In this chapter you will learn:

- to integrate using known derivatives
- to integrate by reversing the chain rule
- to integrate using a change of variable (substitution)
- to apply integration to problems involving motion (kinematics)
- how to find volumes of rotationally symmetrical shapes.

15 Further integration

Introductory problem

Prove that the area of a circle with radius r is πr^2.

Having extended the range of functions we can differentiate, we now do the same for integration. Sometimes we will be able to use results from previous chapters directly, but in other cases we have to develop new techniques. In this chapter we look at the most commonly used integration method – substitution or reversing the chain rule – and show how integration can be used to solve some problems that arise in applications.

15A Reversing standard derivatives

In chapter 13 we reversed a number of standard derivatives that had been established in chapter 12, thus obtaining the following list of basic integrals:

$$\int x^n \, dx = \frac{1}{n+1}x^{n+1} + c, \qquad n \neq -1$$

$$\int e^x \, dx = e^x + c$$

$$\int \frac{1}{x} \, dx = \ln x + c, \qquad x > 0$$

$$\int \sin x \, dx = -\cos x + c$$

$$\int \cos x \, dx = \sin x + c$$

The chain rule for differentiation (section 14A) allows us to go further and deal with integrals such as $\int 2\cos(2x)\,dx$. Here the idea is to integrate cos to sin and then think about what the chain rule would give us if we differentiated back. In this case, $\frac{d}{dx}(\sin 2x) = 2\cos 2x$, and the factor 2 (the derivative of $2x$) that came from applying the chain rule happens to match precisely the factor in the original integral, so we have the correct answer straight away:

$$\int 2\cos(2x)\,dx = \sin 2x + c$$

However, in a similar situation, we may find that the factor coming from the chain rule does not match the factor in the integral. If these factors are constants, we can compensate for the discrepancy by cancelling out the constant factor generated by the chain rule. For example, to find $\int \sin(3x)\,dx$ we proceed as before, integrating sin to –cos; but this time, when we differentiate back, the chain rule gives us an unwanted factor of 3: $\frac{d}{dx}(-\cos 3x) = 3\sin 3x$. Therefore, we simply divide by 3 to cancel it out:

$$\int \sin(3x)\,dx = -\frac{1}{3}\cos 3x + c$$

This method can be used with any of the standard derivatives and integrals in the list above.

Worked example 15.1

Find the following integrals.

(a) $\int (2x-3)^4\,dx$ (b) $\int (1-4x)^{-\frac{2}{3}}\,dx$

Integrate $(\ )^4$ to $\frac{1}{5}(\ )^5$ and then consider the effect of the chain rule.

We know that $\frac{d}{dx}\left(\frac{1}{5}(2x-3)^5\right) = 2(2x-3)^4$, so multiply by $\frac{1}{2}$ to remove the unwanted factor of 2.

(a)

$$\int (2x-3)^4\,dx = \frac{1}{2}\times\frac{1}{5}(2x-3)^5 + c$$
$$= \frac{1}{10}(2x-3)^5 + c$$

Integrate $(\ )^{-\frac{2}{3}}$ to $3(\ )^{\frac{1}{3}}$ and then consider the effect of the chain rule.

We know that $\frac{d}{dx}\left(3(1-4x)^{\frac{1}{3}}\right) = (-4)(1-4x)^{-\frac{2}{3}}$ so multiply by $-\frac{1}{4}$ to remove the unwanted –4.

(b)

$$\int (1-4x)^{-\frac{2}{3}}\,dx = \left(-\frac{1}{4}\right)3(1-4x)^{\frac{1}{3}} + c$$
$$= -\frac{3}{4}(1-4x)^{\frac{1}{3}} + c$$

You may notice a pattern here: we always divide by the coefficient of *x*. This is indeed a general rule, which comes from reversing the special case of the chain rule given in Key point 14.2.

KEY POINT 15.1

If a and b are constants, then

$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b)+c$$

where $F(x)$ is the integral of $f(x)$

EXAM HINT

This rule only applies when the 'inside' function is of the form $ax+b$, where a and b are constants.

With this shortcut in hand, we no longer need to consciously consider the effect of the chain rule every time.

Worked example 15.2

Find the following:

(a) $\int \frac{1}{2}e^{4x}\,dx$ (b) $\int \frac{2}{6x+5}\,dx$

Integrate e$^{(\,)}$ to e$^{(\,)}$ and divide by the coefficient of x.

(a)

$$\int \frac{1}{2}e^{4x}dx = \frac{1}{2}\times\frac{1}{4}e^{4x}+c$$

$$=\frac{1}{8}e^{4x}+c$$

Integrate $\frac{1}{(\,)}$ to ln( ) and divide by the coefficient of x.

(b)

$$\int \frac{2}{6x+5}dx = 2\times\frac{1}{6}\ln(6x+5)+c$$

$$=\frac{1}{3}\ln(6x+5)+c$$

You may need to use algebra to rewrite the expression in the form of a standard derivative which can then be reversed.

Worked example 15.3

Find the integral

$$\int \frac{x+4}{12-5x-2x^2}\,dx$$

If there are polynomials in the expression, it is generally a good idea to check whether they factorise.

$$\int \frac{x+4}{12-5x-2x^2}dx = \int \frac{x+4}{(3-2x)(x+4)}dx$$

Now the function resembles one of the standard derivatives.

$$=\int \frac{1}{3-2x}dx$$

So we can use the rule in Key point 15.1.

$$=-\frac{1}{2}\ln(3-2x)+c$$

Exercise 15A

1. Find the following integrals.

(a) (i) $\int 5(x+3)^4\,dx$ (ii) $\int (x-2)^5\,dx$

(b) (i) $\int (4x-5)^7\,dx$ (ii) $\int \left(\frac{1}{8}x+1\right)^3 dx$

(c) (i) $\int (4-x)^8\,dx$ (ii) $\int 4\left(3-\frac{1}{2}x\right)^6 dx$

(d) (i) $\int \sqrt{2x-1}\,dx$ (ii) $\int 7(2-5x)^{\frac{3}{4}}\,dx$

(e) (i) $\int \frac{6}{(4-3x)^2}\,dx$ (ii) $\int \frac{1}{\sqrt[4]{2+\frac{x}{3}}}\,dx$

