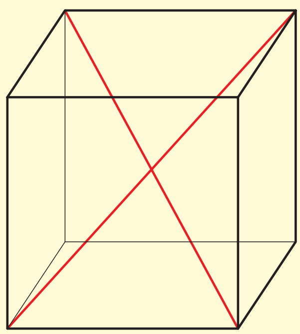
11 Vectors

In this chapter you will learn:

- to use vectors to represent displacements and positions in two and three dimensions
- to perform algebraic operations with vectors, and understand their geometric interpretation
- how to calculate the distance between two points
- how to use vectors to calculate the angle between two lines
- a new operation on vectors, called the scalar product
- how to describe a straight line using vectors
- to find intersections and angles between lines using vector methods.

Introductory problem

What is the angle between the diagonals of a cube?

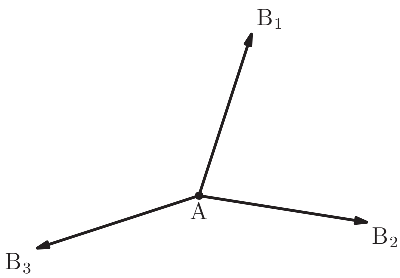


Solving problems in three dimensions can be difficult, as it is not always straightforward to visualise the geometry. Vectors provide a useful tool for translating geometrical properties into equations, which can often be analysed more easily. In this chapter we will develop techniques to calculate angles and distances in two and three dimensions; we will also look at how vectors can be used to describe lines in three dimensions and to find their intersections.

Vectors are an example of abstraction in mathematics – a single concept that can be applied to many different situations. Forces, velocities and displacements appear to have little in common, yet they can all be described and manipulated using the rules of vectors. In the words of the French mathematician and physicist Henri Poincaré (1854–1912), 'Mathematics is the art of giving the same name to different things.'

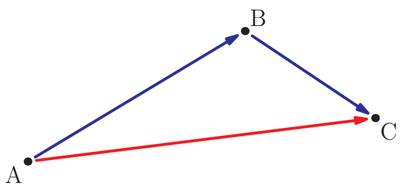
11A Positions and displacements

You may know from studying physics that **vectors** are used to represent quantities which have both magnitude (size) and direction, such as force or velocity. **Scalar** quantities, in contrast, are fully described by a single number. In pure mathematics, vectors are also used to represent displacements from one point to another, and thus to describe geometrical figures.

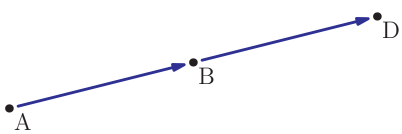


Consider a fixed point A and another point B that is 10 cm away from it. This information alone does not tell you where B is; for example, it could be any of the three positions shown in the diagram.

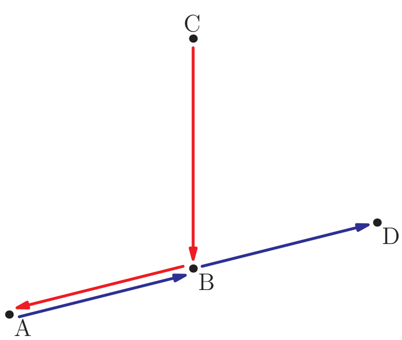
The position of B relative to A can be represented by the **displacement vector** $\overrightarrow{AB}$. The vector contains both distance and direction information. We can think of $\overrightarrow{AB}$ as describing a way of getting from A to B.



If we now add a third point, C, then there are two ways of getting from A to C: either directly, or via B. To express the second possibility using vectors, we write $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$; the addition sign means that moving from A to B is followed by moving from B to C.



Remember that while a vector represents a way of getting from one point to another, it does not tell us anything about the position of the starting point or end point; nor does it provide any information about what route was taken. If getting from B to D involves moving the same distance and in the same direction as getting from A to B, then the displacement vectors are the same: $\overrightarrow{BD} = \overrightarrow{AB}$.



To return from the end point to the starting point, we have to reverse direction; this is represented by a minus sign, so $\overrightarrow{BA} = -\overrightarrow{AB}$. We can also use a subtraction sign between two vectors; for example, $\overrightarrow{CB} - \overrightarrow{AB} = \overrightarrow{CB} + \overrightarrow{BA}$.

To get from A to D we need to move in the same direction, but twice as far, as in getting from A to B. We express this as $\overrightarrow{AD} = 2\,\overrightarrow{AB}$ or, equivalently, $\overrightarrow{AB} = \frac{1}{2}\overrightarrow{AD}$.

To refer to vectors conveniently, we often give them letters as names, just as we do with variables in algebra. To emphasise that something is a vector rather than a scalar (number), we use either bold type or an arrow on top. When writing by hand, we use underlining instead of bold type. For example, we can denote vector $\overrightarrow{AB}$ by $\boldsymbol{a}$ (you may also see $\vec{a}$ used in some texts). Then in the diagrams above, $\overrightarrow{BD} = \boldsymbol{a}$, $\overrightarrow{BA} = -\boldsymbol{a}$ and $\overrightarrow{AD} = 2\boldsymbol{a}$.

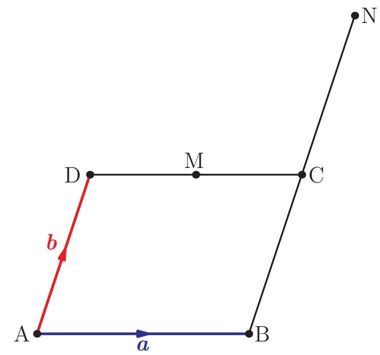
EXAM HINT

Fractions of a vector are usually written as multiples, e.g. $\frac{1}{2}\overrightarrow{AD}$ rather than $\frac{\overrightarrow{AD}}{2}$.

Worked example 11.1

The diagram shows a parallelogram ABCD. Let $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{AD} = \boldsymbol{b}$. M is the midpoint of [CD], and N is the point on (BC) such that CN = BC.

Express the vectors $\overrightarrow{CM}$, $\overrightarrow{BN}$ and $\overrightarrow{MN}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.



Think of $\overrightarrow{CM}$ as describing a way of getting from C to M by moving only along the directions of $\boldsymbol{a}$ and $\boldsymbol{b}$. Going from C to M is the same as going half way from B to A, and we know $\overrightarrow{BA} = -\overrightarrow{AB}$.

$$\overrightarrow{CM} = \frac{1}{2}\overrightarrow{BA} = -\frac{1}{2}\underline{a}$$

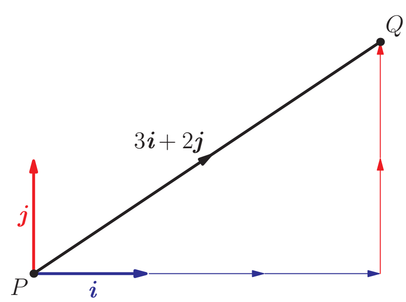
Going from B to N involves moving twice the distance in the same direction as from B to C, and $\overrightarrow{BC} = \overrightarrow{AD}$.

$$\overrightarrow{BN} = 2\overrightarrow{BC} = 2\underline{b}$$

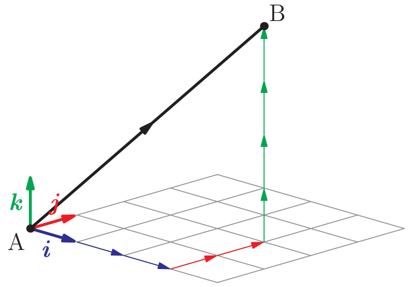
To get from M to N, we can go from M to C and then from C to N. $\overrightarrow{MC} = -\overrightarrow{CM}$ and $\overrightarrow{CN} = \overrightarrow{BC}$.

$$\begin{aligned}\overrightarrow{MN} &= \overrightarrow{MC} + \overrightarrow{CN} \\ &= -\overrightarrow{CM} + \overrightarrow{BC} \\ &= \frac{1}{2}\underline{a} + \underline{b}\end{aligned}$$

To make it easier to do further calculations with vectors, we need a way of describing them using numbers, not just diagrams. You are already familiar with coordinates, which are used to represent positions of points. A similar idea can be used to represent vectors.



Let us start by looking at displacements in the plane. Select two directions perpendicular to each other, and let $\boldsymbol{i}$ and $\boldsymbol{j}$ denote vectors of length 1 in those two directions. Then any vector in the plane can be expressed in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$, as shown in the diagram. The vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ are called **base vectors**.



$\vec{AB} = 3\boldsymbol{i} + 2\boldsymbol{j} + 4\boldsymbol{k}$

To represent displacements in three-dimensional space, we need three base vectors, all perpendicular to each other. They are conventionally called $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$, where $\boldsymbol{i}$ represents one unit in the x direction, $\boldsymbol{j}$ represents one unit in the y direction and $\boldsymbol{k}$ represents one unit in the z direction. In the diagram alongside, $\overrightarrow{AB} = 3\boldsymbol{i} + 2\boldsymbol{j} + 4\boldsymbol{k}$.

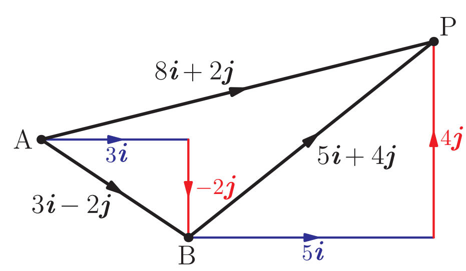
Alternatively, displacements can be written as **column vectors**. In this notation, the displacements in the diagrams above would be expressed as $\overrightarrow{PQ} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$.

The numbers in each column are called the **components** of the vector.

EXAM HINT

You need to be familiar with both base vector and column vector notation, as both will be used in exam questions. When writing answers, you can use whichever notation you prefer.

Using components makes it easy to add displacements. In the diagram below, to get from A to B we need to move 3 units in the $\boldsymbol{i}$ direction, and to get from B to P we need to move 5 units in the $\boldsymbol{i}$ direction; thus, getting from A to P requires moving a total of 8 units in the $\boldsymbol{i}$ direction. Similarly, in the $\boldsymbol{j}$ direction we move -2 units from A to B and 4 units from B to P, making the total movement in the $\boldsymbol{j}$ direction from A to P equal to $+2$ units. As the total displacement from A to P is $\overrightarrow{AP} = \overrightarrow{AB} + \overrightarrow{BP}$, we can write $(3\boldsymbol{i} - 2\boldsymbol{j}) + (5\boldsymbol{i} + 4\boldsymbol{j}) = 8\boldsymbol{i} + 2\boldsymbol{j}$ or, in column vector notation, $\begin{pmatrix} 3 \\ -2 \end{pmatrix} + \begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$.



Reversing the direction of a vector is also simple in component notation: to get from B to A we need to move -3 units in the $\boldsymbol{i}$ direction and 2 units in the $\boldsymbol{j}$ direction; thus $\overrightarrow{BA} = -\overrightarrow{AB} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$.

Analogous rules for adding and subtracting vectors – that is, doing so component by component – apply in three dimensions as well.

Worked example 11.2

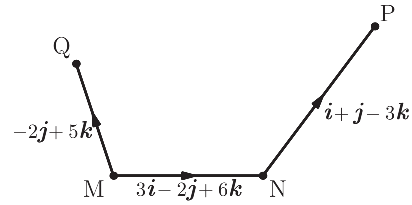
The diagram shows points M, N, P and Q such that $\overrightarrow{MN} = 3\boldsymbol{i} - 2\boldsymbol{j} + 6\boldsymbol{k}$, $\overrightarrow{NP} = \boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$ and $\overrightarrow{MQ} = -2\boldsymbol{j} + 5\boldsymbol{k}$.

Write the following vectors in component form:

(a) $\overrightarrow{MP}$

(b) $\overrightarrow{PM}$

(c) $\overrightarrow{PQ}$



We can get from M to P via N.

(a) $\overrightarrow{MP} = \overrightarrow{MN} + \overrightarrow{NP}$

$= (3\underline{i} - 2\underline{j} + 6\underline{k}) + (\underline{i} + \underline{j} - 3\underline{k})$

$= 4\underline{i} - \underline{j} + 3\underline{k}$

We have already found $\overrightarrow{MP}$.

(b) $\overrightarrow{PM} = -\overrightarrow{MP} = -4\underline{i} + \underline{j} - 3\underline{k}$

We can get from P to Q via M, using the answers from the previous parts.

(c) $\overrightarrow{PQ} = \overrightarrow{PM} + \overrightarrow{MQ}$

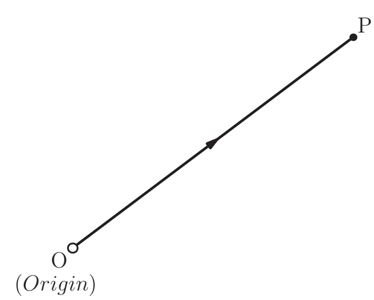
$= (-4\underline{i} + \underline{j} - 3\underline{k}) + (-2\underline{j} + 5\underline{k})$

$= -4\underline{i} - \underline{j} + 2\underline{k}$

EXAM HINT

As you can see from this example, vector diagrams do not have to be accurate or to scale to be useful: a two-dimensional sketch of a 3D situation is often enough to show you what is going on.

We have been speaking of vectors as representing displacements, but they can also be used to represent positions of points. To do this, we fix one particular point, called the *origin*; then the position of any point can be thought of as its displacement from the origin. For example, the position of point P in the diagram can be described by its **position vector** $\overrightarrow{OP}$.



If we know the position vectors of two points A and B, we can find the displacement $\overrightarrow{AB}$ as shown in the diagram in Key point 11.1 on the next page.

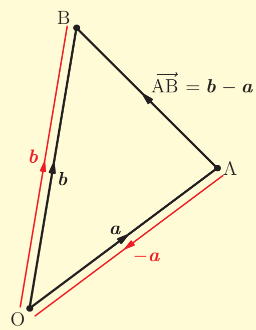
Taking the route from A to O and then on to B, we get $\overrightarrow{AB}=\overrightarrow{AO}+\overrightarrow{OB}$. But since $\overrightarrow{AO}=-\overrightarrow{OA}$, we find that $\overrightarrow{AB}=-\overrightarrow{OA}+\overrightarrow{OB}=\overrightarrow{OB}-\overrightarrow{OA}$.

EXAM HINT

The position vector of point A is usually denoted by $\boldsymbol{a}$.

