In this chapter you will learn:

- about the concept of function
- the notation used to represent functions
- about what happens when one function acts after another function
- how to reverse the effect of a function
- about the reciprocal function, simple rational functions and their graphs.

4 The theory of functions

Introductory problem

Think of any number. Add on 3. Double your answer. Take away 6. Divide by the number you first thought of. Is your answer always prime? Why?

Doubling, adding five, finding the largest prime factor, … these are all actions that can be applied to numbers, producing a result. This idea of performing operations on numbers to get another value out comes up a lot in mathematics, and its formal study leads to the concept of functions.

4A Function notation

A function is a rule where for each value you put in there is exactly one value that comes out. In this section we will see how to describe functions using mathematical expressions.

Suppose we have the rule 'add 3 to the input', and we call this rule 'f'. In function notation, this is written as:

$$f : x \mapsto x+3$$

We read this as 'the function f transforms x into $x+3$'. An alternative way of writing this is:

$$f(x) = x+3$$

Although conventionally the letters f and x are often used, they are not intrinsically special. We could replace f with any other name and x with any input value we like; whatever we call this function and its input, the function will always do the same thing to the input (e.g. add 3). The input is sometimes called the **argument** of the function and the output is called the image of the input.

Is the notation $f(x)$ just a label for a rule, or does it help to open up new techniques and new knowledge? It may surprise you to learn that the latter is actually the case. Particularly in many applications of calculus, we do not need to know exactly what the rule is, but simply that it depends on 'x', which may stand for time, or height, or some other quantity of interest in the application.

Worked example 4.1

Given a function defined by $g(x) = x^2 + x$, find and simplify the following:

(a) $g(2)$ (b) $g(y)$ (c) $g(x+1)$ (d) $g(3x)$ (e) $4g(x-1)-3$

Replace x with 2.

(a) $g(2) = 2^2 + 2 = 6$

Replace x with y.

(b) $g(y) = y^2 + y$

Replace x with $(x+1)$. Don't forget brackets!

(c) $g(x+1) = (x+1)^2 + (x+1)$
$= x^2 + 2x + 1 + x + 1$
$= x^2 + 3x + 2$

Replace x with $(3x)$. Don't forget brackets!

(d) $g(3x) = (3x)^2 + (3x)$
$= 9x^2 + 3x$

Replace x with $(x-1)$ and then multiply by 4 and subtract 3.

(e) $4g(x-1) - 3 = 4((x-1)^2 + (x-1)) - 3$
$= 4(x^2 - 2x + 1 + x - 1) - 3$
$= 4(x^2 - x) - 3$
$= 4x^2 - 4x - 3$

Exercise 4A

1. If $h(x) = 3x^2 - x$, find and simplify the following.

(a) (i) $h(3)$ (ii) $h(7)$

(b) (i) $h(-2)$ (ii) $h(-1)$

(c) (i) $h(z)$ (ii) $h(a)$

(d) (i) $h(x+1)$ (ii) $h(x-2)$

(e) (i) $\frac{1}{2}(h(x) - h(-x))$ (ii) $3h(x) + 4h(2x)$

(f) (i) $h\left(\frac{1}{x}\right)$ (ii) $h(\sqrt{x})$

2. If $g : x \mapsto 1 + \log_{10} x$, find and simplify the following.

(a) (i) $g(100)$ (ii) $g(1\,000\,000)$

(b) (i) $g(0.1)$ (ii) $g(1)$

(c) (i) $g(y)$ (ii) $g(z)$

(d) (i) $g(10x)$ (ii) $g(100y)$

(e) (i) $g(x) + g(x^2)$ (ii) $\frac{1}{2}\left(g(x) + g\left(\frac{1}{x}\right)\right)$

Calculations with logarithms were covered in chapter 2.

3. If $u(x) = 3x + 1$ and $v(x) = -\sqrt{x}$, find and simplify the following.

(a) (i) $u(2) + v(9)$ (ii) $u(1)v(4)$

(b) (i) $u(x) + v(y)$ (ii) $u(2x+1) - v(4x)$

(c) (i) $2u(4x) + 3v(4x)$ (ii) $v(x^2+1) + xu(2x)$

4B Domain and range

A function is a rule that tells you what to do with the input, but, to be completely defined, it also needs to specify what type of input values are allowed.

KEY POINT 4.1

The set of allowed input values is called the **domain** of the function. Conventionally, we write it after the rule using set notation or inequalities.

Worked example 4.2

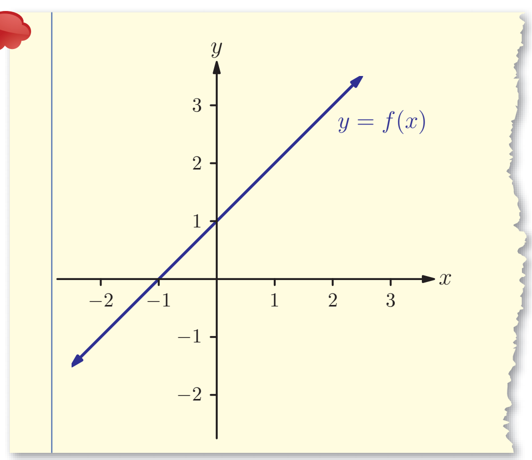
Sketch the graph of $f(x) = x + 1$ over the domain

(a) $x \in \mathbb{R}, x > 1$ (b) $x \in \mathbb{Z}$

EXAM HINT

If you are instructed to sketch the graph of $f(x)$, this means the graph of $y = f(x)$.

First, sketch the graph for $x \in \mathbb{R}$ (all real numbers).



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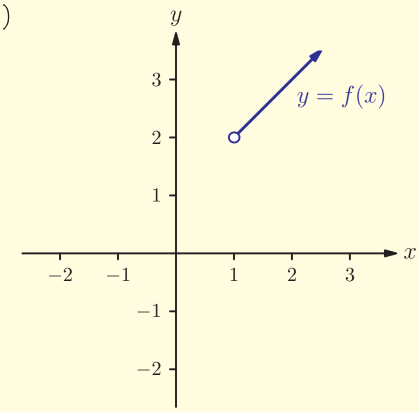
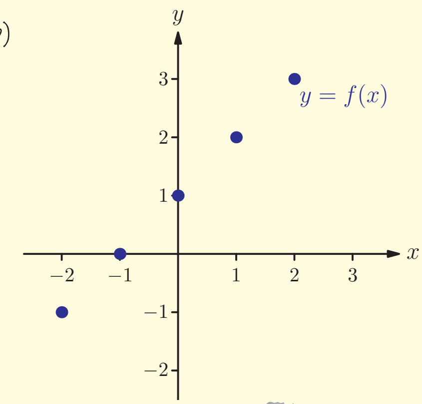
Discard the part of the graph which is outside the given domain. Since '$x > 1$' does not include the endpoint 1, we label the point (1, 2) with an open circle.

