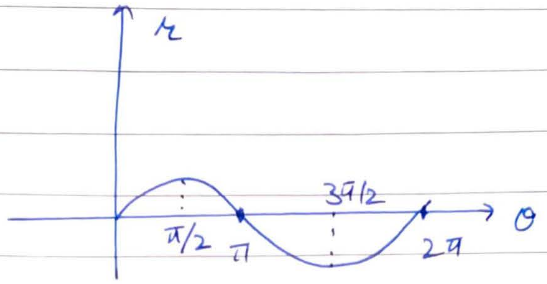
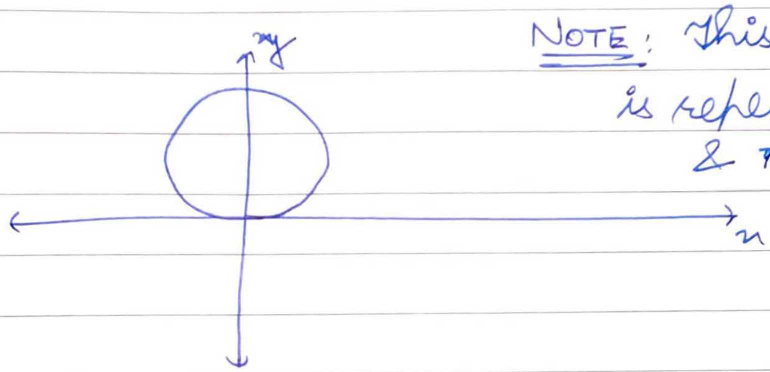
STEPS TO PLOT POLAR FORM →

i) First plot r vs θ graph

ii) now using it plot x ~~vs~~ & y graph

eg – $r = \sin\theta$

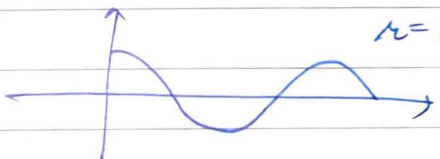


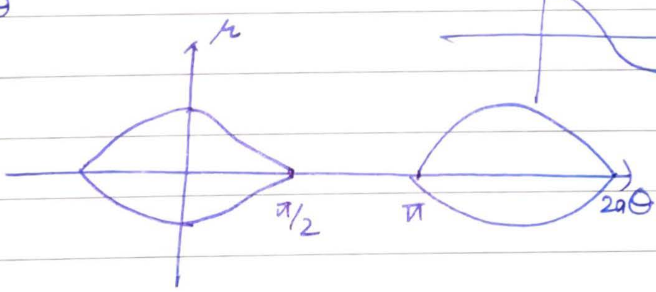


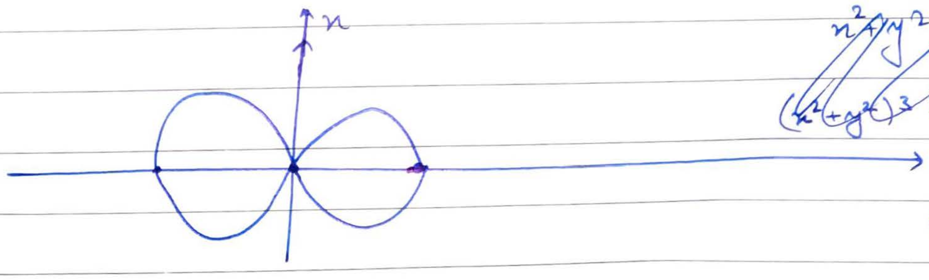
<u>NOTE</u>: This same circle is repeated in $0 - \pi$ & $\pi - 2\pi$ & so on.

