MA 201 (Part II)
Partial Differential Equations
Session July-Nov, 2014
Solutions to Tutorial Problems - 7

Topics: Orthogonal surface, 1st order non-linear PDE
Classification of 2nd order PDEs, Canonical/Normal forms, The wave equation: Infinite string problem

1. Find the surface which is orthogonal to the one-parameter system

$$u = cxy(x^2 + y^2)$$

and which passes through the hyperbola $x^2 - y^2 = a^2$, $u = 0$.

Solution: Write $f(x, y, u) \equiv \dfrac{u}{xy(x^2 + y^2)} = c$ so that

$$\frac{\partial f}{\partial x} = -\frac{u}{y}\frac{3x^2 + y^2}{x^2(x^2 + y^2)^2}; \quad \frac{\partial f}{\partial y} = -\frac{u}{x}\frac{3y^2 + x^2}{y^2(x^2 + y^2)^2}; \quad \frac{\partial f}{\partial u} = \frac{1}{xy(x^2 + y^2)}.$$

The auxiliary equations can be written and adjusted as

$$\frac{x(x^2 + y^2)dx}{3x^2 + y^2} = \frac{y(x^2 + y^2)dy}{3y^2 + x^2} = -udu.$$

Adding first and second and equating to third,

$$\frac{(x^2 + y^2)(xdx + ydy)}{4(x^2 + y^2)} = -udu.$$

giving $x^2 + y^2 + 4u^2 = c_1$. Similarly, on subtraction and using $x^2 + y^2 = c_1 - 4u^2$:

$$\frac{(x^2 + y^2)(xdx - ydy)}{2(x^2 - y^2)} = -udu, \quad \Rightarrow x^2 - y^2 = c_2\sqrt{c_1 - 4u^2}.$$

Using the given conditions: $a^4 = c_2^2 c_1$ which ultimately gives
$(x^2 + y^2)a^4 = (x^2 + y^2 + 4u^2)(x^2 - y^2)^2$.

2. Show that the equations $xp - yq = x$ and $x^2p + q = xu$ are compatible, and find a one–parameter family of common solutions.

Solution: Check the compatibility condition for first part. Then from the given equations find

$$p = \left(1 + \frac{y(u - x)}{1 + xy}\right), \quad q = \frac{xu - x^2}{1 + xy}.$$

From the integrability condition $du = pdx + qdy$, get $du = dx + \dfrac{(u - x)(ydx + xdy)}{1 + xy}$ which on integration gives $u = x + c(1 + xy)$.

3. Determine the general solution of the following non-linear partial differential equations:

$$\text{(i) } p^2 + qy - u = 0; \quad \text{(ii) } (p^2 + q^2)y - qu = 0$$

Solution: (i) $u = ay + \dfrac{1}{4}(x - b)^2$, (ii) $u^2 = a^2y^2 + (ax + b)^2$.

4. Solve the following partial differential equations:

$$\text{(i) } pq + p + q = 0; \quad \text{(ii) } u = px + qy + \sqrt{1 + p^2 + q^2};$$

$$\text{(iii) } upq - p - q = 0; \quad \text{(iv) } p^2 + q^2 = x + y; \quad \text{(v) } p^2 y(1 + x^2) = qx^2.$$

Solution: (i) $u = ax - \dfrac{a}{a+1}y + b$, (Eqn independent of x, y, z)

(ii) $u = ax + by + \sqrt{1 + a^2 + b^2}$, (Eqn Clairaut's type)

(iii) $u^2 = \dfrac{2(1+a)}{a}[ax + y] + b$, (Eqn not containing x, y)

(iv) $u = \dfrac{2}{3}(x+a)^{3/2} + \dfrac{2}{3}(y-a)^{3/2} + b$, (Separable equations)

(v) $u = \sqrt{a(1+x^2)} + \dfrac{1}{2}ay^2 + b$, (Separable equations)

5. Find an integral surface of

$$y(u_x^2 - u_y^2) + uu_y = 0$$

containing the initial curve

$$u = 3t, \quad \text{on } x = 2t, \quad y = t.$$

Solution: Here, $f = y(p^2 - q^2) + uq$, so that

$$f_x = 0, \; f_y = p^2 - q^2, \quad f_u = q, \quad f_p = 2yp, \quad f_q = -2yq + u.$$

