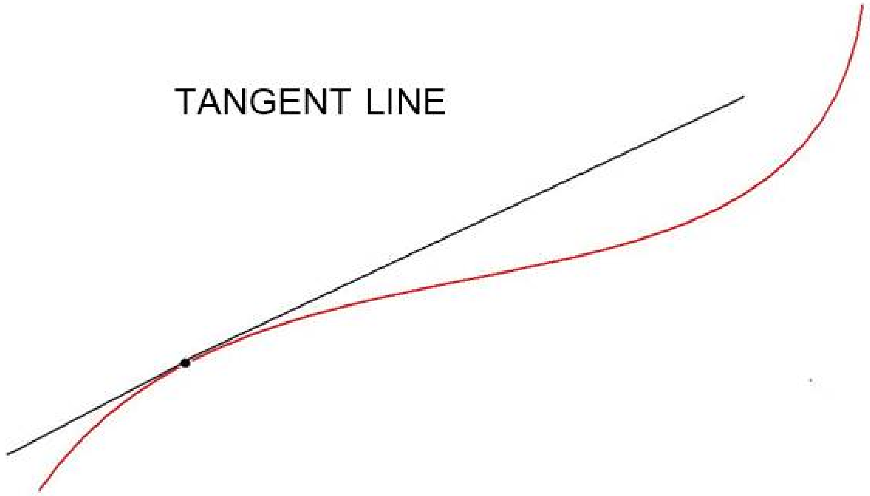
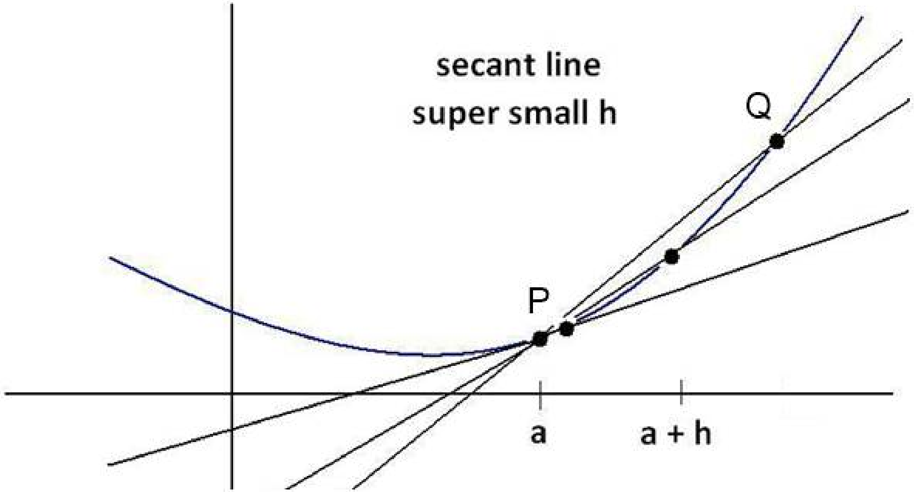
Chapter 2

