In this chapter you will learn:

- about different units for measuring angles, and how measuring angles is related to distance travelled around a circle
- the definitions of the sine, cosine and tangent functions, their basic properties and their graphs
- how to calculate certain special values of trigonometric functions
- how to apply your knowledge of transformations of graphs to sketch more complicated trigonometric functions
- how to use trigonometric functions to model periodic phenomena.

8 Circular measure and trigonometric functions

Introductory problem

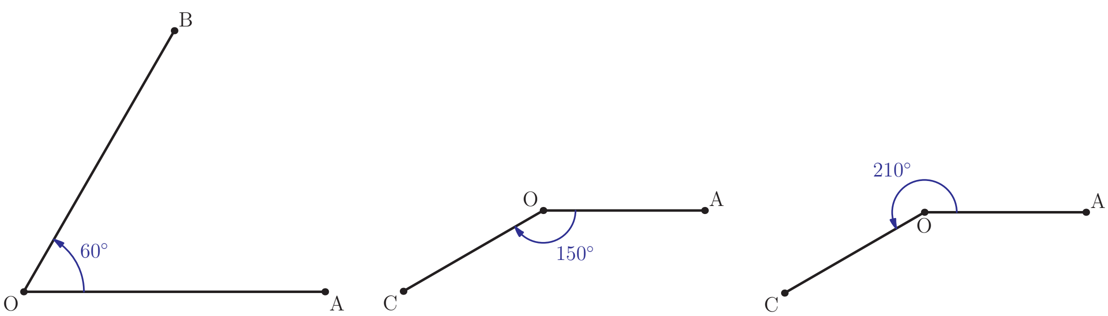
A clock has an hour hand of length 10 cm, and the centre of the clock is 4 m above the floor. Find an expression for the height, *h* metres, of the tip of the hour hand above the floor at a time *t* hours after midnight.

Measuring angles is related to measuring lengths around the perimeter of a circle. This observation leads to a new unit for measuring angles, the **radian**, which will turn out to be more useful than the degree as a unit of angle measurement in advanced mathematics.

Periodic motion is motion that repeats after a fixed time interval. Motion in a circle is just one example of this; other examples include oscillation of a particle attached to the end of a spring or vibration of a guitar string. There are also periodic phenomena where a pattern is repeated in space rather than in time – for example, the shape of a water wave. All of these can be modelled using **trigonometric functions**.

8A Measuring angles

An angle measures the amount of rotation between two straight lines. You are already familiar with measuring angles in degrees, where a full turn measures 360°. There are two directions (or senses) of rotation: clockwise and anti-clockwise (counter-clockwise). In mathematics, the convention is to take anti-clockwise as the positive direction. In the first diagram below, the line OA rotates 60° anti-clockwise into position OB. Therefore $A\hat{O}B = 60°$.



In the second diagram, the line OA rotates 150° clockwise into position OC. Clockwise rotations are represented by negative angle measures; therefore $A\hat{O}C = -150°$. Note that we can also move OA to position OC by rotating 210° anti-clockwise, as shown in the third diagram, so it is equally correct to say that $A\hat{O}C = 210°$. In this book, as well as in exam questions, it will always be made clear which of the two angles is required.

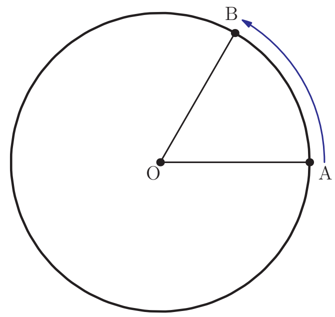
You may wonder why there are 360 degrees in a full turn. Division of a full rotation into 360 equal angles seems already to have been standard for geometers in ancient Babylon, Greece and India. It is believed to have come from ancient astrologers, who noticed that the stars appeared to rotate in the sky, returning to their original positions after around 360 days. They therefore divided a full rotation into 360, so that each day's rotation would be equal to one unit. We now know that there are 365.24 days in a year, so these ancient astrologers were quite accurate – you may want to think about how one might measure the number of days in a year.

Defining a full turn to be 360° is somewhat arbitrary, and there are other ways of measuring sizes of angles. In advanced mathematics, the most useful unit of angle measurement is the **radian**. This measure relates the size of the angle to the distance moved by a point around a circle.

The unit circle will be used when defining trigonometric functions in Section 8B.

Consider a circle with centre O and radius 1; this is called the **unit circle**. Let A and B be two points on its circumference. As the line OA rotates into position OB, point A moves a distance equal to the length of the arc AB. The measure of the angle $A\hat{O}B$ in radians is defined to be this arc length.

Recall that an arc is the path joining two points on a curve.



If point A makes a full rotation around the circle, it will cover a distance equal to the length of the circumference of the circle. As the radius of the circle is 1, the length of the circumference is 2π. Hence a full turn measures 2π radians. From this we can deduce the sizes in radians of other angles; for example, a right angle is one quarter of a full turn, so it measures $2\pi \div 4 = \frac{\pi}{2}$ radians. Although the sizes of common angles measured in radians are often expressed as fractions of π, we can also use decimal approximations. Thus a right angle measures approximately 1.57 radians. The fact that a full turn measures 2π radians can be used to convert any angle measurement from degrees to radians, and vice versa.

Worked example 8.1

(a) Convert 75° to radians.

(b) Convert 2.5 radians to degrees.

What fraction of a full turn is 75°?

(a) $\frac{75}{360} = \frac{5}{24}$

Calculate the same fraction of 2π.

$\frac{5}{24} \times 2\pi = \frac{5\pi}{12}$

$\therefore 75° = \frac{5\pi}{12}$ radians

This is the *exact* answer. Using a calculator, we can find the decimal equivalent to 3 significant figures.

75° = 1.31 radians (3 SF)

What fraction of a full turn is 2.5 radians?

(b) $\frac{2.5}{2\pi} (\approx 0.3979)$

Calculate the same fraction of 360°.

$\frac{2.5}{2\pi} \times 360 = 143.239\ldots$

2.5 radians = 143° (3 SF)

KEY POINT 8.1

$$\text{full turn} = 360° = 2\pi \text{ radians}$$

$$\text{half turn} = 180° = \pi \text{ radians}$$

To convert from degrees to radians, divide by 180 and multiply by π:

$$\text{radians} = \frac{\pi \times \text{degrees}}{180}$$

To convert from radians to degrees, divide by π and multiply by 180:

$$\text{degrees} = \frac{180 \times \text{radians}}{\pi}$$

There are various other measures of angle. For example, a unit that originated in France when the metric system was introduced is the gradian, which is one-hundredth of a right angle. In most countries today, use of the gradian continues only within a few specialised fields, such as architectural surveying and artillery. Since there are so many different ways of defining angle units, does this mean that the facts you have learnt – such as there being 180 degrees in a triangle – are purely consequences of particular definitions and have no link to truth?

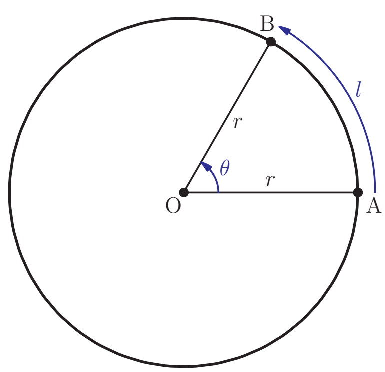
Our definition of radian measure used the unit circle. However, we can consider a point moving around a circle of any radius.

Think about what happens as the line OA rotates into position OB.

The distance covered by point A will be the arc length, l.

Since the entire circumference is $2\pi r$, the fraction of the circle travelled is $\frac{l}{2\pi r}$.

Let θ be the measure of $A\hat{O}B$ in radians; this corresponds to a fraction $\frac{\theta}{2\pi}$ of a full rotation.



Now we have two expressions for the proportion of the circle covered as we move from OA to OB: one from considering the length of the arc, and the other from considering the size of the angle. Both give the fraction of the circle covered by the sector AOB, so they must be equal:

$$\frac{\theta}{2\pi} = \frac{l}{2\pi r} \Rightarrow \theta = \frac{l}{r}$$

KEY POINT 8.2

The radian measure of an angle in a circle is given by

$$\theta = \frac{l}{r}$$

That is, the measure of the angle is the ratio of the length of the arc to the radius of the circle.

In particular, an angle of 1 radian corresponds to an arc whose length is equal to the radius of the circle.

EXAM HINT

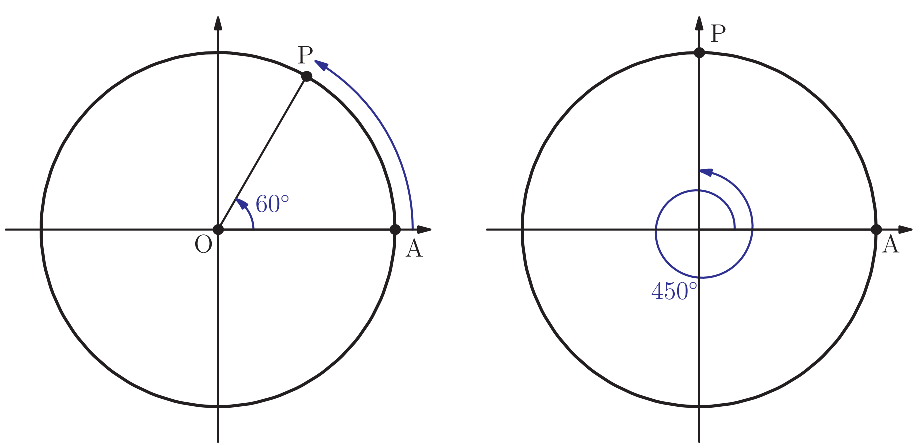
Some people write angles in radians as, for instance, $\theta = 1.31\,\text{rad}$ or $\theta = 1.31^c$, but the International Baccalaureate does not use this notation.

Radians will be used whenever we differentiate or integrate trigonometric functions (from chapter 12 onwards).

In the unit circle (where $r = 1$), the size of an angle is numerically equal to the length of the arc; however, these two quantities have different units. If we think of the size of an angle as a ratio of two lengths (as in Key point 8.2), then it should have no units. This is why the radian is said to be a *dimensionless unit* and, when writing the size of an angle in radians, we need only give the angle as a number, for example $\theta = 1.31$.

All this may sound complicated, and you may wonder why we cannot just use degrees to measure angles. You will see in the next two sections that the formulas for calculating lengths and areas of parts of circles are much simpler when radians are used, and the advantages of using radians will become even clearer when you study calculus.

