1.4 Parametric Equations

Relations

A **relation** is a set of ordered pairs (x, y) of real numbers. The **graph of a relation** is the set of points in the plane that correspond to the ordered pairs of the relation. If x and y are *functions* of a third variable t, called a *parameter*, then we can use the *parametric mode* of a grapher to obtain a graph of the relation.

DEFINITIONS Parametric Curve, Parametric Equations

If x and y are given as functions

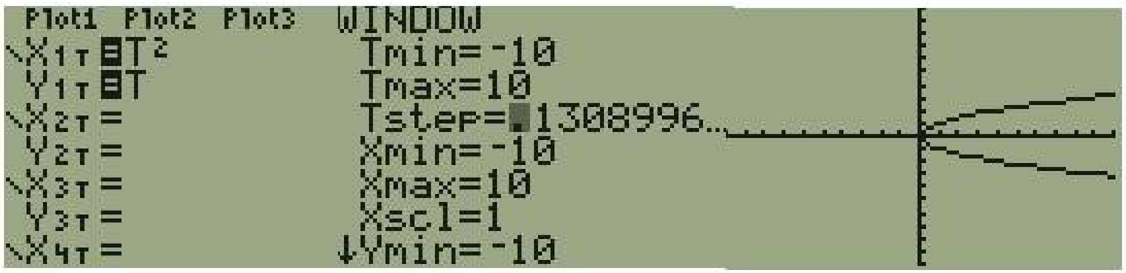
$$x = f(t), \quad y = g(t)$$

over an interval of t-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a **parametric curve**. The equations are **parametric equations** for the curve.

The variable t is a **parameter** for the curve and its domain I is the **parameter interval.** If I is a closed interval, $a \leq t \leq b$, the point $(f(a), g(a))$ is the **initial point of the curve** and the point $(f(b), g(b))$ is the **terminal point of the curve.** When we give parametric equations and a parameter interval for a curve, we say that we have **parametrized** the curve. The equations and interval constitute a **parametrization of the curve.**

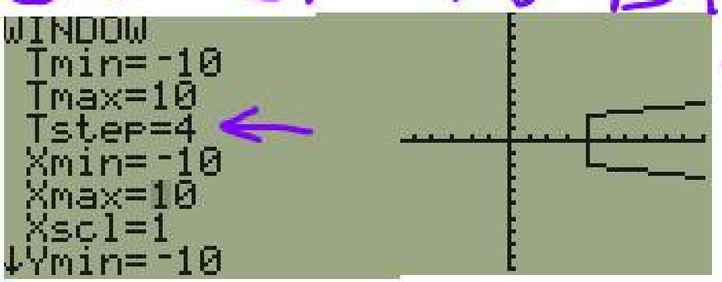
Example:
Graph $x = y^2$
(Note: NOT A function)

