3.2 Differentiability

How $f'(a)$ Might Fail to Exist

- Differentiability Implies Local Linearity
- Derivatives on a Calculator
- Differentiability Implies Continuity
- Intermediate Value Theorem for Derivatives

… and why

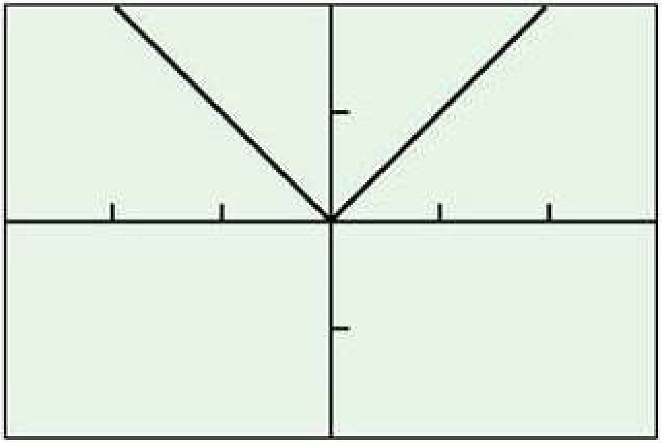
Graphs of differentiable functions can be approximated by their tangent lines at points where the derivative exists.

1. a corner, where the one-sided derivatives differ;

$f(x)=|x|$

$f(x)=|x|$

$$f(x)=\begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$



[−3, 3] by [−2, 2]

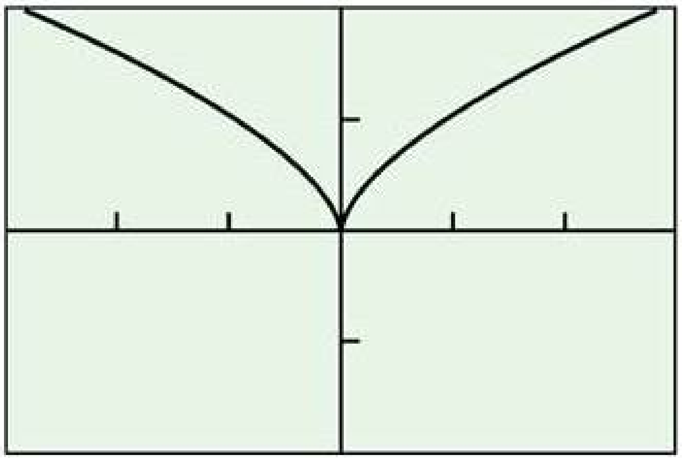
$f'(0)$ fails to exist

$f'(x)\Big|_{x \to 0^+} = 1$

$f'(x)\Big|_{x \to 0^-} = -1$

2. a cusp, where the slopes of the secant lines approach ∞ from one side and approach $-\infty$ from the other (an extreme case of a corner);

$f(x) = x^{\frac{2}{3}}$



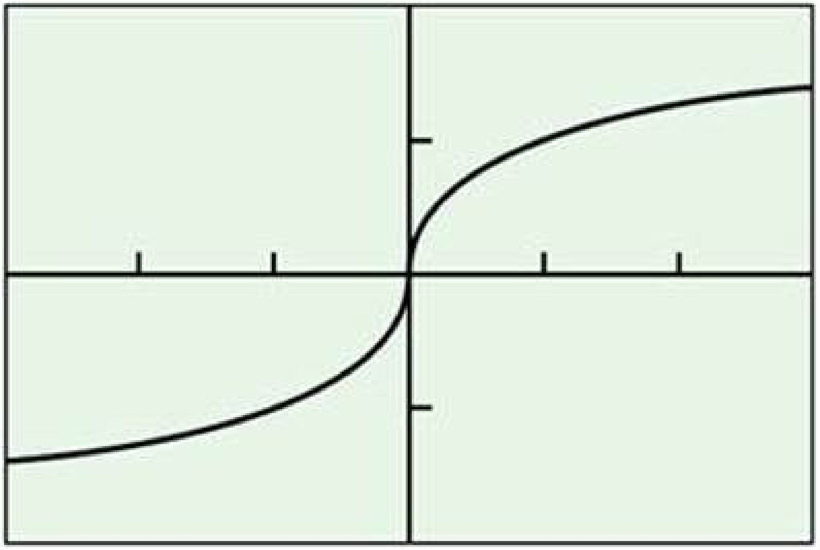
[–3, 3] by [–2, 2]