If we think of angles as measuring the amount of rotation around the unit circle, then we can represent an angle of any size by its corresponding point on the unit circle. As mentioned earlier, the convention in mathematics is to measure positive angles by anti-clockwise rotations. Another convention is that we consider the unit circle as having its centre at the origin of the coordinate system, and start measuring from the point on the circle which lies on the positive x-axis. In the first diagram below, the starting point is labelled A, and the point P corresponds to the angle 60°. In other words, to get from the starting point to point P, we need to rotate 60°, or one-sixth of a full turn, anti-clockwise around the circle.



We can also represent negative angles (by clockwise rotations) and angles larger than 360° (by rotating through more than one full turn). The second diagram above shows point P representing the angle of 450°, or one and a quarter turns.

There is a three-dimensional analogue of an angle, called a solid angle. The size of a solid angle has units of *steradians*, which measure the fraction of the surface area of a sphere covered. Many aspects of two-dimensional trigonometry can be transferred to these solid angles.

Worked example 8.2

(a) Mark on the unit circle the points corresponding to the following angles, measured in degrees:

A: $135°$ B: $270°$ C: $-120°$ D: $765°$

(b) Mark on the unit circle the points corresponding to the following angles, measured in radians:

A: π B: $-\frac{\pi}{2}$ C: $\frac{5\pi}{2}$ D: $\frac{13\pi}{3}$

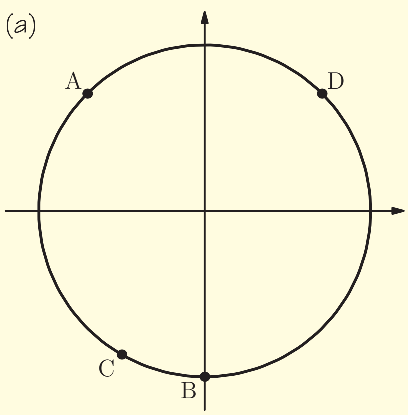
$135 = 90 + 45$, so point A represents quarter plus another eighth of a full turn.

$270 = 3 \times 90$, so point B represents rotation through three right angles.

$120 = 360 \div 3$, so point C represents a third of a full turn, but clockwise (because of the minus sign).

$765 = 2 \times 360 + 45$, so point D represents two full turns plus one half of a right angle.

(a)



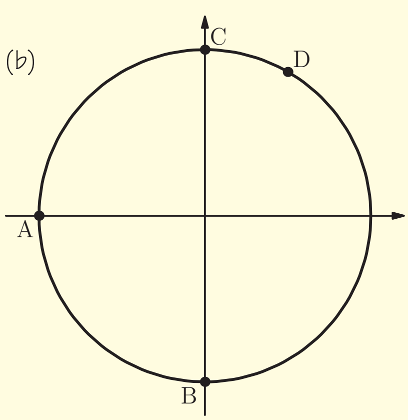
π radians is one half of a full turn so point A represents half a turn.

$\frac{\pi}{2} = \frac{1}{4} \times 2\pi$, so point B represents one quarter of a full turn in the clockwise direction.

$\frac{5\pi}{2} = 2\pi + \frac{\pi}{2}$, so point C represents a full turn followed by another quarter of a turn.

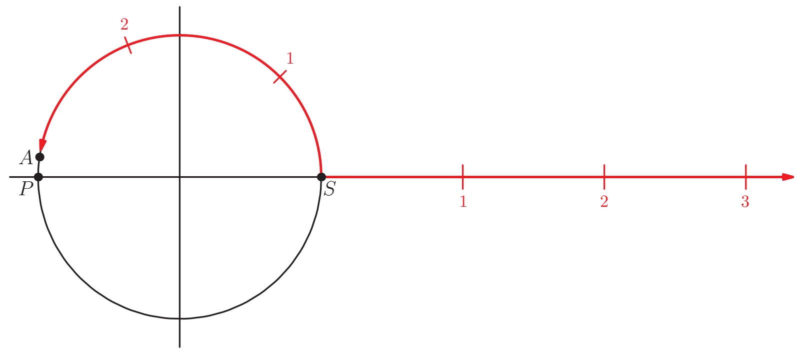
$\frac{13\pi}{3} = 4\pi + \frac{\pi}{3}$, so point D represents two full turns followed by another $\frac{1}{6}$ of a turn.

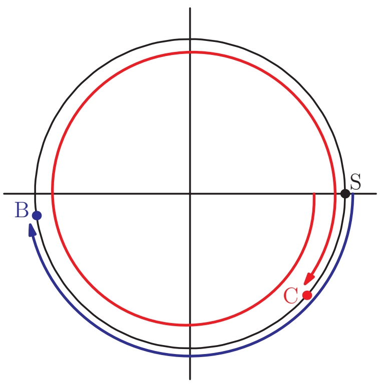
(b)



Looking at the number line in this way should suggest why radians are a more natural measure of angle than degrees. While radians are rarely used outside the mathematical community, within mathematics they are the unit of choice.

The idea of representing angles by points on the unit circle can be applied to numbers too: instead of representing numbers by points on a number line, we can represent them by points on the unit circle. To do this, imagine wrapping the number line around the unit circle: start by placing zero on the positive x-axis (point S in the diagram) and then, going anti-clockwise, lay positive numbers on the circle. As the circumference of the circle has length 2π, the numbers $2\pi, 4\pi$ and so on will also be represented by point S. The numbers $\pi, 3\pi, 5\pi$ and so on are represented by point P. The number 3, which is a little less than π, is represented by point A.





Similarly, we can represent negative numbers by wrapping the negative part of the number line clockwise around the circle. For example, the number -3 is represented by point B and the number -7 by point C (7 is a bit bigger than 2π, which means wrapping once and a bit around the circle).

It is useful to describe where points are on the unit circle by using quadrants. A **quadrant** is one quarter of the circle, and conventionally quadrants are labelled going anti-clockwise, starting from the top right one. So in part (a) of Worked example 8.2, point A is in the second quadrant, point B in the third quadrant and point D in the first quadrant.

Exercise 8A

1. Draw a unit circle for each part and mark the points corresponding to the given angles:

   (a) (i) 60° (ii) 150°

   (b) (i) –120° (ii) –90°

   (c) (i) 495° (ii) 390°

2. Draw a unit circle for each part and mark the points corresponding to the following angles:

(a) (i) $\frac{\pi}{4}$ (ii) $\frac{\pi}{3}$

(b) (i) $\frac{4\pi}{3}$ (ii) $\frac{3\pi}{4}$

(c) (i) $-\frac{\pi}{3}$ (ii) $-\frac{\pi}{6}$

(d) (i) -2π (ii) -4π

3. Express the following angles in radians, giving your answers in terms of π:

(a) (i) 135° (ii) 45°

(b) (i) 90° (ii) 270°

(c) (i) 120° (ii) 150°

(d) (i) 50° (ii) 80°

4. Express the following angles in radians, correct to 3 decimal places:

(a) (i) 320° (ii) 20°

(b) (i) 270° (ii) 90°

(c) (i) 65° (ii) 145°

(d) (i) 100° (ii) 83°

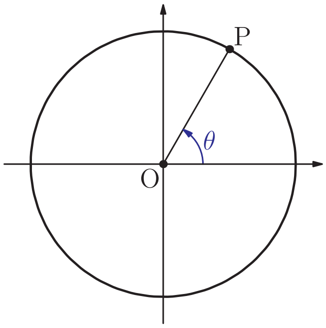
5. Express the following angles in degrees:

(a) (i) $\frac{\pi}{3}$ (ii) $\frac{\pi}{4}$

(b) (i) $\frac{5\pi}{6}$ (ii) $\frac{2\pi}{3}$

(c) (i) $\frac{3\pi}{2}$ (ii) $\frac{5\pi}{3}$

(d) (i) 1.22 (ii) 4.63



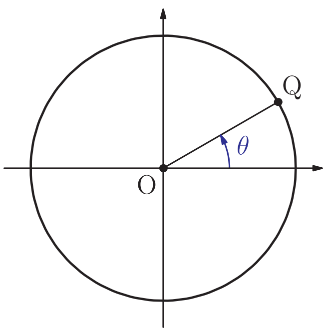
6. The diagram shows point P on the unit circle corresponding to angle θ (measured in degrees). Copy the diagram and mark the points corresponding to the following angles.

(a) (i) $180° - \theta$ (ii) $180° + \theta$

(b) (i) $\theta + 180°$ (ii) $\theta + 90°$

(c) (i) $90° - \theta$ (ii) $270° - \theta$

(d) (i) $\theta - 360°$ (ii) $\theta + 360°$



7. The diagram shows point Q on the unit circle corresponding to the real number θ.

Copy the diagram and mark the points corresponding to the following real numbers.

(a) (i) $2\pi - \theta$ (ii) $\pi - \theta$

(b) (i) $\theta + \pi$ (ii) $-\pi - \theta$

(c) (i) $\frac{\pi}{2} + \theta$ (ii) $\frac{\pi}{2} - \theta$

(d) (i) $\theta - 2\pi$ (ii) $\theta + 2\pi$

8B Definitions and graphs of the sine and cosine functions

We now define trigonometric functions.

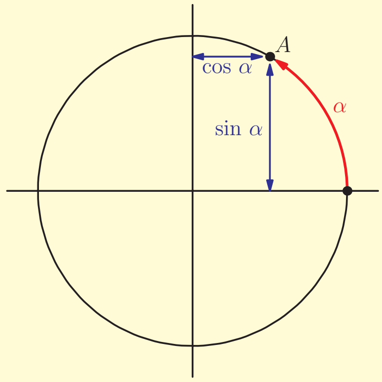
For a real number α, mark the point A on the unit circle that represents the number α (or, equivalently, the angle α radians).

The sine and cosine of α are defined in terms of the distance from point A to the x- and y-axes. (Remember that, by convention, the unit circle has its centre at the origin of the coordinate axes.)

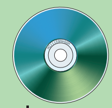
KEY POINT 8.3

The **sine** of the number α, written $\sin\alpha$, is the distance of the point A above the horizontal axis (its y-coordinate).

The **cosine** of the number α, written $\cos\alpha$, is the distance of the point A to the right of the vertical axis (its x-coordinate).



You have previously seen sine and cosine defined using right-angled triangles. See Prior Learning section U on the CD-ROM for a reminder.

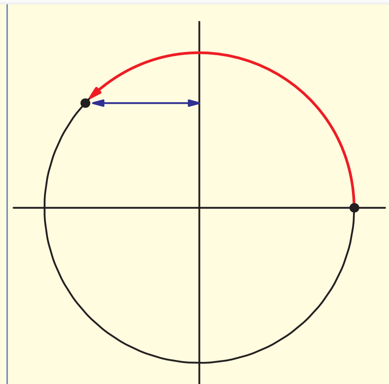


The definition in Key point 8.3 is consistent with the definition by right-angled triangles, but it further allows us to define sine and cosine for angles beyond 90°. This raises the question of how to decide which of several alternative definitions to use – should we go with the one that came first historically, the one that is more understandable, or the one that is more general?

