MA-111 Calculus II
(D1 & D2)

Lecture 11

Saurav Bhaumik

Department of Mathematics
Indian Institute of Technology Bombay
Powai, Mumbai - 76

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Recap

Let S be an oriented surface. Let Φ_1 and Φ_2 be two $\mathcal{C}^1$ non-singular parametrisations of S and let F be a continuous vector field on S.

- If Φ_1 and Φ_2 are orientation preserving, then

$$\iint_{\Phi_1} \mathsf{F} \cdot d\mathsf{S} = \iint_{\Phi_2} \mathsf{F} \cdot d\mathsf{S}.$$

- If Φ_1 is orientation preserving and Φ_2 is orientation reversing, then

$$\iint_{\Phi_1} \mathsf{F} \cdot d\mathsf{S} = -\iint_{\Phi_2} \mathsf{F} \cdot d\mathsf{S}.$$

For an oriented surface, the notation

$$\iint_S \mathsf{F} \cdot d\mathsf{S} = \iint_S \mathsf{F} \cdot \hat{\mathsf{n}}\, dS,$$

is unambiguous.

Homeomorphism

We now introduce the notion of 'Homeomorphism'.

Let ψ be function from $U_1 \subset \mathbb{R}^n$ to $U_2 \subset \mathbb{R}^m$.
We call the mapping $\psi : U_1 \longrightarrow U_2$ is a homeomorphism if ψ is continuous, bijective map such that $\psi^{-1} : U_2 \to U_1$ is also continuous.

Example. $U_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1, \quad z > 0\}$. Consider $U_1 = \{(x, y) \mid x^2 + y^2 < 1\} \subseteq \mathbb{R}^2$ and the mapping

$$\psi(x, y) = (x, y, \sqrt{1 - x^2 - y^2}), \quad \forall\, (x, y) \in U_1.$$

Then this is a homeomorphism. Check.

Example. The spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic unless $n = m$.

Example: The ball or radius $r > 0$, denoted $B(0, r) = \{\mathrm{x} \in \mathbb{R}^n : ||\mathrm{x}|| < r\}$ is homeomorphic to $\mathbb{R}^n$ via

$$\psi : B(0, r) \to \mathbb{R}^n,\ \psi(\mathrm{x}) = \frac{\mathrm{x}}{r - ||\mathrm{x}||},\ \psi^{-1}(\mathrm{y}) = \frac{r\mathrm{y}}{1 + ||\mathrm{y}||}.$$

Under homeomorphism many properties of a domain, are preserved.

• **Surfaces without boundary** A subset $S \subset \mathbb{R}^3$ is called a surface without boundary if for every point $P \in S$ there is a open subset $U \subseteq \mathbb{R}^3$ containing P such that $U \cap S$ is homeomorphic to $\mathbb{R}^2$.

• **Surfaces with boundary** A surface with boundary is a subset $S \subset \mathbb{R}^3$ such that for every $P \in S$ there is an open subset $U \subset \mathbb{R}^3$ containing P such that $U \cap S$ is homeomorphic to either $\mathbb{R}^2$ or the upper half plane $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$. A point $P \in S$ lies in the *boundary* ∂S if there is an open subset $U \subset \mathbb{R}^3$, and a homeomorphism $\psi : U \cap S \to \mathbb{H}$ such that $\psi(P) \in \partial\mathbb{H} = \{(x, y) \in \mathbb{R}^2 : y = 0\}$. The boundary ∂S is a curve.

• **Remark** The boundary ∂S defined as above is in general not the same as the set of boundary points. The open unit disk $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < 1, z = 0\}$ is a surface without boundary, but the set of boundary points is not empty. Indeed, the unit circle is the set of boundary points. If the surface S is a closed subset of $\mathbb{R}^3$, then the boundary ∂S defined above is indeed the set of boundary points. In general, $S - \partial S$ is the set of interior points.

