3. Probability: Widget production. Consider a factory that produces widgets. These widgets can have one (or more) of three different types: A, B, and C. Suppose that 20% of these widgets have type A, 40% have type B, 10% have both type A and B, and 50% have type C. Any widget of type C only has one type (i.e., there are no widgets of types A and C, B and C, or A, B, and C). Widgets can either be defective (D) or functional (D^c). Denote by $\Pr(D|X)$ the probability that a widget that has type X is defective. The factory knows that $\Pr(D|A) = 0.25$, $\Pr(D|B) = 0.6$, $\Pr(D|A \cap B) = 0.4$, and $\Pr(D|C) = 0.2$.

   a. What is the probability that a widget is defective, $\Pr(D)$? (Hint: Recall the Law of Total Probability.)

3) Law of Total Probability is for mutually exclusive events.
That is not the case here.

a) $Pr(D) = Pr(A) Pr\left(\frac{D}{A}\right) + Pr(B) Pr\left(\frac{D}{B}\right) + Pr(C)\, Pr(D/C)$
$- Pr(A \cap B) \cdot Pr\left(\frac{D}{A \cap B}\right)$.

$P(A) = 0.2$ $\left(20\% = \frac{20}{100} = 0.20 \right)$
$Pr(B) = 0.4$
$Pr(C) = 0.5$
$Pr(A \cap B) = 0.1$.

Given $Pr\left(\frac{D}{A}\right) = 0.25$ $Pr\left(\frac{D}{B}\right) = 0.6$
$Pr\left(\frac{D}{C}\right) = 0.2$ $Pr\left(\frac{D}{A \cap B}\right) = 0.4$

$Pr(D) = 0.25 \times 0.2 + 0.6 \times 0.4 + 0.5 \times 0.2$
$- 0.1 \times 0.4$

$= 0.05 + 0.24 + 0.1 - 0.04$
$= \underline{0.35}$.

Prob widget is defective ; $Pr(D) = 0.35$

b. What is the probability that a defective widget is of type B, or $\Pr(B|D)$?

If $Pr(D) = 0.35$

$Pr(D') = 1 - Pr(D) = 0.65.$

b). $Pr\left(\frac{B}{D}\right) = Pr\left(\frac{D}{B}\right) \cdot \frac{Pr(B)}{Pr(D)}.$

(Bayes Theoren)

$= \frac{0.6 \times 0.4}{0.35.} = \underline{0.6857}.$

c. What is the probability that a non-defective (i.e., functional) widget is either type A or type B (or both), i.e., what is $\Pr(A \cup B|D^c)$?

c). To find $Pr\left(\frac{A \cup B}{D^c}\right)$

$= \frac{P_r(A \cup B)}{P_r(D^c)} \cdot Pr\left(\frac{D^c}{A \cup B}\right)$

(Bayes Theorem).

$P_r(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.2 + 0.4 - 0.1 = 0.5$

Now,

$$P\left(\frac{D'}{A\cup B}\right) = 1 - P\left(\frac{D}{A\cup B}\right)$$

$$= 1 - \left(\frac{P(D\cap A) + P(D\cap B) - P(D\cap A\cap B)}{P(A\cup B).}\right)$$

Now $P(D\cap A) = P\left(\frac{D}{A}\right) P(A)$

$$= 0.25 \times 0.2 = 0.05$$

$$P(D\cap B) = P\left(\frac{D}{B}\right) P(B) = 0.4 \times 0.6$$

$$= 0.24$$

$$P(D\cap A\cap B) = P\left(\frac{D}{A\cap B}\right) \cdot P(A\cap B)$$

$$= 0.4 \times 0.1 = 0.04$$

$$P\left(\frac{D'}{A\cup B}\right) = 1 - \left[\frac{0.05 + 0.24 - 0.04}{0.5}\right]$$

$$= 1 - \frac{0.25}{0.5} = \underline{0.5}$$

$$P\left(\frac{A \cup B}{D^c}\right) = \frac{P(A \cup B) \cdot P\left(\frac{D'}{A \cup B}\right)}{P(D')}$$

$$= \frac{0.5 \times 0.5}{0.65} = 0.384615$$

Probability that a non defective widget is either A or B or Both is

$$Pr\left(\frac{A \cup B}{D^c}\right) = 0.384615$$

4. Probability: Inclusion-exclusion. Recall that the additive rule tells us for events A and B that are not mutually exclusive that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. We can extend this additive rule to more than two events, which gives us the general inclusion-exclusion identity as follows:

$$P(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) + ... + (-1)^{n+1} P(A_1 \cap A_2 \cap ... \cap A_n)$$

a. Explicitly write the inclusion-exclusion identity for $n = 3$ events, A_1, A_2, A_3 (i.e., reduce down so that there aren't summations).

