Gauss's Principle of Least Constraint

① Consider a system 3R Manipulator

$$M\ddot{q} = \tau$$ ← unconstrained dynamics

$$q = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \end{bmatrix}$$

$$J\ddot{q} = a^*$$

cartesian acceleration

joint to cartesian jacobian

joint acceleration

$$\underbrace{3 \times n \quad n \times 1}\ = 3$$

② if you add a cartesian acceleration constraint a^*

$n = 3$

Understanding:

what is the $\ddot{q}$ that satisfies constraint?

it is some $\underline{\ddot{q}}$ that minimizes the distance from actual unconstrained acceleration

(or) The constrained system evolves in as close a proximity to the unconstrained system as possible

$$\dot{v} = \min_{\underline{\ddot{q}}} \| \underline{\ddot{q}} - M^{-1}\tau \|_M$$

Subject to $J\underline{\ddot{q}} = a^*$

Primal formulation also known as gauss principle of least constraint

Where,

$$\|x\|_M^2 = \sqrt{x^T M x}$$

Thus the constraint evaluates the smallest deviation from unconstrained acceleration.

Primal Problem :

$$(\dot{v}, \dot{w}) = \underset{(x,y)}{\text{argmin}} \; \| x - M^{-1}(\tau - c) \|_M^2 + \| y - a^* \|_{R^{-1}}$$

gen acceleration

constraint deformation acceleration

Subject to

(1) equality constraints $J_e x_e - y_e = 0$
(2) friction constraints $J_f x_f - y_f = 0$
(3) contact constraints $J_c x_c - y_c \in K^*$

R : makes constraints soft

a^* : stabilizes constraints

solves for

con

Reduced Primal Problem :

$$\underset{x}{\text{argmin}} \; \| x - M^{-1}(\tau - C) \|_M^2 + s(Jx - a^*)$$

$S(\cdot)$ constraint softening

solves for constraint acceleration

Dual form

$$f = \underset{\lambda}{\arg\min} \ \frac{1}{2} \lambda^T (A+R) \lambda + \lambda^T (a^0 - a^*)$$

subject to $\lambda \in \Omega$

directly solves for constraint force f

where R = regularizer matrix

a^* = constraint stabilization acceleration

note that the dual problem is strictly convex

Parameter computation :

how are R and a^* computed ??

a^0 - unconstrained acceleration

a^* - reference acceleration (in constraint space) constraint,

$$a' \in [a^0, a^*]$$

decided by R, given by

$$\boxed{a' = A(A+R)^{-1} a^* + R(A+R)^{-1} a^0}$$

$$R_{ii} = \left(\frac{1-d_i}{d_i}\right) \hat{A}_{ii} \quad , \text{ where } \hat{A} \cong A$$

** $\boxed{a'_i = d_i a^*_i + (1-d_i) a^0_i}$ — 1

** $\boxed{a^*_i = -b_i (J v)_i - k r_i}$ — 2

Summary;

$d \in (0,1)$

$b > 0$

$K > 0$

aside

$$\boxed{m\ddot{x} + \tilde{b}\dot{x} + \tilde{K}x = Q}$$

$$\ddot{x} = Q - \frac{\tilde{b}}{m}(\dot{x}) - \frac{\tilde{K}x}{m}$$

$$\boxed{\ddot{x} = Q - b\,(Jv) - Kr}$$

Settings in mujoco

You can set d, b, K through solref and solimp which are available in all MJCF elements involving constraints (see ① in parameter comp section)

meaning equation ① in parameter comp section

$$a_1 + d\,(\underbrace{bv + Kr}_{-a^*}) = (1-d)\,a_0$$

$$a_1 - d a^* = (1-d)\,a_0$$

d = constraint impedance

b = damping

K = stiffness

a_0 = unconstrained acceleration

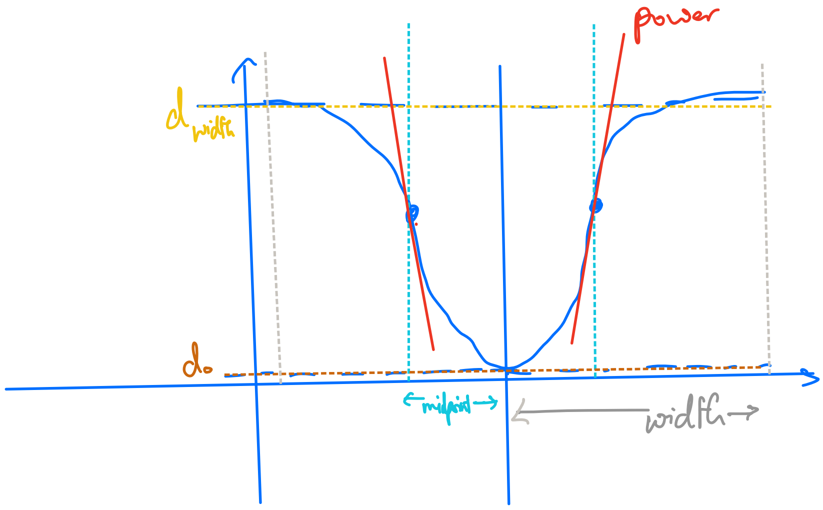
a_1 = constrained acceleration

a^* = reference acceleration

solimp,

$d \in (0,1)$

constraints   ability to generate force



solimp = " d_0 d_{width} width midpoint power "

midpoint, width $\in \mathbb{R}^+$

power $\in \mathbb{R}^+$

$d_0, d_{width} \in (0,1)$

Solref,

(timeconstant, dampratio)

(-stiffness, -damping)

$b = 2/(d_{width} \cdot \text{timeconst})$

(or)

$b = (\text{damping}/d_{width})$

$k = d(r)/(d_{width}^2 \cdot \text{timeconst}^2 \cdot \text{damping}^2)$

(or)

$= \text{stiffness}/d^2_{width}$

elastic collision $\Rightarrow$ damping $= 0$