(a)



If the domain is $x \in \mathbb{Z}$, the graph exists only at whole numbers, which we label with closed circles.

(b)



If the domain is not explicitly stated, you can assume that it consists of all real numbers. You may wonder why we would ever need to consider any other domain. One reason might be that we are modelling a physical situation where the variables can only take particular values; for example, if x represented the age of humans, it would not make sense for x to be negative or much greater than 120.

It is quite tempting to say that dividing by zero results in infinity. However, doing this leads to some unfortunate consequences, such as all numbers being 'equal'!

Another reason is that the function may involve a mathematical operation that cannot handle certain types of numbers. For example, if we want to find the largest prime factor of a number, we should only be looking at positive integers. When working with real numbers, the three most common reasons to restrict the domain are:

- you cannot divide by zero
- you cannot take the square root of a negative number
- you cannot take the logarithm of a negative number or zero.

Worked example 4.3

What is the largest possible domain of $h: x \mapsto \frac{1}{x-2} + \sqrt{x+3}$?

Check for division by zero.

There will be division by zero when $x - 2 = 0$.

Check for square rooting of a negative number.

There will be square rooting of a negative number when $x + 3 < 0$.

Decide what can be allowed in the function.

$x \geq -3$ and $x \neq 2$

We can also use interval notation to write domains; in the previous example we could write the answer as $x \in [-3, 2[\cup]2, \infty[$.

Once we have specified what can go into a function, it is interesting to see what values can come out of the function.

EXAM HINT

Remember that you can restrict the viewing window to the x-values in the domain.

KEY POINT 4.2

The set of all possible *outputs* of a function (y-values on the graph) is called the **range** of the function. The easiest way of finding the range is to sketch the graph (possibly using your GDC). Be aware that the range depends on the domain.

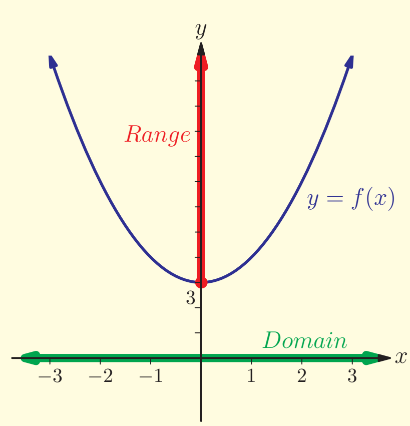
Worked example 4.4

Find the range of $f(x) = x^2 + 3$ if the domain is

(a) $x \in \mathbb{R}$ (b) $x > 2$

Sketch the graph of $y = x^2 + 3$ for $x \in \mathbb{R}$.

(a)



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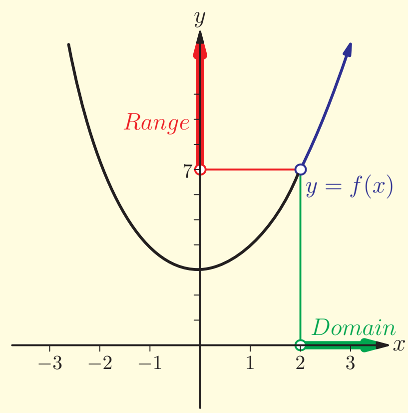
From the graph, observe which y-values can occur.

Range: $y \geq 3$

Sketch the graph of $y = x^2 + 3$ for $x > 2$.

(b)



From the graph, observe which y-values can occur.

Range: $y > 7$

Exercise 4B

1. State the largest possible domain and range of the following functions:

(a) $f(x) = 2^x$.

(b) $f(x) = a^x,\ a > 0$.

State the largest possible range and domain of the following functions:

(c) $f(x) = \log_{10} x$.

(d) $f(x) = \log_b x, b > 0$.

Exponential and logarithm functions were covered in chapter 2.

What are the largest possible domain and range of $f(x) = (-2)^x$? This function illustrates why it is important to be careful in deciding how to define a continuous function – an important concept in higher mathematics.

2. Find the largest possible domain of the following functions.

(a) (i) $f(x) = \dfrac{1}{x+2}$ (ii) $f(x) = \dfrac{5}{x-7}$

(b) (i) $f(x) = \dfrac{3}{(x-2)(x+4)}$ (ii) $g(x) = \dfrac{x}{x^2-9}$

(c) (i) $r(y) = \sqrt{y^3 - 1}$ (ii) $h(x) = \sqrt{x+3}$

(d) (i) $f(a) = \dfrac{1}{\sqrt{a-1}}$ (ii) $f(x) = \dfrac{5x}{\sqrt{2-5x}}$

(e) (i) $a(x)=\frac{1}{x}+\frac{2}{x+1}$ (ii) $f(x)=\sqrt{x+1}+\frac{1}{x+2}$

(f) (i) $f(x)=\sqrt{x^2-5}$ (ii) $f(x)=4\sqrt{x^2+2x-3}$

(g) (i) $f(x)=\sqrt{x}+\frac{1}{x+7}-x^3+5$

(ii) $f(x)=e^x+\sqrt{2x+3}-\frac{1}{x^2+4}-2$

3. Find the range of the following functions.

(a) (i) $f(x)=7-x^2,\ x\in\mathbb{R}$ (ii) $f(x)=x^2+3,\ x\in\mathbb{R}$

(b) (i) $g(x)=x^2+3,\ x\geq 3$ (ii) $h(x)=x+1,\ x>3,\ x\in\mathbb{Z}$

(c) (i) $f(x)=|x-1|$ (ii) $f(x)=|2x+3|$

(d) (i) $d(x)=\frac{1}{x},\ x\geq -1,\ x\neq 0$ (ii) $q(x)=3\sqrt{x},\ x>0$

See Prior Learning section I on the CD-ROM for the definition of the modulus function, $|x|$.

