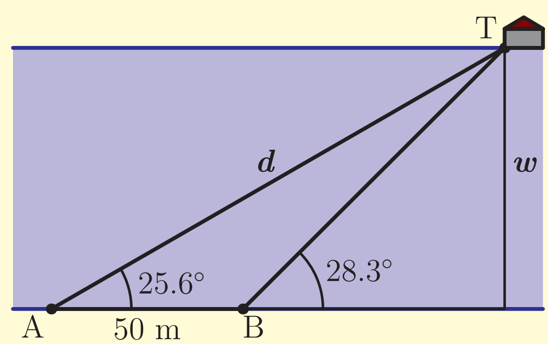
10 Geometry of triangles and circles

In this chapter you will learn:

- some important trigonometric relations in right-angled triangles
- how to use the sine rule to find sides and angles of a triangle
- how to use the cosine rule to find sides and angles of a triangle
- an alternative formula for the area of a triangle
- techniques for solving geometry problems in two and three dimensions
- how to calculate the length of an arc of a circle
- how to calculate the area of a sector of a circle
- to use trigonometry to solve problems involving circles and triangles.

Introductory problem

Two people are trying to measure the width of a river. There is no bridge across the river, but they have instruments for measuring lengths and angles. When they stand 50 m apart on the same side of the river, at points A and B, the person at A measures that the angle between line (AB) and the line from A to the tower on the other side of the river is 25.6°. The person at B finds the corresponding angle to be 28.3°, as shown in the diagram. Use this information to calculate the width of the river.



The first steps in developing trigonometry were taken by Babylonian astronomers as early as the second millennium BCE. It is thought that the Egyptians used trigonometric calculations when building their pyramids. Trigonometry was further developed by the Greeks and Indians. Some of the most significant contributions were made by Islamic mathematicians in the second half of the first millennium BCE.

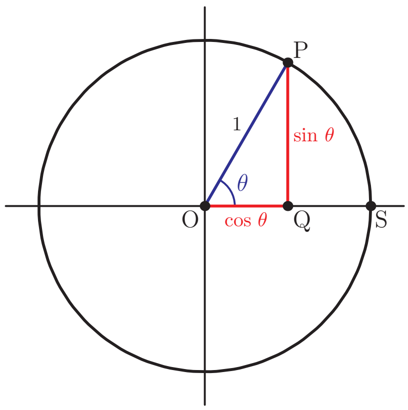
The problem above involves finding lengths and angles in triangles. Such problems can be solved using trigonometric functions. In fact, the word *trigonometry* means 'measuring triangles', and historically trigonometry was used to solve similar problems in land measurement, building and astronomy. In this chapter we will use what we have already learned about trigonometric functions and develop some new results to enable us to calculate lengths and angles in triangles.

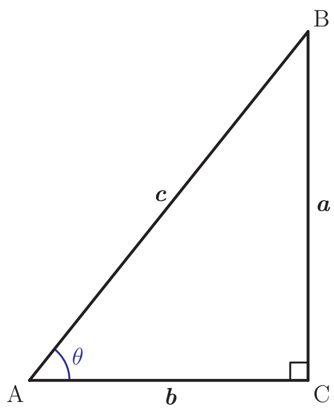
10A Right-angled triangles

In previous courses, you may have been introduced to the sine, cosine and tangent functions in the context of a right-angled triangle. We will now briefly discuss how this view relates to the definitions given in chapter 8.

Recall how sine and cosine were defined using the unit circle. Let $0° < \theta < 90°$, and let P be the point on the unit circle such that $S\hat{O}P = \theta$. Then $PQ = \sin\theta$ and $OQ = \cos\theta$.

Trigonometric functions were defined in sections 8B and 8C.





Now consider a right-angled triangle ABC with right angle at C and $B\hat{A}C = \theta$.

The triangles OPQ and ABC have the same angles, so they are similar triangles. Therefore

$$\frac{a}{\sin\theta} = \frac{b}{\cos\theta} = \frac{c}{1}$$

See Prior Learning section T on the CD-ROM for a reminder about similar triangles.

By rearranging these equations we find that the ratios of sides in a right-angled triangle are trigonometric functions of the angle θ.

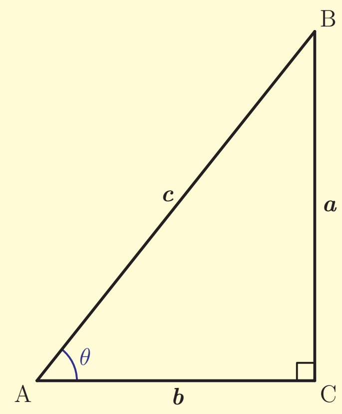
KEY POINT 10.1

In a right-angled triangle:

$$\frac{a}{c} = \sin\theta$$

$$\frac{b}{c} = \cos\theta$$

$$\frac{a}{b} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$



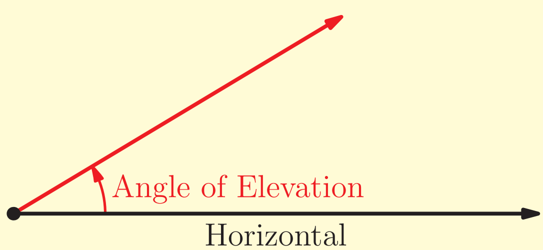
See Prior Learning section U on the CD-ROM for practice questions on right-angled triangles.

These equations apply only to acute angles. In the rest of this chapter we shall deal with triangles in general, including those containing obtuse angles, in which case we would need to use the definitions of trigonometric functions based on the unit circle.

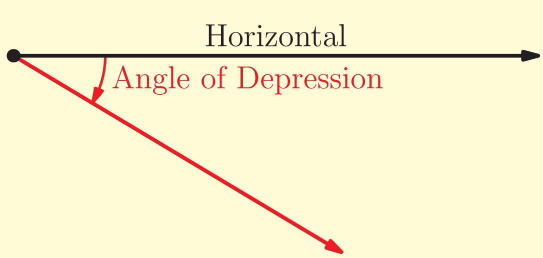
Two terms that come up frequently in trigonometry are the **angle of elevation** and the **angle of depression**.

KEY POINT 10.2

The angle of elevation is the angle above the horizontal.



The angle of depression is the angle below the horizontal.



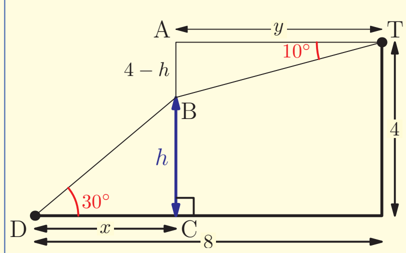
Worked example 10.1

Daniel and Theo are trying to work out the height of a bird's nest in their garden. From Theo's bedroom window, which is 4 m above the ground, the angle of depression of the nest is 10°. From Daniel's position at the end of the flat garden, 8 m away from the house, the angle of elevation is 30°. Find the height of the nest above the ground.

EXAM HINT

If a diagram is not given, it is always a good idea to sketch one, labelling any points that you refer to in your working.

Sketch a diagram. The letters B, D and T represent the positions of the bird's nest, Daniel and Theo, respectively. The letters A and C refer to points directly above and below the nest.



Apply trigonometry to the right-angled triangles to find the horizontal distances x and y in terms of the height h.

From triangle TAB,

$$\tan 10° = \frac{4-h}{y}$$

$$\Rightarrow y = \frac{4-h}{\tan 10°}$$

From triangle BCD,

$$\tan 30° = \frac{h}{x}$$

$$\Rightarrow x = \frac{h}{\tan 30°}$$

Now use the length of the garden, which is the total horizontal distance.

$$x + y = 8$$

Substitute the trigonometric expressions for x and y.

$$\therefore \frac{4-h}{\tan 10°} + \frac{h}{\tan 30°} = 8$$

continued . . .

Rearrange the equation to find h. If you find it cumbersome to write out tan 10° and tan 30° repeatedly, you can define a short name for each.

Let $a = \tan 10°$ and $b = \tan 30°$.

Then $\frac{4-h}{a} + \frac{h}{b} = 8$

$\Rightarrow (4-h)b + ha = 8ab$

$\Rightarrow \quad h(a-b) = 8ab - 4b$

$\Rightarrow \quad h = \frac{8ab-4b}{a-b}$

Finally, evaluate a and b and hence find h.

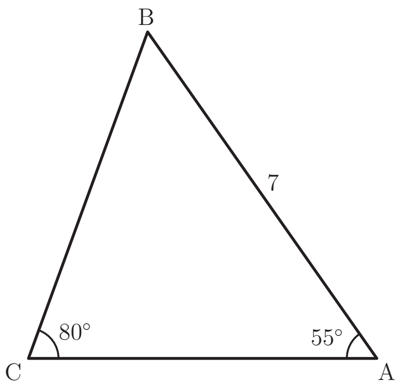
$a = 0.176$, $b = 0.577$
$\therefore h = 3.73\text{m}$ (3 SF)

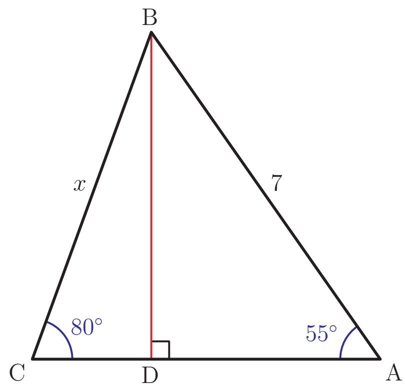
EXAM HINT

While it is acceptable to put in the values for $\tan 10°$ and $\tan 30°$ before rearranging the equation, this can easily lead to arithmetic errors. It is safer to rearrange first and calculate values at the end. Remember that you can save values to each of the lettered memory sites in your calculator, so you could assign $\tan 10°$ to memory A and $\tan 30°$ to memory B, for example; this would make it much faster to evaluate the final expression for h.

10B The sine rule

Can we use trigonometry to calculate sides and angles of triangles that do not have a right angle? The answer is yes, and one way to do this is by using a set of identities called the **sine rule**. In the diagram on the left at the top of the next page, the triangle has AB = 7, $B\hat{A}C = 55°$ and $A\hat{C}B = 80°$. Can we find the length BC?





There are no right angles in the diagram, but we can create some by drawing the line [BD] perpendicular to [AC], as shown in the second diagram.

We now have two right-angled triangles: ABD and BCD.

In triangle ABD, $\frac{BD}{7} = \sin 55°$, so $BD = 7\sin 55°$.

In triangle BCD, $\frac{BD}{x} = \sin 80°$, so $BD = x\sin 80°$.

Comparing the two expressions for BD, we get

$$x \sin 80° = 7 \sin 55° \qquad (*)$$

and rearranging gives

$$x = \frac{7\sin 55°}{\sin 80°} = 5.82$$

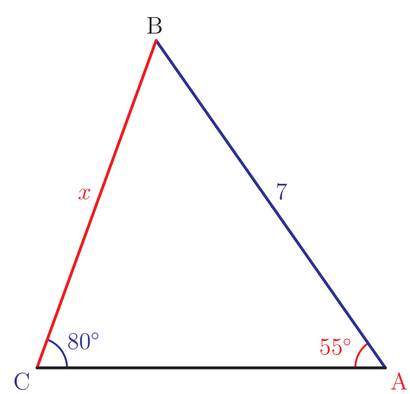
In fact, we did not even need to write down expressions for BD but can go straight to equation (*). This equation can also be written as

$$\frac{x}{\sin 55°} = \frac{7}{\sin 80°}$$

which is an example of the sine rule. Note that the length of each side is divided by the sine of the angle *opposite* that side.

EXAM HINT

Remember the International Baccalaureate® notation for lines and line segments: (AB) stands for the straight line through A and B that extends indefinitely in both directions, [AB] denotes the line segment between A and B, and AB is the length of [AB].

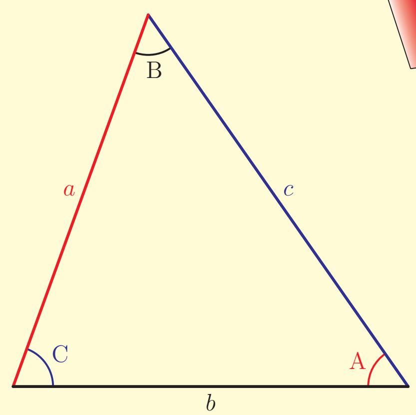


We can repeat the same process, dropping a perpendicular from A or from C, to obtain a general formula relating the three side lengths and three angles.

