MA 108 - Ordinary Differential Equations

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Outline of the lecture

- Method of undetermined coefficients
- Annihilator Method
- Laplace transform

Undetermined Coefficients

We have seen the method of undetermined coefficients for the second order case. For the higer order case also, form of the particular solution is as in the second order case.

$r(x) =$	$y_p =$
$x^n e^{ax}$, $a \in \mathbb{R}$ and a is not a root of the charact. eq.	$A_0 e^{ax} + A_1 x e^{ax} + \cdots + A_n x^n e^{ax}$
$x^n e^{ax}$, $a \in \mathbb{R}$ and a is a root of the charact. eq. with multiplicity μ	$x^\mu (A_0 e^{ax} + A_1 x e^{ax} + \cdots + A_n x^n e^{ax})$
$x^n e^{ax} \cos bx \ / \ x^n e^{ax} \sin bx$ $a + \imath b$ is not a root the charact. eq.	$e^{ax}\left(\sum_{k=0}^{n} A_k x^k \cos bx + \sum_{k=0}^{n} B_k x^k \sin bx\right)$
$x^n e^{ax} \cos bx \ / \ x^n e^{ax} \sin bx$ $a + \imath b$ is a root of charact. eq.	$x^\mu e^{ax}\left(\sum_{k=0}^{n} A_k x^k \cos bx + \sum_{k=0}^{n} B_k x^k \sin bx\right)$

Example 1

Find a particular solution of

$$y^{(4)} + 2y'' + y = 3\sin x - 5\cos x.$$

Set $\mathcal{L}y = y^{(4)} + 2y'' + y$. The characteristic equation is

$$m^4 + 2m^2 + 1 = (m^2 + 1)^2 = 0,$$

and therefore a basis of $\mathcal{L}y = 0$ is

$$\{\cos x, \sin x, x\cos x, x\sin x\}.$$

Note tha $\imath$ is a root of multiplicity 2. So the candidate is

$$y = x^2(A\cos x + B\sin x).$$

Substituting, we get:

$$-8A\cos x - 8B\sin x = 3\sin x - 5\cos x \implies$$

$$y(x) = x^2(\frac{5}{8}\cos x - \frac{3}{8}\sin x).$$

Example 2

Find the candidate solution for

$$y^{(4)} - y^{(3)} - y'' + y' = x^2 + 4 + x\sin x.$$

Note that the characteristic equation is

$$m^4 - m^3 - m^2 + m = m(m-1)^2(m+1) = 0.$$

Thus a basis of $\mathcal{L}y = 0$ is $\{1, e^x, xe^x, e^{-x}\}$.
For $x^2 + 4$, form of the particular solution is $x(A_0 + A_1x + A_2x^2)$, since $m = 0$ is a root of the characteristic equation with multiplicity 1.
For $x\sin x$, the form is

$$(B_0 + B_1x)\cos x + (C_0 + C_1x)\sin x.$$

So our final candidate is

$$x(A_0 + A_1x + A_2x^2) + (B_0 + B_1x)\cos x + (C_0 + C_1x)\sin x.$$

Example 3

Find the candidate solution for

$$y^{(4)} - 2y'' + y = x^2 e^x + e^{2x}.$$

Note that the characteristic equation is

$$m^4 - 2m^2 + 1 = (m-1)^2(m+1)^2 = 0.$$

So a basis of $\mathcal{L}y = 0$ is $\{e^x, xe^x, e^{-x}, xe^{-x}\}$.
e^{2x} on the right hand side suggests

$$Ae^{2x}$$

and
$x^2 e^x$ suggests

$$x^2(Bx^2 + Cx + D)e^x,$$

since $m = 1$ is root with multiplicity 2. So final candidate would be

$$Ae^{2x} + x^2(Bx^2 + Cx + D)e^x.$$

Review Question 1

For the non-homogeneous equation $y'' - 5y' + 6y = x\cos x$, the form of y_p using the method of undetermined coefficients is

__________.

(a) $A_1 x\,\cos x + B_1 x\,\sin x$
(b) $(A_0 + A_1 x)\cos x + (B_0 + B_1 x)\sin x$
(c) $x((A_0 + A_1 x)\cos x + (B_0 + B_1 x)\sin x)$
(d) None of the above.

