Chapter - INTEGRATION

DARBOUX INTEGRAL ⟶

- Let f be bounded on $[a, b]$ on any sets S
- ~~[illegible]~~

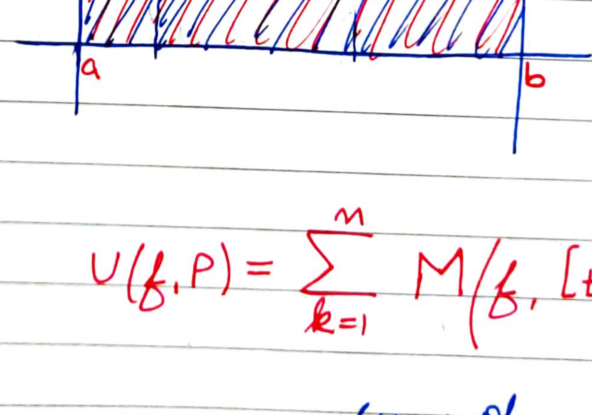
$$M(f, S) = \sup\{f(x) : x \in S\}$$

$$m(f, S) = \inf\{f(x) : x \in S\}$$

- We now partition ~~[illegible]~~ $[a, b]$ in parts like

$$\text{Partition} = \{a = t_0 < t_1 < t_2 < \cdots < t_n = b\}$$

of $[a, b]$

- 

UPPER DARBOUX SUM

$$U(f, P) = \sum_{k=1}^{n} M(f, [t_{k-1}, t_k])\,(t_k - t_{k-1})$$

* This represents sum of areas under supremas in the intervals

NOTE: The width of the intervals need not be homogenous. They can be different.

LOWER DARBOUX SUM

$$L(f,P) = \sum_{k=1}^{M} m(f,[t_{k-1},t_k])\,(t_k - t_{k-1})$$

* This represents sum of areas under infimas in the interval

we can say
$$L(f,P) < \int_a^b f\,dx < U(f,P)$$

$$\therefore \boxed{U(f) = \inf U(f,P) = \overline{\int_a^b} f}$$

Here we make U indep of P

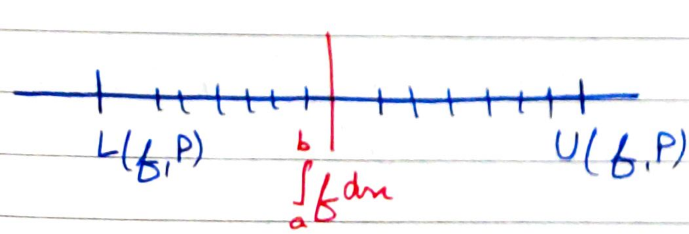
$$\boxed{L(f) = \sup L(f,P) = \underline{\int_a^b} f}$$

NOTE: If $U(f) = L(f)$, Darboux Integral exists.

. when no of partitions increase both $U(f,P)$ & $L(f,P)$ will tend towards the integral $\therefore$ for many such cases,

$U(f)$ would be minimum of such upper darboux sums (ie when we vary P)

Same case for $L(f)$



- Consider

$$Q > P$$

$$\therefore \; U(f,P) \geq U(f,Q)$$

$$L(f,P) < L(f,Q)$$

$$\therefore \; \boxed{L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P)}$$

- Now if $U(f) = L(f)$, then Darboux Integral will exist ie

$$U(f) = L(f) = \int_a^b f(x)\,dx$$

- **Lemma:** Let P, Q such that $P \subseteq Q$ be two partitions of $[a,b]$ and f be bounded on $[a,b]$, then

$$L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P)$$

1 2 3

PROOF: • To prove this, we need to prove 1, 2, 3 individually.

- For 2, we can simply prove using property $\inf \leq \sup$.

