17 Using Graphs

17.1 Transformations of Graphs

There are 4 basic transformations of the graph of a function that are considered in this section. These are explored in the following worked examples and then summarised.

Worked Example 1

The function f is defined as $f(x) = x^2$. Plot graphs of each of the following and describe how they are related to the graph of $y = f(x)$:

(a) $y = f(x) + 2$ (b) $y = f(x+1)$

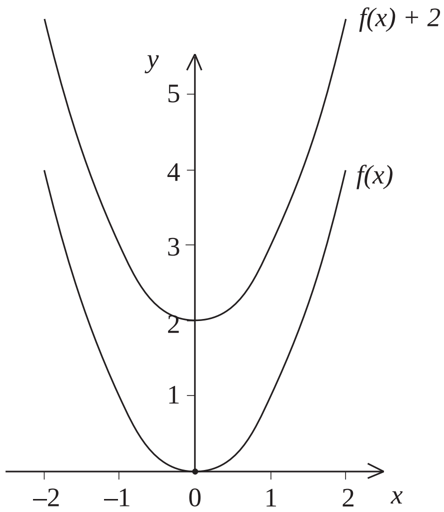
(c) $y = f(2x)$ (d) $y = 2f(x)$

Solution

The table below gives the values needed to plot these graphs.

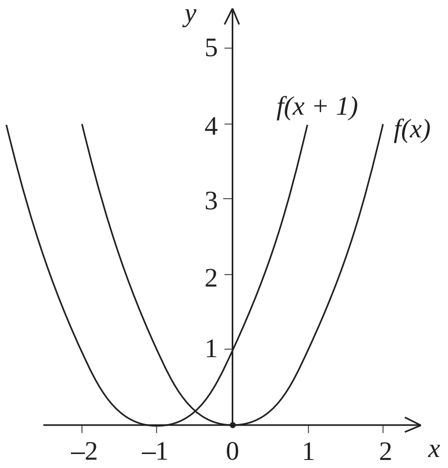
x	–2	–1	0	1	2
$f(x)$	4	1	0	1	4
$f(x) + 2$	6	3	2	3	6
$f(x+1)$	1	0	1	4	9
$f(2x)$	16	4	0	4	16
$2f(x)$	8	2	0	2	8

The graphs below show how each graph relates to $f(x)$.



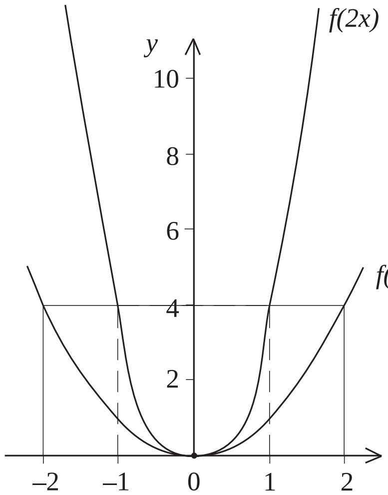
The graph of $y = f(x)$ is mapped onto the graph of $y = f(x) + 2$ by translating it up 2 units.

> In general $f(x) + a$ moves a curve up a units and $f(x) - a$ moves it down a units.



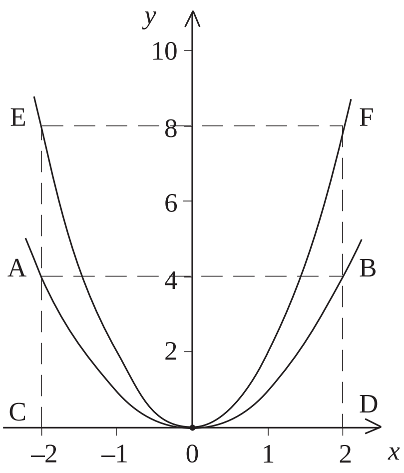
The graph of $y = f(x)$ is mapped onto $f(x+1)$ by a translation of 1 unit to the left.

> In general $f(x + a)$ translates a curve a units to the left and $f(x - a)$ translates a curve a units to the right.



The curve for $f(2x)$ is much steeper than for $f(x)$. This is because the curve has been compressed by a factor of 2 in the *x*-direction. Compare the rectangles ABCD and EFGH.

> In general the curve of $y = f(kx)$ will be compressed by a factor of k in the *x*-direction



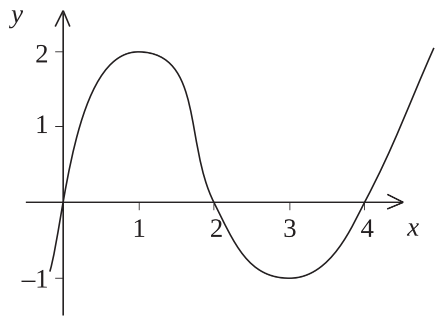
Here the curve $y = f(x)$ has been stretched by a factor of 2 in the vertical or *y*-direction to obtain the curve $y = 2f(x)$. Compare the rectangles ABCD and CDFE.

> In general the curve of $y = kf(x)$ stretches the graph of $y = f(x)$ by a factor of k in the *y*-direction.

Note that if k is negative the curve will be stretched and reflected in the *x*-axis.

Worked Example 2

The graph below shows $y = g(x)$.



On separate diagrams show:

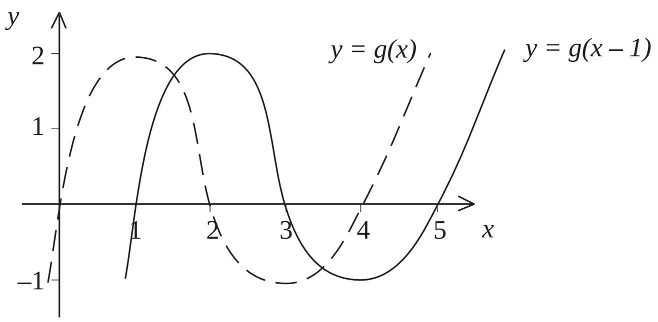
(a) $y = g(x)$ and $y = g(x-1)$

(b) $y = g(x)$ and $y = g(2x)$

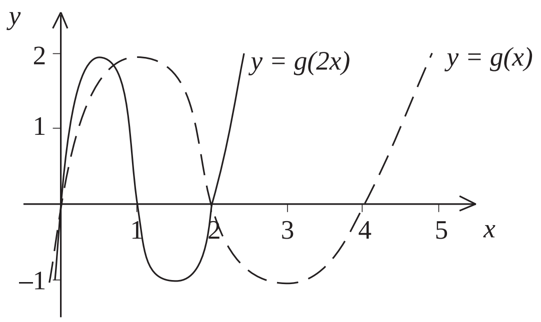
(c) $y = g(x)$ and $y = 3g(x)$

Solution

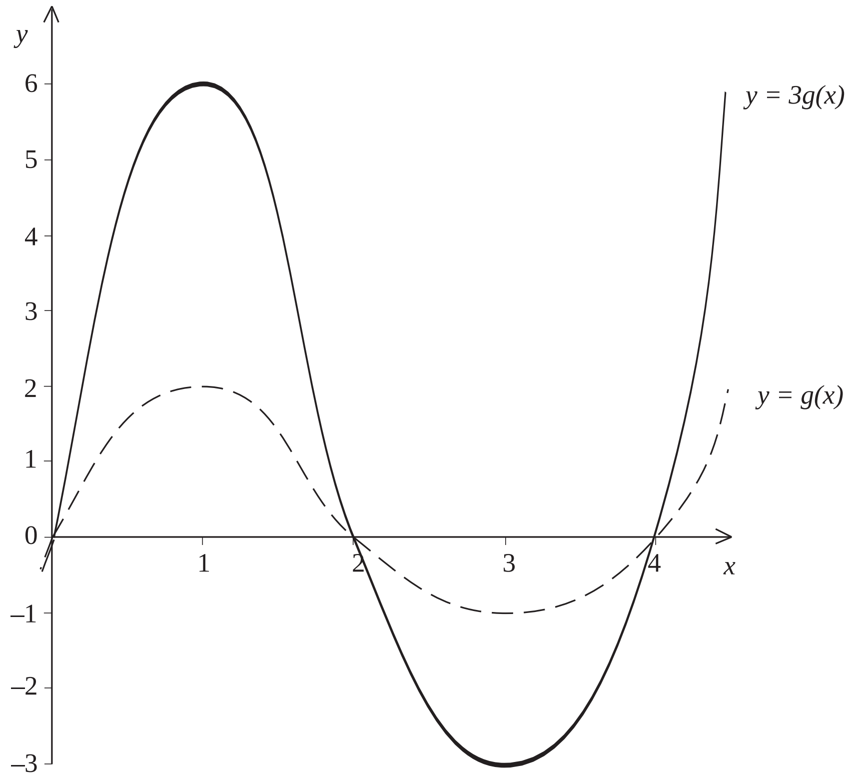
(a) To obtain $y = g(x-1)$ translate $y = g(x)$ 1 unit to the right.



