17 Probability

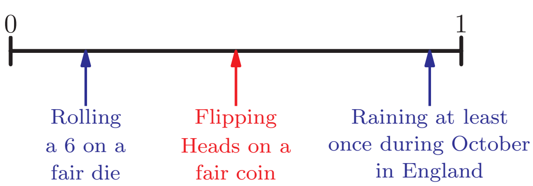
In this chapter you will learn:

- how probability can be estimated from data
- how probability can be predicted theoretically
- how to work out probabilities when you are interested in more than one outcome
- how to work out the probability of a sequence of events occurring
- how to work out probabilities of simple functions of independent random events
- how being given additional information changes our estimate of a probability.

Introductory problem

A woman gives birth to non-identical twins. One of them is a girl. What is the probability that the other one is a girl?

In real life we often deal with uncertain events, but not all events are equally uncertain. It is not certain that the next Steven Spielberg film will be a big hit, nor is it certain that it will snow in India next summer. However, intuitively, these two events are not equally likely. We can put events on a scale with impossible at one end and certain at the other. To indicate where an event lies on this scale, we assign it a number between 0 and 1; this number is called the **probability** of the event.



For the concept of probability to be useful, it must be more than just a reflection of past experience. We also have to be able to predict probabilities of events in the future, and use these predictions to make decisions. This is the focus of this chapter.

Probability is one of the most recent additions to the field of mathematics. It was formalised by the French mathematician Pierre de Fermat (1601–1655) in response to a request from his patron, the Chevalier de Méré, a notorious gambler who wanted help at the gambling table.

17A Empirical probability

In this section we shall look at how we can use data to estimate the probability of an event occurring. A probability that is estimated from observations is called *empirical* or *experimental probability*.

Out of 291 240 births in Australia in 2009, there were 4444 multiple births (twins, triplets, etc.). The fraction of multiple births is $\frac{4444}{291240}$ or about 0.015; this is an estimate of the probability of a pregnancy being a multiple birth in Australia.

KEY POINT 17.1

The probability of an event A occurring is denoted by $\mathrm{P}(A)$.

From observation, we can estimate $\mathrm{P}(A)$ as:

$$\mathrm{P}(A) = \frac{\text{number of times } A \text{ occurs}}{\text{number of times } A \text{ could have occurred}}$$

Notice that, according to this definition, $\mathrm{P}(A)$ must be between 0 and 1 inclusive.

We can turn this idea around or, in other words, use a rearranged form of the equation: if we are given a probability and the number of times the event could have occurred, we can estimate how many times we think that the event might occur.

Worked example 17.1

A six-sided die is rolled 200 times and a '6' comes up 40 times.

(a) Use this data to estimate the probability of rolling a '6'.

(b) Suppose the die is rolled a further 35 times. Assuming that the probability remains the same, estimate the number of '6's that would occur.

Use the formula in Key point 17.1.

(a) $\mathrm{P}(6) = \frac{40}{200} = \frac{1}{5}$

Rearrange the formula.

(b) Expected number of '6's that occur

$= \text{number of attempts} \times \mathrm{P}(6)$

$= 35 \times \frac{1}{5}$

$= 7$

Suppose that in part (b) of the above example, the first seven outcomes were all '6'. Would we then expect to get no more '6's in the next 28 rolls? This idea, though tempting, is incorrect: the outcome of the first 7 rolls does not affect the remaining rolls.

This misconception illustrates how a conflict between language and logic can sometimes lead to incorrect conclusions; it is so famous that it has been given a special name – the Gambler's Fallacy.

Exercise 17A

1. In this question, give your answers correct to three significant figures.

   (a) (i) 600 boxes of cereal are opened and 400 contain plastic toys. Estimate the probability that a randomly opened box contains a plastic toy.

   (ii) 2000 people are surveyed and 300 say that they will vote for the Green party in an upcoming election. Estimate the probability that a randomly chosen voter will vote for the Green party.

   (b) (i) In a board game some cards are drawn: 4 of the cards give positive effects and 3 of them give negative effects. Estimate the probability that a randomly chosen card gives a positive effect.

   (ii) While practising basketball free throws, Brian scores 12 and misses 18. Estimate the probability that Brian scores in his next attempt.

   (c) (i) From a large bag of marbles several are drawn: 3 are red, 6 are blue and 5 are green. Estimate the probability that the next randomly drawn marble is blue.

   (ii) Yuna picked 20 DVDs randomly from her collection: 12 were comedy, 6 were action and the remainder were documentaries. Estimate the probability that another randomly chosen DVD from her collection is a documentary.

2. (a) The probability of getting a heart when a card is drawn from a normal pack of cards is $\frac{1}{4}$. If 20 cards are chosen with replacement (i.e. put back into the pack each time), how many hearts would you expect to draw?

   (b) A cancer drug shows a positive effect within one year in 15% of cases. If 500 cancer patients are put on a trial of the drug, how many would you expect to show a positive effect in one year?

3. In one clinical trial a drug was found to have a positive effect in 18 out of 20 cases. A larger study was then conducted, in which the drug showed a positive effect in 68 cases, a negative effect in 10 cases and no effect in 30 cases. Based on all the available evidence, estimate the probability that in a randomly chosen patient the drug will have a positive effect. *[3 marks]*

4. Two independent studies examined the number of light bulbs that last over 100 hours. In the first study, 40% of light bulbs passed the test. In the second study, 60% of light bulbs passed the test. Is it true that if the studies are combined, it will be shown that 50% of light bulbs pass the test? Justify your answer.

17B Theoretical probability

We can always estimate the probability of an event empirically by performing an experiment repeatedly, but in some situations it is possible to predict the probability *before* doing any experiments. To do this we need to be able to make a list of all possible outcomes, called the **sample space**.

For example, when you toss a coin, there are two possible outcomes: heads and tails. Since the two are equally likely, the probability of each must be one half. This leads to the theoretical definition of probability.

There are two possible outcomes if you enter a lottery: either you win or you don't win. But this does not mean that the probability of winning is one half, because there is no reason to believe that the outcomes are *equally likely*. Many mistakes in probability come from making such an erroneous assumption.

KEY POINT 17.2

$$\mathrm{P}(A)=\frac{\text{number of times } A \text{ occurs in the sample space}}{\text{number of items in the sample space}}$$

In the formula booklet:

$$\mathrm{P}(A)=\frac{n(A)}{n(U)}$$

This definition again ensures that $0 \le \mathrm{P}(A) \le 1$. It also gives the following interpretation to $\mathrm{P}(A)$: if $\mathrm{P}(A)=0$, the event A is impossible; if $\mathrm{P}(A)=1$, the event A is certain; as $\mathrm{P}(A)$ rises, the likelihood of A occurring increases.

You might consider it obvious that the definitions of empirical and theoretical probability are equivalent. However, this is quite tricky to prove. If you would like to see how it is done, do some research on the 'law of large numbers'.

Worked example 17.2

(a) For a family with two children, list the sample space for the sexes of the children, assuming no twins.

(b) Hence find the theoretical probability that the two children are a boy and a girl.

A systematic way to list the possibilities is to make a table.

(a)

First child	Second Child
Boy	Boy
Boy	Girl
Girl	Boy
Girl	Girl

EXAM HINT

Although the 'boy first, girl second' and 'girl first, boy second' cases can be described as a 'boy and a girl', we must remember to count them separately in the sample space.

There are four outcomes, two of which consist of a boy and a girl.

(b) $P(\text{a boy and a girl}) = \frac{2}{4} = \frac{1}{2}$

When the sample space is more complicated, it is important to list the possible outcomes in a systematic way, so that you do not miss any. For example, consider a board game in which players face a penalty whenever the sum of two rolled dice is 7. It does not matter how the 7 is achieved – it may be from a 1 and a 6, or a 3 and a 4, among other possibilities. In this situation it helps to make a probability grid diagram, in which you list each possibility of the first die on one axis, each possibility of the second die on the other axis, and the result of interest (the sum of numbers on the dice) in each cell of the grid.

EXAM HINT

Expressions such as 'fair die' and 'fair coin' mean that all possible outcomes of rolling the die (singular for dice) or tossing the coin are equally likely.

Worked example 17.3

What is the probability of getting a sum of 7 when two fair dice are rolled?

Draw a probability grid diagram showing all possible totals when two dice are rolled.

		Die A					
		1	2	3	4	5	6
Die B	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

EXAM HINT

Notice that a '3' on die A and a '1' on die B has to be counted as a separate outcome from a '1' on die A and a '3' on die B. However, a '3' on die A and a '3' on die B is only one outcome. This often causes confusion.

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Count how many items are in the sample space.

36 items in the sample space, each of equal probability.

Count how many sums of 7 there are.

6 sums of 7, therefore the probability of getting a sum of 7 is

$$\frac{6}{36}=\frac{1}{6}$$

Saying that an event either happens or does not happen is called the 'law of the excluded middle', and it is a basic axiom of standard logic. However, there is an alternative logical system called 'fuzzy logic', where an event could be in a state of 'maybe happening'. Fuzzy logic is used in a philosophical physics problem called 'Schrödinger's cat', and also has many real-world applications.

An event either happens or does not happen. Everything other than the event happening is called the **complement** of the event. For example, the complement of rolling a 6 on a die is rolling a 1, 2, 3, 4 or 5. The complement of event A is denoted by A'.

