Theory of Electromagnetic Waves

The Wave Equation

- The following equation is called the wave equation

$$\nabla^2 f - \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = 0 \tag{1}$$

  which denotes the propagation of a wave through the given medium with velocities $\pm$v. f is the quantity, often called the wave function, which oscillates. It could be the displacement of a string from its equilibrium position, or the density of a fluid, etc.

- It can be shown in a mathematatcally rigorous manner that the solution of Eq. 1 is of the form

$$f = g(r - vt) + h(r + vt),$$

  where g and h are two vector functions. The exact nature of g/h depends upon the physical situation, initial values, and the boundary values.

Wave Eqn (Contd.)

- Let us assume that the wave is propagating with a speed v in the positive z direction so that the wave equation is

$$\frac{\partial^2 \mathrm{f}}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 \mathrm{f}}{\partial t^2} = 0.$$

- One possible solution for this equation, called the sinusoidal wave, is of the form

$$\mathrm{f}(\mathrm{r}, t) = \mathrm{A}\cos(k(z - vt) + \delta),$$

where A is the amplitude, $k = 2\pi/\lambda$ is the wave vector, and δ is the phase. Obviously $kv = 2\pi v/\lambda = 2\pi\nu = \omega$

- So that

$$\mathrm{f}(\mathrm{r}, t) = \mathrm{A}\cos(kz - \omega t + \delta)$$

- The 3D version of this solution is

$$\mathrm{f}(\mathrm{r}, t) = \mathrm{A}\cos(\mathrm{k}\cdot\mathrm{r} - \omega t + \delta),$$

where now we have $\mathrm{k} = (2\pi/\lambda)\hat{\mathrm{k}}$, where $\hat{\mathrm{k}}$ denotes the direction of the propagation of the wave.

- We can alternatively write

$$\mathrm{f}(\mathrm{r}, t) = \mathrm{Re}(\mathrm{A} e^{i(\mathrm{k}\cdot\mathrm{r} - \omega t + \delta)})$$

where $\mathrm{Re}(z)$ denotes the real part of the complex number z.
- Or we introduce the complex function $\tilde{\mathrm{f}}(\mathrm{r}, t)$

$$\tilde{\mathrm{f}}(\mathrm{r}, t) = \tilde{\mathrm{A}} e^{i(\mathrm{k}\cdot\mathrm{r} - \omega t)})$$

in which $\tilde{\mathrm{A}} = \mathrm{A} e^{i\delta}$ has absorbed the phase constant, and obviously $\mathrm{f}(\mathrm{r}, t) = \mathrm{Re}(\tilde{\mathrm{f}}(\mathrm{r}, t))$.
- It is obvious that $\tilde{\mathrm{f}}(\mathrm{r}, t)$ will also satisfy the wave equation, and because of its simpler form, we will work with it.
- Direction of oscillations, i.e., the direction of A is called the polarization direction of the wave. If polarization and propagation directions are the same, the wave is called longitudinally polarized. If they are mutually pependicular it is said to be transversely polarized.

- Let us consider free space, with no charges or currents ($\rho = 0$, $\mathrm{J} = 0$), so that the Maxwell's Equations take the form

$$\nabla \cdot \mathrm{E} = 0$$
$$\nabla \cdot \mathrm{B} = 0$$

and

$$\nabla \times \mathrm{E} = -\frac{\partial \mathrm{B}}{\partial t}$$
$$\nabla \times \mathrm{B} = \mu_0 \varepsilon_0 \frac{\partial \mathrm{E}}{\partial t}$$

- We take the curl of the second set of Maxwell's Equations, and use the curl equations again on the RHS

$$\nabla \times \nabla \times \mathrm{E} = -\nabla \times \frac{\partial \mathrm{B}}{\partial t} = -\frac{\partial (\nabla \times \mathrm{B})}{\partial t} = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathrm{E}}{\partial t^2} \tag{2}$$

$$\nabla \times \nabla \times \mathrm{B} = \nabla \times \mu_0 \varepsilon_0 \frac{\partial \mathrm{E}}{\partial t} = \mu_0 \varepsilon_0 \frac{\partial (\nabla \times \mathrm{E})}{\partial t} = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathrm{B}}{\partial t^2} \tag{3}$$

