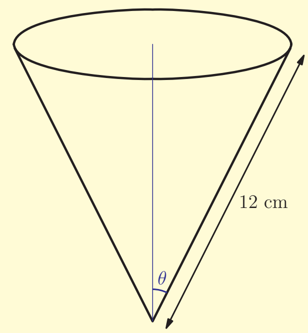
In this chapter you will learn:

- how to differentiate composite functions
- how to differentiate products of functions
- how to differentiate quotients of functions
- how to maximise or minimise functions with constraints.

14 Further differentiation

Introductory problem

If a cone has a fixed slant height of 12 cm, find the maximum volume it can have as the angle θ varies.



In this chapter we will build on the techniques developed in chapter 12 and learn new tools for differentiating a wider range of functions. Much of the work here will also be used in chapter 15, where we expand our integration techniques.

14A Differentiating composite functions using the chain rule

We can already differentiate functions such as $y=(3x^2+5x)^2$ by expanding the brackets and differentiating term by term:

$$\begin{aligned} y &= (3x^2)^2+2(3x^2)(5x)+(5x)^2 \\ &= 9x^4+30x^3+25x^2 \\ \therefore \frac{\mathrm{d}y}{\mathrm{d}x} &= 36x^3+90x^2+50x \\ &= 2x(18x^2+45x+25) \end{aligned}$$

But what if the function is more complicated – for instance, having a higher power or more terms in the brackets? While the same method would work, it is clearly not efficient to expand, say, $y=(3x^2+5x+2)^7$ and then differentiate each term. And what about functions such as $y = \sin 3x$ or $y = e^{x^2}$? Although we know how to differentiate $y = \sin x$ and $y = e^x$, we do not yet know any rules that tell us what to do when the argument of the function is changed to $3x$ or x^2.

The functions $y = (3x^2 + 5x + 2)^7$, $y = \sin 3x$ and $y = e^{x^2}$ may seem quite different but do have something in common – they are all *composite functions*:

We met composite functions in section 4C.

$y = (3x^2 + 5x + 2)^7$	is $y = u^7$	where $u(x) = 3x^2 + 5x + 2$
$y = \sin 3x$	is $y = \sin u$	where $u(x) = 3x$
$y = e^{x^2}$	is $y = e^u$	where $u(x) = x^2$

There is a general rule for differentiating any composite function.

KEY POINT 14.1

The **chain rule**:

If $y = f(u)$ where $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

The proof of the chain rule is very technical and involves differentiation from first principles. In this course we will just accept the rule and learn to use it in a variety of situations. Let us first apply it to the three functions above.

Worked example 14.1

Differentiate the following functions.

(a) $y = (3x^2 + 5x + 2)^7$ (b) $y = \sin 3x$ (c) $y = e^{x^2}$

As these are all composite functions, we can use the chain rule on each of them.

(a) $y = u^7$ where $u = 3x^2 + 5x + 2$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 7u^6 \times (6x + 5)$$

Write the answer in terms of x.

$$= 7(3x^2 + 5x + 2)^6(6x + 5)$$

continued . . .

(b) $y = \sin u$ where $u = 3x$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= (\cos u) \times 3$$

Write the answer in terms of x and rearrange into the conventional form.

$$= 3\cos 3x$$

(c) $y = e^u$ where $u = x^2$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= e^u \times 2x$$

Write the answer in terms of x and rearrange into the conventional form.

$$= 2xe^{x^2}$$

Part (b) of Worked example 14.1 illustrates a special case of the chain rule, where the 'inside' function is of the form $ax + b$.

KEY POINT 14.2

$$\frac{d}{dx} f(ax+b) = af'(ax+b)$$

For example,

$$\frac{d}{dx}(4x+1)^7 = 4 \times 7(4x+1)^6 \text{ and } \frac{d}{dx}(e^{3-2x}) = -2e^{3-2x}.$$

It is useful to remember this shortcut and it is not necessary to write down the 'inner function' u each time. The chain rule calculation can then be done more concisely, as demonstrated in the next example. The basic idea is to imagine brackets around the inner function, differentiate the outer function as if the bracketed expression were a single argument, and then multiply by the derivative of the bracketed expression.

Worked example 14.2

Differentiate the following composite functions.

(a) $y = e^{x^2-3x}$

(b) $y = \dfrac{3}{\sqrt{x^3-5}}$

$e^{()}$ differentiates to $e^{()}$
x^2-3x differentiates to $2x-3$.

(a) $y = e^{(x^2-3x)}$

$$\frac{dy}{dx} = (2x-3)e^{(x^2-3x)}$$

First, rewrite the square root as a power.

