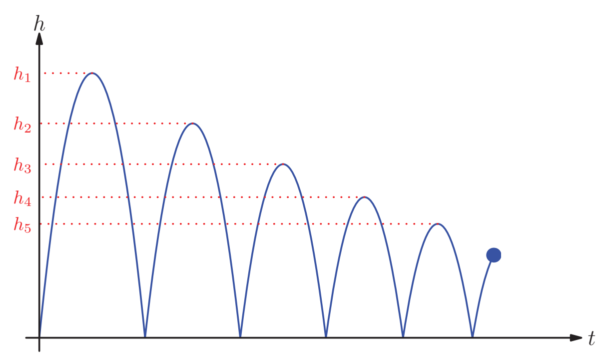
6 Sequences and series

In this chapter you will learn:

- how to describe sequences mathematically
- a way to describe sums of sequences
- about sequences with a constant difference between terms
- about finite sums of sequences with a constant difference between terms
- about sequences with a constant ratio between terms
- about finite sums of sequences with a constant ratio between terms
- about infinite sums of sequences with a constant ratio between terms
- how to apply sequences to real-life problems.

Introductory problem

A mortgage of \$100 000 has a fixed rate of 5% compound interest. It needs to be paid off in 25 years by fixed annual instalments. Interest is debited at the end of each year, just before the payment is made. How much should be paid each year?



If you drop a ball, it will bounce a little lower each time it hits the ground. The heights that the ball reaches after each bounce form a sequence. Although the study of sequences may seem to be just about abstract number patterns, it actually has a remarkable number of applications in the real world – from calculating mortgages to estimating harvests on farms.

6A General sequences

A **sequence** is a list of numbers in a specified order. Examples include:

1, 3, 5, 7, 9, 11, ...

1, 4, 9, 16, 25, ...

100, 50, 25, 12.5, ...

The numbers in a sequence are called **terms**; so in the first sequence above, the first term is 1, the second term is 3, and so on. To study sequences, it is useful to have some special notation to describe them.

KEY POINT 6.1

u_n denotes the value of the nth term of a sequence.

In the first sequence above, we would write $u_1 = 1$, $u_2 = 3$, $u_5 = 9$, etc.

We are mainly interested in sequences with well-defined mathematical rules. There are two types of rules for defining a sequence: recursive rules and deductive rules.

A **recursive rule** links new terms to previous terms in the sequence. For example, if each term is three times the previous term, we would write $u_{n+1} = 3u_n$.

EXAM HINT

While u_n is a conventional symbol for a sequence, there is nothing special about the letters used. We could also call a sequence t_x or a_h. The important thing to remember is that the subscript (n) tells us where the term is in the sequence, and the letter together with the subscript (u_n) represents the value of that term.

Worked example 6.1

A sequence is defined by $u_{n+1} = u_n + u_{n-1}$ with $u_1 = 1$ and $u_2 = 1$. What is the fifth term of this sequence?

The sequence is defined by a recursive rule, so we have to work our way up to u_5. We are given u_1 and u_2. To find u_3 set $n = 2$ in the inductive formula.

$$u_3 = u_2 + u_1$$
$$= 1 + 1$$
$$= 2$$

To find u_4 set $n = 3$.

$$u_4 = u_3 + u_2$$
$$= 2 + 1$$
$$= 3$$

To find u_5 set $n = 4$.

$$u_5 = u_4 + u_3$$
$$= 3 + 2$$
$$= 5$$

You may recognise the sequence in Worked example 6.1 as the famous 'Fibonacci sequence', based on a model Leonardo Fibonacci proposed for rabbit populations. This sequence has arisen in many applications, from the arrangement of seeds in pine cones to a proof of the infinity of prime numbers. There is also a beautiful link to a special number called the 'golden ratio', $\frac{1+\sqrt{5}}{2}$.

A **deductive rule** links the value of the term to where it is in the sequence. For example, if each term is the square of its position in the sequence, we would write $u_n = n^2$.

EXAM HINT

There are several alternative names for deductive and recursive rules. A recursive rule may also be referred to as a 'term-to-term rule', 'recurrence relation' or 'inductive rule'. A deductive rule may also be called a 'position-to-term rule', '*n*th term rule' or simply 'formula'.

Worked example 6.2

A sequence is defined by $u_n = 2n-1$.

(a) Find the fourth term of this sequence.

(b) Find and simplify an expression for $u_{n+1} - u_n$.

With a deductive rule, we can jump straight to the fourth term by setting $n = 4$ in the formula.

(a) $u_4 = 2\times 4 - 1$
$= 7$

To get u_{n+1}, put $n + 1$ in place of n in the formula.

(b) $u_{n+1} = 2(n+1)-1$

$u_{n+1} - u_n = (2(n+1)-1)-(2n-1)$
$= 2n+2-1-2n+1$
$= 2$

(You might be intersted to know that the answer to part (b) is the difference between each term in the sequence.)