- On the LHS of Eqs 2 and 3 we use the relation $\nabla \times \nabla \times \mathrm{V} = \nabla(\nabla \cdot \mathrm{V}) - \nabla^2 \mathrm{V}$, and the Maxwell Eqs. $\nabla \cdot \mathrm{E} = \nabla \cdot \mathrm{B} = 0$ to obtain

$$\nabla^2 \mathrm{E} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathrm{E}}{\partial t^2} = 0, \tag{4}$$

$$\nabla^2 \mathrm{B} - \mu_0 \varepsilon_0 \frac{\partial^2 \mathrm{B}}{\partial t^2} = 0, \tag{5}$$

- Comparing 4 and 5 with Eq. (1), we conclude that E and B follow the wave equation with speed $v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3.0 \times 10^8 \mathrm{m/s}$. In other words, electro-magnetic field can not only exist independent of charges and currents in the free space, it can also propagate as a wave with the speed of light. That is a remarkable result
- This derivation of Maxwell showed that EM waves are possible, and Heinrich Hertz demonstrated their existence for the first time experimentally in his lab.

Monochromatic Plane EM Waves

- We look for sinusoidal solutions of EM wave equations (Eqs 4 and 5) of the form

$$\tilde{\mathbf{E}}(\mathbf{r}, t) = \tilde{\mathbf{E}}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{6}$$

$$\tilde{\mathbf{B}}(\mathbf{r}, t) = \tilde{\mathbf{B}}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{7}$$

  with the understanding that the physical solutions correspond to the real parts of these fields.
- This solution is called monochromatic because it has only one frequency ω. Recall that the frequency ω determines the colour of light.
- When the amplitudes $\tilde{\mathbf{E}}_0/\tilde{\mathbf{B}}_0$ are constants, the wave is called a plane wave. It can be shown that $\tilde{\mathbf{E}}(\mathbf{r}, t)$ and $\tilde{\mathbf{B}}(\mathbf{r}, t)$ are uniform in a plane perpendicular to the direction of propagation $\hat{\mathbf{k}}$. This can be seen easily by choosing $\hat{\mathbf{k}}$ to be along a Cartesian axis.

- Let us substitute Eqs. 6 and 7 into the divergence Maxwell's equations
- Easy to see that

$$\nabla \cdot \tilde{\mathrm{E}}(\mathrm{r}, t) = 0 \implies i\mathrm{k} \cdot \tilde{\mathrm{E}} = 0 \implies \mathrm{k} \cdot \tilde{\mathrm{E}}_0 = 0$$

- Similarly $\nabla \cdot \tilde{\mathrm{B}} = 0$ leads to

$$\mathrm{k} \cdot \tilde{\mathrm{B}}_0 = 0$$

- Thus both the electric and magnetic fields of the EM wave are perpendicular to the direction of propagation. Thus EM waves are transversely polarized.
- Let us substitute Eq. 6 in $\nabla \times \mathrm{E} = -\frac{\partial \mathrm{B}}{\partial t}$, and integrate to obtain B.

- Easy to show that for $\tilde{\mathrm{E}}$ given by Eq. 6

$$\nabla \times \tilde{\mathrm{E}} = i\{(\tilde{E}_{0z}k_y - \tilde{E}_{0y}k_z)\hat{i} + (\tilde{E}_{0x}k_z - \tilde{E}_{0z}k_z)\hat{j} + (\tilde{E}_{0y}k_x - \tilde{E}_{0x}k_y)\hat{k}\}e^{i(\mathrm{k}\cdot\mathrm{r}-\omega t)}$$

$$\nabla \times \tilde{\mathrm{E}} = i\mathrm{k} \times \tilde{\mathrm{E}}_0 e^{i(\mathrm{k}\cdot\mathrm{r}-\omega t)} = -\frac{\partial \tilde{\mathrm{B}}}{\partial t}$$

- Upon integrating the differential equation for $\tilde{\mathrm{B}}$, we easily obtain

$$\tilde{\mathrm{B}}(\mathrm{r},t) = \frac{\mathrm{k}}{\omega} \times \tilde{\mathrm{E}}_0 e^{i(\mathrm{k}\cdot\mathrm{r}-\omega t)} = \frac{\mathrm{k}}{\omega} \times \tilde{\mathrm{E}}(\mathrm{r},t) = \frac{\hat{\mathrm{k}}}{c} \times \tilde{\mathrm{E}}(\mathrm{r},t),$$

where we used the relation $k/\omega = c$.