(b) $y = 3(x^3-5)^{-\frac{1}{2}}$

$3(\ )^{-\frac{1}{2}}$ differentiates to $-\frac{3}{2}(\ )^{-\frac{3}{2}}$
x^3-5 differentiates to $3x^2$.

$$\frac{dy}{dx} = -\frac{3}{2}(x^3-5)^{-\frac{3}{2}}(3x^2)$$

$$= -\frac{9}{2}x^2(x^3-5)^{-\frac{3}{2}}$$

EXAM HINT

It takes practice to be able to apply the chain rule without writing down u explicitly. Don't worry if you find this method confusing; although it is often quicker, you will never be forced to use it.

Sometimes it is necessary to apply the chain rule more than once.

Worked Example 14.3

Differentiate $y = \cos^3(\ln 2x)$.

Remember that $\cos^3 A$ means $(\cos A)^3$. You can think of this as a composite of three functions.

$y = (\cos(\ln 2x))^3$

$(\ )^3$ differentiates to $3(\ )^2$
$\cos(\ )$ differentiates to $-\sin(\ )$
$\ln 2x$ differentiates to $2 \times \frac{1}{2x} = \frac{1}{x}$ (by Key point 14.2).

$$\frac{dy}{dx} = 3(\cos(\ln 2x))^2 \times (-\sin(\ln 2x)) \times \frac{1}{x}$$

$$= -\frac{3}{x}\cos^2(\ln 2x)\sin(\ln 2x)$$

Exercise 14A

1. Use the chain rule to differentiate the following expressions with respect to x.

(a) (i) $(3x+4)^5$ (ii) $(5x+4)^7$

(b) (i) $\sqrt{3x-2}$ (ii) $\sqrt{x+1}$

(c) (i) $\frac{1}{3-x}$ (ii) $\frac{1}{(2x+3)^2}$

(d) (i) e^{10x+1} (ii) e^{4-3x}

(e) (i) $\sin 4x$ (ii) $\cos(3x+\pi)$

(f) (i) $\ln(5-x)$ (ii) $\ln(3-2x)$

2. Use the chain rule to differentiate the following expressions with respect to x.

(a) (i) $(x^2-3x+1)^7$ (ii) $(x^3+1)^5$

(b) (i) e^{x^2-2x} (ii) e^{4-x^3}

(c) (i) $(2e^x+1)^{-3}$ (ii) $(2-5e^x)^{-4}$

(d) (i) $\sin(3x^2+1)$ (ii) $\cos(x^2+2x)$

(e) (i) $\cos^3 x$ (ii) $\sin^4 x$

(f) (i) $\ln(2x-5x^3)$ (ii) $\ln(4x^2-1)$

(g) (i) $(4\ln x-1)^4$ (ii) $(\ln x+3)^{-5}$

3. Differentiate the following using the short cut from Key point 14.2.

(a) (i) $(2x+3)^5$ (ii) $(4x-1)^8$

(b) (i) $(5-x)^{-4}$ (ii) $(1-x)^{-7}$

(c) (i) $\cos(1-4x)$ (ii) $\cos(2-x)$

(d) (i) $\ln(5x+2)$ (ii) $\ln(x-4)$

4. Differentiate the following using the chain rule twice.

(a) (i) $\frac{1}{\cos^2 3x}$ (ii) $\tan^2 2x$

(b) (i) $e^{\sin^2 3x}$ (ii) $e^{(\ln 2x)^2}$

(c) (i) $(1-2\sin^2 2x)^2$ (ii) $(4\cos 3x+1)^2$

(d) (i) $\ln(1-3\cos 2x)$ (ii) $\ln(2-\cos 5x)$

5. Find the equation of the tangent to $y=(3x+5)^2$ at the point where $x=2$. *[7 marks]*

6. Find the equation of the normal to the curve $y = \dfrac{1}{\sqrt{4x^2+1}}$ at the point where $x = \sqrt{2}$. *[7 marks]*

7. Find the exact coordinates of stationary points on the curve $y = e^{\sin x}$ for $x \in [0, 2\pi]$. *[5 marks]*

8. For what values of x does the function $f(x) = \ln(x^2 - 35)$ have a gradient of 1? *[5 marks]*

9. A non-uniform chain hangs from two posts. Its height h above the ground is described by the equation

$$h = e^x + \frac{1}{e^{2x}}, \ -1 \le x \le 2$$

The left post is positioned at $x = -1$, and the right post is at $x = 2$.

(a) State, with reasons, which post is taller.

(b) Show that the minimum height occurs at $x = \frac{1}{3}\ln 2$.

(c) Find the exact value of the minimum height of the chain. *[8 marks]*

Many people think that a chain fixed at both ends will hang as a parabola, but it can be proved that it hangs in the shape of the curve in question 9, called a *catenary*. The proof of this requires techniques from a mathematical area called *differential geometry*.

