Introduction: Markov Switching Auto-regressive

In this section, we describe how markov-switching models arise in the context of modelling abrupt change in financial time series. In large part our explanation below is motivated by an exposition by Hamilton [1], although some modifications and extra content are added to suit our specific context.

Let begin by considering how we might describe the consequences of a dramatic change in the behavior of a single variable y_t. Suppose that the typical historical behavior could be described with a zero-mean first-order autoregression,

$$y_t = \phi_1 y_{t-1} + \epsilon_t, \tag{1}$$

$\epsilon_t \sim N(0, \sigma^2)$, which adequately describe the observed data up for $t = 1, 2, \dots, t_0$. Suppose that at date t_0 there was a significant change in the mean reverting rate of the series, so that we would instead wish to describe the data according to

$$y_t = \phi_2 y_{t-1} + \epsilon_t \tag{2}$$

for $t = t_0 + 1, t_0 + 2, \dots$. On one hand, we could say that the series are governed by two different models (1), (2) and there is a determistic shift in parameter shift after day t_0, but this is not satisfactory as a probability law that genereate data. The idea of regime switching model is that we, alternatively, encompass both models by a single, larger one

$$y_t = \phi_{s_t} y_{t-1} + \epsilon_t, \tag{3}$$

in which we introducing a new variable s_t, which indicates whether or not we are in the first regime, or in the second regime, according to whether or not t is before or after t_0. Specifically,

$$s_t = \begin{cases} 1, & \text{if } t \le t_0, \\ 2, & \text{otherwise.} \end{cases}$$

A complete description of the probability law governing the observed data would then require a probabilistic model of what caused the change from $s_t = 1$ to $s_t = 2$. The simplest such specification is that s_t is the realization of a two-state Markov chain with

$$\mathbb{P}(s_t = i | s_{t-1} = j, s_{t-2}, \dots, y_{t-1}, y_{t-2}, \dots) = \mathbb{P}(s_t = i | s_{t-1} = j) := p_{ij}. \tag{4}$$

The specification in (4) assumes that the probability of a change in regime depends on the past only through the value of the most recent regime. Although there are some excellent discussions (e.g., [2], [3]) about time-varying transitional probability, of which the dependency of past observation $y_{t-1}, y_{t-2}, \dots$ are maintained as in LHS., it is found in most applications that regime switching models work well under markov assumption.

