(4) $\xi \sim R(\theta, 2\theta)$

$$f(x) = \frac{1}{\theta}\{(\theta, 2\theta)\}$$

$$M\xi = \int_{\theta}^{2\theta} x \cdot \frac{1}{\theta}\,dx = \frac{1}{\theta}\,\frac{1}{2}(4\theta^2 - \theta^2) = \frac{3}{2}\theta$$

$$M\xi^2 = \int_{\theta}^{2\theta} x^2\,\frac{1}{\theta}\,dx = \frac{1}{\theta}\,\frac{1}{3}(8\theta^3 - \theta^3) = \frac{7}{3}\theta^2$$

$$D\xi = \frac{7}{3}\theta^2 - \frac{9}{4}\theta^2 = \frac{1}{12}\theta^2$$

a. оценка θ: по выборке $\vec{x}_n$

I. ОММ:

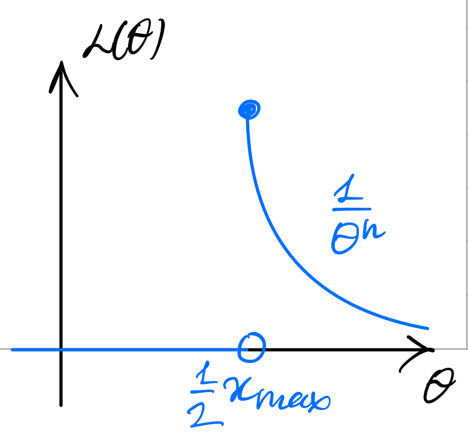
$$\alpha_1 = M\xi = \frac{3}{2}\theta$$

$$\tilde{\alpha}_1 = \frac{1}{n}\sum x_i = \bar{x}$$ — несмещ. состоят. оценка

$$\Rightarrow \alpha_1 = \tilde{\alpha}_1 \Rightarrow \frac{3}{2}\theta = \bar{x} \Rightarrow$$

$$\Rightarrow \boxed{\tilde{\theta}_1 = \frac{2}{3}\bar{x}}$$

II. ОМП:



$$L(\theta) = \frac{1}{\theta^n}\{(\theta < x_i < 2\theta)\} \Rightarrow$$

$$\Rightarrow \begin{cases} \theta < x_{max} \\ x_{max} < 2\theta \end{cases} \Rightarrow \frac{1}{2}x_{max} < \theta < x_{max}$$

$$\Rightarrow L(\theta) \to \sup \Rightarrow \boxed{\tilde{\theta}_2 = \frac{1}{2}x_{max}}$$

б. анализ оценок:

I. ОММ: $\tilde{\theta}_1 = \frac{3}{2}\bar{x}$ $\quad \overline{x_n}$

$R(\theta, 2\theta)$

$$M[\tilde{\theta}_1] = \frac{2}{3} M[\bar{x}] = \frac{2}{3} M\xi = \theta$$ — несмещ

$$D[\tilde{\theta}_1] = \frac{4}{9n^2} D[\bar{x}] = \frac{4}{9n^2} n D\xi = \frac{\theta^2}{27n} \xrightarrow[n\to\infty]{} 0$$

по дост усл состоят:

$\tilde{\theta}_1$ — несмещ

$D[\tilde{\theta}_1] \xrightarrow[n\to\infty]{} 0$ $\Rightarrow$ $\tilde{\theta}_1$ — состоят

II. ОМП: $\tilde{\theta}_2 = \frac{1}{2} x_{max}$

x_{max}:

$\Psi(y) = (F(y))^n$

$g(y) = \Psi'(y) =$

$= n\left(\frac{y}{\theta} - 1\right)^{n-1} \frac{1}{\theta}$

$(\theta, 2\theta)$

$$M[\tilde{\theta}_2] = \frac{1}{2} M[x_{max}] =$$

$$= \frac{1}{2} \int_{-\infty}^{+\infty} y g(y)\, dy =$$

$$= \frac{1}{2} \int_{\theta}^{2\theta} y n \left(\frac{y}{\theta} - 1\right)^{n-1} \frac{1}{\theta}\, dy = \left\{ t = \frac{y}{\theta} - 1 \right\} =$$

$\theta dt = dy$

$$= \frac{1}{2} \int_0^1 n\theta (t+1) \cdot t^{n-1}\, dt =$$

$$= \frac{\theta n}{2} \left(\int_0^1 t^n dt + \int_0^1 t^{n-1} dt \right) = \frac{\theta n}{2} \left(\frac{1}{n+1} + \frac{1}{n} \right) =$$

$$= \frac{2n+1}{2(n+1)} \theta \neq \theta \Rightarrow$$ $\tilde{\theta}_2$ — смещенная

$$\Rightarrow \tilde{\theta}_2^* = \frac{2(n+1)}{2n+1}\,\tilde{\theta}_2 = \frac{n+1}{2n+1}\, x_{max}$$