MODE



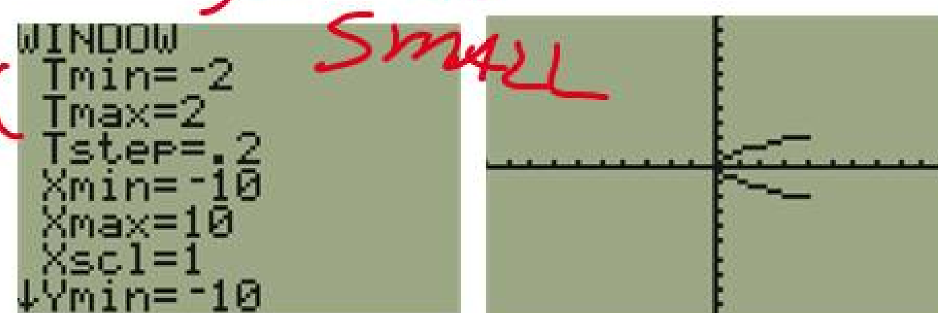
Possible mistakes

① Step to Big

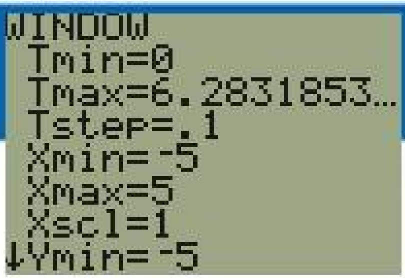


② No negative T's

③ T range too small



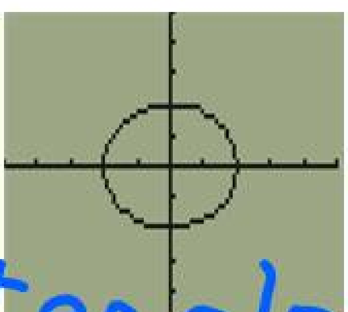
EXPLORATION 1 Parametrizing Circles

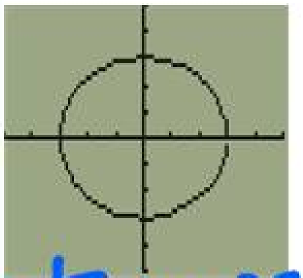


Let $x = a \cos t$ and $y = a \sin t$.

1. Let $a = 1, 2$, or 3 and graph the parametric equations in a *square viewing window* using the parameter interval $[0, 2\pi]$. How does changing a affect this graph?
2. Let $a = 2$ and graph the parametric equations using the following parameter intervals: $[0, \pi/2]$, $[0, \pi]$, $[0, 3\pi/2]$, $[2\pi, 4\pi]$, and $[0, 4\pi]$. Describe the role of the length of the parameter interval.
3. Let $a = 3$ and graph the parametric equations using the intervals $[\pi/2, 3\pi/2]$, $[\pi, 2\pi]$, $[3\pi/2, 3\pi]$, and $[\pi, 5\pi]$. What are the initial point and terminal point in each case?
4. Graph $x = 2 \cos(-t)$ and $y = 2 \sin(-t)$ using the parameter intervals $[0, 2\pi]$, $[\pi, 3\pi]$, and $[\pi/2, 3\pi/2]$. In each case, describe how the graph is traced.

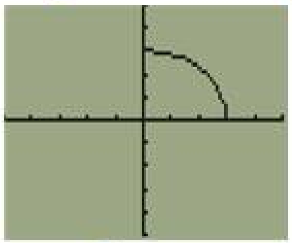
1. a=1,2,3





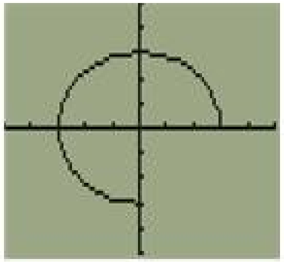
counter clockwise

2. $0 \le t \le \pi/2$



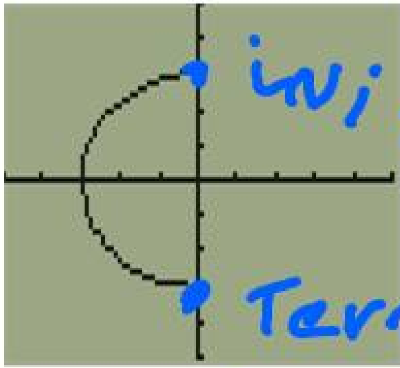
CC

$0 \le t < \frac{3}{2}\pi$



3. Let $a = 3$ and graph the parametric equations using the intervals $[\pi/2, 3\pi/2]$, $[\pi, 2\pi]$, $[3\pi/2, 3\pi]$, and $[\pi, 5\pi]$. What are the initial point and terminal point in each case?

