3.1 Derivative of a Function

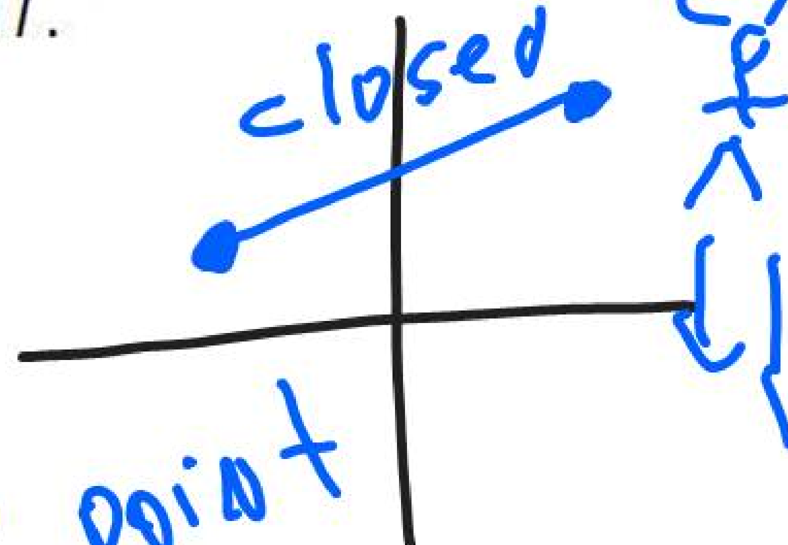
The **derivative** of the function f is the function f' given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

The domain of f is the set of all x's for which this limit exists.

The process of computing a derivative is called **differentiation.** Further, f is **differentiable on an open interval** I if it is differentiable at every point in I.

ahh!



Limits do not exist at end point

EXAMPLE

Finding the Derivative of a Simple Rational Function

If $f(x) = \dfrac{1}{x}$ $(x \neq 0)$, find $f'(x)$.

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \to 0} \frac{x-(x+h)}{(x+h)x} \cdot \frac{1}{h} = \lim_{h \to 0} \frac{-h}{(x+h)x} \frac{1}{h}$$

$$\lim_{h \to 0} \frac{-1}{(x+h)x} = \frac{-1}{x^2}$$

$$f'(x) = -\frac{1}{x^2}$$

The **derivative** of the function f at the point $x = a$ is defined as

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

provided the limit exists. If the limit exists, we say that f is **differentiable** at $x = a$.

An alternative form is

$$f'(a) = \lim_{b \to a} \frac{f(b) - f(a)}{b - a}.$$

EXAMPLE Finding the Derivative at a Point

Compute the derivative of $f(x) = 3x^3 + 2x - 1$ at $x = 1$.

$a = 1$

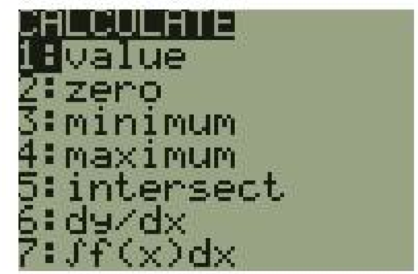
$$f'(a) = f'(1) = \lim_{h \to 0} \frac{\left[3(h+1)^3 + 2(h+1) - 1\right] - \left[3(1)^3 + 2(1) - 1\right]}{h}$$

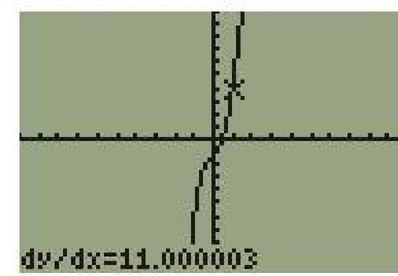
$$= \lim_{h \to 0} \frac{3(h^3 + 3h^2 + 3h + 1) + 2h + 2 - 1 - 4}{h}$$

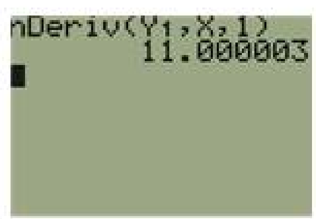
$$= \lim_{h \to 0} \frac{3h^3 + 9h^2 + 9h + \cancel{3} + 2h - \cancel{3}}{h}$$