Exercise 6A

1. Write out the first five terms of the following inductively defined sequences.

(a) (i) $u_{n+1} = u_n + 5,\ u_1 = 3.1$ (ii) $u_{n+1} = u_n - 3.8,\ u_1 = 10$

(b) (i) $u_{n+1} = 3u_n + 1,\ u_1 = 0$ (ii) $u_{n+1} = 9u_n - 10,\ u_1 = 1$

(c) (i) $u_{n+2} = u_{n+1} \times u_n,\ u_1 = 2, u_2 = 3$

(ii) $u_{n+2} = u_{n+1} \div u_n,\ u_1 = 2, u_2 = 1$

(d) (i) $u_{n+2} = u_n + 5,\ u_1 = 3, u_2 = 4$

(ii) $u_{n+2} = 2u_n + 1,\ u_1 = -3, u_2 = 3$

(e) (i) $u_{n+1} = u_n + 4,\ u_4 = 12$ (ii) $u_{n+1} = u_n - 2,\ u_6 = 3$

2. Write out the first five terms of the following deductively defined sequences.

(a) (i) $u_n = 3n + 2$ (ii) $u_n = 1.5n - 6$

(b) (i) $u_n = n^3 - 1$ (ii) $u_n = 5n^2$

(c) (i) $u_n = 3^n$ (ii) $u_n = 8 \times (0.5)^n$

(d) (i) $u_n = n^n$ (ii) $u_n = \sin(90n°)$

3. Give a possible inductive definition for each of the following sequences.

(a) (i) $7, 10, 13, 16, \ldots$ (ii) $1, 0.2, -0.6, -1.4, \ldots$

(b) (i) $3, 6, 12, 24, \ldots$ (ii) $12, 18, 27, 40.5, \ldots$

(c) (i) $1, 3, 6, 10, \ldots$ (ii) $1, 2, 6, 24, \ldots$

4. Give a deductive definition for each of the following sequences.

(a) (i) $2, 4, 6, 8, \ldots$ (ii) $1, 3, 5, 7, \ldots$

(b) (i) $2, 4, 8, 16, \ldots$ (ii) $5, 25, 125, 625, \ldots$

(c) (i) $1, 4, 9, 16, \ldots$ (ii) $1, 8, 27, 64, \ldots$

(d) (i) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$ (ii) $\frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \frac{7}{16}, \ldots$

5. The sequence u_n is defined by $u_n = n2^n$.

(a) Write down u_1.

(b) Find and simplify an expression for $\frac{u_{n+1}}{u_n}$. *[4 marks]*

6. A sequence $\{u_n\}$ is defined by $u_0 = 1$, $u_1 = 2$, $u_{n+1} = 3u_n - 2u_{n-1}$ where $n \in \mathbb{Z}$.

(a) Find u_2, u_3 and u_4.

(b) (i) Using the results in part (a), suggest an expression for u_n in terms of n.

(ii) Verify that your answer to part (b)(i) satisfies the equation $u_{n+1} = 2(3u_n - 2u_{n-1})$. *[6 marks]*

In science we may state an observed pattern as a law if there is no contradictory evidence, but this is not the case in mathematics. For instance, given the first four terms of a sequence, we cannot be sure that the sequence will continue on for ever with the same pattern. Nevertheless, a principle in philosophy called 'Occam's Razor' suggests that the simplest answer is often the right one.

6B General series and sigma notation

If 10% interest is paid on money in a bank account each year, the amounts paid form a sequence. While it is good to know how much interest is paid each year, you may be even more interested in knowing how much interest will be paid in total over a certain number of years. This is an example of a situation where we may want to add up the terms of a sequence. The sum of a sequence up to a certain point is called a **series**. We often use the symbol S_n to denote the sum of the first n terms of a sequence.

Worked example 6.3

The sum of consecutive odd numbers, starting from 1, forms a series. Let S_n denote the sum of the first n terms. List the first five terms of the sequence S_n and suggest a rule for it.

Start by examining the first few terms.

$S_1 = 1$
$S_2 = 1 + 3 = 4$
$S_3 = 1 + 3 + 5 = 9$
$S_4 = 1 + 3 + 5 + 7 = 16$
$S_5 = 1 + 3 + 5 + 7 + 9 = 25$

Do we recognise these numbers?

It seems that $S_n = n^2$

You will learn how to prove this result in section 6D.

EXAM HINT

Do not be intimidated by this complicated-looking notation. If you struggle with an expression given in sigma notation, try writing out the first few terms of the series.

To work with series mathematically, it is often too tedious to specify adding up a defined sequence from a given start point to a certain end point, or to write $u_1 + u_2 + u_3 + u_4 + \cdots + u_n$. The same thing can be expressed in a shorter (although not necessarily simpler) way by using **sigma notation**:

KEY POINT 6.2

This is the last value taken by r; where counting ends

r is a placeholder; it shows what changes with each new term

Capital Greek letter sigma means 'add up'

$$\sum_{r=1}^{r=n} f(r) = f(1) + f(2) + \cdots + f(n)$$

This is the first value taken by r; where counting starts

Note that although in Key point 6.2 the counting started at $r = 1$, this does not necessarily have to be the case: you can replace 1 by any other starting value. However, r always increases by 1 until it reaches the end value. If there is only one variable in the expression being summed, it is acceptable to omit the '$r =$' above and below the sigma.

Worked example 6.4

Given that $T_n = \sum_{2}^{n} r^2$, find the value of T_4.

Put the starting value, $r = 2$ into the expression being summed, r^2.

$T_4 = 2^2 + \dots$

We've not reached the end value 4, so put in $r = 3$.

$T_4 = 2^2 + 3^2 + \dots$

We've still not reached the end value, so put in $r = 4$.

$T_4 = 2^2 + 3^2 + 4^2$

Now we have reached the end value and can proceed to evaluate the sum.

$T_4 = 4 + 9 + 16$
$= 29$

In the example, both the letters n and r are unknowns, but they are not the same type of unknown. If we replace r by any other letter (except n), the sum will keep the same value. For example, $T_4 = \sum_{k=2}^{k=4} k^2 = 29$; $\sum_{r=2}^{r=n} r^2$ and $\sum_{k=2}^{k=n} k^2$ have exactly the same meaning. Thus, r is called a 'dummy variable'. However, if we replace n with anything else, the value of the expression may change; for example, $T_3 = 13$, $T_4 = 29$.

Worked example 6.5

Write the series $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$ in sigma notation.

We must describe each term of the series using a dummy variable r.

General term: $\frac{1}{r}$

What is the first value of r?

Starts from $r = 2$

What is the final value of r?

Ends at $r = 6$

Express in sigma notation.

