MA 108 - Ordinary Differential Equations

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Outline of the lecture

- Integrating factors
- Bernoulli equation
- Orthogonal Trajectories
- Lipschitz continuity

Warm up!

1. The value of b that makes $(xy^2 + bx^2y)dx + (x + y)x^2dy = 0$ exact is ....
2. The value of r for which the DE $y'' + y' - 6y = 0$ has solutions of the form $y = e^{rt}$ are ....
   (Ans. $(r^2 + r - 6)e^{rt} = 0$. So, $r = 2, -3$)
3. The solution of $-ydx + (x + \sqrt{xy})dy = 0$ is ....
4. The solution of $y' = e^{3x} - y$ is ....
   (Ans. $ye^x = \int Q(x)e^x\, dx + C$ where $Q(x) = e^{3x}$)
5. Let $L := x^2\dfrac{d^2}{dx^2} + 2x\dfrac{d}{dx} + 2$ be a differential operator. $L(x^3)$ is ....
   (Ans. $L(x^3) = 6x^3 + 6x^3 + 2x^3 = 14x^3$)

In the last Lecture we looked at the ODE

$$(3x + y^2)dx + (x^2 + xy)dy = 0$$

and found that ODE is not exact.
Question is how to solve first order ODEs which are not in exact form?

Integrating Factors

Suppose the first order ODE

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

is not exact; that is, $M_y \neq N_x$. In this situation, we try to find a function $\mu(x, y)$ such that

$$\mu \cdot M + \mu \cdot N\frac{dy}{dx} = 0$$

is exact; i.e.,

$$(\mu \cdot M)_y = (\mu \cdot N)_x.$$

Thus,

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x.$$

That is, $\mu(x, y)$ satisfies the DE

$$\boxed{\mu_y M - \mu_x N + (M_y - N_x)\mu = 0}.$$

Such a function $\mu(x, y)$ is called an integrating factor of the given ODE.

In practice, we start by looking for an IF which depends only on one variable x or y, because it may be difficult to solve the PDE $\mu_y M - \mu_x N + (M_y - N_x)\mu = 0$.

Case 1 :
Suppose $\boxed{\mu \text{ is a function of } x \text{ alone}}$. That is, $\mu = \mu(x), \mu_y = 0$.
Then, the PDE above reduces to

$$\mu_x N = (M_y - N_x)\,\mu.$$

Thus,

$$\frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N}\right)\mu.$$

If further, $\boxed{\frac{M_y - N_x}{N} \text{ is a function of } x}$ then the above DE is separable & we try to solve it to find $\mu(x)$.

$$\boxed{\mu(x) = e^{\int \frac{M_y - N_x}{N}\,dx}.}$$

Integrating Factor - function of y alone

Case 2 :

If we assume $\boxed{\mu \text{ to be a function of } y \text{ alone}}$ in the PDE

$$\mu_y M - \mu_x N + (M_y - N_x)\mu = 0,$$

then we get an analogous equation:

$$\frac{d\mu}{dy} = \left(\frac{N_x - M_y}{M}\right)\mu.$$

If further, $\boxed{\frac{N_x - M_y}{M} \text{ is a function of } y}$ then the above DE is separable & we try to solve it to find $\mu(y)$.

$$\boxed{\mu(y) = e^{\int \frac{N_x - M_y}{M}\, dy}.}$$

Example 1

Solve the ODE:

$$(8xy - 9y^2) + (2x^2 - 6xy)\frac{dy}{dx} = 0.$$

Let $M = 8xy - 9y^2$ and $N = 2x^2 - 6xy$.
Thus, $M_y = 8x - 18y$ and $N_x = 4x - 6y$. As $M_y \neq N_x$, the given ODE is not exact.
We first try to find an IF depending only upon one variable.
Note that

$$\frac{M_y - N_x}{N} = \frac{4x - 12y}{2x(x - 3y)} = \frac{2}{x}, \text{ a function of } x \text{ alone.}$$

Hence by the earlier discussion, we have:

$$\frac{d\mu}{dx} = \frac{2}{x}\mu.$$

Solving this separable ODE, we get $\ln|\mu| = \ln x^2$. Hence,
$\boxed{\mu(x) = x^2}$ can be chosen as an IF for the given ODE.

