12 Basic differentiation and its applications

In this chapter you will learn:

- how to find the gradients of curves from first principles – a process called differentiation
- to differentiate x^n
- to differentiate $\sin x$, $\cos x$ and $\tan x$
- to differentiate e^x and $\ln x$
- how to find the equations of tangents and normals to curves at given points
- how to find maximum and minimum points on curves, as well as points of inflexion.

Introductory problem

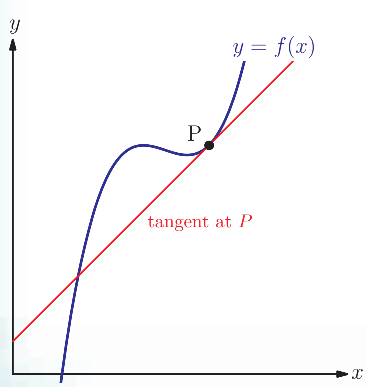
The cost of petrol consumed by a car is £$\left(12+\frac{v^2}{100}\right)$ per hour, where the speed $v\ (>0)$ is measured in miles per hour. If Daniel wants to travel 50 miles as cheaply as possible, what speed should he go at?

In real life things change: planets move, babies grow and prices rise. Calculus is the study of change, and one of its most important tools is differentiation, that is, finding the rate at which the y-coordinate of a curve is changing when the x-coordinate changes. For a straight-line graph, this rate of change was given by the gradient of the line. In this chapter we apply the same idea to curves, where the gradient is different at different points.

We already met tangents in chapters 3 and 9.

12A Sketching derivatives

Our first task is to establish exactly what is meant by the gradient of a function. We are clear on what is meant by the gradient of a straight line, so we will use this idea to make a more general definition. A **tangent** to a curve is a straight line which touches the curve without crossing it. We define the **gradient** of a function at a point P to be the gradient of the tangent to the function's graph at that point.

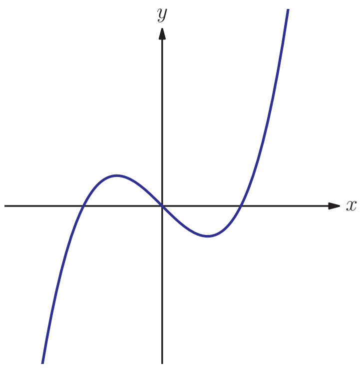


Note that when we say the tangent at P does not cross the curve, we mean this in a 'local' sense – that is, the tangent does not cross the curve close to the point P; it can intersect a different part of the curve (as shown in the diagram).

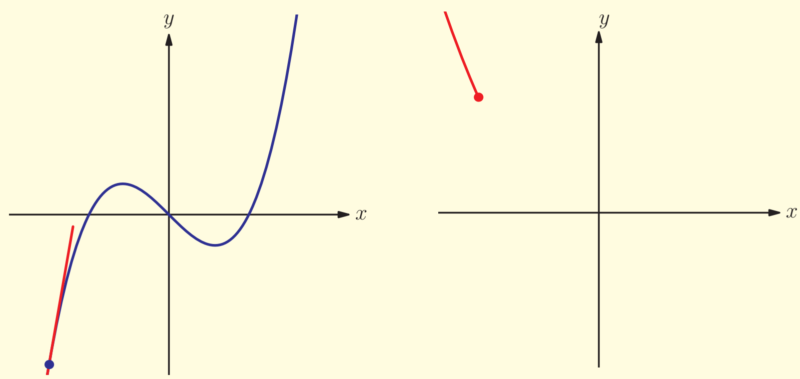
The **derivative** of a function $f(x)$ is a function which gives the gradient of $y = f(x)$ at any point x in the domain. It is often useful to be able to roughly sketch the derivative of a given function.

Worked example 12.1

Sketch the derivative of this function.

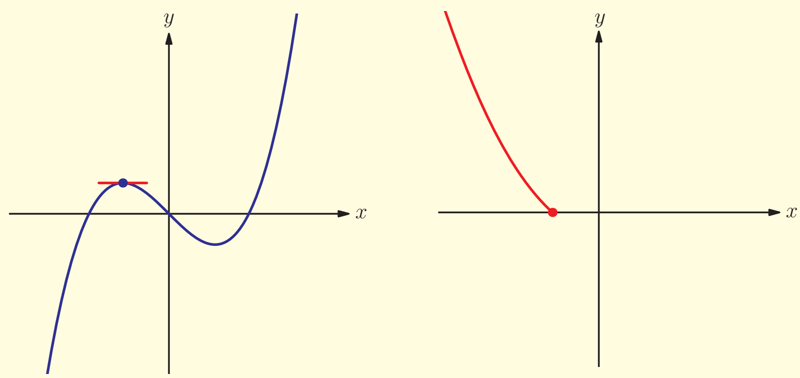


Imagine a point moving along the curve from left to right; we will track the tangent to the curve at the moving point and form the graph of its gradient.



The curve is increasing, but more and more slowly …

… so the gradient is positive and falling.



The tangent is horizontal …

… so the gradient is zero.

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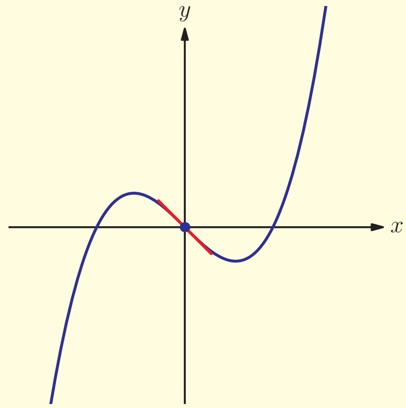
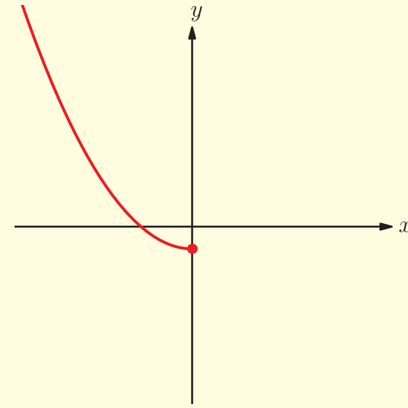
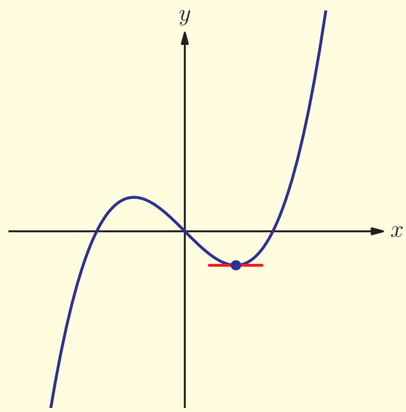


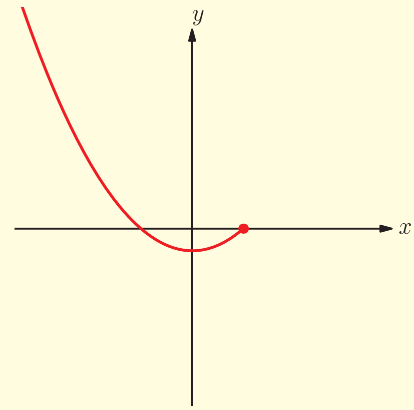
The curve is now decreasing …



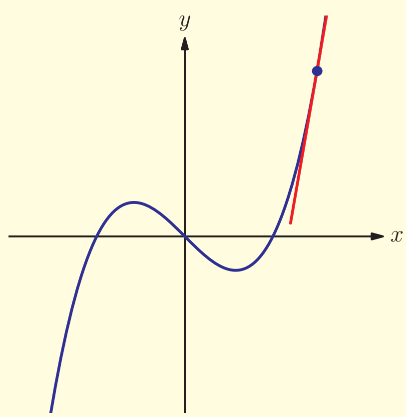
… so the gradient is negative.



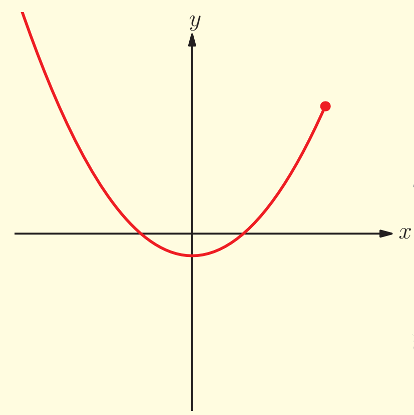
The tangent becomes horizontal again …



… so the gradient is zero.



The curve increases again, and does so faster and faster …

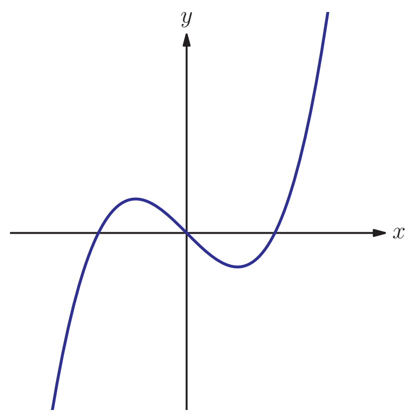


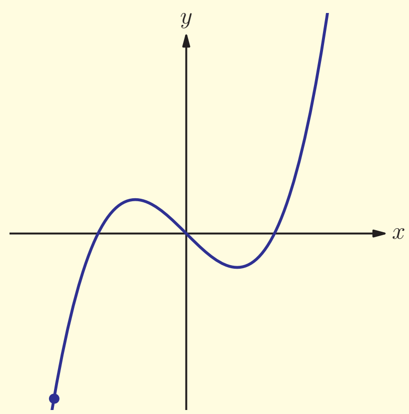
… so the gradient is positive and getting larger.

We can also apply the same reasoning backwards.

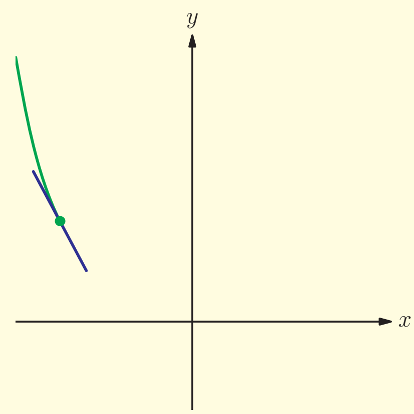
Worked example 12.2

The graph shows the derivative of a function. Sketch a possible graph of the original function.

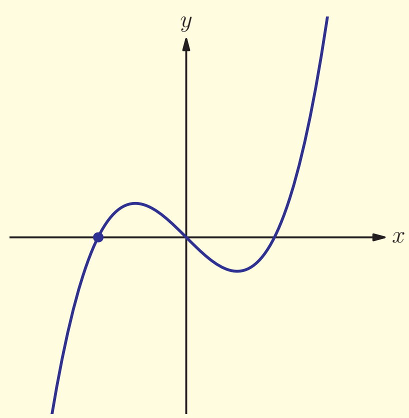




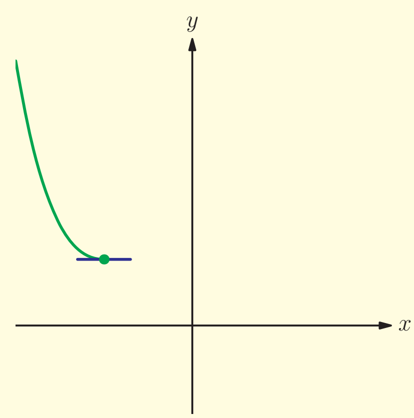
The gradient is negative …



… so the curve is decreasing.



