13 Graphs, Equations and Inequalities

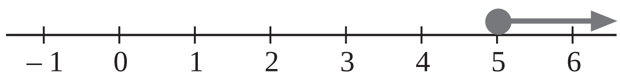
13.1 Linear Inequalities

In this section we look at how to solve linear inequalities and illustrate their solutions using a number line.

When using a number line, a small solid circle is used for $\leq$ or $\geq$ and a hollow circle is used for $>$ or $<$.

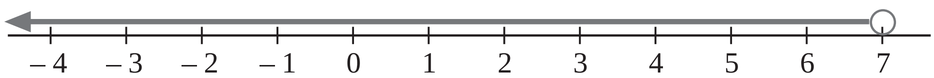
For example,

$x \geq 5$



Here the solid circle means that the value 5 is included.

$x < 7$



Here the hollow circle means that the value 7 is not included.

When solving linear inequalities we use the same techniques as those used for solving linear equations. The important exception to this is that when *multiplying or dividing by a negative number*, you must *reverse the direction of the inequality*. However, in practice, it is best to try to avoid doing this.

Example 1

Solve the inequality $x + 6 > 3$ and illustrate the solution on a number line.

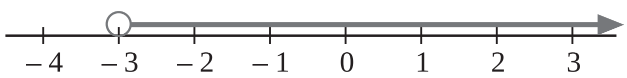
Solution

$$x + 6 > 3$$

$$x > 3 - 6 \qquad \textit{Subtracting 6 from both sides of the inequality}$$

$$x > -3$$

This can be illustrated as shown below:



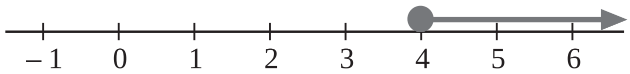
Example 2

Solve the inequality $3x + 7 \geq 19$ and illustrate the solution on a number line.

Solution

$$
\begin{aligned}
3x + 7 &\geq 19 \\
3x &\geq 12 && \textit{Subtracting 7 from both sides} \\
x &\geq 4 && \textit{Dividing both sides by 3}
\end{aligned}
$$

This can now be shown on a number line.



Example 3

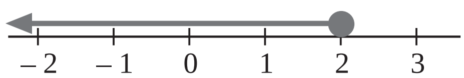
Illustrate the solution to the inequality $12 - 3x \geq 6$.

Solution

Because this inequality contains the term '$-3x$', first add $3x$ to both sides to remove the – sign.

$$
\begin{aligned}
12 - 3x &\geq 6 \\
12 &\geq 6 + 3x && \textit{Adding } 3x \textit{ to both sides} \\
6 &\geq 3x && \textit{Subtracting 6 from both sides} \\
2 &\geq x && \textit{Dividing both sides by 3} \\
\text{or} \quad x &\leq 2
\end{aligned}
$$

This is illustrated below.



Example 4

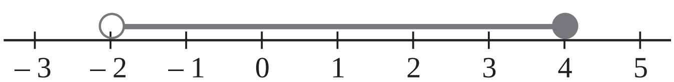
Solve the equation $-7 < 5x + 3 \leq 23$.

Solution

In an inequality of this type you must apply the same operation to each of the 3 parts.

$$
\begin{aligned}
-7 < \; &5x + 3 \leq 23 \\
-10 < \; &5x \leq 20 && \textit{Subtracting 3 from both sides} \\
-2 < \; &x \leq 4 && \textit{Dividing both sides by 5}
\end{aligned}
$$

This can then be illustrated as below.



Exercises

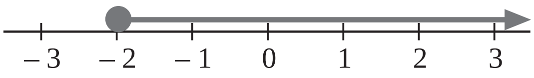
1. Draw diagrams to illustrate the following inequalities:

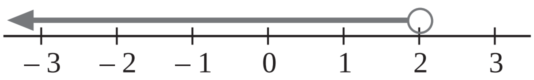
(a) $x > 3$ (b) $x \le 4$

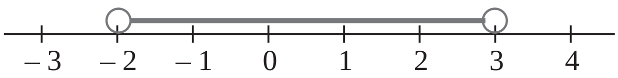
(c) $x \le -2$ (d) $x \ge -3$

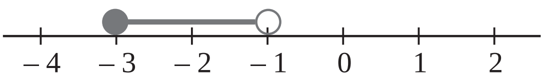
(e) $-2 \le x < 4$ (f) $0 \le x \le 3$

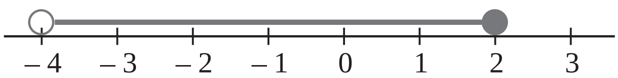
2. Write down the inequality represented by each of the following diagrams:

(a)


(b)


(c)


(d)


(e)


3. Solve each of the following inequalities and illustrate the results on a number line.

(a) $x + 7 > 12$ (b) $x - 6 > 3$ (c) $4x \le 20$

(d) $5x \ge 10$ (e) $x + 6 \ge 8$ (f) $x - 6 \le -3$

(g) $x + 8 \le 5$ (h) $\dfrac{x}{2} \ge 3$ (i) $\dfrac{x}{4} \le -1$

4. Solve each of the following inequalities and illustrate your solutions on a number line.

(a) $6x + 2 \ge 8$ (b) $5x - 3 > 7$ (c) $3x - 9 < 6$

(d) $4x + 2 \le 30$ (e) $5x + 9 \le -1$ (f) $4x + 12 > 4$

(g) $\dfrac{x}{2} + 4 > 3$ (h) $\dfrac{x}{5} - 1 \le -3$ (i) $\dfrac{x}{4} + 6 \le 5$

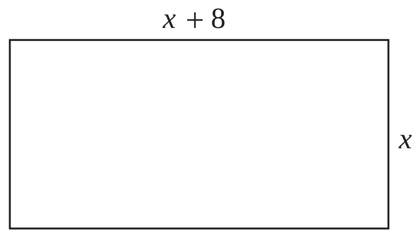
5. Solve the following inequalities, illustrating your solutions on a number line.

(a) $-1 \le 3x + 2 \le 17$ (b) $4 - 6x < 22$ (c) $5 - 3x \ge -1$

(d) $14 \le 4x - 2 \le 18$ (e) $20 - 8x < 4$ (f) $32 - 9x \ge -4$

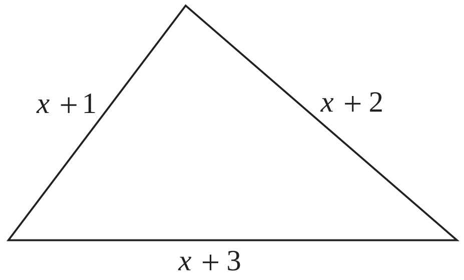
(g) $11 - 3x \le 20$ (h) $-11 \le 3x - 2 \le -5$ (i) $-7 < 2x + 5 \le 1$

6. Given that the perimeter of the rectangle shown is less than 44, form and solve an inequality.



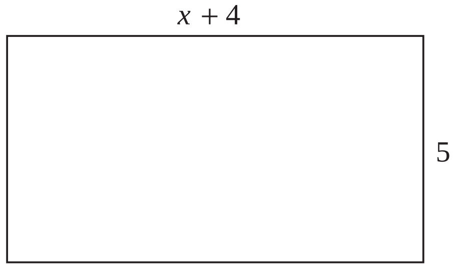
7. The perimeter of the triangle shown is greater than 21 but less than or equal to 30.

   Form and solve an inequality using this information.



8. The area of the rectangle shown must be less than 50 but greater than or equal to 10.

   Form and solve an inequality for x.

