Undirected graphs

Simple Graphs

Multi Edged Graphs

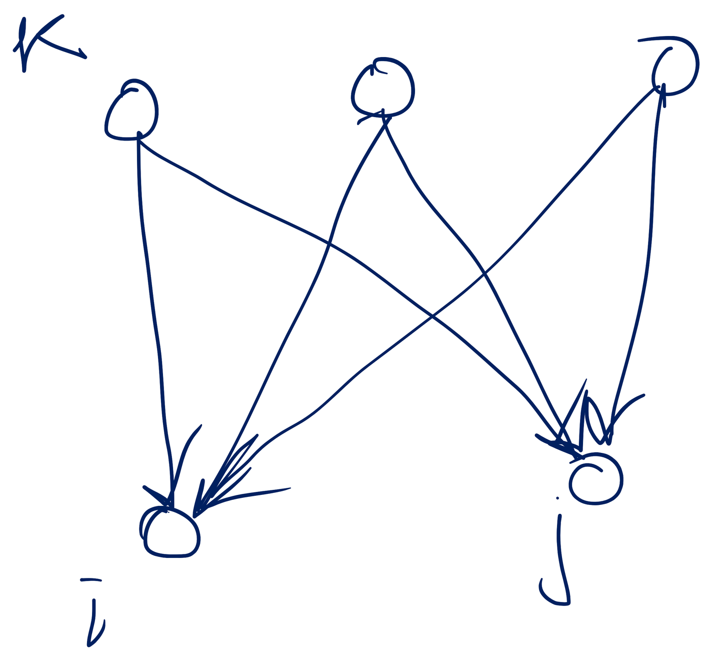
Directed graph

$A \rightarrow$ Adjacency matrix

A_{ij} ~~...~~ $= 1$ if $\exists$ edge j to i

Use the direction information for finding different coupling between nodes.

Co-citation Coupling



Let A represent the adjacency matrix ~~of the~~ of the network.

$A_{ik} = 1$

$A_{ik} A_{jk} = 1$

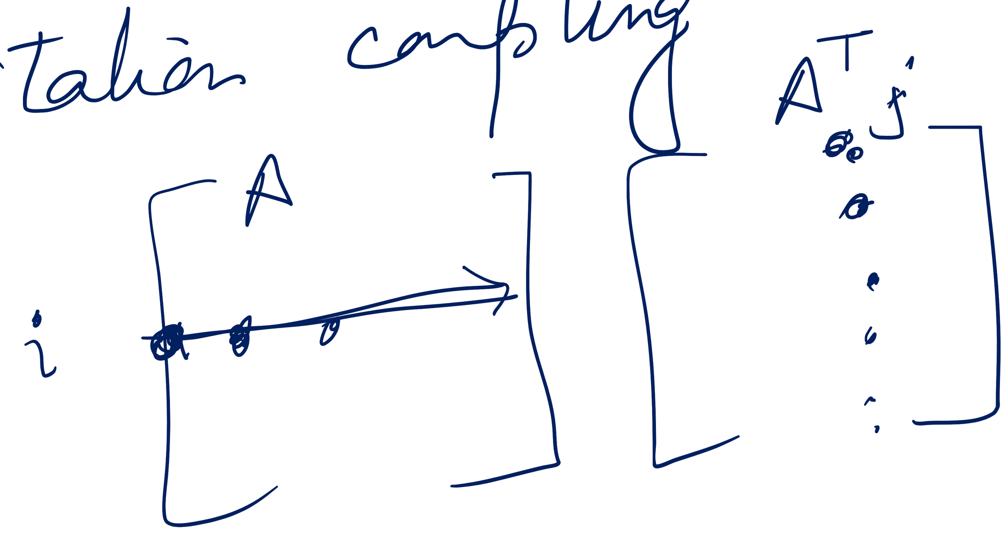
Total co-citations for nodes i & j = $\sum_{k=1}^{n} A_{ik} A_{jk}$

$$c_{ij} = \sum_{k=1}^{n} A_{ik} A_{jk}$$

Let C be a matrix of the ~~Co~~ Co-citation coupling

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & \cdots \\ \vdots & & & \\ c_{n1} & c_{n2} & \cdots & \end{bmatrix}$$

$$c_{ij} = \sum_{k=1}^{n} [A]_{ik} [A^T]_{kj}$$



$$C = AA^T$$

$$C_{ii} = \sum_{k=1}^{n} A_{ik} A_{ik} = \sum_{k=1}^{n} (A_{ik})^2 = \sum_{k=1}^{n} A_{ik} = K_i^{(in)}$$

