1) Fourier Series representations

- Another series representation of a function, like the Taylor series.

- Taylor series: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

- Fourier series: $$\boxed{\begin{aligned} f(x) &= \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \\ &= A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\end{aligned}}$$

  will get Fourier series representation (FSR) on the interval $\boxed{-L \leq x \leq L}$

- can only represent periodic functions as FS, but can shift non periodic functions.

When $f(x)$ is:

		Even	Odd
$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$	→	$\frac{1}{L}\int_0^L f(x)\,dx$	0
$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$	→	$\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$	0
$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$	→	0	$\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

$f(x) = x$ FSR on $-L \leq x \leq L$

↑ violates periodic rule, but ignoring that for now

$f(x)$ is odd $\therefore A_0 = 0 \quad A_n = 0$

$B_n = \frac{2}{L}\int_0^L x\sin\left(\frac{n\pi x}{L}\right)dx$

$u = x \qquad v = -\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)$

$du = dx \qquad dv = \sin\left(\frac{n\pi x}{L}\right)dx$

$$\begin{aligned} B_n &= \frac{2}{L}\left[-x\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{L}{n\pi}\int \cos\left(\frac{n\pi x}{L}\right)dx\right] \\ &= \frac{2}{\cancel{L}}\left[-x\frac{\cancel{L}}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{L^{\cancel{2}}}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\right]_0^L \\ &= -\frac{2x}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{2L}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\Bigg|_0^L \\ &= -\frac{2L}{n\pi}\cos(n\pi) + \frac{2L}{(n\pi)^2}\sin(n\pi)\end{aligned}$$

n is an integer b/c its an indexing variable.

$n = 1, 2, 3, \ldots$

$\cos\pi = -1$	$\sin\pi = 0$
$\cos(2\pi) = 1$	$\sin(2\pi) = 0$
$\cos(3\pi) = -1$	$\sin(3\pi) = 0$
$\ldots$	$\ldots$

$\therefore \boxed{\cos(n\pi) = (-1)^n} \qquad \boxed{\sin(n\pi) = 0}$

$B_n = -\frac{2L}{n\pi}(-1)^n + \cancel{\frac{2L}{(n\pi)^2}(0)}$

$\therefore B_n = \frac{2L}{n\pi}(-1)^{n+1}$

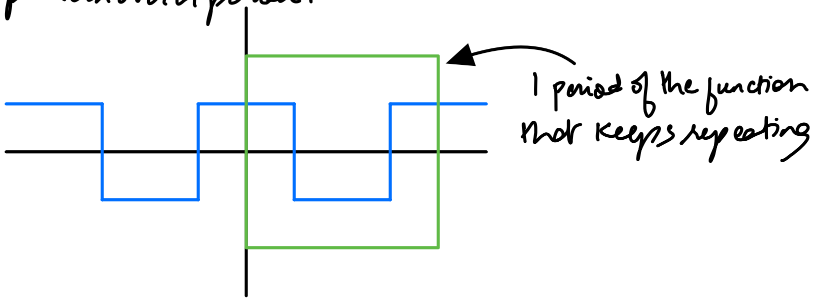
$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n\sin\left(\frac{n\pi x}{L}\right)$$

$$x = 0 + 0 + \sum_{n=1}^{\infty}\frac{2L}{n\pi}(-1)^{n+1}\sin\left(\frac{n\pi x}{L}\right)$$

$$x = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}2L}{n\pi}\sin\left(\frac{n\pi x}{L}\right) \quad \Leftarrow \text{FSR of } f(x) = x$$

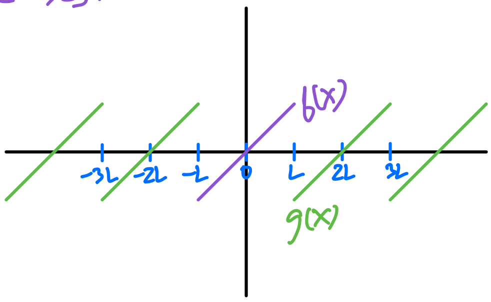
2) Periodic functions and periodic extensions

- Periodic function - repeating pattern over a period:



- To make a non periodic function periodic, limit it to some symmetric interval ($-L$ to L here), and then take that section of the graph & repeat infinitely in both directions, to the left & right.