4. The function f is given by $f(x)=\sqrt{\ln(x-4)}$. Find the domain of the function. *[4 marks]*

5. Find the largest possible domain of the function

$$f(x)=\frac{4^{\sqrt{x-1}}}{x+2}-\frac{1}{x^2-5x+6}+x^2+1$$ *[5 marks]*

6. Find the largest possible domain of the function $g(x)=\ln(x^2+3x+2)$. *[5 marks]*

7. Find the largest set of values of x such that the function f given by $f(x)=\sqrt{\frac{8x-4}{x-12}}$ takes real values. *[5 marks]*

8. Define $f(x)=\sqrt{x-a}+\ln(b-x)$.

(a) State the domain of the function if

(i) $a<b$ (ii) $a>b$

(b) Evaluate $f(a)$. *[6 marks]*

4C Composite functions

We can apply one function to a number and then apply another function to that result. The overall rule linking the original input value with the final output value is called a **composite function**.

KEY POINT 4.3

If we first apply the function g to x and then apply the function f to the result, we write this composite function as

$$f(g(x)) \quad or \quad fg(x) \quad or \quad f \circ g(x)$$

As we shall see later, it is useful to refer to $g(x)$ as the **inner function** and $f(x)$ as the **outer function**.

For the composite function $fg(x)$ to exist, the range of $g(x)$ must lie entirely within the domain of $f(x)$, otherwise we would be trying to put values into $f(x)$ which cannot be calculated.

EXAM HINT

None of these three expressions for a composite function is 'better' than the others. Use whichever one suits you, but be aware that in the exam you must be able to interpret any of them. Remember the correct order: the function nearest to x acts first!

Worked example 4.5

If $f(x) = x^2$ and $g(x) = x - 3$, find

(a) $f \circ g(1)$ (b) $f \circ g(x)$ (c) $gf(x)$

We need to evaluate $g(1)$ and then apply f to the result.

(a) $g(1) = 1 - 3 = -2$

$f(-2) = (-2)^2 = 4$

$\therefore f(g(1)) = 4$

EXAM HINT

Notice that to work this out we did not need to find the general expression for $f \circ g(x)$

Replace the x in $f(x)$ with the expression for $g(x)$.

(b) $f(g(x)) = f(x-3) = (x-3)^2$

$= x^2 - 6x + 9$

Replace the x in $g(x)$ with the expression for $f(x)$.

(c) $g(f(x)) = g(x^2)$

$= x^2 - 3$

Notice that $f(g(x))$ and $g(f(x))$ are not the same function.

It is more difficult to recover one of the original functions from a composite function. The best way to do this is by using a substitution.

Worked example 4.6

If $f(x+1)=4x^2+x$, find $f(x)$.

Substitute y = inner function.	Let $y=x+1$
Rearrange to make x the subject.	$x=y-1$
Replace all the x's by $y-1$.	$f(y)=4(y-1)^2+(y-1)$ $=4y^2-8y+4+y-1$ $=4y^2-7y+3$
We were asked to write the answer in terms of x.	$f(x)=4x^2-7x+3$

Exercise 4C

1. If $f(x)=x^2+1$ and $g(x)=3x+2$, find:

(a) (i) $g(f(0))$ (ii) $fg(1)$

(b) (i) $gg(x)$ (ii) $f\circ g(x)$

(c) (i) $gg(\sqrt{a}+1)$ (ii) $f\bullet f(y-1)$

(d) (i) $ggf(y)$ (ii) $gfg(z)$

2. Find $f(x)$ given the following conditions:

(a) (i) $f(2a)=4a^2$ (ii) $f\left(\frac{b}{3}\right)=\frac{b^3}{27}$

(b) (i) $f(x+1)=3x-2$ (ii) $f(x-2)=x^2+x$

(c) (i) $f(1-y)=5-y$ (ii) $f(y^3)=y^2$

(d) (i) $f(e^k)=\ln k$ (ii) $f(3n+2)=\ln(n+1)$

3. If $f(x)=x^2+1$ and $g(x)=3x+2$, solve $fg(x)=gf(x)$. *[4 marks]*

4. If $f(x)=3x+1$ and $g(x)=\dfrac{x}{x^2+25}$, solve $gf(x)=0$. *[5 marks]*

5. Functions g and h are defined by $g(x)=\sqrt{x}$ and

$h(x)=\dfrac{2x-3}{x+1}$.

(a) Find the range of h.

(b) Solve the equation $h(x)=0$.

(c) Find the domain and range of $g \circ h$. *[6 marks]*

6. The function f is defined by $f: x \to x^3$. Find an expression for $g(x)$ in terms of x in each of the following cases:

(a) $(f \circ g)(x)=2x+3$

(b) $(g \circ f)(x)=2x+3$ *[6 marks]*

7. Functions f and g are defined by $f(x)=\sqrt{x^2-2x}$ and $g(x)=3x+4$. The composite function $f \circ g$ is *undefined* for $x \in]a,b[$.

(a) Find the value of a and the value of b.

(b) Find the range of $f \circ g$. *[7 marks]*

8. Define $f(x)=x-1,\ x>3$ and $g(x)=x^2,\ x \in \mathbb{R}$.

(a) Explain why $g \circ f$ exists but $f \circ g$ does not.

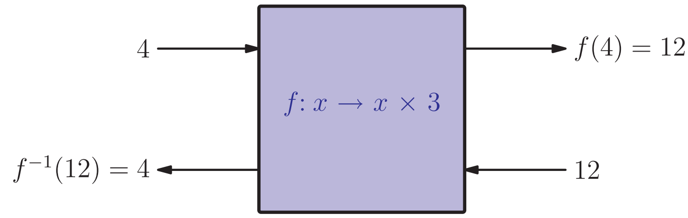
(b) Find the largest possible domain for g so that $f \circ g$ is defined. *[6 marks]*

9. Let f and g be two functions. Given that $(f \circ g)(x)=\dfrac{x+2}{3}$ and $g(x)=2x+5$, find $f(x-1)$. *[6 marks]*

4D Inverse functions

Functions transform an input into an output, but sometimes we need to reverse this process – that is, to find out what input produced a particular output. When it is possible to reverse the actions of a function f, we can define its **inverse function**, usually written as f^{-1}.

For example, if $f(x)=3x$, then $f^{-1}(12)$ is a number which when put into f produces the output 12; in other words, we are looking for a number x such that $f(x)=12$. In this case, $f^{-1}(12)=4$.



EXAM HINT

Make sure you do not get confused by this notation. With numbers, the superscript '-1' denotes the reciprocal, e.g. $3^{-1}=\frac{1}{3}$ and $x^{-1}=\frac{1}{x}$. With functions, f^{-1} denotes the inverse function of f.

To find a formula for the inverse function, you must rearrange the formula of the original function to find the input (x) in terms of the output (y).