2. Find the following integrals.

(a) (i) $\int 3e^{3x}\,dx$ (ii) $\int e^{2x+5}\,dx$

(b) (i) $\int e^{\frac{1}{2}x}\,dx$ (ii) $\int 4e^{\frac{2x-1}{3}}\,dx$

(c) (i) $\int -6e^{-3x}\,dx$ (ii) $\int \frac{1}{e^{4x}}\,dx$

(d) (i) $\int e^{-\frac{2}{3}x}\,dx$ (ii) $\int \frac{-2}{e^{x/4}}\,dx$

3. Find the following integrals.

(a) (i) $\int \frac{1}{x+4}\,dx$ (ii) $\int \frac{5}{5x-2}\,dx$

(b) (i) $\int \frac{2}{3x+4}\,dx$ (ii) $\int \frac{-8}{2x-5}\,dx$

(c) (i) $\int \frac{1}{7-2x}\,dx$ (ii) $\int \frac{-3}{1-4x}\,dx$

4. Integrate the following.

(a) $\int \sin(2-3x)\,dx$ (b) $\int 2\cos 4x\,dx$

5. By first simplifying, find the following integrals.

(a) $\int \frac{(4x^2-9)^2}{(2x+3)^2}\,dx$

(b) $\int e^{2x}e^{4x}\,dx$

(c) $\int \frac{x+3}{6-13x-5x^2}\,dx$

6. Two students integrate $\int \frac{1}{3x}\,\mathrm{d}x$ in different ways.
Marina writes

$$\int \frac{1}{3x}\,\mathrm{d}x = \frac{1}{3}\int \frac{1}{x}\,\mathrm{d}x = \frac{1}{3}\ln x + c$$

while Jack uses the rule from Key point 15. 1 and divides by the coefficient of x:

$$\int \frac{1}{3x}\,\mathrm{d}x = \frac{1}{3}\ln(3x) + c$$

Who has the right answer?

7. Given that $0 < a < 1$ and that the area enclosed between the graph of $y = \frac{1}{1-x}$, the x-axis, and the lines $x = a^2$ and $x = a$ is 0.4, find the value of a correct to 3 significant figures. *[5 marks]*

15B Integration by substitution

The shortcut for reversing the chain rule (Key point 15.1) works only when the derivative of the 'inside' function is a constant. This is because a constant factor can 'move through the integral sign' (see Key point 13.2 from section 13C); for example,

$$\begin{aligned}\int \cos 2x\,\mathrm{d}x &= \int \frac{1}{2}\times 2\cos 2x\,\mathrm{d}x\\ &= \frac{1}{2}\int 2\cos 2x\,\mathrm{d}x\\ &= \frac{1}{2}\sin 2x + c\end{aligned}$$

However, any expression containing the integration variable *cannot* be moved across the integral sign: $\int x\sin x\,\mathrm{d}x$ is not the same as $x\int \sin x\,\mathrm{d}x$. So we need different ways to integrate a product of two functions. Some products of a special form can be integrated by extending the principle of reversing the chain rule, which leads to the method of **integration by substitution**.

When we use the chain rule to differentiate a composite function, we differentiate the outer function and multiply by the derivative of the inner function; for example,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin\left(x^2+2\right)\right) = \cos(x^2+2)\times 2x$$

We can think of this as using a substitution $u = x^2 + 2$, and then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

Now look at $\int x \cos(x^2 + 2)\,dx$. Since $\cos(x^2 + 2)$ is a composite function, we can write it as $\cos u$ where $u = x^2 + 2$. Thus our integral becomes $\int x \cos u\,dx$. We know how to integrate $\cos u$ with respect to u, so we would like our integration variable to be u; in other words, we want du instead of dx. These two are not the same (so we cannot simply replace dx by du), but they are related because $u = x^2 + 2 \Rightarrow \frac{du}{dx} = 2x$. We can then 'rearrange' the derivative equation to get $dx = \frac{1}{2x}\,du$.

Substituting all this into our integral gives

$$\begin{aligned}\int x \cos(x^2 + 2)\,dx &= \int x \cos u \left(\frac{1}{2x}\right) du \\ &= \int \frac{1}{2} \cos u\,du \\ &= \frac{1}{2} \sin u + c \\ &= \frac{1}{2} \sin(x^2 + 2) + c\end{aligned}$$

Note that after obtaining the integral with respect to u we should rewrite it in terms of the original variable x.

A word of warning here: $\frac{du}{dx}$ is not really a fraction (as we saw in section 12D; it means the differentiation operator $\frac{d}{dx}$ applied to u), so it is not clear that 'rearranging' the equation $\frac{du}{dx} = f'(x)$ is valid. Nevertheless, it can be shown that as a consequence of the chain rule, dx *can be replaced by* $\frac{1}{f'(x)}\,du$.

We summarise the method of integration by substitution as follows.

KEY POINT 15.2

Integration by substitution

1. Select a substitution u = inner function (if not already given one).
2. Differentiate the substitution and rearrange to write dx in terms of du.
3. In the integral, replace dx by the expression from step 2, and replace any obvious occurrences of u.
4. Simplify as far as possible.
5. If any terms with x remain, write them in terms of u.
6. Find the new integral with respect to u.
7. Write the answer in terms of x.

Worked example 15.4

Find $\int x\,e^{x^2}\,dx$ using the substitution $u = x^2$.