An event and its complement are said to be *mutually exclusive*, which means that they cannot both happen. We also know that one of the two must happen, so they have a total probability of 1. From this we can deduce a formula linking the probability of an event and the probability of its complement.

KEY POINT 17.3

$$\mathrm{P}(A)+\mathrm{P}(A')=1$$

This looks like a simple and trivial formula, but it can be really useful in simplifying probability calculations, because there are many situations in which the probability of the complement of an event is much easier to find than the probability of the event itself. The next example illustrates one such case.

Worked example 17.4

Two fair dice are rolled. What is the probability that the sum of the scores is greater than 2?

The sum of the scores must be at least 2, and we can get a total of 2 in just one way, so it is easier to find this probability first, which is the probability of the complement of the required event.

Let A be the event 'the sum of the scores is greater than 2'.
Then A' is the event 'the sum of the scores is 2'.
The total number of possible outcomes is 36.

continued . . .

Out of those, only (1, 1) has sum 2.

$$\therefore \mathrm{P}(A') = \frac{1}{36}$$

Use $\mathrm{P}(A) + \mathrm{P}(A') = 1$.

$$\mathrm{P}(A) = 1 - \mathrm{P}(A') = \frac{35}{36}$$

Often we are interested in more complicated events, such as 'today is Tuesday *and* it is raining' or 'getting an A in Extended Essay *or* TOK'. When describing such combined events we need to be very clear about the meaning of the words used. The word 'and' indicates that the desired outcome is *both* of the component events happening.

In the next two sections we will see how the probability of a combined event is related to the probabilities of the individual events.

In everyday language the word 'or' is used ambiguously. If you say 'Peter is a doctor or a lawyer', you generally do not mean that he could be both. However, in a game, if you say 'I win if I get a black number or an even number', you would expect to win if you got a black even number. In probability we use the word 'or' in the second sense: 'A or B' means A or B *or both* happening.

Worked example 17.5

When a die is rolled, find the probability that the outcome is

(a) odd *and* a prime number

(b) odd *or* a prime number.

List the sample space.

Possible outcomes: 1, 2, 3, 4, 5, 6

List the outcomes that satisfy *both* of the given conditions.

(a) Odd and prime: 3, 5

Probability $= \frac{2}{6} = \frac{1}{3}$

List the outcomes that satisfy *one or both* of the conditions.

(b) Odd or prime: 1, 2, 3, 5

Probability $= \frac{4}{6} = \frac{2}{3}$

Exercise 17B

1. List the sample space for each of the following:
   (a) a fair six-sided die
   (b) rearrangements of the word RED
   (c) the sexes of three children in a family
   (d) a six-sided die with three sides labelled 1 and the remaining sides labelled 2, 3 and 4.

2. In a standard pack of 52 playing cards there are four different suits (red hearts, red diamonds, black clubs and black spades). Each suit consists of number cards from 2 to 10 and four 'picture cards': jack, queen, king and ace. Find the probability that a randomly chosen card is
   (a) (i) red (ii) a spade
   (b) (i) a jack (ii) a picture card
   (c) (i) a black number card (ii) a club picture card
   (d) (i) not a heart (ii) not a picture card
   (e) (i) a club or a picture card (ii) a red card or a number card
   (f) (i) a red number card strictly between 3 and 9
   (ii) a picture card that is not a jack

3. A bag contains marbles of three different colours: six are red, four are blue and five are yellow. One marble is taken from the bag. Calculate the probability that it is
   (a) (i) red (ii) yellow
   (b) (i) not blue (ii) not red
   (c) (i) blue or yellow (ii) red or blue
   (d) (i) green (ii) not green
   (e) (i) neither red nor yellow (ii) neither yellow nor blue
   (f) (i) red or green (ii) neither blue nor green
   (g) (i) red and green (ii) red and blue

4. By means of an example, show that $\mathrm{P}(A)+\mathrm{P}(B)=1$ does not mean that B is the complement of A.

5. Two fair six-sided dice numbered 1 to 6 are rolled. By drawing probability grid diagrams, find the probability that
   (a) the sum is 8
   (b) the product is greater than or equal to 8
   (c) the product is 24 or 12
   (d) the maximum value is 4
   (e) the larger value is more than twice the other value
   (f) the value on the first die divided by the value on the second die is a whole number.

6. A fair four-sided die (numbered 1 to 4) and a fair eight-sided die (numbered 1 to 8) are rolled. Find the probability that
   (a) the sum is 8
   (b) the product is greater than or equal to 8
   (c) the product is 24 or 12
   (d) the maximum value is 4
   (e) the larger value is more than twice the other value
   (f) the value on the eight-sided die divided by the value on the four-sided die is a whole number.

7. Two fair six-sided dice are thrown, and the score is the highest common factor of the two outcomes. If this were done 180 times, how many times would you expect the score to be 1? *[5 marks]*

8. Three fair six-sided dice are thrown, and the score is the sum of the three results. What is the probability that the score is less than 6? *[5 marks]*

17C Combined events and Venn diagrams

In this section we shall generalise the sample space method to efficiently calculate probabilities of combined events.

Which is more likely when you roll a die once:

- getting a prime number *and* an odd number
- getting a prime number *or* an odd number?

The first possibility is more restrictive – we have to satisfy both conditions. For the second, we can satisfy either condition, so it must be more likely.

The $\cap$ and $\cup$ symbols are part of set notation, covered in Prior Learning section G on the CD-ROM.

The 'and' and 'or' are two of the most common ways of combining events, known respectively as 'intersection' and 'union' in mathematical language. These are given the following symbols:

$A \cap B$ is the **intersection** of A and B, meaning that *both* A *and* B happen.

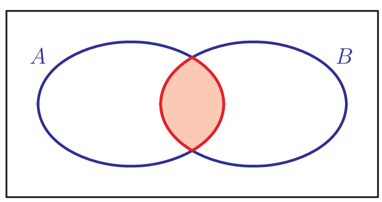
$A \cup B$ is the **union** of A and B, meaning that *either* A happens, *or* B happens, *or both* happen.

If you have neither apples nor pears, then you have no apples *and* no pears. In set notation this can be written as $(A \cup B)' = A' \cap B'$. This is one of De Morgan's laws – a description of the algebraic rules obeyed by sets and hence events in probability.

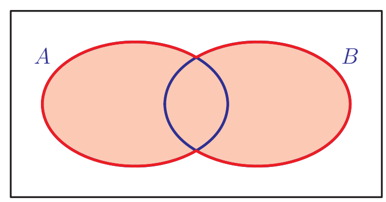
Why do mathematicians use complicated words such as 'union' and 'intersection'? One reason is that everyday language can be ambiguous. If I say that I play football or hockey, some people will take this to mean that I do not play both. Mathematicians hate ambiguity.

See Prior Learning section J on the CD-ROM for explanation of Venn diagrams.

We can use **Venn diagrams** to illustrate the concepts of union and intersection:

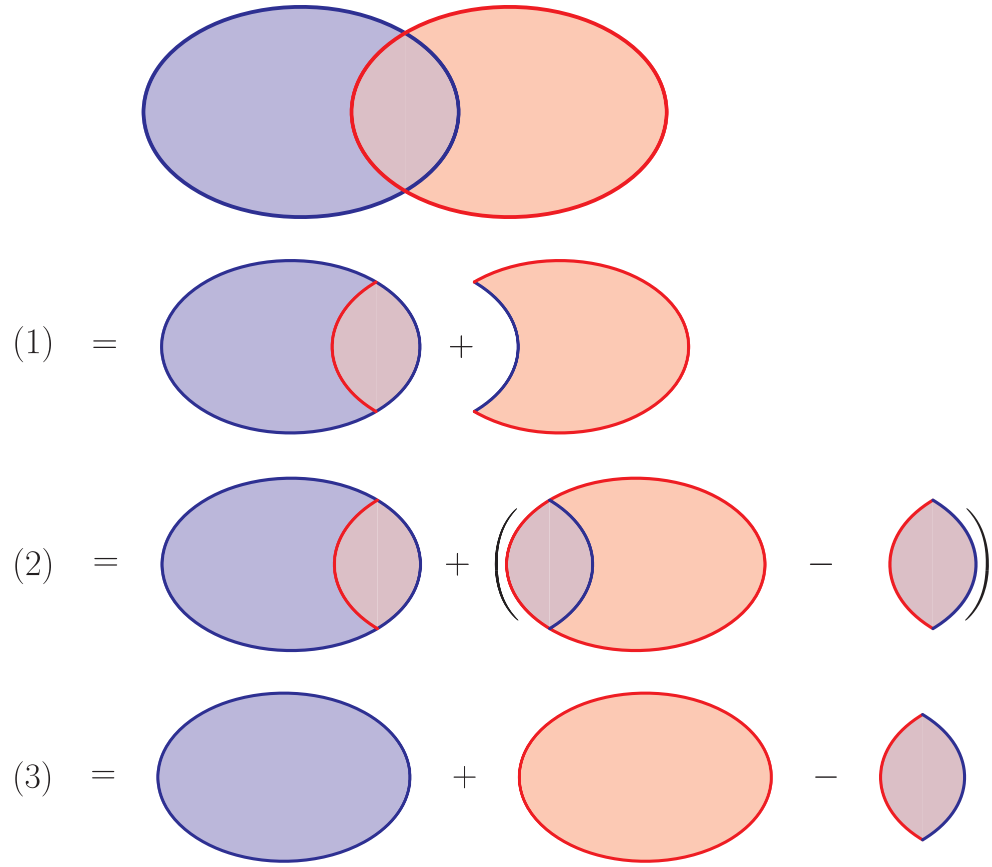


$A \cap B$



$A \cup B$

Venn diagrams can help us calculate the probability of the union of two events. The diagrams below show that the union of two sets which have some overlap can be viewed as the sum of the two separate sets minus the region of overlap.



