1. Consider random variables X and Y. Calculate $\text{Var}(3X + 2Y)$ given the following information. (Hint: At some point, you may need to use the fact that variance cannot be negative.)

- $E(3X + 2) = 8$
- $E(4X + 2Y) = 14$
- $E(2Y(X + 1)) = 28$
- $E(X^2Y^2) = 144$
- $\text{Cov}(X^2, Y^2) = 36$
- $E(X^2 + 2Y^2) = 33$

1) Given:

$E(3X+2) = 8$ — ①
$E(4X+2Y) = 14$ — ②
$E(2Y(X+1)) = 28$ — ③
$E(X^2 Y^2) = 144$ — ④
$Cov(X^2, Y^2) = 36$ — ⑤
$E(X^2+2Y^2) = 33$ — ⑥

To find: $Var(3X+2Y)$.

We know, $E(aX+b) = aE(X)+b$

$E(3X+2) = 3E(X)+2 = 8$

$3E(X) = 6$

$E(X) = 2$ //

Also, $E(aX+bY) = aE(X) + bE(Y)$.

$E(4X+2Y) = 14$

$4E(X) + 2E(Y) = 14$

$4 \times 2 + 2E(Y) = 14$

$2E(Y) = 6$

$E(Y) = 3$ //.

Now

$Var(3X+2Y) = 9Var(X) + 4Var(Y) + 12 Cov(X,Y)$

because

$Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$

and $Var(aX+bY) = a^2 Var(X) + b^2 Var(Y) + 2ab\, Cov(X,Y)$.

$$E(2XY + 2Y) = 28$$

$$2E(XY) + 2E(Y) = 28$$

$$E(Y) = 3$$

$$2E(XY) + 6 = 28$$

$$E(XY) = 11$$

$$\mathrm{cov}(X, Y) = E(XY) - E(X)E(Y)$$

$$= 11 - 2 \times 3$$

$$= 11 - 6 = 5$$

$$\mathrm{cov}(X, Y) = 5$$

$$\mathrm{Cov}(X^2, Y^2) = E(X^2 Y^2) - E(X^2)E(Y^2)$$

$$\cancel{144} \; 36 = 144 - E(X^2)E(Y^2)$$

$$\therefore \; E(X^2)E(Y^2) = 108 \quad - (7)$$

Also:

$$E(X^2 + 2Y^2) = 33$$

$$E(X^2) + 2E(Y^2) = 33 \quad - (8)$$

Solving (7) & (8)

$E(x^2) = 33 - 2E(Y^2)$.

let $E(y^2) = z$.

$(33 - 2z) \cdot z = 108$

$2z^2 - 33z + 108 = 0$

$z = 12$ or $4 \cdot 5$.

$E(Y^2) = 12$	$4 \cdot 5$ $\cancel{\times}$
$E(x^2) = 9$	24

$E(Y^2) = 12$ & $E(x^2) = 9$

Calculating $var(Y) = E(Y^2) - [E(Y)]^2$
$= 12 - 9 = 3$

$var(x) = E(x^2) - [E(x)]^2$
$= 9 - 4 = 5$

$V(3x + 2Y) = 9\,var(x) + 4\,var(x) + 12\,cov(X, Y)$
$= 9 \times 5 + 4 \times 3 + 12 \times 5$
$= 45 + 12 + 60$
$= 117$

$Var(3X + 2Y) = 117$ //

2. The density function of X is given by $f_X(x) = ax^3 + bx + \frac{2}{3}$ for $x \in [0, 1]$, and $\mathrm{E}(X) = \frac{7}{15}$.

a. Find a and b.

b. Calculate the CDF, $F(X)$.

c. Calculate $\Pr(X > 0.75)$

d. Suppose $Y = 1.5X + 2$. Calculate $\mathrm{E}(Y)$.

A2). $f_x(X) = ax^3 + bx + 2/3 \qquad x \in [0,1]$

$$E(X) = \frac{7}{15}$$

a)

We know.

$$\int_n^m f(x)\,dx = 1$$

$$\int_n^m f(x)\,x\,dx = E(X).$$

Using first.

$$\int_0^1 \left(ax^3 + bx + \frac{2}{3}\right)dx = 1$$

$$\left[\frac{ax^4}{4} + \frac{bx^2}{2} + \frac{2x}{3}\right]_0^1 = 1$$

$$\frac{a}{4} + \frac{b}{2} + \frac{2}{3} = 1$$

Taking LCM

$$3a + 6b = 4 \quad -(1)$$

b. Using second.

$$\int_0^1 \left(ax^3 + bx + \frac{2}{3}\right)x\,dx = \frac{7}{15}$$

$$\int_0^1 \left(ax^4 + bx^2 + \frac{2}{3}x\right)dx = \frac{7}{15}$$

$$\left[\frac{ax^5}{5} + \frac{bx^3}{3} + \frac{x^2}{3}\right]_0^1 = \frac{7}{15}$$

$$\frac{a}{5} + \frac{b}{3} + \frac{1}{3} = \frac{7}{15}$$

Taking LCM

$$3a + 5b = 2 \quad \text{--} ②$$

from ① & ②.

$$3a + 6b = 4$$
$$-(3a + 5b = 2)$$

$$b = 2$$
$$a = -8/3 \quad //$$

b).

$$y(x) = \int_0^x \left(-\frac{8}{3}t^3 + 2t + \frac{2}{3}\right) dt.$$

$$= -\frac{2}{3}x^4 + x^2 + \frac{2}{3}x.$$

CDF:

$$F(x) = \begin{cases} 0 & x < 0 \\ -\frac{2}{3}x^4 + x^2 + \frac{2}{3}x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

c). $P(x > 0.75)$

$= 1 - P(x \leq 0.75)$

$= 1 - \int_0^{0.75} \left(-\frac{8}{3}x^3 + 2x + \frac{2}{3}\right) dx$

$= 1 - \left[-\frac{2}{3}x^4 + x^2 + \frac{2}{3}x\right]_0^{0.75}$

$= 1 - \left[-0.2109375 + 0.5625 + 0.5\right]$

$= 0.148375$

d) $Y = 1.5X + 2$. $\qquad E(x) = \frac{7}{15}$

$E(Y) = \frac{3}{2}E(x) + 2$.

$= \frac{3}{2} \times \frac{7}{15} + 2$.

$= \frac{27}{10} = \underline{2.7}$