$\pi/2 \le t \le 3\pi/2$

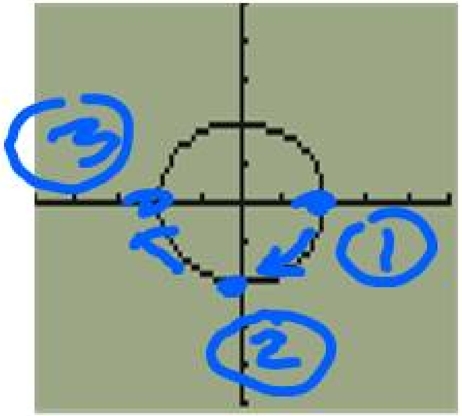


initial

Terminal

t	x	y
	$3\cos t$	$3\sin t$
$\pi/2$	0	3
$3\pi/2$	0	-3

4. Graph $x = 2\cos(-t)$ and $y = 2\sin(-t)$ using the parameter intervals $[0, 2\pi]$, $[\pi, 3\pi]$, and $[\pi/2, 3\pi/2]$. In each case, describe how the graph is traced.



traced clockwise

T	x	y	
0	2	0	1
$\pi/2$	0	-2	2
π	-2	0	3

Describe the graph of the relation determined by

$$x = 2\cos t, \quad y = 2\sin t, \quad 0 \leq t \leq 2\pi.$$

Find the initial and terminal points, if any, and indicate the direction in which the curve is traced. Find a Cartesian equation for a curve that contains the parametrized curve.

SOLUTION

Figure 1.29 shows that the graph appears to be a circle with radius 2. By watching the graph develop we can see that the curve is traced exactly once counterclockwise. The initial point at $t = 0$ is $(2, 0)$, and the terminal point at $t = 2\pi$ is also $(2, 0)$.

Next we eliminate the variable t.

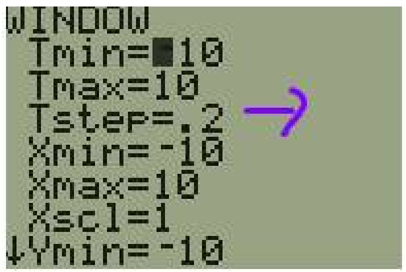
$$\begin{aligned} x^2 + y^2 &= 4\cos^2 t + 4\sin^2 t \\ &= 4(\cos^2 t + \sin^2 t) \\ &= 4 \qquad \text{Because } \cos^2 t + \sin^2 t = 1 \end{aligned}$$

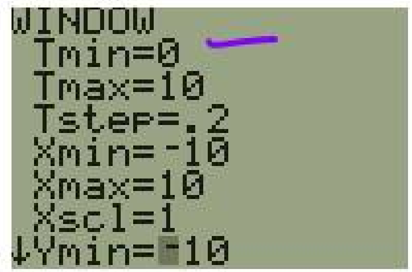
Graph $x = y^2 + y - 6$

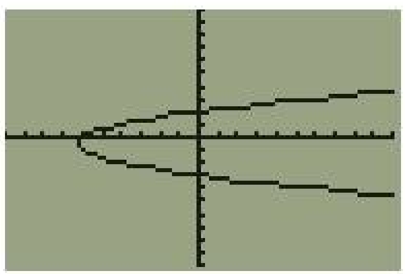
Not a function

$X_T = T^2 + T - 6$

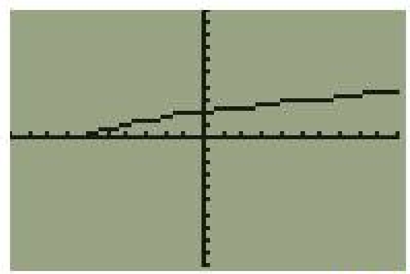
$Y_T = T$



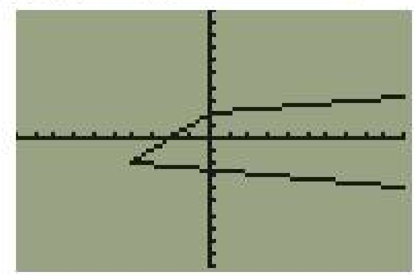




good



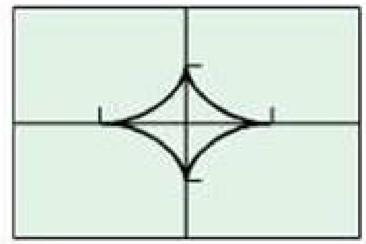
incomplete



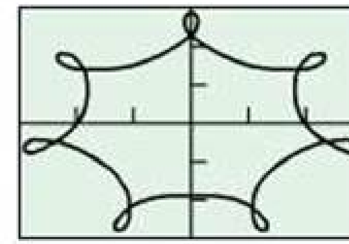
Not very good

In Exercises 1–4, match the parametric equations with their graph. State the approximate dimensions of the viewing window. Give a parameter interval that traces the curve exactly once.