This gives a very important formula.

KEY POINT 17.4

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

In section 17B we already met one example of mutually exclusive events: A and its complement A′.

Essentially, this formula is saying that if you want to count the number of ways of getting A or B, count the number of ways of getting A and add to that the number of ways of getting B; however, the number of ways of getting A *and* B would have been counted twice, so we compensate by subtracting it.

If there is no possibility of A and B occurring at the same time, then $P(A \cap B) = 0$. These events are said to be *mutually exclusive*, and in this case the formula reduces to $P(A \cup B) = P(A) + P(B)$.

EXAM HINT

Be careful that you do not use this formula for mutually exclusive events unless you are sure that the events cannot both occur at the same time.

Worked example 17.6

A chocolate is selected randomly from a box. The probability of it containing nuts is $\frac{1}{4}$. The probability of it containing caramel is $\frac{1}{3}$. The probability of it containing both nuts and caramel is $\frac{1}{6}$. What is the probability of a randomly chosen chocolate containing either nuts or caramel or both?

The event we are interested in is 'nuts $\cup$ caramel', so we can use the formula from Key point 17.4.

P(nuts $\cup$ caramel)

$= \text{P(nuts)} + \text{P(caramel)} - \text{P(nuts} \cap \text{caramel)}$

$= \frac{1}{4} + \frac{1}{3} - \frac{1}{6}$

$= \frac{5}{12}$

Exercise 17C

1. (a) (i) $P(A) = 0.4$, $P(B) = 0.3$ and $P(A \cap B) = 0.2$.
   Find $P(A \cup B)$.

   (ii) $P(A) = \frac{3}{10}$, $P(B) = \frac{4}{5}$ and $P(A \cap B) = \frac{1}{10}$.
   Find $P(A \cup B)$.

   (b) (i) $P(A) = \frac{2}{3}$, $P(B) = \frac{1}{8}$ and $P(A \cup B) = \frac{5}{8}$.
   Find $P(A \cap B)$.

   (ii) $P(A) = 0.2$, $P(B) = 0.1$ and $P(A \cup B) = 0.25$.
   Find $P(A \cap B)$.

   (c) (i) $P(A \cap B) = 20\%$, $P(A \cup B) = 0.4$ and $P(A) = \frac{1}{3}$.
   Find $P(B)$.

   (ii) $P(A \cup B) = 1$, $P(A \cap B) = 0$ and $P(B) = 0.8$.
   Find $P(A)$.

   (d) (i) Find $P(A \cup B)$ if $P(A) = 0.4$, $P(B) = 0.3$ and A and B are mutually exclusive.

   (ii) Find $P(A \cup B)$ if $P(A) = 0.1$, $P(B) = 0.01$ and A and B are mutually exclusive.

2. (a) (i) When a fruit pie is selected at random,

$\mathrm{P}(\text{it contains pears}) = \frac{1}{5}$ and $\mathrm{P}(\text{it contains apples}) = \frac{1}{4}$.
10% of the pies contain both apples and pears. What is the probability that a randomly selected pie contains apples or pears?

(ii) In a library, 80% of books are classed as fiction and 70% are classed as 20th century. Half of the books are 20th-century fiction. What proportion of the books are fiction or from the 20th century?

(b) (i) 25% of students in a school play football or tennis. The probability of a randomly chosen student playing football is $\frac{3}{5}$, and the probability that they play tennis is $\frac{5}{8}$. What percentage of students play both football and tennis?

(ii) Two in five people at a school study Spanish, and one in three study French. Half of the school study French or Spanish. What fraction study both French and Spanish?

(c) (i) 90% of students in a school have a Facebook account, and three out of five have a Twitter account. One-twentieth of students have neither a Facebook account nor a Twitter account. What percentage of students are on both Facebook and Twitter?

(ii) 25% of teams in a football league have French players and a third have Italian players. 60% have neither French nor Italian players. What percentage have both French and Italian players?

3. Simplify the following expressions where possible.

(a) $\mathrm{P}(x > 2 \cap x > 4)$

(b) $\mathrm{P}(y \leq 3 \cup y < 2)$

(c) $\mathrm{P}(a < 3 \cap a > 4)$

(d) $\mathrm{P}(a < 5 \cup a \geq 0)$

(e) $\mathrm{P}(\text{apple} \cup \text{fruit})$

(f) $\mathrm{P}(\text{apple} \cap \text{fruit})$

(g) $\mathrm{P}(\text{multiple of } 4 \cap \text{multiple of } 2)$

(h) $\mathrm{P}(\text{square} \cup \text{rectangle})$

(i) $\mathrm{P}(\text{blue} \cap (\text{blue} \cup \text{red}))$

(j) $\mathrm{P}(\text{blue} \cap (\text{blue} \cap \text{red}))$

4. In a survey, 60% of people are in favour of a new primary school and 85% are in favour of a new library. Half of all those surveyed would like both a new primary school and a new library. What percentage supported neither a new library nor a new primary school? *[5 marks]*

5. If $\mathrm{P}(A) = 0.2$, $\mathrm{P}(A \cap B) = 0.1$ and $\mathrm{P}(A \cup B) = 0.7$ find $\mathrm{P}(B')$. *[5 marks]*

6. Events A and B satisfy $\mathrm{P}((A \cup B)') = 0.2$ and $\mathrm{P}(A) = P(B) = 0.5$. Find $\mathrm{P}(A \cap B')$. *[5 marks]*

7. An integer is chosen at random from the first one thousand positive integers. Find the probability that the integer chosen is

(a) a multiple of 6

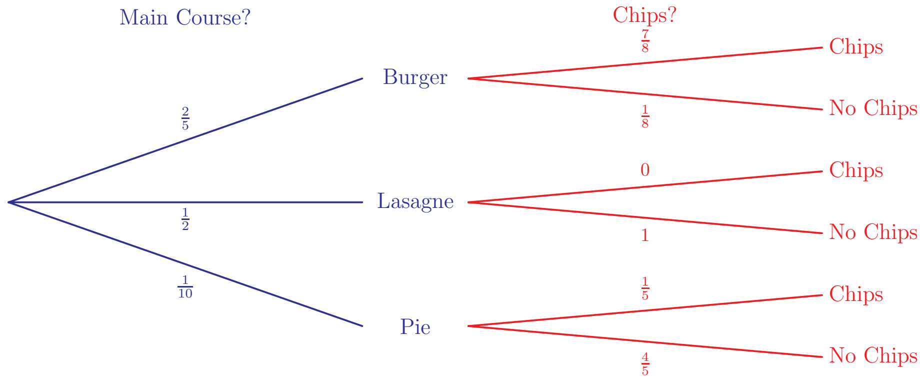
(b) a multiple of *both* 6 and 8. *[5 marks]*

17D Tree diagrams and finding intersections

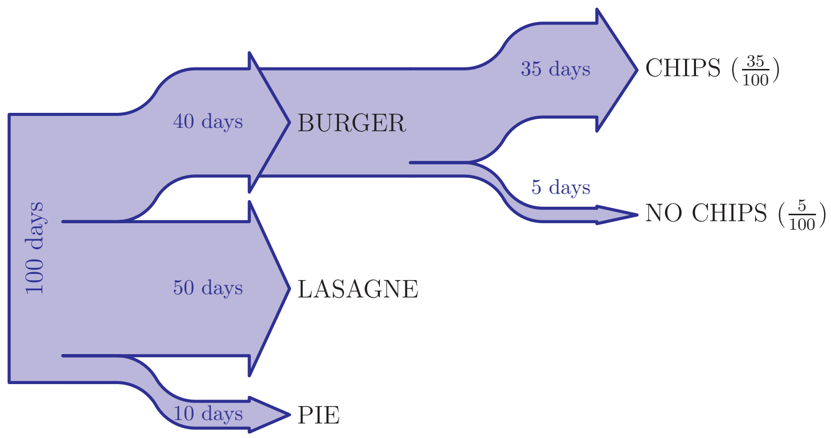
Venn diagrams have given us a useful formula connecting the intersection and union of two events. However, they are not of much help in calculating the probability of an intersection or union by itself. Another method – tree diagrams – is more useful for doing this.

When several events happen, either in succession or simultaneously, a tree diagram displays all the possible combinations of outcomes. It starts out with branches representing all the possible outcomes for one of the events; then from each branch we draw further branches that represent all possible outcomes for the next event. Along each branch we write the probability of taking that branch. Tree diagrams have an advantage over the sample space method as they can accommodate outcomes that are not equally likely.

The following is an example of a tree diagram, showing meal combinations at a school.



We can think of there being a filtering process at each branching point. Consider 100 days of school food. On $\frac{2}{5}$ of these days (i.e. 40 days) there will be burgers. On $\frac{7}{8}$ of these burger days there will also be chips. So overall there will be 35 days out of the 100 that have both burgers and chips, equivalent to a probability of $\frac{7}{20}$.



