4.2 Implicit Differentiation

Explicit vs Implicit Functions

Explicit: $y = x^2 - 2x$

Implicit: $x^2y^2 + y = 1$

EXAMPLE 1 **Differentiating Implicitly**

Find dy/dx if $y^2 = x$.

SOLUTION

To find dy/dx, we simply differentiate both sides of the equation $y^2 = x$ with respect to x, treating y as a differentiable function of x and applying the Chain Rule:

$$y^2 = x$$

$$2y\frac{dy}{dx} = 1 \qquad \frac{d}{dx}(y^2) = \frac{d}{dy}(y^2)\cdot\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{2y}.$$

Now Try Exercise 3.

Differentiate

3. $y^2 = \dfrac{x-1}{x+1}$

$$\frac{d}{dx}[y^2] = \frac{d}{dx}\left[\frac{x-1}{x+1}\right]$$

$$2y\left(\frac{dy}{dx}\right) = \frac{(1)(x+1)-(x-1)(1)}{(x+1)^2} = \frac{x+1-x+1}{(x+1)^2} = \frac{2}{(x+1)^2}$$

$$\frac{2y\left(\frac{dy}{dx}\right)}{2y} = \frac{2}{(x+1)^2}\,\frac{1}{2y} \Rightarrow \frac{dy}{dx} = \frac{1}{y(x+1)^2}$$

EXAMPLE 2 Finding Slope on a Circle

Find the slope of the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

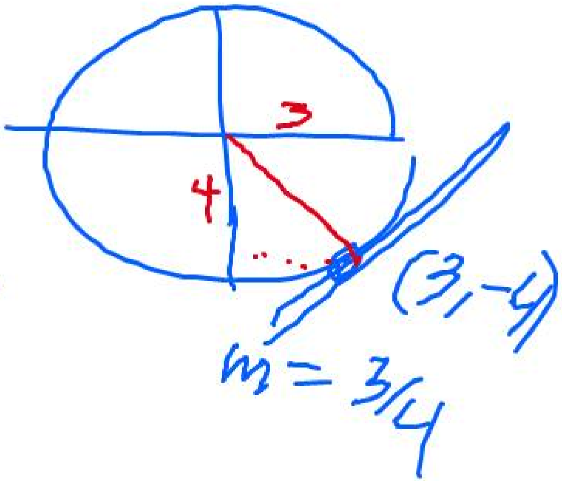
$$\frac{d}{dx}\left[x^2+y^2=25\right]$$

$$2x+2y\left(\frac{dy}{dx}\right)=0$$

$$\frac{2y\left(\frac{dy}{dx}\right)}{2y}=\frac{-2x}{2y}$$

$$\frac{dy}{dx}=\frac{-x}{y}=\frac{-3}{-4}=3/4$$

$$m=3/4$$



11. $(x-1)^2 + (y-1)^2 = 13, \quad (3, 4)$

$$\frac{d}{dx}\left[(x-1)^2+(y-1)^2=13\right]$$

$$\frac{2(x-1)}{2}+\frac{2(y-1)}{2}\,dy/dx=\frac{0}{2}$$

$$x-1+(y-1)\,dy/dx=0$$

$$(y-1)\frac{dy}{dx}=1-x$$

$$\frac{dy}{dx}=\frac{1-x}{y-1}$$

$$m=\frac{1-3}{4-1}=-\frac{2}{3}$$

EXAMPLE 3 Solving for dy/dx

Show that the slope dy/dx is defined at every point on the graph of $2y = x^2 + \sin y$.

$$\frac{d}{dx}\left[2y = x^2 + \sin y\right]$$

$$2\frac{dy}{dx} = 2x + \cos y\left(\frac{dy}{dx}\right)$$

$$2\frac{dy}{dx} - \cos y\left(\frac{dy}{dx}\right) = 2x$$

$$\frac{dy}{dx}(2 - \cos y) = 2x$$

$$\frac{dy}{dx} = \frac{2x}{2 - \cos y}$$

exist for all (x, y)

because $2 - \cos y \neq 0$

In Exercises 13–16, find where the slope of the curve is defined.

13. $x^2y - xy^2 = 4$

$$\frac{d}{dx}\left[x^2y - xy^2 = 4\right]$$

$$2xy + x^2y' - \left((1)y^2 + x\,2y\,y'\right) = 0$$

$$2xy + x^2y' - y^2 - 2xyy' = 0$$

$$x^2y' - 2xyy' = y^2 - 2xy$$

$$y'(x^2 - 2xy) = y^2 - 2xy$$

$$y' = \frac{y^2 - 2xy}{x^2 - 2xy}$$

$x^2 - 2xy \neq 0$; $x(x - 2y) \neq 0$

y' is define $\forall$ (x, y) except

1. $x = 0$
2. $x = 2y$

Implicit Differentiation Process

1. Differentiate both sides of the equation with respect to x.
2. Collect the terms with dy/dx on one side of the equation.
3. Factor out dy/dx.
4. Solve for dy/dx.