KEY POINT 4.4

To find the inverse function $f^{-1}(x)$ given an expression for $f(x)$:

1. Start off with $y=f(x)$.
2. Rearrange to get x (the input) in terms of y (the output).
3. This gives us $f^{-1}(y)$, but often we are asked to find $f^{-1}(x)$; we do this by replacing every occurrence of y in the expression with x.

Worked example 4.7

Find the inverse function of $f(x)=\frac{1+x}{3-x}$.

Write $y=f(x)$.

$$y=\frac{1+x}{3-x}$$

Make x the subject of the formula. Then the right-hand side expression is $f^{-1}(y)$.

$$\begin{aligned}
& y(3-x)=1+x \\
\Leftrightarrow\ & 3y-yx=1+x \\
\Leftrightarrow\ & 3y-1=x+xy \\
\Leftrightarrow\ & 3y-1=x(1+y) \\
\Leftrightarrow\ & x=\frac{3y-1}{1+y}
\end{aligned}$$

$$\therefore\ f^{-1}(y)=\frac{3y-1}{1+y}$$

Replace y with x to get $f^{-1}(x)$.

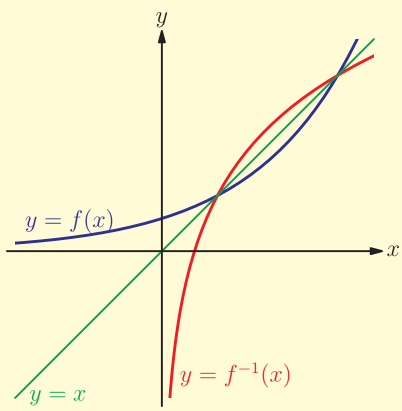
$$f^{-1}(x)=\frac{3x-1}{1+x}$$

Now we have learned how to find inverse functions, there are some important facts we need to know about them:

- The inverse function switches inputs and outputs; graphically this is equivalent to switching the x- and y-axes.

KEY POINT 4.5

The graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ in the line $y = x$.



- If you apply a function and then undo it, you get back to where you started.

KEY POINT 4.6

$$f^{-1}(f(x)) = f(f^{-1}(x)) = x$$

The function $g(x) = x$, for which the output is always the same as the input, is called the **identity function**. Thus $f^{-1} \circ f$ and $f \circ f^{-1}$ are both equal to the identity function.

EXAM HINT

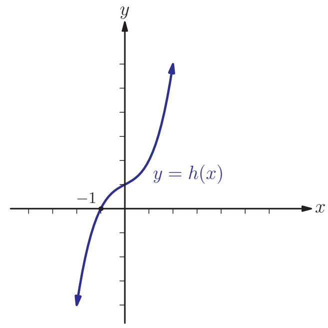
See Calculator Skills sheet 7 on the CD-ROM for how to sketch the graph of an inverse function on your calculator.

EXAM HINT

If you have found an algebraic expression for $f^{-1}(x)$, you can check whether it is correct by plotting both $f(x)$ and $f^{-1}(x)$ on the same axes and looking for symmetry.

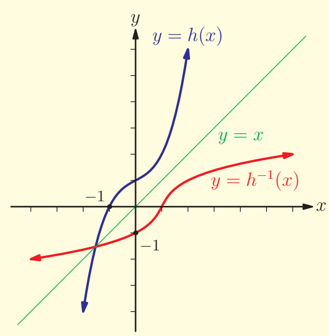
Worked example 4.8

The graph of $y = h(x)$ is shown. Sketch the graphs of $y = h^{-1}(x)$ and $y = h \circ h^{-1}(x)$.



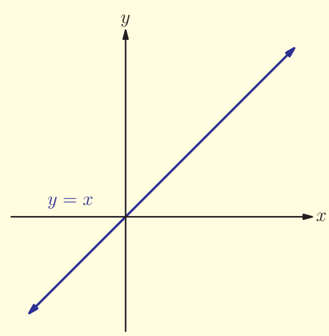
The graph of $y = h^{-1}(x)$ is obtained by reflecting the graph of $y = h(x)$ in the line $y = x$.

(a)



$h \circ h^{-1}$ is the identity function, so $y = h \circ h^{-1}(x)$ simplifies to $y = x$.

(b)



- Reflection in the line $y = x$ swaps the domain and the range of a function (because it swaps the x- and y-coordinates).

KEY POINT 4.7

> The domain of $f^{-1}(x)$ is the same as the range of $f(x)$.
>
> The range of $f^{-1}(x)$ is the same as the domain of $f(x)$.

Exercise 4D

1. Find $f^{-1}(x)$ if

(a) (i) $f(x)=3x+1$ (ii) $f(x)=7x-3$

(b) (i) $f(x)=\dfrac{2x}{3x-2},\ x\neq\dfrac{2}{3}$ (ii) $f(x)=\dfrac{x}{2x+1},\ x\neq-\dfrac{1}{2}$

(c) (i) $f(x)=\dfrac{x-a}{x-b},\ x\neq b$ (ii) $f(x)=\dfrac{ax-1}{bx-1},\ x\neq\dfrac{1}{b}$

(d) (i) $f(a)=1-a$ (ii) $f(y)=3y+2$

(e) (i) $f(x)=\sqrt{3x-2},\ x\geq\dfrac{2}{3}$ (ii) $f(x)=\sqrt{2-5x},\ x\leq\dfrac{2}{5}$

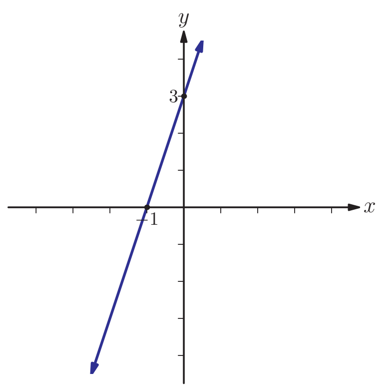
(f) (i) $f(x)=\ln(1-5x),\ x<0.2$ (ii) $f(x)=\ln(2x+2),\ x>-1$

(g) (i) $f(x)=7e^{\frac{x}{2}}$ (ii) $f(x)=9e^{10x}$

(h) (i) $f(x)=x^2-10x+6,\ x<5$ (ii) $f(x)=x^2+6x-1,\ x>0$

2. Sketch the inverse functions of the following functions.

(a)



