In this chapter you will learn:

- how to solve equations involving trigonometric functions
- about relationships between different trigonometric functions, called identities
- how to use identities to solve more complicated trigonometric equations
- about relationships between trigonometric functions of an angle and trigonometric functions of twice that angle.

9 Trigonometric equations and identities

Introductory problem

The original Ferris Wheel was constructed in 1893 in Chicago. It was just over 80 m tall and could complete one full revolution in 9 minutes. During each revolution, how much time did the passengers spend more than 50 m above the ground?

Often, when using trigonometric functions to model real-life situations we need to solve equations where the unknown is in the argument of a trigonometric function; for example, $5\sin(2x+1)=3$ or $\cos 2x-\sin^2 x=-2$. Because trigonometric functions are periodic, such equations may have more than one solution. In this chapter you will see how to find all solutions in a given interval. You will also learn some trigonometric identities – relationships between different trigonometric functions – which can be very useful in transforming more complicated equations into simpler ones.

9A Introducing trigonometric equations

To solve trigonometric equations it is important that we can 'undo' trigonometric functions. If you were told that the sine of a value is $\frac{1}{2}$, you would know from section 8D that the original value could be $\frac{\pi}{6}$, but if you were told that the sine of a value is 0.8 the original value would not be so easy to find. The answer is given by the inverse function of sine, written as $\arcsin x$ or $\sin^{-1} x$.

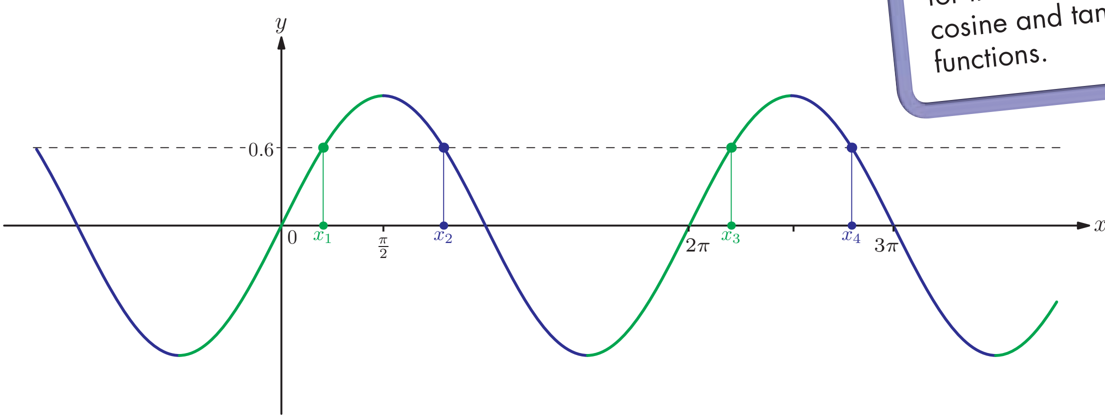
Inverse functions and their graphs were covered in section 4D.

The inverse function of cosine is denoted by $\arccos x$ or $\cos^{-1} x$, and the inverse tangent function is $\arctan x$ or $\tan^{-1} x$.

Suppose we want to find the values of x which satisfy $\sin x = 0.6$. Applying the inverse sine function, we get $x = \sin^{-1} 0.6$; then, using a calculator we find $\sin^{-1} 0.6 = 0.644$ (3 SF). The $\sin^{-1}$ function gives us only one solution. However, from the graph of $y = \sin x$ we can see that there are many x-values that satisfy the equation (these correspond to the intersections of the curve $y = \sin x$ and the line $y = 0.6$).

EXAM HINT

Calculators usually do not have a button labelled arcsin. Use the $\sin^{-1}$ button, which is usually obtained by pressing SHIFT and sin; similarly for the inverse cosine and tangent functions.



The solutions come in pairs – one in the green sections of the graph and one in the blue sections. The $\sin^{-1}$ function will always give us only one solution: the one in the green section closest to zero (x_1). To find the solution x_2 in the blue section we use the fact that the graph has a line of symmetry at $x = \frac{\pi}{2}$, so x_2 is as far below π as x_1 is above zero; in our example, this means that $x_2 = \pi - 0.644 = 2.50$. Once we have this pair of solutions, we can use the fact that the sine graph repeats with period 2π to find the other solutions: $x_3 = x_1 + 2\pi = 6.93$, $x_4 = x_2 + 2\pi = 8.78$, and so on.

EXAM HINT

If you have found a value using an inverse trigonometric function and need to do further calculations with it, save the value in your calculator's memory and always use the stored value or the ANS button in subsequent calculations – not the rounded answer. See Calculator Skills sheet 1 on the CD-ROM for how to do this.

KEY POINT 9.1

To find the possible values of x which satisfy $\sin x = a$:

- Use the calculator to find $x_1 = \sin^{-1} a$.
- A second solution is given by $x_2 = \pi - x_1$ (or $180° - x_1$ if working in degrees).
- Other solutions are found by adding (or subtracting) multiples of 2π (or $360°$) to x_1 or x_2.

In the International Baccalaureate you will only be asked to find solutions in a given interval.

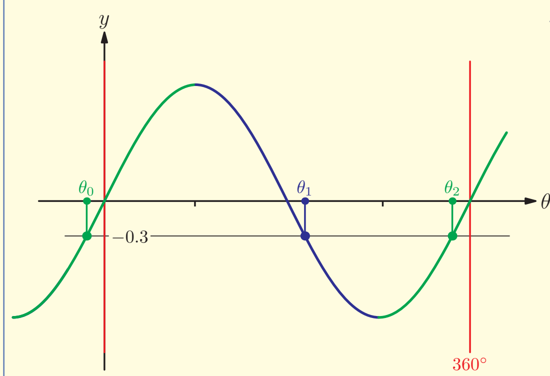
Worked example 9.1

Find the possible values of angle $\theta \in [0°, 360°]$ for which $\sin\theta = -0.3$.

Put the calculator in degree mode.

$\sin^{-1}(-0.3) = -17.5°$

Look at the graph to see how many solutions there are in the required interval.



There are two solutions.

The second solution is given by $180° - \theta$.

$180° - (-17.5°) = 197.5°$

The solution $-17.5°$ is not in the required interval, so add $360°$.

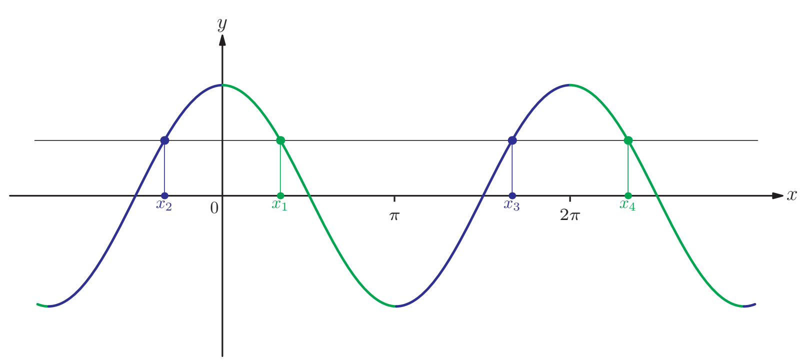
$\theta_1 = 197.5°$

$\theta_2 = -17.5° + 360° = 342.5°$

EXAM HINT

Always make sure that your calculator is in the appropriate mode, degree or radian, as indicated in the question.
See Calculator Skills sheet 1 on the CD-ROM for how to set the mode.

We can solve the equation $\cos x = k$ in a similar way.



The function $\cos^{-1} k$ gives the solution x_1 in the green region closest to zero. We use the symmetry of the cosine graph to find x_2: it is simply the negative of x_1. Once we have this pair of solutions, we can use the fact that the cosine graph repeats with period 2π to find the other solutions.

KEY POINT 9.2

To find the possible values of x which satisfy $\cos x = a$:

- Use the calculator to find $x_1 = \cos^{-1} a$.
- A second solution is given by $x_2 = -x_1$.
- Other solutions are found by adding (or subtracting) multiples of 2π (or 360°) to x_1 or x_2.

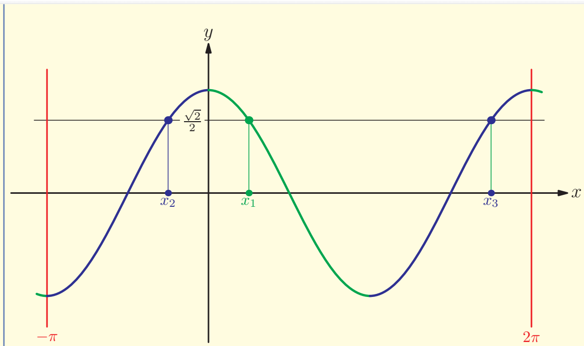
EXAM HINT

It is useful to remember that the first two positive solutions will be $\cos^{-1} a$ and $2\pi - \cos^{-1} a$.

Worked example 9.2

Find the values of x between $-\pi$ and 2π for which $\cos x = \frac{\sqrt{2}}{2}$.

Sketch the graph.



Note how many solutions there are.

