-st} bg(t)dt \\ &= aL(f)(s) + bL(g)(s). \end{aligned}$$

Example 1

For $s > 0$,

$$
\begin{aligned}
L(e^{i\omega t})(s) &= \int_0^\infty e^{-st} e^{i\omega t}\, dt \\
&= \int_0^\infty e^{-(s-i\omega)t}\, dt \\
&= \left. \frac{e^{-(s-i\omega)t}}{-(s-i\omega)} \right|_{t=0}^{\infty} = \frac{1}{s-i\omega} \\
&= \frac{s+i\omega}{s^2+\omega^2}.
\end{aligned}
$$

Hence,

$$L(\cos\omega t + i\sin\omega t)(s) = \frac{s+i\omega}{s^2+\omega^2}.$$

Using linearity,

$$\boxed{L(\cos\omega t)(s) = \frac{s}{s^2+\omega^2}, \qquad L(\sin\omega t)(s) = \frac{\omega}{s^2+\omega^2}.}$$

Example 2

For $a \geq 0$,

$$
\begin{aligned}
L(\cosh at)(s) &= L(\frac{e^{at}+e^{-at}}{2})(s), s > a, \\
&= \frac{1}{2}L(e^{at})(s) + \frac{1}{2}L(e^{-at})(s) \\
&= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) = \frac{s}{s^2-a^2}.
\end{aligned}
$$

Hence, $\boxed{L(\cosh at)(s) = \frac{s}{s^2-a^2}, s > a.}$

Similarly, $\boxed{L(\sinh at)(s) = \frac{a}{s^2-a^2}, s > a.}$

Example 3

For $s > 0$,

$$
\begin{aligned}
L(te^{i\omega t})(s) &= \int_0^\infty e^{-st} te^{i\omega t}\, dt = \int_0^\infty te^{-(s-i\omega)t}\, dt \\
&= \left.\frac{te^{-(s-i\omega)t}}{-(s-i\omega)}\right|_{t=0}^{\infty} + \int_0^\infty \frac{e^{-(s-i\omega)t}}{s-i\omega}\, dt \\
&= \frac{1}{s-i\omega} \times \left.\frac{e^{-(s-i\omega)t}}{-(s-i\omega)}\right|_{t=0}^{\infty} \\
&= \frac{1}{(s-i\omega)^2} = \frac{1}{s^2-\omega^2-2is\omega}\,\frac{s^2-\omega^2+2is\omega}{s^2-\omega^2+2is\omega} \\
&= \frac{s^2-\omega^2}{(s^2-\omega^2)^2+4s^2\omega^2} + i\frac{2s\omega}{(s^2-\omega^2)^2+4s^2\omega^2} \\
&= \frac{s^2-\omega^2}{(s^2+\omega^2)^2} + i\frac{2s\omega}{(s^2+\omega^2)^2}.
\end{aligned}
$$

Hence,

$$
\boxed{L(t\cos\omega t)(s) = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}, \qquad L(t\sin\omega t)(s) = \frac{2s\omega}{(s^2+\omega^2)^2}, s > 0.}
$$

If $L(f(t))(s) = F(s)$, then $L(e^{at}f(t))(s) = F(s-a)$.

Proof :

$$\begin{aligned} L(e^{at}f(t))(s) &= \int_0^\infty e^{-st}e^{at}f(t)\,dt \\ &= \int_0^\infty e^{-(s-a)t}f(t)\,dt \\ &= F(s-a). \end{aligned}$$

Examples :

1. $L(t^2)(s) = \dfrac{2}{s^3}, s > 0 \Longrightarrow L(e^{-t}t^2)(s) = \dfrac{2}{(s+1)^3}, s > -1.$
2. $L(\cos\omega t)(s) = \dfrac{s}{s^2+\omega^2}, s > 0 \Longrightarrow L(e^{at}\cos\omega t)(s) = \dfrac{s-a}{(s-a)^2+\omega^2}, s > a.$

