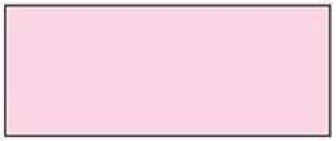
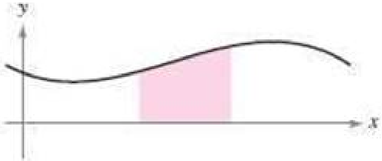
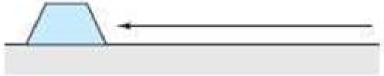
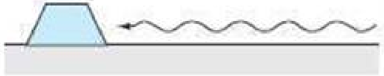
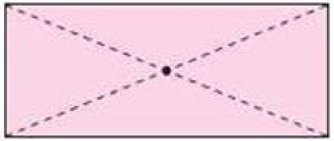
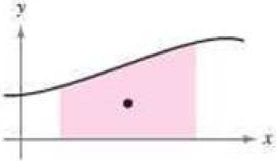
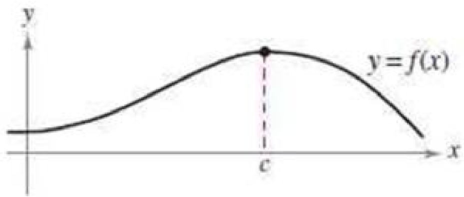
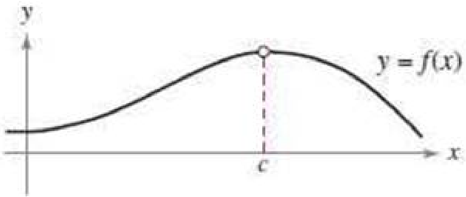
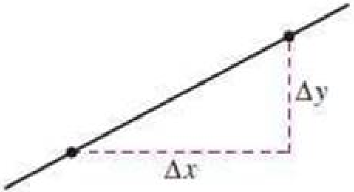
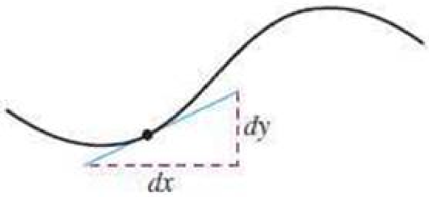
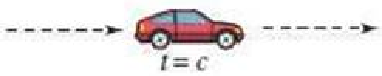
What Is Calculus?

Calculus is the mathematics of change—velocities and accelerations. Calculus is also the mathematics of tangent lines, slopes, areas, volumes, arc lengths, centroids, curvatures, and a variety of other concepts that have enabled scientists, engineers, and economists to model real-life situations.

Although precalculus mathematics also deals with velocities, accelerations, tangent lines, slopes, and so on, there is a fundamental difference between precalculus mathematics and calculus. Precalculus mathematics is more static, whereas calculus is more dynamic. Here are some examples.

- An object traveling at a constant velocity can be analyzed with precalculus mathematics. To analyze the velocity of an accelerating object, you need calculus.
- The slope of a line can be analyzed with precalculus mathematics. To analyze the slope of a curve, you need calculus.
- A tangent line to a circle can be analyzed with precalculus mathematics. To analyze a tangent line to a general graph, you need calculus.
- The area of a rectangle can be analyzed with precalculus mathematics. To analyze the area under a general curve, you need calculus.

Without Calculus	With Integral Calculus
Area of a rectangle 	Area under a curve 
Work done by a constant force 	Work done by a variable force 
Center of a rectangle 	Centroid of a region 
Length of a line segment 	Length of an arc 
Surface area of a cylinder 	Surface area of a solid of revolution 

Without Calculus		With Differential Calculus	
Value of $f(x)$ when $x = c$		Limit of $f(x)$ as x approaches c	
Slope of a line		Slope of a curve	
Secant line to a curve		Tangent line to a curve	
Average rate of change between $t = a$ and $t = b$		Instantaneous rate of change at $t = c$	

Average and Instantaneous Speed

AVERAGE RATE OF CHANGE

$$\frac{\Delta y}{\Delta t} = \frac{f(t+h) - f(t)}{h}.$$

EXAMPLE 1 **Finding an Average Speed**

A rock breaks loose from the top of a tall cliff. What is its average speed during the first 2 seconds of fall?