To find the probability of getting to each branch tip at the far right of the diagram, we calculate a fraction of a fraction; in other words, we multiply the probabilities on the branches we travelled along to reach that branch tip. What do these two probabilities represent in our burger and chips example? Clearly the $\frac{2}{5}$ represents the probability of getting burgers, and it might be tempting to say that the $\frac{7}{8}$ is the probability of getting chips – but this is not true. If we complete the filtering diagram above, we will see that chips are served on 37 out of 100 days, and this is certainly not $\frac{7}{8}$. Rather, the $\frac{7}{8}$ represents the probability of having chips *if you already know* that burgers are being served; this is called a **conditional probability**. The conditional probability in this case is written P(chips|burgers), read as 'the probability of chips given burgers'. Tree diagrams therefore lead us to the following important formula for the probability of an intersection of two events.

KEY POINT 17.5

$$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B|A)$$

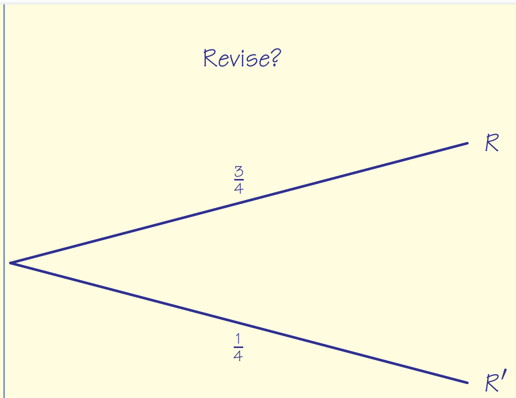
Worked example 17.7

If I revise, there is an 80% chance I will pass the test; but if I do not revise, there is only a 30% chance of passing. I revise for ¾ of tests. What proportion of tests can I expect to pass?

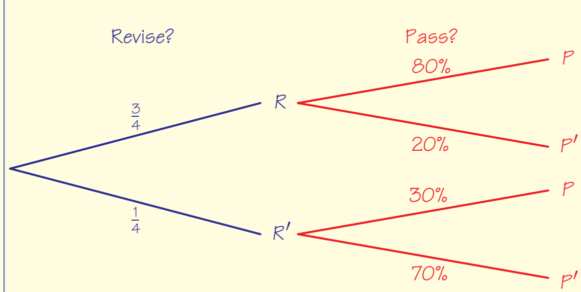
EXAM HINT

This question uses the words 'chance' and 'proportion'. These are just other ways of referring to probability. Try not to get put off by unusually worded questions.

Decide which of the given probabilities is not conditional. Start the tree diagram with this event. Since the probability of passing the test is conditional on revision, the revision branches should come first.



Add the conditional event: passing or not passing the test.



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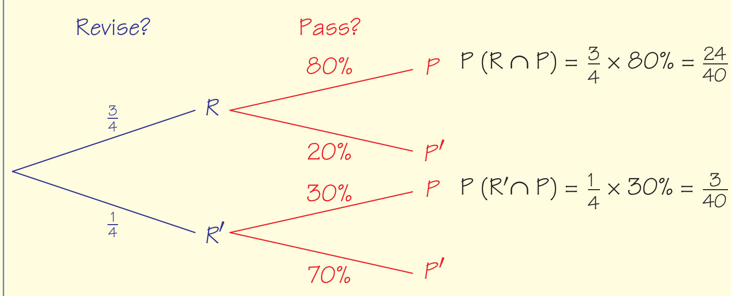
Identify which branch sequences result in passing the test.

Multiply the probabilities along each branch sequence to find the probability at the end.



Add up the end probabilities associated with passing the test.

$$P(\text{passing}) = P(\text{revising} \cap \text{passing}) + P(\text{not revising} \cap \text{passing})$$

$$= \frac{24}{40} + \frac{3}{40}$$

$$= \frac{27}{40}$$

EXAM HINT

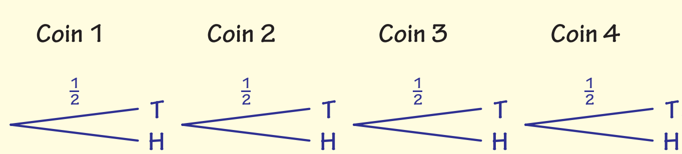
Frequently, when a question says to find the probability of 'at least …' or 'at most …', looking at the complement is a good idea.

As we saw in section 17B, sometimes the probability we are asked for may be quite difficult to find directly, but the probability of its complement is much easier to work out.

Worked example 17.8

Find the probability of getting at least one head when you toss four fair coins.

We could draw a tree diagram, find the probabilities of getting 1, 2, 3 or 4 heads, and add them up. But it is easier to look at the complement – getting 0 heads (all tails).



$$P(0 \text{ heads}) = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$$

$$P(\text{at least 1 head}) = 1 - P(0 \text{ heads})$$

$$= \frac{15}{16}$$

Exercise 17D

1. (a) (i) $P(A)=0.4$ and $P(B|A)=0.3$. Find $P(A\cap B)$.

(ii) $P(X)=\frac{3}{5}$ and $P(Y|X)=0$. Find $P(X\cap Y)$.

(b) (i) $P(A)=0.3$, $P(B)=0.2$ and $P(B|A)=0.8$. Find $P(A\cap B)$.

(ii) $P(A)=0.4$, $P(B)=0.8$ and $P(A|B)=0.3$. Find $P(A\cap B)$.

(c) (i) $P(A)=\frac{2}{5}$, $P(B)=\frac{1}{3}$ and $P(A|B)=\frac{1}{4}$. Find $P(A\cup B)$.

(ii) $P(A)=\frac{3}{4}$, $P(B)=\frac{1}{4}$ and $P(B|A)=\frac{1}{3}$. Find $P(A\cup B)$.

2. A class contains 6 boys and 8 girls. If two students are picked at random, what is the probability that they are both boys? *[4 marks]*

3. A bag contains 4 red balls, 3 blue balls and 2 green balls. A ball is chosen at random from the bag and is not replaced; then a second ball is chosen. Find the probability of choosing one green ball and one blue ball in any order. *[5 marks]*

4. Given that $P(X)=\frac{1}{3}$, $P(Y|X)=\frac{2}{9}$ and $P(Y|X')=\frac{1}{3}$, find

(a) $P(Y')$

(b) $P(X'\cup Y')$ *[6 marks]*

5. A factory has two machines for making widgets. The older machine has larger capacity, so it makes 60% of the widgets, but 6% are rejected by quality control. The newer machine has only a 3% rejection rate. Find the probability that a randomly selected widget is rejected. *[5 marks]*

6. The school tennis league consists of 12 players. Daniel has a 30% chance of winning any game against a higher-ranked player, and a 70% chance of winning any game against a lower-ranked player. If Daniel is currently in third place, find the probability that he wins his next game against a random opponent. *[5 marks]*

7. There are 36 disks in a bag. Some of them are black and the rest are white. Two are simultaneously selected at random. Given that the probability of selecting two disks of the same colour is equal to the probability of selecting two disks of different colour, how many black disks are there in the bag? *[6 marks]*

17E Independent events

We can now evaluate the probabilities of the intersection and union of two events A and B if we know $\mathrm{P}(A)$, $\mathrm{P}(B)$ and $\mathrm{P}(A|B)$ or $\mathrm{P}(B|A)$, but finding the conditional probabilities can be quite difficult. There is one important exception: if the two events do not affect each other. Such events are called **independent events**. In this case, knowing that B has occurred has no impact on the probability of A occurring; in probability notation, $\mathrm{P}(A|B)=\mathrm{P}(A)$. Since $\mathrm{P}(A\cap B)=\mathrm{P}(A|B)\mathrm{P}(B)$, we get the following relationship that applies to independent events.

KEY POINT 17.6

For independent events,

$$\mathrm{P}(A\cap B)=\mathrm{P}(A)\mathrm{P}(B)$$

As well as being true for all independent events, this equation is actually a defining feature of independent events. So, in order to prove that two events are independent, we only need to show that they satisfy the equation in Key point 17.6.

All strawberries are red, but not all red things are strawberries. It can be difficult to distinguish between a property and a defining feature. A circle has a constant width, but is every plane shape with constant width necessarily a circle?

Worked example 17.9

$P(A) = 0.5$, $P(B) = 0.2$ and $P(A \cup B) = 0.6$. Are the events A and B independent?

Use the given information to find $P(A \cap B)$.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$0.6 = 0.5 + 0.2 - P(A \cap B)$$
$$\therefore P(A \cap B) = 0.1$$

Evaluate $P(A)P(B)$.

$$P(A)P(B) = 0.5 \times 0.2 = 0.1$$

Compare the two.

$$P(A \cap B) = P(A)P(B)$$

So A and B are independent.

If we know that two events are independent, we can use this fact to help calculate other probabilities.

Worked example 17.10

A and B are independent events with $P(A \cup B) = 0.8$ and $P(A) = 0.2$. Find $P(B)$.

Write $P(A \cap B)$ in terms of other probabilities.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$0.8 = 0.2 + P(B) - P(A \cap B)$$

Use independence.

$$= 0.2 + P(B) - 0.2 \times P(B)$$
$$= 0.2 + 0.8P(B)$$

Solve the equation for $P(B)$.

$$\Rightarrow 0.6 = 0.8P(B)$$
$$\therefore P(B) = \frac{3}{4} \text{ (or } 0.75)$$

Exercise 17E

1. In this question, assume that events A and B are independent.
   (a) (i) $P(A) = 0.3$ and $P(B) = 0.7$. Find $P(A \cap B)$.
   (ii) $P(A) = \frac{1}{5}$ and $P(B) = \frac{1}{3}$. Find $P(A \cap B)$.

