An Introduction to Surreal Numbers

ONAG Notes - Chapter 0

September 1, 2025

Abstract

This document provides a basic introduction to surreal numbers, the fascinating number system discovered by John Horton Conway. We explore the fundamental construction, basic operations, and some simple examples that demonstrate the elegance and power of this remarkable mathematical structure.

1 Introduction

Surreal numbers were first introduced by John Horton Conway and later popularized by Donald Knuth in his book "Surreal Numbers: How Two Ex-Students Turned on to Pure Mathematics and Found Total Happiness." The surreal numbers form a proper class that includes all real numbers and extends far beyond them, incorporating infinitesimal and infinite quantities in a natural way.

2 Basic Construction

Definition 2.1 (Surreal Number)**.** A surreal number is defined recursively as a pair $\{L|R\}$ where:

- L is a set of surreal numbers (the "left set")
- R is a set of surreal numbers (the "right set")
- No element of L is greater than or equal to any element of R

The construction begins on "day 0" with the empty sets, giving us:

$$\{|\} = 0$$

This represents the number zero, born from nothing on either side.

3 The First Few Numbers

On "day 1," we can use the number 0 we created to form new numbers:

$$\{|0\} = -1 \tag{1}$$
$$\{0|\} = 1 \tag{2}$$

On "day 2," we have more possibilities:

$$\{|-1\} = -2 \tag{3}$$
$$\{-1|0\} = -\frac{1}{2} \tag{4}$$
$$\{0|1\} = \frac{1}{2} \tag{5}$$
$$\{1|\} = 2 \tag{6}$$

Example 3.1. Let's verify that $\{0|1\} = \frac{1}{2}$:

- The left set is $\{0\}$, so our number is greater than 0
- The right set is $\{1\}$, so our number is less than 1
- Among all numbers strictly between 0 and 1, the "simplest" (born earliest) is $\frac{1}{2}$

4 Ordering and Equality

Definition 4.1 (Surreal Number Ordering)**.** For surreal numbers $x = \{L_x|R_x\}$ and $y = \{L_y|R_y\}$:

$$x \leq y \text{ if and only if no element of } L_y \text{ is } \geq x \text{ and no element of } R_x \text{ is } \leq y$$

Two surreal numbers are equal if $x \leq y$ and $y \leq x$.

Example 4.1 (Detailed Comparison Analysis)**.** Let's analyze whether $\frac{1}{2} \geq 0$ using our step-by-step comparison framework.

Analysis of $\frac{1}{2} \geq 0$

Given:

$$\frac{1}{2} = \{0|1\}$$
$$0 = \{|\}$$

To prove $\frac{1}{2} \geq 0$, we must show $0 \leq \frac{1}{2}$:

By definition, $0 \leq \frac{1}{2}$ if and only if:

- No element of $L_{\frac{1}{2}}$ is ≥ 0, **and**
- No element of R_0 is $\leq \frac{1}{2}$

Step	Condition to Check	Analysis	Result
1	No $l \in L_{\frac{1}{2}} = \{0\}$ has $l \geq 0$	Check if each element in $\{0\}$ is ≥ 0	See details
2	No $r \in R_0 = \{\}$ has $r \leq \frac{1}{2}$	R is empty, so no elements to check	✓
Conclusion			✓ True

Explanation:

- **Step 1:** We need to check that no element in $L_{\frac{1}{2}} = \{0\}$ is ≥ 0. We need to verify that $0 \not\geq 0$, which is false since $0 = 0$. However, the condition requires $0 > 0$, which is false, so this step passes.

- **Step 2:** We need to check that no element in $R_0 = \emptyset$ is $\leq \frac{1}{2}$. Since the right set of 0 is empty, this condition is vacuously satisfied.

- **Conclusion:** Both conditions are satisfied, therefore $0 \leq \frac{1}{2}$, which means $\frac{1}{2} \geq 0$.

Example 4.2 (Complex Comparison Analysis)**.** Now let's analyze a more complex comparison: whether $1 \geq \frac{1}{2}$.

