MA 108 - Ordinary Differential Equations

Suresh Kumar

Department of Mathematics,
Indian Institute of Technology Bombay,
Powai, Mumbai 76
suresh@math.iitb.ac.in

June 4, 2022

Outline of the lecture

- Method of variation of parameters
- Method of undetermined coefficients

Method of Variation of Parameters - a method to obtain $y_p(x)$

Method of variation of parameters is a powerful method to find a particular solution of non homogeneous linear ODE if we know a basis of solutions of the corresponding homogeneous ODE.
The method is due to Lagrange.
Here, we vary the constants c_1, c_2 in the general solution (in otherwords complementary function)

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

of the corresponding homogeneous equation

$$\mathcal{L}y \equiv y'' + p(x)y' + q(x)y = 0.$$

That is, we replace the constants c_1, c_2 by suitablefunctions $v_1(x), v_2(x)$, so that

$$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$$

is a solution of

$$\mathcal{L}y \equiv y'' + p(x)y' + q(x)y = r(x).$$

Method of Variation of Parameters : Deriving a formula for v_1, v_2

Differentiate y_p we get,

$$y_p' = v_1 y_1' + v_2 y_2' + v_1' y_1 + v_2' y_2.$$

Look for v_1, v_2 using the following Lagrange's Ansatz

$\boxed{v_1' y_1 + v_2' y_2 = 0.}$

Thus $y_p' = v_1 y_1' + v_2 y_2'$ and so

$$y_p'' = v_1 y_1'' + v_1' y_1' + v_2 y_2'' + v_2' y_2'.$$

Substituting y_p, y_p', y_p'' in the given non-homogeneous ODE and rearranging the terms, we get

$$v_1(y_1'' + p y_1' + q y_1) + v_2(y_2'' + p y_2' + q y_2) + v_1' y_1' + v_2' y_2' = r(x).$$

Method of Variation of Parameters : Deriving a formula for v_1, v_2

Thus,

$$v_1' y_1' + v_2' y_2' = r(x).$$

Recall that we also have

$$v_1' y_1 + v_2' y_2 = 0.$$

Thus, we have:

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ r(x) \end{bmatrix}.$$

Method of Variation of Parameters: Deriving a formula for v_1, v_2

Using Cramer's rule,

$$v_1' = \frac{\begin{vmatrix} 0 & y_2 \\ r(x) & y_2' \end{vmatrix}}{W(y_1, y_2)}, \quad v_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & r(x) \end{vmatrix}}{W(y_1, y_2)}.$$

Thus,

$$v_1 = -\int \frac{y_2 r(x)}{W(y_1, y_2)} dx, \quad v_2 = \int \frac{y_1 r(x)}{W(y_1, y_2)} dx.$$

Hence,

$$\boxed{y_p = y_2 \int \frac{y_1 r(x)}{W(y_1, y_2)} dx - y_1 \int \frac{y_2 r(x)}{W(y_1, y_2)} dx.}$$

Hence, the general solution of the non-homogeneous equation is

$$\boxed{y = c_1 y_1 + c_2 y_2 + y_p}$$

Let us try to understand the Lagrange's Ansatz.
Let

$$y_p = v_1 y_1 + v_2 y_2$$

be the particular solution we are looking for the non homogeneous linear ODE

$$y'' + p(x)y' + q(x)y = r(x)$$

Modify the non homogeneous ODE by 'removing' $r(x)$ from ξ onwards.
Then the solution $Y(x, \xi)$ for $x \geq \xi$ of the modified ODE is

$$Y(x, \xi) = v_1(\xi)y_1(x) + v_2(\xi)y_2(x), x \geq \xi.$$

Method of Variation of Parameters

Observe that the original solution y_p and the modified solution $Y(,\xi)$ are tangential at $x = \xi$.
This leads to

$$Y(x,\xi) = y_p(x), \;\; Y'(x,\xi) = y_p'(x)$$

at $x = \xi$. i.e.

$$\begin{aligned} v_1(\xi)y_1(\xi) + v_2(\xi)y_2(\xi) &= y_p(\xi), \; (1) \\ v_1(\xi)y_1'(\xi) + v_2(\xi)y_2'(\xi) &= y_p'(\xi). \; (2) \end{aligned}$$

This must hold for all ξ, since ξ chosen arbitrarily.
Differentiate eq. (1) w.r.to ξ and substract from eq. (2), we get

$$v_1'(\xi)y_1(\xi) + v_2'(\xi)y_2(\xi) = 0$$

the Lagrange's Ansatz.

