Quantum Information

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Homework 2

1. Distinguishing two non-orthogonal states

$$|\psi_1> = |0>, |\psi_2> = |+> = \frac{1}{\sqrt{2}}(|0> + |1>)$$
$$\mathrm{E}_+ = \alpha|1><1|, E_0 = \alpha|-><-|, E_\eta = \mathbb{I} - E_+ - E_0$$
$$|-> = \frac{1}{\sqrt{2}}(|0> - |1>)$$

1-1. Find the maximum of α

We suppose $\mathrm{E} = \{E_+, E_0, E_\eta\}$ is POVM. There are two constraints in POVM:

A. $\Sigma_\mathrm{m} E_m = \mathbb{I}$, and this is obviously true in E.

B. All measurement is PSD, and it means that all the eigenvalues of measurement are non-negative.

Therefore,

$$<\psi_1|E_+|\psi_1> = 0, \quad <\psi_2|E_+|\psi_2> = \frac{\alpha}{2}$$
$$<\psi_1|E_0|\psi_1> = \frac{\alpha}{2}, \quad <\psi_2|E_0|\psi_2> = 0$$
$$<\psi_1|E_\eta|\psi_1> = 1 - \frac{\alpha}{2}, \quad <\psi_2|E_\eta|\psi_2> = 1 - \frac{\alpha}{2}$$

These are possibility of measurements, and we know that possibility must be non-negative, so we can get these results:

$$2 \geq \alpha \geq 0$$

Now we calculate the E_η.

$$\mathrm{E}_\eta = \mathbb{I} - E_+ - E_0$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \alpha \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \frac{\alpha}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{\alpha}{2} & -\frac{\alpha}{2} \\ -\frac{\alpha}{2} & 1 - \frac{3\alpha}{2} \end{pmatrix}$$

Because of the constraint B, we can do this.

$$\begin{vmatrix} 1-\frac{\alpha}{2}-\lambda & -\frac{\alpha}{2} \\ -\frac{\alpha}{2} & 1-\frac{3\alpha}{2}-\lambda \end{vmatrix} = 0, for\ \lambda \geq 0$$

$$\left(1-\frac{\alpha}{2}-\lambda\right)\left(1-\frac{3\alpha}{2}-\lambda\right)-\frac{\alpha^2}{4} = 1-2\alpha+\frac{\alpha^2}{2}+2\alpha\lambda-2\lambda+\lambda^2$$

$$= (\alpha+\lambda)^2-2(\alpha+\lambda)+1-\frac{\alpha^2}{2} = \left(\frac{\sqrt{2}+1}{\sqrt{2}}\alpha+\lambda+1\right)\left(\frac{\sqrt{2}-1}{\sqrt{2}}\alpha+\lambda+1\right) = 0$$

We want to find the biggest α , therefore we need to put the smallest $\lambda = 0$ into the equation.

$$\alpha = 2-\sqrt{2} \text{ or } 2+\sqrt{2}$$

We already know that $2 \geq \alpha \geq 0$, so the maximum of α is:

$$\alpha = 2-\sqrt{2}$$

1-2. What is $\mathrm{P}(\mathrm{E}_\eta)$ on $|\psi_1>$ if α is maximum?

$$\mathrm{P}(\mathrm{E}_\eta) = <\psi_1|E_\eta|\psi_1>$$
$$=<\psi_1||\psi_1>+<\psi_1|E_+|\psi_1>+<\psi_1|E_0|\psi_1>$$
$$= 1+0-\frac{\alpha}{2} = 1-\left(1-\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$$

2. Stern-Gerlach experiment

$$M_{\uparrow} = |\uparrow><\uparrow|, M_{\downarrow} = |\downarrow><\downarrow|$$

$$M_{\uparrow}^{\Delta t} = \sqrt{\frac{1+\Delta t}{2}}|\uparrow><\uparrow| + \sqrt{\frac{1-\Delta t}{2}}|\downarrow><\downarrow|$$

$$M_{\downarrow}^{\Delta t} = \sqrt{\frac{1-\Delta t}{2}}|\uparrow><\uparrow| + \sqrt{\frac{1+\Delta t}{2}}|\downarrow><\downarrow|$$

2-1.

The constraint of measurement is $\Sigma_j M_j^{\dagger} M_j = \mathbb{I}$.

$$(M_{\uparrow}^{\Delta t})^{\dagger} M_{\uparrow}^{\Delta t} + (M_{\downarrow}^{\Delta t})^{\dagger} M_{\downarrow}^{\Delta t} = |\uparrow><\uparrow| + |\downarrow><\downarrow| = \mathbb{I}$$

2-2.

