

MA111 (IIT Bombay) Tutorial Sheet 5 : Green's theorem, February 22, 2021

1. Verify Green's theorem in each of the following cases:

- (i) $F_1(x, y) = -xy$; $F_2(x, y) = xy$; $R: x \geq 0, 0 \leq y \leq 1 - x^2$;
- (ii) $F_1(x, y) = 2xy$; $F_2(x, y) = e^x + x^2$; where R is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$.

2. Use Green's theorem to evaluate the integral $\oint_{\partial R} y^2 dx + x dy$, where

- (i) R is the square with vertices $(0, 0)$, $(2, 0)$, $(2, 2)$, $(0, 2)$.
- (ii) R is the square with vertices $(\pm 1, \pm 1)$.
- (iii) R is the disc of radius 2 and center $(0, 0)$ oriented clock-wise.

3. For a simple closed curve given in polar coordinates show using Green's theorem that the area enclosed is given by

$$A = \frac{1}{2} \oint_C r^2 d\theta.$$

Use this to compute the area enclosed by the following curves:

- (i) The cardioid: $r = a(1 - \cos \theta)$, $0 \leq \theta \leq 2\pi$.
- (ii) The lemniscate: $r^2 = a^2 \cos 2\theta$, $-\pi/4 \leq \theta \leq \pi/4$.

4. Find the area of the following regions:

- (i) The area lying in the first quadrant of the cardioid $r = a(1 - \cos \theta)$.
- (ii) The region under one arch of the cycloid

$$\mathbf{r} = a(t - \sin t)\mathbf{i} + a(1 - \cos t)\mathbf{j}, \quad 0 \leq t \leq 2\pi.$$

- (iii) The region bounded by the limaçon on

$$r = 1 - 2 \cos \theta, \quad 0 \leq \theta \leq \pi/2$$

and the two axes.

5. Let $D = \{(x, y) \in \mathbb{R}^2 \mid a^2 \leq x^2 + y^2 \leq b^2\}$, where $0 < a < b$. Evaluate

$$\int_{\partial D} x e^{-y^2} dx + [-x^2 y e^{-y^2} + 1/(x^2 + y^2)] dy,$$

where ∂D is positively oriented.

6. Let C be a simple closed curve in the xy -plane. Show that

$$3I_0 = \oint_C x^3 dy - y^3 dx,$$

where $I_0 = \frac{1}{3} \int_D r^2 dx dy$, D is the region enclosed by C . This I_0 is often called 'polar moment of inertia' of the region D .

7. If C is the line segment connecting (x_1, y_1) to the point (x_2, y_2) , show that

$$\int_C x dy - y dx = x_1 y_2 - x_2 y_1.$$

8. Let C be any counterclockwise closed curve in the plane and let \mathbf{n} be the outward unit normal to the curve C . Compute $\oint_C \nabla(x^2 - y^2) \cdot \mathbf{n} ds$.
9. Let D be a region in \mathbb{R}^2 with boundary ∂D satisfying the hypothesis stated in the ‘Green’s theorem’. Let $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a C^2 function.

- (i) Show that $\nabla^2 \phi = \text{div}(\text{grad } \phi)$, where the operator ∇^2 is defined by

$$\nabla^2 \phi(x, y) = \frac{\partial^2 \phi}{\partial^2 x}(x, y) + \frac{\partial^2 \phi}{\partial^2 y}(x, y).$$

The operator ∇^2 is called ‘Laplace operator’.

- (ii) Show that the Green’s Identity holds:

$$\iint_D \nabla^2 \phi d(x, y) = \oint_{\partial D} \frac{\partial \phi}{\partial \mathbf{n}} ds,$$

where \mathbf{n} is the outward unit normal to the curve ∂D .

(Hint. Use the divergence form of Green’s theorem for the vector field $\mathbf{F} = \text{grad } \phi$)

- (iii) Using the above identity, compute

$$\oint_C \frac{\partial \phi}{\partial \mathbf{n}} ds$$

for $\phi = e^x \sin y$, and D the triangle with vertices $(0, 0), (4, 2), (0, 2)$.

10. Let us consider the region $\Omega = \{(x, y) \mid x^2 + y^2 > 1\}$ and the vector field be defined on Ω . Evaluate the following line integrals where the loops are traced in the counter clockwise sense

- (i)

$$\oint_C \frac{y dx - x dy}{x^2 + y^2}$$

where C is any simple closed curve in Ω enclosing the origin.

- (ii)

$$\oint_C \frac{y dx - x dy}{x^2 + y^2}$$

where C is any simple closed curve in Ω not enclosing the origin.

- (iii) Let C be a smooth simple closed curve lying in Ω . Find

$$\oint_C \frac{\partial(\ln r)}{\partial y} dx - \frac{\partial(\ln r)}{\partial x} dy.$$

11. Is there a vector field \mathbf{G} in \mathbb{R}^3 such that

i) $\text{curl } \mathbf{G}(x, y, z) = (x \sin y)\mathbf{i} + (\cos y)\mathbf{j} + (z - xy)\mathbf{k}.$

ii) $\text{curl } \mathbf{G}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{j}.$

12. Show that any vector field defined of the form

$$\mathbf{F}(x, y, z) = f(x) \mathbf{i} + g(y) \mathbf{j} + h(z) \mathbf{k}, \quad \text{in } \mathbb{R}^3,$$

where f, g, h are differentiable functions, is irrotational, i.e., $\text{curl } \mathbf{F} = 0$.

13. Show that any vector field defined of the form

$$\mathbf{F}(x, y, z) = f(y, z) \mathbf{i} + g(x, z) \mathbf{j} + h(x, y) \mathbf{k}, \quad \text{in } \mathbb{R}^3,$$

where f, g, h are differentiable functions, is incompressible, i.e., $\text{div } \mathbf{F} = 0$.