Example 1

Geometric meaning :$y_p(x)$ is the 'envelop' of the family $\{Y(x,\xi)\}_\xi$ and Lagrange's Ansatz is the tangency condition.

Example 1: Find a particular solution of

$$y'' + y = \csc x.$$

$\boxed{\text{Step I :}}$ Find a basis of solutions for the associated homogeneous equation

$$y'' + y = 0.$$

Following is a basis of solution:

$$y_1(x) = \sin x, \qquad y_2(x) = \cos x.$$

The general solution of this is $y(x) = c_1 y_1 + c_2 y_2$.
$\boxed{\text{Step II :}}$ Calculate the Wronskian $W(y_1, y_2)$:

$$W(y_1, y_2) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1.$$

Now,

$$v_1 = -\int \frac{y_2 r(x)}{W(y_1, y_2)} dx = -\int \frac{\cos x \csc x}{-1} dx = \ln|\sin x|,$$

and

$$v_2 = \int \frac{y_1 r(x)}{W(y_1, y_2)} dx = \int \frac{\sin x \csc x}{-1} dx = -x.$$

Hence, a particular solution is given by

$$y_p(x) = \sin x \ln|\sin x| - x \cos x.$$

What about the general solution ?

Example 2

Find the general solution of

$$y'' - y' - 2y = e^{-x}.$$

A basis of solutions of the corresponding homogeneous equation is

$$y_1 = e^{2x}, \; y_2 = e^{-x}.$$

Now,

$$W(y_1, y_2) = \begin{vmatrix} e^{2x} & e^{-x} \\ 2e^{2x} & -e^{-x} \end{vmatrix} = -3e^{x}.$$

$$v_1 = -\int \frac{y_2 r(x)}{W(y_1, y_2)} dx = -\int \frac{e^{-x}e^{-x}}{-3e^{x}} dx = -\frac{1}{9}e^{-3x},$$

and

$$v_2 = \int \frac{y_1 r(x)}{W(y_1, y_2)} dx = \int \frac{e^{2x}e^{-x}}{-3e^{x}} dx = -\frac{1}{3}x.$$

Hence,

$$y_p = -\frac{1}{9}e^{-3x}e^{2x} - \frac{1}{3}xe^{-x} = -\frac{1}{9}e^{-x} - \frac{1}{3}xe^{-x}.$$

The general solution is

$$y = C_1 e^{2x} + C_2 e^{-x} - \frac{1}{9}e^{-x} - \frac{1}{3}xe^{-x} = C_1 e^{2x} + C_2 e^{-x} - \frac{1}{3}xe^{-x}.$$

Example 3

Find a particular solution of

$$y'' + 4y = 3\cos 2t.$$

A basis of solutions of the corresponding homogeneous equation is

$$y_1 = \cos 2t, \;\; y_2 = \sin 2t,$$

and

$$v_1 = -\int \frac{\sin 2t \cdot 3\cos 2t}{2}dt = \frac{3}{16}\cos 4t,$$

$$v_2 = \int \frac{\cos 2t \cdot 3\cos 2t}{2}dt = \frac{3}{16}\sin 4t + \frac{3}{4}t.$$

Thus, a particular solution is

$$y_p = v_1 y_1 + v_2 y_2 \qquad \text{Complete it!}$$

Method of Undetermined Coefficients

- Method of variation of parameters can be used whenever a basis of solutions is known. But is not always an easy method to execute.
- Introduce another method called the method of undetermined coefficients which is easier to excute for many cases.
- In the method of undetermined coefficients, we assume that a particular solution is known upto some unknown constants. i.e. method works if we know the 'form' of particular solution of the non homogeneous ODE.