SOLUTION

Experiments show that a dense solid object dropped from rest to fall freely near the surface of the earth will fall

$$y = 16t^2$$

feet in the first t seconds. The average speed of the rock over any given time interval is the distance traveled, Δy, divided by the length of the interval Δt. For the first 2 seconds of fall, from $t = 0$ to $t = 2$, we have

$$\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(0)^2}{2 - 0} = 32\,\frac{\text{ft}}{\text{sec}}.$$

Now Try Exercise 1.

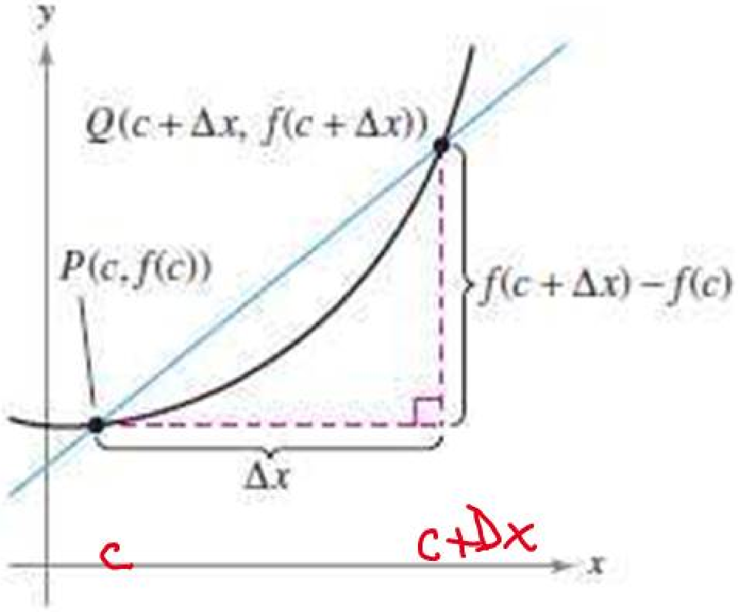
In Exercises 1–4, an object dropped from rest from the top of a tall building falls $y = 16t^2$ feet in the first t seconds.

1. Find the average speed during the first 3 seconds of fall.

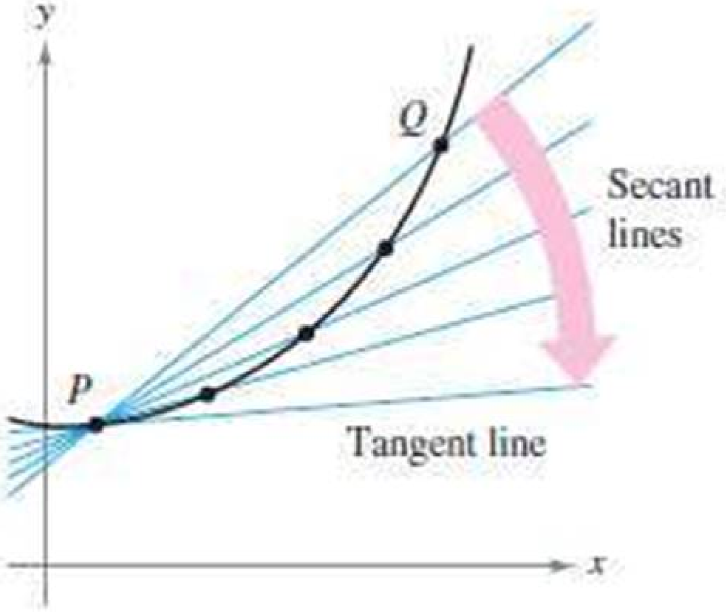
$$\text{average speed} = \frac{16(3)^2 - 16(0)^2}{3 - 0}$$

$$= \frac{16(9)}{3} = 16(3) = 48 \text{ ft/sec}$$

$$m_{sec} = \frac{f(c + \Delta x) - f(c)}{c + \Delta x - c} = \frac{f(c + \Delta x) - f(c)}{\Delta x}. = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x+h) - f(x)}{h}$$



