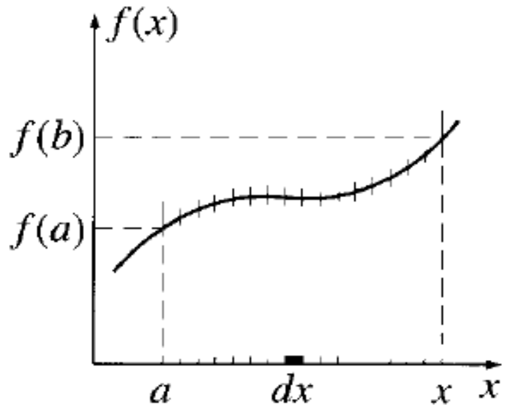
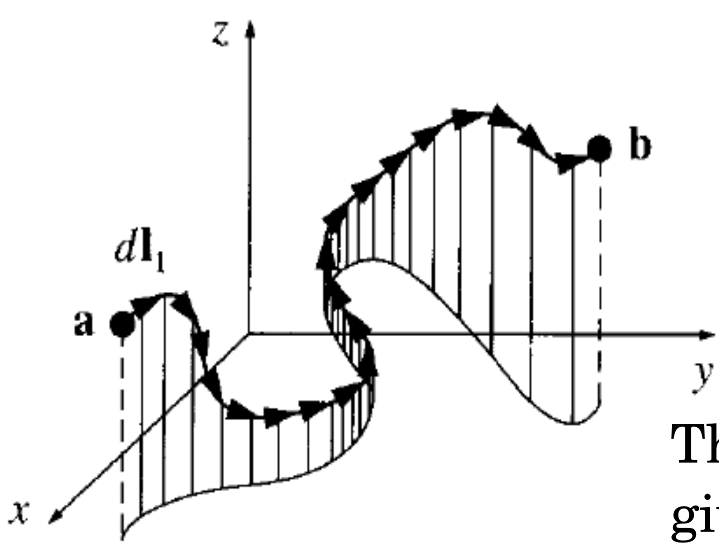
FUNDAMENTAL THEOREM OF GRADIENTS



$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a)$$

The integral of a derivative over an interval is the value of the function at the end points or boundaries.

Scalar field T(x,y,z) $\quad dT = (\nabla T).(d\vec{l})$



$$\int_{a\,\mathcal{P}}^{b} (\nabla T).(d\vec{l}) = T(b) - T(a)$$

The line integral of a gradient of a scalar field is given by the value of the function at the boundaries.

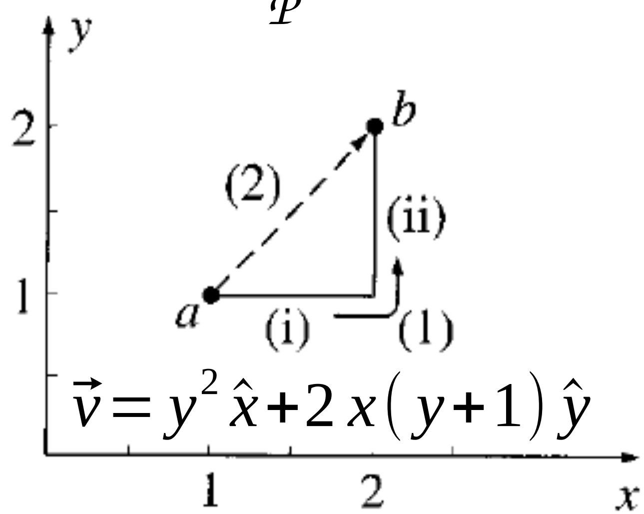
Path Independence - Is this obvious?

LINE INTEGRALS

For any general vector field $\vec{v}$

$$\int_{a\,\mathcal{P}}^{b} \vec{v}.(d\vec{l})$$ depends on the path $\mathcal{P}$!