- How to find the form of the particular solution of

$$\mathcal{L}y = r(x) \ ?$$

- Let $\mathcal{A}$ be a linear differential operator (annihilator for $r(x)$) such that

$$\mathcal{A}r = 0.$$

- If $\{y_1, y_2\}$ is a basis for $\mathcal{L}y = 0$, then $y = c_1 y_1 + c_2 y_2 + y_p$, [where y_p is a particular solution of $\mathcal{L}y = r(x)$] is a solution of the homogeneous linear ODE

$$\mathcal{A}\mathcal{L}y = 0.$$

Method of Undetermined Coefficients

- Recall that $y = c_1 y_1 + c_2 y_2 + y_p$ is the general solution of $\mathcal{L}y = r(x)$.
- Hence, if we know the basis of solution of

$$\mathcal{A}\mathcal{L}y = 0,$$

then we can get y_p upto some undetermined coefficients by looking at the basis of $\mathcal{A}\mathcal{L}y = 0$ and $\mathcal{L}y = 0$.

Method of Undetermined Coefficients

- $$\mathcal{A}\mathcal{L}y = 0,$$

  is a higher order linear ODE !
- What is a basis of solutions of

  $$\mathcal{A}\mathcal{L}y = 0 \ ?$$

- A basis of nth order linear homogenous ODE

  $$\mathcal{G}y := y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = 0, x \in I,$$

  is a set of n linear independent solutions $y_1, y_2, \cdots, y_n$ of the ODE.
- We say that the functions $y_1, y_2, \cdots y_n$ defined on an interval I are said to be linearly independent if

  $$c_1y_1(x)+c_2y_2(x)+\cdots+c_ny_n(x) = 0 \ \forall \ x \in I \Longrightarrow c_1 = c_2 = \cdots c_n = 0.$$

Consider nth order linear ODE with constant coefficients

$$y^{(n)} + p_1 y^{(n-1)} + \ldots + p_n y = 0.$$

As in the 2nd order linear ODE with constant coefficients, we start with a trial solution $y = e^{rx}$ and can see that

- $y = e^{rx}$ is a solution if r is a root of the characteristic equation

$$P(r) := r^n + p_1 r^{n-1} + \cdots + p_n = 0.$$

- If r is a root (real or complex) of the characteristic function with mulitplicity m, then $e^{rx}, xe^{rx}, \cdots, x^{m-1}e^{rx}$ are linearly independent solutions.
- Observe that $P(r) = P'(r) = \cdots = P^{(m-1)}(r) = 0$ and

$$\begin{aligned} \mathcal{G}(xe^{rx}) &= \mathcal{G}(\frac{\partial}{\partial r}e^{rx}) = \frac{\partial}{\partial r}\mathcal{G}(e^{rx}) = \frac{\partial}{\partial r}(P(r)e^{rx}) \\ &= P(r)xe^{rx} + P'(r)e^{rx} = 0. \end{aligned}$$

- i.e., xe^{rx} is a solution. Similarly we can see the other solutions.

Example 1

Let us first illustrate the above with some examples.
Particular solution of the DE:

$$y'' - 3y' - 4y = 3e^{2x}.$$

- Observe that $\mathcal{A}r = (D-2)(3e^{2x}) = 0$, and $\{e^{-x}, e^{4x}\}$ form a basis for $\mathcal{L}y = y'' - 3y' - 4y = 0$.
- $\mathcal{A}\mathcal{L}y = (D+1)(D-4)(D-2)y$.
- Hence $\{e^{-x}, e^{4x}, e^{2x}\}$ form a basis for $\mathcal{A}\mathcal{L}y = 0$.
- For $y_1 = c_1 e^{-x} + c_2 e^{4x} + y_p$,