KEY POINT 11.1

If points A and B have position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, then $\overrightarrow{AB}=\boldsymbol{b}-\boldsymbol{a}$.



Position vectors are closely related to coordinates. As the base vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are chosen to have directions along the coordinate axes, the components of the position vector will simply be the coordinates of the point.

Worked example 11.3

Points A and B have coordinates (3,−1,2) and (5,0,3), respectively. Write as column vectors

(a) the position vectors of A and B

(b) the displacement vector $\overrightarrow{AB}$.

The components of the position vectors are the coordinates of the point.

(a) $\underline{a}=\begin{pmatrix}3\\-1\\2\end{pmatrix}$ $\quad \underline{b}=\begin{pmatrix}5\\0\\3\end{pmatrix}$

(b) $\overrightarrow{AB}=\underline{b}-\underline{a}$

$=\begin{pmatrix}5\\0\\3\end{pmatrix}-\begin{pmatrix}3\\-1\\2\end{pmatrix}$

$=\begin{pmatrix}2\\1\\1\end{pmatrix}$

Worked example 11.4

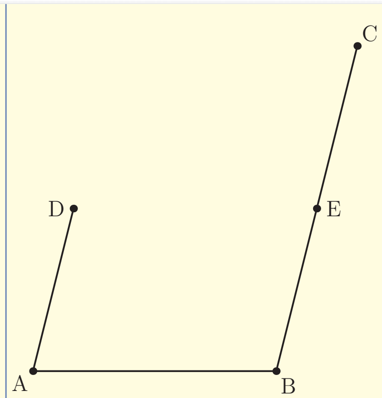
Points A, B, C and D have position vectors $\boldsymbol{a} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}, \boldsymbol{b} = \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix}, \boldsymbol{c} = \begin{pmatrix} 7 \\ 8 \\ -3 \end{pmatrix}, \boldsymbol{d} = \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix}$.

Point E is the midpoint of [BC].

(a) Find the position vector of E.

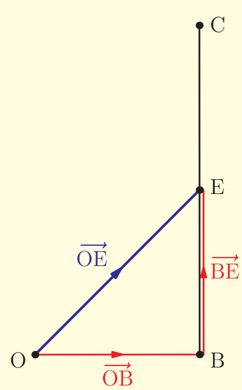
(b) Show that ABED is a parallelogram.

Make a sketch to try to see what is going on.



For part (a), we only need to look at points B, C and E. As we are given the position vectors, it will help to show the origin on the diagram.

(a)



$$\overrightarrow{OE} = \overrightarrow{OB} + \overrightarrow{BE}$$

$$= \overrightarrow{OB} + \frac{1}{2}\overrightarrow{BC}$$

Use relationship $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$

$$= \underline{b} + \frac{1}{2}(\underline{c} - \underline{b})$$

$$= \frac{1}{2}\underline{b} + \frac{1}{2}\underline{c}$$

$$= \begin{pmatrix} 2.5 \\ 0 \\ 1.5 \end{pmatrix} + \begin{pmatrix} 3.5 \\ 4 \\ -1.5 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 0 \end{pmatrix}$$

continued . . .

In a parallelogram, opposite sides are parallel and of the same length, which means that the vectors corresponding to those sides are equal. So we need to show that $\overrightarrow{AD} = \overrightarrow{BE}$.

(b) $\overrightarrow{AD} = \underline{d} - \underline{a}$

$$= \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$$

$\overrightarrow{BE} = \underline{e} - \underline{b}$

$$= \begin{pmatrix} 6 \\ 4 \\ 0 \end{pmatrix} - \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$$

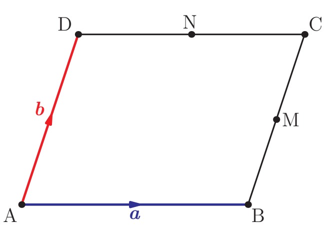
$\overrightarrow{AD} = \overrightarrow{BE}$, so ABED is a parallelogram.

In part (a) of the above example we derived a general formula for the position vector of the midpoint of a line segment.

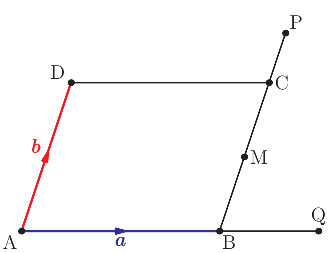
KEY POINT 11.2

The position vector of the midpoint of [AB] is $\frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$.

Exercise 11A



1. The diagram shows a parallelogram ABCD with $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{AD} = \boldsymbol{b}$. M is the midpoint of [BC] and N is the midpoint of [CD]. Express the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.
   (a) (i) $\overrightarrow{BC}$ (ii) $\overrightarrow{AC}$
   (b) (i) $\overrightarrow{CD}$ (ii) $\overrightarrow{ND}$
   (c) (i) $\overrightarrow{AM}$ (ii) $\overrightarrow{MN}$



2. In the parallelogram ABCD, $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{AD} = \boldsymbol{b}$. M is the midpoint of [BC], Q is the point on (AB) such that $BQ = \frac{1}{2}AB$, and P is the point on (BC) such that $BC : CP = 3 : 1$, as shown in the diagram.

Express the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

(a) (i) $\overrightarrow{AP}$ (ii) $\overrightarrow{AM}$

(b) (i) $\overrightarrow{QD}$ (ii) $\overrightarrow{MQ}$

(c) (i) $\overrightarrow{DQ}$ (ii) $\overrightarrow{PQ}$

3. Write the following vectors in three-dimensional column vector notation.

(a) (i) $4\boldsymbol{i}$ (ii) $-5\boldsymbol{j}$

(b) (i) $3\boldsymbol{i}+\boldsymbol{k}$ (ii) $2\boldsymbol{j}-\boldsymbol{k}$

4. Three points O, A and B are given. Let $\overrightarrow{OA}=\boldsymbol{a}$ and $\overrightarrow{OB}=\boldsymbol{b}$.

(a) Express $\overrightarrow{AB}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

(b) C is the midpoint of [AB]. Express $\overrightarrow{OC}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

(c) Point D lies on the line (AB), on the same side of B as A, so that AD = [3AB]. Express $\overrightarrow{OD}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$. *[5 marks]*

5. Points A and B lie in a plane and have coordinates (3, 0) and (4,2) respectively. C is the midpoint of [AB].

(a) Express $\overrightarrow{AB}$ and $\overrightarrow{AC}$ as column vectors.

(b) Point D is such that $\overrightarrow{AD}=\begin{pmatrix}7\\-2\end{pmatrix}$. Find the coordinates of D. *[5 marks]*

6. Points A and B have position vectors $\overrightarrow{OA}=\begin{pmatrix}3\\1\\-2\end{pmatrix}$ and $\overrightarrow{OB}=\begin{pmatrix}4\\-2\\5\end{pmatrix}$.

(a) Write $\overrightarrow{AB}$ as a column vector.

(b) Find the position vector of the midpoint of [AB]. *[5 marks]*

7. Point A has position vector $\boldsymbol{a}=2\boldsymbol{i}-3\boldsymbol{j}$, and point D is such that $\overrightarrow{AD}=\boldsymbol{i}-\boldsymbol{j}$. Find the position vector of point D. *[4 marks]*

8. Points A and B have position vectors $\boldsymbol{a}=\begin{pmatrix}2\\2\\1\end{pmatrix}$ and $\boldsymbol{b}=\begin{pmatrix}1\\-1\\3\end{pmatrix}$.

Point C lies on [AB] so that AC : BC = 2 : 3. Find the position vector of C. *[5 marks]*

9. Points P and Q have position vectors $\boldsymbol{p} = 2\boldsymbol{i} - \boldsymbol{j} - 3\boldsymbol{k}$ and $\boldsymbol{q} = \boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k}$.

(a) Find the position vector of the midpoint M of [PQ].

(b) Point R lies on the line (PQ) such that QR = QM. Find the coordinates of R if R and M are distinct points. *[6 marks]*

10. Points A, B and C have position vectors $\boldsymbol{a} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$. Find the position vector of point D such that ABCD is a parallelogram. *[5 marks]*

EXAM HINT

The ability to switch between diagrams and equations is essential for solving harder vector problems.

EXAM HINT

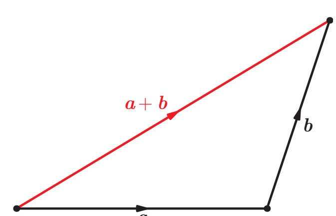
Remember that vectors only show the *relative* positions of two points; they don't have a fixed starting point. This means we are free to 'move' the second vector so that its starting point coincides with the end point of the first.

11B Vector algebra

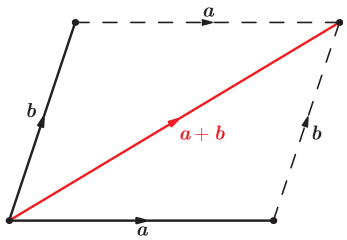
In the previous section we used vectors to describe positions and displacements of points in space; we also mentioned that vectors can represent quantities other than displacements, for example velocities or forces. Whatever the vectors represent, they always follow the same algebraic rules. In this section we will summarise those rules, which can be expressed using either diagrams or equations.

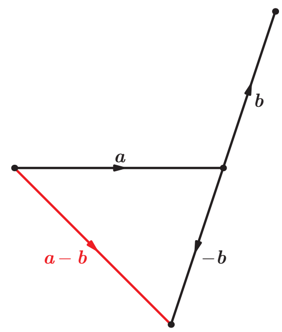
Vector addition can be done on a diagram by joining the starting point of the second vector to the end point of the first; the sum of the two vectors is the vector which starts at the starting point of the first vector and ends at the end point of the second vector. In component form, we add vectors by adding their corresponding components. When the vectors describe displacements, addition represents one displacement followed by another.

$$\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix}$$



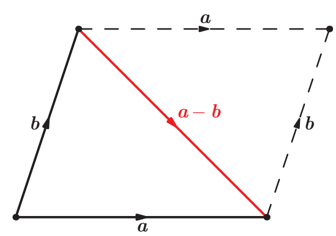
Another way of visualising the sum of two vectors is as the diagonal of the parallelogram formed by the two vectors being added.





As we saw in section 11A, reversing the direction of a vector is represented by taking its negative; in component form, this means switching the signs of all the components. *Subtracting* a vector is the same as adding its negative. It is carried out in component form by subtracting corresponding components. When the vectors describe displacements, subtracting a vector represents moving along the vector from the end point back to the starting point.

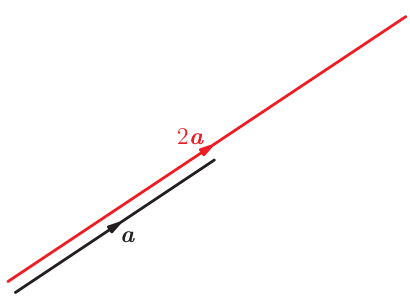
$$\begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix} + \begin{pmatrix} -3 \\ -3 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

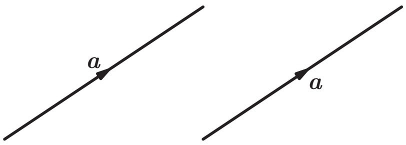


The difference of two vectors can be represented by the other diagonal of the parallelogram formed by the two vectors.

Scalar multiplication changes the magnitude (length) of the vector, leaving the direction the same. In component form, each component is multiplied by the scalar. For a displacement vector ***a***, k***a*** represents a displacement in the same direction but with distance multiplied by k.

$$2\begin{pmatrix} 3 \\ -5 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ -10 \\ 0 \end{pmatrix}$$



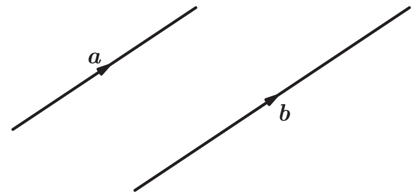


Two vectors are *equal* if they have the same magnitude and direction. All their components are equal. They represent the same displacements but may have different start and end points.

If two vectors are in the same direction, then they are **parallel**. Parallel vectors are scalar multiples of each other, since multiplying a vector by a scalar does not change its direction.

$\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ is parallel to $\begin{pmatrix} 6 \\ -9 \\ 3 \end{pmatrix}$

because $\begin{pmatrix} 6 \\ -9 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$



KEY POINT 11.3

If vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel, we can write $\boldsymbol{b} = t\boldsymbol{a}$ for some scalar t.

The following example illustrates the vector operations we have just described.

Worked example 11.5

Given the vectors $\boldsymbol{a} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} -2 \\ p \\ q \end{pmatrix}$

(a) Find $2\boldsymbol{a} - 3\boldsymbol{b}$.

(b) Find the values of p and q such that $\boldsymbol{c}$ is parallel to $\boldsymbol{a}$.

(c) Find the value of the scalar k such that $\boldsymbol{a} + k\boldsymbol{b}$ is parallel to vector $\begin{pmatrix} 0 \\ 10 \\ 23 \end{pmatrix}$.

(a) $2\underline{a} - 3\underline{b} = 2\begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} - 3\begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ 4 \\ 14 \end{pmatrix} - \begin{pmatrix} -9 \\ 12 \\ 6 \end{pmatrix} = \begin{pmatrix} 11 \\ -8 \\ 8 \end{pmatrix}$

continued . . .

If two vectors are parallel we can write $\boldsymbol{v}_2 = t\boldsymbol{v}_1$.

(b) Write $\underline{c} = t\underline{a}$ for some scalar t.
Then

$$\begin{pmatrix} -2 \\ p \\ q \end{pmatrix} = t\begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} t \\ 2t \\ 7t \end{pmatrix}$$

Two vectors being equal means that all their components are equal.

$$\Leftrightarrow \begin{cases} -2 = t \\ p = 2t \\ q = 7t \end{cases}$$

$\therefore p = -4, q = -14$

We can write vector $\boldsymbol{a} + k\boldsymbol{b}$ in terms of k and then solve $\boldsymbol{a} + k\boldsymbol{b} = t\begin{pmatrix} 0 \\ 10 \\ 23 \end{pmatrix}$.