Series is $\sum_{2}^{6} \frac{1}{r}$

Exercise 6B

1. Evaluate the following expressions.

(a) (i) $\sum_{2}^{4} 3r$ (ii) $\sum_{5}^{7} (2r+1)$

(b) (i) $\sum_{3}^{6} (2^r - 1)$ (ii) $\sum_{-1}^{4} 1.5^r$

(c) (i) $\sum_{a=1}^{a=4} b(a+1)$ (ii) $\sum_{q=-3}^{q=2} pq^2$

2. Write the following expressions in sigma notation. Be aware that in each case there is more than one correct answer.

(a) (i) $2+3+4+\cdots+43$ (ii) $6+8+10+\ldots+60$

(b) (i) $\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots+\frac{1}{128}$ (ii) $2+\frac{2}{3}+\frac{2}{9}+\cdots+\frac{2}{243}$

(c) (i) $14a+21a+28a+\cdots+70a$ (ii) $0+1+2^b+3^b+\cdots+19^b$

6C Arithmetic sequences

We now focus on a particular type of sequence – one where there is a constant difference between consecutive terms. Such sequences are called **arithmetic sequences** or **arithmetic progressions**. The constant difference between consecutive terms is called the *common difference*, usually denoted by d, so arithmetic sequences obey the recursive rule

$$u_{n+1} = u_n + d$$

This formula is not enough to fully define the sequence. There are many different sequences with common difference 2, for example $1,3,5,7,9,11,\ldots$ and $106,108,110,112,\ldots$. To fully define the sequence we also need to specify the first term, u_1. So the second sequence above is defined by $u_1 = 106$, $d = 2$.

Worked example 6.6

What is the fourth term of an arithmetic sequence with $u_1 = 300$, $d = -5$?

Use the recursive rule to find the first four terms.

$u_1 = 300$
$u_2 = u_1 - 5 = 295$
$u_3 = u_2 - 5 = 290$
$u_4 = u_3 - 5 = 285$

In the above example it did not take long to find the first four terms, but what if you had been asked to find the hundredth term? To do this efficiently, we need to move from the inductive definition of an arithmetic sequence to a deductive rule. Think about how arithmetic sequences are built up:

$$u_2 = u_1 + d$$
$$u_3 = u_2 + d = u_1 + d + d$$
$$u_4 = u_3 + d = u_1 + d + d + d$$

and so on. To get to the nth term, we start at the first term and add on the common difference $n-1$ times. This suggests the following formula.

KEY POINT 6.3

$$u_n = u_1 + (n-1)d$$

Worked example 6.7

The fifth term of an arithmetic sequence is 7 and the eighth term is 16. What is the 100th term?

We need to find a deductive rule for u_n, which we can do once we know u_1 and d.
So let's write down the information given and relate it to u_1 and d.
Write an expression for the 5th term in terms of u_1 and d.

$u_5 = u_1 + 4d$

We are told that $u_5 = 7$

$7 = u_1 + 4d \quad (1)$

Repeat for the 8th term.

$16 = u_1 + 7d \quad (2)$

Solve this pair of equations simultaneously.

$(2)-(1)$ gives
$9 = 3d$
$\therefore d = 3$
and hence
$u_1 = -5$

Write down the general term and use it to answer the question.

$u_n = -5 + (n-1)\times 3$
$u_{100} = -5 + 99 \times 3$
$= 292$

EXAM HINT

Many exam questions on this topic involve writing the given information in the form of simultaneous equations and then solving them.

Worked example 6.8

An arithmetic progression has first term 5 and common difference 7. What is the term number corresponding to the value 355?

The question is asking us to find n when $u_n = 355$. Write this as an equation.

$$355 = u_1 + (n-1)d$$
$$= 5 + 7(n-1)$$

Solve this equation.

$$\Leftrightarrow 350 = 7(n-1)$$
$$\Leftrightarrow 50 = n-1$$
$$\Leftrightarrow n = 51$$

So 355 is the 51st term.

EXAM HINT

'Arithmetic progression' is just another way of saying 'arithmetic sequence'. Make sure you are familiar with these different expressions for the same thing.

Exercise 6C

1. Find the general formula for the arithmetic sequence that satisfies the following conditions.

(a) (i) First term 9, common difference 3

(ii) First term 57, common difference 0.2

(b) (i) First term 12, common difference −1

(ii) First term 18, common difference $-\dfrac{1}{2}$

(c) (i) First term 1, second term 4

(ii) First term 9, second term 19

(d) (i) First term 4, second term 0

(ii) First term 27, second term 20

(e) (i) Third term 5, eighth term 60

(ii) Fifth term 8, eighth term 38

2. How many terms are there in each of the following sequences?
   (a) (i) 1,3,5,…,65 (ii) 18,13,8,…,−122
   (b) (i) First term 8, common difference 9, last term 899
   (ii) First term 0, ninth term 16, last term 450

3. An arithmetic sequence has 5 and 13 as its first term and second term, respectively.
   (a) Write down, in terms of n, an expression for the nth term a_n.
   (b) Find the number of terms of the sequence which are less than 400. *[8 marks]*

4. The 10th term of an arithmetic sequence is 61 and the 13th term is 79. Find the value of the 20th term. *[4 marks]*

5. The 8th term of an arithmetic sequence is 74 and the 15th term is 137. Which term has the value 227? *[4 marks]*

6. The heights above ground of the rungs in a ladder form an arithmetic sequence. The third rung is 70 cm above the ground and the tenth rung is 210 cm above the ground. If the top rung is 350 cm above the ground, how many rungs does the ladder have? *[5 marks]*

7. The first four terms of an arithmetic sequence are 2, $a - b$, $2a + b + 7$ and $a - 3b$, where a and b are constants. Find a and b. *[5 marks]*

8. A book starts at page 1 and is numbered on every page.
   (a) Show that the first eleven pages contain thirteen digits.
   (b) If the total number of digits used is 1260, how many pages are in the book? *[8 marks]*

6D Arithmetic series

When you add up the terms of an arithmetic sequence, you get an arithmetic series. There is a formula for the arithmetic series of n terms. See the Fill-in proof 3 'Arithmetic series and the story of Gauss' on the CD-ROM if you are interested in how it is derived, though you are not required to know this derivation for the International Baccalaureate.

There are two different forms of the formula.

KEY POINT 6.4

If you know the first and last terms:

$$S_n = \frac{n}{2}(u_1 + u_n)$$

If you know the first term and the common difference:

$$S_n = \frac{n}{2}\left(2u_1 + (n-1)d\right)$$

Worked example 6.9

Find the sum of the first 30 terms of an arithmetic progression with first term 8 and common difference 0.5.

We have all of the information needed to use the second sum formula.

$$S_{30} = \frac{30}{2}\left(2\times 8 + (30-1)\times 0.5\right)$$
$$= 457.5$$

Sometimes you have to interpret the question carefully to realise that it is about an arithmetic sequence.

