# 4.1 Chain Rule

## Derivative of a Composite Function

**EXAMPLE 2** **Relating Derivatives**

The polynomial $y = 9x^4 + 6x^2 + 1 = (3x^2 + 1)^2$ is the composite of $y = u^2$ and $u = 3x^2 + 1$. Calculating derivatives, we see that

$$\begin{aligned}\frac{dy}{du}\cdot\frac{du}{dx} &= 2u\cdot 6x\\ &= 2(3x^2+1)\cdot 6x\\ &= 36x^3 + 12x.\end{aligned}$$

Also,

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(9x^4 + 6x^2 + 1)\\ &= 36x^3 + 12x.\end{aligned}$$

**RULE 8** **The Chain Rule**

If $f$ is differentiable at the point $u = g(x)$, and $g$ is differentiable at $x$, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at $x$, and

$$(f \circ g)'(x) = f'(g(x))\cdot g'(x).$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx},$$

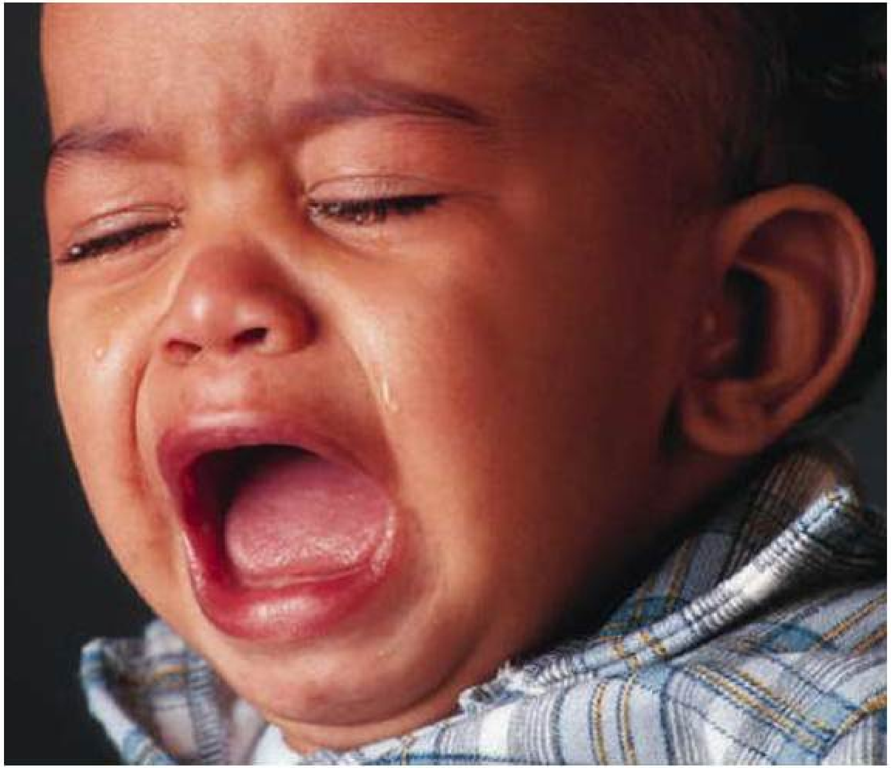
where $dy/du$ is evaluated at $u = g(x)$.

CHAIN RULE (for composite functions)

$$\frac{d}{dx}(f \circ g)(x) = f'(g(x)) \cdot g'(x)$$

## Three Steps

1. Take the derivative of the outside function.
2. Copy the inside function.
3. Multiply by the derivative of the inside function (the “baby”).



Don’t forget your baby!

Given: $y = (2 - 5x)^3$ , find dy/dx.

$\frac{dy}{dx} = 3(\ ?\ )^2$ (step 1 – derivative of outer)

$\frac{dy}{dx} = 3(2 - 5x\ )^2$ (step 2 – copy inner)

$\frac{dy}{dx} = 3(2 - 5x)^2 \cdot (-5)$ (step 3 – **multiply by the baby**)

In Exercises 1–8, use the given substitution and the Chain Rule to find $dy/dx$.