$f(x) = x$ on $[-L, L]$:

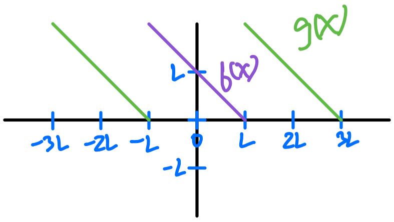


- $f(x)$ – original non-periodic function.
- $g(x)$ – its periodic extension, that continues infinitely on both sides.

- $g(x)$ is a periodic function and we're actually finding its Fourier Series representation (FSR).

- f & g are equivalent everywhere b/w $[-L, L]$, so that's how we can find the FSR of ANY function.

$f(x) = L - x$ on $[-L, L]$

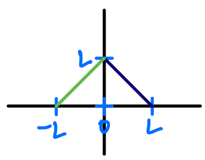


- Even extension:

$$\boxed{g(x) = \begin{cases} f(x) & 0 \leq x \leq L \\ f(-x) & -L \leq x \leq 0 \end{cases}}$$

$f(x) = L - x \qquad f(-x) = L - (-x) = L + x$

$$\rightarrow g(x) = \begin{cases} L - x & 0 \leq x \leq L \\ L + x & -L \leq x \leq 0 \end{cases} \rightarrow$$

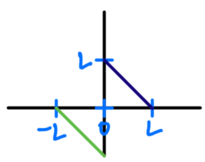


- Odd extension:

$$\boxed{g(x) = \begin{cases} f(x) & 0 \leq x \leq L \\ -f(-x) & -L \leq x \leq 0 \end{cases}}$$

$f(x) = L - x \qquad -f(-x) = -($

$$\rightarrow g(x) = \begin{cases} L - x & 0 \leq x \leq L \\ -L - x & -L \leq x \leq 0 \end{cases} \rightarrow$$



3) Representing piecewise functions

- Almost same finding FSR of regular function except need to take the integral of each piece separately.

$$f(x) = \begin{cases} 1 & -L \le x \le 0 \\ x & 0 \le x \le L \end{cases}$$

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$A_0 = \frac{1}{2L}\int_{-L}^{0} 1\,dx + \frac{1}{2L}\int_{0}^{L} x\,dx$$

$$= \frac{1}{2L}x\Big|_{-L}^{0} + \frac{1}{2L}\,\frac{1}{2}x^2\Big|_{0}^{L}$$

$$A_0 = \frac{1}{2} + \frac{L}{4} = \frac{L+2}{4}$$

$$A_0 = \frac{L+2}{4}$$

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$A_n = \frac{1}{L}\int_{-L}^{0} 1\cos\left(\frac{n\pi x}{L}\right)dx + \frac{1}{L}\int_{0}^{L} x\cos\left(\frac{n\pi x}{L}\right)dx$$

$u = x \quad v = \frac{L}{n\pi}\sin(\dots)$

$du = dx \quad dv = \cos\left(\frac{n\pi x}{L}\right)dx$

$$A_n = \frac{1}{\cancel{L}}\left(\frac{\cancel{L}}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right)\Big|_{-L}^{0} + \frac{1}{L}\left[x\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{0}^{L} - \frac{L}{n\pi}\int_{0}^{L}\sin\left(\frac{n\pi x}{L}\right)dx\right]$$

$$A_n = \cancel{\frac{1}{n\pi}\overset{0}{\sin(-n\pi)}} + \cancel{x\frac{1}{n\pi}\overset{0}{\sin(n\pi)}} + \frac{L}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right)\Big|_{0}^{L}$$

$$A_n = \frac{L}{(n\pi)^2}\,\cancel{\overset{(-1)^n}{\cos(n\pi)}} - \frac{L}{(n\pi)^2}$$

$$A_n = \frac{L}{(n\pi)^2}\left((-1)^n - 1\right) = \frac{(-1)^n L - L}{(n\pi)^2}$$

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{-L}^{0}\sin\left(\frac{n\pi x}{L}\right)dx + \frac{1}{L}\int_{0}^{L} x\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{\cancel{L}}\left[-\frac{\cancel{L}}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right]_{-L}^{0} + \frac{1}{\cancel{L}}\left[-x\frac{\cancel{L}}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{L^{\cancel{2}}}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\right]_{0}^{L}$$

$$B_n = -\frac{1}{n\pi} + \frac{1}{n\pi}(-1)^n - \frac{L}{n\pi}(-1)^n \Rightarrow B_n = \frac{(-1)^n(1-L) - 1}{n\pi}$$

$$f(x) = \frac{L+2}{4} + \sum_{n=1}^{\infty}\frac{(-1)^n L - L}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty}\frac{(-1)^n(1-L)-1}{n\pi}\sin\left(\frac{n\pi x}{L}\right)$$