Worked example 6.10

Find the sum of all the multiples of 3 between 100 and 1000.

Write out a few terms to see what is happening.

Sum $= 102 + 105 + 108 + \cdots + 999$

This is an arithmetic series with $u_1 = 102$ and $d = 3$.

Since we know the first and last terms, we can use the first formula; but in order to do so, we also need to know how many terms are in the sequence. We find the number of terms by solving $u_n = 999$.

$$\Leftrightarrow \quad 999 = 102 + 3(n-1)$$
$$\Leftrightarrow \quad 897 = 3(n-1)$$
$$\Leftrightarrow \quad n = 300$$

Use the first sum formula.

$$S_{300} = \frac{300}{2}(102 + 999)$$
$$= 165\,150$$

You need to be able to work backwards too – for example, to find out how many terms are in a series given its sum and some other information. Remember that the number of terms can only be a positive integer.

Worked example 6.11

An arithmetic sequence has first term 5 and common difference 10. If the sum of all the terms is 720, how many terms are in the sequence?

We know u_1, d and S_n, so in the second sum formula n is the only unknown.

$$720 = \frac{n}{2}(2\times 5+(n-1)\times 10)$$
$$= \frac{n}{2}(10+10n-10)$$
$$= 5n^2$$

Solve this equation to find n.

$$n^2 = 144$$

Therefore

$$n = \pm 12$$

But n must be a positive integer, so $n = 12$.

Exercise 6D

1. Find the sum of the following arithmetic sequences.

(a) (i) $12, 33, 54, \ldots$ (17 terms)

(ii) $-100, -85, -70, \ldots$ (23 terms)

(b) (i) $3, 15, \ldots, 459$ (ii) $2, 11, \ldots, 650$

(c) (i) $28, 23, \ldots, -52$ (ii) $100, 97, \ldots, 40$

(d) (i) $15, 15.5, \ldots, 29.5$ (ii) $\frac{1}{12}, \frac{1}{6}, \ldots, 1.5$

2. An arithmetic sequence has first term 4 and common difference 8. Find the number of terms required to get a sum of:

(a) (i) 676 (ii) 4096 (iii) 11236

(b) x^2, $x > 0$

3. The second term of an arithmetic sequence is 7. The sum of the first four terms of the sequence is 12. Find the first term, a, and the common difference, d, of the sequence. *[5 marks]*

4. Consider the arithmetic series $2 + 5 + 8 + \ldots$

(a) Find an expression for S_n, the sum of the first n terms.

(b) Find the value of n for which $S_n = 1365$. *[5 marks]*

5. The sum of the first n terms of a series is given by $S_n = 2n^2 - n$, where $n \in \mathbb{Z}^+$.

(a) Find the first three terms of the series.

(b) Find an expression for the nth term of the series, giving your answer in terms of n. *[7 marks]*

6. Find the sum of the positive terms of the arithmetic sequence 85, 78, 71, ... *[6 marks]*

7. The second term of an arithmetic sequence is 6, and the sum of the first four terms of the sequence is 8. Find the first term, a, and the common difference, d, of the sequence. *[6 marks]*

8. Consider the arithmetic series $-6+1+8+15+\cdots$

Find the least number of terms so that the sum of the series is greater than 10 000. *[6 marks]*

9. The sum of the first n terms of an arithmetic sequence is $S_n = 3n^2 - 2n$. Find the nth term, u_n. *[6 marks]*

10. A circular disc is cut into twelve sectors whose angles are in an arithmetic sequence. The angle of the largest sector is twice the angle of the smallest sector. Find the size of the angle of the smallest sector. *[6 marks]*

11. The ratio of the fifth term to the twelfth term of a sequence in an arithmetic progression is $\frac{6}{13}$. If each term of this sequence is positive, and the product of the first term and the third term is 32, find the sum of the first 100 terms of the sequence. *[7 marks]*

12. What is the sum of all three-digit numbers which are multiples of 14 but not 21? *[8 marks]*

6E Geometric sequences

A **geometric sequence** has a constant ratio between terms. To get from one term (u_n) to the next (u_{n+1}), you always multiply by the same number, which is called the *common ratio* and usually denoted by r. So geometric sequences obey the recursive rule

$$u_{n+1} = r\,u_n$$

Here are some examples of geometric sequences:

$$1, 2, 4, 8, 16, \ldots \quad (r = 2)$$

$$100, 50, 25, 12.5, 6.25, \ldots \quad \left(r = \frac{1}{2}\right)$$

$$1, -3, 9, -27, 81, \ldots \quad (r = -3)$$

As with arithmetic sequences, to fully define a geometric sequence we also need to know the first term, u_1.

To obtain the deductive rule, observe that in order to get to the nth term starting from the first term, you need to multiply by the common ratio $n-1$ times. For example, $u_2 = ru_1$, $u_3 = ru_2 = r^2u_1$, and so on.

KEY POINT 6.5

$$u_n = u_1\,r^{n-1}$$

Worked example 6.12

The 7th term of a geometric sequence is 13. The 9th term is 52. What values could the common ratio take?

Write an expression for the 7th term in terms of u_1 and r.

$u_7 = u_1r^6$

But we know $u_7 = 13$.

$13 = u_1r^6 \quad (1)$

Repeat the same process for the 9th term.

$52 = u_1r^8 \quad (2)$

Divide the two equations to eliminate u_1 and hence solve for r.

$(2) \div (1)$ gives

$4 = r^2$

$r = \pm 2$

EXAM HINT

Notice that the question asked for *values* rather than a value. This is a big hint that there is more than one answer.

When questions on geometric sequences ask which term satisfies a particular condition, you can either generate the sequence with your calculator and inspect the values, or use logarithms to solve an equation.

Worked example 6.13

A geometric sequence has first term 2 and common ratio −3. Which term has the value −4374?

Write down the formula for the *n*th term.

$u_n = 2\times(-3)^{n-1}$

Use a table on your GDC to list the terms of the sequence and search for −4374. (See calculator skills sheet 10 on the CD-ROM.)

