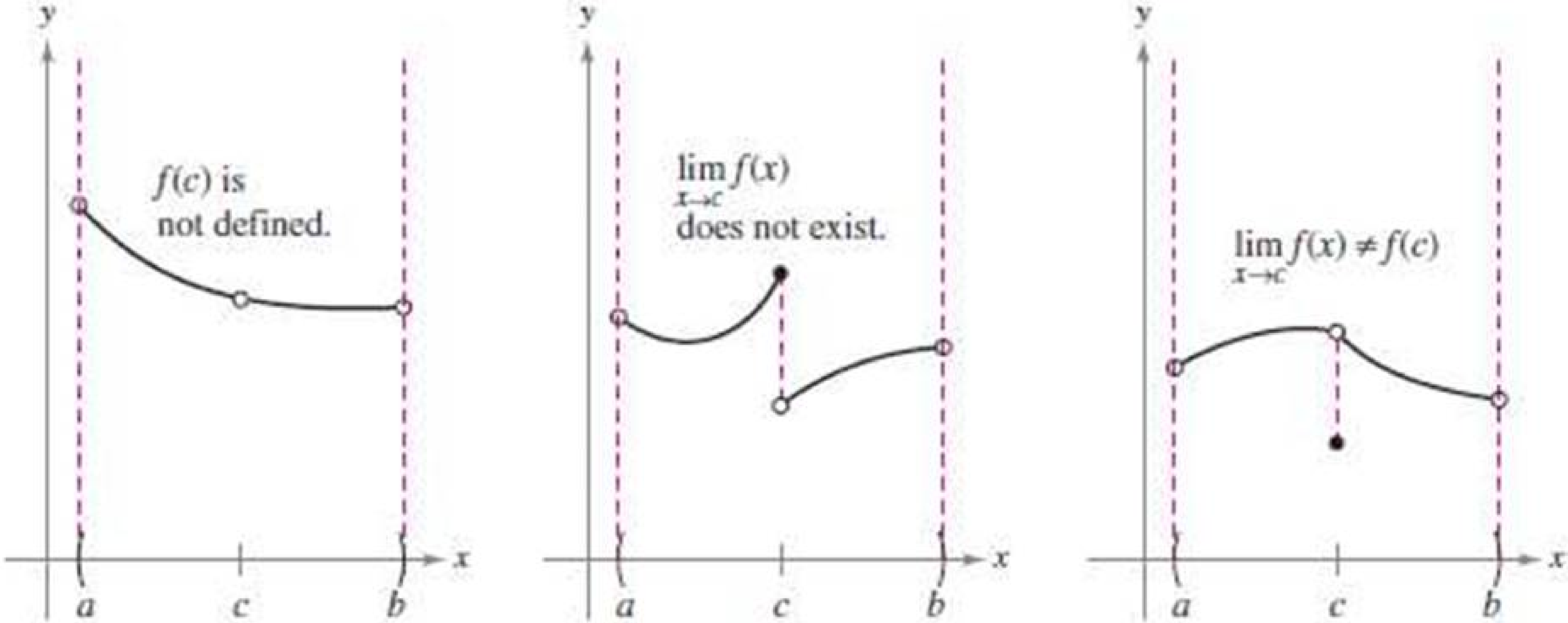
2.3 Continuity



Three conditions exist for which the graph of f is not continuous at $x = c$.

DEFINITION Continuity at a Point

Interior Point: A function $y = f(x)$ is **continuous at an interior point c** of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

Endpoint: A function $y = f(x)$ is **continuous at a e t endpoint a** or is **continuous at a rig t endpoint b** of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x \to b^-} f(x) = f(b), \quad \text{respectively.}$$