(Indegree of node i)

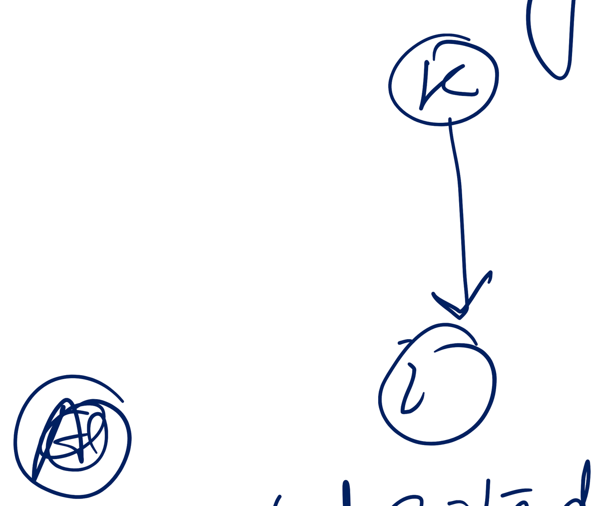
Co-citation on Bibliographic Records

- Indicate the similarity of topics of 2 papers

<u>Problem</u>

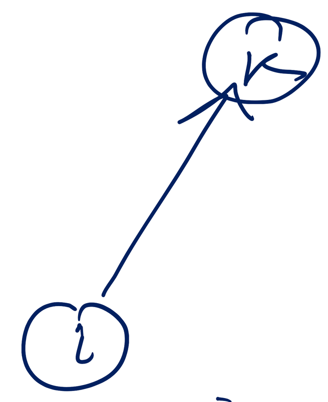
Gaining citation takes time
hence the adjacency matrix A may be very sparse

So rather than 'cited by', consider the 'references' of a paper



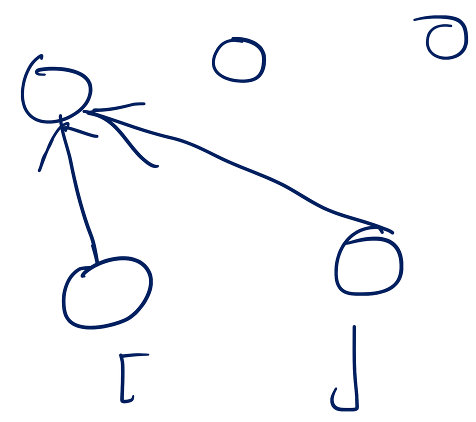
i is ~~cited~~ cited by k

(Co-citation coupling)



i cites k

(Bibliographic Coupling)



Bibliographic Coupling

$$B_{ij} = \sum_{k=1}^{n} A_{ki} A_{kj} = \sum_{k=1}^{n} [A^T]_{ik} A_{kj}$$

$$B = A^T A$$

Both B & C are symmetric matrix, however they indicate very different measures of similarity.

$$B_{ii} = \sum_{k=1}^{n} A_{ki} A_{ki} = \sum_{k=1}^{n} A_{ki} = K_i^{out} \text{ (outdegree)}$$

Problem 1 in slide

$$A = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 & 0 \\ 4 & 1 & 1 & 1 & 0 & 0 \\ 5 & 1 & 1 & 0 & 0 & 0 \end{array}$$

