MA-111 Calculus II (D1 & D2)

Lecture 12

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Recap: Consequences of Gauss' theorem

Let $\mathbf{F}$ be a C^1 vector field defined on $\mathbb{R}^3$, suppose $\operatorname{div}\mathbf{F} = 0$. Suppose there is a region W (satisfying the hypothesis of Gauss's theorem) whose boundary ∂W is the union of two smooth surfaces S_1 and S_2 that do not intersect each other except along their common boundary, which is C. Give S_1, S_2 orientations in such a way that the unit normal field for S_1 is away from the region W and the unit normal field of S_2 into the region W. Let us use the notation $\partial W = S_1 \cup (-S_2)$, where $-S_2$ means S_2 with the reverse orientation. Then

$$\iint_{S_1} \mathbf{F} \cdot \mathbf{dS} = \iint_{S_2} \mathbf{F} \cdot \mathbf{dS}$$

.

• This follows from Gauss's theorem, because

$$0 = \iiint_W div(\mathbf{F})\, dV = \iint_{\partial W} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} - \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}.$$

• This also follows from Stokes' theorem in nice situations. Assume that S_1, S_2 intersect along a picewise smooth curve C (which need not be connected). In most situations, the orientations induced on C by S_1 and S_2 will agree. If $div(\mathbf{F}) = 0$ on the whole of $\mathbb{R}^3$, there is some $\mathbf{G}$ such that $\mathbf{F} = curl(\mathbf{G})$, so the result follows from Stokes' theorem:

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \nabla \times \mathbf{G} \cdot d\mathbf{S} = \oint_C \mathbf{G} \cdot d\mathbf{r} = \iint_{S_2} \nabla \times \mathbf{G} \cdot d\mathbf{S} = \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}.$$

• Remember this typical example. Take the closed lower hemisphere $S_1 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, z \leq 0\}$ with the outward orientation (call it 'the bowl') and the disc $S_2 = \{(x, y, 0) \in \mathbb{R}^3 : x^2 + y^2 \leq 1\}$ with the *downward* orientation (call it 'the lid'). Then they both induce the clockwise orientation (as seen from the positive z-axis) on their intersection circle $C = \{(x, y, 0) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$.

Examples

Example 1 Let $\boldsymbol{F}(x, y, z) := (y, -x, e^{xz})$ for $(x, y, z) \in \mathbb{R}^3$, and let $S := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + (z)^2 = 1 \text{ and } z \geq 0\}$, be oriented by the outward unit normal vectors. Find $\iint_S (\text{curl}\boldsymbol{F}) \cdot d\boldsymbol{S}$.

Method 1: Replacing the surface

Since $\text{div} \cdot \text{curl} = 0$, by the above observation, we can replace the oriented surface S with the disc $D = \{(x, y, 0) : x^2 + y^2 \leq 1\}$ with the orientation given by *upward* unit normal $= \mathbf{k}$. We calculate $curl(\mathbf{F}) \cdot \mathbf{k} = -2$. Therefore,

$$\iint_S curl(\mathbf{F}) \cdot d\mathbf{S} = \iint_D curl(\mathbf{F}) \cdot d\mathbf{S} = \iint_D curl(\mathbf{F}) \cdot \mathbf{k} dS = -2 \iint_D dS = -2\pi.$$

Example 1 continued

Method 2: Stokes' theorem

Note

$$\partial S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1 \text{ and } z = 0\}$$

is anticlockwise as seen from the point $(0, 0, 4)$.

∂S is parametrized by $\mathbf{c}(t) = (\cos t, \sin t, 0)$ for all $t \in [0, 2\pi]$ and hence the outward normal to the curve ∂S is $\mathbf{n}(t) = (-\sin t, \cos t, 0)$.

By the Stokes theorem,

$$\begin{aligned}
\iint_S (\text{curl}\boldsymbol{F})\cdot d\boldsymbol{S} &= \int_{\partial S} \boldsymbol{F}\cdot d\boldsymbol{s} \\
&= \int_{-\pi}^{\pi} (\sin t, -\cos t, e^{\cos t \cdot 0})\cdot(-\sin t, \cos t, 0)dt \\
&= \int_{-\pi}^{\pi} -(\sin^2 t + \cos^2 t)dt = -2\pi.
\end{aligned}$$

Examples of Stokes' theorem

Example 2 Find the integral of $\mathbf{F}(x, y, z) = z\mathbf{i} - x\mathbf{j} - y\mathbf{k}$ around the triangle with vertices $(0, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 2)$.