From GDC: $u_8 = -4374$

It is the 8th term.

It may be that the value of *n* you seek is large, in which case it could be impractical to search through a table on your calculator. Instead, try to set up an equation and solve it using logarithms.

Worked example 6.14

A geometric sequence has first term 5000 and common ratio 0.2.

Which term is equal to 3.36×10^{-15}, correct to three significant figures?

Express the condition as an equation.

$5000\times(0.2)^{n-1} = 3.36\times10^{-15}$

The unknown is in the exponent, so solve using logarithms.

$$(0.2)^{n-1} = \frac{3.36\times10^{-15}}{5000} = 6.72\times10^{-19}$$

$$\log\left((0.2)^{n-1}\right) = \log\left(6.72\times10^{-19}\right)$$

$$(n-1)\log 0.2 = -18.17\ldots$$

$$n-1 = \frac{-18.17\ldots}{\log 0.2} = 25.9991\ldots$$

$$\therefore n = 27$$

See section 2G if you need a reminder of how to solve exponential equations using logarithms.

Exercise 6E

1. Find an expression for the nth term of the following geometric sequences.

(a) (i) $6, 12, 24, \ldots$ (ii) $12, 18, 27, \ldots$

(b) (i) $20, 5, 1.25, \ldots$ (ii) $1, \frac{1}{2}, \frac{1}{4}, \ldots$

(c) (i) $1, -2, 4, \ldots$ (ii) $5, -5, 5, \ldots$

(d) (i) $a, ax, ax^2, \ldots$ (ii) $3, 6x, 12x^2, \ldots$

2. Find the number of terms in each of the following geometric sequences.

(a) (i) $6, 12, 24, \ldots, 24576$ (ii) $20, 50, \ldots, 4882.8125$

(b) (i) $1, -3, \ldots, -19683$ (ii) $2, -4, 8, \ldots, -1024$

(c) (i) $\frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{1024}$ (ii) $3, 2, \frac{4}{3}, \ldots, \frac{128}{729}$

3. How many terms are needed in the following geometric sequences to get within 10^{-9} of zero?

(a) (i) $5, 1, \frac{1}{5}, \ldots$ (ii) $0.6, 0.3, 0.15, \ldots$

(b) (i) $4, -2, 1, \ldots$ (ii) $-125, 25, -5, \ldots$

4. The second term of a geometric sequence is 6 and the fifth term is 162. Find the tenth term. *[5 marks]*

5. The third term of a geometric sequence is 112 and the sixth term is 7168. Which term takes the value 1 835 008? *[5 marks]*

6. Which is the first term of the sequence $\frac{2}{5}, \frac{4}{25}, \ldots, \frac{2^n}{5^n}$ that is less than 10^{-6}? *[6 marks]*

7. The difference between the fourth and the third term of a geometric sequence is $\frac{75}{8}$ times the first term. Find the common ratio given that $r > 0$. *[6 marks]*

8. The third term of a geometric progression is 12 and the fifth term is 48. Find the two possible values of the eighth term. *[6 marks]*

9. The first three terms of a geometric sequence are $a, a+14, 9a$. Find the value of a. *[6 marks]*

10. The three terms a, 1, b are in arithmetic progression. The three terms 1, a, b are in geometric progression. Find the values of a and b given that $a \neq b$. *[7 marks]*

11. The sum of the first n terms of an arithmetic sequence $\{u_n\}$ is given by $S_n = 4n^2 - 2n$. Three terms of this sequence, u_2, u_m and u_{32}, are consecutive terms in a geometric sequence. Find m. *[7 marks]*

6F Geometric series

When you add up the terms of a geometric sequence, you get a geometric series. As with arithmetic series, there is a formula for the sum of n terms in a geometric sequence. See the Fill-in proof 4 'Self-similarity and geometric series' on the CD-ROM if you are interested in learning where this formula comes from.

KEY POINT 6.6

$$S_n = \frac{u_1(1-r^n)}{1-r} \quad (r \neq 1)$$

or equivalently

$$S_n = \frac{u_1(r^n-1)}{r-1} \quad (r \neq 1)$$

EXAM HINT

Generally we use the first of these formulas when the common ratio is less than one, the second when the common ratio is greater than one. This way we can avoid working with negative numbers as much as possible.

Worked example 6.15

Find the exact value of the sum of the first 6 terms of the geometric sequence with first term 8 and common ratio $\frac{1}{2}$.

Since $r < 1$, we use the first sum formula.

$$S_6 = \frac{8\left(1-\left(\frac{1}{2}\right)^6\right)}{1-\frac{1}{2}}$$

$$= \frac{8\left(1-\frac{1}{64}\right)}{\frac{1}{2}}$$

$$= 16\left(\frac{63}{64}\right)$$

$$= \frac{63}{4}$$

We may be given information about the sum and have to deduce other information.

Worked example 6.16

How many terms are needed for the sum of the geometric series $3+6+12+24+\ldots$ to exceed 100 000?

We need to find n, but first we need the values of u_1 and r.

$u_1 = 3$

$r = 2$

As $r > 1$, use the second sum formula and express the condition as an inequality.

$$S_n = \frac{3(2^n - 1)}{2-1} > 100\,000$$

We can use a calculator to generate values for S_n.

From GDC:

$S_{15} = 98\,301$

$S_{16} = 196\,605$

So 16 terms are needed.

Exercise 6F

1. Find the sums of the following geometric series. (Some of these may have more than one possible answer.)

(a) (i) $7, 35, 175, \ldots$ (10 terms)

(ii) $1152, 576, 288, \ldots$ (12 terms)

(b) (i) 16, 24, 36, …, 182.25

(ii) 1, 1.1, 1.21, …, 1.771561

(c) (i) First term 8, common ratio −3, last term 52 488

(ii) First term −6, common ratio −3, last term 13 122

(d) (i) Third term 24, fifth term 6, 12 terms

(ii) Ninth term 50, thirteenth term 0.08, last term 0.0032

2. Find the value of the common ratio if

(a) (i) the first term is 11, sum of the first 12 terms is 2 922 920

(ii) the first term is 1, sum of the first 6 terms is 1.24992

(b) (i) the first term is 12, sum of the first 6 terms is −79 980

(ii) the first term is 10, sum of the first 4 terms is 1

3. The nth term, u_n, of a geometric sequence is given by $u_n = 3\times 5^{n+2}$.

(a) Find the common ratio r.

