UNIT 11 Fractions and Percentages *NC: 2b and 3d*

		St	Ac	Ex	Sp
TOPICS (Text and Practice Books)					
11.1	*Fractions, Decimals and Percentages*	✓	-	-	-
11.2	*Fractions and Percentages of Quantities*	✓	✓	-	-
11.3	*Quantities as Percentages*	✓	✓	-	-
11.4	*More Complex Percentages*	✓	✓	✓	-
11.5	*Percentage Increase and Decrease*	✓	✓	✓	✓
11.6	*Addition and Subtraction of Fractions*	✓	✓	✓	✓
11.7	*Multiplication and Division of Fractions*	✓	✓	✓	✓
11.8	*Compound Interest and Depreciation*	✗	✓	✓	✓
11.9	*Revise Percentage Problems*	✗	✓	✓	✓
Activities (* particularly suitable for coursework tasks)					
11.1	*Currency Exchange*	✓	✓	-	-
11.2	*Taxi Fares*	✓	✓	-	-
11.3	*VAT Problems*	✗	✓	✓	✓
11.4	*Premium Bonds*	✗	✓	✓	✓
11.5	*APR*	✗	✗	✓	✓
OH Slides					
11.1	*Percentages and Fractions*	✓	✓	-	-
11.2	*Fraction Number Lines*	✓	✓	-	-
11.3	*Equivalent Fractions 1*	✓	✓	✓	✓
11.4	*Equivalent Fractions 2*	✓	✓	✓	✓
11.5	*Percentages of Quantities*	✓	✓	✓	✓
11.6	*Adding and Subtracting Fractions*	✓	✓	✓	✓
11.7	*Multiplying Fractions*	✓	✓	✓	✓
Revision Tests					
11.1		✓	-	-	-
11.2		✗	✓	-	-
11.3		✗	✗	✓	✓
Mental Tests					
11.1		✓	✓	-	-
11.2		✓	✓	-	-
11.3		✗	✓	✓	✓
11.4		✗	✓	✓	✓

UNIT 11 *Fractions and Percentages* Teaching Notes

Background and Preparatory Work

In everyday life we almost never think about how we speak or write in terms of 'grammar' – 'parts of speech', tenses', or 'correct sentence construction'. Yet those who never go beyond instinctive, colloquial speech, and who have little feeling for the way the language works, cannot escape from the fact that their possibilities in life are restricted by their limited means of expression.

Mathematics – as Galileo observed – is 'the language in which the Book of Nature is written'. And as our daily lives come to depend more and more on the control we exert on the world around us, it is ever more important for ordinary people to have a deeper understanding for the simple 'grammar' which underpins all mathematics.

Colloquial mathematics is limited to addition. Mathematics proper begins with *multiplication* (and division), and with the associated themes of *ratio*, *fractions* and *proportion* – though this fact could easily be missed by someone reading the English National Curriculum and examining the associated guidance and assessment materials (and their mark schemes)!

The idea of counting has arisen naturally in many, if not most, cultures. The act of counting tends to highlight the fact that the counting sequence is based on repetition of a single step. In its crudest form this step is not yet mathematical – being close to the call 'next please' that one hears in queues the world over. However, once one begins to think in terms of *quantity*, it is fairly natural to re-interpret the pure 'sequencing in time' of successive *numbers* in terms of repeatedly 'increasing each quantity by the *first* number in the sequence': that is, 'add one'. From there it is a relatively short step to primitive addition and subtraction – based on the idea of 'adding on' (as used by infants with their fingers, and at check-outs the world over for giving change). These procedures have built-in limitations, which teachers have to help pupils to transcend; but they illustrate the universal 'colloquial' character of addition.