Thus, Charpit's equations are given by

$$\frac{dx}{2yp} = \frac{dy}{-2yq + u} = \frac{du}{2yp^2 + q(u - 2yq)} = \frac{dp}{-pq} = \frac{dq}{-p^2}.$$

Last pair of terms gives

$$p^2 - q^2 = a. \tag{1}$$

This together with given equation, we obtain

$$ay + uq = 0. \tag{2}$$

Solving for p and q, we obtain

$$q = -\frac{ay}{u} \;\; \& \;\; p = \pm\sqrt{a + \frac{a^2y^2}{u^2}}.$$

This leads to

$$\frac{udu + aydy}{\sqrt{u^2 + ay^2}} = \pm\sqrt{a}dx.$$

Integrating, we obtain

$$\sqrt{u^2 + ay^2} = \pm\sqrt{a}x \pm b \;\text{ or }\; u^2 = (b + \sqrt{a}x)^2 - ay^2.$$

Then apply initial data to have $b = 0$ and $a = 3$. An integral surface is

$$u^2 = 3(x^2 - y^2).$$

6. Classify the following second-order partial differential equations:
(i) $u_{xx} + 4u_{xy} + 4u_{yy} - 12u_y + 7u = x^2 + y^2$; (ii) $u_{xx} + 4u_{xy} + (x^2 + 4y^2)u_{yy} = \sin(x + y)$
(iii) $(x+1)u_{xx} - 2(x+2)u_{xy} + (x+3)u_{yy} = 0$; (iv) $yu_{xx} + (x+y)u_{xy} + xu_{yy} = 0$.

Solution: (i) Parabolic, (ii) Parabolic on the ellipse $\dfrac{x^2}{4} + y^2 = 1$, hyperbolic inside the ellipse and elliptic outside the ellipse, (iii) Hyperbolic, (iv) hyperbolic if $x \neq y$, parabolic for $x = y$.

7. Reduce the following equations to canonical form and hence solve them:
(i) $u_{xx} + 4u_{xy} + 3u_{yy} = 0$; (ii) $4u_{xx} - 12u_{xy} + 9u_{yy} = e^{3x+2y}$,
(iii) $u_{xx} + 2u_{xy} + u_{yy} = x^2 + 3\sin(x - 4y)$.

Solution: (i) Equation is hyperbolic. Characteristics are given by $\xi = 3x - y$ and $\eta = x - y$. Canonical form is $u_{\xi\eta} = 0$. The solution is $u = f(3x - y) + g(x - y)$.
(ii) Equation is parabolic. Characteristics are given by $\xi = x$ and $\eta = 2y + 3x$. Canonical form is $u_{\xi\xi} = \frac{1}{4}e^{\eta}$. The solution is $u = \frac{x^2}{8}e^{3x+2y} + xf(3x + 2y) + g(3x + 2y)$.
(iii) Equation is parabolic. Characteristics are given by $\xi = y$ and $\eta = y - x$. Canonical form is $u_{\xi\xi} = (\xi - \eta)^2 - 3\sin(3\xi + \eta)$. Solution is $u = \frac{x^4}{12} + \frac{1}{3}\sin(x - 4y) + yf(y - x) + g(y - x)$.

8. Find D'Alembert solution of one-dimensional wave equation with the following initial conditions:

$$\text{(a) } u(x,0) = \sin x, \quad u_t(x,0) = 0, \qquad \text{(b) } u(x,0) = \sin x, \quad u_t(x,0) = \cos x.$$

Solution:
(a) $u(x,t) = \sin x \cos ct$, (b) $u(x,t) = \sin x \cos ct + \frac{1}{c}\sin ct \cos x$.

