2 Exponents and logarithms

In this chapter you will learn:

- some rules for dealing with exponents
- about a class of functions where the unknown is in the exponent
- about the number e and some of its properties
- about the inverse of an exponential function, called a logarithm
- the rules of logarithms
- about graphs of logarithms
- how to use logarithms to find solutions to simple exponential equations.

Introductory problem

A radioactive substance has a half-life of 72 years (this is the time it takes for half of the mass, and hence radioactivity, to decay). A 1 kg block of the substance is found to have a radioactivity of 25 million becquerels (Bq). How long, to the nearest 10 years, would it take for the radioactivity to fall to 10 000 Bq?

Many mathematical models (of biological, physical and financial phenomena, for instance) involve the concept of continuous growth or decay, where the rate of growth or decay of the population of interest depends on the size of that population. You may have encountered similar situations already – for example, for a bank account earning compound interest, the increase in the amount of money each year is given by the interest rate multiplied by the starting balance. Such situations are governed by exponential functions, which you will learn about in this chapter.

See Prior Learning section C on the CD-ROM for a review of positive exponents.

2A Laws of exponents

The exponent of a number tells you how many times the number is to be multiplied by itself. You will have met some of the rules for dealing with exponents before, and in this section we shall revisit and extend these rules.

An exponent is also called a 'power' or an 'index' (plural: indices). The operation of raising to a power is sometimes referred to as 'exponentiating'.

A number written in **exponent form** is one which explicitly looks like:

$$a^n$$

where n is referred to as the **exponent**
a is referred to as the **base**

The expression a^n is read as 'a to the exponent n' or, more simply, 'a to the n'.

Although we said earlier that an exponent shows how many times a number is multiplied by itself, if the exponent is a negative integer this interpretation would not apply. Nevertheless, we would like to give a meaning to negative integer exponents; to do this, we first need to look at the rules of exponents in more detail.

EXAM HINT

Remember that $a^1 = a$.

If the exponent is increased by one, the value of the expression is multiplied by the base. If the exponent is decreased by one, the value of the expression is divided by the base. It follows that the value of a^0 must be consistent with the value of a^1 divided by a, that is, $a^0 = a^{1-1} = a^1 \div a = 1$.

Mathematics is often considered a subject without ambiguity. However, if you ask what the value of 0^0 is, the answer is undetermined – it depends upon how you get there!

KEY POINT 2.1

$$a^0 = 1$$

If we continue dividing by the value of the base, we move into negative exponents:

$$a^{-1} = a^0 \div a = \frac{1}{a} \qquad \text{For example, } 3^{-1} = \frac{1}{3}$$

$$a^{-2} = a^{-1} \div a = \frac{1}{a^2} \qquad \text{For example, } 3^{-2} = \frac{1}{9}$$

Generalising this gives the following formula:

KEY POINT 2.2

$$a^{-n} = \frac{1}{a^n}$$

*$\frac{1}{x}$ is called the **reciprocal** of x. We will study this in more detail in section 4E.*

We now look at an example of what can happen when we apply operations to exponents.

Worked example 2.1

Simplify:

(a) $a^3 \times a^4$ (b) $a^3 \div a^4$ (c) $\left(a^4\right)^3$ (d) $a^4 + a^3$

The exponent counts the number of times you multiply the base to itself.

(a) $a^3 \times a^4 = (a \times a \times a \times a) \times (a \times a \times a) = a^7$

Divide top and bottom of the fraction by *a* three times.

(b) $a^3 \div a^4 = \dfrac{\overset{1}{\cancel{a}} \times \overset{1}{\cancel{a}} \times \overset{1}{\cancel{a}}}{\underset{1}{\cancel{a}} \times \underset{1}{\cancel{a}} \times \underset{1}{\cancel{a}} \times a} = \dfrac{1}{a} = a^{-1}$

Use the same idea as in part (a).

(c) $\left(a^4\right)^3 = a^4 \times a^4 \times a^4 = a^{12}$

All we can do here is factorise.

(d) $a^4 + a^3 = a^3(a+1)$

It is questionable whether in part (d) of the example we have actually simplified the expression. Sometimes the way mathematicians choose to 'simplify' expressions is governed by aesthetics, i.e. how the result looks, as well as how useful it might be.

Worked example 2.1 suggests some rules of exponents.

KEY POINT 2.3

$$a^m \times a^n = a^{m+n}$$

KEY POINT 2.4

$$a^m \div a^n = a^{m-n}$$

KEY POINT 2.5

$$(a^m)^n = a^{m\times n}$$

With these rules, we can interpret exponents that are fractions (rational numbers) as well.