1. $x = 3 \sin(2t), \quad y = 1.5 \cos t$
2. $x = \sin^3 t, \quad y = \cos^3 t$
3. $x = 7 \sin t - \sin(7t), \quad y = 7 \cos t - \cos(7t)$
4. $x = 12 \sin t - 3 \sin(6t), \quad y = 12 \cos t + 3 \cos(6t)$



(a)



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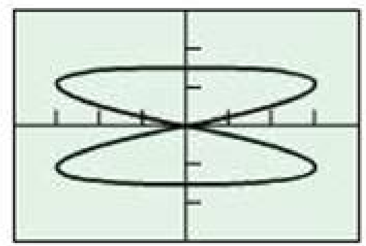
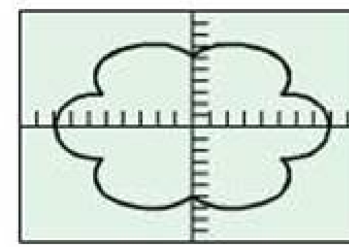
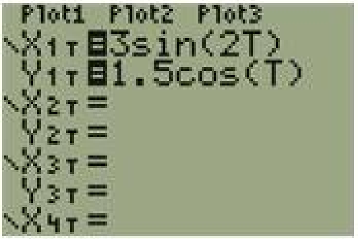
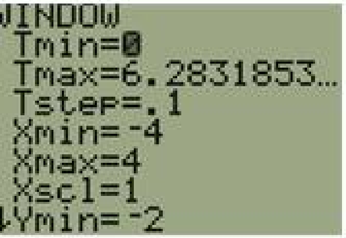


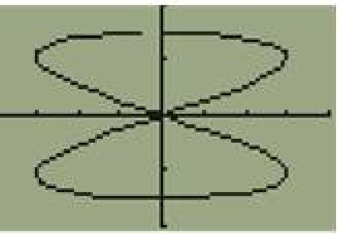
(c)

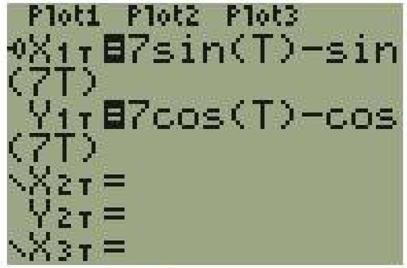


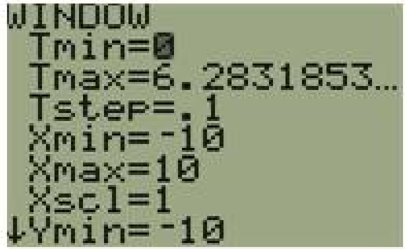
(d)