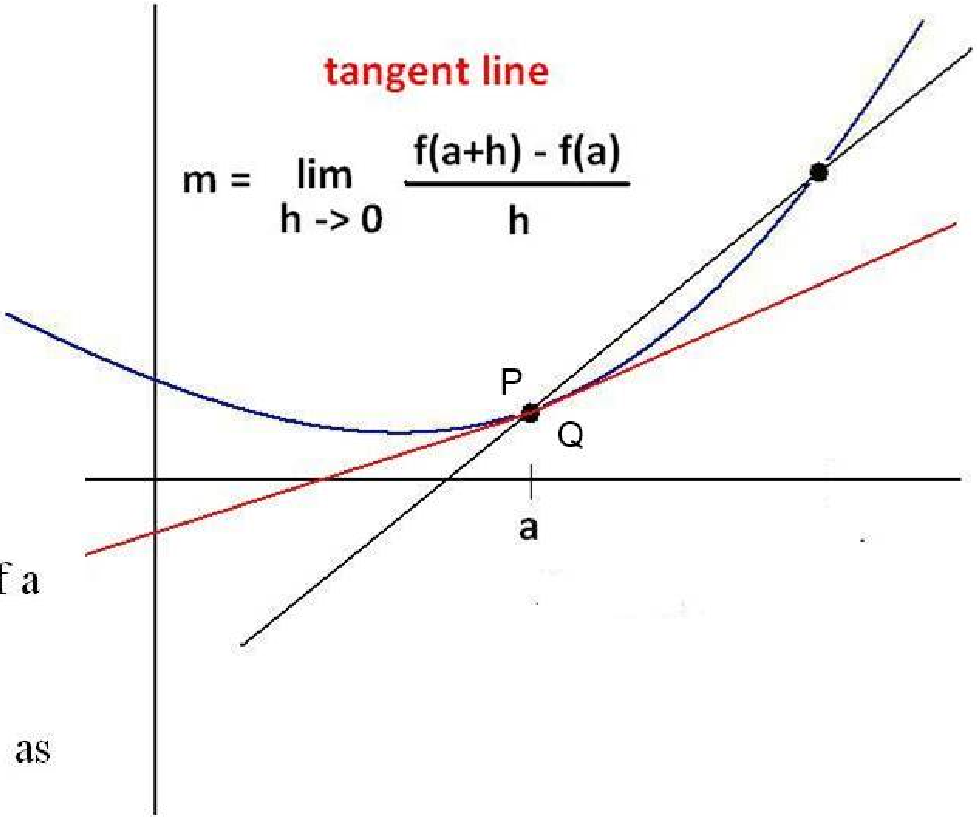
Limits and Continuity

Section 2.4

Rates of Change and Tangent Lines







The process becomes:

1. Start with what can be calculated, namely, the slope of a secant through P and a point Q nearby on the curve.

2. Find the limiting value of the secant slope (if it exists) as Q approaches P along the curve.

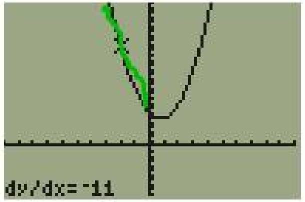
3. Define the *slope of the curve* at P to be this number and *define the tangent to the curve* at P to be the line through P with this slope.

Find the average rate of change of $f(x)=2x^2-3x+7$

over the interval $[-2,4]$ slope of secant line

$$\frac{f(-2)-f(4)}{-2-4}=\frac{\left(2(-2)^2-3(-2)+7\right)-\left(2(4)^2-3(4)+7\right)}{-2-4}$$

$$=\frac{21-27}{-6}=\frac{-6}{-6}=1$$



Find the instantaneous rate of change at $x=-2$

$$\lim_{h\to 0}\frac{f(-2+h)-f(-2)}{-2+h--2}=\lim_{h\to 0}\frac{2(-2+h)^2-3(-2+h)+7-(21)}{h}$$

$$\lim_{h\to 0}\frac{2(\cancel{4}-4h+h^2)+\cancel{6}-3h+\cancel{7}-\cancel{21}}{h}=\lim_{h\to 0}\frac{-8h+2h^2-3h}{h}=\lim_{h\to 0}-11+2h$$

$$=-11$$

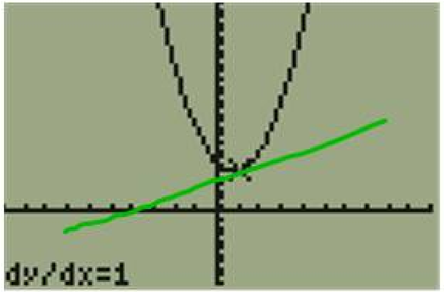
Find the instantaneous rate of change at x=1 $f(x)=2x^2-3x+7$

$f(1)$

$$\lim_{h \to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h \to 0} \frac{2(1+2h+h^2)-3(1+h)+7-6}{h}$$

$$\lim_{h \to 0} \frac{\cancel{2}+4h+2h^2-\cancel{3}-3h+\cancel{7}-\cancel{6}}{h} = \lim_{h \to 0} \frac{h+2h^2}{h}$$

$$= \lim_{h \to 0} 1+2h = 1$$



Given $y=x^2+2$ at $x=-1$ find:

the slope of the curve and an equation of the tangent line.