Example: Let $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = a^2, \mid 0 \leq z \leq h\}$, be a cylinder of height h. Then ∂S is the union of the following two sets
$\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = a^2, \quad z = 0\}$ and
$\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = a^2, \quad z = h\}$.
Example: Let $S = \{(x, y, z) \mid x^2 + y^2 + z^2 = a^2\}$, a sphere. Then $\partial S = \varnothing = \text{empty}$. Why? A sphere, and a torus have no boundary.
What about an upper hemisphere?

Induced orientation of the boundary of a surface

Let S be an oriented surface with a boundary that is a disjoint union of simple, closed, piecewise non-singular parametrized curves. (For instance, the cylinder of height h or the upper hemisphere.)

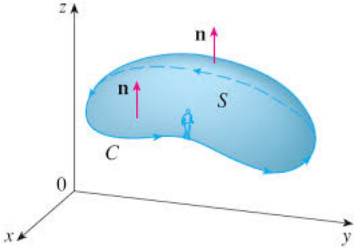
Let $\mathrm{n}(P)$ be the prescribed unit normal vector field defined at all interior points $P \in S$. An orientation on the interior of S induces a positive orientation on the boundary ∂S, which is a finite union of simple closed curves. This is called the *induced orientation* of the boundary.

Convention for the induced orientation

Two ways to remember the convention for the induced orientation, both similar to the situation in Green's theorem.

• The convention is that if you walk in the positive direction around ∂S with your head pointing in the direction of n, then the surface will always be on your left.

• The right hand rule is that the thumb points in the chosen normal direction n, and the fingers curl in the direction of the induced orientation of the boundary curve.



Note: If D is a path-connected subset on $\mathbb{R}^2$ and $\Phi : D \to \mathbb{R}^3$ is a smooth orientation-preserving parametrization of the surface S, then $\Phi(\partial D) = \partial S$ and the induced orientation of ∂S corresponds to the positive orientation of ∂D with respect to D.

Example. Let $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1, \quad z \geq 0\}$, the unit upper hemisphere. Let S be oriented by

$$\mathsf{n}(P) := (x, y, z), \quad \text{for} \quad P := (x, y, z) \in S.$$

Let $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ and $\Phi(x, y) := (x, y, \sqrt{1 - x^2 - y^2})$ for all $(x, y) \in D$. Note the boundary of the hemisphere S is the circle in x-y plane

$$\partial S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1, \quad z = 0\}.$$

The induced orientation ∂S by the oriented-parametrization Φ corresponds to the counter clock-wise orientation of

$$\partial D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

Stokes theorem

A surface S is called piecewise smooth if it is a finite union of smooth surfaces joining along smooth curves. Smooth here means n-times continuously differentiable for al n.

Theorem

1. *Let S be a bounded piecewise smooth oriented surface with non-empty boundary ∂S. Suppose S is a closed subset of $\mathbb{R}^3$.*
2. *Let ∂S, the boundary of S, be the disjoint union of simple closed curves each of which is a piecewise non-singular parametrized curve with the induced orientation.*
3. *Let $\mathsf{F} = F_1\mathsf{i} + F_2\mathsf{j} + F_3\mathsf{k}$ be a C^1 vector field defined on an open set containing S.*

Then

$$\int_{\partial S} \mathsf{F}.\mathsf{ds} = \int\int_S (\nabla \times \mathsf{F}).\mathsf{dS}.$$

Remark: It is sufficient to assume the surface is C^2 for this theorem.

Remarks

- If two different oriented surfaces S_1 and S_2 have the same boundary C, then it follows from Stokes theorem that

$$\int\int_{S_1} (\nabla \times \mathrm{F}).\mathrm{dS} = \int\int_{S_2} (\nabla \times \mathrm{F}).\mathrm{dS},$$

  where the surfaces are correctly oriented.

- Green's theorem in the tangential form is the special case of Stokes theorem for the planar regions.