Ans To use Stokes' theorem for the given triangle C, consider S is the surface enclosed by the triangle C. Stokes' theorem says

$$\int_T \mathbf{F} \cdot d\mathbf{s} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} dS,$$

where $\mathbf{n}$ is the unit normal vector in the direction of positive orientation. The given triangle lies in the yz-plane. If the surface is to lie to the left of an observer walking around the triangle in the order described, the surface must be oriented so that the unit normal points in the direction of the positive x-axis. So $\mathbf{n} = \mathbf{i}$.
Calculating the curl of $\mathbf{F}$, we get

$$\nabla \times \mathbf{F} = -\mathbf{i} + \mathbf{j} - \mathbf{k}.$$

Hence,

$$(\nabla \times \mathbf{F}) \cdot \mathbf{n} = -1,$$

and

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} dS = -1 \times A(S) = -2.$$

Examples of flux

Example 3 Let $\mathbf{F} = (x^3, y^3, z^2)$, and consider the cylindrical volume $x^2 + y^2 \leq 9$, $0 \leq z \leq 2$. Find the flux with respect to the boundary surface with outward orientation.

Answer. For the surface we need three integrals. The top of the cylinder can be represented by $\Phi(v, u) = (v \cos u, v \sin u, 2)$; $\Phi_v \times \Phi_u = (0, 0, v)$, which points away from the cylinder. Then

$$\int_0^{2\pi} \int_0^3 (v^3 \cos^3 u, v^3 \sin^3 u, 4) \cdot (0, 0, v)\, dv\, du = \int_0^{2\pi} \int_0^3 4v\, dv\, du = 36\pi.$$

For the bottom choose the parametrization $\Phi(u, v) = (v \cos u, v \sin u, 0)$; $\Phi_u \times \Phi_v = (0, 0, -v)$ and

$$\int_0^{2\pi} \int_0^3 (v^3 \cos^3 u, v^3 \sin^3 u, 0) \cdot (0, 0, -v)\, dv\, du = \int_0^{2\pi} \int_0^3 0\, dv\, du = 0.$$

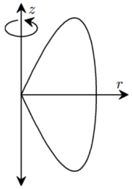
The side of the cylinder: continue the parametrization $\Phi(u, v) = (3\cos u, 3\sin u, v)$; $\Phi_u \times \Phi_v = (3\cos u, 3\sin u, 0)$ which does point outward, so

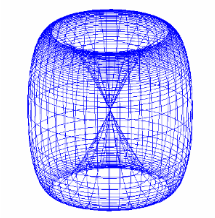
$$\int_0^{2\pi}\int_0^2 (27\cos^3 u, 27\sin^3 u, v^2)\cdot(3\cos u, 3\sin u, 0)\,dv\,du$$
$$= \int_0^{2\pi}\int_0^2 81\cos^4 u + 81\sin^4 u\,dv\,du = 243\pi.$$

The total surface integral is thus $36\pi + 0 + 243\pi = 279\pi$.

Application of Gauss's theorem : surface of revolution

Example 4 Let S be the surface obtained by rotating the curve $x = \cos u$, $z = \sin 2u$, $-\pi/2 \le u \le \pi/2$, around the z-axis.





Answer. We wish to evaluate $\iiint_W dV$, where W is the region inside of S. By the divergence theorem, this is equal to $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}$ is any vector field whose divergence is 1. Because of the cylindrical symmetry, $x\mathbf{i}$ and $y\mathbf{j}$ are poor choices for $\mathbf{F}$. We therefore let $\mathbf{F} = z\mathbf{k}$.

Now we parametrize the surface of revolution as $\Phi(t,u) = (x,y,z)$, where $x = \cos u \cos t$, $y = \cos u \sin t$, $z = \sin 2u$, $0 \le t \le 2\pi$, $-\pi/2 \le u \le \pi/2$.
We compute $d\mathbf{S} = \Phi_t \times \Phi_u dtdu =$ $(2\cos u \cos 2u \cos t, 2\cos u \cos 2u \sin t, \cos u \sin u)dtdu$. Therefore,

$$\iint_S z\mathbf{k} \cdot d\mathbf{S} = \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \sin 2u \, \cos u \, \sin u \, dtdu = \pi^2/2.$$

Flux in Physics : Fourier's Law

• Let $T(x, y, z)$ denote the temperature at a point of a surface S. The famous law of heat flow due to Fourier says that the rate of heat transfer through a material is proportional to the negative gradient in the temperature and to the area (normal to the gradient) through which the heat flows. Its differential form is: $\mathbf{q} = -k\nabla T$, where $\mathbf{q}$ is the vector of local heat flux density, k is the material's conductivity, T is the temperature.

• The negative sign here means that heat flows in the direction of decreasing temperature i.e. from hotter to a colder body.