Find $\dfrac{dy}{dx}$ if $3y^2+2y=5x$

To find $\dfrac{dy}{dx}$ differentiate both sides of the equation with respect to x, treating y as a differentiable function of x and applying the Chain Rule.

$$3y^2+2y=5x$$

$$6y\frac{dy}{dx}+2\frac{dy}{dx}=5 \qquad \left\{\begin{aligned}\frac{d}{dx}\left(3y^2\right)&=\frac{d}{dy}\left(3y^2\right)\frac{dy}{dx}\\ \frac{d}{dx}(2y)&=\frac{d}{dy}(2y)\frac{dy}{dx}\end{aligned}\right.$$

$$\frac{dy}{dx}(6y+2)=5$$

$$\frac{dy}{dx}=\frac{5}{6y+2}$$

EXAMPLE 4 **Tangent and normal to an ellipse**

Find the tangent and normal to the ellipse $x^2 - xy + y^2 = 7$ at the point $(-1, 2)$. (See Figure 4.11.)

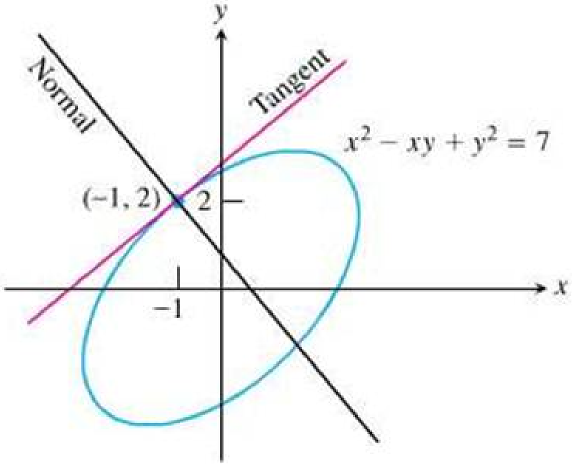


Figure 4.11 Tangent and normal lines to the ellipse $x^2 - xy + y^2 = 7$ at the point $(-1, 2)$. (Example 4)

$$\frac{d}{dx}\left[x^2 - xy + y^2 = 7\right]$$

$$2x - (y + xy') + 2y\,y' = 0$$

$$2x - y - xy' + 2yy' = 0$$

$$2yy' - xy' = y - 2x$$

$$y'(2y - x) = y - 2x$$

$$y' = \frac{y - 2x}{2y - x}$$

$$m = \frac{2+2}{4+1}$$

$$m = 4/5$$

Tangent +

$$y - 2 = 4/5\,(x+1)$$

$$y - 2 = -\frac{5}{4}(x+1)$$

Find the equations of the tangent and normal lines to the graph

given by $x^4 + x^2y^2 - y^2 = 0$ at the point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

$$\frac{d}{dx}\left[x^4 + x^2y^2 - y^2 = 0\right]$$

$$4x^3 + 2xy^2 + x^2\,2y\,y' - 2yy' = 0$$

$$4x^3 + 2xy^2 = 2yy' - 2x^2yy'$$

$$4x^3 + 2xy^2 = y'(2y - 2x^2y)$$

$$y' = \frac{4x^3 + 2xy^2}{2y - 2x^2y}$$

$$m = \frac{4\left(\frac{\sqrt{2}}{2}\right)^3 + 2\left(\frac{\sqrt{2}}{2}\right)^3}{2\left(\frac{\sqrt{2}}{2}\right) - 2\left(\frac{\sqrt{2}}{2}\right)^3} = \frac{4\left(\frac{2\sqrt{2}}{8}\right) + 2\left(\frac{2\sqrt{2}}{8}\right)}{\sqrt{2} - 2\left(\frac{2\sqrt{2}}{8}\right)} = \frac{\sqrt{2} + \sqrt{2}/2}{\sqrt{2} - \sqrt{2}/2}\left(\frac{2}{2}\right)$$

$$\frac{2\sqrt{2} + \sqrt{2}}{2\sqrt{2} - \sqrt{2}} = \frac{3\sqrt{2}}{\sqrt{2}}$$

$$3$$

Tangent

$$y - \sqrt{2}/2 = 3\left(x - \sqrt{2}/2\right)$$

Normal

$$y - \sqrt{2}/2 = -1/3\left(x - \sqrt{2}/2\right)$$

EXAMPLE 5 **Finding a Second Derivative Implicitly**

Find d^2y/dx^2 if $2x^3 - 3y^2 = 8$.

$$\frac{d}{dx}\left[2x^3 - 3y^2 = 8\right]$$

$$6x^2 - 6y\left(\frac{dy}{dx}\right) = 0$$

$$\frac{dy}{dx} = \frac{-6x^2}{-6y} \; ; \; \frac{dy}{dx} = \frac{x^2}{y}$$

$$\frac{d}{dx}\left[\frac{dy}{dx} = \frac{x^2}{y}\right]$$

$$\frac{d^2y}{dx^2} = \frac{2xy - x^2\frac{dy}{dx}}{y^2}$$

$$\frac{d^2y}{dx^2} = \frac{2xy - x^2\left(\frac{x^2}{y}\right)}{y^2} \quad \frac{(y)}{(y)}$$

$$\frac{d^2y}{dx^2} = \frac{2xy^2 - x^4}{y^3}$$

In Exercises 27–30, use implicit differentiation to find dy/dx and then d^2y/dx^2.

