April 13

Q1)

$$A = \begin{bmatrix} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \end{bmatrix} \qquad A^t = \begin{bmatrix} a_1 & b_1 \\ \vdots & \\ a_n & b_n \end{bmatrix}$$

$$AA^t = \begin{bmatrix} a_1^2 + \cdots + a_n^2 & a_1b_1 + \cdots + a_nb_n \\ a_1b_1 + \cdots + a_nb_n & b_1^2 + \cdots + b_n^2 \end{bmatrix}$$

$$\det(AA^t) = \sum_{i=1}^{n} (a_i b_i)^2 + \sum_{\substack{i \neq j \\ i,j \in \{1,\ldots,n\}}} (a_i b_j)^2 - \sum_{i=1}^{n} (a_i b_i)^2 - \sum_{\substack{i \neq j \\ i,j \in \{1,\ldots,n\}}} a_i a_j b_i b_j$$

$$P = A^t A = \begin{bmatrix} a_1 & b_1 \\ & \\ a_n & b_n \end{bmatrix} \begin{bmatrix} a_1 & \cdots & a_n \\ b_1 & & b_n \end{bmatrix}$$

For $i<j$

$$\begin{vmatrix} P_{ii} & P_{ij} \\ P_{ji} & P_{jj} \end{vmatrix} = \begin{vmatrix} a_i^2+b_i^2 & a_ia_j+b_ib_j \\ a_ia_j+b_ib_j & a_j^2+b_j^2 \end{vmatrix}$$

$$= a_i^2b_j^2 + b_i^2a_j^2 - 2a_ia_jb_ib_j$$

$$= (a_ib_j)^2 - a_ia_jb_ib_j + (a_jb_i)^2 - a_ia_jb_ib_j$$

$$\sum_{i<j} \overset{2\times 2}{\text{Principal minors}} = \sum_{\substack{i\neq j \\ i,j\in\{1,\dots,n\}}} \left((a_ib_j)^2 - a_ia_jb_ib_j\right)$$

$$= \det(AA^t)$$

2)

let $A_1 = \begin{bmatrix} \cos\alpha \end{bmatrix}$

$$A_2 = \begin{bmatrix} \cos\alpha & 1 \\ 1 & 2\cos\alpha \end{bmatrix}$$

for $n > 2$

$$A_n = n\left\{ \underbrace{\begin{bmatrix} \cos\alpha & 1 & & & \\ 1 & 2\cos\alpha & 1 & & \\ & & \ddots & & \\ 0 & & 1 & 2\cos\alpha & 1 \\ & & & 1 & 2\cos\alpha \end{bmatrix}}_{n} \right.$$

let $B_1 = \begin{bmatrix} 1 \end{bmatrix}$

$$B_2 = \begin{bmatrix} \cos\alpha & 0 \\ 1 & 1 \end{bmatrix}$$

for $n > 2$

$$B_n = n\left\{ \underbrace{\begin{bmatrix} \cos\alpha & 1 & & & \\ 1 & 2\cos\alpha & 1 & & \\ & & \ddots & & \\ & & 1 & 2\cos\alpha & 0 \\ & & & 1 & 1 \end{bmatrix}}_{n} \right.$$

$\det(A_{n+1}) = 2\cos\alpha \det(A_n) - \det(B_n)$

(Expanding along the bottom row)

$\det(B_{n+1}) = \cancel{\det(A_n)} \det(A_n) - 0$

as cofactor of $[n+1, n]$ ~~...~~ has the last column as 0

$[\cancel{B_{n+1}}]$

$\det(A_2) = 2\cos^2\alpha - 1 = \cos 2\alpha$

$\det(A_1) = \cos\alpha$ $\longrightarrow$ $\therefore$ for $n = 1, 2$ it is true

