13 Basic integration and its applications

In this chapter you will learn:

- how to reverse differentiation, a process called integration
- how to find the equation of a curve given its derivative and a point on the curve
- to integrate $\sin x$ and $\cos x$
- to integrate e^x and $\frac{1}{x}$
- how to find the area between a curve and the x-axis
- how to find the area enclosed between two curves.

Introductory problem

The amount of charge stored in a capacitor is given by the area under the graph of current (I) against time (t). For alternating current the relationship between I and t is $I = \sin t$; for direct current the relationship is $I = k$, where k is a constant. For what value of k is the amount of charge stored in the capacitor from $t = 0$ to $t = \pi$ the same whether alternating or direct current is used?

As in many areas of mathematics, as soon as we learn a new process we must then learn how to undo it. It turns out that undoing the process of differentiation opens up a way to solve a seemingly unconnected problem: how to calculate the area under a curve.

13A Reversing differentiation

We saw in chapter 12 how differentiation gives us the gradient of a curve or the rate of change of one quantity with another. What if we already know the function describing a curve's gradient, or the expression for a rate of change, and want to find the original function? This is the same as asking how we can 'undo' the differentiation that has already taken place; the process of reversing differentiation is known as **integration**.

Let us look at two particular cases to get a feel for this process. Each time we will be given $\frac{dy}{dx}$ and need to answer the question, 'what function was differentiated to give this?'

Suppose that $\frac{dy}{dx} = 2x$.

Since $\frac{dy}{dx} = 2x$, the original function y must contain x^2, as we know that differentiation decreases the power by 1. In fact, differentiating x^2 gives exactly $2x$, so we can say that

if $\frac{dy}{dx} = 2x$, then $y = x^2$.

Now suppose that $\frac{dy}{dx} = x^{\frac{1}{2}}$.

Using the same reasoning as above, since $\frac{dy}{dx} = x^{\frac{1}{2}}$, we deduce that the original function y must contain $x^{\frac{3}{2}}$. But if we differentiate $x^{\frac{3}{2}}$ we will get $y = \frac{3}{2}x^{\frac{1}{2}}$, so there is an extra factor of $\frac{3}{2}$ which we do not want. However, if we multiply $x^{\frac{3}{2}}$ by $\frac{2}{3}$, then when we differentiate, the coefficient will cancel to leave 1.

Therefore we can say that if $\frac{dy}{dx} = x^{\frac{1}{2}}$, then $y = \frac{2}{3}x^{\frac{3}{2}}$.

Writing out 'if $\frac{dy}{dx} = x^{\frac{1}{2}}$ then $y = \frac{2}{3}x^{\frac{3}{2}}$' is rather laborious, so we use a shorthand notation for integration:

$$\int x^{\frac{1}{2}}\,dx = \frac{2}{3}x^{\frac{3}{2}}$$

The integration symbol comes from the old English way of writing the letter 'S'. Originally it stood for the word 'sum' (or rather, $\int$um). As you will see later, the integral does indeed represent a sum of infinitesimally small quantities.

The dx states that integration is taking place with respect to the variable x, in exactly the same way that $\frac{d}{dx}$ tells us that the differentiation is taking place with respect to x. We could equally well write, for example,

$$\int t^{\frac{1}{2}}\,dt = \frac{2}{3}t^{\frac{3}{2}}$$

Exercise 13A

1. Find a possible expression for y in terms of x for each of the following.

(a) (i) $\frac{dy}{dx} = 3x^2$ (ii) $\frac{dy}{dx} = 5x^4$

(b) (i) $\frac{dy}{dx} = -\frac{1}{x^2}$ (ii) $\frac{dy}{dx} = -\frac{4}{x^5}$

(c) (i) $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$ (ii) $\frac{dy}{dx} = \frac{1}{3\sqrt[3]{x^2}}$

(d) (i) $\frac{dy}{dx} = 10x^4$ (ii) $\frac{dy}{dx} = 12x^2$

You may have heard of the term 'differential equation'. The equations in Question 1 are the simplest types of differential equation.

13B Constant of integration

We have seen how to integrate some functions of the form x^n by reversing the effects of differentiation. However, the process as carried out above was not quite complete.

Let us consider again the example $\frac{dy}{dx} = 2x$, where we stated that

$$\int 2x \, dx = x^2$$

because the derivative of x^2 is $2x$. But besides x^2, there are other functions which when differentiated give $2x$, for example $x^2 + 1$ or $x^2 - \frac{3}{5}$. This is because when we differentiate the additional constant ($+1$ or $-\frac{3}{5}$) we just get zero. So we could write

$$\int 2x \, dx = x^2 + 1$$

or

$$\int 2x \, dx = x^2 - \frac{3}{5}$$

and both of these answers would be just as valid as $\int 2x dx = x^2$. In fact, we could have added any constant to x^2; without further information we cannot know what constant term the original function had before it was differentiated.

Therefore, the complete answers to the integrals considered in section 13A should be

$$\int 2x \, dx = x^2 + c$$

$$\int x^{\frac{1}{2}} \, dx = \frac{2}{3}x^{\frac{3}{2}} + c$$

where c represents an unknown **constant of integration**. We will see later that, given further information, we can find the value of this constant.

We will see how to determine the constant of integration in section 13F.

Exercise 13B

1. Give three possible functions which when differentiated with respect to x give the following.

   (a) $3x^3$ (b) 0

2. Find the following integrals.

(a) (i) $\int 7x^4\,dx$ (ii) $\int \frac{1}{3}x^2\,dx$

(b) (i) $\int \frac{1}{2t^2}\,dt$ (ii) $\int \frac{8}{y^3}\,dy$

13C Rules of integration

First, let us think about how to reverse the general rule of differentiating a power function.

We know that if $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$, or in words:

To differentiate x^n, multiply by the old power and then decrease the power by 1.

The reverse of this process is:

To integrate x^n, increase the power by 1 and then divide by the new power.

Using integral notation, the general rule for integrating x^n is expressed as follows.

We will see how to integrate x^{-1} in section 13D.

KEY POINT 13.1

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c$$

This holds for any rational power $n \neq -1$.

EXAM HINT

The $+c$ is an essential part of the answer; you must write it every time.

Note the condition $n \neq -1$ which ensures that we are not dividing by zero.

It is worth remembering the formula for integrating a constant: $\int k\,dx = kx + c$, which is a special case of the above rule with $n = 0$: $\int k\,dx = \int kx^0\,dx = \frac{k}{1}x^1 + c$.

EXAM HINT

This rule holds only if k is a constant – a number or expression that does not change with the integration variable.