14B Differentiating products using the product rule

We now move on to products of two functions. We can already differentiate some products, such as $y = x^4(3x^2 - 5)$, by expanding and then differentiating term by term. However, as for composite functions, this method is rather impractical when the function is more complicated, for example $y = x^4(3x^2 - 5)^9$; moreover, expanding does not help with functions such as $y = x^2 \cos x$ or $y = x \ln x$.

Just as there is a rule for differentiating composite functions, there is a rule for differentiating products.

KEY POINT 14.3

The product rule

If $y = u(x)v(x)$, then

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

See Fill-in proof sheet 15 on the CD-ROM if you are interested in how this rule is derived.

Worked example 14.4

Differentiate $y = x^4(3x^2 - 5)$ using the product rule.

It doesn't make any difference which function you call u and which you call v.

Let $u = x^4$ and $v = 3x^2 - 5$. Then

$$\frac{du}{dx} = 4x^3$$

and

$$\frac{dv}{dx} = 6x$$

Apply the product rule.

$$\begin{aligned}\frac{dy}{dx} &= v\frac{du}{dx} + u\frac{dv}{dx}\\ &= (3x^2 - 5)4x^3 + x^4 \times 6x\\ &= 12x^5 - 20x^3 + 6x^5\\ &= 18x^5 - 20x^3\end{aligned}$$

EXAM HINT

After applying the product rule, you do not need to simplify the resulting expression unless the question explicitly tells you to do so.

When differentiating a more complicated product, be aware that the chain rule may be needed as well as the product rule.

Worked example 14.5

Differentiate $y = x^4(3x^2-5)^5$ and factorise your answer.

This is a product of two functions x^4 and $(3x^2-5)^5$. It does not matter which we take as u and which as v.

Let $u = x^4$ and $v = (3x^2-5)^5$.

Then

$$\frac{du}{dx} = 4x^3$$

$v(x)$ is a composite function, so use the chain rule to find its derivative.

and

$$\frac{dv}{dx} = 5(3x^2-5)^4(6x)$$
$$= 30x(3x^2-5)^4$$

Now apply the product rule.

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$
$$= (3x^2-5)^5\, 4x^3 + x^4\, 30x(3x^2-5)^4$$

We are asked to factorise the answer, so instead of expanding the brackets, look for common factors.

$$= 2x^3(3x^2-5)^4\left[2(3x^2-5)+15x^2\right]$$
$$= 2x^3(3x^2-5)^4(6x^2-10+15x^2)$$
$$= 2x^3(3x^2-5)^4(21x^2-10)$$

Exercise 14B

1. Differentiate the following using the product rule.

(a) (i) $y = x(1+x)^3$ (ii) $y = 4x^2(x+3)^4$

(b) (i) $x^2 \sin x$ (ii) $5x \tan x$

(c) (i) $e^x \ln x$ (ii) $e^x \sin x$

2. Differentiate the following using the product rule.

(a) (i) $y = x^2 \cos x$ (ii) $y = x^{-1} \sin x$

(b) (i) $y = x^{-2} \ln x$ (ii) $y = x^3 \ln x$

(c) (i) $y = x^3\sqrt{2x+1}$ (ii) $y = x^{-1}\sqrt{4x}$

(d) (i) $e^{2x} \tan x$ (ii) $e^{x+1} \sin 3x$

3. Find $f'(x)$ and fully factorise your answer.

(a) (i) $f(x) = (x+1)^4(x-2)^5$ (ii) $f(x) = (x-3)^7(x+5)^4$

(b) (i) $f(x) = (2x-1)^4(1-3x)^3$ (ii) $f(x) = (1-x)^5(4x+1)^2$

4. Differentiate $y=(3x^2-x+2)e^{2x}$, giving your answer in the form $P(x)e^{2x}$. *[4 marks]*

5. Find the equation of the tangent to $y=xe^x$ where $x=1$. *[7 marks]*

6. Given that $f(x)=x^2e^{3x}$, find $f''(x)$ in the form $(ax^2+bx+c)e^{3x}$. *[4 marks]*

7. Find the x-coordinates of the stationary points on the curve $y=(2x+1)^5 e^{-2x}$. *[5 marks]*

8. Find the coordinates of the stationary point on the graph of $y=x\sqrt{x+1}$. *[3 marks]*

9. Find the exact values of the x-coordinates of the stationary points on the curve $y=(3x+1)^5(3-x)^3$. *[6 marks]*

10. Consider the graph of $y=x\sin 2x$ for $x\in[0,2\pi]$.

(a) Show that the x-coordinates of the points of inflexion satisfy $\cos 2x = x\sin 2x$.

