MA-111 Calculus II
(D1 & D2)

Lecture 5

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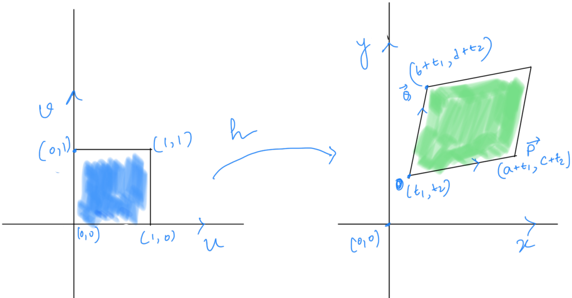
Change of variables in $\mathbb{R}^2$

Suppose we have a change in coordinates given by linear functions composed with translations (such functions are called affine linear functions):

$$x = au + bv + t_1 \quad \text{and} \quad y = cu + dv + t_2.$$

How does the area of the image of a rectangle under this map compare with the area of the original rectangle?

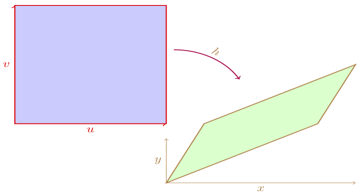


A linear change of coordinates

First, let us write down the affine map in a more compact notation:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$

Clearly, a rectangle $[1,0] \times [0,1]$ in the u-v plane is mapped to a parallelogram in the x-y plane. The sides of the parallelogram are given by $(a + t_1, c + t_2)$ and $(b + t_1, d + t_2)$.



How does one compute the area of this parallelogram?

This is given by the absolute value of the cross product of the vectors,

$$(a, c, 0) \times (b, d, 0) = (ad - bc) \cdot \mathbf{k} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \cdot \mathbf{k}.$$

The area element for a change of coordinates

Let us now suppose that we have a general (not linear any more) change of coordinates given by $x = h_1(u, v)$ and $y = h_2(u, v)$.
How does the area of a rectangle in the u-v plane change? In order to compute the change we need to know the partial derivatives exist.

Let us assume h is a one-one continuously differentiable function .

Noting

$$\Delta x = h_1(u+\Delta u, v+\Delta v) - h_1(u, v), \quad \Delta y = h_2(u+\Delta u, v+\Delta v) - h_2(u, v),$$

and using the chain rule for functions of two variables we see that

$$\Delta x \sim \frac{\partial h_1}{\partial u}\Delta u + \frac{\partial h_1}{\partial v}\Delta v$$

and

$$\Delta y \sim \frac{\partial h_2}{\partial u}\Delta u + \frac{\partial h_2}{\partial v}\Delta v.$$

Using our previous notation, we can write

$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} \frac{\partial h_1}{\partial u} & \frac{\partial h_1}{\partial v} \\ \frac{\partial h_2}{\partial u} & \frac{\partial h_2}{\partial v} \end{pmatrix} \begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix}$$

The Jacobian

The matrix

$$J(h) = \begin{pmatrix} \frac{\partial h_1}{\partial u} & \frac{\partial h_1}{\partial v} \\ \frac{\partial h_2}{\partial u} & \frac{\partial h_2}{\partial v} \end{pmatrix}$$

that appears in the preceding formula is the Jacobian matrix for the function $h = (h_1, h_2) : \mathbb{R}^2 \to \mathbb{R}^2$.

By Taylor's theorem, the Jacobian $J(h)$ is a first order approximation (also called linear approximation) to the function h in a neighborhood of a point, say (u_0, v_0).

In particular, under the change of coordinates h, the area of a small rectangle changes by $|\det J(h)|$.

Theorem (Change of Variables theorem)

Let $h : A \to B$ be a C^1 diffeomorphism of bounded open sets in $\mathbb{R}^n$. Let $f : B \to \mathbb{R}$ be a continuous function. Then f is integrable over B if and only if the function $(f \circ h)|\det J(h)|$ is integrable over A. In this case,

$$\int_B f = \int_A (f \circ h)|\det J(h)|.$$

- h is a C^1 if it is differentiable and $J(h)$ is continuous. h is a C^1 diffeomorphism if h is one-one, onto, differentiable, while $J(h)$ is continuous and invertible on A. In this case $h^{-1} : B \to A$ is also a C^1 diffeomorphism.

