5 Transformations of graphs

In this chapter you will learn:

- how certain changes to functions affect their graphs
- how to sketch complicated functions by considering them as transformations of simpler functions.

Introductory problem

Sketch the graph of $y = \log(x^2 - 6x + 9)$ without using a calculator.

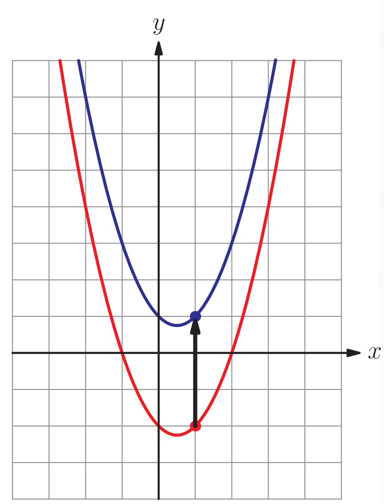
You have met various transformations which can be applied to two-dimensional shapes: translations, enlargements, reflections and rotations. In this chapter you will learn how translations, stretches and reflections of a graph relate to changing parts of its equation.

Self-Discovery Worksheet 2 'Changing functions and their graphs' on the CD-ROM guides you in discovering these rules for yourself.

5A Translations

Compare these graphs of two functions which differ only by a constant:

$$y = x^2 - x + 1$$
$$y = x^2 - x - 2$$



When the x-coordinates on the two graphs are the same (where $x = x$) the y-coordinates differ by 3 ($y = y + 3$). We can interpret this as meaning that at the same x-coordinate, the blue graph is three units above the red graph – it has been translated vertically.

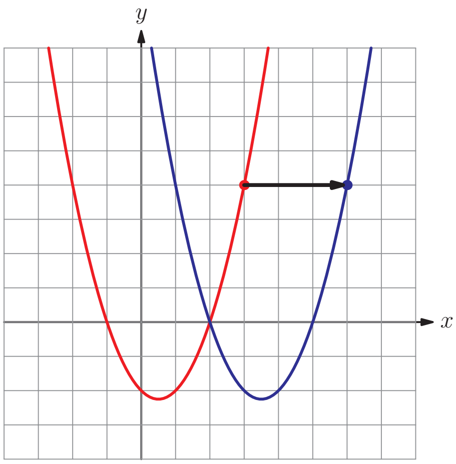
KEY POINT 5.1

The graph of $y = f(x) + c$ is the graph of $y = f(x)$ moved up by c units. If c is negative, the graph is moved down.

Vector notation is explained in more detail in chapter 11.

It is common to use vector notation to describe translations.

A translation by c units up is described by $\begin{pmatrix}0\\c\end{pmatrix}$.



In this next case, the blue graph is obtained from the red graph upon replacing x by $x - 3$ in the function.

$$y = x^2 - x - 2$$

$$y = (x-3)^2 - (x-3) - 2$$

Here, when $x = x$ there is nothing obvious we can say about the relationship between y and y. However, note that a way of getting $y = y$ is to have $x = x - 3$ or, equivalently, $x = x + 3$. We can interpret this as meaning that the two graphs are at the same height when the blue graph is three units to the right of the red graph – it has been translated horizontally.

KEY POINT 5.2

The graph $y = f(x+d)$ is the same as the graph of $y = f(x)$ moved *left* by d units. If d is negative, the graph is moved *right*.

In vector notation, a translation to the left by d units is written $\begin{pmatrix}-d\\0\end{pmatrix}$.

Worked example 5.1

The graph of $y = x^2 + 2x$ is translated 5 units to the left. Find the equation of the resulting graph in the form $y = ax^2 + bx + c$.

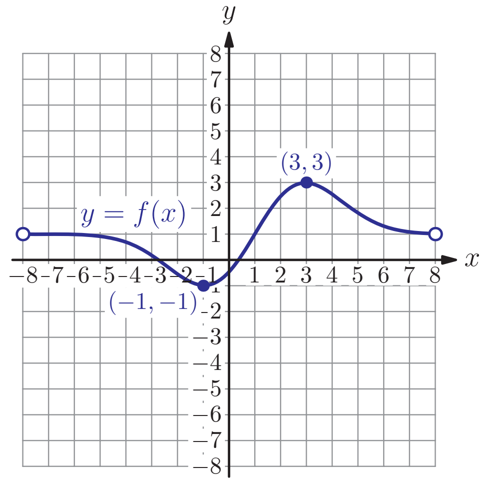
Relate the transformation to function notation.

If $f(x) = x^2 + 2x$, then the new graph is $y = f(x+5)$.

Replace all occurrences of x by $(x + 5)$ in the expression for $f(x)$.

$$y = (x+5)^2 + 2(x+5)$$
$$= x^2 + 12x + 35$$

Exercise 5A



1. Given the graph of $y = f(x)$, sketch the graph of the following functions, indicating the positions of the minimum and maximum points.