9. A string stretching to infinity in both directions is given the initial displacement

$$\phi(x) = \frac{1}{1 + 4x^2}$$

and released from rest. Find its subsequent motion as a function of x and t.
Solution: Recall D'Alembert solution for one-dimensional wave equation. Here initial displacement $u(x,0) = \phi(x) = \frac{1}{1+4x^2}$ and initial velocity $u_t(x,0) = \psi(x) = 0$. The required expression for $u(x,t)$ is

$$u(x,t) = \frac{1 + 4(x^2 + c^2t^2)}{[1 + 4(x + ct)^2][1 + 4(x - ct)^2]}$$

MA 201 (PART II)
PARTIAL DIFFERENTIAL EQUATIONS
SESSION JULY-NOV, 2016
SOLUTIONS TO TUTORIAL PROBLEMS - 6

Derivation of PDEs, General integrals, Integral surface through given curves, Cauchy problems

1. Find the partial differential equation arising from each of the following surfaces:

   (a) $u = f(x-y)$, (b) $2u = (ax+y)^2 + b$, (c) $\log u = a \log x + \sqrt{1-a^2}\log y + b$,

   (d) $f(x^2+y^2,\, x^2-u^2) = 0$.

   Solution: (a) $u_x + u_y = 0$.
   (Differentiating w.r.t. x gives $u_x = f'(x-y)$ and then w.r.t. y gives $u_y = -f'(x-y)$. Eliminating f' gives the PDE)
   (b) $xu_x + yu_y = (u_y)^2$.
   (Differentiating w.r.t. x gives $u_x = a(ax+y)$ and then w.r.t. y gives $u_y = ax+y$. $\Rightarrow a = u_x/u_y \Rightarrow u_y = (u_x/u_y)x + y$ which gives the PDE.)
   (c) $(u_x)^2x^2 + (u_y)^2y^2 = u^2$.
   (Differentiating w.r.t. x gives $u_x/u = a/x$, $\Rightarrow a = (u_x x/u)$ and then w.r.t. y gives $u_y/u = \sqrt{1-a^2}/y$.
   Putting value of a from the first one into the second one gives the PDE.)
   (d) $u_x y - u_y x = \dfrac{xy}{u}$.
   (Take $x^2+y^2 = u_1, x^2-u^2 = u_2$). With $u_x = p$ and $u_y = q$, we obtain

$$\begin{aligned}
\frac{\partial f}{\partial u_1}\left\{\frac{\partial u_1}{\partial x} + p\frac{\partial u_1}{\partial u}\right\} + \frac{\partial f}{\partial u_2}\left\{\frac{\partial u_2}{\partial x} + p\frac{\partial u_2}{\partial u}\right\} &= 0,\\
\frac{\partial f}{\partial u_1}\left\{\frac{\partial u_1}{\partial y} + q\frac{\partial u_1}{\partial u}\right\} + \frac{\partial f}{\partial u_2}\left\{\frac{\partial u_2}{\partial y} + q\frac{\partial u_2}{\partial u}\right\} &= 0.
\end{aligned}$$

   (Find the partial derivatives as:
   $\frac{\partial u_1}{\partial x} = 2x$, $\frac{\partial u_1}{\partial y} = 2y$, $\frac{\partial u_2}{\partial x} = 2x$, $\frac{\partial u_2}{\partial y} = 0$, $\frac{\partial u_1}{\partial u} = 0$, $\frac{\partial u_2}{\partial u} = -2u$.
   Eliminate the partial derivatives of f by considering

$$\begin{vmatrix} 2x & (2x-2up) \\ 2y & (2y-2uq) \end{vmatrix} = 0$$

   which gives the PDE.

2. Find the general integral of the following partial differential equations, where $u_x = p$ and $u_y = q$.

   (a) $x^2p + y^2q + u^2 = 0$, (b) $(p-q)u = u^2 + (x+y)$, (c) $x^2(y-u)p + y^2(u-x)q = u^2(x-y)$.

   Solution:
   (a) $F(1/x - 1/y, 1/y + 1/u) = 0$.
   (Write the auxiliary equations as $dx/x^2 = dy/y^2 = du/(-u^2)$. Take the first two fractions to get $1/x - 1/y = c_1$ and take the second and third fractions to get $1/y + 1/u = c_2$. [You can also take the first and third fractions.])
   (b) $F(x+y, 2x - \log(u^2+x+y)) = 0$.

(Write the auxiliary equations as $dx/u = dy/(-u) = du/(u^2 + x + y)$. Take the first two fractions to get $x + y = c_1$. Now take the first and third fractions and write $x + y = c_1$ there to get $dx/u = du/(u^2 + c)$.)

(c) $F(1/x + 1/y + 1/u, xyu) = 0$.

