3 Algebraic structures

Introductory problem

Solve the equation $\frac{1}{x^2} = \ln x$.

Equations are the building blocks of mathematics. There are many different types: some have no solutions; some have many solutions; some have solutions which cannot be expressed in terms of any function you have met.

Graphs are an alternative way of expressing a relationship between two variables. If you understand the connection between graphs and equations, and can switch readily between the two representations, you will have a wider variety of tools for solving mathematical problems. The International Baccalaureate® places great emphasis on using graphical calculators to analyse graphs.

Identities are used to rewrite an expression in a different form, which can be very useful when solving equations. In much of mathematics we do not distinguish between equations and identities, although they are fundamentally different. In this chapter we shall explore some of these differences and look at the different techniques we can apply to equations and identities.

Very few examination questions are set on this topic alone, but the techniques of this chapter are involved in virtually every examination question.

In this chapter you will learn:

- some standard algebraic strategies for solving equations
- how to sketch graphs, and some limitations of graphical calculators
- how to use a graphical calculator to solve equations
- how to work with identities.

3A Solving equations by factorising

We start by looking at some common algebraic methods for solving equations. You have already seen how factorising can be used to help sketch the graphs of quadratic functions; in this chapter we will apply the same principles in a wider context.

This is true for real numbers, but if the numbers you are looking for are integers, then knowing that they multiply together to give a non-zero constant is extremely useful. For example, can you solve $xy = 14$ if x and y are known to be integers? This type of equation is called a *Diophantine equation*.

If two numbers multiply together to give 5, what can you say about those two numbers? They could be 1 and 5, 10 and $\frac{1}{2}$, π and $\frac{5}{\pi}$, …. In fact, there are an infinite number of possibilities. So, just knowing that two numbers multiply together to give 5 is not of much help in determining what the numbers actually are.

However, if you know that two (or more) numbers multiply together to give zero, then you can deduce that at least one of those numbers must be zero. Therefore, if we can factorise the expression we would have a powerful tool for solving equations that take the form of an expression equal to zero.

KEY POINT 3.1

For factorising to be useful in solving an equation, one side of the equation must be zero.

Worked example 3.1

Solve the equation $e^x(\ln(x)-1)(2x-1)=0$.

If a product is equal to zero, then one of the factors must be zero.

Either	$e^x = 0$	(1)
or	$\ln(x)-1=0$	(2)
or	$2x-1=0$	(3)

From (1): $e^x = 0$ has no solution

From (2): $\ln(x) = 1$

$\Leftrightarrow x = e$

From (3): $2x = 1$

$\Leftrightarrow x = \frac{1}{2}$

$\therefore x = e$ or $\frac{1}{2}$

If, instead of factorising an expression, you divide it by another expression, it is possible that you will lose solutions. For example, consider the equation $x^3 = x$. If you divide both sides by x, you get $x^2 = 1$ and hence $x = \pm 1$. But, by substituting $x = 0$ into the original equation, you can see that it is also a solution, which was missed by the method of dividing through by x. The correct way to solve this equation is:

$$
\begin{aligned}
& x^3 = x \\
\Leftrightarrow\quad & x^3 - x = 0 \\
\Leftrightarrow\quad & x(x^2-1) = 0 \\
\Leftrightarrow\quad & x(x-1)(x+1) = 0 \\
\Leftrightarrow\quad & x = 0 \text{ or } x = 1 \text{ or } x = -1
\end{aligned}
$$

EXAM HINT

Whenever you are tempted to divide both sides by an expression involving x, rearrange the equation so that one side is zero and then factorise the other side.

Exercise 3A

1. Solve the following equations.

(a) (i) $3(x-3)^3 = 0$ (ii) $-4(x+1)^5 = 0$

(b) (i) $7(2x-1)(5x+3)^2 = 0$ (ii) $5(3-x)^2(2x+6)^2 = 0$

(c) (i) $(\log_3 x - 3)(3^x - 3) = 0$ (ii) $(\sqrt{x}-4)(9\sqrt{x}-1) = 0$

(d) (i) $x(x^2-3) = 7(x^2-3)$

(ii) $5x(x^2-5x+4) = 6(x^2-5x+4)$

2. Solve the equation $6^x - 4\times 3^x = 0$. *[5 marks]*

3. Find the exact solution to the equation $2\times 5^x - 7\times 10^x = 0$. *[6 marks]*

4. Solve $(3x-1)^{x^2-4} = 1$. *[4 marks]*

3B Solving equations by substitution

There are certain types of equation which you should know how to solve. In this section we shall focus particularly on quadratic equations, since we have a formula for solving them. We will see how some complicated-looking equations can be turned into quadratic equations by means of a substitution.

