MA-111 Calculus II (D1 & D2)

Lecture 4

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Existence of Integrals over bounded sets in $\mathbb{R}^2$

Boundary of a subset Let $D \subseteq \mathbb{R}^2$ be a bounded set. A point $x \in \mathbb{R}^2$ is in the boundary of D if there is a sequence $\{x_n\}_n$ in D and a sequence $\{y_n\}_n$ in $\mathbb{R}^2 - D$, such that $\{x_n\}_n \to x$, $\{y_n\}_n \to x$. The set of boundary points of D is denoted by ∂D.

Example. $D = \{(x, y) \mid x^2 + y^2 \leq r^2\}$. The boundary of D, $\partial D = \{(x, y) \mid x^2 + y^2 = r^2\}$.

Example. $R = [a, b] \times [c, d]$. The boundary of rectangle R, $\partial R = \{(a, y) \in \mathbb{R}^2 \mid c \leq y \leq d\} \cup \{(b, y) \in \mathbb{R}^2 \mid c \leq y \leq d\} \cup \{(x, c) \in \mathbb{R}^2 \mid a \leq x \leq b\} \cup \{(x, d) \in \mathbb{R}^2 \mid a \leq x \leq b\}$.

Example. Let $S = \{(x, y) \mid x, y \in \mathbb{Q}\}$. Then $\partial S = \mathbb{R}^2$.

Suppose D is a bounded subset contained in a rectangle R, and $f : D \to \mathbb{R}$ is a bounded *continuous* function extended to $f^* : R \to \mathbb{R}$ by defining $f^* = 0$ on the complement of D. Then all the discontinuities of f^* lie on the boundary ∂D.

Example $\mathrm{D} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$. Let $f(x, y) = x^2 + y^2$ on disk D. Extend the function f to f^* on the rectangle $R = [-2, 2] \times [-2, 2]$.

The points of discontinuity of f^* lie on the the boundary of D i.e., $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.

Convention : A *path* γ in $\mathbb{R}^2$ (or $\mathbb{R}^3$) will mean a continuous function $\gamma : [a, b] \to \mathbb{R}^2$ (or $\gamma : [a, b] \to \mathbb{R}^3$) for $a, b \in \mathbb{R}$. It is said to be *closed* if $\gamma(a) = \gamma(b)$.

By a *curve* γ we mean the image of a path γ in $\mathbb{R}^2$ (or $\mathbb{R}^3$). A "good" curve is always of measure zero (hence content zero).

Theorem

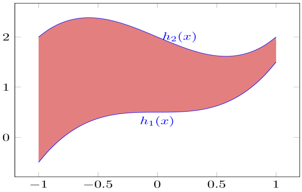
Let $D \subset \mathbb{R}^2$ be a bounded set whose boundary ∂D is given by the finitely many continous closed curves then any bounded and continuous function $f : D \to \mathbb{R}$ is integrable over D.

We will now discuss two classes of regions as in the theorem. They will be called elementary regions of type I and type II. *Continuous functions defined on elementary regions of types I & II are always integrable by the theorem.*

Elementary region: Type 1

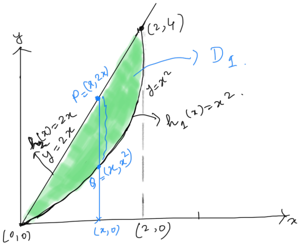
Let $h_1, h_2 : [a, b] \to \mathbb{R}$ be two continuous functions such that $h_1(x) \leq h_2(x)$ for all $x \in [a, b]$. Consider the set of points

$$D_1 = \{(x, y) \mid a \leq x \leq b \text{ and } h_1(x) \leq y \leq h_2(x)\}.$$



Such a region is said to be of *Type 1* and for every $x \in \mathbb{R}$ vertical cross-section of D_1 is an interval.

Example. $D_1 = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2, \quad x^2 \leq y \leq 2x\}$. Here for all $x \in [0, 2]$, $h_1(x) = x^2$ and $h_2(x) = 2x$. Note $h_1(x) \leq h_2(x)$ for $x \in [0, 2]$.