The gradient is zero …



… so the tangent is horizontal.

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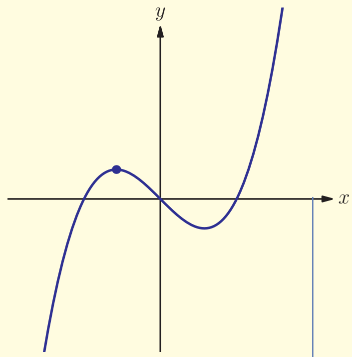
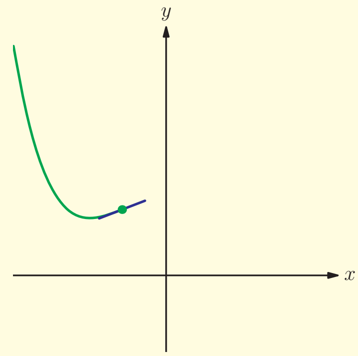
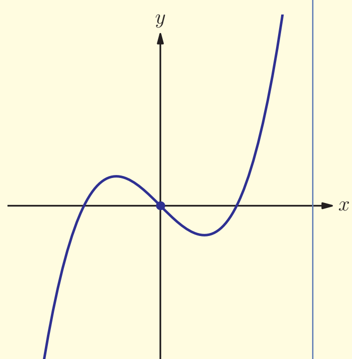


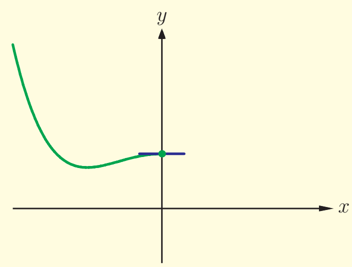
The gradient is positive . . .



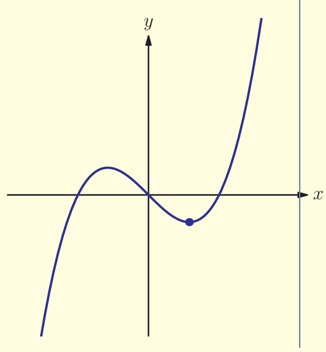
. . . so the curve is increasing.



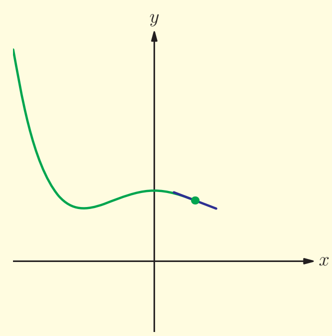
The gradient is zero . . .



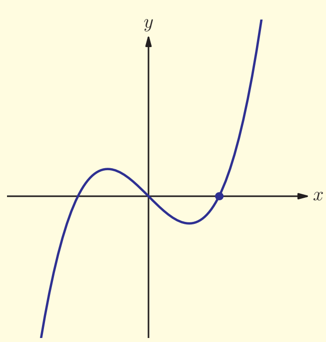
. . . so the tangent is horizontal.



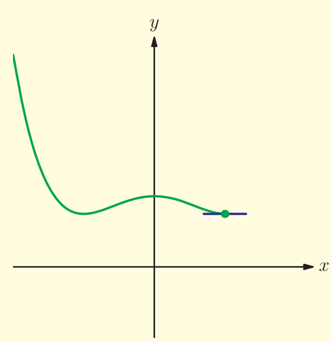
The gradient is negative . . .



. . . so the curve is decreasing.



The gradient is zero . . .



. . . so the tangent is horizontal.

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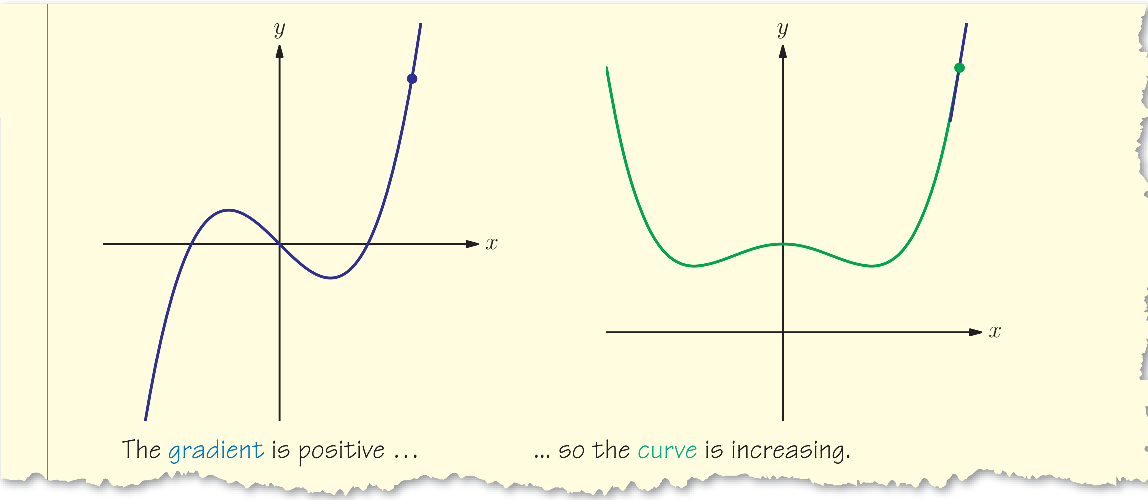


Note that in Worked example 12.2 there was more than one possible graph we could have drawn, depending on where we started the sketch. In chapter 13 you will see more about this ambiguity when you learn how to 'undo' differentiation.

The relationship between a graph and its derivative can be summarised as follows.

KEY POINT 12.1

When the curve is increasing the gradient is positive.

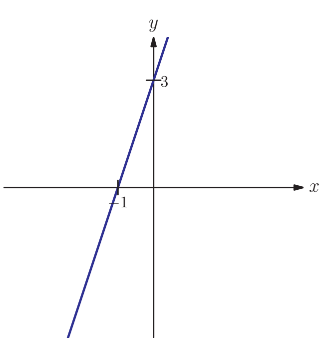
When the curve is decreasing the gradient is negative.

When the tangent is horizontal the gradient is zero. A point on the curve where this occurs is called a **stationary point** or **turning point**.

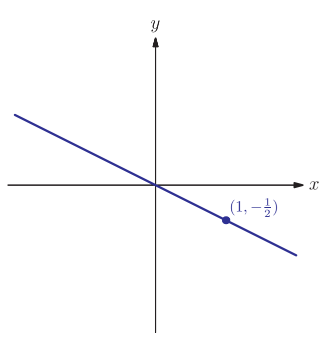
Exercise 12A

1. Sketch the derivatives of the following, showing intercepts with the x-axis.

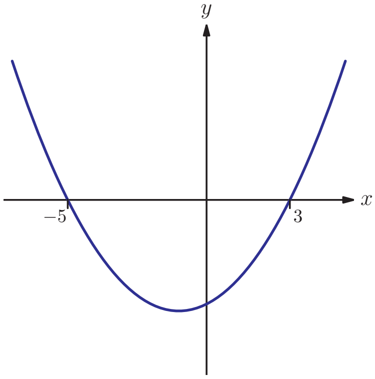
(a) (i)



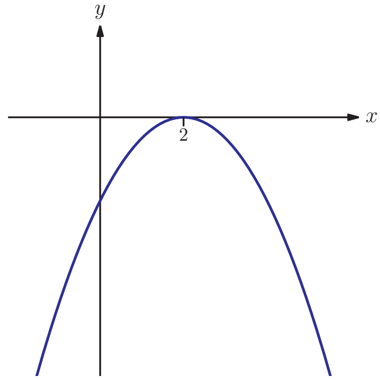
(ii)



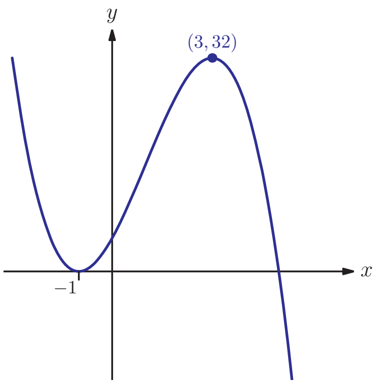
(b) (i)



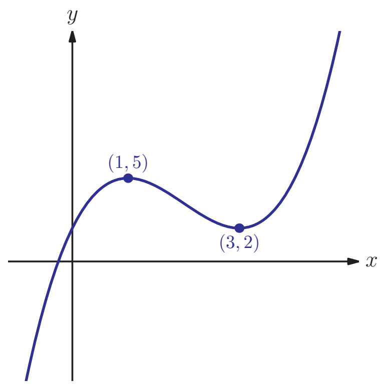
(ii)



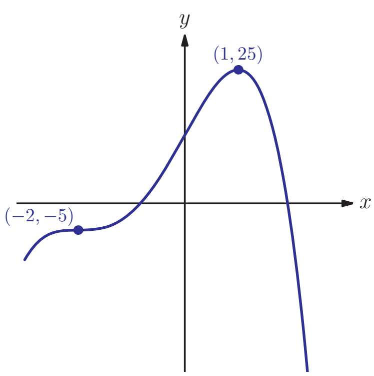
(c) (i)



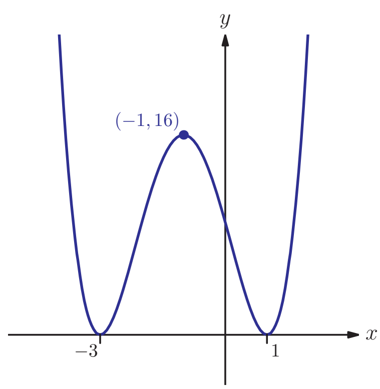
(ii)



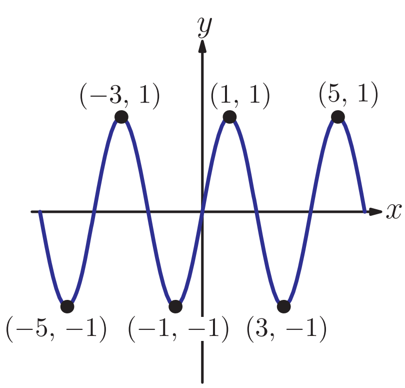
(d) (i)



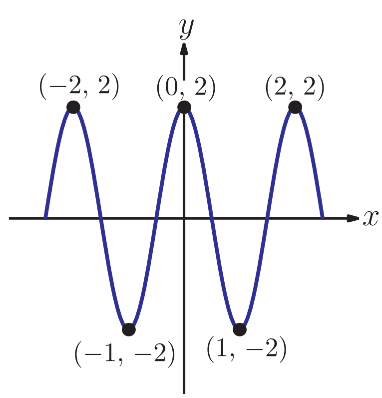
(ii)



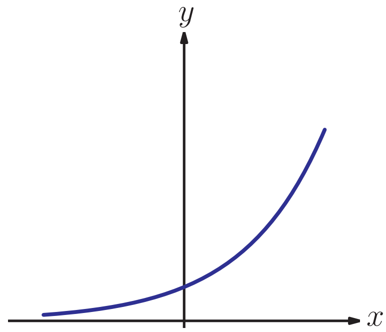
(e) (i)



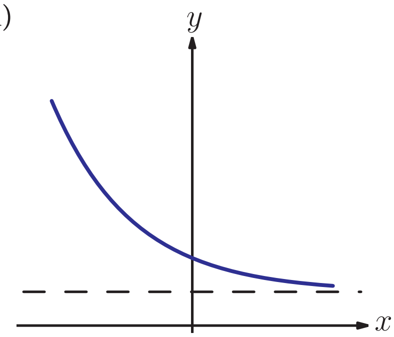
(ii)



(f) (i)

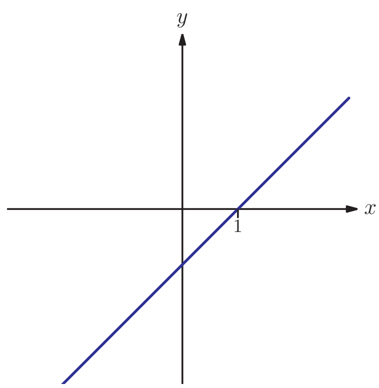


(ii)

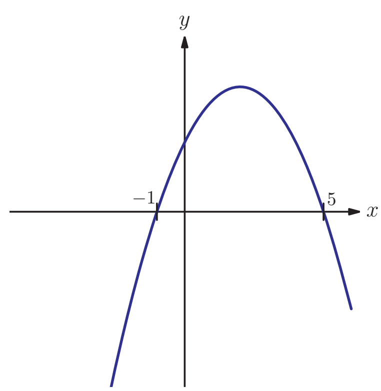


2. Each of the following graphs represents a function's derivative. Sketch a possible graph for the original function, indicating any stationary points.