KEY POINT 10.3

The sine rule:

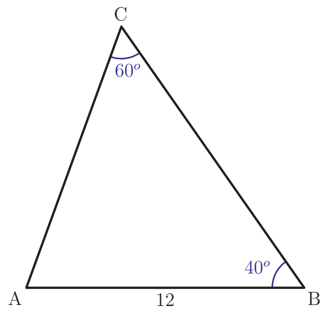
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



Normally, you would use only two of the three ratios in the sine rule. To decide which ones to use, look at what information is given in the question. In any case, you always need to know one angle and its opposite side to be able to apply the sine rule.

Worked example 10.2

Find the length of side [AC].



We are given the angles opposite [AB] and [AC], so use the sine rule with those two sides.

$$\frac{AB}{\sin C} = \frac{AC}{\sin B}$$

$$\frac{12}{\sin 60°} = \frac{AC}{\sin 40°}$$

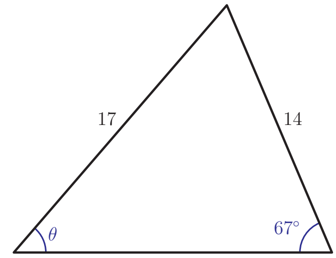
$$\Rightarrow AC = \frac{12 \sin 40°}{\sin 60°}$$

$$= 8.91 \text{ (3 SF)}$$

We can use the sine rule to find angles as well as side lengths.

Worked example 10.3

Find the size of the angle marked θ.



We can use the sine rule because we know an angle together with its opposite side, are given one of the other sides, and want to find the angle opposite that side.

$$\frac{17}{\sin 67^\circ} = \frac{14}{\sin \theta}$$

$$\Rightarrow \sin\theta = \frac{14 \sin 67^\circ}{17} = 0.758$$

$$\therefore \quad \theta = \sin^{-1} 0.758 = 49.3^\circ$$

In the above example, once we have found θ we can deduce the size of the third angle even though we do not know the length of the side opposite it: the third angle must be $180^\circ - 67^\circ - \theta = 63.7^\circ$. So, provided that we know one side length and the angle opposite it, we can use either of the remaining side lengths to calculate both remaining angles.

You should remember from chapter 9 that there is more than one value of θ with $\sin\theta = 0.758$; besides 49.3°, another solution of the equation is 180° – 49.3° = 130.7°. Does this mean that Worked example 10.3 actually has more than one possible answer?

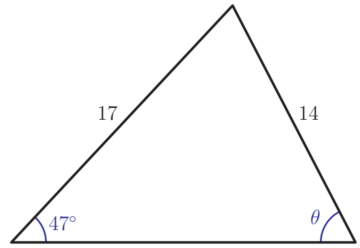
See section 9A on solving trigonometric equations.

Note, however, that because the given angle is 67°, the solution 130.7° for θ is impossible: the three angles of the triangle must add up to 180°, but 67 + 130.7 = 190.7 is already greater than 180. All other solutions of $\sin\theta = 0.758$ are outside the interval $]0^\circ, 180^\circ[$, so cannot be angles of a triangle. Therefore, in Worked example 10.3, there is only one possible value for the angle θ.

The next example shows that this is not always the case.

Worked example 10.4

Find the size of the angle marked θ, giving your answer to the nearest degree.



Use the sine rule with the two given sides.

$$\frac{17}{\sin\theta} = \frac{14}{\sin 47°}$$

$$\Rightarrow \sin\theta = \frac{17\sin 47°}{14} = 0.888$$

Find the two possible values of θ in $]0°, 180°[$.

$$\sin^{-1} 0.888 = 62.6°$$

$$\Rightarrow \theta = 62.6° \text{ or } 180° - 62.6° = 117.4°$$

Check whether each solution is possible: do the two known angles add up to less than 180°?

$$62.6 + 47 = 109.6 < 180$$

$$117.4 + 47 = 164.4 < 180$$

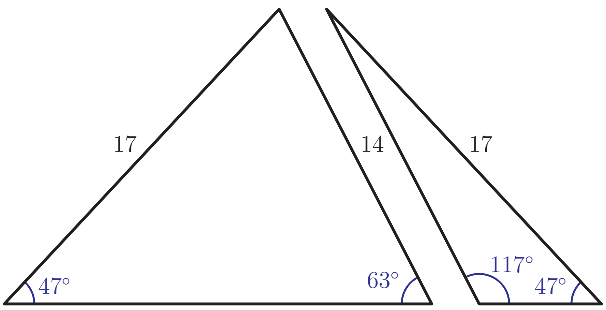
Both solutions are possible.

$$\therefore \theta = 63° \text{ or } 117°$$

EXAM HINT

In the examination, a question will often alert you to look for two possible answers, or instruct you which one to choose, for example by specifying that θ is obtuse. However, if the question gives no clue, you should always check whether the second solution is possible by finding the sum of the known angles.

The diagram below shows the two possible triangles for Worked example 10.4. In both triangles, the side of length 14 is opposite the angle 47°, with another side having length 17, which could be opposite an angle of either 63° or 117°. If these two triangles are placed adjacent to each other, they would form an isosceles triangle with base angles 47° and equal sides of length 17.



Exercise 10B

1. Find the lengths of the sides marked with letters.

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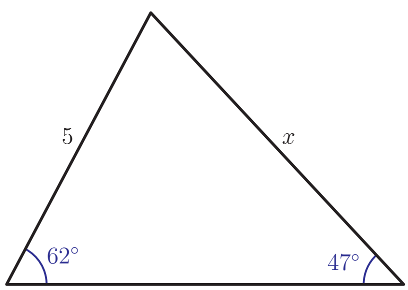
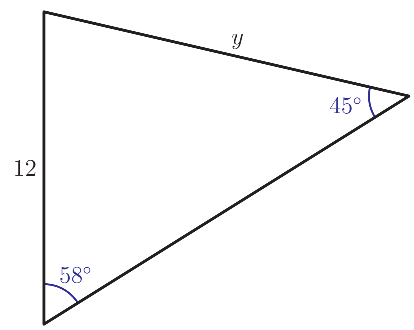
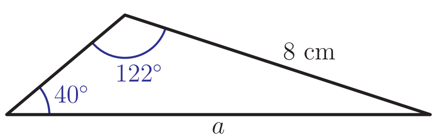


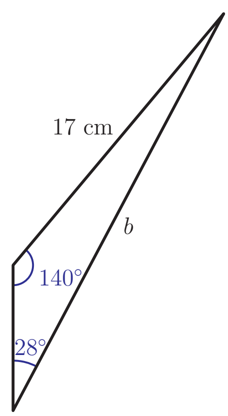
(ii)



(b) (i)

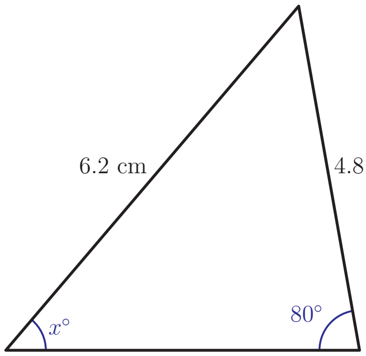


(ii)

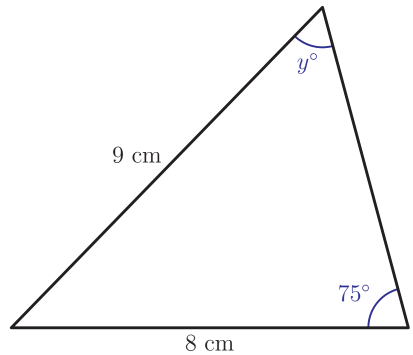


2. In each triangle, find the size of the angle marked with a letter, checking whether there is more than one solution.

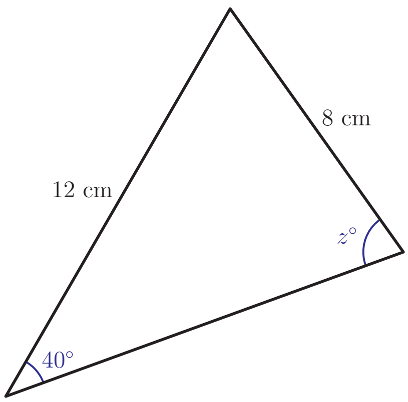
(a) (i)



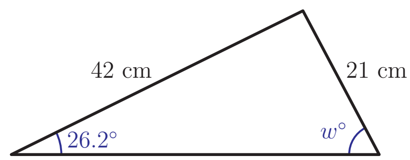
(ii)



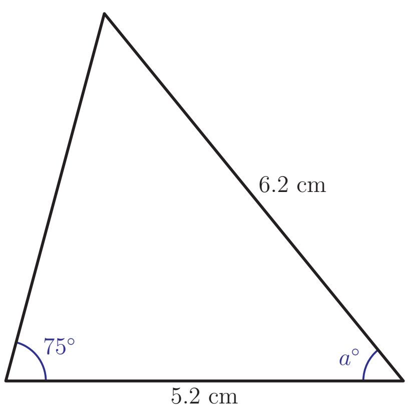
(b) (i)



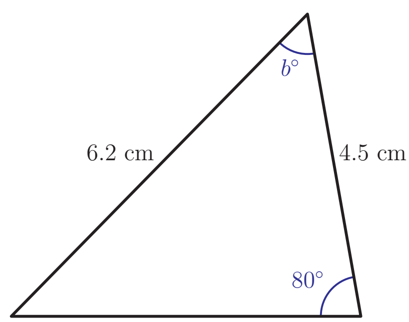
(ii)



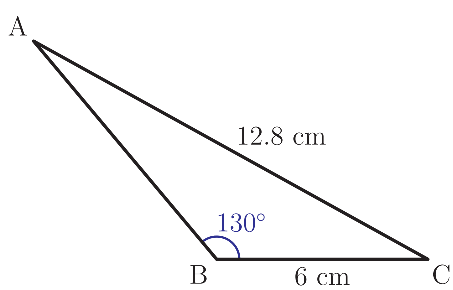
(c) (i)



(ii)

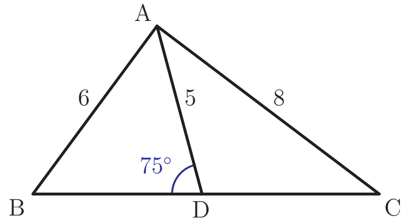


3. Find all the unknown sides and angles in triangle ABC.



4. In triangle ABC, $AB = 6$ cm, $BC = 8$ cm, $A\hat{C}B = 35°$. Show that there are two possible triangles with these measurements, and find the remaining side and angles for each. *[4 marks]*

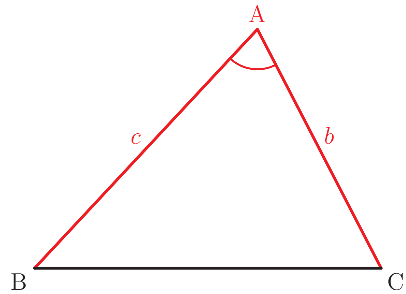
5. In the triangle shown in the diagram alongside, $AB = 6$, $AC = 8$, $AD = 5$ and $A\hat{D}B = 75°$. Find the length of the side [BC]. *[5 marks]*

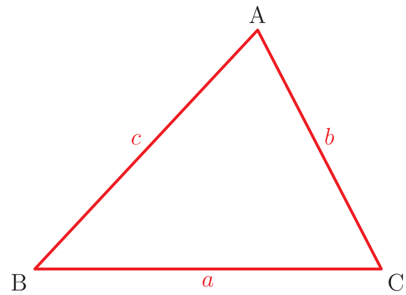


6. A balloon is tethered to a peg in the ground by a 20 m string, which makes an angle of 72° to the horizontal. An observer notes that the angle of elevation from him to the balloon is 41° and his angle of depression to the peg is 10°. Find the horizontal distance of the observer from the peg. *[6 marks]*

7. Show that it is impossible to draw a triangle ABC with $AB = 12$ cm, $AC = 8$ cm and $A\hat{B}C = 47°$. *[5 marks]*

10C The cosine rule



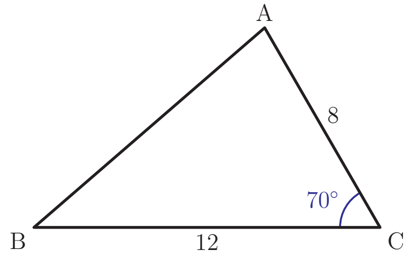


The sine rule allows us to calculate angles and side lengths of a triangle provided that we know one side and its opposite angle, together with one other angle or side length. If we know two sides and the angle between them (the upper diagram alongside), or if we know all three side lengths but no angles (the lower diagram alongside), then we cannot use the sine rule. However, in these cases we can still find the remaining angles and side lengths of the triangle – we just need to apply a different rule, called the cosine rule.