When we differentiate a function multiplied by a constant k, we get k times the derivative of the function (Key point 12.4). Reversing this gives the following rule for integration:

KEY POINT 13.2

$$\int kf(x)\,dx = k\int f(x)\,dx$$

As we can differentiate term by term (Key point 12.4), we can also split up integrals of sums and do them term by term.

KEY POINT 13.3

For the sum of integrals:

$$\int f(x)+g(x)\,dx=\int f(x)\,dx+\int g(x)\,dx$$

EXAM HINT

Be warned – you cannot integrate products or quotients by integrating each part separately.

By combining Key point 13.3 and Key point 13.2 with $k=-1$, we can also show that the integral of a difference is the difference of the integrals of the separate terms.

These ideas are demonstrated in the following examples.

Worked example 13.1

Find (a) $\int 6x^{-3}\,dx$ (b) $\int\left(3x^4-8x^{-\frac{4}{3}}+2\right)dx$

Add one to the power and divide by the new power.

(a) $\int 6x^{-3}\,dx=\frac{6}{-3+1}x^{-3+1}+c$

$=\frac{6}{-2}x^{-2}+c$

$=-3x^{-2}+c$

Go through term by term, adding one to the power of x and dividing by the new power. Remember the special rule for integrating a constant.

(b) $\int 3x^4-8x^{-\frac{4}{3}}+2\,dx$

$=\frac{3}{4+1}x^{4+1}-\frac{8}{-\frac{4}{3}+1}x^{-\frac{4}{3}+1}+2x+c$

$=\frac{3}{5}x^5-\frac{8}{-\frac{1}{3}}x^{-\frac{1}{3}}+2x+c$

$=\frac{3}{5}x^5+24x^{-\frac{1}{3}}+2x+c$

Just as for differentiation, it may be necessary to manipulate terms into the form kx^n before integrating.

Worked example 13.2

Find (a) $\int 5x^2\sqrt[3]{x}\,dx$ (b) $\int\frac{(x-3)^2}{\sqrt{x}}\,dx$

Write the cube root as a power and use the laws of exponents to combine the two powers.

(a) $\int 5x^2\sqrt[3]{x}\,dx=\int 5x^2x^{\frac{1}{3}}\,dx$

$=\int 5x^{\frac{7}{3}}\,dx$

continued . . .

Dividing by $\frac{10}{3}\left(\text{which is from } \frac{7}{3}+1\right)$ is the same as multiplying by $\frac{3}{10}$.

$$= 5 \times \frac{3}{10} x^{\frac{10}{3}} + c$$

$$= \frac{3}{2} x^{\frac{10}{3}} + c$$

Expand the brackets first, then use rules of exponents to write as a sum of powers.

(b) $\int \frac{(x-3)^2}{\sqrt{x}} dx = \int \frac{x^2 - 6x + 9}{x^{\frac{1}{2}}} dx$

$$= \int \frac{x^2}{x^{\frac{1}{2}}} - \frac{6x}{x^{\frac{1}{2}}} + \frac{9}{x^{\frac{1}{2}}} dx$$

$$= \int x^{\frac{3}{2}} - 6x^{\frac{1}{2}} + 9x^{-\frac{1}{2}} dx$$

Dividing by a fraction is the same as multiplying by its reciprocal.

$$= \frac{2}{5} x^{\frac{5}{2}} - 6 \times \frac{2}{3} x^{\frac{3}{2}} + 9 \times 2x^{\frac{1}{2}} + c$$

$$= \frac{2}{5} x^{\frac{5}{2}} - 4x^{\frac{3}{2}} + 18x^{\frac{1}{2}} + c$$

Exercise 13C

1. Find the following integrals

(a) (i) $\int 9x^8 \, dx$ (ii) $\int 12x^{11} \, dx$

(b) (i) $\int x \, dx$ (ii) $\int x^3 \, dx$

(c) (i) $\int 9 \, dx$ (ii) $\int \frac{1}{2} \, dx$

(d) (i) $\int 3x^5 \, dx$ (ii) $\int 9x^4 \, dx$

(e) (i) $\int 3\sqrt{x} \, dx$ (ii) $\int 3\sqrt[3]{x} \, dx$

(f) (i) $\int \frac{5}{x^2} \, dx$ (ii) $\int \frac{2}{x^3} \, dx$

EXAM HINT

Do not neglect the dx or equivalent in the integral; it tells you what letter represents the variable – see Questions 2 and 3, for example. We will make more use of it later. You can think of the function you are integrating as being multiplied by 'dx', so sometimes you will see integrals written as, for instance, $\int \frac{2\,dx}{x^3}$.

2. Find the following integrals

(a) (i) $\int 3 \, dt$ (ii) $\int 7 \, dz$

(b) (i) $\int q^5 \, dq$ (ii) $\int r^{10} \, dr$

(c) (i) $\int 12g^{\frac{3}{5}} \, dg$ (ii) $\int 5y^{\frac{7}{2}} \, dy$

(d) (i) $\int 4\frac{dh}{h^2}$ (ii) $\int \frac{dp}{p^4}$

3. Find the following integrals.

(a) (i) $\int x^2 - x^3 + 2 \, dx$ (ii) $\int x^4 - 2x + 5 \, dx$

(b) (i) $\int \frac{1}{3t^3} + \frac{1}{4t^4} \, dt$ (ii) $\int 5 \times \frac{1}{v^2} - 4 \times \frac{1}{v^5} \, dv$

(c) (i) $\int x\sqrt{x} \, dx$ (ii) $\int \frac{3\sqrt{x}}{\sqrt[3]{x}} \, dx$

(d) (i) $\int (x+1)^3 \, dx$ (ii) $\int x(x+2)^2 \, dx$

4. Find $\int \frac{1+x}{\sqrt{x}} \, dx$. *[4 marks]*

13D Integrating x^{-1} and e^x

We can now integrate x^n for any rational power n with one exception: in the formula $\int x^n \, dx = \frac{1}{n+1} x^{n+1} + c$ we had to exclude $n = -1$. How, then, do we cope with this case?

In section 12F we learned that $\frac{d}{dx}(\ln x) = \frac{1}{x}$ (Key point 12.10).

Reversing this gives the integration rule for x^{-1}.

KEY POINT 13.4

For $x > 0$,

$$\int x^{-1} \, dx = \ln x + c$$

We also learned in section 12F that $\frac{d}{dx}(e^x) = e^x$ (Key point 12.10), which gives the following formula for integration.

KEY POINT 13.5

$$\int e^x \, dx = e^x + c$$

Worked example 13.3

Find the integral $\int \frac{1+x}{x}\,dx$.

