Homework 5
Quantum Information

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1 Singular Value Decomposition

In this question, My code is based on the code which you uploaded on Moodle. Therefore I skip the same parts in my code, so that my homework can be more readable.

1.1

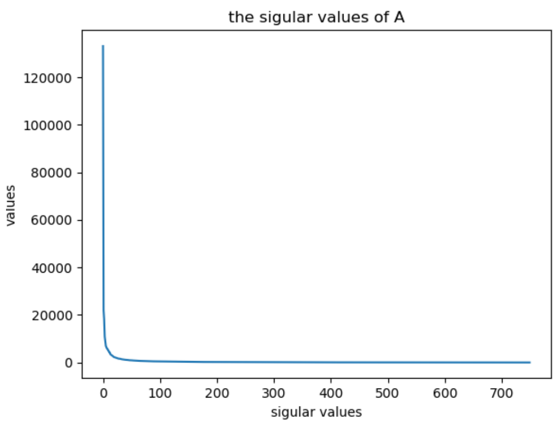
In python, I use the function called *svd* in *numpy.linalg*. It can find the matrix of SVD, which includes two unitaries and singular values. The graph of singular values is figure 1*a*.

```
u, s, v = np.linalg.svd(A) #find the SVD of A
plt.title("the sigular values of A")
plt.xlabel("sigular values")
plt.ylabel("values")
plt.plot(s)
plt.show()
```

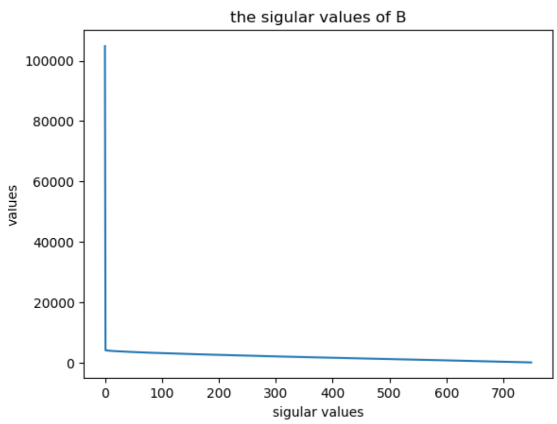
1.2

Now, we create a random integer matrix B which is the same size of A. We find SVD of B, and draw its singular values as a plot to compare with A's. The result is figure 1*b*.

```
N,M = A.shape #find the size of A
B = np.random.random_integers(0,255,(N,M)) #create a random integer matrix w
u,s,v = np.linalg.svd(B) #find SVD of B
plt.title("the sigular values of B")
plt.xlabel("sigular values")
plt.ylabel("values")
plt.plot(s)
plt.show()
```



(a) The singular values of A.

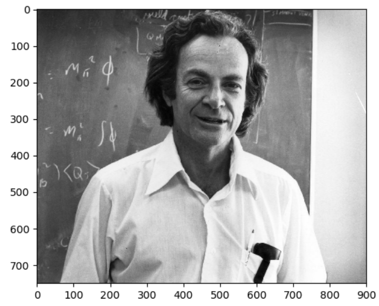


(b) The singular values of B.

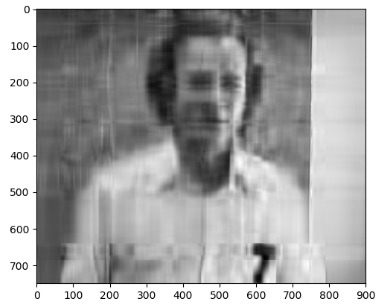
Figure 1: The plot of sigular values. A is the matrix of picture, and B is the random integer matrix.

There are two difference in Figure 1. One is that the graph (b) is dropping sharply than (a), and that's because the singular values of B are even distribution, and it makes large number like a singularity in matrix. However, the singular values of A are not distributed, and it seems like there are some large number in matrix, which makes graph drop smoothly. The other is that after dropping, the values in graph (a) don't change too much, and in graph (b) still decrease. That is the same reason as last difference. Because B is distributed and A is not, the values of B keeps going down and A remains at same value.