(a) The secant line through $(c, f(c))$ and $(c + \Delta x, f(c + \Delta x))$



(b) As Q approaches P, the secant lines approach the tangent line.

Figure 1.2

EXAMPLE 2 Finding an Instantaneous Speed

Find the speed of the rock in Example 1 at the instant $t = 2$.

SOLUTION

Solve Numerically We can calculate the average speed of the rock over the interval from time $t = 2$ to any slightly later time $t = 2 + h$ as

a little after $t=2$

$$\frac{\Delta y}{\Delta t} = \frac{16(2+h)^2 - 16(2)^2}{h}. \tag{1}$$

We cannot use this formula to calculate the speed at the exact instant $t = 2$ because that would require taking $h = 0$, and $0/0$ is undefined. However, we can get a good idea of what is happening at $t = 2$ by evaluating the formula at values of h *close* to 0. When we do, we see a clear pattern (Table 2.1). As h approaches 0, the average speed approaches the limiting value 64 ft/sec.

Confirm Algebraically If we expand the numerator of Equation 1 and simplify, we find that

$$\frac{\Delta y}{\Delta t} = \frac{16(2+h)^2 - 16(2)^2}{h} = \frac{16(4 + 4h + h^2) - 64}{h}$$

$$= \frac{64h + 16h^2}{h} = 64 + 16h.$$

$h=0$

For values of h different from 0, the expressions on the right and left are equivalent and the average speed is $64 + 16h$ ft/sec. We can now see why the average speed has the limiting value $64 + 16(0) = 64$ ft/sec as h approaches 0. *Now Try Exercise 3.*

$y = 16t^2$

3. Find the speed of the object at $t = 3$ seconds and confirm your answer algebraically.

$$\text{speed}\Big|_{t=3} = \frac{16(3+h)^2 - 16(3)^2}{3+h-3} = \frac{\cancel{16(9}+6h+h^2) - \cancel{16(9)}}{h}$$

$(3, y(3))$

$(3+h, y(3+h))$

$$\lim_{h\to 0} \frac{16(6h)+16h^2}{h} = \lim_{h\to 0} 16(6)+16\cancel{h}^{\,0}$$

$$= 16(6) = 96 \text{ ft/s}$$

Precalculus mathematics $\Rightarrow$ Limit process $\Rightarrow$ Calculus

I Like Pushing Things to the Limits

$$\frac{d}{dx} f(x) = \lim_{\Delta \to 0} \frac{f(x - \Delta) - f(x)}{\Delta}$$

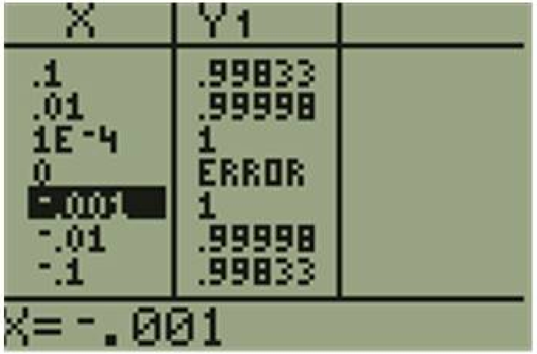
DEFINITION Limit

Assume f is defined in a neighborhood of c and let c and L be real numbers. The function ***f* has limit *L* as *x* approaches *c*** if, given any positive number ε, there is a positive number δ such that for all x,