Worked example 8.3

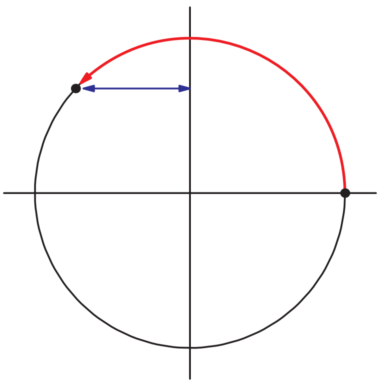
By marking the corresponding point on the unit circle, estimate the value of $\cos 2.4$.

$\frac{\pi}{2} \approx 1.6$ and $\pi \approx 3.1$, so the point is around the middle of the second quadrant.



$\cos 2.4 \approx -0.7$

Notice that in the above example cos 2.4 was a negative number. This is because the point corresponding to 2.4 lies to the left of the vertical axis, so we take the distance to be negative.



You can find sine and cosine values, such as the ones in the previous example, using your calculator.

Most GDC calculators have buttons for the sine, cosine (and tan) functions. If you were to work out the answer to the question in Worked example 8.3 using your GDC, you would simply press [cos] [2][.][4][EXE] and you would get the answer –0.737... (provided your calculator was in radian mode). As the question used an angle measured in radians, you would need to make sure your calculator was set to radians. If your calculator was in degree mode, your calculator would interpret cos 2.4 as the cosine of 2.4° rather than 2.4 radians and you would get an answer of 0.999..., which is incorrect.

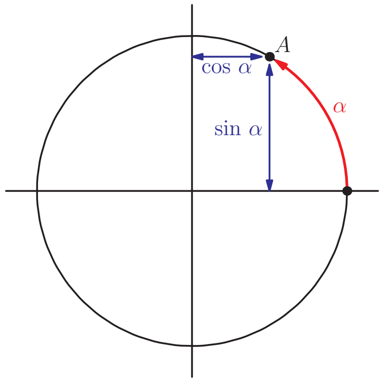
See Calculator skills sheet 1 for how to determine if your calculator is in radians or degree mode, and how to change between them.

Key point 8.1 describes how to change from radians to degrees and vice versa.

Note however, that if your calculator had been in degree mode and you had keyed in the cosine of the equivalent angle in degrees, i.e. 137.5° (which is approximately 2.4 radians), then you would have got the correct answer of –0.737.... This demonstrates that it does not matter whether you find the sine or cosine of an angle in degrees or radians (the answer will be the same) provided that your calculator is in the correct mode according to the form of the angle you enter into your calculator.

We can use the unit circle to get information about functions of angles related to one we already know about.

Remember that $\sin \alpha$ is the distance of the point of interest (e.g. A) on the circumference above the horizontal axis (its y-coordinate) and $\cos \alpha$ is the distance of A to the right of the vertical axis (its x-coordinate).



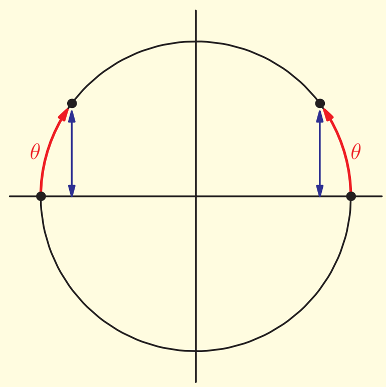
Worked example 8.4

Given that $\sin \theta = 0.6$, find the values of

(a) $\sin(\pi - \theta)$ (b) $\sin(\theta + \pi)$

Mark the points corresponding to θ and $\pi - \theta$ on the circle.

(a)



Since sine of a number is defined to be the y-coordinate of the corresponding point on the unit circle, compare the positions of the points relative to the horizontal axis.

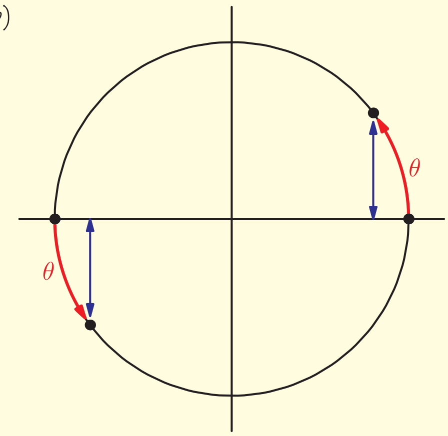
The points are the same distance from and on the same side of the horizontal axis, so

$\sin(\pi - \theta) = 0.6$

continued . . .

Mark the points corresponding to θ and $\theta+\pi$ on the circle.

(b)



Compare the positions of the points relative to the horizontal axis.

The points are the same distance from but on opposite sides of the horizontal axis, so

$\sin(\theta+\pi) = -0.6$

The above example illustrates some of the properties of the sine function. Similar properties hold for the cosine function. The symmetry results summarised below are useful to remember. They can all be derived using circle diagrams.

EXAM HINT

These results are not given in the Formula booklet, so make sure that you are able to work them out from a quick sketch of a circle by considering the symmetries involved.

KEY POINT 8.4

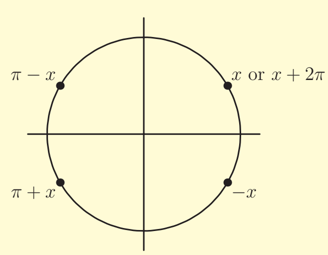
For any real number x:

$\sin x = \sin(\pi - x) = \sin(x + 2\pi)$

$\sin(\pi + x) = \sin(-x) = -\sin x$

$\cos x = \cos(-x) = \cos(x + 2\pi)$

$\cos(\pi - x) = \cos(\pi + x) = -\cos x$

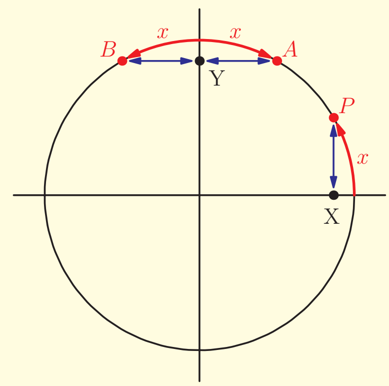


Worked example 8.5

Given that $\sin x = 0.4$, find the value of:

(a) $\cos\left(\frac{\pi}{2} - x\right)$ (b) $\cos\left(x + \frac{\pi}{2}\right)$

Mark on the unit circle the points corresponding to x, $\frac{\pi}{2} - x$ and $x + \frac{\pi}{2}$. The point P represents the number x, and we know that $PX = \sin x = 0.4$.



$\frac{\pi}{2} - x$ is represented by point A, and AY = PX.

(a) $\cos\left(\frac{\pi}{2} - x\right) = 0.4$

$x + \frac{\pi}{2}$ is represented by point B, and BY = PX, but B is to the left of the vertical axis.

(b) $\cos\left(x + \frac{\pi}{2}\right) = -0.4$

The above example illustrates a relationship between sine and cosine functions. It will be useful to remember the following results, or be able to derive them from a circle diagram.

KEY POINT 8.5

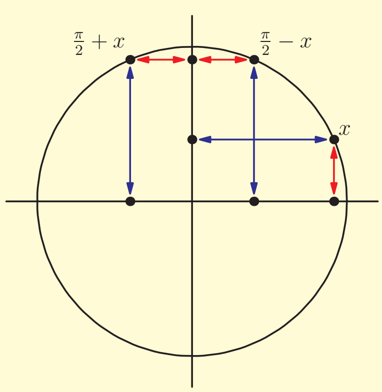
For any real number x:

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x$$

$$\cos\left(\frac{\pi}{2} + x\right) = -\sin x$$

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x$$

$$\sin\left(\frac{\pi}{2} + x\right) = \cos x$$



In chapter 9 you will see another connection between the sine and cosine functions, which arises from Pythagoras' Theorem.

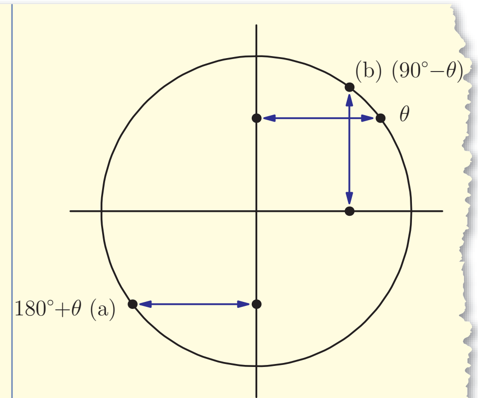
In Key points 8.4 and 8.5, the variable x was an angle measured in radians. Analogous results can be derived when the variable represents an angle measured in degrees.

Worked example 8.6

Given that θ is an angle measured in degrees such that $\cos\theta = 0.8$, find the value of

(a) $\cos(180° + \theta)$ (b) $\sin(90° - \theta)$

Mark on the circle the points corresponding to angles θ, $180° + \theta$ and $90° - \theta$. We know that the point representing θ is at distance 0.8 from the vertical axis.



For the point representing $180° + \theta$, its distance from the vertical axis is also 0.8, but it lies to the left of the axis.

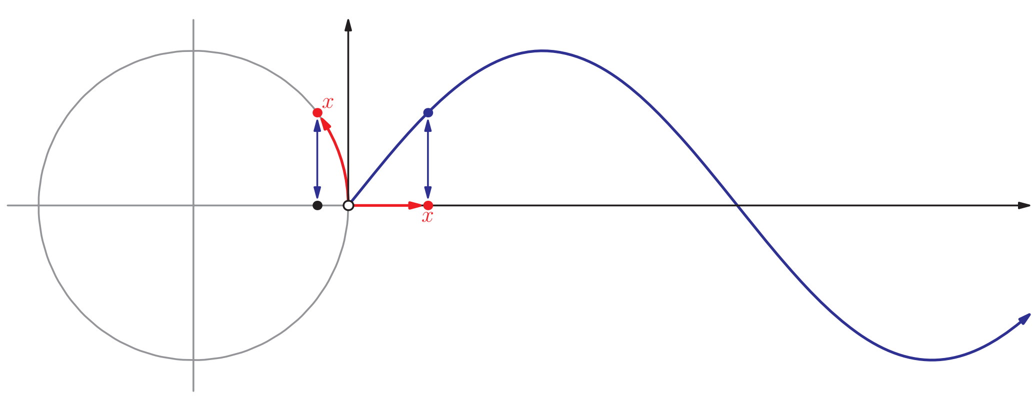
(a) $\cos(180° + \theta) = -0.8$

The point representing $90° - \theta$ is the reflection of the point representing θ in the diagonal line $y = x$, so the distance from the *horizontal* axis is 0.8.