(b) To obtain $y = g(2x)$ compress $y = g(x)$ by a factor of 2 horizontally.

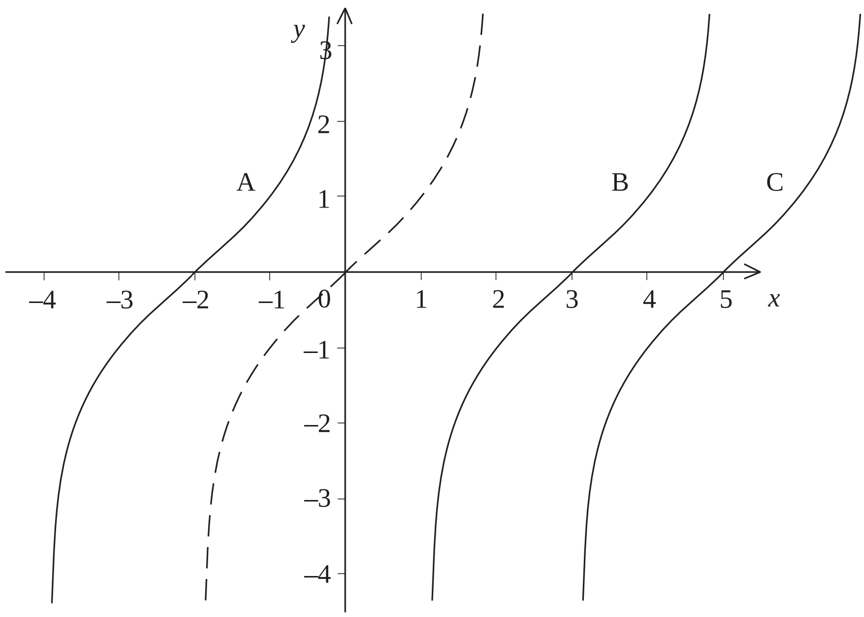


(c) To obtain the graph of $y = 3g(x)$ stretch the graph by a factor of 3 vertically.

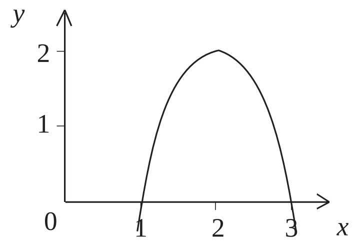


Exercises

1. The graph below shows $y = f(x)$ by a dashed curve.
Write down the equation of each other curve.



2. The graph below shows $y = h(x)$



On separate diagrams show:

(a) $y = h(x)$, $y = h(x) + 1$ and $y = h(x) - 2$

(b) $y = h(x)$ and $y = 2h(x)$

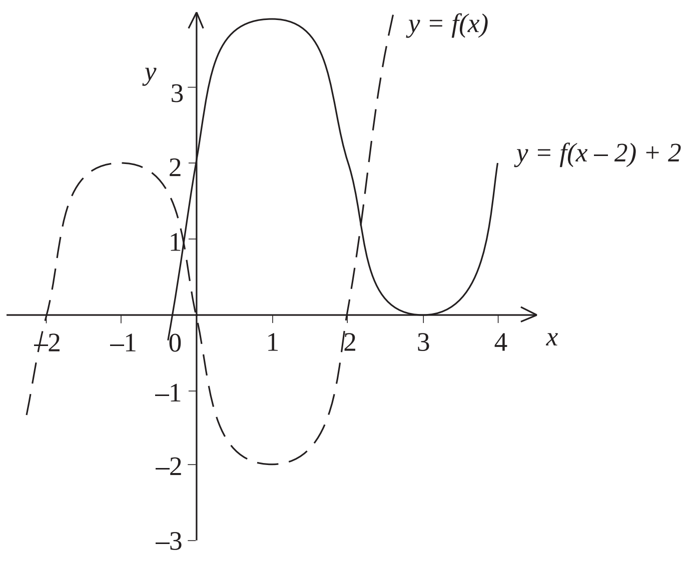
(c) $y = h(x)$ and $y = 3h(x)$

(d) $y = h(x)$ and $y = h(2x)$

3. On the same set of axes sketch the curves;

$$y = x^2,\ y = (x+3)^2,\ y = (x-4)^2 \text{ and } y = (x+1)^2.$$

4. The graph below shows $y = f(x)$ and $y = f(x-2) + 2$.



(a) Describe how to obtain the curve for $y = f(x-2) + 2$ from the curve for $y = f(x)$.

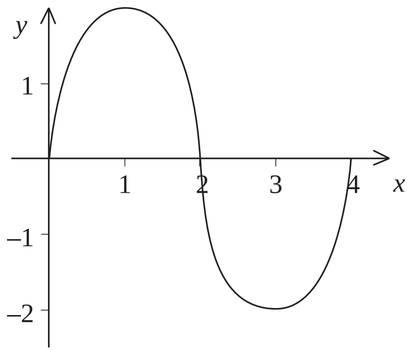
(b) On a set of axes sketch $y = f(x)$, $y = f(x-2) - 1$ and $y = f(x-1) + 1$.

5. On the same set of axes sketch

$$y = x^2,\ y = (x-2)^2 + 1,\ y = (x-3)^2 - 1 \text{ and } y = (x+3)^2 - 2 .$$

6. Draw the graphs of $y = x^2$, $y = 3x^2$, $y = -x^2$ and $y = -3x^2$. Describe how they compare.

7. The graph below shows $y = g(x)$.



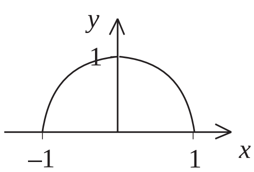
On separate sets of axes plot:

(a) $y = g(x)$ and $y = -g(x)$

(b) $y = g(x)$ and $y = -2g(x)$

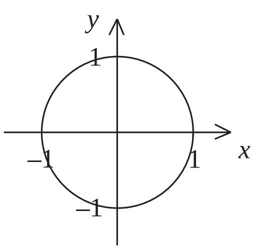
(c) $y = g(x)$ and $y = -\frac{1}{2}g(x)$

8. The function *f(x)* is such that the graph of $y = f(x)$ produces a graph as shown below, in the shape of a semi-circle.

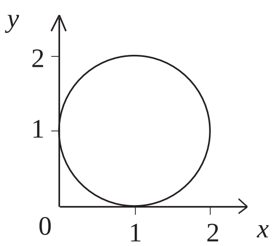


List the pairs of functions that should be plotted to produce the circles below.

(a)

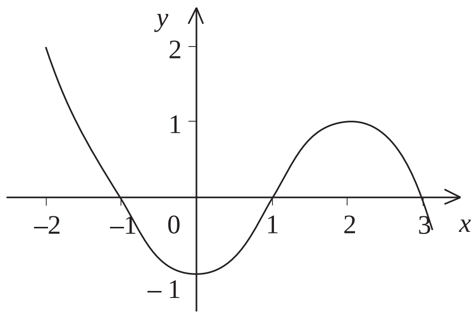


(b)



9. (a) Draw the graphs of $y = f(x)$ and $y = f(-x)$ if $f(x) = x^3$, and describe how the graphs are related.

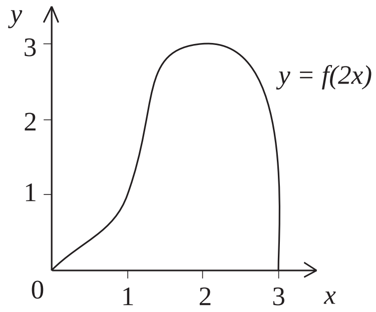
(b) The graph below shows $y = g(x)$.