There are 3 solutions

$\cos^{-1}\frac{\sqrt{2}}{2}$ is a value that we should know.

$$x_1 = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

Use the symmetry of the graph to find the other solutions.

$$x_2 = -\frac{\pi}{4}$$

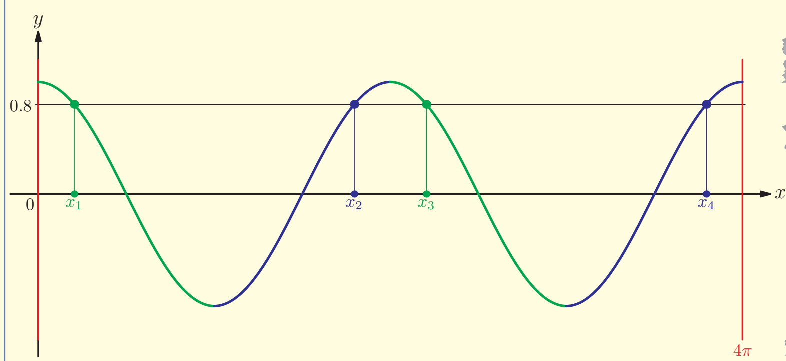
$$x_3 = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$$

It can be difficult to know how many times to add or subtract 2π to make sure that we have found all the solutions in a given interval. Sketching a graph can help, as we can then see how many solutions we are looking for and approximately where they are. A good rule of thumb is that, apart from maximum and minimum values, there are two solutions within each period for sin and cos.

Worked example 9.3

Find all the values of x between 0 and 4π for which $\cos x = 0.8$.

Sketch the graph.



Note how many solutions there are.

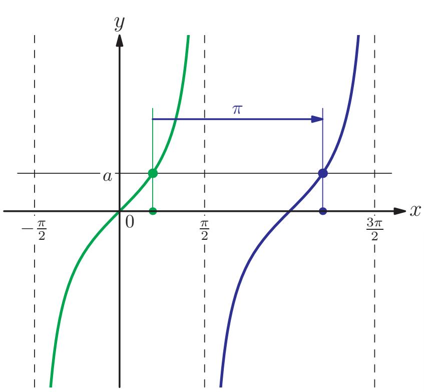
4 solutions

$\cos^{-1}$ on the calculator will give the first value of x.

$x_1 = \cos^{-1} 0.8 = 0.644$ (3 SF)

Use the symmetry of the graph to find the other values.

$x_2 = 2\pi - 0.644 = 5.64$ (3SF)
$x_3 = x_1 + 2\pi = 6.93$ (3SF)
$x_4 = x_2 + 2\pi = 11.9$ (3SF)



The procedure for solving equations of the type $\tan x = a$ is slightly different because the tangent function has period π rather than 2π. It is best understood by looking at the graph of the tangent function.

KEY POINT 9.3

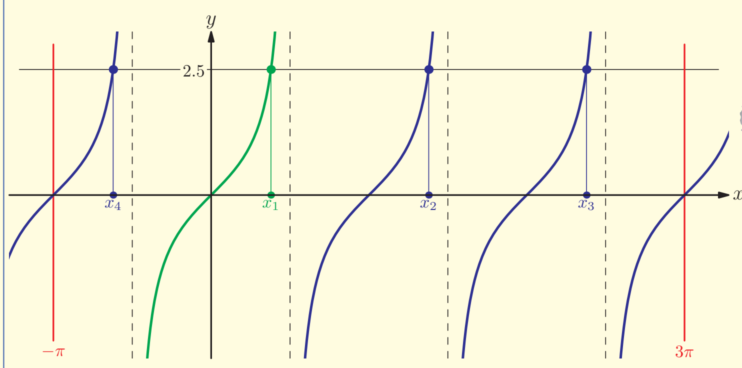
To find the possible values of x which satisfy $\tan x = a$:

- Use the calculator to find $x_1 = \tan^{-1} a$.
- Other solutions are found by adding (or subtracting) multiples of π (or 180°).

Worked example 9.4

Find all real values of $x \in [-\pi, 3\pi]$ such that $\tan x = 2.5$.

Sketch the graph.



Note the number of solutions.

There are four solutions.

Use a calculator to find $\tan^{-1}$.

$x_1 = \tan^{-1} 2.5 = 1.19$

The other solutions are found by adding or subtracting π.

$x_2 = x_1 + \pi = 4.33$
$x_3 = x_2 + \pi = 7.47$
$x_4 = x_1 - \pi = -1.95$

Exercise 9A

1. Use your calculator to evaluate the following in radians, correct to three significant figures.

(a) (i) $\cos^{-1} 0.6$ (ii) $\sin^{-1}(0.2)$

(b) (i) $\tan^{-1}(-3)$ (ii) $\sin^{-1}(-0.8)$

2. Evaluate the following in radians without using a calculator.

(a) (i) $\sin^{-1}\left(\frac{1}{2}\right)$ (ii) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

(b) (i) $\tan^{-1}(-\sqrt{3})$ (ii) $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