4) Convergence of a Fourier series

Using prev example $f(x) = \begin{cases} 1 & -L \le x \le 0 \\ x & 0 \le x \le L \end{cases}$

$$f(x) = \frac{L+2}{4} + \sum_{n=1}^{\infty}\frac{(-1)^n L - L}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty}\frac{(-1)^n(1-L)-1}{n\pi}\sin\left(\frac{n\pi x}{L}\right)$$

Just b/c we find FSR, it doesn't definitively prove that the series representation is equal to $f(x)$ at every single value of x.

How to determine whether FSR actually converges to $f(x)$.

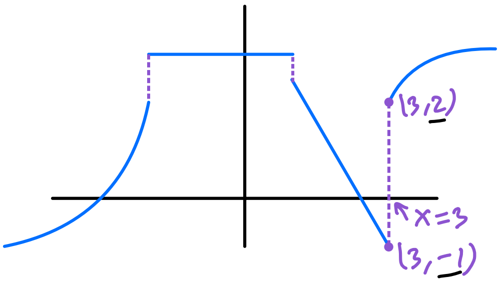
In order for the FSR to converge to $f(x)$ on some symmetric interval $[-L$ to $L]$, the function itself has to be at least piecewise smooth on that interval.

Piecewise smooth – $f(x)$ & $f'(x)$ both continuous on interval, no discontinuities except jump discontinuities & there can only be a finite number of them. Basically, each part of the graph must be smooth. No point discontinuities, everything is smooth & continuous, except for a finite no. of jump discontinuities.

So, if a function is pw smooth, then it can have a periodic extension & we can find its FSR.

Then FSR will actually converge to $f(x)$ wherever f is continuous. FSR value at some x should be equal to $f(x)$ value at same x value.

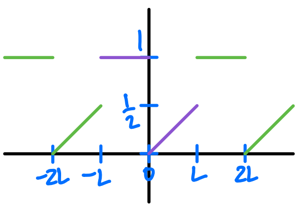
If FSR evaluated at x where there's a jump discontinuity, then the FSR will converge to the avg of the one sided limits.



FSR converges to $\frac{-1+2}{2} = \frac{1}{2}$

If there's a jump discontinuity at $x=a$, then the FSR will converge to:

$$\frac{\lim_{x\to a^-} g(x) + \lim_{x\to a^+} g(x)}{2}$$



$$f(x) = \begin{cases} 1 & -L \le x \le 0 \\ x & 0 \le x \le L \end{cases}$$

FSR conv. to $\frac{1+0}{2} = \frac{1}{2}$ at $x=0$

Assuming $L \neq 1$, the periodic extension $g(x)$ will also have jump discontinuities at $x=-L$ and $x=L$.

At both $x = \pm L$, the FSR converges to $\frac{L+1}{2}$

5) Fourier cosine series

FSR w/ only cosine terms – no sine terms.

FCSR terms for an even function:

Even $f(-x) = f(x)$

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \rightarrow A_0 = \frac{1}{L}\int_0^L f(x)\,dx$$

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \rightarrow A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \rightarrow B_n = 0$$

Then, FCSR:

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$$

↑ Special case of FSR.

$f(x) = x^2 \quad -L \le x \le L$ ← even function, so FSR will be FCSR anyway b/c $B_n = 0$.

$$A_0 = \frac{1}{L}\int_0^L x^2\,dx = \frac{1}{3L}x^3\Big|_0^L = \frac{L^2}{3}$$

$$A_n = \frac{2}{L}\int_0^L x^2\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\left[x^2\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L - \frac{2L}{n\pi}\int_0^L x\sin\left(\frac{n\pi x}{L}\right)dx\right]$$

$+\frac{2}{n\pi}\left(+x\frac{L}{n\pi}\cos(\sim)\right.$
$\left.-+\frac{L^2}{(n\pi)^2}\sin\left(\frac{n\pi}{L}\right)\right)$