Answer-4

a) $P(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \cdots$

$\cdots + (-1)^{n+1} P(A_1 \cap A_2 \cdots \cap A_n)$.

For $n=3$.

$P(\cup_{i=1}^{3} A_i) = \underbrace{\sum_{i=1}^{3} P(A_i)}_{\text{Part 1}} - \underbrace{\sum_{1 \le i < j \le 3} P(A_i \cap A_j)}_{\text{Part 2}} + \underbrace{\sum_{1 \le i < j < k \le 3} P(A_i \cap A_j \cap A_k)}_{\text{Part 3}}$

$P(A \cup B \cup C) = \sum_{i=1}^{3} P(A_i) - \sum_{1 \le i < j \le 3} P(A_i \cap A_j) + \sum_{1 \le i < j < k \le 3} P(A_i \cap A_j \cap A_k)$

Part-1.

$\sum_{i=1}^{3} P(A_i) = P(A) + P(B) + P(C)$.

Part-2.

$\sum_{1 \le i \le j \le 3} P(A_i \cap A_j) = P(A \cap B) + P(B \cap C) + P(A \cap C)$

Part-3.

$\sum_{1 \le i < j < k \le 3} P(A_i \cap A_j \cap A_k) = P(A \cap B \cap C)$.

$\therefore P(A \cup B \cup C) = P(A) + P(B) + P(C)$

$- P(A \cap B) - P(B \cap C) - P(A \cap C)$.

$+ P(A \cap B \cap C)$.

Alternatively,

for $n=2$ we know

$$P(A\cup B) = P(A) + P(B) - P(A\cap B)$$

we have to find $P(A\cup B\cup C)$

Now assume $P(A\cup B) = P(D)$.

$$\therefore \quad P(D) = P(A) + P(B) - P(A\cap B)$$

so, $P(D\cup C) = P(D) + P(C) - P(D\cap C)$

Substituting $P(A\cup B)$ in place of $P(D)$

$$P(A\cup B) = P(A) + P(B) - P(A\cap B) + P(C) - P(A\cap C) - P(B\cap C) + P(A\cap B\cap C).$$

b. Suppose an integer from 1 to 1000 (inclusive) is chosen uniformly at random (i.e., with equal probability). What is the probability that the integer is divisible by 5, 7, or 13?

b) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

To find : prob of integer is divisible by 5, 7 or 13.

$1 \leq n \leq 1000$, Let $A = 5$, $B = 7$ & $C = 13$

Step-1:

Using AP we find number of int divisible by 5 in first 1000 natural numbers

$1000 = 5 + (n-1)5$

$n = 199 + 1 = \underline{200}$

$P(A) = \frac{200}{1000} = \underline{0.2}$

Similarly for 7 & 13

$P(B) = \frac{142}{1000} = 0.142$

$P(C) = \frac{76}{1000} = 0.076$

Step -2:

Now for $P(A \cap B)$

we take LCM of A & B

$LCM(5,7) = 35 = \underline{A \cap B}$

so $P(A \cap B) = P(35) =$

Similarly using AP.

$1000 = 35 + (n-1)35$

$n = 28.57 = \underline{28}$

$P(A \cap B) = P(35) = \frac{28}{1000} = 0.028$

Similarly for $P(B \cap C)$ & $P(C \cap A)$

$P(B \cap C) =$ LCM of 7 & 13 $\Rightarrow 91$

$P(B \cap C) = \frac{10}{1000} = 0.01$

$P(C \cap A) =$ LCM of 5 & 13 $\Rightarrow 65$

$P(C \cap A) = \frac{15}{1000} = 0.015.$

Step-3

$P(A \cap B \cap C) \Rightarrow$ LCM of 5, 7 & 13

$= 455$

$P(A \cap B \cap C) = \frac{2}{1000} = 0.002.$

substitute values into formula

$$= 0.2 + 0.142 + 0.076 - 0.028 - 0.010 - 0.015 + 0.002$$

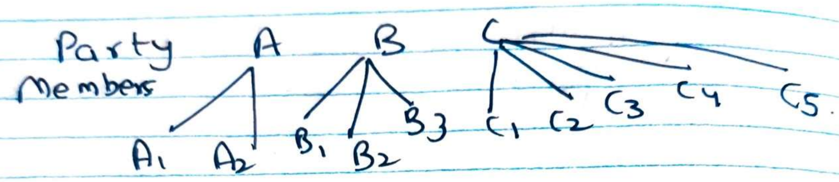
$$= \underline{0.367}$$

Prob that the integer is divisible by 5, 7 or 13 is $\underline{0.367}$.