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

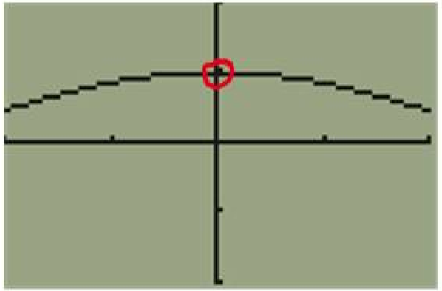
We write

$$\lim_{x \to c} f(x) = L.$$



X	Y1
.1	.99833
.01	.99998
1E-4	1
0	ERROR
-.001	1
-.01	.99998
-.1	.99833

X=-.001



If $f(x)=\frac{\sin x}{x}$ then $f(0)=\emptyset$ $\nexists$

und.

$$\lim_{x\to 0} f(x) = 1$$

THEOREM 1 Properties of Limits

If L, M, c, and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \text{ then}$$

1. *Sum Rule:* $$\lim_{x \to c} (f(x) + g(x)) = L + M$$

   The limit of the sum of two functions is the sum of their limits.

2. *Difference Rule:* $$\lim_{x \to c} (f(x) - g(x)) = L - M$$

   The limit of the difference of two functions is the difference of their limits.

3. *Product Rule:* $$\lim_{x \to c} (f(x) \cdot g(x)) = L \cdot M$$

   The limit of a product of two functions is the product of their limits.

4. *Constant Multiple Rule:* $$\lim_{x \to c} (k \cdot f(x)) = k \cdot L$$

   The limit of a constant times a function is the constant times the limit of the function.

5. *Quotient Rule:* $$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \; M \neq 0$$

   The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

6. *Power Rule:* If r and s are integers, $s \neq 0$, then

   $$\lim_{x \to c} (f(x))^{r/s} = L^{r/s}$$

   provided that $L^{r/s}$ is a real number.

   The limit of a rational power of a function is that power of the limit of the function, provided the latter is a real number.

In Exercises 7–14, determine the limit by substitution. Support graphically.

7. $\lim_{x \to -1/2} 3x^2(2x - 1)$

8. $\lim_{x \to -4} (x + 3)^{1998}$

9. $\lim_{x \to 1} (x^3 + 3x^2 - 2x - 17)$

10. $\lim_{y \to 2} \dfrac{y^2 + 5y + 6}{y + 2}$

11. $\lim_{y \to -3} \dfrac{y^2 + 4y + 3}{y^2 - 3}$

12. $\lim_{x \to 1/2} \operatorname{int} x$

13. $\lim_{x \to -2} (x - 6)^{2/3}$

14. $\lim_{x \to 2} \sqrt{x + 3}$

Practice (See next slides)

7. $\lim_{x \to -1/2} 3x^2(2x - 1)$

$= 3\left(-\tfrac{1}{2}\right)^2\left(2\left(-\tfrac{1}{2}\right) - 1\right)$

$= \frac{3}{4}(-2)$

$= -\frac{3}{2}$

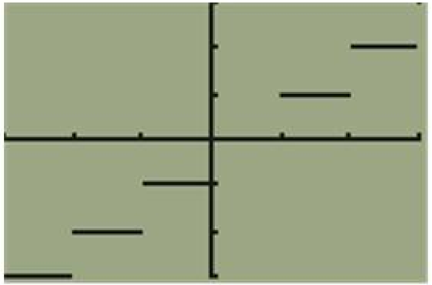
8. $\lim_{x \to -4} (x + 3)^{1998}$

$= (-4 + 3)^{1998}$

$= (-1)^{1998}$

$= 1$

12. $\lim\limits_{x \to 1/2} \operatorname{int} x$ $= \lim\limits_{x \to 1/2} [x] = 0$