Then draw a graph of the curve and tangent line in the same viewing window.

(a) Write an expression for the slope of the secant line and find the limiting value of the slope as Q approaches P along the curve.

When $x=-1$, $y=x^2+2=3$ so $=P(-1,3)$ point of tangency

$$\lim_{h\to 0}\frac{y(-1+h)-y(-1)}{h}=\lim_{h\to 0}\frac{(-1+h)^2+2-\left[(-1)^2+2\right]}{h}$$

$$\lim_{h\to 0}\frac{3-2h+h^2-3}{h}=\lim_{h\to 0}\frac{h^2-2h}{h}=\lim_{h\to 0}\frac{h(h-2)}{h}=\lim_{h\to 0}(h-2)=-2$$

(b) The tangent line has slope -2 and passes through $(-1,3)$.

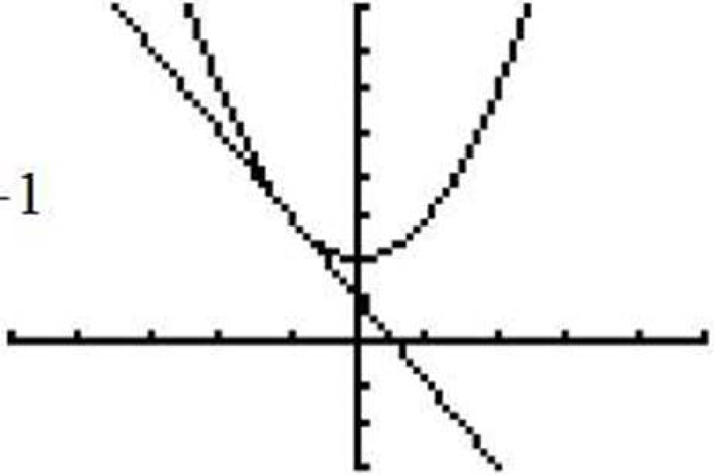
The equation of the tangent line is

$$y-3=-2(x-(-1))$$

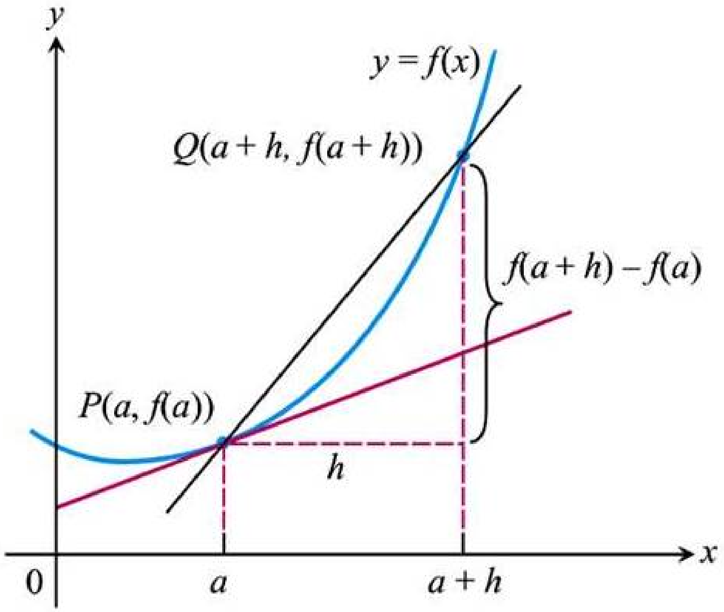
$$y=-2(x+1)+3$$

$$y=-2x-2+3$$

$$y=-2x+1$$



Slope of a Curve



The **slope of the curve** $y = f(x)$ at the point $P(a, f(a))$ is the number

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided the limit exists.