Divide each term of the numerator by the denominator x to split it into two terms.

$$\int \frac{1+x}{3x}\,dx = \int \frac{1}{3x} + \frac{1}{3}\,dx$$

$\frac{1}{3x}$ is the same as $\frac{1}{3} \times \frac{1}{x}$.

$$= \int \frac{1}{3} \times \frac{1}{x} + \frac{1}{3}\,dx$$

Use Key point 13.4 to integrate the first term.

$$= \frac{1}{3}\ln x + \frac{x}{3} + c$$

Exercise 13D

1. Find the following integrals.

(a) (i) $\int \frac{2}{x}\,dx$ (ii) $\int \frac{3}{x}\,dx$

(b) (i) $\int \frac{1}{2x}\,dx$ (ii) $\int \frac{1}{3x}\,dx$

(c) (i) $\int \frac{x^2-1}{x}\,dx$ (ii) $\int \frac{x^3+5}{x}\,dx$

(d) (i) $\int \frac{3x+2}{x^2}\,dx$ (ii) $\int \frac{x-\sqrt{x}}{x^2}\,dx$

2. Find the following integrals.

(a) (i) $\int 5e^x\,dx$ (ii) $\int 9e^x\,dx$

(b) (i) $\int \frac{2e^x}{5}\,dx$ (ii) $\int \frac{7e^x}{11}\,dx$

(c) (i) $\int \frac{(e^x+3x)}{2}\,dx$ (ii) $\int \frac{(e^x+x^3)}{5}\,dx$

13E Integrating trigonometric functions

We will expand the set of functions that we can integrate by continuing to refer back to chapter 12. In section 12E we saw that $\frac{d}{dx}(\sin x) = \cos x$ (Key point 12.8), which means that

$\int \cos x \, dx = \sin x + c$. Similarly, since $\frac{d}{dx}(\cos x) = -\sin x$, we also have $\int \sin x \, dx = -\cos x + c$.

KEY POINT 13.6

$$\int \sin x \, dx = -\cos x + c$$
$$\int \cos x \, dx = \sin x + c$$

Exercise 13E

1. Evaluate the following integrals.

(a) (i) $\int \sin x - \cos x \, dx$ (ii) $\int 3\cos x + 4\sin x \, dx$

(b) (i) $\int \frac{x + \sin x}{7} \, dx$ (ii) $\int \frac{\sqrt{x} + \cos x}{6} \, dx$

(c) (i) $\int 1 - (\cos x + \sin x) \, dx$

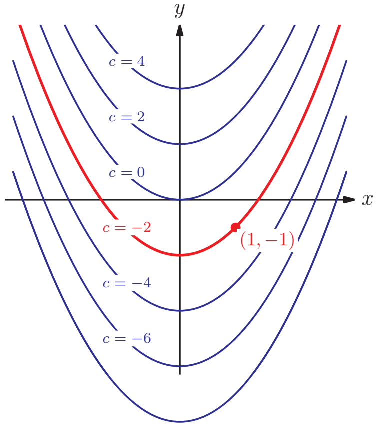
(ii) $\int \cos x - 2(\cos x - \sin x) \, dx$

2. Find $\int \pi(\cos x - 1) \, dx$. *[3 marks]*

3. Find $\int \frac{\cos 2x}{\cos x - \sin x} \, dx$. *[5 marks]*

13F Finding the equation of a curve

We have seen how, given $\frac{dy}{dx}$, we can integrate it to find the equation of the original curve, except for the unknown constant of integration. Geometrically, this means that the gradient determines the shape of the curve, but not exactly where it is. However, if we also know the coordinates of a point on the curve (essentially 'fixing' the curve at a certain position), then we can determine the constant and hence specify the original function precisely.



Consider again the example $\frac{dy}{dx} = 2x$ discussed in sections 13A and 13B. We found that the original curve must have equation $y = x^2 + c$ for some value of the constant c. Each different value of c gives a different curve, but all these graphs have the same shape (a parabola symmetric about the y-axis) and are related to one another by vertical translations; they form a *family* of curves. Now, if we are also told that the curve passes through the point $(1,-1)$, then we can substitute these x and y values into the equation, find c, and thus specify which curve of the family our function corresponds to.

Worked example 13.4

The gradient of a curve is given by $\frac{dy}{dx} = 3x^2 - 8x + 5$, and the curve passes through the point $(1,-4)$. Find the equation of the curve.

To find y from $\frac{dy}{dx}$ we need to integrate. Don't forget + c.

$$y = \int 3x^2 - 8x + 5 \, dx$$
$$= x^3 - 4x^2 + 5x + c$$

The coordinates of the given point must satisfy this equation, so we can find c.

When $x = 1$, $y = -4$, so

$$-4 = (1)^3 - 4(1)^2 + 5(1) + c$$
$$\Rightarrow -4 = 1 - 4 + 5 + c$$
$$\Rightarrow c = -6$$
$$\therefore y = x^3 - 4x^2 + 5x - 6$$

The above example illustrates the general procedure for finding the equation of a curve from its gradient and a point on the curve.

KEY POINT 13.7

To find the equation for y given the gradient $\frac{dy}{dx}$ and one point (p,q) on the curve:

- Integrate $\frac{dy}{dx}$ to get an equation for y in terms of x, remembering $+c$.
- Find the value of c by substituting $x = p$ and $y = q$ into the equation.
- Rewrite the equation, putting in the value of c that was found.

Exercise 13F

1. Find the equation of the original curve if:

(a) (i) $\frac{dy}{dx} = x$ and the curve passes through $(-2,7)$

(ii) $\frac{dy}{dx} = 6x^2$ and the curve passes through (0, 5)

(b) (i) $\frac{dy}{dx} = \frac{1}{\sqrt{x}}$ and the curve passes through (4, 8)

(ii) $\frac{dy}{dx} = \frac{1}{x^2}$ and the curve passes through (1,3)

(c) (i) $\frac{dy}{dx} = 2e^x + 2$ and the curve passes through (1, 1)

(ii) $\frac{dy}{dx} = e^x$ and the curve passes through $(\ln 5, 0)$

(d) (i) $\frac{dy}{dx} = \frac{x+1}{x}$ and the curve passes through (e, e)

(ii) $\frac{dy}{dx} = \frac{1}{2x}$ and the curve passes through $(e^2, 5)$

(e) (i) $\frac{dy}{dx} = \cos x + \sin x$ and the curve passes through $(\pi, 1)$

(ii) $\frac{dy}{dx} = -3\sin x$ and the curve passes through (0, 4)

2. The derivative of the function $f(x)$ is $\frac{1}{2x}$.

(a) Find an expression for all possible functions $f(x)$.