(c) $$\underline{a} + k\underline{b} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} + \begin{pmatrix} -3k \\ 4k \\ 2k \end{pmatrix} = \begin{pmatrix} 1-3k \\ 2+4k \\ 7+2k \end{pmatrix}$$

Parallel to $\begin{pmatrix} 0 \\ 10 \\ 23 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1-3k \\ 2+4k \\ 7+2k \end{pmatrix} = t\begin{pmatrix} 0 \\ 10 \\ 23 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} 1-3k = 0 \\ 2+4k = 10t \\ 7+2k = 23t \end{cases}$$

We can find k from just the first equation; however, we still need to check that all three equations can be satisfied by this value of k.

$1-3k = 0 \Rightarrow k = \frac{1}{3}$

From 2nd equation,

$2+4\left(\frac{1}{3}\right) = 10t \Rightarrow t = \frac{1}{3}$

Put values into 3rd equation:

$\text{LHS} = 7+2\left(\frac{1}{3}\right) = \frac{23}{3}$

$\text{RHS} = 23\left(\frac{1}{3}\right)$ $\therefore$ satisfied

$\therefore k = \frac{1}{3}$

Exercise 11B

1. Let $\boldsymbol{a} = \begin{pmatrix} 7 \\ 1 \\ 12 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. Find the following vectors.

   (a) (i) $3\boldsymbol{a}$ (ii) $4\boldsymbol{b}$

   (b) (i) $\boldsymbol{a} - \boldsymbol{b}$ (ii) $\boldsymbol{b} + \boldsymbol{c}$

   (c) (i) $2\boldsymbol{b} + \boldsymbol{c}$ (ii) $\boldsymbol{a} - 2\boldsymbol{b}$

   (d) (i) $\boldsymbol{a} + \boldsymbol{b} - 2\boldsymbol{c}$ (ii) $3\boldsymbol{a} - \boldsymbol{b} + \boldsymbol{c}$

2. Let $\boldsymbol{a} = \boldsymbol{i} + 2\boldsymbol{j}$, $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{k}$ and $\boldsymbol{c} = 2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$. Find the following vectors:

   (a) (i) $-5\boldsymbol{b}$ (ii) $4\boldsymbol{a}$

   (b) (i) $\boldsymbol{c} - \boldsymbol{a}$ (ii) $\boldsymbol{a} - \boldsymbol{b}$

   (c) (i) $\boldsymbol{a} - \boldsymbol{b} + 2\boldsymbol{c}$ (ii) $4\boldsymbol{c} - 3\boldsymbol{b}$

3. Given that $\boldsymbol{a} = 4\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$, find the vector $\boldsymbol{b}$ such that

   (a) $\boldsymbol{a} + \boldsymbol{b}$ is the zero vector

   (b) $2\boldsymbol{a} + 3\boldsymbol{b}$ is the zero vector

   (c) $\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{j}$

   (d) $\boldsymbol{a} + 2\boldsymbol{b} = 3\boldsymbol{i}$

4. Given that $\boldsymbol{a} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 5 \\ 3 \\ 3 \end{pmatrix}$, find the vector $\boldsymbol{x}$ such that

   $3\boldsymbol{a} + 4\boldsymbol{x} = \boldsymbol{b}$. *[4 marks]*

5. Given that $\boldsymbol{a} = 3\boldsymbol{i} - 2\boldsymbol{j} + 5\boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{c} = \boldsymbol{i} + \boldsymbol{k}$, find the value of the scalar t such that $\boldsymbol{a} + t\boldsymbol{b} = \boldsymbol{c}$. *[4 marks]*

6. Given that $\boldsymbol{a} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$, find the value of the scalar p

   such that $\boldsymbol{a} + p\boldsymbol{b}$ is parallel to the vector $\begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}$. *[5 marks]*

7. Given that $\boldsymbol{x} = 2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{y} = 4\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$, find the value of the scalar λ such that $\lambda\boldsymbol{x} + \boldsymbol{y}$ is parallel to vector $\boldsymbol{j}$. *[5 marks]*

8. Given that $\boldsymbol{a} = \boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{b} = 2q\boldsymbol{i} + \boldsymbol{j} + q\boldsymbol{k}$, find the values of scalars p and q such that $p\boldsymbol{a} + \boldsymbol{b}$ is parallel to the vector $\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$. *[6 marks]*

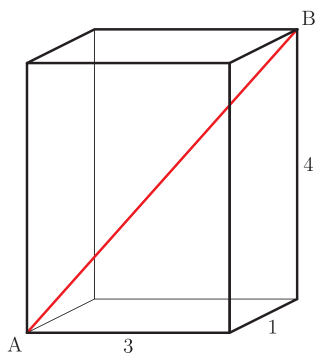
11C Distances

Geometry problems often involve finding distances between points. In this section we will see how to do this using vectors.

Consider two points A and B such that the displacement between them is $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$. The distance AB can be found by using Pythagoras' Theorem in three dimensions:

$\mathrm{AB} = \sqrt{3^2 + 1^2 + 4^2} = \sqrt{26}$.

See section 10E on finding the length of the diagonal of a cuboid.



This quantity is called the **magnitude** of $\overrightarrow{AB}$, and is denoted by $\left|\overrightarrow{AB}\right|$.

If we know the position vectors of A and B, then to find the distance between A and B we need to find the displacement vector $\overrightarrow{AB}$ and then calculate its magnitude.

Worked example 11.6

Points A and B have position vectors $\boldsymbol{a} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$. Find the exact distance AB.

The distance is the magnitude of the displacement vector, so we need to find $\overrightarrow{AB}$ first.

$$\overrightarrow{AB} = \underline{b} - \underline{a}$$
$$= \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -2 \end{pmatrix}$$

Now use the formula for the magnitude.

$$\left|\overrightarrow{AB}\right| = \sqrt{3^2 + 3^2 + (-2)^2} = \sqrt{22}$$

EXAM HINT

Don't forget that squaring a negative number gives a positive value.

KEY POINT 11.4

The magnitude of a vector $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ is $|\boldsymbol{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

A useful point that is not in the Formula booklet is: the distance between points with position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is $|\boldsymbol{b} - \boldsymbol{a}|$.

We saw in section 11B that multiplying a vector by a scalar (other than 0 or 1) produces a vector in the same direction but of different magnitude. In more advanced applications of vectors it will be useful to produce vectors of length 1, called **unit vectors**. The base vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are examples of unit vectors. For any vector $\boldsymbol{v}$, the unit vector in the direction of $\boldsymbol{v}$ is often written as $\hat{\boldsymbol{v}}$.

Worked example 11.7

(a) Find the unit vector in the same direction as $\boldsymbol{a} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

(b) Find a vector of magnitude 5 that is parallel to $\boldsymbol{a}$.

To produce a vector in the same direction as $\boldsymbol{a}$ but with a different magnitude, we need to multiply $\boldsymbol{a}$ by a scalar. We need to find the value of this scalar.

(a) Call the required unit vector $\hat{\underline{a}}$.

Then $\hat{\underline{a}} = k\underline{a}$ and $|\hat{\underline{a}}| = 1$.

$|k\underline{a}| = k|\underline{a}| = 1$

$\Rightarrow k = \dfrac{1}{|\underline{a}|}$

$|\underline{a}| = \sqrt{2^2 + 2^2 + 1^2} = 3$

$\therefore k = \dfrac{1}{3}$

Now find the vector $\hat{\boldsymbol{a}}$.

The unit vector is

$$\hat{\underline{a}} = \frac{1}{3}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$$

To get a vector of magnitude 5, we multiply the unit vector by 5.

(b) Let $\underline{b}$ be parallel to $\underline{a}$ and $|\underline{b}| = 5$.

Then $\underline{b} = 5\hat{\underline{a}} = \begin{pmatrix} \frac{10}{3} \\ -\frac{10}{3} \\ \frac{5}{3} \end{pmatrix}$

EXAM HINT

In fact, part (b) has two possible answers, as $\boldsymbol{b}$ could be in the opposite direction to the one we found. To get the second answer, multiply the unit vector by −5 instead of 5.

Worked example 11.7 showed the general method for finding the unit vector in a given direction.

KEY POINT 11.5

The unit vector in the same direction as $\boldsymbol{a}$ is $\hat{\boldsymbol{a}} = \frac{1}{|\boldsymbol{a}|}\boldsymbol{a}$.

Exercise 11C

1. Find the magnitude of the following vectors in two dimensions:

$\boldsymbol{a} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ $\quad \boldsymbol{b} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$ $\quad \boldsymbol{c} = 2\boldsymbol{i} - 4\boldsymbol{j}$ $\quad \boldsymbol{d} = -\boldsymbol{i} + \boldsymbol{j}$

2. Find the magnitude of the following vectors in three dimensions:

$\boldsymbol{a} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$ $\quad \boldsymbol{b} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ $\quad \boldsymbol{c} = 2\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}$ $\quad \boldsymbol{d} = \boldsymbol{j} - \boldsymbol{k}$

3. Find the distance between the following pairs of points in the plane.

(a) (i) A(1,2) and B(3,7) (ii) C(2,1) and D(1,2)

(b) (i) P(−1,−5) and Q(−4,2) (ii) M(1,0) and N(0,−2)

4. Find the distance between the following pairs of points in three dimensions.

(a) (i) A(1,0,2) and B(2,3,5) (ii) C(2,1,7) and D(1,2,1)

(b) (i) P(3,−1,−5) and Q(−1,−4,2) (ii) M(0,0,2) and N(0,−3,0)

5. Find the distance between the points with the given position vectors:

(a) $\boldsymbol{a} = 2\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} - 2\boldsymbol{j} - 6\boldsymbol{k}$

(b) $\boldsymbol{a} = \begin{pmatrix} 3 \\ 7 \\ -2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix}$

(c) $\boldsymbol{a} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix}$

(d) $\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{j} - \boldsymbol{k}$

6. (a) (i) Find a unit vector parallel to $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$.

(ii) Find a unit vector parallel to $6\boldsymbol{i} + 6\boldsymbol{j} - 3\boldsymbol{k}$.

(b) (i) Find a unit vector in the same direction as $\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$.

(ii) Find a unit vector in the same direction as $\begin{pmatrix} 4 \\ -1 \\ 2\sqrt{2} \end{pmatrix}$.

7. Find the possible values of the constant c such that the vector $\begin{pmatrix} 2c \\ c \\ -c \end{pmatrix}$ has magnitude 12. *[4 marks]*

8. Points A and B have position vectors $\boldsymbol{a} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$. C is the midpoint of [AB]. Find the exact distance AC. *[4 marks]*

9. Let $\boldsymbol{a} = \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$. Find the possible values of λ such that $|\boldsymbol{a} + \lambda\boldsymbol{b}| = 5\sqrt{2}$. *[6 marks]*

10. (a) Find a vector of magnitude 6 that is parallel to $\begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$.

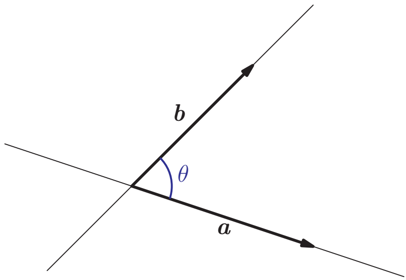
(b) Find a vector of magnitude 3 in the same direction as $2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$. *[6 marks]*

11. Points A and B are such that $\overrightarrow{OA} = \begin{pmatrix} -1 \\ -6 \\ 13 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix}$ where O is the origin. Find the possible values of t such that $AB = 3$. *[5 marks]*

12. Points P and Q have position vectors $\boldsymbol{p} = \boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{q} = (2+t)\boldsymbol{i} + (1-t)\boldsymbol{j} + (1+t)\boldsymbol{k}$. Find the value of t for which the distance PQ is as small as possible and find this minimum distance. *[6 marks]*

11D Angles

In solving geometry problems, we often need to find angles between lines. The diagram shows two lines with angle θ between them; $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors in the directions of the two lines, arranged so that both arrows point away from the intersection point. It turns out that $\cos\theta$ can be expressed in terms of the components of $\boldsymbol{a}$ and $\boldsymbol{b}$.



KEY POINT 11.6

If θ is the angle between vectors $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, then

$$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{|\boldsymbol{a}||\boldsymbol{b}|}.$$

See the Fill-in proof 8 'Deriving scalar product' on the CD-ROM for how to derive this result using the cosine rule.

The expression in the numerator of the fraction in the formula has some very important uses, so it has been given a special name.

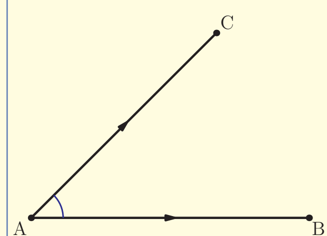
KEY POINT 11.7

The quantity $a_1b_1 + a_2b_2 + a_3b_3$ is called the **scalar product** (or *dot product*) of $\boldsymbol{a}$ and $\boldsymbol{b}$ and is denoted by $\boldsymbol{a} \cdot \boldsymbol{b}$.

Worked example 11.8

Given points A(3,−5,2), B(4,1,1) and C(−1,1,2), find the size of the acute angle B$\hat{\text{A}}$C in degrees.

It is always a good idea to draw a diagram.



We can see that the required angle is between vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Use the formula with $\boldsymbol{a} = \overrightarrow{AB}$ and $\boldsymbol{b} = \overrightarrow{AC}$.

Let $\theta = B\hat{A}C$. Then

$$\cos\theta = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|}$$

First, we need to find the components of vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\overrightarrow{AB} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ -1 \end{pmatrix}$$

$$\overrightarrow{AC} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \\ 0 \end{pmatrix}$$

$$\therefore \cos\theta = \frac{1\times(-4) + 6\times 6 + (-1)\times 0}{\sqrt{1^2+6^2+1^2}\sqrt{4^2+6^2+0^2}}$$

$$= \frac{32}{\sqrt{38}\sqrt{52}}$$

$$= 0.7199$$

$$\theta = \cos^{-1}(0.7199)$$

$$= 44.0°$$

The formula in Key point 11.6 makes it very straightforward to check whether two vectors are perpendicular. If $\theta = 90°$, then $\cos\theta = 0$, and so the numerator of the fraction in the formula must be zero. We do not even need to calculate the magnitudes of the two vectors.

KEY POINT 11.8

Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.

Worked example 11.9

Given that $\boldsymbol{p} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$ and $\boldsymbol{q} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, find the value of the scalar t such that $\boldsymbol{p} + t\boldsymbol{q}$ is perpendicular to $\begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$.