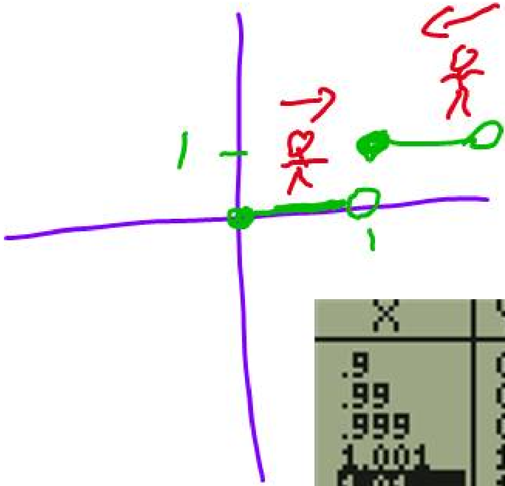
14. $\lim\limits_{x \to 2} \sqrt{x + 3}$ $= \sqrt{5}$

$\lim\limits_{x \to 1} [x] = \nexists$

$\lim\limits_{x \to 1^-} [x] = 0$

$\lim\limits_{x \to 1^+} [x] = 1$

$0 \neq 1$



X	Y_1
.9	0
.99	0
.999	0
1.001	1
1.01	1
1.1	1

X=1.01

$$\lfloor 1 \rfloor = 1$$

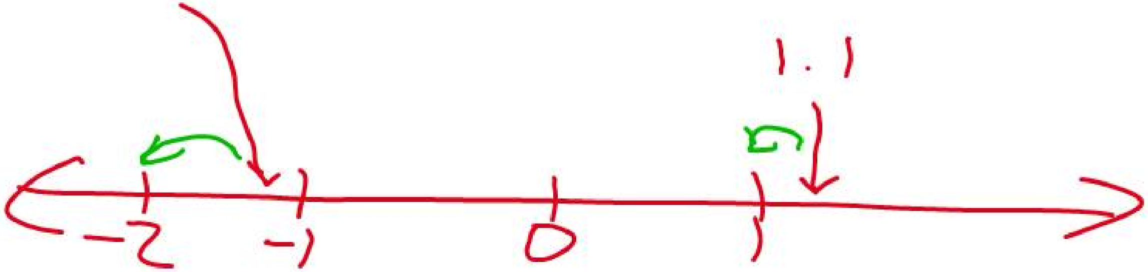
$$\lfloor 1.1 \rfloor = 1$$

$$\lfloor 1.99 \rfloor = 1$$

$$\lfloor 2 \rfloor = 2$$

$$\lfloor -1 \rfloor = -1$$

$$\lfloor -1.1 \rfloor = -2$$



In Exercises 25–34, determine the limit graphically. Confirm algebraically.

25. $\lim_{x \to 1} \dfrac{x-1}{x^2-1}$

26. $\lim_{t \to 2} \dfrac{t^2-3t+2}{t^2-4}$

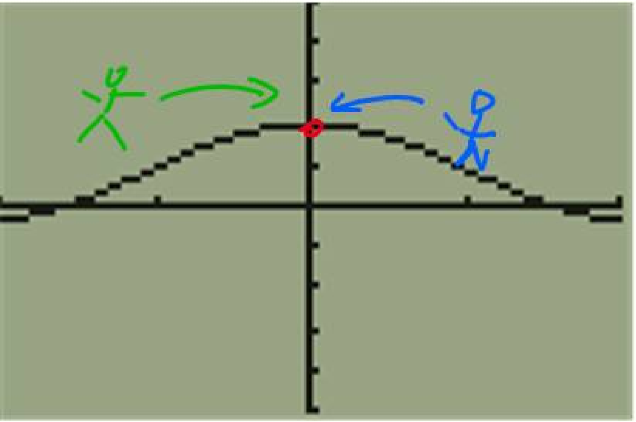
27. $\lim_{x \to 0} \dfrac{5x^3+8x^2}{3x^4-16x^2}$

