# Final Examination for Regular Students

Final Examination of MA 106 for Regular Students

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1.

Consider the $4 \times 4$ matrix $A$ and the $4 \times 1$ column vectors $\mathbf{u}$ and $\mathbf{v}$ defined by

$$A = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 3 & 0 & 0 & 3 \\ 0 & -1 & 0 & 2 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 1 \end{bmatrix}.$$

Let $\mathcal{C}(A)$ denote the column space of $A$. Then which of the following options is correct?

Marks: 2

Type: SINGLE_CORRECT_ANSWER

**Options:**

0) $\mathbf{u} \in \mathcal{C}(A)$ and $\mathbf{v} \in \mathcal{C}(A)$

1) $\mathbf{u} \notin \mathcal{C}(A)$, but $\mathbf{v} \in \mathcal{C}(A)$.

2) $\mathbf{u} \in \mathcal{C}(A)$, but $\mathbf{v} \notin \mathcal{C}(A)$.

3) $\mathbf{u} \notin \mathcal{C}(A)$ and $\mathbf{v} \notin \mathcal{C}(A)$.

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2. The value(s) of $k$ for which the system

Marks: 2

Type: SINGLE_CORRECT_ANSWER

$$\begin{cases} y + 3kz &= 0 \\ x + 2y + 6z &= 2 \\ kx + 2ky + 12z &= -4 \end{cases}$$

has no solution is (are)

**Options:**

0) $k = 2$

1) $k = 4$

2) $k = 2 \text{ and } k = 4$

3) $k = 4 \text{ and } k = 6$

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3.

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ denote the standard basic vectors in $\mathbb{R}^3$ and let $S : \mathbb{R}^3 \to \mathbb{R}^3$ and $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformations for which

$$S(\mathbf{e}_1) = \mathbf{e}_1 - \mathbf{e}_2, \quad S(\mathbf{e}_2) = \mathbf{e}_2 - \mathbf{e}_3, \quad S(\mathbf{e}_3) = \mathbf{e}_3 - \mathbf{e}_1$$

and

$$T(\mathbf{e}_1 + \mathbf{e}_2) = 2\mathbf{e}_1, \quad T(\mathbf{e}_1 - \mathbf{e}_2) = 4\mathbf{e}_2, \quad T(\mathbf{e}_3 - \mathbf{e}_2) = 2\mathbf{e}_1 - 4\mathbf{e}_2 - \mathbf{e}_3.$$

Suppose $C = \mathrm{M}_E^E(T \circ S)$ denotes the matrix of the composite linear map $T \circ S : \mathbb{R}^3 \to \mathbb{R}^3$ with respect to the standard ordered basis $E = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ of $\mathbb{R}^3$. Then the sum of entries of the second column of $C$ is equal to

Marks: 2

Type: SINGLE_CORRECT_ANSWER

**Options:**

0) $7$

1) $-3$

2) $3$

3) None of the above

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4.

Consider the $4 \times 4$ matrix $A$ and the $4 \times 1$ column vectors $\mathbf{u}$ and $\mathbf{v}$ defined by

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 2 & 0 & 2 \\ 1 & 0 & 0 & 2 \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}.$$

Let $\mathcal{C}(A)$ denote the column space of $A$. Then which of the following options is correct?

Marks: 2

Type: SINGLE_CORRECT_ANSWER

**Options:**

0) $\mathbf{u} \in \mathcal{C}(A)$ and $\mathbf{v} \in \mathcal{C}(A)$.

1) $\mathbf{u} \notin \mathcal{C}(A)$, but $\mathbf{v} \in \mathcal{C}(A)$

2) $\mathbf{u} \in \mathcal{C}(A)$, but. $\mathbf{v} \notin \mathcal{C}(A)$.

3) $\mathbf{u} \notin \mathcal{C}(A)$ and $\mathbf{v} \notin \mathcal{C}(A)$.

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5.