**1.** $y = \sin(\overset{u}{3x + 1})$, $u = 3x + 1$

$$\frac{dy}{dx} = \cos(3x+1)\cdot 3$$

**5.** $y = \left(\dfrac{\sin x}{1 + \cos x}\right)^2$

$$\frac{dy}{dx} = 2\left(\frac{\sin x}{1+\cos x}\right)\cdot\left(\frac{\cos x + \overbrace{\cos^2 x + \sin^2 x}^{1}}{}\right)$$

$$\frac{dy}{dx} = 2\left(\frac{\sin x}{1+\cos x}\right)\cdot\left(\frac{\cos x(1+\cos x) - (-\sin x)\sin x}{(1+\cos x)^2}\right)$$

$$= \frac{2\sin x}{(1+\cos x)}\cdot\frac{(1+\cos x)}{(1+\cos x)^2} = \frac{2\sin x}{(1+\cos x)^2}$$

In Exercises 9–12, an object moves along the $x$-axis so that its position at any time $t \geq 0$ is given by $x(t) = s(t)$. Find the velocity of the object as a function of $t$.

**9.** $s = \cos\left(\dfrac{\pi}{2} - 3t\right)$

$$v(t) = s'(t) = -\sin\left(\frac{\pi}{2} - 3t\right) \cdot (-3) = 3\sin\left(\frac{\pi}{2} - 3t\right)$$

In Exercises 13–24, find $dy/dx$. If you are unsure of your answer, use NDER to support your computation.

**13.** $y = (x + \sqrt{x})^{-2}$

**14.** $y = (\csc x + \cot x)^{-1}$

$$\frac{dy}{dx} = -2\left(x + \sqrt{x}\right)^{-3} \cdot \left(1 + \frac{1}{2}x^{-1/2}\right)$$

$$= \frac{-2\left(1 + \frac{1}{2\sqrt{x}}\right)}{\left(x + \sqrt{x}\right)^3} = \frac{-2 - \frac{1}{\sqrt{x}}}{\left(x + \sqrt{x}\right)^3} = \frac{-2\sqrt{x} - 1}{\sqrt{x}\left(x + \sqrt{x}\right)^3}$$

**14.** $y = (\csc x + \cot x)^{-1}$

$$\frac{dy}{dx} = -(\csc x + \cot x)^{-2} \cdot (-\csc x \cot x + -\csc^2 x)$$

$$= \frac{\csc x(\cot x + \csc x)}{(\csc x + \cot x)^2}$$

$$= \frac{\csc x}{\csc x + \cot x}$$

Differentiate $\cos\left(3x^4-2\right)$ with respect to $x$.

$$\frac{d}{dx}\cos\underbrace{\left(3x^4-2\right)}_{\text{inside}}=-\sin\underbrace{\left(3x^4-2\right)}_{\text{inside left alone}}\cdot\underbrace{12x^3}_{\substack{\text{derivative}\\\text{of inside}}}$$

$$=-12x^3\sin\left(3x^4-2\right)$$

Sometimes the chain rule needs to be used more than once to find a derivative.

Find the derivative of $y=\sqrt[3]{\sin x^2}$.

$$y=\sqrt[3]{\sin x^2}=\left(\sin x^2\right)^{\frac{1}{3}}$$

$$\frac{dy}{dx}=\frac{1}{3}\left(\sin x^2\right)^{-\frac{2}{3}}\left(\cos x^2\right)\cdot 2x$$

$$\frac{dy}{dx}=\frac{2}{3}x\left(\sin x^2\right)^{-\frac{2}{3}}\left(\cos x^2\right)$$

$$\frac{d}{dx}\left[\sin x^2\right] = \left(\cos x^2\right) 2x$$

outter: $\sin u$ inner: $u = x^2$

$$\frac{d}{dx}\left[\sin^2 x\right] = 2 \sin x \cos x = \sin 2x$$

outter: $u^2$ inner: $u = \sin x$

**23.** $y = (1 + \cos^2 7x)^3$

$$\frac{dy}{dx} = 3(1+\cos^2(7x))^2 \cdot (0 + 2\cos(7x)) \cdot (-\sin(7x)(7))$$

$$\frac{dy}{dx} = -42(1+\cos^2(7x))^2 \cdot \cos(7x)\sin(7x)$$

$$= -21(1+\cos^2(7x))\sin(14x)$$

**EXAMPLE 5** **A Three-Link "Chain"**

Find the derivative of $g(t) = \tan(5 - \sin 2t)$.