(c) (i) $\sin^{-1}(-1)$ (ii) $\tan^{-1}(1)$

3. Evaluate the following, giving your answer in degrees correct to one decimal place.

(a) (i) $\sin^{-1}0.7$ (ii) $\sin^{-1}0.3$

(b) (i) $\cos^{-1}(-0.62)$ (ii) $\cos^{-1}(-0.75)$

(c) (i) $\tan^{-1}6.4$ (ii) $\tan^{-1}(-7.1)$

4. Find the value of

(a) (i) $\sin(\sin^{-1}0.6)$ (ii) $\cos\left(\cos^{-1}(-0.3)\right)$

(b) (i) $\tan\left(\tan^{-1}(-2)\right)$ (ii) $\sin(\sin^{-1}(-1))$

5. Find the exact values of x between $0°$ and $360°$ which satisfy the following equations.

(a) (i) $\sin x = \frac{1}{2}$ (ii) $\sin x = \frac{\sqrt{2}}{2}$

(b) (i) $\cos x = \frac{1}{2}$ (ii) $\cos x = \frac{\sqrt{3}}{2}$

(c) (i) $\sin x = -\frac{\sqrt{3}}{2}$ (ii) $\sin x = -\frac{1}{2}$

(d) (i) $\tan x = 1$ (ii) $\tan x = \sqrt{3}$

6. Find the exact values of x between 0 and 2π which satisfy the following equations.

(a) (i) $\cos x = \frac{\sqrt{3}}{2}$ (ii) $\cos x = \frac{\sqrt{2}}{2}$

(b) (i) $\cos x = -\frac{1}{2}$ (ii) $\cos x = -\frac{\sqrt{3}}{2}$

(c) (i) $\sin x = \frac{\sqrt{2}}{2}$ (ii) $\sin x = \frac{\sqrt{3}}{2}$

(d) (i) $\tan x = \frac{1}{\sqrt{3}}$ (ii) $\tan x = -1$

7. Solve these equations in the given interval, giving your answers to one decimal place.

(a) (i) $\sin x = 0.45$ for $x \in [0°, 360°]$

(ii) $\sin x = 0.7$ for $x \in [0°, 360°]$

(b) (i) $\cos x = -0.75$ for $-180° \le x \le 180°$

(ii) $\cos x = -0.2$ for $-180° \le x \le 180°$

(c) (i) $\tan\theta = \frac{1}{3}$ for $0° \le \theta \le 720°$

(ii) $\tan\theta = \frac{4}{3}$ for $0° \le \theta \le 720°$

(d) (i) $\sin t = -\frac{2}{3}$ for $t \in [-180°, 360°]$

(ii) $\sin t = -\frac{1}{4}$ for $t \in [-180°, 360°]$

8. Solve these equations in the given interval, giving your answers to three significant figures.

(a) (i) $\cos t = \frac{4}{5}$ for $t \in [0, 4\pi]$

(ii) $\cos t = \frac{2}{3}$ for $t \in [0, 4\pi]$

(b) (i) $\sin\theta = -0.8$ for $\theta \in [-2\pi, 2\pi]$

(i) $\sin\theta = -0.35$ for $\theta \in [-2\pi, 2\pi]$

(c) (i) $\tan\theta = -\frac{2}{3}$ for $-\pi \le \theta \le \pi$

(ii) $\tan\theta = -3$ for $-\pi \le \theta \le \pi$

(d) (i) $\cos\theta = 1$ for $\theta \in [0, 4\pi]$

(ii) $\cos\theta = 0$ for $\theta \in [0, 4\pi]$

9. Solve the following equations in the given interval, giving exact answers.

(a) (i) $\sin x = \frac{1}{2}$ for $-360° \le x \le 360°$

(ii) $\sin x = \frac{\sqrt{2}}{2}$ for $-360° \le x \le 360°$

(b) (i) $\cos x = -1$ for $-180° \le x \le 180°$

(ii) $\sin x = -1$ for $-180° \le x \le 180°$

(c) (i) $\tan x = \sqrt{3}$ for $-360° < x < 0°$

(ii) $\tan x = 1$ for $-360° < x < 0°$

(d) (i) $\cos x = -\frac{\sqrt{2}}{2}$ for $-360° \le x \le 360°$

(ii) $\cos x = -\frac{\sqrt{3}}{2}$ for $-360° \le x \le 360°$

10. Find the exact solutions of the following equations.

(a) (i) $\cos\theta = \frac{1}{2}$ for $-2\pi < \theta < 2\pi$

(ii) $\cos\theta = \frac{\sqrt{3}}{2}$ for $-2\pi < \theta < 2\pi$

(b) (i) $\sin\theta = -\frac{\sqrt{3}}{2}$ for $-\pi < \theta < 3\pi$

(ii) $\sin\theta = -\frac{\sqrt{2}}{2}$ for $-\pi < \theta < 3\pi$

(c) (i) $\tan\theta = -\frac{1}{\sqrt{3}}$ for $-\pi < \theta < \pi$

(ii) $\tan\theta = -1$ for $-\pi < \theta < \pi$

(d) (i) $\cos\theta = 0$ for $0 < \theta < 3\pi$

(ii) $\sin\theta = 0$ for $0 < \theta < 3\pi$

(e) $\sin\theta = \frac{1}{\sqrt{2}}$ for $-2\pi < \theta < 0$

11. Solve the following equations:

(a) (i) $2\sin\theta + 1 = 1.2$ for $0° < \theta < 360°$

(ii) $4\sin x + 3 = 2$ for $-90° < x < 270°$

(b) (i) $3\cos x - 1 = \frac{1}{3}$ for $0 < x < 2\pi$

(ii) $5\cos x + 2 = 4.7$ for $0 < x < 2\pi$

(c) (i) $3\tan t - 1 = 4$ for $-\pi < t < \pi$

(ii) $5\tan t - 3 = 8$ for $0 < t < 2\pi$

12. Find the exact values of $x \in (-\pi, \pi)$ for which $2\sin x + 1 = 0$. *[5 marks]*

13 Show by a counterexample that $\tan^{-1} x \neq \frac{\sin^{-1} x}{\cos^{-1} x}$.

9B Harder trigonometric equations

In this section we shall look at two kinds of trigonometric equations that are more difficult to deal with: equations that need to be rearranged first and equations in which the argument of the trigonometric function is more complicated.

The previous section showed how to solve equations of the form 'trigonometric function = constant'. It is not always obvious how to write an equation in this form. There are three tactics which are often used:

- look for disguised quadratics
- take everything over to one side and factorise
- use trigonometric identities.

See chapter 3 for a reminder on disguised quadratics and solving equations by factorising.

Section 9D covers how to use identities to solve trigonometric equations.

Worked example 9.5

Solve the equation $\cos^2\theta = \frac{4}{9}$ for $\theta \in [0°, 360°]$.

Give answers correct to one decimal place.

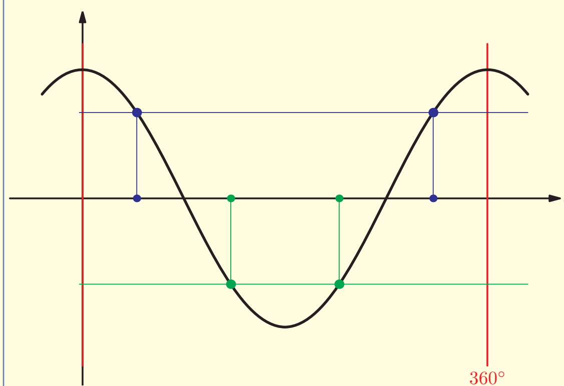
EXAM HINT

$\cos^2\theta$ means $(\cos\theta)^2$.

First, find the possible values of $\cos\theta$. Remember $\pm$ signs when taking the square root.

$\cos^2\theta = \frac{4}{9} \Rightarrow \cos\theta = \pm\frac{2}{3}$

Sketch the graph to see how many solutions there are in the required interval.



2 solutions for each of $\pm\frac{2}{3}$.

Solve each equation separately.

When $\cos\theta = \frac{2}{3}$:

$\cos^{-1}\left(\frac{2}{3}\right) = 48.2°$

$\theta = 48.2°$ or $360° - 48.2° = 311.8°$

When $\cos\theta = -\frac{2}{3}$:

$\cos^{-1}\left(-\frac{2}{3}\right) = 131.8°$

$\theta = 131.8°$ or $360° - 131.8° = 228.2°$

List all the solutions.

$\theta_1 = 48.2°$
$\theta_2 = 131.8°$
$\theta_3 = 228.2°$
$\theta_4 = 311.8°$

In the next example we need to use factorising.

Worked example 9.6

Solve the equation $3\sin x\cos x = 2\sin x$ for $-\pi \le x \le \pi$.

The equation is not in the form 'trig function = constant', so we cannot take the inverse directly. However, both sides have a factor of $\sin x$, so we can rearrange the equation so that the RHS is zero and then factorise the LHS.

$3\sin x\cos x - 2\sin x = 0$

$\Leftrightarrow \sin x(3\cos x - 2) = 0$

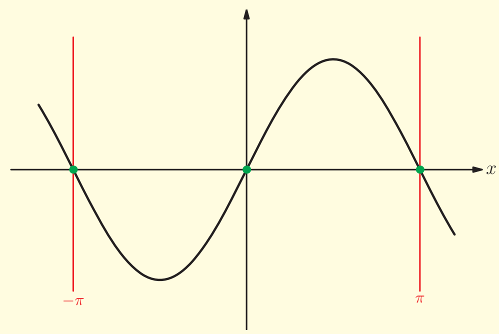
If a product is equal to 0, then one of the factors must be 0.

$\sin x = 0$ or $\cos x = \frac{2}{3}$

Now solve each equation separately. Sketch the graph for each equation to see how many solutions there are.

When $\sin x = 0$:

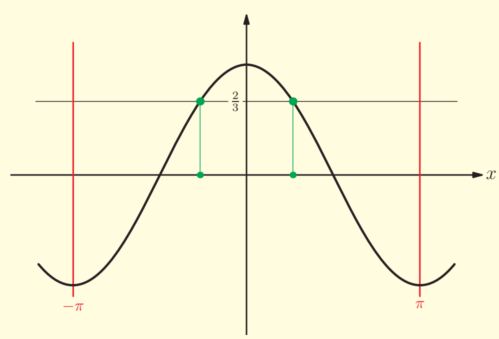
$\sin^{-1} 0 = 0$



$x = 0$ or $\pi - 0 = \pi$ or $0 - \pi = -\pi$

When $\cos x = \frac{2}{3}$:

$\cos^{-1}\left(\frac{2}{3}\right) = 0.841$



$2\pi - 0.841 = 5.44$ is not in the interval,

but $5.44 - 2\pi = -0.841$ is in the interval.

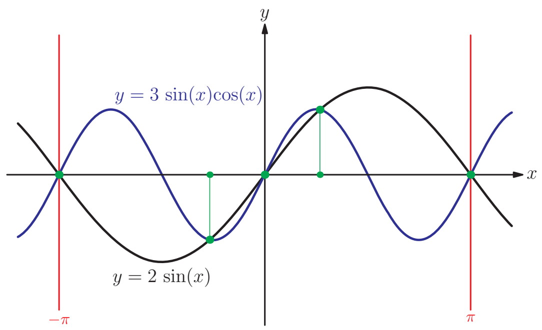
We have found five solutions.

$x = -\pi, -0.841, 0, 0.841, \pi$

EXAM HINT

Do not be tempted to divide both sides of the original equation by sinx as you would lose some solutions (the ones coming from $\sin x = 0$).

Check these solutions by looking at graphs on your calculator. If you plot the graphs of $y = 3\sin x\cos x$ and $y = 2\sin x$ on the same set of axes over the given interval, you will see that they intersect at five points.



Another common type of trigonometric equation is a disguised quadratic.

Worked example 9.7

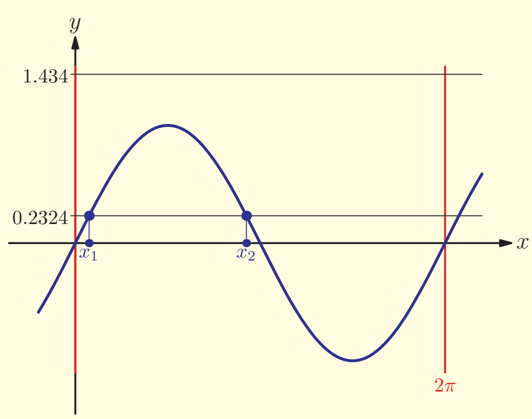
(a) Given that $3\sin^2 x - 5\sin x + 1 = 0$, find the possible values of $\sin x$.
(b) Hence solve the equation $3\sin^2 x - 5\sin x + 1 = 0$ for $0 < x < 2\pi$.

Recognise that this is a quadratic equation in $\sin x$. Since we cannot factorise it, use the quadratic formula.

(a) $\sin x = \dfrac{5 \pm \sqrt{5^2 - 4\times3\times1}}{2\times3}$

$\sin x = 1.434$ *or* 0.2324

Sketch the graph of $\sin x$ to see how many solutions there are to $\sin x = 1.434$ and $\sin x = 0.2324$.



There are two solutions in total.

$\sin x$ is always between -1 and 1, so only $\sin x = 0.2324$ has solutions.

$\sin x = 1.434 > 1$ is impossible.
Hence $\sin x = 0.2324$

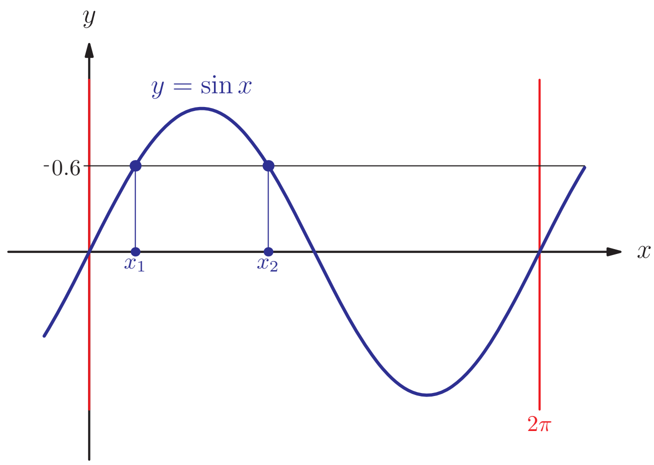
Solve this equation as before.