$r^2 = \cos\theta$

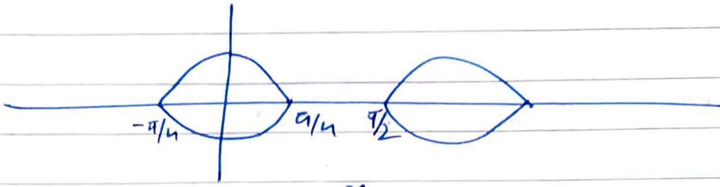
$r = \cos\theta$

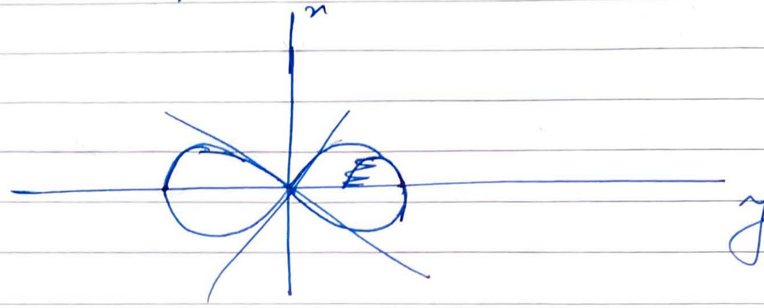






$r^2 = \cos 2\theta$





Chapter-1
Number System & Set System

- Natural nos. - infinite & countable
  Rational nos. - infinite & countable
  Irrational nos. - infinite & uncountable

- Countability of a set - If we can establish one one onto mapping from given set to natural nos. set.

- Types of sets (with elements)
  a) finite b) infinite & countable c) uncountable

*→ ORDERED SET

- Definition - Let S be a set. An order on S is a relation, denoted by <, with the following 2 properties:

a) $x \in S$ & $y \in S$ then one and only one of the statements (Tricotomy Law)

$x \in S$ $y \in S$

$x < y$ $x = y$ $x > y$

b) Transitive Law - $x\, y\, z \in S$ If $x < y$ $y <$ then $x < z$

complex no. is not an ordered set

- **Upper & Lower bounds**

Suppose S is ord. set, and $E \subseteq S$. If there exists a $\beta \in S$ such that $x \le \beta$ for every $x \in E$, we say E is bounded above & call β upper bound of E. Similarly we can have lower bound. There can be ∞ upper & lower bounds.

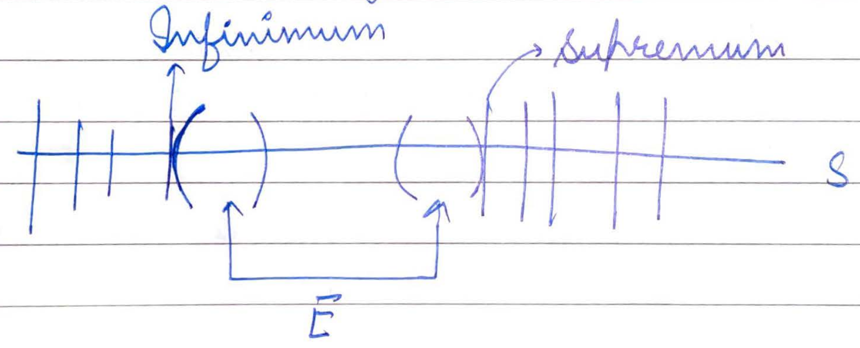
- **Supremum & Infimum.**

Suppose S is ord. set & $E \subset S$ and E is bounded above. Suppose that there exists a $\beta \in S$ with properties

a) β is upper bound

b) $\gamma < \beta$ & γ is not upper bound.

Then β is least upper bound or supremum of E. Similarly we have infimum



→ **Archimedean Property:**

If $x \in R$, $y \in R$ & $x > 0$ then there exists a +ve integer n such that

$nx > y$.

Ordered set holds law of tricotomy & is transitive.

PROOF: Let A be any set defined as

$$A = \{nx : x \in R, x > 0\}$$

If $nx \not> y$ then $nx \le y$

So A is bounded above
& A is not empty
$A \subset R$

$\Rightarrow$ Sup{A} exists (supremum of A)
(l.u.b. property)

let sup $A = \alpha$

$\alpha - x < \alpha$
$\Rightarrow \alpha - x$ is not an upper bound

→ there exists

~~Such that~~ $\exists$ some elements in A such that $a \in A$

$$a > \alpha - x$$
$$a = mx$$
$$mx > \alpha - x$$
$$\alpha < (m+1)x$$
$$(m+1)x \in A$$

$\alpha < (m+1)x$ is contradiction
α is sup A

Hence by contradiction
$$nx > y$$

NOTE: If least upper bound & max lower bound exist then set is ordered.