$u = x^2$, $du = 2x\,dx$; $v = \frac{L}{n\pi}\sin(\sim)$, $dv = \cos(\sim$

$u = x$, $du = dx$; $v = -\frac{L}{n\pi}\cos\sim$, $dv = $ —

$$A_n = \cancel{\frac{2x^2}{n\pi}\sin\left(\frac{n\pi x}{L}\right)} + x\frac{4L}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right) - \frac{4L^2}{(n\pi)^3}\sin\left(\frac{n\pi x}{L}\right)\Bigg|_0^L \qquad A_n = \frac{4L^2}{(n\pi)^2}(-1)^n$$

$$f(x) = \frac{L^2}{3} + \sum_{n=1}^{\infty} \frac{4L^2}{(n\pi)^2}(-1)^n \cos\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \frac{L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos\left(\frac{n\pi x}{L}\right)$$

FCSR for non-even functions: $f(x) = x$

We'll be looking at the even extension of the function.

Even extension:

$$\boxed{g(x) = \begin{cases} f(x) & 0 \le x \le L \\ f(-x) & -L \le x \le 0 \end{cases}}$$

$$g(x) = \begin{cases} x & 0 \le x \le L \\ -x & -L \le x \le 0 \end{cases}$$

← Since we want FCSR, we can only say the FCSR represents the function on the interval $0 \le x \le L$. If $f(x)$ was even, we'd say on $-L$ to L. But b/c f isn't even & we still want its cosine series, we restrict interval to $0 \le x \le L$.

So instead of $\int_{-L}^{L}$, we use $\int_0^L$.

$$A_0 = \frac{1}{2L}\int_0^L x\,dx = \frac{1}{4L}x^2\Big|_0^L = \frac{L}{4}$$

$$A_n = \frac{1}{L}\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\left[x\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right) + \frac{L^2}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right)\right]_0^L$$

$$= \frac{x}{n\pi}\sin\left(\frac{n\pi x}{L}\right) + \frac{L}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right)\Bigg|_0^L = \frac{L}{(n\pi)^2}(-1)^n - \frac{L}{(n\pi)^2}$$

$$A_n = \frac{L}{(n\pi)^2}\left((-1)^n - 1\right)$$

FCSR:

$$f(x) = \frac{L}{4} + \frac{L}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos\left(\frac{n\pi x}{L}\right) \qquad 0 \le x \le L$$

6) Fourier sine series

FSSR for odd functions:

Odd

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \longrightarrow A_0 = 0$$

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \longrightarrow A_n = 0$$

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \longrightarrow B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

FSSR:

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)dx$$

just special case of FSR.

For non odd functions, need to get its odd extension, & find its FSSR over $0 \le x \le L$

$f(x) = x$

$$B_n = \frac{2}{L}\int_0^L x\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\left[-\frac{xL}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{L^2}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\right]_0^L$$

$$= -\frac{2L}{n\pi}(-1)^n \Rightarrow B_n = \frac{2L(-1)^{n+1}}{n\pi}$$

$$f(x) = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{L}\right)$$

$f(x) = x^2$

$$g(x) = \begin{cases} x^2 & 0 \le x \le L \\ -x^2 & -L \le x \le 0 \end{cases}$$

$$B_n = \frac{1}{L}\int_0^L x^2 \sin\left(\frac{n\pi x}{L}\right)dx \qquad u = x^2,\ du = 2x\,dx \qquad v = -\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right),\ dv = \sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{L}\left[-x^2\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{2L}{n\pi}\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx\right] \qquad u = x,\ du = dx \qquad v = \frac{L}{n\pi}\sin(\sim),\ dv = \cos\left(\frac{n\pi x}{L}\right)dx$$

$$= -x^2\frac{1}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L + \frac{2}{n\pi}\left[\frac{xL}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L - \frac{L}{n\pi}\int_0^L \sin\left(\frac{n\pi x}{L}\right)dx\right]$$

$$= -\frac{x^2}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \cancel{\frac{2xL}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)} + \frac{2L^2}{(n\pi)^3}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L$$

$$= \frac{L^2}{n\pi}(-1)^{n+1} + \frac{2L^2}{(n\pi)^3}(-1)^n - \frac{2L^2}{(n\pi)^3}$$

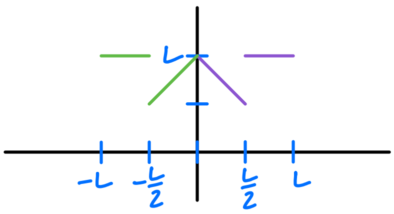
$$B_n = \frac{2L^2(-1)^n + L^2(n\pi)^2(-1)^{n+1} - 2L^2}{(n\pi)^3}$$

$$f(x) = \frac{L^2}{\pi^3}\sum_{n=1}^{\infty}\frac{2(-1)^n + (n\pi)^2(-1)^{n+1} - 2}{n^3}\sin\left(\frac{n\pi x}{L}\right) \qquad 0 \le x \le L$$

7) Cosine and sine series of piecewise functions

$$f(x) = \begin{cases} L - x & 0 \le x \le L/2 \\ L & L/2 \le x \le L \end{cases}$$

For FCSR – even extension. For FSSR – odd extension.