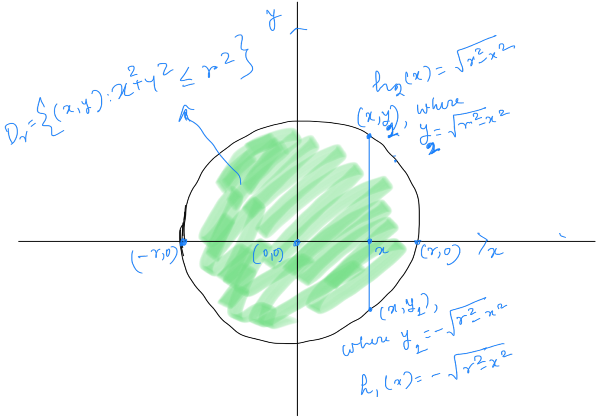


Type 1 contd.

Example. The closed disc D_r of radius r around the origin,

$$D_r := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq r^2\}.$$

Take $h_1(x) = -\sqrt{r^2 - x^2}$ and $h_2(x) = \sqrt{r^2 - x^2}$. We see that D_r is of Type 1.



Evaluating integrals on regions of Type 1

Let D be a region of Type 1 and assume that $f : D \to \mathbb{R}$ is continuous. Let $D \subset R = [\alpha, \beta] \times [\gamma, \delta]$ and let f^* be the corresponding function on R (obtained by extending f by zero).

The region D is bounded by continuous curves (the straight lines $x = a$ and $x = b$ and the graphs of the curves $y = h_1(x)$ and $y = h_2(x)$). Hence we can conclude that f^* is integrable on R. Applying Fubini's theorem on f^* we get,

$$\int\int_D f(x,y)dxdy := \int\int_R f^*(x,y)dxdy = \int_\alpha^\beta \left[\int_\gamma^\delta f^*(x,y)dy\right] dx.$$

In turn, this gives

$$\int_\alpha^\beta \left[\int_{h_1(x)}^{h_2(x)} f^*(x,y)dy\right] dx = \int_a^b \left[\int_{h_1(x)}^{h_2(x)} f(x,y)dy\right] dx,$$

since $f^*(x,y) = 0$ if $y < h_1(x)$ or $y > h_2(x)$. Finally, we get

$$\int\int_D f(x,y)dxdy = \int_a^b \left[\int_{h_1(x)}^{h_2(x)} f(x,y)dy\right] dx.$$

Examples

Example Let $D = \{(x, y) \mid 0 \leq x \leq 2, \quad x^2 \leq y \leq 2x\}$ and $f(x, y) = x + y$. Find $\int\int_D f(x, y) dxdy$.

Ans Note D is a bounded set in $\mathbb{R}^2$ enclosed by the graphs of the curves $y = x^2$ and $y = 2x$ and hence ∂D is of content zero. Since f is continuous over D and D is bounded with ∂D of content zero, f is integrable over D.

$$\int\int_D f(x, y)\, dxdy = \int_0^2 \Big(\int_{x^2}^{2x} (x + y)\, dy \Big) dx = \int_0^2 [xy + \frac{y^2}{2}]_{y=x^2}^{y=2x}\, dx$$
$$= \int_0^2 [2x^2 + 4\frac{x^2}{2} - x^3 - \frac{x^4}{2}]\, dx$$

Example Let $D = \{(x, y) \mid x^2 + y^2 \leq 1, \quad x \geq 0, \quad y \geq 0\}$ and $f(x, y) = \sqrt{1 - y^2}$. Find $\int\int_D f(x, y) dxdy$.

Ans Type 1, i.e, $D = \{(x, y) \mid 0 \leq x \leq 1, \quad 0 \leq y \leq \sqrt{1 - x^2}\}$. Then

$$\int\int_D f(x, y)\, dxdy = \int_0^1 \Big(\int_0^{\sqrt{1-x^2}} \sqrt{1 - y^2}\, dy \Big)\, dx.$$

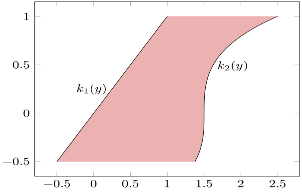
Not easy to compute!