(a) (i) $y = f(x) + 3$ (ii) $y = f(x) + 5$

(b) (i) $y = f(x) - 7$ (ii) $y = f(x) - 0.5$

(c) (i) $y = f(x+2)$ (ii) $y = f(x+4)$

(d) (i) $y = f(x-1.5)$ (ii) $y = f(x-2)$

2. Find the equation of the graph after the given transformation is applied.

(a) (i) $y = 3x^2$ after a translation of 3 units vertically up

(ii) $y = 9x^3$ after a translation of 7 units vertically down

(b) (i) $y = 7x^3 - 3x + 6$ after a translation of 2 units down

(ii) $y = 8x^2 - 7x + 1$ after a translation of 5 units up

(c) (i) $y = 4x^2$ after a translation of 5 units to the right

(ii) $y = 7x^2$ after a translation of 3 units to the left

(d) (i) $y = 3x^3 - 5x^2 + 4$ after a translation of 4 units to the left

(ii) $y = x^3 + 6x + 2$ after a translation of 3 units to the right

3. Find the required translations.

(a) (i) Transforming the graph of $y = x^2 + 3x + 7$ to the graph of $y = x^2 + 3x + 2$

(ii) Transforming the graph of $y = x^3 - 5x$ to the graph of $y = x^3 - 5x - 4$

(b) (i) Transforming the graph of $y = x^2 + 2x + 7$ to the graph of $y = (x+1)^2 + 2(x+1) + 7$

(ii) Transforming the graph of $y = x^2 + 5x - 2$ to the graph of $y = (x+5)^2 + 5(x+5) - 2$

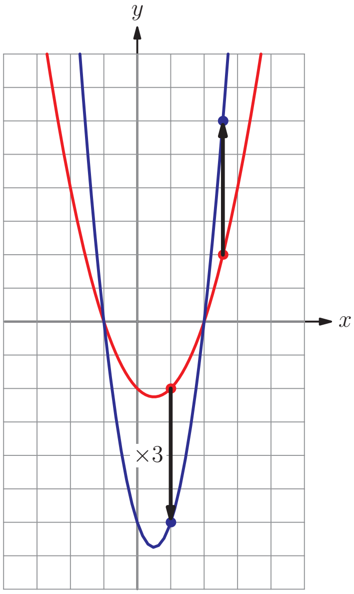
(c) (i) Transforming the graph of $y = e^x + x^2$ to the graph of $y = e^{x-4} + (x-4)^2$

(ii) Transforming the graph of $y = \log(3x) - \sqrt{4x}$ to the graph of $y = \log(3(x-5)) - \sqrt{4(x-5)}$

(d) (i) Transforming the graph of $y = \ln(4x)$ to the graph of $y = \ln(4x+12)$

(ii) Transforming the graph of $y = \sqrt{2x+1}$ to the graph of $y = \sqrt{2x-3}$

5B Stretches



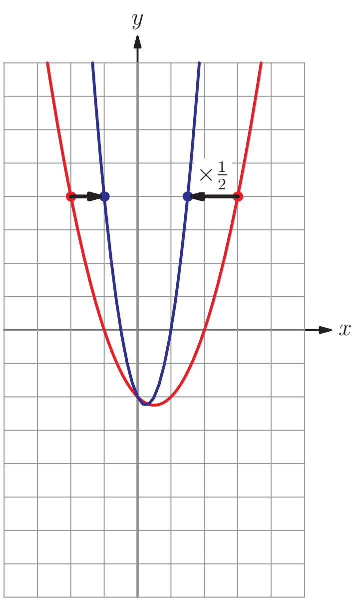
With these two graphs, one function is 3 times the other.

$$y = x^2 - x - 2$$
$$y = 3(x^2 - x - 2)$$

When the x-coordinates on the two graphs are the same ($x = x$), the y-coordinates of the blue graph are three times larger ($y = 3y$). We can interpret this as meaning that at the same x-coordinate, the blue graph is three times further from the x-axis than the red graph – it has been stretched *vertically*.

KEY POINT 5.3

The graph of $y = pf(x)$, with $p > 0$, is the same as the graph of $y = f(x)$ *stretched* vertically relative to the x-axis (away from) with scale factor p. If $0 < p < 1$, then $y = f(x)$ is *compressed* vertically relative to the x-axis (towards). If $p < 0$, you have a negative scale factor $(-p)$ and it might be easier to think of the transformation as a stretch by scale factor p followed by a reflection in the x-axis.



These graphs illustrate what happens when you replace x in the function by $2x$.

$$y = x^2 - x - 2$$
$$y = (2x)^2 - (2x) - 2$$

When $x = x$, there is nothing obvious we can say about the relationship between y and y. But note that a way of getting $y = y$ is to have $x = 2x$ or, equivalently, $x = \frac{x}{2}$. We can interpret this as meaning that the two graphs are at the same height when the distance from the y-axis of the blue graph is half the distance of the red graph – it has been stretched *horizontally*.

KEY POINT 5.4

The graph of $y = f(qx)$ is the same as the graph of $y = f(x)$ stretched horizontally relative to the y-axis by scale factor $\frac{1}{q}$. This can be considered a compression relative to the y-axis (towards) when $q > 0$. When $0 < q < 1$, it is considered a stretch relative to the y-axis (away from) and when $q < 0$ you have a negative scale factor $(-\frac{1}{q})$ and it is easier to think of the transformation as a stretch/compression by scale factor $\frac{1}{q}$ followed by a reflection in the y-axis.

Although we have used the terms 'stretched' and 'compressed', both transformations are generally referred to as 'stretches'.

Worked example 5.2

Describe a transformation which transforms the graph of $y = \ln x - 1$ to the graph of $y = \ln x^4 - 4$.

Try to relate the two equations using function notation.

Let $f(x) = \ln x - 1$.
Then

None of the transformations we know involves raising x to a power, so first think of a different way to write $\ln x^4$.