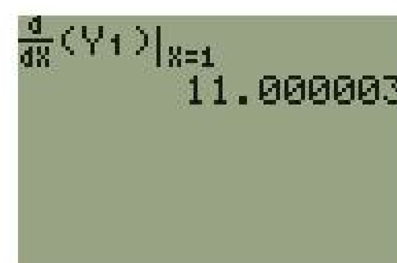
$$= \lim_{h \to 0} 3h^2 + 9h + 11 = 11$$

$$f'(1) = 11$$









$$(A \pm B)^2 = A^2 \pm 2AB + B^2$$

$$(A \pm B)^3 = A^3 \pm 3A^2B + 3AB^2 \pm B^3$$

$$(A+B)^4 =$$

$$A^4 + 4A^3B + 6A^2B^2 + 4AB^3 + B^4$$

...

Binomial Theorem

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

PASCAL $\triangle$

Different Notations for the Derivative:

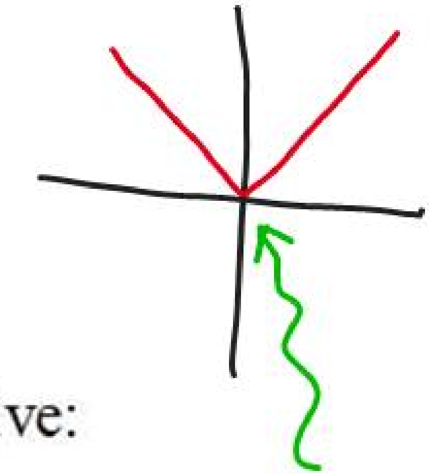
$$\boxed{f'(x),\ \frac{dy}{dx},\ \ y',\ \frac{d}{dx}(f(x)),}\ D_x\ ,\ D_x y\ ,\ \dot{u}$$

Notations for the Derivative at a single point

$$f'(a)\ \ or\ \ \left.\frac{dy}{dx}\right|_{x=a}$$

Example One-sided Derivatives

Show that the following function has left-hand and right-hand derivatives at $x=0$, but no derivative there.



$$y=\begin{cases} -x, & x<0 \\ x, & x\geq 0 \end{cases}$$

NOT differentiable

Left-hand derivative:

$$\lim_{h\to 0^-}\frac{-(0+h)-0}{h}$$

$$=\lim_{h\to 0^-}\frac{-h}{h}=-1$$

Right-hand derivative:

$$\lim_{h\to 0^+}\frac{(0+h)-0}{h}$$

$$=\lim_{h\to 0^+}\frac{h}{h}=1$$

The derivatives are not equal at $x=0$. The function does not have a derivative at 0.

31. Using one-sided derivatives, show that the function

$$f(x) = \begin{cases} x^2 + x, & x \le 1 \\ 3x - 2, & x > 1 \end{cases}$$

does not have a derivative at $x = 1$.

$\lim_{x \to 1^-} f(x) = 1+1 = 2$

$\lim_{x \to 1^+} f(x) = 3-2 = 1$

f is NOT continuous

$$\frac{d}{dx}[f(x)]\Big|_{x \to 1^-} = \lim_{h \to 0} \frac{(h+1)^2 + (h+1) - (1^2+1)}{h}$$

$$= \lim_{h \to 0} \frac{h^2 + 2h \cancel{+1} + h \cancel{+1} \cancel{-2}}{h} = \lim_{h \to 0} \frac{h^2+3h}{h} = \lim_{h \to 0} h+3 = 3$$