5. Combinatorics: Consider a political setting where there are three political parties, A, B, and C vying for seats on a 3-person committee. Party A has 2 members, B has 3 members, and C has 5 members. Members of parties are distinguishable from each other, but positions on the committee are indistinguishable from each other.

a. How many ways are there of forming an unordered 3-person committee?

As).



a) Unordered 3-person comittee.

$n = 10 \quad r = 3$

$$\text{ways} = \frac{n!}{r!(n-r)!} = \frac{10!}{7! \cdot 3!} = \underline{120}$$

120 ways are there.

b. How many different party breakdowns (e.g., ABC, CCC, etc.) are possible when forming an unordered 3-person comittee?

b). Case -1

All diff : A B C => 1 way.

Case-2 2 member of same party

A ~~B~~A _ => 2 ways

B B _ => 2 ways

C C _ => 2 ways

Case-3 3 members of same party

B B B & C C C => 2 ways

Not applicable for Party - A

Total ways = 9

c. How many ways are there of forming an unordered 3-person committee if at least one member must be from party A?

c) Total cases = $^{10}C_3$

No member of Party A cases $= {}^{8}C_3$

(Members in B + members in C = 8)

For Atleast One member of party A.

$= {}^{10}C_3 - {}^{8}C_3$

$= 120 - 56 = \underline{64}$

There are 64 ways.

6. Combinatorics: Miscellaneous counting.

a. There are 20 indistinguishable children who would like to have one ice cream cone each. There are 6 distinct flavors of ice cream. How many distinct collections of ice cream cones are there where at least two children must order each flavor?

A6)

a) Let each dash _ represent children & F_1 = icecream flavor-1, similarly F_2, F_3, F_4, F_5 & F_6.

20 indistinguishable children

_ _ _ _ _ _ _ _ _ _

_ _ _ _ _ _ _ _ _ _

Atleast 2 children must order each flavour, so lets say C_1 & C_2 order F_1.

Similarly C_3 & C_4 order F_2 ...

so on till F_6.

$\boxed{F_1\ F_1}$ $\boxed{F_2\ F_2}$ $\boxed{F_3\ F_3}$ $\boxed{F_4\ F_4}$ $\boxed{F_5\ F_5}$

$\boxed{F_6\ F_6}$ _ _ _ _ _ _ _ _

Now there are 8 children who can chose from 6 flavours

Applying Stars & Bars concept.

For non negative constraints

$$^{n+k-1}C_{k-1}$$

where

$k = 6$ flavours of ice cream

$n = 8$ number of children

$$^{6+8-1}C_{6-1} = {}^{13}C_{5}$$

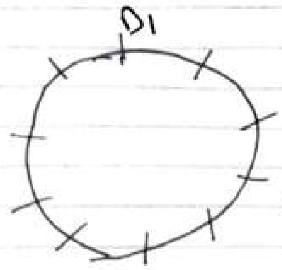
$$= \frac{13!}{8!\,5!} = 1287$$

There are 1287 ways

b. There are five cats and five dogs, all distinguishable from one another. How many distinct ways are there of seating them at a round table such that every cat is adjacent to two dogs and every dog is adjacent to two cats? Note that here two orderings are not considered distinct if it is possible to rotate one and achieve the other. For instance, if there are only four seats at the table, the order Cat 1 - Dog 1 - Cat 2 - Dog 2 is the same as Cat 2 - Dog 2 - Cat 1 - Dog 1.

b) Let there be a circular table
D_1, D_2, D_3, D_4, D_5 be dogs

C_1, C_2, C_3, C_4, C_5 be cats



In order to arrange cats and dogs, we place D_1 at top, which will be starting position.

Now there are 4 dogs and 4 positions.
ways = $4! = \underline{24}$

For cats there are 5 positions and 5 cats
ways = $5! = \underline{120}$

Total ways = $4!\,5!$

$= 24 \times 120$
$= \underline{2880}$.

There are 2880 ways

Short Answers:

- About how long did this assignment take you? Did you feel it was too long, too short, or reasonable?

6-7 hours.

- Who, if anyone, did you work with on this assignment?

No one

- What questions do you have relating to any of the material we have covered so far in class?

Visual interpretation, correlation, sensitivity, specificity, PPV, NPV, Probability and Combinatorics.