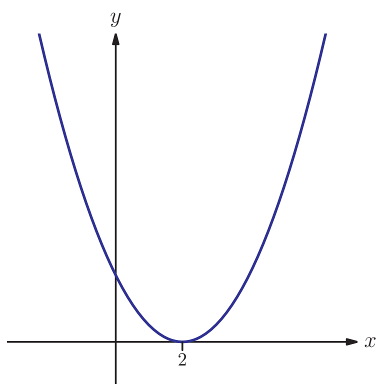
(a)



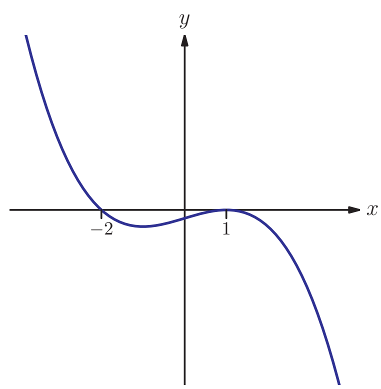
(b)



(c)



(d)



3. Decide whether each of the following statements is always true, sometimes true or always false.

(a) At a point where the derivative is positive the original function is positive.

(b) When the original function is negative the derivative is also negative.

(c) The derivative crossing the *x*-axis corresponds to a stationary point on the function's graph.

(d) When the derivative is zero the graph is at a local maximum or minimum point.

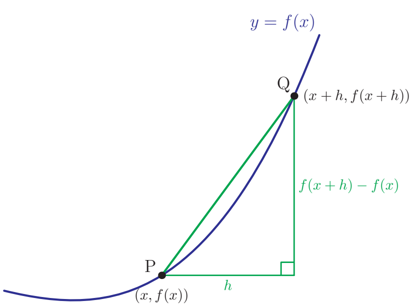
(e) If the derivative is always positive, then part of the original function is above the x-axis.

(f) At the lowest value of the original function the derivative is zero.

12B Differentiation from first principles

You will probably find that drawing a tangent to a graph is quite difficult to do accurately, and that the line you draw typically crosses the curve at two points. The line segment between these two intersection points is called a **chord**. If the two points are close together, the gradient of the chord will be very close to the gradient of the tangent. We can use this geometric insight to develop a method for calculating the derivative of a given function.

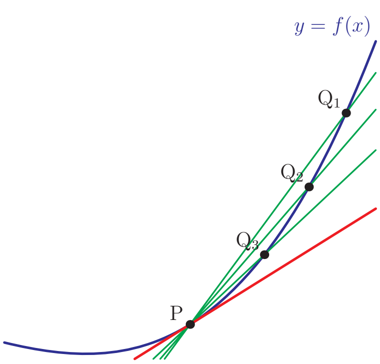
Self-Discovery Worksheet 3 'Investigating derivatives of polynomials' on the CD-ROM leads you through several concrete examples of using this method. Here we summarise the general procedure.



Consider a point $\mathrm{P}(x, f(x))$ on the graph of the function $y = f(x)$, and move a horizontal distance h away from x to the point $\mathrm{Q}(x+h, f(x+h))$.

We can find an expression for the gradient of the chord [PQ]:

$$\begin{aligned} m_{PQ} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{f(x+h) - f(x)}{(x+h) - x} \\ &= \frac{f(x+h) - f(x)}{h} \end{aligned}$$



As the point Q gets closer and closer to P, the gradient of the chord [PQ] approximates the gradient of the tangent at P more and more closely.

To denote the distance h between P and Q approaching zero, we use $\lim_{h \to 0}$, which reads as 'the limit as h tends to zero'. This idea of a limit is very much like that encountered in chapters 2 and 3 with asymptotes on graphs, where the graph approaches the asymptote (the limit) as x tends to ∞.

The process of finding $\lim_{h \to 0}$ of the gradient of the chord [PQ] is called **differentiation from first principles**, and we have the following definition.

KEY POINT 12.2

Differentiation from first principles

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

EXAM HINT

Differentiation from first principles means finding the derivative using the definition in Key point 12.2, rather than any of the rules we will meet in later sections.

The expression $f'(x)$ is referred to as the **derivative** of $f(x)$. It is also written as just f', or as y' or $\frac{dy}{dx}$ where $y = f(x)$. The process of finding the derivative is called **differentiation**.

We can use this definition to find the derivative of simple polynomial functions.

Worked example 12.3

For the function $y = x^2$, find $\frac{dy}{dx}$ from first principles.

Use the formula; here $f(x) = x^2$.

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{(x+h)^2 - x^2}{h}$$

We do not want to have the denominator go to zero, so first try to simplify the numerator and hope that the h in the denominator will cancel out.

$$= \lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}$$

$$= \lim_{h\to 0}\frac{2xh+h^2}{h}$$

Now, as hoped, we can divide top and bottom by h.

$$= \lim_{h\to 0} 2x+h$$

Finally, since we don't have to worry about zero in the denominator, we are free to let $h \to 0$.

$$= 2x$$

Let us see how we can use the formula derived in the above example. On the curve $y = x^2$, at the point where $x = 3$ the y-coordinate is 9. Now we also know from the formula $\frac{dy}{dx} = 2x$ that the gradient at that point is 6. We could, of course, use this formula for $\frac{dy}{dx}$ to find the gradient at any point on the curve $y = x^2$.

Exercise 12B

1. Find the derivatives of the following functions from first principles.

   (a) (i) $f(x) = x^3$ (ii) $f(x) = x^4$

   (b) (i) $f(x) = -4x$ (ii) $f(x) = 3x^2$

   (c) (i) $f(x) = x^2 - 6x$ (ii) $f(x) = x^2 - 3x + 4$

2. Prove from first principles that the derivative of $x^2 + 1$ is $2x$. *[4 marks]*

3. Prove from first principles that the derivative of 8 is zero. *[4 marks]*

4. If k is a constant, prove from first principles that the derivative of $kf(x)$ is $kf'(x)$. *[4 marks]*

12C Rules of differentiation

From Exercise 12B and the results of Self-Discovery Worksheet 3 'Investigating derivatives of polynomials' on the CD-ROM, it seems that we have the following formula for the derivative of a power function.

KEY POINT 12.3

If $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$.

The Fill-in Proof 9 'Differentiating polynomials' on the CD-ROM guides you through deriving this result for positive integer values of n; however, the formula actually holds for all rational powers.

A special case is when $n = 0$: the function is $y = x^0 = 1$, and the formula gives $\frac{dy}{dx} = 0x^{-1} = 0$. The geometric interpretation is that the graph $y = 1$ is a horizontal line and thus has zero gradient everywhere. In fact, the derivative of any constant is zero (see Exercise 12B question 3).

See chapter 2 if you need to review the rules of exponents.

Often you may have to simplify a function using rules of algebra, in particular the laws of exponents, before you can differentiate it.

Worked example 12.4

Find the derivatives of the following functions:

(a) $f(x) = x^2\sqrt{x}$

(b) $g(x) = \dfrac{1}{\sqrt[3]{x}}$

First, rewrite the function in the form x^n using the laws of exponents.

(a)
$$\begin{aligned} f(x) &= x^2\sqrt{x} \\ &= x^2 x^{\frac{1}{2}} \\ &= x^{2+\frac{1}{2}} \\ &= x^{\frac{5}{2}} \end{aligned}$$

Then use the differentiation formula.

$$\begin{aligned} f'(x) &= \frac{5}{2}x^{\frac{5}{2}-1} \\ &= \frac{5}{2}x^{\frac{3}{2}} \end{aligned}$$

Rewrite the function in the form x^n using the laws of exponents.

(b)
$$\begin{aligned} g(x) &= \frac{1}{\sqrt[3]{x}} \\ &= x^{-\frac{1}{3}} \end{aligned}$$

Then use the differentiation formula.

$$\begin{aligned} g'(x) &= -\frac{1}{3}x^{-\frac{1}{3}-1} \\ &= -\frac{1}{3}x^{-\frac{4}{3}} \end{aligned}$$

The results of Exercise 12B questions 1(c) and 4 suggest some properties of differentiation.

KEY POINT 12.4

- If we differentiate $kf(x)$, where k is a constant, we get $kf'(x)$.
- To differentiate a sum, we can differentiate its terms one at a time and then add up the results.

EXAM HINT

Note: you cannot differentiate a product by differentiating each of the factors and multiplying the results together – we will see in chapter 14 that there is a more complicated rule for dealing with products.

See Fill-in Proof 9 'Differentiating polynomials' on the CD-ROM for the derivation of these rules.

The following example illustrates these properties.

Worked example 12.5

Find the derivatives of the following functions.

(a) $f(x) = 5x^3$

(b) $g(x) = x^4 - \frac{3}{2}x^2 + 5x - 4$

(c) $h(x) = \frac{2(2x-7)}{\sqrt{x}}$

Differentiate x^3 and then multiply by 5.

(a) $f'(x) = 5 \times 3x^2$
$= 15x^2$

Differentiate each term separately and add up the results.

(b) $g'(x) = 4x^3 - \frac{3}{2} \times 2x + 5 + 0$
$= 4x^3 - 3x + 5$

We need to write this as a sum of terms of the form x^n.

(c) $h(x) = \frac{2(2x-7)}{\sqrt{x}}$
$= \frac{4x - 14}{x^{\frac{1}{2}}}$

Use the laws of exponents.

$= 4x^{1-\frac{1}{2}} - 14x^{-\frac{1}{2}}$
$= 4x^{\frac{1}{2}} - 14x^{-\frac{1}{2}}$

Now differentiate term by term.