$C = AA^T$

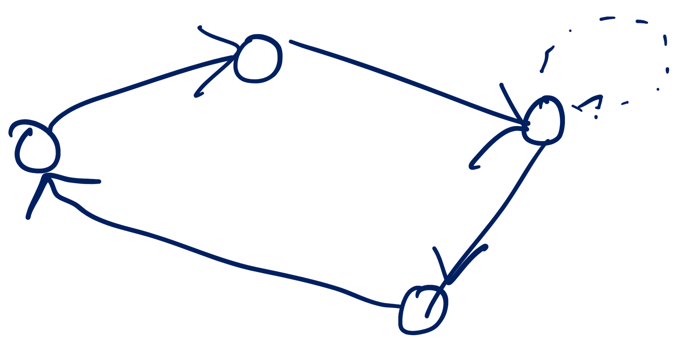
find Thresholding.

find $m = \frac{1}{n^2} \sum C_{ij}$ & find $\sigma = \sqrt{\frac{1}{n^2}(C_{ij} - m)^2}$

Test for values of $C_{ij} \geq (m + \sigma)$. For values of $C_{ij} \geq m + \sigma$ treat them as papers ~~with~~ on similar topics.

Acyclic Directed Networks.

Cycles :- Closed Loops of edges



- Directed Networks without cycles are Acyclic Directed Networks.

Important property of Acyclic Networks

→ You can draw all the edges pointing downwards

$$\begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & \cdots \\ \hline 1 & 0 & 0 & & & & \\ 2 & 0 & 0 & & & & \\ 3 & 0 & 0 & & & & \\ 4 & \vdots & \vdots & & & & \\ 5 & & & & & & \\ 6 & & & & & & \\ 7 & & & & & & \\ 8 & & & & & & \\ 9 & 0 & 0 & & & & \end{array}$$

~~All~~ Some entries may be 1.

A_{ij}, $i > j$

All Entries below diagonal = 0

$A_{ij} = 0$ if $i > j$

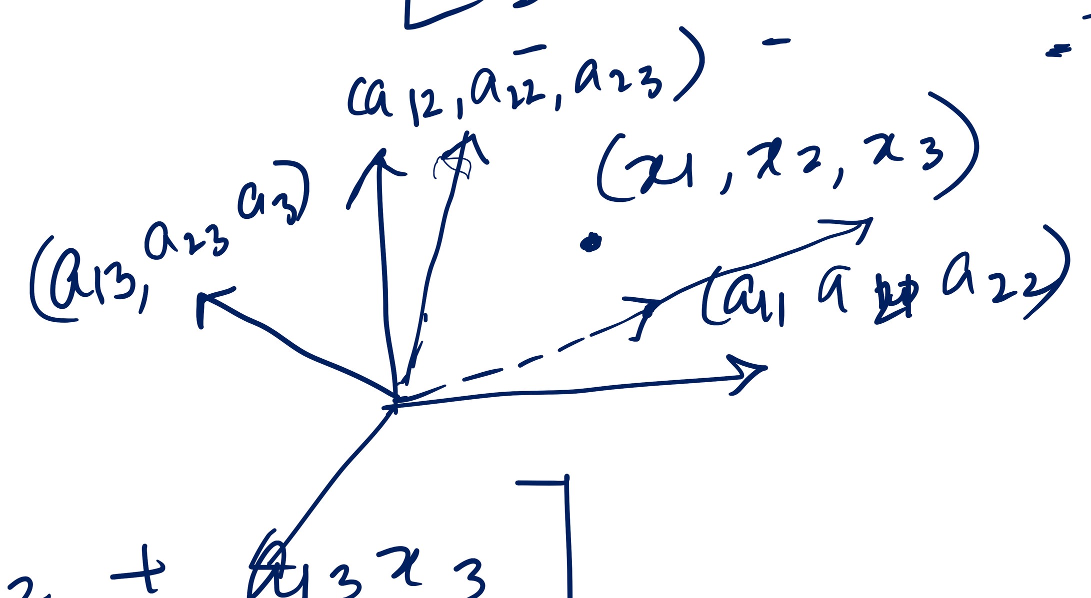
- All of the Eigen values of an adjacency matrix = 0 iff the network is acyclic.