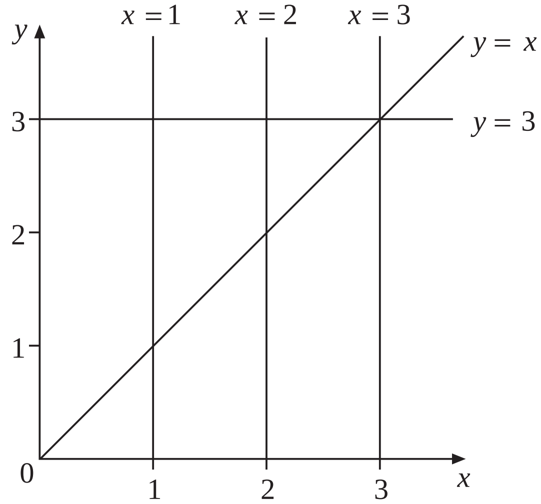


9. A cyclist travels at a constant speed v miles per hour. He travels 30 miles in a time that is greater than 3 hours but less than 5 hours.

   Form an inequality for v.

10. The area of a circle must be greater than or equal to 10 m^2 and less than 20 m^2. Determine an inequality that the radius, r, of the circle must satisfy.

11. The pattern shown is formed by straight lines of equations in the first quadrant.



   (a) One region of the pattern can be described by the inequalities

   $x \le 2$

   $x \ge 1$

   $y \ge x$

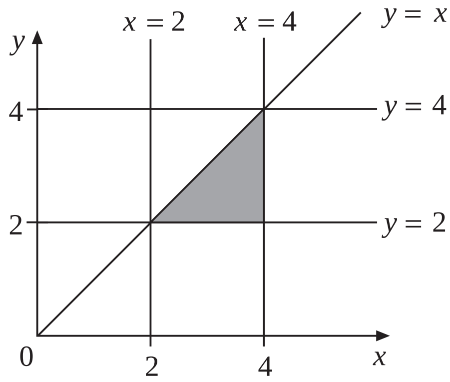
   $y \le 3$

   Copy the diagram and put an R in the single region of the pattern that is described.

This is another pattern formed by straight line graphs of equations in the first quadrant.

(b) The shaded region can be described by *three* inequalities.

Write down these three inequalities.



(KS3/96/Ma/Tier 6-8/P1)

13.2 Graphs of Quadratic Functions

In this section we recap the graphs of straight lines before looking at the graphs of quadratic functions.

A straight line has equation $y = mx + c$ where m is the gradient and c is the y-intercept.

Example 1

(a) Draw the lines with equations,

$$y = x + 8 \qquad \text{and} \qquad y = x + 3.$$

(b) Describe the translation that would move $y = x + 8$ onto $y = x + 3$.

Solution

(a) To plot the graphs, we calculate the coordinates of three points on each line.

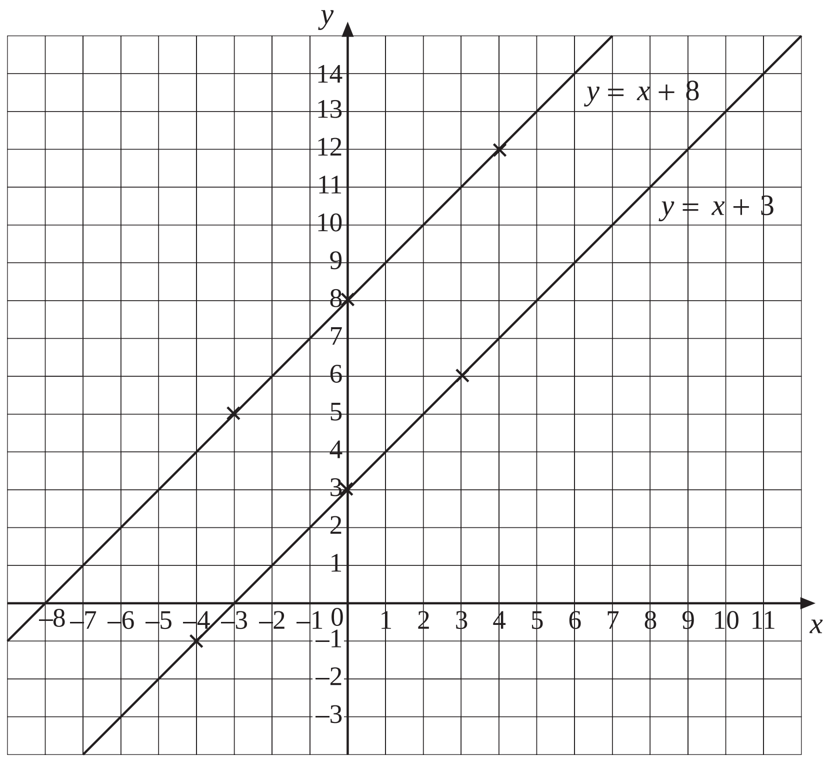
For $y = x + 8$ we have

$(-3, 5)$, $(0, 8)$ and $(4, 12)$.

For $y = x + 3$ we have,

$(-4, -1)$, $(0, 3)$ and $(3, 6)$.

The graphs are shown below.



(b) A translation along the vector $\begin{pmatrix} 0 \\ -5 \end{pmatrix}$ would move the line $y = x + 8$ onto the line $y = x + 3$.

Example 2

Draw the curve with equation $y = x^2$.

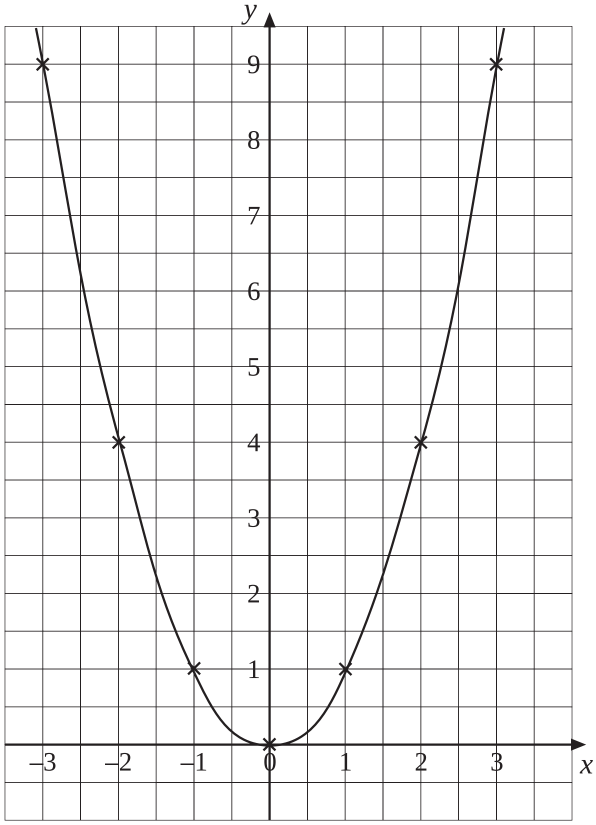
Solution

$y = x^2$ is not a linear equation, so we will have to draw a smooth curve. To do this we need to calculate and plot a reasonable number of points. We begin by drawing up a table of values:

x	-3	-2	-1	0	1	2	3
x^2	9	4	1	0	1	4	9

Using these values the graph can be drawn, as shown:

$y = x^2$



Example 3

(a) Draw the curve with equation $y = x^2 + 2$.

(b) Describe how the curve is related to the curve with equation $y = x^2$.