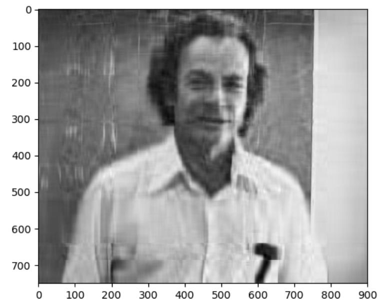
1.3



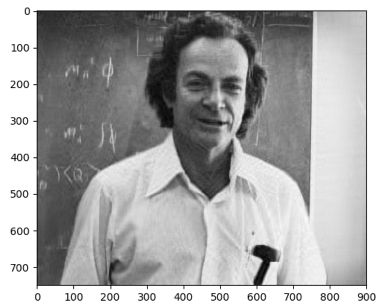
(a) Origin picture



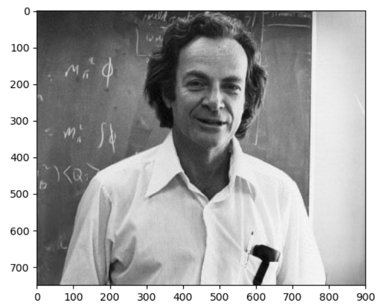
(b) Compressed picture with $n = 10$



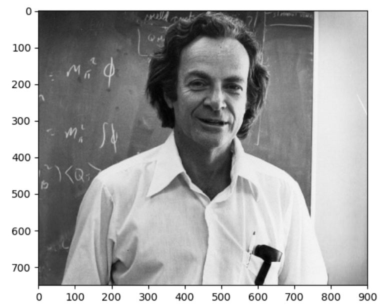
(c) Compressed picture with $n = 20$



(d) Compressed picture with $n = 50$



(e) Compressed picture with $n = 100$



(f) Compressed picture with $n = 150$

Figure 2: This is all the results of 1.3. Except (a) is origin picture, the others are compressed by function $SVDcompress$.

First, we define the function $SVDcompress$.

```
def SVDcompress(A,n): #input is the matrix A and an integer n
    u,s,v = np.linalg.svd(A) #find SVD of A
    #we slice the unitaries and singular values
    d = s[:n]
    Ut = u[:,:n]
    Vt = v[:n,:] #v is already done the conjugate transpose
    return Ut, Vt, d #output is two unitaries and singular values
```

Now, we compress the picture.

```
from functools import reduce

n = [10,20,50,100,150]
for ndx in n:
    Ut ,Vt, d = SVDcompress(A,ndx)
    Ut = np.matrix(Ut)
    Dt = np.diag(d) #become the diagonal matrix
    Vt = np.matrix(Vt)
    At = reduce(np.matmul,[Ut,Dt,Vt]) #=(Ut*Dt)*Vt
    plt.figure(num=1)
    plt.imshow(At, cmap='gray')
    plt.show()
```

The results are in Figure 2

1.4

Let A is the origin matrix, and A_n is the compressed matrix, and we assume that

$$A = UDV^\dagger$$
$$A_n = UD_nV^\dagger$$

U and V are orthogonal unitaries. D and D_n are $L \times L$ diagonal matrices, but D_n only includes n largest singular values of D. Now, we suppose A in Forbenius norm is

$$\|A\|_{Fro} = \sqrt{\sum_{i,j} |A_{ij}|^2} = \sqrt{\sum_{k=1}^{L} |u_k d_k v_k|^2}$$

where u_k and v_k denote the kth column of U and V. Because U and V are unitaries, we can express the norm in only singular values.

$$\|A\|_{Fro} = \sqrt{\sum_{k=1}^{L} |d_k|^2}$$

Therefore, we can do the same thing on the norm of difference between A and A_n.

$$\begin{aligned}\|A - A_n\|_{Fro} &= \sqrt{|\sum_{k=1}^{L} d_k - \sum_{k=1}^{n} d_k|^2} \\ &= \sqrt{\sum_{k=n+1}^{L} d_k^2}\end{aligned}$$

Thus, the error of A_n is

$$e_n = \frac{\|A - A_n\|_{Fro}}{\|A\|_{Fro}} = \frac{\sqrt{\sum_{k=n+1}^{L} d_k^2}}{\sqrt{\sum_{k=1}^{L} d_k^2}}$$