For a gradient field, $\boxed{\oint (\nabla T).(d\vec{l}) = 0}$



$$\vec{v} = y^2\hat{x} + 2x(y+1)\hat{y}$$

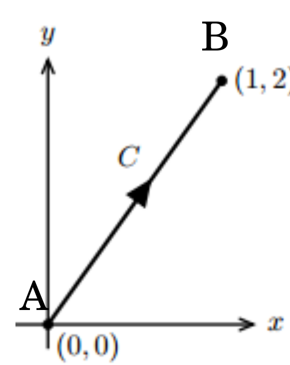
If $\vec{v} \equiv \nabla T$, then $\vec{v}$ is a conservative field
Electric field **E** is conservative.

$$f(x,y) = xy^3 + xy^2$$

$$\int_{a\ 1}^{b} \vec{v}.(d\vec{l}) = 11$$

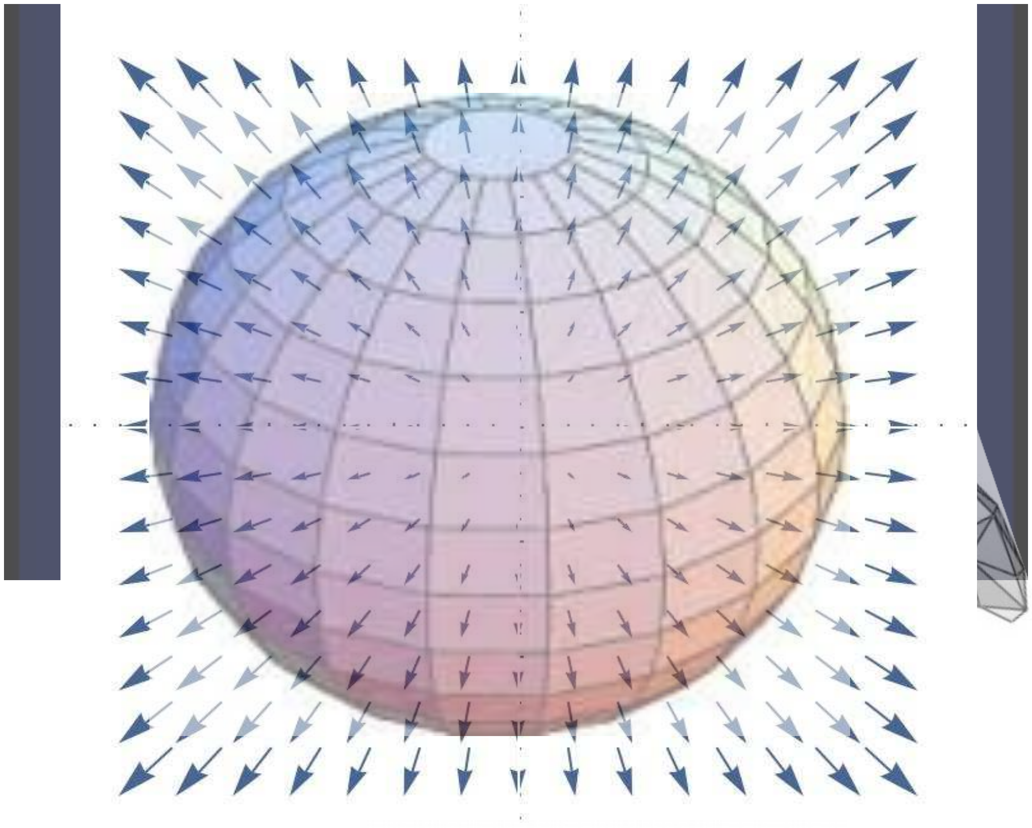
$$\int_{a\ 2}^{b} \vec{v}.(d\vec{l}) = 10$$

$$\Rightarrow \oint \vec{v}.(d\vec{l}) \neq 0$$



$$\int_{A}^{B} \nabla f.(d\vec{l}) = ?$$

FLUX OF A VECTOR FIELD



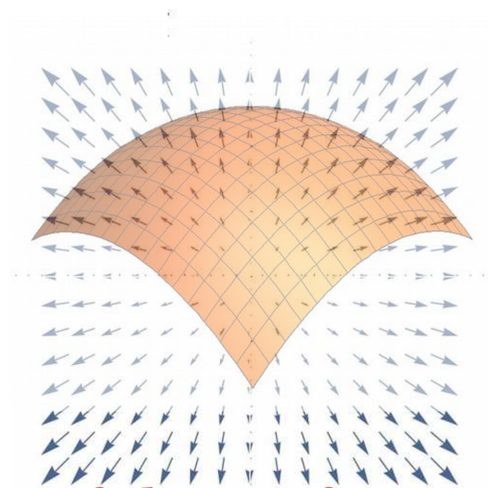
The volume of water flowing out through the surface per unit time,

Flux $\oiint \vec{v} \, . \, d\vec{S}$

$d\vec{S} \Rightarrow$ Infinitesimal area element, direction perpendicular to surface

$\oiint \Rightarrow$ Integral over a closed surface

Sign convention: Outward is positive



For the surface integral over an open surface

$$\iint \vec{v} \, . \, d\vec{S}$$

the sign is arbitrary

Does the value of the surface integral depend on the surface? Yes, except...