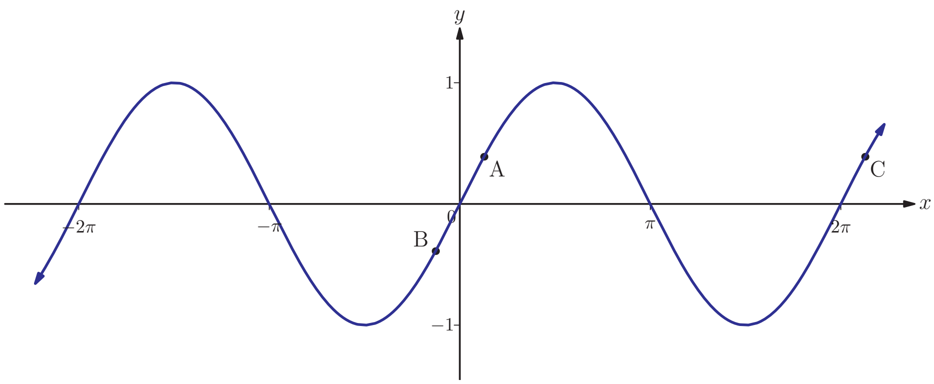
(b) $\sin(90° - \theta) = 0.8$

So far we have focused on the sine and cosine of angles, but in reality the domain of the sine and cosine functions is all real numbers.

Having defined the sine function for all real numbers, we can draw its graph. To do this, let's go back to thinking about the real number line wrapped around the unit circle. Each real number corresponds to a point on the circle, and the value of the sine function is the distance of the point from the horizontal axis.



All of the properties of the sine function discussed above can be seen in its graph. For example, increasing x by 2π corresponds to making a full turn around the circle and returning to the same point; therefore, $\sin(x+2\pi)=\sin x$. We say that the sine function is **periodic** with **period** 2π. Looking at the graph below, by considering points A and B we can see that $\sin(-x)=-\sin x$. We can also see that the minimum possible value of $\sin x$ is -1 and the maximum value is 1. Thus we say that the sine function has **amplitude** 1.

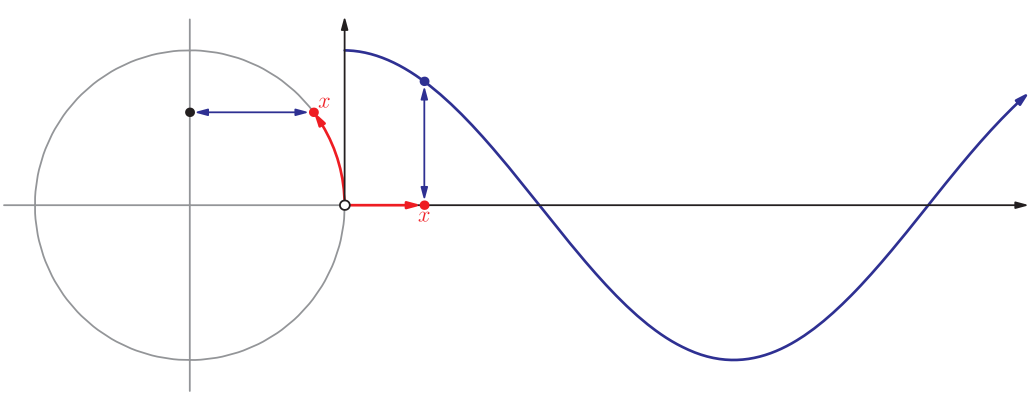


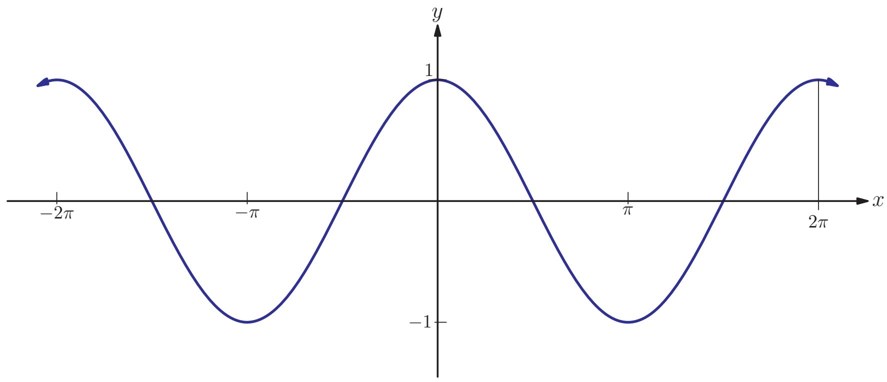
KEY POINT 8.6

A function is **periodic** if its pattern repeats regularly. The interval between the start of two consecutive repeating blocks is called the **period**.

The **amplitude** of a periodic function is half the distance between the maximum and minimum values.

To draw the graph of the cosine function, we look at the distance from the vertical axis of the point on the unit circle representing the real number x.





Again, we can see on this graph many of the properties discussed previously; for example, $\cos(-x) = \cos x$ and $\cos(\pi - x) = -\cos x$. The period and the amplitude are the same as for the sine function. In fact, the graphs of the sine and cosine functions are related to each other in a simple way: the graph of $y = \cos x$ is obtained from the graph of $y = \sin x$ by translating it $\frac{\pi}{2}$ units to the left. This corresponds to one of the properties listed in Key point 8.5: $\cos x = \sin\left(x + \frac{\pi}{2}\right)$.

We discussed the transformations of graphs in chapter 5.

KEY POINT 8.7

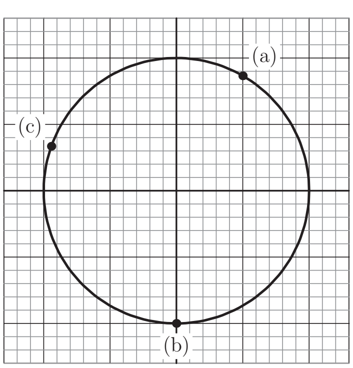
The sine and cosine functions are periodic with period 2π.

The sine and cosine functions have amplitude 1.

Trigonometric functions can be used to define *polar coordinates*. This alternative to the Cartesian coordinate system makes it easier to write equations of certain graphs. Equations in polar coordinates produce some beautiful curves, such as the cardioid and the polar rose.

Exercise 8B

1. Write down the approximate values of $\sin x$ and $\cos x$ for the number x corresponding to each of the points marked on the diagram.



2. Use the unit circle to find the following values:

(a) (i) $\sin\frac{\pi}{2}$ (ii) $\sin 2\pi$

(b) (i) $\cos 0$ (ii) $\cos(-\pi)$

(c) (i) $\sin\left(-\frac{\pi}{2}\right)$ (ii) $\cos\frac{5\pi}{2}$

3. Use the unit circle to find the following values:

(a) (i) $\cos 90°$ (ii) $\cos 180°$

(b) (i) $\sin 270°$ (ii) $\sin 90°$

(c) (i) $\sin 720°$ (ii) $\cos 450°$

4. Given that $\cos\frac{\pi}{5} = 0.809$, find the value of:

(a) $\cos\frac{4\pi}{5}$ (b) $\cos\frac{21\pi}{5}$

(c) $\cos\frac{9\pi}{5}$ (d) $\cos\frac{6\pi}{5}$

5. Given that $\sin\frac{2\pi}{3} = 0.866$, find the value of:

(a) $\sin\left(\frac{-2\pi}{3}\right)$ (b) $\sin\frac{4\pi}{3}$

(c) $\sin\frac{10\pi}{3}$ (d) $\sin\frac{\pi}{3}$

6. Given that $\cos 40° = 0.766$, find the value of:

(a) $\cos 400°$ (b) $\cos 320°$

(c) $\cos(-220°)$ (d) $\cos 140°$

7. Given that $\sin 130° = 0.766$, find the value of:

(a) $\sin 490°$ (b) $\sin 50°$

(c) $\sin(-130°)$ (d) $\sin 230°$

8. Sketch the graph of $y = \sin x$ for:

(a) (i) $0° \le x \le 180°$ (ii) $90° \le x \le 360°$

(b) (i) $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ (ii) $-\pi \le x \le 2\pi$

9. Sketch the graph of $y = \cos x$ for:

(a) (i) $-180° \le x \le 180°$ (ii) $0 \le x \le 270°$

(b) (i) $\frac{\pi}{2} \le x \le \frac{3\pi}{2}$ (ii) $-\pi \le x \le 2\pi$

10. (a) On the unit circle, mark the points representing $\frac{\pi}{6}, \frac{\pi}{3}$ and $\frac{2\pi}{3}$.

(b) Given that $\sin\frac{\pi}{6} = 0.5$, find the value of:

(i) $\cos\frac{\pi}{3}$ (ii) $\cos\frac{2\pi}{3}$

11. Use your calculator to evaluate the following, giving your answers to 3 significant figures:

(a) (i) $\cos 1.25$ (ii) $\sin 0.68$

(b) (i) $\cos(-0.72)$ (ii) $\sin(-2.35)$

12. Use your calculator to evaluate the following, giving your answers to 3 significant figures:

(a) (i) $\sin 42°$ (ii) $\cos 168°$

(b) (i) $\sin(-50°)$ (ii) $\cos(-227°)$

13. Simplify $\cos(\pi + x) + \cos(\pi - x)$. *[3 marks]*

14. Simplify the following expression:

$$\sin x + \sin\left(x + \frac{\pi}{2}\right) + \sin(x + \pi) + \sin\left(x + \frac{3\pi}{2}\right) + \sin(x + 2\pi)$$

[5 marks]

8C Definition and graph of the tangent function

We now introduce another trigonometric function: the **tangent** function. It is defined as the ratio between the sine and the cosine functions.

KEY POINT 8.8

$$\tan x = \frac{\sin x}{\cos x}$$

You may notice that there is a problem with this definition: we cannot divide by $\cos x$ when it is zero. Thus the tangent function is undefined at values of x where $\cos x = 0$; that is, $\tan x$ is undefined for $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$.