28. $\lim_{x \to 0} \dfrac{\frac{1}{2+x}-\frac{1}{2}}{x}$

29. $\lim_{x \to 0} \dfrac{(2+x)^3-8}{x}$

30. $\lim_{x \to 0} \dfrac{\sin 2x}{x}$

30. $\lim\limits_{x \to 0} \dfrac{\sin 2x}{x} = 2$

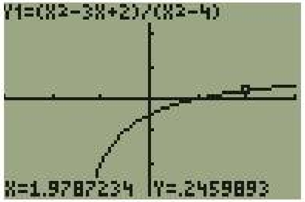


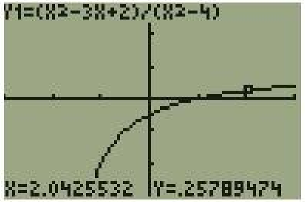
$$\frac{2^2 - 3(2) + 2}{2^2 - 4} = 0/0$$

$$\lim_{t \to 2} \frac{\cancel{(t-2)}(t-1)}{\cancel{(t-2)}(t+2)}$$

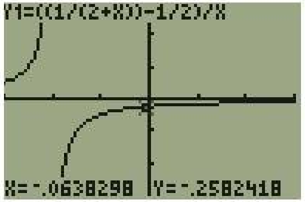
$$\lim_{t \to 2} \frac{t-1}{t+2} = \frac{2-1}{2+2} = \frac{1}{4}$$

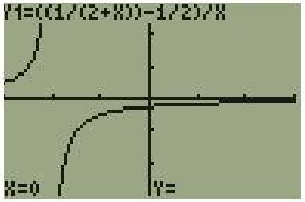
26. $\lim\limits_{t \to 2} \dfrac{t^2 - 3t + 2}{t^2 - 4} = .25$

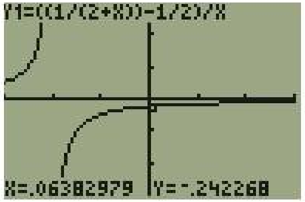




28. $\lim\limits_{x\to 0} \dfrac{\dfrac{1}{2+x}-\dfrac{1}{2}}{x}$ $= -.25$







$$\lim_{x\to 0} \frac{\frac{1}{2+x}-\frac{1}{2}}{x} = \frac{\frac{1}{2}-\frac{1}{2}}{0} = 0/0$$

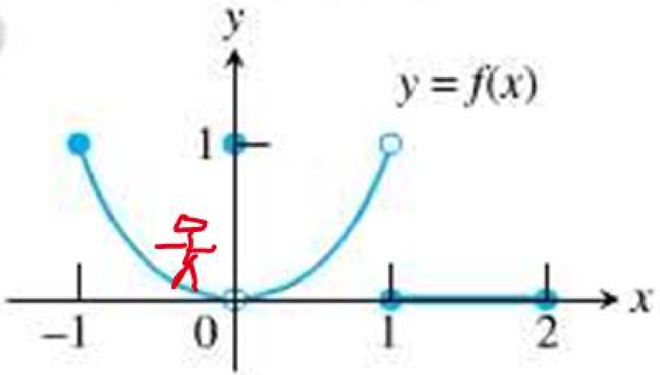
$$\lim_{x\to 0} \frac{\frac{1}{2+x}-\frac{1}{2}}{x}\left(\frac{2(2+x)}{2(2+x)}\right)$$

$$\lim_{x\to 0} \frac{2-(2+x)}{2x(2+x)} = \lim_{x\to 0} \frac{-x}{2x(2+x)}$$

$$= \lim_{x\to 0} \frac{-1}{2(2+x)} = \frac{-1}{2(2)} = -1/4$$

In Exercises 43 and 44, which of the statements are true about the function $y = f(x)$ graphed there, and which are false?

43.