We are given the substitution, so differentiate it and then write dx in terms of du.

$$\frac{du}{dx} = 2x$$

$$\therefore\ dx = \frac{1}{2x}\,du$$

In the integral, replace dx by the above expression, and replace any obvious occurrences of u.

$$\int x\,e^{x^2}\,dx = \int x\,e^{u}\,\frac{1}{2x}\,du$$

$$= \frac{1}{2}\int e^{u}\,du$$

Find the new integral in terms of u.

$$= \frac{1}{2}e^{u} + c$$

Write the answer in terms of x.

$$= \frac{1}{2}e^{x^2} + c$$

We can use integration by substitution with definite integrals as well. It is usually best to wait until the integration has been completed and written in terms of the original variable before you apply the limits (but see the solution to the introductory problem at the end of the chapter for an alternative approach). You can remind yourself of this by writing 'x =' in the limits.

Worked example 15.5

Use the substitution $u = x^2 + 1$ to evaluate the integral $\int_0^1 \frac{x}{\sqrt{x^2+1}}\,dx$.

Differentiate the given substitution and write dx in terms of du.

$$\frac{du}{dx} = 2x$$

$$\therefore\ dx = \frac{1}{2x}\,du$$

Replace dx by the above expression, and replace any obvious occurrences of u.

$$\int_{x=0}^{x=1} \frac{x}{\sqrt{x^2+1}}\,dx = \int_{x=0}^{x=1} \frac{x}{\sqrt{u}}\,\frac{1}{2x}\,du$$

$$= \frac{1}{2}\int_{x=0}^{x=1} \frac{1}{\sqrt{u}}\,du$$

Find the new integral in terms of u.

$$= \frac{1}{2}\int_{x=0}^{x=1} u^{-\frac{1}{2}}\,du$$

$$= \left[\frac{1}{2}\times 2u^{\frac{1}{2}}\right]_{x=0}^{x=1}$$

$$= \left[\sqrt{u}\right]_{x=0}^{x=1}$$

Write the answer in terms of x. Then apply the limits.

$$= \left[\sqrt{x^2+1}\right]_{x=0}^{x=1}$$

$$= \sqrt{1^2+1} - \sqrt{0^2+1}$$

$$= \sqrt{2} - 1$$

You will nearly always be told what substitution to use. If you are not given a substitution, look for a composite function and take u = inner function.

Worked example 15.6

Find the following integrals.

(a) $\int \sin^5 x \cos x\,dx$ (b) $\int x^2 e^{x^3+4}\,dx$

Remember that $\sin^5 x$ means $(\sin x)^5$, which is a composite function with inner function $\sin x$.

(a) Let $u = \sin x$.

Then $\frac{du}{dx} = \cos x$ and so $dx = \frac{1}{\cos x}\,du$

continued . . .

Make the substitution.

$$\int (\sin x)^5 \cos x \, dx = \int u^5 \cos x \frac{1}{\cos x} \, du$$

$$= \int u^5 \, du$$

$$= \frac{1}{6} u^6 + c$$

Write the answer in terms of x.

$$= \frac{1}{6} \sin^6 x + c$$

e^{x^3+4} is a composite function with inner function $x^3 + 4$.

(b) Let $u = x^3 + 4$.

Then $\frac{du}{dx} = 3x^2$ and hence $dx = \frac{1}{3x^2} du$

Make the substitution.

$$\int x^2 e^{x^3+4} dx = \int x^2 e^u \frac{1}{3x^2} du$$

$$= \int \frac{1}{3} e^u \, du$$

$$= \frac{1}{3} e^u + c$$

Write the answer in terms of x.

$$= \frac{1}{3} e^{x^3+4} + c$$

It is quite common to have to integrate a quotient where the numerator is a multiple of the derivative of the denominator. In this situation, use the substitution u = denominator.

Worked example 15.7

Find $\int \frac{x-3}{x^2-6x+7} \mathrm{d}x$.

The derivative of the denominator is $2x - 6$, which is 2 times the numerator, so substitute u = denominator.

Let $u = x^2 - 6x + 7$. Then

$$\frac{du}{dx} = 2x - 6$$

$$\therefore \ dx = \frac{1}{2x-6} du = \frac{1}{2(x-3)} du$$

continued . . .

Make the substitution.

$$\int \frac{x-3}{x^2-6x+7}\,dx = \int \frac{x-3}{u}\frac{du}{2(x-3)}$$
$$= \frac{1}{2}\int \frac{1}{u}\,du$$
$$= \frac{1}{2}\ln u + c$$

Write the answer in terms of x.

$$= \frac{1}{2}\ln(x^2-6x+7)+c$$

Whenever you have a quotient to integrate, always look out for this situation. In fact, if an integral is of the form $\int \frac{f'(x)}{f(x)}\,dx$, you can immediately write down the result as $\ln\left(f(x)\right)+c$.

You may have noticed in all of the above examples that, after making the substitution, the part of the integrand which was still in terms of x cancelled with a similar term coming from $\frac{du}{dx}$. For instance, in Worked example 15.6 part (b),

$$\int x^2 e^{x^3+4}\,dx = \int x^2 e^u \frac{1}{3x^2}\,du = \int \frac{1}{3}e^u\,du.$$

When integrating a product of functions, this cancellation will always happen if one part of the product is a composite function and the other part is a constant multiple of the derivative of the inner function. The explanation comes from the chain rule.

For example, consider the integral

$$\int (2x+3)(x^2+3x-5)^4\,dx$$

To find this integral, think about what we would need to differentiate to get $(2x+3)(x^2+3x-5)^4$. Notice that $(x^2+3x-5)^4$ is a composite function and that $2x+3$ is the derivative of the inner function x^2+3x-5, so we know that we would get $2x+3$ 'for free' when differentiating some power of x^2+3x-5 using the chain rule. To end up with $(x^2+3x-5)^4$, we would have to be differentiating $\frac{1}{5}(x^2+3x-5)^5$. Check that

$$\frac{d}{dx}\left(\frac{1}{5}(x^2+3x-5)^5\right) = (2x+3)(x^2+3x-5)^4$$

Therefore

$$\int (2x+3)(x^2+3x-5)^4 \, dx = \frac{1}{5}(x^2+3x-5)^5 + c$$

This is of course the same answer we would have obtained by using the substitution $u = x^2+3x-5$. However, if you spot that a particular integral can be found by reversing the chain rule, you can just write down the answer without working through the details of the substitution.

