3.4 Velocity and Other Rates of Change

DEFINITION Instantaneous Rate of Change

The **(instantaneous) rate of change** of f with respect to x at a is the derivative

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

provided the limit exists.

1. (a) Write the volume V of a cube as a function of the side length s.
   (b) Find the (instantaneous) rate of change of the volume V with respect to a side s.
   (c) Evaluate the rate of change of V at $s = 1$ and $s = 5$.
   (d) If s is measured in inches and V is measured in cubic inches, what units would be appropriate for dV/ds?

$V(s) = s^3$

$V'(s) = 3s^2$

$V'(1) = 3(1)^2 = 3$

$V'(5) = 3(5)^2 = 75$

$\frac{dV}{ds} = \frac{in^3}{in}$

DEFINITION Instantaneous Velocity

The **(instantaneous) velocity** is the derivative of the position function $s = f(t)$ with respect to time. At time t the velocity is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

DEFINITION Speed

Speed is the absolute value of velocity.

$$\text{Speed} = |v(t)| = \left|\frac{ds}{dt}\right|$$

DEFINITION Acceleration

Acceleration is the derivative of velocity with respect to time. If a body's velocity at time t is $v(t) = ds/dt$, then the body's acceleration at time t is

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

Free-fall Constants (Earth)

English units: $g = 32\dfrac{\text{ft}}{\text{sec}^2}$, $s = \dfrac{1}{2}(32)t^2 = 16t^2$ (s in feet)

Metric units: $g = 9.8\dfrac{\text{m}}{\text{sec}^2}$, $s = \dfrac{1}{2}(9.8)t^2 = 4.9t^2$ (s in meters)

9. *Particle Motion* The accompanying figure shows the velocity $v = f(t)$ of a particle moving on a coordinate line.

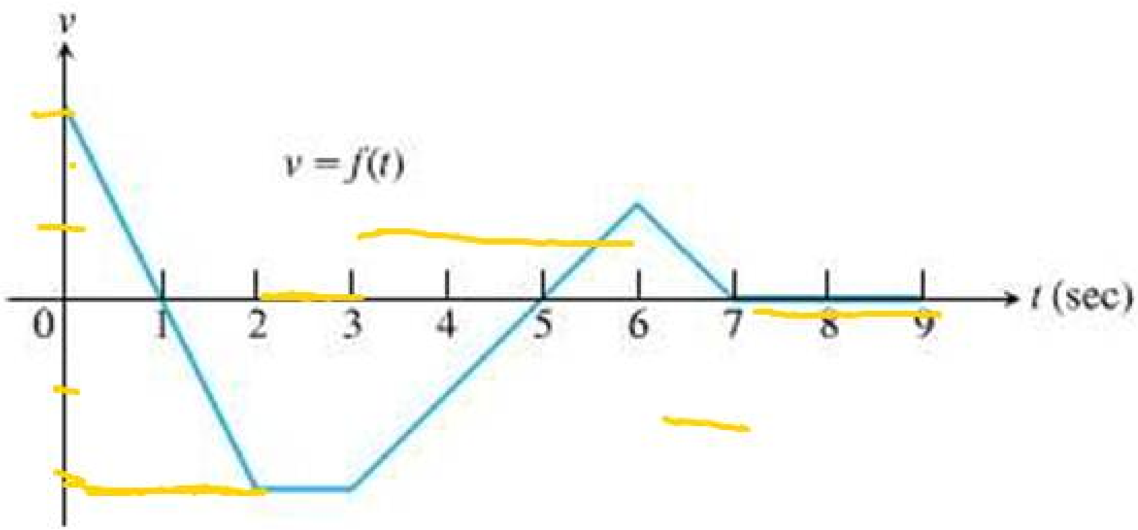
(a) When does the particle move forward? move backward? speed up? slow down?

(b) When is the particle's acceleration positive? negative? zero?

(c) When does the particle move at its greatest speed?