Can we find the length of side [AB] in the triangle below?

The sine rule for this triangle says $\frac{AB}{\sin 70°} = \frac{8}{\sin B} = \frac{12}{\sin A}$,

but since we do not know either of the angles A or B, it is impossible to find the length AB from this formula. However, just like for the sine rule, by creating two right-angled triangles in the original triangle we can derive a different formula for AB.

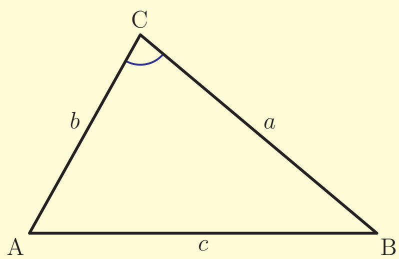


See the Fill-in proof sheet 7 'Cosine rule' on the CD-ROM for details.

KEY POINT 10.4

The cosine rule

$$c^2 = a^2 + b^2 - 2ab\cos C$$



EXAM HINT

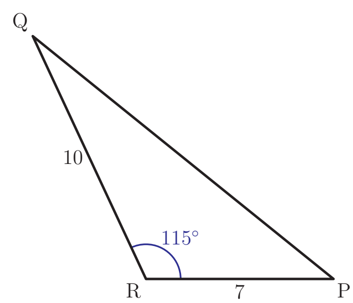
This is the form of the cosine rule given in the Formula booklet, but you can change the names of the variables to whatever you like as long as the angle on the right-hand side (the argument of cosine) corresponds to (i.e. is opposite to) the length on the left-hand side of the equation.

Note that the capital letter C stands for the angle at vertex C, that is, angle AĈB, which lies opposite the side marked c. (Similarly, B would stand for AB̂C opposite b, and A for BÂC opposite a.)

The cosine rule can be used to find the third side of a triangle when we know the other two sides and the angle between them.

Worked example 10.5

Find the length of the side [PQ].



As we are given two sides and the angle between them, we can use the cosine rule.

$PQ^2 = 7^2 + 10^2 - 2 \times 7 \times 10 \times \cos 115°$

$PQ^2 = 208.2$

$\therefore\ PQ = \sqrt{208.2} = 14.4$

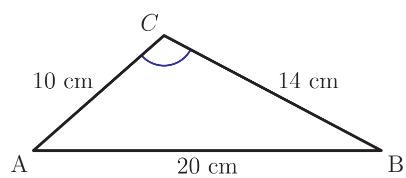
We can also use the cosine rule to find an angle if we know all three sides of a triangle. To do this, we need to rearrange the formula.

KEY POINT 10.5

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Worked example 10.6

Find the size of the angle $\hat{ACB}$ correct to the nearest degree.



Apply the rearranged cosine rule.

$$\cos C = \frac{14^2 + 10^2 - 20^2}{2 \times 14 \times 10}$$
$$= -\frac{104}{280}$$
$$= -0.371$$

Use inverse cosine to find the angle.

$$\therefore C = \cos^{-1}(-0.371) = 112°$$

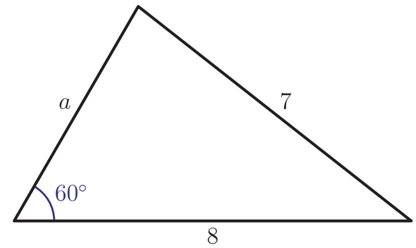
Notice that in both Worked examples 10.5 and 10.6, the angle was obtuse and thus its cosine turned out negative. Note also that, unlike with the sine rule, when using the cosine rule to find an angle there is no second solution: recall that the second solution of $\cos x = k$ is $360° - \cos^{-1} k$, and this will always be greater than 180°, so it cannot be an angle in a triangle.

It is possible to use the cosine rule even when the given angle is not opposite the required side, as illustrated in the next example. This example also reminds you that there are some exact values of trigonometric functions you need to remember.

Exact values of trigonometric functions were covered in section 8D.

Worked example 10.7

Find the possible lengths of the side marked a.



As all three sides feature in the question, we can use the cosine rule. The known angle is opposite the side of length 7.

$$7^2 = a^2 + 8^2 - 2 \times a \times 8 \cos 60°$$

Use the fact that $\cos 60° = \frac{1}{2}$ to simplify.

$$\Leftrightarrow 49 = a^2 + 64 - 8a$$

Recognise that this is a quadratic equation.

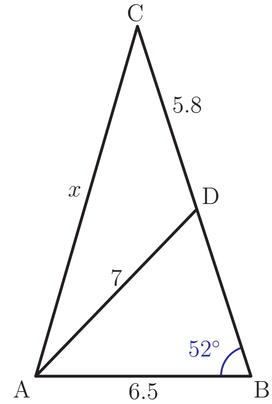
$$\Leftrightarrow \quad a^2 - 8a + 15 = 0$$
$$\Leftrightarrow (a-3)(a-5) = 0$$
$$\Leftrightarrow a = 3 \text{ or } 5$$

It is also possible to answer this question by using the sine rule twice, first to find the angle opposite the side of length 8, and then to find side a. Try this to see if you arrive at the same answers.

The next example illustrates how to select which of the sine rule or cosine rule to use. For both rules, we need to know three measurements in a triangle to find a fourth one.

Worked example 10.8

In the triangle shown in the diagram, AB = 6.5 cm, AD = 7 cm, CD = 5.8 cm, $A\hat{B}C = 52°$ and AC = x cm. Find the value of x correct to one decimal place.



The only triangle in which we know three measurements is ABD. We know two side lengths and an angle opposite one of these sides, so we can use the sine rule to find $A\hat{D}B$.

Sine rule in triangle ABD:

let $A\hat{D}B = \theta$; then

$$\frac{6.5}{\sin\theta} = \frac{7}{\sin 52°}$$

$$\Rightarrow \sin\theta = \frac{6.5\sin 52°}{7} = 0.7317$$

$\sin^{-1} 0.7317 = 47.0°$

Are there two possible solutions?

$180 - 47 = 133$ but $133 + 52 > 180$, so there is only one solution, $\theta = 47°$

In triangle ADC, we know two sides and want to find the third. If we knew $A\hat{D}C$, we could use the cosine rule, but this angle can be found easily.

$A\hat{D}C = 180° - 47° = 133°$

Now we can now apply the cosine rule.

Cosine rule in triangle ADC:

$x^2 = 7^2 + 5.8^2 - 2 \times 7 \times 5.8 \cos 133°$

$x^2 = 137.99$

$x = \sqrt{137.99} = 11.7$ cm

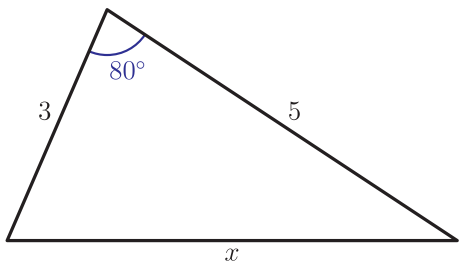
EXAM HINT

In a question like this where the solution has several steps, it is important not to round intermediate calculation results. Use the memory or ANS button on your calculator and round answers only at the end. You do not, however, have to write down all the digits on your calculator display in the intermediate steps of working.

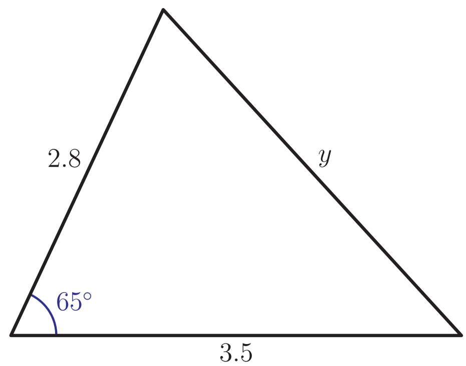
Exercise 10C

1. Find the lengths of the sides marked with letters.

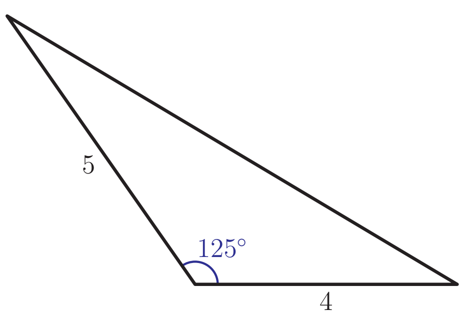
(a) (i)



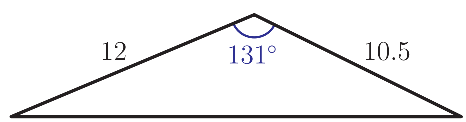
(ii)



(b) (i)

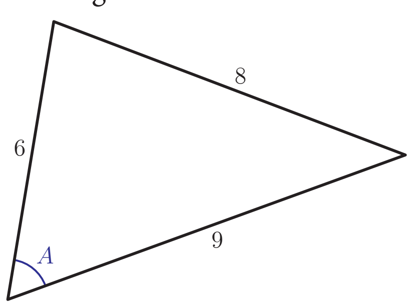


(ii)

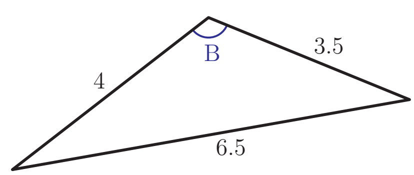


2. Find the angles marked with letters.

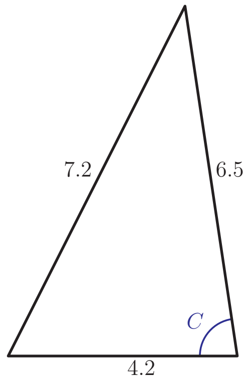
(a) (i)



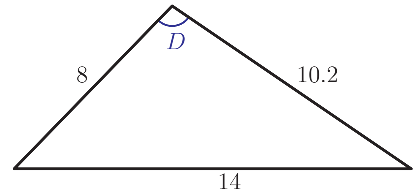
(ii)



(b) (i)



(ii)

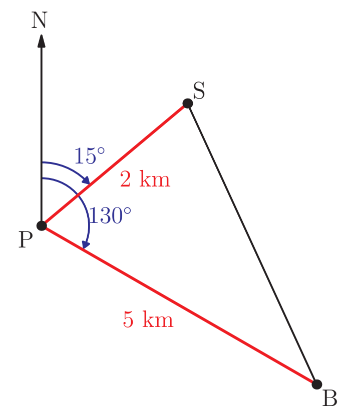


3. (i) Triangle PQR has sides PQ = 8 cm, QR = 12 cm and RP = 7 cm. Find the size of the largest angle.

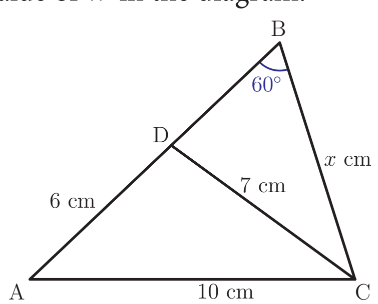
(ii) Triangle ABC has sides AB = 4.5 cm, BC = 6.2 cm and CA = 3.7 cm. Find the size of the smallest angle.

4. Ship S is 2 km from port P on a bearing of 15°, and boat B is 5 km from the port on a bearing of 130°, as shown in the diagram. Find the distance between the ship and the boat.

[6 marks]



5. Find the value of x in the diagram.

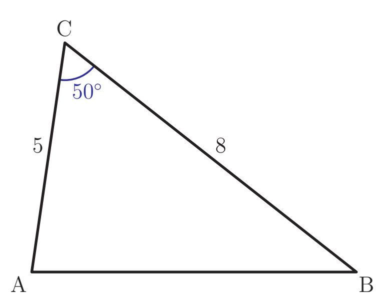


[6 marks]

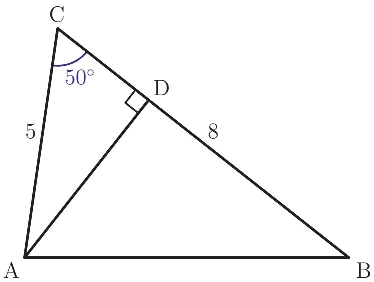
6. In triangle ABC, $AB = (x-3)\text{cm}$, $BC = (x+3)\text{cm}$, $AC = 8\text{ cm}$ and $B\hat{A}C = 60°$. Find the value of x. *[6 marks]*

7. In triangle KLM, KL = 4, LM = 7 and $L\hat{K}M = 45°$. Find the exact length KM. *[6 marks]*

10D Area of a triangle



We have seen how to calculate side lengths and angles of a triangle; another quantity that we might be interested in finding is the area of the triangle. The formula $\frac{1}{2}\text{base} \times \text{perpendicular height}$ should be familiar, and we can use this to find the area of the triangle shown in the diagram.