Two vectors are perpendicular if their scalar product equals 0.

$$(\underline{p} + t\underline{q}) \cdot \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} = 0$$

Write the components of $\boldsymbol{p} + t\boldsymbol{q}$ in terms of t and then form an equation.

$$\underline{p} + t\underline{q} = \begin{pmatrix} 4+2t \\ -1+t \\ 2+t \end{pmatrix}$$

So

$$\begin{pmatrix} 4+2t \\ -1+t \\ 2+t \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} = 0$$

Form and solve the equation.

$$\Leftrightarrow 3(4+2t) + 5(-1+t) + 1(2+t) = 0$$

$$\Leftrightarrow 9 + 12t = 0$$

$$\Leftrightarrow t = -\frac{3}{4}$$

Exercise 11D

1. Calculate the angle between each pair of vectors, giving your answers in radians.

(a) (i) $\begin{pmatrix}5\\1\\2\end{pmatrix}$ and $\begin{pmatrix}1\\-2\\3\end{pmatrix}$ (ii) $\begin{pmatrix}3\\0\\2\end{pmatrix}$ and $\begin{pmatrix}0\\-1\\1\end{pmatrix}$

(b) (i) $2\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k}$ and $\boldsymbol{i}-\boldsymbol{j}+3\boldsymbol{k}$

(ii) $3\boldsymbol{i}+\boldsymbol{j}$ and $\boldsymbol{i}-2\boldsymbol{k}$

(c) (i) $\begin{pmatrix}3\\2\end{pmatrix}$ and $\begin{pmatrix}-1\\4\end{pmatrix}$ (ii) $\boldsymbol{i}-\boldsymbol{j}$ and $2\boldsymbol{i}+3\boldsymbol{j}$

2. The angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is θ. Find the exact value of $\cos\theta$ in the following cases.

(a) (i) $\boldsymbol{a}=2\boldsymbol{i}+3\boldsymbol{j}-\boldsymbol{k}$ and $\boldsymbol{b}=\boldsymbol{i}-2\boldsymbol{j}+\boldsymbol{k}$

(ii) $\boldsymbol{a}=\boldsymbol{i}-3\boldsymbol{j}+3\boldsymbol{k}$ and $\boldsymbol{b}=\boldsymbol{i}+5\boldsymbol{j}-2\boldsymbol{k}$

(b) (i) $\boldsymbol{a}=\begin{pmatrix}2\\2\\3\end{pmatrix}$ and $\boldsymbol{b}=\begin{pmatrix}1\\1\\-2\end{pmatrix}$ (ii) $\boldsymbol{a}=\begin{pmatrix}5\\1\\-3\end{pmatrix}$ and $\boldsymbol{b}=\begin{pmatrix}2\\-1\\2\end{pmatrix}$

(c) (i) $\boldsymbol{a}=-2\boldsymbol{k}$ and $\boldsymbol{b}=4\boldsymbol{i}$ (ii) $\boldsymbol{a}=5\boldsymbol{i}$ and $\boldsymbol{b}=3\boldsymbol{j}$

3. (i) The vertices of a triangle have position vectors $\boldsymbol{a}=\begin{pmatrix}1\\1\\3\end{pmatrix}$, $\boldsymbol{b}=\begin{pmatrix}2\\-1\\1\end{pmatrix}$ and $\boldsymbol{c}=\begin{pmatrix}5\\1\\2\end{pmatrix}$. Find, in degrees, the angles of the triangle.

(ii) Find, in degrees, the angles of the triangle with vertices (2, 1, 2),(4, –1, 5) and (7, 1, –2).

4. Determine whether each pair of vectors is perpendicular.

(a) (i) $\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ (ii) $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 6 \\ 0 \end{pmatrix}$

(b) (i) $5\boldsymbol{i}-2\boldsymbol{j}+\boldsymbol{k}$ and $3\boldsymbol{i}+4\boldsymbol{j}-7\boldsymbol{k}$

(ii) $\boldsymbol{i}-3\boldsymbol{k}$ and $2\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}$

5. Points A and B have position vectors $\overrightarrow{OA}=\begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$ and $\overrightarrow{OB}=\begin{pmatrix} -1 \\ 7 \\ 2 \end{pmatrix}$. Find the angle between $\overrightarrow{AB}$ and $\overrightarrow{OA}$. *[5 marks]*

6. Four points have coordinates A(2, –1, 3), B(1, 1, 2), C(6, –1, 2) and D(7, –3, 3). Find the angle between $\overrightarrow{AC}$ and $\overrightarrow{BD}$. *[5 marks]*

7. Four points have coordinates A(2, 4, 1), B(k, 4, 2k), C($k+4$, $2k+4$, $2k+2$) and D(6, $2k+4$, 3).

(a) Show that ABCD is a parallelogram for all values of k.

(b) When $k=1$, find the angles of the parallelogram.

(c) Find the value of k for which ABCD is a rectangle. *[8 marks]*

8. Vertices of a triangle have position vectors $\boldsymbol{a}=\boldsymbol{i}-2\boldsymbol{j}+2\boldsymbol{k}$, $\boldsymbol{b}=3\boldsymbol{i}-\boldsymbol{j}+7\boldsymbol{k}$ and $\boldsymbol{c}=5\boldsymbol{i}$.

(a) Show that the triangle is right-angled.

(b) Calculate the other two angles of the triangle.

(c) Find the area of the triangle. *[8 marks]*

11E Properties of the scalar product

In section 11D we defined the scalar product of vectors

$$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \text{ and } \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \text{ as}$$

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

and we saw that if θ is the angle between the directions of $\boldsymbol{a}$ and $\boldsymbol{b}$, then

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$$

In this section we will look at various properties of the scalar product in more detail – in particular, the algebraic rules it follows. The scalar product has many properties similar to the multiplication of numbers; these can be proved by using the components of the vectors.

KEY POINT 11.9

Algebraic properties of the scalar product:

$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$$

$$(-\boldsymbol{a}) \cdot \boldsymbol{b} = -(\boldsymbol{a} \cdot \boldsymbol{b})$$

$$(k\boldsymbol{a}) \cdot \boldsymbol{b} = k(\boldsymbol{a} \cdot \boldsymbol{b})$$

$$\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{b}) + (\boldsymbol{a} \cdot \boldsymbol{c})$$

All the operations with vectors work the same way in two and three dimensions. If there were a fourth dimension, so that the position of each point is described using four numbers, we could use analogous rules to calculate 'distances' and 'angles'. Does this mean that we can acquire knowledge about a four-dimensional world which we can't see, or even imagine?

There are also some properties of multiplication of numbers which do not hold for the scalar product. For example, it is not possible to calculate the scalar product of three vectors: the expression $(\boldsymbol{a} \cdot \boldsymbol{b}) \cdot \boldsymbol{c}$ has no meaning, as $\boldsymbol{a} \cdot \boldsymbol{b}$ is a scalar, and the scalar product involves multiplying two vectors.

Another important property of the scalar product concerns parallel vectors.

KEY POINT 11.10

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel vectors, then $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|$.

In particular, $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$.

The next two examples show how you can use the rules discussed in this section.

Worked example 11.10

Given that $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular vectors such that $|\boldsymbol{a}|=5$ and $|\boldsymbol{b}|=3$, evaluate $(2\boldsymbol{a}-\boldsymbol{b})\cdot(\boldsymbol{a}+4\boldsymbol{b})$.

According to Key point 11.9, we can multiply out the brackets just as we would with numbers.

$(2\underline{a}-\underline{b})\cdot(\underline{a}+4\underline{b})=2\underline{a}\cdot\underline{a}+8\underline{a}\cdot\underline{b}-\underline{b}\cdot\underline{a}-4\underline{b}\cdot\underline{b}$

As $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular, $\boldsymbol{a}\cdot\boldsymbol{b}=\boldsymbol{b}\cdot\boldsymbol{a}=0$.

$=2\underline{a}\cdot\underline{a}-4\underline{b}\cdot\underline{b}$

Now use the fact that $\boldsymbol{a}\cdot\boldsymbol{a}=|\boldsymbol{a}|^2$ and similarly for $\boldsymbol{b}$, and then substitute the given magnitudes.

$=2|\underline{a}|^2-4|\underline{b}|^2$

$=2\times5^2-4\times3^2$

$=14$

Worked example 11.11

Points A, B and C have position vectors $\boldsymbol{a}=k\begin{pmatrix}3\\-1\\1\end{pmatrix}$, $\boldsymbol{b}=\begin{pmatrix}3\\4\\-2\end{pmatrix}$ and $\boldsymbol{c}=\begin{pmatrix}1\\1\\5\end{pmatrix}$.

(a) Find $\overrightarrow{BC}$.

(b) Find $\overrightarrow{AB}$ in terms of k.

(c) Find the value of k for which (AB) is perpendicular to (BC).

Use $\overrightarrow{BC}=\boldsymbol{c}-\boldsymbol{b}$.

(a) $\overrightarrow{BC}=\underline{c}-\underline{b}$

$=\begin{pmatrix}1\\1\\5\end{pmatrix}-\begin{pmatrix}3\\4\\-2\end{pmatrix}=\begin{pmatrix}-2\\-3\\7\end{pmatrix}$

Use $\overrightarrow{AB}=\boldsymbol{b}-\boldsymbol{a}$.

(b) $\overrightarrow{AB}=\underline{b}-\underline{a}$

$=\begin{pmatrix}3\\4\\-2\end{pmatrix}-\begin{pmatrix}3k\\-k\\k\end{pmatrix}=\begin{pmatrix}3-3k\\4+k\\-2-k\end{pmatrix}$

continued . . .

For (AB) and (BC) to be perpendicular, we must have $\overrightarrow{AB}\cdot\overrightarrow{BC}=0$.

(c) $\overrightarrow{AB}\cdot\overrightarrow{BC}=0$

$$\Leftrightarrow \begin{pmatrix}3-3k\\4+k\\-2-2k\end{pmatrix}\cdot\begin{pmatrix}-2\\-3\\7\end{pmatrix}=0$$

$\Leftrightarrow -6+6k-12-3k-14-14k=0$

$\Leftrightarrow -11k=32$

$$\Leftrightarrow \quad k=-\frac{11}{32}$$

Exercise 11E

1. Evaluate $\boldsymbol{a}\cdot\boldsymbol{b}$ in the following cases.

(a) (i) $\boldsymbol{a}=\begin{pmatrix}2\\1\\2\end{pmatrix}$ and $\boldsymbol{b}=\begin{pmatrix}5\\2\\2\end{pmatrix}$ (ii) $\boldsymbol{a}=\begin{pmatrix}3\\-1\\2\end{pmatrix}$ and $\boldsymbol{b}=\begin{pmatrix}-12\\4\\-8\end{pmatrix}$

(b) (i) $\boldsymbol{a}=\begin{pmatrix}2\\-1\\2\end{pmatrix}$ and $\boldsymbol{b}=\begin{pmatrix}5\\-2\\2\end{pmatrix}$ (ii) $\boldsymbol{a}=\begin{pmatrix}3\\0\\2\end{pmatrix}$ and $\boldsymbol{b}=\begin{pmatrix}0\\0\\-8\end{pmatrix}$

(c) (i) $\boldsymbol{a}=4\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}$ and $\boldsymbol{b}=\boldsymbol{i}+\boldsymbol{j}+3\boldsymbol{k}$

(ii) $\boldsymbol{a}=4\boldsymbol{i}-2\boldsymbol{j}+\boldsymbol{k}$ and $\boldsymbol{b}=\boldsymbol{i}-\boldsymbol{j}+3\boldsymbol{k}$

(d) (i) $\boldsymbol{a}=-3\boldsymbol{j}+\boldsymbol{k}$ and $\boldsymbol{b}=2\boldsymbol{i}-4\boldsymbol{k}$ (ii) $\boldsymbol{a}=-3\boldsymbol{j}$ and $\boldsymbol{b}=4\boldsymbol{k}$

2. Given that θ is the angle between vectors $\boldsymbol{p}$ and $\boldsymbol{q}$, find the exact value of $\cos\theta$.

(a) (i) $\boldsymbol{p}=\begin{pmatrix}1\\1\\2\end{pmatrix}$ and $\boldsymbol{q}=\begin{pmatrix}2\\1\\2\end{pmatrix}$ (ii) $\boldsymbol{p}=\begin{pmatrix}3\\0\\2\end{pmatrix}$ and $\boldsymbol{q}=\begin{pmatrix}1\\1\\1\end{pmatrix}$

(b) (i) $\boldsymbol{p}=\begin{pmatrix}-1\\1\\2\end{pmatrix}$ and $\boldsymbol{q}=\begin{pmatrix}1\\1\\2\end{pmatrix}$ (ii) $\boldsymbol{p}=\begin{pmatrix}-1\\1\\0\end{pmatrix}$ and $\boldsymbol{q}=\begin{pmatrix}0\\1\\2\end{pmatrix}$

3. (i) Given that $|\boldsymbol{a}|=3, |\boldsymbol{b}|=5$ and $\boldsymbol{a}\cdot\boldsymbol{b}=10$, find, in degrees, the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

(ii) Given that $|\boldsymbol{c}|=9, |\boldsymbol{d}|=12$ and $\boldsymbol{c}\cdot\boldsymbol{d}=-15$, find, in degrees, the angle between $\boldsymbol{c}$ and $\boldsymbol{d}$.

4. (a) Given that $|\boldsymbol{a}|=6, |\boldsymbol{b}|=4$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is 37°, calculate $\boldsymbol{a}\cdot\boldsymbol{b}$.

(b) Given that $|\boldsymbol{a}|=8, \boldsymbol{a}\cdot\boldsymbol{b}=12$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is 60°, find the exact value of $|\boldsymbol{b}|$.