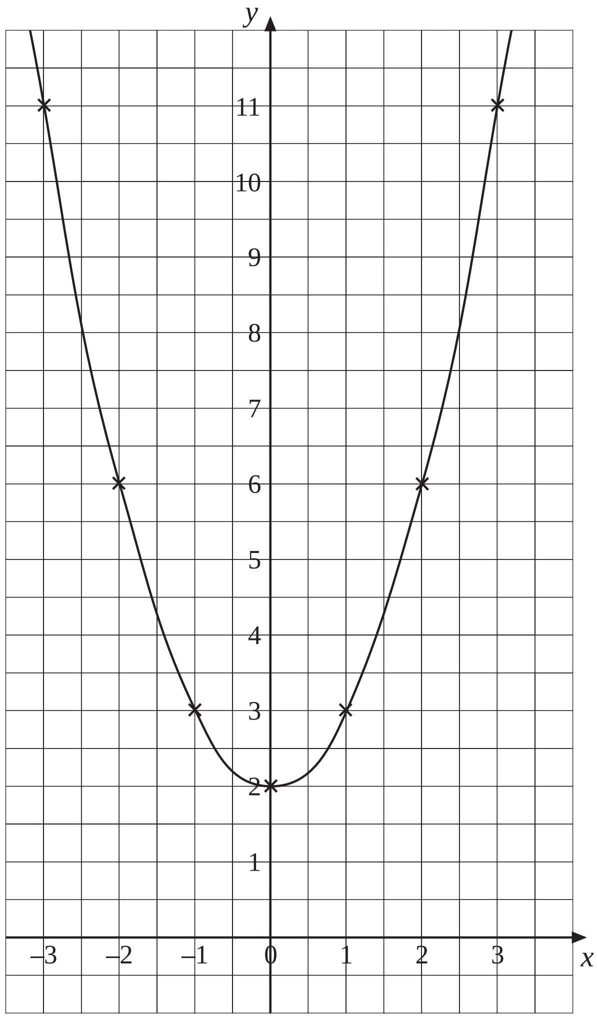
Solution

(a) A table of values has been completed:

x	-3	-2	-1	0	1	2	3
$x^2 + 2$	11	6	3	2	3	6	11

The graph is shown below:

$y = x^2 + 2$



(b) This curve is a translation of the curve $y = x^2$ along the vector $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$.

Exercises

1. (a) Draw the graph with equation $y = 2x + 1$.

   (b) State the gradient of this line.

2. Draw the line that has gradient $-\frac{1}{2}$ and y-intercept 6.

3. (a) Draw the lines with equations $y = 2x + 3$ and $y = -2x + 3$.

   (b) Describe the transformation that would map one line onto the other.

4. (a) Draw the curves with equations $y = x^2$ and $y = x^2 - 1$.

   (b) Describe how the two curves are related.

5. (a) Draw the curves with equations $y = x^2 + 3$ and $y = x^2 - 1$.

(b) Describe the transformation that would map the first curve onto the second.

6. Without drawing any graphs, describe the relationship between the curves with equations,

$$y = x^2 + 1, \qquad y = x^2 - 5 \qquad \text{and} \qquad y = x^2 + 6.$$

7. (a) Copy and complete the following table:

x	-4	-3	-2	-1	0	1	2	3	4
$x^2 + 2x$									

(b) Draw the graph $y = x^2 + 2x$.

(c) Describe how this curve is related to the curve with equation $y = x^2$.

8. (a) Draw the curve $y = 2x^2$.

(b) On the same diagram, also draw the curves with equations $y = 2x^2 - 1$ and $y = 2x^2 + 2$.

(c) Describe how the three curves are related.

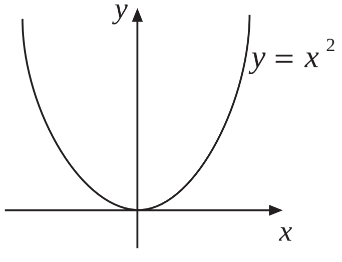
9. (a) Draw the graphs with equations $y = x^2 + 4$ and $y = 2 - x^2$.

(b) Describe the transformation that would map one curve onto the other.

10. (a) Draw the curves with equations $y = x^2 - 4x$ and $y = x^2 + 2x + 3$.

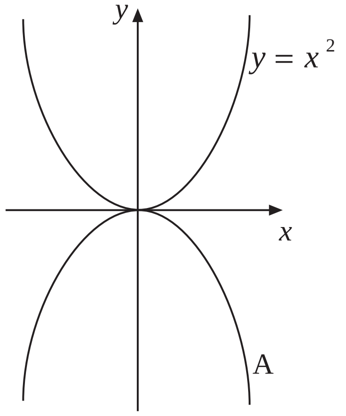
(b) Describe the relationship between the two curves.

11. (a) The diagram shows the graph with equation $y = x^2$.



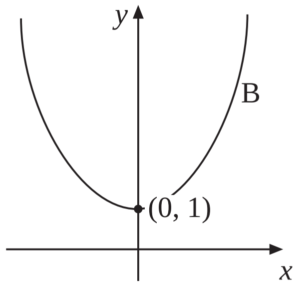
Copy the diagram and, on the same axes, sketch the graph with equation $y = 2x^2$.

(b) Curve A is the reflection in the x-axis of $y = x^2$.



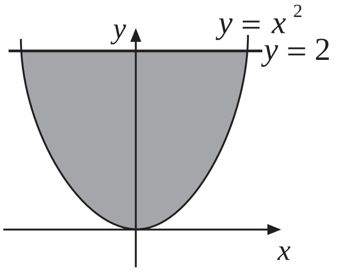
What is the equation of curve A ?

(c) Curve B is the translation, one unit up the y-axis, of $y = x^2$.



What is the equation of curve B ?

(d) The shaded region is bounded by the curve $y = x^2$ and the line $y = 2$.



Write down *two* of the inequalities below which together *fully describe* the shaded region.

$y < x^2$	$x < 0$	$y < 2$	$y < 0$
$y > x^2$	$x > 0$	$y > 2$	$y > 0$

(KS3/98/Ma/Tier 6-8/P1)

13.3 Graphs of Cubic and Reciprocal Functions

In this section we look at the graphs of cubic functions, i.e. the graphs of functions whose polynomial equations contain x^3 and no higher powers of x. We also look at the graphs of reciprocal functions, for example, $y = \frac{1}{2}x$, $y = \frac{3}{x}$ and $y = \frac{-2}{x}$.

Example 1

(a) Draw the graph of $y = x^3$.

(b) Describe the symmetry of the curve.