Worked example 3.2

Solve the equation $x^4 - 4 = 3x^2$.

A substitution $y = x^2$ turns this into a quadratic equation, since $x^4 = y^2$.

Let $y = x^2$. Then the equation becomes

$$y^2 - 4 = 3y$$

This is a standard quadratic equation, which can be factorised.

$$y^2 - 3y - 4 = 0$$
$$\Leftrightarrow (y+1)(y-4) = 0$$
$$\Leftrightarrow y = -1 \text{ or } y = 4$$

Use the substitution to find the corresponding values of x.

$x^2 = -1$ (not possible, reject)

or

$x^2 = 4$

$x = 2$ or $x = -2$

EXAM HINT

Note that you should explicitly state that you have rejected the possibility $x^2 = -1$. Do not just cross it out.

It may not always be obvious what substitution to use. It is quite common to be given an exponential equation which needs a substitution; in such cases, look out for an a^{2x} or $(a^2)^x$ term, both of which can be rewritten as $(a^x)^2$.

Worked example 3.3

Solve the equation $2(4^x + 2) = 9 \times 2^x$.

Note that $4^x = (2^2)^x = 2^{2x} = (2^x)^2$, so a substitution $y = 2^x$ turns this into a quadratic equation.

Let $y = 2^x$. Then the equation becomes

$$2(y^2 + 2) = 9y$$

continued . . .

This is a standard quadratic equation, which can be factorised.

$$2y^2 - 9y + 4 = 0$$
$$\Leftrightarrow (2y-1)(y-4) = 0$$
$$\Leftrightarrow y = \frac{1}{2} \text{ or } y = 4$$

Use the substitution to find the values of x.

Therefore $2^x = \frac{1}{2} \Rightarrow x = -1$

$2^x = 4$

$\Rightarrow x = 2$

Exercise 3B

1. Solve the following equations, giving your answers in exact form.

   (a) (i) $a^4 - 10a^2 + 21 = 0$ (ii) $x^4 - 7x^2 + 12 = 0$

   (b) (i) $2x^6 + 7x^3 = 15$ (ii) $a^6 + 7a^3 = 8$

   (c) (i) $x^2 - 4 = \frac{2}{x^2}$ (ii) $x^2 + \frac{36}{x^2} = 12$

   (d) (i) $x - 6\sqrt{x} + 8 = 0$ (ii) $x - 10\sqrt{x} + 24 = 0$

2. Solve the following equations.

   (a) $e^{2x} + 16e^x = 80$

   (b) $25^x - 15 \times 5^x + 50 = 0$

   (c) $\log_4 x = (\log_4 x^2)^2$

3. Solve the following equations.

   (a) $e^{2x} - 9e^x + 20 = 0$

   (b) $4^x - 7 \times 2^x + 12 = 0$

   (c) $(\log_3 x)^2 - 3\log_3 x + 2 = 0$ *[15 marks]*

4. Solve the equation $9(1 + 9^{x-1}) = 10 \times 3^x$. *[5 marks]*

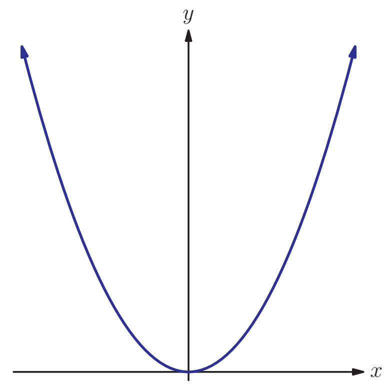
5. If $a^x = \frac{-5}{a^x} + 6$ where $a > 0$ is a constant, solve for x, leaving your answer in terms of a. *[5 marks]*

6. Solve the equation $\log_2 x = 6 - 5\log_x 2$. *[6 marks]*

3C Features of graphs

In this course you will meet many different types of equations and will learn various techniques for solving them. You already encountered quadratic, exponential and logarithmic equations in chapters 1 and 2, and they are very common in applications, but you may also need to solve equations which do not belong to one of these standard types. Approximate solutions to such equations can be found by using graphs.

Graphs are simply another way of representing a relationship between two variables. For example, we can write $y = x^2$ or draw



Graphical calculators can be very helpful in sketching or drawing graphs. In this section we will examine important features you should look for when plotting graphs with your graphical calculator; we will also discuss some limitations of this method.