(b) Hence or otherwise find S_n, the sum of the first n terms of this sequence. *[5 marks]*

4. The sum of the first three terms of a geometric sequence is $23\frac{3}{4}$, and the sum of the first four terms is $40\frac{5}{8}$. Find the first term and the common ratio. *[6 marks]*

5. The first term of a geometric series is 6, and the sum of the first 15 terms is 29. Find the common ratio. *[5 marks]*

6. The sum of the first four terms of a geometric series is 520. The sum of the first five terms is 844. The sum of the first six terms is 1330.

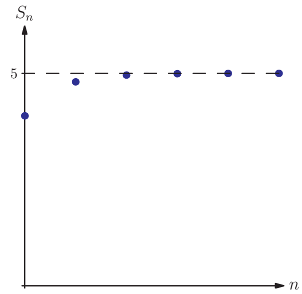
(a) Find the common ratio of the geometric progression.

(b) Find the sum of the first two terms. *[6 marks]*

6G Infinite geometric series

If we keep adding together terms of an arithmetic sequence, the sum will grow (or decrease) without limit, and is said to be **divergent**. This can occur with some geometric series, too, but it could also happen that the sum gets closer and closer to and 'settles down' to a finite number; in this case we say that the geometric series is **convergent**.

The graph shows the values of S_n for a geometric series with first term $u_1 = 4$ and common ratio $r = 0.2$. As n increases, the value of S_n seems to be getting closer and closer to 5; thus we say that the series converges to 5.



Not all geometric series converge. To determine which ones do, we need to look at the formula for a geometric series:

$$S_n = \frac{u_1(1-r^n)}{1-r}$$

We want to know what happens to S_n as n gets large, so we focus on the r^n term. With most numbers, when you raise the number to a larger power the result gets bigger; for example, $1.2^{20} = 38.3$ and $1.2^{30} = 237$. The exception is when r is a number between -1 and 1. In this case, r^n gets smaller as n increases – in fact, it approaches zero; for example, $0.2^2 = 0.04, 0.2^3 = 0.008$ and $0.2^{20} = 1.05\times10^{-14}$. This means that for $-1 < r < 1$, as n increases the value of S_n will get closer and closer to $\frac{u_1}{1-r}$.

When $r = 1$, the geometric series certainly diverges. But when $r = -1$, it is not clear whether the series converges or diverges: the sum could have value 0, u_1 or $\frac{u_1}{2}$ depending on how the terms in the series are grouped. This is an example of a situation where mathematics is open to debate.

KEY POINT 6.7

As n increases, the sum of a geometric series converges to

$$S_\infty = \frac{u_1}{1-r} \text{ if } |r| < 1.$$

This is called the **sum to infinity of the series**.

EXAM HINT

The condition that $|r| < 1$ is just as important as the formula itself.

$|r|$ is the modulus, or absolute value, of r. The modulus leaves positive values unchanged but reverses the sign of negative values. So, $|8| = 8$ and $|-8| = 8$.

Worked example 6.17

The sum to infinity of a geometric sequence is 5. The second term is $-\frac{6}{5}$. Find the common ratio.

Write the given information as equations in u_1 and r.

$$S_\infty = \frac{u_1}{1-r} = 5 \qquad (1)$$

$$u_2 = u_1 r = -\frac{6}{5} \qquad (2)$$

continued . . .

Solve the equations simultaneously.

From (2):

$$u_1 = -\frac{6}{5r}$$

Substituting into (1):

$$-\frac{6}{5r(1-r)} = 5$$

$$-6 = 25(r - r^2)$$

$$25r^2 - 25r - 6 = 0$$

$$(5r-6)(5r+1) = 0$$

Therefore $r = \frac{6}{5}$ or $r = -\frac{1}{5}$

Watch out! Check whether the series actually converges for the r-values found.

But since the sum to infinity exists, we must have $|r| < 1$, so $r = -\frac{1}{5}$

Some questions may focus on the condition for the sequence to converge rather than the value that it converges to.

Worked example 6.18

The geometric series $(2-x)+(2-x)^2+(2-x)^3+\cdots$ converges. What values can x take?

Identify r.

$$r = (2-x)$$

Use the fact that the series converges.

Since the series converges,

$$|2-x| < 1$$

Convert the modulus expression into a double inequality.

$$-1 < 2-x < 1$$

$$\Leftrightarrow -3 < -x < -1$$

Therefore

$$1 < x < 3$$

Exercise 6G

1. Find the value of each of the following infinite geometric series, or state that the series is divergent.

(a) (i) $9+3+1+\frac{1}{3}+\ldots$ (ii) $56+8+1\frac{1}{7}+\ldots$

(b) (i) $0.3+0.03+0.003+\ldots$

(ii) $0.78+0.0078+0.000078+\ldots$

(c) (i) $0.01+0.02+0.04+\ldots$ (ii) $\frac{19}{10000}+\frac{19}{1000}+\frac{19}{100}+\ldots$

(d) (i) $10-2+0.4+\ldots$ (ii) $6-4+\frac{8}{3}+\ldots$

(e) (i) $10-40+160+\ldots$ (ii) $4.2-3.36+2.688+\ldots$

2. Find the values of x which allow the following geometric series to converge.

(a) (i) $9+9x+9x^2+\ldots$ (ii) $-2-2x-2x^2+\ldots$

(b) (i) $1+3x+9x^2+\ldots$ (ii) $1+10x+100x^2+\ldots$

(c) (i) $-2-10x-50x^2+\ldots$ (ii) $8+24x+72x^2+\ldots$

(d) (i) $40+10x+2.5x^2+\ldots$ (ii) $144+12x+x^2+\ldots$

(e) (i) $243-81x+27x^2+\ldots$ (ii) $1-\frac{5}{4}x+\frac{25}{16}x^2+\ldots$

(f) (i) $3-\frac{6}{x}+\frac{12}{x^2}+\ldots$ (ii) $18-\frac{9}{x}+\frac{1}{x^2}+\ldots$