• If S is a surface through which heat is flowing,

$$\iint_S \mathbf{q} \cdot d\mathbf{S}$$

is the total rate of heat flow or flux across S.

• Suppose S is the boundary of a region W, and suppose S is oriented with normal pointing away from W. Then $\iint_S \mathbf{q} \cdot d\mathbf{S}$ is positive if heat flows out, and negative if heat flows into, the region W.

Example

Suppose the scalar field $T(x, y, z) = x^2 + y^2 + z^2$ represents the temperaturre function at each point, and let S be the unit sphere $x^2 + y^2 + z^2 = 1$ oriented with outward normal vector. Find the heat flux across the surface if $k = 1$.

Solution: The heat flow field is given by

$$\mathbf{q} = -\nabla T(x, y, z) = -2x\mathbf{i} - 2y\mathbf{j} - 2z\mathbf{k}.$$

The outward unit normal vector on S is simply given by $\hat{\mathbf{n}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

We have

$$\mathbf{q} \cdot \hat{\mathbf{n}} = -2x^2 - 2y^2 - 2z^2 = -2$$

as the normal component of $\mathbf{q}$. Now the surface integral is given by

$$\iint_S \mathbf{q} \cdot d\mathbf{S} = -2 \iint_S dS = -8\pi.$$

In what direction is the heat flux flowing?

Application of Stokes' theorem: Maxwell's equation

Let $\mathbf{E}$ and $\mathbf{B}$ be time-dependent electric and magnetic fields, respectively. One of the Maxwell's equations is

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

Let S be a surface with boundary C. Define

$$\oint_C \mathbf{E} \cdot d\mathbf{s} = \text{voltage drop around C}$$

and

$$\iint_S \mathbf{B} \cdot d\mathbf{S} = \text{magenetic flux across S.}$$

We will show that Faraday's Law of induction can be derived from this equation of Maxwell.

Faraday's Law

Faraday's Law: The voltage (drop) around C equals the negative rate of change of magnetic flux through S.

Using Stokes' theorem

$$\int_C \mathbf{E} \cdot d\mathbf{s} = \iint_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S}.$$

Now we use Maxwell's equation to obtain

$$\iint_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = \iint_S -\frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = -\frac{\partial}{\partial t} \iint_S \mathbf{B} \cdot d\mathbf{S}.$$

The key observation is that we can move the $\frac{\partial}{\partial t}$ across the integral sign. We can do this because the parameter t is independent of the variables dS occurring in the surface integral. This is a very useful trick called "differentiating under the integral sign".

Application of Gauss's theorem: Maxwell's equations

One of the Maxwell's equations says:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0},$$

where $\mathbf{E}$ is the electric field, ϵ_0 is the vacuum permittivity and ρ is the charge density (scalar field).
From the Maxwell's equation, we derive the Gauss's law of electricity, which is:

$$\oiint_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\epsilon_0},$$

for any closed surface S containing charge Q.
By Gauss's divergence theorem, if V is the region enclosed by S,

$$\oiint_S \mathbf{E} \cdot d\mathbf{S} = \iiint_V \operatorname{div} \mathbf{E} \, dV = \iiint_V \frac{\rho}{\epsilon_0} \, dV = \frac{Q}{\epsilon_0}.$$

An interpretation of divergence

Let $\mathbf{F}(x, y, z)$ be a C^1 vector field. If P_0 is a point, since *div* $\mathbf{F}$ is continuous at P_0, given any epsilon there is some $a > 0$ such that if $B(a)$ is a unit ball with center P_0 with radius a, then $|\mathrm{div}\ \mathbf{F}(P) - \mathrm{div}\ \mathbf{F}(P_0)| < \epsilon$ for all $P \in B(a)$. By Gauss's divergence theorem,

$$\iint_{S(a)} \mathbf{F} \cdot d\mathbf{S} = \iiint_{B(a)} \mathrm{div}\ \mathbf{F}\ dV.$$

Therefore,

$$\left| \frac{\iint_{S(a)} \mathbf{F} \cdot d\mathbf{S}}{vol(B(a))} - \mathrm{div}\, \mathbf{F}(P_0) \right| = \left| \frac{\iiint_{B(a)} (\mathrm{div}\ \mathbf{F}(P) - \mathrm{div}\ \mathbf{F}(P_0))\ dV}{vol(B(a))} \right|$$
$$\leq \epsilon$$

This means,

$$\mathrm{div}\ \mathbf{F}(P_0) = \lim_{a \to 0+} \frac{1}{Vol(B(a))} \iint_{S(a)} \mathbf{F} \cdot d\mathbf{S}.$$

Divergence

• Therefore $\operatorname{div} \mathbf{F}(P_0)$ is the net rate of outward flux per unit volume at P_0.