5. Given that $\boldsymbol{a}=2\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k}, \boldsymbol{b}=\boldsymbol{i}+3\boldsymbol{j}-\boldsymbol{k}, \boldsymbol{c}=5\boldsymbol{i}-3\boldsymbol{k}$ and $\boldsymbol{d}=-2\boldsymbol{j}+\boldsymbol{k}$ verify that

(a) $\boldsymbol{b}\cdot\boldsymbol{d}=\boldsymbol{d}\cdot\boldsymbol{b}$

(b) $\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c})=\boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{a}\cdot\boldsymbol{c}$

(c) $(\boldsymbol{c}-\boldsymbol{d})\cdot\boldsymbol{c}=|\boldsymbol{c}|^2-\boldsymbol{c}\cdot\boldsymbol{d}$

(d) $(\boldsymbol{a}+\boldsymbol{b})\cdot(\boldsymbol{a}+\boldsymbol{b})=|\boldsymbol{a}|^2+|\boldsymbol{b}|^2+2\boldsymbol{a}\cdot\boldsymbol{b}$

6. Find the values of t for which the following pairs of vectors are perpendicular.

(a) (i) $\begin{pmatrix}2t\\1\\-3t\end{pmatrix}$ and $\begin{pmatrix}1\\-2\\2\end{pmatrix}$ (ii) $\begin{pmatrix}t+1\\2t-1\\2t\end{pmatrix}$ and $\begin{pmatrix}2\\6\\0\end{pmatrix}$

(b) (i) $5t\boldsymbol{i}-(2+t)\boldsymbol{j}+\boldsymbol{k}$ and $3\boldsymbol{i}+4\boldsymbol{j}-t\boldsymbol{k}$

(ii) $t\boldsymbol{i}-3\boldsymbol{k}$ and $2t\boldsymbol{i}+\boldsymbol{j}+t\boldsymbol{k}$

7. Given that $\boldsymbol{a}=\begin{pmatrix}2\\-2\\1\end{pmatrix}, \boldsymbol{b}=\begin{pmatrix}1\\1\\2\end{pmatrix}, \boldsymbol{c}=\begin{pmatrix}3\\-5\\1\end{pmatrix}$ and $\boldsymbol{d}=\begin{pmatrix}3\\-3\\2\end{pmatrix}$, calculate

(a) $\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c})$

(b) $(\boldsymbol{b}-\boldsymbol{a})\cdot(\boldsymbol{d}-\boldsymbol{c})$

(c) $(\boldsymbol{b}+\boldsymbol{d})\cdot(2\boldsymbol{a})$ *[7 marks]*

8. (a) If $\boldsymbol{a}$ is a unit vector perpendicular to $\boldsymbol{b}$, find the value of $\boldsymbol{a}\cdot(2\boldsymbol{a}-3\boldsymbol{b})$.

(b) If $\boldsymbol{p}$ is a unit vector making a 45° angle with vector $\boldsymbol{q}$, and $\boldsymbol{p}\cdot\boldsymbol{q}=3\sqrt{2}$, find $|\boldsymbol{q}|$. *[6 marks]*

9. (a) $\boldsymbol{a}$ is a vector of magnitude 3 and $\boldsymbol{b}$ makes an angle of 60° with $\boldsymbol{a}$. Given that $\boldsymbol{a}\cdot(\boldsymbol{a}-\boldsymbol{b})=\frac{1}{3}$, find the value of $|\boldsymbol{b}|$.

(b) Given that $\boldsymbol{a}$ and $\boldsymbol{b}$ are two vectors of equal magnitude such that $(3\boldsymbol{a}+\boldsymbol{b})$ is perpendicular to $(\boldsymbol{a}-3\boldsymbol{b})$, show that $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular. *[6 marks]*

10. Points A, B and C have position vectors $\boldsymbol{a}=\boldsymbol{i}-19\boldsymbol{j}+5\boldsymbol{k}$, $\boldsymbol{b}=2\lambda\boldsymbol{i}+(\lambda+2)\boldsymbol{j}+2\boldsymbol{k}$ and $\boldsymbol{c}=-6\boldsymbol{i}-15\boldsymbol{j}+7\boldsymbol{k}$.

(a) Find the value of λ for which (BC) is perpendicular to (AC).

For the value of λ found above, find

(b) the angles of the triangle ABC

(c) the area of the triangle ABC. *[8 marks]*

11. ABCD is a parallelogram with [AB] parallel to [DC]. Let $\overrightarrow{AB}=\boldsymbol{a}$ and $\overrightarrow{AD}=\boldsymbol{b}$.

(a) Express $\overrightarrow{AC}$ and $\overrightarrow{BD}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

(b) Simplify $(\boldsymbol{a}+\boldsymbol{b})\cdot(\boldsymbol{b}-\boldsymbol{a})$.

(c) Hence show that if ABCD is a rhombus, then its diagonals are perpendicular. *[8 marks]*

12. Points A and B have position vectors $\begin{pmatrix}2\\1\\4\end{pmatrix}$ and $\begin{pmatrix}2\lambda\\\lambda\\4\lambda\end{pmatrix}$.

(a) Show that B lies on the line (OA) for all values of λ.

Point C has position vector $\begin{pmatrix}12\\2\\4\end{pmatrix}$.

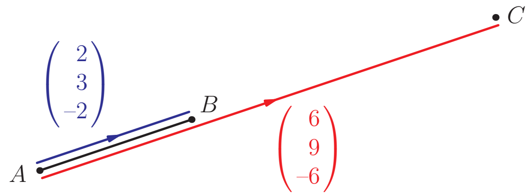
(b) Find the value of λ for which $\text{C}\hat{\text{B}}\text{A}$ is a right angle.

(c) For the value of λ found above, calculate the exact distance from C to the line (OA). *[8 marks]*

11F Vector equation of a line

Before we can solve problems involving lines in space, we need a way of deciding whether a given point lies on a certain straight line.

Consider two points A(−1, 1, 4) and B(1, 4, 2); they determine a unique straight line (by ‘straight line’ we mean a line that extends indefinitely in both directions). If we are given a third point C, how can we check whether it lies on the same line? We can use vectors to answer this question. For example, if C has coordinates (5, 10, –2), then



$$\overrightarrow{AC} = \begin{pmatrix} 6 \\ 9 \\ -6 \end{pmatrix} = 3\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} = 3\overrightarrow{AB}$$

This means that (AC) is parallel to (AB). But since they both contain the point A, (AC) and (AB) must then be the same straight line; in other words, C lies on the line (AB).

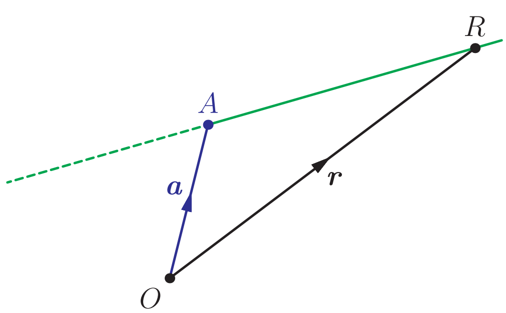
The next question is: how can we characterise all the points on the line (AB)? Following the above reasoning, we realise that a point R lies on (AB) if (AR) and (AB) are parallel; this can be expressed using vectors by saying that $\overrightarrow{AR} = \lambda\overrightarrow{AB}$ for some value of the scalar λ, so $\overrightarrow{AR} = \begin{pmatrix} 2\lambda \\ 3\lambda \\ -2\lambda \end{pmatrix}$ in our example. On the other hand, we also know that $\overrightarrow{AR} = \boldsymbol{r} - \boldsymbol{a}$, where $\boldsymbol{r}$ and $\boldsymbol{a}$ are the position vectors of R and A.

EXAM HINT

Remember the International Baccalaureate ® notation for lines and line segments: (AB) stands for the (infinite) straight line through A and B, [AB] for the line segment between A and B, and AB for the length of [AB].

Recall that a scalar is a number without direction.

See section 11B for a reminder of vector algebra.



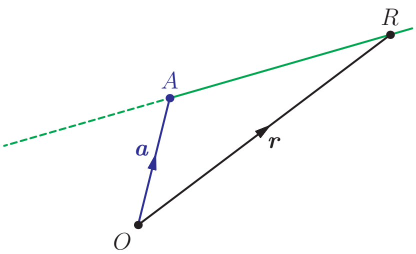
Hence $\boldsymbol{r} = \boldsymbol{a} + \overrightarrow{AR} = \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} + \begin{pmatrix} 2\lambda \\ 3\lambda \\ -2\lambda \end{pmatrix}$ is the position vector of a general point R on the line (AB). In other words, R has coordinates $(-1+2\lambda, 1+3\lambda, 4-2\lambda)$ for some value of λ.

Different values of λ correspond to different points on the line; for example, $\lambda = 0$ corresponds to point A, $\lambda = 1$ to point B and $\lambda = 3$ to point C. The line is parallel to the vector $\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$, so this vector determines the direction of the line. The expression

You will see that there is more than one possible vector equation of a line.

for the position vector of R is usually written in the form

$$\boldsymbol{r} = \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$$ to make it easy to identify the direction vector.



KEY POINT 11.11

The vector equation of a line is of the form $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{d}$ where:

$\boldsymbol{r}$ is the position vector of a general point on the line

$\boldsymbol{d}$ is the direction vector of the line

$\boldsymbol{a}$ is the position vector of one point on the line

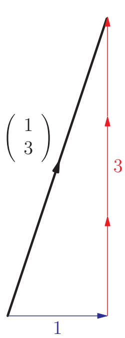
Different values of the parameter λ give the positions of different points on the line.

Worked example 11.12

Write down a vector equation of the line passing through the point $(-1, 1, 2)$ in the direction of the vector $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$.

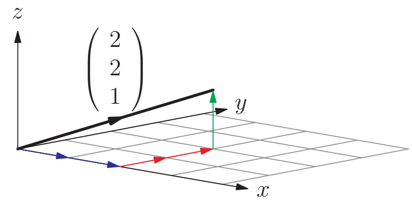
The equation of the line is $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{d}$, where $\boldsymbol{a}$ is the position vector of a point on the line and $\boldsymbol{d}$ is the direction vector.

$$\boldsymbol{r} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$



In two dimensions, a straight line is determined by its gradient and one point. The gradient is a number that tells us the direction of the line. For example, for a line with gradient 3, an increase of 1 unit in x produces an increase of 3 units in y; thus the line is in the direction of the vector $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

In three dimensions, a straight line is still determined by its direction and one point, but it is no longer possible to use a single number to represent the direction vector. The line in Worked example 11.12 had direction vector $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$, which means that an increase of 2 units in x will produce an increase of 2 units in y *and* an increase of 1 unit in z; we cannot describe these different increase amounts by just one number.



As you know, two points determine a straight line. The next example shows how to find a vector equation when two points on the line are given.

Worked example 11.13

Find a vector equation of the line through the points A(−1, 1, 2) and B(3, 5, 4).

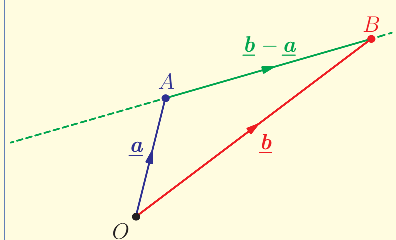
To find an equation of the line, we need to know one point and the direction vector.

$\underline{r} = \underline{a} + \lambda \underline{d}$

The line passes through A(−1, 1, 2).

$\underline{a} = \begin{pmatrix}-1\\1\\2\end{pmatrix}$

Draw a diagram. The line is in the direction of $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$.



$\underline{d} = \overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix}4\\4\\2\end{pmatrix}$

$\therefore \underline{r} = \begin{pmatrix}-1\\1\\2\end{pmatrix} + \lambda \begin{pmatrix}4\\4\\2\end{pmatrix}$

What if, for '$\boldsymbol{a}$' in the formula, we had used the position vector of point B instead? Then we would have got the equation

$\boldsymbol{r} = \begin{pmatrix}3\\5\\4\end{pmatrix} + \lambda \begin{pmatrix}4\\4\\2\end{pmatrix}$. This equation represents the same line as the one given as the answer to Worked example 11.13, but the values of λ corresponding to particular points will be different.

For example, with the equation $\boldsymbol{r} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix}$, point A has $\lambda = 0$ and point B has $\lambda = 1$, while with the equation $\boldsymbol{r} = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix}$ point A has $\lambda = -1$ and point B has $\lambda = 0$.

The direction vector is not unique either: as we are only interested in its direction and not its magnitude, any (non-zero) scalar multiple of the direction vector will also be a direction vector. Hence $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} -6 \\ -6 \\ -3 \end{pmatrix}$ could also be used as direction vectors for the line in Worked example 11.13, and yet another form of the equation of the same line would be $\boldsymbol{r} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -6 \\ -6 \\ -3 \end{pmatrix}$. With this equation, point A has $\lambda = 0$ and point B has $\lambda = -\frac{2}{3}$. To simplify calculations, we usually choose the direction vector to be the one whose components are smallest possible integer values, although sometimes it will be convenient to use the corresponding unit vector.

Worked example 11.4

(a) Show that the equations $\boldsymbol{r} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} 5 \\ 7 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 6 \\ 6 \\ 3 \end{pmatrix}$ represent the same straight line.

(b) Show that the equation $\boldsymbol{r} = \begin{pmatrix} -5 \\ -3 \\ 1 \end{pmatrix} + t \begin{pmatrix} -4 \\ -4 \\ -2 \end{pmatrix}$ represents a different straight line.

EXAM HINT

When a problem involves more than one line, different letters should be used for the parameters in their vector equations. The most commonly used letters are λ (lambda), μ (mu), t and s.

continued . . .

We need to show that the two lines have parallel direction vectors (so that the lines are parallel) and one common point (then they will be the same line). Two vectors are parallel if one is a scalar multiple of the other (Key point 11.3).

(a) Direction vectors are parallel, because

$$\begin{pmatrix}6\\6\\3\end{pmatrix}=3\begin{pmatrix}2\\2\\1\end{pmatrix}$$

We know that the second line contains the point (5, 7, 5). Now check that (5, 7, 5) also lies on the first line: this will be the case if we can find a value of λ which, when substituted in the equation of the first line, will give the position vector of (5, 7, 5).

Show that (5, 7, 5) lies on the first line:

Find the value of λ which gives the first coordinate.

$$-1+2\lambda=5$$
$$\Rightarrow \lambda=3$$

Check whether this value of λ also gives the other two coordinates.

$$\begin{cases}1+3\times 2=7\\2+3\times 1=5\end{cases}$$

so (5, 7, 5) lies on the first line.
Hence the two lines are the same.

Check whether the direction vectors are parallel.

(b) $$\begin{pmatrix}-4\\-4\\-2\end{pmatrix}=-2\begin{pmatrix}2\\2\\1\end{pmatrix}$$

So this line is parallel to the other two.