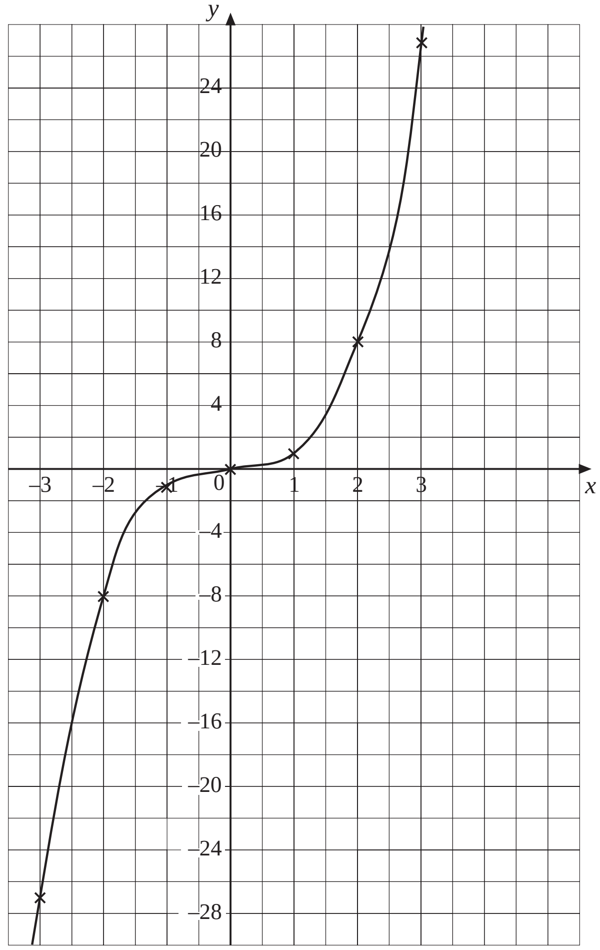
Solution

(a) First complete a table of values:

x	-3	-2	-1	0	1	2	3
$y = x^3$	-27	-8	-1	0	1	8	27

The graph is shown below:

(b) $y = x^3$



(b) The graph has rotational symmetry of order 2 about the point with coordinates $(0, 0)$.

Example 2

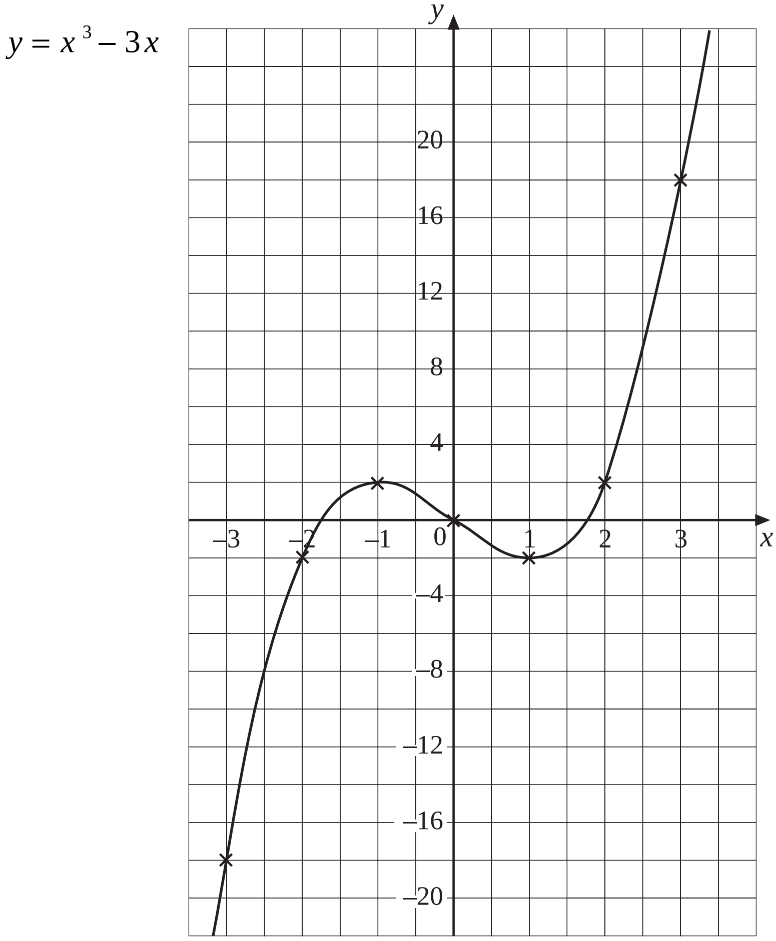
Draw the graph with equation $y = x^3 - 3x$.

Solution

Completing a table of values gives:

x	-4	-3	-2	-1	0	1	2	3	4
$y = x^3 - 3x$	-52	-18	-2	2	0	-2	2	18	52

The graph is shown below:



Example 3

(a) Draw the curve with equation $y = \dfrac{8}{x}$.

(b) On the same diagram, draw the line with equation $y = x + 2$.

(c) Write down the coordinates of the points where the line crosses the curve.

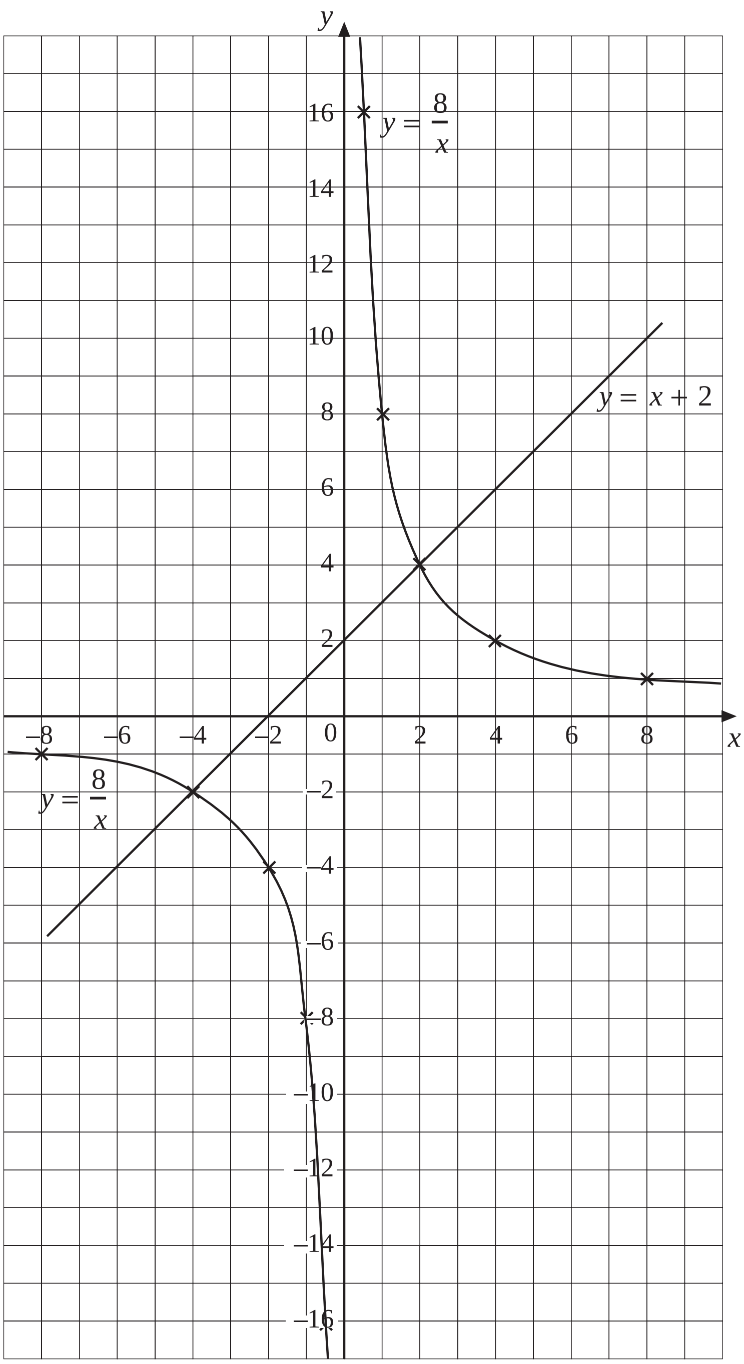
Solution

(a) Completing a table of values gives:

x	-8	-4	-2	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2	4	8
$y = \dfrac{8}{x}$	-1	-2	-4	-8	-16	16	8	4	2	1

Note that $\dfrac{8}{x}$ is not defined when $x = 0$.