For 1 $\quad L(f,P) \leq L(f,Q)$

let $P = \{a = t_0 < t_1 < t_2 < \ldots < t_k < t_{k+1} < \ldots t_n = b\}$

$Q = \{t_0 < t_1 < t_2 < \ldots t_k < u < t_{k+1} < \ldots < t_n = b\}$

What we are trying to do is prove this by induction. First we add 1 extra point and prove it. We then assume it true for n points & then prove it for (n+1) points.

$\therefore$ now $\quad L(f,Q) = \sum_{r=1}^{k} m(f, [t_{r-1}, t_r])(t_r - t_{r-1})$

$$+ \sum_{r=k+1}^{n} m(f, [t_{r-1}, t_r])(t_r - t_{r-1})$$

$$+ m(f, [t_k, u])(u - t_k)$$

$$+ m(f, [u, t_{k+1}])(t_{k+1} - u)$$

$\therefore L(f,Q) - L(f,P) = m(f, [t_k, u])(u - t_k) + m(f, [u, t_{k+1}])(t_{k+1} - u) - m(f, [t_k, t_{k+1}])(t_{k+1} - t_k)$

Comparing the infimas we can write

$m(f, [t_k, u]) \geq m(f, [t_k, t_{k+1}])$ $\quad$ (i)

$m(f, [u, t_{k+1}]) \geq m(f, [t_k, t_{k+1}])$ $\quad$ (ii)

Multiplying (i) by $(u - t_k)$ and (ii) by $(t_{k+1} - u)$ and adding them,

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$$m(f,[t_k,u])(u-t_k) + m(f,[u,t_{k+1}])(t_{k+1}-u)$$
$$\geq m(f,[t_k,t_{k+1}])(t_{k+1}-t_k)$$

$\therefore\ L(f,Q) - L(f,P) \geq 0$ for our induction value $n=1$

Similarly prove for $n+1$ and complete induction.

$$\therefore\ L(f,Q) - L(f,P) \geq 0$$

Hence proved.

imp: • Similarly we also need to prove 3 & then whole expression is proven

• Lemma: when no relation between P and Q is given to us, let P and Q be 2 partitions of $[a,b]$ and f be bounded on $[a,b]$ then

$$\therefore\ L(f,P) \leq U(f,Q)$$

PROOF: now we can write

$$P \subset P\cup Q, \quad Q \subset P\cup Q$$

$$\boxed{\begin{array}{l} L(f,P) \leq L(f,P\cup Q) \leq U(f,P\cup Q) \leq U(f,Q) \\ L(f,Q) \leq L(f,P\cup Q) \leq U(f,P\cup Q) \leq U(f,P) \end{array}}$$

Hence, proved.

• **Lemma-** Let f be bounded on $[a,b]$ then $U(f) \geq L(f)$

PROOF: $L(f,P) \leq U(f,Q)$ [P, Q are any 2 partitions on $[a,b]$]

$$\therefore L(f,P) \leq \inf_{Q} U(f,Q) = U(f)$$

$$\Rightarrow \sup_{P} L(f,P) \leq U(f)$$

$$\therefore L(f) \leq U(f)$$

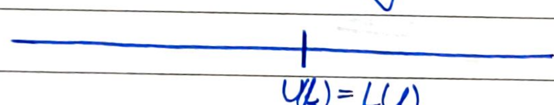
Hence proved.

→ THEOREM

Let f be bounded on $[a,b]$. Then f is integrable (Darboux) on $[a,b]$ if and only if given $\epsilon > 0$ $\exists$ a ~~partition~~ partition P such that

$$\boxed{U(f,P) - L(f,P) < \epsilon}$$

PROOF: Given that f is integrable $\Rightarrow U(f) = L(f)$



given $\epsilon > 0$ $\exists$ a partition P_1 such that

$$L(f,P_1) > L(f) - \frac{\epsilon}{2}$$

Similarly $\exists$ a partition P_2 such that
$$U(f,P_2) < U(f) + \frac{\epsilon}{2}$$