(b) (i) $\mathrm{P}(A)=\frac{4}{5}$ and $\mathrm{P}(A\cap B)=\frac{3}{7}$. Find $\mathrm{P}(B)$.

(ii) $\mathrm{P}(A\cap B)=0.5$ and $\mathrm{P}(B)=0.9$. Find $\mathrm{P}(A)$.

(c) (i) $\mathrm{P}(A)=40\%$ and $\mathrm{P}(B)=16\%$. Find $\mathrm{P}(A\cup B)$.

(ii) $\mathrm{P}(A)=0.2$ and $\mathrm{P}(B)=\frac{1}{4}$. Find $\mathrm{P}(A\cup B)$.

(d) (i) $\mathrm{P}(A\cup B)=0.6$ and $\mathrm{P}(A)=0.4$. Find $\mathrm{P}(B)$.

(ii) $\mathrm{P}(A\cup B)=0.5$ and $\mathrm{P}(A)=0.1$. Find $\mathrm{P}(B)$.

2. Determine which of the following pairs of events are independent.

(a) (i) $\mathrm{P}(A)=0.6$, $\mathrm{P}(B)=0.5$ and $\mathrm{P}(A\cap B)=0.3$.

(ii) $\mathrm{P}(A)=0.8$, $\mathrm{P}(B)=0.1$ and $\mathrm{P}(A\cap B)=0.05$.

(b) (i) $\mathrm{P}(A)=\frac{1}{5}$, $\mathrm{P}(B)=\frac{1}{4}$ and $\mathrm{P}(A\cup B)=\frac{2}{5}$.

(ii) $\mathrm{P}(A)=56\%$, $\mathrm{P}(B)=32\%$ and $\mathrm{P}(A\cup B)=72\%$.

3. The independent events A and B are such that $\mathrm{P}(A)=0.6$ and $\mathrm{P}(A\cup B)=0.72$. Find

(a) $\mathrm{P}(B)$

(b) the probability that either A occurs or B occurs, *but not both.* *[5 marks]*

4. A school has two photocopiers, one for teachers and one for students. The probability of the teachers' photocopier working is 92%. The probability of the students' photocopier working is 68%. The two outcomes do not affect each other. What is the probability that

(a) both photocopiers are working

(b) neither photocopier is working

(c) at least one photocopier is working? *[7 marks]*

5. As part of a promotion a toy is put in each packet of crisps sold. There are eight different toys available. Each toy is equally likely to be found in any packet of crisps.

David buys four packets of crisps.

(a) Find the probability that the four toys in these packets are all different.

(b) Of the eight toys in the packets, his favourites are the yo-yo and the gyroscope. What is the probability that he finds at least one of his favourite toys in the four packets? *[7 marks]*

6. Given that events A and B are independent with $P(A \cap B) = 0.3$ and $P(A \cap B') = 0.6$, find $P(A \cup B)$. *[5 marks]*

17F Conditional probability

Estimate the probability that a randomly chosen person is a dollar millionaire. Would your estimate change if you were told that they live in a mansion?

When we get additional information, probabilities change. In the above example, P(millionaire) is very different from P(millionaire|lives in mansion). The second is a **conditional probability**, which we already used in section 17D when working with tree diagrams.

To find conditional probabilities, one useful method is restricting the sample space: we write out a list of all equally likely possibilities before we know any extra information (the full sample space); then we cross out any possibilities ruled out by the extra information given.

Worked example 17.11

Given that the number rolled on a die is prime, show that the probability that it is odd is $\frac{2}{3}$.

Write out the full sample space for a single roll of a die.	On one roll we could get 1, 2, 3, 4, 5 or 6.
We are told that the number is prime.	If the number is prime it can only be 2, 3 or 5.
Decide how many of the remaining numbers are odd.	Two of these are odd, so the probability is $\frac{2}{3}$.

In section 17D (Key point 17.5) we saw that $P(A \cap B) = P(A)P(B|A)$. Rearranging this equation gives a very important formula for conditional probability.

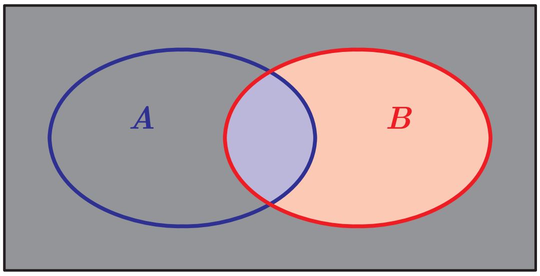
KEY POINT 17.7

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

EXAM HINT

This rearranged form is not given in the Formula booklet.

We can visualise this formula in a Venn diagram.



When calculating a probability it is not always obvious which sample space needs to be used. Supplementary sheet 22 on the CD-ROM explores this difficulty in the context of estimating the risk of a transport accident.

If we know that event B has occurred, we can focus on B and ignore the rest of the Venn diagram. The portion of the region B in which A occurs is $A \cap B$. We will look at such Venn diagrams in more detail in section 17G.

Worked example 17.12

The probability that a randomly chosen resident of a certain city is a millionaire is $\frac{1}{10000}$. The probability that a randomly chosen resident lives in a mansion is $\frac{1}{30000}$. Only 1 in 40 000 residents are millionaires who live in mansions. What is the probability of a randomly chosen individual being a millionaire given that they live in a mansion?

Write the required probability in 'given' notation and apply the formula.

$$P(\text{millionaire}|\text{mansion}) = \frac{P(\text{millionaire} \cap \text{mansion})}{P(\text{mansion})}$$

$$= \frac{\frac{1}{40000}}{\frac{1}{30000}}$$

$$= \frac{3}{4}$$

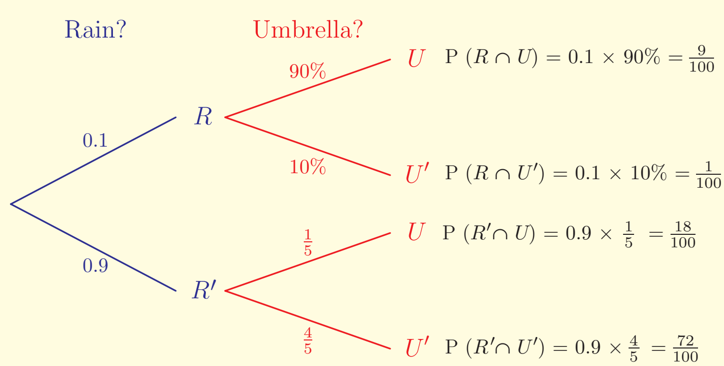
It can be difficult to tell whether a question is asking for conditional probability or combined probability. For example, if the question says that a boy has green eyes and asks what is the probability that he also has brown hair, it is tempting to think this is P(green eyes $\cap$ brown hair). However, the fact that the boy has green eyes has been given, so we should actually find P(brown hair|green eyes).

Sometimes we are given one conditional probability and need to find another.

Worked example 17.13

If it is raining in the morning, there is a 90% chance that I will bring my umbrella. If it is not raining in the morning, there is only a $\frac{1}{5}$ chance of me taking my umbrella. On any given morning the probability of rain is 0.1. If you see me with an umbrella, what is the probability that it was raining that morning?

First draw a tree diagram.



Write the required quantity in probability notation. We realise that it is a conditional probability, so use the formula.

$$P(\text{rain} \mid \text{umbrella}) = \frac{P(\text{rain} \cap \text{umbrella})}{P(\text{umbrella})}$$

Use the tree diagram to find the relevant probabilities.

$$P(\text{rain} \cap \text{umbrella}) = \frac{9}{100}$$

$$P(\text{umbrella}) = \frac{9}{100} + \frac{18}{100} = \frac{27}{100}$$

Put these numbers into the formula.

$$\therefore P(\text{rain} \mid \text{umbrella}) = \frac{P(\text{rain} \cap \text{umbrella})}{P(\text{umbrella})}$$

$$= \frac{\frac{9}{100}}{\frac{27}{100}}$$

$$= \frac{1}{3}$$

Exercise 17F

1. For each of the questions below, write the probability required in mathematical notation; you are not required to calculate the probability. For example, the probability that I revise and pass the test would be written P(revise $\cap$ pass).
   (a) Find the probability that the outcome of rolling a die is a prime and odd number.
   (b) Find the probability that a person is from either Senegal or Taiwan.
   (c) A student is studying the IB. Find the probability that he is also studying French.
   (d) If a playing card is a red card, find the probability that it is a heart.
   (e) What proportion of German people live in Munich?
   (f) What is the probability that someone is wearing neither black nor white socks?
   (g) What is the probability that a vegetable is a potato if it is not a cabbage?
   (h) What is the probability that a ball drawn from a bag is red, given that the ball is either red or blue?

2. (a) (i) If $\mathrm{P}(X)=0.3$ and $\mathrm{P}(X\cap Y)=\frac{1}{5}$, find $\mathrm{P}(Y\mid X)$.
   (ii) If $\mathrm{P}(Y)=0.8$ and $\mathrm{P}(X\cap Y)=\frac{3}{7}$, find $\mathrm{P}(X\mid Y)$.
   (b) (i) If $\mathrm{P}(X)=0.4$, $\mathrm{P}(Y)=0.7$ and $\mathrm{P}(X\cap Y)=\frac{1}{4}$, find $\mathrm{P}(X\mid Y)$.
   (ii) If $\mathrm{P}(X)=0.6$, $\mathrm{P}(Y)=0.9$ and $\mathrm{P}(X\cap Y)=\frac{1}{2}$, find $\mathrm{P}(Y\mid X)$.