→ COROLLARY of ARCHIMEDIAN PROPERTY

- If $y > 0$, $y \in \mathbb{R}$ $\exists$ (there exists) $m_y \in \mathbb{N}$ such that

$$m_y - 1 \leq y < m_y$$

- PROOF

Let $S = \{m \in \mathbb{N} : m > y\}$

S is an ordered set
So there exists a least number in S.
Say that is 'm_y' then $m_y > y$

So where is $m_y - 1$?

$m_y - 1 \notin S$

$\Rightarrow m_y - 1 \leq y$

Hence, proved

PTO

THEOREM → (DENSE'S THEOREM)

If $x \in \mathbb{R}$, $y \in \mathbb{R}$ and $x < y$ then there exists a $p \in \mathbb{Q}$ such that

$$x < p < y$$

PROOF

Given $y > x$

$$y - x > 0$$

Applying Archimedian property on '$y-x$' and '1'

$$n(y-x) > 1 \qquad \because y - x > 0$$

$$1 + nx < ny$$

By previous corollary on "nx"

$\exists$ "m" such that $m - 1 \leq nx < m$

$$\Rightarrow nx < m \leq 1 + nx < ny$$

$$\Rightarrow nx < m < ny$$

$$\Rightarrow x < \frac{m}{n} < y \qquad (\because n > 0)$$

Let $p = m/n$

$$\therefore p \in \mathbb{Q}$$

Problem: Prove $a + \text{Sup } S = \text{Sup}(a+S)$

$$a+S = \{a+x : x \in S, a \in \mathbb{R}\}$$

Ans: ~~Let x be an element of S~~

~~ie $x \in S$~~

~~$\alpha = \text{Sup } S$~~

~~now $x \leq \alpha$~~

~~$x + a \leq \alpha + a$~~

~~$\therefore$ $\alpha + a$ is $\text{Sup}(a+S)$~~

~~$\therefore$ $a + \text{Sup } S = \text{Sup}(a+S)$~~

Let $u = \text{Sup } S$

$u \geq x \quad \forall x \in S$

$a + u \geq a + x$

$\Rightarrow a + u \geq \text{Sup}(a+S)$

$\Rightarrow a + \text{Sup } S \geq \text{Sup}(a+S) \quad -(i)$

Let v be an upper bound of $a+S$

$v \geq a + x \quad \Rightarrow \quad x \leq v - a \quad \forall x \in S$

$\Rightarrow \text{Sup } S \leq v - a$

$a + \text{Sup } S \leq v$

$a + \text{Sup } S \leq \text{Sup}(a+S) \quad -(ii)$

$\therefore$ using (i) & (ii)

NOTE: $a \le b \ \forall a \in A \ \& \ \forall b \in B$

$\sup A \le \inf B$

Q. Let S be a non-empty bounded subset of $\mathbb{R}$

a) If $a > 0$ & $aS = \{ax : x \in S\}$

Show $\sup(aS) = a \sup(S)$

$\inf(aS) = a \inf(S)$

b) If $b < 0$, and $bS = \{bx ; x \in S\}$

Show: $\sup(bS) = b \inf(S)$

$\inf(bS) = b \sup(S)$

a Ans Let $u = \sup S$

$\Rightarrow u \ge x \quad \forall x \in S$

$au \ge ax \qquad [\because a > 0]$

$\Rightarrow au \ge \sup(aS)$

$\Rightarrow a \sup(S) \ge \sup(aS) \qquad (i)$

Let v be an upper bound of aS

$v \ge ax \quad \Rightarrow x \le v/a \quad \forall x \in S$

$\Rightarrow \sup S \le v/a$

$\Rightarrow a \sup S \le v \qquad (ii)$

$\therefore$ using (i) & (ii)