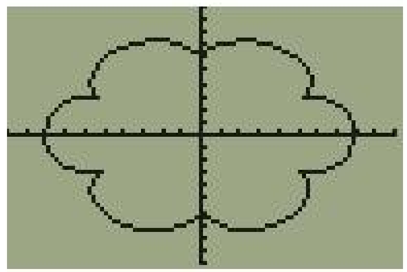


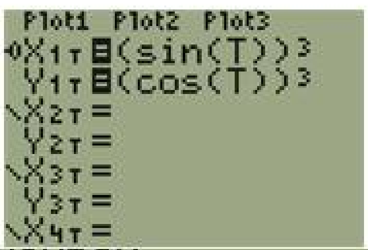


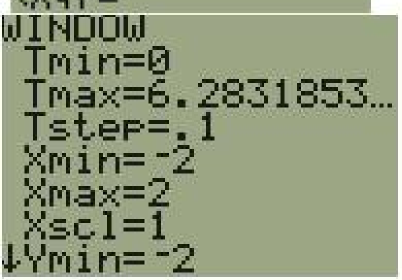


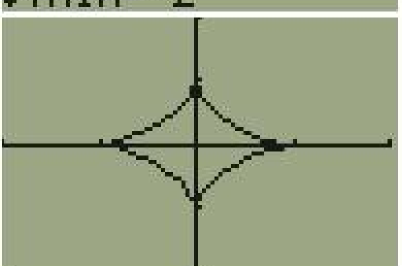


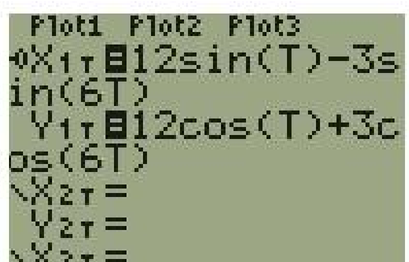


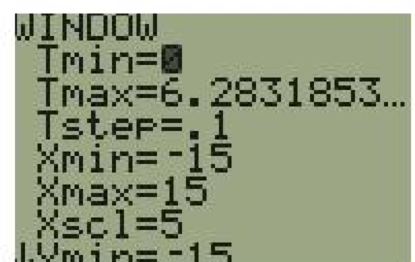


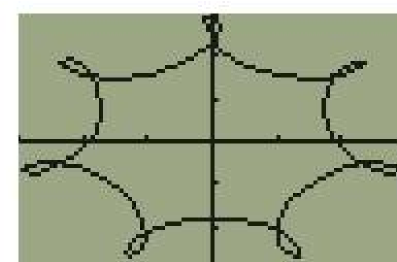












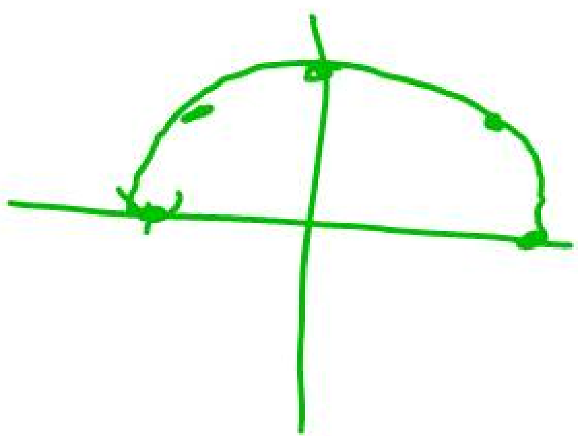
In Exercises 5–22, a parametrization is given for a curve.

(a) Graph the curve. What are the initial and terminal points, if any? Indicate the direction in which the curve is traced.

(b) Find a Cartesian equation for a curve that contains the parametrized curve. What portion of the graph of the Cartesian equation is traced by the parametrized curve?

9. $x = \cos t, \quad y = \sin t, \quad 0 \leq t \leq \pi$

t	x	y
0	1	0
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/2$	0	1
$3\pi/4$	$-\sqrt{2}/2$	$\sqrt{2}/2$
π	-1	0



initial
$(1,0)$
terminal
$(-1,0)$

curve traced CCW

$$x^2 = \cos^2 t$$
$$y^2 = \sin^2 t$$
$$\overline{x^2 + y^2 = 1} \longrightarrow y = \sqrt{1 - x^2}$$

upper half of a unit circle

Ellipses

Parametrizations of ellipses are similar to parametrizations of circles. Recall that the standard form of an ellipse centered at $(0, 0)$ is

t	x	y
0	3	0
$\pi/2$	0	4

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

$\frac{x}{3} = \cos t$

$\frac{y}{4} = \sin t$

EXAMPLE 3 Graphing an Ellipse

Graph the parametric curve $x = 3\cos t$, $y = 4\sin t$, $0 \le t \le 2\pi$.

Find a Cartesian equation for a curve that contains the parametric curve. What portion of the graph of the Cartesian equation is traced by the parametric curve? Indicate the direction in which the curve is traced and the initial and terminal points, if any.