Solution : ✔ (b)

Review Question 2

Solve the IVP : $y'' + (1 + \frac{1}{y})(y')^2 = 0,\ y(0) = 1,\ y'(0) = 1/e.$

- Put $y' = v$. So $y'' = \frac{dv}{dx} = \frac{dy}{dx}\frac{dv}{dy} = v\frac{dv}{dy}$
- The DE becomes $v\frac{dv}{dy} + (1 + \frac{1}{y})v^2 = 0.$
- Solve for v : $v(y) = Ce^{-y}y^{-1}.$
- Use initial conditions to obtain $C = 1.$
- Thus $ye^y v = 1.$ Solve to obtain $ye^y - e^y = x + C_1.$
- Use $y(0) = 1$ to obtain $C_1 = 0.$

Hence, $\boxed{(y - 1)e^y = x}$.

Review Questions 3 & 4

If the Wronskian $W(y_1, y_2)(x) = x^2 e^x,\ x > 0$ and $y_1(x) = x$, then y_2 is ——.

Solve using the method of variation of parameters :

$$(x^2 + 1)(y'' - 2y + 1) = e^x, \qquad y(0) = y'(0) = 1.$$

Annihilator Operator

Recall : If $\mathcal{A}$ is a linear differential operator with constant coefficients and $f(x)$ is a sufficiently smooth differentiable function such that

$$\mathcal{A}f(x) = 0,$$

then $\mathcal{A}$ is said to be the annihilator of the function $f(x)$.
Annihilators for some standard 'basis'functions for nth order linear ODEs with constant coefficients:

1. D^{n+1} annihilates any polynomial of degree n (i.e., it is a solution of DE $D^{n+1}y = 0$.)
2. $D - \alpha$ annihilates $e^{\alpha x}, \alpha \in \mathbb{R}$.
3. $(D - \alpha)^n$ annihilates $x^{n-1}e^{\alpha x}$.
4. $(D^2 - 2\alpha D + (\alpha^2 + \beta^2))^n$ annihilates $e^{\alpha x}\cos\beta x,\ xe^{\alpha x}\cos\beta x, \cdots$ $x^{n-1}e^{\alpha x}\cos\beta x,\ e^{\alpha x}\sin\beta x,\ xe^{\alpha x}\sin\beta x, \cdots\ x^{n-1}e^{\alpha x}\sin\beta x$.

Example 1

Using Annihilation argument, find the form of the particular solution and then compute a particular solution of

$$y^{(4)} - 16y = x^4 + x + 1.$$

Here,

$$\mathcal{L} = D^4 - 16 = (D^2 + 4)(D - 2)(D + 2),$$

and

$$\mathcal{A} = D^5.$$

Basis for $\mathcal{L}y = 0$ is $\{e^{2x}, e^{-2x}, \cos 2x, \sin 2x\}$.

$$\mathcal{A}\mathcal{L} = D^5(D^4 - 16).$$

Hence basis for $\mathcal{A}\mathcal{L}y = 0$ is $\{1, x, x^2, x^3, x^4, e^{2x}, e^{-2x}, \cos 2x, \sin 2x\}$.