Even extension:

$$g(x) = \begin{cases} L - x & 0 \le x \le L/2 \\ L & L/2 \le x \le L \\ L & -L \le x \le -L/2 \\ L + x & -L/2 \le x \le 0 \end{cases}$$

↑ can also get these values using formula for even extension.

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx$$

$$= \frac{1}{L}\int_0^{L/2} L - x\,dx + \frac{1}{L}\int_{L/2}^L L\,dx$$

$$= \frac{1}{L}\left[Lx - \frac{x^2}{2}\right]\Big|_0^{L/2} + \frac{1}{L}\left[Lx\right]\Big|_{L/2}^L$$

$$= \frac{1}{L}\left[\frac{L^2}{2} - \frac{L^2}{8}\right] + \frac{1}{L}\left[L^2 - \frac{L^2}{2}\right]$$

$$A_0 = \frac{3L}{8} + \frac{L}{2} = \frac{7L}{8}$$

$$A_n = \frac{1}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$A_n = \frac{1}{L}\int_0^{L/2}(L-x)\cos\left(\frac{n\pi x}{L}\right)dx + \frac{1}{L}\int_{L/2}^L L\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{L}\left[\frac{L^2}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_0^{L/2} - \int_0^{L/2} x\cos\left(\frac{n\pi x}{L}\right)dx\right] + \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{L/2}^L$$

$u = x$ $v = \frac{L}{n\pi}\sin$ $du = dx$ $dv = \cos$ —

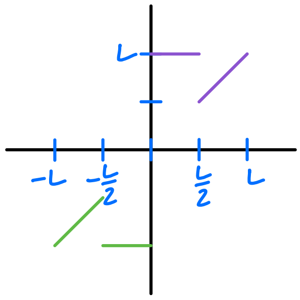
$$= \frac{1}{L}\left[\frac{L^2}{n\pi}\sin\left(\frac{n\pi x}{L}\right) - \cancel{\frac{Lx}{n\pi}\sin\left(\frac{n\pi x}{L}\right)} + \frac{L^2}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right)\right]\Big|_0^{L/2} + \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{L/2}^L$$

$$= \frac{L^2}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^{L/2} + \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{L/2}^L$$

$$A_n = \frac{L}{(n\pi)^2} - \frac{L}{n\pi}\sin\left(\frac{n\pi}{2}\right) - \frac{L}{(n\pi)^2}\cos\left(\frac{n\pi}{2}\right)$$

$$f(x) = \frac{7L}{8} + \frac{L}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1}{n^2} - \frac{\pi}{n}\sin\left(\frac{n\pi}{2}\right) - \frac{1}{n^2}\cos\left(\frac{n\pi}{2}\right)\right]\cos\left(\frac{n\pi x}{L}\right) \qquad 0 \le x \le L$$

FSSR: Odd extension:



$$f(x) = \begin{cases} L & 0 \le x \le L/2 \\ x & L/2 \le x \le L \end{cases}$$

$$g(x) = \begin{cases} L & 0 \le x \le L/2 \\ x & L/2 \le x \le L \\ -L & -L/2 \le x \le 0 \\ x & -L \le x \le -L/2 \end{cases}$$

$$B_n = \frac{1}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{L}\int_0^{L/2} L\sin\left(\frac{n\pi x}{L}\right)dx + \frac{1}{L}\int_{L/2}^{L} x\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= -\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^{L/2} - \frac{x}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{L}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\Big|_{L/2}^{L}$$

$$B_n = \frac{L}{n\pi}\left[1 + (-1)^{n+1} - \frac{1}{2}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right]$$

$$f(x) = \frac{L}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[1 + (-1)^{n+1} - \frac{1}{2}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right]\sin\left(\frac{n\pi x}{L}\right) \qquad 0 \le x \le L$$