**SOLUTION**

Notice here that tan is a function of $5 - \sin 2t$, while sin is a function of $2t$, which is itself a function of $t$. Therefore, by the Chain Rule,

$$g'(t) = \frac{d}{dt}(\tan(5 - \sin 2t))$$

$$= \sec^2(5 - \sin 2t) \cdot \frac{d}{dt}(5 - \sin 2t) \quad \text{Derivative of } \tan u \text{ with } u = 5 - \sin 2t$$

$$= \sec^2(5 - \sin 2t) \cdot \left(0 - \cos 2t \cdot \frac{d}{dt}(2t)\right) \quad \text{Derivative of } 5 - \sin u \text{ with } u = 2t$$

$$= \sec^2(5 - \sin 2t) \cdot (-\cos 2t) \cdot 2$$

$$= -2(\cos 2t)\sec^2(5 - \sin 2t).$$

***Now Try Exercise 23.***

In Exercises 41–48, find the equation of the line tangent to the curve at the point defined by the given value of $t$.

**41.** $x = 2\cos t, \quad y = 2\sin t, \quad t = \pi/4$

$$m = \frac{dy/dt}{dx/dt} = \frac{2\cos t}{-2\sin t} = -1$$

point $\left(2\cos\frac{\pi}{4}, 2\sin\frac{\pi}{4}\right)$

$(\sqrt{2}, \sqrt{2})$

$$y - \sqrt{2} = -1(x - \sqrt{2})$$

### Finding *dy/dx* Parametrically

If all three derivatives exist and $dx/dt \neq 0$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

Note:
(dy/dx)(dx/dt)=dy/dt

**53.** What is the largest value possible for the slope of the curve $y = \sin(x/2)$?

$$\frac{dy}{dx} = \cos\left(\frac{x}{2}\right) \cdot \frac{1}{2}$$

So $\frac{1}{2}$ is the greatest value for the slope.

Find dy/dx for: $y = \overset{u}{(x + x^2)^5}\,\overset{v}{(1 + x^3)^2}$

$$\frac{dy}{dx} = \underbrace{5(x+x^2)^4(1+2x)}_{u'}\,\underbrace{(1+x^3)^2}_{v} + \underbrace{(x+x^2)^5}_{u}\,\underbrace{2(1+x^3)3x^2}_{v'}$$

$$= (x+x^2)^4(1+x^3)\left[5(1+2x)(1+x^3) + 6x^2(x+x^2)\right]$$

Compute the derivative of $g(x) = \dfrac{8x}{(x^3+1)^2}$

$$g'(x) = \frac{8(x^3+1)^2 - 8\underline{x}(\underline{2}(x^3+1)\,\underline{3x^2})}{((x^3+1)^2)^2}$$

$$= \frac{8(x^3+1)\left[(x^3+1) - \underline{6x^3}\right]}{(x^3+1)^4}$$

$$= \frac{8\,(1-5x^3)}{(x^3+1)^3}$$

Find the derivative of $f(x) = \left(\sqrt{x^2+4} - 3x^2\right)^{3/2}$.

$$f'(x) = \frac{3}{2}\left(\sqrt{x^2+4} - 3x^2\right)^{1/2}\left(\frac{1}{2}\left(x^2+4\right)^{-1/2} 2x - 6x\right)$$

$$= \frac{3}{2}\sqrt{\sqrt{x^2+4} - 3x^2}\left(\frac{x}{\sqrt{x^2+4}} - 6x\right)$$

Homework
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4-56 (4x)
70-75 all