(b) Hence find the coordinates of the points of inflexion. *[6 marks]*

11. Find the derivative of $\sin(xe^x)$ with respect to x. *[5 marks]*

12. (a) If $f(x)=x\ln x$, find $f'(x)$.

(b) Hence find $\int \ln x\,dx$. *[5 marks]*

13. Find the exact coordinates of the minimum point on the curve $y=e^{-x}\cos x$, for $0\le x\le \pi$. *[6 marks]*

14. Given that $f(x)=x^2\sqrt{1+x}$, show that $f'(x)=\dfrac{x(a+bx)}{2\sqrt{1+x}}$ where a and b are constants to be found. *[6 marks]*

15. (a) Write $y=x^x$ in the form $y=e^{f(x)}$.

(b) Hence or otherwise, find $\dfrac{dy}{dx}$.

(c) Find the exact coordinates of the stationary points on the curve $y=x^x$. *[8 marks]*

14C Differentiating quotients using the quotient rule

By combining the product rule and the chain rule, we can differentiate quotients such as

$$y = \frac{x^2 - 4x + 12}{(x-3)^2}$$

This function can be expressed as

$$y = (x^2 - 4x + 12)(x-3)^{-2}$$

Then, taking $u = (x^2 - 4x + 12)$ and $v = (x-3)^{-2}$, with $\frac{du}{dx} = 2x - 4$ and $\frac{dv}{dx} = (-2)(x-3)^{-3}$, we get

$$\frac{dy}{dx} = (x-3)^{-2}(2x-4) + (x^2 - 4x + 12)(-2)(x-3)^{-3}$$

After tidying up the negative powers and fractions, this simplifies to $\frac{dy}{dx} = \frac{-2x-12}{(x-3)^3}$.

If we apply the same process to a general function of the form $\frac{u(x)}{v(x)}$, we can derive a new rule for differentiating quotients.

See Fill-in proof 16 on the CD-ROM for details.

In this course you are only expected to know how to use the result.

KEY POINT 14.4

The **quotient rule**

If $y = \frac{u(x)}{v(x)}$, then

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Worked example 14.6

Differentiate $y=\frac{x^2-4x+12}{(x-3)^2}$ using the quotient rule and simplify your answer as far as possible.

The function is a quotient. Make sure to get u and v the right way round.

$y=\frac{u}{v}$ where $u=x^2-4x+12$, $v=(x-3)^2$

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

Use chain rule to differentiate v.

$$=\frac{(x-3)^2(2x-4)-(x^2-4x+12)\times 2(x-3)}{\left[(x-3)^2\right]^2}$$

Notice that we can cancel a factor of $(x-3)$ from top and bottom.

$$=\frac{(x-3)(2x-4)-(x^2-4x+12)\times 2}{(x-3)^3}$$

$$=\frac{2x^2-10x+12-2x^2+8x-24}{(x-3)^3}$$

$$=\frac{-2x-12}{(x-3)^3}$$

$$=\frac{-2(x+6)}{(x-3)^3}$$

In section 12E we stated that the derivative of $\tan x$ is $\frac{1}{\cos^2 x}$.

We can now use the quotient rule, together with the derivatives of $\sin x$ and $\cos x$, to prove this result.

Worked example 14.7

Prove that $\frac{d}{dx}(\tan x)=\frac{1}{\cos^2 x}$.

Express $\tan x$ in terms of $\sin x$ and $\cos x$, whose derivatives we know.

$\tan x=\frac{\sin x}{\cos x}$

Let $u=\sin x$, $v=\cos x$

continued . . .

Use the quotient rule.

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

$$=\frac{\cos x\cos x-\sin x(-\sin x)}{(\cos x)^2}$$

$$=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}$$

Use the identity $\sin^2 x+\cos^2 x=1$ to simplify.

$$=\frac{1}{\cos^2 x}$$

The quotient rule, like the product rule, often leads to a long expression. Sometimes product and quotient rule questions are used to test your skill with fractions and exponents, as in the following example.

Worked example 14.8

Differentiate $\frac{x}{\sqrt{x+1}}$ and give your answer in the form $\frac{x+c}{k\sqrt{(x+1)^p}}$ where $c, k, p \in \mathbb{N}$.

The function is a quotient. Identify u and v.

$$y=\frac{x}{\sqrt{x+1}}$$

Let $u=x$, $v=\sqrt{x+1}=(x+1)^{\frac{1}{2}}$

Use the quotient rule

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

$$=\frac{(x+1)^{\frac{1}{2}}\times 1-x\times\frac{1}{2}(x+1)^{-\frac{1}{2}}}{\left((x+1)^{\frac{1}{2}}\right)^2}$$

As we want a square root in the answer, turn the fractional powers back into roots.

$$=\frac{\sqrt{x+1}-\frac{x}{2\sqrt{x+1}}}{x+1}$$

continued . . .