Let $P = P_1 \cup P_2$

$\therefore$ Clearly using previous proofs

$$U(f, P) \leq U(f, P_2)$$

and $$L(f, P) \geq L(f, P_1)$$

$$\therefore U(f, P) - L(f, P) \leq U(f, P_2) - L(f, P_1) \leq U(f) + \frac{\epsilon}{2} - L(f) +$$

$$U(f, P) - L(f, P) < U(f) - L(f) + \epsilon = \epsilon \quad \{\because U(f) = L(f)$$

$$\Rightarrow U(f, P) - L(f, P) < \epsilon \quad \text{ie} \quad U(f, P) < L(f, P)$$

$$\text{(i)}$$

To prove converse part

$$U(f) \leq U(f, P) \leq L(f, P) + \epsilon \leq L(f) + \epsilon$$

~~Since~~ now ϵ is arbitrarily small

$$\Rightarrow U(f) \leq L(f) \ \& \ L(f) \leq U(f) \text{ (always)}$$

$$\text{(ii)}$$

$\therefore$ From (i) & (ii)

$$U(f) = L(f)$$

Hence, proved.

THEOREM →

~~A bounded function~~

Cauchy's Criterion of Integrability (THEOREM)

- In the previous theorem, we ~~had to~~, have a limitation that we can't find the required partition specifically, even though we know it exists.

- $P = \{a = t_0 < t_1 < \ldots\ldots < t_n = b\}$
  $mesh(P) = \max\{t_k - t_{k-1} : k = 1 \text{ to } n\}$

- Theorem: A bounded function f on $[a, b]$ is integrable if and only if given $\epsilon > 0$

  $\exists$ a $\delta > 0$ such that

$$\boxed{mesh(P) < \delta \implies U(f, P) - L(f, P) < \epsilon}$$

where $\delta = \delta(\epsilon, \;)$

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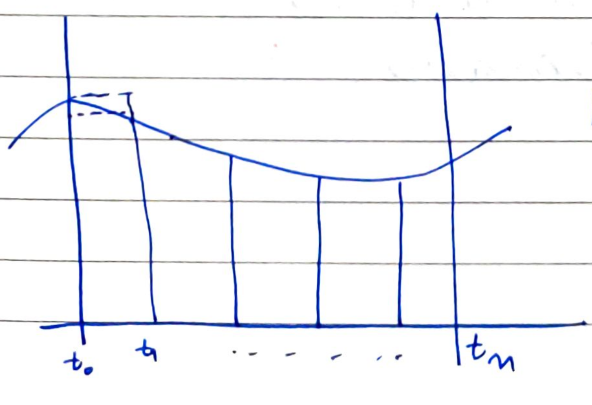
RIEMANN SUM →



$$\boxed{S = \sum_{k=1}^{n} f(x_k)(t_k - t_{k-1})}$$

where $x_k \in [t_k, t_{k-1}]$

and x_k is chosen arbitrarily

- For a given partition, S can have infinitely many values. (Since x_k can be chosen arbitrarily.

→ THEOREM

- A bounded function f on [a, b] is Riemann Integrable if $\exists$ a real number "r" such that the following is true:

given $\epsilon > 0$ $\exists$ $\delta > 0$ such that.

$$\boxed{mesh(P) < \delta \implies |S - r| < \epsilon}$$

where $r = \int_a^b f(x)\,dx$

i.e. for every Riemann Sum "S" for a given partition P where

$S = \sum_{k=1}^{n} f(x_k)(t_k - t_{k-1})$. The number 'r' is value of Riemann Integral.

→ THEOREM

- A function (Bounded) f is Riemann Integrable if and only if ($\Leftrightarrow$) it is Darboux Integrable.

THEOREM

Every monotonic bounded function f on $[a,b]$ is Integrable.