Elementary region: Type 2

Similarly, if $k_1, k_2 : [c, d] \to \mathbb{R}$ are two continuous functions such that $k_1(y) \leq k_2(y)$, for all $y \in [c, d]$. The set of points

$$D_2 = \{(x, y) \,|\, c \leq y \leq d \text{ and } k_1(y) \leq x \leq k_2(y)\}$$

is called a region of Type 2 and for every $y \in \mathbb{R}$ horizontal cross-section of D_2 is an interval.



Example $D_2 = \{(x, y) \mid x^2 + y^2 \leq 1\}$. If we take $k_1(y) = -\sqrt{1 - y^2}$ and $k_2(y) = \sqrt{1 - y^2}$, we see that D_2 is of Type 2.

Evaluating integrals on regions of type 2

Using exactly the same reasoning as in the previous case (basically, interchanging the roles of x and y) we can obtain a formula for regions of Type 2.

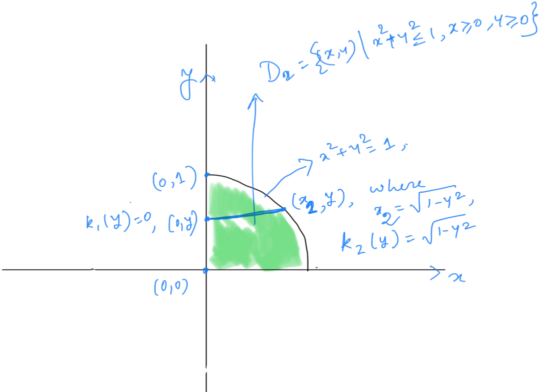
Let D be a bounded set of Type 2 in $\mathbb{R}^2$. Let $f : D \to \mathbb{R}$ be a continuous function on D. We get

$$\int\int_D f(x,y)dxdy = \int_c^d \left[\int_{k_1(y)}^{k_2(y)} f(x,y)dx\right] dy.$$

Example Let $D = \{(x, y) \mid x^2 + y^2 \leq 1, \quad x \geq 0, \quad y \geq 0\}$. Evaluate the integral

$$\int\int_D \sqrt{1-y^2}dxdy.$$

Ans.
$$\begin{aligned}\int\int_D \sqrt{1-y^2}dxdy &= \int_0^1 \left(\int_0^{\sqrt{1-y^2}} \sqrt{1-y^2}\, dx \right) dy \\ &= \int_0^1 [x\sqrt{1-y^2}]_{x=0}^{\sqrt{1-y^2}}\, dy = \int_0^1 (1-y^2)\, dy = \frac{2}{3}.\end{aligned}$$



Remark

There exist bounded subsets of $\mathbb{R}^2$ which are not elementary regions; for example, *star-shaped subset* of $\mathbb{R}^2$ or an *annulus*.

Sometime, we can write D as a union of regions of Types 1 and 2 and then we call it a region of *type 3*.

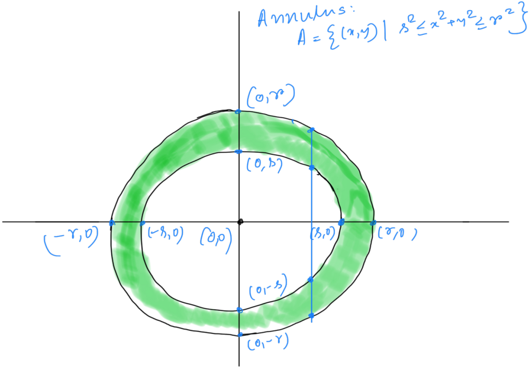
Note the Domain Additivity theorem will then allow us to evaluate integrals which are defined over finite union of such sets.

We could also view the disc as a region of type 3, by dividing it into four quadrants.

Remark contd.

What about the *annulus* $A = \{(x, y) \in \mathbb{R}^2 \mid s^2 \leq x^2 + y^2 \leq r^2\}$?