2 Schmidt Rank and Local Operations

2.1

$$\begin{aligned}|\psi\rangle &= \frac{1}{\sqrt{2}} (|00\cdots 0\rangle + |11\cdots 1\rangle) \\ &= \Sigma_{i=0}^{1} \frac{1}{\sqrt{2}} |i\rangle \otimes |i\rangle \cdots |i\rangle\end{aligned}$$

Therefore, Schmidt rank of $|\psi\rangle$ is 2, that is,

$$SR(\psi) = 2$$

2.2

We can write down $|\psi\rangle$ as this.

$$|\psi\rangle = \frac{1}{\sqrt{2}} |0\rangle \otimes |00\rangle \otimes |0\cdots 0\rangle + \frac{1}{\sqrt{2}} |1\rangle \otimes |11\rangle \otimes |1\cdots 1\rangle$$

Because the measurement of acts on qubits 2 and 3, we just need to focus on these two qubits. We measure the states separately.

$$|\psi_0'\rangle \equiv \frac{\Pi_0 |00\rangle}{\|\Pi_0 |00\rangle\|} = \frac{\frac{1}{2}|00\rangle + \frac{1}{2}|11\rangle}{\frac{1}{2}} = |00\rangle + |11\rangle$$
$$|\psi_1'\rangle \equiv \frac{\Pi_0 |11\rangle}{\|\Pi_0 |11\rangle\|} = \frac{\frac{1}{2}|00\rangle + \frac{1}{2}|11\rangle}{\frac{1}{2}} = |00\rangle + |11\rangle$$

Thus, the cat state after measurement is

$$\begin{aligned}|\psi'\rangle &= \frac{1}{\sqrt{2}} |0\rangle \otimes |\psi_0'\rangle \otimes |0\cdots 0\rangle + \frac{1}{\sqrt{2}} |1\rangle \otimes |\psi_1'\rangle \otimes |1\cdots 1\rangle \\ &= \frac{1}{\sqrt{2}} |0\rangle \otimes (|00\rangle + |11\rangle) \otimes |0\cdots 0\rangle + \frac{1}{\sqrt{2}} |1\rangle \otimes (|00\rangle + |11\rangle) \otimes |1\cdots 1\rangle \\ &= \frac{1}{\sqrt{2}} (|000\cdots 0\rangle + |111\cdots 1\rangle + |0110\cdots 0\rangle + |1001\cdots 1\rangle) \\ &\Rightarrow SR(\psi') = 4\end{aligned}$$

2.3

If U_{456} act on $|\psi\rangle$, then

$$|\psi'\rangle = U_{456}\,|\psi\rangle$$

$$= \frac{1}{\sqrt{2}}\left(|000\rangle \otimes U_{456}\,|000\rangle \otimes |0\cdots 0\rangle + |111\rangle \otimes U_{456}\,|111\rangle \otimes |1\cdots 1\rangle\right)$$

Now we suppose that

$$|\phi_0\rangle = U_{456}\,|000\rangle = \Sigma_{i=0}^{3}\lambda_i\,|A_i\rangle\,|B_i\rangle\,|C_i\rangle$$

$$|\phi_1\rangle = U_{456}\,|111\rangle = \Sigma_{j=0}^{3}\lambda_j\,|A_j\rangle\,|B_j\rangle\,|C_j\rangle$$

This because the fact that $|\phi_0\rangle$ and $|\phi_1\rangle$ are 3-qubits state. Therefore,

$$|\psi'\rangle = \Sigma_{i=0}^{3}\frac{1}{\sqrt{2}}\left(\lambda_i^{(0)}\,|000\rangle \otimes \left|A_i^{(0)}\right\rangle\left|B_i^{(0)}\right\rangle\left|C_i^{(0)}\right\rangle \otimes |0\cdots 0\rangle + \lambda_i^{(1)}\,|111\rangle \otimes \left|A_i^{(1)}\right\rangle\left|B_i^{(1)}\right\rangle\left|C_i^{(1)}\right\rangle \otimes |1\cdots 1\rangle\right)$$

$$\Rightarrow SR(\psi') = SR(\phi_0) + SR(\phi_1) = 3 + 3 = 6$$

So the maximum rank is 6.