GAUSS'S THEOREM / DIVERGENCE THEOREM

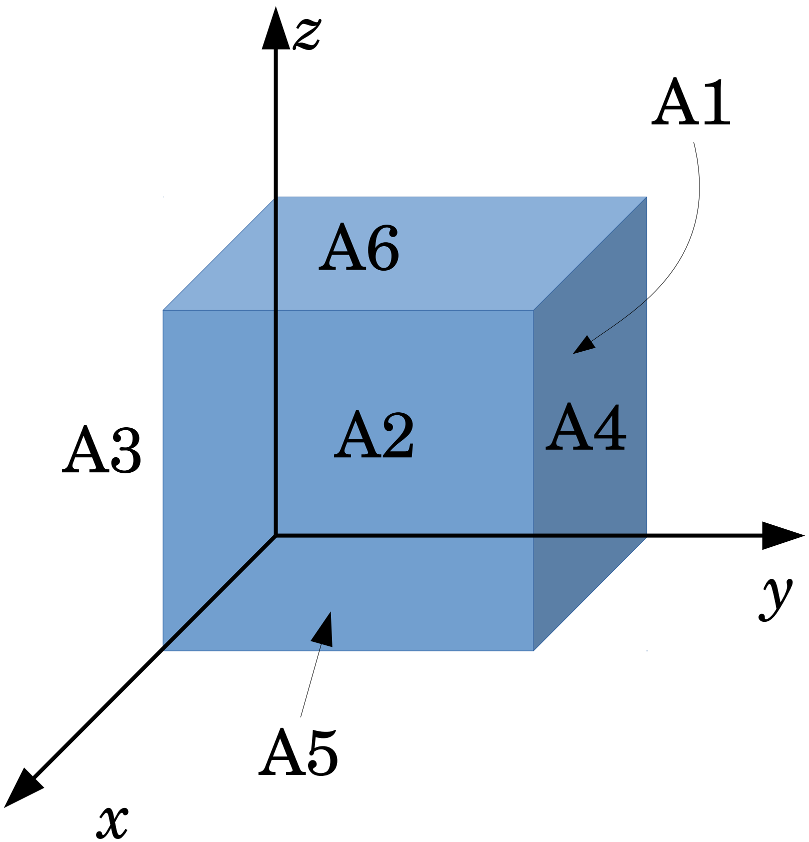
$$\underset{V}{\oiiint} (\nabla . \vec{v}) \; dV = \underset{S}{\oiint} \vec{v} . d\vec{S}$$

where, S is the surface bounding a volume V

The volume integral of the divergence of a function over a region is equal to the value of the function at the boundary (the surface enclosing the volume).

RHS measures the flux through the closed surface S.
LHS counts the sources and sinks within the region bounded by S

GAUSS'S THEOREM / DIVERGENCE THEOREM



$$a \leqslant x \leqslant b, \quad c \leqslant y \leqslant d, \quad e \leqslant z \leqslant f$$

$$\iint_{A_1} \vec{v} \,.\, d\vec{S} + \iint_{A_2} \vec{v} \,.\, d\vec{S}$$

$$= -\int_e^f \int_c^d v_x(a, y, z)\, dy\, dz + \int_e^f \int_c^d v_x(b, y, z)\, dy\, dz$$

$$= \int_e^f \int_c^d \big(v_x(b, y, z) - v_x(a, y, z) \big)\, dy\, dz$$

$$= \int_e^f \int_c^d \int_a^b \left(\frac{\partial v_x}{\partial x}\, dx \right) dy\, dz = \oiiint_V \frac{\partial v_x}{\partial x}\, dV$$

Repeating over the other 4 sides and adding,

$$\oiint_S \vec{v} \,.\, d\vec{S} = \oiiint_V \left(\nabla \,.\, \vec{v} \right)\, dV$$