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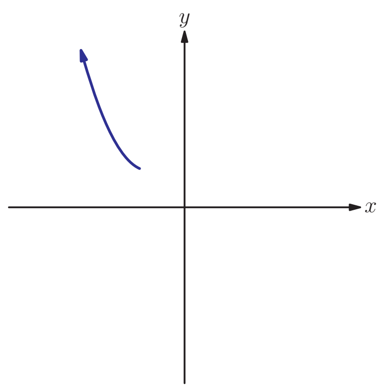
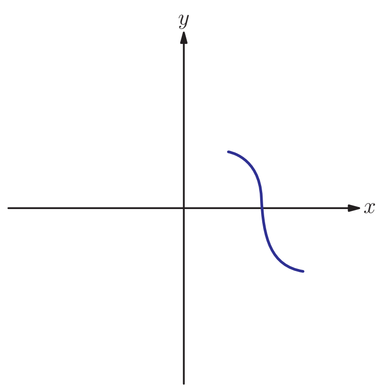
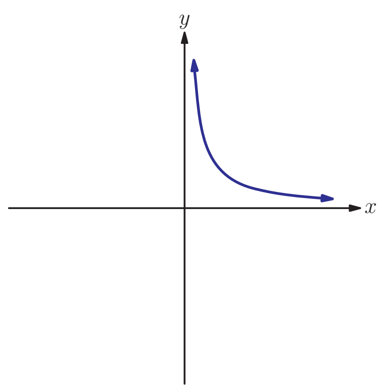


(c)



(d)



3. The following table gives selected values of the function $f(x)$.

x	-1	0	1	2	3	4
$f(x)$	-4	-1	3	0	7	2

(a) Evaluate $ff(2)$.

(b) Evaluate $f^{-1}(3)$. *[4 marks]*

4. The function f is defined by $f: x \mapsto \sqrt{3-2x},\ x \le \frac{3}{2}$.

Evaluate $f^{-1}(7)$. *[4 marks]*

5. Given that $f(x) = 3e^{2x}$, find the inverse function $f^{-1}(x)$. *[4 marks]*

6. Given functions $f: x \mapsto 2x+3$ and $g: x \to x^3$, find the function $(f \circ g)^{-1}$. *[5 marks]*

7. The functions f and g are defined by $f: x \mapsto e^{2x}$ and $g: x \mapsto x+1$.

(a) Calculate $f^{-1}(3) \times g^{-1}(3)$.

(b) Show that $(f \circ g)^{-1}(3) = \ln\sqrt{3} - 1$. *[6 marks]*

8. Let $f(x) = \sqrt{x}$ and $g(x) = 2x$. Solve the equation $(f^{-1} \circ g)(x) = 0.25$. *[5 marks]*

9. The function f is defined for $x \le 0$ by $f(x) = \dfrac{x^2-4}{x^2+9}$.
Find an expression for $f^{-1}(x)$. *[5 marks]*

10. Let $f(x) = \ln(x-1) + \ln 3$ for $x > 1$.

(a) Find $f^{-1}(x)$.

(b) Let $g(x) = e^x$. Find $(g \circ f)(x)$, giving your answer in the form $ax+b$ where $a, b \in \mathbb{Z}$. *[7 marks]*

11. A function is said to be *self-inverse* if $f(x) = f^{-1}(x)$ for all x in the domain.

(a) Show that $f(x) = \dfrac{1}{x}$ is a self-inverse function.

(b) Find the value of the constant k so that $g(x) = \dfrac{3x-5}{x+k}$ is a self-inverse function. *[8 marks]*

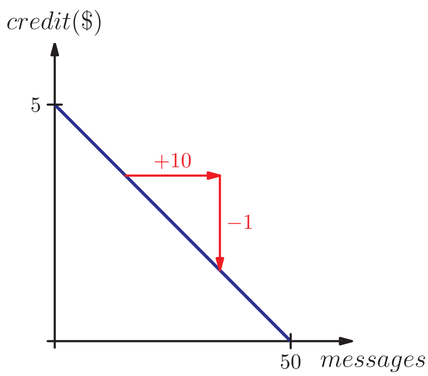
4E Rational functions

There are many situations where one quantity decreases as another increases.

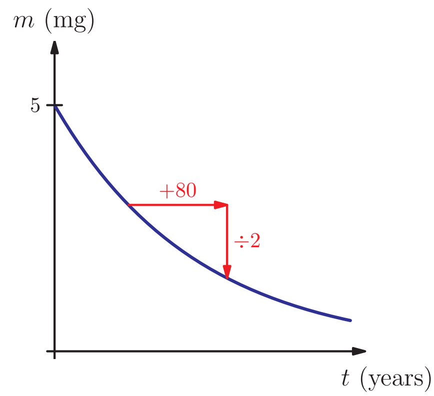
For example, the amount of your phone credit decreases as the number of text messages you send increases; moreover, as the number of messages increases by a fixed number, the credit decreases by a fixed amount – this is called linear decay.

Another example is radioactive decay, where the amount of a radioactive substance halves in a fixed time period, called the half-life; in this case, as time increases by a fixed number, the amount of substance decreases by a fixed factor – this is called exponential decay.

Exponential functions were covered in chapter 2. See Prior Learning section Q on the CD-ROM for a review of linear functions.



linear decay



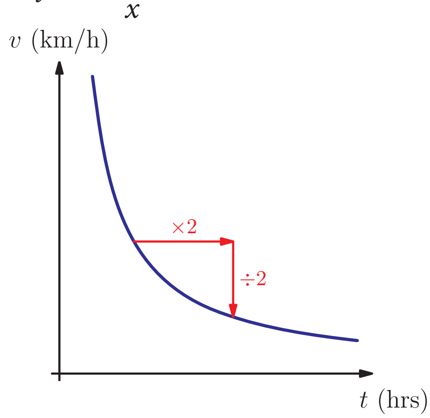
exponential decay

In this section we look at a third type of decay, called inverse proportion, where as one quantity increases by a fixed factor, another decreases by the same factor. For example, if you double your speed, the amount of time it takes to travel a given distance will halve. If the total distance travelled is 12 km, then the equation for travel time (in hours) in terms of speed (in km/h) is $t = \frac{12}{v}$. This is an example of a **reciprocal function**, which has the general form $f(x) = \frac{k}{x}$.

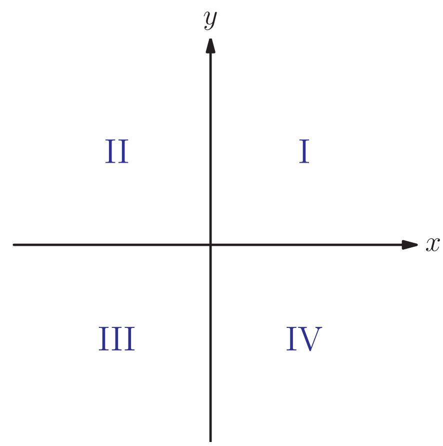
EXAM HINT

The reciprocal of a non-zero real number x is $\frac{1}{x}$.