Multiplying the given ODE by $\mu(x) = x^2$, we get:

$$(8x^3y - 9x^2y^2) + (2x^4 - 6x^3y)\frac{dy}{dx} = 0.$$

Check that this is an exact ODE. (How?)
To solve this exact ODE, we need to find $u(x, y)$ such that

$$8x^3y - 9x^2y^2 = u_x \ \& \ 2x^4 - 6x^3y = u_y.$$

To find $u(x, y)$:
Step I: $u(x, y) = 2x^4y - 3x^3y^2 + k(y)$.
Step II: $2x^4 - 6x^3y = u_y = 2x^4 - 6x^3y + k'(y)$.
Thus, $k'(y) = 0$. Hence,

$$u(x, y) = 2x^4y - 3x^3y^2 = c$$

is a solution of the given ODE.

Example 2

Solve the DE: $-y + x\dfrac{dy}{dx} = 0$.
Check that this is not an exact DE.
Let $M(x,y) = -y$ and $N(x,y) = x$.
To find a possible IF μ: note that $\dfrac{N_x - M_y}{M} = -\dfrac{2}{y}$, a function of y alone.
By the earlier discussion, we obtain :

$$\frac{d\mu}{dy} = -\frac{2}{y}\mu.$$

Thus, $\ln|\mu| = -2\ln|y|$.
So we choose

$$\mu(y) = \frac{1}{y^2}$$

as an IF.
Then, $\dfrac{-y + x\dfrac{dy}{dx}}{y^2} = 0$ is exact. Thus, $d\left(-\dfrac{x}{y}\right) = 0$.
Therefore, solution is given by $\boxed{\dfrac{x}{y} = c.}$

Bernoulli equation

Consider

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \qquad (n = 0,\ 1 \text{ yields linear equations!})$$

For $n = 0$, one can verify that

$$\mu(x) = e^{\int P(x)dx}$$

is an integrating factor.
Now multiply be the integrating factor and solve we get

$$y = e^{-\int P(x)dx}\Big(\int Q(x)e^{\int P(x)dx}dx + c\Big).$$

For $n = 1$ a similar analysis

$\boxed{\text{Claim :}}$ Let $n \neq 0, 1$.

Then the transformation $\boxed{v = y^{1-n}}$ reduces the Bernoulli equation to a linear equation in v. [Leibniz (1696)]

$\boxed{\text{Justification :}}$

Let $v = y^{1-n}$.

$$\frac{dv}{dx} = (1-n)\, y^{\cancel{1}-n-\cancel{1}}\, \frac{dy}{dx}$$

That is,

$$\frac{dy}{dx} = \frac{1}{1-n}\, y^{n}\, \frac{dv}{dx}.$$

Substituting in the DE,

$$\frac{1}{1-n}\, y^n \,\frac{dv}{dx} + P(x)y = Q(x)y^n$$
$$\frac{1}{1-n}\,\frac{dv}{dx} + P(x)v = Q(x) \;\; (\text{assuming } y \neq 0)$$

Hence,

$\dfrac{dv}{dx} + (1-n)P(x)v = Q(x)(1-n),$ which is a linear DE in v.

Example - Bernoulli

Solve : $\dfrac{dy}{dx} + y = xy^3$.

Let $\boxed{v = y^{-2}}$.