$\tilde{\theta}_2^*$ — несмещенная

$$M[x_{max}] = \int_{\theta}^{2\theta} x n \left(\frac{x}{\theta}-1\right)^{n-1} \frac{1}{\theta}\, dx = \frac{2n+1}{n+1}\,\theta$$

$$M[x_{max}^2] = \int_{\theta}^{2\theta} x^2 n \left(\frac{x}{\theta}-1\right)^{n-1} \frac{1}{\theta}\, dx = \frac{2n(2n^2+4n+1)}{n(n+1)(n+2)}\,\theta^2$$

$$D[x_{max}] = \left[\frac{2(2n^2+4n+1)}{(n+1)(n+2)} - \left(\frac{2n+1}{n+1}\right)^2\right]\theta^2$$

$$D[\tilde{\theta}_2^*] = \left(\frac{n+1}{2n+1}\right)^2 D[x_{max}] =$$

$$= \left(\frac{n+1}{2n+1}\right)^2 \left[\frac{2(2n^2+4n+1)}{(n+1)(n+2)} - \left(\frac{2n+1}{n+1}\right)^2\right]\theta^2 =$$

$$= \frac{n\theta^2}{(2n+1)^2(n+2)} \xrightarrow[n\to\infty]{} 0$$

по дост усл сост:

$\tilde{\theta}_2^*$ — несмещ

$D[\tilde{\theta}_2^*] \xrightarrow[n\to\infty]{} 0$ $\Rightarrow$ $\tilde{\theta}_2^*$ — сост

C. сравнить по асимпт эфф:

асимпт эффективность:

$$nD[\tilde{\xi}(\vec{x_n})] \xrightarrow[n\to\infty]{} \frac{g'^2}{I(\theta)} \quad \forall\theta\in\Theta$$

проверим модель на регулярность:

1. $\rho(x,\theta)$ — непр дифф по θ на Θ ✓

2. $$\underbrace{\frac{\partial}{\partial\theta}\int_{-\infty}^{+\infty}\rho(x,\theta)dx}_{0} \neq \int_{-\infty}^{+\infty}\frac{\partial}{\partial\theta}\rho(x,\theta)\,dx =$$

$$= \int_{\theta}^{2\theta}\frac{\partial}{\partial\theta}\frac{1}{\theta}dx = -\frac{1}{\theta}$$

$\Rightarrow$ модель нерегулярна

$$nD[\hat{\theta}_1] = \frac{\theta^2}{27}$$

$$nD[\tilde{\theta}_2^*] = \frac{n^2\theta^2}{(2n+1)^2(n+2)} \xrightarrow[n\to\infty]{} 0$$

θ_2^* асимпт эффективнее $\tilde{\theta}_1$

d. точный доверит. интервал

$\vec{x}_n$, $\xi \sim p(x,\theta) = \frac{1}{\theta}\{[\theta, 2\theta]\}$

1. находим $f(h, \vec{x}_n) \sim q(t)$

$$f(h, \vec{x}_n) = \frac{x_{max}}{\theta} \sim$$

2. график $q(t)$

3. $t_1 = q_{\frac{1-\beta}{2}}$ $\quad t_2 = q_{\frac{1+\beta}{2}}$

$$t_1 < \frac{x_{max}}{\theta} < t_2$$

$$\frac{x_{max}}{t_2} < \theta < \frac{x_{max}}{t_1}$$

$$\frac{x_{max}}{\sqrt[n]{\frac{1+\beta}{2}}+1} < \theta < \frac{x_{max}}{\sqrt[n]{\frac{1-\beta}{2}}+1}$$

точный (центральный) доверит. интервал

е. асимпт доверит инт

$\xi, \vec{x}_n$ $\qquad f(h, \vec{x}_n) \sim g(t)$

$$\lim_{n\to\infty} P(\tilde{h}_1 < h < \tilde{h}_2) \geq \beta$$

ОММ:

$$\sqrt{n}(\tilde{\alpha}_1 - \alpha_1) \sim N(0, \underbrace{\nabla^T \alpha_1 K \nabla \alpha_1}_{\sigma^2(\alpha)})$$

$$\sigma^2(\alpha) = \frac{2}{3}\sqrt{\overline{x^2} - \overline{x}^2}$$

$K_{ij} = \alpha_{s_i+s_j} - \alpha_{s_i}\alpha_{s_j}$

$$\sqrt{n}\left(\frac{2}{3}\overline{x} - \frac{3}{2}\theta\right) \sim N\left(0, \frac{2}{3}\sqrt{\overline{x^2} - \overline{x}^2}\right)$$