(b) If the curve $y = f(x)$ passes through the point (2, 7), find the equation of the curve. *[5 marks]*

3. The gradient of a curve is found to be $\frac{dy}{dx} = x^2 - 4$.

(a) Find the x-coordinate of the maximum point, justifying that it is a maximum.

(b) Given that the curve passes through the point (0, 2), show that the y-coordinate of the maximum point is $7\frac{1}{3}$. *[5 marks]*

4. A curve is defined only for positive values of x, and the gradient of the normal to the curve at any point is equal to the x-coordinate at that point. If the curve passes through the point $(e^2, 3)$, find the equation of the curve in the form $y = \ln g(x)$ where $g(x)$ is a rational function. *[6 marks]*

13G Definite integration

Until now we have been carrying out a process called **indefinite integration**: indefinite in the sense that we have an unknown constant each time, for example $\int x^2\,\mathrm{d}x = \frac{1}{3}x^3 + c$.

There is also a process known as **definite integration**, which yields a numerical answer. To calculate a definite integral, we evaluate the indefinite integral at two points and take the difference of the results:

$$\begin{aligned}\int_2^3 x^2\,\mathrm{d}x &= \left[\frac{1}{3}x^3 + c\right]_2^3 \\ &= \left(\frac{1}{3}3^3 + c\right) - \left(\frac{1}{3}2^3 + c\right) \\ &= 6\frac{1}{3}\end{aligned}$$

Note that the constant of integration, c, cancels out in the subtraction, so we can omit it altogether from the definite integral calculation and just write

$$\begin{aligned}\int_2^3 x^2\,\mathrm{d}x &= \left[\frac{1}{3}x^3\right]_2^3 \\ &= \left(\frac{1}{3}3^3\right) - \left(\frac{1}{3}2^3\right) \\ &= 6\frac{1}{3}.\end{aligned}$$

The numbers 2 and 3 here are known as the **limits of integration**; 2 is the lower limit and 3 is the upper limit.

EXAM HINT

Make sure you know how to evaluate definite integrals on your calculator. See Calculator Skills sheet 9 on the CD-ROM for guidance on how to do this.
Besides saving you time, your calculator can help you evaluate integrals that you don't know how to do algebraically. And even when you are asked to find the exact value of the integral, it is still a good idea to use your calculator to check the answer.

We could calculate the definite integral of x^2 with any numbers a and b as the lower and upper limits; the answer will, of course, depend on a and b:

$$\begin{aligned}\int_a^b x^2\,\mathrm{d}x &= \left[\frac{1}{3}x^3\right]_a^b \\ &= \frac{1}{3}b^3 - \frac{1}{3}a^3\end{aligned}$$

The integration variable x does *not* come into the answer – it has taken on the values of the limits and is referred to as a 'dummy variable'.

The square bracket notation indicates that integration has taken place but the limits have not yet been applied. 'Applying the limits' just means to evaluate the integrated expression at the upper limit and subtract the integrated expression evaluated at the lower limit.

Worked example 13.5

Find the exact value of $\int_1^e \frac{1}{x} + 4\,dx$.

Integrate and write in square brackets.

$$\int_1^e \frac{1}{x} + 4\,dx = [\ln x + 4x]_1^e$$

Evaluate the integrated expression at the upper and lower limits and subtract the lower from the upper.

$$= (\ln(e) + 4(e)) - (\ln(1) + 4(1))$$
$$= (1 + 4e) - (0 + 4)$$
$$= 4e - 3$$

Exercise 13G

1. Evaluate the following definite integrals, giving exact answers.

(a) (i) $\int_2^6 x^3\,dx$ (ii) $\int_1^4 x^2 + x\,dx$

(b) (i) $\int_0^{\pi/2} \cos x\,dx$ (ii) $\int_\pi^{2\pi} \sin x\,dx$

(c) (i) $\int_0^1 e^x\,dx$ (ii) $\int_{-1}^1 3e^x\,dx$

2. Evaluate the following definite integrals.

(a) (i) $\int_{0.3}^{1.4} \sqrt{x}\, dx$ (ii) $\int_{9}^{9.1} \frac{3}{\sqrt{x}}\, dx$

(b) (i) $\int_{0}^{1} e^{x^2}\, dx$ (ii) $\int_{1}^{e} \ln x\, dx$

3. Find the exact value of the integral $\int_0^{\pi} e^x + \sin x + 1\, dx$. *[5 marks]*

4. Show that the value of the integral $\int_k^{2k} \frac{1}{x}\, dx$ is independent of k. *[4 marks]*

5. If $\int_3^9 f(x)\, dx = 7$, evaluate $\int_3^9 2f(x)+1\, dx$. *[4 marks]*

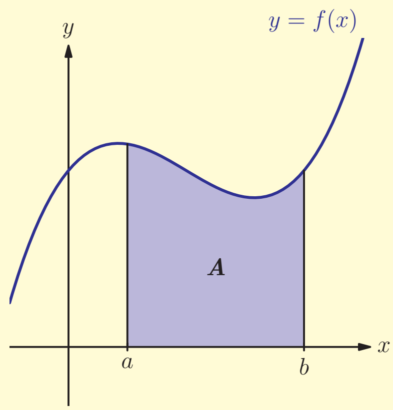
6. Solve the equation $\int_1^a \sqrt{t}\, dt = 42$. *[5 marks]*

13H Geometrical significance of definite integration

Now we have a method that gives a numerical value for an integral, the natural question to ask is: what does this number mean? The answer is that the definite integral represents the area under a curve; more precisely, $\int_a^b f(x)\, dx$ is the area enclosed between the curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$.

KEY POINT 13.8

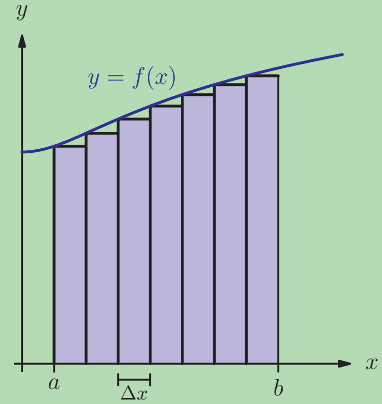
$$A = \int_a^b f(x)\, dx$$



See Fill-in proof 14 'The fundamental theorem of calculus' on the CD-ROM for a justification of this result. Strictly speaking, it holds only if the graph lies above the x-axis, but we shall see later how we can also find areas associated with graphs that go below the x-axis.