- From the equation above we conclude that for EM waves electric field, magnetic field, and the direction of propagation are all mutually perpendicular. So E and B are in a mutually perpendicular direction in a plane perpendicular to k.

- We also obtain the relation between the E and B amplitudes

$$\tilde{B}_0 = \frac{\tilde{E}_0}{c}$$

- By convention the direction of the electric field defines the direction of polarization of the EM wave.
- The real parts of the filed are given as

$$\mathrm{E}(\mathrm{r}, t) = E_0 \cos(\mathrm{k} \cdot \mathrm{r} - \omega t + \delta)\hat{\mathrm{n}}$$
$$\mathrm{B}(\mathrm{r}, t) = \frac{E_0}{c} \cos(\mathrm{k} \cdot \mathrm{r} - \omega t + \delta)(\hat{\mathrm{k}} \times \hat{\mathrm{n}})$$

where $\hat{\mathrm{n}}$ is the polarization vector.

- Energy density for a plane wave

$$u = \frac{1}{2}(\varepsilon_0 E^2 + \frac{B^2}{\mu_0})$$

- But

$$B^2 = \frac{E^2}{c^2} = \varepsilon_0 \mu_0 E^2 \implies \frac{B^2}{\mu_0} = \varepsilon_0 E^2$$

- So that
$$u = \varepsilon_0 E^2 = \varepsilon_0 E_0^2 \cos^2(\mathrm{k}\cdot\mathrm{r} - \omega t + \delta)$$
- And the Poynting vector
$$\mathrm{S} = \frac{\mathrm{E}\times\mathrm{B}}{\mu_0} = \frac{E_0^2}{c\mu_0}\cos^2(\mathrm{k}\cdot\mathrm{r} - \omega t + \delta)(\hat{\mathrm{n}}\times(\hat{\mathrm{k}}\times\hat{\mathrm{n}}))$$
$$= c\varepsilon_0 E_0^2 \cos^2(\mathrm{k}\cdot\mathrm{r} - \omega t + \delta)\hat{\mathrm{k}} = cu\hat{\mathrm{k}}$$
- Above we used the identity
$$\mathrm{A}\times(\mathrm{B}\times\mathrm{C}) = \mathrm{B}(\mathrm{A}\cdot\mathrm{C}) - \mathrm{C}(\mathrm{A}\cdot\mathrm{B})$$

  along with the fact that $\hat{\mathrm{n}}\cdot\hat{\mathrm{k}} = 0$.
- We note that the Poynting vector, which denotes the energy flux, is along the direction of propagation of the wave. This is consistent with the fact that energy flows with the wave along the direction of propagation.

- Intensity associated with the EM wave is defined as the time average of the Poynting vector

$$I = \langle |\mathrm{S}| \rangle = \frac{1}{2} c \varepsilon_0 E_0^2$$

EM Waves through a linear medium

- Maxwell's equations in a charge and current free medium are

$$\nabla \cdot D = 0$$
$$\nabla \cdot B = 0$$
$$\nabla \times E = -\frac{\partial B}{\partial t}$$
$$\nabla \times B = \mu \frac{\partial D}{\partial t}$$

- For a linear medium $D = \varepsilon E$, and $B = \mu H$, where ε/μ are permittivity/permeability of the medium

- So for these systems the Maxwell's equations become

$$\nabla \cdot \mathrm{E} = 0$$
$$\nabla \cdot \mathrm{B} = 0$$
$$\nabla \times \mathrm{E} = -\frac{\partial \mathrm{B}}{\partial t}$$
$$\nabla \times \mathrm{B} = \mu\varepsilon\frac{\partial \mathrm{E}}{\partial t}$$