STOKES THEOREM

$$\iint_{S} (\nabla \times \vec{v}).d\vec{S} = \oint_{\wp} \vec{v}.d\vec{l}$$

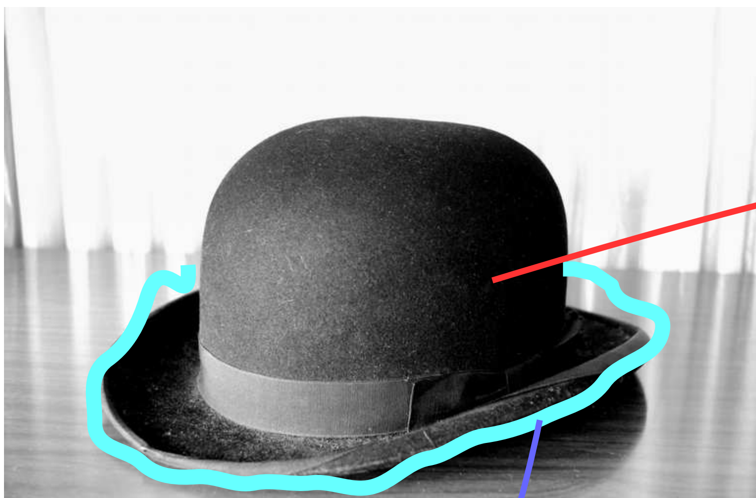
The flux of the curl of a vector through a surface S is equal to the closed line integral of the vector function over the bounding line of the surface.

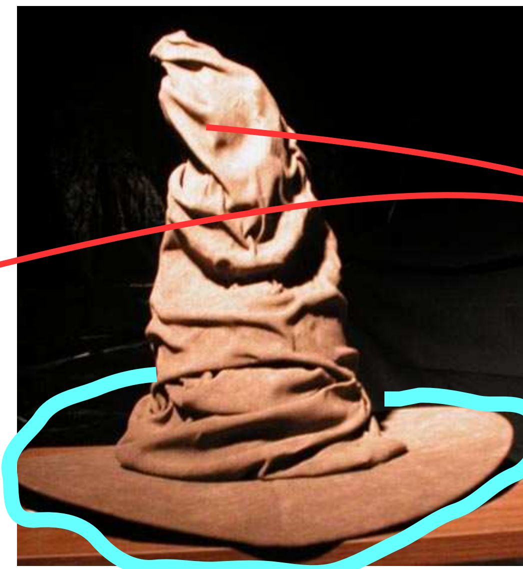
What is the sign of the line integral?
What is the sign of the surface integral?

Right hand rule: If the fingers point in the direction of the line integral, the thumb fixes the direction of $d\vec{S}$

STOKES THEOREM

The same bounding line can enclose many surfaces!





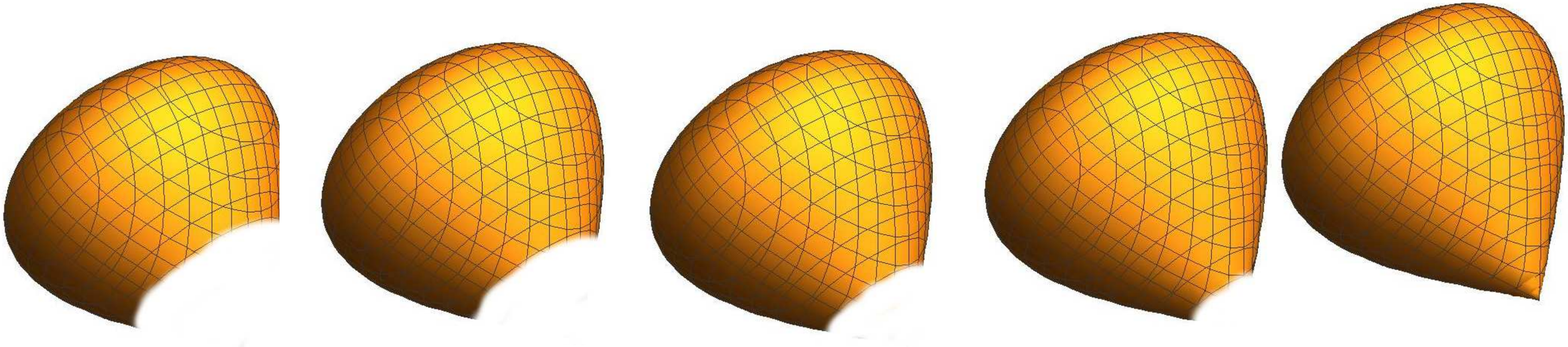
Surface

Bounding line

The surface integral can be over ***ANY*** surface that shares a common bounding line!