(g) (i) $5+5(3-2x)+5(3-2x)^2+\ldots$

(ii) $7+\frac{7(2-x)}{2}+\frac{7(2-x)^2}{4}+\ldots$

(h) (i) $1+\left(3-\frac{2}{x}\right)+\left(3-\frac{2}{x}\right)^2+\ldots$ (ii) $1+\frac{1+x}{x}+\frac{(1+x)^2}{x^2}+\ldots$

(i) (i) $7+7x^2+7x^4+\ldots$ (ii) $12-48x^3+192x^6+\ldots$

3. Find the sum to infinity of the geometric sequence $-18, 12, -8, \ldots$. *[4 marks]*

4. The first and fourth terms of a geometric sequence are 18 and $-\frac{2}{3}$ respectively.

(a) Find the sum of the first n terms of the sequence.

(b) Find the sum to infinity of the sequence. *[5 marks]*

5. $f(x)=1+2x+4x^2+8x^3+\ldots$ is an infinitely long expression. Evaluate

(a) $f\left(\frac{1}{3}\right)$

(b) $f\left(\frac{2}{3}\right)$ *[6 marks]*

6. A geometric sequence has all positive terms. The sum of the first two terms is 15 and the sum to infinity is 27. Find the value of

(a) the common ratio

(b) the first term. *[5 marks]*

7. The sum to infinity of a geometric series is 32. The sum of the first four terms is 30 and all the terms are positive. Find the difference between the sum to infinity and the sum of the first eight terms. *[5 marks]*

8. Consider the infinite geometric series

$$1+\left(\frac{2x}{3}\right)+\left(\frac{2x}{3}\right)^2+\ldots$$

(a) For what values of x does the series converge?

(b) Find the sum of the series if $x=1.2$. *[6 marks]*

9. The sum of an infinite geometric sequence is 13.5, and the sum of the first three terms is 13. Find the first term. *[6 marks]*

10. An infinite geometric series is given by $\sum_{k=1}^{\infty} 2(4-3x)^k$.

(a) Find the values of x for which the series has a finite sum.

(b) When $x=1.2$, find the minimum number of terms needed to give a sum which is greater than 1.328. *[7 marks]*

11. The common ratio of the terms in a geometric series is 2^x.

(a) State the set of values of x for which the sum to infinity of the series exists.

(b) If the first term of the series is 35, find the value of x for which the sum to infinity is 40. *[6 marks]*

6H Mixed questions on sequences and series

Be very careful when dealing with questions on sequences and series. It is vital that you first:

- identify whether the question is about a geometric or an arithmetic sequence
- determine whether you are being asked for a term in the sequence or a sum of terms in the sequence
- translate the information given into equations that you can work with.

One frequently examined topic is **compound interest**. The questions are usually about savings or loans, where the interest added is a percentage of the current amount. As long as no other money is added or removed, the balance of the savings account or loan will follow a geometric sequence. A compound interest rate of $p\%$ is equivalent to a common ratio of $r = 1 + \frac{p}{100}$.

Worked example 6.19

A savings account pays 2.4% annual interest, added at the end of each year. If \$200 is paid into the account at the start of the first year, how much will there be in the account at the start of the 7th year?

Each year the balance of the account is increased by the same percentage, so this gives a geometric sequence.

Geometric sequence with

$$r = 1 + \frac{2.4}{100} = 1.024$$

$$u_1 = 200$$

If the start of the first year is u_1, the start of the 7th year is u_7.

$$u_7 = u_1 r^6$$

$$= 230.58$$

There will be \$230.58.

EXAM HINT

Think carefully about whether the amount you are calculating is for the beginning or the end of a year.

Exercise 6H

1. Philippa invests £1000 at 3% compound interest for 6 years.

(a) How much interest does she get paid in the 6th year?

(b) How much does she get back after 6 years? *[6 marks]*

2. Lars starts a job on an annual salary of \$32 000 and is promised an annual increase of \$1500.

(a) How much will his salary be in the 20th year?

(b) After how many complete years will he have earned a total of \$1 million? *[6 marks]*

3. A sum of \$5000 is invested at a compound interest rate of 6.3% per annum.

(a) Write down an expression for the value of the investment after n full years.

(b) What will be the value of the investment at the end of five years?

(c) The value of the investment will exceed \$10 000 after n full years.

(i) Write an inequality to represent this information.

(ii) Calculate the minimum value of n. *[8 marks]*

4. Suppose that each row of seats in a theatre has 200 more seats than the previous row. There are 50 seats in the front row and the designer wants the theatre's capacity to be at least 8000.

(a) How many rows are required?

(b) Assuming the rows are equally spread, what percentage of people are seated in the front half of the theatre? *[7 marks]*

5. A sum of \$100 is invested.

(a) If the interest is compounded annually at a rate of 5% per year, find the total value V of the investment after 20 years.

(b) If the interest is compounded monthly at a rate of $\frac{5}{12}$% per month, find the minimum number of months for the value of the investment to exceed V. *[6 marks]*

6. A marathon is a 26-mile race. In a training regime for a marathon, a runner runs 1 mile on his first day of training and increases his distance by $\frac{1}{4}$ of a mile each day.

(a) After how many days has he run a total of 26 miles?

(b) On which day does he first run over 26 miles? *[6 marks]*

7. A football is dropped vertically from 2 m above the ground. A model suggests that each time it bounces up to a height of 80% of its previous height.

(a) How high does it bounce on the 4th bounce?

(b) How far has it travelled when it hits the ground for the 9th time?

(c) Give one reason why this model is unlikely to work after 20 bounces. *[7 marks]*

8. Samantha puts \$1000 into a bank account at the beginning of each year, starting in 2010, which corresponds to year 1. At the end of each year, 4% interest is added to the account.

(a) Show that at the beginning of 2012 there is $\$1000 + \$1000 \times 1.04 + \$1000 \times (1.04)^2$ in the account.

(b) Find an expression for the amount in the account at the beginning of year n.