<u>Eigen Value</u>

$A\vec{x}$

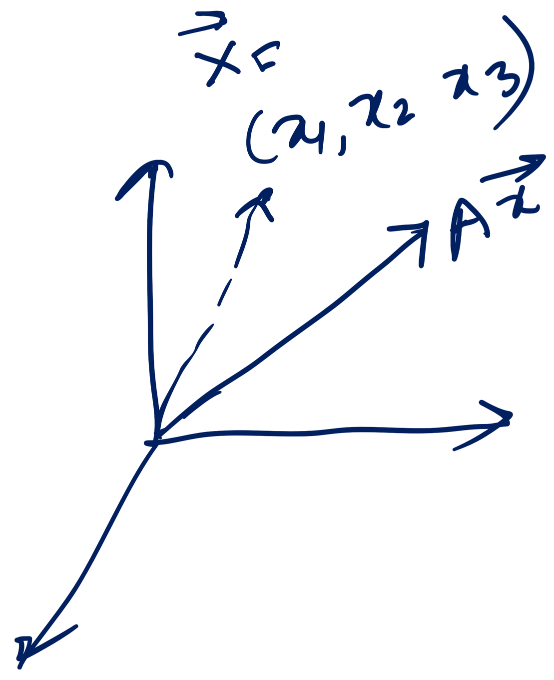
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

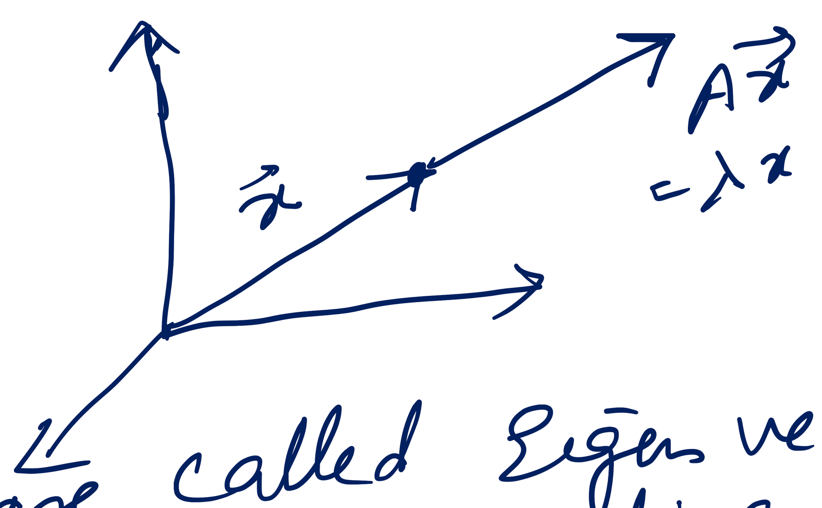


$$A\vec{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{22} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix}$$



$$A\vec{x} = \lambda\vec{x}$$



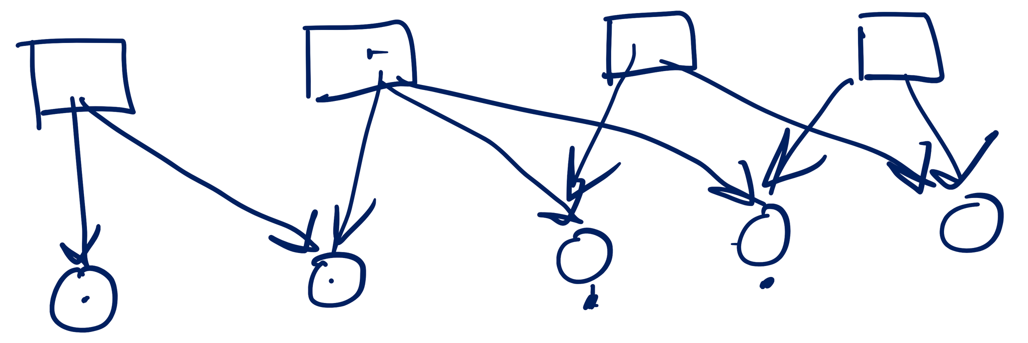
Those vectors $\vec{x}$ that satisfy the condition are called Eigen vectors of A & the corresponding λ are the Eigen values

Bipartite Network

→ 2 mode Networks

(actor)

(groups)



2 types of vertices

Edges run across the types.