For example, the reciprocal of -2 is $-\frac{1}{2}$ and the reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$.



Graphs of reciprocal functions all have the same shape, called a **hyperbola**. A hyperbola is made up of two curves, with the axes as asymptotes. The function is not defined for $x=0$ (the y-axis is a vertical asymptote), and as x gets very large (positive or negative) y approaches zero (the x-axis is a horizontal asymptote). This means that neither x nor y can equal zero. The two parts of the hyperbola can be either in the first and third **quadrants** or in the second and fourth quadrants, depending on the sign of k.



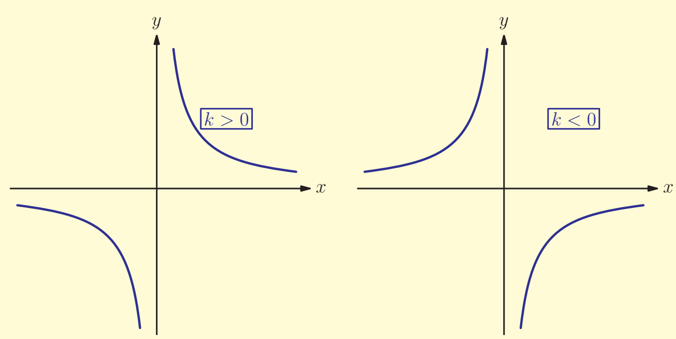
Quadrants

KEY POINT 4.8

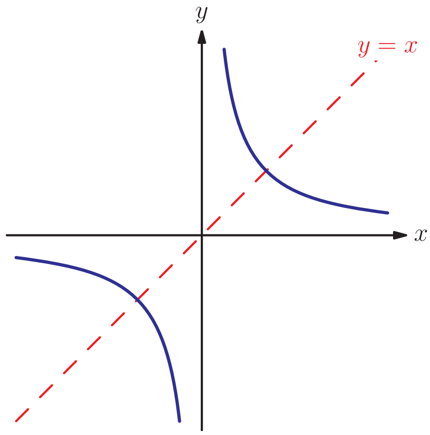
A reciprocal function has the form $f(x)=\dfrac{k}{x}$.

The domain of f is $x \neq 0$ and the range is $y \neq 0$.

The graph of $f(x)$ is a hyperbola.



What is the inverse of a reciprocal function? If $y=\dfrac{k}{x}$, then $xy=k$ and hence $x=\dfrac{k}{y}$. This means that $f^{-1}(x)=\dfrac{k}{x}=f(x)$, so the reciprocal function is its own inverse. We can also see this from the graph: a hyperbola is symmetrical about the line $y = x$, so its reflection in the line is the same as itself.



KEY POINT 4.9

The reciprocal function $f(x) = \frac{k}{x}$ is a **self-inverse function**; that is, $f^{-1}(x) = f(x)$.

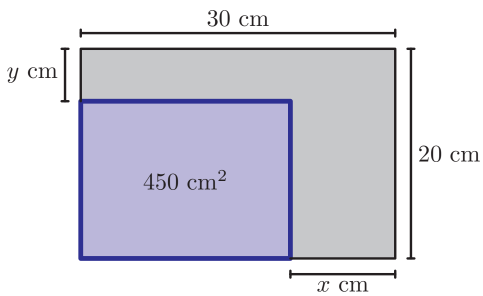
Related to reciprocal functions, **rational functions** are a ratio of two polynomials: $f(x) = \frac{p(x)}{q(x)}$ can be used to model a wider variety of situations where one quantity decreases as another increases. The following example illustrates one such situation.

Worked example 4.9

A rectangular piece of card has dimensions 30 cm by 20 cm. Strips of width x cm and y cm are cut off the ends, as shown in the diagram, so that the remaining card has area 450 cm^2.

(a) Find an expression for y in terms of x in the form $y = \frac{ax - b}{cx - d}$.

(b) Sketch the graph of y against x.



continued . . .

Write the equation for the remaining area in terms of x and y.

(a) $(30-x)(20-y)=450$

we want to make y the subject, so divide by $(30-x)$ (rather than expanding the brackets).

$$20-y=\frac{450}{30-x}$$

$$y=20-\frac{450}{30-x}$$

$$=\frac{20(30-x)-450}{30-x}$$

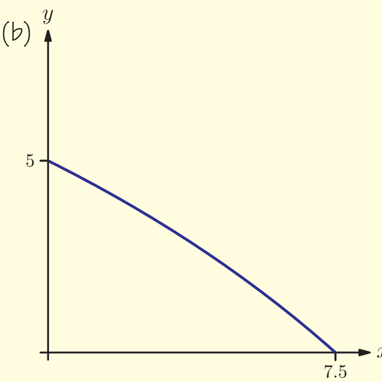
$$=\frac{150-20x}{30-x}$$

To get the expression in the required form, multiply top and bottom by -1.

$$\therefore y=\frac{20x-150}{x-30}$$

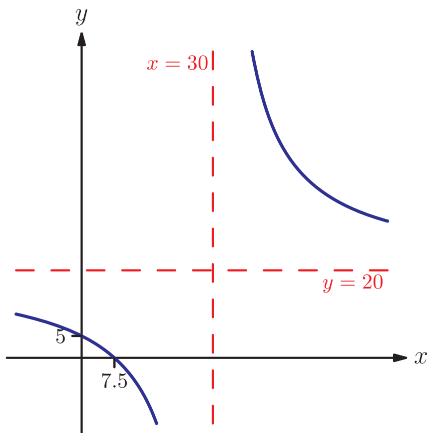
We can use a GDC to sketch the graph. Only positive values of x and y are relevant in this situation.

(b)



Is zero the same as 'nothing'? What happens if you say that the result of dividing by zero is infinity? What would $\frac{0}{0}$ be then?

In the above example only positive values of x and y were relevant, but let us look at the graph of $f(x)=\frac{20x-150}{x-30}$ over the whole of $\mathbb{R}$. The function is not defined at $x=30$ (as we would be dividing by zero), so there is a vertical asymptote there. The y-intercept is (0, 5). We can find the x-intercept by setting the top of the fraction equal to zero, which gives (7.5, 0). The graph looks like a hyperbola with horizontal asymptote $y=20$.