(a) $\lim_{x \to -1^+} f(x) = 1$ T

(b) $\lim_{x \to 0^-} f(x) = 0$ T

(c) $\lim_{x \to 0^-} f(x) = 1$ F

(d) $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$ T

(e) $\lim_{x \to 0} f(x)$ exists T

( ) $\lim_{x \to 0} f(x) = 0$ T

(g) $\lim_{x \to 0} f(x) = 1$ F

( ) $\lim_{x \to 1} f(x) = 1$ F

(i) $\lim_{x \to 1} f(x) = 0$ F

(j) $\lim_{x \to 2^-} f(x) = 2$ F

"Squeeze" Theorem

THEOREM 4 The Sandwich Theorem

If $g(x) \le f(x) \le h(x)$ for all $x \ne c$ in some interval about c, and

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L,$$

then

$$\lim_{x \to c} f(x) = L.$$

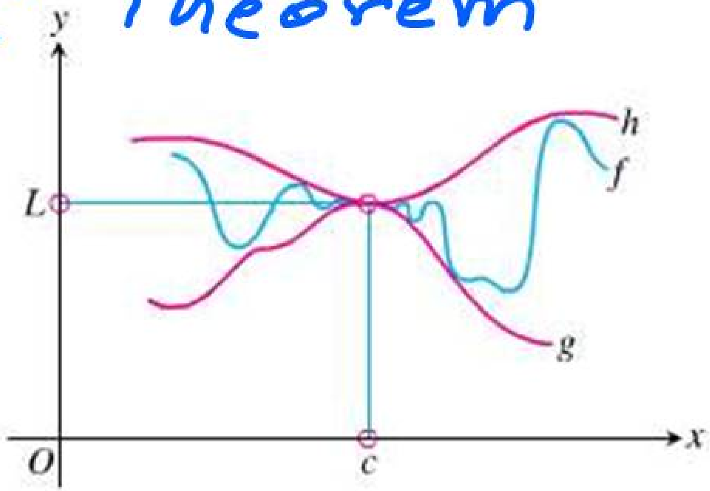


Figure 2.7 Sandwiching f between g and h forces the limiting value of f to be between the limiting values of g and h.

EXAMPLE 9 Using the Sandwich Theorem

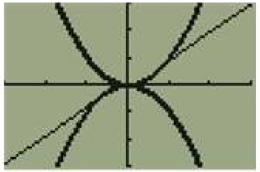
Show that $\lim_{x \to 0} [x^2 \sin(1/x)] = 0$.

$$\left[-1 \le \sin\left(\frac{1}{x}\right) \le 1\right] x^2$$

$$-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2$$

by the Sandwich T.

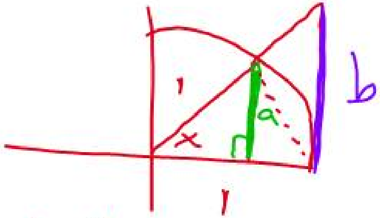
$\lim_{x \to 0} -x^2 = 0$, $\lim_{x \to 0} x^2 = 0$ $\therefore$ $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$