Exercise 15B

1. Find the following integrals using the given substitutions.

(a) (i) $\int x\sqrt{x^2+2}\, dx$; $u = x^2+2$

(ii) $\int (x+3)\sqrt{x^2+6x+4}\, dx$; $u = x^2+6x+4$

(b) (i) $\int \frac{x^2}{x^3+1}\, dx$; $u = x^3+1$

(ii) $\int \frac{3x}{x^2+5}\, dx$; $u = x^2+5$

(c) (i) $\int \sin x \cos^2 x\, dx$; $u = \cos x$

(ii) $\int \frac{1}{x} \ln x\, dx$; $u = \ln x$

2. Find the following integrals using appropriate substitutions.

(a) (i) $\int x(x^2+3)^3\, dx$ (ii) $\int 3x(x^2-1)^5\, dx$

(b) (i) $\int (2x-5)(3x^2-15x+4)^4\, dx$

(ii) $\int (x^2+2x)(x^3+3x^2-5)^3\, dx$

(c) (i) $\int \frac{2x}{x^2+3}\, dx$ (ii) $\int \frac{6x^2-12}{x^3-6x+1}\, dx$

(d) (i) $\int 4\cos^5 3x \sin 3x\, dx$ (ii) $\int \cos 2x \sin^3 2x\, dx$

(e) (i) $\int 3x e^{3x^2-1}\, dx$ (ii) $\int 3x e^{x^2} dx$

(f) (i) $\int \frac{e^{2x+3}}{e^{2x+3}+4}\, dx$ (ii) $\int \frac{\cos x}{3+4\sin x}\, dx$

3. Use the substitution $u = \sin x$ to find the value of $\int_0^{\pi/2} \cos x \, e^{\sin x}\, dx$. *[7 marks]*

4. Use the substitution $u = x^3 + 5$ to find the indefinite integral $\int x^2\sqrt{x^3+5}\,dx$. *[5 marks]*

5. Use the substitution $u = e^x$ to evaluate $\int_0^1 \frac{e^x}{\sqrt{e^x+1}}\,dx$.

6. Find the exact value of $\int_0^2 (2x+1)e^{x^2+x-1}\,dx$. *[6 marks]*

7. Evaluate $\int_2^5 \frac{2x}{x^2-1}\,dx$, giving your answer in the form $\ln k$. *[4 marks]*

8. (a) Write $\tan x$ in terms of $\sin x$ and $\cos x$.
(b) Hence or otherwise, find $\int \tan x\,dx$. *[7 marks]*

9. Evaluate $\int_1^3 \frac{(2x-3)\sqrt{x^2-3x+3}}{x^2-3x+3}\,dx$. *[5 marks]*

10. Three students integrate $\cos x \sin x$ in three different ways.

Amara uses the chain rule in reverse with $u = \sin x$:

$$\frac{du}{dx} = \cos x, \text{ so } \int \cos x \sin x\,dx = \int u\,du = \frac{1}{2}\sin^2 x + c.$$

Ben uses the chain rule in reverse with $u = \cos x$:

$$\frac{du}{dx} = -\sin x, \text{ so } \int \cos x \sin x\,dx = \int -u\,du = -\frac{1}{2}\cos^2 x + c.$$

Carlos uses a double-angle formula:

$$\int \cos x \sin x\,dx = \int \frac{1}{2}\sin 2x\,dx = -\frac{1}{4}\cos 2x + c.$$

Who is right?

15C Kinematics

Kinematics is the study of movement – in particular position, velocity and acceleration. We first need to define some terms carefully.

- *Time* is normally given the symbol t. We can define $t = 0$ to be any convenient time.
- In a standard 400-metre race athletes run a single lap, so despite running 400 m they return to where they started. The distance is how much ground someone has covered,

In the International Baccalaureate® you will only have to deal with movement in one dimension. In reality, however, motion often occurs in two or three dimensions. To deal with this requires a combination of vectors and calculus called (unsurprisingly) vector calculus.

400 m in this case; the **displacement** is how far away they are from a particular position (called the origin), so the athletes' displacement upon finishing the race is 0 m. Displacement is usually represented by the symbol s.

- The rate of change of displacement with respect to time is called **velocity**, usually denoted by v.

KEY POINT 15.3

Velocity is $v = \frac{ds}{dt}$.

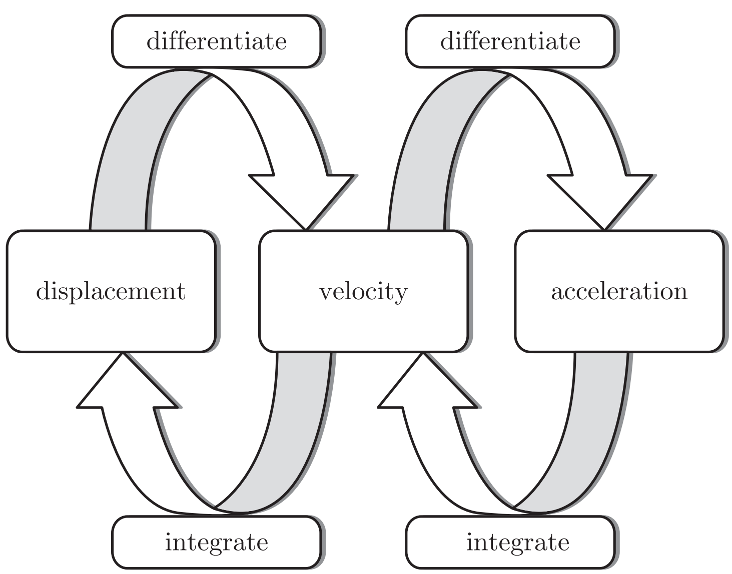
Speed is the magnitude of velocity, $|v|$.