By considering the signs of $\sin x$ and $\cos x$ in different quadrants, we can see that $\tan x$ is positive in the first and third quadrants, and negative in the second and fourth quadrants. It is equal to zero when $\sin x = 0$, that is, at $x = 0, \pi, 2\pi, \dots$

Since $\sin(x+\pi) = -\sin x$ and $\cos(x+\pi) = -\cos x$ (see Key point 8.4), we have

$$\tan(x+\pi) = \frac{\sin(x+\pi)}{\cos(x+\pi)} = \frac{-\sin x}{-\cos x} = \frac{\sin x}{\cos x} = \tan x$$

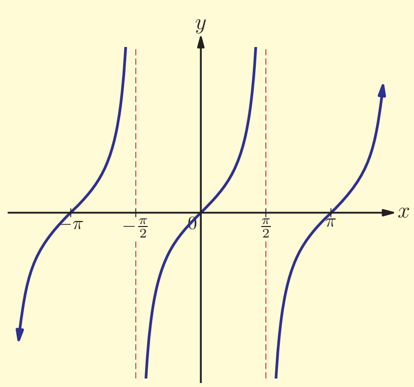
So the tangent function is periodic with period π:

$$\tan x = \tan(x+\pi) = \tan(x+2\pi) = \dots$$

Using the information we have collected above, we can sketch the graph of the tangent function. The graph will have vertical asymptotes at $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$

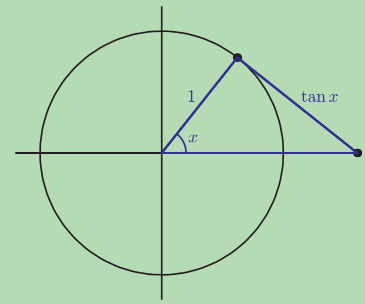
KEY POINT 8.9

The graph of the tangent function is:



Remember that points on the unit circle can also represent angles measured in degrees. Sometimes you may be asked to work in degrees.

You may wonder why the tangent function is given this name. On the unit circle, if we draw a tangent from the point representing x, then $\tan x$ will be the distance along this tangent to the horizontal axis. See if you can prove this based on your understanding of sine and cosine on the unit circle.



Vertical asymptotes were discussed in chapters 2 and 4.

EXAM HINT

You should always use radians unless explicitly told to use degrees.

Worked example 8.7

Sketch the graph of $y = \tan x$ for $-90° < x < 270°$.

The given domain $-90° < x < 270°$ covers the fourth quadrant, first quadrant, second quadrant, and then third quadrant. Find the values of x for which tan x is not defined.

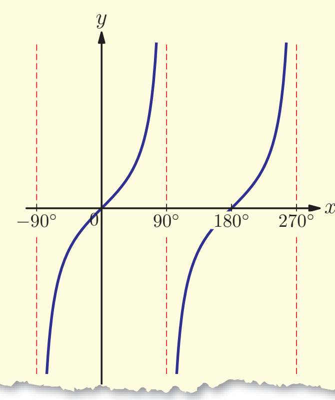
$\tan x$ is undefined when $\cos x = 0$, i.e. at

$x = -90°, 90°, 270°$

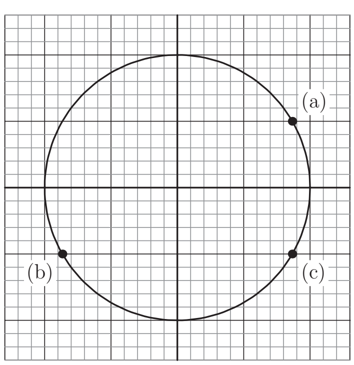
When is the function positive/negative? When is it zero?

$\tan x$ is:
negative in the fourth quadrant,
positive in the first quadrant,
negative in the second quadrant,
positive in the third quadrant.
$\tan x = 0$ at $x = 0°$ and $180°$

Start by marking the asymptotes and zeros.



Exercise 8C



1. By estimating the values of $\sin\theta$ and $\cos\theta$, find the approximate value of $\tan\theta$ for the points shown on the diagram.

2. Sketch the graph of $y = \tan x$ for:
   (a) (i) $0° \le x \le 360°$ (ii) $-90° \le x \le 270°$
   (b) (i) $\frac{\pi}{2} \le x \le \frac{5\pi}{2}$ (ii) $-\pi \le x \le \pi$

3. Use your calculator to evaluate the following, giving your answers to 2 decimal places:
   (a) (i) $\tan 1.2$ (ii) $\tan 4.7$
   (b) (i) $\tan(-0.65)$ (ii) $\tan(-7.3)$

4. Use your calculator to evaluate the following, giving your answers to 3 significant figures:

   (a) (i) $\tan 32°$ (ii) $\tan 168°$

   (b) (i) $\tan(-540°)$ (ii) $\tan(-128°)$

5. Use the properties of sine and cosine to express the following in terms of $\tan x$:

   (a) $\tan(\pi - x)$ (b) $\tan\left(x + \frac{\pi}{2}\right)$

   (c) $\tan(x + \pi)$ (d) $\tan(x + 3\pi)$

6. Use the properties of sine and cosine to express the following in terms of $\tan \theta°$:

   (a) $\tan(-\theta°)$ (b) $\tan(360° - \theta°)$

   (c) $\tan(90° - \theta°)$ (d) $\tan(180° + \theta°)$

7. Sketch the graph of:

   (i) $y = 2\sin x - \tan x$ for $-\frac{\pi}{2} \le x \le 2\pi$

   (ii) $y = 3\cos x + \tan x$ for $-\pi \le x \le \pi$

8. Find the zeros of the following functions:

   (i) $y = 2\tan x° + \sin x°$ for $0 \le x \le 360$

   (ii) $y = 3\cos x° - \tan x°$ for $-180 \le x \le 360$

9. Find the coordinates of the maximum and minimum points on the following graphs:

   (i) $y = 3\sin x - \tan x$ for $0 \le x \le 2\pi$

   (ii) $y = \cos x - 2\sin x$ for $-\pi \le x \le \pi$

10. Find approximate solutions of the following equations, giving your answers correct to 3 significant figures:

   (i) $\cos x - \tan x = 3,\ x \in]0, 2\pi[$

   (ii) $\sin x + \cos x = 1,\ x \in [0, 2\pi]$

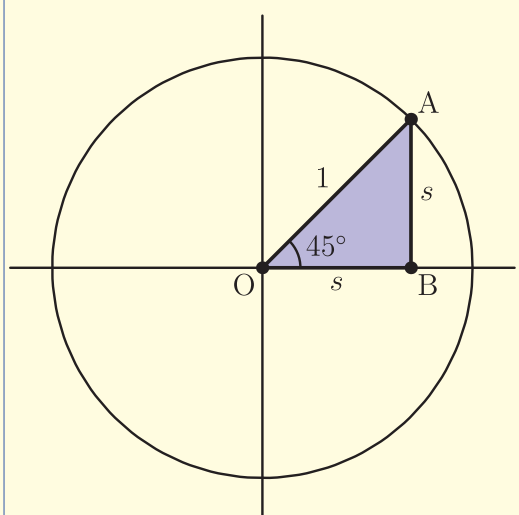
8D Exact values of trigonometric functions

Although generally values of trigonometric functions are difficult to find without a calculator, there are a few special numbers for which exact values can easily be found. The method relies on the properties of two special right-angled triangles.

Worked example 8.8

Find the exact values of $\sin\frac{\pi}{4}$, $\cos\frac{\pi}{4}$ and $\tan\frac{\pi}{4}$.

Mark the point corresponding to $\frac{\pi}{4}$ on the unit circle (point A in the diagram).



Look at the triangle OAB. It has a right angle at B, and the angle at O is equal to 45° (because $\frac{\pi}{4}$ is one-eighth of a full turn).

$s^2 + s^2 = 1$, so $s = \frac{1}{\sqrt{2}}$. Hence

$$\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

We can now use the definition of tan x.

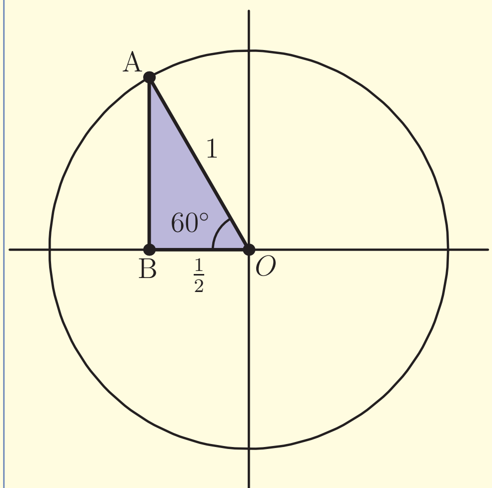
$$\tan\frac{\pi}{4} = \frac{\sin\frac{\pi}{4}}{\cos\frac{\pi}{4}} = 1$$

The other special right-angled triangle is made by cutting an equilateral triangle in half.

Worked example 8.9

Find the exact values of the three trigonometric functions of $\frac{2\pi}{3}$.

Mark the point corresponding to $\frac{2\pi}{3}$ on the unit circle. $\frac{2\pi}{3}$ is one-third of a full turn (120°), so angle $A\hat{O}B$ is 60°.



Triangle AOB is half of an equilateral triangle with side length 1. OB is equal to half the side of the equilateral triangle.

$$OB = \frac{1}{2}$$

Point A is to the left of the vertical axis, so $\cos\frac{2\pi}{3} < 0$

$$\therefore \cos\frac{2\pi}{3} = -\frac{1}{2}$$

To find AB, use Pythagoras' Theorem.

$$AB^2 = 1^2 - \left(\frac{1}{2}\right)^2 = \frac{3}{4}$$

$$\therefore \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$$

Use the definition of $\tan x$.

$$\tan\frac{2\pi}{3} = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3}$$

The sine, cosine and tangent values of other special numbers are summarised below. You should understand how they are derived, as shown in Worked examples 8.8 and 8.9.

KEY POINT 8.10

Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
Degrees	0	30	45	60	90	120	135	150	180
sin *x*	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
cos *x*	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
tan *x*	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0

EXAM HINT

The angles $\frac{\pi}{6}, \frac{\pi}{4}$ and $\frac{\pi}{3}$ come up so frequently in exam questions that it is useful to memorise the results for them, rather than having to repeatedly derive the values from triangles. The table might be easier to remember if you notice the pattern in the values of sin and cos:

$0, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, 1\left(=\frac{\sqrt{4}}{2}\right)$.

Exercise 8D

1. By marking the corresponding points on the unit circle, find the exact values of

(a) $\cos\frac{3\pi}{4}$ (b) $\cos\frac{\pi}{2}$

(c) $\sin\frac{5\pi}{4}$ (d) $\tan\frac{3\pi}{4}$

2. Find the exact values of

(a) $\sin\frac{\pi}{6}$ (b) $\sin\frac{7\pi}{6}$

(c) $\cos\left(\frac{4\pi}{3}\right)$ (d) $\tan\left(-\frac{\pi}{3}\right)$

3. Find the exact values of

(a) $\cos 45°$ (b) $\sin 135°$

(c) $\cos 225°$ (d) $\tan 225°$

4. Find the exact values of

(a) $\sin 210°$ (b) $\cos 210°$

(c) $\tan 210°$ (d) $\tan 330°$

5. Evaluate the following, simplifying as far as possible.

(a) $1-\sin^2\left(\frac{\pi}{6}\right)$

(b) $\sin\left(\frac{\pi}{4}\right)+\sin\left(\frac{\pi}{3}\right)$

(c) $\cos\frac{\pi}{3}-\cos\frac{\pi}{6}$

6. Show that

(a) $\sin 60°\cos 30°+\cos 60°\sin 30°=\sin 90°$

(b) $(\sin 45°)^2+(\cos 45°)^2=1$

(c) $\cos^2\left(\frac{\pi}{6}\right)-\sin^2\left(\frac{\pi}{6}\right)=\cos\left(\frac{\pi}{3}\right)$

(d) $\left(1+\tan\frac{\pi}{3}\right)^2=4+2\sqrt{3}$

8E Transformations of trigonometric graphs

Transformations of functions and graphs were introduced in chapter 5.