The main features you should indicate on your sketch are:

- the y-intercept – this is where $x = 0$
- the x-intercepts (zeros) – these are where $y = 0$
- maximum and minimum points.

If the graph you are sketching is completely unfamiliar, it can be difficult to choose a good viewing window. If the window is too small, you may miss important parts of the graph; if it is too large, you may not be able to distinguish between features that are close together. This is why it is useful to have a general idea of the overall shape of the graph before trying to plot it. It is important to learn about the graphs of different types of functions, even if you have a graphical calculator available.

EXAM HINT

If you are asked to *sketch* a graph, you only need to show the overall shape and indicate the important features. Occasionally you may be asked to *draw* a graph, in which case the graph should be accurate and to scale, and you should use a ruler for straight lines.

EXAM HINT

See Calculator Skills sheets 2 and 4 on the CD-ROM for a guide to sketching graphs and finding the main features.

Worked example 3.4

Sketch the graph of $y = \dfrac{x^3 - 16x}{x^2 + 1}$.

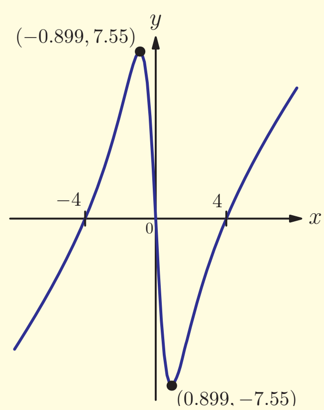
Can we quickly deduce any of the important features?
When $x = 0$, $y = 0$, so the graph goes through (0, 0).
We also know that the zeros (x-intercepts) occur when the numerator is zero, and we can find those points by factorising.

Zeros occur where

$$x^3 - 16x = 0$$
$$x(x^2 - 16) = 0$$
$$x(x-4)(x+4) = 0$$
$$x = 0, 4, -4$$

We don't know much about the shape of the graph, so use a GDC to plot it.
The viewing window should include -4 and 4.
There appear to be one maximum and one minimum point; if needed, we can find their coordinates with the GDC too.

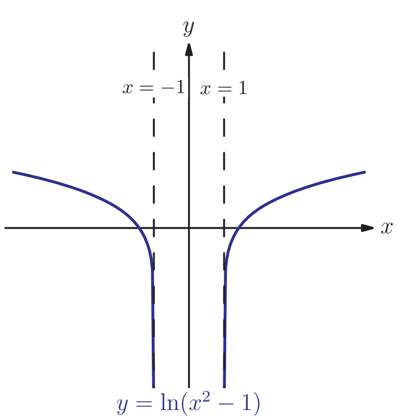
From GDC:



Some graphs may have asymptotes.

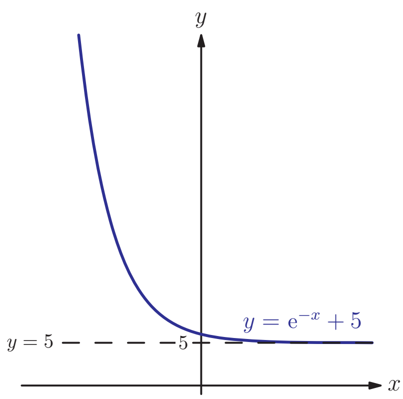
KEY POINT 3.2

An **asymptote** is a straight line which the graph approaches as either x or y gets very large.



Asymptotes are usually shown on graphs as dotted lines.

Vertical asymptotes occur where a function ceases to be defined. They are vertical lines of the form $x = a$. For example, $y = \dfrac{1}{x-3}$ has an asymptote $x = 3$ (because we cannot divide by zero), and $y = \ln(x^2 - 1)$ has asymptotes $x = -1$ and $x = 1$ (because we cannot take a logarithm of zero).



Horizontal asymptotes indicate long-term behaviour of the function; they are lines that the graph approaches for large values of x (positive or negative). They are horizontal lines of the form $y = a$. For example, $y = e^{-x} + 5$ has an asymptote $y = 5$.