Let $n$ be a positive integer and let $\mathcal{P}_n$ denote the vector space over $\mathbb{R}$ of all polynomials in one variable with real coefficients and of degree $\leq n$. Consider the linear map $T : \mathcal{P}_n \to \mathcal{P}_n$ defined by

$$T(p(t)) = p'(t) \quad \text{for } p(t) \in \mathcal{P}_n,$$

where $p'(t)$ denotes the derivative of $p(t)$. Then which of the following options is correct?

Marks: 2

Type: SINGLE_CORRECT_ANSWER

**Options:**

0) $\text{nullity}(T) = 0 \text{ and } \text{rank}(T) = n - 1.$

1) $\text{nullity}(T) = 0 \text{ and } \text{rank}(T) = n.$

2) $\text{nullity}(T) = 1 \text{ and } \text{rank}(T) = n - 1.$

3) $\text{nullity}(T) = 1 \text{ and } \text{rank}(T) = n.$

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6.

Let $V$ be a finite dimensional inner product space over $\mathbb{C}$. For a subset $S$ of $V$, denote by $S^\perp$ the orthogonal complement of $S$ defined by $S^\perp = \{v \in V : \langle v, u \rangle = 0 \text{ for all } u \in S\}$. Consider the following statements and then choose the correct option.

1. $S^\perp \subset S$ for every subspace $S$ of $V$.
2. $S \cap S^\perp = \{0\}$ for every subspace $S$ of $V$.
3. $S + S^\perp = V$ for every subspace $S$ of $V$.
4. $\dim S = \dim S^\perp$ for every subspace $S$ of $V$.

Marks: 2

Type: SINGLE_CORRECT_ANSWER

**Options:**

0) Statements 1 and 2 are true, but 3 and 4 are false.

1) Statements 2 and 3 are true, but 1 and 4 are false.

2) Statements 3 and 4 are true, but 1 and 2 are false.

3) Statements 1 and 4 are true, but 2 and 3 are false.

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7.
Let $A$ be a $2 \times 2$ matrix with entries in $\mathbb{R}$ such that $\det(A) < 0$ and let $B$ be a $5 \times 5$ matrix with entries in $\mathbb{R}$ such that $B^3 = \mathbf{0}$, but $B^2 \neq \mathbf{0}$, where $\mathbf{0}$ denotes the zero matrix of size $5 \times 5$. Then which of the following options is correct?

Marks: 2
Type: SINGLE_CORRECT_ANSWER

**Options:**

0) Both $A$ and $B$ are diagonalisable.

1) $A$ is diagonalisable, but $B$ is not diagonalisable.

2) $B$ is diagonalisable, but $A$ is not diagonalisable.

3) It is possible that neither $A$ nor $B$ is diagonalisable.

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8.
Consider the subset $W$ of the vector space $\mathbb{R}^{2\times 2}$ of all $2 \times 2$ real matrices defined by

$$W = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} : a, b \in \mathbb{R} \right\}.$$

Regard the set $\mathbb{C}$ of all complex numbers as a vector space over $\mathbb{R}$ and let $T : W \to \mathbb{C}$ be the map defined by

$$T\left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\right) = a + ib.$$

Consider the the following statements and then choose the correct option.

1. $W$ is a subspace of $\mathbb{R}^{2\times 2}$ and $T : W \to \mathbb{C}$ is a linear map of vector spaces over $\mathbb{R}$.

2. $T : W \to \mathbb{C}$ is one-one.

3. $T : W \to \mathbb{C}$ is onto.

4. $T: W \to \mathbb{C}$ preserves multiplication, that is, $T(AB) = T(A)T(B)$ for all $A, B \in W$, where $AB$ denotes the product of matrices $A$ and $B$, while $T(A)T(B)$ denotes the product of complex numbers $T(A)$ and $T(B)$.