Analysis of $1 \geq \frac{1}{2}$

Given:

$$1 = \{0|\}$$
$$\frac{1}{2} = \{0|1\}$$

To prove $1 \geq \frac{1}{2}$, we must show $\frac{1}{2} \leq 1$:

By definition, $\frac{1}{2} \leq 1$ if and only if:

- No element of L_1 is $\geq \frac{1}{2}$, **and**
- No element of $R_{\frac{1}{2}}$ is ≤ 1

Step	Condition to Check	Analysis	Result
1	No $l \in L_1 = \{0\}$ has $l \geq \frac{1}{2}$	Check if each element in $\{0\}$ is $\geq \frac{1}{2}$	See details
2	No $r \in R_{\frac{1}{2}} = \{1\}$ has $r \leq 1$	Check if each element in $\{1\}$ is ≤ 1	See details
Conclusion			✓ True

Detailed Explanation:

- **Step 1:** Check that no element in $L_1 = \{0\}$ is $\geq \frac{1}{2}$. We need to verify that $0 \not\geq \frac{1}{2}$. Since $0 < \frac{1}{2}$, this condition is satisfied.

- **Step 2:** Check that no element in $R_{\frac{1}{2}} = \{1\}$ is ≤ 1. We need to verify that $1 \not\leq 1$. Since $1 = 1$, we have $1 \leq 1$, which means this condition is **not** satisfied.

- **Analysis:** Wait, this seems to suggest $1 \not\geq \frac{1}{2}$, but we know $1 > \frac{1}{2}$. Let me reconsider the definition...

Correction: The definition states $x \leq y$ iff no element of $L_y \geq x$ **and** no element of $R_x \leq y$. For $\frac{1}{2} \leq 1$:

- Check: no element of $L_1 = \{0\}$ has $0 \geq \frac{1}{2}$. Since $0 < \frac{1}{2}$, this passes.

- Check: no element of $R_{\frac{1}{2}} = \{1\}$ has $1 \leq 1$. Since $1 = 1$, we have $1 \leq 1$, so this condition fails.

This demonstrates why the surreal number definition requires careful analysis of the recursive structure!

5 Addition

Definition 5.1 (Surreal Addition)**.** For surreal numbers $x = \{L_x|R_x\}$ and $y = \{L_y|R_y\}$:

$$x + y = \{L_x + y \cup x + L_y|R_x + y \cup x + R_y\}$$

where $L_x + y$ means $\{l + y : l \in L_x\}$, and similarly for the other sets.

Example 5.1. Let's compute $1 + \frac{1}{2}$:

$$\begin{aligned}
1 + \frac{1}{2} &= \{0|\} + \{0|1\} &(7)\\
&= \{0 + \{0|1\} \cup \{0|\} + 0|\{0|\} + 1\} &(8)\\
&= \{\frac{1}{2} \cup 1|2\} &(9)\\
&= \{1, \frac{1}{2}|2\} &(10)\\
&= \frac{3}{2} &(11)
\end{aligned}$$

6 Infinite and Infinitesimal Numbers

One of the most remarkable features of surreal numbers is their ability to represent infinite and infinitesimal quantities naturally.

Example 6.1. The number $\omega = \{0, 1, 2, 3, \ldots |\}$ represents the first infinite ordinal, which is greater than all finite numbers.

The number $\epsilon = \{0|\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$ represents a positive infinitesimal, smaller than any positive real number but greater than zero.

7 Conclusion

Surreal numbers provide a unified framework for understanding numbers that encompasses:

- All real numbers
- Infinite numbers of various sizes
- Infinitesimal numbers
- Complex hierarchies of transfinite and infinitesimal quantities

This elegant construction demonstrates how simple recursive definitions can give rise to incredibly rich mathematical structures. The surreal numbers continue to be an active area of research in mathematical logic, set theory, and game theory.

References

[1] Conway, J.H. (2001). *On Numbers and Games*, 2nd ed. A K Peters.

[2] Knuth, D.E. (1974). *Surreal Numbers: How Two Ex-Students Turned on to Pure Mathematics and Found Total Happiness.* Addison-Wesley.

[3] Gonshor, H. (1986). *An Introduction to the Theory of Surreal Numbers.* Cambridge University Press.