не зависит от θ

$$\sqrt{n}\,\frac{g(\tilde{\alpha}) - g(\alpha)}{\sigma(\alpha)} \sim N(0,1)$$

$$\sqrt{n}\,\frac{\frac{2}{3}\overline{x} - \frac{3}{2}\theta}{\frac{2}{3}\sqrt{\overline{x^2} - \overline{x}^2}} \sim N(0,1) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$u_{\frac{1+\beta}{2}} = 1{,}96 \qquad u_{\frac{1-\beta}{2}} = -1{,}96$

$$-1{,}96 < \sqrt{n}\,\frac{\overline{x} - \frac{9}{4}\theta}{\sqrt{\overline{x^2} - \overline{x}^2}} < 1{,}96$$

$$\overline{x} - 1{,}96\,\frac{\sqrt{\overline{x^2} - \overline{x}^2}}{\sqrt{n}} < \theta < +1{,}96\,\frac{\sqrt{\overline{x^2} - \overline{x}^2}}{\sqrt{n}} + \overline{x}$$

асимпт инт ОММ

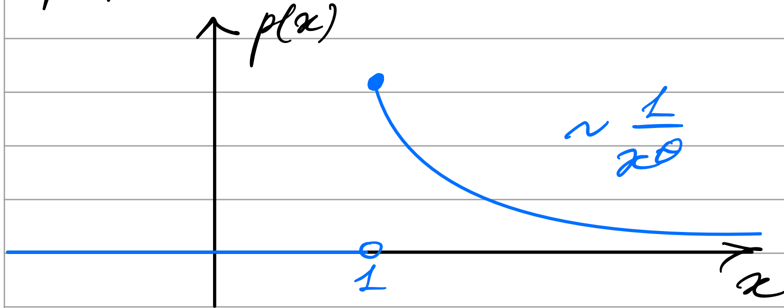
ОМП:

$$\sigma^2 \eta(\theta) = \nabla^T g(\vec{\theta}) \, I^{-1}(\vec{\theta}) \, \nabla g(\vec{\theta})$$

⑤ Распределение Парето: $\theta > 1$

$$p(x) = \begin{cases} \dfrac{\theta - 1}{x^{\theta}}, & x \geq 1 \\ 0, & x < 1 \end{cases}$$

график



$\vec{x}_n$ — выборка

$\xi \sim p(x)$

a. омп:

$$L(\theta) = (\theta - 1)^n \left(\frac{1}{\prod_{i=1}^{n} x_i} \right)^{\theta}$$

$$\ln L(\theta) = n \ln(\theta - 1) - \theta \ln \prod_{i=1}^{n} x_i$$

$$= n \ln(\theta - 1) - \theta \sum_{i=1}^{n} \ln x_i$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = \frac{n}{\theta - 1} - \sum_{i=1}^{n} \ln x_i = 0 \qquad \sup$$

$$\Rightarrow \frac{n}{\theta-1} = \sum_{i=1}^{n} \ln x_i \quad \Rightarrow \quad \tilde{\theta} = \frac{n}{\sum \ln x_i} + 1$$

8. доверительный инт для медианы

$$\tilde{\theta} \to f(\tilde{\theta})$$

$$\int_1^{med} (\theta-1)\frac{1}{x^\theta}\,dx = -x^{1-\theta}\Big|_1^{med} = 1 - med^{1-\theta} = \frac{1}{2}$$

$$\Rightarrow med(\theta) = 2^{\frac{1}{\theta-1}}$$

из ОМП: $\quad med(\tilde{\theta}) = 2^{\frac{1}{\tilde{\theta}-1}}$

$$\nabla med(\theta) = 2^{\frac{1}{\theta-1}} \ln 2 \; \frac{1}{(\theta-1)^2}$$

$$\tilde{\theta} = \frac{n}{\sum \ln x_i} + 1$$

$$I(\theta) = M\left[\left(\frac{\partial \ln p}{\partial \theta}\right)^2\right] = \int_1^{+\infty} \left(\frac{1}{\theta-1} - \ln x\right)^2 \frac{\theta-1}{x^\theta}\,dx =$$

$$= \frac{1}{(\theta-1)^2}$$

считаем оценку $\rightarrow$ $\quad \sigma^2(\tilde{\theta}) = 2^{\frac{1}{\tilde{\theta}-1}} \ln 2 \cdot \frac{1}{(\tilde{\theta}-1)^2} \cdot (\tilde{\theta}-1)^2\, 2^{\frac{1}{\tilde{\theta}-1}} \ln 2 \cdot \frac{1}{(\tilde{\theta}-1)^2}$

