13/02/2020

# Normal Distribution

A RV $x$ is said to follow Normal Distribution :- if

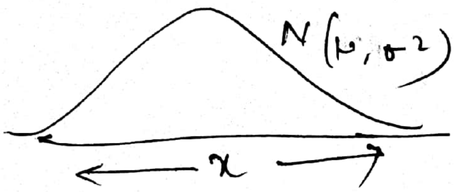
$$f_x(x, \mu, \sigma) = \frac{1 \times e^{-\frac{1}{2\sigma^2}(x-\mu)^2}}{\sqrt{2\pi}\,\sigma}, \quad \begin{array}{l} -\infty < x < \infty \\ -\infty < \mu < \infty \\ 0 < \sigma < \infty \end{array}$$

$$\boxed{X \sim N(\mu, \sigma^2)}$$

$\mu$ - location parameter.

$\sigma$ - scale parameter.

- No shape parameter. Shape of a Normal Distribution is bell-shaped.




Probably the most widely used distribution in literature.

Checking correctness of $f_x(x)$ ⇒

$$\frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}\, dx$$

In general use substitution $\frac{x-\mu}{\sigma} = z$.

$\boxed{dx = \sigma\, dz}$, $z \in (-\infty, \infty)$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}\, dz$$

$$= \frac{2}{\sqrt{2\pi}\,\sigma} \int_{0}^{\infty} e^{-z^2/2}\, dz.$$

Put $z^2/2 = u$ ... continuous R.V, then

(If range is $0 \to \infty$ and there is some exp term in Integral - then try to convert it to $\Gamma$ (Gamma func).

→ Now :- $z\,dz = du$. $\Rightarrow dz = \frac{du}{\sqrt{2u}}$

$$= \frac{2\,\cancel{\sigma}}{\sqrt{2\pi}\,\cancel{\sigma}} \int_0^\infty \frac{e^{-u}\,du}{\sqrt{2u}}$$

$$= \frac{\sqrt{2}\,\cancel{\sigma}}{\sqrt{2\pi}\,\cancel{\sigma}} \int_0^\infty u^{-\frac{1}{2}} e^{-u}\,du.$$

$$= \frac{\sqrt{2}\,\cancel{\sigma}}{\sqrt{2\pi}\,\cancel{\sigma}} \Gamma\tfrac{1}{2} = \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1$$

$$E(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} x\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

→ Put $\frac{x-\mu}{\sigma} = z \Rightarrow dx = dz\,\sigma$

and $x = \sigma\left(z + \frac{\mu}{\sigma}\right)$. ; Now

$$E(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} \sigma\left(z + \frac{\mu}{\sigma}\right) e^{-\frac{z^2}{2}}\,dz$$

$$= \frac{\sigma}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} z e^{-z^2/2}\,dz}_{0} + \mu\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2}\,dz\right)$$

$$E(x) = \mu \times 1 \Rightarrow \boxed{E(x) = \mu}$$

$$E(x^2) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} x^2 \, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \, dx$$

put $\frac{x-\mu}{\sigma} = z \Rightarrow dx = \sigma \, dz$

$$E(x^2) = \frac{1}{\sqrt{2\pi}\,\cancel{\sigma}} \int_{-\infty}^{\infty} (\sigma z + \mu)^2 \, e^{-\frac{z^2}{2}} \, (\cancel{\sigma} \, dz)$$

$\vdots$

$$= \sigma^2 + \mu^2$$

$$\therefore V(x) = E(x^2) - (E(x))^2 = \sigma^2 + \mu^2 - \mu^2$$

$$= \sigma^2$$

Note: Normal distribution is symmetric about point $\mu$,

And

$$\beta_1 = \frac{E(x - E(x))^3}{\sigma^3} = \frac{0}{\sigma^3} = 0.$$

All odd-order central moments is 0, for normal distribution

→ Moment Generating function.

$$M_X(t) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{tx} \, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \, dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x^2 - 2\mu x + \mu^2)} \, e^{tx} \, dx$$