The **tangent line to the curve** at P is the line through P with this slope.

All of the following mean the same:

1. the slope of $y = f(x)$ at $x = a$
2. the slope of the tangent to $y = f(x)$ at $x = a$
3. the (instantaneous) rate of change of $f(x)$ with respect to x at $x = a$
4. $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

The expression $\frac{f(a+h) - f(a)}{h}$ is the **difference quotient** of f at a.

The **normal line** to a curve at a point is the line perpendicular to the tangent at the point.

The slope of the normal line is the negative reciprocal of the slope of the tangent line.

$\perp$ lines

$m_1 \cdot m_2 = -1$

or

$m_1 = -\frac{1}{m_2}$

Given $y = x^2 + 2$ at $x = -1$ write the equation of the normal line.
Draw a graph of the curve, the tangent line and the normal line in the same viewing window.
From an earlier example, the slope of the tangent line was found to be -2 so the slope of the normal is $\frac{1}{2}$.

$$y - 3 = \frac{1}{2}(x - (-1))$$

$$y = \frac{1}{2}(x + 1) + 3$$

$$y = \frac{1}{2}x + \frac{1}{2} + \frac{6}{2}$$

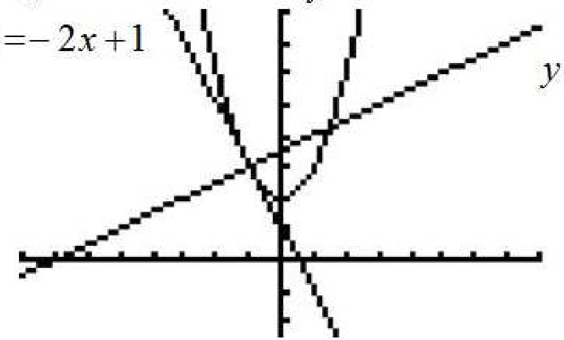
$$y = \frac{1}{2}x + \frac{7}{2}$$

curve

$y = x^2 + 2$

tangent

$y = -2x + 1$



normal line

$y = \frac{1}{2}x + \frac{7}{2}$

a) Determine the slope of the line tangent to $f(x)$ at x=2 if $f(x) = -3x^2 + 15x$

$$f(2) = -3(2)^2 + 15(2)$$
$$= 18$$

$$\lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{-3(4+4h+h^2) + 15(2+h) - 18}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{-12} - 12h - 3h^2 + \cancel{30} + 15h - \cancel{18}}{h}$$

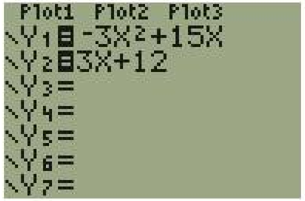
$$= \lim_{h \to 0} \frac{-3h^2 + 3h}{h} = \lim_{h \to 0} -3h + 3 = 3$$

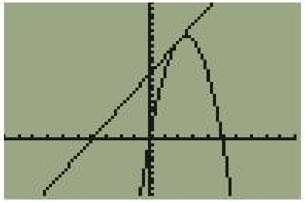
b) Determine the equation of the tangent line

$m = 3$

$(2, 18)$

$$y - 18 = 3(x - 2)$$
$$y = 3x + 12$$





The function $y = 16t^2$ is an object's **position function**. An object's average speed along a coordinate axis for a given period of time is the average rate of change of its position $y = f(t)$.

Its **instantaneous speed** at any time t is the **instantaneous rate of change** of position with respect to time at time t, or $\lim_{h\to 0}\frac{f(t+h)-f(t)}{h}$.