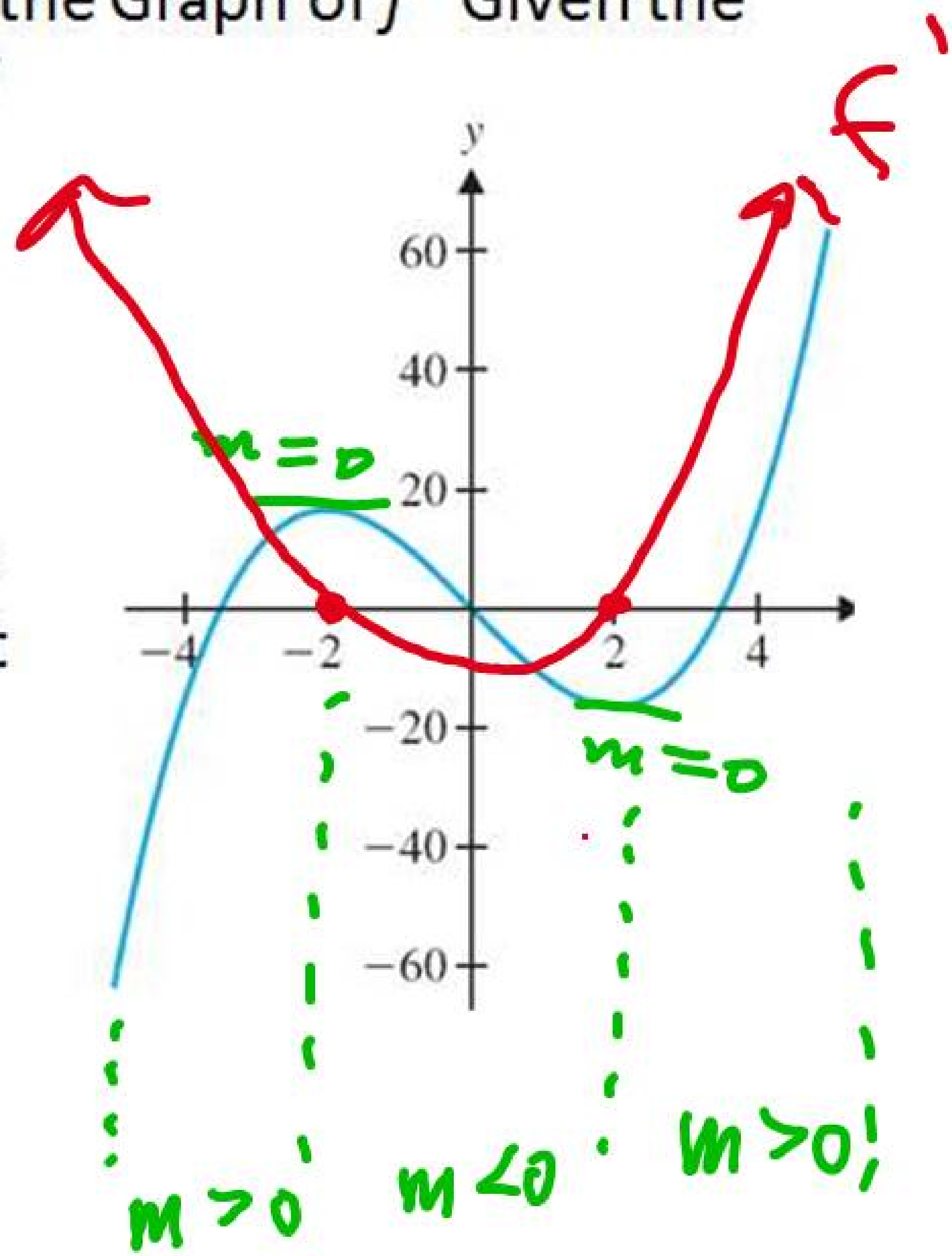
$$\frac{d}{dx}[f(x)]\Big|_{x \to 1^+} = m = 3$$

Even though both one-sided derivatives are equal, f is not differentiable at $x=1$ because it is NOT continuous at $x=1$.