Stokes theorem for closed surface

Corollary

Let S be a closed oriented smooth surface in $\mathbb{R}^3$ (and so $\partial S = \emptyset$). Suppose F *is a smooth vector field on an open subset containing S. Then*

$$\iint_S (curl\boldsymbol{F}) \cdot d\boldsymbol{S} = 0.$$

Proof: Introduce a hole in S by cutting out a small piece along a smooth simple closed curve C on S. Let S_1 denote the part of S cut out, and let S_2 denote the remaining part of S. Then

$$\iint_S (\text{curl}\boldsymbol{F}) \cdot d\boldsymbol{S} = \iint_{S_1} (\text{curl}\boldsymbol{F}) \cdot d\boldsymbol{S} + \iint_{S_2} (\text{curl}\boldsymbol{F}) \cdot d\boldsymbol{S}.$$

by the domain additivity.
Now the Stokes theorem shows that

$$\iint_{S_1} (\text{curl}\boldsymbol{F}) \cdot d\boldsymbol{S} = \int_{\partial S_1} \boldsymbol{F} \cdot d\boldsymbol{s} \text{ and } \iint_{S_2} (\text{curl}\boldsymbol{F}) \cdot d\boldsymbol{S} = \int_{\partial S_2} \boldsymbol{F} \cdot d\boldsymbol{s}.$$

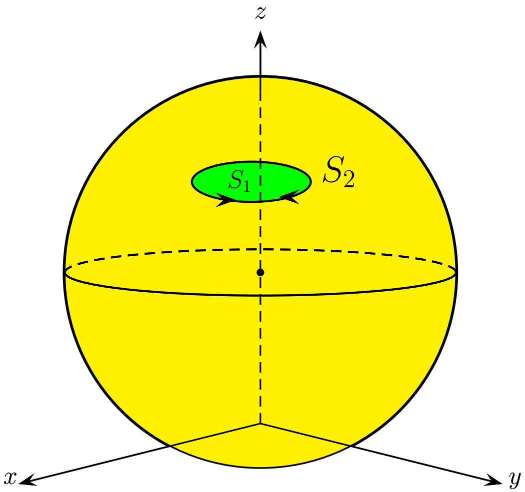


Figure: The Stokes theorem for S with $\partial S = \emptyset$.

We observe that the boundary ∂S_1 of S_1 is the closed curve C with the orientation induced by the orientation on S_1.

Since $\partial S = \emptyset$, the boundary ∂S_2 of S_2 is also C. But the orientations induced on C by the orientations on S_1 and on S_2 are opposite.

Hence

$$\iint_S (\text{curl}\boldsymbol{F}) \cdot d\boldsymbol{S} = \int_{\partial S_1} \boldsymbol{F} \cdot d\boldsymbol{s} + \int_{\partial S_2} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_C \boldsymbol{F} \cdot d\boldsymbol{s} - \int_C \boldsymbol{F} \cdot d\boldsymbol{s} = 0.$$

$\square$

Examples.

Example Calculate

$$\oint_C ydx + zdy + xdz,$$

where C is the intersection of the surface $bz = xy$ and the cylinder $x^2 + y^2 = a^2$, for $b \neq 0, a \neq 0$, oriented counter clockwise as viewed from a point high upon the positive z-axis.