Annihilator Method

Any general solution $y = y_1 + y_p$, where y_1 is the complementary function and y_p a particular solution is a solution of $\mathcal{A}\mathcal{L}y = 0$. Hence we look for a solution of the form

$$y_p = A_1 + A_2 x + A_3 x^2 + A_4 x^3 + A_5 x^4.$$

$$(D^4 - 16)y_p = x^4 + x + 1 \Longrightarrow$$

$$24A_5 - 16(A_1 + A_2 x + A_3 x^2 + A_4 x^3 + A_5 x^4) = x^4 + x + 1.$$

Comparing coefficients, get:

$$A_5 = -\frac{1}{16}, A_4 = 0, A_3 = 0, A_2 = -\frac{1}{16}, A_1 = -\frac{5}{32}.$$

Example

Solve the DE:

$$\mathcal{L}y = (D^2 - 5D + 6)(y) = xe^x.$$

Characteristic polynomial for the associated homogeneous equation is

$$m^2 - 5m + 6 = (m-2)(m-3).$$

Basis of $\mathcal{L}y = 0$ is $\{e^{2x}, e^{3x}\}$. Take,

$$\mathcal{A} = (D-1)^2 \implies \mathcal{A}\mathcal{L} = (D-1)^2(D-2)(D-3).$$

Basis of $\mathcal{A}\mathcal{L}y = 0$ is $\{e^{2x}, e^{3x}, e^x, xe^x\}$.
So $y_p = Ae^x + Bxe^x$, i.e., we need to find A, B such that

$$\mathcal{L}(Ae^x + Bxe^x) = xe^x.$$

This gives $(2A - 3B)e^x + 2Bxe^x = xe^x$. Thus,

$$y = \frac{3}{4}e^x + \frac{1}{2}xe^x$$

is a particular solution.

Solve $x^2y'' + 2xy' - 6y = 10x^2$.
Set

$$\mathcal{L} = x^2D^2 + 2xD - 6.$$

The indicial equation is

$$m(m-1) + 2m - 6 = m^2 + m - 6 = (m+3)(m-2) = 0.$$

Basis for $\mathcal{L}y = 0$ is $\{\frac{1}{x^3}, x^2\}$.
Need to find out an appropriate annihilator. Notice that D^3 is an annihilator for $r(x) = 10x^2$. But we don't know how to find a basis for $D^3(x^2D^2 + 2xD - 6)y = 0$, since it is neither with contstant coefficients nor a Cauchy Euler ODE.
Note that $\mathcal{A} = (xD - 2)$ annihilates $10x^2$.

$$\mathcal{A}\mathcal{L}y = (xD-2)(x^2D^2+2xD-6)y = 0$$

is Cauchy Euler.
Note that $x^2D^2 = xD(xD-1)$.
$x^3D^3 = xD(xD-1)(xD-2)$, etc Thus,

$$\begin{aligned}\mathcal{A}\mathcal{L} &= (xD-2)(xD(xD-1)+2xD-6) \\ &= (xD-2)((xD)^2+xD-6) \\ &= (xD-2)(xD-2)(xD+3).\end{aligned}$$

So the basis for $\mathcal{A}\mathcal{L}y = 0$ is $\{x^2, x^2\ln x, \frac{1}{x^3}\}$. Thus, $y_p = Ax^2\ln x$. Substituting y_p into the ODE we get

$$(xD-2)(xD+3)Ax^2\ln x = 10x^2.$$

Therefore $5Ax^2 = 10x^2$, i. e., $A = 2$. Hence $y_p = 2x^2\ln x$. The general solution is $y = c_1x^{-3} + c_2x^2 + 2x^2\ln x$.

Annihilator Method

Exercise: Get candidate solutions by the annihilator method:

1. $y^{(4)} - y^{(3)} - y'' + y' = x^2 + 4 + x \sin x$.
2. $y^{(4)} - 2y'' + y = x^2 e^x + e^{2x}$.

Let $f\colon (0,\infty) \to \mathbb{R}$ be a function. The Laplace transform $L(f)$ of f is the function defined by

$$\boxed{L(f)(s) = \int_0^\infty e^{-st} f(t)dt = \lim_{a\to\infty} \int_0^a e^{-st} f(t)dt, \qquad s > 0,}$$

for all values of s for which the integral exists. Sometimes we denote $F(s) = L(f)(s)$.
The integral above may not converge for every s. We may impose suitable restrictions on f later under which the intergral exists.