- Note that we require the change of variables map h to be $1-1$ and its Jacobian $J(h)$ invertible *only on open sets*.
- In particular, suppose $D^*, D \subset \mathbb{R}^n$ are closed bounded with interiors respectively A, B. Suppose and $h : D^* \to D$ is surjective, and extends to a C^1 function to an open set containing D^*. Suppose $h(A) = B$ and the restriction $h : A \to B$ satisfies the hypothesis the theorem. Also suppose $\partial D^* = D^* - A$ and $\partial D = D - B$ are of measure zero, and f extends to continuous $f : D \to \mathbb{R}$. Then f is integrable on D, $(f \circ h)|\det J(h)|$ is integrable on D^*, and we have

$$\int_D f = \int_{D^*} (f \circ h)|\det J(h)|.$$

Notation

Often we write $x = x(u, v)$ and $y = y(u, v)$. In this case we use the notation $\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$, for the Jacobian determinant.

Let D be a region in the xy plane and D^* a region in the uv plane such that $\phi(D^*) = D$. Then

$$\int\int_D f(x, y)dxdy = \int\int_{D^*} f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dudv.$$

Remark: Note what we get in the familiar case of polar coordinates: $D = \{(x, y) \mid x^2 + y^2 \leq a^2\}$, $D^* = [0, a] \times [0, 2\pi]$.

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \det \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = r,$$

so that the area element $dxdy$ transforms to $rdrd\theta$.

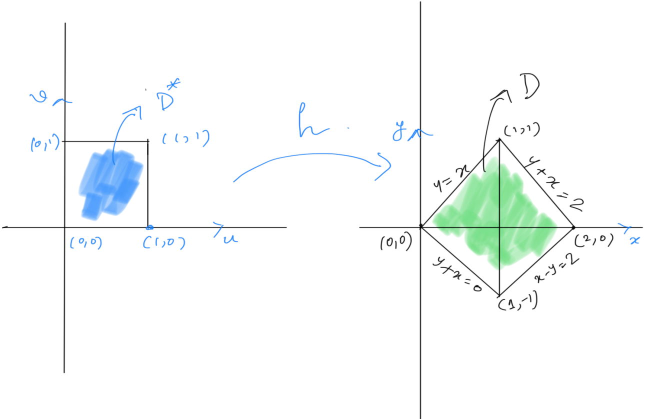
Example Evaluate the integral

$$\iint_D (x^2 - y^2)\,dxdy$$

where D is the square with vertices at $(0,0)$, $(1,-1)$, $(1,1)$ and $(2,0)$.

Solution: Note D is the region in $x-y$ plane bounded by lines $y = x$, $y + x = 0$, $x - y = 2$ and $y + x = 2$.
Put

$$x = u + v, \quad y = u - v,$$



Then the rectangle

$$D^* = \left\{(u, v) \in \mathbb{R}^2 \mid 0 \leq u \leq 1, \;\; 0 \leq v \leq 1\right\}$$

in the uv-plane gets mapped to D, in the xy-plane.
Further,

$$\frac{\partial(x, y)}{\partial(u, v)} = \det\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -2.$$

$$\int\int_D (x^2 - y^2)\, dxdy = \int\int_{D^*} (4uv) \times 2\, dudv$$

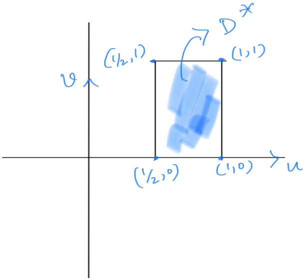
$$= 8\left(\int_0^1 udu\right)\left(\int_0^1 vdv\right) = 2.$$

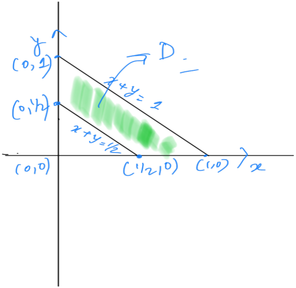
Example

Let D be the region in the first quadrant of the xy-plane bounded by the lines $x + y = \frac{1}{2}$ and $x + y = 1$. Find $\iint_D dA$ by transforming it to $\iint_{D^*} dudv$, where $u = x + y, v = \frac{y}{x+y}$.