(b) $\sin^{-1}(0.2324) = 0.235$
$x = 0.235$ *or* $\pi - 0.235 = 2.91$

EXAM HINT

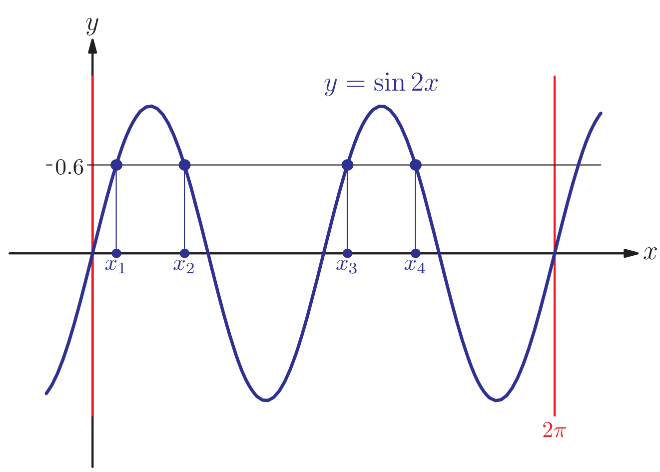
Questions like the one in Worked example 9.7 force you to use algebraic methods to solve the equation, but you should still use a graph on your calculator to check your solution. If the question did not include part (a), then you could solve the equation directly using your GDC, without any algebraic working.

Next, we look at equations where the argument of the trigonometric function is more complicated than just x. For the equation $\sin x = 0.6$ with $0 \le x \le 2\pi$, we can see from the graph that there are two solutions.



Now consider the equation $\sin 2x = 0.6$ for $0 \le x \le 2\pi$. From the graph we can see that there are four solutions.

Notice this is the graph of sinx squashed by a factor of 2. See section 5B.



We need to extend the methods from section 9A to deal with equations like this. A substitution is a useful step in converting such an equation to the form 'trigonometric function = constant'.

Worked example 9.8

Find the zeros of the function $3\sin(2x)+1$ for $x \in [0, 2\pi]$.

Write down the equation to be solved.

$3\sin(2x)+1=0$

Rearrange it into the form $\sin(A)=k$.

$\Leftrightarrow \sin(2x) = -\dfrac{1}{3}$

Make a substitution for the argument.

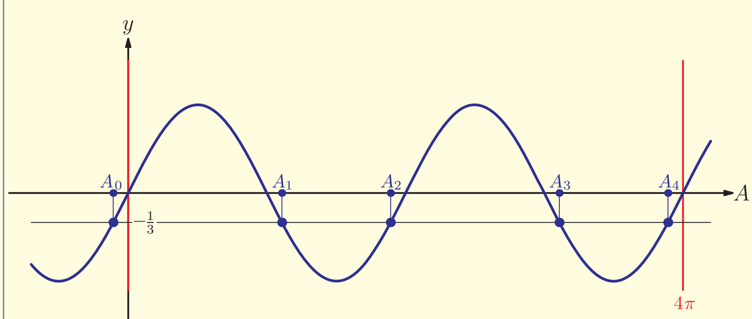
Let $A = 2x$

Rewrite the interval in terms of A.

$x \in [0, 2\pi] \Leftrightarrow A \in [0, 4\pi]$

Solve the equation for A.

$\sin A = -\dfrac{1}{3}$



There are four solutions.

$A_0 = \sin^{-1}\left(-\dfrac{1}{3}\right) = -0.3398$ is outside of the interval.

$A_1 = \pi - A_0 = 3.481$
$A_2 = A_0 + 2\pi = 5.943$
$A_3 = A_1 + 2\pi = 9.764$
$A_4 = A_2 + 2\pi = 12.23$

Transform the solutions back into x.

$x = \dfrac{A}{2}$
$= 1.74, 2.97, 4.88, 6.11$ (3 SF)

This procedure can be summarised in the following four-step process.

KEY POINT 9.4

> To solve trigonometric equations:
> 1. Make a substitution for the argument of the trigonometric function (such as $A = 2x$).
> 2. Change the interval for x into an interval for A.
> 3. Solve the equation for A in the usual way.
> 4. Transform the solutions back into the original variable.

The following example illustrates this method in a more complicated situation.

Worked example 9.9

Solve the equation $3\cos(2x+1) = 2$ for $x \in [-\pi, \pi]$.

Write the equation in the form $\cos(A) = k$.

$$\cos(2x+1) = \frac{2}{3}$$

Make a substitution for the argument.

Let $A = 2x + 1$

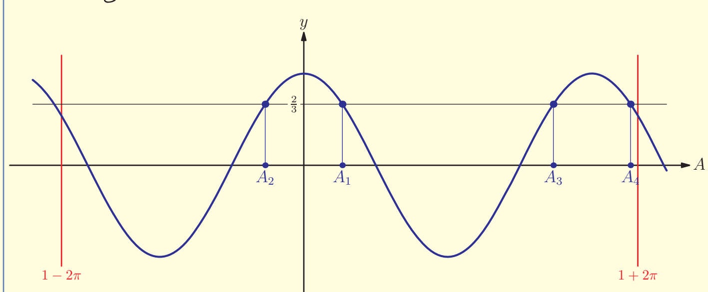
Rewrite the interval in terms of A.

$$-\pi \le x \le \pi$$
$$\Leftrightarrow -2\pi \le 2x \le 2\pi$$
$$\Leftrightarrow -2\pi + 1 \le 2x + 1 \le 2\pi + 1$$

So $A \in [-2\pi + 1, 2\pi + 1]$

Solve the equation for A.

$$\cos A = \frac{2}{3}$$



There are four solutions:

$$A_1 = \cos^{-1}\left(\frac{2}{3}\right) = 0.841$$
$$A_2 = -A_1 = -0.841$$
$$A_3 = A_2 + 2\pi = 5.44$$
$$A_4 = A_1 + 2\pi = 7.12$$

Transform the solutions back into x.

$$x = \frac{A-1}{2} = -0.0795, 2.22, 6.2035, -0.921$$

In the next example we revisit the tangent function, working in degrees and finding exact solutions.

Worked example 9.10

Solve the equation $3\tan\left(\frac{1}{2}\theta° - 30°\right) = \sqrt{3}$ for $0 \le \theta \le 720$.

Rearrange the equation into the form $\tan(A) = k$.

$$\tan\left(\frac{1}{2}\theta° - 30°\right) = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$$

Make a substitution for the argument.

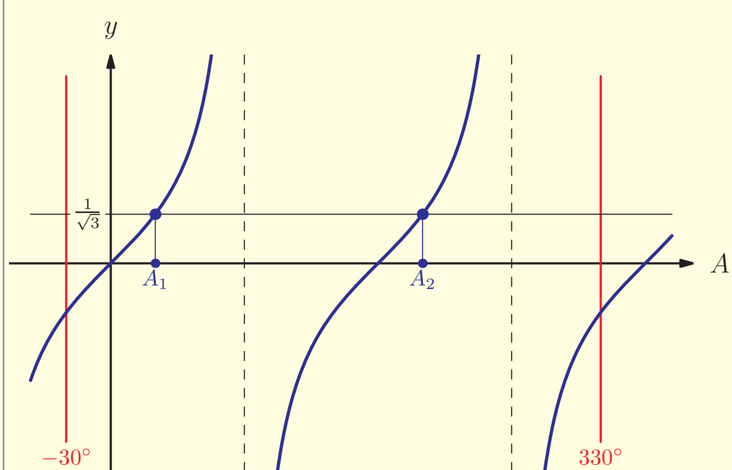
$$A = \frac{1}{2}\theta° - 30°$$

Rewrite the interval in terms of A.

$$0 \le \theta \le 720$$
$$\Leftrightarrow 0 \le \frac{1}{2}\theta \le 360$$
$$\Leftrightarrow -30 \le \frac{1}{2}\theta - 30 \le 330$$
$$\Leftrightarrow -30° \le A \le 330°$$

Solve the equation for A. $\frac{1}{\sqrt{3}}$ is one of the exact values from Section 8D.

$$\tan A = \frac{1}{\sqrt{3}}$$



There are two solutions:

$$A_1 = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30°$$
$$A_2 = A_1 + 180° = 210°$$

Transform the solutions back into θ.

$$\theta° = 2(A + 30°)$$
$$\theta = 120° \text{ or } 480°$$

Exercise 9B

1. Solve the following equations, giving your answers to 3 significant figures.

   (a) (i) $\tan^2 x = 2$ for $-\pi \le x \le \pi$

   (ii) $\sin^2 x = 0.6$ for $-\pi \le x \le \pi$

   (b) (i) $9\cos^2 \theta = 4$ for $0° < \theta < 360°$

   (ii) $3\tan^2 \theta = 5$ for $0° < \theta < 360°$

2. Without using graphs on your calculator, find all solutions of each equation in the given interval. Use graphs on your calculator to check your answers.