(**Step 1:** Write the auxiliary equations as $\frac{dx/x^2}{y-u} = \frac{dy/y^2}{u-x} = \frac{du/u^2}{x-y}$. Then rewrite them as $\frac{(dx/x^2) + (dy/y^2)}{y-x} = \frac{du/u^2}{x-y}$ which gives $\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{du}{u^2} = 0 \Rightarrow 1/x + 1/y + 1/u = c_1$.

Step 2: Rewrite the auxiliary equations as $\frac{dx/x}{x(y-u)} = \frac{dy/y}{y(u-x)} = \frac{du/u}{u(x-y)}$. Then $\frac{(dx/x) + (dy/y)}{u(y-x)} = \frac{du/u}{u(x-y)}$ which gives $\frac{dx}{x} + \frac{dy}{y} + \frac{du}{u} = 0 \Rightarrow \log xyu = \log c_2$.)

These steps give us the solution.

3. Show that the integral surface of the equation $2y(u-3)p + (2x-u)q = y(2x-3)$ that passes through the circle $x^2 + y^2 = 2x$, $u = 0$ is $x^2 + y^2 - u^2 - 2x + 4u = 0$.

Solution: Take the first and third fractions to get the first curve as $x^2 - u^2 - 3x + 6u = C_1$. Then use multipliers $1, 2y, -2$ to write $\frac{dx}{dt} + 2y\frac{dy}{dt} - 2\frac{du}{dt} = 0$ (each fraction is equal to this). This gives $d\{x + y^2 - u\} = 0 \Rightarrow x + y^2 - 2u = c_2$ to get the second curve. Using the given conditions, $x^2 - 3x = c_1$, $x + y^2 = c_2$ adding which we get $x^2 + y^2 - 2x = c_1 + c_2$ and finally the relation $c_1 + c_2 = 0$ which will give the required surface.

4. Find the solution of the following Cauchy problems:

(a) $u_x + u_y = 2, \quad u(x, 0) = x^2$; (b) $5u_x + 2u_y = 0, \quad u(x, 0) = \sin x$.

Solution:(a) The characteristics equations are: $\frac{dx(t,s)}{dt} = 1$, $\frac{dy(t,s)}{dt} = 1$, $\frac{du(t,s)}{dt} = 2$, whose solutions are given by

$$x(t,s) = t + C_1(s), \quad y(t,s) = t + C_2(s), \quad u(t,s) = 2t + C_3(s).$$

Using the parametric initial conditions $x(0, s) = s$, $y(0, s) = 0$, $u(0, s) = s^2$, we obtain

$$x(t,s) = t + s \quad y(t,s) = t \quad u(t,s) = 2t + s^2.$$

Now, writing (t, s) as functions of (x, y), we have $t = y \quad s = x - y$. Thus, the integral surface is given by

$$U(x,y) = u(t(x,y), s(x,y)) = 2y + (x-y)^2.$$

(b) The solution will proceed as in Q.4 (a). The integral surface is given by $U(x, y) = \sin(x - \frac{5}{2}y)$.

5. Show that the Cauchy problem $u_x + u_y = 1, \quad u(x, x) = x$ has infinitely many solution.

Solution: Note that the transversality condition is violated i.e., the Jacobian $J = 0$. Further, the initial curve is a characteristics curve. Therefore, it has infinitely many solution.

6. Consider the PDE $xu_x + yu_y = 4u$, where $x, y \in \mathbb{R}$. Find the characteristics curves for the equation and determine an explicit solution that satisfies $u = 1$ on the circle $x^2 + y^2 = 1$.

Solution: The characteristics equations $x'(t) = x$, $y'(t) = y$, $u'(t) = 4u$ yield the solutions

$$x(t,s) = C_1(s)e^t, \quad y(t,s) = C_2(s)e^t, \quad u(t,s) = C_3(s)e^{4t}.$$

The characteristics curves are given by $\frac{x}{y} = C$. With $x_0(s) = s$, $y_0(s) = (1-s^2)^{1/2}$, $u_0(s) = 1$, it now follows that $U(x, y) = u(t(x, y), s(x, y)) = (x^2 + y^2)^2$, which is the required integral surface.

7. Find a function $u(x,y)$ that solves the Cauchy problem

$$x^2u_x + y^2u_y = u^2, \quad u(x,2x) = x^2, \quad x \in \mathbb{R}.$$

Is the solution defined for all x and y? Check whether the transversality condition holds.