Solution: Put

$$x = u(1 - v), \;\; y = uv.$$





Then the rectangle $D^* = \{(u, v) \in \mathbb{R}^2 \mid \frac{1}{2} \leq u \leq 1, \quad 0 \leq v \leq 1\}$ in the uv-plane gets mapped to D in the xy-plane.
Further,

$$\frac{\partial(x, y)}{\partial(u, v)} = \det\begin{pmatrix} 1 - v & -u \\ v & u \end{pmatrix} = u \neq 0.$$

Hence,

$$\mathrm{Area}(D) = \int\int_D dA = \int\int_{D^*} |u| du dv$$

$$= \left(\int_{\frac{1}{2}}^{1} \frac{u^2}{2} du\right)\left(\int_0^1 dv\right) = \frac{3}{4}.$$

The change of variables formula in three variables

In three variables, we once again have a formula for a change of variables. The formula has the same form as in the two variable case:

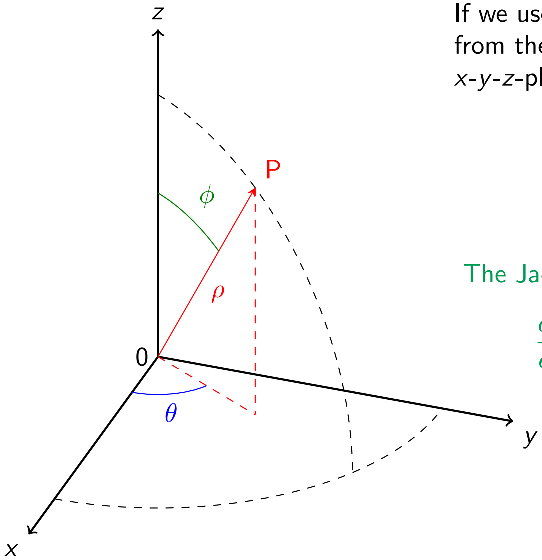
$$\iiint_P f(x,y,z)dxdydz = \iiint_{P^*} g(u,v,w)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw,$$

where $h(P^*) = P$. If the change in coordinates is given by $h = (h_1, h_2, h_3)$, the function g is defined as $g = f(h_1, h_2, h_3)$. The expression

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}$$

is just the Jacobian determinant for a function of three variables.

Spherical Coordinates



If we use (ρ, θ, ϕ) what is the map from these coordinates to the x-y-z-planes?

$$\begin{aligned} x &= \rho \sin\phi \cos\theta \\ y &= \rho \sin\phi \sin\theta \\ z &= \rho \cos\phi. \end{aligned}$$

The Jacobian determinant is

$$\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} x = \rho^2 \sin\phi.$$

Example

Example: It should be much easier computing the volume of the unit sphere now. Let $W = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}$.

Then $W^* = \{(\rho, \theta, \phi) \in \mathbb{R}^3 \mid 0 \leq \rho \leq 1, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi\}$.
Then,

$$\begin{aligned}
\iiint_W dxdydz &= \iiint_{W^*} \rho^2 \sin\phi \; d\rho d\theta d\phi \\
&= \int_0^{\pi} \int_0^{2\pi} \int_0^{1} \rho^2 \sin\phi \; d\rho d\theta d\phi \\
&= \frac{2\pi}{3} \int_0^{\pi} \sin\phi \; d\phi = \frac{4\pi}{3}
\end{aligned}$$

Cylindrical coordinates in formulae

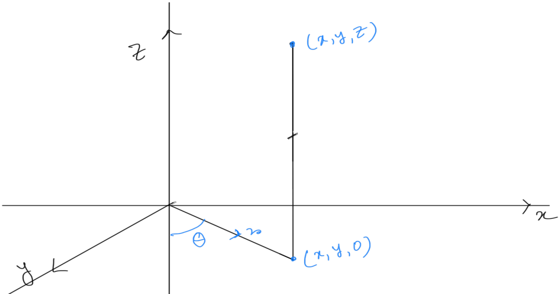
We can also consider a generalization of the polar coordinates. In this case, we use the change of transformation from (r, θ, z) coordinates to $P = (x, y, z) \in \mathbb{R}^3$ given by

$$x = r\cos\theta, \quad y = r\sin\theta \quad \text{and} \quad z = z.$$

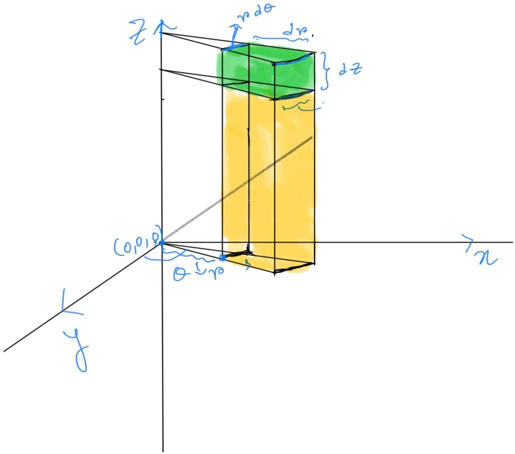
Here $r \geq 0$ and $0 \leq \theta \leq 2\pi$ and the (r, θ, z) are

It is very easy to see that

$$\left|\frac{\partial(x, y, z)}{\partial(r, \theta, z)}\right| = r.$$



The good thing about our convention is that θ means the same thing in both the cylindrical and spherical coordinate systems as well as in the (two-dimensional) polar coordinate system, and r means the same thing in both the cylindrical and (two-dimensional) polar coordinate systems.