- Following the same procedure as for free space, the EM wave equations will be

$$\nabla^2\mathrm{E} - \mu\varepsilon\frac{\partial^2 \mathrm{E}}{\partial t^2} = 0, \tag{8}$$
$$\nabla^2\mathrm{B} - \mu\varepsilon\frac{\partial^2 \mathrm{B}}{\partial t^2} = 0, \tag{9}$$

- So that wave speed in the medium is $v = \sqrt{1/\mu\varepsilon}$, or $v/c = \sqrt{\mu_0\varepsilon_0/\mu\varepsilon} = 1/n$, where n is the refractive index of the medium, leading to the result

$$n = \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}}$$

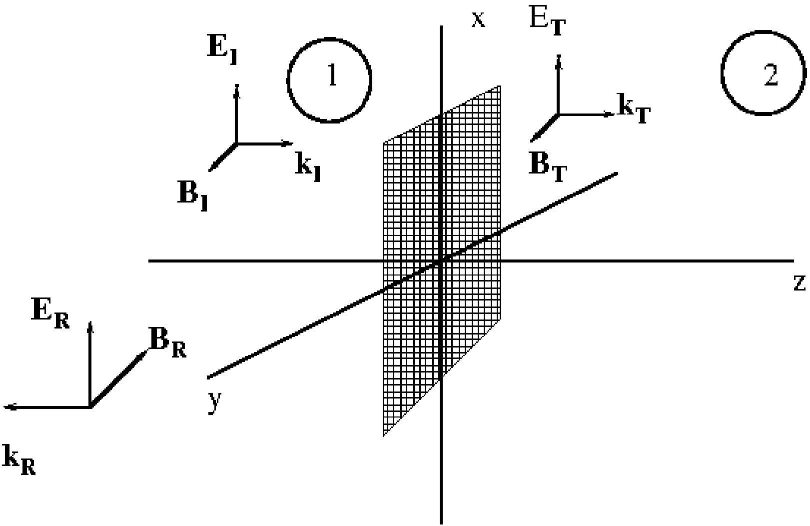
- We will similarly obtain the following results for this case

$$u = \varepsilon E^2 = \varepsilon E_0^2 \cos^2(\mathrm{k}\cdot\mathrm{r} - \omega t + \delta)$$

$$\mathrm{S} = \frac{\mathrm{E}\times\mathrm{B}}{\mu} = vu\hat{\mathrm{k}}$$

$$I = \frac{1}{2}\varepsilon v E_0^2$$

Case of Normal Incidence on an Interface

- 

- The wave is incident from left from a medium (1) with parameters (ε_1, μ_1), and the meshed portion shows the interface with medium (2) with parameters (ε_2, μ_2). So the wave speeds in the two media are $v_1 = 1/\sqrt{\varepsilon_1 \mu_1}$, $v_2 = 1/\sqrt{\varepsilon_2 \mu_2}$. The interface is the plane $z = 0$.
- Incident wave is assumed to travel along $+z$ direction, and is polarized along the $+x$ direction. Because, the angle of incidence is 90^0, the reflected wave travels in $-z$ direction, and transmitted wave in $+z$ direction.
- Polarizations of the reflected and transmitted waves are shown in the figure.
- The incident wave with wave vector $\mathrm{k}_I = k_1 \hat{\mathrm{z}}$ is described by

$$\tilde{\mathrm{E}}_I(\mathrm{r}, t) = \tilde{E}_{0I} e^{i(k_1 z - \omega t)} \hat{\mathrm{x}} \tag{10}$$

$$\tilde{\mathrm{B}}_I(\mathrm{r}, t) = \frac{\tilde{E}_{0I}}{v_1} e^{i(k_1 z - \omega t)} \hat{\mathrm{y}} \tag{11}$$

- The reflected wave is represented by

$$\tilde{\mathrm{E}}_R(\mathrm{r},t) = \tilde{E}_{0R} e^{i(-k_1 z - \omega t)} \hat{\mathrm{x}} \tag{12}$$

$$\tilde{\mathrm{B}}_R(\mathrm{r},t) = -\frac{\tilde{E}_{0R}}{v_1} e^{i(-k_1 z - \omega t)} \hat{\mathrm{y}} \tag{13}$$