In the 17th century the integral was defined as the area under a curve. The region under the curve $y = f(x)$ was broken down into thin rectangles, each with height $f(x)$ and width being a small distance in the x direction, written Δx. The total area can be approximated by the sum of the areas of all these rectangles.

The height of the first rectangle is $f(a)$ and its width is Δx, so its area is $\Delta x f(a)$.
The height of the second rectangle is $f(a+\Delta x)$, so its area is $\Delta x f(a+\Delta x)$.



This pattern continues until the final rectangle with left edge at $b-\Delta x$, which has area $\Delta x f(b-\Delta x)$.

Thus, the area under the curve is approximately

$$\left[f(a)+f(a+\Delta x)+f(a+2\Delta x)+\cdots+f(b-\Delta x)\right]\Delta x$$

which can be written more briefly, in sigma notation (see chapter 6), as

$$\sum_{x=a}^{x=b-\Delta x} f(x)\Delta x$$

This approximation to the area gets better as the rectangles become thinner and more numerous. If we take the limit as the width of the rectangles becomes very small ($\Delta x \to 0$), while their number tends to infinity, we should be able to obtain the exact area under the curve.

Isaac Newton, one of the pioneers of calculus, was a big fan of writing in English rather than Greek. 'Sigma' became the English letter 'S' and 'delta' became the English letter 'd', so when the limit $\Delta x \to 0$ is taken, the expression for the sum of rectangle areas becomes

$$\int_a^b f(x)\,dx$$

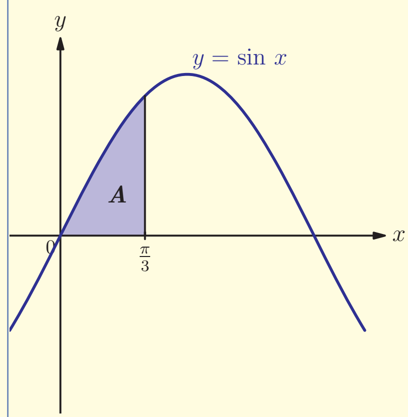
This illustrates another very important interpretation of integration – as the infinite sum of infinitesimally small parts.

The ancient Greeks had already developed ideas of limiting processes similar to those used in calculus, but it took nearly 2000 years for the ideas to be formalised. This was done almost simultaneously by Isaac Newton and Gottfried Leibniz in the 17th century. Was this a coincidence, or is it often the case that a long period of slow progress is needed to reach a stage of readiness for major breakthroughs? Supplementary sheet 13 looks at some other people who made contributions to the development of calculus.

Worked example 13.6

Find the exact area enclosed between the x-axis, the curve $y = \sin x$ and the lines $x = 0$ and $x = \frac{\pi}{3}$.

Sketch the graph and identify the area required.



Integrate and write in square brackets.

$$A = \int_0^{\pi/3} \sin x \, dx = [-\cos x]_0^{\pi/3}$$

Evaluate the integrated expression at the upper and lower limits and subtract the lower from the upper.

$$= \left(-\cos\frac{\pi}{3}\right) - (-\cos 0)$$

$$= -\frac{1}{2} + 1$$

$$= \frac{1}{2}$$

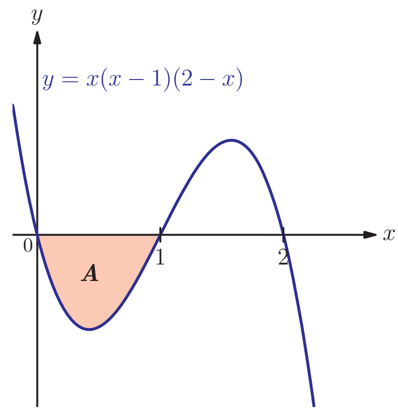
EXAM HINT

If you sketch the graph on a calculator, you can get it to shade and evaluate the required area, as explained on Calculator Skills sheet 9 on the CD-ROM. If it is not already given in the question, you should show the sketch as part of your working.

When the curve is entirely below the x-axis the integral will give us a negative value. In this case, the modulus of the definite integral is the area bounded by the curve and the x-axis.

Worked example 13.7

Find the area A in the diagram.



Write down the integral we need, then use the calculator to evaluate it.

$\int_0^1 x(x-1)(2-x)\,dx = -0.25$ (by GDC)

The area must be positive.

$\therefore A = 0.25$

Unfortunately, the relationship between integrals and areas is not so simple when there are parts of the curve above and below the x-axis. Those parts above the axis contribute positively to the area, but portions below the axis contribute negatively to the area. Therefore, to calculate the total area enclosed between the curve and the x-axis, we must separate out the sections above the axis and those below the axis.

Worked example 13.8

(a) Find $\int_1^4 x^2 - 4x + 3\,dx$.

(b) Find the area enclosed between the x-axis, the curve $y = x^2 - 4x + 3$ and the lines $x = 1$ and $x = 4$.

continued . . .

Integrate and evaluate at the upper and lower limits.

(a) $\int_1^4 x^2 - 4x + 3\,dx$

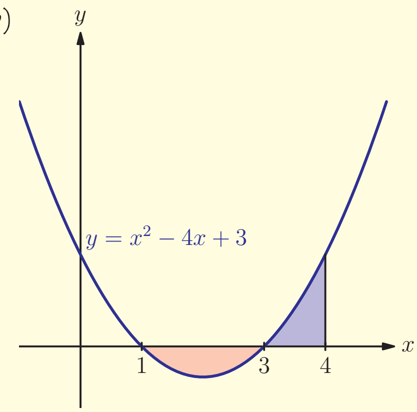
$= \left[\frac{1}{3}x^3 - 2x^2 + 3x\right]_1^4$

$= \left(\frac{1}{3}(4)^3 - 2(4)^2 + 3(4)\right) - \left(\frac{1}{3}(1)^3 - 2(1)^2 + 3(1)\right)$

$= \frac{4}{3} - \frac{4}{3} = 0$

The value found above cannot be the area asked for in part (b). Sketch the curve to see exactly what area we are being asked to find.

(b)



The required area is made up of two parts, one below the axis and one above, so evaluate each of them separately.

$\int_1^3 x^2 - 4x + 3\,dx = \left[\frac{1}{3}x^3 - 2x^2 + 3x\right]_1^3$

$= (0) - \left(\frac{4}{3}\right)$

$= -\frac{4}{3}$

$\therefore$ area below the axis is $\frac{4}{3}$

$\int_3^4 x^2 - 4x + 3\,dx = \left[\frac{1}{3}x^3 - 2x^2 + 3x\right]_3^4$

$= \left(\frac{4}{3}\right) - (0)$

$= \frac{4}{3}$

$\therefore$ area above the axis is $\frac{4}{3}$

Total area $= \frac{4}{3} + \frac{4}{3} = \frac{8}{3}$

The integral being zero in part (a) of the example means that the area above the x-axis is exactly cancelled by the area below the axis.