The rate of change of velocity with respect to time is called **acceleration** and is typically given the symbol a.

KEY POINT 15.4

Acceleration is $a = \frac{dv}{dt}$.

We differentiate to go from displacement to velocity to acceleration. Therefore, reversing the process – going from acceleration to velocity to displacement – is done by integration.



There is an important difference between finding the distance and finding the displacement from time $t = a$ to time $t = b$.

KEY POINT 15.5

Displacement between times $t = a$ and $t = b$ is the integral $\int_a^b v\, dt$.

Distance travelled between times $t = a$ and $t = b$ is the area $\int_a^b |v|\, dt$, that is, the total area enclosed between the graph of v against t, the t-axis, and the lines $t = a$ and $t = b$.

See section 13H if you need to refresh your memory on the relationship and differences between areas and integrals.

Worked example 15.8

The velocity (ms^{-1}) of a car at time t after passing a flag is modelled by $v = 17 - 4t$ for $0 \le t < 5$.

(a) What is the initial speed of the car?

(b) Find the acceleration of the car.

(c) What is the maximum displacement of the car from the flag?

(d) Find the distance that the car travels.

'Initial' means at $t = 0$.

(a) When $t = 0$, $v = 17$, so speed is 17 ms^{-1}.

To find acceleration, differentiate the velocity.

(b) $a = \frac{dv}{dt} = -4\ ms^{-1}$

Maximum displacement occurs at a stationary point of s, i.e. when $\frac{ds}{dt} = 0$, which is the same as $v = 0$.

(c) $v = 0 \Rightarrow t = 4.25$

At this time the displacement is

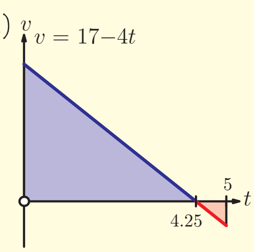
$$s = \int_0^{4.25} v\, dt$$

$$= \int_0^{4.25} 17 - 4t\, dt$$

$$= 36.125\text{ m (from GDC)}$$

Distance is the actual area enclosed between the graph of v against t, the t-axis, and the lines $t = a$ and $t = b$.

(d)



The area above the t-axis is 36.125 from part (c).

The area below the axis is $\int_{4.25}^{5} |v|\ dt = 1.125$ (from GDC)

So total distance is 37.25 m

Exercise 15C

1. Find expressions for the velocity and acceleration in terms of time if the displacement is given by the following.

   (a) (i) $s = 4e^{-2t}$ (ii) $s = 5 - 2e^{3t}$

   (b) (i) $s = 5\sin\left(\frac{t}{2}\right)$ (ii) $s = 2 - 3\cos(2t)$

2. A particle is at the origin ($s = 0$) at $t = 0$ and moves with the given velocity. Find the displacement in terms of t.

   (a) (i) $v = 3t^2 - 1$ (ii) $v = \frac{1}{2}(1 - t^3)$

   (b) (i) $v = 2e^{-t}$ (ii) $v = 1 + e^{2t}$

   (c) (i) $v = \frac{3}{t+2}$ (ii) $v = 3 - \frac{1}{t+1}$

3. For the given velocity function, find the distance travelled between the specified times.

   (a) (i) $v = 2e^{-t}$ between $t = 0$ and $t = 2$

   (ii) $v = 4(\ln t)^3$ between $t = 2$ and $t = 3$

   (b) (i) $v = 1 - 5\cos t$ between $t = 0.2$ and $t = 0.9$

   (ii) $v = 2\cos(3t)$ between $t = 1$ and $t = 1.5$

   (c) (i) $v = t^2 - 2$ between $t = 0$ and $t = 2.3$

   (ii) $v = 5\sin(2t)$ between $t = 0.5$ and $t = 2.5$

4. Use integration to derive the following constant acceleration formulas: for an object moving with constant acceleration a from an initial velocity u, these give the velocity v at time t and the displacement s from the starting position.

   (i) $v = u + at$

   (ii) $s = ut + \frac{1}{2}at^2$

5. The velocity of an object, in metres per second, is given by $v = 5\cos\left(\frac{t}{3}\right)$.

   (a) Find the displacement of the object from the starting point when $t = 6$.

   (b) Find the total distance travelled by the object in the first 6 seconds.

   [6 marks]

6. A ball is projected vertically upwards so that its velocity in metres per second at time t seconds is given by $v = 12 - 9.8t$.

(a) Find the displacement of the ball relative to its starting position after 2 seconds.

(b) Find the distance travelled by the ball in the first 2 seconds of motion. *[5 marks]*

7. An object moves in a straight line so that its velocity at time t is given by $v = \frac{t}{t^2+1}$.

(a) Find an expression for the acceleration of the object at time t.

(b) Given that the object is initially at the origin, find its displacement from the origin when $t = 5$. *[6 marks]*

8. The displacement of an object varies with time as

$$s = -\frac{1}{3}t^3 + \frac{3}{2}t^2 + 4t \text{ for } 0 \le t \le 5.$$

Find the maximum velocity of the object. *[5 marks]*

9. A car leaving a parking space has velocity (in metres per second) given by the equation $v = \frac{1}{2}t^2 - 2t$, where t is the time in seconds. The car is assumed to move along a straight line at all times.

(a) Show that the car is stationary when $t = 0$ and when $t = 4$.

(b) Describe the car's motion during the first eight seconds.

(c) Find the distance travelled in the first four seconds.

(d) Find the distance travelled in the first eight seconds.

(e) Find the displacement after eight seconds.