In order to calculate the area, we need to find a height of the triangle. For instance, we could draw the line (AD) perpendicular to (BC), as in the diagram alongside.

Then $AD = 5\sin 50°$, so the area of the triangle is

$$\frac{1}{2}BC \times AD = \frac{1}{2} \times 8 \times 5\sin 50° = 15.3\text{ cm}^2$$

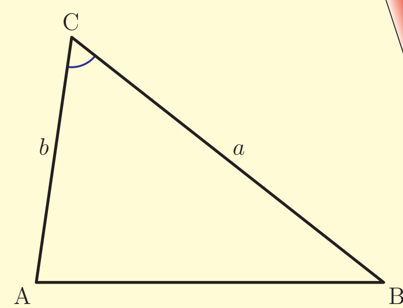
We can use the same method for any triangle, and hence obtain a general formula for the area.

There is also a formula for the area of a triangle involving its three sides; it is called *Heron's formula.*

KEY POINT 10.6

The area of a triangle is given by

$$\text{Area} = \frac{1}{2}ab\sin C$$

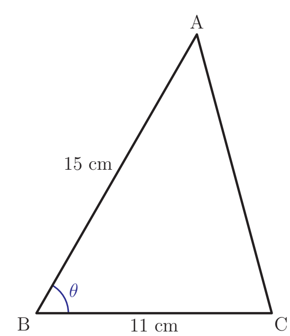


As well as calculating the area given two sides and the angle between them, we could also be asked to find the angle (or one of the side lengths) given the area.

Worked example 10.9

The area of the triangle in the diagram is 52 cm^2.

Find the two possible values of $A\hat{B}C$, correct to one decimal place.



We can directly use the formula from Key point 10.6.

$$\frac{1}{2} \times 11 \times 15\sin\theta = 52$$

$$\Rightarrow \quad \sin\theta = \frac{2 \times 52}{11 \times 15} = 0.6303$$

$$\sin^{-1} 0.6303 = 39.07°$$

$$\therefore\ \theta = 39.1° \text{ or } 180° - 39.1° = 141° \text{ (3SF)}$$

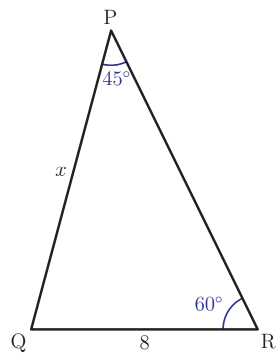
In the next example, we use the sine rule to calculate the information needed, and then apply the formula for the area of the triangle.

Worked example 10.10

Triangle PQR is shown in the diagram.

(a) Calculate the exact value of x.

(b) Find the area of the triangle.



Since we know two angles and a side that is opposite one of the angles, we can use the sine rule.

(a) $\dfrac{8}{\sin 45°} = \dfrac{x}{\sin 60°}$

We are asked to find the *exact* value of x, so use the exact values for $\sin 45°$ and $\sin 60°$.

$\Rightarrow \dfrac{8}{\frac{\sqrt{2}}{2}} = \dfrac{x}{\frac{\sqrt{3}}{2}}$

$\Rightarrow \dfrac{16}{\sqrt{2}} = \dfrac{2x}{\sqrt{3}}$

$\Rightarrow x = \dfrac{16\sqrt{3}}{2\sqrt{2}} = 4\sqrt{6}$

To use the formula for the area of the triangle, we need $P\hat{Q}R$.

(b) $P\hat{Q}R = 180° - 60° - 45° = 75°$

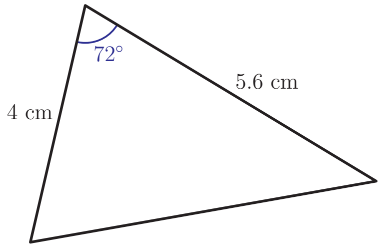
$\text{Area} = \dfrac{1}{2}\left(8 \times 4\sqrt{6}\right)\sin 75°$

$= 37.9\ (3\text{SF})$

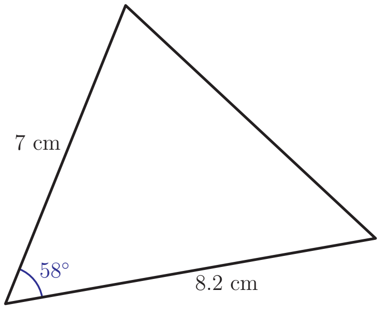
Exercise 10D

1. Calculate the areas of these triangles.

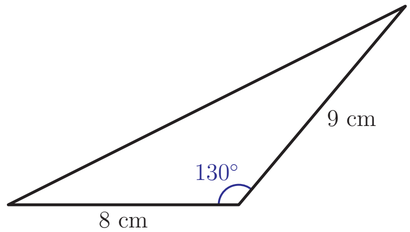
(a) (i)



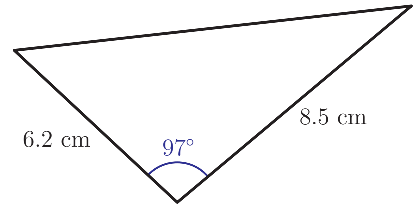
(ii)



(b) (i)

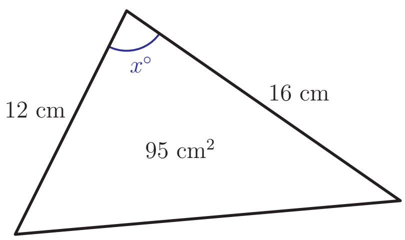


(ii)

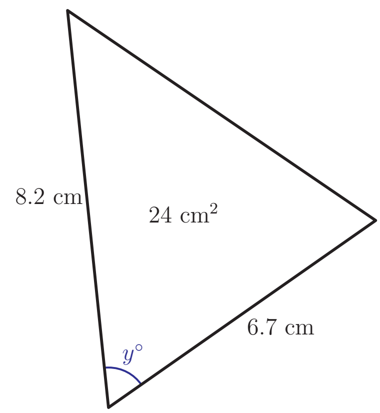


2. Each triangle has the area shown. Find two possible values for each marked angle.

(i)



(ii)



3. In triangle LMN, LM = 12 cm, MN = 7 cm and $\text{L}\hat{\text{M}}\text{N} = 135°$. Find LN and the area of the triangle. *[6 marks]*

4. In triangle PQR, PQ = 8 cm, RQ = 7 cm and $\text{R}\hat{\text{P}}\text{Q} = 60°$. Find the exact difference in areas between the two possible triangles. *[6 marks]*

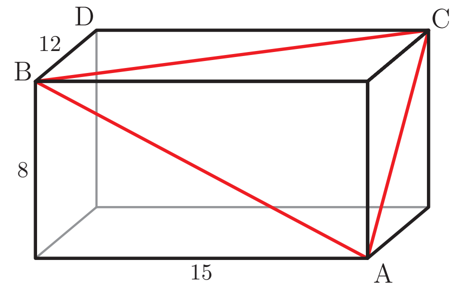
10E Trigonometry in three dimensions

In many applications we need to work with three-dimensional objects. The examples in this section show you how trigonometry can be applied in three dimensions. The general strategy is to identify a suitable triangle and then use one of the rules from the previous sections.

Worked example 10.11

A cuboid has sides of length 8 cm, 12 cm and 15 cm. The diagram shows diagonals of three of the faces.

(a) Find the lengths AB, BC and CA.
(b) Find the size of the angle $A\hat{C}B$.
(c) Calculate the area of the triangle ABC.
(d) Find the length AD.



[AB] is the diagonal of the front face, so it is the hypotenuse of a right-angled triangle with sides 8 and 15.

(a) $AB^2 = 15^2 + 8^2 = 289$

$\therefore AB = \sqrt{298} = 17\ cm$

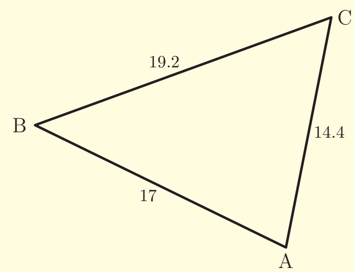
Find BC and CA in a similar way.

$BC = \sqrt{12^2 + 15^2} = 19.2\ cm$

$CA = \sqrt{12^2 + 8^2} = 14.4\ cm$

Now look at triangle ABC. Draw the triangle by itself if it helps.

(b)



We know all three sides and want to find an angle, so we use the cosine rule.

$$\cos C = \frac{14.4^2 + 19.2^2 - 17^2}{2 \times 14.4 \times 19.2} = 0.519$$

$$\therefore C = \cos^{-1} 0.519 = 58.7°$$

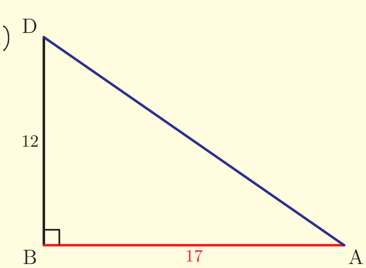
Use the formula for the area, with the angle we found in (b).

(c) $Area = \frac{1}{2}(14.4 \times 19.2)\sin 58.7°$

$= 118\ cm^2$

ABD is a right-angled triangle.

(d)



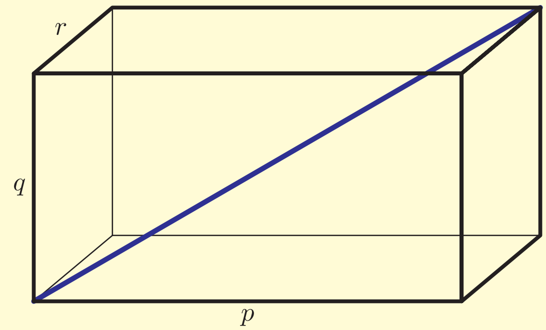
$AD^2 = 12^2 + 17^2$

$\therefore AD = \sqrt{433} = 20.8\,cm$

Part (d) of the previous example illustrates a very useful fact about the diagonal of a cuboid.

KEY POINT 10.7

The diagonal of a cuboid with dimensions p, q and r has length $\sqrt{p^2+q^2+r^2}$



In chapter 11 you will meet vectors, which can also be used to solve three-dimensional problems.

The key to solving many three-dimensional problems is spotting right angles. This is not always easy to do from diagrams that are drawn in perspective, but there are some common configurations to look for; for example, a vertical edge will always meet a horizontal edge at 90°, and the symmetry line of an isosceles triangle is perpendicular to its base.

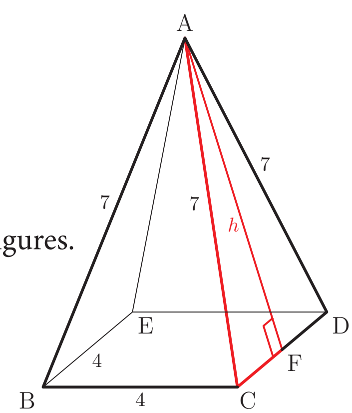
Worked example 10.12

The base of a pyramid is a square of side length 4 cm. The other four faces are isosceles triangles with sides of length 7 cm. The height of one of the side faces is labelled h.

(a) Find the exact value of h.

(b) Find the exact height of the pyramid.

(c) Calculate the volume of the pyramid, correct to 3 significant figures.



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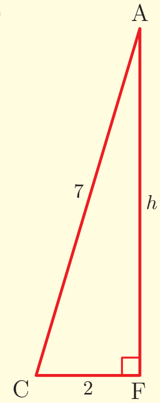
Triangle AFC is right-angled. Draw it separately and label the sides.

(a)



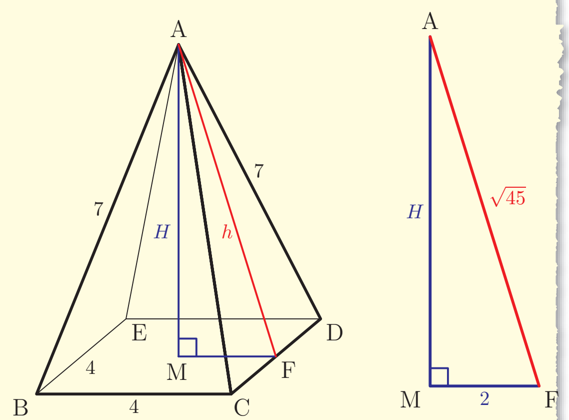
Use Pythagoras' Theorem to find h.