PROOF: Assume $f(t)$ is increasing (non-decreasing)

$$U(f,P) - L(f,P) = \sum_{k=1}^{n} \{M(f,[t_{k-1},t_k]) - m(f,[t_{k-1},t_k])\}(t_k - t_{k-1})$$

Since it is monotonically increasing

$$\therefore M(f,[t_{k-1},t_k]) = f(t_k)$$

$$m(f,[t_{k-1},t_k]) = f(t_{k-1})$$

$\therefore$ from previous equation

$$U(f,P) - L(f,P) = \sum_{k=1}^{n} (f(t_k) - f(t_{k-1})) \times (t_k - t_{k-1})$$

now

$$P = \{a = t_0 < t_1 < \dots < t_n = b\}$$

let mesh$(P) < \delta$ & $\delta = \frac{\epsilon}{f(b)-f(a)}$

& $\delta = \max(t_k - t_{k-1})$ $\therefore (t_k - t_{k-1}) < \delta$

$$\therefore U(f,P) - L(f,P) < \delta \sum_{k=1}^{n} (f(t_k) - f(t_{k-1}))$$

$$\Rightarrow U(f,P) - L(f,P) < \delta(f(b) - f(a)) = \epsilon$$

$$\therefore U(f,P) - L(f,P) < \epsilon$$

$$\& \quad \delta = \frac{\epsilon}{(f(b) - f(a))}$$

given $\epsilon > 0$ $\exists$ $\delta = \frac{\epsilon}{f(b)-f(a)}$

$$\text{mesh}(P) < \delta \Rightarrow U(f,P) - L(f,P) < \epsilon$$

Hence, proved.

→ THEOREM

- continuous function on $[a,b]$ is Integrable.

PROOF: $U(f,P) - L(f,P) = \sum_{k=1}^{n} \left[M(f,[t_{k-1},t_k]) - m(f,[t_{k-1},t_k]) \right](t_k - t_{k-1})$

now

f continuous on $[a,b]$ $\Rightarrow$ f is uniformly continuous on $[a,b]$

$\Rightarrow$ given $\epsilon > 0$ $\exists$ $\delta > 0$ such that

$\forall x, y \in [a,b]$, $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$

Similarly

If mesh $P < \delta \Rightarrow |t_k - t_{k-1}| < \delta$

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THEOREM →

Let f and g be integrable on $[a, b]$ and c be a real number. Then

i) cf is integrable and $\int_a^b cf\,dt = c\int_a^b f\,dt$

(Self Study)

ii) $f+g$ is integrable and $\int_a^b (f+g)\,dt = \int_a^b f\,dt + \int_a^b g\,dt$

PROOF: of ii

Given $\epsilon > 0$ $\exists$ two partitions P_1 and P_2 such that

$$U(f, P_1) - L(f, P_1) < \frac{\epsilon}{2}$$

and $U(g, P_2) - L(g, P_2) < \frac{\epsilon}{2}$

Define $P = P_1 \cup P_2$

$$U(f, P) \leq U(f, P_1), \quad L(f, P) \geq L(f, P_1)$$

$$\Rightarrow U(f, P) - L(f, P) < \frac{\epsilon}{2}$$

Similarly

$$U(g, P) - L(g, P) < \frac{\epsilon}{2}$$

Now we know

$$\inf(f+g, S) \geq \inf(f, S) + \inf(g, S)$$

$$\therefore \Rightarrow m(f+g, [t_{k-1}, t_k]) \geq m(f, [t_{k-1}, t_k]) + m(g, [t_{k-1}, t_k])$$