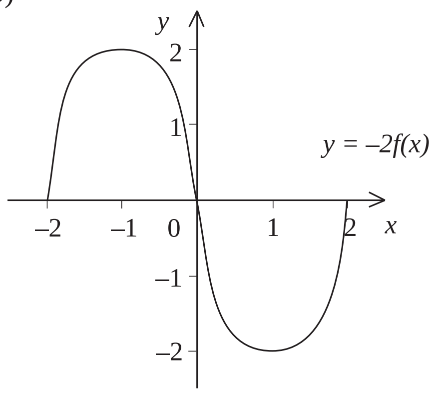
Sketch graphs of $y = g(-x)$, $y = g(-2x)$, $y = g\left(\frac{1}{2}x\right)$ and $y = g\left(-\frac{1}{2}x\right)$.

10. Use each graph below to sketch a graph of $y = f(x)$.

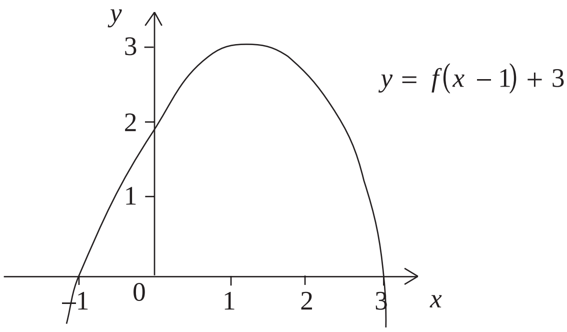
(a)



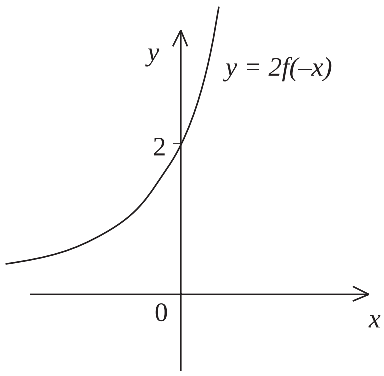
(b)



(c)

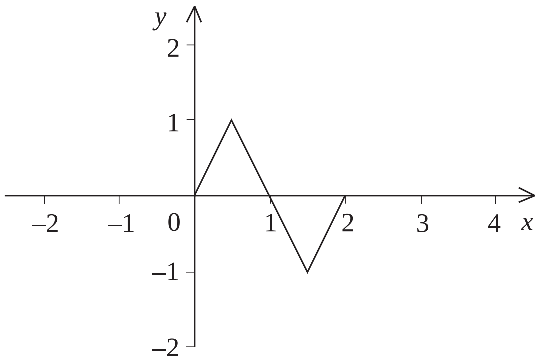


(d)

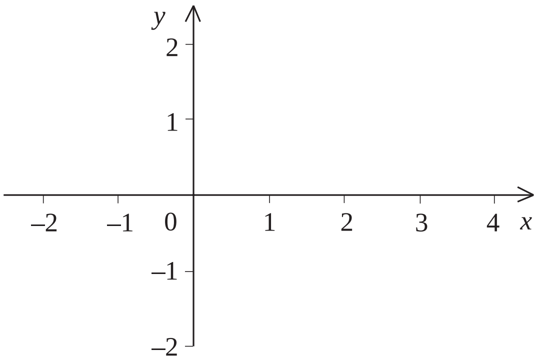


11. The function $y = f(x)$ is defined for $0 \le x \le 2$.

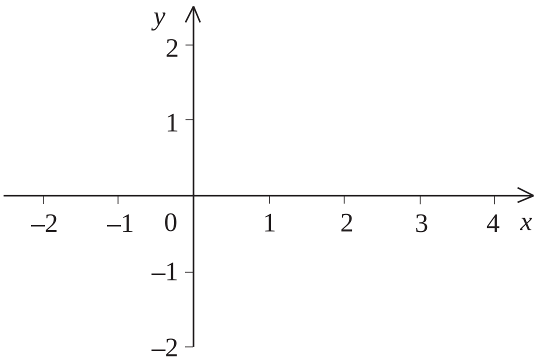
The function is sketched opposite.



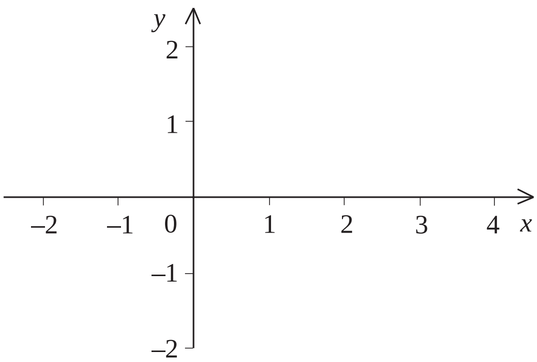
(a) Sketch $y = f(x) + 1$ on axes like the ones below.



(b) Sketch $y = f(x - 1)$ on axes like the ones below.

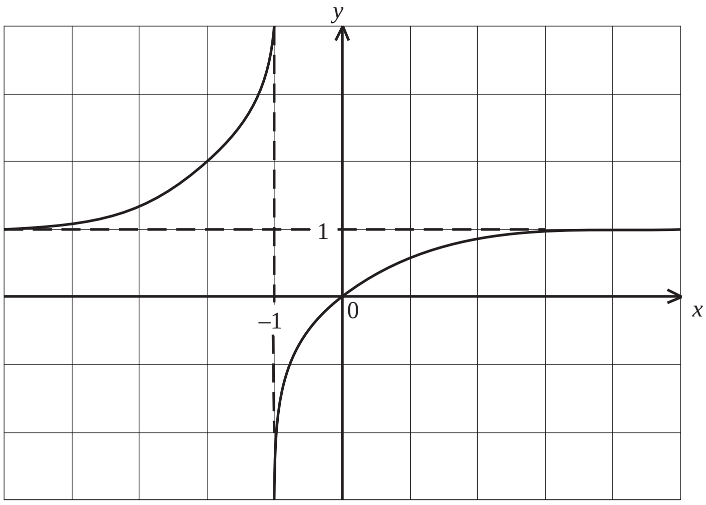


(c) Sketch $y = f\left(\frac{x}{2}\right)$ on axes like the ones below.



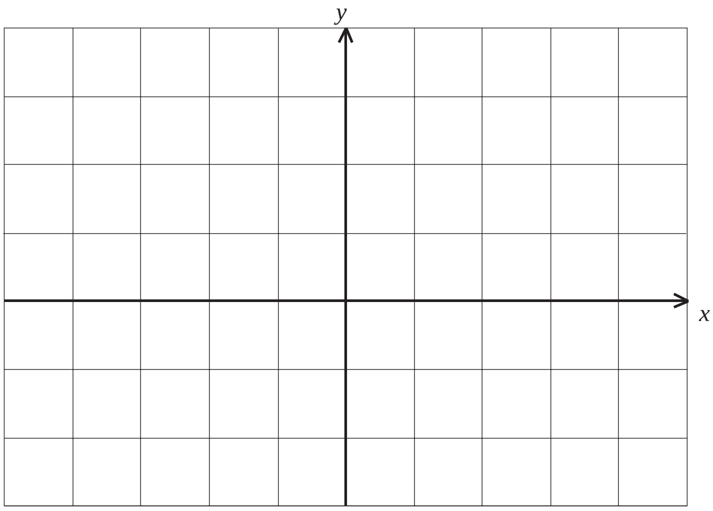
(SEG)