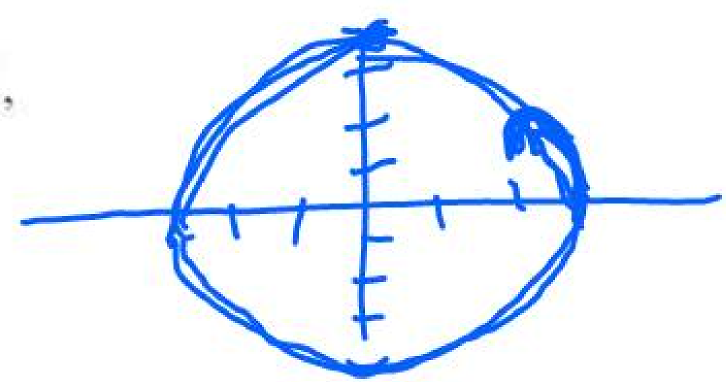
SOLUTION initial = terminal = $(3, 0)$

Figure 1.30 suggests that the curve is an ellipse. The Cartesian equation is

x	y
0	± 4
± 3	0

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = \cos^2 t + \sin^2 t = 1,$$

A full ellipse



In Exercises 5–22, a parametrization is given for a curve.

(a) Graph the curve. What are the initial and terminal points, if any? Indicate the direction in which the curve is traced.

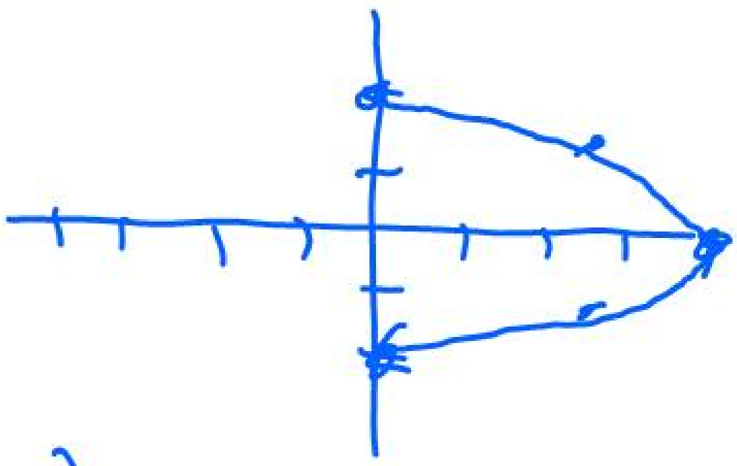
(b) Find a Cartesian equation for a curve that contains the parametrized curve. What portion of the graph of the Cartesian equation is traced by the parametrized curve?

t	x	y
0	0	2
$\pi/4$	$2\sqrt{2}$	$\sqrt{2}$
$\pi/2$	4	0
$3\pi/4$	$2\sqrt{2}$	$-\sqrt{2}$
π	0	-2

13. $x = 4\sin t, \quad y = 2\cos t, \quad 0 \le t \le \pi$

$\frac{x}{4} = \sin t \; ; \; \frac{y}{2} = \cos t$

$\left(\frac{x}{4}\right)^2 = \sin^2 t \; ; \; \left(\frac{y}{2}\right)^2 = \cos^2 t$