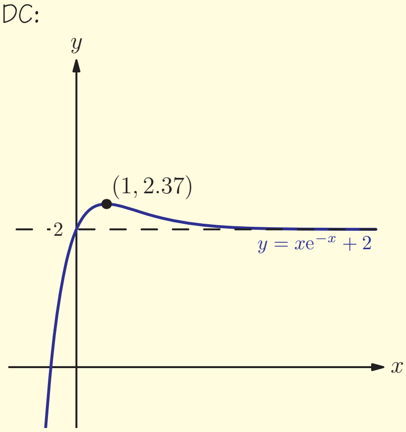
Since asymptotes are not actually part of the graph, they will not show up on your calculator sketch. But you can find approximately where they are by moving the cursor along the graph towards large values of x or y and estimating what y or x value the graph seems to be approaching. To find the exact location of an asymptote, you need to use knowledge of the function in question – for example, the fact that you cannot divide by zero or that e^{-x} approaches zero as x gets large. You will only be asked to identify the exact position of asymptotes for familiar functions.

Worked example 3.5

Sketch the graph of $y = xe^{-x} + 2$.

Start with a standard viewing window with both x and y between −10 and 10.
It looks as if there is something interesting happening for x-values between 0 and 3; zoom in to confirm that there is a maximum point at $x = 1$.
There also seems to be a horizontal asymptote. We know that e^{-x} approaches 0, so it is likely that the asymptote is $y = 2$.

Using GDC:



See Prior learning section I on the CD-ROM for an introduction to the modulus function $|x|$. *Make sure you know where to find it on your calculator.*

Note that in the above example, although the line $y = 2$ is an asymptote, the graph actually crosses it when $x = 0$. This is fine, as the asymptote is only relevant for large values of x.

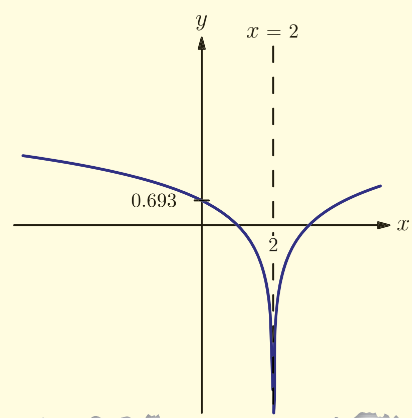
Vertical asymptotes can sometimes be unclear. In the next example, the graph approaches the vertical asymptote from both sides, so your calculator may attempt to 'connect the points' into a 'V' shape. You must be able to identify for yourself that there is a vertical asymptote between the two arms of the 'V' and draw it correctly.

Worked example 3.6

Sketch the graph of $y=\ln|x-2|$.

It looks as if the two branches of the graph join at a point with x-coordinate equal to 2. However, we know that $\ln 0$ is not defined, so there should instead be an asymptote at $x=2$.

Asymptote: $x-2=0 \Leftrightarrow x=2$



Exercise 3C

1. Sketch the following graphs, indicating the axis intercepts, asymptotes, and maximum and minimum points.

(a) (i) $y=x^4-x^3+1$ (ii) x^4-x^2

(b) (i) $y=(x-1)e^x$ (ii) $y=(e^x-1)^2$

(c) (i) $y=\dfrac{\frac{1}{2}e^x-1}{x-1}$ (ii) $y=\ln\left(\dfrac{x+2}{x-1}\right)^2$

(d) (i) $\left|\dfrac{x^2-1}{x+2}\right|$ (ii) $y=\dfrac{|x^2-4|}{x+1}$

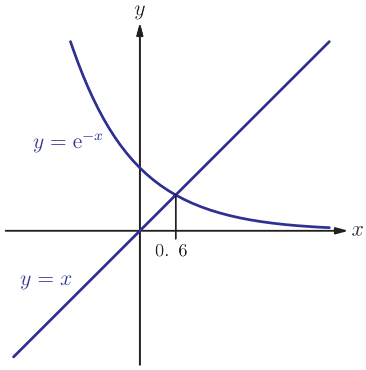
2. Sketch the graph of $y=x\ln x$. *[4 marks]*

3. Sketch the graph of $y=\dfrac{e^x}{\ln x}$. *[6 marks]*

4. Sketch the graph of $y=\dfrac{x^2(x^2-9)}{e^x}$ for $-5\le x\le 6$. Mark the coordinates of all zeros and maximum and minimum points. *[6 marks]*

EXAM HINT
In question 4 you need to explore different viewing windows to locate all the maximum and minimum points.

3D Using a graphical calculator to solve equations



Some equations cannot be solved analytically, that is, you cannot rearrange them to get $x =$ a certain number using the standard set of operations and functions mathematicians allow. Nevertheless, there may still be values of x which satisfy the equation. An example of such an equation is $x = e^{-x}$.