As you can see from the example above, when you are asked to find an area it is essential to sketch the graph and identify exactly where each part of the area is. The area bounded by $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is given by $\int_a^b f(x)\,dx$ only when $f(x)$ is entirely positive between a and b. If the curve crosses the x-axis somewhere between a and b, then we have to split up the integral and find each piece of area separately.

See Prior Learning Section I if you are unfamiliar with the modulus function and Calculator Skills Sheet 3 for how to find it on your calculator.

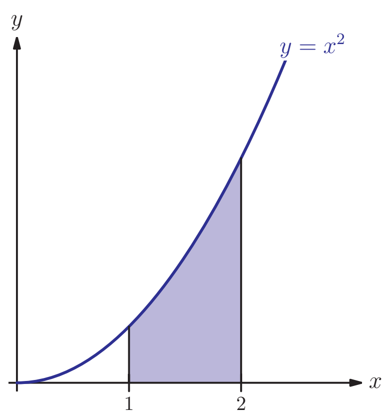
If you are evaluating the area on your calculator, you can use the modulus function to ensure that all parts of the area are counted as positive:

$$\text{Area} = \int_a^b |f(x)|\,dx$$

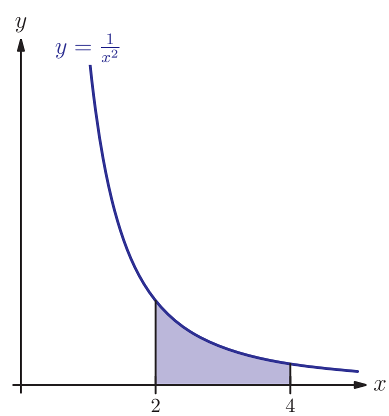
Exercise 13H

1. Find the shaded areas.

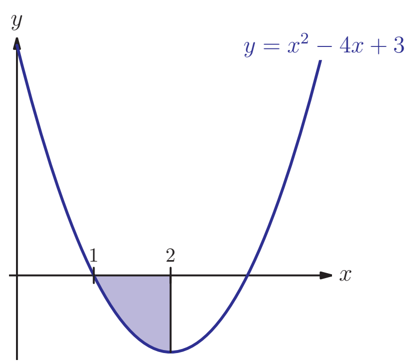
(a) (i)



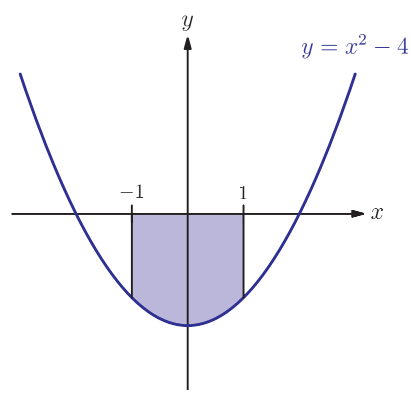
(i)



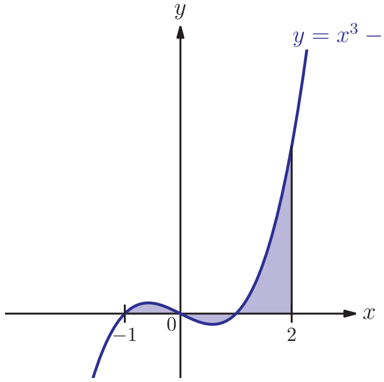
(b) (i)



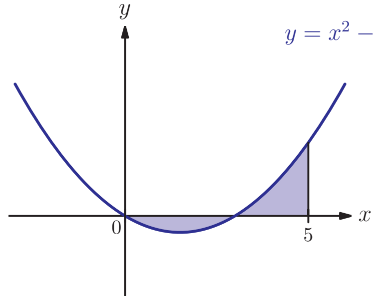
(ii)



(c) (i)



(ii)



2. The area enclosed by the x-axis, the curve $y = \sqrt{x}$ and the line $x = k$ is 18. Find the value of k. *[6 marks]*

EXAM HINT

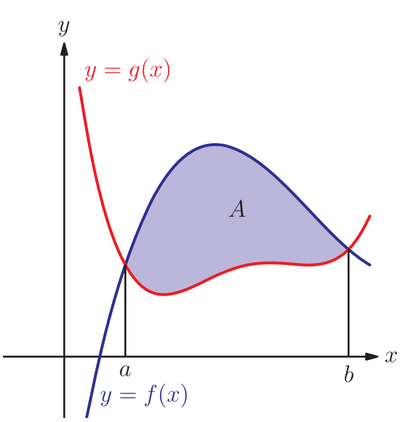
'Find the area enclosed' means to first identify a closed region bounded by the curves mentioned, and then find its area. A sketch is a very useful tool.

3. (a) Find $\int_0^3 x^2 - 1\,dx$.

(b) Find the area between the curve $y = x^2 - 1$ and the x-axis between $x = 0$ and $x = 3$. *[5 marks]*

4. Between $x = 0$ and $x = 3$, the area of the graph $y = x^2 - kx$ below the x-axis equals the area above the x-axis. Find the value of k. *[6 marks]*

5. Find the area enclosed by the curve $y = 7x - x^2 - 10$ and the x-axis. *[7 marks]*



13 I The area between two curves

So far we have considered only areas bounded by a curve and the x-axis, but it is also useful to be able to find areas bounded by two curves.

The area A in the diagram can be found by taking the area under $y = f(x)$ and subtracting the area under $y = g(x)$, that is,

$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

It is usually easier to do the subtraction before integrating, so that we only have to integrate one expression instead of two. This gives an alternative formula for the area.

KEY POINT 13.9

The area bounded above by the curve $y = f(x)$ and below by the curve $y = g(x)$ is

$$A = \int_a^b |f(x) - g(x)|\,dx$$

where a and b are the x-coordinates of the intersection points of the two curves.

Worked example 13.9

Find the area A enclosed between $y = 2x+1$ and $y = x^2 - 3x + 5$.

First find the x-coordinates of intersection.

For intersection:

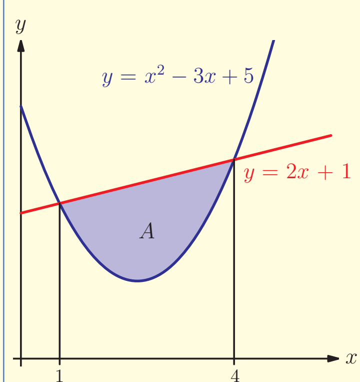
$x^2 - 3x + 5 = 2x + 1$

$\Leftrightarrow x^2 - 5x + 4 = 0$

$\Leftrightarrow (x-1)(x-4) = 0$

$\Leftrightarrow x = 1, 4$

Make a rough sketch to see the relative positions of the two curves.