Example

Evaluate $\int\int\int_W z^2(x^2+y^2)\,dxdydz$, where W is the cylindrical region determined by $x^2+y^2 \leq 1$ and $-1 \leq z \leq 1$.

Solution. The region W is described in cylindrical coordinates as W^*

$$W^* = \{(r,\theta,z) \mid 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi, \quad -1 \leq z \leq 1\}.$$

$$\int\int\int_W z^2(x^2+y^2)\,dxdydz = \int_{z=-1}^{1}\int_{\theta=0}^{2\pi}\int_{r=0}^{1} z^2 r^2\, r\, drd\theta dz$$
$$= \int_{-1}^{1} \frac{2\pi}{4} z^2\,dz = \frac{\pi}{3}.$$

$\mathbb{R}^n$

Let $n \in \mathbb{N}$ and $\mathbb{R}^n$ be the Euclidean space defined by

$$\mathbb{R}^n := \{(x_1, x_2, \cdots, x_n) \mid x_j \in \mathbb{R}; \quad \forall\, j = 1, 2, \cdots, n\},$$

equipped with the norm

$$\|x\| := \Big(\sum_{j=1}^{n} |x_j|^2\Big)^{\frac{1}{2}}.$$

- Any real number is called a scalar.
- For $n \in \mathbb{N}$, any element from $\mathbb{R}^n$ is called vector. Note this means elements of $\mathbb{R}$ can be thought of both as a scalar and vector. To avoid confusion we will talk about vectors in $\mathbb{R}^n$ for $n > 1$.

Basic structure:

For any $x := (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ and $y := (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$ and any $a \in \mathbb{R}$:

$$x+y := (x_1+y_1, x_2+y_2, \cdots, x_n+y_n) \in \mathbb{R}^n, \quad \text{sum of two elements in} \quad \mathbb{R}^n$$

$$ax := (ax_1, ax_2, \cdots, ax_n) \in \mathbb{R}^n, \quad \text{Scalar multiplication.}$$

Scalar fields and Vector fields

Let D be a subset of $\mathbb{R}^n$.

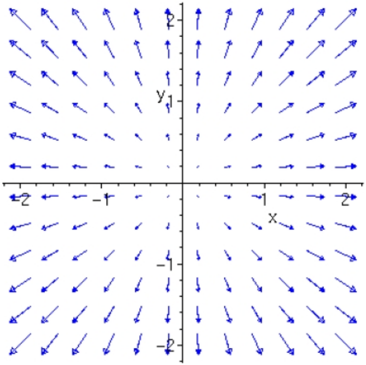
Definition: A scalar field on D is a map $f : D \to \mathbb{R}$.

Definition A vector field on D is a map $\mathbf{F} : D \to \mathbb{R}^n$. We choose $n \geq 2$.

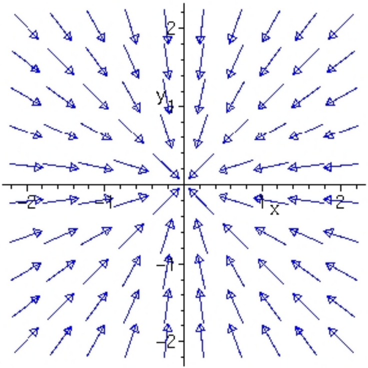
- A scalar field associates a number to each point of D, whereas a vector field associates a vector (of the same space) to each point of D.
- The temperature at a point on the earth is a scalar field.
- The velocity field of a moving fluid, a field describing heat flow, the gravitational field, the magnetic field etc are examples of various vector fields.