12. The graph of $y = f(x)$ where $f(x) = \dfrac{x}{x+1}$ is sketched below.

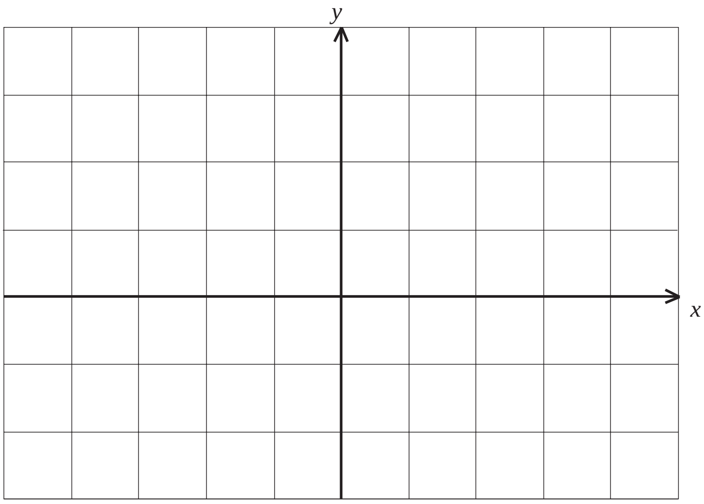


Hence, or otherwise, sketch on an axis like the one below

(a) $y = f(x-1)$

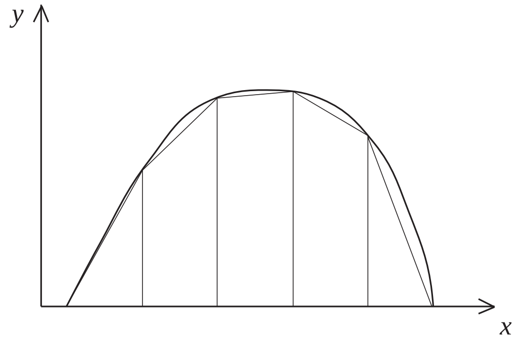


(b) $y = f(2x)$



17.2 Areas Under Graphs

It is possible to estimate the area under a curve by drawing trapezia and finding their areas as shown below.



In this example two triangles are included as well as 3 trapezia. The area given by the method will underestimate the area in this example, but for other curves it may overestimate the area.

Worked Example 1

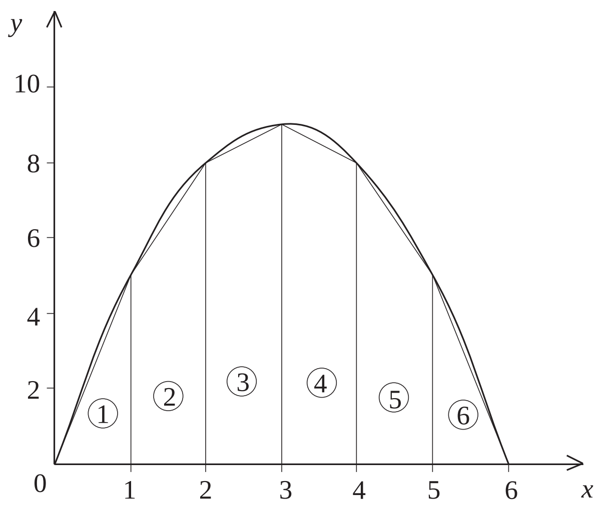
Estimate the area between the curve $y = 6x - x^2$ and the x-axis, by using trapezia, for $0 \le x \le 6$.

Solution

The table below gives the values of y for $0 \le x \le 6$.

x	0	1	2	3	4	5	6
y	0	5	8	9	8	5	0

This allows the graph to be plotted and the trapezia drawn in.



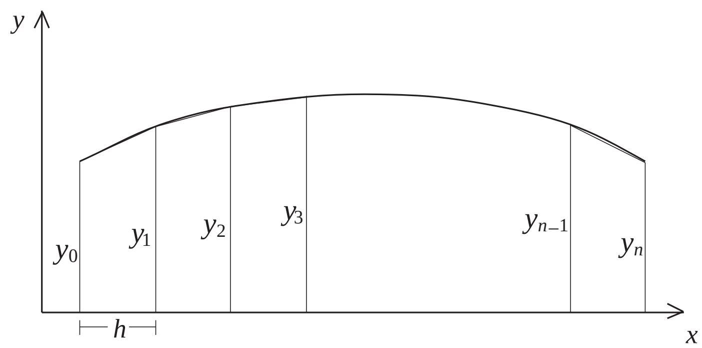
The area of each trapezium can then be calculated.

Trapezium	*Area*
1	$\frac{1}{2}(5+0)\times 1 = 2\frac{1}{2}$
2	$\frac{1}{2}(5+8)\times 1 = 6\frac{1}{2}$
3	$\frac{1}{2}(8+9)\times 1 = 8\frac{1}{2}$
4	$\frac{1}{2}(9+8)\times 1 = 8\frac{1}{2}$
5	$\frac{1}{2}(8+5)\times 1 = 6\frac{1}{2}$
6	$\frac{1}{2}(5+0)\times 1 = 2\frac{1}{2}$

$$\text{Total area} = 2\frac{1}{2} + 6\frac{1}{2} + 8\frac{1}{2} + 8\frac{1}{2} + 6\frac{1}{2} + 2\frac{1}{2}$$
$$= 35$$

The Trapezium Rule

If a number of trapezia are to be used to estimate an area then the trapezium rule can be used.



$$\text{Total area} = \frac{1}{2}(y_0 + y_1)h + \frac{1}{2}(y_1 + y_2)h + \frac{1}{2}(y_2 + y_3)h + \ldots + \frac{1}{2}(y_{n-1} + y_n)h$$
$$= \frac{h}{2}(y_0 + y_1 + y_1 + y_2 + y_2 + y_3 + \ldots + y_n)$$
$$= \frac{h}{2}(y_0 + 2y_1 + 2y_2 + 2y_3 + \ldots + 2y_{n-1} + y_n)$$

Note that y_1, y_2 etc. are all multiplied by 2 except for y_0 and y_n. Also, the narrower the trapezia, the more accurate the result.

Worked Example 2

Use the trapezium rule to find the area under the curve $y = 5x - x^2 + 2$ between $x = 1$ and $x = 5$.

Solution

The table below gives the value of y for $1 \le x \le 5$.

x	1	2	3	4	5
y	6	8	8	6	2

So $y_0 = 6$, $y_1 = 8$, $y_2 = 8$, $y_3 = 6$, $y_4 = 2$ and $h = 1$

Using the trapezium rule gives:

$$\text{Area} = \frac{1}{2}(6 + 2 \times 8 + 2 \times 8 + 2 \times 6 + 2)$$

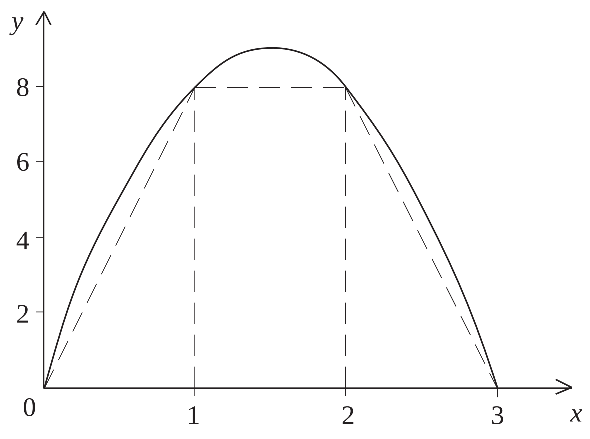
$$= \frac{1}{2}(6 + 16 + 16 + 12 + 2)$$

$$= \frac{1}{2} \times 52$$

$$= 26$$

Exercises

1. The graph of $y = 9x^2 - x^4$ is shown below.



Use the two triangles and the rectangle to estimate the area between the curve and the x-axis.

2. Use the values given in the table below to find the area under the curve $y = 6x - x^2 - 5$ between $x = 1$ and $x = 5$.

x	1	2	3	4	5
y	0	3	4	3	0

3. Estimate the area under the curve $y = 4 - x^2$ between $x = -2$ and $x = 2$ by using

   (a) 4 trapezia,

   (b) 8 trapezia.

4. (a) Draw the lines $x = 1$, $x = 5$ and the curve $y = x^2$.