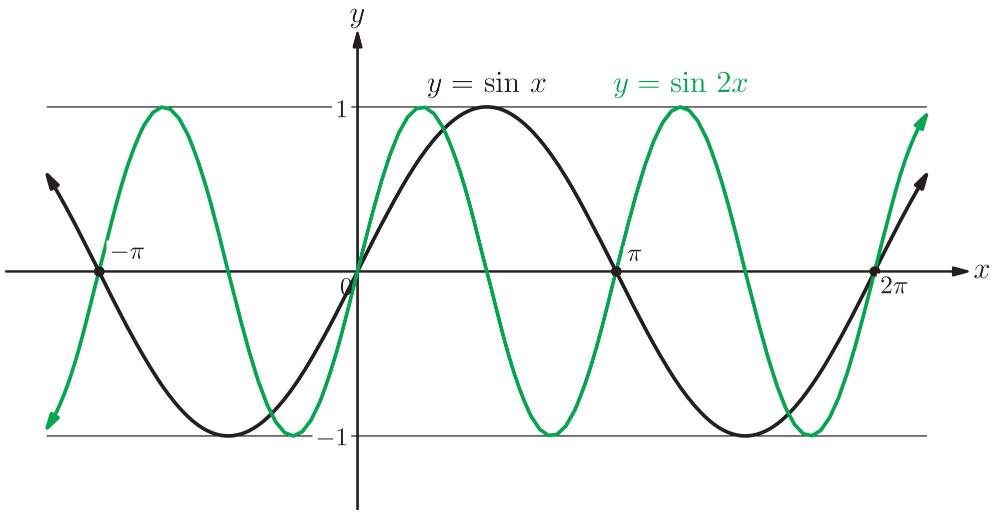
In this section we shall apply the ideas from chapter 5 to the trigonometric graphs we have met. This will enable us to model many real-life situations which show periodic behaviour (see section 8F), and will also be useful in solving equations involving trigonometric functions.

First, consider how we might obtain the graph of $y=\sin 2x$ by using its relationship with $y=\sin x$. The equation $y=\sin 2x$ is of the form $y=f(2x)$ where $f(x)=\sin x$, so we need to apply a horizontal stretch with scale factor $\frac{1}{2}$ to the graph of $y=\sin x$.

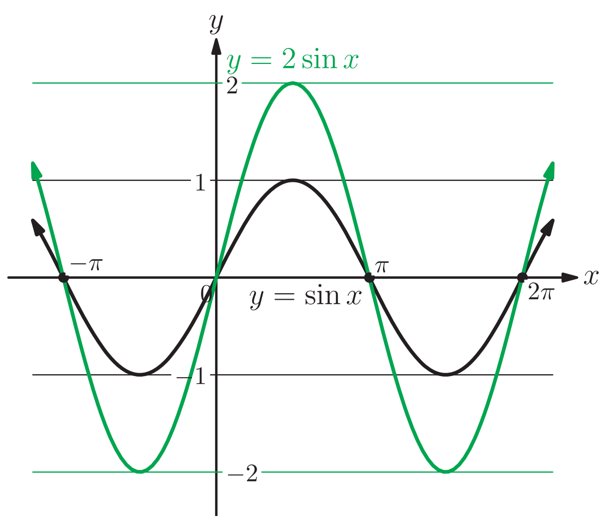
We can see that the amplitude of $\sin 2x$ is still 1, but its period is halved to π.

EXAM HINT

$\sin 2x$ is not the same as $2\sin x$. Importantly, $\frac{\sin 2x}{2}$ *cannot* be simplified to $\sin x$! Plot the functions on your GDC and compare the results.



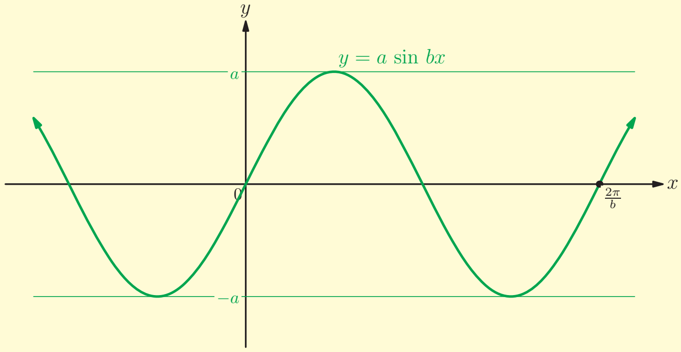
Now let us sketch the graph of $y = 2\sin x$. This is of the form $y = 2f(x)$ where $f(x) = \sin x$, so we need to apply a vertical stretch with scale factor 2 to the graph of $y = \sin x$. The resulting function has amplitude 2, while the period is unchanged.



We can combine these two types of transformation (horizontal and vertical stretches) to change both the amplitude and the period of the sine function. The same transformations can also be applied to the graph of the cosine function.

KEY POINT 8.11

The functions $y = a\sin bx$ and $y = a\cos bx$ have amplitude a and period $\frac{2\pi}{b}$.



Worked example 8.10

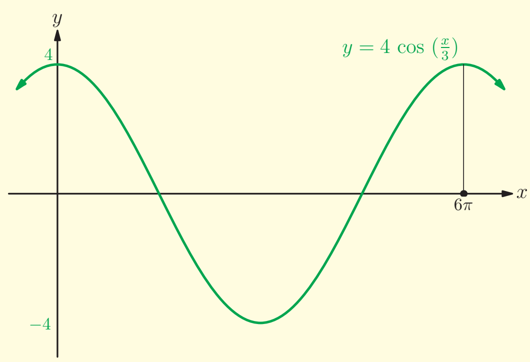
(a) Sketch the graph of $y = 4\cos\left(\frac{x}{3}\right)$ for $0 \le x \le 6\pi$.

(b) Write down the amplitude and the period of the function.

Start with the graph of $y = \cos x$ and think about what transformations to apply to it.

(a) Vertical stretch with scale factor 4

Horizontal stretch with scale factor 3



(b) Amplitude = 4

Period $= \frac{2\pi}{3^{-1}} = 6\pi$

EXAM HINT

'Write down' means that you do not need to show working.

Besides vertical and horizontal stretches, we can also apply translations to graphs of trigonometric functions. They will leave the period and the amplitude unchanged, but will change the positions of maximum and minimum points and the axis intercepts.

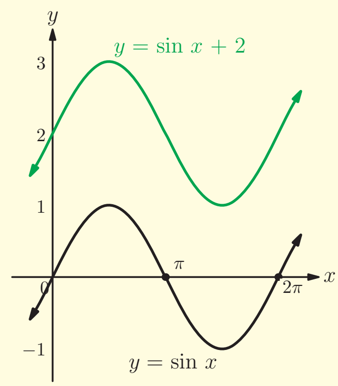
Worked example 8.11

(a) Sketch the graph of $y = \sin x + 2$ for $x \in [0, 2\pi]$.

(b) Find the maximum and the minimum values of the function.

The equation is of the form $y = f(x) + 2$ where $f(x) = \sin x$. What transformation does this correspond to?

(a) Apply vertical translation by 2 units upward to $y = \sin x$



We know that the minimum and maximum values of $\sin x$ are 1 and -1, so add 2 to those values.

(b) Minimum value: $-1 + 2 = 1$

Maximum value: $1 + 2 = 3$

In the next example we consider a horizontal translation.

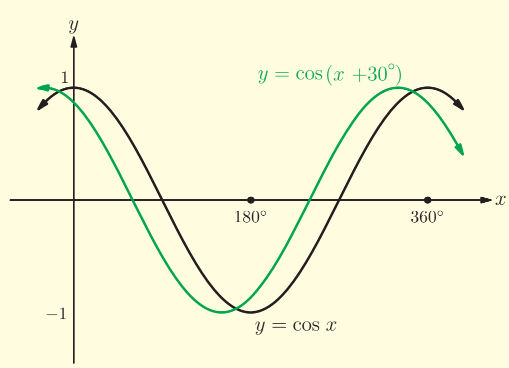
Worked example 8.12

(a) Sketch the graph of $y = \cos(x + 30°)$ for $0° \le x \le 360°$.

(b) State the minimum and maximum values of the function, and the values of x at which they occur.

The equation is of the form $y = f(x + 30)$ where $f(x) = \cos x$. What transformation does this correspond to?

(a) Apply horizontal translation by 30 units leftward to $y = \cos x$.



continued . . .

The minimum value of $\cos x$ is -1, and in the interval $[0°, 360°]$ it occurs at $x = 180°$. Since the graph is translated 30 units to the left, we subtract 30.

(b) Minimum value is -1

It occurs at $x = 180° - 30° = 150°$

The maximum value of $\cos x$ is 1, and in the interval $[0°, 360°]$ it occurs at $x = 0°, 360°$. Pick the value which is in the required interval *after* translation to the left.

Maximum value is 1

It occurs at $x = 360° - 30° = 330°$

The result of applying the two types of translations and two types of stretches to the sine and cosine functions can be summarised as follows.

KEY POINT 8.12

The functions $y = a\sin b(x+c)+d$ and $y = a\cos b(x+c)+d$ have

- amplitude a
- period $\frac{2\pi}{b}$
- minimum value $d-a$ and maximum value $d+a$

Note that the value of d is always half-way between the minimum and maximum values; in other words, $d = \frac{min+max}{2}$.

The amplitude is half the difference between the minimum and maximum values: $a = \frac{max-min}{2}$.

The value of c in the above equation determines the horizontal translation of the graph; therefore it affects the position of the maximum and minimum points. The following example shows how to work out these positions.

Worked example 8.13

Find the exact values of x for which the function $y = \sin 3(x+1)$ attains its maximum value.

When does the sine function attain its maximum value?

$\sin x$ has a maximum when $x = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}$ etc.

The given function is of the form $f(3(x+1))$; this means that x has been replaced by $3(x+1)$.