Vector fields: Examples

$\mathbf{F}_1(x, y) = (2x, 2y)$

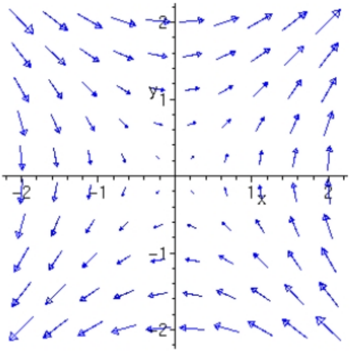


$\mathbf{F}_2(x, y) = \left(\frac{-x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$

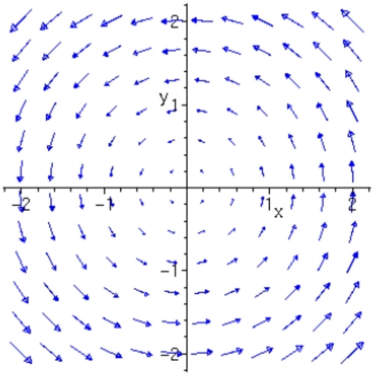


Vector fields: Examples

$\mathbf{F}_3(x, y) = (y, x)$

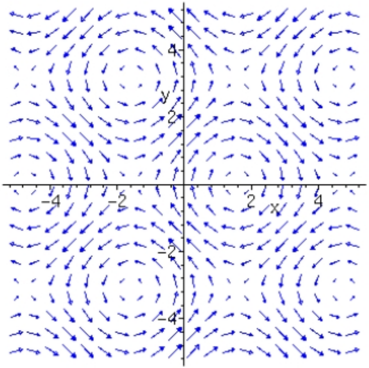


$\mathbf{F}_4(x, y) = (-y, x)$



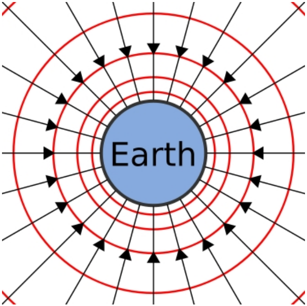
Vector fields: Examples

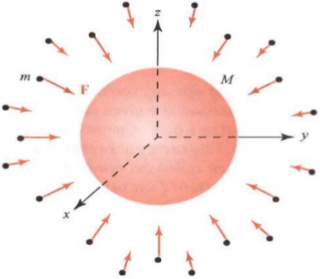
$$\mathbf{F}_5(x, y) = (\sin y, \cos x)$$



The vector fields also occur in nature. Some of this you may have seen in MA 109 as well.

Gravitation fields





The first figure describes the gravitational field of the earth whereas the second one describes that of a body with mass M. The red lines denote the direction of the force exerted on the small particles around the body.

Del operator on Functions

We will assume from now on that our vector fields are smooth wherever they are defined.

One important class of vector fields are those that are given by the gradient of a scalar function. We will study these in some detail later.

The del operator on functions: Gradient field We define the del operator restricting ourselves to the case $n = 3$:

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}.$$

The del operator acts on functions $f : \mathbb{R}^3 \to \mathbb{R}$ to give a gradient vector field :

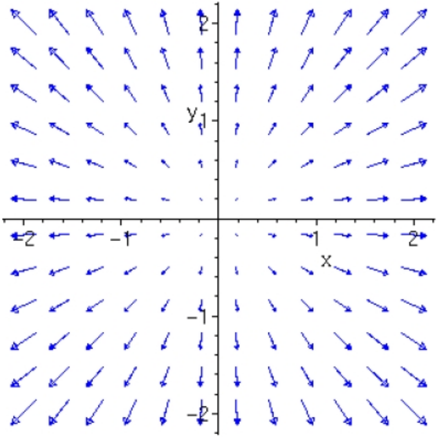
$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

Thus the del operator takes scalar functions to vector fields.

Definition

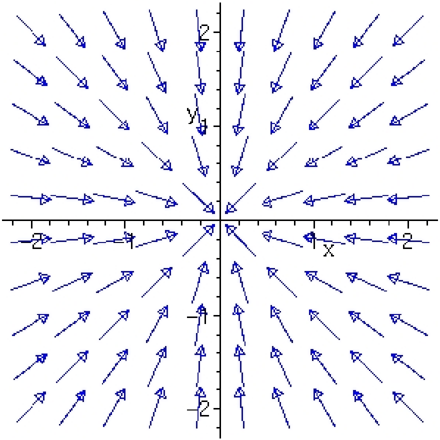
Let $f : D \subset \mathbb{R}^n \to \mathbb{R}$, $n = 2, 3$ be a differentiable function. Then the vector field associated to ∇f is called a gradient vector field.