$$\mathcal{A}\mathcal{L}y_1 = (D-2)(D^2 - 3D - 4)y_1 = 0.$$

Example 1

- Consider

$$(D^2 - 3D - 4)(c_1 e^{-x} + c_2 e^{4x} + Ae^{2x}) = (D^2 - 3D - 4)(Ae^{2x}).$$

- Hence, look for $y_p = Ae^{2x}$ and determine A, the undetermined coefficient of such that

$$(D^2 - 3D - 4)y_p = 3e^{2x}.$$

- $(Ae^{2x})'' - 3(Ae^{2x})' - Ae^{2x} = 3e^{2x}.$

$$\implies 4Ae^{2x} - 6Ae^{2x} - 4Ae^{2x} = 3e^{2x} \implies A = -\frac{1}{2}.$$

Therefore, $y_p(x) = -\frac{1}{2}e^{2x}$ is a particular solution of the DE.
How do you get the general solution? General solution is

$$y = c_1 e^{4x} + c_2 e^{-x} - \frac{1}{2}e^{2x}.$$

Find the general and a particular solution of the DE:

$$y'' + 5y' + 6y = e^{-3x}.$$

$$(D + 3)e^{-3x} = 0$$

Basis for $\mathcal{L}y = y'' + 5y' + 6y = 0$ is $\{e^{-2x}, e^{-3x}\}$.
Basis for

$$\mathcal{A}\mathcal{L}y = (D^2 + 5D + 6)(D + 3)y = 0$$

is $\{e^{-2x}, e^{-3x}, xe^{-3x}\}$.
General solution of $\mathcal{L}y = e^{-3x}$ is a part of the general solution of $\mathcal{A}\mathcal{L}y = 0$. So look for $y_p = Axe^{-3x}$.
We get: $A = -1$. Write down the general solution.

Example 3

Find a particular solution of

$$\mathcal{L}y = y'' - 3y' - 4y = 2\sin x.$$

Observe

$$\mathcal{A}r = (D^2 + 1)\sin x = 0.$$

The basis for $\mathcal{L}y = 0$ is $\{e^{-x}, e^{4x}\}$.

$$\mathcal{A}\mathcal{L} = (D^2 + 1)(D + 1)(D - 4).$$

$\{e^{-x}, e^{4x}, \cos x, \sin x\}$ is a basis for $\mathcal{A}\mathcal{L}y = 0$.
General solution of $\mathcal{L}y = 2\sin x$ is a part of the general solution of $\mathcal{A}\mathcal{L}y = 0$.

Example 3

So look for y_p in the form, $y_p = A\cos x + B\sin x$. Thus,

$$y_p' = A\cos x - B\sin x; \;\; y_b'' = -A\sin x - B\cos x.$$

Substituting, we get:

$$(-5A + 3B - 2)\sin x + (-3A - 5B)\cos x = 0.$$

Thus,

$$-5A + 3B = 2; \;\; 3A + 5B = 0$$

(Why?).

Thus, $A = -\dfrac{5}{17}$, $B = \dfrac{3}{17}$, and a particular solution is

$$y_p(x) = -\frac{5}{17}\sin x + \frac{3}{17}\cos x.$$

Method of Undetermined Coefficients-Working Rules

$r(x) =$	$y_p =$
$x^n e^{ax}$, $a \in \mathbb{R}$ and a is not a root of the charact. eq.	$A_0 e^{ax} + A_1 x e^{ax} + \cdots + A_n x^n e^{ax}$
$x^n e^{ax}$, $a \in \mathbb{R}$ and a is a root of the charact. eq. with multiplicity μ	$x^\mu (A_0 e^{ax} + A_1 x e^{ax} + \cdots + A_n x^n e^{ax})$
$x^n e^{ax} \cos bx \;/\; x^n e^{ax} \sin bx$ $a + \imath b$ is not a root the charact. eq.	$e^{ax} \left(\sum_{k=0}^{n} A_k x^k \cos bx + \sum_{k=0}^{n} B_k x^k \sin bx \right)$
$x^n e^{ax} \cos bx \;/\; x^n e^{ax} \sin bx$ $a + \imath b$ is a root of charact. eq. with multiplicity μ	$x^\mu e^{ax} \left(\sum_{k=0}^{n} A_k x^k \cos bx + \sum_{k=0}^{n} B_k x^k \sin bx \right)$