- And the transmitted wave with wave vector $\mathrm{k}_T = k_2 \hat{\mathrm{z}}$

$$\tilde{\mathrm{E}}_T(\mathrm{r},t) = \tilde{E}_{0T} e^{i(k_2 z - \omega t)} \hat{\mathrm{x}} \tag{14}$$

$$\tilde{\mathrm{B}}_T(\mathrm{r},t) = \frac{\tilde{E}_{0T}}{v_2} e^{i(k_2 z - \omega t)} \hat{\mathrm{y}} \tag{15}$$

- Note that all the three waves have a common frequency, why? Because frequency is a property of the wave originating in its source, so it cannot change.
- Boundary conditions at the interface are

$$E_1^{||} = E_2^{||}$$

$$\frac{B_1^{||}}{\mu_1} = \frac{B_2^{||}}{\mu_2}$$

- Applying the boundary conditions at the interface ($z = 0$)

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T} \tag{16}$$

$$\frac{1}{\mu_1 v_1}(\tilde{E}_{0I} - \tilde{E}_{0R}) = \frac{\tilde{E}_{0T}}{\mu_2 v_2}$$

- The second equation can be rewritten as

$$\tilde{E}_{0I} - \tilde{E}_{0R} = \beta \tilde{E}_{0T} \tag{17}$$

where $\beta = (\mu_1 v_1/\mu_2 v_2) = (\mu_1 n_2/\mu_2 n_1)$.

- Eqs. 16 and 17 can be solved to yield

$$\tilde{E}_{0R} = \left(\frac{1-\beta}{1+\beta}\right)\tilde{E}_{0I}, \qquad \tilde{E}_{0T} = \left(\frac{2}{1+\beta}\right)\tilde{E}_{0I}$$

- We will use an approximation henceforth. Most of the materials through which light can transmit are nonmagnetic so that their magnetic permeabilities are approximately that of the vacuum. So we use

$$\mu_1 \approx \mu_2 \approx \mu_0$$

- With this

$$\beta \approx \frac{v_1}{v_2} = \frac{n_2}{n_1}$$

- And

$$\tilde{E}_{0R} \approx \left(\frac{n_1 - n_2}{n_1 + n_2}\right)\tilde{E}_{0I}, \qquad \tilde{E}_{0T} \approx \left(\frac{2n_1}{n_1 + n_2}\right)\tilde{E}_{0I}$$

- So it is obvious when $n_2 > n_1$, i.e., light is traveling from a rarer medium to a denser one, the reflected wave undergoes a 180^0 phase change.
- Real transmitted and reflected amplitudes are given as

$$E_{0R} \approx \left|\frac{n_1 - n_2}{n_1 + n_2}\right| E_{0I}, \qquad E_{0T} \approx \left(\frac{2n_1}{n_1 + n_2}\right) E_{0I}$$

- Given that the intensity of the wave is $I = \frac{1}{2}\varepsilon v E_0^2$, the reflection coefficient of the system, which is the ratio of the reflected intensity to the incident intensity is given by

$$R = \frac{I_R}{I_I} = \left(\frac{E_{0R}}{E_{0I}}\right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2$$

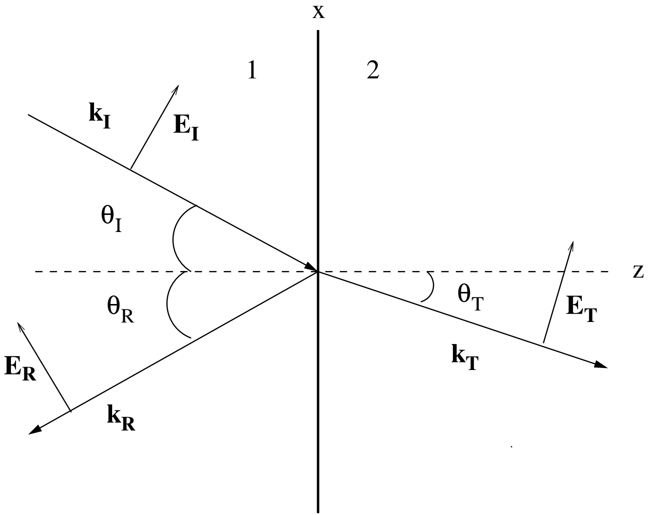
- And the transmission coefficient, which is the ratio of the transmitted intensity to that of the incident one is given by