$h'(x) = 4 \times \frac{1}{2}x^{\frac{1}{2}-1} - 14\left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1}$
$= 2x^{-\frac{1}{2}} + 7x^{-\frac{3}{2}}$

Exercise 12C

1. Differentiate the following.

(a) (i) $y = x^4$ (ii) $y = x$

(b) (i) $y = 3x^7$ (ii) $y = -4x^5$

(c) (i) $y = 10$ (ii) $y = -3$

(d) (i) $y = 4x^3 - 5x^2 + 2x - 8$ (ii) $y = 2x^4 + 3x^3 - x$

(e) (i) $y = \frac{1}{3}x^6$ (ii) $y = -\frac{3}{4}x^2$

(f) (i) $y = 7x - \frac{1}{2}x^3$ (ii) $y = 2 - 5x^4 + \frac{1}{5}x^5$

(g) (i) $y=x^{\frac{3}{2}}$ (ii) $y=x^{\frac{2}{3}}$

(h) (i) $y=6x^{\frac{4}{3}}$ (ii) $y=\frac{3}{5}x^{\frac{5}{6}}$

(i) (i) $y=3x^4-x^2+15x^{\frac{2}{5}}-2$ (ii) $y=x^3-\frac{3}{5}x^{\frac{5}{3}}+\frac{4}{3}x^{\frac{1}{2}}$

(j) (i) $y=x^{-1}$ (ii) $y=-x^{-3}$

(k) (i) $y=x^{-\frac{1}{2}}$ (ii) $y=-8x^{-\frac{3}{4}}$

(l) (i) $y=5x-\frac{8}{15}x^{-\frac{5}{2}}$ (ii) $y=-\frac{7}{3}x^{-\frac{3}{7}}+\frac{4}{3}x^{-6}$

2. Find $\frac{dy}{dx}$ for each of the following.

(a) (i) $y=\sqrt[3]{x}$ (ii) $y=\sqrt[5]{x^4}$

(b) (i) $y=\frac{3}{x^2}$ (ii) $y=-\frac{2}{5x^{10}}$

(c) (i) $y=\frac{1}{\sqrt{x}}$ (ii) $y=\frac{8}{3\sqrt[4]{x^3}}$

(d) (i) $y=x^2(3x-4)$ (ii) $y=\sqrt{x}(x^3-2x+8)$

(e) (i) $y=(x+2)(\sqrt[3]{x}-1)$ (ii) $y=\left(x+\frac{2}{x}\right)^2$

(f) (i) $y=\frac{3x^5-2x}{x^2}$ (ii) $y=\frac{9x^2+3}{2\sqrt[3]{x}}$

3. Find $\frac{dy}{dx}$ if

(a) (i) $x+y=8$ (ii) $3x-2y=7$

(b) (i) $y+x+x^2=0$ (ii) $y-x^4=2x$

12D Interpreting derivatives and second derivatives

The derivative $\frac{dy}{dx}$ has two related interpretations:

- It is the gradient of the graph of y against x.
- It measures how fast y changes when x is changed, that is, the **rate of change** of y with respect to x.

EXAM HINT

We can also write this using function notation: if $f(x) = x^2$, then $f'(x) = 2x$; so $f'(3) = 6$ and $f'(-1) = -2$.

Remember that $\frac{dy}{dx}$ is itself a function whose value depends on x.

For example, if $y = x^2$, then $\frac{dy}{dx} = 2x$; so $\frac{dy}{dx}$ is equal to 6 when $x = 3$, and it is equal to -2 when $x = -1$. This corresponds to the fact that the gradient of the graph of $y = x^2$ changes with x, or that the rate of change of y varies with x.

To calculate the gradient (or the rate of change) at any particular point, we simply substitute the value of x into the expression for the derivative.

Worked example 12.6

Find the gradient of the graph of $y = 4x^3$ at the point where $x = 2$.

The gradient is given by the derivative, so find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = 12x^2$$

Substitute the given value for x.

When $x = 2$,

$$\frac{dy}{dx} = 12 \times 2^2 = 48$$

So the gradient is 48

EXAM HINT

Most calculators are not able to find the general expression for the derivative of a function, but can find the gradient of a curve at a specific point. See Calculator Skills sheet 8 on the CD-ROM for instructions on how to do this.

If we know the gradient of a graph at a particular point, we can find the value of x at that point. This involves solving an equation involving $\frac{dy}{dx}$.

Worked example 12.7

Find the values of x for which the graph of $y = x^3 - 7x + 1$ has gradient 5.

The gradient is given by the derivative.

$$\frac{dy}{dx} = 3x^2 - 7$$

We know the value of $\frac{dy}{dx}$, so we can write down an equation for x and solve it.

$$3x^2 - 7 = 5$$
$$\Leftrightarrow 3x^2 = 12$$
$$\Leftrightarrow x^2 = 4$$
$$\Leftrightarrow x = 2 \text{ or } -2$$

The sign of the gradient tells us whether the function is increasing or decreasing.

KEY POINT 12.5

If $\frac{dy}{dx}$ is positive, the function is increasing: as x gets larger, so does y.

If $\frac{dy}{dx}$ is negative, the function is decreasing: as x gets larger, y gets smaller.

In section 12H we will discuss what happens when $\frac{dy}{dx} = 0$.

Worked example 12.8

Find the range of values of x for which the function $f(x) = 2x^2 - 6x$ is decreasing.

A decreasing function has negative gradient.

$$f'(x) < 0$$
$$\Leftrightarrow 4x - 6 < 0$$
$$\Leftrightarrow x < 1.5$$

You may wonder why it is important to emphasise that we are differentiating with respect to x (or Q or 'monkeys'). In this course we only consider functions of one variable, but it is possible to generalise calculus to deal with functions that depend on several variables. Multivariable calculus has many applications, particularly in physics and engineering.

There is nothing special about the variables y and x. We can just as well say that $\frac{dB}{dQ}$ is the gradient of the graph of B against Q or that $\frac{d(\text{bananas})}{d(\text{monkeys})}$ measures how fast the 'bananas' variable changes when you change the variable 'monkeys'.

To emphasise which variables we are using, we call $\frac{dy}{dx}$ the *derivative of y with respect to x*.

Worked example 12.9

Given that $a = \sqrt{S}$, find the rate of change of a when $S = 9$.

The rate of change is given by the derivative.

$$a = S^{\frac{1}{2}}$$

$$\frac{da}{dS} = \frac{1}{2}S^{-\frac{1}{2}}$$

$$= \frac{1}{2\sqrt{S}}$$

Substitute the given value for S.

When $S = 9$,

$$\frac{da}{dS} = \frac{1}{2\sqrt{9}}$$

$$= \frac{1}{6}$$

$\frac{d}{dx}$ is an example of what is called an *operator* – something that acts on functions to turn them into other functions. In this case, the operation that $\frac{d}{dx}$ performs on a function is differentiation with respect to x. For example, we can think of differentiating $y = 3x^2$ as applying the $\frac{d}{dx}$ operator to both sides of the equation:

$$\frac{d}{dx}(y) = \frac{d}{dx}(3x^2)$$

$$\Rightarrow \frac{dy}{dx} = 6x$$

So $\frac{dy}{dx}$ just means the operator $\frac{d}{dx}$ applied to y.

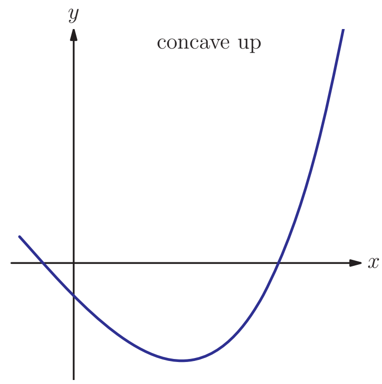
Since, as discussed earlier, the derivative $\frac{dy}{dx}$ is itself a function of x, we can also apply the $\frac{d}{dx}$ operator to it. The result is called the **second derivative**.

KEY POINT 12.6

The second derivative $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is denoted by $\frac{d^2y}{dx^2}$ or $f''(x)$ and measures the rate of change of the gradient.

The sign of the second derivative tells us whether the gradient is increasing or decreasing.

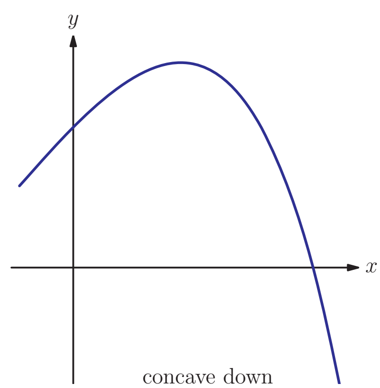
If the gradient is increasing, the curve is said to be 'concave-up'. If the gradient is decreasing, the curve is described as 'concave-down'.



KEY POINT 12.7

If $\frac{d^2y}{dx^2} > 0$, the curve is concave-up.

If $\frac{d^2y}{dx^2} < 0$, the curve is concave-down.



We can differentiate the second derivative to get the third derivative, denoted by $\frac{d^3y}{dx^3}$ or $f'''(x)$, and then differentiate again to find the fourth derivative, written $\frac{d^4y}{dx^4}$ or $f^{(4)}(x)$, and so on.

Worked example 12.10

Let $f(x) = 5x^3 - 4x$.

(a) Find $f''(x)$.

(b) Find the rate of change of the gradient of the graph of $y = f(x)$ at the point where $x = -1$.

Differentiate $f(x)$ and then differentiate the result.

(a) $f'(x) = 15x^2 - 4$

$f''(x) = 30x$

The rate of change of the gradient means the second derivative.

(b) $f''(-1) = -30$

Exercise 12D

1. Write the following rates of change as derivatives.
   (a) The rate of change of z as t changes.
   (b) The rate of change of Q with respect to p.
   (c) How fast R changes when m is changed.
   (d) How quickly the volume of a balloon (V) changes over time (t).
   (e) The rate of increase of the cost of apples (y) as the average weight of an apple (x) increases.
   (f) The rate of change of the rate of change of z as y changes.
   (g) The second derivative of H with respect to m.

2. (a) (i) If $f = 5x^{\frac{1}{3}}$, what is the derivative of f with respect to x?
   (ii) If $p = 3q^5$, what is the derivative of p with respect to q?
   (b) (i) Differentiate $d = 3t + 7t^{-1}$ with respect to t.
   (ii) Differentiate $r = c + \frac{1}{c}$ with respect to c.
   (c) (i) Find the second derivative of $y = 9x^2 + x^3$ with respect to x.
   (ii) Find the second derivative of $z = \frac{3}{t}$ with respect to t.

3. (a) (i) If $y = 5x^2$, find $\frac{dy}{dx}$ when $x = 3$.
   (ii) If $y = x^3 + \frac{1}{x}$, find $\frac{dy}{dx}$ when $x = 1.5$.
   (b) (i) If $A = 7b + 3$, find $\frac{dA}{db}$ when $b = -1$.
   (ii) If $\phi = \theta^2 + \theta^{-3}$, find $\frac{d\phi}{d\theta}$ when $\theta = 0.1$.
   (c) (i) Find the gradient of the graph of $A = x^3$ when $x = 2$.
   (ii) Find the gradient of the tangent to the graph of $z = 2a + a^2$ when $a = -6$.
   (d) (i) How quickly does $f = 4T^2$ change as T changes when $T = 3$?
   (ii) How quickly does $g = y^4$ change as y changes when $y = 2$?
   (e) (i) What is the rate of increase of W with respect to p when p is -3 if $W = -p^2$?
   (ii) What is the rate of change of L with respect to c when $c = 6$ if $L = 7\sqrt{c} - 8$?