$3(x+1) = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots$

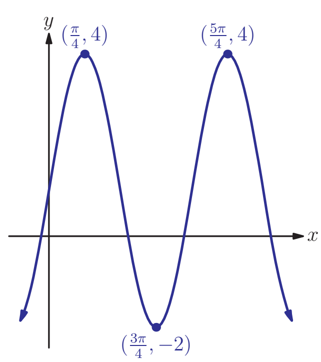
Now solve for x.

$x = \frac{\pi}{6} - 1, \frac{5\pi}{6} - 1$, etc.

We can use our knowledge of transformations of graphs to find an equation of a function given its graph.

Worked example 8.14

The graph shown has equation $y = a\sin(bx) + d$. Find the values of a, b and d.



a is the amplitude, which is half the difference between the minimum and maximum values.

$a = \frac{4-(-2)}{2} = 3$

b is related to the period via the formula $period = \frac{2\pi}{b}$.

The period is also the distance between two consecutive maximum points, which we can find from the graph.

$period = \frac{5\pi}{4} - \frac{\pi}{4} = \pi$

$\therefore \pi = \frac{2\pi}{b}$

Hence $b = 2$

d represents the vertical translation of the graph. It is the value half-way between the minimum and the maximum values.

$d = \frac{4+(-2)}{2} = 1$

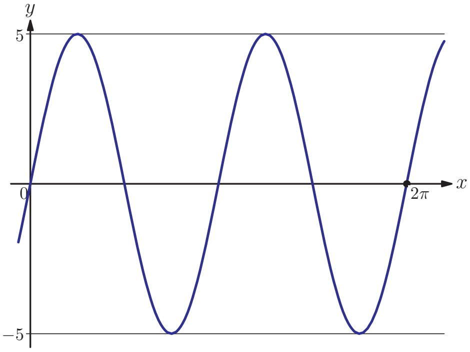
Exercise 8E

1. Sketch the following graphs, indicating any axis intercepts.
   (a) (i) $y=\sin 2x$ for $-180°\le x\le 180°$
   (ii) $y=\cos 3x$ for $0°\le x\le 360°$
   (b) (i) $y=\tan\left(x-\frac{\pi}{2}\right)$ for $0\le x\le \pi$
   (ii) $y=\tan\left(x+\frac{\pi}{3}\right)$ for $0\le x\le \pi$
   (c) (i) $y=3\cos x-2$ for $0°\le x\le 720°$
   (ii) $y=2\sin x+1$ for $-360°\le x\le 360°$

2. Sketch the following graphs, giving the coordinates of maximum and minimum points.
   (a) (i) $y=\cos\left(x-\frac{\pi}{3}\right)$ for $0\le x\le 2\pi$
   (ii) $y=\sin\left(x+\frac{\pi}{2}\right)$ for $0\le x\le 2\pi$
   (b) (i) $y=2\sin(x+45°)$ for $-180°\le x\le 180°$
   (ii) $y=3\cos(x-60°)$ for $-180°\le x\le 180°$
   (c) (i) $y=-3\sin 2x$ for $-\pi\le x\le \pi$
   (ii) $y=3-2\cos x$ for $0°\le x\le 360°$

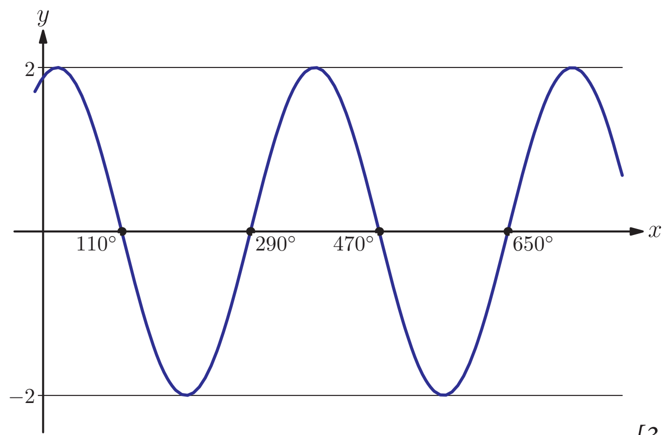
3. Find the amplitude and period of the following functions.
   (a) $f(x)=3\sin 4x$, where x is in degrees
   (b) $f(x)=\tan 3x$, where x is in radians
   (c) $f(x)=\cos 3x$, where x is in degrees
   (d) $f(x)=2\sin \pi x$, where x is in radians

4. The graph shown has equation $y=p\sin(qx)$ for $0\le x\le 2\pi$. Find the values of p and q.



[3 marks]

5. The graph shown has equation $y = a\cos(x - b)$ for $0° \le x \le 720°$. Find the values of a and b.



[3 marks]

6. (a) On the same set of axes sketch the graphs of $y = 1 + \sin 2x$ and $y = 2\cos x$ for $0 \le x \le 2\pi$.

(b) Hence state the number of solutions of the equation $1 + \sin 2x = 2\cos x$ for $0 \le x \le 2\pi$.

(c) Write down the number of solutions of the equation $1 + \sin 2x = 2\cos x$ for $-2\pi \le x \le 6\pi$. *[6 marks]*

7. (a) Sketch the graph of $y = 2\cos(x + 60°)$ for $x \in [0°, 360°]$.

(b) Find the coordinates of the maximum and minimum points on the graph.

(c) Write down the coordinates of the maximum and minimum points on the graph of $y = 2\cos(x + 60°) - 1$ for $x \in [0°, 360°]$. *[6 marks]*

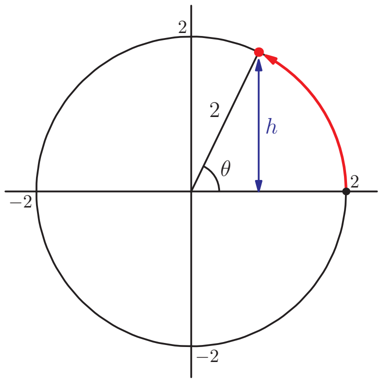
8F Modelling using trigonometric functions

In this section we shall see how trigonometric functions can be used to model real-life situations that show periodic behaviour.

Imagine a point moving with constant speed around a circle of radius 2 cm centred at the origin, starting from the positive x-axis and taking 3 seconds to complete one full rotation.

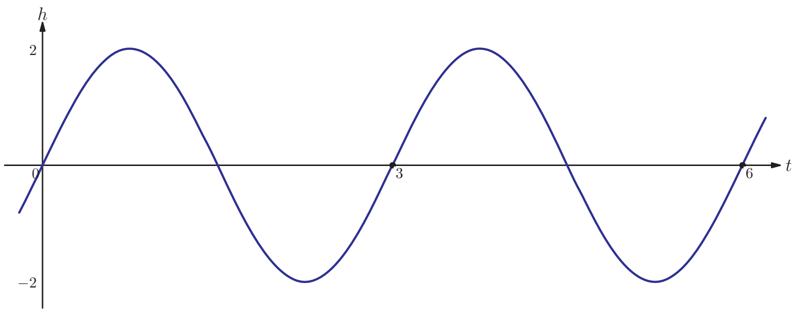
Let h be the height of the point above the x-axis. How does h vary with time t (measured in seconds)?

We know that if the point is moving around the unit circle, the height above the x-axis would be $\sin\theta$, where θ is the angle between the radius and the x-axis. As the circle now has radius 2, the height is $2\sin\theta$.



So, to find how h varies with time, we need to find how θ depends on time. As the point starts on the positive x-axis, $\theta = 0$ when $t = 0$. After one complete rotation, we have $\theta = 2\pi$ and $t = 3$. Because the point is moving with constant speed, we can use ratios to state that $\frac{\theta}{2\pi} = \frac{t}{3}$, so $\theta = \frac{2\pi}{3}t$.

Therefore the equation for the height in terms of time is $h = 2\sin\left(\frac{2\pi}{3}t\right)$. The diagram below shows the graph of this function.



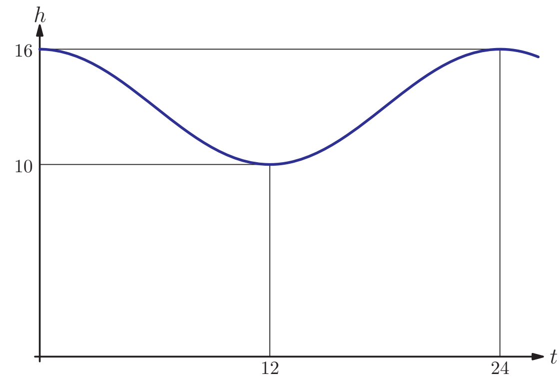
This is an example of **modelling** using trigonometric functions. We can use sine and cosine functions to model periodic motion, such as motion around a circle, oscillation of a particle attached to the end of a spring, water waves, or heights of tides. In practice, we would collect experimental data to sketch a graph and then use our knowledge of trigonometric functions to find its equation. We can then use the equation to do further calculations.

Worked example 8.15

The height of water in a harbour is 16 m at high tide and 10 m at low tide, which occurs 12 hours later. The graph below shows how the height of water changes with time over 24 hours.

(a) Find an equation for the height of water (in metres) in terms of time (in hours) in the form $h = m + a\cos(bt)$.

(b) Find the first two times after the high tide when the height of water is 12 m.



We know from the previous section that m is the value half-way between the minimum and maximum values, a is the amplitude, and b is related to the period.

(a) $m = \dfrac{16+10}{2} = 13$

The amplitude is half the distance between the minimum and maximum values.

$a = \dfrac{16-10}{2} = 3$

The period is $\dfrac{2\pi}{b}$.

period $= 24$

$24 = \dfrac{2\pi}{b} \quad \therefore b = \dfrac{\pi}{12}$

We can now write down the equation for height.

So $h = 13 + 3\cos\left(\dfrac{\pi}{12}t\right)$

Set $h = 12$ and solve for t.

(b) $13 + 3\cos\left(\dfrac{\pi}{12}t\right) = 12$

We can use a calculator to solve the equation.

From GDC, $t = 7.3$ or 16.7

In chapter 9 you will see how to solve the equation algebraically.

Answer the question.

The height of the water will be 12 m at 7.3 hours and 16.7 hours after the high tide.

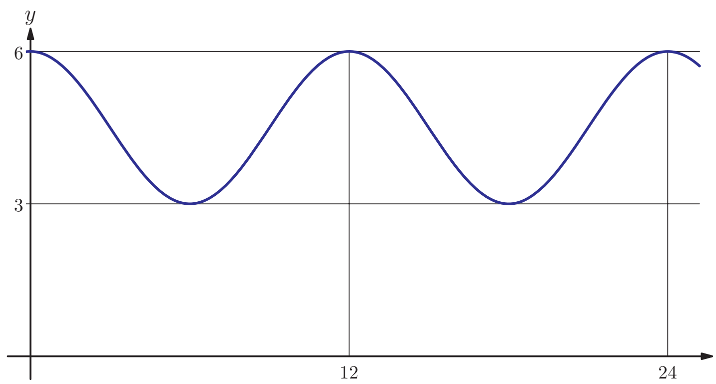
Exercise 8F

1. The depth of water in a harbour varies during the day and is given by the equation $d = 16 + 7\sin\left(\frac{\pi}{12}t\right)$, where d is measured in metres and t is the number of hours after midnight.