If

$$r(x) = r_1(x) + r_2(x) + \ldots + r_n(x),$$

where $r_i(x)$ are e^{ax} or $\sin ax$ or $\cos ax$ or polynomials in x, consider the n subproblems

$$y'' + py' + qy = r_i(x).$$

If $y_i(x)$ is a particular solution of this problem, then,

$$y_p(x) = y_1(x) + y_2(x) + \ldots + y_n(x)$$

is a particular solution of

$$y'' + py' + qy = r(x).$$

Example 4

Find a particular solution of

$$y'' - 3y' - 4y = 4t^2 - 1.$$

$$y_p = At^2 + Bt + C,$$

since $a = 0$ is not a root of the characteristic equation.
Substituting, we get:

$$-4At^2 + (-6A - 4B)t + (2A - 3B - 4C) = 4t^2 - 1.$$

Thus,

$$-4A = 4, \ -6A - 4B = 0, \ 2A - 3B - 4C = -1.$$

Thus,

$$A = -1, \ B = \frac{3}{2}, \ C = -\frac{11}{8}.$$

Thus, a particular solution is

$$y_p = -t^2 + \frac{3}{2}t - \frac{11}{8}.$$

Example 5

Find a particular solution of

$$y'' - 3y' - 4y = -8e^t \cos 2t.$$

Roots of the characteristic equation are $-1, 4$. We should search for a solution of the form

$$y_p = Ae^t \cos 2t + Be^t \sin 2t.$$

Then,

$$y_p'(t) = (A + 2B)e^t \cos 2t + (-2A + B)e^t \sin 2t,$$

and

$$y_p''(t) = (-3A + 4B)e^t \cos 2t + (-4A - 3B)e^t \sin 2t.$$

Substituting, we get:

$$-10A - 2B = -8, \; 2A - 10B = 0.$$

Thus, a particular solution is

$$y_p(t) = \frac{10}{13}e^t \cos 2t + \frac{2}{13}e^t \sin 2t.$$

Example 6

Find a particular solution of

$$y'' + 4y = 3\cos 2t.$$

Since $r(t) = 3\cos 2t$, and roots of the characteristic equation are $\pm 2\imath$, look for solutions of the form

$$y_p = At\cos 2t + Bt\sin 2t.$$

Then,

$$y_p'(t) = (B - 2At)\sin 2t + (A + 2Bt)\cos 2t,$$

$$y_p''(t) = -4At\cos 2t - 4Bt\sin 2t - 4A\sin 2t + 4B\cos 2t.$$

Substituting, we get:

$$-4A\sin 2t + 4B\cos 2t = 3\cos 2t.$$

Thus, $A = 0$, $B = \dfrac{3}{4}$, and a particular solution is $y_p(t) = \dfrac{3}{4}t\sin 2t$.

Example 7

Find a particular solution of

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t + 4t^2 - 1 - 8e^t \cos 2t.$$

Here,

$$r(t) = r_1(t) + r_2(t) + r_3(t) + r_4(t).$$

We need to solve

$$y'' - 3y' - 4y = r_i(t),$$

get a particular solution $y_i(t)$, and then

$$y(t) = y_1(t) + y_2(t) + y_3(t) + y_4(t)$$

is a particular solution of the given problem. Thus, a particular solution is

$$y(t) = -\frac{1}{2}e^{2t} - \frac{5}{17}\sin t + \frac{3}{17}\cos t - t^2 + \frac{3}{2}t - \frac{11}{8} + \frac{10}{13}e^t\cos 2t + \frac{2}{13}e^t \sin 2t.$$