Gradient fields



$$\mathbf{F}_1(x, y) = (2x, 2y) = \nabla(x^2 + y^2)$$

Gradient fields



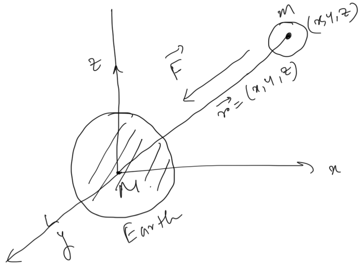
$$\mathbf{F}_2(x,y) = \left(\frac{-x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right) = \nabla\left(-\ln(\sqrt{x^2+y^2})\right)$$

Gradient Vector fields

Gravitational force field is a gradient field: Placing the origin of a coordinate system at the center of the earth (assumed spherical) with mass M, the force of attraction of the earth on a mass m whose position vector is $\mathbf{r}(x, y, z) = (x, y, z)$ can be given by Newton's Law

$$\mathbf{F}(x, y, z) = -\frac{mMG}{|\mathbf{r}(x, y, z)|^3}\mathbf{r}(x, y, z) = -\nabla V(x, y, z),$$

where $V(x, y, z) = -\frac{mMG}{|\mathbf{r}(x,y,z)|}$. Note that the gravitational force exerted on the mass m acts towards the origin.

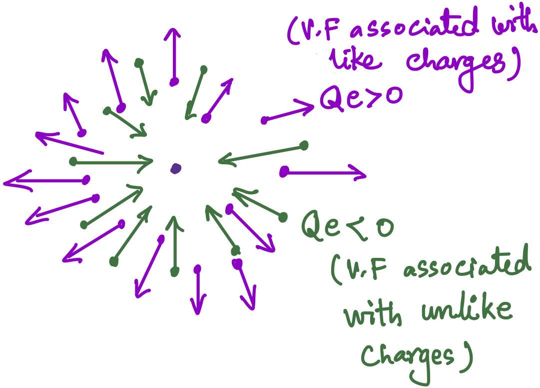


Gradient Vector fields contd.

Coulomb's law says that the force acting on a charge e at a point $\mathbf{r}(x, y, z) = (x, y, z)$ due to a charge Q at the origin is

$$\mathbf{F}(x, y, z) = -\nabla V(x, y, z)$$

where $V(x, yz) = \epsilon Qe/|\mathbf{r}(x, y, z)|$ is the potential. For like charges $Qe > 0$ force is repulsive and for unlike charges $Qe < 0$ the force is attractive.



Definition (Conservative vector field)

A vector field **F** is called a conservative vector field if it is a gradient of some scalar function, i..e, there exists a differentiable scalar function U such that $\mathbf{F} = -\nabla U$. In this case, U is called a potential function for **F**.

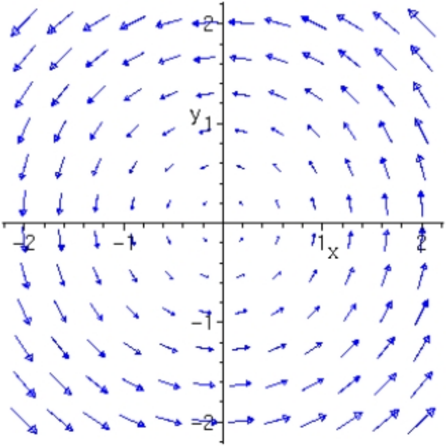
Conservative forces are important as work done along a path will be only dependent on the end points.

Several of the examples we have seen turn out to be gradient vector fields/ conservative vector field. The natural question to ask is which vector field is a gradient field.

There is a neat answer to the above question, which we will see later. Application of the 'Fundamental theorem for line integrals'.

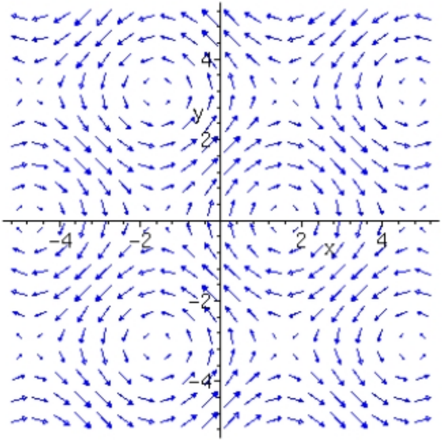
Not all vector fields will turn out to be gradient vector field.

Not gradient fields



$\mathbf{F}_4(x, y) = (-y, x)$, this vector field is not ∇f for any f.

Not gradient fields



$\mathbf{F}_5(x, y) = (\sin y, \cos x)$, this vector field is not ∇f for any f.

How do you check this?