$$\begin{aligned} T = \frac{I_T}{I_I} &= \left(\frac{\varepsilon_2 v_2 E_{0T}^2}{\varepsilon_1 v_1 E_{0I}^2}\right) \\ &= \left(\frac{\varepsilon_2 \sqrt{\varepsilon_1 \mu_1}}{\varepsilon_1 \sqrt{\varepsilon_2 \mu_2}}\right)\left(\frac{2n_1}{n_1 + n_2}\right)^2 \\ &= \sqrt{\frac{\varepsilon_2 \mu_1}{\varepsilon_1 \mu_2}}\left(\frac{4n_1^2}{(n_1 + n_2)^2}\right) \approx \left(\frac{v_1}{v_2}\right)\left(\frac{4n_1^2}{(n_1 + n_2)^2}\right) \end{aligned}$$

where we used $\mu_1 \approx \mu_2 \approx \mu_0$.

- Or
$$T = \left(\frac{n_2}{n_1}\right)\left(\frac{4n_1^2}{(n_1+n_2)^2}\right) = \frac{4n_1 n_2}{(n_1+n_2)^2}$$
- We note the important result that
$$R + T = 1,$$
which is a consequence of the conservation of energy.
- Let us consider a light beam incident from air into glass so that $n_1 = 1$ and $n_2 = 1.5$. We obtain $R = 0.04$ and $T = 0.96$, consistent with the fact that most of light is transmitted through the glass.

The Case of Oblique Incidence



- We assume oblique incidence as shown in the figure above.

- The incident, reflected, and transmitted beams, respectively, will be represented by

$$\tilde{\mathrm{E}}_I(\mathrm{r},t) = \tilde{\mathrm{E}}_{0I} e^{i(\mathrm{k}_I\cdot\mathrm{r}-\omega t)}, \qquad \tilde{\mathrm{B}}_I(\mathrm{r},t) = \frac{1}{v_1}(\hat{\mathrm{k}}_I \times \tilde{\mathrm{E}}_I)$$

$$\tilde{\mathrm{E}}_R(\mathrm{r},t) = \tilde{\mathrm{E}}_{0R} e^{i(\mathrm{k}_R\cdot\mathrm{r}-\omega t)}, \qquad \tilde{\mathrm{B}}_R(\mathrm{r},t) = \frac{1}{v_1}(\hat{\mathrm{k}}_R \times \tilde{\mathrm{E}}_R)$$

$$\tilde{\mathrm{E}}_T(\mathrm{r},t) = \tilde{\mathrm{E}}_{0T} e^{i(\mathrm{k}_T\cdot\mathrm{r}-\omega t)}, \qquad \tilde{\mathrm{B}}_T(\mathrm{r},t) = \frac{1}{v_2}(\hat{\mathrm{k}}_T \times \tilde{\mathrm{E}}_T)$$

- Because, all the three waves share a common frequency, the following relation holds

$$\omega = k_I v_1 = k_R v_1 = k_T v_2$$

- so that

$$k_I = k_R = \frac{v_2}{v_1} k_T = \frac{n_1}{n_2} k_T$$

- Now that the wave is incident obliquely, all the three waves will have both normal and tangential components for the E and B fields. Therefore, the boundary conditions (BCs) will be more complicated.