A) Find the average speed between t=3 and t=5

$$\bar{v} = \frac{f(5)-f(3)}{5-3} = \frac{16(25)-16(9)}{2} = 128$$

B) Find the instantaneous speed at t=3

$$v\Big|_{t=3} = \lim_{h\to 0}\frac{16(3+h)^2-16(3)^2}{h} = \lim_{h\to 0}\frac{16(9+6h+h^2)-16(9)}{h}$$

$$= \lim_{h\to 0}\frac{16(6h+h^2)}{h} = \lim_{h\to 0} 16(6+h) = 16(6)$$

$$= 96$$

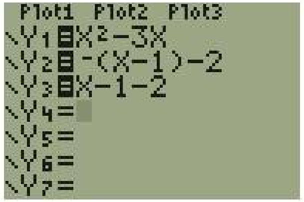
Let $f(x)=x^2-3x$ and $P=(1, f(1))$. $=(1,-2)$

Find (a) the slope of the curve $y=f(x)$ at P, (b) an equation of the tangent at P and (c) an equation of the normal at P.

(a) $m\Big|_{x=1} = \lim_{h\to 0} \frac{(1+h)^2-3(1+h)-(-2)}{h}$

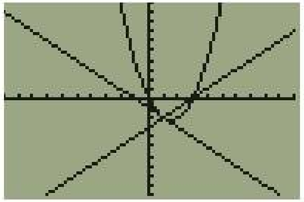
$= \lim_{h\to 0} \frac{\cancel{1}+2h+h^2-\cancel{3}-3h+\cancel{2}}{h} = \lim_{h\to 0} \frac{h^2-h}{h} = \lim_{h\to 0} h-1 = -1$

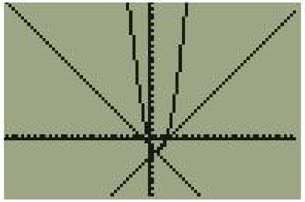
(b) $y - -2 = -1(x-1)$



(c) $y+2 = x-1$

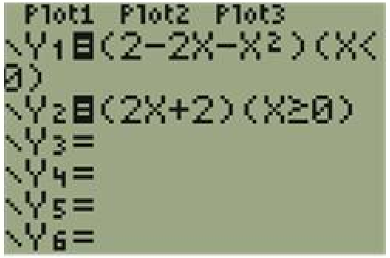
do not look $\perp$



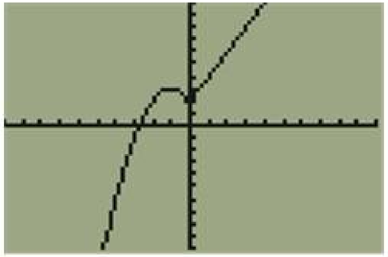


Given $f(x) = \sqrt{x+2}$, find the equation of the line normal to $f(x)$ at $x = 7$

In Exercises 15–18, determine whether the curve has a tangent at the indicated point. If it does, give its slope. If not, explain why not.



15. $f(x) = \begin{cases} 2 - 2x - x^2, & x < 0 \\ 2x + 2, & x \geq 0 \end{cases}$ at $x = 0$



For a f(x) to have a tangent line at x=p

1) f(x) must be continuous at x=p
   - *f(x) exist at x=p $f(0) = 2$
   - *lim of f(x) as x->p exist $\lim_{x\to 0^-} f(x) = 2 = \lim_{x\to 0^+} f(x) = f(0)$
   - *lim f(x) as x->p = f(p)

2) instantaneous rate of chage at x=p from the left and from the right must be equal.

i.r.c. $_{x\to 0^-}$ $\lim_{h\to 0} \frac{(2-2(0+h)-(0+h)^2)-2}{h} = \lim_{h\to 0} \frac{-2h-h^2}{h} = \lim_{h\to 0} -2-h = -2$

i.r.c. $_{x\to 0^+}$ $\lim_{h\to 0} \frac{2(0+h)+2-2}{h} = \lim_{h\to 0} 2h/h = 2$

Conclusion: f(x) does NOT have a tangent line at $x=0$

17. $f(x) = \begin{cases} 1/x, & x \leq 2 \\ \dfrac{4-x}{4}, & x > 2 \end{cases}$ at $x = 2$

Homework
page 93
10-36 even
39-44 all
AP quiz 1-4