Marks: 2

Type: SINGLE_CORRECT_ANSWER

**Options:**

0) Statement 1 is false.
1) Statements 1 and 2 are true, but 3 and 4 are false.
2) Statements 1, 2 and 3 are true, but 4 is false.
3) Statements 1, 2 , 3 and 4 are true.

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9.

Suppose the ternary quadratic form $3x^2 - 12xy + 12yz - 3z^2$ is transformed to $9(v^2 - w^2)$ by the transformation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = C \begin{bmatrix} u \\ v \\ w \end{bmatrix},$$

where $C = [c_{jk}]$ is a $3 \times 3$ orthogonal matrix. Then the sum of the absolute values of the diagonal entries of $C$, that is, $|c_{11}| + |c_{22}| + |c_{33}|$, is equal to

Marks: 2

Type: SINGLE_CORRECT_ANSWER

**Options:**

0) $0$
1) $2$
2) $6$
3) None of the above

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10.

Let $V$ denote the set of all $3 \times 3$ matrices $A$ with entries in $\mathbb{C}$ such that $A$ is self-adjoint, i.e., $A^* = A$. Then which of the following options is correct?

Marks: 2

Type: SINGLE_CORRECT_ANSWER

**Options:**

0) $V$ is a real vector space and the dimension of $V$ over $\mathbb{R}$ is 6.

1) $V$ is a real vector space and the dimension of $V$ over $\mathbb{R}$ is 9.

2) $V$ is a complex vector space and the dimension of $V$ over $\mathbb{C}$ is 3.

3) $V$ is a complex vector space and the dimension of $V$ over $\mathbb{C}$ is 6.

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11.

For a positive integer $n$, let $V_n$ and $W_n$ denote the subspaces of the vector space $\mathbb{R}^{n\times n}$ of all $n \times n$ real matrices defined by

$$V_n = \{A \in \mathbb{R}^{n\times n} : \text{trace}(A) = 0\}$$

and

$$W_n = \{A \in \mathbb{R}^{n\times n} : A \text{ is skew-symmetric}\}.$$

Then $\dim V_3 + \dim W_4$ is equal to

Marks: 2

Type: SINGLE_CORRECT_ANSWER

**Options:**

0) $14$

1) $15$

2) $18$

3) $19$

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12.

For a positive integer $n$, let $V_n$ and $W_n$ denote the subspaces of the vector space $\mathbb{R}^{n\times n}$ of all $n \times n$ real matrices defined by

$$V_n = \{A \in \mathbb{R}^{n\times n} : \text{trace}(A) = 0\}$$

and

$$W_n = \{A \in \mathbb{R}^{n\times n} : A \text{ is skew-symmetric}\}.$$

Then $\dim V_4 + \dim W_3$ is equal to

Marks: 2

Type: SINGLE_CORRECT_ANSWER

**Options:**

0) $14$

1) $15$

2) $18$

3) $19$

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13.

Let $A$ and $B$ be the $4 \times 4$ matrices defined by

$$A = \begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 4 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Consider the following statements and then choose the correct option.

1. Both $A$ and $B$ have the same eigenvalues and their algebraic multiplicities are also the same.

2. Both $A$ and $B$ have the same eigenvalues and their geometric multiplicities are also the same.

3. $A$ is similar to $B$.

4. $A$ is not similar to $B$.

Marks: 2

Type: SINGLE_CORRECT_ANSWER

**Options:**

0) Statements 1, 2 and 3 are true, but 4 is false.

1) Statements 1, 2 and 4 are true, but 3 is false.

2) Statements 1 and 4 are true, but 2 and 3 are false.

3) Statements 2 and 4 are true, but 1 and 3 are false.

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14.

Let $A$ and $B$ be the $4 \times 4$ matrices defined by

$$A = \begin{bmatrix} 3 & 4 & 0 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \quad \text{and.} \quad B = \begin{bmatrix} 3 & 3 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Consider the following statements and then choose the correct option.