с дельтой

$$= 2^{\frac{1}{\tilde{\theta}-1}} \ln 2 \, \frac{1}{(\tilde{\theta}-1)}$$

$$\underbrace{\boxed{\sqrt{n}\,\frac{med(\tilde{\theta}) - med(\theta)}{\sigma^2(\theta)}}}_{\xrightarrow{P} N(0,1)} \cdot \underbrace{\boxed{\frac{\sigma^2(\theta)}{\sigma^2(\tilde{\theta})}}}_{\xrightarrow{P} 1} \sim N(0,1)$$

по лемме Слуцкого

$u_{0,025} = -1,96$ $\qquad$ $u_{0,975} = 1,96$

$$-1,96 < \sqrt{n}\,\frac{2^{\frac{1}{\tilde\theta-1}} - med(\theta)}{2^{\frac{1}{\tilde\theta-1}} \ln 2 \frac{1}{\tilde\theta-1}} < 1,96$$

$$-1,96 \cdot \frac{2^{\frac{1}{\tilde\theta-1}} \ln 2 \frac{1}{\tilde\theta-1}}{\sqrt{n}} + 2^{\frac{1}{\tilde\theta-1}} < med(\theta) <$$

$$< 1,96\,\frac{2^{\frac{1}{\tilde\theta-1}} \ln 2 \frac{1}{\tilde\theta-1}}{\sqrt{n}} + 2^{\frac{1}{\tilde\theta-1}}$$

доверит инт для med

6. асимпт. довер. инт

$$\tilde{\theta} = \frac{n}{\sum \ln x_i} + 1$$

модель регулярна $\quad x \neq 0$

$p(x,\theta)$ беск раз дифференц $\forall \theta \in \Theta$

$$\underbrace{\frac{\partial}{\partial\theta}\int_1^{+\infty} p(x,\theta)\,dx}_{0} = \int_1^{+\infty}\frac{\partial}{\partial\theta} p(x,\theta)\,dx =$$

$$= \int_1^{+\infty}\frac{\partial}{\partial\theta}\;\frac{\theta-1}{x^\theta}\,dx =$$

$$= \int_1^{+\infty}(-1)\;\frac{\ln x}{x^\theta}\,dx = \ldots = x^{1-\theta}\ln x\Big|_1^{\infty} = 0$$

$I(\theta) = \frac{1}{(\theta-1)^2} > 0$ и непр

$$\underbrace{\frac{\partial^2}{\partial\theta^2}\int_1^{+\infty} p(x,\theta)\,dx}_{0} = \int_1^{+\infty}\frac{\partial^2}{\partial\theta^2} p(x,\theta)\,dx =$$

$$= \int_1^{+\infty}\frac{\partial^2}{\partial\theta^2}\;\frac{\theta-1}{x^\theta}\,dx =$$

$$= \int_1^{+\infty}(-1)\;\frac{\ln x}{x^\theta}\,dx = \ldots = x^{1-\theta}\ln x\Big|_1^{\infty} = 0$$

$$\int_1^{+\infty}\frac{-2\ln x + \theta\ln^2 x - \ln^2 x}{x^\theta}\,dx$$

значит, модель сильно регулярна

$\Rightarrow$ 4 св-ва гарант (сост, асимпт. эфф, несм, норм)

$$\tilde{\theta} = \frac{n}{\sum \ln x_i} + 1$$

с фокусом:

$$f(\theta) = \theta \qquad \nabla f = 1$$

$$\tau(\tilde{\theta}) = \tilde{\theta} - 1$$

$$f(\tilde{\theta}) = \tilde{\theta} \qquad I^{-1} = (\theta - 1)^2$$

$$-1{,}96 < \sqrt{n}\,\frac{\tilde{\theta} - \theta}{\tilde{\theta} - 1} < 1{,}96$$

$$-1{,}96 < \frac{\frac{n}{\sum \ln x_i} + 1 - \theta}{\sqrt{n}} \sum \ln x_i < 1{,}96$$

$$\frac{n}{\sum \ln x_i} + 1 - 1{,}96\frac{\sqrt{n}}{\sum \ln x_i} < \theta <$$

$$< \frac{n}{\sum \ln x_i} + 1 + 1{,}96\frac{\sqrt{n}}{\sum \ln x_i}$$

асимпт инт дов θ
по ОМП