Solution: Solving the characteristics equations: $\frac{dx(t,s)}{dt} = x^2$, $\frac{dy(t,s)}{dt} = y^2$, $\frac{du(t,s)}{dt} = u^2$, we get

$$-\frac{1}{x} = t + C_1(s), \quad -\frac{1}{y} = t + C_2(s), \quad -\frac{1}{u} = t + C_3(s).$$

Using the parametric initial conditions $x(0,s) = s$, $y(0,s) = 2s$, $u(0,s) = s^2$, we obtain

$$\frac{1}{x} = \frac{1}{s} - t, \quad \frac{1}{y} = \frac{1}{2s} - t, \quad \frac{1}{u} = \frac{1}{s^2} - t.$$

Now, writing (t,s) as functions of (x,y), we obtain $s = \frac{xy}{2(y-x)}$ $t = \frac{y-2x}{xy}$. Thus, the integral surface is given by

$$U(x,y) = u(t(x,y), s(x,y)) = \frac{s^2}{1-ts^2} = \frac{x^2y^2}{\{4(y-x)^2 - xy(y-2x)\}}.$$

The solution is not defined on the curve $4(y-x)^2 = xy(y-2x)$ that passes through the origin.

8. Find a function $u(x,y)$ that satisfies the PDE $-yu_x + xu_y = 0$ subject to the side condition $u(x,x^2) = x^3$, $(x > 0)$.

Solution

$$\text{PDE:} \qquad -yu_x + xu_y = 0 \tag{1}$$
$$\text{Side Condition:} \qquad u(x,x^2) = x^3, \quad (x > 0). \tag{2}$$

Step 1. (Finding characteristic curves $(x(t,s), y(t,s), u(t,s))$)

Solve

$$\frac{d}{dt}x(t,s) = -y(t,s), \quad \frac{d}{dt}y(t,s) = x(t,s), \quad \frac{d}{dt}u(t,s) = 0.$$

with initial conditions $x(0,s) = s$, $y(0,s) = s^2$, $u(0,s) = s^3$. The general solution is

$$x(t,s) = c_1(s)\cos(t) + c_2(s)\sin(t), \quad y(t,s) = c_1(s)\sin(t) - c_2(s)\cos(t).$$

Step 2. (Applying IC)

Using ICs, we find that

$$c_1(s) = s, \quad c_2(s) = -s^2,$$

and hence

$$x(t,s) = s\cos(t) - s^2\sin(t) \quad \text{and} \quad y(t,s) = s\sin(t) + s^2\cos(t).$$

Step 3. (Writing the parametric form of the solution)
Note that $c(x,y) = 0$ and $d(x,y) = 0$. Therefore, it follows that

$$d(t,s) = 0, \quad \mu(t,s) = 1.$$

In view of the given initial curve and $u = u(t,s)$, we obtain

$$u(x(0,s), y(0,s)) = u(s,s^2) = g(s) = s^3, \quad u(t,s) = s^3.$$

Thus, the parametric form of the solution of the problem is given by

$$x(t,s) = s\cos(t) - s^2\sin(t), \quad y(t,s) = s\sin(t) + s^2\cos(t), \quad u(t,s) = s^3.$$

Step 4. (Expressing $u(s,t)$ in terms of $U(x,y)$) It is left as an exercise to show that

$$U(x,y) = \frac{1}{\sqrt{8}}\left[-1 + \sqrt{1 + 4(x^2+y^2)}\right]^{3/2}.$$

Assignment 1

Definition:

We say that a function $f : \mathbb{R}^2 \to \mathbb{R}$ of two variable x and y is diff at (x_0, y_0) if

(1) Both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at the point (x_0, y_0),

(2)
$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - f(x_0,y_0) - \left[\frac{\partial f}{\partial x}(x_0,y_0)\right](x-x_0) - \left[\frac{\partial f}{\partial x}(x_0,y_0)\right](y-y_0)}{\left|(x-x_0)^2 + (y-y_0)^2\right|^{1/2}} = 0$$

Q. 1

let's prove that f is diff at $(0,0)$ but partial derivatives are not conti at $(0,0)$. We have

$$\frac{f(x,0) - f(0,0)}{x-0} = \frac{(x^2+0)\sin\frac{1}{x-0} - 0}{x-0} = x \sin\frac{1}{a} \to 0$$

as $x \to 0$, sin stay bdd. Hence $\frac{\partial f}{\partial x}(0,0) = 0$.