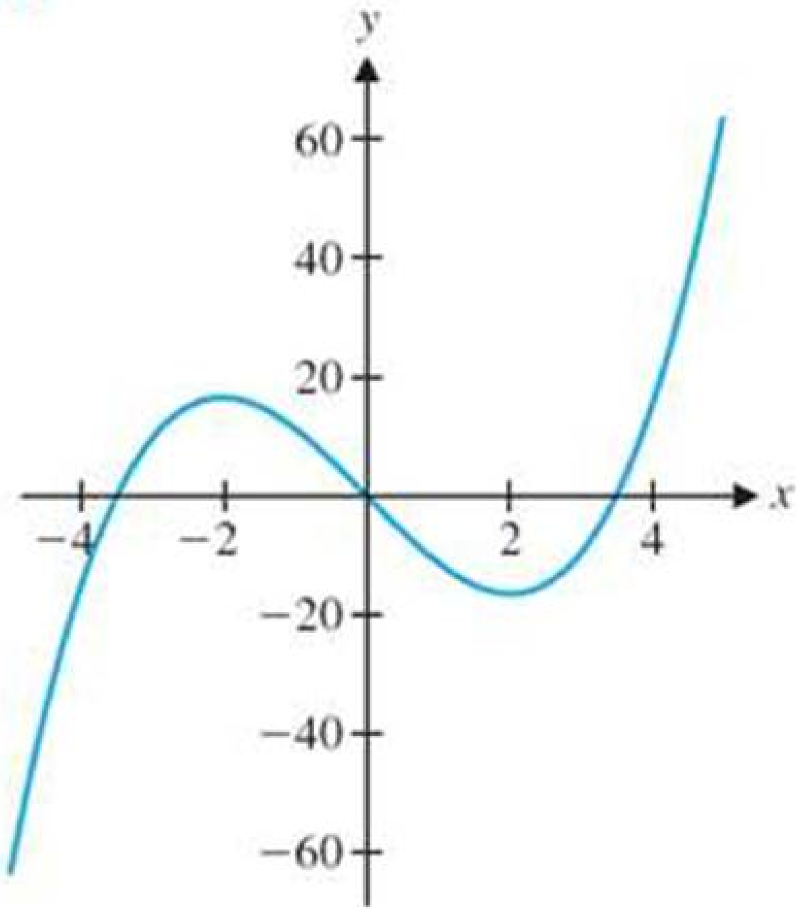
EXAMPLE Sketching the Graph of f' Given the Graph of f

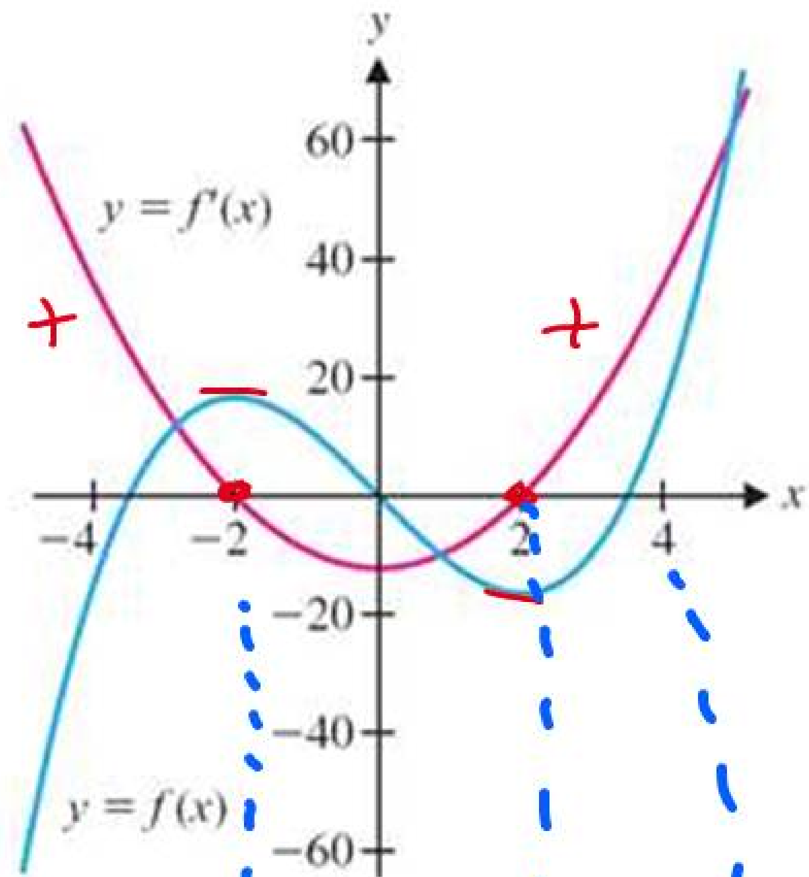
Given the graph of f in the figure, sketch a plausible graph of f'.