- However, we can extract a lot of information from the phase part of the BCs. The form of the BCs at the interface ($z = 0$) is

$$()e^{i(\mathbf{k}_I \cdot \mathbf{r} - \omega t)} + ()e^{i(\mathbf{k}_R \cdot \mathbf{r} - \omega t)} = ()e^{i(\mathbf{k}_T \cdot \mathbf{r} - \omega t)}$$

- This equality is possible only if the phase parts are equal. But the time-dependent part of the phase is trivially equal, therefore

$$\mathbf{k}_I \cdot \mathbf{r} = \mathbf{k}_R \cdot \mathbf{r} = \mathbf{k}_T \cdot \mathbf{r}, \quad \text{for } z = 0$$
$$\text{or } k_{Ix}x + k_{Iy}y = k_{Rx}x + k_{Ry}y = k_{Tx}x + k_{Ty}y$$

- Since x and y are independent variables, so the x and y components of $\mathbf{k}$ must be separately equal

$$k_{Ix} = k_{Rx} = k_{Tx}, \quad k_{Iy} = k_{Ry} = k_{Ty} \tag{18}$$

- If we orient the axis in a way such that $k_{Iy} = 0$, i.e., $\mathbf{k}_I$ lies in the xz plane, then so will $\mathbf{k}_R$ and $\mathbf{k}_T$.

- So we conclude that the incident wave, the reflected wave, and the transmitted wave all lie in the same plane, which also includes the normal to the surface (z axis here)
- From Eq. 18 we conclude

$$k_I \sin\theta_I = k_R \sin\theta_R \tag{19}$$

$$k_I \sin\theta_I = k_T \sin\theta_T \tag{20}$$

- Using the fact that $k_I = k_R$, and $k_I/k_T = v_2/v_1 = n_1/n_2$, we obtain from Eqs. 19 and 20

$$\sin\theta_I = \sin\theta_R \implies \theta_I = \theta_R$$

which is the law of reflection, and

$$\frac{\sin\theta_T}{\sin\theta_I} = \frac{n_1}{n_2}$$

which is nothing but Snell's law.

- Now that the phases are same for all the waves, the BCs for the normal and tangential components of E and B can be applied in a straight forward manner to the amplitudes. .
- Thus

$$\varepsilon_1(\tilde{\mathrm{E}}_{0I}+\tilde{\mathrm{E}}_{0R})_z=\varepsilon_2(\tilde{\mathrm{E}}_{0T})_z \tag{21}$$
$$(\tilde{\mathrm{B}}_{0I}+\tilde{\mathrm{B}}_{0R})_z=(\tilde{\mathrm{B}}_{0T})_z \tag{22}$$
$$(\tilde{\mathrm{E}}_{0I}+\tilde{\mathrm{E}}_{0R})_{x,y}=(\tilde{\mathrm{E}}_{0T})_{x,y} \tag{23}$$
$$\frac{1}{\mu_1}(\tilde{\mathrm{B}}_{0I}+\tilde{\mathrm{B}}_{0R})_{x,y}=\frac{1}{\mu_2}(\tilde{\mathrm{B}}_{0T})_{x,y} \tag{24}$$

- As per the figure, we have

$$\mathrm{k}_I=-k_I\sin\theta_I\hat{\mathrm{x}}+k_I\cos\theta_I\hat{\mathrm{z}}$$
$$\mathrm{k}_R=-k_I\sin\theta_I\hat{\mathrm{x}}-k_I\cos\theta_I\hat{\mathrm{z}}$$
$$\mathrm{k}_T=-k_T\sin\theta_T\hat{\mathrm{x}}+k_T\cos\theta_T\hat{\mathrm{z}}$$

- and

$$\tilde{\mathrm{E}}_{0I} = \tilde{E}_{0I}(\cos\theta_I \hat{\mathrm{x}} + \sin\theta_I \hat{\mathrm{z}})$$
$$\tilde{\mathrm{E}}_{0R} = \tilde{E}_{0R}(\cos\theta_I \hat{\mathrm{x}} - \sin\theta_I \hat{\mathrm{z}})$$
$$\tilde{\mathrm{E}}_{0T} = \tilde{E}_{0T}(\cos\theta_T \hat{\mathrm{x}} + \sin\theta_T \hat{\mathrm{z}})$$