STOKES THEOREM

What happens if we shrink the boundary line?



$$\oiint_S (\nabla \times \vec{v}).d\vec{S} = 0$$

COROLLARIES

$\nabla \times \nabla \varphi = ?$

$$\iint_S (\nabla \times \vec{v}) . d\vec{S} = \oint \vec{v} . d\vec{l}$$ Stokes Theorem

If, $\vec{v} = \nabla \varphi$

$$\oint \nabla \varphi . d\vec{l} = 0$$ Gradient Theorem

Combining, we get,

$$\iint_S (\nabla \times \nabla \varphi) . d\vec{S} = 0$$ for any surface S

$$\Rightarrow \nabla \times \nabla \varphi = 0$$ for any scalar function φ

$\nabla . \nabla \times \vec{A} = ?$

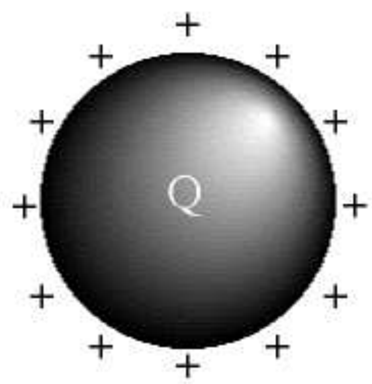
$$\oiint_S (\nabla \times \vec{A}) . d\vec{S} = 0$$ Stokes Theorem

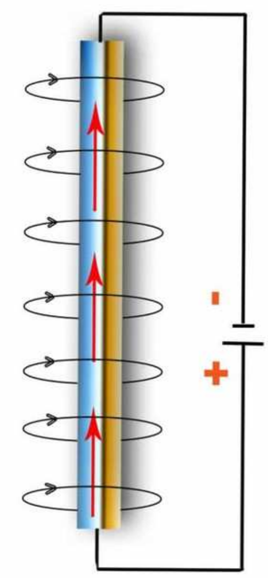
$$\oiiint \nabla . (\nabla \times \vec{A}) dV = \oiint (\nabla \times \vec{A}) . d\vec{S} = 0$$ Gauss's Theorem

$$\Rightarrow \nabla . (\nabla \times \vec{A}) = 0$$ for any vector function $\vec{A}$

CURVILINEAR COORDINATE SYSTEMS

The symmetries of a problem can dictate the most efficient *choice* of the coordinate system.





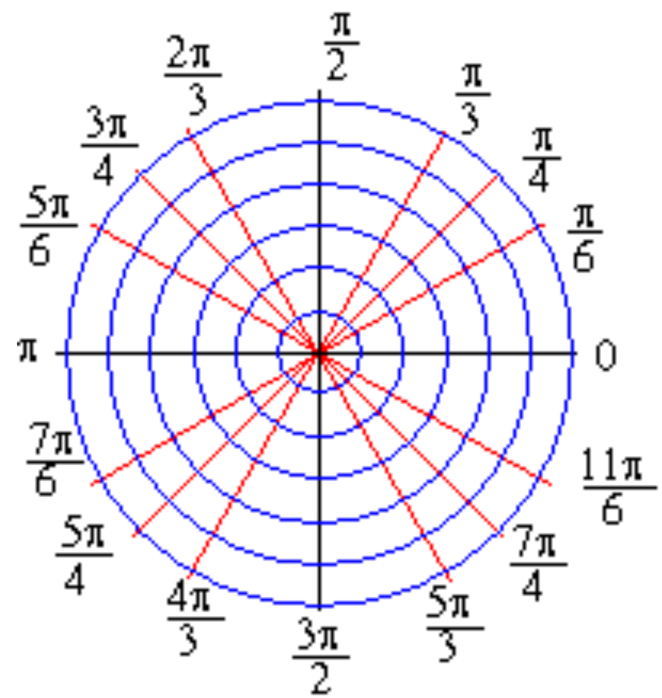
Any coordinate system will do for any problem, however, some coordinate systems will be easier!

PLANE POLAR COORDINATES

STEP 1: Write down the relation with (x,y) co-ordinates

$$x = r \cos\theta$$
$$y = r \sin\theta$$

STEP 2: Draw the coordinate grid



How do r=constant lines look?
How do θ= constant lines look?