In our desire to encourage those who find mathematics difficult (and perhaps also to ease the lot of those faced with the difficult task of *teaching* mathematics effectively) we have fallen into the trap of ignoring the nature of the discipline. The National Curriculum suggests that it is enough to *'use appropriate methods to solve problems'*; but the associated bureaucracy too often interprets this to mean 'any method that will obtain an acceptable answer'; thus attention shifts from the appropriateness of the *method* to the acceptability of the answer. This effectively encourages teachers and pupils to be satisfied with any method that seems to work (since this is enough to obtain almost all the available marks). We have therefore got out of the habit of exercising that judgement, which is an essential part of all good teaching, as to whether the method is really acceptable.

The consequences are now clear for all to see (except for those who prefer to close their eyes). For example, in the *Third International Mathematics and Science Study*, a large random sample of Year 4 primary pupils (aged 9) in 29 countries were asked to write the addition sum '$4+4+4+4+4=20$ as a multiplication'. Despite the fact that pupils in most countries start school later (often considerably later) than they do in England, only 39% of English pupils managed what should have been an automatic response. In contrast, the appropriate response was given by 90% of pupils in the top scoring country and by 63% of pupils in the median country.

Our failure to teach pupils to see the multiplicative structure of so many elementary problems, and our willingness to accept inappropriate 'additive' strategies, has profound consequences. In particular, when we encourage pupils to use primitive addition to solve what should be multiplication problems (simply because this appears to allow more of them to obtain the right answer without having to master any new tricks), we effectively convince them that *all problems can be solved using additive strategies.* Thus when faced with ratio problems, which are unavoidably multiplicative, pupils try using additive strategies and are doomed to fail.

Thus, while it is true that a price increase of 10% may at first be worked out by calculating 10% of the original price and then adding, the goal must be to ensure that pupils understand that the result is bound to be 110% of the original price, and so can be obtained *in one step* by multiplying by 1.1.

The advantage of this way of thinking becomes even more pronounced when trying to analyse a problem such as the following:

> *'One third of the class got As. One quarter of the remainder got Bs. What fraction of the class got Cs or worse?'*

The answer can be obtained in lots of ways; but these approaches miss the underlying structure of the problem *which is what makes the problem important*! (Since 'one quarter of the remainder got Bs', the required group is '$\frac{3}{4}$ of those who did not get As'. And since 'one third of the whole group got As', precisely $\frac{2}{3}$ of the whole group did not get As. Thus the answer is $\frac{3}{4}$ of $\frac{2}{3}$ – which simplifies to 'one half'.) Once one learns to think this way, all sorts of other similar problems can be solved very quickly.

The lesson here is that mathematics teachers need a clear *mental map* of elementary school mathematics, in which multiplication and its associated themes (fractions, ratio and proportion) are firmly centre-stage. Thus we need to build systematically and purposefully from

- *multiplication* ($4 \times 3 = 12$, $6 \times 3 = 18$, etc.)

via

- *division* with integer answers ($\frac{12}{3} = 4$, $\frac{18}{3} = 6$, etc.)
- with associated *ratio* problems ('12 pies cost £18, what do 4 pies cost?)

to

- the manipulation and simplification of simple ratios and equivalent fractions ($\frac{18}{12} = \frac{3}{2}$)

- and the complete arithmetic of fractions ('$\dfrac{\left[\frac{2}{3} + \frac{7}{4}\right]}{\left[\frac{5}{6} - \frac{3}{8}\right]}$ = ?').

In practice, an important ingredient in this sequence is the ability to handle *percentages.* But too often percentages are seen as something separate – almost a subject in their own right – when they are in fact a simple application of the multiplicative principle (albeit with a notation of their own).

