MA 108 - Ordinary Differential Equations

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Outline of the lecture

- Lipschitz continuity
- Existence & uniqueness
- Picard's iteration

Recall

- Let f be defined on D, where D is either a domain or a closed domain of the xy- plane. The function f is said to satisfy Lipschitz condition (with respect to y) in D if $\exists$ a constant $M > 0$ such that

$$|f(x, y_1) - f(x, y_2)| \leq M|y_1 - y_2|$$

  for every $(x, y_1), (x, y_2)$ which belong to D. The constant M is called the Lipschitz constant.
- If f satisfies Lipschitz condition with respect to y in D, then for each fixed x, the resulting function of y is a continuous function of y, for all (x, y) in D.

Does Continuity w.r.t. second variable $\Longrightarrow$ Lipschitz condtn. w.r.t. second variable?

Continuity w.r.t. second variable $\not\Rightarrow$ Lipschitz condtn. w.r.t. second variable.

Example : Consider $f(x, y) = \sqrt{|y|}$.

f is continuous for all y.
Note that f doesn't satisfy Lipschitz condition in any region that includes $y = 0$ as for $y_1 = 0,\ y_2 > 0$, we have

$$\frac{|f(x, y_1) - f(x, y_2)|}{|y_1 - y_2|} = \frac{\sqrt{y_2}}{|y_2|} = \frac{1}{\sqrt{y_2}}$$

which can be made as large as we want by making y_2 smaller. The Lipschitz condition requires that the quotient should be bounded by a fixed constant K.

Result : If f is such that $\frac{\partial f}{\partial y}$ exists and is bounded for all $(x, y) \in D$, then f satisfies Lipschitz condition w.r.t. y in D, where the Lipschitz constant

$$M = l.u.b._{(x,y)\in D} \left|\frac{\partial f}{\partial y}(x, y)\right|.$$

Proof : Mean value theorem

$$\implies f(x, y_1) - f(x, y_2) = (y_1 - y_2)\frac{\partial f}{\partial y}(x, \xi),\ \xi \in (y_1, y_2).$$

$$\begin{aligned} |f(x, y_1) - f(x, y_2)| &= |y_1 - y_2|\left|\frac{\partial f}{\partial y}(x, \xi)\right| \\ &\leq |y_1 - y_2| \ \ l.u.b._{(x,y)\in D}\left|\frac{\partial f}{\partial y}(x, y)\right|. \end{aligned}$$

That is, f satisfies Lipschitz condition.

Example

Consider

$$f(x,y) = y^2 \text{ defined in } D : |x| \le a, \; |y| \le b.$$

$f_y = 2y$ is bounded in D. The Lipschitz contant is

$$M = \; l.u.b._{(x,y)\in D} |\frac{\partial f}{\partial y}(x,y)| = \; l.u.b._{(x,y)\in D} |2y| = 2b.$$

(Verify Lipschitz condition directly!)

Bounded derivative - sufficient condition

Consider

$$f(x,y) = x|y| \ \text{ defined in } D : |x| \le a, \ |y| \le b.$$

$\dfrac{\partial f}{\partial y}$ doesn't exist for any point $(x,0) \in D$. (Why?)
Now f satisfies Lipschitz condition :

$$\begin{aligned} |f(x,y_1) - f(x,y_2)| &= \big|x|y_1| - x|y_2|\big| \\ &= |x| \, \big||y_1| - |y_2|\big| \\ &\le |x| \, |y_1 - y_2| \\ &\le a|y_1 - y_2| \end{aligned}$$

Existence of bounded derivative f_y is a sufficient condition for Lipschitz condition to hold true (not necessary).

Existence - Uniqueness Theorem

Let R be a rectangle containing (x_0, y_0) in the domain D,

- $f(x, y)$ be continuous at all points (x, y) in $R : |x - x_0| < a, \ |y - y_0| < b$ and
- bounded in R, that is, $|f(x, y)| \leq K \ \forall (x, y) \in R$.

Then, the IVP $\boxed{y' = f(x, y), \ y(x_0) = y_0}$ has at least one solution $y(x)$ defined for all x in the interval $|x - x_0| < \alpha$, where

$$\alpha = \min \left\{ a, \frac{b}{K} \right\}.$$

In addition to the above conditions, if f satisfies the Lipschitz condition with respect to y in R, that is,

$$|f(x, y_1) - f(x, y_2)| \leq M|y_1 - y_2| \ \ \forall (x, y_1), (x, y_2) \text{ in } R,$$

then, the solution $y(x)$ defined at least for all x in the interval $|x - x_0| < \alpha$, with α defined above is unique [1].

[1] Existence - Peano, Existence & uniqueness -Picard

A quick check!

1. Is $f(x) = \sin x$ Lipschitz continuous over $\mathbb{R}$? Yes.
2. Is $f(x) = x^2$ globally Lipschitz continuous over $\mathbb{R}$? No.
   (Hint: $\left|\dfrac{f(x_1) - f(x_2)}{x_1 - x_2}\right| = |x_1 + x_2|$)
   However, it is Lipschitz continuous over any closed interval of $\mathbb{R}$. We say that it is locally Lipschitz continuous over $\mathbb{R}$.
3. Is $f(x) = \dfrac{1}{x^2}$ globally Lipschitz continuous on $[\alpha, \infty)$ for any $\alpha > 0$? Yes.