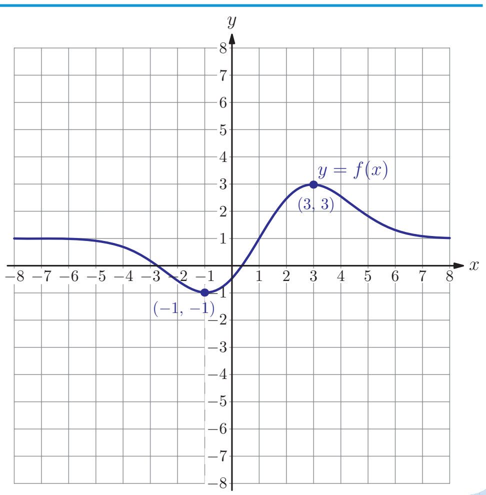
$$\begin{aligned}\ln x^4 - 4 &= 4\ln x - 4\\ &= 4(\ln x - 1)\\ &= 4f(x)\end{aligned}$$

Relate the function notation to the transformation.

It is a vertical stretch with scale factor 4.

Exercise 5B

1. Given the graph of $y = f(x)$, sketch the graph of the following functions, indicating the positions of the minimum and maximum points.



(a) (i) $y = 3f(x)$ (ii) $y = 5f(x)$

(b) (i) $y = \dfrac{f(x)}{4}$ (ii) $y = \dfrac{f(x)}{2}$

(c) (i) $y = f(2x)$ (ii) $y = f(6x)$

(d) (i) $y = f\left(\dfrac{2x}{3}\right)$ (ii) $y = f\left(\dfrac{5x}{6}\right)$

2. Find the equation of the graph after the given transformation is applied.
   (a) (i) $y = 3x^2$ after a vertical stretch 7 relative to the x-axis
   (ii) $y = 9x^3$ after a vertical stretch 2 relative to the x-axis
   (b) (i) $y = 7x^3 - 3x + 6$ after a vertical stretch factor $\frac{1}{3}$ relative to the x-axis
   (ii) $y = 8x^2 - 7x + 1$ after a vertical stretch factor $\frac{4}{5}$ relative to the x-axis
   (c) (i) $y = 4x^2$ after a horizontal stretch factor 2 relative to the y-axis
   (ii) $y = 7x^2$ after a horizontal stretch factor 5 relative to the y-axis
   (d) (i) $y = 3x^3 - 5x^2 + 4$ after a horizontal stretch factor $\frac{1}{2}$ relative to the y-axis
   (ii) $y = x^3 + 6x + 2$ after a horizontal stretch factor $\frac{2}{3}$ relative to the y-axis

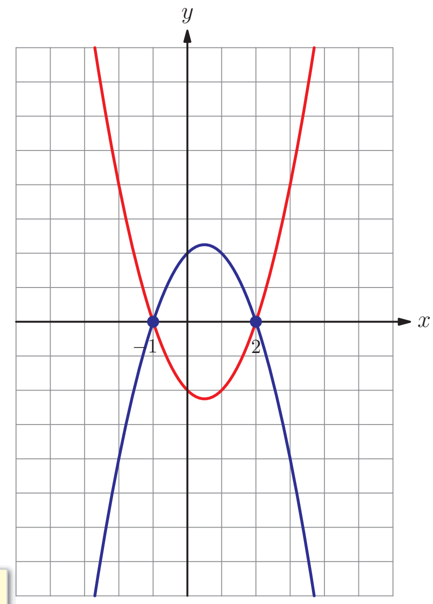
3. Describe the following stretches.
   (a) (i) Transforming the graph of $y = x^2 + 3x + 7$ to the graph of $y = 4x^2 + 12x + 28$
   (ii) Transforming the graph of $y = x^3 - 5x$ to the graph of $y = 6x^3 - 30x$
   (b) (i) Transforming the graph of $y = x^2 + 2x + 7$ to the graph of $y = (3x)^2 + 2(3x) + 7$
   (ii) Transforming the graph of $y = x^2 + 5x - 2$ to the graph of $y = (4x)^2 + 5(4x) - 2$
   (c) (i) Transforming the graph of $y = \mathrm{e}^x + x^2$ to the graph of $y = \mathrm{e}^{\frac{x}{2}} + \left(\frac{x}{2}\right)^2$
   (ii) Transforming the graph of $y = \log(3x) - \sqrt{4x}$ to the graph of $y = \log\left(\frac{3x}{5}\right) - \sqrt{\frac{4x}{5}}$
   (d) (i) Transforming the graph of $y = \ln(4x)$ to the graph of $y = \ln(12x)$
   (ii) Transforming the graph of $y = \sqrt{2x+1}$ to the graph of $y = \sqrt{x+1}$

5C Reflections

Compare these two graphs and their equations:

$$y = x^2 - x - 2$$

$$y = -(x^2 - x - 2)$$



When the x-coordinates on the two graphs are the same ($x = x$), the y-coordinates are negatives of each other ($y = -y$). We can interpret this as meaning that at the same x-coordinate, the blue graph is the same vertical distance from the x-axis as the red graph but on the opposite side of the axis – it has been reflected vertically.