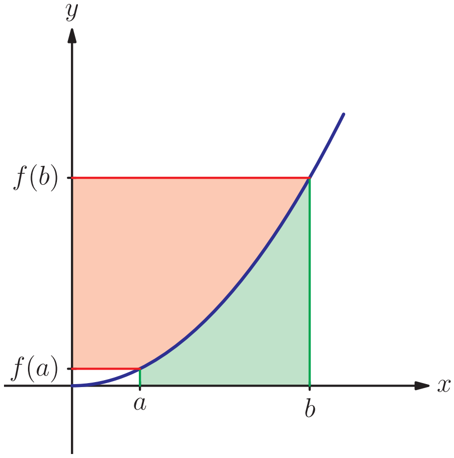
(f) At what time is the acceleration $5\ \text{m/s}^2$?

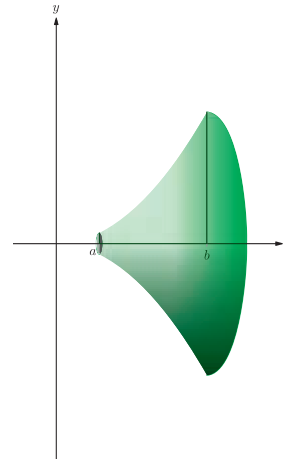
(g) Find the maximum speed in the first eight seconds. *[20 marks]*

15D Volumes of revolution

In chapter 13 you saw that the area between a curve and the x-axis from $x = a$ to $x = b$ is given by $\int_a^b y \, dx$ as long as $y > 0$. In this section we shall find a similar formula to calculate the volume of a shape that can be formed by rotating a curve around the x-axis.

Imagine a curve $y = f(x)$ being rotated fully around the x-axis; the resulting shape is called a **volume of revolution**.





See Fill-in proof sheet 17 'Volumes of revolution' on the CD-ROM for the derivation of this formula.

KEY POINT 15.6

The volume of revolution formed by rotating a portion of a curve between $x = a$ and $x = b$ around the x-axis is given by $V = \int_{x=a}^{x=b} \pi y^2 \, dx$

for clarity

Worked example 15.9

The graph of $y = \sqrt{\sin 2x}$, $0 \leq x \leq \frac{\pi}{2}$, is rotated 360° around the x-axis. Find, in terms of π, the volume of the solid generated.

Use the formula for the volume of revolution.

$$V = \int_0^{\frac{\pi}{2}} \pi\left(\sqrt{\sin 2x}\right)^2 dx$$

$$= \int_0^{\frac{\pi}{2}} \pi \sin 2x \, dx$$

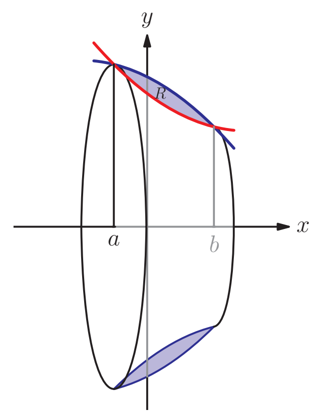
$$= \pi\left[-\frac{1}{2}\cos 2x\right]_0^{\pi/2}$$

$$= \pi\left(-\frac{1}{2}(-1) + \frac{1}{2}\right)$$

$$= \pi$$

We can extend this idea to find a volume of revolution formed by rotating a region between two curves. The argument is similar to the one we used for calculating the area enclosed between two curves in section 13I.

The volume formed when the region R is rotated around the x-axis is given by the volume of revolution obtained from rotating $y = g(x)$ minus the volume of revolution obtained from rotating $y = f(x)$.



KEY POINT 15.7

The volume of revolution formed by rotating the region bounded above by the curve $y = f(x)$ and below by the curve $y = g(x)$ (i.e. where $g(x)$ is above $f(x)$) around the x-axis is

$$\int_a^b \pi\left[g(x)^2 - f(x)^2\right]\mathrm{d}x$$

where a and b are the x-coordinates of the points of intersection between the curves.

EXAM HINT

Be careful to square each term *within* the bracket. Do not fall into the trap of squaring the whole bracket, saying that the volume is $\int_a^b \pi\left[g(x) - f(x)\right]^2 \mathrm{d}x$.

Worked example 15.10

The region bounded by the curves $y = x^2 + 6$ and $y = 8x - x^2$ is rotated 360° about the x-axis.

(a) Show that the volume of revolution is given by $4\pi\int_1^3 13x^2 - 4x^3 - 9\,\mathrm{d}x$.

(b) Evaluate this volume, correct to 3 significant figures.

The limits of integration are the intersection points.

(a)

Intersections:

$x^2 + 6 = 8x - x^2$

$\Leftrightarrow 2x^2 - 8x + 6 = 0$

$\Leftrightarrow 2(x-1)(x-2) = 0$

$\Leftrightarrow x = 1$ or 3

Use $V = \pi\int_b^a [g(x)]^2 - [f(x)]^2 \,\mathrm{d}x$. Draw a sketch to see which curve is above.

continued . . .

$$V = \pi\int_1^3 (8x - x^2)^2 - (x^2 + 6)^2 dx$$

$$= \pi\int_1^3 (64x^2 - 16x^3 + x^4) - (x^4 + 12x^3 + 36)\, dx$$

$$= \pi\int_1^3 52x^2 - 16x^3 - 36\, dx$$

$$= 4\pi\int_1^3 13x^2 - 4x^3 - 9\, dx$$

We can evaluate the integral using the GDC.

(b)

Using GDC,

$V = 184$ (3 SF)

EXAM HINT

Notice that, due to the possibility of cancelling or combining terms, subtracting the squares of the two functions first and then integrating (i.e. doing $\int [g(x)]^2 - [f(x)]^2 \, dx$) is often easier than integrating $f(x)^2$ and $g(x)^2$ separately and then subtracting (i.e. doing $\int [g(x)]^2 \, dx - \int [f(x)]^2 \, dx$).

There are also formulas for finding the surface area of a solid formed by rotating a region around an axis. Some particularly interesting examples arise if we allow one end of the region to extend infinitely far; for example, rotating the region to the right of $x = 1$ between the curve $y = \frac{1}{x}$ and the x-axis gives a solid called Gabriel's horn or Torricelli's trumpet. Surface areas and volumes generated by such unbounded regions can still be calculated, by using so-called improper integrals, and it turns out that it is possible to have a solid with finite volume but infinite surface area!