(c) When Samantha has a total of at least \$50 000 in her account at the beginning of a year she will start looking for a house to buy. In which year will this happen? *[7 marks]*

Summary

- Sequences can be described by either **recursive** (term to term) or **deductive** (nth term) rules.
- A **series** is a sum of terms in a sequence; it can be described concisely using **sigma notation**:

$$\sum_{r=1}^{r=n} f(r) = f(1) + f(2) + \ldots + f(n)$$

- One very important type of sequence is an **arithmetic sequence**, which has a constant difference, d, between consecutive terms. The relevant formulas are given in the information booklet:
  - if you know the first term u_1, the nth term in the sequence is: $u_n = u_1 + (n-1)d$
  - if you know the first and last terms, the sum to the nth term in the series is: $S_n = \frac{n}{2}(u_1 + u_n)$

– if you know the first term and the common difference, the sum to the nth term in the series is: $S_n = \frac{n}{2}(2u_1 + (n-1)d)$

- Another frequently encountered type of sequence is a **geometric sequence**, which has a constant ratio, r, between consecutive terms. The following formulas are also given in the information booklet:

  – if you know the first term u_1, the nth term in the sequence is: $u_n = u_1 r^{n-1}$

  – the sum of the first n terms in the series is: $S_n = \frac{u_1(1-r^n)}{1-r}$ or $\frac{u_1(r^n-1)}{r-1} (r \neq 1)$

- As the number of terms being added increases, a series can be **convergent** (the sum gets closer and closer to a single finite value) or **divergent** (the sum increases or decreases without bound).

  – if $|r| < 1$, the **sum to infinity** of a geometric series is given by

$$S_\infty = \frac{u_1}{1-r}$$

Introductory problem revisited

A mortgage of $100 000 has a fixed rate of 5% compound interest. It needs to be paid off in 25 years by fixed annual instalments. Interest is debited at the end of each year, just before the payment is made. How much should be paid each year?

Imagine two separate accounts: one in which the debt is accumulating interest, and another in which you deposit your payments, where they acquire interest at the same rate. The first payment you make will have interest paid on it 24 times, the second payment will have 23 interest payments, and so on.

After 25 years, the amount in the debt account will be $100\,000 \times 1.05^{25}$.

If the annual payment is $$x$, the amount in the credit account will be:

$$x \times 1.05^{24} + x \times 1.05^{23} + x \times 1.05^{22} + \ldots + x \times 1.05^{1} + x$$

This is a finite geometric series with 25 terms, where the first term is x and the common ratio is 1.05. Therefore, using the second sum formula in Key point 6.6, it can be simplified to

$$\frac{x(1.05^{25}-1)}{1.05-1}$$

If the debt is to be paid off, the amount in the credit account must equal the amount in the debt account; that is,

$$100\,000 \times 1.05^{25} = \frac{x(1.05^{25}-1)}{1.05-1}$$

Solving this gives $x = 7095.25$, so the annual instalment should be $7095.25.

Mixed examination practice 6

Short questions

1. The fourth term of an arithmetic sequence is 9.6 and the ninth term is 15.6. Find the sum of the first nine terms. *[5 marks]*

2. Which is the first term of the sequence $\frac{1}{3}, \frac{1}{9}, \ldots, \frac{1}{3^n}$ that is less than 10^{-6}? *[5 marks]*

3. The fifth term of an arithmetic sequence is three times larger than the second term. Find the ratio $\frac{\text{common difference}}{\text{first term}}$. *[6 marks]*

4. Evaluate $\sum_{n=0}^{n=\infty} \frac{(2^n + 4^n)}{6^n}$ *[6 marks]*

5. Find the sum of all the integers between 300 and 600 which are divisible by 7. *[7 marks]*

6. A geometric sequence and an arithmetic sequence both have 1 as their first term. The third term of the arithmetic sequence is the same as the second term of the geometric sequence. The fourth term of the arithmetic sequence is the same as the third term of the geometric sequence. Find the possible values of the common difference of the arithmetic sequence. *[7 marks]*

7. Find an expression for the sum of the first 23 terms of the series

$$\ln\frac{a^3}{\sqrt{b}} + \ln\frac{a^3}{b} + \ln\frac{a^3}{b\sqrt{b}} + \ln\frac{a^3}{b^2} + \cdots$$

giving your answer in the form $\ln\left(\frac{a^m}{b^n}\right)$ where $m, n \in \mathbb{Z}$. *[7 marks]*

Long questions

1. Kenny is offered a choice of two investment plans, each requiring an initial investment of \$10 000.

Plan A offers a fixed return of \$800 per year.

Plan B offers a return of 5% each year, reinvested in the plan.

(a) Find an expression for the amount in plan A after n years.

(b) Find an expression for the amount in plan B after n years.

(c) Over what period of time is plan A better than plan B? *[10 marks]*

2. Ben builds a pyramid out of toy bricks. The top row contains one brick, the second row contains three bricks, and each row beneath that contains two more bricks than the row above.

(a) How many bricks does the nth row (from the top) contain?

(b) If a total of 36 bricks are used, how many rows are there?

(c) In Ben's largest ever pyramid, he noticed that the total number of bricks was four more than four times the number of bricks in the bottom row. What is the total number of bricks in this pyramid? *[10 marks]*

3. A student writes '1' on the first line of a page, then the next two integers '2,3' on the second line of the page, then the next three integers '4,5,6' on the third line. She continues this pattern.

(a) How many integers are there on the nth line?

(b) What is the last integer on the nth line?

(c) What is the first integer on the nth line?

(d) Show that the sum of all the integers on the nth line is $\frac{n}{2}(n^2+1)$.

(e) The sum of all the integers on the last line of the page is 16 400. How many lines are on the page? *[10 marks]*

4. Selma has taken out a mortgage for £150 000. At the end of each year, 6% interest is added, and then Selma pays £10 000.

(a) Explain why at the end of the third year the amount still owed is

$$150\,000\times(1.06)^3-10\,000\times(1.06)^2-10\,000\times1.06-10\,000$$

(b) Find an expression for how much is owed at the end of the nth year.

(c) After how many years will the mortgage be paid off? *[10 marks]*