Exercises

Find the Laplace transforms of

1. $t^n e^{at}$
2. $\cosh at \cos at$
3. $e^{-t} \sin^2 t$

Property 3 : Scaling

$$\boxed{\text{If } L(f)(s) = F(s), \text{ then } L(f(ct))(s) = \frac{1}{c}F\left(\frac{s}{c}\right), \ c > 0.}$$

Proof: Let $\xi = ct$. Then, $d\xi = c\,dt$.

$$\begin{aligned} L(f(ct)) &= \int_0^\infty e^{-st} f(ct)\,dt \\ &= \int_0^\infty e^{-\left(\frac{s\xi}{c}\right)} \frac{1}{c} f(\xi)\,d\xi \\ &= \frac{1}{c}\int_0^\infty e^{-\left(\frac{s\xi}{c}\right)} f(\xi)\,d\xi \\ &= \frac{1}{c}F\left(\frac{s}{c}\right). \end{aligned}$$

Example :

$$L(e^t)(s) = \frac{1}{s-1}, s > 1 \Longrightarrow L(e^{at})(s) = \frac{1}{a}\frac{1}{\left(\frac{s}{a}-1\right)} = \frac{1}{s-a}, s > a.$$

I.

(i) Suppose f is continuous,

(ii) f' is piecewise continuous on $[0, a]$ for all $a > 0$,

(iii) $|f(t)| \leq Ke^{\alpha t}$, for $t \geq t_0 > 0$, where $K > 0$, t_0, $\alpha \in \mathbb{R}$.

Then, $L(f')(s)$ exists for $s > \alpha$ and

$$L(f') = sL(f) - f(0).$$

II.

(a) Suppose $f, f', \ldots, f^{(n-1)}$ are continuous

(b) $f^{(n)}$ is piecewise continuous on $[0, a]$, for all $a > 0$,

(c) For all $t \geq t_0 > 0$, $|f^{(i)}(t)| \leq Ke^{\alpha t}$, $0 \leq i \leq n-1$, where $K > 0$, t_0, $\alpha \in \mathbb{R}$.

Then, $L(f^{(n)})(s)$ exists for all $s > \alpha$ and

$$L(f^{(n)}) = s^n L(f) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

Proof of Property 4

Consider the interval $[0, \beta]$. Let f' be discontinuous at $t_1, t_2, \ldots, t_n$, where $0 = t_0 < t_1 < t_2 < \ldots < t_n < t_{n+1} = \beta$. Then,

$$\int_0^\beta e^{-st} f'(t)dt = \int_0^{t_1} e^{-st} f'(t)dt + \int_{t_1}^{t_2} e^{-st} f'(t)dt + \ldots + \int_{t_n}^{\beta} e^{-st} f'(t)dt.$$

Integrating by parts, we get:

$$\begin{aligned}
\int_0^\beta e^{-st} f'(t)dt &= \sum_{i=1}^{n+1} [e^{-st} f(t)]_{t_{i-1}}^{t_i} + s \sum_{i=1}^{n+1} \int_{t_{i-1}}^{t_i} e^{-st} f(t)dt \\
&= e^{-s\beta} f(\beta) - f(0) + s \int_0^\beta e^{-st} f(t)dt.
\end{aligned}$$

Taking limit as $\beta \to \infty$, we get:

$$L(f')(s) = sL(f) - f(0),$$

for $s > \alpha$.

Proof of Corollary

Induction. $n = 1$ is already done. Assume that the result is true up to $n - 1$. Then, for $s > \alpha$,

$$\begin{aligned} L(f^{(n)})(s) &= L((f^{(n-1)})')(s) \\ &= sL(f^{(n-1)}) - f^{n-1}(0) \\ &= s(s^{n-1}L(f) - s^{n-2}f(0) - \ldots - f^{(n-2)}(0)) - f^{(n-1)}(0) \\ &= s^n L(f) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0). \end{aligned}$$

- ✔ First derivative :

$$L(f')(s) = sL(f) - f(0).$$

- ✔ Second derivative :

$$L(f'')(s) = s^2 L(f) - sf(0) - f'(0).$$