Q1) $\hat{a}, \hat{b}$ are unit vectors in $\mathbb{R}^3$

From the definition of cross product

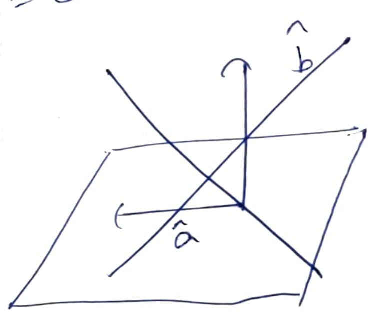
$\hat{a} \times \hat{x}$ (If $\hat{x}$ is not parallels to $\hat{a}$)

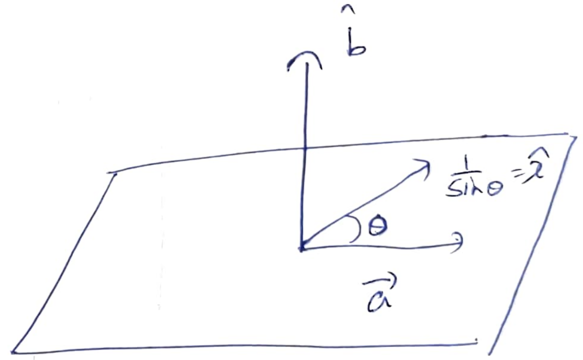
is a vector which ~~lies~~ is perpendicular to the plane containing $\hat{a}$, and $\hat{x}$.

$\therefore$ If a solution exists then $\hat{a}$ and $\hat{b}$ are perpendicul.

and if $\hat{b}$ is not perpendicular to $\hat{a}$ no solution exists.

Case
$\hat{a} \cdot \hat{b} = 0$





For different values of θ (~~#~~ which are not integral multiples of π) we get different ~~values~~ solutions.

$|\hat{a}| \, |\hat{x}| (\sin\theta) = |\hat{b}| \quad : \quad |\hat{x}| = \frac{1}{\sin\theta}$

Q2)

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Claim $\quad A^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}$

Proof $\quad n=1$ it is true, say it is true for $n=k$

Then $\quad A^{k+1} = A^k \, A = \begin{bmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$= \begin{bmatrix} \cos(k+1)\theta & -\sin(k+1)\theta \\ \sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}$$

we are done by induction.

as $|P|=1$ let $x_1 = \cos\alpha$, $x_2 = \sin\alpha$.

Suppose $(A^{n_1})P = (A^{n_2})P$ for $n_1, n_2 \in \mathbb{N}$, $n_1 \neq n_2$

Then $\begin{bmatrix} \cos n_1\theta & -\sin n_1\theta \\ \sin n_1\theta & \cos n_1\theta \end{bmatrix} \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} = \begin{bmatrix} \cos(n_1\theta+\alpha) \\ \sin(n_1\theta+\alpha) \end{bmatrix}$

$\therefore A^{n_1}P = A^{n_2}P \iff n_1\theta + \alpha = n_2\theta + \alpha + 2n_3\pi$

$n_3 \in \mathbb{Z}$

$\therefore \theta = \dfrac{2n_3\pi}{(n_1-n_2)}$ $\quad \therefore \theta$ is a rational multiple of π.

$\therefore$ If θ is not a rational multiple of π then all $A^n P$ is are distinct and the set is infinite. If θ is a rational multiple of π. $\exists\ n \in \mathbb{N}$ such that $n\theta = 2k\pi$ for some $k \in \mathbb{Z}$.

$$\therefore (A^{x+n})P = (A^x)(A^n)P = (A^x P)$$

$\cancel{A^n = Id}$ (As $A^n = Id_n$)

$\therefore$ The set is finite

$\cancel{\{P, AP, \ldots, A^{n-1}B\}}$

Q3)

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a_1 & b_1 & c_1 \\ b_1 & a_2 & b_2 \\ c_1 & b_2 & a_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\therefore = a_1 x^2 + a_2 y^2 + a_3 z^2 + 2b_1 xy + 2b_2 yz + 2c_1 xz$$

$$\therefore a_1 = 1, \quad a_2 = 1, \quad a_3 = -1, \quad b_1 = \frac{7}{2}, \quad b_2 = -\frac{3}{2}, \quad c_1 = 3$$

And A is unique if it is symmetric.

