19 Questions crossing chapters

One of the hardest features of the International Baccalaureate is that examination questions can draw on different areas of the syllabus. This chapter brings together questions of this type, so you may find it very challenging. Indeed, some of the questions are at the highest difficulty level that could be expected to come up in the exam. Do not be disheartened if you cannot finish a question; try to extract as many marks from it as possible – that is how to get a top grade! Although some questions may look unfamiliar and daunting, if you can avoid being intimidated, we hope you will find that they are not impossible. In the long questions you will see that later parts are sometimes easier than earlier ones, particularly after 'show that' parts.

On your first pass through this chapter, you may find it a useful exercise to simply identify which topics each question links together – it is not always obvious! There is an index of what topics are covered in what questions available online (education.cambridge.org/standard).

Some of these questions are on the border of the International Baccalaureate syllabus, but then again, so are some recent examination questions!

Some exam tips

- Remember to use your reading time effectively:
  - Decide which order to do the questions in. In particular, several long questions can be easier than the last few short questions, and are worth more marks!
  - Think about which questions can be done on the calculator.
  - Practise using your reading time on your practice papers.
  - Try to classify which section of the course each question is about.

- Just because you cannot do part (a) of a question does not mean you cannot do later parts. As you have seen, sometimes later parts are easier than earlier parts.
- If you cannot do an early part of a question, you can show how you would have used the answer in later parts; you will still gain marks.
- Look for links between parts of multi-part questions. They often act as hints.
- Plan your time before you go into the exam. Decide whether you work better quickly with lots of time for checking, or working slowly but not having much checking time.
- Do not get distracted if you cannot do some questions; for whatever grade you want, you do not have to get 100%!
- Practise checking your answers; this is not as easy a skill as it seems. Particularly in the calculator paper; you should be able to use your calculator to check your work.
- Scavenge for marks. A blank response is guaranteed to score zero. If you have a sensible idea write it down as some marks may be awarded for what might seem to be a minor point. However, do not waste too much time on a question where you feel uncertain about your method. Leave these questions to the end.

Short questions

1. What is the probability of getting an average of 3 on two rolls of a fair die? *[6 marks]*

2. The sum of the first n terms of an arithmetic sequence is $S_n = 3n + 2n^2$. Find the common difference of the sequence. *[6 marks]*

3. If $f(x) = |x|$, sketch $f'(x)$ for $-2 \le x \le 2$, $x \ne 0$. *[5 marks]*

4. If $\boldsymbol{u} = \begin{pmatrix} x \\ x \\ 1 \end{pmatrix}$ and $\boldsymbol{v} = \begin{pmatrix} x \\ 1 \\ x \end{pmatrix}$, find $\int \boldsymbol{u}.\boldsymbol{v}\, \mathrm{d}x$. *[6 marks]*

5. What is the average value of the first n terms of a geometric progression with first term a and common ratio r? *[4 marks]*

6. Three data items are collected: 3, x^2 and x. Find the smallest possible value of the mean. *[6 marks]*

7. The discrete random variable X has the probability distribution $\mathrm{P}(X = x) = \ln kx$ for $x = 1,2,3,4$. Find the exact value of k. *[6 marks]*

8. If $f(x) = \sin(x)$, give the single transformation which maps $f(x)$ to $g(x) = 1 - \cos(x)$. *[4 marks]*

9. The graph of $y = \ln x$ can be transformed into the graph of $y = \ln kx$ by either a horizontal stretch or a vertical translation.

(a) State the stretch factor of the horizontal stretch.

(b) Find the vertical translation vector. *[4 marks]*

10. The sequence u_n is defined by $u_n = 0.5^n$.

(a) Find the exact value of $\sum_{0}^{10} u_r$

(b) Find the exact value of $\sum_{0}^{10} \ln(u_r)$ *[7 marks]*

11. The functions f and g are defined by $f(x) = 3x + 1$ and $g(x) = ax^2 - x + 5$. Find the value of a such that $f(g(x)) = 0$ has equal roots. *[7 marks]*

12. Find the positive solution of the equation $\int_0^y x^2 + 1\, \mathrm{d}x = 4$. *[5 marks]*

13. For what values of x is the series $x^2 - x + (x^2 - x)^2 + (x^2 - x)^3 + \cdots$ convergent? *[6 marks]*

14. $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors such that $\boldsymbol{u} \cdot \boldsymbol{v} = 0$. If $|\boldsymbol{u}| = 2$ and $|\boldsymbol{v}| = 3$, find $|\boldsymbol{u} - \boldsymbol{v}|$. *[4 marks]*

15. Theo repeatedly rolls a fair die until he gets a six.

(a) Show that the probability of him getting the six on the third roll is $\frac{25}{216}$.