Consider $a^{\frac{1}{2}}$. If the laws of exponents hold true for fractional exponents, then we must have

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$$

In other words, $\left(a^{\frac{1}{2}}\right)^2 = a$; so $a^{\frac{1}{2}}$ is a value which, when squared, gives a. This is, by definition, the square root of a.

By a similar argument, $a^{\frac{1}{3}}$ is a value such that $\left(a^{\frac{1}{3}}\right)^3 = a$, and so $a^{\frac{1}{3}}$ is the same as the cube root of a. We can continue on to larger powers and roots; for example, $32^{\frac{1}{5}} = \sqrt[5]{32} = 2$ since $2^5 = 32$.

KEY POINT 2.6

$a^{\frac{1}{n}}$ means the nth root of a, that is, $\sqrt[n]{a}$.

We can combine Key points 2.5 and 2.6 to interpret $a^{\frac{m}{n}}$. We can express $a^{\frac{m}{n}}$ in two different ways, $a^{m\times\frac{1}{n}}$ or $a^{\frac{1}{n}\times m}$, and thus obtain the following rule.

KEY POINT 2.7

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$$

EXAM HINT

These rules are *not* given in the Formula booklet, so make sure that you learn them and can use them in both directions. For example, if you see 2^6, know that you can rewrite it as either $(2^3)^2$ or $(2^2)^3$; and if you see $(2^3)^2$, recognise it as 2^6. Both ways will be important!

Worked example 2.2

Evaluate $64^{\frac{2}{3}}$.

Use $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$

$$64^{\frac{2}{3}} = \left(\sqrt[3]{64}\right)^2$$
$$= 4^2$$
$$= 16$$

You must take care when expressions with *different* bases are to be combined by multiplication or division, for example $2^3 \times 6^2$. The rule 'multiplication means add the exponents together' (Key point 2.3) is true only when *the bases are the same*. You cannot use the rule to simplify $2^3 \times 6^2$.

There is however, another rule that works when the bases are different but *the exponents are the same*. Consider the following example:

$$\begin{aligned} 3^2 \times 5^2 &= 3\times 3\times 5\times 5 \\ &= 3\times 5\times 3\times 5 \\ &= 15\times 15 \\ &= 15^2 \end{aligned}$$

This suggests the following rule.

KEY POINT 2.8

$$a^n \times b^n = (ab)^n$$

Similarly, we can divide two expressions with the same exponent:

$$\frac{3^2}{5^2} = \frac{3\times 3}{5\times 5} \quad = \frac{3}{5}\times\frac{3}{5} \quad = \left(\frac{3}{5}\right)^2$$

KEY POINT 2.9

$$a^n \div b^n = \left(\frac{a}{b}\right)^n$$

Below is a summary of the rules of exponents covered so far.

If the base is the same and the exponents are different:	
Multiplication: keep the base the same, add the exponents	$a^m \times a^n = a^{m+n}$
Division: keep the base the same, subtract the exponents	$a^m \div a^n = a^{m-n}$
If the exponent is the same and the bases are different:	
Multiplication: keep the exponent the same, multiply the bases	$a^n \times b^n = (ab)^n$
Division: keep the exponent the same, divide the bases	$a^n \div b^n = \left(\frac{a}{b}\right)^n$
Any base to the exponent zero equals one	$a^0 = 1$
Any base to the exponent one is equal to the base itself	$a^1 = a$
Any base to a negative exponent is the same as the reciprocal of the same base to the equivalent positive exponent	$a^{-n} = \frac{1}{a^n}$
Any base to an exponent, all raised to another exponent, is the same as that base to the product of the exponents	$(a^m)^n = a^{m \times n}$
A base to a fractional exponent is a combination of an nth root and a power	$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$

You can use these rules to solve equations where the unknown is in the exponent.

Worked example 2.3

Solve the equation $4^{2x+1} = 8^{7-x}$.