If not then it is not symmetric as

$$\begin{bmatrix} x & y & z \end{bmatrix} B \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$ also ~~is~~ gives the same expression.

where $B = \begin{bmatrix} 1 & 7 & 6 \\ 0 & 1 & -3 \\ 0 & 0 & -1 \end{bmatrix}$

Q4)

$A(\mathrm{Adj}(A)) = (\det(A))\, Id$.

$\therefore$ If $\det A \neq 0$ then $A(\underbrace{(\det(A))^{-1}\,\mathrm{Adj}(A)}_{A^{-1}}) = Id$

If ~~AB = Id~~ $\therefore$ A is invertible

If $\det A = 0$

Then A cannot be invertible by contradiction

as $\det(AB) = \det(I)$

$\Rightarrow 0 = 1 \Rightarrow\Leftarrow$.

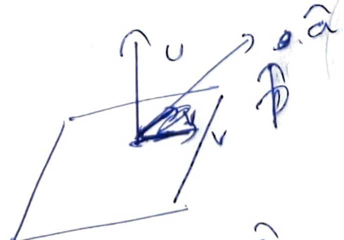
$A = I - uu^T$ is not invertible

as $Au = u - u(u^Tu) = 0$ $\quad u \neq 0$

$(A^{-1}A)u = u = A^{-1}(Au) = 0 \Rightarrow\Leftarrow$.

$f(u) = -u$

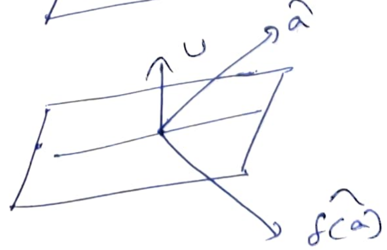
$f(v) = v$



$\hat{a} = \hat{b} + \hat{c}$

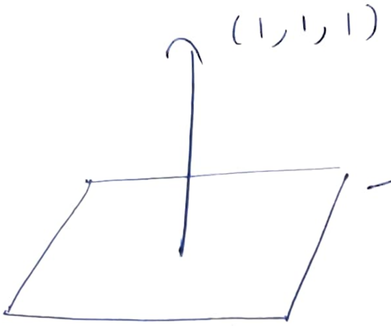
$\downarrow$ $\parallel$ to u $\quad$ $\downarrow$ $\perp$ to u

$f(a) = -\hat{b} + \hat{c}$



~~Q4)~~

Q5)



plane
$\rightarrow P = (x+y+z=0)$

take some vector $\hat{a}_1 = (1,-1,0)$ on P

Now let $\hat{b}_1 = (1,1,x)$ on P

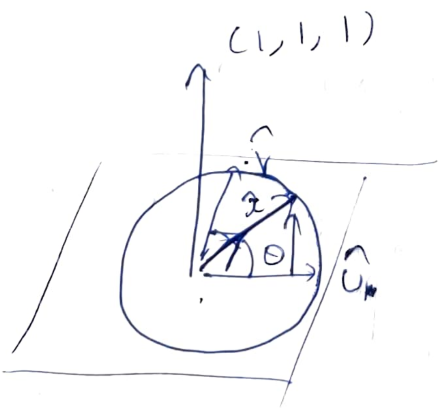
then $1+1+x=0 \quad \therefore x=-2$

clearly $\hat{a}_1 \cdot \hat{b}_1 = 0$.

By normalizing $\hat{u}_1 = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right)$

$\hat{v} = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)$

are such vectors



$\hat{x} = \hat{u}_1 \cos\theta + \hat{v}\sin\theta$

Gives a parametrization