$\dfrac{dv}{dx} = -2y^{-3}\dfrac{dy}{dx} \implies -\dfrac{1}{2}\dfrac{dv}{dx} + v = x$

That is, $\dfrac{dv}{dx} - 2v = -2x$ (linear equation in v)

Integrating factor is e^{-2x}.

$$\begin{aligned} ve^{-2x} &= -\int 2xe^{-2x}\,dx + C \\ &= \frac{2xe^{-2x}}{-2} - \int \cancel{2}\frac{e^{-2x}}{\cancel{2}} + C \\ &= xe^{-2x} + \frac{e^{-2x}}{2} + C \end{aligned}$$

$$\implies \frac{1}{y^2} = x + \frac{1}{2} + Ce^{2x}.$$

Consider

$$\boxed{\frac{d}{dy}(f(y))\frac{dy}{dx} + P(x)f(y) = Q(x),}$$

where f is an unknown function of y.
Set $\boxed{v = f(y)}$.
Then,

$$\frac{dv}{dx} = \frac{dv}{dy} \cdot \frac{dy}{dx} = \frac{d}{dy}(f(y))\frac{dy}{dx}.$$

Hence the given equation is

$$\frac{dv}{dx} + P(x)v = Q(x), \quad \text{which is linear in } v.$$

Remark : Bernoulli DE is a special case when $f(y) = y^{1-n}$.

Example

Solve : $\cos y \dfrac{dy}{dx} + \dfrac{1}{x} \sin y = 1.$

Set $\boxed{v = \sin y}$.

Then,

$$\frac{dv}{dx} = \frac{dv}{dy} \cdot \frac{dy}{dx} = \cos y \frac{dy}{dx}.$$

Hence the given equation is

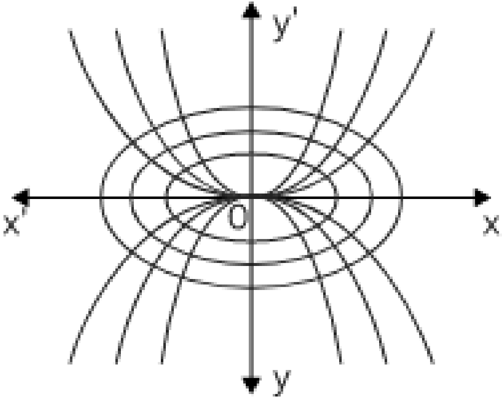
$$\frac{dv}{dx} + \frac{1}{x} v = 1, \quad \text{which is linear in } v.$$

That is,

$$\begin{aligned} e^{\int \frac{1}{x}\, dx} v(x) &= \int e^{\int \frac{1}{x}\, dx} dx + C \\ \Longrightarrow x\, v(x) &= \frac{x^2}{2} + C \\ \sin y &= \frac{x}{2} + \frac{C}{x}. \end{aligned}$$

Orthogonal Trajectories

If two families of curves always intersect each other at right angles, then they are said to be orthogonal trajectories of each other.



To find the OT of a family of curves

$$F(x, y, c) = 0.$$

An example :

$$F(x, y, c) = x^2 - 4cy = 0$$

defines a family of parabolas given in the previous slide.

$$G(x, y, c) = x^2 + 2y^2 - 2c^2 = 0$$

defines the family of ellipses given in the previous slide.

Working Rule

OT of a family of curves $F(x, y, c) = 0$.

- Find the DE $\frac{dy}{dx} = f(x, y)$.
  How ?
  Differentiate $F(x, y, c) = 0$ and elliminate the parameter c
- Slopes of the OT's are given by

$$\frac{dy}{dx} = -\frac{1}{f(x, y)}.$$

  (Only for those (x, y) for which $f(x, y) \neq 0$)
- Obtain a one parameter family of curves $G(x, y, c) = 0$ as solutions of the above DE.

(Leaving a part certain trajectories that are vertical lines!)

Example

Find the set of OT's of the family of circles $x^2 + y^2 = c^2$.

$$x + y\frac{dy}{dx} = 0 \Longrightarrow \frac{dy}{dx} = -\frac{x}{y}$$

The slope of OT's are $\dfrac{dy}{dx} = \dfrac{y}{x}\ (x \neq 0) \Longrightarrow y = kx$.
Hence the orthogonal trajectories are given by $\boxed{y = kx}$.