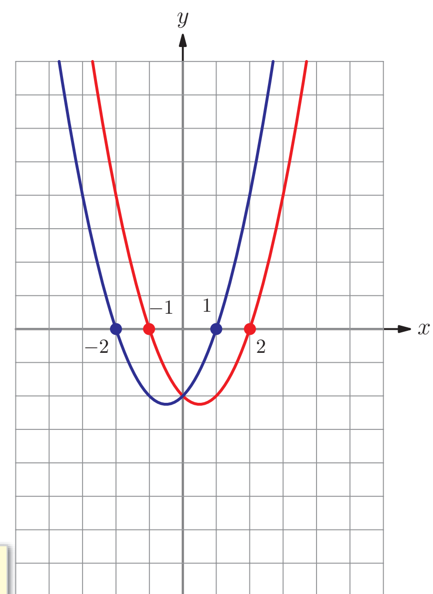
KEY POINT 5.5

> The graph of $y = -f(x)$ is the same as the graph of $y = f(x)$ reflected in the x-axis.

Next, see what happens when we replace x by $-x$ in the equations:

$$y = x^2 - x - 2$$

$$y = (-x)^2 - (-x) - 2$$



When $x = x$, there is nothing obvious we can say about the relationship between y and y. But note that for $x = -x$ we have $y = y$. This means that the heights of the two graphs are the same when the blue graph is at the same position relative to the y-axis as the red graph, but on the opposite side of the y-axis – it has been reflected horizontally.

KEY POINT 5.6

> The graph of $y = f(-x)$ is the same as the graph of $y = f(x)$ reflected in the y-axis.

Worked example 5.3

The graph of $y = f(x)$ has a single maximum point with coordinates $(4, -3)$. Find the coordinates of the maximum point on the graph of $y = f(-x)$.

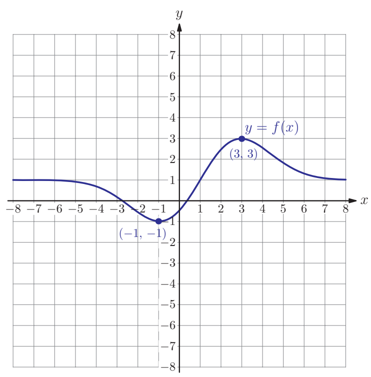
Relate the function notation to an appropriate transformation.

The transformation taking $y = f(x)$ to $y = f(-x)$ is reflection in the y-axis.

Reflection in the y-axis leaves y-coordinates unchanged but switches the sign of x-coordinates.

The maximum point is $(-4, -3)$.

Exercise 5C



1. Given the graph of $y = f(x)$, sketch the graph of the following functions, indicating the positions of the minimum and maximum points.
   (a) $y = -f(x)$
   (b) $y = f(-x)$

2. Find the equation of the graph after the given transformation is applied.
   (a) (i) $y = 3x^2$ after reflection in the x-axis
   (ii) $y = 9x^3$ after reflection in the x-axis
   (b) (i) $y = 7x^3 - 3x + 6$ after reflection in the x-axis
   (ii) $y = 8x^2 - 7x + 1$ after reflection in the x-axis
   (c) (i) $y = 4x^2$ after reflection in the y-axis
   (ii) $y = 7x^3$ after reflection in the y-axis
   (d) (i) $y = 3x^3 - 5x^2 + 4$ after reflection in the y-axis
   (ii) $y = x^3 + 6x + 2$ after reflection in the y-axis

3. Describe the following transformations
   (a) (i) Transforming the graph of $y = x^2 + 3x + 7$ to the graph of $y = -x^2 - 3x - 7$
   (ii) Transforming the graph of $y = x^3 - 5x$ to the graph of $y = 5x - x^3$
   (b) (i) Transforming the graph of $y = x^2 + 2x + 7$ to the graph of $y = x^2 - 2x + 7$
   (ii) Transforming the graph of $y = x^2 - 5x - 2$ to the graph of $y = x^2 + 5x - 2$

(c) (i) Transforming the graph of $y = e^x + x^2$ to the graph of $y = e^{-x} + x^2$

(ii) Transforming the graph of $y = \log(3x) - \sqrt{4x}$ to the graph of $y = \sqrt{4x} - \log(3x)$

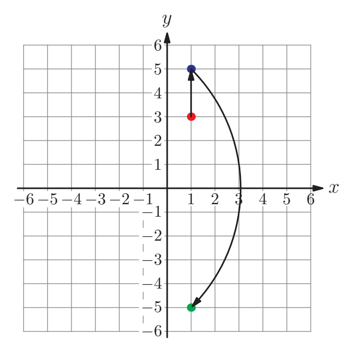
(d) (i) Transforming the graph of $y = \ln(4x)$ to the graph of $y = \ln(-4x)$

(ii) Transforming the graph of $y = \sqrt{2x-1}$ to the graph of $y = \sqrt{-1-2x}$

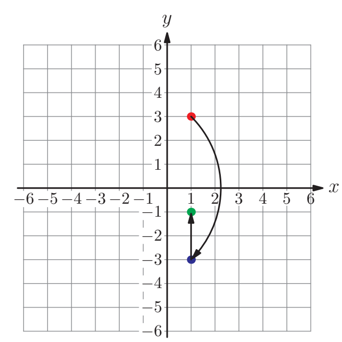
5D Consecutive transformations

In this section we look at what happens when we apply two transformations in succession.