~~an~~ $\sup(aS) = a\sup(S)$

Similarly we prove for infimum

$i \leq x \quad \forall x \in S$

$\Rightarrow ai \leq ax$

$\Rightarrow a\,\mathrm{Inf}(S) \leq \mathrm{Inf}(aS) \quad$ (i)

Now let m be a lowerbound of aS

$\therefore m \leq ax$

$\Rightarrow \frac{m}{a} \leq x$

$\Rightarrow \frac{m}{a} \leq \mathrm{Inf}(S)$

$\Rightarrow m \leq a\,\mathrm{Inf}(S)$

$\Rightarrow \mathrm{Inf}(aS) \leq a\,\mathrm{Inf}(S) \quad$ (ii)

$\therefore$ From (i) & (ii)

$\mathrm{Inf}(aS) = a\,\mathrm{Inf}(S)$

b Ans

Let

$\Rightarrow$

M

b Ans.

Let $u = \text{Sup}(S)$

$\Rightarrow \quad u \geq x \quad \forall x \in S$

$\Rightarrow \quad bu \leq bx \quad \forall x \in S \qquad [\because b<0]$

$\Rightarrow bu \leq \text{Inf}(bS)$

$\Rightarrow b\,\text{Sup}(S) \leq \text{Inf}(bS) \qquad (i)$

Now let v be a lower bound of bS

$\therefore \quad v \leq bx$

$\Rightarrow \quad \frac{v}{b} \geq x$

$\Rightarrow \quad \frac{v}{b} \geq \text{Sup}(S)$

$\Rightarrow \quad v \leq b\,\text{Sup}(S)$

$\Rightarrow \quad \text{Inf}(bS) \leq b\,\text{Sup}(S) \qquad (ii)$

$\therefore$ using (i) & (ii)

we have $\text{Inf}(bS) = b\,\text{Sup}($

~~Here~~ Now again

Let $i = \text{Inf}(S)$

$\Rightarrow \quad i \leq x \quad \forall x \in S$

$\Rightarrow \quad bi \geq bx$

$b \, Inf(S) \geq Sup(bS)$ (i)

now let ~~\$~~ m = upper bound of bS

$\Rightarrow m \geq bx$

$\Rightarrow \frac{m}{b} \leq x$

$\Rightarrow \frac{m}{b} \leq Inf(S)$

$\Rightarrow Sup(bS) \geq b \, Inf(S)$ (ii)

$\therefore$ Using (i) & (ii)

$$Sup(bS) = b \, Inf(S)$$

Hence, proved.

imp

Q→ Let A, B be bounded subsets of $\mathbb{R}$ and let

$$A+B = \{a+b : a \in A, b \in B\}$$

then

$$Sup(A+B) = Sup A + Sup B$$
$$Inf(A+B) = Inf A + Inf B$$

Ans: Let $\alpha = Sup(A)$ & $\beta = Sup(B)$

$\Rightarrow \alpha \geq a \quad \forall a \in A$

& $\beta \geq b \quad \forall b \in B$

$\Rightarrow \alpha + \beta \geq a+b$

$\Rightarrow \alpha + \beta \geq Sup(A+B)$ (i) now let v be an upper bound of $A+B$

$\Rightarrow v \geq a+b$

~~$\Rightarrow v \geq \sup(A+B)$~~

$\Rightarrow v - b \geq \sup(A)$

$\Rightarrow v - \sup(A) \geq b$

$\Rightarrow v - \sup(A) \geq \sup(B)$

$\Rightarrow v \geq \sup(A) + \sup(B)$

$\Rightarrow \sup(A+B) \geq \sup A + \sup B$ (ii)

$\therefore$ From (i) & (ii)

$\sup(A+B) = \sup(A) + \sup(B)$

Imp **NOTE:** (i) $f(x) \leq g(x) \ \forall x \in D$ then $\sup f(x) \leq \sup g$
& $\inf(f(x)) \leq \inf(g(x))$

