Quantum Information 116031 – Homework №5
Mixed states and Entanglement

Due date: 21-May-2018

1 Singular Value Decomposition (40 points)

In this exercise we will compress a picture (of Richard Feynman) using SVD. You may work in Python, Matlab or any other programming language of your choice. For your convenience, we have added in Moodle short scripts to read and display the picutre file in Matlab or Python.

1.1 (10 points) Calculate the SVD of the attached image A, and *draw* the graph of its singular values. Here, $A \in \mathbb{R}^{N\times M}$ is a matrix of integers in the range $[0, 255]$.

1.2 (10 points) Take a random integer matrix B, which is of the size of A, with values in $[0, 255]$, and calculate its SVD. Draw the graph of its singular values. What is the difference between this graph and the one you got in question 1.1?

1.3 (10 points) Write a function – called `SVDcompress` – that gets an image A and a scalar n, and outputs two matrices $\tilde{U}, \tilde{V}$ and a vector d using the following conventions.

- Denote the SVD of $A \in \mathbb{R}^{N\times M}$ by $A = USV^\dagger$.
- The vector $d \in \mathbb{R}^n$ contains the n largest singular values of A.
- The matrix $\tilde{U} \in \mathbb{R}^{N\times n}$ contains the first n columns of U.
- The matrix $\tilde{V} \in \mathbb{R}^{M\times n}$ contains the first n columns of V.

Apply `SVDcompress` on A for $n = 10, 20, 50, 100, 150$. Show the results in your solutions.

1.4 (10 points) One may identify the function `SVDcompress`(A, n) as a function that approximates an image A by its first n singular values. Let us define the error by the Frobenius norm,

$$\|A\|_{\text{Fro}} \stackrel{\text{def}}{=} \sqrt{\sum_{i,j} |A_{ij}|^2}.$$

The error, using the aforementioned approximation by n singular values is then given by

$$e_n \stackrel{\text{def}}{=} \frac{\|A - A_n\|_{\text{Fro}}}{\|A\|_{\text{Fro}}},$$

where A_n is the output of `SVDcompress`(A, n). Prove that

$$e_n = \frac{\sqrt{\sum_{k=n+1}^{L} d_k^2}}{\sqrt{\sum_{k=1}^{L} d_k^2}},$$

where d_k are the singular values and L is the index of the smallest one.

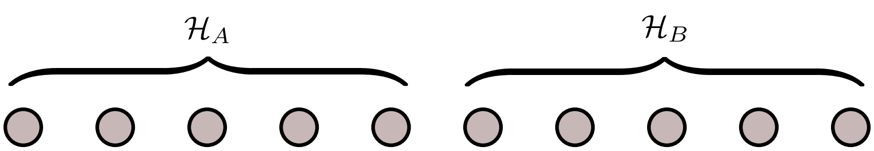
The solutions you submit must contain: (i) a documented code of `SVDcompress`; (ii) a documented script that uploads the original image and produces the compressed images; (iii) two graphs of singular values with explanations; (iv) five compressed images; and (v) the proof of question 1.4.

2 Schmidt rank and local operations (35 points)

Consider a system with 10 qubits on a line that is a cat state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\big(|00\cdots 0\rangle + |11\cdots 1\rangle\big)$$

The Hilbert space of the system can be written as $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_A$ is the Hilbert space of the first five qubits and $\mathcal{H}_B$ is the Hilbert space of the five qubits on the right, as shown below



2.1 (5 points) Calculate $\mathrm{SR}(\psi)$ with respect to the $\mathcal{H}_A \otimes \mathcal{H}_B$ partition.

2.2 (15 points) Assume we perform a 2-qubit measurement on qubits 2, 3 defined by the measurement operators $\Pi_0 = |\phi\rangle\langle\phi|$ and $\Pi_1 = \mathbb{1} - \Pi_0$, where $|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Let $|\psi'\rangle$ be the state after the measurement. What is $\mathrm{SR}(\psi')$ if we measured Π_0?

2.3 (15 points) Let U_{456} be an arbitrary unitary acting on qubits 4, 5, 6, and let $|\psi'\rangle = U|\psi\rangle$. What is the maximal possible $\mathrm{SR}(\psi')$?

2.4 (10 **bonus** points) Find a unitary U_{456} that achieves the maximal SR from the previous question.

3 Von Neumann entropy (25 points)

Given a mixed state ρ, its von Neumann entropy is defined by $S(\rho) \stackrel{\text{def}}{=} -\operatorname{Tr}(\rho \log_2 \rho)$.

3.1 (5 points) Consider a tensor-product Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, and let ρ be a density matrix over H which is a product of two density matrices: $\rho = \rho_A \otimes \rho_B$. Show that

$$S(\rho) = S(\rho_A) + S(\rho_B).$$

3.2 (10 points) Calculate the von Neumann entropy of a single qubit in a state $\rho = \frac{1}{2}\left[\mathbb{1} + \boldsymbol{r} \cdot \boldsymbol{\sigma}\right]$ (where $|\boldsymbol{r}| \leq 1$).

3.3 (10 points) Let ρ be a mixed state in a Hilbert space of dimension D. Show that $S(\rho) \leq \log_2(D)$. **Hint:** Use Lagrange multipliers.