Including

multiplying $[t_k - t_{k-1}]$ and taking summation on both sides

$$\Rightarrow L(f+g,P) \geq L(f,P) + L(g,P)$$

Similarly

$$U(f+g,P) \leq U(f,P) + U(g,P)$$

$$\therefore U(g+f,P) - L(f+g,P) \leq \underbrace{U(f,P) - L(f,P)}_{<\frac{\epsilon}{2}} + \underbrace{U(g,P) - L(g,P)}_{+\frac{\epsilon}{2}}$$

$$\therefore U(f+g,P) - L(f+g,P) < \epsilon$$

$\Rightarrow$ $f+g$ is integrable (Darboux)

now

$$\int_a^b (f+g)\,dt = U(f+g) \leq U(f+g,P) \leq U(f,P) + U(g,P)$$

$$< L(f,P) + \frac{\epsilon}{2} + L(g,P) + \frac{\epsilon}{2}$$

$$\Rightarrow \int_a^b (f+g)\,dt < L(f,P) + L(g,P) + \epsilon < L(f) + L(g) + \epsilon$$

$$\Rightarrow \int_a^b (f+g)\,dt < \int_a^b f + \int_a^b g + \epsilon$$

Since ϵ is arbitrarily small,

$$\Rightarrow \int_a^b (f+g)\,dt \leq \int_a^b f + \int_a^b g \qquad (i)$$

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Similarly now

$$\int_a^b (f+g)\,dt = L(f+g) \geq L(f,P) + L(g,P)$$

$$> U(f,P) + U(g,P) - \epsilon$$

Since $U(f,P) \geq U(g)$ & $U(g,P) \geq U(g)$

$$\Rightarrow \int_a^b (f+g)\,dt > U(f) + U(g) - \epsilon$$

$$\Rightarrow \int_a^b (f+g)\,dt \geq \int_a^b f\,dt + \int_a^b g\,dt \quad \text{(ii)}$$

$\therefore$ from (i) & (ii)

$$\int_a^b (f+g)\,dt = \int_a^b f\,dt + \int_a^b g\,dt$$

Hence, proved.

→ THEOREM →

- If f and g are integrable and $f \leq g$ $\forall x \in [a,b]$ then $\int_a^b f\,dx \leq \int_a^b g\,dx$

PROOF: Let $h(x) = g(x) - f(x) \geq 0$

$$\Rightarrow L(h,P) \geq 0$$

$\therefore\ L(h) = \sup_P L(h,P) \geq 0$

$\Rightarrow L(h) \geq 0$

$\Rightarrow \int_a^b h\,dx \geq 0$

$\Rightarrow \int_a^b (g-f)\,dx \geq 0$

$\Rightarrow \int_a^b g\,dx - \int_a^b f\,dx \geq 0$

$\Rightarrow \int_a^b f\,dx \leq \int_a^b g\,dx$

Hence, proved.

→ THEOREM ⟶

- If f is integrable, then $|f|$ is integrable and $\left|\int_a^b f\,dx\right| \leq \int_a^b |f|\,dx$

PROOF: As we know

$-|f| \leq f \leq |f|$

$\Rightarrow -\int_a^b |f|\,dx \leq \int_a^b f\,dx \leq \int_a^b |f|\,dx$

$$\Rightarrow \left|\int_a^b f\,dx\right| \leq \int_a^b |f|\,dx$$

Hence, proved.

Let f be integrable (given) and $\therefore$ given $\epsilon > 0$ $\exists$ P such that

$$U(f,P) - L(f,P) < \epsilon$$

now

$$M(|f|,[t_{k-1},t_k]) - m(|f|,[t_{k-1},t_k]) \leq M(f,[t_{k-1},t_k]) - m(f,[t_{k-1},t_k])$$

Multiplying ~~$(t_k - t_{k-1})$~~ $(t_k - t_{k-1})$ both sides

$$\left[M(|f|,[t_{k-1},t_k]) - m(|f|,[t_{k-1},t_k])\right](t_k - t_{k-1}) \leq \left[M(f,[t_{k-1},t_k]) - m(f,[t_{k-1},t_k])\right] \times (t_k - t_{k-1})$$

Taking summation both sides

$$U(|f|,P) - L(|f|,P) \leq U(f,P) - L(f,P) < \epsilon$$

$\therefore$ $|f|$ is integrable.

Hence proved.