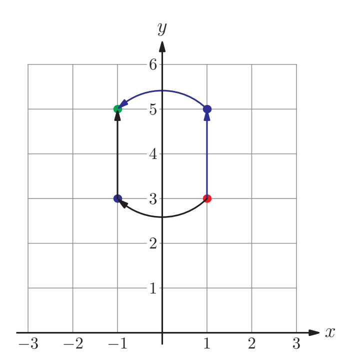
If the point (1, 3) is translated two units up and then reflected in the x-axis, there have been two vertical transformations, and the new point is (1, −5).



If (1, 3) is first reflected in the x-axis and then translated two units up, there have been two vertical transformations, and the new point is (1, −1).



However, if the two transformations were a translation by two units up and a reflection in the y-axis, there has been one vertical transformation and one horizontal transformation, and the new point will be (−1, 5) regardless of the order in which the transformations were applied.



In advanced mathematics, algebra involves much more than using letters to represent numbers. Unknowns can be transformations and many other things besides numbers. As you can see in this section, the rules for transformations are different from the rules for numbers, but there are certain similarities too. The study of this more general form of algebra is called group theory, and it has many applications, ranging from particle physics to painting polyhedra.

EXAM HINT

You may find it helpful to remember that when resolving vertical transformations we follow the normal order of operations, while horizontal transformations are resolved in the opposite order.

KEY POINT 5.7

> When two vertical transformations or two horizontal transformations are combined, the order in which they are applied affects the outcome.
>
> When one vertical and one horizontal transformation are combined, the outcome does not depend on the order.

There is a very important rule to remember when resolving horizontal or vertical transformations:

- vertical transformations follow the 'normal' order of operations as applied to arithmetic
- horizontal transformations are resolved in the opposite order to the 'normal' order of operations.

This is demonstrated below:

First, let us consider how we could combine two vertical transformations to transform the graph of $y = f(x)$ into the graph of

$$y = pf(x) + c$$

We can achieve this by first multiplying $f(x)$ by p and then adding on c, so this process is composed of a stretch (and/or reflection if $p < 0$) followed by a translation.

$$\boxed{f(x)} \xrightarrow[\text{multiply function by } p]{\text{Stretch Factor } p} \boxed{pf(x)} \xrightarrow[\text{add } c \text{ to function}]{\text{Translate } \begin{pmatrix}0\\c\end{pmatrix}} \boxed{pf(x)+c}$$

This follows the order of operations as you would expect.

Next, think about how we could combine two horizontal transformations to transform the graph of $y = f(x)$ into the graph of

$$y = f(qx + d)$$

We can achieve this by first replacing x with $x + d$ and then replacing all occurrences of x by qx, so this process consists of a translation followed by a stretch (and/or reflection if $q < 0$).

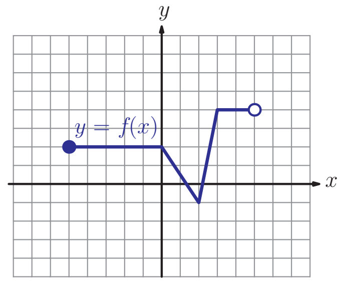
$$\boxed{f(x)} \xrightarrow[\text{Replace } x \text{ with } x+d]{\text{Translate } \begin{pmatrix}-d\\0\end{pmatrix}} \boxed{f(x+d)} \xrightarrow[\text{Replace } x \text{ with } qx]{\text{Stretch Factor } q} \boxed{f(qx+d)}$$

Following the normal order of operations, you would expect to resolve 'qx' before '$+d$' but you resolve the transformation in the opposite order.

The first question in Exercise 5D asks you what happens if you apply these transformations in the reverse order.

Worked example 5.4

Given the graph of $y = f(x)$, sketch the corresponding graph of $y = 3 - 2f(2x+1)$.



Break down the changes to the function into components.

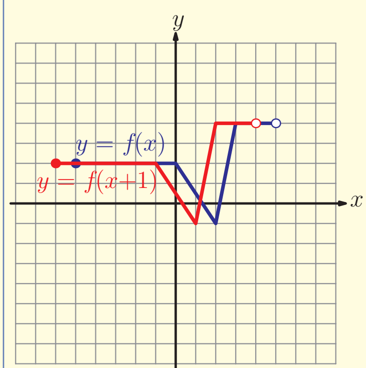
$(2x+1)$ is two horizontal transformations (changes x); $3 - 2f(2x+1)$ is two vertical transformations (changes y)

Changing x: add 1, then multiply x by 2

Changing y: multiply by -2, then add 3

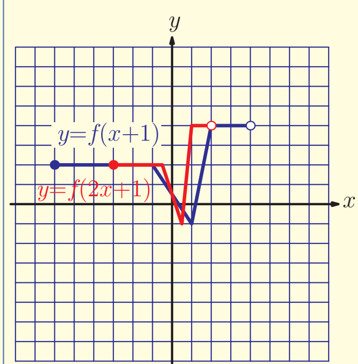
Relate each component to a transformation of the graph.
Replace x with $x+1$;
change $y = f(x)$ to $y = f(x+1)$.

Horizontal translation by $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$



Replace x with $2x$;
change $y = f(x+1)$ to $y = f(2x+1)$.