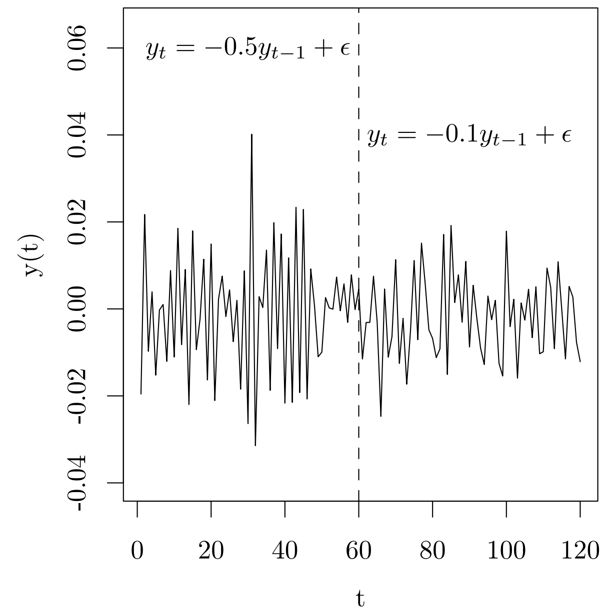

Figure 1: Simple Example

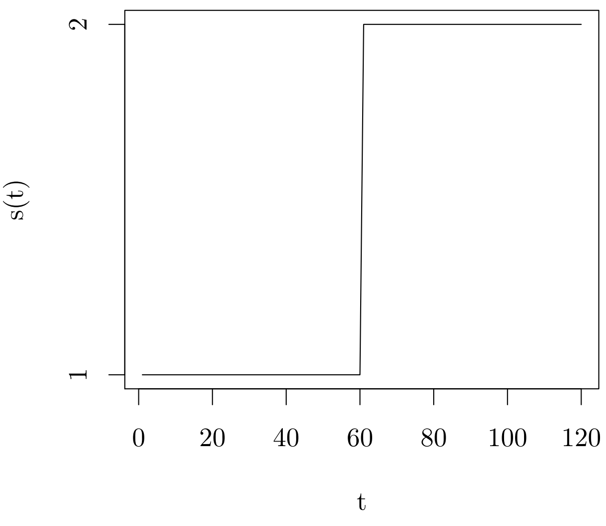

Figure 2: Simple Example

Formal Definitions

$$y_t = c_{s_t} + \sum_{i=1}^{L} \phi_i^{s_t} y + \epsilon_t \tag{5}$$

$$A = \begin{bmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{1S} \\ p_{21} & p_{22} & x_{23} & \cdots & p_{2S} \\ \multicolumn{5}{c}{\dots\dots\dots\dots\dots\dots\dots\dots} \\ p_{S1} & p_{S2} & p_{S3} & \cdots & p_{SS} \end{bmatrix} \tag{6}$$

Learning and Inferences

EM algorithm

To do.

Forward Filtering

Next, we discuss how do we infer which regime our time series y_t is falling in on a particular day t. Equations (7), (8) and (9) below are the derivation of probability that the regime variable s_t equals to regime i, conditioning on all the past observations $y_t, y_{t-1}, \dots$ that are available up to day t.

$$\mathbb{P}(s_t = i|y_t, y_{t-1}, \dots) = \frac{\mathbb{P}(y_t|s_t = i, y_{t-1}, y_{t-2}, \dots)\mathbb{P}(s_t = i|y_{t-1}, \dots)}{\mathbb{P}(y_t|y_{t-1}, \dots)} \tag{7}$$

$$\mathbb{P}(s_t = i|y_{t-1}, \dots) = \sum_{j=1}^{S} \mathbb{P}(s_t = i|s_{t-1} = j, y_{t-1}, \dots)\mathbb{P}(s_{t-1} = j|y_{t-1}, \dots) \tag{8}$$

By applying markov assumption[1] equations (7) and (8) become

$$\mathbb{P}(s_t = i|y_t, \dots) \propto^2 \mathbb{P}(y_t|s_t = i, y_{t-1}, \dots) \sum_{j=1}^{S} p_{ij}\mathbb{P}(s_{t-1} = j|y_{t-1}, \dots). \tag{9}$$

Training

References

[1] Hamilton, J.D. (2008). Regime-switching models. In: S. Durlauf and L. Blume (eds.), New Palgrave dictionary of economics, 2nd edition. Palgrave McMillan Ltd.

[1] $p_{ij} = \mathbb{P}(s_t = i|s_{t-1} = j) = \mathbb{P}(s_t = i|s_{t-1} = j, \cancel{y_{t-1}}, \dots)$.

[2] The normalizing constant, $\mathbb{P}(y_t|y_{t-1}, \dots) = \sum_{i=1}^{2} \mathbb{P}(y_t|s_t = i, y_{t-1}, y_{t-2}, \dots)\mathbb{P}(s_t = i|y_{t-1}, \dots)$, is actually the summation of the RHS. of (9) over $i \in \{1, \dots, S\}$.

[2] Diebold, F.X., J.H. Lee, and G.C. Weinbach. Regime Switching with Time-Varying Transition Probabilities. In: C. Hargreaves, ed., Nonstationary Time Series Analysis and Cointegration. Oxford, UK: Oxford University Press, 1994, pp. 283-302.

[3] Filardo, A.J., 1994, Business cycle phases and their transitional dynamics. Journal of Business and Economic Statistics 12, 299 308.

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