Ans: We have

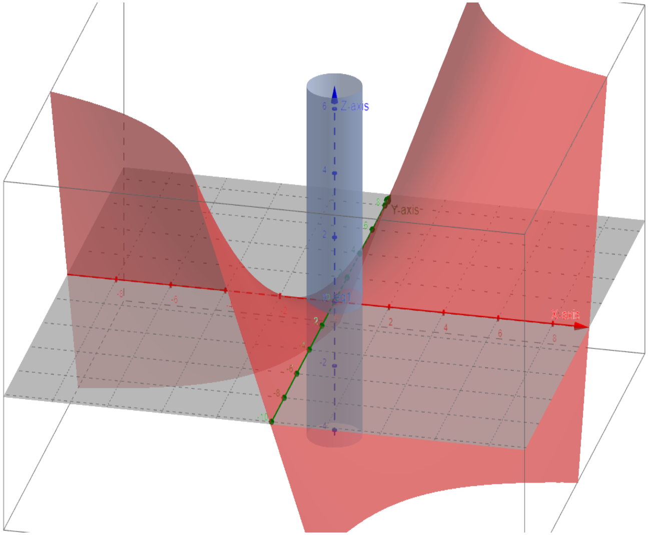
$$\mathsf{F} = y\mathsf{i} + z\mathsf{j} + x\mathsf{k} \quad \text{and} \quad \mathrm{curl}\ \mathsf{F} = -(\mathsf{i} + \mathsf{j} + \mathsf{k}).$$

Parametrize the surface lying on the hyperbolic paraboloid $z = xy/b$ and bounded by the curve C as

$$S = \{(x, y, z) \mid \mathbb{R}^3 \mid x^2 + y^2 \leq a^2, \quad z = \frac{xy}{b}\}$$

Then $\mathsf{n}\, dS = (-\frac{y}{b}, -\frac{x}{b}, 1)dxdy$ and

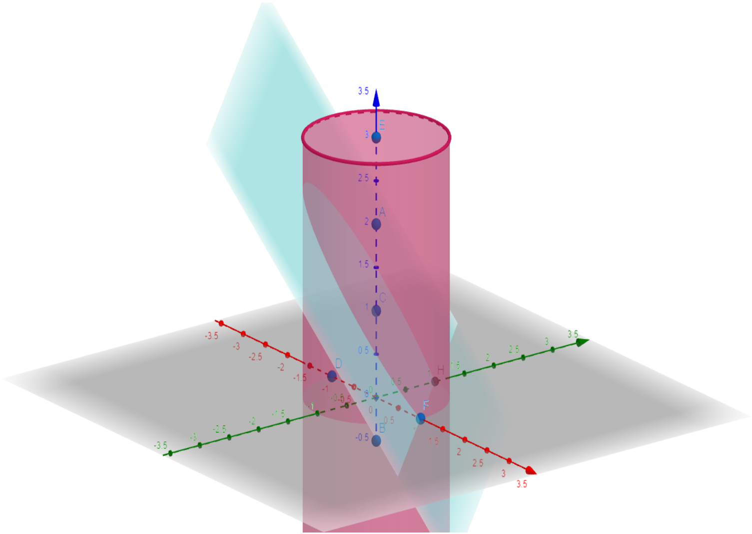
$$\iint_S \mathrm{curl}\ \mathsf{F} \cdot \mathsf{n}\, dS = \frac{1}{b} \iint_{x^2+y^2 \leq a^2} (y + x - b)dxdy$$
$$= \frac{1}{b} \int_0^{2\pi} \int_0^a (r \sin\theta + r\cos\theta - b)rdrd\theta = -\pi a^2.$$


Z-axis
Y-axis
X-axis

Examples: Homework

Example Let C be the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 1$. Let C be oriented so that when it is projected onto the xy-plane the resulting curve is traversed counterclockwise. Evaluate

$$\int_C -y^3 dx + x^3 dy - z^3 dz.$$


E
A
C
D
H
F
B

A more involved example

Example Evaluate the surface integral

$$\iint_S (\nabla \times \mathsf{F}) \cdot \mathsf{n} dS$$

where S is the portion of the surface of a sphere defined by $x^2 + y^2 + z^2 = 1$ and $x + y + z \geq 1$, $\mathsf{F} = \mathsf{r} \times (\mathsf{i} + \mathsf{j} + \mathsf{k})$ and $\mathsf{r} = x\mathsf{i} + y\mathsf{j} + z\mathsf{k}$. The outward normal is used to orient S.

How does one proceed? One can do this directly as a surface integral or use Stokes' theorem but in either case the evaluation is quite tedious.