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Let X be a continuous R.V, then

$$= \frac{e^{-\frac{\mu^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma} \cdot \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x^2 - 2\mu x - 2\sigma^2 t x)} dx$$

Now, we have :-

$$x^2 - 2x(\mu + \sigma^2 t) + (\mu + \sigma^2 t)^2 - (\mu + \sigma^2 t)^2$$

$$\cancel{x - \mu t} = \left(x - (\mu + \sigma^2 t)\right)^2 - (\mu + \sigma^2 t)^2$$

Substituting back :-

$$= e^{-\frac{\mu^2}{2\sigma^2}} e^{\frac{(\mu + \sigma^2 t)^2}{2\sigma^2}} \underbrace{\left[\frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x - (\mu + \sigma^2 t))^2} dx\right]}_{\text{will become } 1}$$

put $\frac{x - (\mu + \sigma^2 t)}{\sigma} = z$. if this substitution is put

then,

$$= e^{-\frac{\mu^2}{2\sigma^2}} \cdot e^{\frac{(\mu + \sigma^2 t)^2}{2\sigma^2}}$$

$$= e^{\frac{-\mu^2 + \sigma^4 t^2 + \mu^2 + 2\mu\sigma^2 t}{2\sigma^2}}$$

$$M_X(t) = e^{\frac{1}{2}(\sigma^2 t + 2\mu t)} = \underline{e^{\mu t + \frac{\sigma^2 t^2}{2}}}$$

Hence

$$\boxed{M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}}.$$

Ex : If $M_X(t) = e^{2t + t^2}$ then $\underline{X \sim N(2, 2)}$.

## Mode :-

Solve $f_X'(x) = 0$ for $x$

to get $\boxed{x = \mu}$

Note :- For continous symmetric distribution — mean, median, mode are all same.

CDF of $N(\mu, \sigma^2)$.

$$F_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{x} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2} dt$$

Note $\Rightarrow$ put $\frac{t-\mu}{\sigma} = z$.

$$= \frac{\sigma}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\frac{x-\mu}{\sigma}} e^{-\frac{1}{2}z^2} dz$$

$$= \int_{-\infty}^{\frac{x-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$= \Phi\left(\frac{x-\mu}{\sigma}\right)$$

$$F_X(x) = \int_{-\infty}^{\frac{x-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$= \Phi\left(\frac{x-\mu}{\sigma}\right).$$

Let $X$ be a continuous R.V, then

where $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-v^2/2} \, dv = \int_{-\infty}^{z} \phi(v) \, dv$

$\Rightarrow$ ~~$\Phi(z) \Rightarrow$~~ $\phi(v) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2}, \quad v \in R$

In $\phi(v) = f_X(v, 0, 1) \; (\underset{\text{mean}}{\mu = 0}, \; \underset{\text{variance}}{\sigma = 1})$.

$\rightarrow$ Now, $$Z = \frac{X - E(X)}{\sqrt{V(X)}}, \quad X \sim N(\mu, \sigma^2)$$

$$E(Z) = E\left(\frac{X - E(X)}{\sqrt{V(X)}}\right) = 0$$

$$V(Z) = V\left(\frac{X - E(X)}{\sqrt{V(X)}}\right) = \frac{1}{V(X)} V(X) = 1$$

$Z$ is called as a standard variable of $X$ ($\because$ of $X$ mean $(Z)$ or $E(Z) = 0$; and $V(Z) = 1$).

$$Z = \frac{X - \mu}{\sigma}, \quad X \sim N(\mu, \sigma^2)$$

then $Z$ is standard normal variable.