People → organization

or Actor – Movies

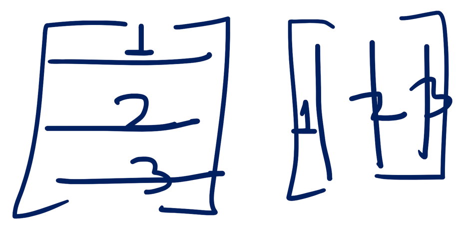
Bipartite network is represented using incidence matrix B

If there are n number of actors and g groups

$$B_{ij} = \begin{cases} 1 & \text{if vertex } j \text{ belongs to group } i \\ 0 & \text{otherwise.} \end{cases} \qquad B \in \mathbb{R}^{g \times n}$$

One mode projection

Use the bipartite network to infer connections between nodes of same type.



Mathematical representation of one mode projection

Two actors i & j will belong to group k iff $B_{ki} B_{kj} = 1$

So the total no. of groups to which i & j belong $= \sum_{k=1}^{g} B_{ki} B_{kj}$

$$= \sum [B^T]_{ik} B_{kj} = P_{ij}$$

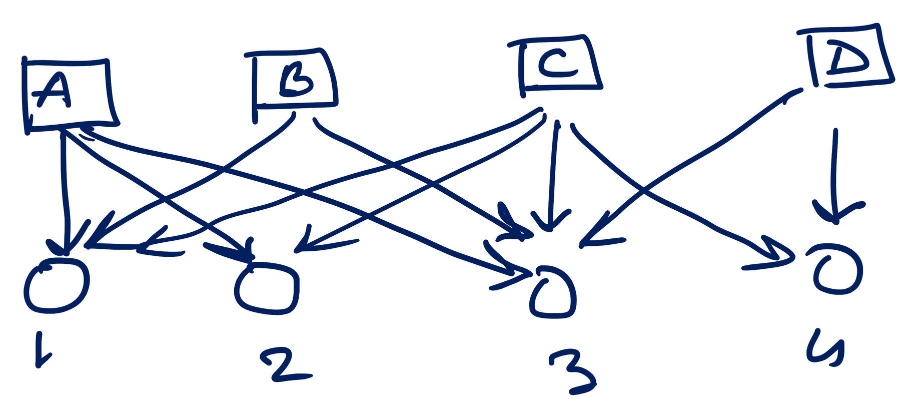
So the projection matrix

$$P = B^T B$$

Problem 2 in slide

Users

Items



$B_{ij} = 1$ if user j has purchased item i

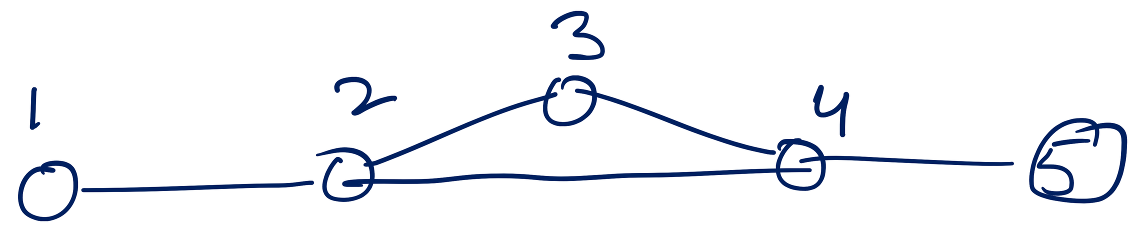
$$\boxed{P = B^T B}$$

$P_{ij} = 1$ if $B_{ki} B_{kj} = 1$

Trees

~~a~~ Mathematical expressions for degrees, indegree, outdegree & average degree of nodes

Walks in a graph.