(d) When does the particle stand still for more than an instant?

a) Foward
(0,1)U(5,7)
Backward
(1,5)
Speedup (3,6)
Slows down
(0,2)U(6,7)



b) positive acceleration (3,6)
negative acceleration (0,2)U(6,7)
acceleration is zero (2,3)U(7,9)

c) At t=0 or in the interval (2,3) is the greatest speed.
d) (7,9) particle is at rest.

13. *Lunar Projectile Motion* A rock thrown vertically upward from the surface of the moon at a velocity of 24 m/sec (about 86 km/h) reaches a height of $s = 24t - 0.8t^2$ meters in t seconds.

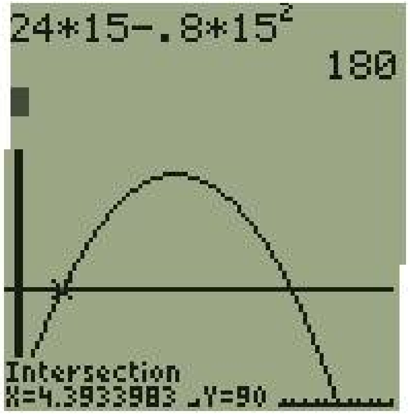
(a) Find the rock's velocity and acceleration as functions of time. (The acceleration in this case is the acceleration of gravity on the moon.)

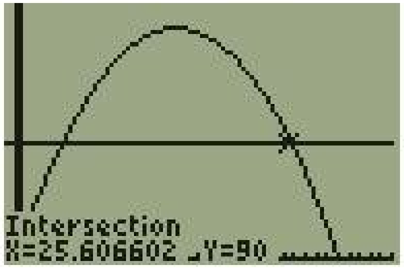
(b) How long did it take the rock to reach its highest point?

(c) How high did the rock go?

(d) When did the rock reach half its maximum height?

(e) How long was the rock aloft?





a) $v(t) = \frac{ds}{dt} = 24 - 1.6t$

$a(t) = \frac{dv}{dt} = -1.6$

b) $v(t) = 0 = 24 - 1.6t$

$t = 24/1.6 = 15\text{ sec}$

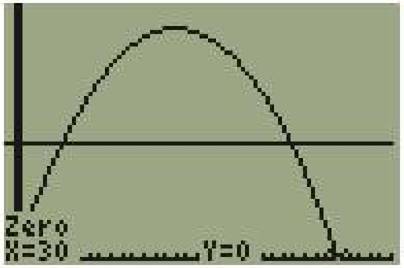
c) $s(15) = 24(15) - .8(15)^2$

$s(15) = 180\text{ m}$

d) $90 = 24t - .8t^2$

$t = 4.4\text{ sec}$

$t = 25.6\text{ sec}$



e) $2(15) = 30\text{ sec}$

19. *Particle Motion* A particle moves along a line so that its position at any time $t \geq 0$ is given by the function

$$s(t) = t^2 - 3t + 2,$$

$v(t) = 2t - 3$

where s is measured in meters and t is measured in seconds.

$a(t) = 2$

(a) Find the displacement during the first 5 seconds.

(b) Find the average velocity during the first 5 seconds.

(c) Find the instantaneous velocity when $t = 4$.

(d) Find the acceleration of the particle when $t = 4$.

(e) At what values of t does the particle change direction?

(f) Where is the particle when s is a minimum?

(a) displacement $= S_{final} - S_{initial}$

$S(5) - S(0) = (25 - 15 + 2) - (2) = 10\,m$

(b) $\bar{v} = \dfrac{S(5) - S(0)}{5 - 0} = \dfrac{10}{5} = 2\,m/s$

(c) $V(4) = 2(4) - 3 = 5\,m/s$

(d) $a(4) = 2\,m/s^2$

(e)

$0 = 2t - 3$

$t = 3/2$ sec

(f) vertex $\rightarrow$

$s(3/2) = \dfrac{9}{4} - \dfrac{9}{2} + 2 = \dfrac{-1}{4}\,m$

25. *Draining a Tank* It takes 12 hours to drain a storage tank by opening the valve at the bottom. The depth y of fluid in the tank t hours after the valve is opened is given by the formula

$$y = 6\left(1 - \frac{t}{12}\right)^2 \text{ m.} = 6\left(1 - \frac{t}{6} + \frac{t^2}{144}\right)$$

(a) Find the rate dy/dt (m/h) at which the water level is changing at time t.