$Z \sim N(0, 1)$ and pdf $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad z \in R$

17/02/2020

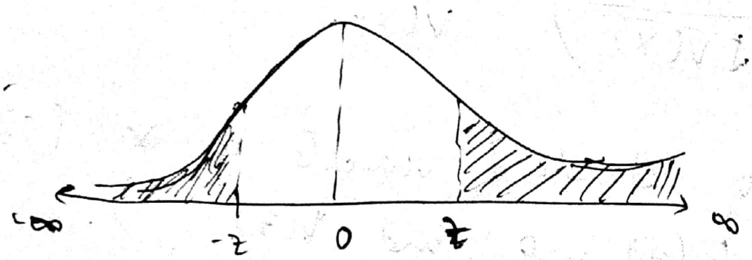
Some properties of Standard Normal Distribution:-

(i) $P(z \leq -z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-z} e^{-t^2/2}\, dt$

$= \frac{1}{\sqrt{2\pi}} \int_{z}^{\infty} e^{-t^2/2}\, dt$

$= P(z \geq z) = 1 - \Phi(z)$

$\Phi(-z) = 1 - \Phi(z) \quad \forall\, z \in R$




$\Phi(0) = 1/2$ - True.

Now, Median = ? For median

$\underbrace{\Phi\left(\frac{x-\mu}{\sigma}\right)}_{\text{any normal } (\mu, \sigma^2)} = \frac{1}{2}$ ; but we know only $\Phi(0) = \frac{1}{2}$

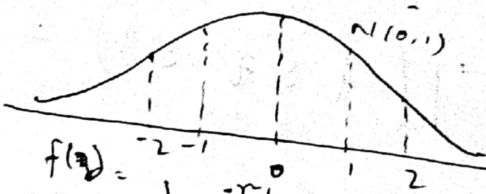
hence $\frac{x-\mu}{\sigma} = 0 \Rightarrow \boxed{x = \mu}$

'$\mu$' is the mean, median, mode.

Let $X$ be a continuous R.V, then

Area Property of $N(0,1)$.

$$P(-1 \le z \le 1) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-z^2/2} \, dz$$




$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

$$P(-1 \le z \le 1) = \Phi(1) - \Phi(-1) = 0.6826$$

$$P(-2 \le z \le 2) = \Phi(2) - \Phi(-2) = 0.9544.$$

$$P(-3 \le z \le 3) = \Phi(+3) - \Phi(-3) = 0.9974.$$

→ For $N(\mu, \sigma^2)$.

$$P(a \le X \le b) = P\left(\frac{a-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{b-\mu}{\sigma}\right).$$

$$= P\left(\frac{a-\mu}{\sigma} \le z \le \frac{b-\mu}{\sigma}\right)$$ standard normal.

$$= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right).$$

Example :- $X \sim N(-1, 3)$

$$P(-2 \le X \le 1)$$

$$= P\left(\frac{-2+1}{\sqrt{3}} \le z \le \frac{2}{\sqrt{3}}\right) \qquad z = \frac{x+1}{\sqrt{3}}$$

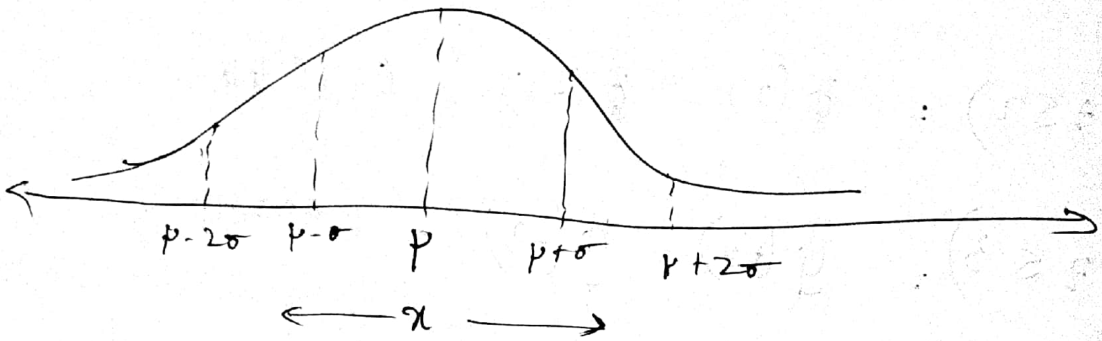
$$= \Phi\left(\frac{2}{\sqrt{3}}\right) - \Phi\left(\frac{-1}{\sqrt{3}}\right).$$ (Compute this from the table).