STEP 3: What happens when the independent variables are changed infinitesimally?

$$\delta x = \cos\theta\,\delta r - r\sin\theta\,\delta\theta$$
$$\delta y = \sin\theta\,\delta r + r\cos\theta\,\delta\theta$$

PLANE POLAR COORDINATES

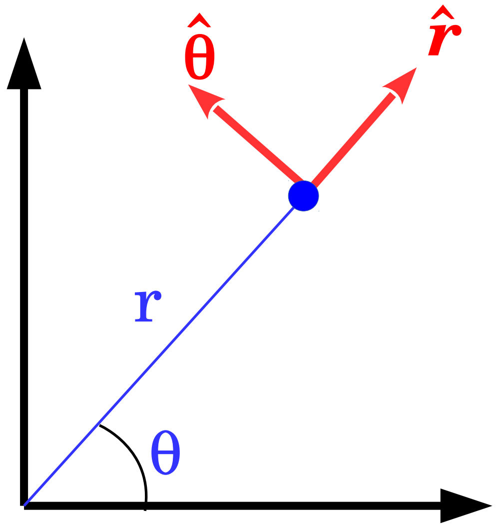
STEP 4: Which direction would we move, if only one variable was changed?

$\underline{\delta\theta=0}$

$$\hat{x}\,\delta x+\hat{y}\,\delta y = (\hat{x}\cos\theta+\hat{y}\sin\theta)\,\delta r = \hat{\boldsymbol{r}}\,\delta r$$

$\underline{\delta r=0}$

$$\hat{x}\,\delta x+\hat{y}\,\delta y = (-\hat{x}\sin\theta+\hat{y}\cos\theta)\,r\,\delta\theta = \hat{\boldsymbol{\theta}}\,r\,\delta\theta$$



$$\begin{pmatrix}\hat{r}\\ \hat{\theta}\end{pmatrix}=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix}$$

$$\hat{r}\,.\,\hat{\theta}=(-\cos\theta\sin\theta)+(\sin\theta\cos\theta)=0$$

Curvilinear, but still orthogonal

PLANE POLAR COORDINATES

STEP 5: What happens to an element of distance, or arclength?

$$d\vec{l} = \delta r\,\hat{r} + r\,\delta\theta\,\hat{\theta}$$

Compare with the Cartesian case, $d\vec{l} = \delta x\,\hat{x} + \delta y\,\hat{y}$

Can it be $d\vec{l} = \delta r\,\hat{r} + \delta\theta\,\hat{\theta}$? Why not?

In general, for a curvilinear coordinate system, the scale factors are not unity!

Scale factor gives a measure of how much a change in the coordinate changes the position of a point.

$$h_r = \sqrt{\left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2} = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$h_\theta = \sqrt{\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2} = \sqrt{r^2\sin^2\theta + r^2\cos^2\theta} = r$$

$$ds^2 = d\vec{l}\,.\,d\vec{l} = \delta r^2 + r^2\,\delta\theta^2$$

There are no cross terms in the arclength expression $O(\delta r\,\delta\theta)$ for orthogonal coordinate systems.

PLANE POLAR COORDINATES

STEP 6: What happens to the element of area?

We take a small step in the $\hat{r}$ direction, and a small step in the $\hat{\theta}$ direction. What is the infinitesimal area enclosed by these two perpendicular vectors?

$$d l_r = \delta r \qquad d l_\theta = r \delta \theta$$

$$dA = d l_r \, d l_\theta = r \delta \theta \delta r$$

STEP 7: What is the gradient?

$$\begin{aligned} dT &= \frac{\partial T}{\partial r} \delta r + \frac{\partial T}{\partial \theta} \delta \theta \\ &= [some \quad fn] . d\vec{l} \\ &= [\nabla T] . (\delta r \hat{r} + r \delta \theta \hat{\theta}) \end{aligned}$$

$$\boxed{\nabla T = \frac{\partial T}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial T}{\partial \theta} \hat{\theta}}$$

$$\nabla = \frac{\partial}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial}{\partial \theta} \hat{\theta}$$

$$dT = \left(\frac{\partial T}{\partial x}\right) dx + \left(\frac{\partial T}{\partial y}\right) dy + \left(\frac{\partial T}{\partial z}\right) dz$$