$h^2 = 7^2 - 2^2$

$\therefore h = \sqrt{45}\ cm$

Add the height of the pyramid to the diagram. It is the length of a vertical line [AM] which is perpendicular to the base. The point M is the centre of the base, so MF = 2 cm, and $A\hat{M}F$ is a right angle. Draw the triangle AMF.

(b)



Use Pythagoras' Theorem to find H.

$H^2 = \left(\sqrt{45}\right)^2 - 2^2$

$\therefore H = \sqrt{41}\ cm$

The formula for the volume of a pyramid (from the Formula booklet) is

$\frac{1}{3} \times$ area of base $\times$ vertical height.

(c) $V = \frac{1}{3} \times 4^2 \times \sqrt{41})$

$= 34.1\ cm^3 (3SF)$

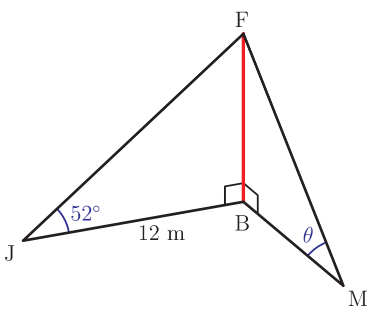
See Prior Learning section V on the CD-ROM for the volumes of three-dimensional shapes.

Exercise 10E

1. Find the length of the diagonal of the cuboid with the following dimensions.

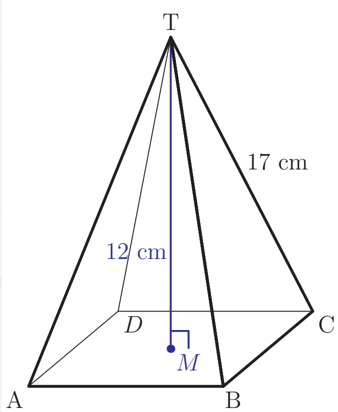
   (i) 3 cm × 5 cm × 10 cm (ii) 4 cm × 4 cm × 8 cm

2. A cuboid has sides of length 12.5 cm, 10 cm and 7.3 cm. It is intersected by a plane passing through vertices A, B and C. Find the angles and the area of triangle ABC. *[8 marks]*



3. John stands 12 m from the base of a flagpole and sees the top of the pole at an angle of elevation of 52°. Marit stands 8 m from the flagpole. At what angle of elevation does she see the top? *[6 marks]*

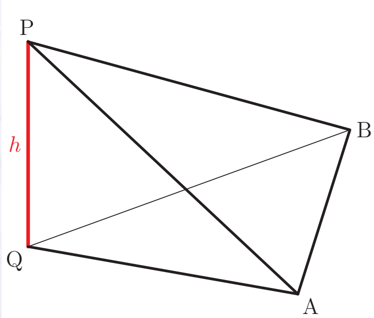
4. A square-based pyramid has a base of side length $a = 8$ cm and height $H = 12$ cm. Find the length l of the sloping side. *[6 marks]*



5. The base of pyramid TABCD is a square. The height of the pyramid is TM = 12 cm, and the length of a sloping side is TC = 17 cm.

   (a) Calculate the length MC.

   (b) Find the length of the side of the base. *[6 marks]*



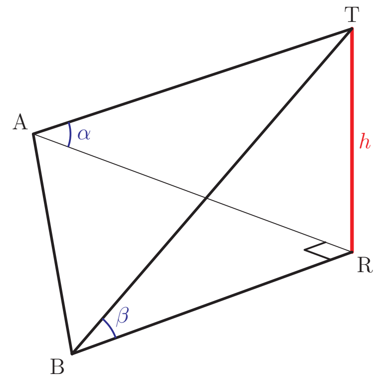
6. The diagram shows a vertical tree PQ and two observers, A and B, standing on horizontal ground. The following quantities are known:

$$AQ = 25 \text{ m},\ Q\hat{A}P = 37°,\ Q\hat{B}P = 42°,\ A\hat{Q}B = 75°$$

   (a) Find the height of the tree, h.

   (b) Find the distance between the two observers. *[8 marks]*

7. Annabel and Bertha are trying to measure the height, h, of a vertical tree RT. They stand on horizontal ground, distance d apart, so that $A\hat{R}B$ is a right angle. From where Annabel is standing, the angle of elevation of the top of the tree is α and from where Bertha is standing, the angle of elevation of the top of the tree is β.



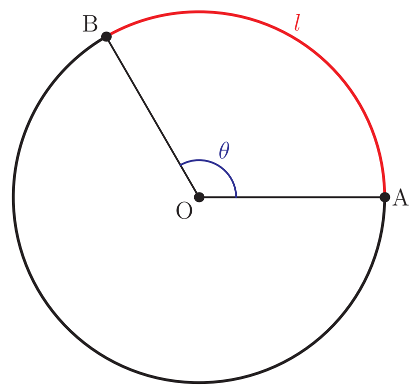
(a) Find expressions for RA and RB in terms of h, α and β, and hence show that

$$h^2\left(\frac{1}{\tan^2\alpha}+\frac{1}{\tan^2\beta}\right)=d^2$$

(b) Given that $d = 26$ m, $\alpha = 45°$ and $\beta = 30°$, find the height of the tree. *[8 marks]*

10F Length of an arc

The diagram shows a circle with centre O and radius r, and points A and B on its circumference. The part of the circumference between points A and B is called an **arc** of the circle. As you can see, there are in fact two arcs: the shorter one is called the *minor arc*, and the longer one the *major arc*. We say that the minor arc AB **subtends** angle θ at the centre of the circle; that is, the angle $A\hat{O}B$ beneath the arc is θ.



You learned in chapter 8 that the measure of angle θ in radians is equal to the ratio of the length of the arc AB to the radius of the circle; in other words, $\theta=\frac{l}{r}$. This gives us a very simple formula for the length of an arc of a circle, if we know the angle it subtends at the centre.

KEY POINT 10.8

The length of an arc of a circle is

$$l = r\theta$$

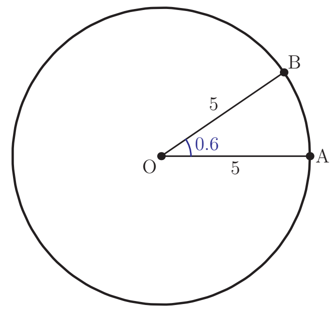
where r is the radius of the circle and θ is the angle subtended at the centre, measured in radians.

Worked example 10.13

Arc AB of a circle with radius 5 cm subtends an angle of 0.6 at the centre, as shown in the diagram.

(a) Find the length of the minor arc AB.

(b) Find the length of the major arc AB.



Use the formula for the length of an arc.

(a) $l = r\theta$

$= 5 \times 0.6$

$= 3 \text{ cm}$

The angle subtended by the major arc is equal to a full turn (2π radians) minus the smaller angle.

(b) $\theta_1 = 2\pi - 0.6$

$= 5.683$

$l = r\theta_1$

$= 5 \times 5.683$

$= 28.4 \text{ cm} (3 \text{ SF})$

We could have done part (b) differently, by finding the length of the whole circumference and then taking away the minor arc: the circumference is $2\pi r = 2\pi \times 5 = 31.42$, so the length of the major arc is $31.42 - 3 = 28.4$ cm (3 SF).

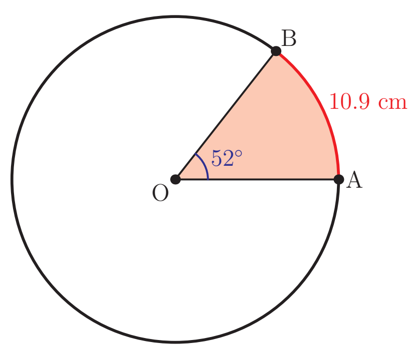
If the angle is given in degrees, we must convert to radians before using the formula for arc length.

Worked example 10.14

Two points A and B lie on the circumference of a circle of radius r cm. The arc AB has length 10.9 cm and subtends an angle of 52° at the centre of the circle.

(a) Find the value of r.

(b) Calculate the perimeter of the shaded region.



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We know the arc length and the angle, so we can find the radius.

(a) $l = r\theta \Rightarrow r = \dfrac{l}{\theta}$

Remember that in order to use the arc length formula, the angle must be in radians.

$$\theta = 52 \times \frac{\pi}{180} = 0.908$$

$$\therefore r = \frac{10.9}{0.908} = 12.0 \text{ cm}$$

The perimeter is made up of two radii and the arc length.

(b) $p = 2r + l$

$= 2 \times 12.0 + 10.9$

$= 34.9$ cm

Exercise 10F

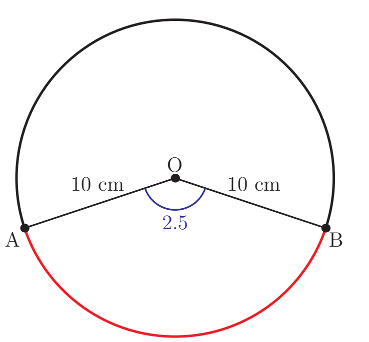
1. Calculate the length of the minor arc subtending an angle of θ radians at the centre of the circle of radius r cm.

(i) $\theta = 1.2,\ r = 6.5$ (ii) $\theta = 0.4,\ r = 4.5$

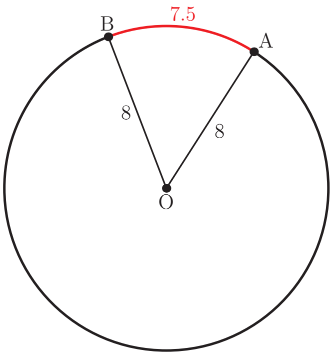
2. Points A and B lie on the circumference of a circle with centre O and radius r cm. Angle $A\hat{O}B$ is θ radians. Calculate the length of the major arc AB.

(i) $r = 15,\ \theta = 0.8$ (ii) $r = 1.4,\ \theta = 1.4$

3. Calculate the length of the minor arc AB in the diagram below.



[4 marks]



4. In the diagram alongside, the radius of the circle is 8 cm and the length of the minor arc AB is 7.5 cm. Calculate the size of the angle $A\hat{O}B$

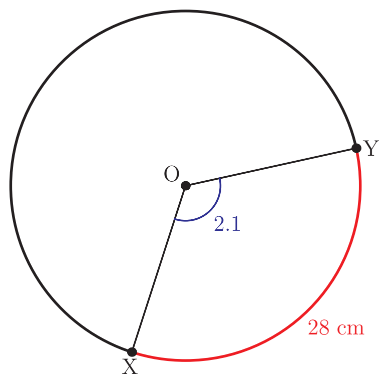
(a) in radians

(b) in degrees. *[5 marks]*

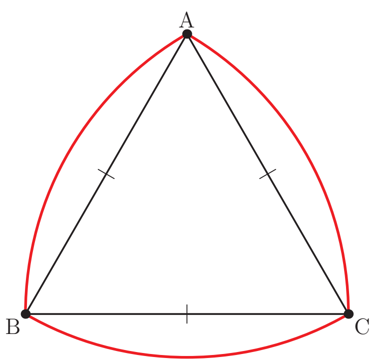
5. Points M and N lie on the circumference of a circle with centre C and radius 4 cm. The length of the *major* arc MN is 15 cm. Calculate the size of the *smaller* angle $M\hat{C}N$. *[4 marks]*

6. Points P and Q lie on the circumference of a circle with centre O. The length of the minor arc PQ is 12 cm and $P\hat{O}Q = 1.6$. Find the radius of the circle. *[4 marks]*

7. In the diagram below, the length of the major arc XY is 28 cm. Find the radius of the circle.

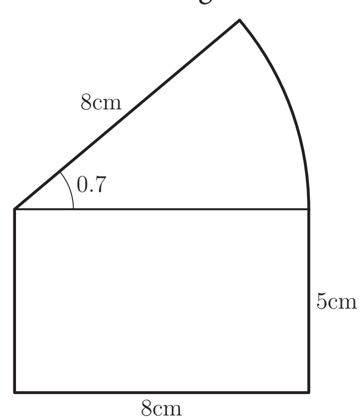


[4 marks]