• In the context of a fluid flow, if $\mathbf{F}$ is the velocity vector field, and if $\operatorname{div} \mathbf{F}(P_0) > 0$, the net flow is outward near P_0 and P_0 is called a *source*. If $\operatorname{div} \mathbf{F}(P_0) < 0$, the net flow is inward near P_0 and P_0 is called a *sink*. If $\operatorname{div} \mathbf{F} = 0$ then the fluid is incompressible.

• In general, we have the **continuity equation** in fluid dynamics, which says that the rate at which mass enters a system is equal to the rate at which mass leaves the system plus the accumulation of mass within the system. Mathematically,

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0,$$

where ρ is the fluid density and $\mathbf{v}$ is the flow velocity vector.

Curl and divergence

Theorem

1. *If* $\mathbf{F} = \nabla \times \mathbf{G}$, *where* $\mathbf{G}$ *is a* C^2 *vector field defined on an open set* W *in* $\mathbb{R}^3$, *then*
$$\mathit{div}\,\mathbf{F} = 0 \quad \text{on} \quad W.$$
2. *If* $\mathbf{F}$ *is a* C^1 *vector field defined on* $\mathbb{R}^3$ *satisfying* $\mathit{div}\,\mathbf{F} = 0$ *on* $\mathbb{R}^3$, *then there exists a* C^2 *vector filed* $\mathbf{G}$ *defined on* $\mathbb{R}^3$ *such that*
$$\mathbf{F} = \mathit{curl}\,\mathbf{G}, \quad \text{on} \quad \mathbb{R}^3.$$

If $\operatorname{div}\mathbf{F} = 0$ in $\mathbb{R}^3$, how to find $\mathbf{G}$ such that $\mathbf{F} = \operatorname{curl}\mathbf{G}$?

Example Is $\mathbf{F}(x, y, z) = x\mathbf{i} - 2y\mathbf{j} + z\mathbf{k}$ defined in $\mathbb{R}^3$ the curl of a vector filed?
Check $\mathbf{F}$ is smooth vector field satisfying $\operatorname{div}\mathbf{F} = 0$ in $\mathbb{R}^3$. So there exists a smooth vector filed $\mathbf{G}$ such that $\mathbf{F} = \mathbf{G}$ in $\mathbb{R}^3$.

Example contd.

To find **G**: Let us assume $\mathbf{G}(x,y,z) = G_1(x,y,z)\mathbf{i} + G_2(x,y,z)\mathbf{j}$ for all $(x,y,z) \in \mathbb{R}^3$. Then solve G_1 and G_2 in such a way that $\operatorname{curl}\mathbf{G} = \mathbf{F}$, i.e.,

$$\frac{\partial G_2}{\partial z}(x,y,z) = -F_1(x,y,z) = -x, \quad \frac{\partial G_1}{\partial z}(x,y,z) = F_2(x,y,z) = -2y,$$

$$\Big(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\Big)(x,y,z) = F_3(x,y,z) = z.$$

Now solving the equations, $G_2(x,y,z) = -xz + g(x,y)$ and $G_1(x,y) = -2yz + h(x,y)$. Using the 3rd equation,

$$-z + \partial_x g(x,y) + 2z - \partial_y h(x,y) = z.$$

It yields $\partial_x g(x,y) - \partial_y h(x,y) = 0$. Choosing, $g \equiv 0 \equiv h$, we get

$$\mathbf{G}(x,y,z) = -2yz\mathbf{i} - xz\mathbf{j}, \quad \text{in}\,\mathbb{R}^3.$$

In general, $\operatorname{div} \mathbf{F} = 0$ does not imply that $\mathbf{F}$ is the curl of a vector field.

Example Note that for any $\mathbf{G}$, $\oiint_S \mathit{curl}\, \mathbf{G} \cdot d\mathbf{S} = 0$ for any *closed* surface S, by Stokes' theorem because S does not have any boundary.

Take $\mathbf{F} = \frac{\mathbf{r}}{r^3}$ on $\mathbb{R}^3 - \{(0,0,0)\}$. It is easy to see that $\operatorname{div} \mathbf{F} = 0$. But let S be the unit sphere with outward orientation. Then on S, $\mathbf{F} \cdot d\mathbf{S} = dS$, and

$$\oiint_S \mathbf{F} \cdot d\mathbf{S} = \oiint_S dS = 4\pi \neq 0.$$

Note that we cannot use Gauss's divergence theorem here because $\mathbf{F}$ is not defined everywhere.