Similarly, $\frac{\partial f}{\partial y}(0,0) = 0$.

Now,
$$\frac{f(x,y) - f(0,0) - \frac{\partial f}{\partial x}(0,0)(x-0) - \frac{\partial f}{\partial y}(0,0)(y-0)}{\sqrt{(x-0)^2 + (y-0)^2}}$$

$$= \begin{cases} \dfrac{(x^2+y^2)\sin\frac{1}{x+y}}{\sqrt{x^2+y^2}} & x+y \neq 0 \\ 0 & \text{othe} \end{cases}$$

$\to 0$ when $x \to 0$, $y \to 0$.

So, f is differentiable at $(0,0)$.

on other hand if $x+y \neq 0$,

$$\frac{\partial f}{\partial x}(x,y) = (2x+0)\sin\frac{1}{x+y} + (x^2+y^2)\cos\left(\frac{1}{x+y}\right)\left(-\frac{1}{(x+y)^2}\right)$$

$$\frac{\partial f}{\partial y}(x,y) = (0+2y)\sin\frac{1}{x+y} + (x^2+y^2)\cos\frac{1}{x+y}\left(-\frac{1}{(x+y)^2}\right)$$

taking $x=y$ gives

$$\frac{\partial f}{\partial x}(x,x) = 2x\sin\frac{1}{2x} - \cos\frac{1}{2x}$$

$$\frac{\partial f}{\partial y}(x,x) = 2x\sin\frac{1}{2x} - \cos\frac{1}{2x}$$

Which have no limit as $x\to 0$. thus $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are not contn at $(0,0)$.

Q.2 Directional derivative:

The directional derivative of f at x_0 in the direction $\vec{v}$ is defined by

$$\frac{\partial f}{\partial \vec{v}}(x_0) = \lim_{t\to 0}\frac{f(x_0+t\vec{v})-f(x_0)}{t},$$ provided

the limit exists in $\mathbb{R}$.

In the special case in which $\vec{v}=e_i$ the directional derivative $\frac{\partial f}{\partial e_i}(x_0)$ is called partial derivative of f with respect to x_i and denoted by $\frac{\partial f}{\partial x_i}(x_0)$.

So, Directional derivative $\Rightarrow$ partial derivative

but converse is not true.

Ex:
$$f(x,y)=\begin{cases}\frac{x}{y} & \text{if } y\neq 0,\\ 0 & \text{if } y=0\end{cases}$$

Now, partial derivative,

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= 0.$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{k\to 0} \frac{f(0,y_0+k) - f(0,0)}{k}$$

$$= 0.$$

but take $\vec{v} = (1,1)$.

$$\text{So, } \frac{\partial f}{\partial \vec{v}}(0,0) = \lim_{t\to 0} \frac{f(x_0 + t\vec{v}) - f(x_0)}{t}$$

$$= \lim_{t\to 0} \frac{f(t,t) - f(0,0)}{t}$$

$$= \lim_{t\to 0} \frac{1-0}{t} = \infty.$$

<u>Q. 3</u>

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

$v = (h,k)$.

$$\text{Now, } D_{\vec{v}} f(0) = \lim_{t\to 0} \frac{f(0+t\vec{v}) - f(0)}{t}$$

$$= \lim_{t\to 0} \frac{f(th, tk) - f(0)}{t}$$

$$= \lim_{t\to 0} \left(\frac{h^2 k}{t^2h^2+k^2}\right) = \begin{cases} \dfrac{h^2}{k} & \text{if } k \neq 0 \\ 0 & \text{if } k = 0. \end{cases}$$

So, Directional derivative exists $\Rightarrow$ partial derivative exist.

Now, $$\frac{\partial f}{\partial x} = \begin{cases} \frac{(x^4+y^2)2xy - 4x^5y}{(x^4+y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$\frac{\partial f}{\partial y} = \begin{cases} \frac{(x^4+y^2)x^2 - 2x^2y^2}{(x^4+y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

then f is partially differentiable everywhere in $\mathbb{R}^2$.

but f is not continue.