Horizontal stretch with scale factor $\frac{1}{2}$



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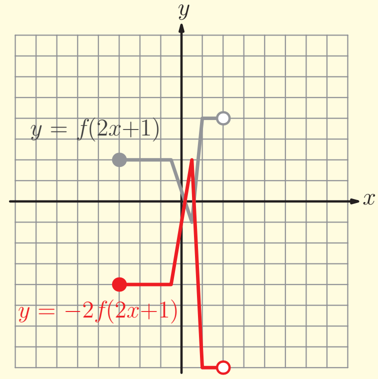
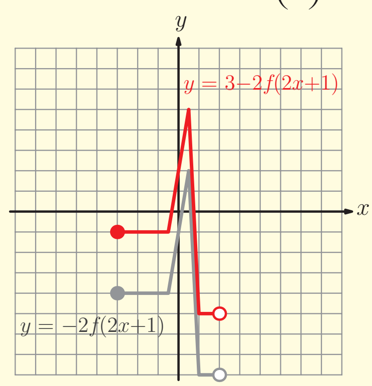
Multiply RHS by −2;
change $y = f(2x+1)$ to $y = -2f(2x+1)$.

Reflection in the x-axis and vertical stretch with scale factor 2



Add 3 to RHS;
change $y = -2f(2x+1)$ to $y = 3 - 2f(2x+1)$.

Vertical translation by $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$

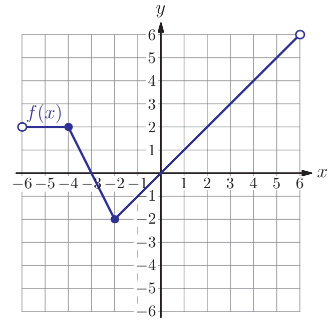


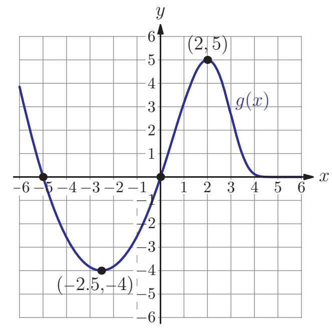
Exercise 5D

1. (a) The graph of $y = f(x)$ is transformed by applying first a vertical translation by c units up and then a vertical stretch with factor p relative to the x-axis. What is the equation of the resulting graph?

   (b) The graph of $y = f(x)$ is transformed by applying first a horizontal stretch with factor q relative to the y-axis, then a horizontal translation by d units to the left. What is the equation of the resulting graph?

2. The graphs of $y = f(x)$ and $y = g(x)$ are given.
Sketch the graphs of the following.





(a) (i) $2f(x)-1$ (ii) $\frac{1}{2}g(x)+3$

(b) (i) $4-f(x)$ (ii) $2-2g(x)$

(c) (i) $3(f(x)-2)$ (ii) $\frac{1-g(x)}{2}$

(d) (i) $f\left(\frac{x}{2}-1\right)$ (ii) $g(2x+3)$

(e) (i) $f\left(\frac{4-x}{5}\right)$ (ii) $g\left(\frac{x-3}{2}\right)$

3. If $f(x)=x^2$, express each of the following functions as $af(x)+b$ and hence describe the transformation(s) mapping $f(x)$ to the given function.

(a) (i) $k(x)=2x^2-6$ (ii) $k(x)=5x^2+4$

(b) (i) $h(x)=5-3x^2$ (ii) $h(x)=4-8x^2$

4. If $f(x)=2x^2-4$, write down the function $g(x)$ which gives the graph of $f(x)$ after:

(a) (i) translation $\begin{pmatrix}0\\2\end{pmatrix}$, followed by a vertical stretch of scale factor 3

(ii) translation $\begin{pmatrix}0\\6\end{pmatrix}$, followed by a vertical stretch of scale factor $\frac{1}{2}$

(b) (i) vertical stretch of scale factor $\frac{1}{2}$, followed by a translation $\begin{pmatrix}0\\6\end{pmatrix}$

(ii) vertical stretch of scale factor $\frac{7}{2}$, followed by a translation $\begin{pmatrix}0\\10\end{pmatrix}$

(c) (i) reflection through the horizontal axis

(ii) reflection through the horizontal axis followed by a translation $\begin{pmatrix}0\\2\end{pmatrix}$

(d) (i) reflection through the horizontal axis and vertical stretch of scale factor $\frac{1}{2}$, followed by a translation $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$

(ii) reflection through the horizontal axis followed by a translation $\begin{pmatrix} 0 \\ -6 \end{pmatrix}$ followed by a vertical stretch, scale factor $\frac{3}{2}$

5. If $f(x) = x^2$, express each of the following functions as $f(ax+b)$ and hence describe the transformation(s) mapping $f(x)$ to the given function.

(a) (i) $g(x) = x^2 + 2x + 1$ (ii) $g(x) = x^2 - 6x + 9$

(b) (i) $h(x) = 4x^2$ (ii) $h(x) = \dfrac{x^2}{9}$

(c) (i) $k(x) = 4x^2 + 8x + 4$ (ii) $k(x) = 9x^2 - 6x + 1$

6. If $f(x) = 2x^2 - 4$, write down the function $g(x)$ which gives the graph of $f(x)$ after:

(a) (i) translation $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ followed by a horizontal stretch of scale factor $\frac{1}{4}$

(ii) translation $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$, followed by a horizontal stretch of scale factor $\frac{1}{2}$

(b) (i) horizontal stretch of scale factor $\frac{1}{2}$, followed by a translation $\begin{pmatrix} -4 \\ 0 \end{pmatrix}$

(ii) horizontal stretch of scale factor $\frac{2}{3}$, followed by a translation $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(c) (i) translation $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$ followed by a reflection through the vertical axis

(ii) reflection through the vertical axis followed by a translation $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$

7. For each of the following functions $f(x)$ and $g(x)$, express $g(x)$ in the form $a\colon f(x+b)+c$ for some values a, b and c, and hence describe a sequence of horizontal and vertical transformations which map $f(x)$ to $g(x)$.