Idea: Change the surface, keeping the boundary (and its orientation) unchanged!

Example contd.

With this idea in mind, we let C be the curve of intersection of the sphere and the plane $x + y + z = 1$, and we let S_1 be the region of this plane enclosed by C which is just a disc. We have to make sure that we orient S_1 so that C has the same orientation as in the given problem. The normals to S_1 are given by

$$\mathsf{n}_1 = \pm \frac{1}{\sqrt{3}}(1, 1, 1).$$

Which normal should we take for orienting S_1? Clearly $\frac{1}{\sqrt{3}}(1, 1, 1)$. Now $\nabla \times \mathsf{F} = -2\mathsf{i} - 2\mathsf{j} - 2\mathsf{k}$ and $(\nabla \times \mathsf{F}) \cdot \mathsf{n}_1 = -2\sqrt{3}$. Hence

$$\iint_{S_1} (\nabla \times \mathsf{F}) \cdot d\mathsf{S} = \iint_{S_1} -2\sqrt{3} dS = -2\sqrt{3} A(S_1)$$

where $A(S_1)$ is the surface area of the surface S_1 which we can easily compute!

Consequences of Stokes theorem

Proposition

Let $\mathbf{F}$ be a smooth vector field on an open subset D of $\mathbb{R}^3$ such that curl$\mathbf{F} = \mathbf{0}$ on D.

1. *Suppose S is a bounded oriented piecewise C^2 surface in D, and let ∂S denote its boundary with the induced orientation, as in the Stokes theorem. Then $\int_{\partial S} \mathbf{F} \cdot d\mathbf{s} = 0$.*

   In particular, if $\partial S = C_1 \cup (-C_2)$, so that C_1 and $-C_2$ have the induced orientation, then $\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_{C_2} \mathbf{F} \cdot d\mathbf{s}$.

2. *If F is a vector field defined on $\mathbb{R}^3$, then $\mathbf{F}$ is a gradient field on D.*

Gauss's divergence theorem

- How to generalize Green's theorem in the normal form to 3 dimensions?

- Closed surfaces We define a closed surface S in $\mathbb{R}^3$ to be a surface which is bounded, whose complement is open and boundary of S is empty.

Examples: 1. Sphere, 2. the union of two concenctric spheres of different radii, 3. ellipsoid.

(This is analogous to the closed curve.)

- If S is a connected closed surface, then it encloses a bounded 3-dimensional region. Call it W, and then $S = \partial W$ is the boundary surface.

This is analogous to a simple closed curve being boundary of a region D in $\mathbb{R}^2$.

- Let us consider a region W in $\mathbb{R}^3$ which is simultaneously Type 1, Type 2, Type 3 and the boundary of the region as a subset of $\mathbb{R}^3$ is a closed surface. We call such region in $\mathbb{R}^3$ as simple solid region.

Examples regions bounded by ellipsoids, spheres, or rectangular boxes are simple solid regions.

Gauss's divergence theorem

If W is a simple solid region, W is a closed and bounded region in $\mathbb{R}^3$.

Theorem (Gauss's Divergence Theorem)

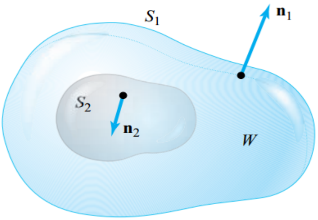
1. *Let W be a simple solid region of $\mathbb{R}^3$ whose boundary $S = \partial W$ is a closed surface.*
2. *Suppose ∂W is positively oriented.*
3. *Let $\boldsymbol{F}$ be a smooth vector field on an open subset of $\mathbb{R}^3$ containing W.*

Then

$$\iint_{\partial W} \boldsymbol{F} \cdot d\boldsymbol{S} = \iiint_W (div\boldsymbol{F})dxdydz.$$

• Clearly, the importance of Gauss's theorem is that it converts surface integrals to volume integrals and vice-versa. Depending on the context one may be easier to evaluate than the other.