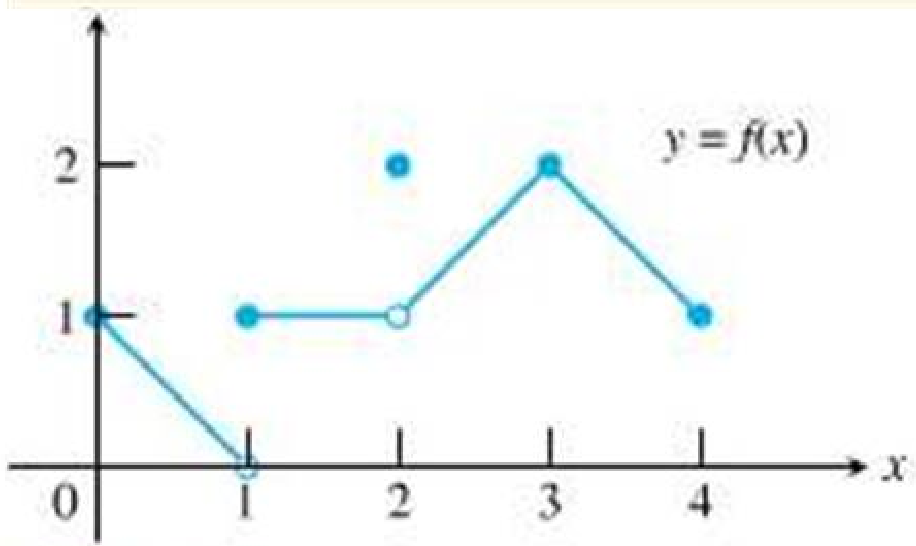


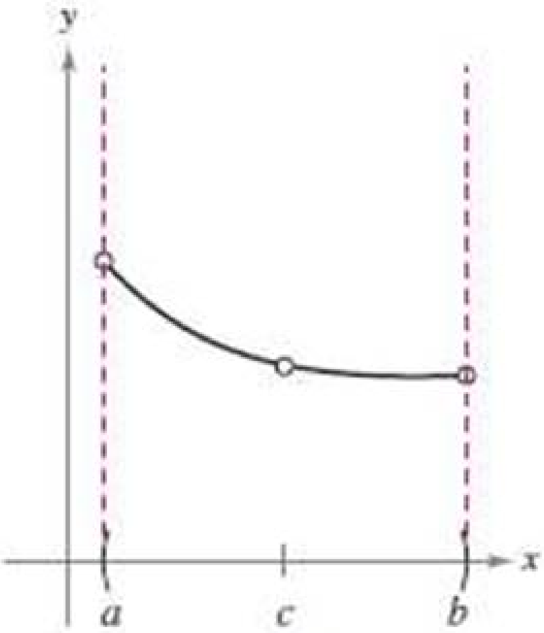
Figure 2.18 The function is continuous on $[0, 4]$ except at $x = 1$ and $x = 2$. (Example 1)

Definition of Continuity

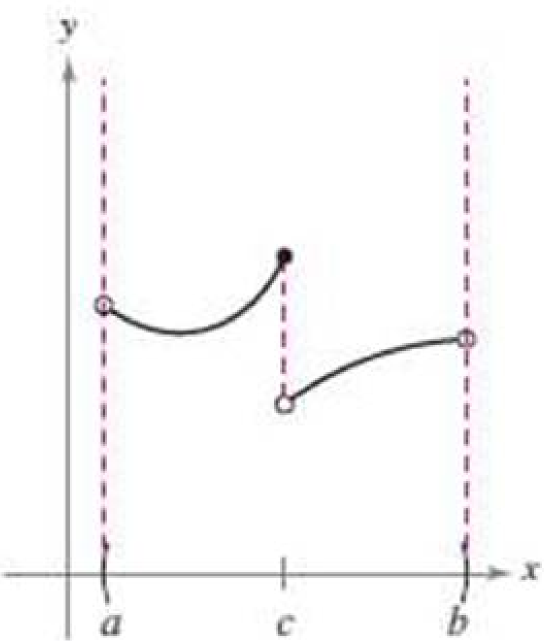
Continuity at a Point: A function f is **continuous at c** if the following three conditions are met.

1. $f(c)$ is defined.
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(c)$.

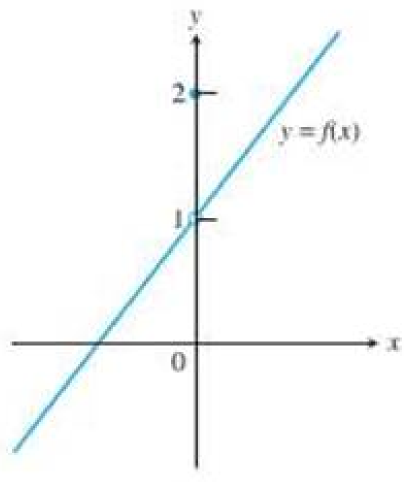
Continuity on an Open Interval: A function is **continuous on an open interval** (a, b) if it is continuous at each point in the interval. A function that is continuous on the entire real line $(-\infty, \infty)$ is **everywhere continuous.**