- and using the relation $\tilde{\mathrm{B}} = \frac{1}{v}(\hat{\mathrm{k}} \times \tilde{\mathrm{E}})$ we obtain

$$\tilde{\mathrm{B}}_{0I} = \frac{\tilde{E}_{0I}}{v_1}\hat{\mathrm{y}}$$
$$\tilde{\mathrm{B}}_{0R} = -\frac{\tilde{E}_{0R}}{v_1}\hat{\mathrm{y}}$$
$$\tilde{\mathrm{B}}_{0T} = \frac{\tilde{E}_{0T}}{v_2}\hat{\mathrm{y}}$$

- We can now apply BCs of Eqs. 21–24 to obtain

$$\varepsilon_1(\tilde{E}_{0I} - \tilde{E}_{0R})\sin\theta_I = \varepsilon_2 \tilde{E}_{0T}\sin\theta_T \tag{25}$$

$$(\tilde{E}_{0I} + \tilde{E}_{0R})\cos\theta_I = \tilde{E}_{0T}\cos\theta_T \tag{26}$$

$$\frac{1}{\mu_1 v_1}(\tilde{E}_{0I} - \tilde{E}_{0R}) = \frac{1}{\mu_2 v_2}\tilde{E}_{0T}\sin\theta_T \tag{27}$$

- On using the Snell's law, Eqs. 25 and 27 yield the same equation

$$\tilde{E}_{0I} - \tilde{E}_{0R} = \beta \tilde{E}_{0T} \tag{28}$$

where $\beta = \mu_1 v_1/\mu_2 v_2$. While Eq. 26 yields

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \alpha \tilde{E}_{0T} \tag{29}$$

where $\alpha = \cos\theta_T/\cos\theta_I$.

- Eqs. 28 and 29 can be solved to obtain

$$\tilde{E}_{0R} = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)\tilde{E}_{0I}, \quad \tilde{E}_{0T} = \left(\frac{2}{\alpha+\beta}\right)\tilde{E}_{0I}.$$

- These are called Fresnel's equations for the polarization of the light in the plane of incidence. Note that for $\beta > \alpha$, reflected wave is out of phase with the incident wave.
- Furthermore

$$\alpha = \frac{\cos\theta_T}{\cos\theta_I} = \frac{\sqrt{1-\sin^2\theta_T}}{\cos\theta_I} = \frac{\sqrt{1-(n_1/n_2)^2\sin^2\theta_I}}{\cos\theta_I}.$$

- For normal incidence $\theta_I = 0^0$, $\alpha = 1$ and we recover the previous results.
- For $\theta_I = 90^0$ (grazing incidence) $\alpha \to \infty$, and there is no transmitted wave.

- For $\alpha = \beta$, the reflected wave vanishes. The angle of incidence for which it happens is called θ_B, the Brewster's angle.
- It is easy to see

$$\sin^2 \theta_B = \frac{1-\beta^2}{(n_1/n_2)^2 - \beta^2}$$

- For light incident from air into glass ($n_1 = 1.0$, $n_2 = 1.5$), $\theta_B \approx 56^0$.
- For oblique incidence, the power per unit area will be given by $\mathbf{S} \cdot \hat{\mathbf{z}}$. Thus the incident intensity $I_I = \frac{1}{2}\varepsilon_1 v_1 E_{0I}^2 \cos\theta_I$, reflected intensity $I_R = \frac{1}{2}\varepsilon_1 v_1 E_{0R}^2 \cos\theta_I$, and transmitted intensity $I_T = \frac{1}{2}\varepsilon_2 v_2 E_{0T}^2 \cos\theta_T$

- So the reflection coefficient will be

$$R = \frac{I_R}{I_I} = \frac{E_{0R}^2}{E_{0I}^2} = \left(\frac{\alpha - \beta}{\alpha + \beta}\right)^2$$

- And the transmission coefficient

$$T = \frac{I_T}{I_I} = \frac{\varepsilon_2 v_2 E_{0T}^2 \cos\theta_T}{\varepsilon_1 v_1 E_{0I}^2 \cos\theta_I} = \alpha\beta \left(\frac{2}{\alpha + \beta}\right)^2$$

- Easy to verify

$$R + T = 1$$