1. Both $A$ and $B$ have the same eigenvalues and their algebraic multiplicities are also the same.
2. Both $A$ and $B$ have the same eigenvalues and their geometric multiplicities are also the same.
3. $A$ is similar to $B$.
4. $A$ is not similar to $B$.

Marks: 1
Type: SINGLE_CORRECT_ANSWER

**Options:**

0) Statements 1, 2 and 3 are true, but 4 is false.
1) Statements 1, 2 and 4 are true, but 3 is false.
2) Statements 1 and 4 are true, but 2 and 3 are false.
3) Statements 2 and 4 are true, but 1 and 3 are false.

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15.
Let $V$ be a finite dimensional vector space over $\mathbb{R}$ and let $P : V \to V$ be a linear map such that $P$ is not the zero map, $P$ is not the identity map, and $P$ satisfies $P^2 = P$, that is, $P \circ P = P$. Consider the following statements and then choose the correct option.

1. $P$ must be invertible.
2. $P$ cannot be invertible.
3. The only possible eigenvalues of $P$ are $0$ and. $1$.
4. The null space of $P$ and the image space of $P$ have a nonzero vector in common.

Marks: 2
Type: SINGLE_CORRECT_ANSWER

**Options:**

0) Statements 1, 3 and 4 are true, but 2 is false.

1) Statements 2, 3 and 4 are true, but 1 is false.

2) Statements 1 and 3 are true, but 2 and 4 are false.

3) Statements 2 and $3$ are true, but 1 and $4$ are false.

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16.

Suppose a $3 \times 3$ matrix $A$ with real entries satisfies $A^3 - 2A^2 = A - 2I$ and has the property that $\det(A) < 0$ and $\mathrm{trace}(A) > 2$. Then the characteristic polynomial $p_A(t) = \det(A - tI)$ of $A$ is given by

Marks: 2

Type: SINGLE_CORRECT_ANSWER

**Options:**

0) $-t^3 + 2t^2 + t - 2$

1) $-t^3 + 3t^2 - 4$

2) $-t^3 + 3t^2 - 8t - 4$

3) $-t^3 + t^2 + t - 1$

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17.

Consider the $3 \times 2$ matrix $A$ and the $3 \times 1$ column vector $\mathbf{b}$ given by

$$A = \begin{bmatrix} 3 & 6 \\ 4 & 8 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix}.$$

Then the best approximation to $\mathbf{b}$ from the column space of $A$ is given by

Marks: 2

Type: SINGLE_CORRECT_ANSWER

**Options:**

0) $\begin{bmatrix} 3/5 \\ 4/5 \\ 4 \end{bmatrix}$

1) $\begin{bmatrix} 21/5 \\ 28/5 \\ 4 \end{bmatrix}$

2)

$\begin{bmatrix} 9 \\ 4 \\ 4 \end{bmatrix}$

3) $\begin{bmatrix} 6 \\ 4 \\ 4 \end{bmatrix}$

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18.

Let $C^1[-\pi,\pi]$ denote the vector space of continuously differentiable real valued functions defined on the interval $[-\pi,\pi]$, and let $V$ denote the subspace of $C^1[-\pi,\pi]$ spanned by the functions $f_0, f_1, f_2$, where

$$f_0(x) = 1, \quad f_1(x) = \cos x \quad \text{and} \quad f_2(x) = \sin x \quad \text{for all } x \in [-\pi,\pi].$$

Consider the linear map $T : V \to V$ defined by $T(f) = f'$, where $f'$ denotes the derivative of $f$. If $A = \mathrm{M}_E^E(T)$ denotes the matrix of $T$ with respect to the ordered basis $E = (f_0, f_1, f_2)$ of $V$, then which of the following options is correct?

Marks: 2

Type: SINGLE_CORRECT_ANSWER

**Options:**

0) $A$ is invertible.

1) $A$ is orthogonal

2) $A$ is symmetric

3) $A$ is skew-symmetric.

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