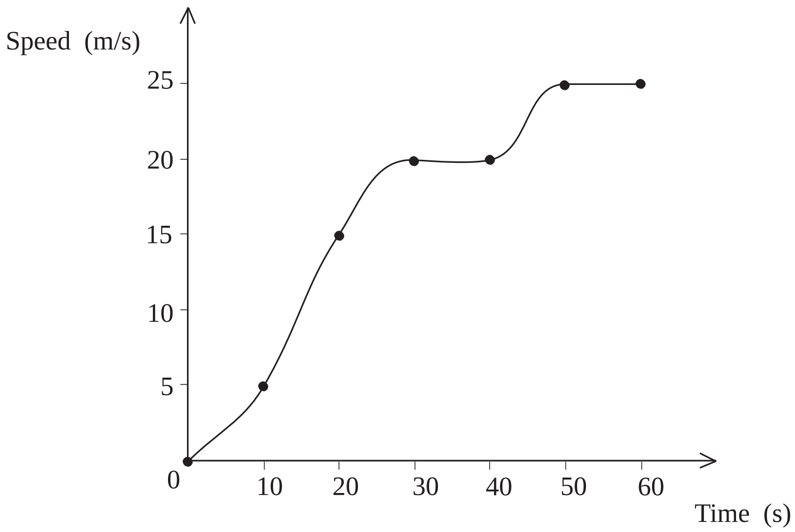
   (b) Estimate the area enclosed by the lines, the curve and the *x*-axis.

   (c) Explain whether the estimate is an underestimate or an overestimate.

5. Draw the graph of $y = x^2 - 4x + 5$ and the line $y = 2x$. Find the area between the curve and the line by using 4 trapezia.

6. A badge is made by cutting out the material between the curve $y = x^2$ and the curve $y = \frac{1}{2}x^2 + 2$. Draw these curves and estimate the area of the badge by using 4 trapezia. Comment on how well the trapezia estimate the area.

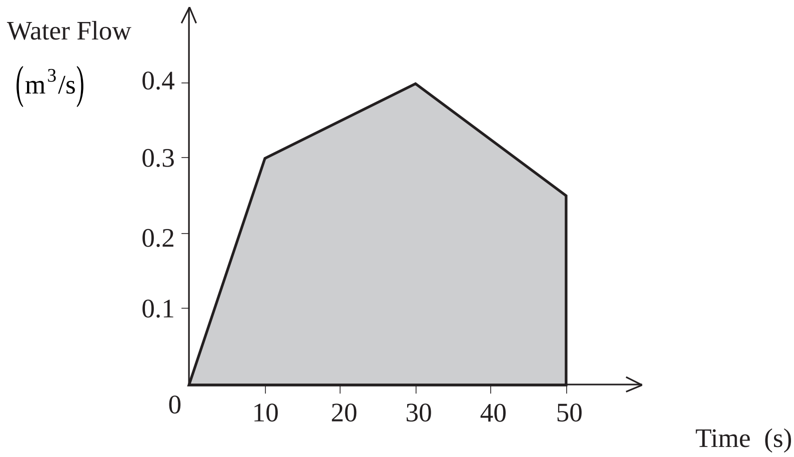
7. The graph below shows how the speed of a car varies as it sets off from rest.



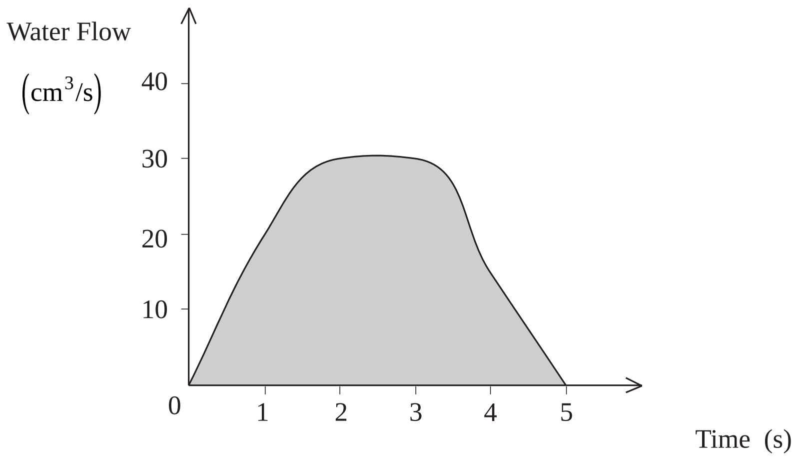
Estimate the distance travelled after 60 seconds.

8. The graphs below show how the rate at which water flows through a pipe varies with time. Find or estimate the volume of water represented by the shaded areas.

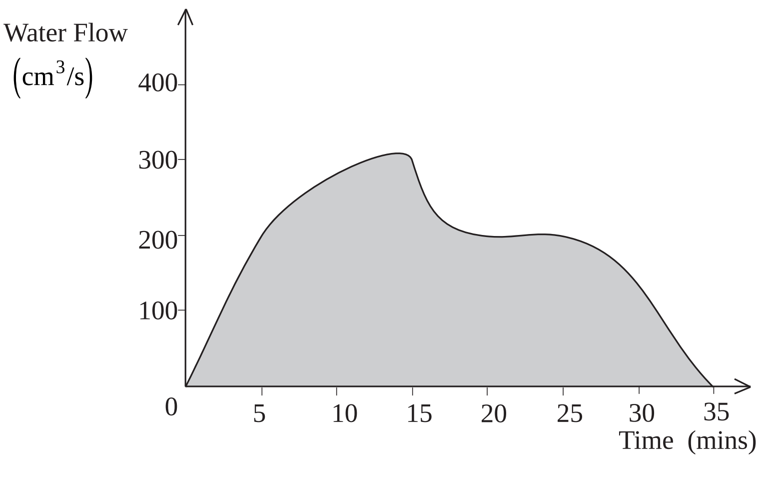
(a)



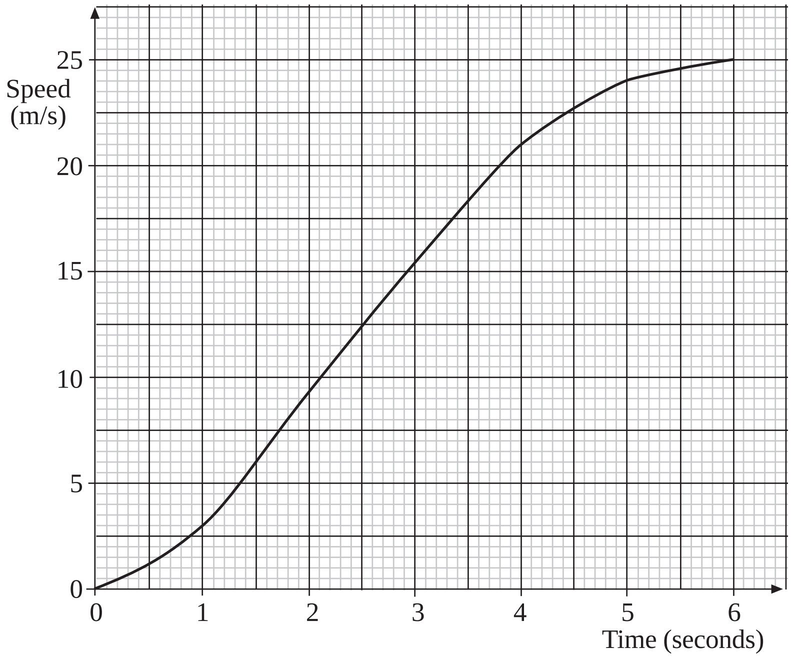
(b)



(c)



9.



The graph shows how a car's speed, measured in metres per second, varies in the first 6 seconds after the car moves away from some traffic lights.

(a) (i) Draw the tangent at the point on the curve where $t = 5$ seconds.

(ii) Find the gradient of this tangent.

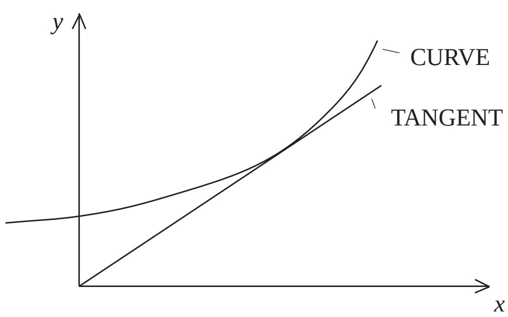
(iii) What does this gradient represent?

(b) Making your method clear, estimate the area beneath the graph between $t = 0$ and $t = 6$. Hence estimate the distance travelled by the car in the first 6 seconds.

(*MEG*)

17.3 Tangents to Curves

A tangent is a line that just touches a curve at one point, as shown opposite.