You may consider it paradoxical to talk about the rate of change of y as x changes when we are fixing x at a certain value; think of it as the rate at which y is changing at the instant when x is passing through this particular point.

4. (a) (i) If $y = ax^2 + (1-a)x$ where a is a constant, find $\frac{dy}{dx}$.

(ii) If $y = x^3 + b^2$ where b is a constant, find $\frac{dy}{dx}$.

(b) (i) If $Q = \sqrt{ab} + \sqrt{b}$ where b is a constant, find $\frac{dQ}{da}$.

(ii) If $D = 3(av)^2$ where a is a constant, find $\frac{dD}{dv}$.

5. (a) (i) If $y = x^3 - 5x$, find $\frac{d^2y}{dx^2}$ when $x = 9$.

(ii) If $y = 8 + 2x^4$, find $\frac{d^2y}{dx^2}$ when $x = 4$.

(b) (i) If $S = 3A^2 + \frac{1}{A}$, find $\frac{d^2S}{dA^2}$ when $A = 1$.

(ii) If $J = v - \sqrt{v}$, find $\frac{d^2J}{dv^2}$ when $v = 9$.

(c) (i) Find the second derivative of B with respect to n if $B = 8n$ and $n = 2$.

(ii) Find the second derivative of g with respect to r if $g = r^7$ and $r = 1$.

6. (a) (i) Given that $y = 3x^3$ and $\frac{dy}{dx} = 36$, find x.

(ii) Given that $y = x^4 + 2x$ and $\frac{dy}{dx} = 6$, find x.

(b) (i) If $y = 2x + \frac{8}{x}$ and $\frac{dy}{dx} = -30$, find y.

(ii) If $y = \sqrt{x} + 3$ and $\frac{dy}{dx} = \frac{1}{6}$, find y.

7. (a) (i) Find the interval in which $x^2 - x$ is an increasing function.

(ii) Find the interval in which $x^2 + 2x - 5$ is a decreasing function.

(b) (i) Find the interval in which $y = x^3 - 3x^2$ is concave-up.

(ii) Find the interval in which $y = x^3 + 5x$ is concave-down.

8. Show that $y = x^3 + kx + c$ is always increasing if $k > 0$.

[4 marks]

9. Find all points on the graph of $y = x^3 - 2x^2 + 1$ where the gradient equals the y-coordinate. *[5 marks]*

10. In what interval is the gradient of the graph of $y = 7x - x^2 - x^3$ decreasing? *[5 marks]*

11. Find an alternative expression for $\frac{d^n}{dx^n}(x^n)$.

12E Differentiating trigonometric functions

Using the techniques from section 12A we can sketch the derivative of the graph of $y = \sin x$. The result is a graph that looks just like $y = \cos x$. See Fill-in Proof 11 'Differentiating trigonometric functions' on the CD-ROM to find out why this is the case.

The derivatives of $y = \cos x$ and $y = \tan x$ can be established in a similar manner, giving the following results.

EXAM HINT

Whenever you are doing calculus you must work in radians.

KEY POINT 12.8

$$\frac{d}{dx}(\sin x) = \cos x$$

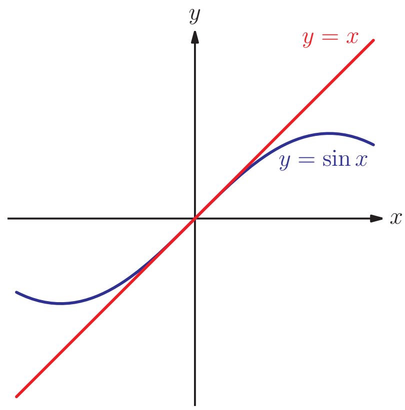
$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x}$$

In section 14C, we will derive the result for $\tan x$ by using the results for differentiating $\sin x$ and $\cos x$.

It is important to remember that these formulas are valid only if x is measured in radians. This is because they are based on the assumption that the value of $\sin x$ is very close to x when x is small; you can use your calculator to confirm that this is true for x in radians but not for x in degrees.

See Fill-in Proof 10 'Small angle approximations' on the CD-ROM for a derivation of the $\sin x \approx x$ approximation, which is also shown on the graph on the next page.



It is possible to do calculus using degrees, or any other unit for measuring angles, but using radians gives the simplest rules, which is why they are the unit of choice for mathematicians.

All the rules of differentiation from section 12C apply to trigonometric functions as well.

Worked example 12.11

Differentiate $y = 3\tan x - 2\cos x$.

Differentiate term by term, using the formulas in Key point 12.8.

$$\frac{dy}{dx} = 3\left(\frac{1}{\cos^2 x}\right) - 2(-\sin x)$$

$$= \frac{3}{\cos^2 x} + 2\sin x$$

Exercise 12E

1. Differentiate the following.

(a) (i) $y = 3\sin x$ (ii) $y = 2\cos x$

(b) (i) $y = 2x - 5\cos x$ (ii) $y = \tan x + 5$

(c) (i) $y = \dfrac{\sin x + 2\cos x}{5}$ (ii) $y = \dfrac{1}{2}\tan x - \dfrac{1}{3}\sin x$

2. Find the gradient of $f(x) = \sin x + x^2$ at the point $x = \dfrac{\pi}{2}$. *[5 marks]*

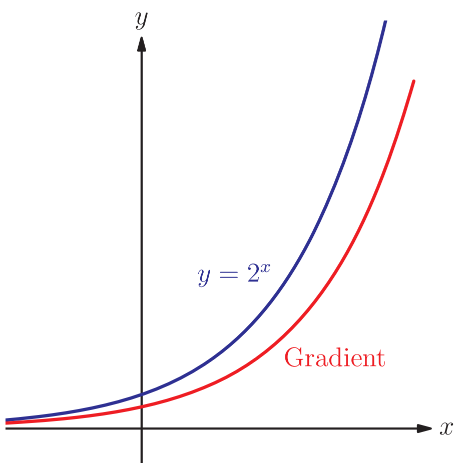
3. Find the gradient of $g(x) = \dfrac{1}{4}\tan x - 3\cos x - x^3$ at the point $x = \dfrac{\pi}{6}$. *[5 marks]*

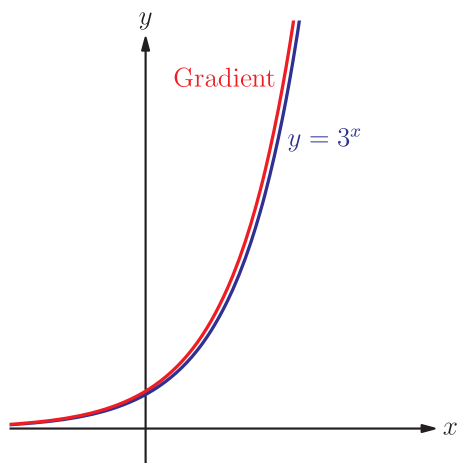
4. Given $h(x) = \sin x + \cos x$ for $0 \le x < 2\pi$, find the values of x for which $h'(x) = 0$. *[6 marks]*

5. Given $y = \frac{1}{4}\tan x + \frac{1}{x^2}$ for $0 < x \le 2\pi$, solve the equation $\frac{dy}{dx} = 1 - \frac{2}{x^3}$. *[6 marks]*

12F Differentiating exponential and natural logarithm functions

Use your calculator to sketch the graphs of $y = 2^x$ and $y = 3^x$ and their derivatives. The derivative graphs look like exponential functions too.





Notice that the graph of the derivative of $y = 2^x$ lies below the graph of $y = 2^x$ itself, whereas the graph of the derivative of $y = 3^x$ lies slightly above the graph of $y = 3^x$ itself. It seems that there should be a number *a* somewhere between 2 and 3 for which the graph of the derivative of $y = a^x$ would be exactly the same as the graph of $y = a^x$ itself. It turns out that this number '*a*' is e = 2.71828… which we met in section 2C.

KEY POINT 12.9

$$\frac{d}{dx}(e^x) = e^x$$

The derivative of the natural logarithm (the logarithm with base e) $y = \ln x$ is somewhat surprising, being of a completely different form from the original function.

KEY POINT 12.10

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

See Fill-in Proof 12 ‘Differentiating logarithmic functions graphically’ on the CD-ROM for a derivation of this result.

Worked example 12.12

Differentiate $y = 2e^x + 3\ln x + 4x$.

Differentiate term by term, using the formulas in Key points 12.9 and 12.10.

$$\frac{dy}{dx} = 2e^x + \frac{3}{x} + 4$$

Exercise 12F

1. Differentiate the following.

(a) (i) $y = 3e^x$ (ii) $y = \frac{2e^x}{5}$

(b) (i) $y = -2\ln x$ (ii) $y = \frac{1}{3}\ln x$

(c) (i) $y = \frac{\ln x}{5} - 3x + 4e^x$ (ii) $y = 4 - \frac{e^x}{2} + 3\ln x$

2. Find the value of the gradient of the graph of $f(x) = \frac{1}{2}e^x - 7\ln x$ at the point $x = \ln 4$. *[2 marks]*

3. Find the exact value of the gradient of the graph of $f(x) = e^x - \frac{\ln x}{2}$ when $x = \ln 3$. *[2 marks]*

4. Find the value of x where the gradient of $f(x) = 5 - 2e^x$ is -6.

[4 marks]

5. Find the interval in which $e^x - 2x$ is an increasing function.

[5 marks]

There is an easier way to do some of the parts in this question, using a method from chapter 14. For now, you will have to rely on your algebra skills!

6. Find the value of x at which the gradient of $g(x) = x^2 - 12\ln x$ is 2.

[4 marks]

7. Differentiate the following.

(a) (i) $y = \ln x^3$ (ii) $y = \ln 5x$

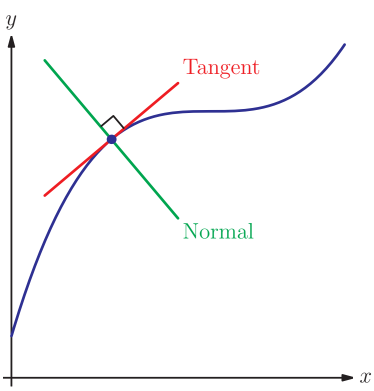
(b) (i) $y = e^{x+3}$ (ii) $y = e^{x-3}$

(c) (i) $y = e^{2\ln x}$ (ii) $y = e^{3\ln x + 2}$

12G Tangents and normals

See Prior Learning section R on the CD-ROM for how to find the equation of a straight line from its gradient and a point on the line.

The tangent to a curve at a given point is a straight line which touches the curve at that point and (by definition) has the same gradient as the curve at that point. Therefore, if we need to find the equation of the tangent, we first have to know the gradient of the curve at that point, which can be obtained by differentiating the curve function. Once we have both the gradient of the tangent and the coordinates of the point where it touches the curve, we can apply the standard procedure for finding the equation of a straight line.



The **normal** to a curve at a given point is a straight line which crosses the curve at that point and is perpendicular to the tangent at that point. Normals have many uses, such as in finding the centres of curvature of shapes and in working out how light is reflected from curved mirrors. In the International Baccalaureate® you are only likely to be expected to find their equations. To do this, use the fact that if two (non-horizontal, non-vertical) lines are perpendicular, their gradients m_1 and m_2 are related by $m_1 m_2 = -1$.

Worked example 12.13

(a) Find the equation of the tangent to the function $f(x) = \cos x + e^x$ at the point $x = 0$.