(a) (i) $f(x)=x^2,\ g(x)=2x^2+4x$

(ii) $f(x)=x^2,\ g(x)=3x^2-24x+8$

(b) (i) $f(x)=x^2+3,\ g(x)=x^2-6x+8$

(ii) $f(x)=x^2-2,\ g(x)=2+8x-4x^2$

8. If $f(x)=2^x+x$, give in simplest terms the formula for $h(x)$, which is obtained from transforming $f(x)$ by the following sequence of transformations:

- vertical stretch, scale factor 8 relative to $y = 0$
- translation by $\begin{pmatrix}1\\4\end{pmatrix}$
- horizontal stretch, scale factor $\frac{1}{2}$ relative to $x = 0$ *[6 marks]*

9. Sketch the following graphs. In each case, indicate clearly the positions of the vertical asymptote and the x-intercept.

(a) $y=\ln x$

(b) $y=3\ln(x+2)$

(c) $y=\ln(2x-1)$ *[6 marks]*

10. (a) The graph of the function $f(x)=ax+b$ is transformed by the following sequence:

- translation by $\begin{pmatrix}1\\2\end{pmatrix}$
- reflection in $y=0$
- horizontal stretch, scale factor $\frac{1}{3}$ relative to $x = 0$

The resultant function is $g(x)=4-15x$. Find a and b.

(b) The graph of the function $f(x) = ax^2 + bx + c$ is transformed by the following sequence:

- reflection in $x = 0$
- translation by $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$
- horizontal stretch, scale factor 2 relative to $y = 0$

The resultant function is $g(x) = 4x^2 + ax - 6$. Find a, b and c. *[10 marks]*

Summary

- Here are the most important transformations that you need to know:

Transformation of $y = f(x)$	Transformation of the graph
$y = f(x) + c$	Translation $\begin{pmatrix} 0 \\ c \end{pmatrix}$
$y = f(x + d)$	Translation $\begin{pmatrix} -d \\ 0 \end{pmatrix}$
$y = pf(x)$	Vertical stretch, scale factor p: when $p > 0$ stretches away from the x-axis; when $0 < p < 1$ stretches towards x-axis; when $p < 0$ stretch by factor p then reflect in x-axis.
$y = f(qx)$	Horizontal stretch, scale factor $\frac{1}{q}$: when $q > 0$ stretches towards the y-axis; when $0 < q < 1$ stretches away from the y-axis; when $q < 0$ stretch by $-\frac{1}{q}$ then reflect in the y-axis.
$y = -f(x)$	Reflection in the x-axis.
$y = f(-x)$	Reflection in the y-axis.

- When combining two or more transformations,
  - when two horizontal transformations, or two vertical transformations, are combined, the order in which they are applied will change the outcome
  - when one horizontal and one vertical transformation are combined, the outcome will be the same regardless of the order in which the transformations were applied.

Introductory problem revisited

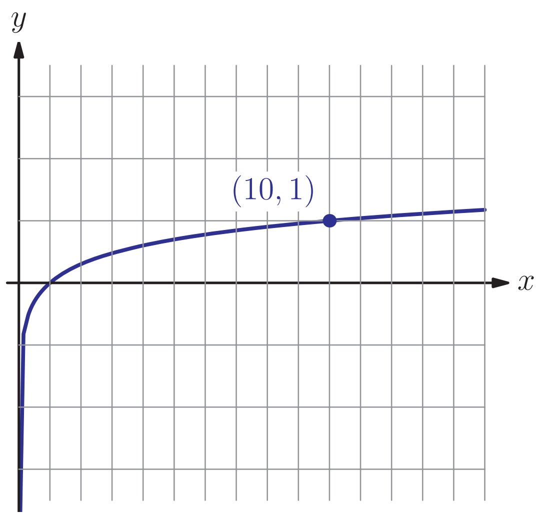
Sketch the graph of $y = \log(x^2 - 6x + 9)$ without using a calculator.

First, simplify the equation algebraically:

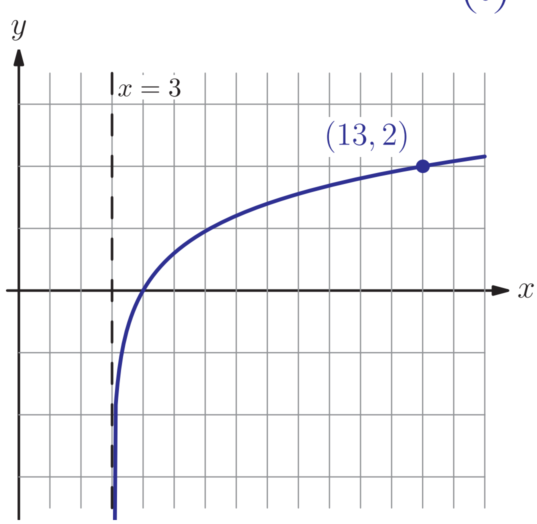
$$\begin{aligned} y &= \log(x^2 - 6x + 9) \\ &= \log(x-3)^2 \\ &= 2\log(x-3) \end{aligned}$$

We can relate this to a graph we know:

$$y = f(x) = \log x$$



The required graph is $y = 2f(x-3)$, which is obtained from the graph of $y = f(x)$ by applying a vertical stretch with scale factor 2 and a translation $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$.