(b) When is the fluid level in the tank falling fastest? slowest? What are the values of dy/dt at these times?

(c) Graph y and dy/dt together and discuss the behavior of y in relation to the signs and values of dy/dt.

a) $\frac{dy}{dt} = 6\left(-\frac{1}{6} + \frac{t}{72}\right) = -1 + \frac{t}{12}$

b)

Derivatives in Economics

Engineers use the terms *velocity* and *acceleration* to refer to the derivatives of functions describing motion. Economists, too, have a specialized vocabulary for rates of change and derivatives. They call them *marginals.*

If $C(x)$ is the cost of producing x units, the $C'(x)$ is the marginal cost.

Something similar happends to the revenues, $r(x)$ and $r'(x)$.

Marginal $\rightarrow$ instantaneous $\neq$ Average

(tangent slope) (secant slope)

27. *Marginal Cost* Suppose that the dollar cost of producing x washing machines is $c(x) = 2000 + 100x - 0.1x^2$.

$c'(x) = 100 - .2x$

(a) Find the average cost of producing 100 washing machines.

(b) Find the marginal cost when 100 machines are produced.

(c) Show that the marginal cost when 100 washing machines are produced is approximately the cost of producing one more washing machine after the first 100 have been made, by calculating the latter cost directly.

Solution (a)
C(100)/100=
$110/machine

a) $\dfrac{C(100) - C(0)}{100 - 0} = 90 \ \dfrac{\text{dollars}}{\text{machine}}$

b) $C'(100) = 100 - 0.2(100) = 100 - 20 = 80 \ \dfrac{\text{dollars}}{\text{machine}}$

c) $C(101) - C(100)$

≈ 80

28. *Marginal Revenue* Suppose the weekly revenue in dollars from selling x custom-made office desks is

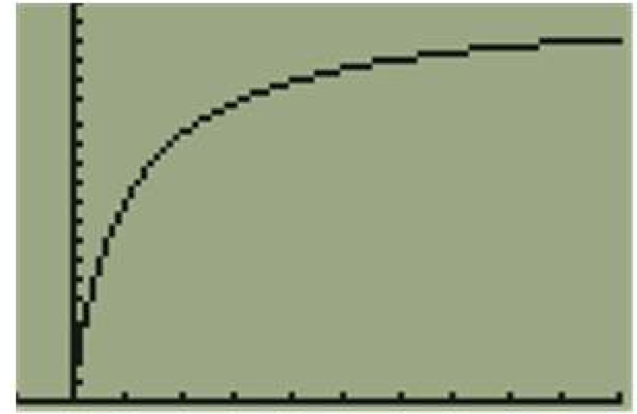
$$r(x) = 2000\left(1 - \frac{1}{x+1}\right).$$

(a) Draw the graph of r. What values of x make sense in this problem situation? $x \in \mathbb{N}$

(b) Find the marginal revenue when x desks are sold.

(c) Use the function $r'(x)$ to estimate the increase in revenue that will result from increasing sales from 5 desks a week to 6 desks a week.

(d) **Writing to Learn** Find the limit of $r'(x)$ as $x \to \infty$. How would you interpret this number?



WINDOW
Xmin=-1
Xmax=10
Xscl=1
Ymin=-1
Ymax=2000
Yscl=100
↓Xres=1

$$r(x) = 2000\left(1 - (x+1)^{-1}\right)$$

$$r'(x) = 2000\left(-(-1)(x+1)^{-2}\right) = \frac{2000}{(x+1)^2}$$

$$r'(5) = \frac{2000}{36} = \frac{\text{dollars}}{\text{desk}}$$

(d) Limit goes to zero. As x gets large very little revenue is expected.

Homework
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exer:
4-24(4x), 29, 40-45 all