Is it a type 3 region? yes



Polar Coordinates

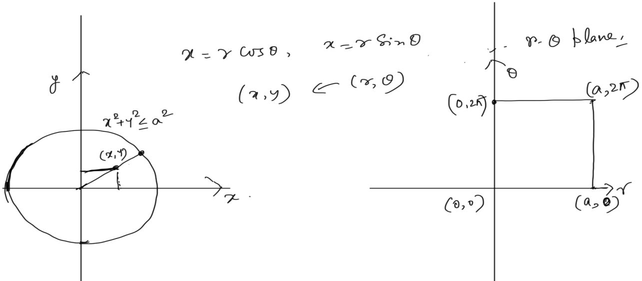
Change of variables from Cartesian coordinate system to polar coordinate system, any $(x, y) \in \mathbb{R}^2$ in Cartesian coordinate can be written as

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad r > 0, \theta \in [0, 2\pi].$$

Transformation of region under change of variables:

Ex. $D := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq a^2\}$ is transformed in polar coordinate system as a rectangle

$$D^* = \{(r, \theta) \mid 0 \leq r \leq a, \quad \theta \in [0, 2\pi]\}.$$



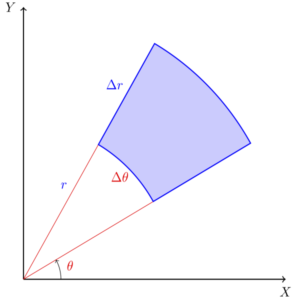
The integral in polar coordinates

Let $D^* = \{(r, \theta) \in [0, \infty) \times [0, 2\pi) : (r\cos(\theta), r\sin(\theta)) \in D\}$,

$$g(r, \theta) := f(r\cos(\theta), r\sin(\theta)), \quad (r, \theta) \in D^*.$$

To integrate the function g on a domain D^* we need to cut up D^* into small rectangles, but these will be rectangles in the r-θ coordinate system.

What shape does a rectangle $[r, r + \Delta r] \times [\theta, \theta + \Delta\theta]$ represent in the x-y plane? A part of a sector of a circle.



Then we will be integrating over this sector instead of rectangle.

What is the area of this part of a sector?

Ans: It is $\frac{1}{2} \cdot [(r+\Delta r)^2 \Delta\theta - r^2\Delta\theta] \sim r^* \Delta r \Delta\theta$, $r \leq r^* \leq r + \Delta r$.

Partitioning the region into subrectangles is equivalent to partitioning the region into parts of sectors as shown earlier.

It follows that the integral we want is approximated by a sum of the form

$$\sum_i \sum_j g(r_i^*, \theta_j^*) r_i^* \Delta r_i \Delta\theta_j,$$

where $\{(r_i^*, \theta_j^*\}$ is a tag for the partition of the "rectangle" in polar coordinates and

$$\int\int_D f(x,y)\, dxdy = \int\int_{D^*} f(r\cos\theta, r\sin\theta)\, rdrd\theta,$$

where D is the image of the region D^*.

This is the change of variable formula for polar coordinates.

Examples

Example1: Integrate $f(x, y) = x^2 + y^2$ on $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$.

Solution: Let us use polar coordinates. Let

$$D^* = \{(r, \theta) \mid 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi\}.$$

Denoting $x = r\cos\theta$ and $y = r\sin\theta$, the polar coordinates will transform D^* to D and

$$g(r, \theta) = f(r\cos\theta, r\sin\theta) = r^2.$$

$$\begin{aligned} \int\int_D f(x, y)\ dxdy &= \int\int_{D^*} g(r, \theta)\ r\ drd\theta = \int\int_{[0,1]\times[0,2\pi]} r^2.r\ drd\theta \\ &= \int_0^{2\pi}\int_0^1 r^3\ drd\theta = \int_0^{2\pi} \frac{r^4}{4}\Big|_0^1\ d\theta = \frac{\pi}{2} \end{aligned}$$

Examples contd.