29. $y^2 = x^2 + 2x$

$$\frac{d}{dx}\left[y^2 = x^2 + 2x\right]$$

$$\frac{2y}{2y}\left(\frac{dy}{dx}\right) = \frac{2x+2}{2y}$$

$$\frac{d}{dx}\left[\frac{dy}{dx} = \frac{x+1}{y}\right]$$

$$\frac{d^2y}{dx} = \frac{(1)y - (x+1)\frac{dy}{dx}}{y^2}$$

$$\frac{d^2y}{dx^2} = \frac{y - (x+1)\left(\frac{x+1}{y}\right)}{y^2}\left(\frac{y}{y}\right)$$

$$\frac{d^2y}{dx^2} = \frac{y^2 - (x+1)^2}{y^3} = \frac{y^2 - (x^2+2x+1)}{y^3}$$

$$\frac{d^2y}{dx^2} = \frac{y^2 - (y^2+1)}{y^3}$$

$$\frac{d^2y}{dx^2} = \frac{-1}{y^3}$$

If n is any rational number, then

$$\frac{d}{dx}x^n = nx^{n-1}.$$

If $n < 1$, then the derivative does not exist at $x = 0$.

In Exercises 31–42, find dy/dx.

31. $y = x^{9/4}$

33. $y = \sqrt[3]{x}$ $= x^{1/3}$

33 $\frac{dy}{dx} = \frac{1}{3}X^{-2/3}$

31 $\frac{dy}{dx} = \frac{9}{4}X^{5/4}$

EXAMPLE 7 Finding the Second Derivative Implicitly

Cute answers

Given $x^2 + y^2 = 25$, find $\frac{d^2y}{dx^2}$.

$$\frac{d}{dx}\left[x^2+y^2=25\right]$$

$$2x+2y\,y'=0$$

$$y'=\frac{-2x}{2y}$$

$$\frac{d}{dx}\left[y'=-x/y\right]$$

$$y''=\frac{-(1)y-(-x)y'}{y^2}$$

$$=\frac{-y+x\left(\frac{-x}{y}\right)}{y^2}\left[\frac{y}{y}\right]$$

$$y''=\frac{-y^2-x^2}{y^3}$$

$$y''=\frac{-y^2-x^2}{y^3}$$

$$=\frac{-(x^2+y^2)}{y^3}$$

$$y''=-25/y^3$$

EXAMPLE 8 Finding a Tangent Line to a Graph

Find the tangent line to the graph given by $x^2(x^2 + y^2) = y^2$ at the point $(\sqrt{2}/2, \sqrt{2}/2)$, as shown in Figure 2.32.

$$\frac{d}{dx}\left[x^4 + \underset{\text{prod}}{x^2y^2} = y^2 \right]$$

$$4x^3 + 2xy^2 + x^2 2y\,y' = 2yy'$$

$$4x^3 + 2xy^2 = 2yy' - x^2 2yy'$$

$$4x^3 + 2xy^2 = y'(2y - x^2 2y)$$

$$\frac{4x^3 + 2xy^2}{2y - x^2(2y)} = y'$$

$$y' = \frac{2x^3 + xy^2}{y - x^2y} \Rightarrow \frac{2\left(\frac{\sqrt{2}}{2}\right)^3 + \frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}\right)^2}{\sqrt{2}/2 - \left(\frac{\sqrt{2}}{2}\right)^2\frac{\sqrt{2}}{2}} = \frac{\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{4}}{\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{4}}\left(\frac{4}{4}\right) = \frac{2\sqrt{2} + \sqrt{2}}{2\sqrt{2} - \sqrt{2}}$$

$$y' = \frac{3\sqrt{2}}{\sqrt{2}} = 3$$

$$y - \sqrt{2}/2 = 3\left(x - \sqrt{2}/2\right)$$

$$\frac{d}{dx}\left[y^3\right] = 3y^2 \cdot \underbrace{y'}_{\text{"baby"}}$$

$$\frac{d}{dx}\left[\sin y\right] = \cos y \cdot \underbrace{y'}_{\text{"baby"}}$$

$$\frac{d}{dx}\left[\sin(3x^4)\right] = \cos(3x^4) \cdot \underbrace{12x^3}_{\text{baby}}$$

Homework
page 167
4-44 (4x),
59-64 all
AP Quiz 1-4