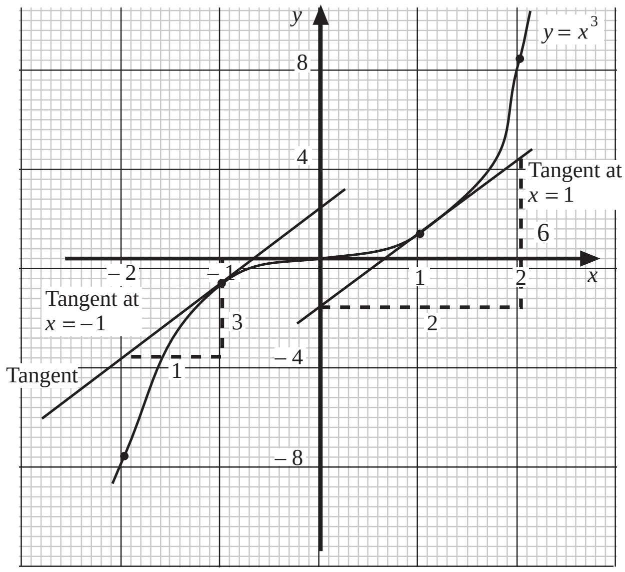
The gradient of the tangent gives the *gradient of the curve* at that point. The gradient of the curve gives the *rate* at which a quantity is changing. For example, the gradient of a distance-time curve gives the rate of change of distance with respect to time, which gives the velocity.

Worked Example 1

Draw the graph of $y = x^3$ for $-2 \le x \le 2$. Draw tangents to the curve at $x = -1$ and $x = 1$. Find the gradients of these tangents.

Solution

The graph of $y = x^3$ is shown below. The tangents have been drawn at $x = -1$ and $x = 1$.



Using the triangles shown under each tangent, show that the gradients of both tangents are 3.

Worked Example 2

The height, h, of a ball thrown straight up in the air varies so that at time, t, $h = 8t - 5t^2$.

Plot a graph of h against t and use it to find:

(a) the speed of the ball when $t = 0.6$,

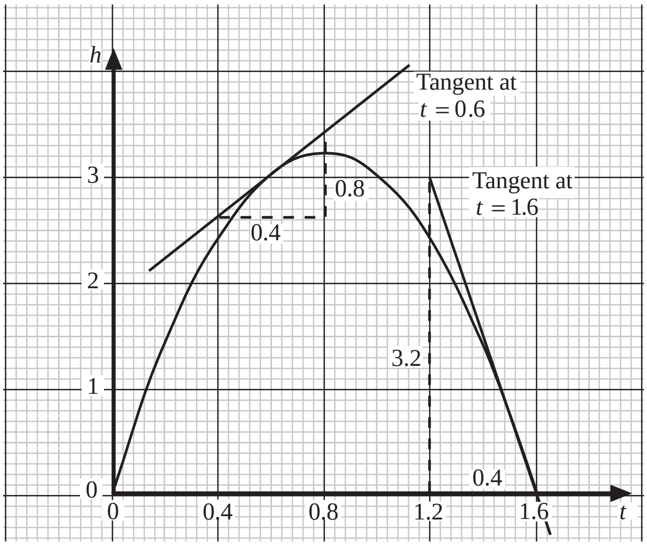
(b) the greatest speed of the ball.

Solution

The table below gives the values needed to plot the graph.

t (s)	0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6
h (m)	0	1.4	2.4	3.0	3.2	3.0	2.4	1.4	0

The graph is shown on the following page.



(a) A tangent has been drawn at the point where $t = 0.6$. The gradient of this tangent is $\frac{0.8}{0.4} = 2$. So the speed of the ball is 2 m/s.

(b) The speed of the ball is a maximum when the curve is steepest, that is at $t = 0$ and $t = 1.6$. At $t = 1.6$ the gradient is $\frac{-3.2}{0.4} = -8$. So the speed is 8 m/s.

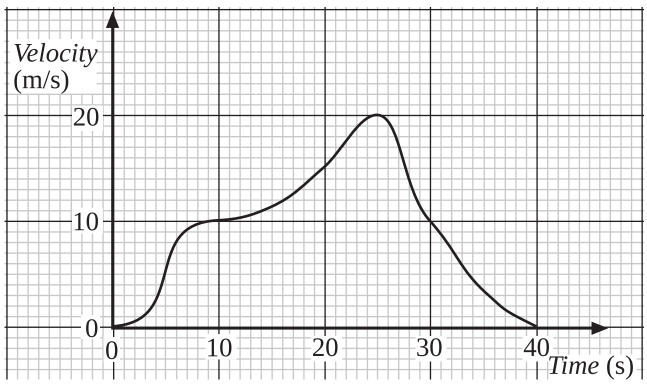
The '–' sign indicates that the ball is moving down rather than up. You can say the ball moves down with speed 8 m/s or that the velocity of the ball is –8 m/s.

Worked Example 3

The graph shows how the velocity of a car changes. Find:

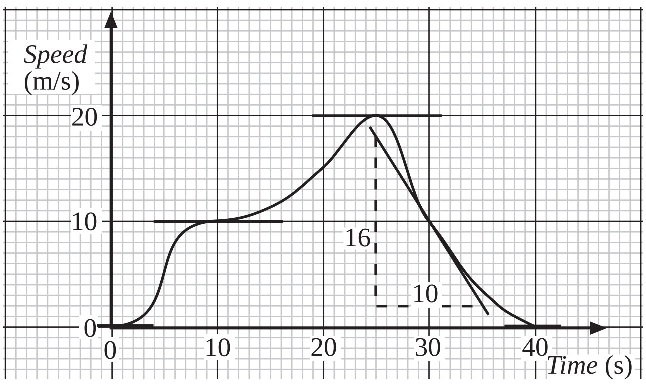
(a) the time when the acceleration of the car is zero,

(b) the acceleration when $t = 30$.



Solution

The acceleration of the car is given by the gradient of the velocity-time graph. There are 4 points where the gradient is zero, at $t = 0$, $t = 10$, $t = 25$ and $t = 40$. At each of these points a horizontal tangent can be drawn to the curve as shown below.



A tangent has been drawn to the curve at $t = 30$.

The gradient of this curve is $\dfrac{-16}{10} = -1.6$, so the acceleration is -1.6 m/s^2.

Exercises

1. (a) Draw the graph of $y = x^2$ for $0 \le x \le 4$.

   (b) By drawing tangents find the gradient of the curve at $x = 0$, $x = 1$, $x = 2$, $x = 3$ and $x = 4$.

   (c) Comment on any patterns that are present in your answers.

2. The height, h, of a ball at time, t, is given by, $h = 10t - 5t^2$.

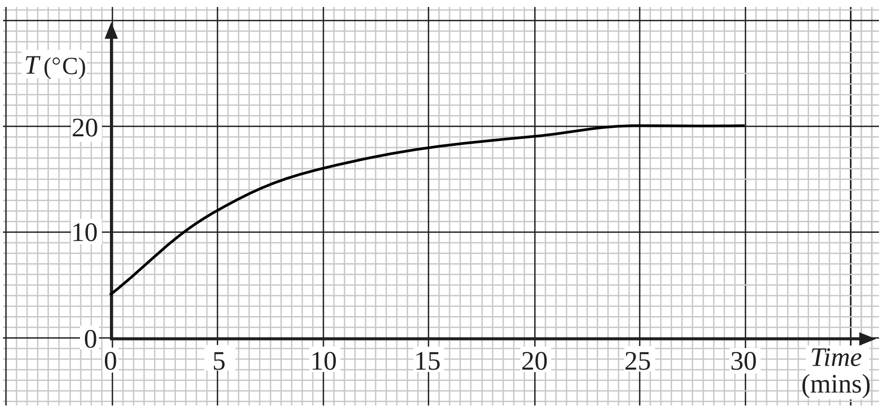
   The ball travels straight up and down.

   (a) Draw a graph of h against t for $0 \le t \le 2$.

   (b) Use the graph to find the velocity of the ball when $t = 0.5$ and $t = 1.2$.

   (c) Find the maximum speed of the ball.