Keep in mind that the value of the derivative function at a point is the slope of the tangent line at that point.



Solution

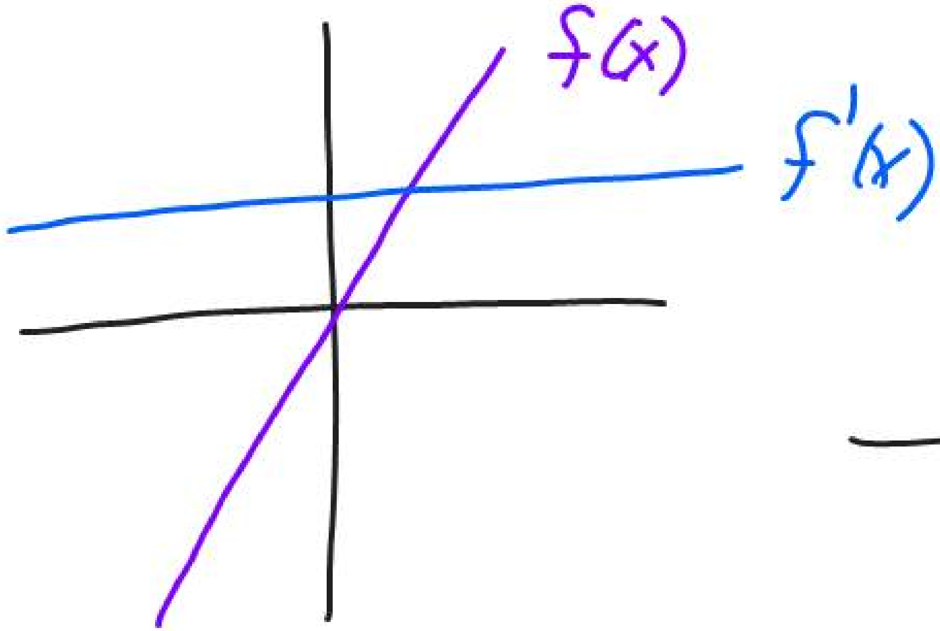




