Exercise 2.

Group name: TAU03E

Task 1

a)

It's known that $m = M^1L^0T^0$ and $l = M^0L^1T^0$. Also it's known that $g = \frac{\Delta v}{\Delta t}$, where $\Delta v = \frac{\Delta s}{\Delta t}$. Thus $g = \frac{\Delta s}{\Delta t^2} \Rightarrow \ g = M^0L^1T^{-2}$.

The period P is a measure of time so its dimension is $P = M^0L^0T^1$. Therefore, the matrix is $\begin{pmatrix} m \\ l \\ g \\ P \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$.

b)

It's known that $\pi = m^\alpha l^\beta g^\gamma P^\delta$, where $g = LT^{-2}$. Thus,
$\pi = M^\alpha L^\beta \left(LT^{-2}\right)^\gamma T^\delta = M^\alpha L^{\beta+\gamma} T^{\delta-2\gamma}$. For the quantities to be dimensionless (i.e. $M^0L^0T^0$) the following has to apply
$\alpha = 0, \beta + \gamma = 0 \text{ and } \delta - 2\gamma = 0 \ \Rightarrow \ \alpha = 0 \text{ and } \beta = -\gamma = \frac{\delta}{2}$. Hence,
$\pi = l^\beta g^{-\beta} P^{2\beta}$, where $\beta \in R$.

Task 2

Equation of motion

We know that $\rho\frac{\partial v}{\partial t} + \rho v\frac{\partial v}{\partial x} + \frac{\partial p}{\partial x} = \mu\frac{\partial^2 v}{\partial x^2}$, where $\rho\frac{\partial v}{\partial t} = \frac{M}{L^3}\frac{L}{T^2} = \frac{M}{L^2T^2}$,
$\rho v\frac{\partial v}{\partial x} = \frac{M}{L^3}\frac{L}{T}\frac{\frac{L}{T}}{L} = \frac{M}{L^2T^2}$, $\frac{\partial p}{\partial x} = \frac{\frac{M}{LT^2}}{L} = \frac{M}{L^2T^2}$ and $\mu\frac{\partial^2 v}{\partial x^2} = \mu\frac{1}{LT}$.

We get $\frac{M}{L^2T^2} + \frac{M}{L^2T^2} + \frac{M}{L^2T^2} = \mu\frac{1}{LT}$ which is $3 \cdot \frac{M}{L^2T^2} = \mu\frac{1}{LT}$ and we get $\mu = \frac{M}{LT}$ in the space $\{M, L, T\}$.

We can derive the dimensions in space $\{F, L, T\}$ from above. We know that force $F = \frac{ML}{T^2} \Leftrightarrow M = \frac{FT^2}{L}$ and we can derive $\mu = \frac{FT^2}{L}\frac{1}{LT} = \frac{FT}{L^2}$

Dimensionless Reynold's number

We know that $Re = \frac{\rho v d}{\mu}$ where Reynold's number is dimensionless. This means that the numerator and denumerator must have equal dimensions that are linearly dependent on each other.

So we get $\mu = \rho v d = \frac{M}{L^3}\frac{L}{T}\frac{L}{1} = \frac{M}{LT}$. This is the same as the dimensions in space $\{M, L, T\}$ from the first method. Now using the same inference as in the first method, the dimensions in the $\{F, L, T\}$ space can be derived as $\mu = \frac{FT}{L^2}$.

Task 3

a)

Let's form the dimension matrix in the space $\{M, L, T\}$

We get $\begin{bmatrix} A & M & L & T \\ v & 0 & 1 & -1 \\ d & 0 & 1 & 0 \\ g & 0 & 1 & -2 \\ \rho_k & 1 & -3 & 0 \\ \rho_p & 1 & -3 & 0 \end{bmatrix}$, where $A = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & -2 \\ 1 & -3 & 0 \\ 1 & -3 & 0 \end{pmatrix}$.

b)

Let's solve for the linearly independent powers $\begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & -2 \\ 1 & -3 & 0 \\ 1 & -3 & 0 \end{pmatrix}^T \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \end{pmatrix} = 0.$

We get $k_4 + k_5 = 0$, $k_1 + k_2 + k_3 - 3k_4 - 3k_5 = 0$ and $-k_1 - 2k_3 = 0$, which leads to $k_4 = -k_5$, $k_1 = -2k_3$ and thus $k_1 + k_2 + k_3 = 0$.

We need to find one solution where we have factor v, so let's set $k_1 = 1$ for this problem.

Thus $k_3 = -\frac{1}{2}$ and $1 + k_2 - \frac{1}{2} = 0 \Leftrightarrow k_2 = \frac{1}{2}$. Set $k_4 = 0$ as it does not affect the result leading to $k_5 = 0$.

We get a linearly independent solution with velocity factor v : $\begin{pmatrix} k_1 = 1 \\ k_2 = \frac{1}{2} \\ k_3 = -\frac{1}{2} \\ k_4 = 0 \\ k_5 = 0 \end{pmatrix}$.