$$P(-1 \le z \le 1) = P\left(-1 \le \frac{X-\mu}{\sigma} \le 1\right)$$

$$\Rightarrow P(\mu-\sigma < X \le \mu+\sigma) = 0.6826.$$

Also

$$P(\mu-2\sigma < X \le \mu+2\sigma) = 0.9544$$




In table given by sir :-

for computing $P(-a \le z \le a) = \Phi(a) - \Phi(-a)$

$$= \Phi(a) - (1 - \Phi(a)) = \underline{\underline{2\Phi(a) - 1}}$$

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Log Normal Distribution :-

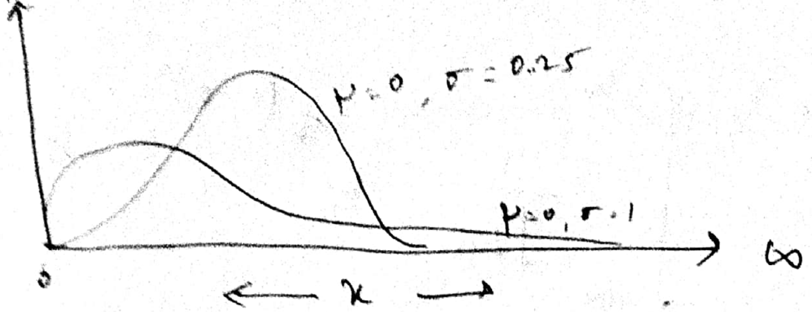
A RV X is reffered as lognormal provided $\ln(X)$ is normally distributed.

The pdf of X is

$$f_X(x, \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma x} \times e^{-\frac{1}{2\sigma^2}(\ln(x)-\mu)^2}, \quad x > \ldots,\ \mu \in \ldots,\ \sigma > \ldots$$

$$F_X(x) = \Phi\left(\frac{\ln x - \mu}{\sigma}\right).$$

MGF doesn't exist for a lognormal variable.




$$(X) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_0^\infty x\left(\frac{1}{x}\right) e^{-\frac{1}{2\sigma^2}(\ln(x)-\mu)^2}\, dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_0^\infty e^{-\frac{1}{2\sigma^2}(\ln(x)-\mu)^2}\, dx$$

put $\frac{\ln x - \mu}{\sigma} = t \Rightarrow \frac{1}{x}\frac{dx}{\sigma} = dt$.

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^\infty e^{-\frac{t^2}{2}}\, \sigma \left(e^{\sigma t + \mu}\right) dt$$

$$= \frac{e^\mu}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^\infty e^{-t^2/2}\, e^{\sigma t}\, dt.$$

$$= \frac{e^\mu}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^\infty e^{\frac{-t^2 + 2\sigma t}{2}}\, dt.$$

$$= \frac{e^\mu}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^\infty e^{\left(-\frac{1}{2}\right)\left(t^2 + \sigma^2 - \sigma^2 - 2\sigma t\right)}\, dt$$

$$= \frac{e^\mu}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^\infty e^{-\frac{1}{2}\left((t-\sigma)^2 - \sigma^2\right)}\, dt$$

$$= e^{\mu + \frac{\sigma^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(t-\sigma)^2} dt$$

$$= E(X) = e^{\mu + \frac{\sigma^2}{2}}$$

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Alternative approach :-

$X \sim LN(\mu, \sigma^2)$

$Y = LN(X) \sim N(\mu, \sigma^2)$

$X = e^{Y}$

$$E(X^k) = E(e^{Yk}) = e^{k\mu + \frac{1}{2}\sigma^2 k^2}$$

↳ This is like MGF of normal distrib

$V(X) = E(X^2) - E(X)^2$

Mode $(X)$ :- Solve $\frac{d}{dx} f_X(x) = 0$ for log normal.