3. The events A and B are such that $\mathrm{P}(A)=0.6$, $\mathrm{P}(B)=0.2$ and $\mathrm{P}(A\cup B)=0.7$.
   (a) (i) Find $\mathrm{P}(A\cap B)$.
   (ii) Hence show that A and B are not independent.
   (b) Find $\mathrm{P}(B\mid A)$. *[7 marks]*

4. Let A and B be events such that $\mathrm{P}(A)=\frac{2}{3}$, $\mathrm{P}(B|A)=\frac{1}{2}$ and $\mathrm{P}(A\cup B)=\frac{4}{5}$.

(a) Find $\mathrm{P}(A\cap B)$.

(b) Find $\mathrm{P}(B)$.

(c) Show that A and B are not independent. *[7 marks]*

5. Box A contains 6 red balls and 4 green balls. Box B contains 5 red balls and 3 green balls. A standard fair cubical die is thrown. If a '6' is obtained, a ball is selected from box A; otherwise a ball is selected from box B.

(a) Calculate the probability that the ball selected was red.

(b) Given that the ball selected was red, calculate the probability that it came from box B. *[7 marks]*

6. Robert travels to work by train every weekday from Monday to Friday. The probability that he catches the 7.30 a.m. train on Monday is $\frac{2}{3}$. The probability that he catches the 7.30 a.m. train on any other weekday is 90%. A weekday is chosen at random.

(a) Find the probability that Robert catches the 7.30 a.m. train on that day.

(b) Given that he catches the 7.30 a.m. train on that day, find the probability that the chosen day is Monday. *[7 marks]*

7. Bag 1 contains 6 red cubes and 10 blue cubes. Bag 2 contains 7 red cubes and 3 blue cubes.

Two cubes are drawn at random, the first from bag 1 and the second from bag 2.

(a) Find the probability that the cubes are of the same colour.

(b) Given that the cubes selected are of different colours, find the probability that the red cube was selected from bag 1. *[8 marks]*

8. On any day in April, there is a $\frac{2}{3}$ chance of rain in the morning. If it is raining, there is a $\frac{4}{5}$ chance I will remember my umbrella, but if it is not raining, there is only a $\frac{2}{5}$ chance of my remembering to bring an umbrella.

(a) On a random day in April, what is the probability I have my umbrella with me?

(b) Given that I have an umbrella on a day in April, what is the probability that it was raining? *[6 marks]*

9. The probability that a man leaves his umbrella in any shop he visits is $\frac{1}{5}$. After visiting two shops in succession, he finds he has left his umbrella in one of them. What is the probability that he left his umbrella in the second shop? *[4 marks]*

10. $P(A)=\frac{2}{3}, P(A|B)=\frac{1}{5}$ and $P(A\cup B)=\frac{4}{5}$. Find $P(B)$. *[6 marks]*

17G Further Venn diagrams

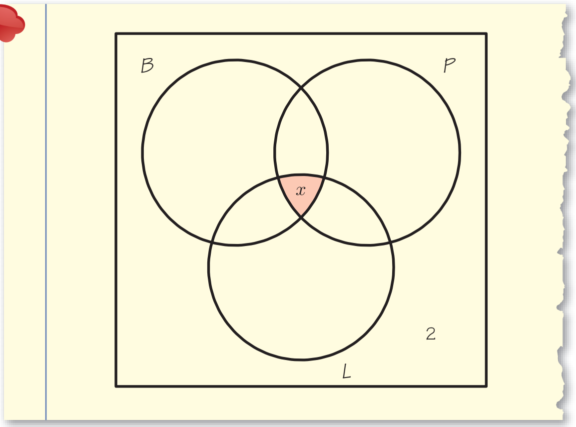
A Venn diagram is a very helpful way of representing information about events or groups which overlap one another.

When labelling the various events or groups in the diagram, the convention is that the number we put into each region is the probability or size unique to that region. It is therefore a good idea to label the intersection of all the regions first, assigning it a variable if necessary, and then work outwards. Do not try to label the total for all the regions joined together.

Worked example 17.14

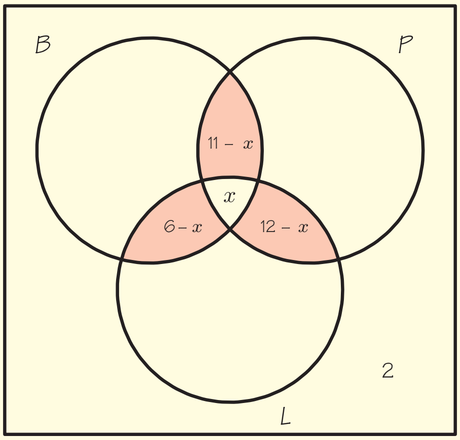
In a class of 32, 19 have a bicycle, 21 have a mobile phone and 16 have a laptop computer; 11 have both a bike and a phone, 12 have both a phone and a laptop, and 6 have both a bike and a laptop. Two have none of these objects. How many people have a bike, a phone and a laptop?

Draw a Venn diagram, showing three overlapping groups.
Label the size of the central region (where all three groups overlap) as x. We know that 2 people are outside all three groups.

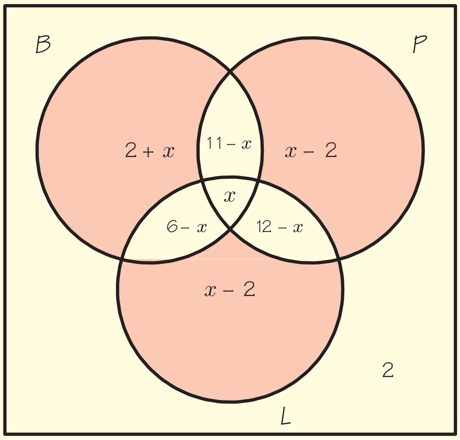


continued . . .

Work outwards from the centre, and look at the three regions where only two of the groups overlap. For example, the number who have a bicycle and a phone but not a laptop is $11 - x$.



Continue working outwards; now consider the regions which belong to one group only. For example, a total of 19 people have a bicycle, so the number who have a bicycle but neither a phone nor a laptop should be $19-(11-x)-(6-x)-x=2+x$.



Use the fact that there are 32 people in the class to form an equation.

$$(2+x)+(11-x)+x+(6-x)$$
$$+(x-2)+(12-x)+(x-2)+2=32$$
$$\Leftrightarrow \quad 29+x=32$$
$$\Leftrightarrow \quad x=3$$

Therefore three people have a bicycle, a phone and a laptop.

Venn diagrams are particularly useful when thinking about conditional probability. We can use the given information to exclude parts of the Venn diagram that are not relevant.

Worked example 17.15

Daniel has 18 toys: 12 are made of plastic and 13 are red; 2 are neither red nor plastic.

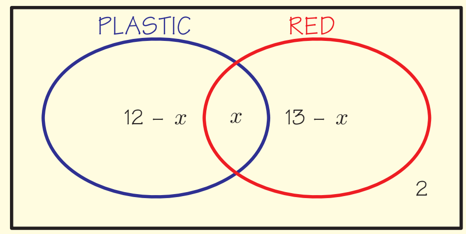
Daniel chooses a toy at random.

(a) Find the probability that it is a red plastic toy.

(b) If it is a red toy, find the probability that it is plastic.

We need to find the size of the intersection. We can do this by putting the given information into a Venn diagram.

(a)

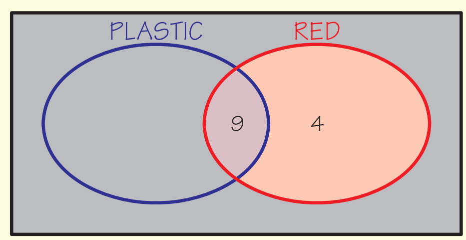


We know that there are 18 toys in total.

$$(12 - x) + x + (13 - x) + 2 = 18$$
$$\Leftrightarrow 27 - x = 18$$
$$\Leftrightarrow x = 9$$
$$\therefore \text{P}(\text{plastic and red}) = \frac{9}{18} = \frac{1}{2}$$

This is asking for a conditional probability. We can focus on the red toys.

(b)



9 out of 13 red toys are plastic

$$\therefore \text{P}(\text{plastic} \mid \text{red}) = \frac{9}{13}$$

When calculating conditional probabilities, it is often easier to use Venn diagrams rather than the formula in Key point 17.7. It is acceptable to do this in the examination.

In the previous two examples we used Venn diagrams to represent frequencies, but they can also represent probabilities, and the same methods work.

Some people find Venn diagrams to be a useful way of understanding the formula $\text{P}(A \mid B) = \frac{\text{P}(A \cap B)}{\text{P}(B)}$. Do visual arguments tend to be clearer than mathematical arguments? If so, why?

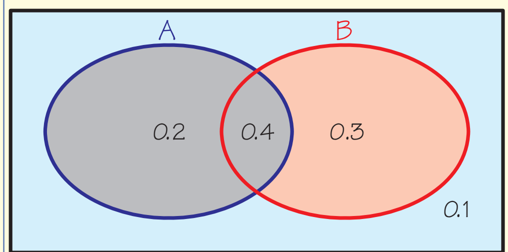
Worked example 17.16

Events A and B are such that $P(A) = 0.6$, $P(B) = 0.7$ and $P(A \cup B) = 0.9$. Find $P(B' \mid A')$.