   (a) (i) $3\sin x - 2\sin x \cos x = 0$ for $0° \le x \le 360°$

   (ii) $4\cos x - \sin x \cos x = 0$ for $0° \le x \le 360°$

   (b) (i) $4\sin^2 \theta = 3\sin\theta$ for $\theta \in [-\pi, \pi]$

   (ii) $3\cos^2 \theta = -\cos\theta$ for $\theta \in [-\pi, \pi]$

   (c) (i) $\tan^2 t - 5\tan t + 5 = 0$ for $t \in]0, 2\pi[$

   (ii) $2\tan^2 t + \tan t - 1 = 0$ for $t \in]0, 2\pi[$

   (d) (i) $\sin\theta \tan\theta + \frac{1}{2}\tan\theta = 0$ for $\theta \in [0, 2\pi]$

   (ii) $2\cos\theta \tan\theta - 3\cos\theta = 0$ for $\theta \in [0, 2\pi]$

   (e) (i) $2\cos^2 x + 3\cos x = 2$ for $0° < x < 180°$

   (ii) $\cos^2 x - 2\cos x = 3$ for $0° < x < 180°$

3. Solve the following equations in the given interval, giving your answers to 3 significant figures.

   (a) (i) $\cos 2x = \frac{1}{3}$ for $0° \le x \le 360°$

   (ii) $\cos 3x = \frac{2}{5}$ for $0° \le x \le 360°$

   (b) (i) $\sin(3x - 1) = -0.2$ for $0 \le x \le \pi$

   (ii) $\sin(2x + 1) = \frac{2}{3}$ for $0 \le x \le 2\pi$

   (c) (i) $\tan(x - 45°) = 2$ for $-180° \le x \le 180°$

   (ii) $\tan(x + 60°) = -3$ for $-180° \le x \le 180°$

4. Find the exact solutions in the given interval.

   (a) (i) $\sin 2x = \frac{1}{2}$ for $0 \le x \le 2\pi$

   (ii) $\sin 3x = -\frac{1}{2}$ for $0 \le x \le 2\pi$

(b) (i) $\cos 2x = -\frac{\sqrt{2}}{2}$ for $0° \le x \le 360°$

(ii) $\cos 3x = \frac{1}{2}$ for $-180° \le x \le 180°$

(c) (i) $\tan 4x = \sqrt{3}$ for $0 \le x \le \pi$

(ii) $\tan 2x = \frac{1}{\sqrt{3}}$ for $0 \le x \le \pi$

5. Find the exact solutions of the following in the given interval.

(a) (i) $\cos(x + 60°) = \frac{\sqrt{3}}{2}$ for $0° \le x \le 360°$

(ii) $\cos\left(x - \frac{\pi}{3}\right) = \frac{1}{2}$ for $-\pi \le x \le \pi$

(b) (i) $\sin\left(x - \frac{\pi}{3}\right) = -\frac{1}{2}$ for $-\pi \le x \le \pi$

(ii) $\sin(x - 120°) = -\frac{\sqrt{2}}{2}$ for $0° \le x \le 360°$

(c) (i) $\tan\left(x + \frac{\pi}{2}\right) = 1$ for $0 < x < 2\pi$

(ii) $\tan\left(x - \frac{\pi}{4}\right) = -1$ for $0 < x < 2\pi$

6. Solve the equation $3\cos x = \tan x$ for $0 \le x \le 2\pi$. *[8 marks]*

7. (a) Given that $2\sin^2 x - 3\sin x = 2$, find the *exact* value of $\sin x$.

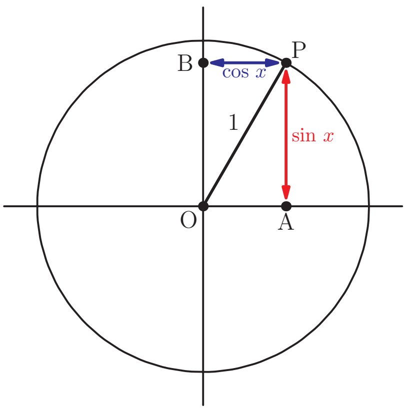
(b) Hence solve the equation $2\sin^2 x - 3\sin x = 2$ for $0 < x < 360°$. *[6 marks]*

8. Solve the equation $\sin x \tan x = \sin^2 x$ for $-\pi \le x \le \pi$. *[8 marks]*

9. Find the exact solutions of the equation $\sin(x^2) = \frac{1}{2}$ for $-\pi < x < \pi$. *[5 marks]*

9C Trigonometric identities

We have already seen one example of a trigonometric identity: $\frac{\sin x}{\cos x} = \tan x$. The two sides are equal for all values of x (except when $\cos x = 0$ and $\tan x$ is undefined). There are many other identities involving trigonometric functions, and we will meet some of them in this section.



EXAM HINT

The identity symbol ≡ means that the equality holds for all values of the variable(s). Many people write $\frac{\sin x}{\cos x} \equiv \tan x$ to emphasise that the statement is an identity (true for all values of x) rather than an equation (true only for some values of x, which need to be found). However, the IB syllabus and most exam questions use the equals sign in identities, so we will do the same in this book, except where there is a possibility of confusion.

Consider again the unit circle diagram, with point P representing number x. According to the definitions of the sine and cosine functions, $\text{AP} = \sin x$ and $\text{BP} = \text{OA} = \cos x$.

Note that the triangle is right-angled, with hypotenuse 1, so using Pythagoras' Theorem we get the following relation between sine and cosine.

KEY POINT 9.5

Pythagorean identity:

$$\sin^2 x + \cos^2 x = 1$$

Worked example 9.11

Given that $\sin x = \frac{1}{3}$, find the possible values of $\cos x$ and $\tan x$.

Think of an identity relating sin and cos.

$$\sin^2 x + \cos^2 x = 1$$

Put in the known value.

$$\left(\frac{1}{3}\right)^2 + \cos^2 x = 1$$

Find the value of the other function. Remember ± signs when taking the square root.

$$\cos^2 x = 1 - \frac{1}{9} = \frac{8}{9}$$

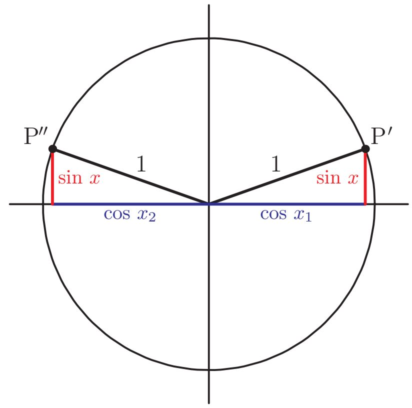
$$\therefore \cos x = \pm\sqrt{\frac{8}{9}} = \pm\frac{2\sqrt{2}}{3}$$

We know how to find tan given sin and cos.

$$\tan x = \frac{\sin x}{\cos x}$$

$$\therefore \tan x = \frac{\frac{1}{3}}{\pm\frac{2\sqrt{2}}{3}} = \pm\frac{\sqrt{2}}{4}$$

We have already seen examples where the values of $\sin x$ and $\cos x$ were used to find the value of $\tan x$. Using the Pythagorean identity, we only need to know the value of one of the functions to find the values of the other two.



Notice that for a given value of $\sin x$, there are two possible values of $\cos x$. The circle diagram makes this clear: points P′ and P″ are the same distance, namely $\sin x$, above the horizontal axis, but have different values of $\cos x$ (equal in size but opposite in sign).

Notice also that we do not need to know what x is to find the possible values of $\cos x$ given $\sin x$ (and vice versa). However by restricting x to a particular quadrant we can select one of the two possible values.

Worked example 9.12

If $\tan x = 2$ and $\frac{\pi}{2} < x < \pi$, find the value of $\cos x$.

We need a relationship between $\cos x$ and $\tan x$.

The only two identities we know so far are $\sin^2 x + \cos^2 x = 1$ and $\frac{\sin x}{\cos x} = \tan x$. We can substitute $\sin x$ from the second identity into the first.

$$\frac{\sin x}{\cos x} = \tan x$$

$$\Rightarrow \sin x = \tan x \cos x$$

So the Pythagorean identity becomes

$$\tan^2 x \cos^2 x + \cos^2 x = 1$$

Now put in the given value of $\tan x$.

$$2^2 \cos^2 x + \cos^2 x = 1$$

$$\Leftrightarrow 5\cos^2 x = 1$$

$$\Leftrightarrow \cos x = \pm\frac{\sqrt{5}}{5}$$

We are told that x is in the second quadrant, so $\cos x$ is negative.

$$\cos x < 0$$

$$\therefore \cos x = -\frac{\sqrt{5}}{5}$$

Exercise 9C

1. Find the exact values of $\cos x$ and $\tan x$ given that

(i) $\sin x = \frac{1}{3}$ and $0° < x < 90°$ (ii) $\sin x = \frac{4}{5}$ and $0° < x < 90°$

2. Find the exact values of $\sin\theta$ and $\tan\theta$ given that

(i) $\cos\theta = -\frac{1}{3}$ and $180° < \theta < 270°$

(ii) $\cos\theta = -\frac{3}{4}$ and $180° < \theta < 270°$

3. (a) Find the exact value of $\cos x$ if

(i) $\sin x = \frac{1}{5}$ and $\frac{\pi}{2} < x < \pi$

(ii) $\sin x = -\frac{1}{2}$ and $\frac{3\pi}{2} < x < 2\pi$

(b) Find the exact value of $\tan x$ if

(i) $\cos x = \frac{3}{5}$ and $-\frac{\pi}{2} < x < 0$

(ii) $\cos x = -1$ and $\frac{\pi}{2} < x < \frac{3\pi}{2}$

4. (i) Find the possible values of $\cos x$ if $\tan x = \frac{2}{3}$.