Mixed examination practice 5

Short questions

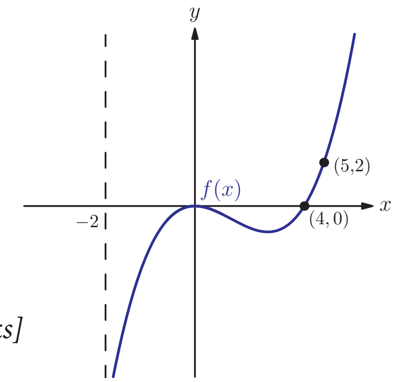
1. The graph of $y = f(x)$ is shown.

Sketch on separate diagrams the graphs of

(a) $y = 3f(x-2)$

(b) $f\left(\frac{x}{3}\right) - 2$

Indicate clearly the positions of any x-intercepts and asymptotes. *[6 marks]*

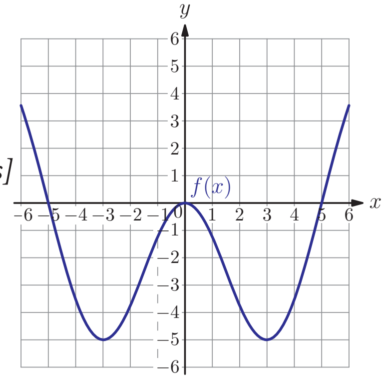


2. The graph of $y = x^3 - 1$ is transformed by applying a translation with vector $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ followed by a vertical stretch with scale factor 2. Find the equation of the resulting graph in the form $y = ax^3 + bx^2 + cx + d$. *[4 marks]*

3. The graph of $y = f(x)$ is shown.

(a) Sketch the graph of $y = f(x-1) + 2$.

(b) State the coordinates of the minimum points on $y = f(x-1) + 2$. *[5 marks]*



4. Find two transformations whose composition transforms the graph of $y = (x-1)^2$ to the graph of $y = 3(x+2)^2$. *[4 marks]*

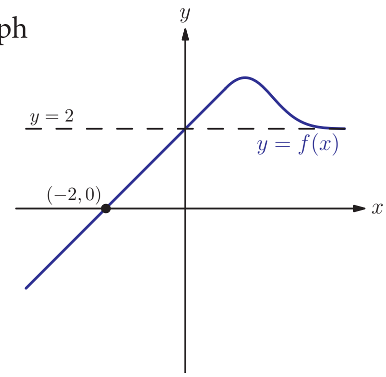
5. (a) Describe two transformations whose composition transforms the graph of $y = f(x)$ to the graph of $y = 3f\left(\frac{x}{2}\right)$.

(b) Sketch the graph of $y = 3\ln\left(\frac{x}{2}\right)$.

(c) Sketch the graph of $y = 3\ln\left(\frac{x}{2} + 1\right)$, marking clearly the positions of any asymptotes and x-intercepts. *[7 marks]*

6. The diagram on the right shows a part of the graph of $y = f(x)$.

Sketch the graph of $y = f(3x-2)$. *[4 marks]*



Long questions

1. (a) Describe two transformations which transform the graph of $y = x^2$ to the graph of $y = 3x^2 - 12x + 12$.

(b) Describe two transformations which transform the graph of $y = x^2 + 6x - 1$ to the graph of $y = x^2$.

(c) Hence describe a sequence of transformations which transform the graph of $y = x^2 + 6x - 1$ to the graph of $y = 3x^2 - 12x + 12$. *[10 marks]*

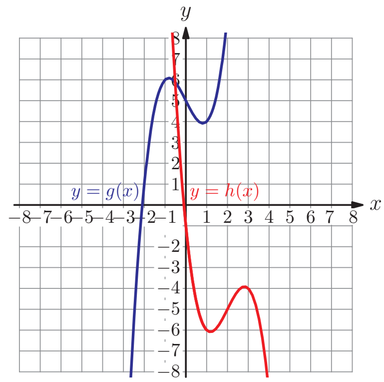
2. (a) Describe a transformation which transforms the graph of $y = f(x)$ to the graph of $y = f(x+2)$.

(b) Sketch on the same diagram the graphs of

(i) $y = \ln(x+2)$ (ii) $y = \ln(x^2 + 4x + 4)$

Mark clearly any asymptotes and x-intercepts on your sketches.

(c) The graph of the function $y = g(x)$ has been translated and then reflected in the x-axis to produce the graph of $y = h(x)$.



(i) State the translation vector.

(ii) If $g(x) = x^3 - 2x + 5$, find constants a, b, c, d such that $h(x) = ax^3 + bx^2 + cx + d$. *[12 marks]*

3. Let $f(x) = \dfrac{3x-5}{x-2}$.

(a) Write down the equation of the horizontal asymptote of the graph of $y = f(x)$.

(b) Find the value of constants p and q such that $f(x) = p + \dfrac{q}{x-2}$.

(c) Hence describe a single transformation which transforms the graph of $y = \dfrac{1}{x}$ to the graph of $y = f(x)$.

(d) Find an expression for $f^{-1}(x)$ and state its domain.

(e) Describe the transformation which transforms the graph of $y = f(x)$ to the graph of $y = f^{-1}(x)$. *[11 marks]*