If both sides had the same base, we could compare the exponents. Notice that 4 and 8 are both powers of 2.

$$4^{2x+1} = 8^{7-x}$$
$$(2^2)^{2x+1} = (2^3)^{7-x}$$
$$2^{4x+2} = 2^{21-3x}$$

Now we can equate the exponents.

$$4x + 2 = 21 - 3x$$
$$\Leftrightarrow \quad 7x = 19$$
$$\Leftrightarrow \quad x = \frac{19}{7}$$

Exercise 2A

1. Simplify the following, leaving your answer in exponent form.

(a) (i) $6^4 \times 6^3$ (ii) $5^3 \times 5^5$

(b) (i) $a^3 \times a^5$ (ii) $x^6 \times x^3$

(c) (i) $7^{11} \times 7^{-14}$ (ii) $5^7 \times 5^{-2}$

(d) (i) $x^4 \times x^{-2}$ (ii) $x^8 \times x^{-3}$

(e) (i) $g^{-3} \times g^{-9}$ (ii) $k^{-2} \times k^{-6}$

2. Simplify the following, leaving your answer in exponent form.

(a) (i) $6^4 \div 6^3$ (ii) $5^3 \div 5^5$

(b) (i) $a^3 \div a^5$ (ii) $x^6 \div x^3$

(c) (i) $5^7 \div 5^{-2}$ (ii) $7^{11} \div 7^{-4}$

(d) (i) $x^4 \div x^{-2}$ (ii) $x^8 \div x^{-3}$

(e) (i) $2^{-5} \div 2^{-7}$ (ii) $3^{-6} \div 3^8$

(f) (i) $g^{-3} \div g^{-9}$ (ii) $k^{-2} \div k^6$

3. Express the following in the specified form.

(a) (i) $(2^3)^4$ as 2^n (ii) $(3^2)^7$ as 3^n

(b) (i) $(5^{-1})^4$ as 5^n (ii) $(7^{-3})^2$ as 7^n

(c) (i) $(11^{-2})^{-1}$ as 11^n (ii) $(13^{-3})^{-5}$ as 13^n

(d) (i) $4 \times (2^5)^3$ as 2^n (ii) $3^{-5} \times (9^{-1})^{-4}$ as 3^n

(e) (i) $(4^2)^3 \times 3^{12}$ as 6^n (ii) $(6^3)^2 \div (2^2)^3$ as 3^n

4. Simplify each of the following, leaving your answer in exponent form with a prime number as the base.

(a) (i) 4^5 (ii) 9^7

(b) (i) 8^3 (ii) 16^5

(c) (i) $4^2 \times 8^3$ (ii) $9^5 \div 27^2$

(d) (i) $4^{-3} \times 8^5$ (ii) $3^7 \div 9^{-2}$

(e) (i) $\left(\frac{1}{4}\right)^3$ (ii) $\left(\frac{1}{9}\right)^3$

(f) (i) $\left(\frac{1}{8}\right)^2 \div \left(\frac{1}{4}\right)^4$ (ii) $9^7 \times \left(\frac{1}{3}\right)^4$

5. Write the following without brackets or negative exponents.

(a) (i) $(2x^2)^3$ (ii) $(3x^4)^2$

(b) (i) $2(x^2)^3$ (ii) $3(x^4)^2$

(c) (i) $\dfrac{(3a^3)^4}{9a^2}$ (ii) $\dfrac{(4x)^4}{8(2x)^4}$

(d) (i) $(2x)^{-1}$ (ii) $\left(\dfrac{3}{y}\right)^{-2}$

(e) (i) $2x^{-1}$ (ii) $\dfrac{3}{y^{-2}}$

(f) (i) $5\div\left(\dfrac{3}{xy^2}\right)^2$ (ii) $\left(\dfrac{ab}{2}\right)^3\div\left(\dfrac{a}{b}\right)^2$

(g) (i) $\left(\dfrac{2}{q}\right)^2\div\left(\dfrac{p}{2}\right)^{-3}$ (ii) $\left(\dfrac{6}{x}\right)^4\div\left(2\times\dfrac{3^2}{x}\right)^{-3}$

6. Evaluate the following, leaving your answer in simplified rational form where appropriate.

(a) (i) 3×2^{-2} (ii) 7×3^{-4}

(b) (i) $(3\times2)^{-2}$ (ii) $(5\times2)^{-3}$

(c) (i) $10^3\times5^{-2}$ (ii) $12^2\times4^{-5}$

(d) (i) $6\div2^3$ (ii) $6^{-3}\div2^{-5}$

7. Evaluate the following, leaving your answers as a fraction where appropriate.

(a) (i) $4^{\frac{1}{2}}$ (ii) $8^{\frac{1}{3}}$

(b) (i) $10000^{0.5}$ (ii) $81^{0.25}$

(c) (i) $\left(\dfrac{1}{25}\right)^{\frac{1}{2}}$ (ii) $\left(\dfrac{9}{16}\right)^{\frac{1}{2}}$