(a) Find the depth of the water at low and at high tide.

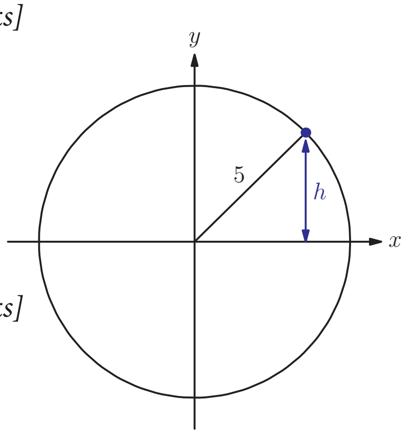
(b) At what times does high tide occur? *[6 marks]*

2. The graph shows the depth of water below a walkway as a function of time. The equation of the graph is of the form $y = a\cos(bt) + m$. Find the values of a, b and m.



[4 marks]

3. A point moves around a circle of radius 5 cm, as shown in the diagram. It takes 10 seconds to complete one revolution.



(a) The height of the point above the x-axis is given by $h = a\sin(kt)$, where t is time measured in seconds. Find the values of a and k.

(b) Find the times during the first revolution when the point is 3 cm below the x-axis. *[6 marks]*

4. A ball is attached to one end of an elastic string; the other end of the string is held fixed above the ground. When the ball is pulled down and released, it starts moving up and down, so that the height of the ball above the ground is given by the equation $h = 120 - 10\cos 400t$, where h is measured in centimetres and t is time in seconds.

(a) Find the least and greatest heights of the ball above the ground.

(b) Find the time required to complete one full oscillation.

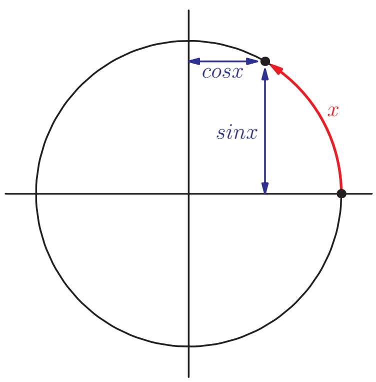
(c) Find the first time after the ball is released at which it reaches its greatest height. *[8 marks]*

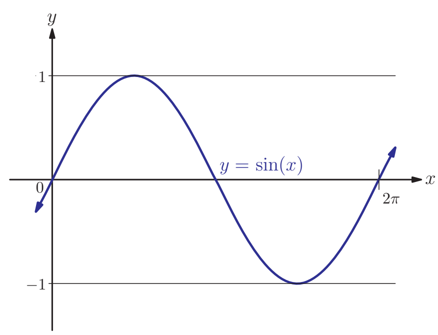
Summary

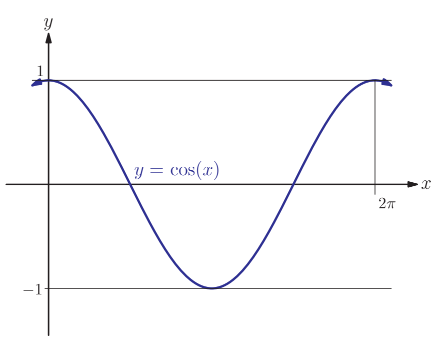
- The unit circle is a circle with centre 0 and radius 1.
- Radian measure for angles is defined in terms of distance travelled around the unit circle, so that a full turn = 2π radians.
- To convert from degrees to radians, divide by 180 and multiply by π.
  To convert from radians to degrees, divide by π and multiply by 180.
- Using the unit circle and the real number α, the **sine** and **cosine** of this number is defined in terms of distance to the axes:

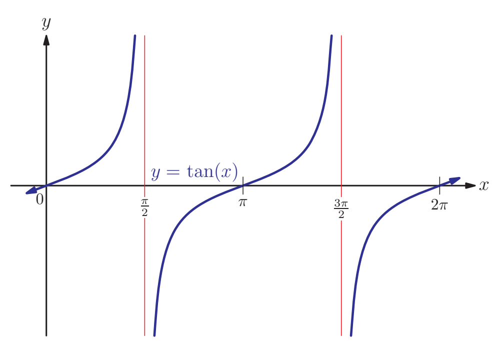
  $\sin\alpha$ is the distance of a point from the horizontal axis

  $\cos\alpha$ is the distance of the point from the vertical axis
- Useful properties of the sine and cosine function are summarised in Key point 8.4.
- The relationship between the sine and cosine function is summarised in Key point 8.5.
- The **tangent** function is another trigonometric function. It is defined as the ratio between the sine and cosine functions: $\tan x = \dfrac{\sin x}{\cos x}$
- The sine and cosine functions can be defined for all real numbers:









- For some real numbers trigonometric functions have exact values, which are useful to remember (see Key point 8.10).

- The sine and cosine functions are periodic with period 2π and amplitude 1.
- The tangent function is periodic with period π.

$$\tan(x) = \tan(x \pm \pi) = \tan(x \pm 2\pi) = \dots$$

- We can apply translations and stretches to the sine and cosine functions. The resulting functions $y = a\sin b(x+c)+d$ and $y = a\cos b(x+c)+d$ are very useful in modelling periodic phenomena; they have:
  - amplitude a
  - period $\frac{2\pi}{b}$
  - minimum value $d-a$ and maximum value $d+a$ (the value d is half-way between the minimum and maximum values).

Introductory problem revisited

A clock has an hour hand of length 10 cm, and the centre of the clock is 4 m above the floor. Find an expression for the height, h metres, of the tip of the hour hand above the floor at a time t hours after midnight.

We can model the height using a cosine function (because when $t = 0$, the graph should be at the maximum height). The period is 12 hours, the amplitude is 0.1 m (the length of the hand), and the half-way height is 4 m (the position of the centre of the clock). Therefore the function is

$$h = 4 + 0.1\cos\left(\frac{\pi}{6}t\right)$$

Mixed examination practice 8

Short questions

1. The height of a wave, in metres, at a distance x metres from a buoy is modelled by the function $f(x) = 1.4\sin(3x - 0.1) - 0.6$.

(a) State the amplitude of the wave.

(b) Find the distance between consecutive peaks of the wave. *[4 marks]*

2. A runner is jogging around a level circular track. His distance north of the centre of the track in metres is given by $60\cos 0.08t$, where t is measured in seconds.

(a) How long does is take the runner to complete one lap?

(b) What is the length of the track?

(c) At what speed is the runner jogging? *[7 marks]*

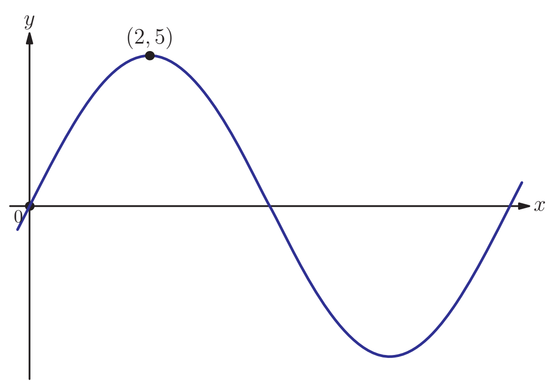
3. Let $f(x) = 3\sin 2\left(x - \frac{\pi}{3}\right)$.

(a) State the period of the function.

(b) Find the coordinates of the zeros of $f(x)$ for $x \in [0, 2\pi]$.

(c) Hence sketch the graph of $y = f(x)$ for $x \in [0, 2\pi]$, showing the coordinates of the maximum and minimum points. *[7 marks]*

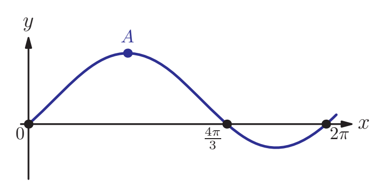
4. The diagram shows the graph of the function $f(x) = a\sin(bx)$. Find the values of a and b.



[4 marks]

Long questions

1. The graph shows the function $f(x) = \sin(x - k) + c$.



(a) (i) Write down the coordinates of A.

(ii) Hence find the values of k and c.

(b) Find all the zeros of the function in the interval $[-4\pi, 0]$.

(c) Consider the equation $f(x) = k$ with $-0.5 < k < 0$.

(i) Write down the number of solutions of this equation in the interval $[0, 9\pi]$.

(ii) Given that the smallest positive solution is α, write the next two solutions in terms of α. *[11 marks]*

2. (a) (i) Sketch the graph of $y = \tan x$ for $0 \le x \le 2\pi$.

(ii) On the same graph, sketch the line $y = \pi - x$.

(b) Consider the equation $x + \tan x = \pi$. Denote by x_0 the solution of this equation in the interval $]0, \frac{\pi}{2}[$.

(i) Find, in terms of x_0 and π, the remaining solutions of the equation in the interval $[0, 2\pi]$.

(ii) How many solutions does the equation $x + \tan x = \pi$ have for $x \in \mathbb{R}$?

(c) Given that $\cos A = c$ and $\sin A = s$:

(i) Write down the values of $\cos\left(\frac{\pi}{2} - A\right)$ and $\sin\left(\frac{\pi}{2} - A\right)$.

(ii) Hence show that $\tan\left(\frac{\pi}{2} - A\right) = \frac{1}{\tan A}$.

(iii) Given that $\tan A + \tan\left(\frac{\pi}{2} - A\right) = \frac{4}{\sqrt{3}}$, find the possible values of $\tan A$.

(iv) Hence find the values of $x \in]0, \frac{\pi}{2}[$ for which

$$\tan A + \tan\left(\frac{\pi}{2} - A\right) = \frac{4}{\sqrt{3}}.$$

[16 marks]

3. (a) Write down the minimum value of $\cos x$ and the smallest positive value of x (in radians) for which the minimum occurs.

(b) (i) Describe two transformations which transform the graph of $y = \cos x$ to the graph of $y = 2\cos\left(x + \frac{\pi}{6}\right)$.

(ii) Hence state the minimum value of $2\cos\left(x + \frac{\pi}{6}\right)$ and find the value of $x \in [0, 2\pi]$ for which the minimum occurs.

(c) The function f is defined for $x \in [0, 2\pi]$ by $f(x) = \dfrac{5}{2\cos\left(x + \frac{\pi}{6}\right) + 3}$.

(i) State, with a reason, whether f has any vertical asymptotes.

(ii) Find the range of f. *[13 marks]*