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{\sqrt{2\pi}\,\sigma} \frac{1}{x} e^{-\frac{1}{2\sigma^2}(\ln(x)-\mu)^2}\right)$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma}\left[-\frac{1}{x^2} e^{-\frac{1}{2\sigma^2}(\ln(x)-\mu)^2} + \frac{e^{-\frac{1}{2\sigma^2}(\ln(x)-\mu)^2}}{x^2} - \frac{1}{\sigma^2}(\ln\right.$$

$$\Rightarrow \boxed{x = e^{\mu - \sigma^2}}$$

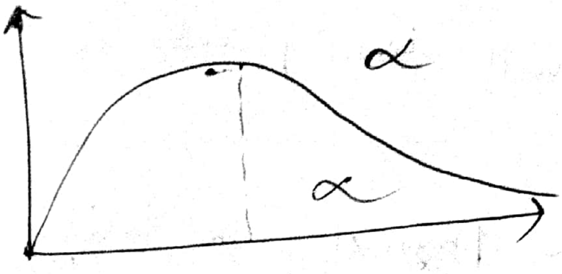
Median $(x)$:

$$\Phi\left(\frac{\ln(x)-\mu}{\sigma}\right) = \frac{1}{2} \Rightarrow x = e^{\mu}$$

$\| \; 0 \Rightarrow \ln(x) - \mu = 0 \Rightarrow$

Mean $= e^{\mu+\sigma^2/2}$ ; Mode : $e^{\mu-\sigma^2}$ ; Median : $e^{\mu}$

Now,





Mode < Median < ~~Mode~~ Mean. (This is true in general for any positively skewed distribution).

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Ex: Let $X \sim N(\mu, \mu^2)$ $\mu > 0$.
Write the probability $P(X < -\mu \mid X < \mu)$ in term of CDF of a standard normal variable.

Soln: Convert normal to standard normal form

$$Z = \frac{X-\mu}{\sigma} = \frac{X-\mu}{\mu}.$$

$$P(X<-\mu \mid X<\mu) = \frac{P(X<-\mu)}{P(X<\mu)} = \frac{P\left(\frac{X-\mu}{\mu} < \frac{-\mu-\mu}{\mu}\right)}{P\left(\frac{X-\mu}{\mu} < \frac{\mu-\mu}{\mu}\right)}$$

$$= \frac{P(Z<-2)}{P(Z<0)} = \frac{\Phi(-2)}{\Phi(0)} = \frac{\Phi(-2)}{\left(\frac{1}{2}\right)}$$

$2\Phi(-z) = 2(1-\Phi(z))$ ; $\Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-z^2/2}\,dz$

$$\Phi(z) = \int_{-\infty}^{0} \text{---} + \frac{1}{\sqrt{2\pi}}\int_{0}^{z} e^{-z^2/2}\,dz$$

$$= \frac{1}{2} + P(0 \le Z \le z).$$

- Table given in example will have probabs of the form

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## Weibull Distn (Swedish Physicist Wal. Weibull (1939))

$$f_X(x) = \frac{\alpha}{\beta^{\alpha}}\, x^{\alpha-1}\, e^{-(x/\beta)^{\alpha}}, \quad x > 0,\ \alpha > 0,\ \beta > 0.$$

$$X \sim \text{Weibull}(\alpha, \beta).$$

$$E(X) = \beta\,\Gamma\!\left(1+\frac{1}{\alpha}\right) ; \quad V(X) = \beta^2\left\{\Gamma\!\left(1+\frac{2}{\alpha}\right) - \left(\Gamma\!\left(1+\frac{1}{\alpha}\right)\right.\right.$$

$\Gamma \to$ Gamma func

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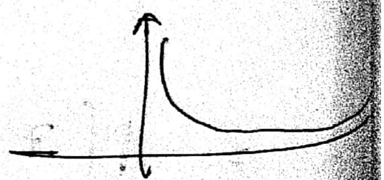
## Pareto Distn

$$f_X(x) = \frac{\alpha\,\beta^{\alpha}}{x^{\alpha+1}}, \quad x \ge \beta,\ \alpha > 0,\ \beta > 0$$

$$F_X(x) = 1 - \left(\frac{\beta}{x}\right)^{\alpha}$$



$$E(X^n) = \beta^n\,\frac{\alpha}{\alpha-n}, \quad n < \alpha.$$