Check whether (–5, –3, 1) lies on the first line. Find the value of λ which gives the first coordinate.

$$-1+2\lambda=-5$$
$$\Rightarrow \lambda=-2$$

Check whether this value of λ also gives the other two coordinates.

$$\begin{cases}1+(-2)\times 2=-3\\2+(-2)\times 1=0\neq 1\end{cases}$$

so (–5, –3, 1) does not lie on the line.
Hence the line is not the same as the first line.

In the above example we used the coordinates of the point to find the corresponding value of λ. Sometimes, however, we know only that a point lies on a given line, but not its precise coordinates. The next example shows how we can work with a general point on the line (with an unknown value of λ).

Worked example 11.15

Point B(3, 5, 4) lies on the line with equation $\boldsymbol{r} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$. Find the possible positions of a point Q on the line such that BQ = 15.

We know that Q lies on the line, so it has position vector $\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ for some value of λ. We will find the possible values of λ and hence the possible position vectors of Q.

$$\underline{q} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1+2\lambda \\ 1+2\lambda \\ 2+\lambda \end{pmatrix}$$

Express vector $\overrightarrow{BQ}$ in terms of λ and then set its magnitude equal to 15.

$$\overrightarrow{BQ} = \underline{q} - \underline{b}$$
$$= \begin{pmatrix} -1+2\lambda \\ 1+2\lambda \\ 2+\lambda \end{pmatrix} - \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 2\lambda-4 \\ 2\lambda-4 \\ \lambda-2 \end{pmatrix}$$

It is easier to work without square roots, so let us square the magnitude equation $\left|\overrightarrow{BQ}\right| = 15$.

$$\left|\overrightarrow{BQ}\right|^2 = 15^2$$
$$\Leftrightarrow (2\lambda-4)^2 + (2\lambda-4)^2 + (\lambda-2)^2 = 15^2$$
$$\Leftrightarrow 9\lambda^2 - 36\lambda - 189 = 0$$
$$\Leftrightarrow \lambda = -3 \text{ or } 7$$

Now we can find the position vector of Q.

$$\therefore \underline{q} = \begin{pmatrix} -7 \\ -5 \\ -1 \end{pmatrix} \text{ or } \begin{pmatrix} 13 \\ 15 \\ 9 \end{pmatrix}$$

Exercise 11F

1. Find a vector equation of the line in the given direction through the given point.

(a) (i) Direction $\begin{pmatrix}1\\4\end{pmatrix}$, point $(4,-1)$

(ii) Direction $\begin{pmatrix}2\\-3\end{pmatrix}$, point $(4,1)$

(b) (i) Point $(1,0,5)$, direction $\begin{pmatrix}1\\3\\-3\end{pmatrix}$

(ii) Point $(-1,1,5)$, direction $\begin{pmatrix}3\\-2\\2\end{pmatrix}$

(c) (i) Point $(4,0)$, direction $2\boldsymbol{i}+3\boldsymbol{j}$

(ii) Point $(0,2)$, direction $\boldsymbol{i}-3\boldsymbol{j}$

(d) (i) Direction $\boldsymbol{i}-3\boldsymbol{k}$, point $(0,2,3)$

(ii) Direction $2\boldsymbol{i}+3\boldsymbol{j}-\boldsymbol{k}$, point $(4,-3,0)$

2. Find a vector equation of the line through the two given points (there is more than one right answer for each part).

(a) (i) $(4,1)$ and $(1,2)$ (ii) $(2,7)$ and $(4,-2)$

(b) (i) $(-5,-2,3)$ and $(4,-2,3)$ (ii) $(1,1,3)$ and $(10,-5,0)$

3. Decide whether or not the given point lies on the given line.

(a) (i) Line $\boldsymbol{r}=\begin{pmatrix}2\\1\\5\end{pmatrix}+t\begin{pmatrix}-1\\2\\2\end{pmatrix}$, point $(0,5,9)$

(ii) Line $\boldsymbol{r}=\begin{pmatrix}-1\\0\\3\end{pmatrix}+t\begin{pmatrix}4\\1\\5\end{pmatrix}$, point $(-1,0,3)$

(b) (i) Line $\boldsymbol{r}=\begin{pmatrix}-1\\5\\1\end{pmatrix}+t\begin{pmatrix}0\\0\\7\end{pmatrix}$, point $(-1,3,8)$

(ii) Line $\boldsymbol{r}=\begin{pmatrix}4\\0\\3\end{pmatrix}+t\begin{pmatrix}4\\0\\3\end{pmatrix}$, point $(0,0,0)$

4. (a) Show that the points $A(4,-1,-8)$ and $B(2,1,-4)$ lie on the line l with equation $\boldsymbol{r}=\begin{pmatrix}2\\1\\-4\end{pmatrix}+t\begin{pmatrix}-1\\1\\2\end{pmatrix}$.

(b) Find the coordinates of the point C on the line l such that $AB = BC$. *[6 marks]*

5. (a) Find a vector equation of the line l through points $P(7,1,2)$ and $Q(3,-1,5)$.

(b) Point R lies on l and $PR = 2\,PQ$. Find the possible coordinates of R. *[6 marks]*

6. (a) Write down a vector equation of the line l through the point $A(2,1,4)$ parallel to the vector $2\boldsymbol{i}-3\boldsymbol{j}+6\boldsymbol{k}$.

(b) Calculate the magnitude of the vector $2\boldsymbol{i}-3\boldsymbol{j}+6\boldsymbol{k}$.

(c) Find possible coordinates of the point P on l such that $AP = 35$. *[8 marks]*

11G Solving problems involving lines

In this section we will use vector equations of lines to solve problems about angles and intersections.

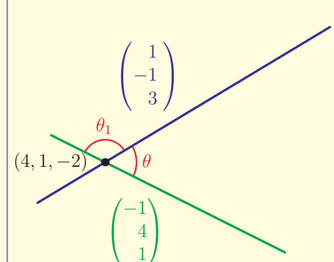
Worked example 11.16

Find the acute angle between the lines with equations $\boldsymbol{r}=\begin{pmatrix}4\\1\\-2\end{pmatrix}+t\begin{pmatrix}1\\-1\\3\end{pmatrix}$ and $\boldsymbol{r}=\begin{pmatrix}4\\1\\-2\end{pmatrix}+\lambda\begin{pmatrix}-1\\4\\1\end{pmatrix}$.

We know a formula for the angle between two vectors (Key point 11.6).
The question is which vectors to choose as our '$\boldsymbol{a}$' and '$\boldsymbol{b}$'.

$$\cos\theta=\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$$

Drawing a diagram is a good way of identifying which two vectors make the required angle.
This indicates that we should take $\boldsymbol{a}$ and $\boldsymbol{b}$ to be the direction vectors of the two lines.



continued . . .

The two vectors are in the directions of the two lines, so we take $\boldsymbol{a}$ and $\boldsymbol{b}$ to be the direction vectors of the two lines.

$$\underline{a} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \quad \underline{b} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$$

Now use the formula to calculate the angle.

$$\cos\theta = \frac{-1-4+3}{\sqrt{1+1+9}\sqrt{1+16+1}} = \frac{-2}{\sqrt{11}\sqrt{18}}$$

$$\therefore \theta = 98.2°$$

Note, however, that the angle we found is obtuse – it is the angle marked θ_1 in the diagram. The question asked for the acute angle.

$$\text{acute angle} = 180° - 98.2° = 81.8°$$

The example above illustrates the general approach to finding an angle between two lines.

KEY POINT 11.12

The angle between two lines is the angle between their direction vectors.

Since we only need to look at direction vectors to determine the angle between two lines, it is easy to identify parallel and perpendicular lines.

Parallel and perpendicular vectors were covered in sections 11B and 11E.

KEY POINT 11.13

Two lines with direction vectors $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ are

- parallel if $\boldsymbol{d}_1 = k\boldsymbol{d}_2$
- perpendicular if $\boldsymbol{d}_1 \cdot \boldsymbol{d}_2 = 0$.

Worked example 11.17

Decide whether the following pairs of lines are parallel, perpendicular, or neither:

(a) $\boldsymbol{r}=\begin{pmatrix}2\\-1\\5\end{pmatrix}+\lambda\begin{pmatrix}4\\-1\\2\end{pmatrix}$ and $\boldsymbol{r}=\begin{pmatrix}-2\\0\\3\end{pmatrix}+\mu\begin{pmatrix}1\\-2\\-3\end{pmatrix}$

(b) $\boldsymbol{r}=\begin{pmatrix}0\\0\\1\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\1\end{pmatrix}$ and $\boldsymbol{r}=\begin{pmatrix}2\\1\\2\end{pmatrix}+t\begin{pmatrix}1\\0\\-3\end{pmatrix}$

(c) $\boldsymbol{r}=\begin{pmatrix}2\\-1\\5\end{pmatrix}+t\begin{pmatrix}4\\-6\\2\end{pmatrix}$ and $\boldsymbol{r}=\begin{pmatrix}-2\\0\\3\end{pmatrix}+s\begin{pmatrix}-10\\15\\-5\end{pmatrix}$

Is $\boldsymbol{d}_1$ a multiple of $\boldsymbol{d}_2$?

(a)

If $\begin{pmatrix}4\\-1\\2\end{pmatrix}=k\begin{pmatrix}1\\-2\\-3\end{pmatrix}$

then

$$\begin{cases}4=k\times 1 \Rightarrow k=4\\ -1=k\times(-2) \Rightarrow k=\dfrac{1}{2}\end{cases}$$

$4\neq\dfrac{1}{2}$

$\therefore$ the lines are not parallel.

Is $\boldsymbol{d}_1\cdot\boldsymbol{d}_2=0$?

$$\begin{pmatrix}4\\-1\\2\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\-3\end{pmatrix}=4+2-6=0$$

$\therefore$ they are perpendicular.

Is $\boldsymbol{d}_1$ a multiple of $\boldsymbol{d}_2$?

(b)

If $\begin{pmatrix}2\\1\\1\end{pmatrix}=k\begin{pmatrix}1\\0\\-3\end{pmatrix}$

then

$$\begin{cases}2=k\times 1 \Rightarrow k=2\\ 1=k\times 0 \text{ impossible}\end{cases}$$

$\therefore$ the lines are not parallel.

continued . . .

Is $\boldsymbol{d_1} \cdot \boldsymbol{d_2} = 0$?

$$\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = 2 + 0 + 6 = 8 \neq 0$$

$\therefore$ they are not perpendicular.

The lines are neither parallel nor perpendicular.

Is $\boldsymbol{d_1}$ a multiple of $\boldsymbol{d_2}$?

(c)

If $\begin{pmatrix} 4 \\ -6 \\ 2 \end{pmatrix} = k \begin{pmatrix} -10 \\ 15 \\ -5 \end{pmatrix}$

then

$$\begin{cases} 4 = k \times (-10) \Rightarrow k = -\frac{2}{5} \\ -6 = k \times 15 \Rightarrow k = -\frac{2}{5} \\ 2 = k \times (-5) \Rightarrow k = -\frac{2}{5} \end{cases}$$

$\therefore$ the lines have parallel directions.

But they could be the same line, so we need to check this.

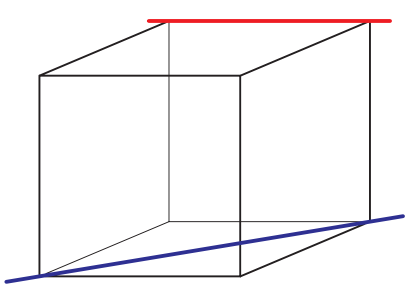
If the point $\begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}$ on the second line also lies on the first line then

$$\begin{cases} 2 + 4t = -2 \Rightarrow t = -1 \\ -1 - 6t = 0 \Rightarrow t = -\frac{1}{6} \end{cases}$$

$\therefore$ they are not the same line.

The lines are parallel.

We will now see how to find the point of intersection of two lines. Suppose two lines l_1 and l_2 have vector equations $\boldsymbol{r}_1 = \boldsymbol{a}_1 + \lambda \boldsymbol{d}_1$ and $\boldsymbol{r}_2 = \boldsymbol{a}_2 + \mu \boldsymbol{d}_2$. If they intersect, then there is a point which lies on both lines. As the position vector of a general point on a line is given by $\boldsymbol{r}$, finding the intersection of l_1 and l_2 means finding values of λ and μ which make $\boldsymbol{r}_1 = \boldsymbol{r}_2$.



In a plane, two different straight lines either intersect or are parallel. However, in three dimensions it is possible to have lines which are not parallel but do not intersect either, like the red and blue lines in the diagram.

Such non-parallel, non-intersecting lines are called **skew lines**.

As we shall see, if l_1 and l_2 are skew lines, we will not be able to find values of λ and μ such that $\boldsymbol{r}_1 = \boldsymbol{r}_2$.

Worked example 11.18

Find the coordinates of the point of intersection of the following pairs of lines.

(a) $\boldsymbol{r}_1 = \begin{pmatrix} 0 \\ -4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\boldsymbol{r}_2 = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix}$

(b) $\boldsymbol{r}_1 = \begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$ and $\boldsymbol{r}_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$

Try to make $\boldsymbol{r}_1 = \boldsymbol{r}_2$.

(a)

$$\begin{pmatrix} 0 \\ -4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 0+\lambda \\ -4+2\lambda \\ 1+\lambda \end{pmatrix} = \begin{pmatrix} 1+4\mu \\ 3-2\mu \\ 5-2\mu \end{pmatrix}$$

For two vectors to be equal, all their components must be equal.

$$\Leftrightarrow \begin{cases} 0+\lambda = 1+4\mu \\ -4+2\lambda = 3-2\mu \\ 1+\lambda = 5-2\mu \end{cases}$$

$$\Leftrightarrow \begin{cases} \lambda - 4\mu = 1 & (1) \\ 2\lambda + 2\mu = 7 & (2) \\ \lambda + 2\mu = 4 & (3) \end{cases}$$

We know how to solve two simultaneous equations in two variables. Pick any two of the three equations. Let us use the first and third (because subtracting them eliminates λ).

$(3)-(1) \Leftrightarrow 6\mu = 3$

$\therefore \mu = \frac{1}{2}, \lambda = 3$

The values of λ and μ that we have found must also satisfy the remaining (i.e. second) equation. Check whether this is the case.