Remove 'fractions within fractions' by multiplying top and bottom by $2\sqrt{x+1}$.

$$= \frac{2(x+1)-x}{2(x+1)\sqrt{x+1}}$$

Notice that $x\sqrt{x} = x^{\frac{3}{2}} = \sqrt{x^3}$.

$$= \frac{x+2}{2\sqrt{(x+1)^3}}$$

Exercise 14C

1. Differentiate the following using the quotient rule.

(a) (i) $y = \dfrac{x-1}{x+1}$ (ii) $y = \dfrac{x+2}{x-3}$

(b) (i) $y = \dfrac{\sqrt{2x+1}}{x}$ (ii) $y = \dfrac{x^2}{\sqrt{x-1}}$

(c) (i) $y = \dfrac{1-2x}{x^2+2}$ (ii) $y = \dfrac{4-x^2}{1+x}$

(d) (i) $y = \dfrac{\ln 3x}{x}$ (ii) $y = \dfrac{\ln 2x}{x^2}$

2. Find the coordinates of the stationary points on the graph of $y = \dfrac{x^2}{2x-1}$. *[5 marks]*

3. Find the equation of the normal to the curve $y = \dfrac{\sin x}{x}$ at the point where $x = \dfrac{\pi}{2}$, giving your answer in the form $y = mx + c$ where m and c are exact. *[7 marks]*

4. The graph of $y = \dfrac{x-a}{x+2}$ has gradient 1 at the point $(a,0)$, where $a \neq -2$. Find the value of a. *[5 marks]*

5. Find the exact coordinates of the stationary point on the curve $y = \dfrac{\ln x}{x}$ and determine its nature. *[6 marks]*

6. Find the range of values of x for which the function $f(x) = \dfrac{x^2}{1-x}$ is increasing. *[6 marks]*

7. Given that $y = \frac{x^2}{\sqrt{x+1}}$, show that $\frac{dy}{dx} = \frac{x(ax+b)}{2(x+1)^p}$, stating clearly the value of the constants a, b and p. *[6 marks]*

8. Show that if the curve $y = f(x)$ has a maximum stationary point at $x = a$, then the curve $y = \frac{1}{f(x)}$ has a minimum stationary point at $x = a$, provided $f(a) \neq 0$. *[7 marks]*

14D Optimisation with constraints

In this section we shall look at how to maximise or minimise functions which at first sight appear to depend on two different variables. However, the two variables will be related by a constraint, which allows us to eliminate one of them; we can then follow the usual procedure for finding maxima or minima.

We looked at optimisation in section 12J.

Worked example 14.9

Find the maximum value of $xy - y$ given that $x + 3y = 7$.

Give the function a name.

We wish to maximise $F = xy - y$

Use the constraint to write one variable in terms of the other, and hence express F as a function of one variable only.

$$x + 3y = 7$$
$$\Rightarrow x = 7 - 3y$$
$$\therefore F = (7 - 3y)y - y$$
$$= 6y - 3y^2$$

Find stationary points.

$$\frac{dF}{dy} = 6 - 6y$$

At a stationary point $\frac{dF}{dy} = 0$:

$$6 - 6y = 0$$
$$\Leftrightarrow \quad y = 1$$

and so $x = 7 - 3y = 4$

The value of F at this point is

$$F = 4 \times 1 - 1 = 3$$

Classify the stationary point.

$$\frac{d^2F}{dx^2} = -6 < 0$$

So $F = 3$ is a local maximum.

continued . . .

Check end points and asymptotes.

There are no asymptotes, and as $|y|$ gets large $F = 6y - 3y^2$ becomes large and negative.
So 3 is the global maximum value.

Sometimes the constraint is not explicitly given, and needs to be deduced from the context. Two common types of constraints are:

- A shape has a fixed perimeter, area or volume – this gives an equation relating different variables (height, length, radius, etc.).
- A point lies on a given curve – this gives a relationship between x and y.

Worked example 14.10

Find the point on the curve $y = x^3$ that is closest to the point (2, 0).

Give a name to the quantity we need to optimise.

Let L be the distance from (2, 0) to the point $P(x, y)$.

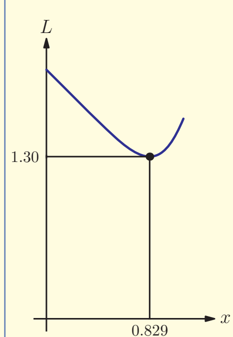
Then $L = \sqrt{(x-2)^2 + y^2}$

Write the function in terms of one variable only.