These values can then be used to draw the graph below.



(b) The line $y = x + 2$, which passes through $(-6, -4)$, $(0, 2)$ and $(8, 10)$. has been added to the graph above.

(c) The coordinates of these points are $(-4, -2)$ and $(2, 4)$.

Exercises

1. (a) On the same set of axes, draw the graphs with equations,

$$y = x^3 + 5, \qquad y = x^3 - 1 \qquad \text{and} \qquad y = x^3 - 4.$$

(b) Describe how the graphs are related.

2. (a) Draw the graph of the curve with equation $y = x^3 - x$.

(b) Describe the symmetry of the curve.

3. (a) Draw the graph of the curve with equation $y = (x+1)^3$.

(b) Describe how the curve relates to the graph of $y = x^3$.

(c) Describe the symmetry of the curve $y = (x+1)^3$.

4. (a) Draw the graph of the curve with equation $y = (x-2)^3$.

(b) Describe the symmetry of this curve.

5. (a) Draw the graphs of the curves with equations $y = 2 - x^3$ and $y = x^3$.

(b) Describe the transformation that would map one curve onto the other.

6. (a) Copy and complete the following table:

x	-4	-2	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2	4
$\frac{1}{x}$								

(b) Use these values to draw the graph of the curve with equation $y = \dfrac{1}{x}$.

(c) Describe the symmetry of this curve.

7. On the same set of axes, draw the curves with equations,

$$y = \frac{1}{x}, \qquad y = \frac{2}{x} \quad \text{and} \quad y = \frac{4}{x}.$$

8. (a) On the same set of axes, draw the curve with equation $y = \frac{6}{x}$ and the line with equation $y = 7 - x$, for values of x from $\frac{1}{2}$ to 7.

   (b) Write down the coordinates of the points where the line intersects the curve.

9. Determine, by drawing a graph, the coordinates of the points where the line with equation $y = x - 3$ intersects the curve with equation $y = \frac{10}{x}$. Use values of x from – 4 to 6.

10. Determine, graphically, the coordinates of the points where the curve $y = \frac{1}{x}$ intersects the curves with equations,

   (a) $y = x^2$, (b) $y = x^3$.

13.4 Solving Non-Linear Equations

In this section we consider how to solve equations by using graphs, trial and improvement or a combination of both.

Example 1

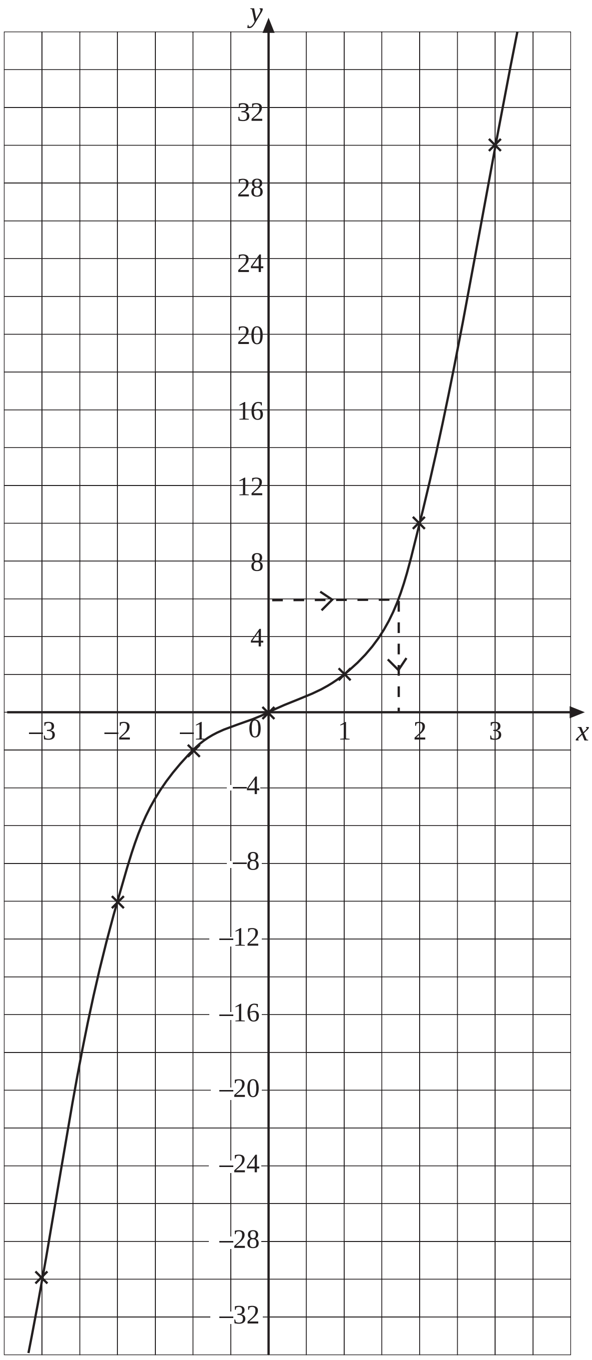
Solve the equation $x^3 + x = 6$ by using a graph.

Solution

The graph $y = x^3 + x$ should be drawn first, as shown.

A line can then be drawn on the graph from 6 on the y-axis, across to the curve and down to the x-axis. This gives an approximate solution between 1.6 and 1.7, so graphically we might estimate x to be 1.65.

$y = x^3 + x$



Example 2

Determine a solution to the equation $x^3 + x = 6$ correct to 2 decimal places.

Solution

The previous example suggested graphically that there is a solution of the equation between $x = 1.6$ and $x = 1.7$. We will now use a trial and improvement method to find x to 2 decimal places, using $x = 1.6$ as a starting value in the process.

Trial x	$x^3 + x$	*Comment*
1.6	5.696	$1.6 < x$
1.7	6.613	$1.6 < x < 1.7$
1.65	6.142125	$1.6 < x < 1.65$
1.63	5.960747	$1.63 < x < 1.65$
1.64	6.050944	$1.63 < x < 1.64$
1.635	6.00572288	$1.63 < x < 1.635$

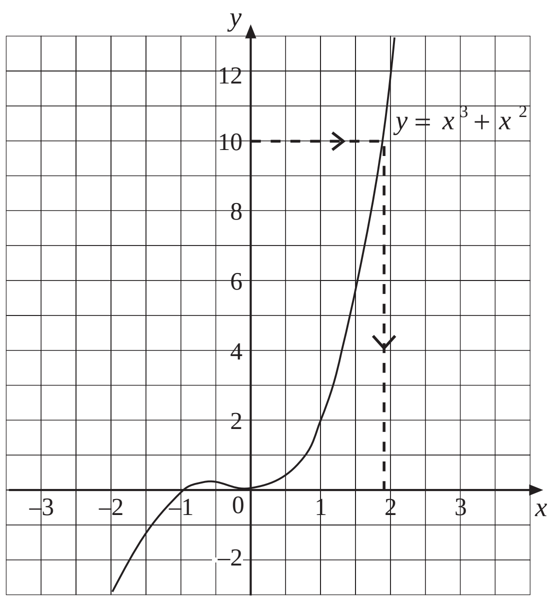
At this stage we can conclude that the solution is $x = 1.63$ correct to 2 decimal places.

Example 3

Use a graph and trial and improvement to solve the equation $x^3 + x^2 = 10$.

Solution

The graph indicates that there will be a solution that is a little less than 2, approximately 1.9.