Subtract the lower curve from the higher before integrating.

$$A = \int_1^4 (2x+1) - (x^2 - 3x + 5)\,dx$$

$$= \int_1^4 -x^2 + 5x - 4\,dx$$

$$= \left[-\frac{x^3}{3} + \frac{5x^2}{2} - 4x\right]_1^4$$

$$= \frac{8}{3}\left(-\frac{11}{6}\right) = \frac{9}{2}$$

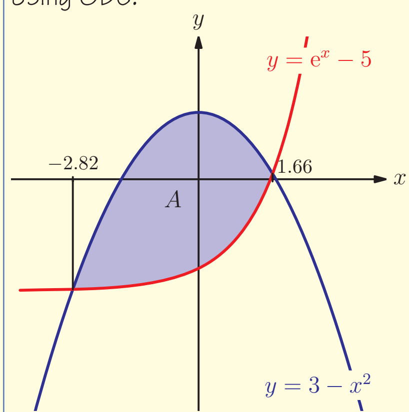
Subtracting the two equations before integrating is particularly convenient when one of the curves is partially below the x-axis: as long as $f(x)$ is above $g(x)$, the expression we are integrating, $f(x)-g(x)$, will always be positive, so we do not have to worry about the signs of $f(x)$ and $g(x)$ themselves.

Worked example 13.10

Find the area bounded by the curves $y = e^x - 5$ and $y = 3 - x^2$.

Sketch the graph to see the relative positions of the two curves.

Using GDC:



Find the intersection points – use the calculator.

Intersections: $x = -2.818$ and 1.658

Write down the integral that represents the area.

$$\text{Area} = \int_{-2.818}^{1.658} (3 - x^2) - (e^x - 5)\,dx$$

$$= \int_{-2.818}^{1.658} (8 - x^2 - e^x)\,dx$$

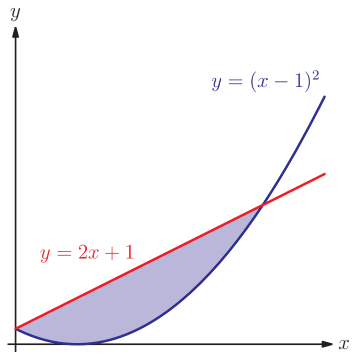
Evaluate the integral using the calculator.

$= 21.6$ (3 SF) using GDC

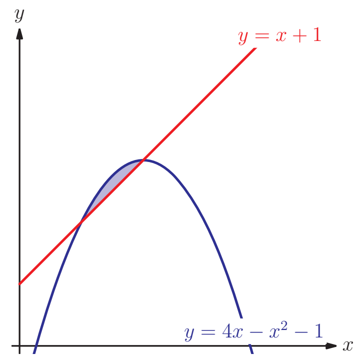
Exercise 13I

1. Find the shaded areas.

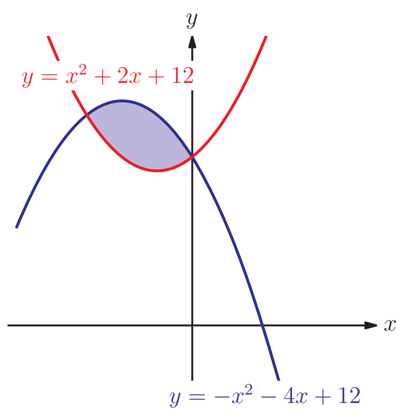
(a) (i)



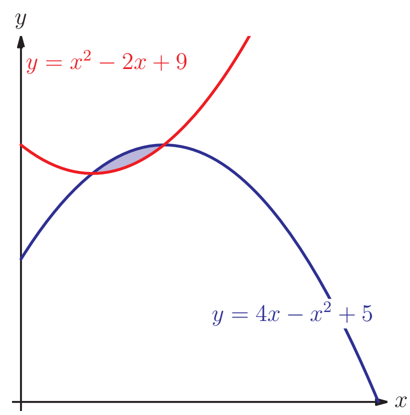
(ii)



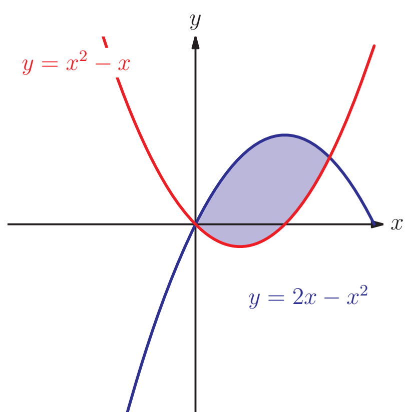
(b) (i)



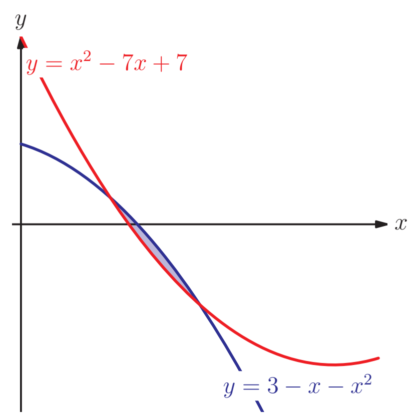
(ii)



(c) (i)

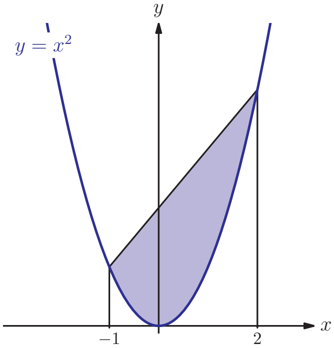


(ii)



2. Find the area enclosed between the graphs of $y = x^2 + x - 2$ and $y = x + 2$. *[6 marks]*

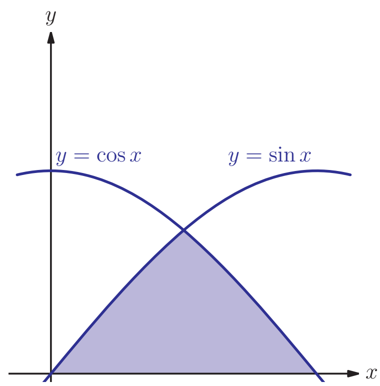
3. Find the area enclosed by the curves $y = e^x$ and $y = x^2$, the y-axis and the line $x = 2$. *[6 marks]*