(ii) Find the possible values of $\sin x$ if $\tan x = -\frac{1}{2}$.

5. Find the exact value of:

(a) $3\sin^2 x + 3\cos^2 x$ (b) $\sin^2 5x + \cos^2 5x$

(c) $-2\cos^2 2x - 2\sin^2 2x$ (d) $2\tan^2 2x - \frac{2}{\cos^2 2x}$

(e) $\frac{1}{\sin^2 x} - \frac{1}{\tan^2 x}$ (f) $\frac{3}{2\sin^2 4x} - \frac{3}{2\tan^2 4x}$

6. (i) Express $3\sin^2 x + 4\cos^2 x$ in terms of $\sin x$ only.

(ii) Express $\cos^2 x - \sin^2 x$ in terms of $\cos x$ only.

7. (a) Express $3 - 2\tan^2 x$ in terms of $\cos x$ only.

(b) Express $\frac{1+\tan^2 x}{\cos^2 x}$ in terms of $\sin x$ only, simplifying your answer as fully as possible. *[7 marks]*

8. If $t = \tan x$, express the following in terms of t:

(a) $\cos^2 x$ (b) $\sin^2 x$ (c) $\cos^2 x - \sin^2 x$ (d) $\frac{2}{\sin^2 x} + 1$

[8 marks]

9D Using identities to solve equations

We can use trigonometric identities to solve more complicated equations. Usually we start by replacing $\tan x$ by $\frac{\sin x}{\cos x}$ or by using the Pythagorean identity. The latter can only be used if the equation contains squares and typically results in a quadratic equation.

Worked example 9.13

Solve the equation $4\sin x = \tan x$ in the interval $0 \le x \le 2\pi$.

The applicable identity here is the one for tan.

$$4\sin x = \frac{\sin x}{\cos x}$$

To eliminate fractions, multiply both sides by $\cos x$.

$$\Leftrightarrow 4\sin x\cos x = \sin x$$

Both sides contain $\sin x$, so rearrange the equation to make one side zero and factorise.

$$\Leftrightarrow 4\sin x\cos x - \sin x = 0$$
$$\Leftrightarrow \sin x(4\cos x - 1) = 0$$

One of the factors must be equal to zero.

$$\sin x = 0 \text{ or } \cos x = \frac{1}{4}$$

Now solve each equation separately.

When $\sin x = 0$:
$x = 0, \pi, 2\pi$

When $\cos x = \frac{1}{4}$:
$x = \cos^{-1}\left(\frac{1}{4}\right) = 1.32\,(3SF)$
or $x = 2\pi - 1.32 = 4.97\,(3SF)$

List all the solutions.

$\therefore x = 0, 1.32, \pi, 4.97, 2\pi$

The next example shows how using the Pythagorean identity can lead to a quadratic equation. You could solve the resulting equation using a graph on your calculator, but the question may require you to use an algebraic method, for instance by asking you to find possible values of $\cos\theta$ first.

Worked example 9.14

Find all values of θ in the interval $[-180°, 180°]$ which satisfy the equation $2\sin^2\theta + 3\cos\theta = 1$.

The equation contains both sin and cos terms. The sin term is squared, so replace $\sin^2\theta$ by $1-\cos^2\theta$.

$$2(1-\cos^2\theta) + 3\cos\theta = 1$$
$$\Leftrightarrow 2 - 2\cos^2\theta + 3\cos\theta = 1$$

This is a quadratic equation in $\cos\theta$. Write it in the standard form and then solve it.

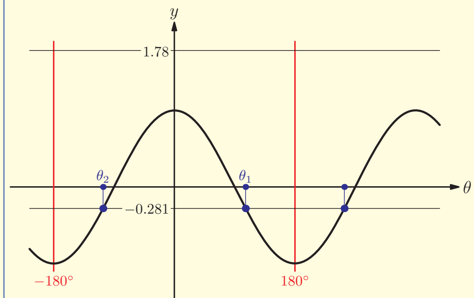
$$\Leftrightarrow 2\cos^2\theta - 3\cos\theta - 1 = 0$$
$$\cos\theta = 1.78 \text{ or } -0.281$$

continued . . .

cos values must be between −1 and 1.

$\cos\theta = 1.78$ is impossible

So $\cos\theta = -0.281$



There are 2 solutions.

$\theta_1 = \cos^{-1}(-0.281) = 106°$

$360° - 106° = 254°$ is not in the interval.

The second solution is

$\theta_2 = 254° - 360° = -106°$

Exercise 9D

1. By using the identity $\tan x = \dfrac{\sin x}{\cos x}$, solve the following equations.

(a) (i) $3\sin x = 2\cos x$ for $0° \le x \le 180°$

(ii) $3\sin x = 5\cos x$ for $0° \le x \le 180°$

(b) (i) $\cos x = 3\sin x$ for $0 \le x \le \dfrac{\pi}{2}$

(ii) $3\cos x = -\sin x$ for $0 \le x \le \pi$

(c) (i) $3\sin x + 5\cos x = 0$ for $0 \le x \le \pi$

(ii) $4\cos x + 3\sin x = 0$ for $0 \le x \le 2\pi$

(d) (i) $7\cos x - 3\sin x = 0$ for $-180° \le x \le 180°$

(ii) $\sin x - 5\cos x = 0$ for $-180° \le x \le 180°$

2. Solve the following equations in the given interval, giving exact answers.

(a) (i) $\sin 3\theta = \cos 3\theta$ for $0 < \theta < \dfrac{\pi}{2}$

(ii) $\sin 2t = \sqrt{3}\cos 2t$ for $t \in [0, \pi]$

(b) (i) $\sin 2x + \sqrt{3}\cos 2x = 0$ for $0 \le x \le 2\pi$

(ii) $\sin 3a + \cos 3a = 0$ for $a \in \left[0, \dfrac{\pi}{2}\right]$

3. Use trigonometric identities to solve these equations. (Do not use graphs or the equation solver function on your calculator.)

(a) $\sin x+\dfrac{\sin^2 x}{\cos x}=0$ for $0°\le x\le 360°$

(b) $3\sin^2 x=2\sin x\cos x$ for $x\in[-\pi,\pi]$

(c) $\dfrac{\cos\theta}{\sin\theta}-2=0$ for $\theta\in[-90°,90°]$

(d) $3\cos^2\theta+4\sin\theta\cos\theta=0$ for $0\le\theta\le 2\pi$

4. Use the identity $\sin^2 x+\cos^2 x=1$ to solve the following equations in the interval $[0°,360°]$.

(a) (i) $7\sin^2 x+3\cos^2 x=5$ (ii) $\sin^2 x+4\cos^2 x=2$

(b) (i) $3\sin^2 x-\cos^2 x=1$ (ii) $\cos^2 x-\sin^2 x=1$

5. Use an algebraic method to solve the equation $5\sin^2\theta=4\cos^2\theta$ for $-180°\le\theta\le 180°$. *[4 marks]*

6. Solve the equation $2\cos^2 t-\sin t-1=0$ for $0\le t\le 2\pi$. *[4 marks]*

7. Solve the equation $4\cos^2 x-5\sin x-5=0$ for $x\in[-\pi,\pi]$. *[4 marks]*

8. Given that $\cos^2 t+5\cos t=2\sin^2 t$, find the *exact* value of $\cos t$. *[4 marks]*

9. (a) Given that $6\sin^2 x+\cos x=4$, find the exact values of $\cos x$.

(b) Hence solve the equation $6\sin^2 x+\cos x=4$ for $0°\le x\le 360°$. *[6 marks]*

10. (a) Show that the equation $2\sin^2 x-3\sin x\cos x+\cos^2 x=0$ can be written in the form $2\tan^2 x-3\tan x+1=0$.

(b) Hence solve the equation $2\sin^2 x-3\sin x\cos x+\cos^2 x=0$, giving all solutions in the interval $-\pi<x<\pi$. *[6 marks]*

9E Double angle identities

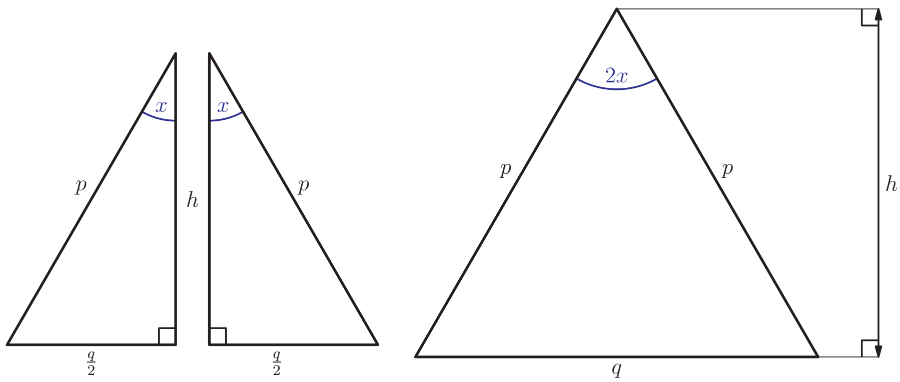
This section looks at the relationships between trigonometric functions of a certain argument and trigonometric functions with double that argument.

Working in radians, use your calculator to find

$\sin 1.2$ and $\sin 2.4$

$\cos 1.2$ and $\cos 2.4$

Are there any rules that relate $\sin 2x$ and $\cos 2x$ to $\sin x$ and $\cos x$? At first glance it may appear that there is no connection between the values of trigonometric functions of an angle and those of twice that angle. To try to discover any relationships that might exist, a sensible starting point would be the familiar right-angled triangle containing the angle x. We are interested in the angle $2x$, which can be formed by adjoining an identical right-angled triangle as shown below.



