Quantum Information 116031 – Homework №2
General Measurements and POVM

Due date: 20-Apr-2018

1 Distinguishing two non-orthogonal states

In class, we have seen that given two non-orthogonal states,

$$|\psi_1\rangle = |0\rangle \ , \qquad\qquad |\psi_2\rangle = |+\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \, ,$$

one can distinguish them using the following POVM:

$$E_+ \stackrel{\text{def}}{=} \alpha \, |1\rangle \langle 1| \, , \qquad\qquad E_0 \stackrel{\text{def}}{=} \alpha \, |-\rangle \langle -| \, , \qquad\qquad E_\eta \stackrel{\text{def}}{=} \mathbb{1} - E_+ - E_0 \, ,$$

where above $|-\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, and α is a sufficiently small positive constant.

The above measurements outputs either '+', '0', or 'η' with the promise that whenever '+' is outputted, we know for sure that the measured state was $|+\rangle$, and similarly when '0' is output the measured state was necessary $|0\rangle$. Intuitively, we would like to maximize α in order to minimize the chances of outputting 'η'.

1.1 What is the maximal α that one can use?

1.2 For that α, what is the probability of outputting 'η' when measuring $|0\rangle$?

2 A gentle-measurement toy-model for the Stern-Gerlach experiment

The Stern-Gerlach experiment is often viewed as the poster-child of a quantum projective measurement. In this experiment, a (electrically neutral) spin-$\frac{1}{2}$ particle passes through an inhomogeneous magnetic field $\boldsymbol{B}$, which deflects it either up or down according to its spin component in the direction of the magnetic field. After the particle has been deflected, its position is measured by a detector screen. In a typical experiment, the detector screen will have two spots where the particles hit it,

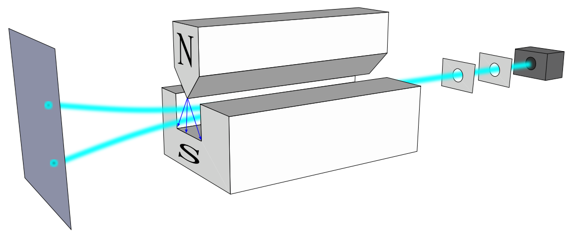


Figure 1: An illustration of the Stern-Gerlach experimental setup (taken from Wikipedia)

corresponding to spin up and spin down measurements. This process corresponds to the projective measurement

$$M_\uparrow \stackrel{\text{def}}{=} |\uparrow\rangle\langle\uparrow|$$
$$M_\downarrow \stackrel{\text{def}}{=} |\downarrow\rangle\langle\downarrow| \,.$$

However, it is clear that the measurement process isn't immediate; it takes a finite amount of time as the particle travels through the magnetic field. At every fraction of time, the particle's wave function changes *gradually*, as the environment (as well as the spatial part of the particle's wave function) acquires more and more information about the particle's spin. The projective measurement can therefore be seen as the accumulative result of many weak measurements that only slightly change the state of the particle. We can understand this process using the following toy model.

We assume that at very short intervals of time $\Delta t \ll 1$, the measuring apparatus performs a 'weak measurement' on the particle that is described by the operators

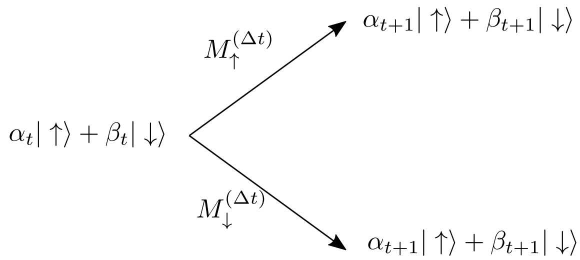
$$M_\uparrow^{(\Delta t)} \stackrel{\text{def}}{=} \sqrt{\frac{1+\Delta t}{2}}\, |\uparrow\rangle\langle\uparrow| + \sqrt{\frac{1-\Delta t}{2}}\, |\downarrow\rangle\langle\downarrow| \,,$$
$$M_\downarrow^{(\Delta t)} \stackrel{\text{def}}{=} \sqrt{\frac{1-\Delta t}{2}}\, |\uparrow\rangle\langle\uparrow| + \sqrt{\frac{1+\Delta t}{2}}\, |\downarrow\rangle\langle\downarrow|$$

2.1 Show that $\{M_\uparrow^{(\Delta t)}, M_\downarrow^{(\Delta t)}\}$ are indeed legal measurement operators.

2.2 Show that when $\Delta t \to 0$, the measurement process does not change the state of the particle, but it also does not acquire any information about its spin. On the other hand, show that when $\Delta t \to 1$, we recover the projective measurement, which gives us the maximal information about the particle, at the cost of drastically changing its state.