$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$

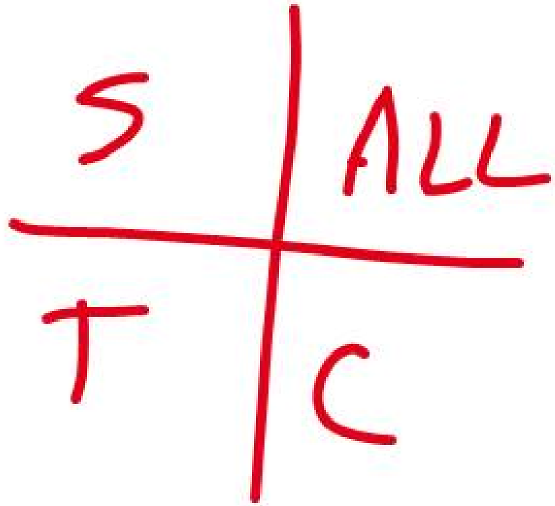
initial $\Rightarrow (0,2)$

Half the ellipse

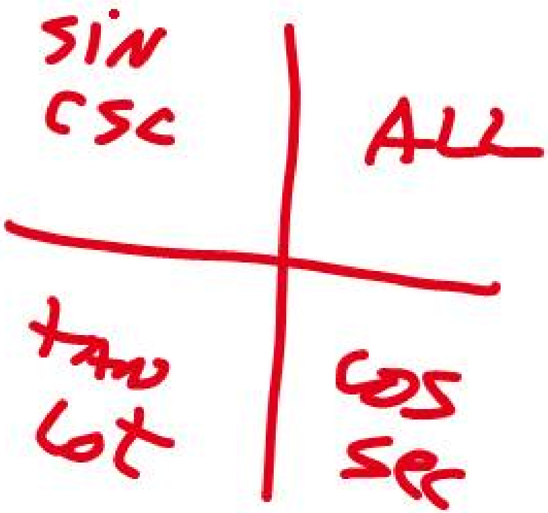
terminal $= (0,-2)$

QI & QIV

traced CW



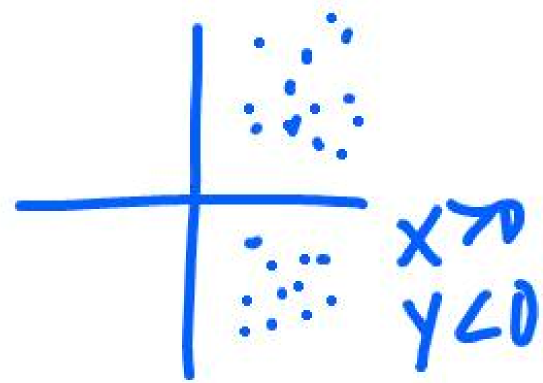
(+)
positives



$x \sim \sin$

$y \sim \cos$

$0 < t < \pi$



EXAMPLE 4 Graphing a Line Segment

Draw and identify the graph of the parametric curve determined by

$$x = 3t, \quad y = 2 - 2t, \quad 0 \leq t \leq 1.$$

$$\frac{x}{3} = t$$

SOLUTION

The graph (Figure 1.31) appears to be a line segment with endpoints $(0, 2)$ and $(3, 0)$.

Confirm Algebraically When $t = 0$, the equations give $x = 0$ and $y = 2$. When $t = 1$, they give $x = 3$ and $y = 0$. When we substitute $t = x/3$ into the y equation, we obtain

$$y = 2 - 2\left(\frac{x}{3}\right) = -\frac{2}{3}x + 2.$$

Thus, the parametric curve traces the segment of the line $y = -(2/3)x + 2$ from the point $(0, 2)$ to $(3, 0)$. *Now Try Exercise 17.*

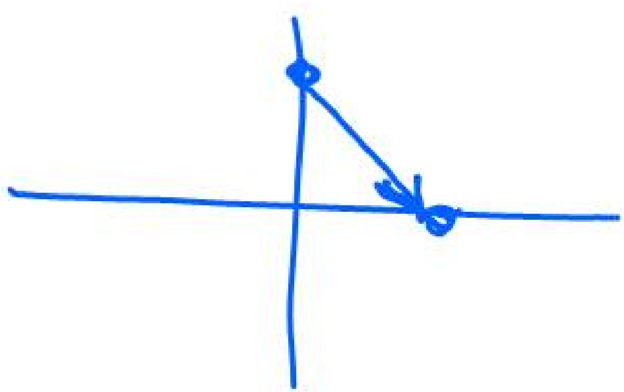
In Exercises 5–22, a parametrization is given for a curve.

(a) Graph the curve. What are the initial and terminal points, if any? Indicate the direction in which the curve is traced.

(b) Find a Cartesian equation for a curve that contains the parametrized curve. What portion of the graph of the Cartesian equation is traced by the parametrized curve?