If you have not seen this formula before, you will meet it in chapter 10.

First consider the whole isosceles triangle. Using the formula Area $= \frac{1}{2}ab\sin C$, we find:

$$\text{Area} = \frac{1}{2}p^2\sin(2x)$$

We can also calculate the area from the base and the height of the triangle (Area $= \frac{1}{2}bh$). To find the height h and the length of the base b, look at one of the right-angled triangles with angle x. We have

$$\frac{h}{p} = \cos x \quad \Rightarrow \quad h = p\cos x$$

$$\frac{\frac{q}{2}}{p} = \sin x \quad \Rightarrow \quad q = 2p\sin x$$

So another expression for the area of the triangle is:

$$\begin{aligned}\text{Area} &= \frac{1}{2}(2p\sin x)(p\cos x)\\ &= p^2\sin x\cos x\end{aligned}$$

Comparing the two expressions for the area, we get

$$\frac{1}{2}p^2\sin(2x) = p^2\sin x\cos x$$

and rearranging this equation gives the **double angle identity** for sine.

KEY POINT 9.6

$$\sin(2x) = 2\sin x\cos x$$

Although x was assumed to be an acute angle in our derivation of this formula, the identity actually holds for all values of x.

Working from the same triangle, it is possible to find a double angle identity for cosine:

$$\cos(2x) = \cos^2 x - \sin^2 x$$

Substituting $\sin^2 x = 1 - \cos^2 x$ or $\cos^2 x = 1 - \sin^2 x$ in this formula gives us two further ways of expressing the double angle identity for cosine.

KEY POINT 9.7

$$\cos(2x) = \begin{cases} 2\cos^2 x - 1 \\ 1 - 2\sin^2 x \\ \cos^2 x - \sin^2 x \end{cases}$$

One application of double angle identities is finding the exact values of half-angles.

EXAM HINT

You will often see sin, cos and tan of a multiple of x written without brackets: $\sin 2x$, $\cos 5x$, etc.

Worked example 9.15

Using the exact value of $\cos 30°$, show that $\sin 15° = \sqrt{\frac{2-\sqrt{3}}{4}}$.

We know that $\cos 30° = \frac{\sqrt{3}}{2}$, and we need to relate this to $\sin 15°$.
Since $30 = 2 \times 15$, the obvious choice is the cosine double angle identity that involves sine: $\cos(2x) = 1 - \sin^2 x$.

Using $\cos(2x) = 1 - \sin^2 x$:

$$\cos(2 \times 15°) = 1 - 2\sin^2 15°$$

$$\cos 30° = 1 - 2\sin^2 15°$$

Using $\cos 30° = \frac{\sqrt{3}}{2}$:

$$\frac{\sqrt{3}}{2} = 1 - 2\sin^2 15°$$

$$\Leftrightarrow \sqrt{3} = 2 - 4\sin^2 15°$$

$$\Leftrightarrow 4\sin^2 15° = 2 - \sqrt{3}$$

$$\Leftrightarrow \sin^2 15° = \frac{2-\sqrt{3}}{4}$$

We have to choose between the positive and negative square root. Since 15° is in the first quadrant, $\sin 15° > 0$ and so we take the positive square root.

$$\therefore \sin 15° = \sqrt{\frac{2-\sqrt{3}}{4}} \quad (\text{as } \sin 15° > 0)$$

In chapter 15 we will see how double angle identities can be used to integrate some trigonometric functions.

Double angle identities are also very useful in proving more complex trigonometric identities and solving equations. Although they are called double angle identities, these formulas can be applied to any multiple of an angle.

Worked example 9.16

Find an expression for $\cos 4x$ in terms of

(a) $\cos 2x$ (b) $\cos x$

Notice that $4x = 2 \times (2x)$, so one of the cosine double angle identities seems suitable.
Since we want an expression involving only cos, the most appropriate formula is $\cos(2x) = 2\cos^2 x - 1$, with $x = 2x$.

(a) Using $\cos(2x) = 2\cos^2 x - 1$:

$$\cos(2(2x)) = 2\cos^2(2x) - 1$$

$$\cos(4x) = 2\cos^2(2x) - 1$$

continued . . .

Can we use the answer from the part (i)? Yes – we just need to replace $\cos(2x)$ in the previous result with an expression involving only $\cos x$.
Replace $\cos(2x)$ in the answer to part (a) with an expression involving only $\cos x$.

(b) From part (i),

$$\cos(4x) = 2\cos^2(2x) - 1$$
$$= 2(2\cos^2 x - 1)^2 - 1$$
$$(\text{as } \cos(2x) = 2\cos^2 x - 1)$$

EXAM HINT

In the exam, any equivalent form of the answer would be acceptable; therefore, unless explicitly asked to do so, you need not go any further than this, for example by expanding the brackets.

Recognising the form of double angle formulas can be helpful in solving trigonometric equations.

Worked example 9.17

Solve the equation $6\sin x\cos x = 1$ for $-\pi < x < \pi$.

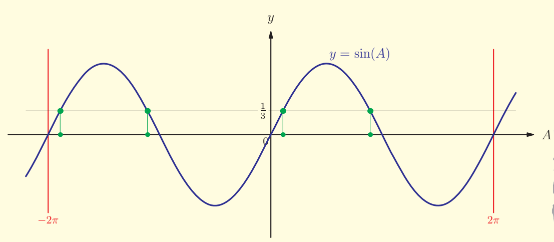
To solve this equation we need to rewrite it in the form 'trigonometric function = constant'.
Here the $\sin x \cos x$ on the LHS should remind us of the $\sin(2x)$ identity.

$$6\sin x\cos x = 1$$
$$\Leftrightarrow 2\sin x\cos x = \frac{1}{3}$$
$$\Leftrightarrow \sin(2x) = \frac{1}{3} \quad (\text{as } 2\sin x\cos x = \sin(2x))$$

Now we can follow the standard procedure.
First, make a substitution for the argument.
Sketching the graph, we see that there are 4 solutions in the given domain.

Let $A = 2x$

$-\pi < x < \pi \Leftrightarrow -2\pi < A < 2\pi$



$\sin^{-1}\left(\frac{1}{3}\right) = 0.3398$

so the 4 solutions are

$A = 0.3398, 2.802, -5.943, -3.481$

and hence

$x = 0.170, 1.40, -2.97, -1.74$ (3 SF)

If an equation contains both $\cos 2\theta$ and $\cos\theta$, we need to use identities to turn it into a quadratic equation involving only $\cos\theta$.

Worked example 9.18

Solve the equation $\cos 2x = \cos x$ for $0° \le x \le 360°$.

We need to write the equation in terms of only one trig function. (Watch out: $\cos 2x$ and $\cos x$ are not the same function!)
Use the identity for $\cos(2x)$ that involves just $\cos x$.

$$\cos(2x) = \cos x$$
$$\Leftrightarrow 2\cos^2 x - 1 = \cos x$$
$$(\text{as } \cos(2x) = 2\cos^2 x - 1)$$

Recognise this as a quadratic equation in $\cos x$. Try to factorise.

$$2\cos^2 x - \cos x - 1 = 0$$
$$(2\cos x + 1)(\cos x - 1) = 0$$
$$\cos x = -\frac{1}{2} \text{ or } \cos x = 1$$

Solve each equation separately.

When $\cos x = -\frac{1}{2}$:

$x = 120°$ or $360° - 120° = 240°$

When $\cos x = 1$:

$x = 0°$ or $360°$

List all the solutions.

$x = 0°, 120°, 240°, 360°$

We can use the double angle identities for sine and cosine to derive a double angle identity for the tangent function, $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$.

Worked example 9.19

Express $\tan 2x$ in terms of $\tan x$.

First, write $\tan 2x$ in terms of $\sin 2x$ and $\cos 2x$.

$$\tan 2x = \frac{\sin 2x}{\cos 2x}$$

Then, express in terms of $\sin x$ and $\cos x$ by using the sin and cos double angle identities.
We have to decide which of the $\cos 2x$ identities to use. If we use $\cos(2x) = \cos^2 x - \sin^2 x$, we can divide through by $\cos^2 x$ to leave $1 - \tan^2 x$ in the denominator.

$$= \frac{2\sin x\cos x}{\cos^2 x - \sin^2 x}$$
$$(\text{as } \cos(2x) = \cos^2 x - \sin^2 x)$$

Dividing top and bottom by $\cos^2 x$:

$$\tan 2x = \frac{2\left(\frac{\sin x}{\cos x}\right)}{1 - \frac{\sin^2 x}{\cos^2 x}}$$

So

$$\tan(2x) = \frac{2\tan x}{1-\tan^2 x}$$

It may not be obvious which version of the $\cos(2x)$ identity to use. The good news is that, in this example and most other cases, even if you choose the 'wrong' version to begin with you will still be able to complete the calculation – it may just take a little longer.

Exercise 9E

1. (a) (i) Given that $\cos\theta = -\frac{1}{4}$, find the exact value of $\cos 2\theta$.

(ii) Given that $\sin A = -\frac{2}{3}$, find the exact value of $\cos 2A$.

(b) (i) Given that $\sin x = \frac{1}{3}$ and $0 < x < \frac{\pi}{2}$, find the exact value of $\cos x$.