3. The graph below shows how the temperature of a can of drink increases after it has been taken out of a fridge.



   Find the rate of change of temperature with respect to time, when;

   (a) $t = 0$,

   (b) $t = 10$,

   (c) $t = 15$.

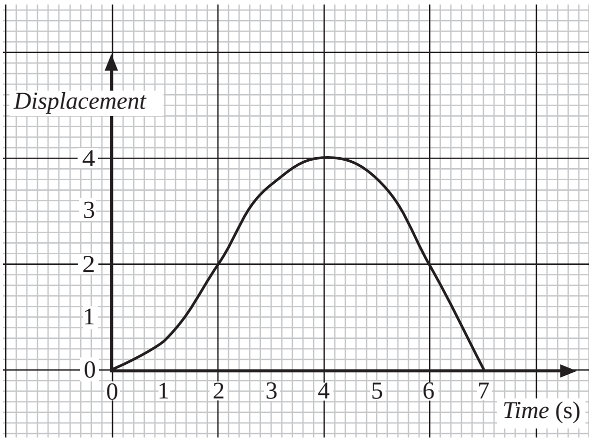
4. A car moves so that its velocity, v, and time, t, is given by $v = t^2 - 8t + 16$.

   (a) Plot a graph of velocity against time for $0 \le t \le 4$.

   (b) Find the gradient of the curve when $t = 0, 1, 2, 3$ and 4.

   (c) Use your results to (b) to sketch a graph of acceleration against time for the graph.

5.



The graph shows how the displacement of an object varies with time.

(a) Copy the graph and by drawing tangents estimate the velocity of the object when $t = 0, 1, 2, 3, 4, 5, 6$ and 7.

(b) Use your results to part (a) to draw a velocity-time graph.

(c) Consider your velocity-time graph and sketch an acceleration-time graph.

6. Draw the graph of $y = x^3$ for $-3 \le x \le 3$.

(a) Find the gradient of the curve at $x = -3, -2, -1, 0, 1, 2$ and 3.

(b) Can you predict how to calculate the gradient of $y = x^3$ for any value of x?

7. Draw a graph of $y = \sin x$ for $0 \le x \le 360°$.

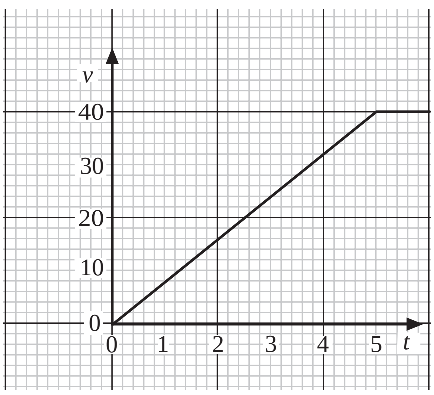
(a) For what values of x is the gradient of the curve zero?

(b) Find the maximum and minimum values of the gradient of the curve $y = \sin x$, and state the values of x for which they are obtained.

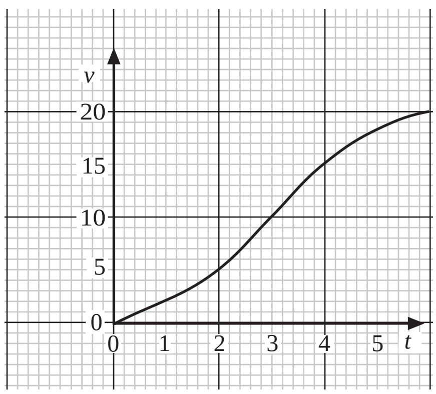
(c) Use these results to draw a graph of the gradient of $y = \sin x$ against x.

8. For each of the following velocity-time graphs, sketch an acceleration-time graph showing the maximum and minimum values of the acceleration.

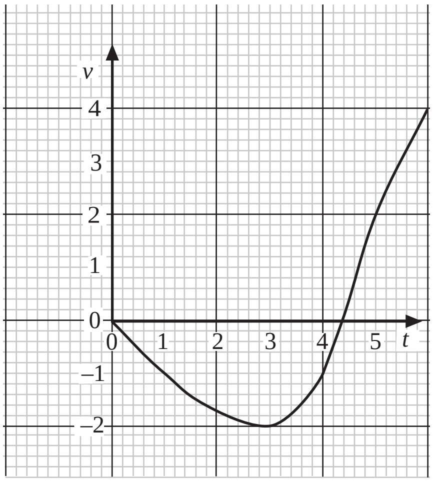
(a)



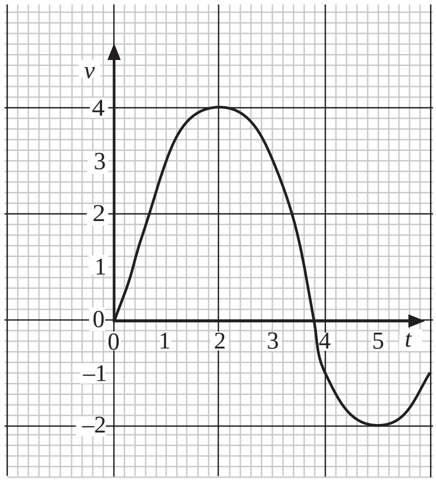
(b)



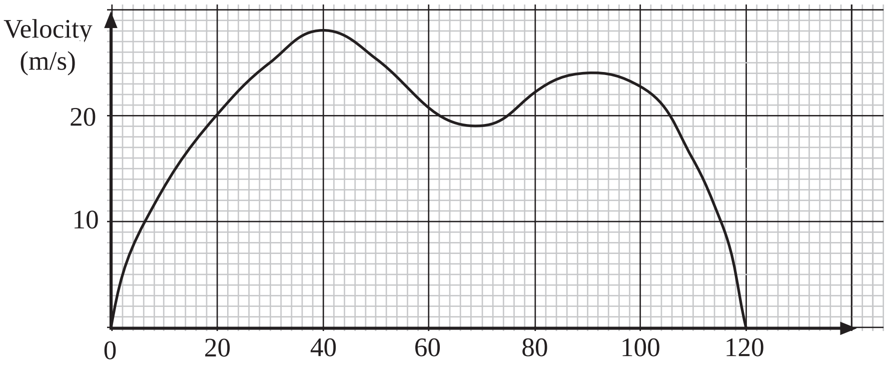
(c)



(d)



9. Here is a velocity-time graph of a car travelling between two sets of traffic lights.



Calculate an estimate for the acceleration of the car when the time is equal to 20 seconds.

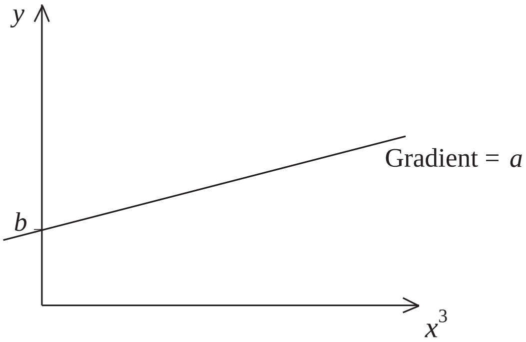
(*LON*)

17.4 Finding Coefficients

If you expect there to be a certain type of relationship between two variables, it is possible to test that relationship and determine any coefficients involved graphically. For example, if you expect the variables x and y to be related so that $y = ax^3 + b$, you would

plot a graph of y against x^3, which would give a straight line graph.

The gradient of this line is a and the intercept is b.