All of these notions are inter-linked, and introduce key ideas which underpin powerful aspects of elementary mathematics. For example,

- the idea that any ratio problem can be solved by the 'unit' method, or by the 'rule of three';
- the fact that ratios and fractions open the door to the exact solution of any linear equation;
- the central notion that *meaning* demands *simplification* (so that one is never satisfied with '$\frac{18}{36}$' as an answer;
- the subtle (but crucial) advantage of using *exact* fraction notation, rather than converting automatically to approximate, ugly decimals;
- the unstated, but important idea, that the *rational numbers* form a number system which is 'closed' under all four operations (with division by zero forbidden);
- the fact that fractions force one to master, to understand, and then to trust, procedures (to evaluate expressions like $\dfrac{\left(\frac{7}{3}\right)}{\left(\frac{35}{12}\right)}$, or $\left[\dfrac{\left(\frac{2}{3} + \frac{7}{4}\right)}{\left(\frac{5}{6} + \frac{3}{8}\right)}\right]$);
- the way in which *proportion* underlies (a) all measurement, (b) *similarity* in geometry, (c) the definitions of the *trigonometric* functions sin, cos and tan;
- the way proportion is reflected in formulae for lengths and perimeters, for areas, and for volumes, and so on.

Teaching Points

Introduction

Much of this Unit will be revision of what pupils have, in theory, done before, although it is a topic that seems to give many of them great problems. As with all maths topics though, it should be stressed that there are logical rules to be obeyed at all times and if these rules are followed there should be no difficulties!

Part of the problem may lie in the fact that a fraction (e.g. $\frac{1}{5}$) is both a number in its own right (with a unique place on the number line) and also an operation when written as 'one fifth' of a quantity. As a number it has a decimal equivalent (i.e. 0.2), and as an operation it is equivalent to a percentage (20%). OS11.1

It is crucial that all pupils are familiar and confident in moving between fractions, decimals and percentages.

It should also be noted that percentages are a key concept used extensively in the outside world; for example,

- 10% sale reduction
- VAT at $17\frac{1}{2}\%$ A11.3
- interest rates for Banks and Building Societies
- rate of inflation
- APR for loans. A11.5

Language / Notation

Important language used includes

- equivalent fractions OS11.3 and 11.4
- percentage increases/decreases OS11.5
- compound interest. A11.5

It is important always to use the % symbol when finding percentage changes, and it is recommended that fractions should be written as, for example, $\frac{4}{5}$, and not 4/5. The second version can so easily lead to errors when, for example, multiplying fractions together.

Key Points

	St	A	E	Sp
• Equivalent fractions, decimals and percentages.	✓	✓	✓	✓

Fractions	*Decimals*	*Percentages*
$\frac{1}{10}$	0.1	10%
$\frac{1}{8}$	0.125	12.5%
$\frac{1}{5}$	0.2	20%
$\frac{1}{4}$	0.25	25%
$\frac{1}{3}$	$0.\dot{3}$	$33\frac{1}{3}\%$
$\frac{1}{2}$	0.5	50%
$\frac{2}{3}$	$0.\dot{6}$	$66\frac{2}{3}\%$
$\frac{3}{4}$	0.75	75%
1	1.0	100%

- Equivalent fractions,

  e.g. $\frac{1}{3} = \frac{2}{6} = \frac{3}{9} = \frac{4}{12}$, etc. ✓ ✓ ✓ ✓ OS11.3 and 11.4

- Percentage increase (decrease)

  $= \frac{\text{actual increase (decrease)}}{\text{initial value}} = 100$ ✓ ✓ ✓ ✓

- Multiplying fractions,

  $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$ ✓ ✓ ✓ ✓ OS11.7

- Dividing fractions,

  $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$ ✓ ✓ ✓ ✓

- Compound Interest,

  $A_n = \left(1 + \frac{r}{100}\right)^n A_0$ ✗ ✓ ✓ ✓ A11.5

Misconceptions

There are many; for example,

- that 5% is equivalent to 0.05, not 0.5
- that $\frac{1}{4} \div \frac{1}{2} = \frac{1}{2}$ (not 2).

Most pupils will also be unaware that

- $12\frac{1}{2}$% is equivalent to 0.125 and to $\frac{1}{8}$
- when the price of an article is firstly increased by 10% and then decreased by 10%, its final price is not the same as its initial price.