If P lies on the curve, then $y = x^3$, so

$L = \sqrt{(x-2)^2 + x^6}$

Find stationary points. This function looks complicated and the question does not require exact answers, so we can use the GDC.



From GDC, the minimum occurs at $x = 0.829$ (3 SF).
The corresponding y-value is $y = 0.569$ (3 SF).
So the coordinates are (0.829, 0.569).

Exercise 14D

1. (a) (i) Find the maximum possible value of xy given that $x+2y=4$.

   (ii) Find the maximum possible value of xy given that $3x+y=7$.

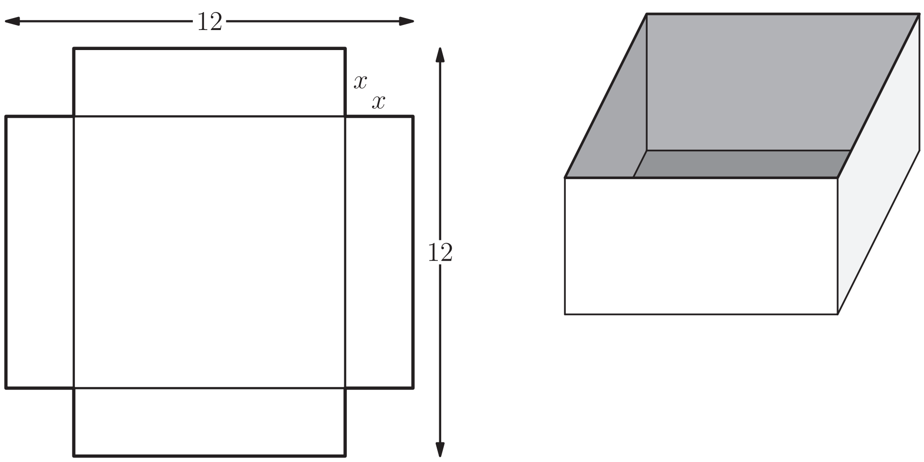
   (b) (i) Find the minimum possible value of $a+b$ given that $ab=3$ and $a,b>0$.

   (ii) Find the minimum possible value of $2a+b$ given that $ab=4$ and $a,b>0$.

   (c) (i) Find the maximum possible value of $4r^2h$ if $2r^2+rh=3$ and $r,h>0$.

   (ii) Find the maximum possible value of rh^2 if $4r^2+3h^2=12$ and $r,h>0$.

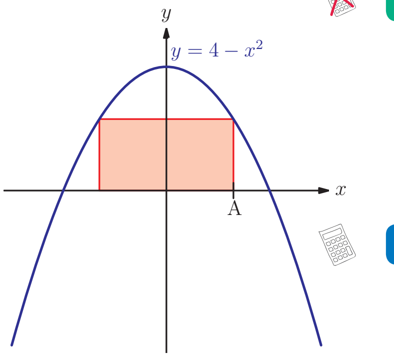
2. A square sheet of card with 12 cm sides has four squares of side x cm cut from the corners. The sides are then folded to make a small open box.



   (a) Find an expression for the volume of the box in terms of x.

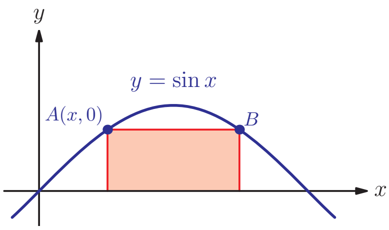
   (b) Find the value of x for which the volume is maximum possible, and prove that it is a maximum. *[6 marks]*

3. An open box in the shape of a square-based prism has volume 32 cm². Find the minimum possible surface area of the box. *[6 marks]*



4. A rectangle is drawn inside the region bounded by the curve $y = 4 - x^2$ and the x-axis, so that two of its vertices lie on the x-axis and the other two lie on the curve.

Find the x-coordinate of vertex A so that the area of the rectangle is the maximum possible. *[6 marks]*



5. A rectangle is drawn inside the region bounded by the curve $y = \sin x$ and the x-axis, as shown in the diagram. The vertex A has coordinates $(x, 0)$.

(a) (i) Write down the coordinates of point B.

(ii) Find an expression for the area of the rectangle in terms of x.

(b) Show that the rectangle has maximum area when $2\tan x = \pi - 2x$.

(c) Find the maximum possible area of the rectangle. *[8 marks]*

6. What is the largest possible capacity of a closed cylindrical can with surface area $450\,\text{cm}^2$? *[6 marks]*

7. What is the largest possible capacity of a closed square-based can with surface area $450\,\text{cm}^2$? *[6 marks]*

8. The sum of two numbers x and y is 6, and $x, y \geq 0$. Find the two numbers if the sum of their squares is

(a) the minimum possible

(b) the maximum possible. *[7 marks]*

9. A cone of radius r and height h has volume 81π.

(a) Show that the curved surface area of the cone is given by $S = \frac{\pi}{r}\sqrt{r^6 + 243^2}$.