PROOF: $f(x) \leq g(x) \leq \sup(g(x))$

$\Rightarrow f(x) \leq \sup(g(x))$

$\Rightarrow \boxed{\sup f(x) \leq \sup(g(x))}$

now

$\inf(f(x)) \leq f(x) \leq g(x)$

$\Rightarrow \inf f(x) \leq g(x) \Rightarrow \boxed{\inf f(x) \leq \inf g(x)}$

ii) $f(x) \le g(x) \quad \forall x \in D$

then $\boxed{\sup f \le \inf g \longrightarrow \text{false}}$

iii) $f(x) \le g(y) \quad \forall x, y \in D$

$$\sup f(x) \le \inf g(y)$$

iv) $\sup f + \sup g \ge \sup (f+g)$

v) $\inf f + \inf g \le \inf (f+g)$

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Tutorial I

1Q→ Let $x_0 \in \mathbb{R}$ & $x_0 \geq 0$. If $x_0 < \varepsilon$, for every +ve real nos ε, show that $x_0 = 0$

Ans. If $x_0 \neq 0$

Then $x_0 > 0$

Then by density property of $\mathbb{R}$

$\exists\ x_1 \in \mathbb{R}$ such that $0 < x_1 < x_0$

$\therefore \Rightarrow\Leftarrow$ contradiction

$\therefore x_0 = 0$

2Q→

Ans. Let α & β be 2 supremum of S

Then by definition of supremum

$\alpha \leq \beta$ & $\beta \leq \alpha$

Hence, $\alpha = \beta$

3Que:

Ans Let $\sqrt{2}$ be a rational number

Then $\sqrt{2} = \frac{p}{q}$ where ~~p & q~~

$(p, q \in \mathbb{Z}$ & $q \neq 0$

$\Rightarrow 2q^2 = p^2$ $(p, q$ have gcd

$\Rightarrow$ 2 is fator of p^2 $\therefore \Rightarrow$ 2 must divide p

Also

$\Rightarrow p = 2k$

$\Rightarrow 2q^2 = 4k^2$

$\Rightarrow q^2 = 2k^2$

$\therefore$ 2 is fator of q^2

$\therefore$ 2 must be factor of q

$\therefore$ gcd = 2

$\therefore$ contradiction $\therefore \sqrt{2}$ is not rational.

now

for $\sqrt{2}+\sqrt{3}$) ~~an~~ is not rational

$\sqrt{2}+\sqrt{3} = p/q \quad (p, q \in \mathbb{Z} \;\&\; p, q \neq 0)$

$\Rightarrow (\sqrt{2} - p/q)^2 = (\sqrt{3})^2$

$\Rightarrow 2 + \frac{p^2}{q^2} - 2\sqrt{2}\,p/q = 3$

$\Rightarrow \frac{p^2}{q^2} - 2\sqrt{2}\,p/q = 1$

$\Rightarrow 2\sqrt{2}\,p/q = \frac{p^2}{q^2} - 1$

$\Rightarrow \underset{\text{irrational}}{\sqrt{2}} = \underbrace{\left(\frac{p^2}{q^2} - 1\right)\frac{q}{2p}}_{\text{rati...}}$

4 Que
Ans

5 Que
Ans

$\therefore$ not true ie contradiction.

4Que:

Ans: Done earlier

5Que:

Ans: ~~Do at~~ Try yourself

given $p, q > 0$

$\Rightarrow \frac{1}{p} \in (0, 1] \longrightarrow Sup = 1$

& $\frac{1}{q} \in (0, 1] \longrightarrow Sup = 1$

where $S = \{$

clearly using proh. $Sup\ S = sup\ S_1 + sup\ S_2$

$\therefore \frac{1}{p} + \frac{1}{q} \in (0, 2] \longrightarrow$

$\therefore$ For $S = \{x : x = \frac{1}{p} + \frac{1}{q}\}$

we have $sup\ S = 2$ & $Inf\ S = 0$.

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6 Que:
Ans: For each $x_0 \in X$, $x_0 = \pi - \frac{1}{n}$ for some $n \in N$

~~Th~~ Then $x_0 < \pi$ (i)

For each $\epsilon > 0$ by Archimedian property

$n\epsilon > 1$ (Taking $x = \epsilon$ & $y = 1$, Arch prop.)