4. Find the area between the curves $y=\frac{1}{x}$ and $y=\sin x$ in the region $0<x<\pi$. *[6 marks]*

5. Show that the area of the shaded region in the diagram on the left is $\frac{9}{2}$. *[6 marks]*

6. The diagram below shows the graphs of $y=\sin x$ and $y=\cos x$. Find the shaded area.



[6 marks]

7. Find the total area enclosed between the graphs of $y=x(x-4)^2$ and $y=x^2-7x+15$. *[6 marks]*

8. The area enclosed between the curve $y=x^2$ and the line $y=mx$ is $10\frac{2}{3}$. Find the value of m if $m>0$. *[7 marks]*

Summary

- **Integration** is the reverse process of differentiation.
- If we know $\frac{dy}{dx}$, the **indefinite integral** gives the original function which has this gradient, with an unknown **constant of integration** c.
- To find c and hence determine the original function precisely, we need to know one point (p,q) on the curve.
- The indefinite integrals of some common functions are:

$$\int x^n \, dx = \frac{1}{n+1}x^{n+1}+c \quad \text{for } n \neq -1$$

$$\int x^{-1} \, dx = \ln x + c \quad \text{for } x>0$$

$$\int e^x \, dx = e^x + c$$

$$\int \sin x \, dx = -\cos x + c$$

$$\int \cos x \, dx = \sin x + c$$

- The **definite integral** $\int_a^b f(x)\,dx$ is found by evaluating the integrated expression at the upper limit b and then subtracting the integrated expression evaluated at the lower limit a.
- The area between the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is given by

  $$A = \int_a^b f(x)\,dx$$

  provided that the curve lies entirely above the x-axis between $x = a$ and $x = b$.

  If the curve goes below the x-axis, then the integral of the part below the axis will be *negative*. On a calculator we can use the modulus function to ensure we are always integrating a positive function.
- The area bounded above by the curve $y = f(x)$ and below by the curve $y = g(x)$ is

  $$A = \int_a^b |f(x) - g(x)|\,dx$$

  where a and b are the x-coordinates of the intersection points of the two curves.

Introductory problem revisited

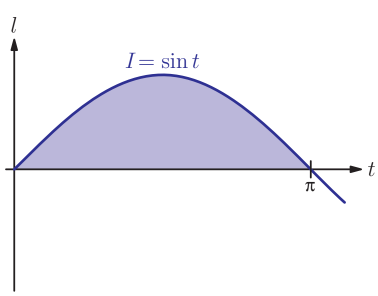
The amount of charge stored in a capacitor is given by the area under the graph of current (I) against time (t). For alternating current the relationship between I and t is $I = \sin t$; for direct current the relationship is $I = k$, where k is a constant. For what value of k is the amount of charge stored in the capacitor from $t = 0$ to $t = \pi$ the same whether alternating or direct current is used?

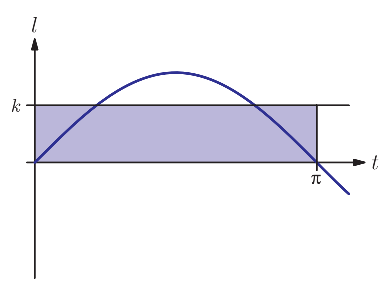
For alternating current, the area under the curve of I against t is $\int_0^\pi \sin t\,dt = [-\cos t]_0^\pi = 2$.

For direct current, the area is $\int_0^\pi k\,dt = [kt]_0^\pi = k\pi$.

These two quantities are equal if $k = \dfrac{2}{\pi}$.

In this example, $k = \dfrac{1}{\pi}\int_0^\pi \sin t\,dt$ can be interpreted as the average value of the current I over the time interval $0 \le t \le \pi$. In fact, integration is a sophisticated way of finding the average value of a quantity.





Mixed examination practice 13

Short questions

1. If $f'(x) = \sin x$ and $f\left(\dfrac{\pi}{3}\right) = 0$, find $f(x)$. *[4 marks]*

2. Find the area enclosed between the graph of $y = k^2 - x^2$ and the x-axis, giving your answer in terms of k. *[6 marks]*

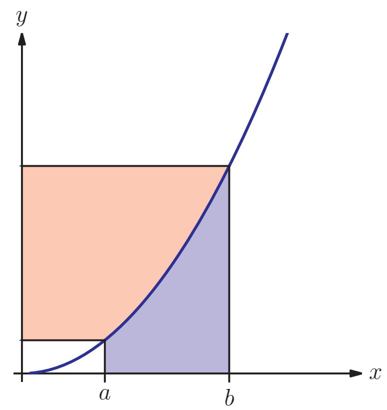
3. Find the indefinite integral $\int \dfrac{1 + x^2\sqrt{x}}{x}\,dx$. *[5 marks]*

4. (a) Solve the equation

$$\int_0^a x^3 - x\,dx = 0, a > 0$$

(b) For this value of a, find the total area enclosed between the x-axis and the curve $y = x^3 - x$ for $0 \le x \le a$. *[6 marks]*

5. The diagram shows the graph of $y = x^n$ for $n > 1$.



(a) (i) Write down an expression for the area of the white rectangle with vertices $x = 0$ and $x = a$.

(ii) If B is the area of the blue shaded region, find an expression for B in terms of a, b and n.

(b) If the red area is three times larger than the blue area, find the value of n. *[6 marks]*

6. Find the area enclosed between the graphs of $y = \sin x$ and $y = 1 - \sin x$ for $0 < x < \pi$. *[3 marks]*

7. The function $f(x)$ has a stationary point at (3, 19), and $f''(x) = 6x + 6$.

(a) What kind of stationary point is at (3, 19)?

(b) Find $f(x)$. *[5 marks]*

Long questions

1. The derivative of $f(x)$ is $f'(x) = e^x + c$. The line $y = 3x + 2$ is tangent to the graph $y = f(x)$ at $x = 1$.

   (a) Find the value of c.

   (b) Find the value of $f(1)$.

   (c) Find an expression for $f(x)$.

   (d) Find the area under the graph $y = f(x)$ between $x = 0$ and $x = 1$.

   [11 marks]

2. (a) Show that $5a^2 + 4ax - x^2 = (5a - x)(x + a)$.

   (b) Find the coordinates of the points of intersection of the graphs $y = 5a^2 + 4ax - x^2$ and $y = x^2 - a^2$.

   (c) Find the area enclosed between these two graphs.

   (d) Show that the fraction of this area above the x-axis is independent of a, and state the value that this fraction takes. *[10 marks]*