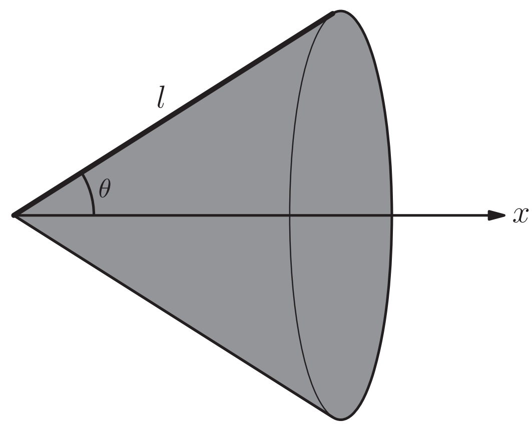
(b) Let p_r be the probability of getting the first six on the rth roll. Find an expression for p_r in terms of r.

(c) Prove algebraically that $\sum_{r=1}^{r=\infty} p_r = 1$. *[9 marks]*

16. If $0 < x < 1$, evaluate exactly the quantity

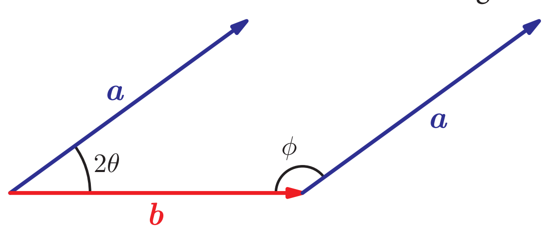
$$\int_0^{1/2} \left(\sum_{i=0}^{i=\infty} x^i \right) \mathrm{d}x$$ *[6 marks]*

17. A rod of length l is inclined at an angle θ to the x-axis. A cone is formed by rotating this rod around the x-axis. Find the maximum possible volume of this cone as θ varies.



[8 marks]

18. The diagram shows two unit vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$, with angle 2θ between them.



(a) Write down the size of the angle ϕ in terms of θ.

(b) Express $|\boldsymbol{a}+\boldsymbol{b}|^2$ in the form $k\cos^2\theta$. *[6 marks]*

19. The function f is defined for $x > 3$ by $f(x) = \ln(x^2 - 9) - \ln(x+3) - \ln x$.

(a) Express $f(x)$ in the form $\ln(g(x))$.

(b) Find an expression for $f^{-1}(x)$. *[6 marks]*

20. (a) What transformation is required to go from the graph of $y = \ln x$ to the graph of $y = \ln(x^2)$?

(b) What transformation is required to go from the graph of $y = \ln x$ to the graph of $y = \log_{10} x$? *[6 marks]*

21. The probability distribution of a discrete random variable X is given by

$$\mathrm{P}(X = x) = \frac{4(p)^x}{5} \text{ for } x \in \mathbb{N}$$

Find the value of p. *[6 marks]*

22. (a) By considering $(1+x)^n$ or otherwise, prove that

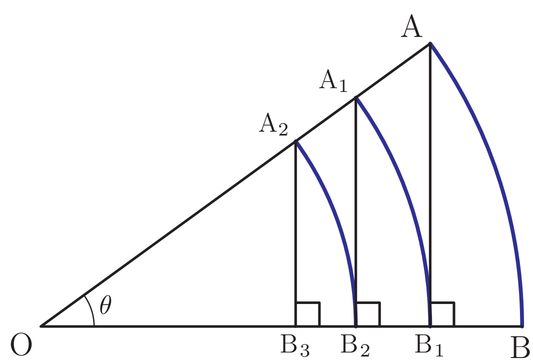
$$\sum_{r=0}^{r=n} \binom{n}{r} = 2^n$$

(b) Evaluate

$$\sum_{r=0}^{r=n} (-1)^n \binom{n}{r}$$

[8 marks]

23. The diagram shows a sector AOB of a circle of radius 1 and centre O, where $\mathrm{A}\hat{\mathrm{O}}\mathrm{B} = \theta$. The lines $[\mathrm{AB_1}]$, $[\mathrm{A_1B_2}]$, $[\mathrm{A_2B_3}]$ etc. are perpendicular to (OB), and $\mathrm{A_1B_1}$, $\mathrm{A_2B_2}$ etc. are all arcs of circles with centre O.



Calculate the sum to infinity of the arc lengths
$\mathrm{AB} + \mathrm{A_1B_1} + \mathrm{A_2B_2} + \mathrm{A_3B_3} + \cdots$.

[7 marks]

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24. (a) If $y = \mathrm{e}^x + \mathrm{e}^{-x}$, express x in terms of y.