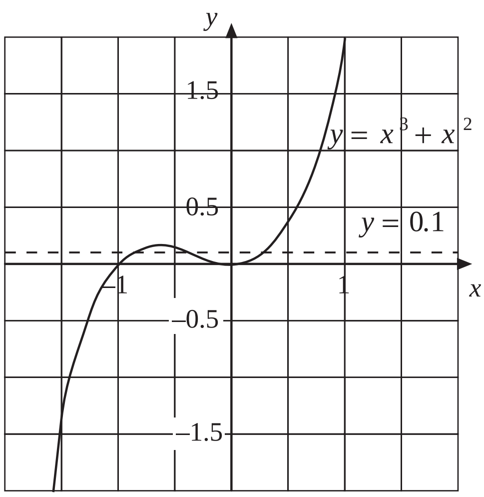


A trial and improvement approach is now used to determine x to a greater degree of accuracy.

Trial x	$x^3 + x^2$	*Comment*
1.9	10.469	$x < 1.9$
1.8	9.072	$1.8 < x < 1.9$
1.85	9.754125	$1.85 < x < 1.9$
1.88	10.179072	$1.85 < x < 1.88$
1.87	10.036103	$1.85 < x < 1.87$
1.86	9.894456	$1.86 < x < 1.87$
1.865	9.96511463	$1.865 < x < 1.87$

The solution is $x = 1.87$ correct to 2 decimal places.

Note: The equation $x^3 + x^2 = 10$ had just one solution. However, in general, there may be more than one solution. For example, the diagram shows that the equation $x^3 + x^2 = 0.1$ has three solutions.



Exercises

1. Use a graph to determine the 2 solutions to the equation $x^2 + x = 6$.

2. (a) Draw the graph $y = 2x^2 - x$.

   (b) Use the graph to determine approximate solutions to the equations:

   (i) $2x^2 - x = 8$,

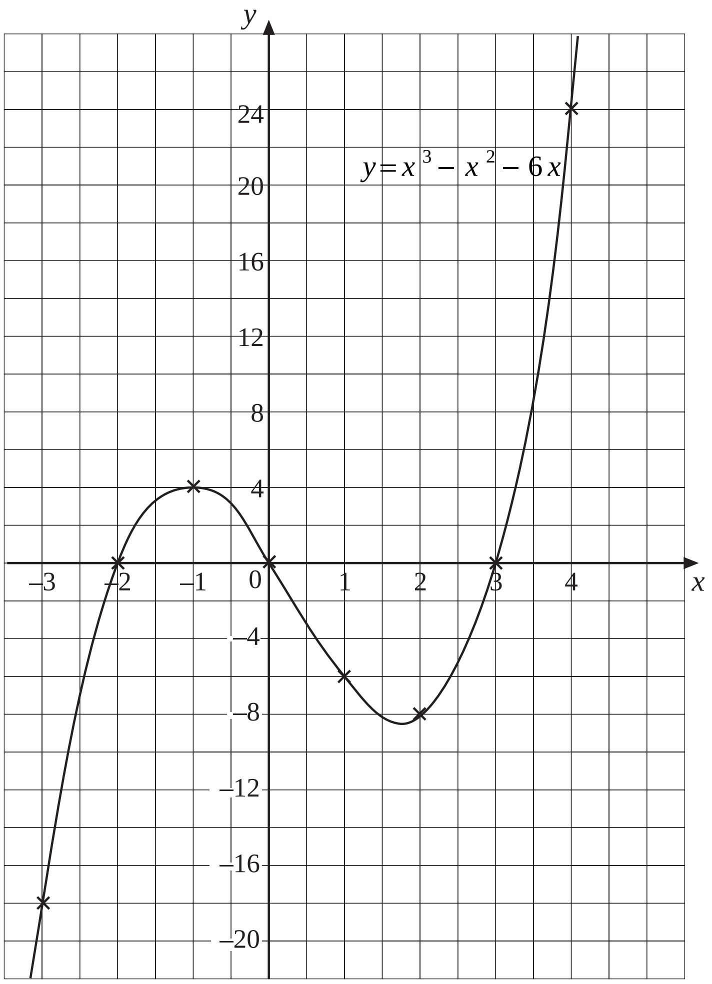
   (ii) $2x^2 - x = 5$.

3. The following graph is for $y = x^3 - x^2 - 6x$.

   Use the graph to solve the following equations:

   (a) $x^3 - x^2 - 6x = 8$ (b) $x^3 - x^2 - 6x = -10$

   (c) $x^3 - x^2 - 6x = 2$ (d) $x^3 - x^2 - 6x = -4$



4. The equation $x^3 + x = 1000$ has a solution close to $x = 10$.

   Use trial and improvement to obtain the solution correct to 2 decimal places.

5. The equation $x + \sqrt{x} = 5$ has a solution between $x = 3$ and $x = 4$. Find this solution correct to 2 decimal places.

6. Use a graphical method followed by trial and improvement to determine both solutions of the equation $x^2 + 6x = 8$, correct to 2 decimal places.

7. The equation $x + \dfrac{1}{x} = 8$ has 2 solutions.

   (a) Use a graph to determine approximates values for these solutions.

   (b) Determine these solutions correct to 2 decimal places using trial and improvement.

8. The equation $8x^2 - x^3 = 5$ has 3 solutions. Determine each solution correct to 1 decimal place.

9. The equation $\frac{1}{x} + x^2 = 1$ has 1 solution. Determine this solution correct to 2 decimal places.

10. Determine each of the solutions of the equation $x^3 - 4x = 2$ correct to 2 decimal places.

11. The table below shows values of *x* and *y* for the equation $y = x^2 + x - 5$.

(a) Copy and complete the table.

x	-2	-1	0	1	2	3
y				-3	1	7

The value of *y* is 0 for a value of *x* between 1 and 2.

(b) Find the value of *x*, to 1 decimal place, that gives the value of *y* closest to 0.

You may use trial and improvement, as shown.

x	y
1	-3
2	1

(KS3/96/Ma/Tier 6-8/P1)

12. Enid wants to find the roots of the equation $2x^2 = 10x - 5$.
The roots are the values of *x* which make the equation correct.

Enid works out values of $2x^2$ and $10x - 5$.
She also works out the difference between each pair of values by subtracting the value $10x - 5$ from the value of $2x^2$.
Enid notes whether this difference is positive or negative.

x	$2x^2$	$10x - 5$	*Difference*	
-2	8	-25	$+33$	Positive
-1	2	-15	$+17$	Positive
0	0	-5	$+5$	Positive
1	2	5	-3	Negative
2	8	15	-7	Negative

(a) One root of the equation $2x^2 = 10x - 5$ lies between $x = 0$ and $x = 1$.
Use the table to explain why.

Enid then tries *1 decimal place* numbers for *x*.

x	$2x^2$	$10x - 5$	*Difference*
0.3	0.18	– 2	+ 2.18
0.4	0.32	– 1	+ 1.32
0.5	0.50	0	+ 0.50
0.6	0.72	1	– 0.28
0.7	0.98	2	– 1.02

(b) Between which two *1 decimal place* numbers does the root lie?

(c) Try some *2 decimal place* numbers for *x*.
Show all your trials in a table like the one below.