(ii) Given that $\sin x = \frac{3}{5}$ and $0 < x < \frac{\pi}{2}$, find the exact value of $\cos x$.

(c) (i) Given that $\sin x = \frac{1}{3}$ and $0 < x < \frac{\pi}{2}$, find the exact value of $\sin 2x$.

(ii) Given that $\sin x = \frac{3}{5}$ and $0 < x < \frac{\pi}{2}$, find the exact value of $\sin 2x$.

2. Find the exact value of

(a) $\sin^2 22.5°$ (b) $\cos^2 75°$

(c) $\cos^2\left(\frac{\pi}{12}\right)$

3. Find the exact value of $\tan 22.5°$.

4. Simplify the following by using double angle identities.

(a) $2\cos^2(3A) - 1$ (b) $4\sin 5x \cos 5x$

(c) $3 - 6\sin^2\left(\frac{b}{2}\right)$ (d) $5\sin\left(\frac{x}{3}\right)\cos\left(\frac{x}{3}\right)$

5. Use an algebraic method to solve each of the following equations.

(a) $\sin 2x = 3\sin x$ for $x \in [0, 2\pi]$

(b) $\cos 2x - \sin^2 x = -2$ for $0° \le x \le 180°$

(c) $5\sin 2x = 3\cos x$ for $-\pi < x < \pi$

(d) $\tan 2x - \tan x = 0$ for $0° \le x \le 360°$

6. Find all the values of $\theta \in [-\pi, \pi]$ which satisfy the equation $\cos^2\theta + \cos 2\theta = 0$. *[5 marks]*

7. Show that $\dfrac{1-\cos 2\theta}{1+\cos 2\theta} = \tan^2\theta$. *[4 marks]*

8. Express $\cos 4\theta$ in terms of

(a) $\cos\theta$ (b) $\sin\theta$ *[7 marks]*

9. (a) Show that

(i) $\cos^2\left(\dfrac{1}{2}x\right) = \dfrac{1}{2}(1+\cos x)$

(ii) $\sin^2\left(\dfrac{1}{2}x\right) = \dfrac{1}{2}(1-\cos x)$

(b) Express $\tan^2\left(\dfrac{1}{2}x\right)$ in terms of $\cos x$. *[7 marks]*

10. Given that $a\sin 4x = b\sin 2x$ and $0 < x < \dfrac{\pi}{2}$, express $\sin^2 x$ in terms of a and b. *[6 marks]*

Summary

- To solve trigonometric equations, follow this procedure:
  - First, rearrange the equation into the form $\sin A = k$, $\cos A = k$ or $\tan A = k$.
  - Make a substitution for the argument of the trigonometric function, e.g. $A = 2x + 1$ then sketch a graph over the required interval to see how many solutions there are (it is not necessary to substitute A for x in simple trigonometric equations).
  - Find solutions in the interval $[0, 2\pi]$:

$$\sin A = k \Rightarrow A_1 = \sin^{-1} k,\ A_2 = \pi - A_1$$

$$\cos A = k \Rightarrow A_1 = \cos^{-1} k,\ A_2 = 2\pi - A_1 \text{ (i.e. the reflection of } A_1\text{)}$$

$$\tan A = k \Rightarrow A_1 = \tan^{-1} k,\ A_2 = \pi + A_1$$

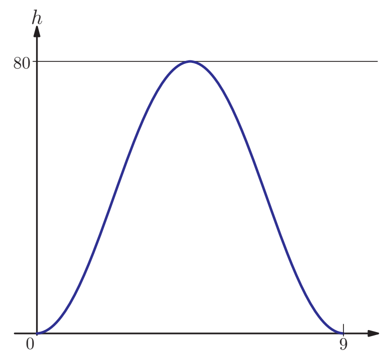
  - Find other solutions in the required interval by adding or subtracting multiples of 2π for sine and cosine and multiples of π for tangent.
- Use the values of A to find the value of x, i.e. transform the solutions back into the original variable.
- Trigonometric functions are related through identities, e.g.
  - $\tan x = \dfrac{\sin x}{\cos x}$
  - Pythagorean identity: $\sin^2 x + \cos^2 x = 1$
  - double angle identities: $\sin 2\theta = 2\sin\theta\cos\theta$;

$$\cos(2\theta) = \begin{cases} 2\cos^2\theta - 1 \\ 1 - 2\sin^2\theta \\ \cos^2\theta - \sin^2\theta \end{cases}$$

Introductory problem revisited

The original Ferris Wheel was constructed in 1893 in Chicago. It was just over 80 m tall and could complete one full revolution in 9 minutes. During each revolution, how much time did the passengers spend more than 5 m above the ground?

A car on the Ferris Wheel moves in a circle. Its height above the ground can therefore be modelled by a suitably transformed sine or cosine function. The car starts on the ground, climbs to a maximum height of 80 m and then returns to the ground; this takes 9 minutes. We can sketch a graph showing the height of the car above the ground as a function of time:



The period of the function is 9 and the amplitude is 40, so the function should involve $40\sin\left(\frac{2\pi}{9}t\right)$ or $40\cos\left(\frac{2\pi}{9}t\right)$. Either the sine or the cosine function can be used as the model; let us choose the cosine function, so that no horizontal translation is needed. The centre of the circle is at height 40, which means the graph is translated up by 40 units. Thus, an appropriate equation for height in terms of time is

$$h = 40 - 40\cos\left(\frac{2\pi}{9}t\right)$$

The minus sign in front of the cosine term ensures that when $t = 0$ we have $h = 0$.

To find the amount of time that the car spends more than 50 m above the ground, we need to find the times at which the height is exactly 50 m. This involves solving the equation

$$40 - 40\cos\left(\frac{2\pi}{9}t\right) = 50$$

Using the methods in this chapter, we get

$$\cos\left(\frac{2\pi}{9}t\right) = -0.25$$

$$\cos^{-1}(-0.25) = 1.82$$

$$\therefore \frac{2\pi}{9}t = 1.82 \text{ or } 2\pi - 1.82 = 4.46$$

$$\Rightarrow t_1 = 2.61, t_2 = 6.39$$

Therefore the time spent more than 50 m above ground is $6.39 - 2.61 \approx 3.78$ minutes.

Mixed examination practice 9

Short questions

1. Solve the equation $\tan x° = -0.62$ for $x \in]-90°, 270°[$. *[4 marks]*

2. Given that $0 < \theta < \frac{\pi}{2}$ and $\sin\theta = \frac{2}{3}$, find the exact value of
   (a) $\cos\theta$
   (b) $\cos 2\theta$ *[6 marks]*

3. Solve the equation $5\sin^2\theta = 4\cos^2\theta$ for $-\pi \le \theta \le \pi$. *[5 marks]*

4. Sketch the graph of $y = \sin(2x) + 2\sin(6x)$ and hence find the exact period of the function. *[4 marks]*

5. Prove the identity $\frac{2}{\cos^2 x} - \tan^2 x = 2 + \tan^2 x$. *[5 marks]*

6. Solve the equation $\cos\theta - 2\sin^2\theta + 2 = 0$ for $\theta \in [0, 2\pi]$. *[6 marks]*

7. Use an algebraic method to solve the equation $6\sin^2 x + \cos x = 4$ for $0° \le x \le 360°$. *[6 marks]*

8. Solve the equation $\sin 2\theta = \cos\theta$ for $0 \le \theta \le 2\pi$. *[7 marks]*

Long questions

1. The shape of a small bridge can be modelled by the equation $y = 1.8\sin\left(\frac{x}{3}\right)$, where y is the height of the bridge above the water and x is the distance from one river bank, both measured in metres. The bridge is just long enough to span the river.
   (a) Find the width of the river.
   (b) A barge has height 1.2 metres above the water level. Find the maximum possible width of the barge so that it can pass under the bridge.
   (c) Another barge has width 3.5 m. What is the maximum possible height of the barge so that it can pass under the bridge? *[10 marks]*

2. (a) Sketch the graph of the function $C(x) = \cos x + \frac{1}{2}\cos 2x$ for $-2\pi \le x \le 2\pi$.

(b) Prove that the function $C(x)$ is periodic and state its period.

(c) For what values of x, with $-2\pi \le x \le 2\pi$, is $C(x)$ a maximum?

(d) Let $x = x_0$ be the smallest positive value of x for which $C(x) = 0$. Find an approximate value of x_0 which is correct to two significant figures.

(e) (i) Prove that $C(x) = C(-x)$ for all x.

(ii) Let $x = x_1$ be that value of x with $\pi < x < 2\pi$ for which $C(x) = 0$. Find the value of x_1 in terms of x_0. *[16 marks]*

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3. (a) Find the value of k for which the equation $4x^2 - kx + 1 = 0$ has a repeated root.

(b) Show that the equation $4\sin^2\theta = 5 - k\cos\theta$ can be written as $4\cos^2\theta - k\cos\theta + 1 = 0$.

(c) Let $f_k(\theta) = 4\cos^2\theta - k\cos\theta + 1$.

(i) State the number of values of $\cos\theta$ which satisfy the equation $f_4(\theta) = 0$.

(ii) Find all the values of $\theta \in [-2\pi, 2\pi]$ which satisfy the equation $f_4(\theta) = 0$.

(iii) Find the value of k for which $x = 1$ is a solution of the equation $4x^2 - kx + 1 = 0$.

(iv) For this value of k, find the number of solutions in $[-2\pi, 2\pi]$ of the equation $f_k(\theta) = 0$. *[14 marks]*