tangent line becomes vertical

$m \to \infty$

3. A vertical tangent, where the slopes of the secant lines approach either ∞ or $-\infty$ from both sides;

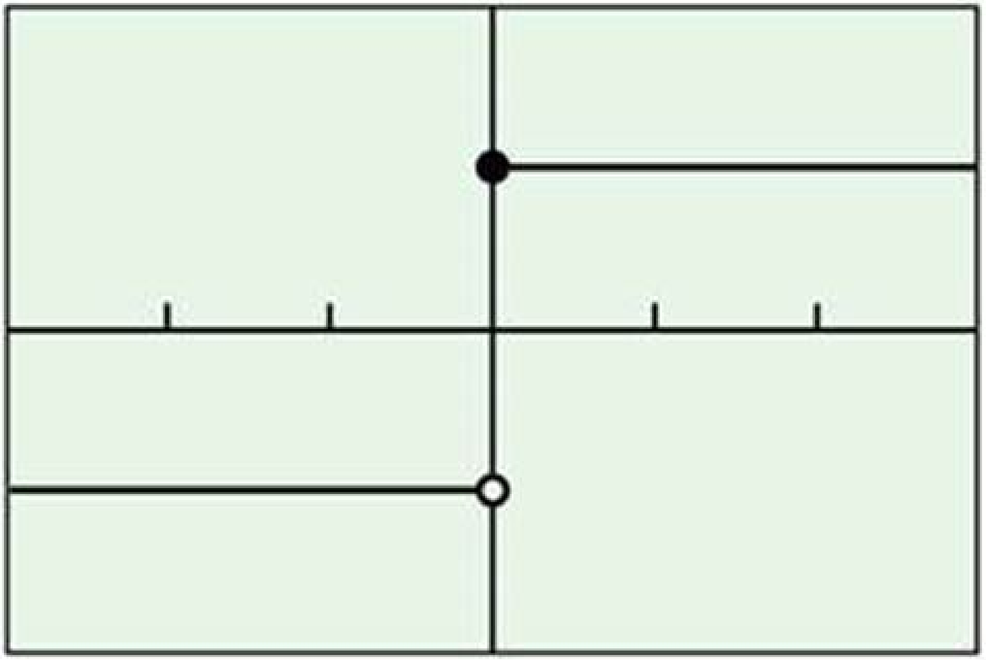
$f(x) = \sqrt[3]{x}$



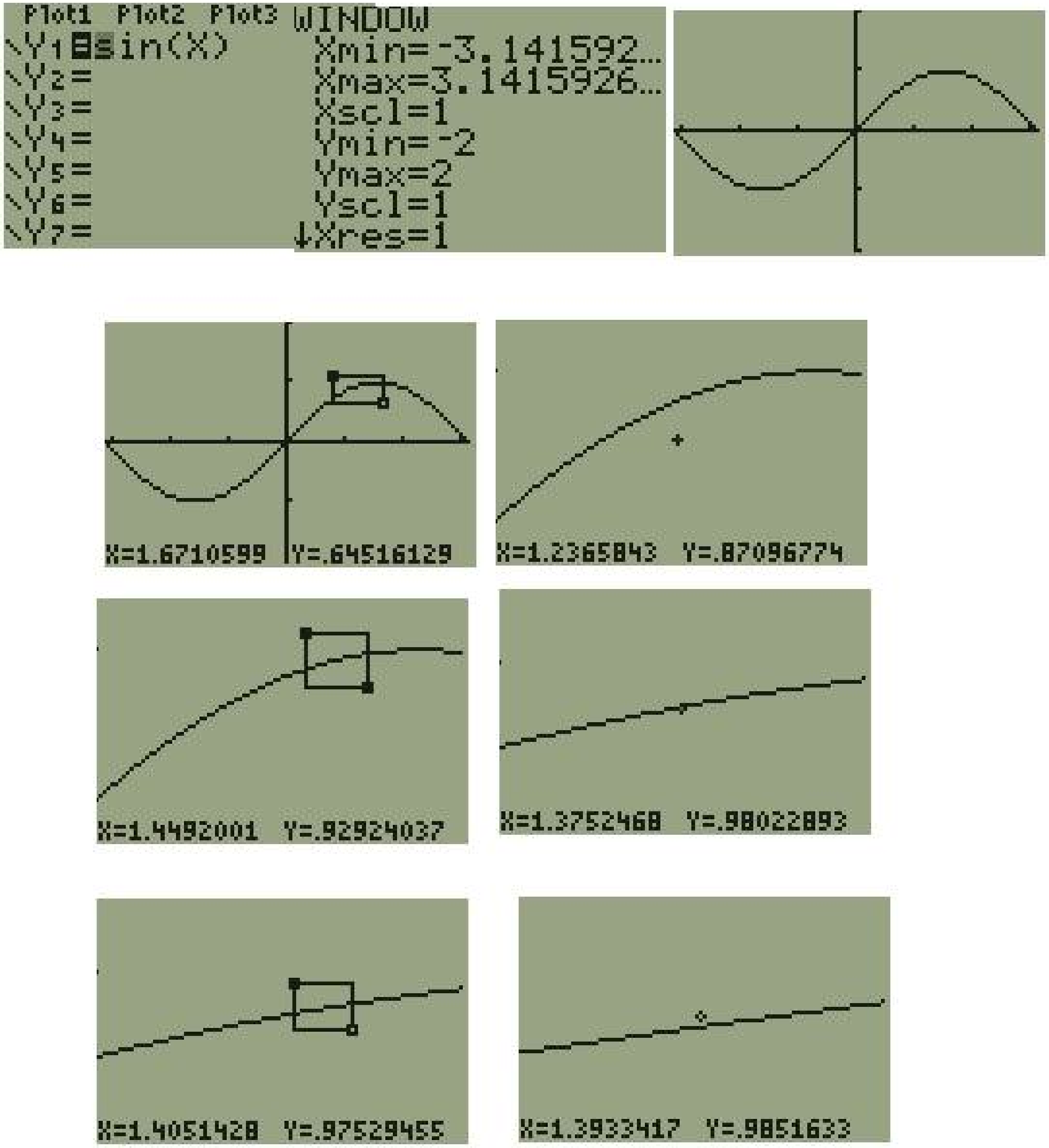
[–3, 3] by [–2, 2]

4. a discontinuity (which will cause one or both of the one-sided derivatives to be nonexistent).

$$U(x)=\begin{cases}-1, & x<0\\ 1, & x\geq 0\end{cases}$$

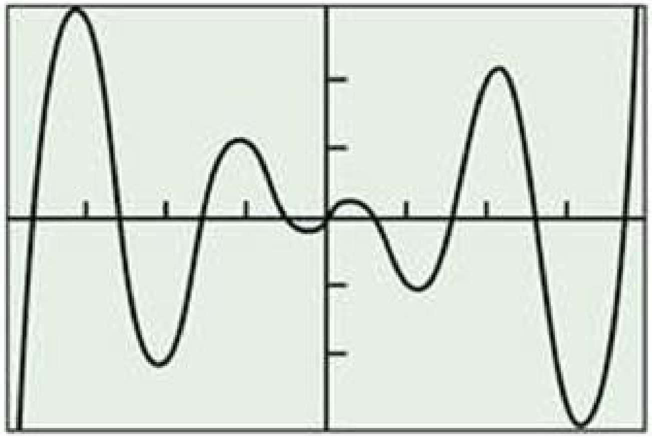


[–3, 3] by [–2, 2]