(b) Find the radius and height of the cone that make the curved surface area of the cone as small as possible. *[7 marks]*

10. A 20 cm piece of wire is bent to form an isosceles triangle with base b.

(a) Show that the area of the triangle is given by $A = \frac{b}{2}\sqrt{100 - 10b}$.

(b) Show that the area of the triangle is the largest possible when the triangle is equilateral. *[6 marks]*

11. The sum of the squares of two positive numbers is a. Prove that their product is the maximum possible when the two numbers are equal. *[6 marks]*

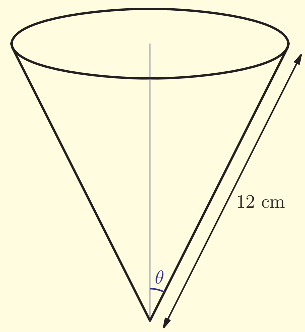
12. Find the coordinates of the point on the curve $y = x^2$, $x \geq 0$, that is closest to the point (0, 4). *[7 marks]*

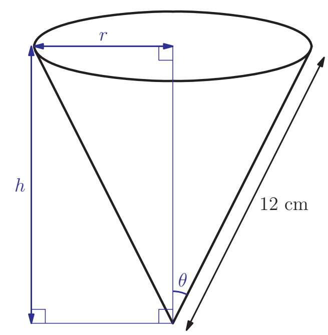
Summary

- The **chain rule** is used to differentiate composite functions.
  If $y = f(u)$ where $u = g(x)$, then
  $$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
- The **product rule** is used to differentiate two functions multiplied together.
  If $y = u(x)v(x)$, then
  $$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$
- The **quotient rule** is used to differentiate one function divided by another.
  If $y = \frac{u(x)}{v(x)}$, then
  $$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
- When solving optimisation problems that involve a function which depends on two variables, the variables will be related by a constraint. It is possible to use the constraint to express the quantity we wish to minimise or maximise in terms of one variable only (by eliminating the other variable); then we can differentiate this function and find its stationary points.
- Two common types of constraint are:
  - a shape with a fixed perimeter, area or volume (this gives an equation relating different variables)
  - a point that lies on a given curve (this gives a relationship between x and y).

Introductory problem revisited

If a cone has a fixed slant height of 12 cm, find the maximum volume it can have as the angle θ varies.





The first thing we need to do is write an expression for the volume V of the cone. Then we will aim to differentiate with respect to θ and solve $\frac{dV}{d\theta} = 0$ to find the value of θ at which the maximum occurs.

The formula for the volume of a cone is

$$V = \frac{1}{3}\pi r^2 h$$

Using the right-angled triangle shown in the diagram, we have

$$r = 12\sin\theta$$

$$h = 12\cos\theta$$

Substituting these into the formula for V gives

$$\begin{aligned} V &= \frac{1}{3}\pi(12\sin\theta)^2(12\cos\theta) \\ &= \frac{12^3}{3}\pi\sin^2\theta\cos\theta \end{aligned}$$

Now that V is expressed in terms of θ only, we can differentiate:

$$\begin{aligned} \frac{dV}{d\theta} &= \frac{12^3}{3}\pi\left[(2\sin\theta\cos\theta)\cos\theta + \sin^2\theta(-\sin\theta)\right] \\ &= \frac{12^3}{3}\pi\left[2\sin\theta\cos^2\theta - \sin^3\theta\right] \end{aligned}$$

For stationary points, $\frac{dV}{d\theta} = 0$:

$$\frac{12^3}{3}\pi[2\sin\theta\cos^2\theta - \sin^3\theta] = 0$$

$$\Leftrightarrow 2\sin\theta\cos^2\theta - \sin^3\theta = 0$$

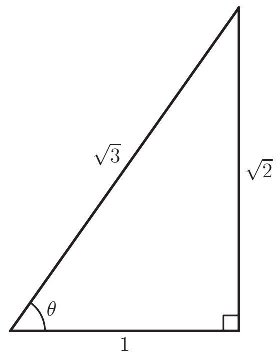
$$\Leftrightarrow \sin\theta(2\cos^2\theta - \sin^2\theta) = 0$$

$$\Leftrightarrow \sin\theta = 0 \quad \text{or} \quad 2\cos^2\theta - \sin^2\theta = 0$$

$\sin\theta = 0$ has no valid solutions, as $0° < \theta < 90°$ for the cone.