Now, take, $y = x^2$.

So, $(x,y) \to (0,0)$ $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{x^4}{2x^2} = \frac{1}{2}$.

Again take, $y = mx$,

$$\lim_{x\to 0} \frac{mx^3}{x^2(x^2+m^2)} = 0.$$

So. $f(x,y)$ is not continue.

(4) Home work.

(5) Same a question no (3).

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$f(x+y+z,\ x^2+y^2+z^2) = 0 \Rightarrow F(u,v) = 0$

$u = x+y+z, \quad v = x^2+y^2+z^2$

$$f_x = 0 = \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x}$$

$$f_y = 0 = \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} = (1+p).$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y} = (1+q),$$

$$\frac{\partial v}{\partial x} = (2x+2zp)$$

$$\frac{\partial v}{\partial y} = (2y+2zq).$$

So,

$$0 = \frac{\partial F}{\partial u}(1+p) + \frac{\partial F}{\partial v}(2x+2pz)$$

$$0 = \frac{\partial F}{\partial u}(1+q) + \frac{\partial F}{\partial v}(2y+2qz)$$

(1) $u = e^{ay} f(x+by)$

$$\frac{\partial u}{\partial y} = e$$

Q.3 Let the required eqn of the plane is

$z = lx + my + n.$

$lx + my - z + n = 0$

So, $K = \dfrac{|n|}{\sqrt{l^2+m^2+1}}$

$\Rightarrow K = \pm\dfrac{n}{\sqrt{l^2+m^2+1}}$

$lx + my - z \pm k\sqrt{l^2+m^2+1} = 0$

So, $l - \frac{\partial z}{\partial x} = 0$. $\Rightarrow l = p$

$m - \frac{\partial z}{\partial y} = 0$ $\Rightarrow \frac{\partial z}{\partial y} = q = m$.

So, $px + qy - z \pm k\sqrt{p^2+q^2+1} = 0$

(2)(i) $v = e^{ay} f(x+by)$

$\frac{\partial v}{\partial x} = e^{ay} f'(x+by)$.

$\frac{\partial v}{\partial y}$

Traversality condi.

$x = x_0(t)$, $y = y_0(t)$, $u = u_0(t)$

$\frac{dy_0}{dt} a(x_0, y_0, u_0) - \frac{dy_0}{dt} b(x_0, y_0, u_0) \neq 0$. $\forall t \in I$.

$1 - 1 = 0$

then uniq solution.

Existance of unique solution of Cauchy problem

$a(x,y,u) u_x + b(x,y,u) u_y = c(x,y,u)$

Initial Cu Γ: $x = f(s)$, $y = g(s)$, $z = h(s)$.

$s \in I$

(i) $x_0(t)$, $y_0(t)$, $u_0(t)$ is Continu

(ii) $a(x,y,u)$, $b(x,y,u)$, $c(x,y,u)$ is C'.

(iii) traversibility condition satisfy

then unique solution.

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$x(y^2+u)p - y(x^2+u)q = u(x^2-y^2)$.

The Lagrange Auxiliary eqn is

$$\frac{dx}{x(y^2+u)} = \frac{dy}{-y(x^2+u)} = \frac{du}{u(x^2-y^2)}.$$

Choosing $x, y, -1$ as multipliers of (1).

$$\frac{x\,dx + y\,dy - du}{x^2(y^2+u) - y^2(x^2+u) - u(x^2-y^2)} = \frac{x\,dx + y\,dy - du}{0}$$

$x^2+y^2-2u = c_1$,

Again, choosing $\frac{1}{x}, \frac{1}{y}, \frac{1}{u}$ as multiplies

So,
$$\frac{dx/x + dy/y + du/u}{y^2+u-(x^2+u)+x^2-y^2} = \frac{dx/x + dy/y + du/u}{0}$$

So, $xyu = c_2$.

So, soln is $\phi(x^2+y^2-2u, xyu) = 0$.

Remark 1 If number of arbitrary constants = the number of independent variables then ellimination of arbitrary contants will give rise to a PDE of first order.

Remark 2 If number of arbitrary Contants is less than the number of independent variable, then will rise two distinct partial diff eqn of first order.

Remark 3 If number of arbitrary Constant is more than the number of independent variables then pde rise of order more that One.