$(2): 2\times 3 + 2\times\frac{1}{2} = 7$

$\therefore$ the lines intersect

continued . . .

The position of the intersection point is given by $\mathbf{r}_1$ with the value of λ we found (or $\mathbf{r}_2$ with the value of μ we found – they should be the same).

$$\mathbf{r}_1 = \begin{pmatrix} 0 \\ -4 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$$

The lines intersect at the point (3, 2, 4).

Repeat the same procedure for the second pair of lines.

(b)

$$\begin{pmatrix} -4 \\ 3 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} t - 2\lambda = 6 & (1) \\ t + 3\lambda = -2 & (2) \\ 4t - 2\lambda = -2 & (3) \end{cases}$$

Solve for t and λ from the first two equations.

$$(2) - (1) \Rightarrow \lambda = -\frac{8}{5}, t = \frac{14}{5}$$

The values found should also satisfy the third equation.

$$(3): 4 \times \frac{14}{5} - 2 \times \left(-\frac{8}{5}\right) = \frac{72}{5} \neq -2$$

This tells us that it is impossible to find t and λ that make $\mathbf{r}_1 = \mathbf{r}_2$.

$\therefore$ the two lines do not intersect.

In vector problems you often need to find a point on a given line which satisfies certain conditions. We have already seen (in Worked example 11.15) how to use the position vector $\mathbf{r}$ of a general point on the line together with the given condition to write an equation for the parameter λ. In the next example we use more complicated conditions.

EXAM HINT

You may be able to use your calculator to solve simultaneous equations. See Calculator Skills sheet 6 on the CD-ROM for guidance on how to do this.

Worked example 11.19

Line l has equation $\boldsymbol{r} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, and point A has coordinates $(3, 9, -2)$.

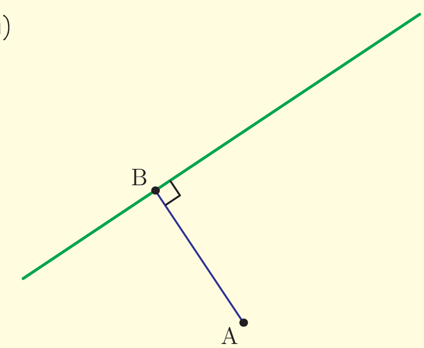
(a) Find the coordinates of point B on l such that (AB) is perpendicular to l.

(b) Hence find the shortest distance from A to l.

(c) Find the coordinates of the reflection of the point A in l.

Draw a diagram. The line (AB) should be perpendicular to the direction vector of l.

(a)



$$\overrightarrow{AB} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0$$

We know that B lies on l, so its position vector is given by $\boldsymbol{r}$.

$$\overrightarrow{OB} = \underline{r} = \begin{pmatrix} 3+\lambda \\ -1-\lambda \\ \lambda \end{pmatrix}$$

$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$

$$\therefore \overrightarrow{AB} = \begin{pmatrix} 3+\lambda \\ -1-\lambda \\ \lambda \end{pmatrix} - \begin{pmatrix} 3 \\ 9 \\ -2 \end{pmatrix} = \begin{pmatrix} \lambda \\ -10-\lambda \\ \lambda+2 \end{pmatrix}$$

We can now find the value of λ for which the two lines are perpendicular.

$$\begin{pmatrix} \lambda \\ -10-\lambda \\ \lambda+2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0$$

$$\Leftrightarrow (\lambda) + (10+\lambda) + (\lambda+2) = 0$$

$$\Leftrightarrow \lambda = -4$$

Substitute the value we found for λ in the equation of the line to get the position vector of B.

$$\therefore \underline{r} = \begin{pmatrix} -1 \\ 3 \\ -4 \end{pmatrix}$$

B has coordinates $(-1, 3, -4)$

continued . . .

The shortest distance from a point to a line is the perpendicular distance, that is, the distance AB.

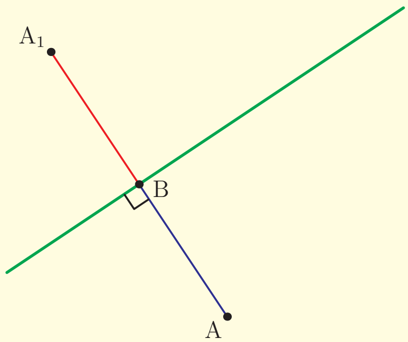
(b)

$$\overrightarrow{AB} = \begin{pmatrix} -1 \\ 3 \\ -4 \end{pmatrix} - \begin{pmatrix} 3 \\ 9 \\ -2 \end{pmatrix} = \begin{pmatrix} -4 \\ -6 \\ -2 \end{pmatrix}$$

$$\therefore \left|\overrightarrow{AB}\right| = \sqrt{16+36+4}$$
$$= 2\sqrt{14}$$

The reflection A_1 lies on the line (AB), with $BA_1 = AB$. As A_1 and A are on opposite sides of the line l, we have $\overrightarrow{BA_1} = \overrightarrow{AB}$ (draw a diagram to make this clear).

(c)



$$\overrightarrow{BA_1} = \overrightarrow{AB}$$
$$\underline{a}_1 - \underline{b} = \overrightarrow{AB}$$

$$\underline{a}_1 = \begin{pmatrix} -4 \\ -6 \\ -2 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \\ -4 \end{pmatrix}$$

So A_1 has coordinates $(-5, -3, -6)$.

Part (c) of the above example illustrates the power of vectors: as vectors contain both distance and direction information, just one equation ($\overrightarrow{BA_1} = \overrightarrow{AB}$) was needed to express both the fact that A_1 is on the line (AB) and that $BA_1 = AB$.

One of the common applications of vectors is in mechanics. You may encounter questions in which the velocity of a moving object is given as a vector, and you have to use the information to find positions. In such a situation, remember that the position $\boldsymbol{r}$ of the object can be expressed as a vector equation where the parameter represents time (and hence is usually denoted by t) and the direction vector is the velocity vector:

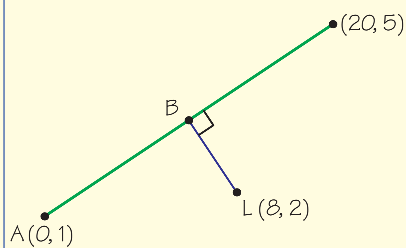
$$\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{v}$$

In this equation, $\boldsymbol{a}$ is the position vector of the object at time $t = 0$.

Worked example 11.20

A ship leaves a port at (0, 1) and moves in a straight line so that after 5 hours it has reached point (20, 5). A lighthouse is situated at point (8, 2). At what time is the ship closest to the lighthouse, and what is the distance between them at that time?

Sketch a diagram. Mark the start position A, the lighthouse L, and the point B at which the boat is closest to the lighthouse. The line (BL) should be perpendicular to (AB).



Write down the equation of the ship's path in terms of time t.

The start point has position vector $\boldsymbol{a}=\begin{pmatrix}0\\1\end{pmatrix}$, and the direction vector is the velocity.

In 5 hours, the ship has moved from (0, 1) to (20, 5), so its velocity is $v=\frac{1}{5}\begin{pmatrix}20\\4\end{pmatrix}=\begin{pmatrix}4\\0.8\end{pmatrix}$

At time t hours, the ship has position

$$\boldsymbol{r}=\begin{pmatrix}0\\1\end{pmatrix}+t\begin{pmatrix}4\\0.8\end{pmatrix}=\begin{pmatrix}4t\\1+0.8t\end{pmatrix}$$

The lighthouse has position vector

$$\underline{l}=\begin{pmatrix}8\\2\end{pmatrix}$$

When the ship is at B, $\boldsymbol{r}-\boldsymbol{l}$ is perpendicular to the path of the ship.

$$\boldsymbol{r}-\boldsymbol{l}=\begin{pmatrix}4t-8\\0.8t-1\end{pmatrix}$$

When the ship is at B,

$$(\boldsymbol{r}-\boldsymbol{l}).\begin{pmatrix}4\\0.8\end{pmatrix}=0$$

Now we can solve for the value of t at which the ship is at B.

$$\begin{pmatrix}4t-8\\0.8t-1\end{pmatrix}.\begin{pmatrix}4\\0.8\end{pmatrix}=0$$

$\Leftrightarrow 16t-32+0.64t-0.8=0$
$\Leftrightarrow 16.64t=32.8$
$\Leftrightarrow t=1.971$

The ship is closest to the lighthouse 1.97 hours (1 hour and 58 minutes) after leaving port.

continued . . .

Substitute this value for t in the equation of the line to find the position vector of B.

When $t = 1.971$, $\boldsymbol{r} - \boldsymbol{l} = \begin{pmatrix} -0.115 \\ 0.577 \end{pmatrix}$

So the distance from the boat to the lighthouse at that time is

$$\left|\begin{pmatrix} -0.115 \\ 0.577 \end{pmatrix}\right| = \sqrt{0.115^2 + 0.577^2} = 0.588$$

Exercise 11G

1. Find the acute angle between the following pairs of lines, giving your answer in degrees.

   (a) (i) $\boldsymbol{r} = \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$

   (ii) $\boldsymbol{r} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix}$

   (b) (i) $\boldsymbol{r} = (2\boldsymbol{i} + \boldsymbol{k}) - t\,\boldsymbol{i}$ and $\boldsymbol{r} = (\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k}) + 5\,(4\boldsymbol{i} + 2\boldsymbol{k})$

   (ii) $\boldsymbol{r} = (6\boldsymbol{i} + 6\boldsymbol{j} + 2\boldsymbol{k}) + t\,(-\boldsymbol{i} + 3\boldsymbol{k})$ and $\boldsymbol{r} = \boldsymbol{i} + 5\,(4\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})$

2. For each pair of lines state whether they are parallel, perpendicular, the same line, or none of the above.

   (a) $\boldsymbol{r} = \boldsymbol{k} + \lambda(2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k})$ and $\boldsymbol{r} = \boldsymbol{k} + \mu\,(2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k})$

   (b) $\boldsymbol{r} = (4\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}) + 5\,(-\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k})$ and $\boldsymbol{r} = (2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$
   $+\, \boldsymbol{t}\,(2\boldsymbol{i} - 4\boldsymbol{i} + 4\boldsymbol{k})$

(c) $\boldsymbol{r}=\begin{pmatrix}1\\5\\2\end{pmatrix}+\lambda\begin{pmatrix}3\\3\\1\end{pmatrix}$ and $\boldsymbol{r}=\begin{pmatrix}0\\0\\1\end{pmatrix}+t\begin{pmatrix}3\\3\\1\end{pmatrix}$

(d) $\boldsymbol{r}=\begin{pmatrix}2\\2\\1\end{pmatrix}+t\begin{pmatrix}1\\-1\\3\end{pmatrix}$ and $\boldsymbol{r}=\begin{pmatrix}5\\-1\\10\end{pmatrix}+s\begin{pmatrix}1\\-1\\3\end{pmatrix}$

3. Determine whether the following pairs of lines intersect; if they do, find the coordinates of the intersection point.

(a) (i) $\boldsymbol{r}=\begin{pmatrix}6\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\1\end{pmatrix}$ and $\boldsymbol{r}=\begin{pmatrix}2\\1\\-14\end{pmatrix}+\mu\begin{pmatrix}2\\-2\\3\end{pmatrix}$

(ii) $\boldsymbol{r}=\begin{pmatrix}4\\-1\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\-4\end{pmatrix}$ and $\boldsymbol{r}=\begin{pmatrix}6\\-2\\0\end{pmatrix}+\mu\begin{pmatrix}3\\-4\\0\end{pmatrix}$

(b) (i) $\boldsymbol{r}=(\boldsymbol{i}-2\boldsymbol{j}+3\boldsymbol{k})+t(-\boldsymbol{i}+\boldsymbol{j}+2\boldsymbol{k})$ and $\boldsymbol{r}=(-4\boldsymbol{i}-4\boldsymbol{j}-11\boldsymbol{k})+5(5\boldsymbol{i}+\boldsymbol{j}+2\boldsymbol{k})$

(ii) $\boldsymbol{r}=(4\boldsymbol{i}+2\boldsymbol{k})+t(2\boldsymbol{i}+\boldsymbol{k})$ and $\boldsymbol{r}=(-\boldsymbol{i}+2\boldsymbol{j}+3\boldsymbol{k})+5(\boldsymbol{i}-2\boldsymbol{i}-2\boldsymbol{j})$

4. Line l has equation $\boldsymbol{r}=\begin{pmatrix}4\\2\\-1\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\2\end{pmatrix}$, and point P has coordinates (7,2,3). Point C lies on l and [PC] is perpendicular to l. Find the coordinates of C. *[6 marks]*

5. Find the shortest distance from the point $(-1,1,2)$ to the line with equation $\boldsymbol{r}=(\boldsymbol{i}+2\boldsymbol{k})+t(-3\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k})$ *[6 marks]*

6. Two lines are given by $l_1 : \boldsymbol{r} = \begin{pmatrix} -5 \\ 1 \\ 10 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}$ and

$l_2 : \boldsymbol{r} = \begin{pmatrix} 3 \\ 0 \\ -9 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 7 \end{pmatrix}$.

(a) l_1 and l_2 intersect at P. Find the coordinates of P.

(b) Show that the point Q(5, 2, 5) lies on l_2.

(c) Find the coordinates of the point M on l_1 such that [QM] is perpendicular to l_1.

(d) Find the area of the triangle PQM. *[10 marks]*

7. Find the distance of the line with equation $\boldsymbol{r} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$

from the origin. *[7 marks]*

8. Two lines $l_1 : \boldsymbol{r} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$ and $l_2 : \boldsymbol{r} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$

intersect at point P.

(a) Find the coordinates of P.

(b) Find, in degrees, the acute angle between the two lines.

Point Q has coordinates $(-1,5,10)$.

(c) Show that Q lies on l_2.

(d) Find the distance PQ

(e) Hence find the shortest distance from Q to the line l_1. *[12 marks]*

9. Consider the line $\boldsymbol{r} = (5\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}) + \lambda\,(2\boldsymbol{i} - 3\boldsymbol{j} + 3\boldsymbol{k})$ and the point P(21, 5, 10).

(a) Find the coordinates of point M on l such that [PM] is perpendicular to l.

(b) Show that the point $Q(15,-14,17)$ lies on l.