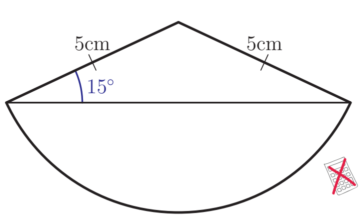


8. The diagram alongside shows an equilateral triangle ABC with side length $a = 5$ cm. A figure (outlined in red) is made up of arcs of three circles whose centres are at the vertices of the triangle. Calculate the perimeter of the figure. *[5 marks]*

9. Calculate the perimeter of the figure shown in the diagram below.



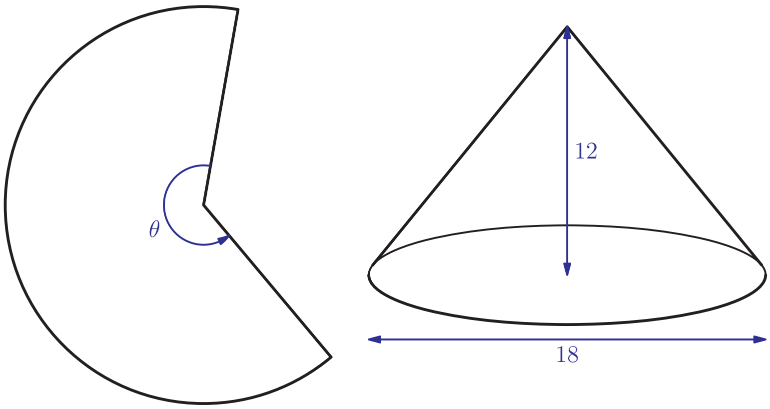
[6 marks]



10. Find the exact perimeter of the figure shown in the diagram alongside. *[6 marks]*

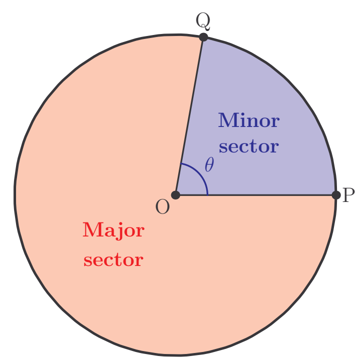
11. A sector of a circle has perimeter $p = 12$ cm and angle $\theta = 0.4$ at the centre. Find the radius of the circle. *[5 marks]*

12. A cone is made by rolling a piece of paper shown in the diagram below.



If the cone is to have height 12 cm and base diameter 18 cm, find the size of the angle marked θ. *[6 marks]*

10G Area of a sector



A **sector** is a part of a circle bounded by two radii and an arc. As with arcs, we can distinguish between a *minor sector* and a *major sector*.

Consider the blue-shaded region. What fraction of the circle does this account for?

Thinking in terms of angles, the fraction should be $\frac{\theta}{2\pi}$.

If we compare areas, then since we know that the total circle area is πr^2, the fraction can also be expressed as $\frac{A}{\pi r^2}$ where A represents the area of the sector.

Both expressions define the same fraction, so they must be equal:

$$\frac{A}{\pi r^2} = \frac{\theta}{2\pi}$$

Rearranging this equation gives a formula for the area of the sector.

KEY POINT 10.9

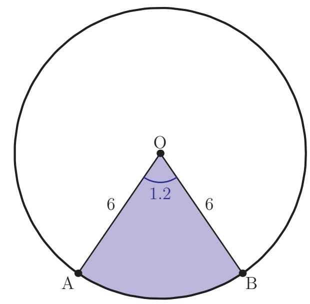
The area of a sector of a circle is

$$A = \frac{1}{2} r^2 \theta$$

where r is the radius of the circle and θ is the angle subtended at the centre, measured in radians.

Worked example 10.15

The diagram shows a circle with centre O and radius 6 cm, and two points A and B on its circumference such that $A\hat{O}B = 1.2$. Find the area of the minor sector AOB.



Use the formula for the area of a sector.

$$A = \frac{1}{2}r^2\theta$$
$$= \frac{1}{2} \times 6^2 \times 1.2$$
$$= 21.6 \text{ cm}^2$$

We may have to use the formulas for arc length and area in reverse.

Worked example 10.16

A sector of a circle has perimeter $p = 12$ cm and angle $\theta = 50°$ at the centre. Find the area of the sector.

To find the area we can use $A = \frac{1}{2}r^2\theta$, but first we need to find r.

We are given the perimeter, which is the sum of the arc length and two radii. We can use this equation to find r.

$$p = r\theta + 2r$$

Remember that the angle needs to be in radians.

and,

$$\theta = 50 \times \frac{\pi}{180} = 0.873$$

Substitute the values into the formula for perimeter to find r.

So,

$$12 = 0.873r + 2r = 2.873r$$
$$\Leftrightarrow r = \frac{12}{2.873} = 4.177$$

Substitute θ and r into the formula for sector area.

$$\therefore \quad A = \frac{1}{2}(4.177)^2(0.873)$$
$$= 7.62 \text{ cm}^2 \text{ (3SF)}$$

Exercise 10G

1. Points M and N lie on the circumference of a circle with centre O and radius r cm, and $M\hat{O}N = \alpha$. Calculate the area of the minor sector MON if

   (i) $r = 5,\ \alpha = 1.3$ (ii) $r = 0.4,\ \alpha = 0.9$

2. Points A and B lie on the circumference of a circle with centre C and radius r cm. The size of the angle $A\hat{C}B$ is θ radians. Calculate the area of the major sector ACB if

   (i) $r = 13,\ \theta = 0.8$ (ii) $r = 1.4,\ \theta = 1.4$

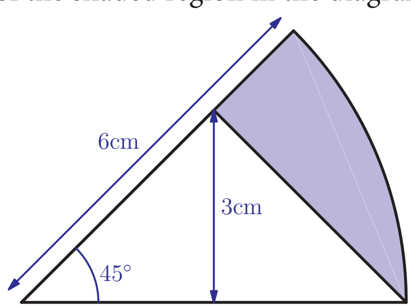
3. A circle has centre O and radius 10 cm. Points A and B lie on the circumference so that the area of the minor sector AOB is 40 cm^2. Calculate the size of the acute angle $A\hat{O}B$. *[5 marks]*

4. Points P and Q lie on the circumference of a circle with radius 21 cm. The area of the *major* sector POQ is 744 cm^2. Find the size of the *smaller* angle $P\hat{O}Q$ in *degrees*. *[5 marks]*

5. A sector of a circle with angle 1.2 radians has area 54 cm^2. Find the radius of the circle. *[4 marks]*

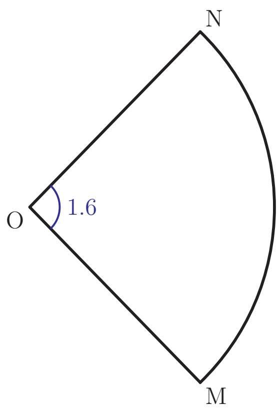
6. A sector of a circle with angle 162° has area 180 cm^2. Find the radius of the circle. *[4 marks]*

7. Find the area of the shaded region in the diagram below.



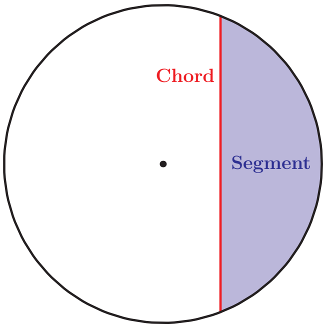
[6 marks]

8. The perimeter of the sector MON shown in the diagram alongside is 28 cm. Find its area. *[5 marks]*



9. A sector of a circle has perimeter 7 cm and area 3 cm^2. Find the possible values of the radius of the circle. *[6 marks]*

10. Points P and Q lie on the circumference of a circle with centre O and radius 5 cm. The difference between the areas of the major sector POQ and the minor sector POQ is 15 cm^2. Find the size of the angle $P\hat{O}Q$. *[5 marks]*



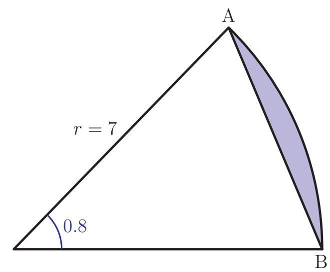
10H Triangles and circles

Besides arcs and sectors, there are two other important parts of circles that you need to know about: **chords** and **segments**.

Worked example 10.17

The diagram shows a sector of a circle with radius 7 cm; the angle at the centre is 0.8 radians. Find

(a) the perimeter of the shaded region
(b) the area of the shaded region.



The perimeter is made up of the arc AB and the chord [AB].

(a) $p = arc + chord$

The formula for the arc length is $l = r\theta$.

$l = 7 \times 0.8$
$= 5.6\ cm$

The chord [AB] is the third side of the isosceles triangle ABC. We can split ABC into two identical right-angled triangles with base $\frac{AB}{2}$ and hypotenuse 7; therefore $\frac{AB}{2} = 7\sin\frac{0.8}{2}$. (Alternatively, you can use the cosine rule on triangle ABC to find AB.)

$AB = 2 \times 7 \times \sin 0.4$
$= 5.45\ cm$

Now we can calculate the perimeter.

$\therefore p = 5.6 + 5.45$
$= 11.1\ cm$

If we calculate the area of the sector and then subtract the area of triangle ABC, we are left with the area of the segment.

(b) $A = sector - triangle$

continued . . .

The formula for the area of a sector is $\frac{1}{2}r^2\theta$.

$sector = \frac{1}{2}(7^2 \times 0.8)$

$c = 19.6\,\text{cm}^2$

The formula for the area of a triangle is $\frac{1}{2}ab\sin C$.

$triangle = \frac{1}{2}(7 \times 7)\sin 0.8$

$= 17.58\,\text{cm}^2$

We can now find the area of the segment.

$\therefore A = 19.6 - 17.58$

$= 2.02\,\text{cm}^2$

Following the method used in Worked example 10.17, we can derive general formulas for the length of a chord and the area of a segment.

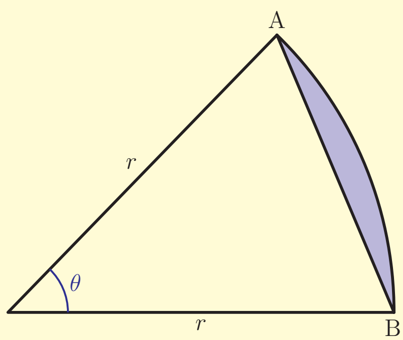
KEY POINT 10.10

The length of a chord of a circle is given by

$$AB = 2r\sin\left(\frac{\theta}{2}\right)$$

and the area of the shaded segment is

$$\frac{1}{2}r^2(\theta - \sin\theta)$$



where the angle θ is measured in radians and is the angle subtended at the centre.

EXAM HINT

These formulae are not given in the Formula booklet, so you need to know how to derive them.

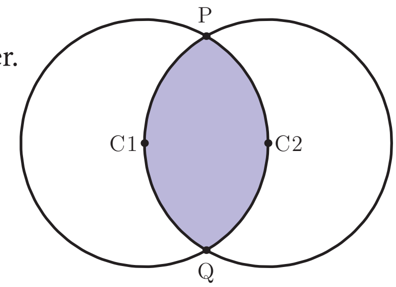
The next example shows how we can solve more complex geometry problems by splitting up the figure into basic shapes such as triangles and sectors.

Worked example 10.18

The diagram shows two identical circles of radius 12 such that the centre of one circle is on the circumference of the other.

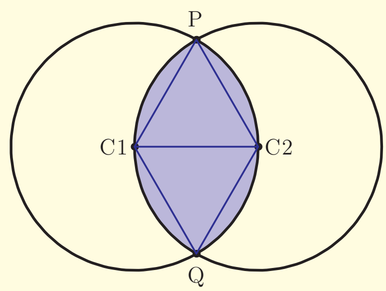
(a) Find the exact size of angle $P\hat{C}_1Q$ in radians.

(b) Calculate the exact area of the shaded region.



The only thing we know is the radius of the circles, so draw in all the lengths which are equal to the radius.

(a)

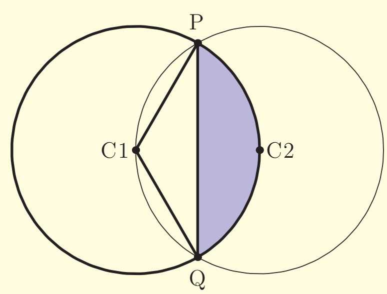


The lengths C_1P, C_2P *and* C_1C_2 are all equal to the radius of the circle, so triangle PC_1C_2 is equilateral; similarly for triangle QC_1C_2.

$$P\hat{C}_1C_2 = \frac{\pi}{3} = Q\hat{C}_1C_2$$

$$\therefore\ P\hat{C}_1Q = \frac{2\pi}{3}$$

The shaded area is made up of two equal segments, one for each circle, each with angle $\frac{2\pi}{3}$ at the centre.