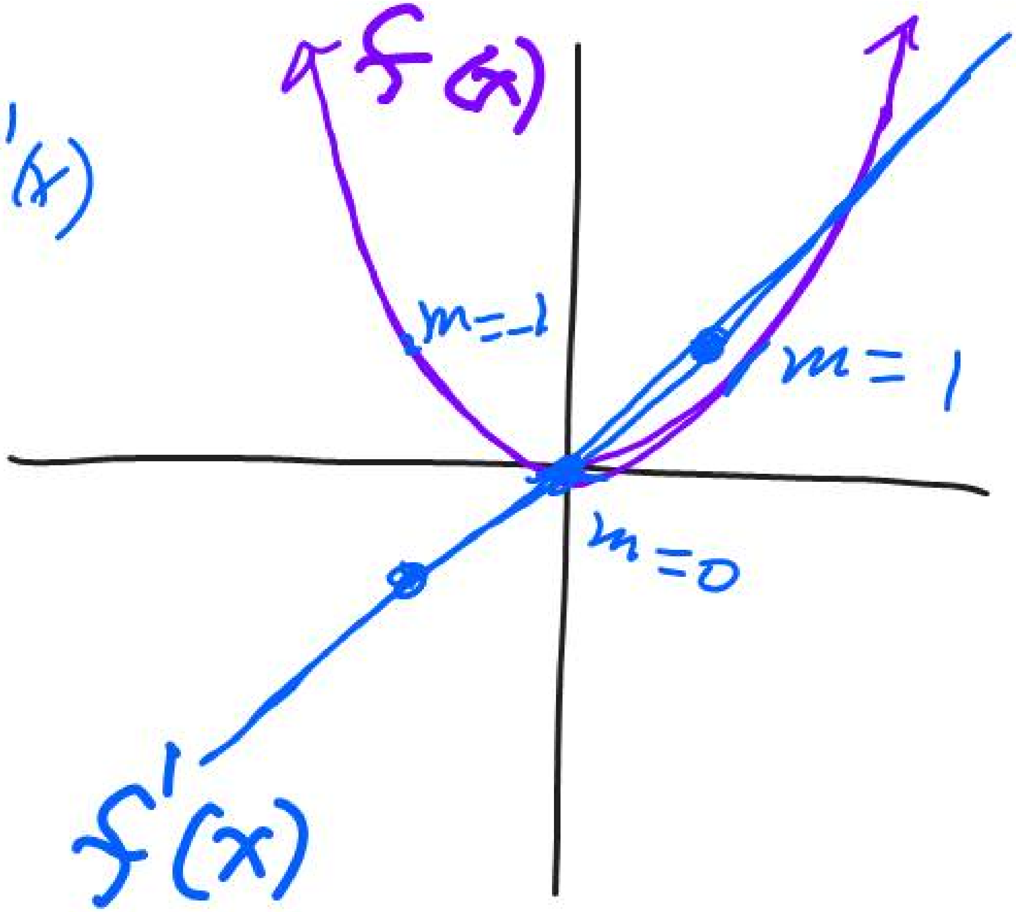
f inc | f dec | f inc

f = 0 = 0

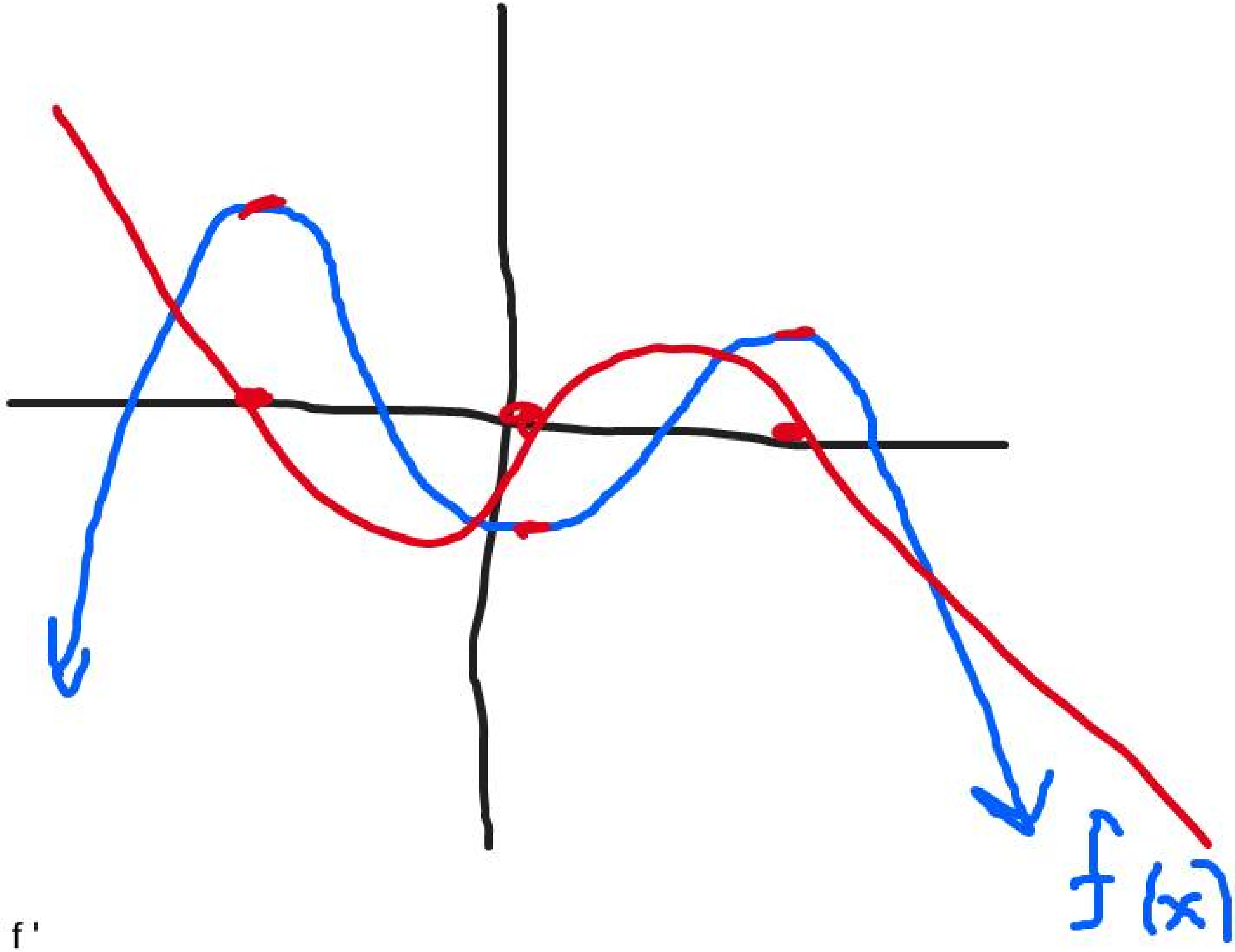
f' + 0 − 0 +



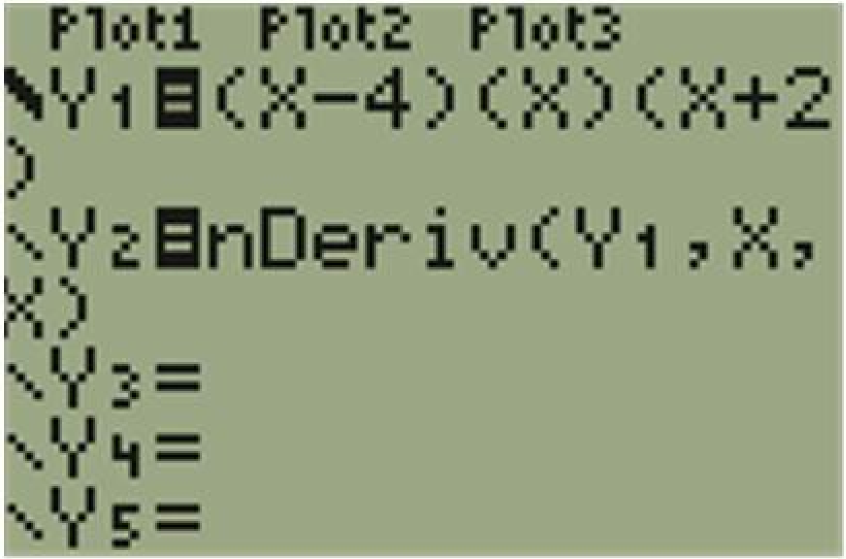
f(x)
f'(x)