• We can extend Gauss's theorem to any region that can be broken up into symmetric elementary regions. This includes all regions of interest to us. An example of a region to which Gauss's theorem applies is the region between two closed surfaces, one inside the other. The surface of this region consists of two pieces oriented as shown in the following figure.



Consequences of Gauss' theorem

Theorem

1. *Let W be a simple solid region in $\mathbb{R}^3$ with positively oriented boundary ∂W.*
2. *Let F be a smooth vector field on an open set in $\mathbb{R}^3$ containing W satisfying $div\,\mathsf{F} = 0$ on W.*

Then, the following holds: $\displaystyle\int\int_{\partial W} \mathsf{F}.\mathsf{dS} = 0.$

Using the above theorem, we have following result:

Corollary

Let F be a vector field defined on $\mathbb{R}^3$. If $div\,\mathsf{F} = 0$, then

$$\iint_{S_1} \mathsf{F} \cdot \mathsf{dS} = \iint_{S_2} \mathsf{F} \cdot \mathsf{dS}$$

whenever S_1, S_2 are oriented surfaces in $\mathbb{R}^3$ with $\partial S_1 = \partial S_2$ and S_1 and S_2 does not intersect each other except along their common boundary and there exists a region W in $\mathbb{R}^3$ with boundary $S_1 \cup S_2$ satisfying the hypothesis in Gauss divergence theorem.

The flux of a vector field $\boldsymbol{F}$ across an oriented surface S is

$$\int\int_S \boldsymbol{F} \cdot d\mathsf{S}.$$

Example 1 Calculate the flux of $\mathsf{F}(x, y, z) = x^3\mathsf{i} + y^3\mathsf{j} + z^3\mathsf{k}$ through the unit sphere (oriented outwards/positively).

Solution: Using Gauss's theorem, we see that we need only evaluate

$$\iint_{\partial W} \mathsf{F} \cdot d\mathsf{S} = \iiint_W (\nabla \cdot \mathsf{F}) dV = \iiint_W 3(x^2 + y^2 + z^2) dxdydz,$$

where W is the unit ball.
This problem is clearly ideally suited to the use of spherical coordinates.
Making a change of variables, we get

$$\int_0^{2\pi} \int_0^{\pi} \int_0^1 3\rho^4 \sin\phi \, d\rho d\phi d\theta = \frac{12\pi}{5}$$

Examples

Example 2: Let $\mathsf{F} = 2x\mathsf{i} + y^2\mathsf{j} + z^2\mathsf{k}$, and let S be the unit sphere. Calculate $\iint_S \mathsf{F} \cdot d\mathsf{S}$.

Solution: Using Gauss' theorem we see that

$$\iint_S \mathsf{F} \cdot d\mathsf{S} = \iiint_W (\nabla \cdot \mathsf{F}) dV,$$

where W is the unit ball bounded by the sphere. Since $\nabla \cdot \mathsf{F} = 2(1 + y + z)$ we get

$$2 \iiint_W (1 + y + z) dV = 2 \iiint_W dV + 2 \iiint_W y dV + 2 \iiint_W z dV.$$

Notice that the last two integrals above are 0, by symmetry. Hence, the flux is simply

$$2 \iiint_W dV = \frac{8\pi}{3}.$$

The flux in physics

- Fluid flow If F is the velocity field desribing the flow of a fluid with density 1, then the flux represents the rate of flow through the surface S (in units of mass per unit time).

- Electricity If E is an electric field, then the surface integral $\int\int_S \mathsf{E} \cdot d\mathsf{S}$ is called the *electric flux* of E through the surface S. Then Gauss's Law says that the net charge enclosed by a *closed* surface S is given by

$$Q = \varepsilon_0 \int\int_S \mathsf{E} \cdot d\mathsf{S},$$

where ε_0 is the permitivity of the free space.