** Prove integrability first.

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→ THEOREM →

- Let f be defined on $[a,b]$ and $c \in (a,b)$. Let f be integrable on $[a,c]$ and $[c,b]$ then f is integrable on $[a,b]$ and

$$\int_a^c f\,dx + \int_c^b f\,dx = \int_a^b f\,dx$$

PROOF: We know

f is integrable on $[a,c]$

$$\Rightarrow U_a^c(f,P_1) - L_a^c(f,P_1) < \frac{\epsilon}{2}$$, P_1 is par[tition] on $[a,c]$

f is integrable on $[c,b]$

$$\Rightarrow U_c^b(f,P_2) - L_c^b(f,P_2) < \frac{\epsilon}{2}$$, P_2 is partition on

$P = P_1 \cup P_2$ is a partition on $[a,b]$

$$U_a^b(f,P) = U_a^c(f,P_1) + U_c^b(f,P_2)$$

$$L_a^b(f,P) = L_a^c(f,P_1) + L_c^b(f,P_2)$$

now

$$U_a^b(f,P) - L_a^b(f,P) = U_a^c(f,P_1) - L_a^c(f,P_1) + U_c^b(f,P_2) - L_c^b(f,P_2)$$

$\Rightarrow U_a^b(f,P) - L_a^b(f,P) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

$\Rightarrow$ f is integrable on $[a,b]$

$$\int_a^b f\,dt = U_a^b(f) \leq U_a^b(f,P) = U_a^c(f,P_1) + U_c^b(f,P_2)$$

$$< L_a^c(f,P_1) + \frac{\epsilon}{2} + L_c^b(f,P_2) + \frac{\epsilon}{2}$$

$$\Rightarrow \int_a^b f\,dt < L_a^c(f) + L_c^b(f) + \epsilon = \int_a^c f\,dt + \int_c^b f\,dt + \epsilon$$

$$\Rightarrow \int_a^b f\,dt \leq \int_a^c f\,dt + \int_c^b f\,dt$$

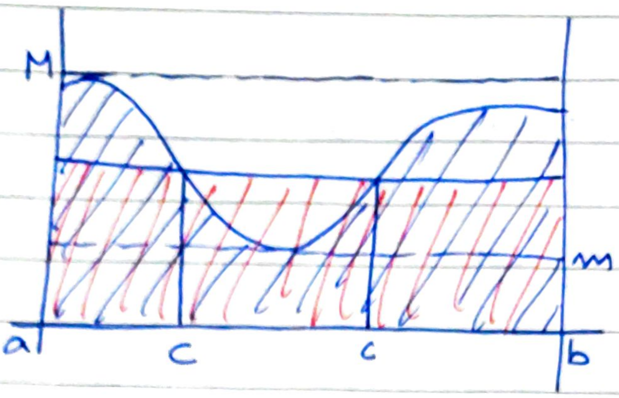
Similarly

$$\int_a^b f\,dt \geq \int_a^c f\,dt + \int_c^b f\,dt$$

$$\therefore \int_a^b f\,dt = \int_a^c f\,dt + \int_c^b f\,dt$$

Hence proved.

→ INTERMEDIATE VALUE THEOREM FOR INTEGRALS

- 

~~This theorem states that if f there exists a c b/w a and b~~

- This theorem states that if f is bounded and integrable on $[a,b]$ and then there exists $c \in [a,b]$ such that

$$f(c)\,(b-a) = \int_a^b f\,dx$$

** → FUNDAMENTAL THEOREM OF CALCULUS I

- Let g be continuous on $[a,b]$ and differentiable on (a,b). If g' is integrable on $[a,b]$ then

$$\boxed{\int_a^b g'(t)\,dt = g(b) - g(a)}$$

PROOF: Let $\epsilon > 0$. Given that g' is Integrable $\exists$ a partition P such that $U(g', P) - L(g', P) < \epsilon$