Examples

Laplace transform:

(1) $\boxed{f(t) = c \text{ for all } t \geq 0}$.

$$L(c)(s) = \int_0^\infty ce^{-st}dt = c\left[-\frac{e^{-st}}{s}\right]_0^\infty = \frac{c}{s}, s > 0.$$

(2) $\boxed{f(t) = e^{at}, t \geq 0, \ a \text{ being a constant.}}$

$$L(e^{at})(s) = \int_0^\infty e^{-st}e^{at}dt = \int_0^\infty e^{(a-s)t}dt = \frac{1}{s-a},$$

for $s > a$.

(3) $\boxed{f(t) = \sin at, t \geq 0.}$

$$
\begin{aligned}
L(\sin at)(s) &= \int_0^{\infty} e^{-st} \sin at \, dt \\
&= \lim_{b\to\infty} \int_0^{b} e^{-st} \sin at \, dt \\
&= \lim_{b\to\infty} \left[-\frac{e^{-st}\cos at}{a} \right]_0^b - \frac{s}{a} \int_0^{\infty} e^{-st} \cos at \, dt \\
&= \frac{1}{a} - \frac{s^2}{a^2} \int_0^{\infty} e^{-st} \sin at \, dt \\
&= \frac{1}{a} - \frac{s^2}{a^2} L(\sin at)(s)
\end{aligned}
$$

Therefore,

$$
L(\sin at)(s) = \frac{a}{s^2 + a^2}, \ \text{for } s > 0.
$$

Exercises

(4) $\boxed{f(t) = t^2, t \geq 0.}$

$$L(t^2)(s) = \frac{2}{s^3}, \text{ for } s > 0.$$

(5) $\boxed{L(\cos at)(s) = \frac{s}{s^2 + a^2}, s > 0.}$

Example

Prove that $L(t^n) = \dfrac{n!}{s^{n+1}},\ n \in \mathbb{N}.$

We show this by induction.

Show that $L(t) = \dfrac{1}{s^2}$.

$$\begin{aligned}
L(t^{n+1}) &= \int_0^\infty e^{-st} t^{n+1}\, dt \\
&= \cancel{t^{n+1} \frac{e^{-st}}{-s}\Big|_0^\infty} - \int_0^\infty (n+1) t^n \frac{e^{-st}}{-s}\, dt \\
&= \frac{n+1}{s} L(t^n) = \frac{n+1}{s} \frac{n!}{s^{n+1}} \\
&= \frac{(n+1)!}{s^{n+2}}.
\end{aligned}$$

Hence, $\boxed{L(t^n) = \dfrac{n!}{s^{n+1}}.}$

Recap

Function	Laplace transform
c	$\frac{c}{s}$
e^{at}	$\frac{1}{s-a}, \quad s > a$
t^n	$\frac{n!}{s^{n+1}}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$

Existence of Laplace transforms

- For a given f, $L(f)$ may or may not exist.
- Sufficient conditions under which convergence is guaranteed for the integral in the definition of the Laplace transform is that f is piecewise continuous and is of exponential order.
- Piecewise continuity - The function is continuous except possibly for finitely many jump discontinuities.
- A function f is said to be of exponential order if there exists $a \in \mathbb{R}$ and positive constants t_0 and K such that

$$|f(t)| \leq Ke^{at},$$

for all $t \geq t_0 > 0$.

Exponential Order

- In other words, we say f is of exponential order if there exists a constant a such that $e^{-at}|f(t)|$ is bounded for all sufficiently large values of t.
- That is, if f is of exponential order and the values $f(t)$ of f become infinite as $t \to \infty$, these values cannot become infinite more rapidly than a multiple of K of the corresponding e^{at} values of some exponential function.