(b) Find the equation of the normal to the function $g(x) = x^3 - 5x^2 - x^{\frac{3}{2}} + 22$ at $(4, -2)$.

In each case give your answer in the form $ax + by + c = 0$, where a, b and c are integers.

We need the gradient at $x = 0$, which is $f'(0)$.

(a)

$$f'(x) = -\sin x + e^x$$
$$\therefore f'(0) = -\sin 0 + e^0 = 1$$

We also need coordinates of the point at which the tangent touches the graph. This is where $x = 0$. The corresponding y-coordinate is $f(0)$.

When $x = 0$,

$$\begin{aligned} y &= f(0) \\ &= \cos 0 + e^0 \\ &= 1 + 1 \\ &= 2 \end{aligned}$$

Put the information into the general equation of a line.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 2 &= 1(x - 0) \\ \Leftrightarrow \quad y &= x + 2 \\ \Leftrightarrow x - y + 2 &= 0 \end{aligned}$$

The normal is perpendicular to the tangent, so find the gradient of the tangent at $x = 4$ first.

(b)

$$f'(x) = 3x^2 - 10x - \frac{3}{2}x^{\frac{1}{2}}$$
$$\begin{aligned} \therefore f'(4) &= 3(4)^2 - 10(4) - \frac{3}{2}(4)^{\frac{1}{2}} \\ &= 48 - 40 - 3 \\ &= 5 \end{aligned}$$

For perpendicular lines, $m_1 m_2 = -1$.

Therefore gradient of normal is

$$m = \frac{-1}{5}$$

Both x- and y-coordinates of the point are given, so we can put all the information into the general equation of a line.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-2) &= \frac{-1}{5}(x - 4) \\ \Leftrightarrow \quad 5y + 10 &= -x + 4 \\ \Leftrightarrow x + 5y + 6 &= 0 \end{aligned}$$

EXAM HINT

Your calculator may be able to find the equation of the tangent at a given point.

We summarise the procedure for finding equations of tangents and normals as follows.

KEY POINT 12.11

At the point on the curve $y = f(x)$ with $x = a$:

- the gradient of the tangent is $f'(a)$
- the gradient of the normal is $\frac{-1}{f'(a)}$
- the coordinates of the point are $x_1 = a, y_1 = f(a)$

To find the equation of the tangent or the normal, use $y - y_1 = m(x - x_1)$ with the appropriate gradient.

Sometimes you may be given limited information about the tangent and have to use this to find out other information.

Worked example 12.14

The tangent at point P on the curve $y = x^2 + 1$ passes through the origin. Find the possible coordinates of P.

We can find the equation of the tangent at P, but we need to use unknowns for the coordinates of P.	Let P have coordinates (p,q).
As P lies on the curve, (p,q) must satisfy $y = x^2 + 1$.	Then $q = p^2 + 1$
The gradient of the tangent is given by $\frac{dy}{dx}$ with $x = p$.	$\frac{dy}{dx} = 2x$ When $x = p$, $\frac{dy}{dx} = 2p$ $\therefore m = 2p$
Write down the equation of the tangent.	Equation of the tangent: $y - q = 2p(x - p)$ $y - (p^2 + 1) = 2p(x - p)$
The tangent passes through the origin, so set $x = 0$ and $y = 0$ in the equation.	Passes through (0,0): $0 - (p^2 + 1) = 2p(0 - p)$ $\Leftrightarrow -p^2 - 1 = -2p^2$ $\Leftrightarrow p^2 = 1$ Hence $p = 1$ or -1

continued . . .

Now find the corresponding y-coordinate q.

When $p=1$, $q=2$
When $p=-1$, $q=2$
So the coordinates of P are $(1,2)$ or $(-1,2)$.

Exercise 12G

1. Find the equations of the tangent and the normal to each of the following curves at the given point. Write your equations in the form $ax+by+c=0$.

   (a) $y=\dfrac{x^2+4}{\sqrt{x}}$ at $x=4$

   (b) $y=3\tan x-2\sqrt{2}\sin x$ at $x=\dfrac{\pi}{4}$

2. Find the equation of the normal to the curve $y=3-\dfrac{1}{5}\mathrm{e}^x$ at $x=2\ln 5$. *[7 marks]*

3. Find the coordinates of the point at which the tangent to the curve $y=x^3-3x^2$ at $x=2$ meets the curve again. *[6 marks]*

4. Find the x-coordinates of the points on the curve $y=x^3-3x^2$ where the tangent is parallel to the normal to the curve at $(1,-2)$. *[6 marks]*

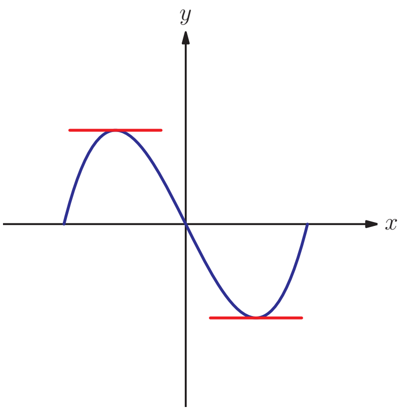
5. Find the equation of the tangent to the curve $y=\mathrm{e}^x+x$ which is parallel to $y=3x$. *[4 marks]*

6. Find the coordinates of the point on the curve $y=(x-1)^2$ for which the normal passes through the origin. *[5 marks]*

7. A tangent is drawn on the graph $y=\dfrac{k}{x}$ at the point where $x=a$ $(a>0)$. The tangent intersects the y-axis at P and the x-axis at Q. If O is the origin, show that the area of the triangle OPQ is independent of a. *[8 marks]*

8. Show that any tangent to the curve $y=x^3-x$ at the point with x-coordinate a meets the curve again at a point with x-coordinate $-2a$. *[6 marks]*

12H Stationary points

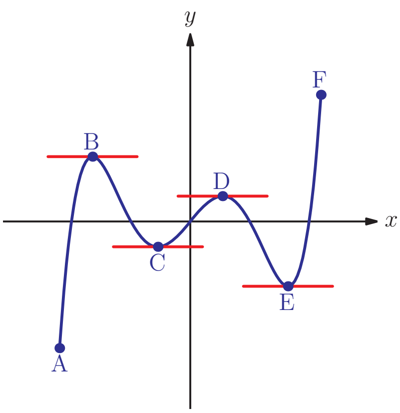


In real life people might be interested in maximising profits or minimising the drag on a car. We can use calculus to describe such goals mathematically as points on a graph.

Note that at both the maximum point and the minimum point on the graph, the gradient is zero.

KEY POINT 12.12

To find local maximum and local minimum points, solve the equation $\frac{dy}{dx} = 0$.



We use the phrases **local maximum** and **local minimum** because it is possible that the overall largest or smallest value of the function occurs at an endpoint of the graph, or that there are multiple peaks or troughs on the graph. The word 'local' means that each peak or trough just represents the largest or smallest y-value in that particular 'neighbourhood' on the graph.

Points at which the graph has a gradient of zero are called **stationary points**.

Worked example 12.15

Find the coordinates of the stationary points of $y = 2x^3 - 15x^2 + 24x + 8$.

Stationary points have $\frac{dy}{dx} = 0$, so we need to differentiate and then set the derivative equal to zero.

$$\frac{dy}{dx} = 6x^2 - 30x + 24$$

For stationary points $\frac{dy}{dx} = 0$:

$$6x^2 - 30x + 24 = 0$$
$$\Leftrightarrow x^2 - 5x + 4 = 0$$
$$\Leftrightarrow (x-4)(x-1) = 0$$
$$\Leftrightarrow x = 1 \text{ or } x = 4$$

Remember to find the y-coordinate for each point.

When $x = 1$:

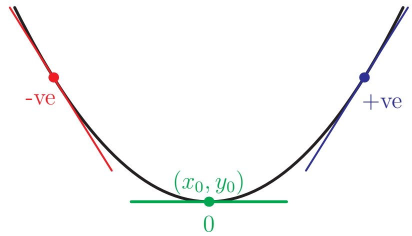
$$y = 2(1)^3 - 15(1)^2 + 24(1) + 8 = 19$$

When $x = 4$:

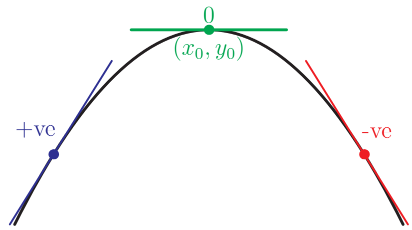
$$y = 2(4)^3 - 15(4)^2 + 24(4) + 8 = -8$$

Therefore, stationary points are $(1, 19)$ and $(4, -8)$

The calculation in Worked example 12.15 does not tell us whether the stationary points we found are maximum or minimum points. One way of testing for the nature of a stationary point is to check the gradient on either side of the point by substituting nearby x-values into the expression for $\frac{dy}{dx}$. For a minimum point, the gradient goes from negative to positive as x moves from left to right through the stationary point; for a maximum point, the gradient changes from positive to negative.



We can also interpret these conditions in terms of the rate of change of the gradient – that is, the second derivative $\frac{d^2y}{dx^2}$. At a minimum point, the gradient is increasing (changing from negative to positive) and hence $\frac{d^2y}{dx^2}$ is positive; at a maximum point, the gradient is decreasing (changing from positive to negative) and so $\frac{d^2y}{dx^2}$ is negative. This leads to the following test.



KEY POINT 12.13

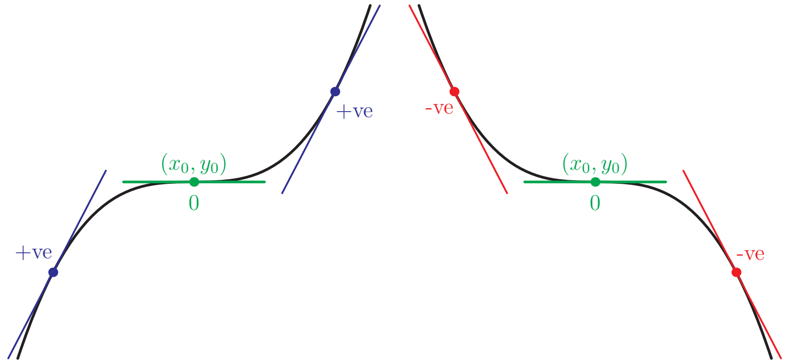
Given a stationary point (x_0, y_0) of a function $y = f(x)$:

- if $\frac{d^2y}{dx^2} < 0$ at x_0, then (x_0, y_0) is a *maximum*
- if $\frac{d^2y}{dx^2} > 0$ at x_0, then (x_0, y_0) is a *minimum*
- if $\frac{d^2y}{dx^2} = 0$ at x_0, then no conclusion can be drawn, so check the sign of the gradient $\frac{dy}{dx}$ on either side of (x_0, y_0)

EXAM HINT

You can find maximum and minimum points on your calculator, as described on Calculator Skills sheet 4. However, questions of this type often appear on the non-calculator paper.

All local maximum and local minimum points have $\frac{dy}{dx} = 0$, but the converse is not true: a point with $\frac{dy}{dx} = 0$ does not have to be a maximum or a minimum point. There are two other possibilities:



The stationary points labelled (x_0, y_0) on the above graphs are called **points of inflexion**. Note that at these points, the line with zero gradient actually passes through the curve. The gradient of the curve is either positive on both sides of a point of inflexion (which is then called a positive point of inflexion, like the one on the left-hand graph), or negative on both sides (in which case we have a negative point of inflexion, like the one on the right-hand graph).