We can find the area of one segment by using the formula. Since we are asked for the exact area, we should use the exact value of $\sin\left(\frac{2\pi}{3}\right)$.

(b) Area of one segment

$$= \frac{1}{2}\times 12^2\left(\frac{2\pi}{3} - \sin\left(\frac{2\pi}{3}\right)\right)$$

$$= 72\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$$

$$= 48\pi - 36\sqrt{3}$$

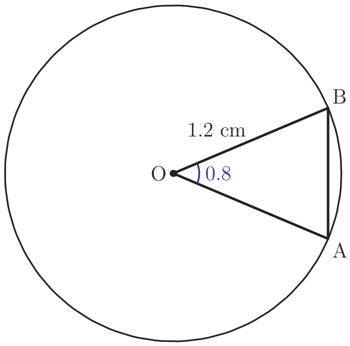
Remember that the shaded area consists of two segments.

$$\therefore \text{shaded area} = 96\pi - 72\sqrt{3}$$

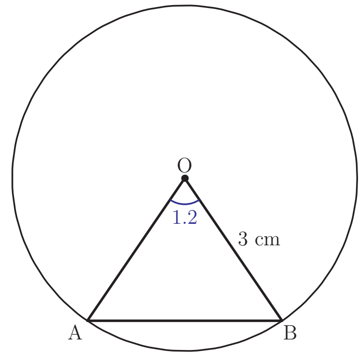
Exercise 10H

1. Find the length of the chord AB in each diagram.

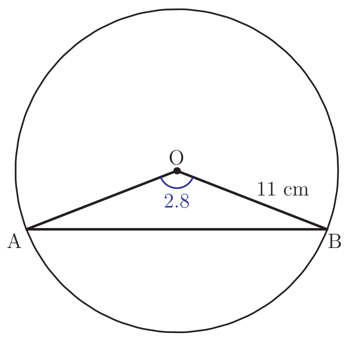
(a) (i)



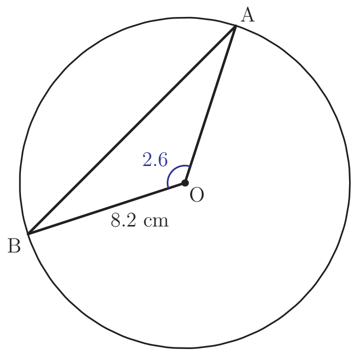
(ii)



(b) (i)



(ii)



2. Find the perimeters of the minor segments from Question 1.

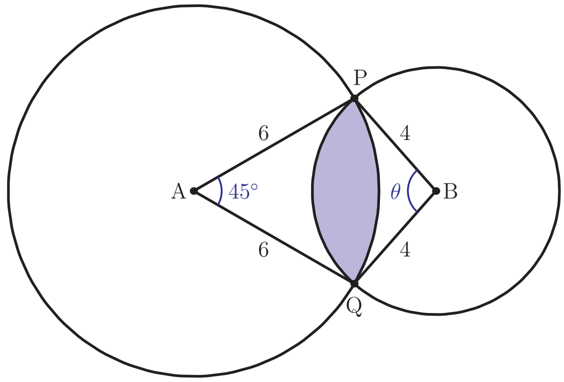
3. Find the areas of the minor segments from Question 1.

4. A circle has centre O and radius 5 cm. Chord PQ subtends angle θ at the centre of the circle.

(a) Write down an expression for the area of the minor segment.

(b) Given that the area of the minor segment is 15 cm^2, find the value of θ. *[6 marks]*

5. Two circles, with centres A and B, intersect at points P and Q. The radii of the circles are 6 cm and 4 cm, and $P\hat{A}Q = 45°$.



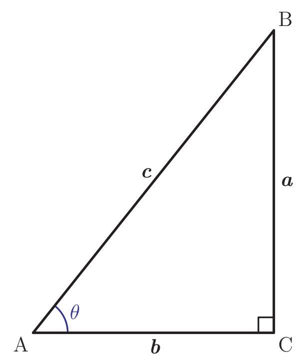
(a) Show that $PQ = 6\sqrt{2-\sqrt{2}}$.

(b) Find the size of $P\hat{B}Q$.

(c) Find the area of the shaded region. *[9 marks]*

Summary

- In a right-angled triangle: $\frac{a}{c} = \sin\theta$; $\frac{b}{c} = \cos\theta$; $\frac{a}{b} = \frac{\sin\theta}{\cos\theta} = \tan\theta$



- The angle of elevation is the angle above the horizontal.
- The angle of depression is the angle below the horizontal.
- To find a side length of a triangle when two angles and a side are given, or to find an angle when two sides and a non-included angle are given, we can use the **sine rule**:

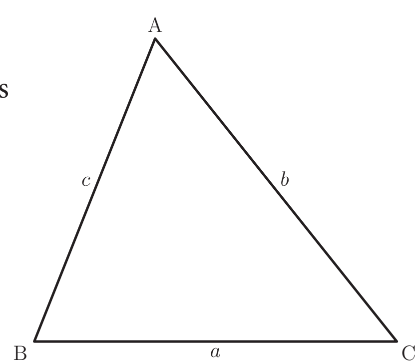
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

- To find a side length of a triangle when two sides and the angle between them are given, or to find an angle when all three sides are given, we can use the **cosine rule**:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Or:

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$



- The diagonal of a cuboid, $p \times q \times r$, has length $\sqrt{p^2 + q^2 + r^2}$.

- An alternative formula for the area of a triangle is:

$$\text{Area} = \frac{1}{2}ab\sin C$$

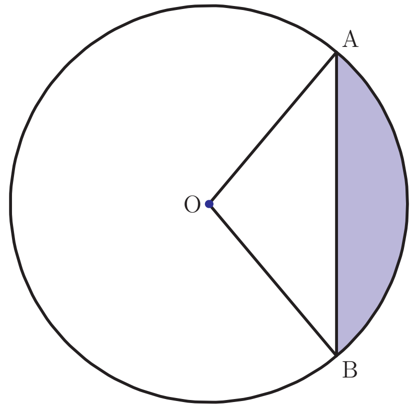
- To solve geometry problems in three dimensions, try to find a suitable triangle and use one of the above formulas or Pythagoras' Theorem. Look out especially for right angles.

- In a circle of radius r, for an angle θ in radians subtended at the centre:
  - arc length $l = r\theta$
  - area of sector $A = \frac{1}{2}r^2\theta$

- Know how to derive the formulae for the length of a chord AB and the area of a segment subtending an angle θ radians at the centre:

$$AB = 2r\sin\left(\frac{\theta}{2}\right)$$

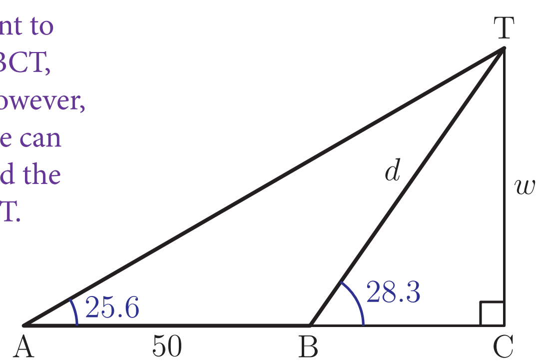
$$\text{Area} = \frac{1}{2}r^2(\theta - \sin\theta)$$



Introductory problem revisited

Two people are trying to measure the width of a river. There is no bridge across the river, but they have instruments for measuring lengths and angles. When they stand 50 m apart on the same side of the river, at points A and B, the person at A measures that the angle between line (AB) and the line from A to the tower on the other side of the river is 25.6°. The person at B finds the corresponding angle to be 28.3°, as shown in the diagram. Use this information to calculate the width of the river.

Draw a diagram of the triangles in the problem. We want to find w. There are two right-angled triangles, ACT and BCT, but we do not know the lengths of any of their sides. However, in triangle ABT we know two angles and one side, so we can calculate the remaining sides. In particular, once we find the length of BT, we will be able to find w from triangle BCT.



In order to use the sine rule in triangle ABT, we need to know the size of the angle opposite [AB], namely $A\hat{T}B$. Since $A\hat{B}T = 180° - 28.3° = 151.7°$, we find that $A\hat{T}B = 180° - 25.6° - 151.7° = 2.7°$.

Now we apply the sine rule:

$$\frac{d}{\sin 25.6°} = \frac{50}{\sin 2.7°}$$

$$\Rightarrow \quad d = \frac{50 \sin 25.6°}{\sin 2.7°}$$

$$= 458.6$$

Finally, use the right-angled triangle BCT to find the width of the river:

$$\begin{aligned} w &= d \sin 28.3° \\ &= 217 \text{ m} (3\ SF) \end{aligned}$$

Mixed examination practice 10

Short questions

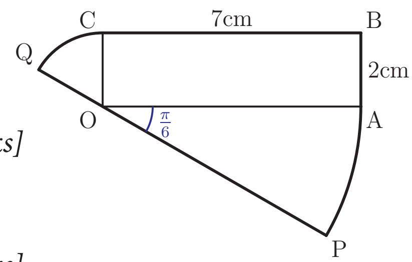
1. In the diagram, OABC is a rectangle with sides 7 cm and 2 cm. PQ is a straight line. AP and CQ are circular arcs, and $A\hat{O}P = \frac{\pi}{6}$.

(a) Write down the size of $C\hat{O}Q$.

(b) Find the area of the whole shape.

(c) Find the perimeter of the whole shape.

[9 marks]

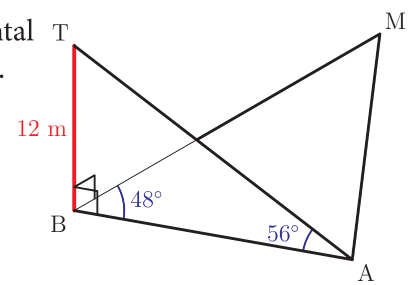


2. A sector has perimeter 36 cm and radius 10 cm. Find its area. *[6 marks]*

3. In triangle ABC, AB = 6.2 cm, CA = 8.7 cm and $A\hat{C}B = 37.5°$. Find the two possible values of $A\hat{B}C$. *[6 marks]*

4. A vertical tree of height 12 m stands on horizontal ground. The bottom of the tree is at the point B. Observer A, standing on the ground, sees the top of the tree at an angle of elevation of 56°.

(a) Find the distance of A from the bottom of the tree.



Another observer, M, stands the same distance away from the tree, with $A\hat{B}M = 48°$.

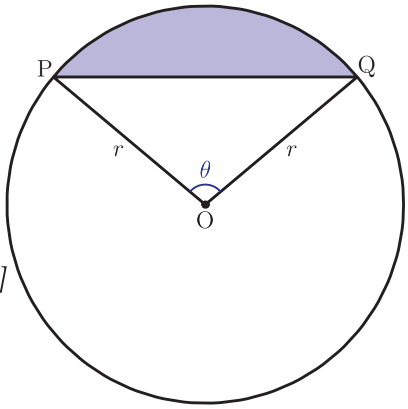
(b) Find the distance AM. *[6 marks]*

5. The diagram shows a circle with centre O and radius $r = 7$ cm. The chord PQ subtends angle $\theta = 1.4$ radians at the centre of the circle.

(a) Find the area of the shaded region.

(b) Find the perimeter of the shaded region.

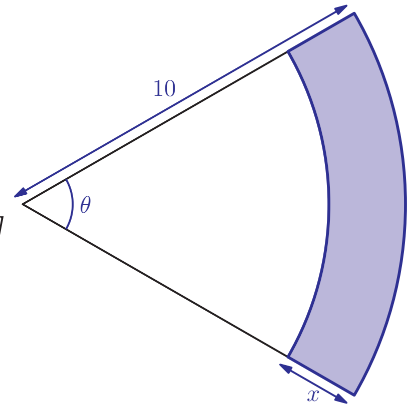
[6 marks]



6. The diagram shows two circular sectors with angle θ at the centre. The radius of the larger sector is 10 cm, the radius of the smaller sector is x cm shorter.