The position of the horizontal asymptote can be discovered by looking at the first equation we found for y in Worked example 4.9, which can also be written as $y = 20 + \frac{450}{x-30}$. As x gets very large (either positive or negative), $x-30$ gets very large, so $\frac{450}{x-30}$ gets very small. Therefore the value of y gets closer and closer to 20. Another way to find the asymptote is to think about what happens as x gets very large in the equation $y = \frac{20x-150}{x-30}$: the terms containing x in the numerator and denominator become much larger than 150 and 30, so these two constant terms can be ignored, leaving $y \approx \frac{20x}{x} = 20$.

The example $f(x) = \frac{20x-150}{x-30}$ illustrates all the important properties of rational functions of the form $f(x) = \frac{ax+b}{cx+d}$.

EXAM HINT

In the examination you can just find the horizontal asymptote by dividing the two coefficients of x in the numerator and denominator.

KEY POINT 4.10

The graph of a rational function of the form $f(x) = \frac{ax+b}{cx+d}$ is a hyperbola which has

- vertical asymptote $x = -\frac{d}{c}$ (where $cx + d = 0$)
- horizontal asymptote $y = \frac{a}{c}$
- x-intercept at $x = -\frac{b}{a}$ (where $ax + b = 0$)
- y-intercept at $y = \frac{b}{d}$ (where $x = 0$)

EXAM HINT

Make sure you include all asymptotes and intercepts when sketching graphs of rational functions. The intercepts should help you determine which quadrants the graph lies in.

Knowing the position of the asymptotes tells you the domain and range of the function.

Worked example 4.10

Find the domain and range of the function $f(x)=\dfrac{3x-4}{2x+1}$.

The only value excluded from the domain is where the denominator is zero.

$2x+1=0$

$\Leftrightarrow x=-\dfrac{1}{2}$

The domain is $x\in\mathbb{R},\ x\neq-\dfrac{1}{2}$

Sketching the graph can show us the range. Find the horizontal asymptote by dividing the coefficients of x.

Horizontal asymptote:

$y=\dfrac{3}{2}$

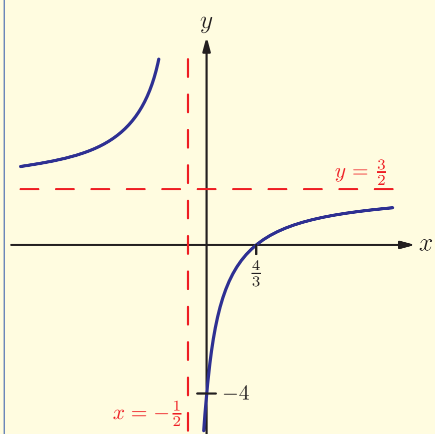
Find the intercepts to decide which quadrants the graph lies in.

Intercepts:

$x=0\Rightarrow y=-4$

$y=0\Rightarrow 3x-4=0\Rightarrow x=\dfrac{4}{3}$

Sketch the graph.



The range is $y\in\mathbb{R}, y\neq\dfrac{3}{2}$

Exercise 4E

1. Find the coordinates of the axes intercepts of the following rational functions.

(a) (i) $f(x)=\dfrac{3x+1}{x+3}$ (ii) $f(x)=\dfrac{2x+5}{x+1}$

(b) (i) $f(x)=\dfrac{2x-3}{2x+7}$ (ii) $f(x)=\dfrac{3x-5}{x+2}$

2. Find the equations of the asymptotes of the following graphs.

(a) (i) $y=\frac{4x+3}{x-1}$ (ii) $y=\frac{2x+1}{x-7}$

(b) (i) $y=\frac{3x+2}{2x-1}$ (ii) $y=\frac{4x+1}{3x-5}$

(c) (i) $y=\frac{3-x}{2x+5}$ (ii) $y=\frac{2x+1}{2-3x}$

(d) (i) $y=\frac{3}{x-2}$ (ii) $y=\frac{2}{2x+1}$

3. Sketch the graphs of the following rational functions, labelling all the axes intercepts and asymptotes.

(a) (i) $y=\frac{2x+1}{x-2}$ (ii) $y=\frac{3x+1}{x-3}$

(b) (i) $y=\frac{x-3}{4-x}$ (ii) $y=\frac{5-x}{x-2}$

(c) (i) $y=\frac{2}{x+3}$ (ii) $y=\frac{1}{x-2}$

(d) (i) $y=-\frac{3}{x}$ (ii) $y=-\frac{2}{x}$

4. Find the domain, range and inverse function of the following rational functions.

(a) (i) $f(x)=\frac{3}{x}$ (ii) $f(x)=\frac{7}{x}$

(b) (i) $f(x)=\frac{2}{x-3}$ (ii) $f(x)=\frac{5}{x+1}$

(c) (i) $f(x)=\frac{2x+1}{3x-1}$ (ii) $f(x)=\frac{4x-5}{2x+1}$

(d) (i) $f(x)=\frac{5-2x}{x+2}$ (ii) $f(x)=\frac{3x-1}{4x-3}$

5. Find the equations of the asymptotes of the graph of $y=\frac{3x-1}{4-5x}$. *[3 marks]*

6. Let $f(x)=\frac{1}{x+3}$.

(a) Find the domain and range of $f(x)$.

(b) Find $f^{-1}(x)$. *[5 marks]*

7. (a) Sketch the graph of $y = -\dfrac{3}{x}$.

(b) Let $f(x) = -\dfrac{3}{x}$, $x \neq 0$. Write down an equation for $f^{-1}(x)$. *[4 marks]*

8. Sketch the graph of $y = \dfrac{3x-1}{x-5}$. *[5 marks]*

9. A function is defined by $f: x \mapsto \dfrac{ax+3}{2x-8}$, $x \neq 4$, where $a \in \mathbb{R}$.

(a) Find, in terms of a, the range of f.

(b) Find the inverse function $f^{-1}(x)$.