17. $x = t, \quad y = 1 - t, \quad 0 \le t \le 1$

t	x	y	
0	0	1	initial
1	1	0	terminal



left to right

$y = 1 - t$

$y = 1 - x$

$x \in [0, 1]$

or $0 \le x \le 1$

segment

EXAMPLE 5 Parametrizing a Line Segment

Find a parametrization for the line segment with endpoints $(-2, 1)$ and $(3, 5)$.

SOLUTION

$t = 0$

Using $(-2, 1)$ we create the parametric equations

$$x = -2 + at, \qquad y = 1 + bt.$$

These represent a line, as we can see by solving each equation for t and equating to obtain

$$\frac{x+2}{a} = \frac{y-1}{b}.$$

This line goes through the point $(-2, 1)$ when $t = 0$. We determine a and b so that the line goes through $(3, 5)$ when $t = 1$.

$$3 = -2 + a \quad \Rightarrow \quad a = 5 \qquad x = 3 \text{ when } t = 1.$$

$$5 = 1 + b \quad \Rightarrow \quad b = 4 \qquad y = 5 \text{ when } t = 1.$$

Therefore,

$$x = -2 + 5t, \quad y = 1 + 4t, \quad 0 \le t \le 1$$

is a parametrization of the line segment with initial point $(-2, 1)$ and terminal point $(3, 5)$.

Now Try Exercise 23.

In Exercises 23–28, find a parametrization for the curve.

23. the line segment with endpoints $(\overset{t=0}{-1, -3})$ and $(\overset{t=1}{4, 1})$

$\boxed{t=0}$

$x_t = -1 + at \quad ; \quad y_t = -3 + bt$

$\boxed{t=1}$

$1 = -3 + b$

$4 = -1 + a \qquad \therefore 4 = b$

$\therefore a = 5$

$x_t = -1 + 5t$

$y_t = -3 + 4t$

$0 \le t \le 1$

In Exercises 23–28, find a parametrization for the curve.

25. the lower half of the parabola $x - 1 = y^2$

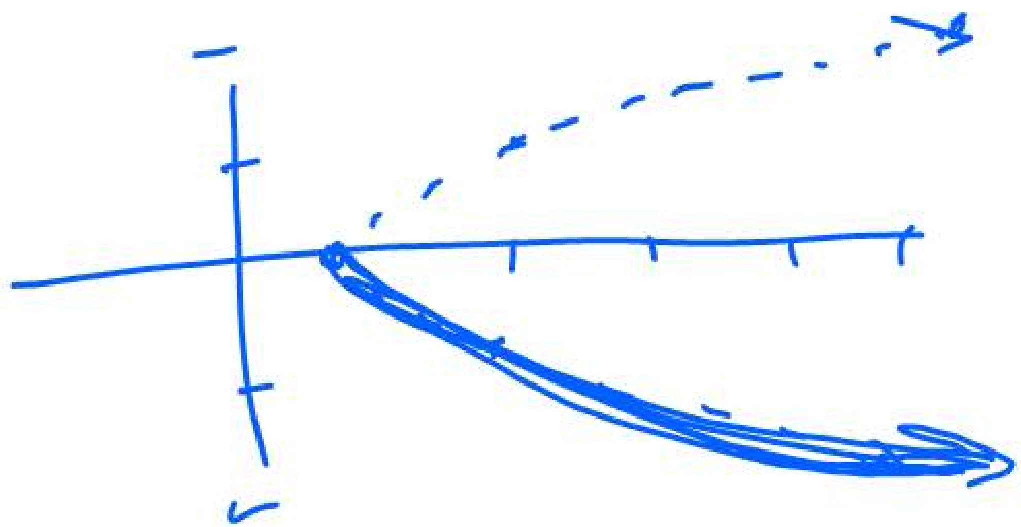
$x = y^2 + 1$

$y_t = t$

$x_t = t^2 + 1$

$t \leq 0$

x	y
1	0
2	± 1
5	± 2



HOMEWORK
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1-4 ALL,
6-28 EVEN
37-42 ALL