(b) Show that the sum of all possible values of x is zero. *[8 marks]*

25. Find the volume of revolution formed when the region enclosed by the graph of $y = \mathrm{e}^x$ and the lines $y = 1$ and $x = 1$ is rotated by 2π around the line $y = 1$. *[6 marks]*

26. The probability of an event occurring is found to be $\frac{1}{7}(x^2 - 14x + 38)$ where x is known to be an integer parameter. Find all possible values of x. *[6 marks]*

27. Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ have the same non-zero length, x.
If $(\boldsymbol{a}+\boldsymbol{b})\bullet(\boldsymbol{a}+\boldsymbol{b}) = 6x$, find the smallest possible value of x. *[6 marks]*

Long questions

1. Daniel and Theo play a game with a biased coin. There is a probability of $\frac{1}{5}$ of the coin showing a head and a probability of $\frac{4}{5}$ of it showing a tail. The boys take it in turns to toss the coin. If the coin shows a head, the player who tossed the coin wins the game. If the coin shows a tail, the other player has the next toss. Daniel plays first and the game continues until there is a winner.

(a) Write down the probability that Daniel wins on his first toss.

(b) Calculate the probability that Theo wins on his first toss.

(c) Calculate the probability that Daniel wins on his second toss.

(d) Show that the probability of Daniel winning is $\frac{5}{9}$.

(e) State the probability of Theo winning.

(f) The boys play the game with a different coin and find that the probability of Daniel winning is twice the probability of Theo winning. Find the probability of this coin showing a head. *[14 marks]*

2. The table shows the values and gradient of $f(x)$ at various points.

x	0	1	2	3	4
$f(x)$	4	2	3	4	6
$f'(x)$	7	9	−3	4	2

(a) Evaluate $f \circ f(3)$.

(b) The graph $y = g(x)$ is formed by translating the graph of $y = f(x)$ by a vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and then reflecting it in the x-axis. Find $g'(2)$. *[6 marks]*

3. A function is defined by

$$f(x) = 2x + \frac{1}{2}\sin 2x - \tan x \text{ for } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

(a) Find $f'(x)$.

(b) Show that the stationary points of $f(x)$ satisfy the equation $2\cos^4 x + \cos^2 x - 1 = 0$.

(c) Hence show that the function has exactly two stationary points. *[11 marks]*

4. (a) The value of the infinite series $\sum_{r=0}^{\infty} a^r$ is 1.5. Find a.

(b) Prove that $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$ for $|x| < k$, stating the value of k.

(c) Show that $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \cdots$

(d) Evaluate $\ln 1.1$ to 3 decimal places. *[12 marks]*

5. At a building site, the probability P(A) that all materials arrive on time is 0.85; the probability P(B) that the building will be completed on time is 0.60. The probability that the materials arrive on time and the building is completed on time is 0.55.

(a) Show that events A and B are *not* independent.

(b) All the materials arrive on time. Find the probability that the building will not be completed on time.

(c) A team of ten people is working on the building, including two plumbers. The number of hours a week worked by the people in the team is normally distributed with a mean of 42 hours, and 10% of the team work 48 hours or more a week. Find the probability that *both* plumbers work more than 40 hours in a given week. *[15 marks]*

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6. The function f is given by $f(x) = \frac{2x+1}{x-3}$, $x \in \mathbb{R}$, $x \neq 3$.

(a) (i) Show that $y = 2$ is an asymptote of the graph of $y = f(x)$.

(ii) Find the vertical asymptote of the graph.

(iii) Write down the coordinates of the point P at which the asymptotes intersect.

(b) Find the points of intersection of the graph and the axes.

(c) Hence sketch the graph of $y = f(x)$, showing the asymptotes by dotted lines.

(d) Show that $f'(x) = \frac{-7}{(x-3)^2}$, and hence find the equation of the tangent at the point S where $x = 4$.

(e) The tangent at the point T on the graph is parallel to the tangent at point S. Find the coordinates of T.

(f) Show that P is the midpoint of [ST].

[24 marks]

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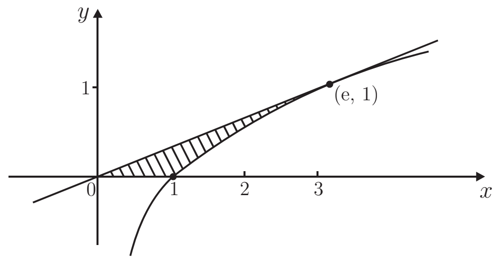
7. (a) Find the equation of the tangent line to the curve $y = \ln x$ at the point (e, 1), and verify that the origin is on this line.

(b) Show that $\frac{\mathrm{d}}{\mathrm{d}x}(x\ln x - x) = \ln x$

(c) The diagram shows the region enclosed by the curve $y = \ln x$, the tangent line in part (a), and the line $y = 0$.