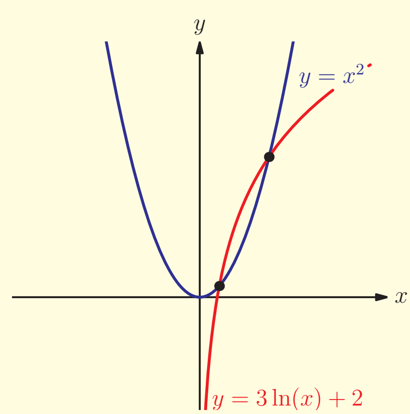
One good way to find these solutions is by plotting both sides of the equation with a graphical calculator. The x-coordinate of the intersection point gives the solution to the equation – in this case 0.567 to three significant figures.

The solution to $x = e^{-x}$ can actually be written in terms of the 'Lambert W Function'. In fact, it is $W(1)$. However, does knowing this actually give us any more information about the solution to the above equation?

Worked example 3.7

Solve the equation $x^2 = 3\ln x + 2$.

There is no obvious substitution or factorisation, so plot both sides of the equation on the calculator and make a sketch.



Use the calculator to find the intersection points.

$x = 2.03$ or $x = 0.573$ (3SF)

(from GDC)

EXAM HINT

When using your calculator to solve an equation, you must show a sketch of the graph and round the answer to an appropriate degree of accuracy (usually 3 SF).

Solving equations graphically has some problems: you may not know how many solutions to look for, or how to set the viewing window so you can see them all; if two solutions are very close together, you may miss one of them. This is where you need to rely on your knowledge of the graphs of different types of functions to make sure that the calculator is showing all the important features.

Many graphical calculators have functions for solving special types of equations. In particular, you may be able to solve polynomial equations (those involving only positive integer powers of x) without having to graph them. Your calculator may also have some sort of equation solver tool, although this has drawbacks similar to those associated with graphical methods.

See Calculator skills sheet 5 and 6 on the CD-ROM for more details.

EXAM HINT

If the question does not ask for an exact answer, you can take it as an indication that a graphical solution might be appropriate. If you cannot see another quick way to solve the equation, try using your calculator.

Worked example 3.8

Solve the equation $x^3 = 5x^2 - 2$.

As we don't yet know any algebraic methods for solving cubic equations, let's try using the polynomial equation solver on the calculator. First, the equation needs to be rearranged.

$$x^3 - 5x^2 + 2 = 0$$

$$x = -0.598, 0.680, 4.92 \text{ (3SF)}$$

(using GDC)

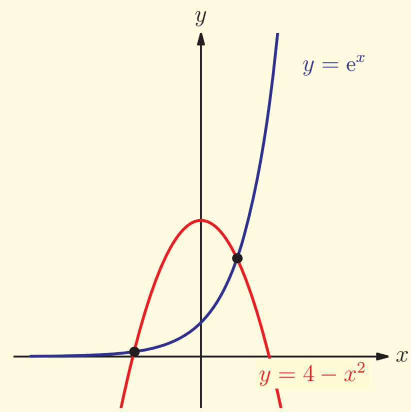
EXAM HINT

You can use the graphical method in the non-calculator paper, too. Questions in the non-calculator paper usually just ask you to find the number of solutions, rather than to actually solve the equation.

Worked example 3.9

Find the number of solutions of the equation $e^x = 4 - x^2$.

Sketch graphs of both sides of the equation; these are graphs you should know how to sketch without a calculator.



The solutions correspond to the intersections of the two curves.

There are 2 solutions.

Exercise 3D

1. Solve the following equations, giving your answers to 3 significant figures.
   (a) (i) $x^3 = 3x - 1$ (ii) $x^3 + 4x^2 = 2x - 1$
   (b) (i) $e^x = x + 1$ (ii) $e^{2x} = x^2 - 3$
   (c) (i) $e^x = \ln x$ (ii) $e^x \ln x = x^3 - x$

2. Solve the equation $x \ln x = 3 - x^2$. *[4 marks]*

3. The equation $\ln x = kx$, where $k > 0$, has one solution. How many solutions do the following equations have?
   (a) $\ln x^2 = kx$
   (b) $\ln\left(\dfrac{1}{x}\right) = kx$
   (c) $\ln\sqrt{x} = kx$ *[6 marks]*

4. Find a value of k such that the equation $\sin x = kx$ has 7 solutions, where x is measured in degrees. *[6 marks]*

3E Working with identities

KEY POINT 3.3

An **identity** is an equation which is true for all values of x, for example $x \times x \equiv x^2$.

The identity sign $\equiv$ is used to emphasise that the left-hand side and right-hand side are equal for all values of x, but frequently an equals sign is written instead.