$\Rightarrow 0 < \frac{1}{n} < \epsilon$

$\Rightarrow \pi - \epsilon < \pi - \frac{1}{n} < \pi$ (ii)

From (i) & (ii)

$\sup X = \pi$

17 Que
Ans: using Mathematical Induction,

For $n = 1$

$1 + x \geq 1 + x$

$\therefore$ property holds for $n = 1$

Now we assume it true for $n = k$

$\Rightarrow (1+x)^k \geq 1 + kx$

$\Rightarrow (1+x)^{k+1} \geq (1+kx)(1+x)$ $[\because (1+x) \geq 0]$

$\geq 1 + x + kx + kx^2$

$\geq [1 + (k+1)x] + kx^2$

7 Que.
Ans.

8 Que.
Ans

Hence, proved.

2 Que.

Ans Let l.u.b of E is y. And given that $\alpha \in E$ is an upper bound of E

If possible let $\alpha \neq y$ then $\alpha > y$ (For every $x \in E$)

~~but α is upper bound of E~~

but $\alpha \in E$

$\Rightarrow \Leftarrow$

$\therefore \alpha = y$

3 Que.

Ans ~~ty~~ Given $\alpha < \beta$

$\Rightarrow \beta - \alpha > 0$

By Archimedian Property

$n(\beta - \alpha) > 1$

$\Rightarrow \beta - \alpha > 1/n$ (i)

$\alpha + \frac{1}{n} < \beta$

$\Rightarrow \alpha < \alpha + \frac{1}{n} < \beta$

From (i), put $\alpha = s$ $\beta = t$ in i

$t - s > 1/n \Rightarrow t - \frac{1}{n} > s$

$\Rightarrow s < t - 1/n < t$

9 1-10 Done (except 9)
11-try 12, 13 in rough copy

Q 10:

Ans. Done in rough copy

11 Que

Ans. $A = (P_1 \cup P_2) \cup (P_2 \cup P_3) \cup (P_3 \cup P_4) \cup \ldots$

$\& \; P_1 < P_2 < P_3 <$ (incr. seq.)

It is easily seen that P_1 is lower bound

12 Que

Ans. ro

13 Que In

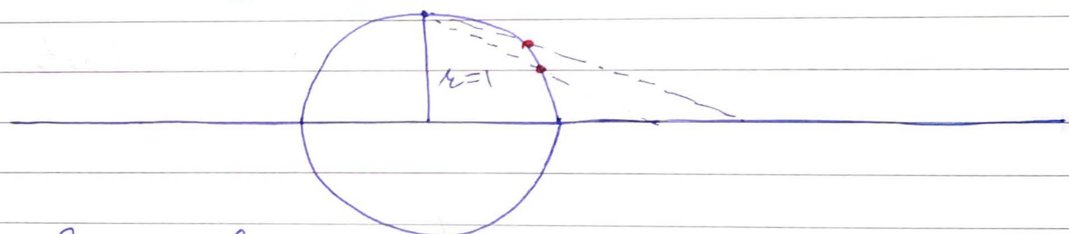
NOTE:

12 Que:
Ans: rough copy

13 Que: In rough copy.

bound

NOTE: A circle and a real line have same number of points.



To show this we draw lines from the upper end of circle. Now note that for every (red) point on the circle there is a point on real line. ∴ we can map every point of real line on circle. Also note that $+\infty$ & $-\infty$ are mapped together on top when a tangent line is drawn.

classmate

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→ EXTENDED REAL NUMBER SYSTEM

i) If A is a non-empty set of real numbers which has no upper bound and ∴ no l.u.b in R, we express this by writing

$$\sup A = +\infty$$

imp

ii) If A is empty ^sub-set ~~of~~ R we put

$$\boxed{\sup A = -\infty}$$

(since every real no. is an upper bound.)

iii) If A is non-empty set of real numbers which has no lower bound and ∴ no greatest lower bound in R, we express this as

$$\inf(A) = -\infty$$

imp

iv) If A is empty subset of R, we put

$$\boxed{\inf A = +\infty}$$

(since every real no. is a lower bound)