3 Von Neumann entropy

3.1

$$\rho_A = \Sigma_i \lambda_i^{(A)}\left|\psi_i^{(A)}\right\rangle\left\langle\psi_i^{(A)}\right|$$

$$\rho_B = \Sigma_j \lambda_j^{(B)}\left|\psi_j^{(B)}\right\rangle\left\langle\psi_j^{(B)}\right|$$

$$\rho = \rho_A \otimes \rho_B = \Sigma_{i,j}\lambda_i^{(A)}\lambda_j^{(B)}\left|\psi_i^{(A)}\right\rangle\left|\psi_j^{(B)}\right\rangle\left\langle\psi_i^{(A)}\right|\left\langle\psi_j^{(B)}\right|$$

$$S(\rho) = -Tr\left(\rho\log_2\rho\right)$$

$$= -\Sigma_{i,j}\lambda_i^{(A)}\lambda_j^{(B)}\log_2\lambda_i^{(A)}\lambda_j^{(B)}$$

$$= -\Sigma_{i,j}\lambda_i^{(A)}\lambda_j^{(B)}\log_2\lambda_i^{(A)} - \Sigma_{i,j}\lambda_i^{(A)}\lambda_j^{(B)}\log_2\lambda_j^{(B)}$$

$$= -\Sigma_i\lambda_i^{(A)}\log_2\lambda_i^{(A)} - \Sigma_j\lambda_j^{(B)}\log_2\lambda_j^{(B)}$$

$$= S(\rho_A) + S(\rho_B)$$

3.2

Let's find the eigenvalues of ρ.

$$\begin{vmatrix} \frac{1+r_z}{2} - \lambda & \frac{r_x - ir_y}{2} \\ \frac{r_x + iry}{2} & \frac{1-r_z}{2} - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \left(\frac{1+r_z}{2} - \lambda\right)\left(\frac{1-r_z}{2} - \lambda\right) = \frac{1}{4}\left(r_x^2 + r_y^2\right)$$

$$\Rightarrow \left(\frac{1}{2} - \lambda\right)^2 = \frac{\mathbf{r}^2}{4}$$

$$\Rightarrow \lambda = \frac{1 \pm \mathbf{r}}{2}$$

Therefore, we write down the entropy.

$$S(\rho) = -Tr(\rho\log_2\rho) = -\Sigma_i\lambda_i\log_2\lambda_i$$

$$= -\frac{1+\mathbf{r}}{2}\log_2\left(\frac{1+\mathbf{r}}{2}\right) - \frac{1-\mathbf{r}}{2}\log_2\left(\frac{1-\mathbf{r}}{2}\right)$$

$$= 1 - \frac{(1+\mathbf{r})\log_2(1+\mathbf{r})}{2} + \frac{(1-\mathbf{r})\log_2(1-\mathbf{r})}{2}$$

3.3

We assume that

$$\rho = \Sigma_{i=0}^{D} \lambda_i \left|\psi_i\right\rangle \left\langle\psi_i\right|$$

we know that $\Sigma_{i=1}^{D} \lambda_i = 1$,so we can write down function with Lagrange mulitplier g.

$$\mathcal{L} = S(\rho) - g(\Sigma_{i=1}^{D} \lambda_i - 1)$$

$$= \Sigma_{i=1}^{D} \left(-\lambda_i \log_2(\lambda_i) - g(\lambda_i - 1)\right)$$

Let's find the maximum of entropy.

$$\partial_{\lambda_i} \mathcal{L} = \partial_{\lambda_i} \left(-\Sigma_{i=1}^{D} \lambda_i \log_2(\lambda_i) - g(\Sigma_{i=1}^{D} \lambda_i - 1)\right) = 0$$

$$\Rightarrow \Sigma_{i=1}^{D} \left(-(\log_2(\lambda_i) + \frac{1}{\ln(2)}) - g\right) = 0$$

$$\Rightarrow \ln(\lambda_i) = -1 - g$$

$$\Rightarrow \lambda_i = \exp(-1 - g)$$

We know that the constraint of density matrix.

$$Tr(\rho) = \Sigma_{i=1}^{D} \lambda_i = \Sigma_{i=1}^{D} \exp(-1 - g) = 1$$

$$\Rightarrow \exp(-1 - g) = \frac{1}{D} = \lambda_i$$

Therefore, we know that entropy is maximum if $\lambda_i = \frac{1}{D}$, that is

$$S(\rho) \leq -\Sigma_{i=1}^{D} \frac{1}{D} \log_2(\frac{1}{D}) = \log_2(D)$$