Definitions

1. Let f be a real function defined on D, where D is either a region (or termed as domain) or a closed region of the xy plane. The function f is said to be bounded in D if there exists a positive number M such that
$$|f(x, y)| \leq M$$
for all (x, y) in D.
2. Let f be defined and continuous on a closed rectangle $R : a \leq x \leq b, \; c \leq y \leq d$. Then, f is bounded in R.
3. Let f be defined on D, where D is either a domain or a closed domain of the xy- plane. The function f is said to satisfy Lipschitz condition (with respect to y) in D if $\exists$ a constant $M > 0$ such that
$$|f(x, y_1) - f(x, y_2)| \leq M|y_1 - y_2|$$
for every $(x, y_1), (x, y_2)$ which belong to D. The constant M is called the Lipschitz constant.

Understanding the Lipschitz condition - $y = g(x)$

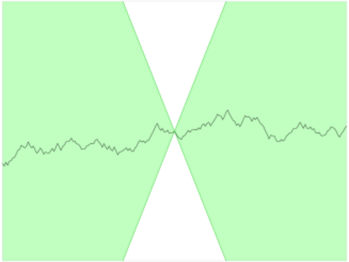
Consider

$$|g(x_2) - g(x_1)| \leq M|x_2 - x_1| \ \ \forall x_1, \ x_2 \text{ in the domain of } g.$$

This condition in the form $\dfrac{|g(x_2) - g(x_1)|}{|x_2 - x_1|} \leq M$ can be interpreted as follows :

At each point $(a, g(a))$, the entire graph of g lies between the lines

$$y = g(a) - M(x - a) \ \& y = g(a) + M(x - a).$$



Example : x^2 is Lipschitz in $[1, 2]$.

Understanding Lipschitz condition - $z = f(x, y)$

- Let (x, y_1) and (x, y_2) be any two points in D having the same abscissa x.
- Consider the corresponding points

$$P_1(x, y_1, f(x, y_1)) \;\&\; P_2(x, y_2, f(x, y_2))$$

  on the surface $z = f(x, y)$, and let α $(0 \leq \alpha \leq \pi/2)$ denote the angle that the chord joining P_1 and P_2 makes with the xy-plane.
- Then if the condition

$$|f(x, y_1) - f(x, y_2)| \leq M|y_1 - y_2|$$

  holds in D, then $\tan\alpha$ is bounded in absolute value.
- That is, the chord joining P_1 and P_2 is bounded away from being perpendicular to the xy- plane.
- Further, this bound is independent of the points (x, y_1) and (x, y_2) belonging to D.

If f satisfies Lipschitz condition with respect to y in D, then for each fixed x, the resulting function of y is a continuous function of y, for all (x, y) in D.

Example : Let $f(x, y) = y + [x]$ where $g(x) = [x]$ is the greatest integer function. For fixed x,

$$\begin{aligned} f(x, y_1) - f(x, y_2) &= y_1 + [x] - y_2 - [x] \\ &= y_1 - y_2 \end{aligned}$$

That is, $|f(x, y_1) - f(x, y_2)| = |y_1 - y_2| \leq 1 \cdot |y_1 - y_2|$
But we know that f is discontinuous w.r.t. x for every integral value of x.

Note that the condition of Lipschitz continuity implies nothing concerning the continuity of f with respect to x.

Does Continuity w.r.t. second variable $\Longrightarrow$ Lipschitz condtn. w.r.t. second variable?

Continuity w.r.t. second variable DOES NOT imply Lipschitz condtn. w.r.t. second variable.

Example : Consider $f(x, y) = \sqrt{|y|}$.

f is continuous for all y.
Note that f doesn't satisfy Lipschitz condition in any region that includes $y = 0$ as for $y_1 = 0,\ y_2 > 0$, we have

$$\frac{|f(x, y_1) - f(x, y_2)|}{|y_1 - y_2|} = \frac{\sqrt{y_2}}{|y_2|} = \frac{1}{\sqrt{y_2}}$$

which can be made as large as we want by making y_2 smaller.
The Lipschitz condition requires that the quotient should be bounded by a fixed constant M.