(d) (i) $8^{\frac{2}{3}}$ (ii) $25^{\frac{3}{2}}$

(e) (i) $100^{2.5}$ (ii) $81^{0.75}$

(f) (i) $\left(\dfrac{1}{16}\right)^{\frac{5}{4}}$ (ii) $\left(\dfrac{8}{27}\right)^{\frac{5}{3}}$

(g) (i) $8^{-\frac{1}{3}}$ (ii) $49^{-\frac{1}{2}}$

(h) (i) $\left(\dfrac{16}{9}\right)^{-\frac{1}{2}}$ (ii) $\left(\dfrac{9}{16}\right)^{-\frac{3}{2}}$

8. Simplify the following.

(a) (i) $(x^6)^{\frac{1}{2}}$ (ii) $(x^9)^{\frac{4}{3}}$

(b) (i) $\left(4x^{10}\right)^{0.5}$ (ii) $\left(8x^{12}\right)^{-\frac{1}{3}}$

(c) (i) $\left(\dfrac{27x^9}{64}\right)^{-\frac{1}{3}}$ (ii) $\left(\dfrac{x^4}{y^8}\right)^{-1.5}$

In section 2G you will see that there is another (often easier) way to solve equations like these, by using logarithms on a calculator.

9. Solve for x, giving your answer as a rational value.

(a) (i) $8^x = 32$ (ii) $25^x = \dfrac{1}{125}$

(b) (i) $\dfrac{1}{49^x} = 7$ (ii) $\dfrac{1}{16^x} = 8$

(c) (i) $2 \times 3^x = 162$ (ii) $3 \div 5^x = 0.12$

10. Suppose that a computer program is able to sort n input values in $k \times n^{1.5}$ microseconds. Observations show that it sorts a million values in half a second. Find the value of k. *[3 marks]*

11. A square-ended cuboid has volume xy^2, where x and y are its lengths. Such a cuboid for which $x = 2y$ has volume $128\,\text{cm}^3$. Find x. *[3 marks]*

12. The volume and surface area of a family of regular solid shapes are related by the formula $V = kA^{1.5}$, where V is given in cubic units and A in square units.

(a) For one such shape, $A = 81$ and $V = 243$. Find k.

(b) Hence determine the surface area of a shape with volume $\dfrac{64}{3}\ \text{cm}^3$. *[4 marks]*

13. Solve $2 \times 5^{x-1} = 250$ for x. *[5 marks]*

14. Solve $5 + 3^{x+2} = 14$ for x. *[5 marks]*

15. Solve $100^{x+5} = 10^{3x-1}$ for x. *[6 marks]*

16. Solve $16 + 2^x = 2^{x+1}$ for x. *[6 marks]*

17. Solve $6^{x+1} = 162 \times 2^x$ for x. *[6 marks]*

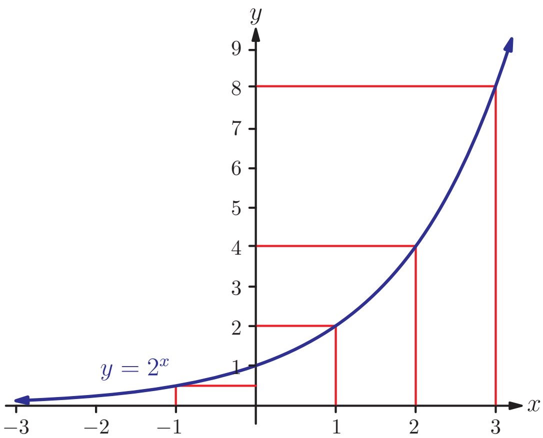
18. Solve $4^{1.5x} = 2 \times 16^{x-1}$ for x. *[6 marks]*

2B Exponential functions

In an exponential function, the unknown appears in the exponent. The general form of a simple exponential function is $f(x) = a^x$.

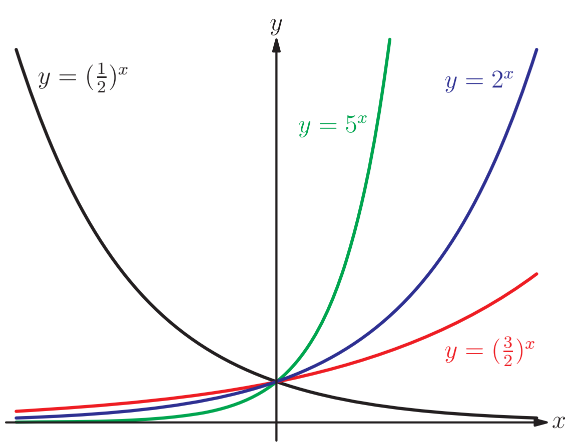
We will only consider situations where the base a is positive, because otherwise some exponents cannot easily be defined (for example, we cannot square root a negative number).