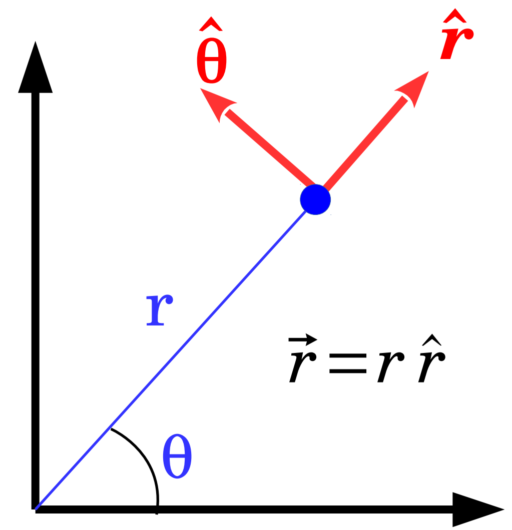
$$dT = \left(\frac{\partial T}{\partial x} \hat{x} + \frac{\partial T}{\partial y} \hat{y} + \frac{\partial T}{\partial z} \hat{z}\right) . \left(dx \, \hat{x} + dy \, \hat{y} + dz \, \hat{z}\right)$$

$$\boxed{\nabla T = \left(\frac{\partial T}{\partial x} \hat{x} + \frac{\partial T}{\partial y} \hat{y} + \frac{\partial T}{\partial z} \hat{z}\right)}$$

PLANE POLAR COORDINATES

STEP 8: What are the velocity components, when a particle's motion is described using polar coordinates?

$$\vec{v}=\frac{d}{dt}(r\,\hat{r})=\dot{r}\,\hat{r}+r\,\dot{\hat{r}}$$



The unit vectors here are not constant, unlike for cartesian coordinates, and must be themselves differentiated.

$$\begin{pmatrix}\hat{r}\\ \hat{\theta}\end{pmatrix}=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix}$$

$$\begin{pmatrix}\dot{\hat{r}}\\ \dot{\hat{\theta}}\end{pmatrix}=\dot{\theta}\begin{pmatrix}-\sin\theta & \cos\theta\\ -\cos\theta & -\sin\theta\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\hat{r}\\ \hat{\theta}\end{pmatrix}$$

$$\Rightarrow\vec{v}=\dot{r}\,\hat{r}+r\,\dot{\hat{r}}=\dot{r}\,\hat{r}+r\,\dot{\theta}\,\hat{\theta}$$

$v_r=\dot{r}\rightarrow$ *radial* velocity component

$v_\theta=r\,\dot{\theta}\rightarrow$ *circumferential* velocity component

$$\vec{v}\;=\;\frac{\delta\vec{l}}{\delta t}=\;\frac{\delta}{\delta t}(\delta r\,\hat{r}+r\,\delta\theta\,\hat{\theta})=\;\frac{dr}{dt}\hat{r}+r\frac{d\theta}{dt}\hat{\theta}$$

PLANE POLAR COORDINATES

STEP 9: What are the acceleration components, when a particle's motion is described using polar coordinates?

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\dot{r}\,\hat{r} + r\,\dot{\theta}\,\hat{\theta}\right)$$

$$= \dot{\theta}\ \dot{r}\,\hat{\theta} + \ddot{r}\,\hat{r} - \dot{\theta}\,r\,\dot{\theta}\,\hat{r} + \dot{r}\,\dot{\theta}\,\hat{\theta} + r\,\ddot{\theta}\,\hat{\theta}\text{¿}$$

$$= \underbrace{\left(\ddot{r} - \dot{\theta}^2 r\right)}_{\text{Radial acceleration}}\hat{r} + \underbrace{\left(r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta}\right)}_{\text{Circumferential acceleration}}\hat{\theta}$$

If the force on the particle is central, then which quantity is conserved?

What can you say about the matrix connecting the two sets and the inverse relation ?

CURVILINEAR COORDINATE SYSTEMS - PRESCRIPTION

Write down the relation with (x,y) co-ordinates

Draw the coordinate grid

What happens when the independent variables are changed infinitesimally?

Which direction would we move, if only one variable was changed?

What happens to an element of distance?

What happens to an element of area?

What is the gradient?

What are the velocity components?

What are the acceleration components?

You're now set to solve problems in this coordinate system