$\det(B_1) = \cos 0 = 1$

By Induction hypothesis it

$\det(A_i) = \cos(i\alpha)$ for $i \in \{1, \dots, n\}$

Then

$\det(A_{n+1}) = 2\cos\alpha \cos n\alpha - \det(B_n)$

$\det(B_n) = \det(A_{n-1}) = \cos(n-1)\alpha$

$\therefore \det(A_{n+1}) = 2\cos\alpha \cos n\alpha - \cancel{\det} \cos(n-1)\alpha$

$= \cos\alpha \cos n\alpha - \sin\alpha \sin n\alpha = \cos(n+1)\alpha$

$\therefore$ for $n+1$ it is true

$\therefore$ By Induction we are done

Q3)

RHS = $\langle x+y, x+y\rangle - \langle x-y, x-y\rangle$
$+ i\langle x+iy, x+iy\rangle - i\langle x-iy, x-iy\rangle$

$= \cancel{2\langle x,y\rangle + 2\langle y,x\rangle}$

$= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle$
$- \langle x,x\rangle + \langle y,x\rangle + \langle x,y\rangle - \langle y,y\rangle$
$+ i\langle x,x\rangle + i\langle x,iy\rangle + i\langle iy,x\rangle + i\langle iy,iy\rangle$
$- i\langle x,x\rangle - i\langle -iy,iy\rangle + i\langle iy,x\rangle + i\langle x,iy\rangle$

$= 2\langle x,y\rangle + 2\overline{\langle x,y\rangle} - 2i^2\langle x,y\rangle$
$+ 2i^2\overline{\langle x,y\rangle}$

$= 4\langle x,y\rangle$

Q4)

$$\text{Proj}_u(v) = \frac{\langle u, v\rangle}{\langle u, u\rangle}\vec{u}$$

v_1, v_2, v_3, v_4 are given

$u_1 = v_1 = [1, 1, 0, 0]$

$u_2 = v_2 - \left(\frac{1}{2}\right)v_1 = \left[\frac{1}{2}, -\frac{1}{2}, 1, 0\right]$

$$u_3 = v_3 - \frac{\langle v_3, u_2\rangle}{\langle u_2, u_2\rangle}u_2 - \frac{\langle v_3, u_1\rangle}{\langle u_1, u_1\rangle}u_1$$

$$= v_3 - \frac{\left(\frac{1}{2}\right)}{\left(\frac{3}{2}\right)}u_2 - \frac{u_1}{2}$$

$$= [1, 0, 0, 1] + \left[-\frac{1}{6}, +\frac{1}{6}, -\frac{1}{3}, 0\right] + \left[-\frac{1}{2}, -\frac{1}{2}, 0, 0\right]$$

$$= \left[\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, 1\right]$$

$$u_4 = v_4 - \frac{\left(-\frac{2}{3}\right)}{\left(\frac{4}{3}\right)}u_3 - \frac{\left(\frac{1}{2}\right)}{\left(\frac{3}{2}\right)}u_2 - \frac{u_1}{2}$$

$$u_4 = \underset{[0,1,1,0]}{v_4} + \left[\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}, \frac{1}{2}\right] + \left[-\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}, 0\right] + \left[-\frac{1}{2}, -\frac{1}{2}, 0, 0\right]$$

$$= \left[-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right]$$

As $u_1, u_2, u_3, u_4 \neq 0$ and orthogonal $\therefore$ $u_5 = u_6 = 0$

$\downarrow$

Gives an orthogonal Basis

Ps (7)

Q8) C^5

$$\{\underbrace{[1, i, 0, 0, 0]}_{v_1}, \underbrace{[0, 1, i, 0, 0]}_{v_2}, \underbrace{[0, 0, 1, i, 0]}_{v_3}, \underbrace{[0, 0, 0, 1, i]}_{v_4}\}$$

$$w_1 = v_1 = [1, i, 0, 0, 0]$$

$$w_2 = v_2 - \left(\frac{v_2^* w_1}{w_1^* w_1}\right) w_1$$

$$= v_2 - \frac{i}{2} w_1 = \frac{1}{2}[i, 1, 2i, 0, 0]$$

$$w_3 = v_3 - \left(\frac{v_3^* w_2}{w_2^* w_2}\right) w_2 - \left(\frac{v_3^* w_1}{w_1^* w_1}\right) w_1$$