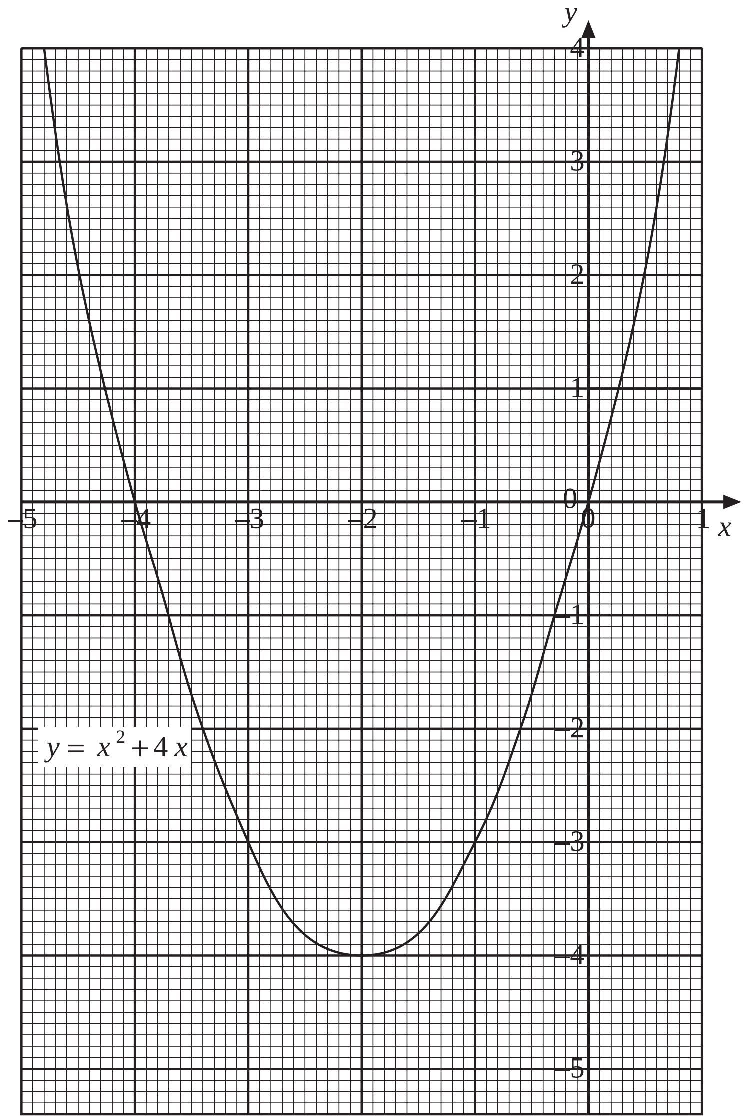
Find the two values of *x* between which the root lies.
Write down *all the digits* you get for the values of $2x^2$, $10x - 5$ and Difference.

x	$2x^2$	$10x - 5$	*Difference*

(d) Between which two *2 decimal place* numbers does the root lie?

(KS3/95/Ma/Levels 5-7/P1)

13.



The graph shows $y = x^2 + 4x$.

(a) Solve the equation $x^2 + 4x - 2 = 0$ using the graph.
Give your answers to 2 decimal places.

(b) Give an example of another equation you could solve in a similar way using the graph.

(c) The equation $x^2 + 4x + 5 = 0$ cannot be solved using the graph.
Why not?

Kelly used an iterative method to find a more accurate solution to the equation $x^2 + 4x - 2 = 0$.

Kelly's method was $x_{n+1} = \dfrac{2}{x_n + 4}$

(d) Explain how Kelly's method relates to the equation $x^2 + 4x - 2 = 0$.

Kelly started with $x_1 = 1$ used her iterative method four times.
She got these results.

x_1	x_2	x_3	x_4	x_5
1	0 4	0.4545455	0.4489796	0.4495413

(e) Steve used a different iterative method.

Steve's method was $x_{n+1} = \dfrac{2 - 4x_n}{x_n}$.

He started with $x_1 = 1$.
Work out x_2, x_3, x_4 and x_5 and write them showing all the digits on your calculator.

(KS3/95/Ma/Levels 9-10)

13.5 Quadratic Inequalities

In the first section of this unit we considered *linear inequalities*. In this section we will consider *quadratic inequalities* and make use of both graphs and the factorisation that you used in Unit 11.

We begin with a graphical approach.

Example 1

(a) Draw the graph $y = x^2 + 3x - 10$.

(b) Use the graph to solve the inequality $x^2 + 3x - 10 \ge 0$.

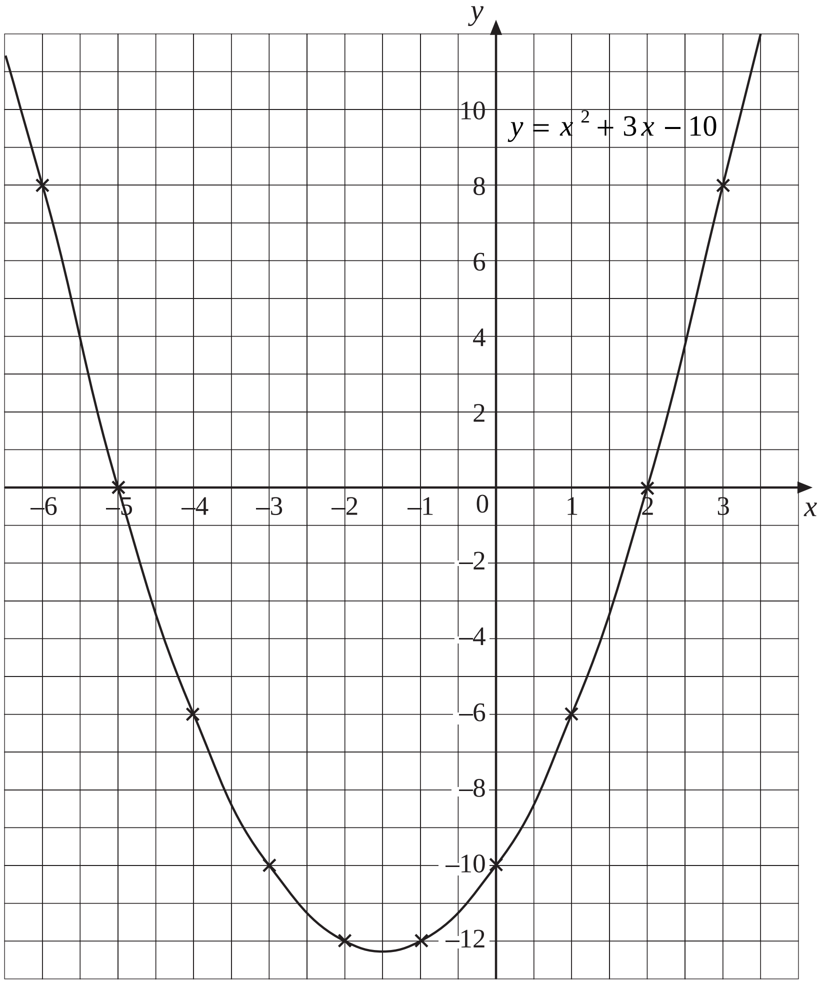
Solution

The graph is shown below. Note that $x^2 + 3x - 10 = 0$ at $x = -5$ and $x = 2$.

The graph shows that

$$x^2 + 3x - 10 \ge 0$$

when $x \le -5$ or $x \ge 2$.



Example 2

Solve the inequality

$$x^2 - 6x < 0$$

Solution

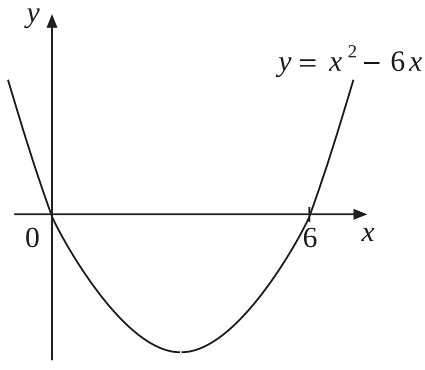
Factorising gives:

$$x^2 - 6x = x(x - 6)$$

So $x^2 - 6x = 0$ when $x = 0$ or $x = 6$.