In UK English, 'inflexion' might be spelt 'inflection'.

If a question asks you to 'find and classify' the stationary points on a curve, it means you have to find the coordinates of all points which have $\frac{dy}{dx} = 0$ and decide whether each one is a maximum point, minimum point or point of inflexion. This may also be referred to as the 'nature' of the stationary points.

Worked example 12.16

Find and classify the stationary points of $y = 3 + 4x^3 - x^4$.

Stationary points have $\frac{dy}{dx} = 0$.

$$\frac{dy}{dx} = 12x^2 - 4x^3$$

For stationary points $\frac{dy}{dx} = 0$:

$$12x^2 - 4x^3 = 0$$
$$\Leftrightarrow 4x^2(3 - x) = 0$$
$$\Leftrightarrow x = 0 \text{ or } x = 3$$

continued . . .

Find the y-coordinates.

When $x = 0$:

$$y = 3 + 4(0)^3 - (0)^4 = 3$$

When $x = 3$:

$$y = 3 + 4(3)^3 - (3)^4 = 30$$

Therefore, stationary points are $(0,3)$ and $(3,30)$

Use the second derivative to determine the nature of each stationary point.

$$\frac{d^2y}{dx^2} = 24x - 12x^2$$

At $x = 0$:

$$\frac{d^2y}{dx^2} = 24(0) - 12(0)^2 = 0$$

As $\frac{d^2y}{dx^2} = 0$, we need to check the gradient on either side of the stationary point.

Inconclusive, so examine $\frac{dy}{dx}$ on either side of $x = 0$:

At $x = -1$:

$$\frac{dy}{dx} = 12(-1)^2 - 4(-1)^3 = 12 + 4 = 16 > 0$$

At $x = 1$:

$$\frac{dy}{dx} = 12(1)^2 - 4(1)^3 = 12 - 4 = 8 > 0$$

$\therefore$ $(0, 3)$ is a positive point of inflexion.

At $x = 3$:

$$\frac{d^2y}{dx^2} = 24(3) - 12(3)^2 = 72 - 108 = -36 < 0$$

$\therefore$ $(3, 30)$ is a local maximum.

When $\frac{d^2y}{dx^2} = 0$, be careful not to jump to the conclusion that the stationary point is a point of inflexion – this is not necessarily the case, as the next example shows.

Worked example 12.17

Find the coordinates and nature of the stationary points of $f(x) = x^4$.

Stationary points have $f'(x) = 0$.

$f'(x) = 4x^3$

For stationary points $f'(x) = 0$:

$$4x^3 = 0$$
$$\Leftrightarrow x = 0$$

Find the y-coordinate.

$$f(0) = 0$$

Therefore, stationary point is $(0, 0)$

Look at $f''(x)$ to determine the nature of the stationary point..

$$f''(x) = 12x^2$$
$$f''(0) = 0$$

As $f''(0) = 0$, we need to check the gradient on either side of $x = 0$.

Therefore, examine $f'(x)$:

$f'(-1) = 4(-1)^3$
$= -4 < 0$

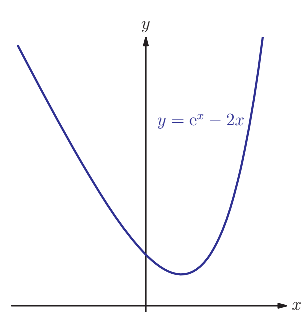
$f'(1) = 4(1)^3$
$= 4 > 0$

$\therefore$ $(0, 0)$ is a local minimum.

You can use stationary points to determine the range of a function.

Worked example 12.18

The graph shows $f(x) = e^x - 2x$. Find the range of $f(x)$.



continued . . .

From the graph it is clear that $f(x)$ can take any value above the minimum point. So we need to find the minimum point.

$$f'(x) = e^x - 2$$

At the minimum point $f'(x) = 0$:

$$e^x - 2 = 0$$
$$\Leftrightarrow e^x = 2$$
$$\Leftrightarrow x = \ln 2$$

When $x = \ln 2$,

$$f(x) = e^{\ln 2} - 2\ln 2$$
$$= 2 - 2\ln 2$$

This is the minimum value of $f(x)$, so the range of $f(x)$ is

$$f(x) \geq 2 - 2\ln 2$$

See section 4B for a reminder of range and domain.

Exercise 12H

1. Find and classify the stationary points on the following curves.

(a) (i) $y = x^3 - 5x^2$ (ii) $y = x^4 - 8x^2$

(b) (i) $y = \sin x + \frac{x}{2}, \ -\pi \leq x \leq \pi$

(ii) $y = 2\cos x + 1, \ 0 \leq x < 2\pi$

(c) (i) $y = \ln x - \sqrt{x}$ (ii) $y = 2e^x - 5x$

2. Find and classify the stationary points on the curve $y = x^3 + 3x^2 - 24x + 12$. *[6 marks]*

3. Find the coordinates of the stationary point on the curve $y = x - \sqrt{x}$ and determine its nature. *[6 marks]*

4. Find and classify the stationary points on the curve $y = \sin x + 4\cos x$ in the interval $0 < x < 2\pi$. *[6 marks]*

5. Show that the function $f(x) = \ln x + \frac{1}{x^k}$ has a stationary point with y-coordinate $\frac{\ln(k) + 1}{k}$. *[6 marks]*

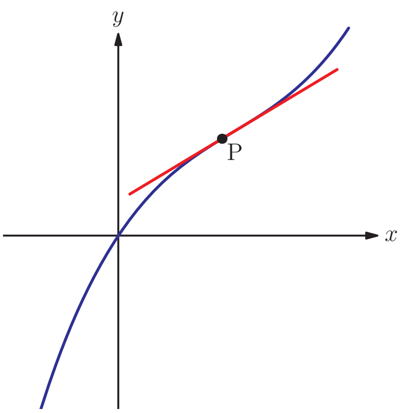
6. Find the range of the function $f : x \mapsto 3x^4 - 16x^3 + 18x^2 + 6$. *[5 marks]*

7. Find the range of the function $f : x \mapsto e^x - 4x + 2$. *[5 marks]*

8. Find in terms of k the stationary points on the curve $y = kx^3 + 6x^2$ and determine their nature. *[6 marks]*

12I General points of inflexion

In the previous section we met stationary points of inflexion, but the idea of a point of inflexion is more general than this.



One definition of a point of inflexion is as a point where the tangent to the curve crosses the curve at that point. This does not require the point to be a stationary point. Geometrically, the portion of the graph around a point of inflexion can be interpreted as an 'S-bend' – a curve whose gradient goes from decreasing to increasing (as shown in the diagram) or vice versa; this is equivalent to the curve switching from concave-down to concave-up or vice versa. At the point of inflexion itself, the gradient is neither decreasing nor increasing.

EXAM HINT

Although the red line actually crosses the graph at P, it is still referred to as the tangent, because it has the same gradient as the curve at P.

KEY POINT 12.14

At a point of inflexion, $\frac{d^2y}{dx^2} = 0$.

Although a point of inflexion must have zero second derivative, the converse is not true: just because a point has $\frac{d^2y}{dx^2} = 0$, it is not necessarily a point of inflexion. You can see this from the function $f(x) = x^4$ of Worked example 12.17: at $x = 0$ we have $f''(x) = 0$, but $f''(x) = 12x^2$ is positive on *both* sides of $x = 0$, which means that the gradient is increasing on both sides; so $x = 0$ is not a point of inflexion.

Worked example 12.19

Find the coordinates of the point of inflexion on the curve $y = x^3 - 3x^2 + 5x - 1$.

Find $\frac{d^2y}{dx^2}$.

$$\frac{dy}{dx} = 3x^2 - 6x + 5$$

$$\frac{d^2y}{dx^2} = 6x - 6$$

At a point of inflexion $\frac{d^2y}{dx^2} = 0$:

$$6x - 6 = 0$$

$$\Leftrightarrow x = 1$$

Remember to calculate the other coordinate!

When $x = 1$, $y = 1 - 3 + 5 - 1 = 2$

So point of inflexion is at (1, 2)

EXAM HINT

If a question states that a curve has a point of inflexion and you find only one solution to the equation $\frac{d^2y}{dx^2} = 0$, you can then assume that you have found the point of inflexion, with no need to check the sign of $\frac{d^2y}{dx^2}$ on either side.

Exercise 12I

1. Find the coordinates of the point of inflexion on the curve $y = e^x - x^2$. *[5 marks]*

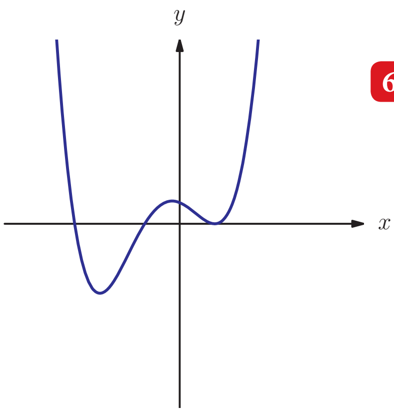
2. The curve $y = x^4 - 6x^2 + 7x + 2$ has two points of inflexion. Find their coordinates. *[5 marks]*

3. Show that all points of inflexion on the curve $y = \sin x$ lie on the x-axis. *[6 marks]*

4. Find the coordinates of the points of inflexion on the curve $y = 2\cos x + x$ for $0 \le x \le 2\pi$. Justify carefully that these points are points of inflexion. *[5 marks]*

5. The point of inflexion on the curve $y = x^3 - ax^2 - bx + c$ is a stationary point of inflexion. Show that $b = -\frac{a^2}{3}$. *[6 marks]*

6. The graph shows $y = f'(x)$.



On a copy of the graph:

(a) mark points corresponding to a local minimum of $f(x)$ with an A

(b) mark points corresponding to a local maximum of $f(x)$ with a B

(c) mark points corresponding to a point of inflexion of $f(x)$ with a C. *[6 marks]*

12J Optimisation

We now have enough tools to start using differentiation to maximise or minimise quantities.

KEY POINT 12.15

To maximise or minimise quantity A by changing quantity B, we follow a four-stage procedure:

1. Find the relationship between A and B.
2. Solve the equation $\frac{\mathrm{d}A}{\mathrm{d}B} = 0$ to find stationary points.
3. Determine whether each stationary point is a local maximum, local minimum or point of inflexion by checking $\frac{\mathrm{d}^2A}{\mathrm{d}B^2}$ and, if necessary, the sign of $\frac{\mathrm{d}A}{\mathrm{d}B}$ on either side of the point.
4. Check whether each end point of the domain is actually a global maximum or global minimum point, and check that there are no vertical asymptotes.

The first stage of this process is often the most difficult, and there are many situations in which we have to make this link in a geometric context. Fortunately, in many questions this relationship is given to you.

Worked example 12.20

The height h in metres of a swing above the ground at time t seconds is given by $h = 2 - 1.5\sin t$ for $0 < t < 3$. Find the minimum and maximum heights of the swing.

Find stationary points.

$\frac{dh}{dt} = -1.5\cos t = 0$ at a stationary point

$\Rightarrow \cos t = 0$

$0 < t < 3 \therefore t = \frac{\pi}{2}$ (only one solution)

Classify stationary points.