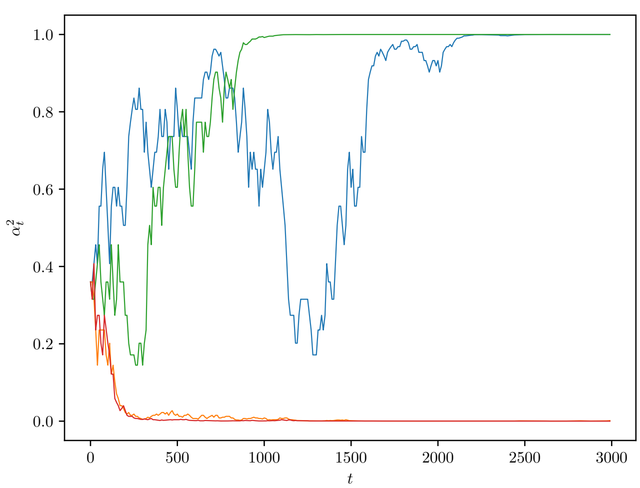
2.3 Let $|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$. Calculate $P_\uparrow(\alpha,\beta)$ and $P_\downarrow(\alpha,\beta)$, the probabilities for the two possible outcomes, and the corresponding states of the particle after each measurement. Show that both probabilities actually depend only on $|\alpha|$, and thus can be written as $P_\uparrow(|\alpha|)$ and $P_\downarrow(|\alpha|)$.

2.4 Let us now divide the measurement process into $T \to \infty$ short intervals of Δt time, and label the time steps of the particle as it goes through the Stern-Gerlach apparatus by $t_0 = 0,\ t_1 = \Delta t,\ t_2 = 2\Delta t,\ t_3 = 3\Delta t, \ldots$ Let $|\psi_t\rangle = \alpha_t\,|{\uparrow}\rangle + \beta_t\,|{\downarrow}\rangle$ describe the state of the particle at time t. Then $|\psi_{t+1}\rangle = \alpha_{t+1}\,|{\uparrow}\rangle + \beta_{t+1}\,|{\downarrow}\rangle$ is the result of the general measurement on $|\psi_t\rangle$. Find $\alpha_{t+1}, \beta_{t+1}$ as a function of α_t, β_t for the two possible outcomes.



2.5 The transition of $|\psi_t\rangle$ to the two possible $|\psi_{t+1}\rangle$ that you have calculated in Question 2.4, together with the corresponding probabilities for these transitions, which were calculated in Question 2.3, define a random walk. Starting with some $|\psi_0\rangle = \alpha_0\,|{\uparrow}\rangle + \beta_0\,|{\downarrow}\rangle$, the state will randomly change from measurement to measurement. Show that this walk has only two stable points: $|\psi_\infty\rangle = |{\uparrow}\rangle$ (i.e., $\alpha = 1, \beta = 0$) and $|\psi_\infty\rangle = |{\downarrow}\rangle$ ($\alpha = 0, \beta = 1$). Hint: in a stable point, the we would like to have $|\psi_t\rangle = |\psi_{t+1}\rangle$ with probability 1.

Typical walks with $\Delta t = 0.05$, $T = 3,000$ and $\alpha_0 = 0.6$ are shown below.



2.6 Starting from $|\psi_0\rangle = \alpha_0\,|{\uparrow}\rangle + \beta_0\,|{\downarrow}\rangle$, we can now define $P_\uparrow^\infty(\alpha_0, \beta_0)$ as the probability that we end up in the fixed point $\uparrow$ as $t \to \infty$, and $P_\downarrow^\infty(\alpha_0, \beta_0) = 1 - P_\uparrow^\infty(\alpha_0, \beta_0)$ as the probability that we arrive to $\downarrow$. Write a short computer program (Python, Matlab, or any other language of your choice) to simulate such random walk with $T = 5,000$ and $\Delta t = 0.05$. Run it many times (say, $N = 10,000$ times) starting from a fixed α_0, β_0 and show that $P_\uparrow^\infty(\alpha_0, \beta_0) \simeq |\alpha_0|^2$. To decide if the walk ended up at $|\uparrow\rangle$ or $|\downarrow\rangle$ you may look at the final $|\alpha_T|^2$ and see if it is greater or smaller than 0.5.

2.7 **(Bonus)** To show that $P_\uparrow^\infty(\alpha_0, \beta_0) = |\alpha_0|^2$ analytically, let us assume without proof that $P_\uparrow^\infty(\alpha_0, \beta_0) = F(|\alpha_0|)$ for some unknown function $F(x)$.

(a) Show then that $F(x)$ must satisfy the equation

$$\begin{aligned} F(x) = & \frac{1}{2}\left[1 + \Delta t\left(2x^2 - 1\right)\right] \cdot F\left(\frac{x\sqrt{1+\Delta t}}{1 + \Delta t\,(2x^2 - 1)}\right) \\ & + \frac{1}{2}\left[1 - \Delta t\left(2x^2 - 1\right)\right] \cdot F\left(\frac{x\sqrt{1-\Delta t}}{1 - \Delta t\,(2x^2 - 1)}\right) \end{aligned}$$

(b) Show that any $F(x)$ of the form $F(x) = ax^2 + b$ solves this equation, but as we demand $F(0) = 0$ and $F(1) = 1$, it follows that $F(x) = x^2$.