$$= \frac{1}{3}[-1, i, 1, 3i, 0]$$

and we set

$$w_4 = \frac{1}{4}[-i, -1, i, 1, 4i]$$

cs

Q8)

$$\{[1, i, 0, 0, 0], [0, 1, i, 0, 0], [0, 0, 1, i, 0], [0, 0, 0, 1, i]\}$$

$$\underset{v_1}{[1, i, 0, 0, 0]} \quad \underset{v_2}{[0, 1, i, 0, 0]} \quad \underset{v_3}{[0, 0, 1, i, 0]} \quad \underset{v_4}{[0, 0, 0, 1, i]}$$

$$w_1 = v_1 = [1, i, 0, 0, 0]$$

$$w_2 = v_2 - \left(\frac{v_2^* w_1}{w_1^* w_1}\right) w_1$$

$$= v_2 - \frac{i}{2} w_1 = \frac{1}{2}[i, 1, 2i, 0, 0]$$

$$w_3 = v_3 - \left(\frac{v_3^* w_2}{w_2^* w_2}\right) w_2 - \left(\frac{v_3^* w_1}{w_1^* w_1}\right) w_1$$

$$= \cancel{v_3 - \frac{i}{\frac{3}{2}} w_2 - 0}$$

$$= \cancel{[0, 0, 1, i, 0] - \frac{i}{3}[i, 1, 2i, 0, 0]}$$

$$= \cancel{\frac{1}{3}[}$$

$$= \frac{1}{3}[-1, i, 1, 3i, 0]$$

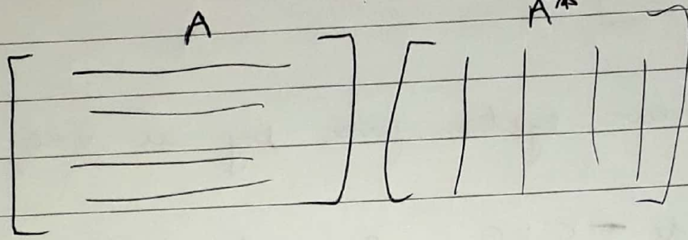
and we set

$$w_4 = \frac{1}{4}[-i, -1, i, 1, 4i]$$

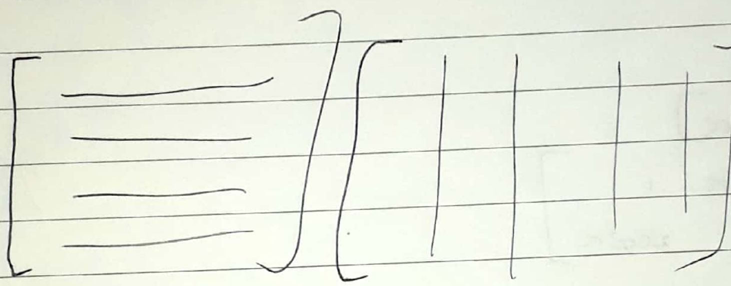
5 Rows are orthonormal

$\Rightarrow AA^* = I$

A $\qquad\qquad$ A^*



$\Rightarrow A^*A = I$



A^T $\qquad\qquad$ A

6 $x = (1,0,0)$

$$\begin{vmatrix} 1/\sqrt{3} & 1/\sqrt{6} & 1 \\ i/\sqrt{3} & -2i/\sqrt{6} & 0 \\ -1/\sqrt{3} & -1/\sqrt{6} & 0 \end{vmatrix} \neq 0$$

$\therefore$ x is not in span of u, v.

Now apply Gram Schmidt.

$u' = x - \langle x, v\rangle v - \langle x, w\rangle w$

Normalize u' to get a vector orthonormal to v, w.

Alternate method

Let $u = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Solve the linear system given by $u \cdot v = 0$, $u \cdot w = 0$

$$a + ib + c = 0, \quad a - 2ib - c = 0$$

Find a solution and normalize it.