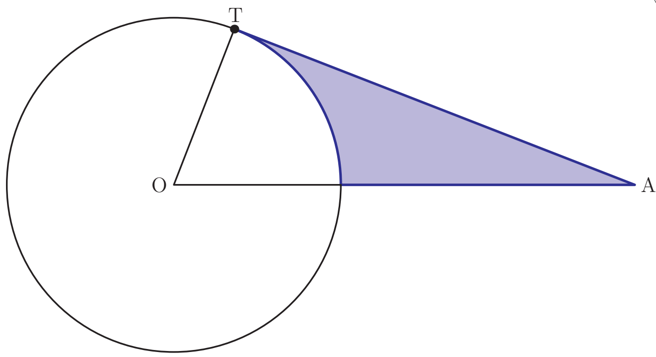
(a) Show that the area of the shaded region is given by $\frac{x(20-x)\theta}{2}$.

(b) If $\theta = 1.2$, find the value of x such that the area of the shaded region is equal to 54.6 cm^2. *[8 marks]*



7. In the diagram, O is the centre of the circle and (AT) is the tangent to the circle at T.

See Prior Learning sections U and W on the CD-ROM for a review of properties of circles and basic trigonometry.



If OA = 12 cm and the circle has a radius of 6 cm, find the area of the shaded region. *[4 marks]*

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8. The diagram shows a triangle and two arcs of circles.

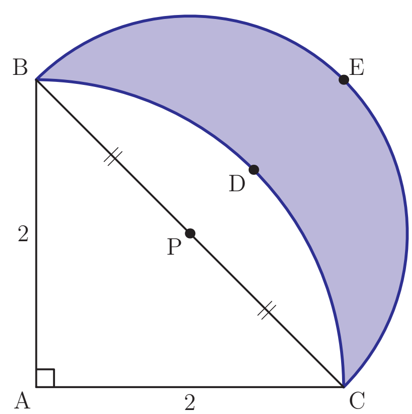
The triangle ABC is a right-angled isosceles triangle, with AB = AC = 2. The point P is the midpoint of [BC].

The arc BDC is part of a circle with centre A.

The arc BEC is part of a circle with centre P.

(a) Calculate the area of the segment BDCP.

(b) Calculate the area of the shaded region BECD.



[6 marks]

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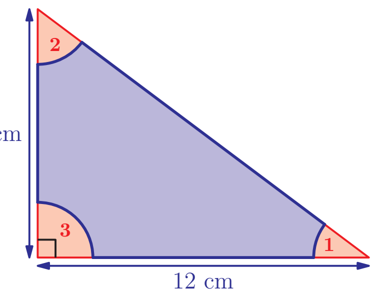
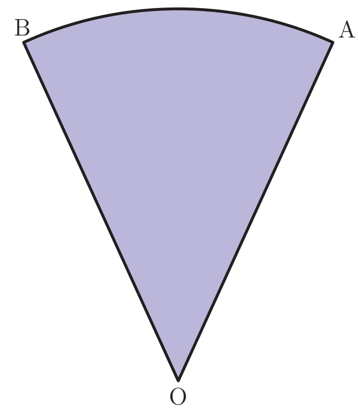
9. A right-angled triangle has sides 12 cm and 9 cm. At each vertex, a sector of radius 2 cm is cut out, as shown in the diagram. The angle of sector 1 is θ radians.

(a) Write down an expression for the area of sector 2 in terms of θ.

(b) Find the remaining area after the triangle has had the corners removed. *[6 marks]*



10. The perimeter of the sector shown in the diagram is 34 cm and its area is 52 cm^2. Find the radius of the sector. *[6 marks]*

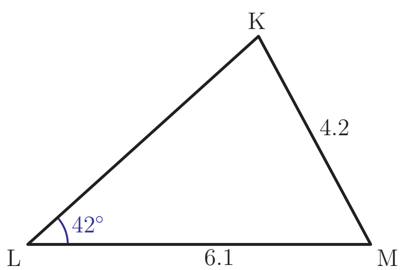


11. In triangle ABC, $\text{AB} = 2\sqrt{3}$, $\text{AC} = 10$ and $\text{B}\hat{\text{A}}\text{C} = 150°$. Find the exact length BC. *[6 marks]*

12. In the obtuse-angled triangle KLM,

LM = 6.1 cm, KM = 4.2 cm and $\text{K}\hat{\text{L}}\text{M} = 42°$.

Find the area of the triangle. *[6 marks]*



13. In triangle ABC, AB = 10 cm, BC = 8 cm and CA = 7 cm.

(a) Find the exact value of $\cos \text{A}\hat{\text{B}}\text{C}$.

(b) Find the exact value of $\sin \text{A}\hat{\text{B}}\text{C}$.

(c) Find the exact value of the area of the triangle. *[8 marks]*

Long questions

1. In triangle ABC, AB = 5, AC = x, and $\text{B}\hat{\text{A}}\text{C} = \theta$; M is the midpoint of the side [AC].

(a) Use the cosine rule to find an expression for MB^2 in terms of x and θ.

(b) Given that BC = MB, show that $\cos\theta = \frac{3x}{20}$.

(c) If $x = 5$, find the value of the angle θ such that MB = BC. *[9 marks]*

2. Two circles have equal radius r and intersect at points S and T. The centres of the circles are A and B, and $A\hat{S}B = 90°$.

(a) Explain why $S\hat{A}T$ is also 90°.

(b) Find the length AB in terms of r.

(c) Find the area of the sector AST.

(d) Find the area of overlap of the two circles. *[10 marks]*

3. The diagram shows a circle with centre O and radius r. Chord AB subtends an angle at the centre of size θ radians. The minor segment and the major sector are shaded.

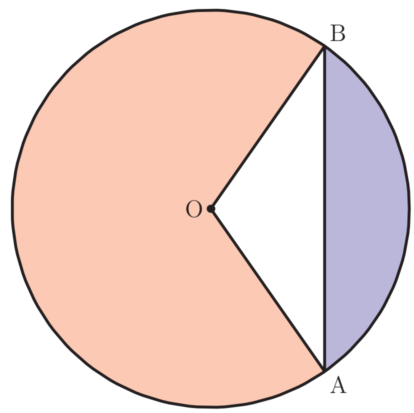
(a) Show that the area of the minor *segment* is $\frac{1}{2}r^2(\theta - \sin\theta)$.

(b) Find the area of the major *sector*.

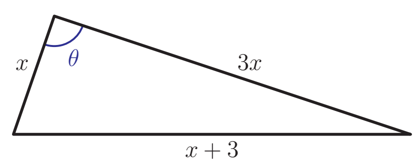
(c) Given that the ratio of the area of the blue region to the area of the pink region is 1:2, show that

$$\sin\theta = \frac{3}{2}\theta - \pi$$

(d) Find the value of θ. *[10 marks]*



4. The area of the triangle shown is 2.21 cm². The length of the shortest side is x cm and the other two sides are $3x$ cm and $(x+3)$ cm.



(a) Using the formula for the area of a triangle, write down an expression for $\sin\theta$ in terms of x.

(b) Using the cosine rule, write down and simplify an expression for $\cos\theta$ in terms of x.

(c) Using your answers to parts (a) and (b), show that,

$$\left(\frac{3x^2 - 2x - 3}{2x^2}\right)^2 = 1 - \left(\frac{4.42}{3x^2}\right)^2$$

(d) Hence find

(i) the possible values of x

(ii) the corresponding values of θ, in *radians*, using your answer to part (b) above. *[10 marks]*

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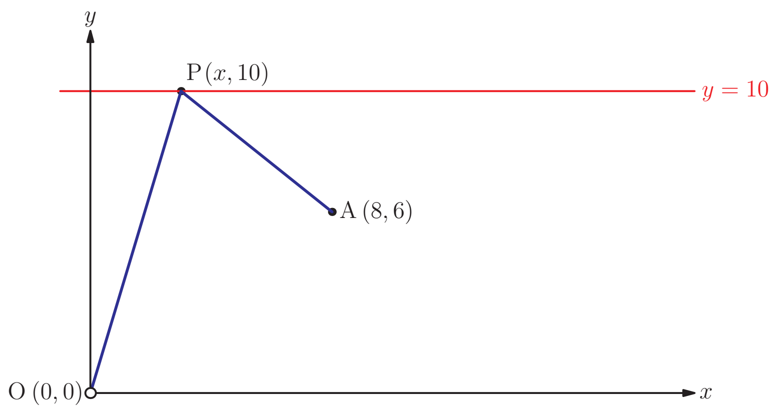
5. In triangle ABC, AB = 5, BC = 10, CA = x and $A\hat{C}B = \theta$.

(a) Show that $x^2 - 20x\cos\theta + 75 = 0$.

(b) Find the range of values of $\cos\theta$ for which the above equation has real solutions for x.

(c) Hence find the set of values of θ for which it is possible to construct triangle ABC with the given measurements. *[8 marks]*

6. In the diagram, the points O(0,0) and A(8,6) are fixed. The angle $O\hat{P}A$ varies as the point $P(x,10)$ moves along the horizontal line $y = 10$.



(a) (i) Show that $AP = \sqrt{x^2 - 16x + 80}$.

(ii) Write down a similar expression for OP in terms of x.

(b) Hence show that

$$\cos O\hat{P}A = \frac{x^2 - 8x + 40}{\sqrt{(x^2 - 16x + 80)(x^2 + 100)}}$$

(c) Find, in degrees, the angle $O\hat{P}A$ when $x = 8$.

(d) Find the positive value of x such that $O\hat{P}A = 60°$.

Let the function f be defined by

$$f(x) = \cos O\hat{P}A = \frac{x^2 - 8x + 40}{\sqrt{(x^2 - 16x + 80)(x^2 + 100)}},\ 0 \le x \le 15$$

(e) Consider the equation $f(x) = 1$.

(i) Explain, in terms of the position of the points O, A and P, why this equation has a solution.

(ii) Find the *exact* solution to the equation. *[17 marks]*

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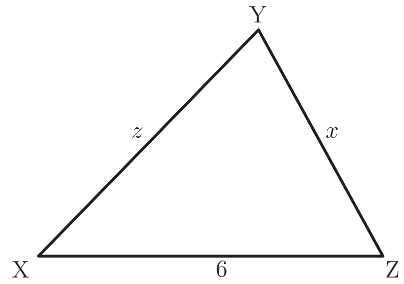
7. (a) Let $y = -16x^2 + 160x - 256$. Given that y has a maximum value, find

(i) the value of x that gives the maximum value of y

(ii) this maximum value of y.

The triangle XYZ has XZ = 6, YZ = x and XY = z, as shown in the diagram. The perimeter of triangle XYZ is 16.



(b) (i) Express z in terms of x.

(ii) Using the cosine rule, express z^2 in terms of x and $\cos Z$.

(iii) Hence show that $\cos Z = \dfrac{5x-16}{3x}$.

Let the area of triangle XYZ be A.

(c) Show that $A^2 = 9x^2 \sin^2 Z$.

(d) Hence show that $A^2 = -16x^2 + 160x - 256$.

(e) (i) Hence write down the maximum area for triangle XYZ.

(ii) What type of triangle is the triangle with maximum area? *[15 marks]*

8. Two circular cogs are connected by a chain as shown in Diagram 1. The radii of the cogs are 3 cm and 8 cm, and the distance between their centres is 25 cm.

Diagram 2 shows the quadrilateral O_1ABO_2. Line O_2P is drawn parallel to AB.

Diagram 1

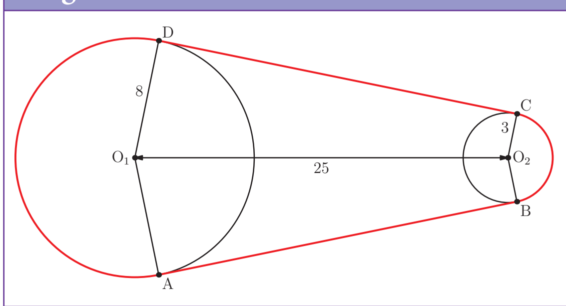
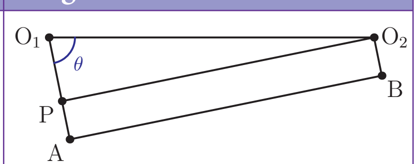


Diagram 2



(a) Write down the size of $O_1\hat{A}B$ in radians, giving a reason for your answer.

(b) Explain why $PO_2 = AB$.

(c) Hence find the length AB.

(d) Find the size of the angle marked θ, giving your answer in radians correct to 4 SF.

(e) Calculate the length of the chain (shown in red in Diagram 1). *[11 marks]*