First, let's suppose $\Delta t \to 0$.

$$\lim_{\Delta t \to 0} M_{\uparrow}^{\Delta t} = \sqrt{\frac{1}{2}}|\uparrow><\uparrow| + \sqrt{\frac{1}{2}}|\downarrow><\downarrow|$$

$$\lim_{\Delta t \to 0} M_{\downarrow}^{\Delta t} = \sqrt{\frac{1}{2}}|\uparrow><\uparrow| + \sqrt{\frac{1}{2}}|\downarrow><\downarrow|$$

We can see that the measurements of spin up and spin down are the same; therefore, it will not give us any information about the state of particle. We can test it in formula of Shannon entropy.

$$H = -\Sigma_{\mathrm{i}} p_i \ln(p_i) = -\ln\left(\frac{1}{2}\right) = \ln(2)$$

Therefore, $\ln(2)$ is the biggest entropy in two system, and it means that it cannot tell what the measurement will be.

The process of measurement will not change the spin, because the possibilities of spin up and down are the same.

Next, let's suppose $\Delta t \rightarrow 1$.

$$\lim_{\Delta t \rightarrow 1} M_{\uparrow}^{\Delta t} = |\uparrow><\uparrow| = M_{\uparrow}$$

$$\lim_{\Delta t \rightarrow 1} M_{\downarrow}^{\Delta t} = |\downarrow><\downarrow| = M_{\downarrow}$$

We can see that measurements become projectors, and it can clearly tell us whether the spin of particle is up or down. However, after measurement, the particle will change the state into only spin up or spin down. We can test in formula of Shannon entropy.

$$H = -\Sigma_{\mathrm{i}} p_i \ln(p_i) = -\ln(1) = 0$$

It doesn't have any entropy, and it means it can clearly tell us the measurement result.

2-3.

Let $|\psi> = \alpha|\uparrow> + \beta|\downarrow>$

$$P_{\uparrow}(\alpha, \beta) = <\psi \left| M_{\uparrow}^{\dagger} M_{\uparrow} \right| \psi> = |\alpha|^2$$
$$P_{\downarrow}(\alpha, \beta) = <\psi \left| M_{\downarrow}^{\dagger} M_{\downarrow} \right| \psi> = |\beta|^2$$

However, we know that $P_{\uparrow}(\alpha, \beta) + P_{\downarrow}(\alpha, \beta) = 1$.

$$P_{\downarrow}(\alpha, \beta) = |\beta|^2 = 1 - |\alpha|^2$$

Therefore, the possibility can be written as below:

$$P_{\uparrow}(\alpha, \beta) = P_{\uparrow}(|\alpha|) = |\alpha|^2$$
$$P_{\downarrow}(\alpha, \beta) = P_{\downarrow}(|\alpha|) = 1 - |\alpha|^2$$

2-4.

Let $|\psi_t> = \alpha_t|\uparrow> + \beta_t|\downarrow>$, and the result of measurement on $|\psi_t>$ is $|\psi_{t+1}> = \alpha_{t+1}|\uparrow>$ $+\beta_{t+1}|\downarrow>$

First, we measure spin up.

$$<\psi_t|(M_\uparrow^{\Delta t})^\dagger M_\uparrow^{\Delta t}|\psi_t> = \left(\frac{1+\Delta t}{2}\right)|\alpha_t|^2 + \left(\frac{1-\Delta t}{2}\right)|\beta_t|^2 = \frac{1}{2}(1+\Delta t(|\alpha_t|^2 - |\beta_t|^2))$$

$$M_\uparrow^{\Delta t}|\psi_t> = \sqrt{\frac{1+\Delta t}{2}}\alpha_t|\uparrow> + \sqrt{\frac{1-\Delta t}{2}}\beta_t|\downarrow>$$

$$|\psi_{t+1,\uparrow}> = \frac{M_\uparrow^{\Delta t}|\psi_t>}{\sqrt{(<\psi_t|(M_\uparrow^{\Delta t})^\dagger M_\uparrow^{\Delta t}|\psi_t>)}} = \frac{(\sqrt{1+\Delta t})\alpha_t|\uparrow> + (\sqrt{1-\Delta t})\beta_t|\downarrow>}{\sqrt{1+\Delta t(|\alpha_t|^2 - |\beta_t|^2)}}$$

$$\alpha_{t+1} = \frac{(\sqrt{1+\Delta t})\alpha_t}{\sqrt{1+\Delta t(|\alpha_t|^2 - |\beta_t|^2)}}, \quad \beta_{t+1} = \frac{(\sqrt{1-\Delta t})\beta_t}{\sqrt{1+\Delta t(|\alpha_t|^2 - |\beta_t|^2)}}$$