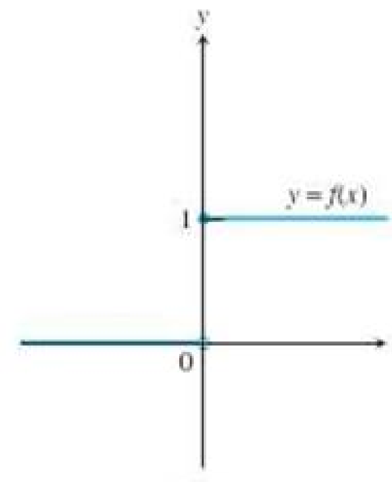
(a) Removable discontinuity



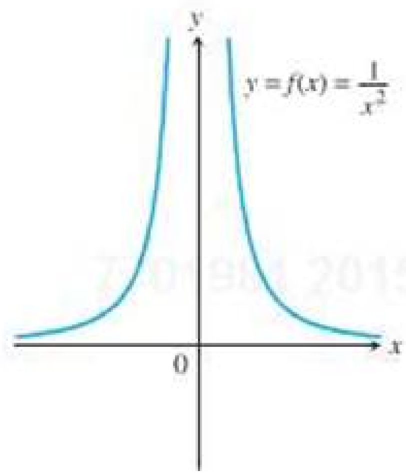
(b) Nonremovable discontinuity

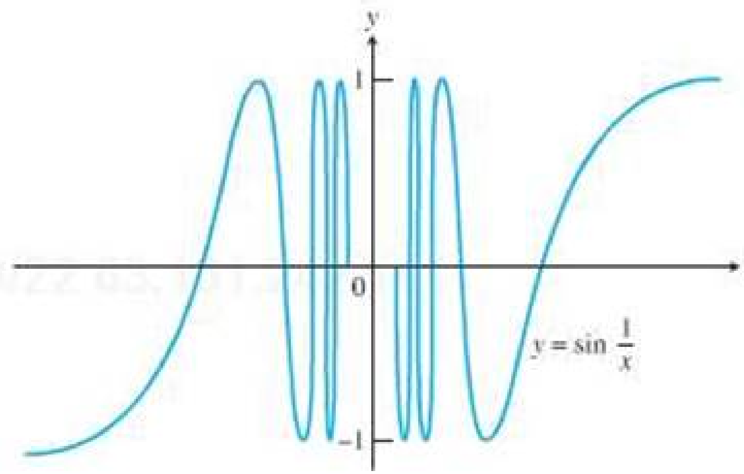


(c)



(d)

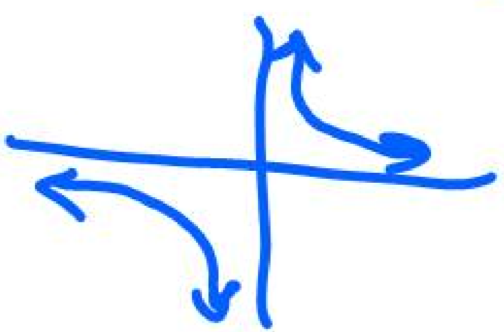




Discuss the continuity of each function.

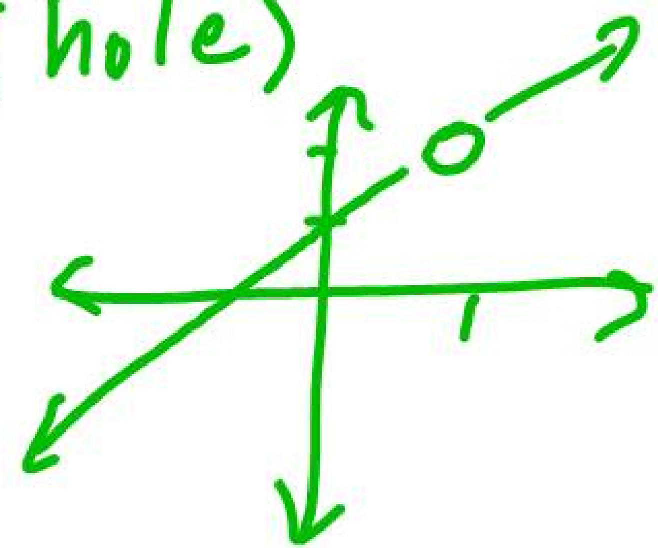
a. $f(x) = \dfrac{1}{x}$ **b.** $g(x) = \dfrac{x^2 - 1}{x - 1}$ **c.** $h(x) = \begin{cases} x + 1, & x \le 0 \\ x^2 + 1, & x > 0 \end{cases}$ **d.** $y = \sin x$

a) f(x) is discontinuous at $x = 0$ because $f(0)$ doesn't exist. Infinite discontinuity.



b) $\dfrac{x^2-1}{x-1} \Rightarrow x+1$

g(x) is discontinuos at $x=1$. It has a removable discontinuity. (hole)



Discuss the continuity of each function.

a. $\cancel{f(x) = \frac{1}{x}}$ b. $\cancel{g(x) = \frac{x^2 - 1}{x - 1}}$ c. $h(x) = \begin{cases} x + 1, & x \le 0 \\ x^2 + 1, & x > 0 \end{cases}$ d. $y = \sin x$

c) $h(0) = 1$ $\therefore$ defined

$\lim_{x \to 0^-} h(x) = 1$

$\lim_{x \to 0^+} h(x) = 1$

$\lim_{x \to 0} h(x) = 1 = h(0)$

$\therefore$ $h(x)$ is continuous

d) continuous

EXPLORATION 1 Removing a Discontinuity

$x^2 - 9 \rightarrow$

$(x-3)(x+3)$

$\{x \mid x \neq \pm 3\}$

$$\begin{array}{r|rrrr} 3 & 1 & 0 & -7 & -6 \\ & & 3 & 9 & 6 \\ \hline & 1 & 3 & 2 & \boxed{0} \end{array}$$