Exercise 15D

1. Find the volume of revolution formed when the curve $y = f(x)$, with $a \le x \le b$, is rotated through 2π radians about the x-axis.

(a) (i) $f(x) = x^2 + 6;\ a = -1, b = 3$

(ii) $f(x) = 2x^3 + 1;\ a = 0, b = 1$

(b) (i) $f(x) = e^{2x} + 1;\ a = 0, b = 1$

(ii) $f(x) = e^{-x} + 2;\ a = 0, b = 2$

(c) (i) $f(x) = \sqrt{\sin x};\ a = 0, b = \pi$

(ii) $f(x) = \sqrt{\cos x};\ a = 0, b = \frac{\pi}{2}$

2. The part of the curve $y = g(x)$ with $a \le x \le b$ is rotated 360° around the x-axis. Find the volume of revolution generated, correct to 3 significant figures.

(a) (i) $g(x) = 4x^2 + 1; a = 0, b = 2$

(ii) $g(x) = \dfrac{x^2 - 1}{3}; a = 1, b = 4$

(b) (i) $g(x) = \ln x + 1; a = 1, b = 3$

(ii) $g(x) = \ln(2x - 1); a = 1, b = 5$

(c) (i) $g(x) = \cos x; a = 0, b = \dfrac{\pi}{2}$

(ii) $g(x) = \tan x; a = 0, b = \dfrac{\pi}{4}$

3. The part of the graph of $y = \ln x$ between $x = 1$ and $x = 2\mathrm{e}$ is rotated 360° around the x-axis. Find the volume generated. *[4 marks]*

4. The part of the curve $y^2 = \sin x$ between $x = 0$ and $x = \dfrac{\pi}{2}$ is rotated 2π radians around the x-axis. Find the exact volume of the solid generated. *[4 marks]*

5. The part of the curve $y = \sqrt{\dfrac{3}{x}}$ between $x = 1$ and $x = a$ is rotated 2π radians around the x-axis. The volume of the resulting solid is $\pi \ln\left(\dfrac{64}{27}\right)$. Find the exact value of a. *[7 marks]*

6. (a) Find the coordinates of the points of intersection of the curves $y = x^2 + 3$ and $y = 4x + 3$.

(b) Find the volume of revolution generated when the region between the curves $y = x^2 + 3$ and $y = 4x + 3$ is rotated 360° around the x-axis. *[7 marks]*

Summary

- Look for **standard derivatives** before attempting any more complicated methods. A list of standard derivatives and integrals is given in the Formula booklet. You may need to divide by the coefficient of x to get the correct constant factor.
- If the expression to be integrated is a product that contains both a function and (a constant multiple of) its derivative, then you should be able to reverse the chain rule or use a substitution to convert the expression into a standard derivative:
  - a general rule, which comes from reversing the special case of the chain rule in Key point 14.2, is: if a and b are constants, then
    $\int f(ax+b)\,\mathrm{d}x = \dfrac{1}{a}F(ax+b) + c$ where $F(x)$ is the integral of $f(x)$.

– the above shortcut only works when the 'inside' function is a constant. When integrating the product of two functions, you can sometimes use integration by substitution, following the steps in Key point 15.2. This method can also be used with definite integrals: complete the integration first and then apply the limits.

- In **kinematics**,
  - distance is the amount of ground covered; displacement is how far away something is from the origin; velocity is the rate of change of displacement with respect to time ($v = \frac{ds}{dt}$); speed is the magnitude of velocity, $|v|$; and acceleration is the rate of change of velocity with respect to time ($a = \frac{dv}{dt}$).
  - differentiate to go from **displacement** to **velocity** to **acceleration**
  - integrate to go from acceleration to velocity to displacement.

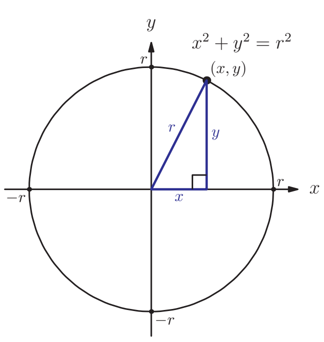
  Do not confuse displacement with distance travelled:
  - displacement between times $t = a$ and $t = b$ is the integral $\int_a^b v \, dt$
  - distance travelled between times $t = a$ and $t = b$ is the area $\int_a^b |v| \, dt$.
- The **volume of revolution** formed by rotating a portion of curve between $x = a$ and $x = b$ around the x-axis is $\int_{x=a}^{x=b} \pi y^2 \, dx$.
- The volume of revolution formed by rotating the region bounded above by curve $y = f(x)$ and below by curve $y = g(x)$ (i.e, $g(x)$ is above $f(x)$) around the x-axis is

$$\int_a^b \pi\left[g(x)^2 - f(x)^2\right] dx$$

where a and b are the x-coordinates of the points of intersection between the curves.

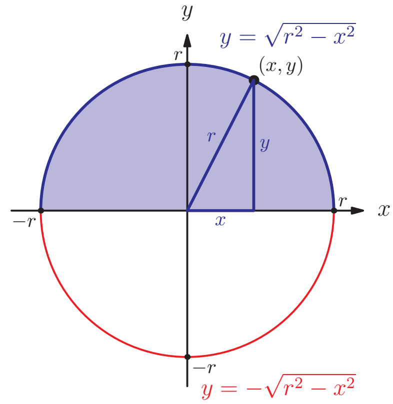
Introductory problem revisited

Prove that the area of a circle with radius r is πr^2.



To do this using integration, we need to find an equation for the circle. Suppose that the circle is centred at the origin of the coordinate axes. By Pythagoras' theorem, any point on the circle must satisfy $x^2 + y^2 = r^2$.