We can find $P(A \cap B)$ from the given information, using Key point 17.4.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$0.9 = 0.6 + 0.7 - P(A \cap B)$$
$$\therefore P(A \cap B) = 0.4$$

Draw a Venn diagram.
Label the central intersection first, and then work outwards.
For example, the region corresponding to A but not B has probability 0.6 – 0.4 = 0.2.



The probability of not being in A is 0.1 + 0.3 = 0.4. Out of this, the probability of not being in B is 0.1. So

$$P(B' \mid A') = \frac{0.1}{0.4} = \frac{1}{4}$$

Exercise 17G

1. Out of 145 students in a college, 34 play football, 18 play badminton, and 5 play both sports.
   (a) Draw a Venn diagram showing this information.
   (b) How many students play neither sport?
   (c) What is the probability that a randomly chosen student plays badminton?
   (d) If we know that the chosen student plays football, what is the probability that they also play badminton? *[6 marks]*

2. Out of 145 students in a college, 58 study Mathematics, 47 study Economics and 72 study neither of the two subjects.
   (a) Draw a Venn diagram to show this information.
   (b) How many students study both subjects?
   (c) A student tells you that she studies Mathematics. What is the probability that she studies both Mathematics and Economics? *[5 marks]*

3. Denise conducts a survey about food preferences in her college. She asks students which of the three meals – spaghetti bolognese, chilli con carne, and vegetable curry – they would eat. She finds that, out of the 145 students:

43 would eat spaghetti bolognese

80 would eat vegetable curry

20 would eat both the spaghetti and the curry

24 would eat both the curry and the chilli

35 would eat both the chilli and the spaghetti

12 would eat all three meals

10 would not eat any of the three meals

(a) Draw a Venn diagram showing this information.

(b) How many students would eat only spaghetti bolognese?

(c) How many students would eat chilli?

(d) What is the probability that a randomly selected student would eat only one of the three meals?

(e) Given that a student would eat only one of the three meals, what is the probability that they would eat curry?

(f) Find the probability that a randomly selected student would eat at least two of the three meals. *[12 marks]*

4. The probability that a person has dark hair is 0.7, the probability that they have blue eyes is 0.4, and the probability that they have both dark hair and blue eyes is 0.2.

(a) Draw a Venn diagram showing this information.

(b) Find the probability that a person has neither dark hair nor blue eyes.

(c) Given that a person has dark hair, find the probability that they also have blue eyes.

(d) Given that a person does not have dark hair, find the probability that they have blue eyes.

(e) Are the characteristics of having dark hair and having blue eyes independent? Explain your answer. *[11 marks]*

5. The probability that it rains on any given day is 0.45, and the probability that it is cold is 0.6. The probability that it is neither cold nor raining is 0.25.

(a) Find the probability that it is both cold and raining.

(b) Draw a Venn diagram showing this information.

(c) Given that it is raining, find the probability that it is not cold.

(d) Given that it is not cold, find the probability that it is raining.

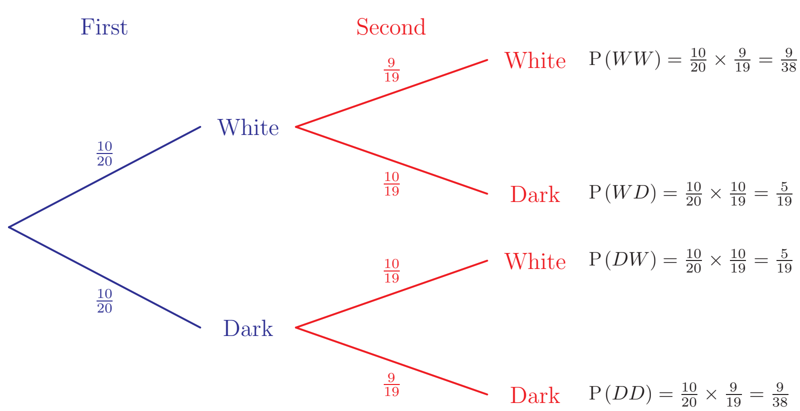
(e) Are the events 'raining' and 'cold' independent? Explain your answer and show any supporting calculations. *[12 marks]*

17H Selections with and without replacement

In this section we look at a particular example which illustrates the difference between dependent and independent events.

Suppose you have a box containing 10 white and 10 dark chocolates, and you pick two chocolates without looking. We can think of this as two events happening one after the other (even if the two chocolates were taken out in one go) and draw a tree diagram to show the outcomes of each event and their probabilities. What is the probability that both chocolates are white? For the first chocolate, the probabilities of getting the white or dark types are the same: $\mathrm{P}(\text{first white}) = \mathrm{P}(\text{first dark}) = \frac{10}{20} = \frac{1}{2}$. However, the probabilities for the second chocolate depend on what the first one was. If the first chocolate was white, then there are 9 white and 10 dark chocolates remaining in the box. So the probability that the second chocolate is also white would be $\frac{9}{19}$. Note that this is *not* P(second white) but rather the conditional probability P(second white | first white). The probability that *both* chocolates are white is therefore

$$\mathrm{P}(\text{first white} \cap \text{second white}) = \mathrm{P}(\text{second white} \mid \text{first white})\ \mathrm{P}(\text{first white})$$
$$= \frac{9}{19} \times \frac{1}{2} = \frac{9}{38}$$

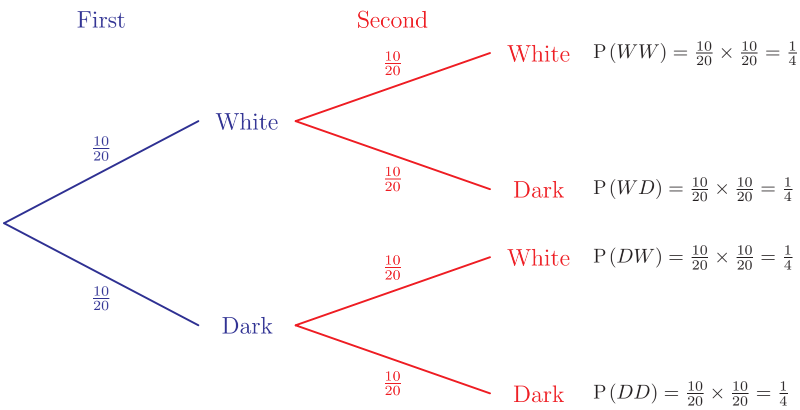


To find the probability of the second chocolate being white, we look at the tree diagram:

$$\begin{aligned}
&\text{P(second white)}\\
&\quad = \text{P(second white} \cap \text{first white)} + \text{P(second white} \cap \text{first dark)}\\
&\quad = \text{P(second white} \mid \text{first white) P(first white)}\\
&\qquad + \text{P(second white} \mid \text{first dark) P(first dark)}\\
&= \frac{9}{19} \times \frac{1}{2} + \frac{10}{19} \times \frac{1}{2}\\
&= \frac{19}{38} = \frac{1}{2}
\end{aligned}$$

Notice that $\text{P(both white)} \neq \text{P(first white)P(second white)}$ because the types of the first and second chocolates are not independent.

Contrast this with the situation where you pick one chocolate, put it back in the box, and then select another one (called selection *with replacement*). This time, $\text{P(second white | first white)} = \frac{10}{20} = \frac{1}{2}$.



Using the new tree diagram, the probability that both chocolates are white is:

$$\begin{aligned}
\text{P(first white} \cap \text{second white)} &= \text{P(second white} \mid \text{first white) P(first white)}\\
&= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}
\end{aligned}$$

The probability that the second chocolate is white is

$$\begin{aligned}
&\mathrm{P}(\text{second white}) \\
&\quad = \mathrm{P}(\text{second white} \mid \text{first white})\,\mathrm{P}(\text{first white}) \\
&\qquad + \mathrm{P}(\text{second white} \mid \text{first dark})\,\mathrm{P}(\text{first dark}) \\
&\quad = \frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2} \\
&\quad = \frac{1}{2}
\end{aligned}$$

Note that this probability is the same as when we did the selections without replacement. This time, however, $\mathrm{P}(\text{both white}) = \mathrm{P}(\text{first white})\mathrm{P}(\text{second white})$, so the types of the two chocolates are independent. This is to be expected, as replacing the first chocolate means that the choice of the second chocolate is not affected by the type of the first.

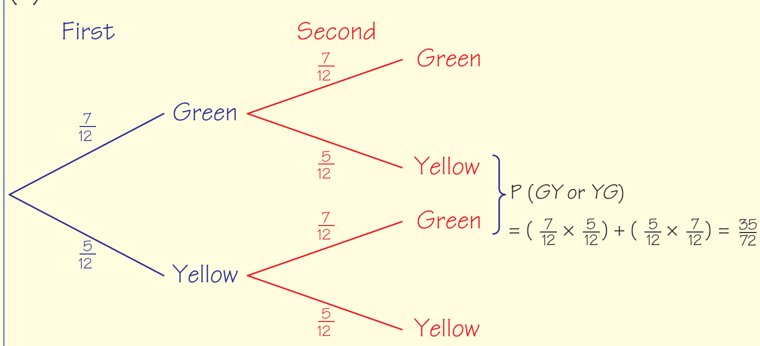
Worked example 17.17

A bag contains 7 green and 5 yellow balls. Two balls are selected at random, one at a time. Find the probability that the two balls are of different colours if the selection is made

(a) with replacement (b) without replacement.