Identities are very useful for manipulating algebraic expressions. Operations such as multiplying out brackets are actually applications of identities; for example, when expanding the expression $(x-2)(x+1)$ to give x^2-x-2, we are actually using the identity $(x+a)(x+b) \equiv x^2+(a+b)x+ab$, which holds for all a, b and x (in particular for $a=-2$, $b=1$ and unknown x).

When solving equations, we can use any known identities or derive a new one where needed. To derive or 'show' an identity, we essentially rewrite a given expression. We must start from one side, and use known rules and identities to transform the expression step by step until we reach the other side of the identity.

You may be unsure about which rules and identities you are allowed to use. Anything listed as a 'Key point' in this book is acceptable, as are basic algebraic manipulations such as multiplying out brackets and simplifying fractions.

Identities are very important in trigonometry; see chapter 9.

EXAM HINT

You can choose which side of the identity to start from. It is usually a good idea to start from the more complicated expression and work towards the simpler one.

Worked example 3.10

Show that $e^{(2\ln x+\ln 3)} = 3x^2$.

Start from the more complicated expression on the left-hand side.

$\text{LHS} = e^{(2\ln x+\ln 3)}$

First, take the multiple inside the log: $p\ln a = \ln(a^p)$.

$= e^{\ln x^2+\ln 3}$

Then $\ln a + \ln b = \ln(ab)$.

$= e^{\ln(3x^2)}$

Finally, apply the cancellation principle.

$= 3x^2$

$= \text{RHS}$

Exercise 3E

1. Show that the following equations are identities.

(a) (i) $(x-y)^2+4xy=(x+y)^2$

(ii) $x^3+y^3=(x+y)(x^2-xy+y^2)$

(b) (i) $\log(xyz)=\log x+\log y+\log z$

(ii) $\log_a b=\dfrac{1}{\log_b a}$

(c) (i) $\dfrac{a^2-b^2}{a-b}=a+b$ (ii) $\dfrac{x-y}{y-x}=-1$

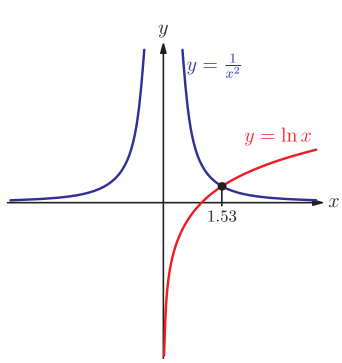
(d) (i) $\sqrt[a]{x^b}=\left(\sqrt[a]{x}\right)^b$ (ii) $2^a+2^a=2^{a+1}$

Summary

- Important methods for solving **equations** include factorising and substitution.
- When sketching graphs on your calculator, you may need to use your knowledge of the shapes of common graphs to make sure you do not miss any important features, such as zeros and asymptotes.
- You can use a graphical calculator to solve equations involving unfamiliar functions by finding intersections of two graphs, one for each side of the equation.
- **Identities** are equations that are true for all values of x. We derive identities by transforming one side into the other, using known identities.

Introductory problem revisited

Solve the equation $\dfrac{1}{x^2}=\ln x$.



If you try to rearrange this equation to isolate x, you will find that it is impossible. It is better to plot the graphs of $y=\dfrac{1}{x^2}$ and $y=\ln x$:

There is one intersection point, and we find its x-coordinate using the calculator. The solution is $x=1.53$ (3 SF).

Mixed examination practice 3

Questions on this topic usually come as parts of longer questions. This exercise is intended to give you a feel for the level of difficulty you may encounter in the examination.

1. (a) Sketch the graphs of $y = 2^x$ and $y = 1 - x^2$ on the same axes.

(b) Hence write down the number of solutions of the equation $2^x = 1 - x^2$.

[6 marks]

2. Find the largest possible value of $y = x^2 e^{-x}$ for $x \in [0,5]$. *[4 marks]*

3. Find the exact solution of the equation $e^x \ln x = 3e^x$. *[5 marks]*

4. Find the equations of the vertical asymptotes of $y = \dfrac{1}{(ax+b)(x-c)}$. *[3 marks]*

5. (a) Sketch the graph of $\dfrac{1}{e^x - 2}$.

(b) State the exact equation of the vertical asymptote. *[6 marks]*

6. By using an appropriate substitution, find the exact solutions to the equation $x^4 + 36 = 13x^2$. *[6 marks]*

7. Solve the equation $x \ln x + 4 \ln x = 0$. *[5 marks]*