$\frac{d^2h}{dt^2} = 1.5\sin t$

When $t = \frac{\pi}{2}$:

$\frac{d^2h}{dt^2} = 1.5 > 0$,

so $t = \frac{\pi}{2}$ is a local minimum.

The minimum height is

$h = 2 - 1.5\sin\frac{\pi}{2} = 0.5$ metres.

There are no vertical asymptotes. Check end points.

When $t = 0, h = 2$

When $t = 3, h = 1.79$

So maximum height is 2 metres.

Exercise 12J

1. What are the minimum and maximum values of e^x for $0 \le x \le 1$? *[4 marks]*

2. A rectangle has width x metres and length $30 - x$ metres.

(a) Find the maximum area of the rectangle.

(b) Show that as x changes the perimeter stays constant, and find the value of this perimeter. *[5 marks]*

3. Find the maximum and minimum values of the function $y = x^3 - 9x$ for $-2 \le x \le 5$. *[4 marks]*

4. What are the maximum and minimum values of $f(x) = e^x - 3x$ for $0 \le x \le 2$? *[5 marks]*

5. What are the minimum and maximum values of $y = \sin x + 2x$ for $0 \le x \le 2\pi$? *[5 marks]*

6. Find the minimum value of the sum of a positive real number and its reciprocal. *[5 marks]*

7. A paper aeroplane of weight $w > 1$ will travel at a constant speed of $1 - \frac{1}{\sqrt{w}}$ metres per second for $\frac{5}{w}$ seconds. What weight will achieve the maximum distance travelled? *[6 marks]*

8. The time t in minutes taken to melt 100 g of butter depends upon the percentage p of the butter that consists of saturated fats, as described by the following function:

$$t = \frac{p^2}{10000} + \frac{p}{100} + 2$$

Find the maximum and minimum times to melt 100 g of butter. *[6 marks]*

9. The volume V of water in a tidal lake, in millions of litres, is modelled by $V = 60\cos t + 100$, where t is the time in days after the tidal lake mechanism is switched on.

(a) What is the smallest volume of the lake?

(b) A hydroelectric plant produces an amount of electricity proportional to the rate of flow of lake water. During the first 6 days, when is the plant producing the maximum amount of electricity? *[6 marks]*

10. A fast-food merchant finds that there is a relationship between the amount of salt, s, that he puts on his fries and his weekly sales of fries, F:

$$F(s) = 4s + 1 - s^2, \quad 0 \le s \le 4.2$$

(a) Find the amount of salt he should put on his fries to maximise sales.

(b) The total cost C associated with selling the fries is given by

$$C(s) = 0.3 + 0.2F(s) + 0.1s$$

Find the amount of salt the merchant should put on his fries to minimise costs.

(c) The profit made on the fries is given by the difference between the sales and the costs. How much salt should the merchant add to maximise profit? *[8 marks]*

11. A car tank is being filled with petrol such that the volume V of petrol in the tank, in litres, after time t minutes is given by

$$V = 300(t^2 - t^3) + 4, \quad 0 < t < 0.5$$

(a) How much petrol was initially in the tank?

(b) After 30 seconds the tank was full. What is the capacity of the tank?

(c) At what time is petrol flowing into the tank at the greatest rate? *[8 marks]*

12. Let x be the surface area of leaves on a tree, in m^2. Because leaves may be shaded by other leaves, the amount of energy produced by the tree is given by $2 - \frac{x}{10}$ kJ per square metre of leaves.

(a) Find an expression for the total energy produced by the tree.

(b) What area of leaves provides the maximum energy for the tree?

(c) Leaves also require energy for maintenance. The total energy requirement is given by $0.01x^3$. The *net* energy produced is the difference between the total energy produced by the leaves and the energy required by the leaves. For what range of x do the leaves produce more energy than they require?

(d) Show that the maximum net energy is produced when the tree has leaves with a surface area of $\frac{10(\sqrt{7} - 1)}{3}$.

[12 marks]

Summary

- The gradient of a function at a point is the gradient of the **tangent** to the function's graph at that point.
- The gradient of a function $f(x)$ at point x is called the **derivative** and is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

  The process of finding the derivative of a function is called **differentiation**.

  The derivative is also written as $\frac{\mathrm{d}}{\mathrm{d}x} f(x)$, where $\frac{\mathrm{d}}{\mathrm{d}x}$ means 'differentiate with respect to x'.
- The derivative of a sum is obtained by differentiating the terms one by one and then adding up the results. If k is a constant, the derivative of $kf(x)$ is $kf'(x)$.
- The derivatives of some common functions are:

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) = \frac{1}{\cos^2 x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x) = \mathrm{e}^x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \frac{1}{x}$$

- At the point on the curve $y = f(x)$ with $x = a$:
  - the gradient of the tangent is $f'(a)$
  - the gradient of the **normal** is $-\frac{1}{f'(a)}$
  - the coordinates of the point are $x_1 = a$, $y_1 = f(a)$

  The equation of the tangent or normal is $y - y_1 = m(x - x_1)$ with the appropriate gradient.
- **Stationary points** of a function are points where the gradient is zero: $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

  There are four types of stationary point:
  - local maximum
  - local minimum
  - positive point of inflexion
  - negative point of inflexion

- To determine the type of a stationary point, we can use the **second derivative**. At a stationary point (x_0, y_0):

  - if $\frac{d^2y}{dx^2} < 0$ at x_0, then (x_0, y_0) is a maximum
  - if $\frac{d^2y}{dx^2} > 0$ at x_0, then (x_0, y_0) is a minimum
  - if $\frac{d^2y}{dx^2} = 0$ at x_0, then no conclusion can be drawn, so check the sign of the gradient $\frac{dy}{dx}$ on either side of (x_0, y_0)

- A **point of inflexion** is a point where the curve switches from being concave-up to being concave-down or vice versa. At a point of inflexion, $\frac{d^2y}{dx^2} = 0$.
- To solve optimisation problems – that is, to maximise or minimise a function – we find and classify the stationary points, and also check the function values at the end points of the domain; the global maximum or minimum of a function may occur at an end point.

Introductory problem revisited

The cost of petrol consumed by a car is £$\left(12 + \frac{v^2}{100}\right)$ per hour, where the speed v (>0) is measured in miles per hour. If Daniel wants to travel 50 miles as cheaply as possible, what speed should he go at?

We know the cost per hour and want the total cost, so we need to find the total time. The time taken is $\frac{50}{v}$ hours, hence the total cost is $C = \frac{50}{v}\left(12 + \frac{v^2}{100}\right) = \frac{600}{v} + \frac{v}{2}$.

We wish to minimise C. To do this, we first look for stationary points by setting $\frac{dC}{dv} = 0$:

$$\frac{dC}{dv} = -\frac{600}{v^2} + \frac{1}{2}$$

$$\therefore -\frac{600}{v^2} + \frac{1}{2} = 0$$

$$\Leftrightarrow -1200 + v^2 = 0$$

$$\Leftrightarrow v = \sqrt{1200} = 34.6 \quad \text{(3SF)}$$

We take the positive square root since $v > 0$.

To check whether we have found a minimum point, calculate $\frac{d^2C}{dv^2} = 1200v^{-3}$. This is positive for any positive v, so the stationary point is a local minimum.

Next, to see if it is in fact the global minimum, we must consider the end points of the domain. Although v is never actually zero, as it gets close to zero the $\frac{600}{v}$ term becomes very large (the function has a vertical asymptote at $v = 0$). At the other end, when v gets very large, the $\frac{v}{2}$ term gets very large. Therefore the global minimum cannot be found at either end, and so the minimum cost is achieved at speed 34.6 miles per hour.

Mixed examination practice 12

Short questions

1. Find the equation of the tangent to the curve $y = e^x + 2\sin x$ at the point where $x = \frac{\pi}{2}$. *[5 marks]*

2. Find the equation of the normal to the curve $y = (x-2)^3$ at the point where $x = 2$. *[5 marks]*

3. $f(x)$ is a quadratic function taking the form $x^2 + bx + c$. If $f(1) = 2$ and $f'(2) = 12$, find the values of b and c. *[5 marks]*

4. Find the coordinates of the point on the curve $y = \sqrt{x} + 3x$ where the gradient is 5. *[4 marks]*

5. Find the coordinates of the point of inflexion on the graph of $y = \frac{x^3}{6} - x^2 + x$. *[5 marks]*

6. Find and classify the stationary points on the curve $y = \tan x - \frac{4x}{3}$. *[6 marks]*

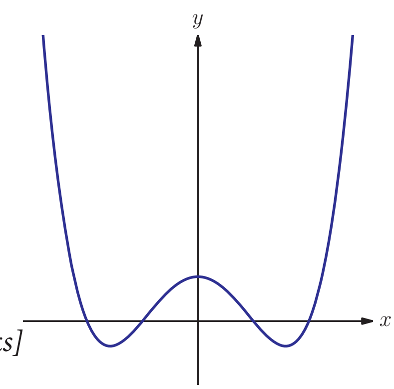
7. The graph shows $y = f'(x)$.

On a copy of this graph:

(a) mark points corresponding to a local minimum of $f(x)$ with an A

(b) mark points corresponding to a local maximum of $f(x)$ with a B

(c) mark points corresponding to a point of inflexion of $f(x)$ with a C. *[6 marks]*



8. On the curve $y = x^3$, a tangent is drawn from the point (a, a^3) and a normal is drawn from the point $(-a, -a^3)$. The tangent and the normal meet on the y-axis. Find the value of a. *[6 marks]*

Long questions

1. The line $y = 24(x-1)$ is tangent to the curve $y = ax^3 + bx^2 + 4$ at $x = 2$.

(a) Use the fact that the tangent meets the curve to show that $2a + b = 5$.

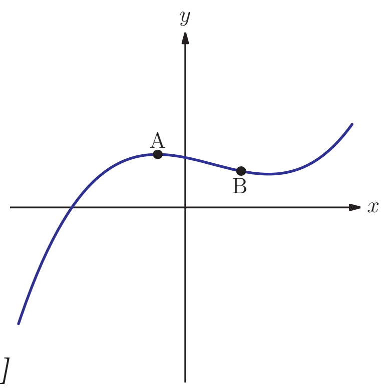
(b) Use the fact that the tangent has the same gradient as the curve to find another relationship between a and b.

(c) Hence find the values of a and b.

(d) The line meets the curve again. Find the coordinates of the other point of intersection. *[12 marks]*

2. The curve shown is part of the graph of $y = x^3 - x^2 - x + 3$.

The point A is a local maximum and the point B is a point of inflexion.



(a) (i) Find the coordinates of A.

(ii) Find the coordinates of B.

(b) (i) Find the equation of the line containing both A and B.

(ii) Find the equation of a tangent to the curve which is parallel to this line. *[10 marks]*

3. The population P of bacteria in thousands at a time t in hours is modelled by

$$P = 10 + e^t - 3t,\ t \geq 0$$

(a) (i) Find the initial population of bacteria.

(ii) At what time does the number of bacteria reach 14 million?

(b) (i) Find $\frac{dP}{dt}$.

(ii) Find the time at which the bacteria are growing at a rate of 6 million per hour.

(c) (i) Find $\frac{d^2P}{dt^2}$ and explain the physical significance of this quantity.

(ii) Find the minimum number of bacteria, justifying that it is a minimum. *[12 marks]*