Next, we measure spin down.

$$<\psi_t|(M_\downarrow^{\Delta t})^\dagger M_\downarrow^{\Delta t}|\psi_t> = \left(\frac{1-\Delta t}{2}\right)|\alpha_t|^2 + \left(\frac{1+\Delta t}{2}\right)|\beta_t|^2 = \frac{1}{2}(1+\Delta t(-|\alpha_t|^2 + |\beta_t|^2))$$

$$M_\downarrow^{\Delta t}|\psi_t> = \sqrt{\frac{1-\Delta t}{2}}\alpha_t|\uparrow> + \sqrt{\frac{1+\Delta t}{2}}\beta_t|\downarrow>$$

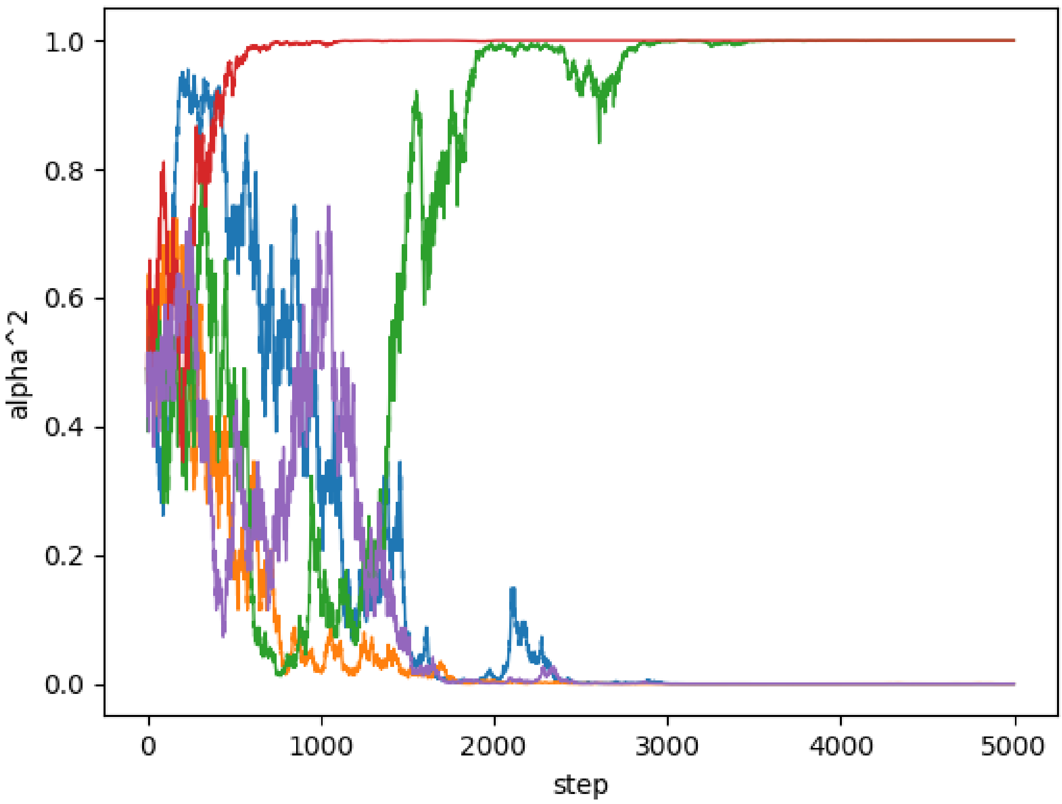
$$|\psi_{t+1,\downarrow}> = \frac{M_\downarrow^{\Delta t}|\psi_t>}{\sqrt{<\psi_t|(M_\downarrow^{\Delta t})^\dagger M_\downarrow^{\Delta t}|\psi_t>}} = \frac{(\sqrt{1-\Delta t})\alpha_t|\uparrow> + (\sqrt{1+\Delta t})\beta_t|\downarrow>}{\sqrt{1+\Delta t(|\beta_t|^2 - |\alpha_t|^2)}}$$

$$\alpha_{t+1} = \frac{(\sqrt{1-\Delta t})\alpha_t}{\sqrt{1+\Delta t(|\beta_t|^2 - |\alpha_t|^2)}}, \quad \beta_{t+1} = \frac{(\sqrt{1+\Delta t})\beta_t}{\sqrt{1+\Delta t(|\beta_t|^2 - |\alpha_t|^2)}}$$

2-5.