Thus $\pi_1 = \frac{v}{\sqrt{dg}}$

Next, we need to find a solution without v. Let's set $k_1 = 0$. Similarly as above, we get

solution without velocity factor v : $\begin{pmatrix} k_1 = 0 \\ k_2 = 0 \\ k_3 = 0 \\ k_4 = -k_5 \\ k_5 = -k_4 \end{pmatrix}$.

Let's set $k_4 = 1$ and we get a solution $\pi_2 = \frac{\rho_k}{\rho_p}$.

Found π values:

$\pi_1 = \frac{v}{\sqrt{dg}}$

$\pi_2 = \frac{\rho_k}{\rho_p}$

According to Buckingham's π theorem, we can say that the 2 independent dimensionless combinations are found that must be found due to having 5 variables and 3 dimensions (5-3=2).

Due to these answers we can say that the velocity of the bubble depends on the diameter of the tube and the gravitational force downwards. This is clearly rational, as the diameter and gravitational pull affect the bubble's rise in the tube. We can also say that the ratio of the densities between the liquid and air in bubble are proportional to each other. If density of the liquid decreases, the density of the air inside the bubble decreases proportionally. This can be thought as the liquid keeping the bubble's walls together and once it loosens the bubble's size increases, thus decreasing its air density.

Task 4

Capillarity phenomenon systems variables are (h, r, γ, ρ, g), thus $h = f(h, r, \gamma, \rho, g)$.

The dimension matrix of the system includes quantities M, L, T, and from the lectures dimension matrix is

$$\begin{bmatrix} A & M & L & T \\ h & 0 & 1 & 0 \\ r & 0 & 1 & 0 \\ \rho & 1 & -3 & 0 \\ \gamma & 1 & 0 & -2 \\ g & 0 & 1 & -2 \end{bmatrix}, \text{ where } A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & -3 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & -2 \end{pmatrix}.$$

Now, as there are 5 variables and 3 dimensions, we must find $5 - 3 = 2$ independent dimensionless combinations. Two independent solutions are presented in lecture and they are

$\pi_1 = \frac{h}{r}$, $\pi_2 = \frac{r^2 \rho g}{\gamma}$.

Applying the Buckingham's π therem, the $\pi_1 = \phi(\pi_2)$, where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is function.

Thus $\pi_1 = \phi(\frac{r^2\rho g}{\gamma}) \Rightarrow \frac{h}{r} = \phi(\frac{r^2\rho g}{\gamma}) \Rightarrow h = r\phi(\frac{r^2\rho g}{\gamma})$.

Task 5

Divide dimension L into R (radial) and Z (vertical). The new dimension matrix includes quantities M, R, Z, T:

$$\begin{bmatrix} A & M & R & Z & T \\ h & 0 & 0 & 1 & 0 \\ r & 0 & 1 & 0 & 0 \\ \rho & 1 & -2 & -1 & 0 \\ \gamma & 1 & -1 & 1 & -2 \\ g & 0 & 0 & 1 & -2 \end{bmatrix}$$

, where height h is converted into vertical, $r \rightarrow radial$, $g \rightarrow vertical$ and ρ, γ lengths iare divided into both R and Z.

Now, there are 5 variables and 4 dimensions. By Buckingham's theorem, there exist one dimensionless product π_1 such $F(\pi_1) = 0$, where $\pi_1 = C$, where $C > 0$ is some constant.

Products:

$\pi = h^a r^b \rho^c \gamma^d g^e$

, where a, b, c, d, e are exponents which make ϕ dimensionless. Input dimensions to variables, now

$\pi = (Z)^a (R)^b (MR^{-2}Z^{-1})^c (MT^{-2})^d (ZT^{-2})^e$.

Exponents need to be zero as previously presented,

Thus for M: $c + d = 0 \Rightarrow d = -c$,

R: $b - 2c + c = 0 \Rightarrow b = c$,

Z: $a - c + d + e0 \Rightarrow a - c - (-c) + e = 0 \Rightarrow a - 2c + e \Rightarrow a = 2c - e \Rightarrow a = c$,

T: $-2d - 2e = 0 \Rightarrow -2(-c) - 2e = 0 \Rightarrow 2c - 2e = 0 \Rightarrow c = e$.

Now include exponent values $a = c, b = c, d = -c, e = c$ into π:

$\pi = h^c r^c \rho^c \gamma^{-c} g^c = (\frac{hr\rho g}{\gamma})^c$

, from the exponent c, can be dropped as the π is dimensionless.

Thus

$\pi = \frac{hr\rho g}{\gamma}$

Solve for h:

$$C = \frac{hr\rho g}{\gamma} \Rightarrow h = C\frac{\gamma}{r\rho g}$$

Formula is seemingly credible. The surface of water in glass near its edge can be bit approximated since now the r is applied in divider. However, it still takes the radius of the glass into action as well as the other relevant variables.