$\lim_{x \to 3} f(x) = 20/6$

Let $f(x) = \dfrac{x^3 - 7x - 6}{x^2 - 9}$. $= \dfrac{(x-3)(x^2+3x+2)}{(x-3)(x+3)}$

1. Factor the denominator. What is the domain of f?
2. Investigate the graph of f around $x = 3$ to see that f has a removable discontinuity at $x = 3$.
3. How should f be defined at $x = 3$ to remove the discontinuity? Use ZOOM-IN and tables as necessary.
4. Show that $(x - 3)$ is a factor of the numerator of f, and remove all common factors. Now compute the limit as $x \rightarrow 3$ of the reduced form for f.
5. Show that the *extended function*

$$g(x) = \begin{cases} \dfrac{x^3 - 7x - 6}{x^2 - 9}, & x \neq 3 \\ 10/3, & x = 3 \end{cases}$$

is continuous at $x = 3$. The function g is the **continuous extension** of the original function f to include $x = 3$.

50. *Continuous Function* Find a value for a so that the function

$$f(x) = \begin{cases} x^2 + x + a, & x < 1 \\ x^3, & x \geq 1 \end{cases}$$

is continuous.

$$\lim_{x \to 1^-} f(x) = 1^2 + 1 + a = 2 + a$$

$$\lim_{x \to 1^+} f(x) = 1^3 = 1 \qquad \therefore \quad 1 = 2 + a$$

$$a = -1$$

$$f(1) = 1$$

III. What value of d will make the function continuous?

$$g(x) = \begin{cases} x^2 - d^2, & x < 4 \\ dx + 20, & x \geq 4 \end{cases}$$

$$\lim_{x \to 4^-} g(x) = 4^2 - d^2$$

$$\lim_{x \to 4^+} g(x) = 4d + 20$$

$$4d + 20 = 16 - d^2$$

$$d^2 + 4d + 4 = 0$$

$$(d+2)^2 = 0 \quad \therefore \quad d = -2$$

Check

$$g(x) = \begin{cases} x^2 - 4 \ ; & x < 4 \\ -2x + 20 \ ; & x \geq 4 \end{cases}$$

THEOREM 6 Properties of Continuous Functions

If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. *Sums:* $f + g$
2. *Differences:* $f - g$
3. *Products:* $f \cdot g$
4. *Constant multiples:* $k \cdot f$, for any number k
5. *Quotients:* f/g, provided $g(c) \neq 0$

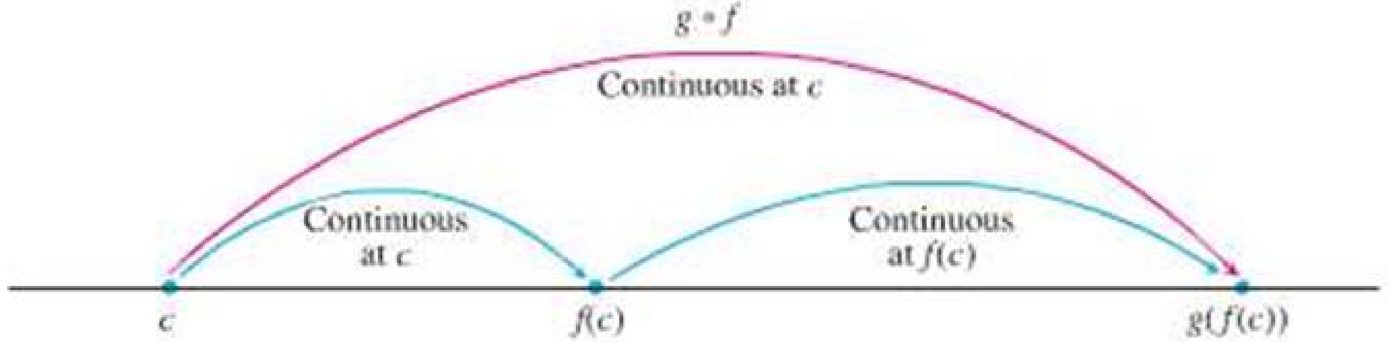


Figure 2.23 Composites of continuous functions are continuous.

THEOREM 7 Composite of Continuous Functions

If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.

HW
page 84
exer. 4-44 (4x)
47-50 all
54-59 all