Example 2: Integrate $f(x,y) = e^{x^2+y^2}$ on $D = \{(x,y) \mid x^2+y^2 \le 1\}$.
Solution: Using the same transformation as above

$$x = r\cos\theta, \quad y = r\sin\theta,$$

we get

$$\begin{aligned}
\int\int_D f(x,y)\ dxdy &= \int\int_{D^*} g(r,\theta)\ r\ drd\theta = \int\int_{[0,1]\times[0,2\pi]} e^{r^2} r\ drd\theta \\
&= \int_0^{2\pi}\int_0^1 e^{r^2} r\ drd\theta = \int_0^{2\pi} \frac{e^{r^2}}{2}\Big|_0^1\ d\theta = \pi(e-1)
\end{aligned}$$

An Applicaton:The integral of the Gaussian

The following improper integral

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx.$$

can be defined as

$$I = \lim_{T \to \infty} \int_{-T}^{T} e^{-x^2} dx.$$

Consider

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \cdot \int_{-\infty}^{\infty} e^{-y^2} dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} e^{-y^2} dxdy.$$

But this iterated integral can be viewed as a double integral on the whole plane. Now under polar coordinates, the plane is sent to the plane.

Hence, we can write this as

$$\int_0^{2\pi} \left[\int_0^{\infty} e^{-r^2} r dr \right] d\theta.$$

But we can now evaluate the inner integral. Hence, we get

$$\int_0^{2\pi} \left[-\frac{1}{2} e^{-r^2} \Big|_0^{\infty} \right] d\theta = \int_0^{2\pi} \frac{1}{2} d\theta = \pi$$

Since $I^2 = \pi$, we see that $I = \sqrt{\pi}$.

Using the above result you can easily conclude that

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}.$$

The integral above arises in a number of places in mathematics - in probability, the study of the heat equation, the study of the Gamma function (next semester) and in many other contexts.

Example

Example Evaluate $\int\int_D (3x+4y^2)\,dxdy$, where D is the region in the upper half-plane bounded by the circled $x^2+y^2=1$ and $x^2+y^2=4$.
Ans The region

$$D = \{(x,y) \mid y \geq 0, \quad 1 \leq x^2+y^2 \leq 4\}.$$

In polar coordinate, after using change of variables $x = r\cos\theta$ and $y = r\sin\theta$, in $r-\theta$ plane, D becomes

$$D^* = \{(r,\theta) \mid 1 \leq r \leq 2, \quad 0 \leq \theta \leq \pi\}.$$

$$\int\int_D (3x+4y^2)\,dxdy = \int_{\theta=0}^{\pi}\int_{r=1}^{2} (3r\cos\theta + 4r^2\sin^2\theta)r\,drd\theta$$
$$= \int_0^{\pi} [r^3\cos\theta + r^4\sin^2\theta]_{r=1}^{2}\,d\theta = \int_0^{\pi} [7\cos\theta + 15\sin^2\theta]\,d\theta = \frac{15\pi}{2}.$$

Triple integrals

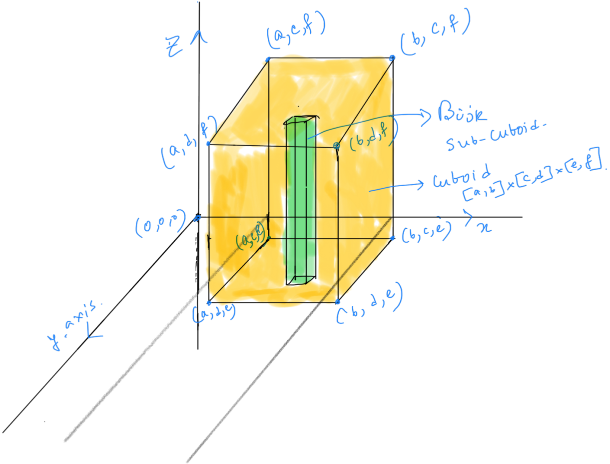
Suppose we have a bounded function $f : B = [a, b] \times [c, d] \times [e, f] \to \mathbb{R}$. We divide the rectangular cuboid B into smaller ones B_{ijk}, and choose tags $t_{ijk} \in B_{ijk}$. The Riemann sum

$$S(f, P, t) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} f(t_{ijk}) \Delta B_{ijk},$$

where ΔB_{ijk} is the volume of B_{ijk}. We say f is Riemann integrable over B with the value of integral S if for every $\epsilon > 0$ there is $\delta > 0$ such that $|S(f, P, t) - S| < \epsilon$ for any tagged partition P with $||P|| < \delta$. We have similar notions of the Darboux integrals $U(f), L(f)$, etc. Riemann integrability and Darboux integrability are again equivalent.
The triple integral is denoted by

$$\iiint_B f dV, \iiint_B f(x, y, z) dV \quad \text{or} \quad \iiint_B f(x, y, z) dxdydz.$$

All the theorems for integrals over rectangles go through for integrals over rectangular cuboids.