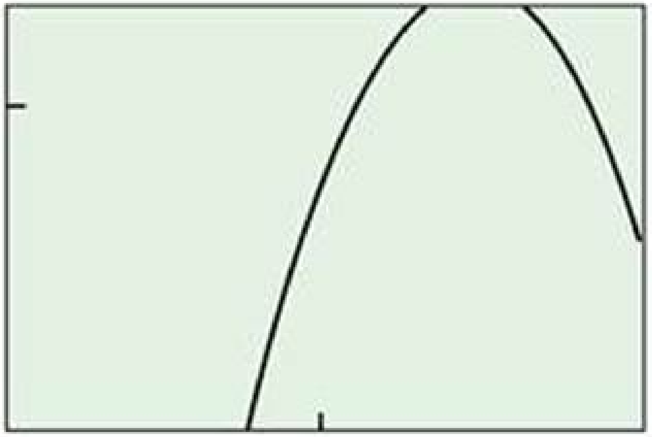


Local Linearity

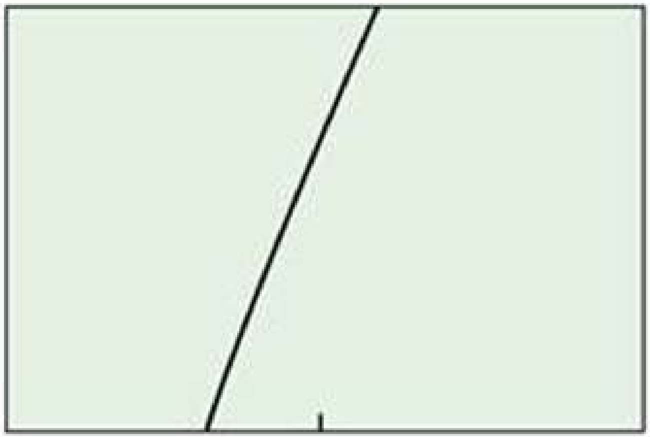
Differentiability Implies Local Linearity



[–4, 4] by [–3, 3]
(a)



[1.7, 2.3] by [1.7, 2.1]
(b)



[1.93, 2.07] by [1.85, 1.95]
(c)

Show that the function is not differentiable at $x=0$.

$$f(x)=\begin{cases} x^3, & x\le 0 \\ 4x, & x>0 \end{cases}$$

$\lim_{x\to 0^-} f(x) = x^3 = 0^3 = 0$

$\lim_{x\to 0^+} f(x) = 4x = 4(0) = 0$

$f(x)$ is continuous

$f'(x) = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \Rightarrow \lim_{h\to 0} \frac{\cancel{x^3} + 3x^2\cancel{h} + 3x\cancel{h}^2 + \cancel{h}^3 - \cancel{x^3}}{\cancel{h}}$

$\Rightarrow \lim_{h\to 0} 3x^2 + 3xh + h^2 \Rightarrow 3x^2$

$\lim_{x\to 0^-} f'(x) = 3x^2 = 0$

$\lim_{x\to 0^+} f'(x) = 4$

$4 \neq 0$

Not differentiable

DEFINITION The Numerical Derivative

The **numerical derivative of f at a,** which we will denote NDER $(f(x), a)$, is the *number*

$$\frac{f(a + 0.001) - f(a - 0.001)}{0.002}.$$

The **numerical derivative of f,** which we will denote NDER $(f(x), x)$, is the *function*

$$\frac{f(x + 0.001) - f(x - 0.001)}{0.002}.$$

EXAMPLE 2 Computing a Numerical Derivative

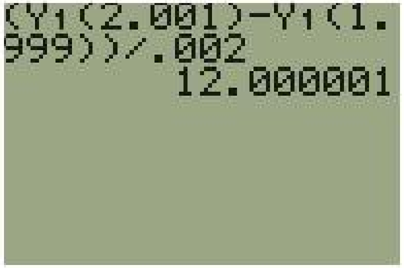
If $f(x) = x^3$, use the numerical derivative to approximate $f'(2)$.

slope

SOLUTION

$$f'(2) = \frac{d}{dx}(x^3)\bigg|_{x=2} \approx \text{NDER}(x^3, 2) = \frac{(2.001)^3 - (1.999)^3}{0.002} = 12.000001.$$

Now Try Exercise 17.



In Exercises 17–26, find the numerical derivative of the given function at the indicated point. Use $h = 0.001$. Is the function differentiable at the indicated point?

17. $f(x) = 4x - x^2, x = 0$

18. $f(x) = 4x - x^2, x = 3$

```
Plot1 Plot2 Plot3
\Y1=4X-X²
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

```
(Y1(3.001)-Y1(2.
999))/.002
                -2
```

Numerical estimate using $Dx = .002$

```
(Y1(.001)-Y1(-.0
01))/.002
                 4
```

$\frac{d}{dx}\left[4x-x^2\right]\Big|_{x=0}$

```
nDeriv(4X-X²,X,0
)
                 4
nDeriv(4X-X²,X,3
)
                -2
```

THEOREM 1 Differentiability Implies Continuity

If f has a derivative at $x = a$, then f is continuous at $x = a$.

THEOREM 2 Intermediate Value Theorem for Derivatives

If a and b are any two points in an interval on which f is differentiable, then f' takes on every value between $f'(a)$ and $f'(b)$.

HW (3.2)
pg 114
2-22 even
39-45 all