Example 1

Consider

$$y' = y^{1/3} \qquad y(0) = 0 \text{ in } R : |x| \leq a, |y| \leq b.$$

$f(x, y)$ is continuous in R and hence existence is guaranteed.

But $\phi_1(x) = 0$ and $\phi_2(x) = \begin{cases} (\frac{2}{3}x)^{3/2} & \text{if } x \geq 0 \\ 0 & \text{if } x \leq 0 \end{cases}$ are solutions in $-\infty < x < \infty$.

Does this imply Lipschitz condition won't be satisfied?

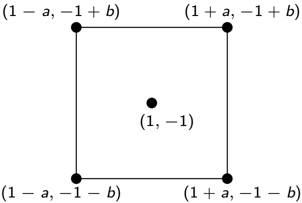
$$\frac{|f(x, y_1) - f(x, y_2)|}{|y_1 - y_2|} = \frac{|y_1^{1/3} - y_2^{1/3}|}{|y_1 - y_2|}.$$

Choosing $y_1 = \delta,\ y_2 = -\delta$, we see that the quotient is unbounded for small values of δ and hence Lipschitz condition is not satisfied.

Solution exists, but not unique.

Example 2

Consider $y' = y^2, \; y(1) = -1$. Find α in the existence & uniqueness theorem.



$f(x, y) = y^2, \; f_y = 2y$ are continuous in the closed rectangle $R: |x-1| \leq a, |y+1| \leq b$.

$$|f(x, y)| = |y|^2 \leq |(-b-1)|^2 \leq (b+1)^2 \tag{1}$$

Now, $\alpha = \min\left\{a, \dfrac{b}{(b+1)^2}\right\}$.

Consider

$$F(b) = \frac{b}{(b+1)^2}.$$

$F'(b) = \dfrac{1-b}{(b+1)^3} \implies$ the maximum value of $F(b)$ for $b > 0$ occurs at $b = 1$ (Why?); and we find $F(1) = \dfrac{1}{4}$.

Hence, if $\boxed{a \geq 1/4}$, $F(b) = \dfrac{b}{(b+1)^2} \leq a$ for all $b > 0$ and

$\alpha = \min\{a, F(b)\} = F(b) = \dfrac{b}{(b+1)^2} \leq 1/4$, whatever be a.

If $\boxed{a < 1/4}$, then certainly $\alpha < 1/4$. Thus in any case, $\boxed{\alpha \leq 1/4}$.

For $b = 1, a \geq 1/4, \alpha = \min\{a, 1/4\} = 1/4$.

Thus the best possible α from the theorem gives that the IVP has a unique solution in $|x-1| \leq 1/4 \implies \boxed{3/4 \leq x \leq 5/4}$.

Example 2 - Remarks

1. The theorem guarantees existence and uniqueness only in a very small interval!
2. The theorem DOES NOT give the largest interval where the solution is unique.
3. What is the solution in this case by separation of variables and where is it valid? Can you think of extending the solution to a larger interval?
   (Ans. $xy = -1$. Largest interval where solution exist is $(0, \infty)$)

[2] AIM : To solve

$$y' = f(x, y), \; y(x_0) = y_0 \tag{2}$$

METHOD

1. Integrate both sides of (2) to obtain

$$y(x) - y(x_0) = \int_{x_0}^{x} f(t, y(t))\, dt$$

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt \tag{3}$$

Note that any solution of (2) is a solution of (3) and vice-versa.

[2] Picard used this in his existence-uniqueness proof

2. Solve (3) by iteration:

$$\begin{aligned} y_1(x) &= y_0 + \int_{x_0}^{x} f(t, y_0)\, dt \\ y_2(x) &= y_0 + \int_{x_0}^{x} f(t, y_1(t))\, dt \\ &\vdots \\ y_n(x) &= y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\, dt \end{aligned}$$

3. Under the assumptions of existence-uniqueness theorem, the sequence of approximations converges to the solution $y(x)$ of (2). That is,

$$y(x) = \lim_{n\to\infty} y_n(x).$$

Example : Picard's

Solve : $y' = xy, \ y(0) = 1$ using Picard's iteration method.

1. The integral equation is

$$y(x) = 1 + \int_{x_0}^{x} ty\,dt.$$

2. The successive approximations are :

$$\begin{aligned}
y_1(x) &= 1 + \int_0^x t \cdot 1\,dt = 1 + \frac{x^2}{2}.\\
y_2(x) &= 1 + \int_0^x t(1 + \frac{t^2}{2})\,dt = 1 + \frac{x^2}{2} + \frac{x^4}{2 \cdot 4}.\\
&\vdots\\
y_n(x) &= 1 + (\frac{x^2}{2}) + \frac{1}{2!}(\frac{x^2}{2})^2 + \cdots + \frac{1}{n!}(\frac{x^2}{2})^n. \text{ (By induction)}
\end{aligned}$$

3. $\boxed{y(x) = \lim_{n\to\infty} y_n(x) = e^{x^2/2}.}$

1. Does uniform continuity $\implies$ Lipschitz continuity ?
   (No, consider $f(x) = \sqrt{x}, \quad x \in [0, 2].$)
2. The value of n such that the curves $x^n + y^n = C$ are the orthogonal trajectories of the family

$$y = \frac{x}{1 - Kx}$$

   is ....?
   (Ans. DE for the given family of curves is $\frac{dy}{dx} = (\frac{y}{x})^2$. Finally, we get n=3).