Z
(a,c,f)
(b, c, f)
(a, d, f)
(b,d,f)
Book
sub-cuboid.
cuboid
[a,b]×[c,d]×[e,f].
(0,0,0)
(a,c,e)
(b,c,e)
x
(a,d,e)
(b, d, e)
y axis.

Fubini's theorem Let f be integrable on the cuboid B. Then any iterated integral that exists is equal to the triple integral. That is,

$$\iiint_B f(x,y,z)dxdydz = \int_a^b \int_c^d \int_e^f f(x,y,z)dzdydx,$$

whenever the right hand side iterated integral exists.
Similarly for five other possibilities for the iterated integrals.

Proposition Any continuous $f : B \to \mathbb{R}$ is integrable on B and all iterated integrals exist and their values are equal to the integral of f on B.

Integrating over bounded regions Given a bounded region $B \subset \mathbb{R}^3$ and a bounded function $f : B \to \mathbb{R}$, the notion of triple integral of f (i.e. integrability and value of the integral) is defined similarly as in the 2 dimensional case. That is, first realize B as the subset of a rectangular cuboid R, then extend f to a function $f^* : R \to \mathbb{R}$ (by defining $f^* = 0$ outside B). We say f is integrable over B if and only if f^* is integrable over R.

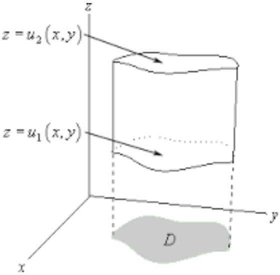
Elementary regions in $\mathbb{R}^3$

The triple integrals that are easiest to evaluate are those for which the region W in space can be described by bounding z between the graphs of two functions in x and y with the domain of these functions being an elementary region in two variables.

For example,

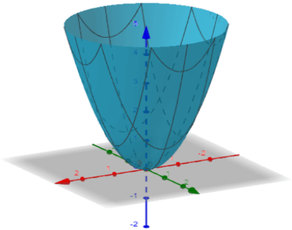
$$W = \{(x, y, z) \in \mathbb{R}^3 \mid \gamma_1(x, y) \leq z \leq \gamma_2(x, y), (x, y) \in D\},$$

where γ_1 and γ_2 are continuous on $D \subset \mathbb{R}^2$ and D is an elementary region in $\mathbb{R}^2$.



Example:

• The region W between the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.



• The region bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x + y = 4$ and $x = z - y - 1$.

Elementary regions (Example)

Suppose that the region W lies between $z = \gamma_1(x, y)$ and $z = \gamma_2(x, y)$. Suppose that the projection of W on the xy plane is bounded by the curves $y = \phi_1(x)$ and $y = \phi_2(x)$ and the straight lines $x = a$ and $x = b$, then for a continuous function f defined over W, we have

$$\iiint_W f(x, y, z)dxdydz = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x, y, z)dzdydx.$$

Example: Let us find the volume of the sphere using the above formula. In other words, let us integrate the function 1 on the region W, where W is the unit sphere, i.e.,

$$\int\int\int_W 1dxdydz = ?, \quad \text{where} \quad W = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}.$$

The volume of the unit sphere

The sphere can be described as the region lying between $z = -\sqrt{1 - x^2 - y^2}$ and $z = \sqrt{1 - x^2 - y^2}$.

The projection of the sphere onto the xy plane gives a disc of unit radius. This can be described as the set of points lying between the curves $-\sqrt{1 - x^2}$ and $\sqrt{1 - x^2}$ and the lines $x = \pm 1$. Thus our triple integral reduces to the iterated integral

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dzdydx.$$

This yields

$$2 \int_{-1}^{1} \left[\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)^{1/2} dy \right] dx.$$

After evaluating the inner integral we obtain

$$2\pi \int_{-1}^{1} \frac{1 - x^2}{2} dx = \frac{4}{3}\pi.$$