By LMVT on g on $[t_{k-1}, t_k]$ where $P = \{a = t_0 < t_1 < \dots t_n = b\}$

$$g(t_k) - g(t_{k-1}) = g'(x_k)(t_k - t_{k-1}) \;;\; x_k \in (t_{k-1}, t_k)$$

$$\Rightarrow g(b) - g(a) = \sum_{k=1}^{n} g(t_k) - g(t_{k-1}) = \sum_{k=1}^{n} g'(x_k)(t_k - t_{k-1})$$

$$\Rightarrow L(g', P) \leq g(b) - g(a) \leq U(g', P) \qquad (i)$$

Also, we know

$$L(g', P) \leq \int_a^b g' dt \leq U(g', P) \qquad (ii)$$

(ii)-(i)

$$\Rightarrow -\left(U(g', P) - L(g', P)\right) \leq \int_a^b g' - \left(g(b) - g(a)\right) \leq \underbrace{U(g', P) - L(g', P)}_{< \epsilon}$$

$$\Rightarrow \left| \int_a^b g' - \left(g(b) - g(a)\right) \right| < \epsilon$$

Since ϵ is arbitrarily small, we can write,

$$\int_a^b g' = g(b) - g(a)$$

Hence, proved.

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FUNDAMENTAL THEOREM OF CALCULUS II

- Let f be integrable on $[a,b]$. For x in $[a,b]$ let $F(x) = \int_a^x f(t)\,dt$. Then $F(x)$ is continuous on $[a,b]$. If f is continuous at x_0 in (a,b), then F is differentiable at x_0 and $F'(x_0) = f(x_0)$

PROOF: If f is integrable, $\Rightarrow$ f is bounded

$\rightarrow |f(x)| < \underset{\text{bound}}{B} \quad \forall x \in [a,b]$

Let $\epsilon > 0$ and $|x-y| < \epsilon/B$ & let $x > y$

$$|F(x) - F(y)| = \left|\int_a^x f\,dt - \int_a^y f\,dt\right| = \left|\int_y^x f\,dt\right|$$

$$\leq \int_y^x |f(t)|\,dt < B(x-y) < \frac{B\epsilon}{B} = \epsilon$$

$\therefore$ given $\epsilon > 0$, $\exists$ $\delta = \frac{\epsilon}{B}$ s.t. $|x-y| < \frac{\epsilon}{B}$

$\Rightarrow |F(x) - F(y)| < \epsilon \Rightarrow F$ is uniformly continuous.

Hence proved

Now we prove the second part ie if f is cont. at x_0, then F is diff. at x_0 and $F'(x_0) = f(x_0)$

using definition of limit (1st principle)

b] $$F'(x_0) = \lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0}$$

now if we prove $F'(x_0) - f(x_0) < \epsilon$, we ~~can~~ will complete the proof.

$\therefore$

$$\frac{F(x) - F(x_0)}{x - x_0} - f(x_0) = \frac{1}{x - x_0}\int_{x_0}^{x} f(t)\,dt - \frac{1}{x - x_0}\int_{x_0}^{x} f(x_0)\,dt$$

$$= \frac{1}{x - x_0}\int_{x_0}^{x} [f(t) - f(x_0)]\,dt$$

given $\epsilon > 0$ $\exists$ δ s.t. $|t - x_0| < \delta$ & $t \in (a, b)$

$$\Rightarrow |f(t) - f(x_0)| < \epsilon \quad \{\text{continuity of } f \text{ at } x_0\}$$

Let $x > x_0$,

$$\left|\frac{F(x) - F(x_0)}{(x - x_0)} - f(x_0)\right| = \frac{1}{x - x_0}\left|\int_{x_0}^{x} (f(t) - f(x_0))\,dt\right|$$

$$\leq \frac{1}{x - x_0}\int_{x_0}^{x} |f(t) - f(x_0)|\,dt < \frac{1}{x - x_0}\int_{x_0}^{x} \epsilon\,dt$$

$$= \epsilon$$

Hence, proved.