The other equation, $2\cos^2\theta - \sin^2\theta = 0$, is satisfied when

$$\sin^2\theta = 2\cos^2\theta$$
$$\Leftrightarrow \tan^2\theta = 2$$
$$\therefore \tan\theta = \sqrt{2}$$



($\tan\theta = -\sqrt{2}$ has no solutions in $0° < \theta < 90°$).

Therefore the maximum volume is attained when $\tan\theta = \sqrt{2}$, which implies that $\sin\theta = \frac{\sqrt{2}}{\sqrt{3}}$ and $\cos\theta = \frac{1}{\sqrt{3}}$ (see the right-angled triangle diagram).

Finally, substitute these values of $\sin\theta$ and $\cos\theta$ into the expression $V = \frac{12^3}{3}\pi\sin^2\theta\cos\theta$ to find the maximum volume:

$$\begin{aligned} V_{\max} &= \frac{12^3}{3}\pi\left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2\left(\frac{1}{\sqrt{3}}\right) \\ &= \frac{12^3 \times 2\sqrt{3}\pi}{3^3} \\ &= 128\sqrt{3}\pi \end{aligned}$$

Mixed examination practice 14

Short questions

1. Find the exact value of the gradient of the curve with equation $y=\frac{1}{4-x^2}$ at the point where $x=\frac{1}{2}$. *[5 marks]*

2. Find $\frac{dy}{dx}$ if

(a) $y=e^{5x}$

(b) $y=\sqrt{3x+2}$

(c) $y=e^{5x}\sqrt{3x+2}$ *[8 marks]*

3. The graph of $y=xe^{-kx}$ has a stationary point at $x=\frac{2}{5}$. Find the value of k. *[4 marks]*

4. The diagram shows a rectangular area ABCD bounded on three sides by 60 m of fencing, and on the fourth by a wall AB.

Find the width of the rectangle (that is, the length AD) that gives its maximum area. *[5 marks]*

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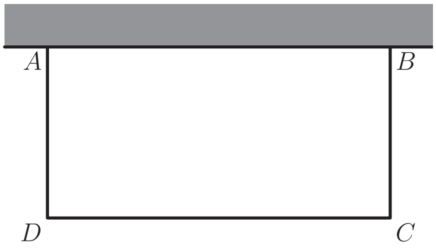

5. A curve has equation

$$f(x)=\frac{a}{b+e^{-cx}} \quad \text{where } a\neq 0 \text{ and } b,c>0$$

(a) Show that

$$f''(x)=\frac{ac^2e^{-cx}\left(e^{-cx}-b\right)}{\left(b+e^{-cx}\right)^3}$$

(b) Find the coordinates of the point on the curve where $f''(x)=0$. *[11 marks]*

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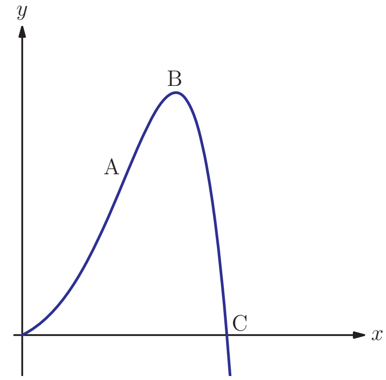
Long questions

1. The function f is defined as $f(x)=e^{x}\sin x$, where x is in radians. Part of the curve of f is shown in the diagram. There is a point of inflexion at A and a local maximum point at B. The graph of f intersects the x-axis at the point C.



   (a) Write down the x-coordinate of the point C. *[1 mark]*

   (b) (i) Find $f'(x)$.

   (ii) Write down the value of $f'(x)$ at the point B.

   (c) Show that $f''(x)=2e^{x}\cos x$.

   (d) (i) Write down the value of $f''(x)$ at A, the point of inflexion.

   (ii) Hence, calculate the coordinates of A. *[11 marks]*

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2. A curve has equation $y=\dfrac{x^2}{1-2x}$.

   (a) Write down the equation of the vertical asymptote of the curve.

   (b) Use differentiation to find the coordinates of stationary points on the curve.

   (c) Determine the nature of the stationary points.

   (d) Sketch the graph of $y=\dfrac{x^2}{1-2x}$. *[17 marks]*

3. The function f is defined by $f(x)=\dfrac{x^2}{2^x}$ for $x>0$.

   (a) (i) Show that $2^x=e^{x\ln 2}$.

   (ii) Hence show that $\dfrac{d}{dx}(2^x)=2^x\ln 2$.

   (b) (i) Show that $f'(x)=\dfrac{2x-x^2\ln 2}{2^x}$.

   (ii) Obtain an expression for $f''(x)$, simplifying your answer as far as possible.

   (c) (i) Find the *exact* value of x satisfying the equation $f'(x)=0$.

   (ii) Show that this value gives a maximum value for $f(x)$. *[14 marks]*