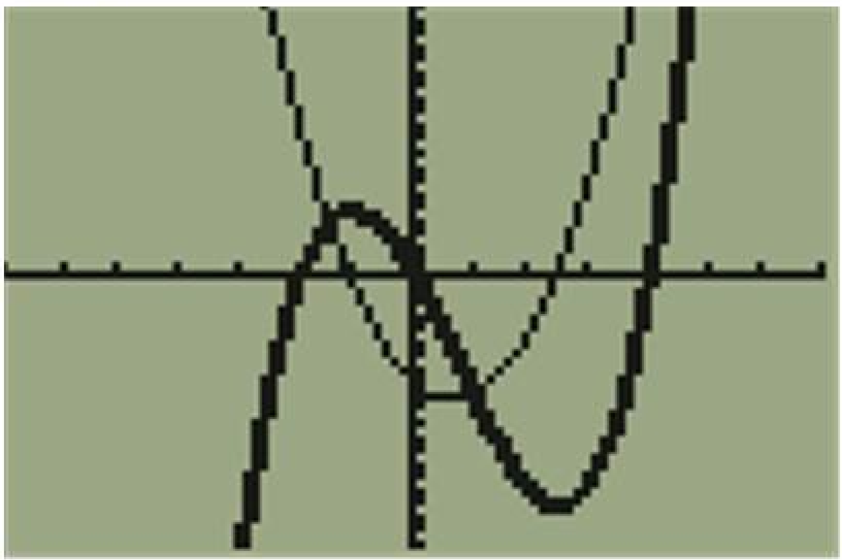


f(x)
m=-1
m=1
m=0
f'(x)



Sketch f '





$$\frac{d}{dx}\left[Y_1\right]\Big|_{x=x}$$

homework
pg 105
4,6,12,
13-16 all,
18-32(2x)
36-41 all