Use the result of part (b) to show that the area of this region is $\frac{1}{2}\mathrm{e} - 1$.

[11 marks]

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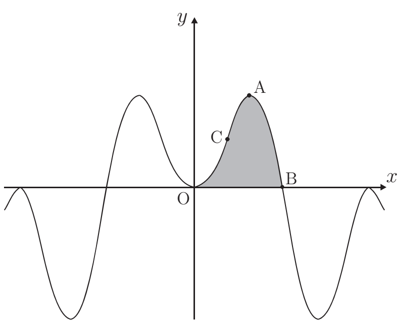
8. The function f is given by $f(x) = (\sin x)^2 \cos x$. The following diagram shows part of the graph of $y = f(x)$.



The point A is a maximum point, the point B lies on the x-axis, and the point C is a point of inflexion.

(a) Give the period of f.

(b) From consideration of the graph of $y = f(x)$, find to an accuracy of **one significant figure** the range of f.

(c) (i) Find $f'(x)$.

(ii) Hence show that at the point A, $\cos x = \sqrt{\frac{1}{3}}$.

(iii) Find the exact maximum value.

(d) Find the exact value of the x-coordinate at the point B.

(e) (i) Find $\int f(x)\,dx$.

(ii) Find the area of the shaded region in the diagram.

(f) Given that $f''(x) = 9(\cos x)^3 - 7\cos x$, find the x-coordinate at the point C.

[20 marks]

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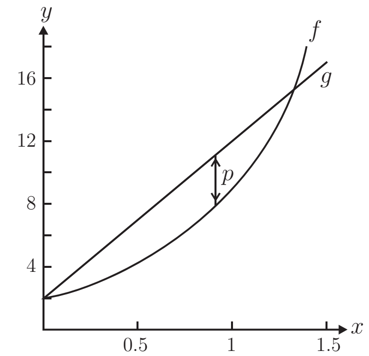
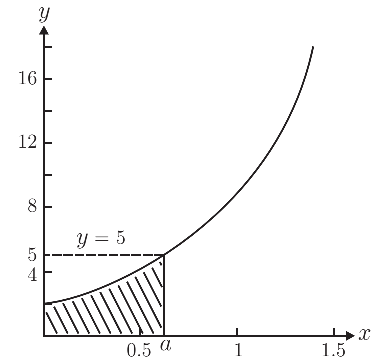
9. The diagram below shows the graphs of $f(x) = 1 + e^{2x}$, $g(x) = 10x + 2$, $0 \le x \le 1.5$.



(a) (i) Write down an expression for the vertical distance p between the graphs of f and g.

(ii) Given that p has a maximum value for $0 \le x \le 1.5$, find the value of x at which this occurs.

The graph of $y = f(x)$ only is shown in the diagram below.
When $x = a$, $y = 5$.



(b) (i) Find $f^{-1}(x)$.

(ii) *Hence* show that $a = \ln 2$.

(c) The region shaded in the diagram is rotated through 360° about the x-axis. Write down an expression for the volume obtained.

[11 marks]

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10. Consider the functions f and g where $f(x) = 3x - 5$ and $g(x) = x - 2$.

(a) Find the inverse function, f^{-1}.

(b) Given that $g^{-1}(x) = x + 2$, find $(g^{-1} \circ f)(x)$.

(c) Given also that $(f^{-1} \circ g)(x) = \dfrac{x+3}{3}$, solve $(f^{-1} \circ g)(x) = (g^{-1} \circ f)(x)$.

Let $h(x) = \dfrac{f(x)}{g(x)}$, $x \neq 2$.

(d) (i) *Sketch* the graph of h for $-3 \leq x \leq 7$ and $-2 \leq y \leq 8$, including any asymptotes.

(ii) Write down the *equations* of the asymptotes.

(e) The expression $\dfrac{3x-5}{x-2}$ may also be written as $3 + \dfrac{1}{x-2}$. Use this to answer the following.

(i) Find $\int h(x)\,\mathrm{d}x$.

(ii) *Hence* calculate the *exact* value of $\int_3^5 h(x)\,\mathrm{d}x$.

(f) On your sketch, shade the region whose area is represented by $\int_3^5 h(x)\,\mathrm{d}x$.

[17 marks]

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11. (a) If p and q are positive integers and $a < b$, find the x-coordinate of the stationary point on the curve $y = (x-a)^p (x-b)^q$ in the interval $a < x < b$.

(b) Sketch the graph in the case where $p = 2$ and $q = 3$.

(c) By considering the graph or otherwise, determine a condition involving p and/or q for this stationary point to be a local maximum.

[12 marks]