(c) Find the coordinates of point R on l such that $PR = PQ$. *[10 marks]*

10. Two lines have equations $l_1 : \boldsymbol{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ and

$l_2 : \boldsymbol{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and intersect at point P.

(a) Show that Q(5, 2, 6) lies on l_2.

(b) R is a point on l_1 such that PR = PQ. Find the possible coordinates of R. *[8 marks]*

Summary

- A vector represents the displacement of one point from another.
- The displacement of a point from the origin is the point's **position vector**. The displacement between points with position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is $\boldsymbol{b} - \boldsymbol{a}$; the midpoint between them has position vector $\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})$.
- Vectors can be expressed in terms of **base vectors** $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ or as column vectors using **components**.
- The vector algebra operations of addition, subtraction and scalar multiplication can be carried out component by component, but it is also important to understand the geometric interpretation of these operations. When solving problems using vectors, drawing diagrams helps us see what calculations we need to do. Three-dimensional situations can be represented by two-dimensional diagrams, which do not have to be accurate to be useful.
- The **magnitude** of a vector can be calculated from the components of the vector:
$$|\boldsymbol{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$
- The distance between points with position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is given by $|\boldsymbol{b} - \boldsymbol{a}|$.
- The unit vector in the direction of $\boldsymbol{a}$ is $|\hat{\boldsymbol{a}}| = \frac{1}{|\boldsymbol{a}|}\boldsymbol{a}$.
- The angle θ between the directions of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is given by $\cos\theta = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$ where $\boldsymbol{a}\cdot\boldsymbol{b}$ is the **scalar product**, defined in terms of the components as
$$\boldsymbol{a}\cdot\boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3$$
- For perpendicular vectors, $\boldsymbol{a}\cdot\boldsymbol{b} = 0$.
- For parallel vectors, $\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|$ and $\boldsymbol{a} = t\boldsymbol{b}$ for some non-zero scalar t.
- A **vector equation** of a line gives the position vectors of points on the line. It is of the form
$$\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{d}$$

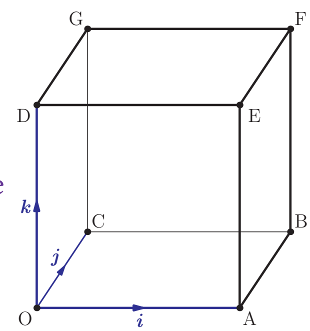
where $\boldsymbol{d}$ is a vector in the direction of the line, $\boldsymbol{a}$ is the position vector of one point on the line, and λ is a parameter whose values correspond to different points on the line.

- The angle between two lines is the angle between their direction vectors.
- Two lines with direction vectors $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ are
  - parallel if $\mathbf{d}_1 = k\mathbf{d}_2$
  - perpendicular if $\boldsymbol{d}_1 \cdot \boldsymbol{d}_2 = 0$
- To find the intersection of two lines with three-dimensional vector equations $\boldsymbol{r}_1 = \boldsymbol{a}_1 + \lambda\boldsymbol{d}_1$ and $\boldsymbol{r}_2 = \boldsymbol{a}_2 + \mu\boldsymbol{d}_2$, set the position vectors equal to each other: $\boldsymbol{a}_1 + \lambda\boldsymbol{d}_1 = \boldsymbol{a}_2 + \mu\boldsymbol{d}_2$. This gives three equations (one for each component); solve two of the them to find λ and μ. If these values also satisfy the remaining equation, they give the point of intersection of the two lines; if not, the lines are **skew**.

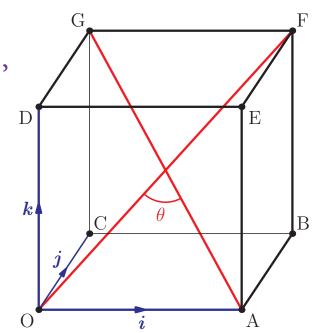
Introductory problem revisited

What is the angle between the diagonals of a cube?

This problem can be solved by applying the cosine rule to one of the triangles made by the diagonals and one side. However, using vectors gives a slightly faster solution, as we do not have to find the lengths of the sides of the triangle.



The angle between two lines can be found from the direction vectors of the lines and the formula involving the scalar product. We do not know the actual positions of the vertices of the cube, or even the length of its sides. But the angle between the diagonals does not depend on the size of the cube, so we can, for simplicity, look at the cube with side length 1 that has one vertex at the origin and sides parallel to the base vectors.



We want to find the angle between the diagonals OF and AG, so we need the coordinates of those four vertices; they are O(0, 0, 0), A(1, 0, 0), F(1, 1, 1) and G(0, 1, 1). The required angle θ is between the lines OF and AG. The corresponding vectors are

$$\overrightarrow{\mathrm{OF}} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ and } \overrightarrow{\mathrm{AG}} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

Now we can use the formula:

$$\cos\theta = \frac{\overrightarrow{\mathrm{OF}} \cdot \overrightarrow{\mathrm{AG}}}{\left|\overrightarrow{\mathrm{OF}}\right|\left|\overrightarrow{\mathrm{AG}}\right|}$$

$$= \frac{-1+1+1}{\sqrt{3}\sqrt{3}} = \frac{1}{3}$$

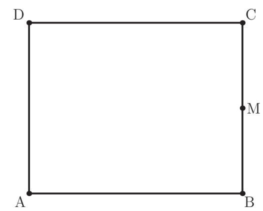
$$\therefore \theta = 70.5°$$

Mixed examination practice 11

Short questions

1. Find a vector equation of the line passing through points $(3,-1,1)$ and $(6,0,1)$. *[4 marks]*

2. The diagram shows a rectangle ABCD. M is the midpoint of [BC].

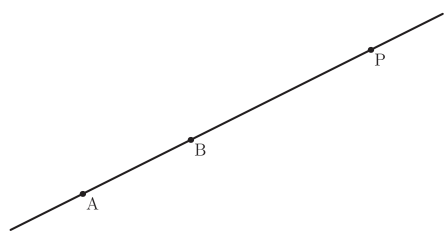


(a) Express $\overrightarrow{MD}$ in terms of $\overrightarrow{AB}$ and $\overrightarrow{AD}$.

(b) Given that $AB = 6$ and $AD = 4$, show that $\overrightarrow{MD} \cdot \overrightarrow{MC} = 4$. *[5 marks]*

3. Points $A(-1,1,2)$ and $B(3,5,4)$ lie on the line with equation

$$\boldsymbol{r} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}.$$ Find the coordinates



of point P on the same line such that $AP = 3\ AB$, as shown in the diagram. *[5 marks]*

4. Point $A(-3,0,4)$ lies on the line with equation $\boldsymbol{r} = -3\boldsymbol{i} + 4\boldsymbol{k} + \lambda(2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})$. Find the coordinates of one point on the line which is 10 units from A. *[6 marks]*

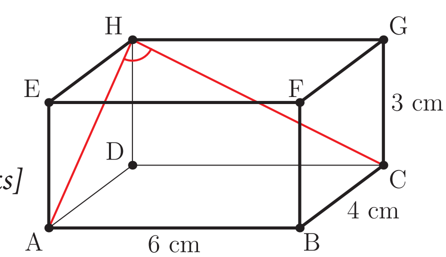
5. Points $A(4,1,12)$ and $B(8,-11,20)$ lie on the line l.

(a) Find an equation for line l, giving your answer in vector form.

(b) The point P is on l such that $\overrightarrow{OP}$ is perpendicular to l. Find the coordinates of P. *[6 marks]*

6. The rectangular box shown in the diagram has dimensions $6\text{cm} \times 5\text{cm} \times 3\text{cm}$.

Find, correct to the nearest one-tenth of a degree, the size of the angle $A\hat{H}C$. *[6 marks]*



7. Let α be the angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, where $\boldsymbol{a} = (\cos\theta)\boldsymbol{i} + (\sin\theta)\boldsymbol{j}$ and $\boldsymbol{b} = (\sin\theta)\boldsymbol{i} + (\cos\theta)\boldsymbol{j}$, with $0 < \theta < \pi/4$. Express α in terms of θ.

[6 marks]

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8. Given two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $|\boldsymbol{a}+\boldsymbol{b}| = |\boldsymbol{a}-\boldsymbol{b}|$, find the value of $\boldsymbol{a}\cdot\boldsymbol{b}$.

[6 marks]

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9. (a) Show that $(\boldsymbol{b}-\boldsymbol{a})\cdot(\boldsymbol{b}-\boldsymbol{a}) = |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2\boldsymbol{a}\cdot\boldsymbol{b}$.

(b) In triangle MNP, $\mathrm{M\hat{P}N} = \theta$. Let $\overrightarrow{\mathrm{PM}} = \boldsymbol{a}$ and $\overrightarrow{\mathrm{PN}} = \boldsymbol{b}$. Use the result from part (a) to derive the cosine rule: $\mathrm{MN}^2 = \mathrm{PM}^2 + \mathrm{PN}^2 - 2\mathrm{PM} \times \mathrm{PN}\cos\theta$.

[6 marks]

Long questions

1. Points A, B and D have coordinates $(1,1,7)$, $(-1,6,3)$ and $(3,1,k)$, respectively. (AD) is perpendicular to (AB).

(a) Write down, in terms of k, the vector $\overrightarrow{\mathrm{AD}}$.

(b) Show that $k = 6$.

Point C is such that $\overrightarrow{\mathrm{BC}} = 2\overrightarrow{\mathrm{AD}}$.

(c) Find the coordinates of C.

(d) Find the exact value of $\cos(\mathrm{A\hat{D}C})$. *[10 marks]*

2. Points A and B have coordinates $(4,1,2)$ and $(0,5,1)$. Line l_1 passes through A and has equation $\boldsymbol{r}_1 = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$. Line l_2 passes through B and has equation $\boldsymbol{r}_2 = \begin{pmatrix} 0 \\ 5 \\ 1 \end{pmatrix} + t\begin{pmatrix} 4 \\ -4 \\ 1 \end{pmatrix}$.

(a) Show that the line l_2 also passes through A.

(b) Calculate the distance AB.

(c) Find the angle between l_1 and l_2 in degrees.

(d) Hence find the shortest distance from A to l_1. *[10 marks]*

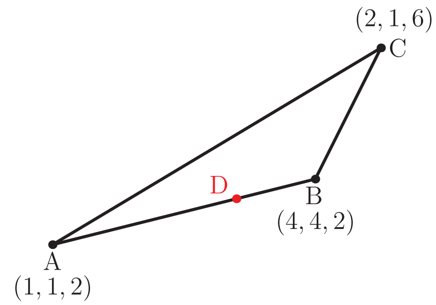
3. A triangle has vertices A(1,1,2), B(4,4,2) and C(2,1,6). Point D lies on the side [AB] and AD : DB = 1 : k.

(a) Find $\overrightarrow{CD}$ in terms of k.

(b) Find the value of k such that [CD] is perpendicular to [AB].

(c) For the above value of k, find the coordinates of D.

(d) Hence find the length of the perpendicular line from vertex C which passes through [AB]. *[10 marks]*



4. Point P lies on the parabola $y = x^2$ and has x-coordinate a ($a > 0$).

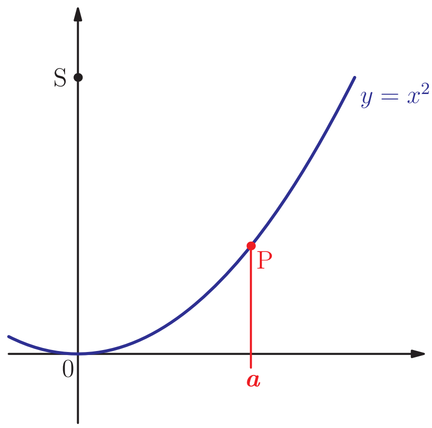
(a) Write down, in terms of a, the coordinates of P.

Point S has coordinates (0, 4).

(b) Write down the vectors $\overrightarrow{PO}$ and $\overrightarrow{PS}$.

(c) Use the scalar product to find the value of a for which [OP] is perpendicular to [PS].

(d) For the value of a found above, calculate the exact area of the triangle OPS. *[10 marks]*



5. *In this question, the base vectors* $\boldsymbol{i}$ *and* $\boldsymbol{j}$ *point due east and due north, respectively.*

A port is located at the origin. One ship starts from the port and moves with velocity $v_1 = (3\boldsymbol{i} + 4\boldsymbol{j})$ km/h.

(a) Write down the ship's position vector at time t hours after leaving port.

A second ship starts at the same time from 18 km north of the port and moves with velocity $v_2 = (3\boldsymbol{i} - 5\boldsymbol{j})$ km/h.

(b) Write down the position vector of the second ship at time t hours.

(c) Show that after half an hour, the distance between the two ships is 13.5 km.

(d) Show that the ships meet, and find the time at which this happens.

(e) How long after the ships meet are they 18 km apart? *[12 marks]*

6. At time $t = 0$ two aircraft have position vectors $5\boldsymbol{j}$ and $7\boldsymbol{k}$. The first moves with velocity $3\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}$ and the second with velocity $5\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$.

(a) Write down the position vector of the first aircraft at time t.

(b) Show that at time t, the distance d between the two aircraft is given by $d^2 = 44t^2 - 88t + 74$.

(c) Show that the two aircraft will not collide.

(d) Find the minimum distance between the two aircraft. *[12 marks]*

7. (a) Show that the lines $l_1 : \boldsymbol{r} = \begin{pmatrix} -3 \\ 3 \\ 18 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -8 \end{pmatrix}$ and $l_2 : \boldsymbol{r} = \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ do not intersect.

(b) Points P and Q lie on l_1 and l_2, respectively. (PQ) is perpendicular to both lines.

(i) Write down $\overrightarrow{PQ}$ in terms of λ and μ.

(ii) Show that $9\mu - 69\lambda + 147 = 0$.

(iii) Find a second equation for λ and μ.

(iv) Find the coordinates of P and the coordinates of Q.

(v) Hence find the shortest distance between l_1 and l_2. *[14 marks]*

8. (a) Find the vector equation of the line L through point $A(-2,4,2)$ parallel to the vector $\boldsymbol{l} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

(b) Point B has coordinates (2,3,3). Find the cosine of the angle between (AB) and the line L.

(c) Calculate the distance AB.

(d) Point C lies on L and [BC] is perpendicular to L. Find the exact distance AC. *[10 marks]*