Sketching the graph as shown indicates that the solution is

$$0 < x < 6.$$



Example 3

Solve the inequality

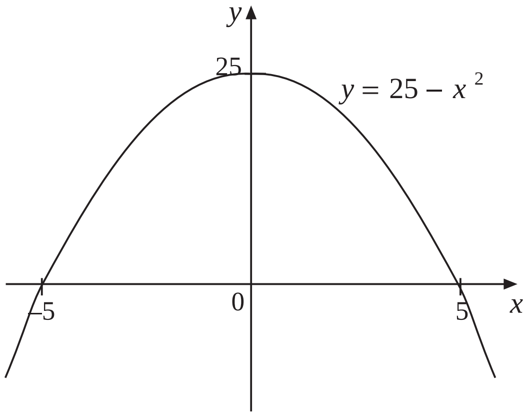
$$25 - x^2 > 0$$

Solution

Factorising gives:

$$25 - x^2 = (5 - x)(5 + x)$$

So $25 - x^2 = 0$ when $x = 5$ or $x = -5$.



The sketch shows that $25 - x^2 > 0$ will be satisfied when $-5 < x < 5$.

Exercises

1. (a) Draw the graph $y = x^2 + 2x - 3$.

   (b) Solve the inequality $x^2 + 2x - 3 \geq 0$

   (c) Solve the inequality $x^2 + 2x - 3 < 0$.

2. Use a graph to solve the inequality $1 - x^2 < 0$.

3. Use a graph to solve the inequality $x^2 + 3x - 4 \leq 0$.

4. (a) Factorise $x^2 - 5x$.

   (b) Sketch the graph of $y = x^2 - 5x$.

   (c) State the solution of the inequality $x^2 - 5x < 0$.

5. Solve the following inequalities:

   (a) $x^2 + 5x > 0$ (b) $x^2 - 3x \leq 0$

   (c) $x - x^2 < 0$ (d) $2x - x^2 \geq 0$

6. (a) Factorise $x^2 - 49$.

   (b) Sketch the graph of $y = x^2 - 49$.

   (c) State the solution of the inequality $x^2 - 49 > 0$.

7. Solve the inequalities:

(a) $x^2 - 36 < 0$ (b) $100 - x^2 \geq 0$

(c) $x^2 - 16 \leq 0$ (d) $81 - x^2 < 0$

8. (a) Factorise $x^2 - 5x - 14$.

(b) Sketch the graph of $y = x^2 - 5x - 14$.

(c) State the solution of the inequality $x^2 - 5x - 14 \geq 0$.

9. Solve the inequalities:

(a) $x^2 - 6x - 27 < 0$ (b) $x^2 + 7x + 12 \leq 0$

(c) $x^2 - 13x + 40 > 0$ (d) $x^2 - 7x - 18 \geq 0$

10. (a) Factorise $-x^2 + 12x - 27$.

(b) Sketch the graph of $y = -x^2 + 12x - 27$.

(c) State the solution of the inequality $x^2 - 12x + 27 < 0$.

11. Denise and Luke are using the expression $\frac{n(n+1)}{2}$ to generate triangular numbers.

For example, the triangular number for $n = 4$ is $\frac{4(4+1)}{2}$, which works out to be 10.

(a) Denise wants to solve the inequality $300 < \frac{n(n+1)}{2} < 360$ to find the two triangular numbers between 300 and 360.

What are these two triangular numbers?

You may use trial and improvement.

(b) Luke wants to find the two smallest triangular numbers which fit the inequality $\frac{n(n+1)}{2} > 2700$.

What are these two triangular numbers?

You may use trial and improvement.

(KS3/95/Ma/Tier 6-8/P1)

13.6 Equations of Perpendicular Lines

In this section we consider the relationship between perpendicular lines.

> If one line has gradient m, $m \neq 0$, a line that is perpendicular to it will have gradient $\dfrac{-1}{m}$.

Note: The examples that follow make use of the general equation $y = mx + c$ for a straight line with gradient m and y-intercept c.

Example 1

A line passes through the origin and is perpendicular to the line with equation $y = 7 - x$. Determine the equation of the line.

Solution

The line's equation $y = 7 - x$ can be rewritten in the form $y = -x + 7$ showing that it has gradient -1.

A perpendicular line will have gradient $\dfrac{-1}{-1} = 1$ and so its equation will be $y = x + c$.

As it passes through the origin, we know $y = 0$ when $x = 0$.

Substituting these values into the equation gives,

$$0 = 0 + c$$

so

$$c = 0$$

Hence the equation is $y = x$.

Example 2

A line passes through the points with coordinates (2, 6) and (5, – 1).
A second line passes through the points with coordinates (0, 3) and (7, 6).
Are the two lines perpendicular?

Solution

$$\text{Gradient of first line} = \frac{(-1) - 6}{5 - 2}$$

$$= \frac{-7}{3}$$

$$\text{Gradient of second line} = \frac{6 - 3}{7 - 0}.$$

$$= \frac{3}{7}.$$

But as $\dfrac{3}{7} = \dfrac{-1}{\left(\dfrac{-7}{3}\right)}$, the lines are perpendicular.

Note: This example illustrates an alternative way of looking at the relationship between the gradients of perpendicular lines, namely that the product of their gradients is -1.

For example, $\dfrac{-7}{3} \times \dfrac{3}{7} = -1$

Exercises

1. (a) Draw the line with equation $y = 2x - 1$.

   (b) Determine the equation of a perpendicular line that passes through the origin.

   (c) Draw this line and check that it is perpendicular.

2. The equations of 5 lines are given below.

   A $\quad y = 8x - 5$

   B $\quad y = 3x + 2$

   C $\quad y = -\dfrac{2x}{16} + 1$

   D $\quad y = 4 - \dfrac{x}{3}$

   E $\quad y = \dfrac{-x}{8} + 7$

   Which line or lines are:

   (a) perpendicular to A, (b) perpendicular to B ?

3. The points A, B, C and D have coordinates (1, 3), (6, 1), (3, 1) and (5, 6) respectively. Show that AB is perpendicular to CD.

4. A quadrilateral has corners at the points with coordinates Q (7, 3), R (6, 5), S (2, 3) and T (3, 1).

   Show that QRST is a rectangle.

5. The line with equation $y = 7 - x$ and a perpendicular line intersect at the point with coordinates (4, 3). Determine the equation of the perpendicular line.

6. Are the lines with equations

$$y = \frac{x - 2}{3} \text{ and } y = 8 - 3x$$

perpendicular?

7. A line passes through the origin and the point (4, 7). Determine the equations of the perpendicular lines that pass through:

(a) the origin, (b) the point (4, 7).

8. A line is drawn perpendicular to the line $y = \frac{1}{2}x + 4$ so that it passes through the point with coordinates (3, 3).

(a) Determine the equation of the perpendicular line.

(b) Determine the coordinates of the point where the two lines intersect.

9. Two perpendicular lines intersect at the point with coordinates (4, 6). One line has gradient -4. Determine the equations of the two lines

10. Two perpendicular lines intersect at the point with coordinates (6, 5). One line passes through the origin. Where does the other line intersect the x-axis?