Worked Example 1

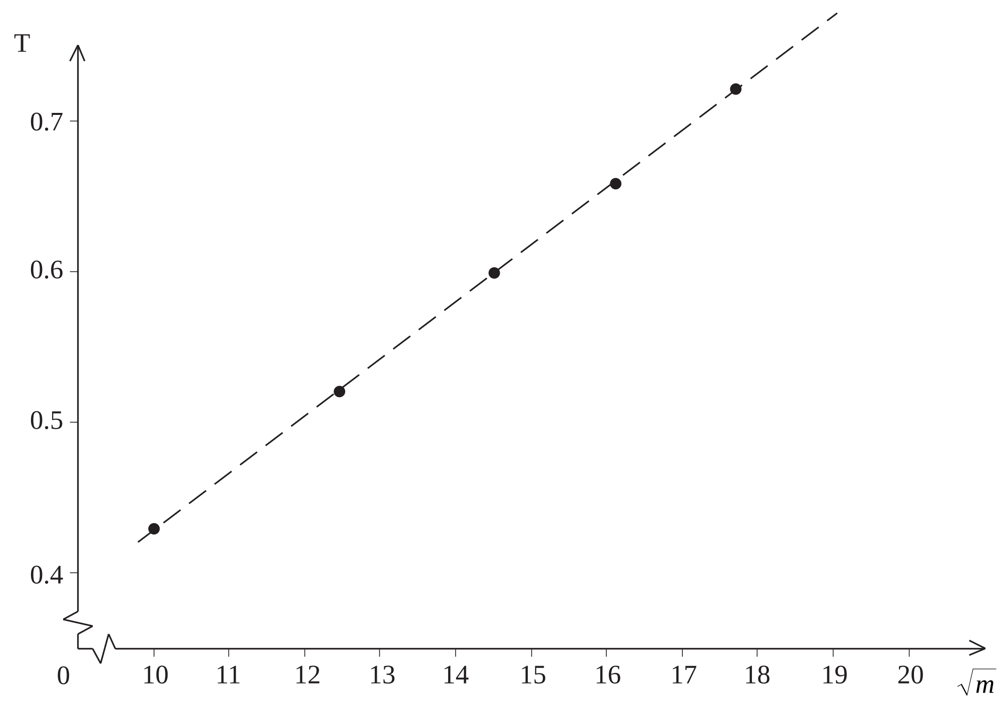
In an experiment different masses were attached to a spring. The masses were pulled down and released, so that they oscillated up and down. The time for one complete oscillation (the period) was recorded for each mass, The results are given in the table below.

Mass (grams)	100	150	200	250	300
Period (seconds)	0.43	0.52	0.60	0.66	0.72

It is suspected that $T = k\sqrt{m}$ where T is the period, m the mass and k is a constant. Show that this relationship is true and find the value of k.

Solution

First plot a graph of T against $\sqrt{m}$. This is shown below.



As the points lie on a straight line this confirms that the relationship is true. The value of k is given by the gradient of the line.

$$= \frac{0.72 - 0.43}{17.3 - 10}$$

$$= \frac{0.29}{7.3}$$

$$= 0.0397$$

Worked Example 2

A builder charges for building square patios using a formula of the form $C = ax^2 + b$ where C is the cost, x is the length of one side of the square and a and b are constants.

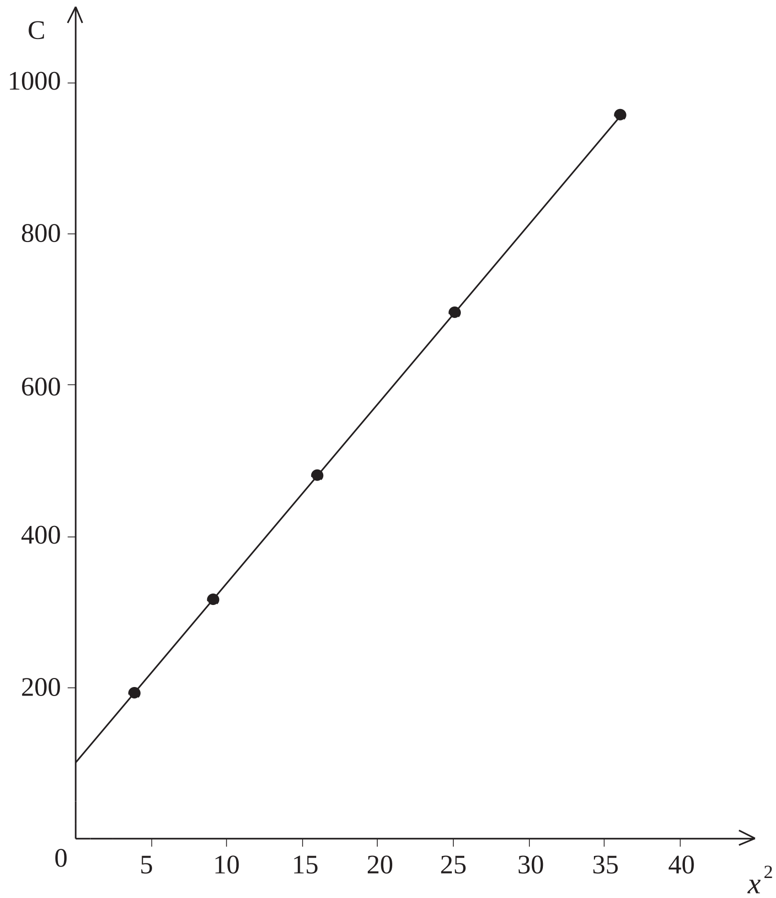
The table below gives the cost of building some square patios.

Length of side (m)	2	3	4	5	6
Cost (£)	196	316	484	700	964

Find the value of a and b.

Solution

First plot a graph of C against x^2.



These points lie on a straight line. The value of a is given by the gradient

$$a = \frac{964 - 196}{36 - 4}$$

$$= \frac{768}{32}$$

$$= 24$$

The value of b is given by the intercept, which is 100 in this case, so $b = 100$. The formula is then

$$C = 24x^2 + 100$$

Exercises

1. Each table below gives a set of data and the form of the relationship between the variables. For each table plot a suitable graph and find estimates for the values of a and b.

(a)

x	1	2	3	4
y	2	$5\frac{1}{2}$	15	$33\frac{1}{2}$

$y = ax^3 + b$

(b)

x	4	9	16	25
y	1	4	7	10

$y = a\sqrt{x} + b$

(c)

x	1	2	3	4	5
y	15.25	7.75	5.25	4.00	3.25

$y = \dfrac{a}{x} + b$

(d)

x	1	4	9	16	25
y	25	10	5	2.5	1

$y = \dfrac{a}{\sqrt{x}} + b$

2. The stopping distance for a car is made up of a thinking distance and a braking distance. These distances are given below.

Speed (mph)	*Thinking Distance* (metres)	*Braking Distance* (metres)
20	6	6
30	9	13.5
40	12	24
50	15	37.5
60	18	54
70	21	73.5

(a) Plot a graph of thinking distance against speed and write down the relationship between these two quantities.

(b) Plot a graph of braking distances against speed squared. Use this to find the relationship between speed and braking distance.

(c) Use your answers to (a) and (b) to write down a formula for the stopping distance for a car travelling at speed v.

3. A simple pendulum consists of a small mass on the end of a string. In an experiment the length of the string is varied and the period recorded. The results are given below.

Length (cm)	10	20	30	40	50	60
Period (seconds)	0.60	0.85	1.04	1.20	1.34	1.47

It is suspected that the period is proportional to the square root of the length of the string. Confirm that this is correct and find the constant of proportionality.

4. The distance travelled by a ball that is falling freely is proportional to the time it has been falling squared. The table gives the data collected from an experiment.

Time (s)	0	1	2	3	4	5
Distance Fallen (m)	0	4.9	19.6	44.1	78.4	122.5

Use a suitable graph to find the constant of proportionality.

5. The current, I, flowing through a resistor is measured in a science experiment. For different resistances, R, the current was recorded and is given in the table below.

R	100	200	500	1000
I	0.1	0.05	0.02	0.01

Plot a graph of R against $\frac{1}{I}$ and use it to write down the relationship between I and R.

6. John thinks that the heat produced, H, by an electric fire is proportional to the electric current, I, squared. He conducts an experiment and obtains the following results.

I	1	1.5	2.0	2.5	3.0	3.5
H	2.4	5.4	9.6	15.0	27.6	29.4

Draw a graph of H against I^2 to test John's theory. State if you think John's idea was correct and comment on the results of the experiment.

7. The height of a tide was recorded during a 12 hour period, to give the results in the table below.

Time	0	1	2	3	4	5	6	7	8	9	10	11	12
Height	5.1	7.1	8.6	9.1	8.6	7.1	5.1	3.1	1.6	1.1	1.6	3.1	5.1

A possible relationship between the height, H metres, and the time, t hours, is

$$H = A\sin(30t) + B$$

Plot a graph of H against $\sin(30t)$ and find the values of A and B.

Use the relationship to find the height of the tide when $t = 2.5$.