Draw a tree diagram. The probabilities for the second ball are the same as for the first.

(a)



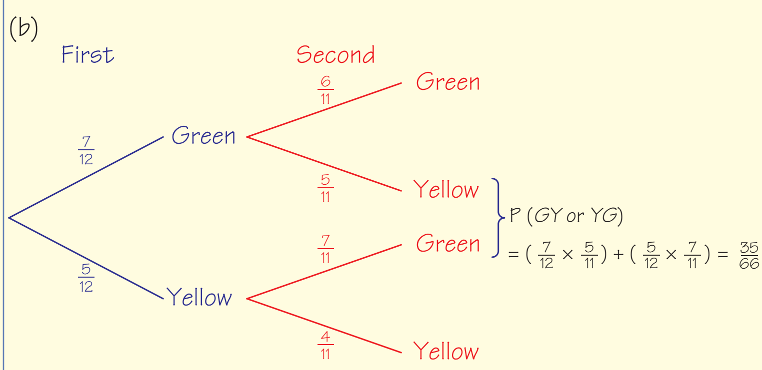
There are two branches corresponding to the two balls being of different colours.

$$\begin{aligned}
P(\textit{different colours}) &= \frac{7}{12}\times\frac{5}{12}+\frac{5}{12}\times\frac{7}{12} \\
&= 0.486\ (3SF)
\end{aligned}$$

continued . . .

Draw a new tree diagram. This time, since the first ball is not put back, the probabilities for the second ball are out of 11.

(b)



$$P(\text{different colours}) = \frac{7}{12} \times \frac{5}{11} + \frac{5}{12} \times \frac{7}{11}$$
$$= 0.530 \text{ (3SF)}$$

Exercise 17H

1. Sumaiya's pencil case contains five blue and three red pens. She selects two pens without looking. What is the probability that they are both blue? *[3 marks]*

2. A bag contains 100 sweets, 35 of which are strawberry flavoured. Ivo picks two sweets at random. Find the probability that at least one is strawberry flavoured. *[4 marks]*

3. Isabelle has three blue scarves and two purple scarves. Each day she picks a scarf at random. Find the probability that she wears a blue scarf on two consecutive days. *[4 marks]*

4. A bag contains 12 black and 15 red counters. A counter is picked from the bag and not replaced. A second counter is then picked from the bag. What is the probability that this second counter is red? *[4 marks]*

5. At the start of each turn in a game, you select two tokens out of a bag containing 13 blue and 13 orange tokens. You win an extra point if the two tokens are of different colours. Are you more likely to win the extra point if you select the tokens with or without replacement? *[7 marks]*

6. A bag contains the same number of red and yellow balls.

(a) One ball is selected and replaced. Another ball is then selected. Show that the probability that the two balls are of different colours is $\frac{1}{2}$.

(b) Two balls are selected without replacement. Find the smallest total number of balls so that the probability of selecting two balls of different colours is less than 0.53. *[9 marks]*

Summary

- Probability is a measure of how likely an event is, varying from 0 for impossible events up to 1 for events that are certain to happen.
- The probability of an event occurring can be estimated from the frequency of occurrence in prior data (empirical probability) or predicted by considering what proportion of the **sample space** – the list of all possible outcomes – corresponds to the event (theoretical probability).
- If the event we are interested in is a function of two outcomes (e.g. if it is the sum or product of two rolled dice), a convenient way of listing the sample space is in a probability grid diagram.
- **Venn diagrams** are a useful tool for understanding how different events can be combined and working out the associated probabilities. When calculating probabilities, label the intersection of all the groups with an unknown and work outwards.
- The probability of event A or event B or both occurring is the union of A and B:
  $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

  If the events are mutually exclusive, then $\mathrm{P}(A \cap B) = 0$ and the formula becomes $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$.

  A particular case of mutually exclusive events is any event A and its **complement** A' (the event of A not happening), and we have $\mathrm{P}(A) + \mathrm{P}(A') = 1$.
- The probability of events A and B both occurring is the intersection of A and B: $\mathrm{P}(A \cap B)$.
- The outcomes of a sequence of events can be displayed in a **tree diagram**. The probability of following the path $A \to B$ in a tree diagram is $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B \mid A)$ where $\mathrm{P}(B \mid A)$ is the probability of B occurring if we already know that A has occurred, a **conditional probability**.

- Conditional probabilities can be found from tree diagrams or Venn diagrams, using the formula

$$\mathrm{P}(B \mid A) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(A)}$$

- If A and B are **independent events**, $\mathrm{P}(B \mid A) = \mathrm{P}(B)$ and $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$.
- Selections with and without replacement are special cases of independent and non-independent events.

Introductory problem revisited

A woman gives birth to non-identical twins. One of them is a girl. What is the probability that the other one is a girl?

There are four equally likely possibilities for the twins' sexes:

First child	Second child
Boy	Boy
Boy	Girl
Girl	Boy
Girl	Girl

If we are told that one of the twins is a girl we can exclude the first case. This leaves three equally likely situations, of which only one consists of two girls; therefore the probability is $\frac{1}{3}$.

A word of warning before we finish: it is very tempting to argue that the probability should be $\frac{1}{2}$, as the probability of the second child being a girl is independent of the sex of the first child. This would indeed be the correct answer if the question had said that the *first child* is a girl, rather than that *one of the children* is a girl. Our intuition about probabilities is often flawed, which is why it is important to develop precise mathematical methods and use the language accurately.

If you find this result intriguing, you may like to explore the famous 'Monty Hall problem'.

If you thought the answer to this question was $\frac{1}{2}$, would you be making a mathematical mistake or an error of interpretation? Are they the same thing?

Mixed examination practice 17

Short questions

1. A drawer contains 6 red socks, 4 black socks and 8 white socks.
Two socks are picked at random. What is the probability that they are of the same colour? *[5 marks]*

2. In a bilingual school there is a class of 21 students. In this class, 15 of the students speak Spanish as their first language, and 12 of these 15 are Argentine. The other 6 students in the class speak English as their first language, and 3 of these 6 are Argentine.

A student is selected at random from the class and is found to be Argentine. Find the probability that the student speaks Spanish as his/her first language. *[4 marks]*

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3. The probability that it rains on a summer's day in a certain town is 0.2. In this town, the probability that the daily maximum temperature exceeds 25°C is 0.3 when it rains and 0.6 when it does not rain. Given that the maximum daily temperature exceeded 25°C on a particular summer's day, find the probability that it rained on that day. *[6 marks]*

4. Given that $(A \cup B)' = \varnothing$, $\mathrm{P}(A' | B) = \frac{1}{5}$ and $\mathrm{P}(A) = \frac{14}{15}$, find $\mathrm{P}(B)$. *[5 marks]*

Long questions

1. (a) A large bag of sweets contains 8 red and 12 yellow sweets. Two sweets are chosen at random from the bag without replacement. Find the probability that two red sweets are chosen.

(b) A small bag contains 4 red and n yellow sweets. Two sweets are chosen without replacement from this bag. If the probability that two red sweets are chosen is $\frac{2}{15}$, show that $n = 6$.

Ayesha has one large bag and two small bags of sweets. She selects a bag at random and then picks two sweets from the bag without replacement.

(c) Calculate the probability that two red sweets are chosen.

(d) Given that two red sweets are chosen, find the probability that Ayesha had selected the large bag. *[15 marks]*

2. (a) If $\mathrm{P}(X)$ represents a probability, state the range of $\mathrm{P}(X)$.

(b) Express $\mathrm{P}(A)-\mathrm{P}(A\cap B)$ in terms of $\mathrm{P}(A)$ and $\mathrm{P}(B|A)$.

(c) (i) Show that $\mathrm{P}(A\cup B)-\mathrm{P}(A\cap B)=\mathrm{P}(A)\left(1-\mathrm{P}(\mathrm{B}|\mathrm{A})\right)+\mathrm{P}(B)\left(1-\mathrm{P}(\mathrm{A}|\mathrm{B}\right)$.

(ii) Hence explain why $\mathrm{P}(A\cup B)\geq \mathrm{P}(A\cap B)$. *[9 marks]*

3. The probability that a student plays badminton is 0.3. The probability that a student plays neither football nor badminton is 0.5, and the probability that a student plays both sports is x.

(a) Draw a Venn diagram showing this information.

(b) Find the probability that a student plays football but not badminton.

Given that a student plays football, the probability that they also play badminton is 0.5.

(c) Find the probability that a student plays both badminton and football.

(d) Hence complete your Venn diagram. What is the probability that a student plays only badminton?

(e) Given that a student plays only one sport, what is the probability that they play badminton? *[13 marks]*

4. Two women, Ann and Bridget, play a game in which they take it in turns to throw an unbiased six-sided die. The first woman to throw a '6' wins the game. Ann is the first to throw.

(a) Find the probability that

(i) Bridget wins on her first throw

(ii) Ann wins on her second throw

(iii) Ann wins on her nth throw.

(b) Let p be the probability that Ann wins the game. Show that $p=\frac{1}{6}+\frac{25}{36}p$.

(c) Find the probability that Bridget wins the game.

(d) Suppose that the game is played six times. Find the probability that Ann wins more games than Bridget. *[17 marks]*

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