We suppose that $\alpha_0 = 0.7, \Delta t(span\ of\ each\ step) = 0.05, T(total\ step) = 5000$

I use Python to simulate the measurement, and this is the result:



The code of this question:

```
def test_1():
    deltat = 0.05
    for jdx in range(5):
        result = []
        num = []
        alpha = 0.7
        for idx in range(5000):
            alsq = pow(alpha, 2)
            result.append(alsq)
            num.append(idx)
            R = random.random()
            prob = 0.5 * math.sqrt(1.0 + deltat * (2 * pow(alpha, 2) - 1))
            if prob >R:
                alpha = (math.sqrt(1.0+deltat)*alpha)/(math.sqrt(1.0+deltat*(2.0*pow(alpha,2)-1.0)))
                continue
            else:
                alpha = (math.sqrt(1.0 - deltat) * alpha) / (math.sqrt(1.0 - deltat * (2.0*pow(alpha,2)-1.0)))
                continue
        plt.plot(num,result,linewidth=1)
    plt.ylabel("alpha^2")
    plt.xlabel("step")
    plt.show()


test_1()
```

2-6.

The code:

```
def test_2():
    deltat = 0.05
    result = []
    for jdx in range(10000):
        alpha = 0.6
        for idx in range(5000):
            R = random.random()
            prob = 0.5*math.sqrt(1.0 + deltat * (2*pow(alpha, 2) - 1))
            if prob > R:
                alpha = (math.sqrt(1.0 + deltat) * alpha) / (math.sqrt(1.0 + deltat * (2*pow(alpha, 2) - 1)))
                continue
            else:
                alpha = (math.sqrt(1.0 - deltat) * alpha) / (math.sqrt(1.0 - deltat * (2*pow(alpha, 2) - 1)))
                continue
        result.append(alpha)
    up = 0
    down = 0
    for kdx in result:
        if pow(kdx,2)>=0.5:
            up +=1
        else:
            down+=1
    print("the value of alpha square: ",pow(0.6,2))
    print("number of spin up: ",up)
    print("error of expected value and real value: ",abs(pow(0.6,2)-up*0.0001)/pow(0.6,2))
test_2()
```

The result (I do it three times):

1st:

The value of alpha square: 0.36

Number of spin up: 4063 => 0.4063

Error of expected value and real value: 12.86%

2nd:

the value of alpha square: 0.36

number of spin up: 4099 =>0.4099

error of expected value and real value: 13.86%

3rd:

the value of alpha square: 0.36

number of spin up: 4069 =>0.4069

error of expected value and real value: 13.03%

2-7.

(1).

We suppose that $P_{\uparrow}^{\infty}(\alpha_0, \beta_0) = F(|\alpha_0|)$, and F(x) must satisfy the equation:

$$F(x) = \frac{1}{2}[1 + \Delta t(2x^2 - 1)] \cdot F\left(\frac{x\sqrt{1+\Delta t}}{1 + \Delta t(2x^2 - 1)}\right) + \frac{1}{2}[1 - \Delta t(2x^2 - 1)] \cdot F\left(\frac{x\sqrt{1-\Delta t}}{1 - \Delta t(2x^2 - 1)}\right)$$

First, we consider the situation that $\lim_{\Delta t \to 0} F(x)$:

$$\lim_{\Delta t \to 0} F(x) = F(x)$$

It cannot tell us any information about the function. Next, we consider $\lim_{\Delta t \to 1} F(x)$:

$$\lim_{\Delta t \to 1} F(x) = x^2 \cdot F\left(\frac{x\sqrt{2}}{2x^2}\right) = x^2 \cdot F\left(\frac{1}{\sqrt{2}x}\right)$$

We want to demand $F(1) = 1$, so we let $x = \frac{1}{\sqrt{2}}$

$$x^2 \cdot F\left(\frac{1}{\sqrt{2}x}\right) = \frac{1}{2}F(1) = \frac{1}{2}$$

$$F\left(\frac{1}{\sqrt{2}x}\right) = \frac{1}{2x^2}$$

$$F(x) = x^2$$

And we can easily see that F(0)=0

(2).

We suppose that the probability is $F(x) = ax^2 + b$

Because the constraint of probability:

$$F(0) = b = 0$$
$$F(1) = a + b = 1$$

Then,

$$a = 1, b = 0$$
$$F(x) = x^2$$