Prove that $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$

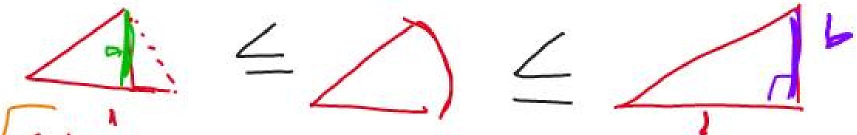
$\frac{1}{2} > \frac{1}{3}$

$2 < 3$



unitary $\odot$

sector



$\sin x = \frac{a}{1}$

$\tan x = \frac{b}{1}$

$$\frac{(1)\sin x}{2} \le \frac{x(1)^2}{2} \le \frac{(1)\tan x}{2}$$

multiply by $2/\sin x$

$$1 \le \frac{x}{\sin x} \le \frac{1}{\cos x}$$

$$1 \ge \frac{\sin x}{x} \ge \cos x$$

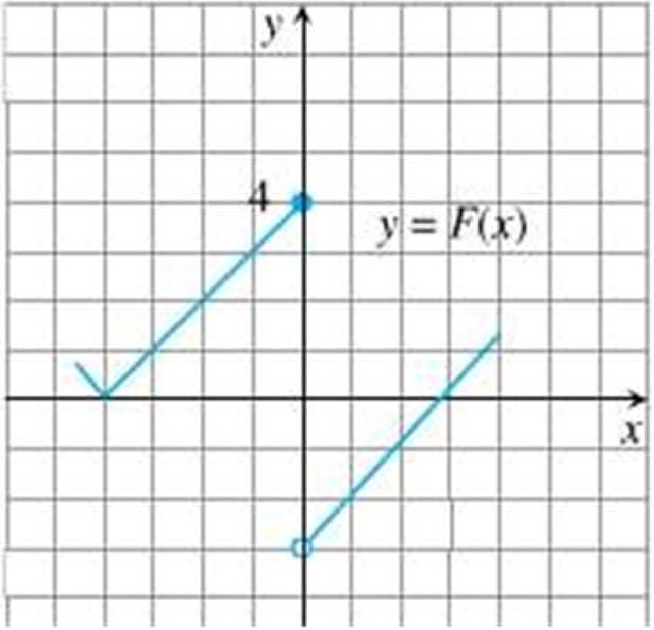
$A_{\odot} = \pi r^2$

$A_{\cup} = \frac{\pi r^2}{2}$

$A_{\triangleleft x} = \frac{x}{2\pi}(\pi r^2) = \frac{x r^2}{2}$

$$\lim_{x \to 0} 1 = 1 \; ; \; \lim_{x \to 0} \cos x = 1 \;\therefore\; \lim_{x \to 0} \frac{\sin x}{x} = 1$$

49.



(a) $\lim\limits_{x \to 0^-} F(x) = 4$

(b) $\lim\limits_{x \to 0^+} F(x) = -3$

(c) $\lim\limits_{x \to 0} F(x) = \nexists$

(d) $F(0) = 4$

Standardized Test Questions

71. True or False If $\lim_{x \to c^-} f(x) = 2$ and $\lim_{x \to c^+} f(x) = 2$, then $\lim_{x \to c} f(x) = 2$. Justify your answer.

True

72. True or False $\lim_{x \to 0} \frac{x + \sin x}{x} = 2$. Justify your answer.

True

$\lim_{x \to 0} \quad 1 + \frac{\sin x}{x} = 1 + 1 = 2$

In Exercises 73–76, use the following function.

$$f(x) = \begin{cases} 2 - x, & x \le 1 \\ \frac{x}{2} + 1, & x > 1 \end{cases}$$

73. Multiple Choice What is the value of $\lim_{x \to 1^-} f(x)$?

(A) 5/2 (B) 3/2 (C) 1 (D) 0 (E) does not exist

74. Multiple Choice What is the value of $\lim_{x \to 1^+} f(x)$?

(A) 5/2 (B) 3/2 (C) 1 (D) 0 (E) does not exist

75. Multiple Choice What is the value of $\lim_{x \to 1} f(x)$?

(A) 5/2 (B) 3/2 (C) 1 (D) 0 (E) does not exist

76. Multiple Choice What is the value of $f(1)$?

(A) 5/2 (B) 3/2 (C) 1 (D) 0 (E) does not exist

$$\lim_{x \to 0} \frac{1-\cos x}{x} = ???$$

Algebraically

$$\sin^2 x + \cos^2 x = 1$$

$$\lim_{x \to 0} \left(\frac{1-\cos x}{x}\right)\frac{(1+\cos x)}{(1+\cos x)} \rightarrow \lim_{x \to 0} \frac{1-\cos^2 x}{x(1+\cos x)}$$

$$\lim_{x \to 0} \frac{\sin^2 x}{x(1+\cos x)} \rightarrow \left(\lim_{x \to 0} \frac{\sin x}{x}\right)\left(\lim_{x \to 0} \frac{\sin x}{1+\cos x}\right)$$

$$(1)\left(\frac{0}{2}\right) =$$

HOMEWORK PAGES 66-68 2-70 EVEN