(c) Find the value of a such that f is a self-inverse function. *[5 marks]*

Summary

- In this chapter we introduced the concept of a **function**: a rule where for each value you put in there is exactly one value that comes out.
- To fully define a function, as well as stating the rule, we also need to specify the **domain** – the set of allowed inputs. Once we know the domain we can also find the **range** – the set of outputs that can be produced.
- A function can act upon the output of another function. The result is called a **composite function** and we write $f(g(x))$ or $fg(x)$ or $f \circ g(x)$ for $g(x)$ followed by $f(x)$.
- Reversing the effect of a function $f(x)$ is done by applying an **inverse function**, $f^{-1}(x)$. The general method of finding an inverse function is:

  1. Start with $y = f(x)$.
  2. Rearrange to get x in terms of y.
  3. Replace each occurrence of y with an x.

- Inverse functions have the following important properties:
  - the graph of an inverse function is the reflection in the line $y = x$ of the graph of the original function
  - the domain of the inverse function is the range of the original function, and the range of the inverse function is the domain of the original function
  - the inverse function and the original function cancel each other out to give the **identity function**: $f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$

- For **rational functions** of the form $f(x)=\frac{ax+b}{cx+d}$, the graph is a **hyperbola** with the following properties:
  - vertical asymptote $x=-\frac{d}{c}$
  - horizontal asymptote $y=\frac{a}{c}$
  - x-intercept at $x=-\frac{b}{a}$
  - y-intercept at $y=\frac{b}{d}$
- A special case of a rational function is a **reciprocal function**, $f(x)=\frac{k}{x}$. This is an example of a **self-inverse** function.

Introductory problem revisited

Think of any number. Add on 3. Double your answer. Take away 6. Divide by the number you first thought of. Is your answer always prime? Why?

We can write the actions on the number as a function:

$$f(x)=\frac{2(x+3)-6}{x}$$

In most cases this expression simplifies to give 2, which is a prime number. However, we now know that a function is more than just a rule; it also needs a domain. The domain for this function must exclude zero, so the function produces a prime number for any input other than zero.

Mixed examination practice 4

Short questions

1. If $f(x) = x^2 + 1$, find $f(2x-1)$. *[3 marks]*

2. If $f(x) = x + 2$ and $g(x) = x^2$, solve the equation $fg(x) = gf(x)$. *[5 marks]*

3. If $f(x) = e^{2x}$, evaluate $f^{-1}(3)$. *[3 marks]*

4. (a) Write down the equations of all asymptotes of the graph of $y = \dfrac{4x-3}{5-x}$.

(b) Find the inverse function of $f(x) = \dfrac{4x-3}{5-x}$. *[6 marks]*

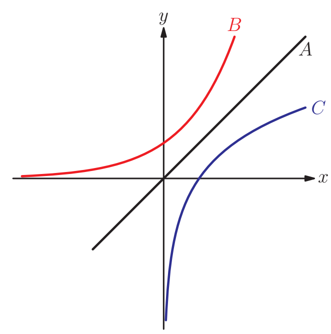
5. Find the inverse of the following functions.

(a) $f(x) = \log_3(x+3),\ x > -3$

(b) $f(x) = 3e^{x^3-1}$ *[4 marks]*

6. The diagram shows three graphs.

A is part of the graph of $y = x$
B is part of the graph of $y = 2^x$
C is the reflection of graph B in line A



Write down:

(a) the equation of C in the form $y = f(x)$

(b) the coordinates of the point where C cuts the x-axis. *[5 marks]*

7. The function f is given by $f(x) = x^2 - 6x + 10$ for $x \geq 3$.

(a) Write $f(x)$ in the form $(x-p)^2 + q$.

(b) Find the inverse function $f^{-1}(x)$.

(c) State the domain of $f^{-1}(x)$. *[6 marks]*

8. The function $f(x)$ is defined by $f(x) = \dfrac{3-x}{x+1}, x \neq -1$.

(a) Find the range of f.

(b) Sketch the graph of $y = f(x)$.

(c) Find the inverse function of f in the form $f^{-1}(x) = \dfrac{ax+b}{cx+d}$, and state its domain and range. *[11 marks]*

9. Let $h(x)=x^2-6x+2$ for $x>3$.

(a) Write $h(x)$ in the form $(x-p)^2+q$.

(b) Hence or otherwise, find the range of $h(x)$.

(c) Find the inverse function $h^{-1}(x)$. *[7 marks]*

10. The functions $f(x)$ and $g(x)$ are given by $f(x)=\sqrt{x-2}$ and $g(x)=x^2+x$. The function $(f\circ g)(x)$ is defined for $x\in\mathbb{R}$ except on the interval $]a,b[$.

(a) Calculate the value of a and of b.

(b) Find the range of $f\circ g$. *[7 marks]*

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Long questions

1. Let $f(x)=x^2+1$, $x\geq 3$ and $g(x)=5-x$.

(a) Evaluate $f(3)$.

(b) Find and simplify an expression for $gf(x)$.

(c) State the geometric relationship between the graphs of $y=f(x)$ and $y=f^{-1}(x)$.

(d) (i) Find an expression for $f^{-1}(x)$.

(ii) Find the range of $f^{-1}(x)$.

(iii) Find the domain of $f^{-1}(x)$.

(e) Solve the equation $f(x)=g(3x)$. *[10 marks]*

2. Define $f(x)=2x+1$ and $g(x)=\dfrac{x+3}{x-1}$, $x\neq 1$.

(a) Find and simplify

(i) $f(7)$ (ii) the range of $f(x)$ (iii) $f(z)$

(iv) $fg(x)$ (v) $ff(x)$

(b) Explain why $gf(x)$ does not exist.

(c) (i) Find an expression for $g^{-1}(x)$.

(ii) State the geometric relationship between the graphs of $y=g(x)$ and $y=g^{-1}(x)$.

(iii) State the domain of $g^{-1}(x)$.

(iv) State the range of $g^{-1}(x)$. *[9 marks]*

3. The functions f and g are defined over the domain of all real numbers by

$$f(x) = x^2 + 4x + 9$$
$$g(x) = e^x$$

(a) Write $f(x)$ in the form $f(x) = (x+p)^2 + q$.

(b) Hence sketch the graph of $y = x^2 + 4x + 9$, labelling carefully all axis intercepts and the coordinates of the turning point.

(c) State the range of $f(x)$ and of $g(x)$.

(d) Hence or otherwise, find the range of $h(x) = e^{2x} + 4e^x + 9$. *[10 marks]*

4. You are given that $(2x+3)(4-y) = 12$ for $x, y \in \mathbb{R}$.

(a) Write y in terms of x, giving your answer in the form $y = \frac{ax+b}{cx+d}$.

(b) Sketch the graph of y against x.

Let $g(x) = 2x + k$ and $h(x) = \frac{8x}{2x+3}$.

(c) Find $h(g(x))$.

(d) Write down the equations of the asymptotes of the graph of $y = h(g(x))$.

(e) Show that when $k = -\frac{19}{2}$, $h(g(x))$ is a self-inverse function. *[17 marks]*