We can rearrange this equation to get $y = \pm\sqrt{r^2 - x^2}$. The positive root represents the top half of the circle and the negative root the bottom half. To find the area of the circle, we can find the area enclosed between the graph of $y=\sqrt{r^2-x^2}$ and the x-axis, and then double the answer.



EXAM HINT

If you have to integrate a function like this in the exam, i.e. one that is not standard you will be given the substitution to use or a hint.

$\sqrt{r^2-x^2}$ is not one of the standard derivatives, but a substitution may help to convert it into a form we recognise. It turns out that a useful substitution is $x=r\cos\theta$. This should make sense, as we know that trigonometric functions are closely related to circles.

Now we can carry out the integration.

Write down the integral to be evaluated.

The area of the top half of the circle is

$$\frac{A}{2}=\int_{-r}^{r}\sqrt{r^2-x^2}\,dx$$

State the method to be used.

Substitute $x=r\cos\theta$

Differentiate the substitution and rearrange.

$$\frac{dx}{d\theta}=-r\sin\theta$$

$$\therefore dx=-r\sin\theta\,d\theta$$

Express the function in terms of θ.

$$r^2-x^2=r^2-(r\cos\theta)^2$$
$$=r^2\left(1-\cos^2\theta\right)=r^2\sin^2\theta$$

Notice that $\sin\theta$ is positive on the top half of the circle.

$$\Rightarrow\sqrt{r^2-x^2}=r\sin\theta$$

It will be easier for later calculations to transform the integration limits for x to limits in θ (so we will not need to change back to the x variable after doing the integration).

Limits:

when $x=-r$, $\cos\theta=-1$, so $\theta=\pi$

when $x=r$, $\cos\theta=1$, so $\theta=0$

Write the integral in terms of θ.

$$\frac{A}{2}=\int_{\pi}^{0}(r\sin\theta)(-r\sin\theta)\,d\theta$$

$$=-r^2\int_{\pi}^{0}\sin^2\theta\,d\theta$$

Use a double angle identity to convert $\sin^2\theta$ to $\frac{1-\cos 2\theta}{2}$, which is the sum of two standard derivatives.

$$=-r^2\int_{\pi}^{0}\frac{1}{2}-\frac{1}{2}\cos 2\theta\,d\theta$$

$$=-r^2\left[\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right]_{\pi}^{0}$$

$$=-r^2\left\{(0-0)-\left(\frac{\pi}{2}-\frac{\sin 2\pi}{4}\right)\right\}$$

$$=\frac{\pi r^2}{2}$$

Hence the area of the whole circle is $2\times\frac{\pi r^2}{2}=\pi r^2$ as required.

Mixed examination practice 15

Short questions

1. Find the following integrals.

(a) $\displaystyle\int \frac{1}{1-3x}\,dx$

(b) $\displaystyle\int \frac{1}{(2x+3)^2}\,dx$ *[4 marks]*

2. Using the substitution $u = e^x + 1$ or otherwise, find the integral

$$\int \frac{e^x}{e^x+1}\,dx$$ *[6 marks]*

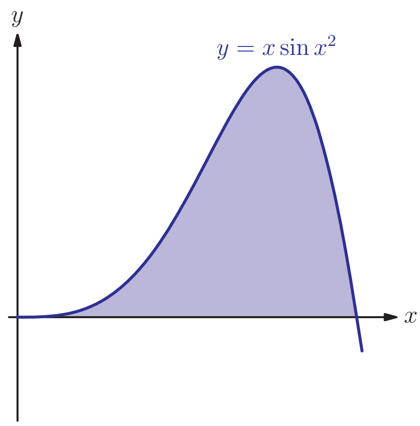
3. Given that $\displaystyle\int_0^m \frac{dx}{3x+1} = 1$, calculate the value of m to 3 significant figures. *[6 marks]*

4. (a) Find the derivative of $x\ln x$.

(b) Hence find $\int \ln x\,dx$. *[5 marks]*

5. Find the volume formed when the graph of $\frac{1}{x}$ between $x = 1$ and $x = 2$ is rotated by 360° around the x-axis.

6. The graph shows $y = x\sin x^2$. Find the shaded area.



[7 marks]

7. The graph of $y = e^x$ between $x = 0$ and $x = a$ is rotated by 360° around the x-axis. The volume of the shape created is $\frac{3\pi}{2}$. Find the exact value of a. *[7 marks]*

8. (a) Show that $\dfrac{1}{x-2}+\dfrac{5}{(x-2)^2}=\dfrac{x+3}{(x-2)^2}$.

(b) Hence, find $\displaystyle\int \frac{x+3}{(x-2)^2}\,dx$. *[4 marks]*

9. Find $\displaystyle\int \frac{1}{x\ln x}\,dx$. *[6 marks]*

Long questions

1. Let $I=\displaystyle\int \frac{\sin x}{\sin x+\cos x}\,dx$ and $J=\displaystyle\int \frac{\cos x}{\sin x+\cos x}\,dx$.

(a) Find $I+J$.

(b) By using the substitution $u=\sin x+\cos x$, find $J-I$.

(c) Hence find $\displaystyle\int \frac{\sin x}{\sin x+\cos x}\,dx$. *[12 marks]*

2. (a) Show that $\cos^2\theta=\dfrac{1}{2}(1+\cos 2\theta)$.

(b) Hence find $\displaystyle\int \cos^2 x\,dx$.

(c) Find the exact value of $\displaystyle\int_0^{\pi} \cos^2(3x)\,dx$. *[12 marks]*

3. Amy does a bungee jump from a platform 50 m above a river. Let h be her height above the river, in metres, at a time t seconds after jumping. Her velocity is given by $v=2t^2-10t$.

(a) What is the initial acceleration that Amy experiences?

(b) At what time is Amy's velocity zero?

(c) How close to the river does Amy get?

(d) What distance does Amy travel in the first seven seconds?

(e) How long does it take for Amy to return to the platform? *[12 marks]*