1) Walk – A sequence of vertices and edges of a graph, i.e, if we traverse a graph, we get a walk. Both vertices and edges can be repeated.

$$1 \to 2 \to 1 \to 2 \to 3 \to 4 \to 2 \to 4$$

(open & closed walks)

2) Trail – Trail is an open walk, where no edges are repeated. Vertices can be repeated

$1 \to 2 \to 3 \to 2 \to 4$ is NOT a trail

$1 \to 2 \to 3 \to 4 \to 2$ is a trail

3) Circuit – A closed trail is a circuit.

$$2 \to 3 \to 4 \to 2$$

4) ~~Tra~~ Path – A trail where neither vertices nor edges are repeated.

$$1 \to 2 \to 3 \to 4 \to 5$$

Walks of length 2

$$N_{ij}^{(2)} = \sum_{k} A_{ik} A_{kj} \qquad N^{(2)} = A^2$$

Walk of length 3

$$N_{ij}^{(3)} = \sum_{k=1}^{n} A_{ik}^{(2)} A_{kj} = [A^3]_{ij}$$

Generalizing

$$N_{ij}^{(r)} = \sum_{k=1}^{n} A_{ik}^{(r)} A_{kj} = [A^r]_{ij}$$

What about $[A^r]_{ii}$? $\to$ Cycles of length r for node i

Total # of cycles of length $r = \sum_i [A^r]_{ii} = Tr[A^r]$

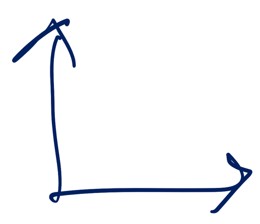
Sum of the diagonal elements. $\Sigma = Tr(A)$

~~B~~ For undirected networks A is symmetric

It will have n Eigen vectors

$A\vec{x} = \lambda \vec{x}$ Here $A \in \{0,1\}^{n\times n}$

There exists n Eigen vectors with n real Eigen values



A can be decomposed as UKU^T

$$U = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix}$$ ← matrix of Eigen vectors of A.

$$K = \begin{bmatrix} \lambda_1 & 0 & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$ Diagonal matrix of Eigen values.

U is orthogonal

Any 2 column vectors of U

$X_i \quad X_j$

$X_i \cdot X_j = 0$

$\vec{X}_1 = \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ \vdots \end{bmatrix}$ $\vec{X}_2 = \begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \\ \vdots \\ x_n^{(2)} \end{bmatrix}$

Then $\vec{X}_1 \cdot \vec{X}_2 = 0$

$= x_1^{(1)} x_1^{(2)} + x_2^{(1)} \cdot x_2^{(2)} + \cdots + x_n^{(1)} x_n^{(2)}$

$= 0$

$$A = UKU^T$$

$$A^r = (UKU^T)^r = U^r K^r (U^T)^r$$

$$U^r = U$$

$$A^r = U K^r U^T$$

$$Tr(A^r) = Tr(U K^r U^T) = Tr(U^T U K^r)$$

Trace is invariant under cyclic permutation.

$$Tr(ABC) = Tr(CAB) = Tr(BCA)$$

$X_i \cdot X_i = X_i$

$|X_i|\,|X_i| \cos\theta$

$= |X_i|\,|X_i|$

If $|X_i| = 1$

$X_i \cdot X_i = 1$

$$U^T U = I$$

Then $Tr(A^r) = Tr(K^r) = \sum_{i=1}^{n} \lambda_i^r$

Sum of the ~~Eigen~~ r^{th} power of the Eigen values of A.

Condition for Hamiltonian Cycle (Travelling Salesperson Prob)

(Ore theorem) :- If G is a simple graph with n vertices $(n > 3)$ and for any non adjacent vertices v & w, ~~the~~ the sum of the degrees $d(v) + d(w) \geq n$, then G has a Hamiltonian cycle

for Hamiltonian path
$$d(v) + d(w) \geq n - 1$$