Here is the graph of $y = 2^x$.



For large positive values of x, the y-value gets very large ('approaches infinity'). For large negative values of x, the y-value approaches (but never reaches) zero. A line that a graph gets increasingly close to (but never touches) is called an **asymptote**. In this case we would say that the x-axis is an asymptote to the graph.

By looking at the graphs of exponential functions with different bases, we can begin to make some generalisations. Try plotting some exponential functions on your calculator; a few examples are shown below.

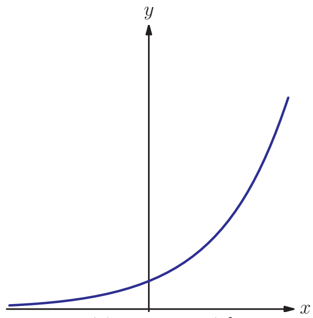


What is the meaning of $(-1)^{\frac{1}{2}}$? What about $\left((-1)^2\right)^{\frac{1}{4}}$? And $(-1)^{\frac{2}{4}}$? Not all mathematics is unambiguous!

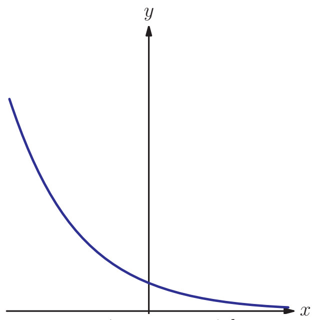
EXAM HINT

See Calculator Skills sheet 2 on the CD-ROM for how to plot graphs on your calculator.

You may notice that the blue curve is a reflection of the black curve in the y-axis. You will see why this is the case in chapter 5.



positive exponential



negative exponential

EXAM HINT

Note that exponential decay can be written either as $y = a^x$ with $0 < a < 1$ or as $y = a^{-x}$ with $a > 1$. This is because, for example,

$$3^{-x} = (3^{-1})^x = \left(\frac{1}{3}\right)^x.$$

In chapter 5 we will investigate how changing the constants in a function affects the shape and position of the graph.

KEY POINT 2.10

For the graphs of $y = a^x$, where $a > 0$:

- The y-intercept is always $(0, 1)$, because $a^0 = 1$.
- The graph of the function lies entirely above the x-axis, since $a^x > 0$ for all values of x.
- The x-axis is an asymptote.
- If $a > 1$, then as x increases, so does y. This is called a **positive exponential**.
- If $0 < a < 1$, then as x increases, y decreases. This is called a **negative exponential**.

Many mathematical models are based on the following property of the exponential function $N = a^t$: as time (t) increases by a *fixed amount*, the quantity we are interested in (N) will rise by a *fixed factor*, called the **growth factor**. Exponential functions can therefore be used to represent many physical, financial and biological forms of **exponential growth** (positive exponential models) and **exponential decay** (negative exponential models).

To model more complex situations we may need to include more constants in our exponential function. A form that is commonly used is

$$N = Ba^{\left(\frac{t}{k}\right)}$$

We can interpret the constants in the following way:

- When $t = 0$ we have $N = B$, so B is the **initial value** of N.
- When $t = k$ we have $N = Ba$, so k is the time taken for N to increase by a factor of a.
- If $k = 1$, then a is the growth factor.
- If k is positive, then with $a > 1$ the function models exponential growth, and with $0 < a < 1$ the function models exponential decay.

When modelling exponential decay, there may be a **background level**, so that N approaches some non-zero value c as t increases. For example, the temperature of a hot drink decays exponentially and approaches room temperature. This means that the asymptote is not necessarily $N = 0$. We can change the asymptote to $N = c$ by adding on a constant to get

$$N = Ba^{\left(\frac{t}{k}\right)} + c$$

In this case, B represents how much N starts above the background level, so the initial value is $B + c$.

KEY POINT 2.11

For $N = Ba^{\left(\frac{t}{k}\right)} + c$:

- the background level is c (i.e. the asymptote is $N = c$)
- the initial value is $B + c$
- k is the time taken for the difference between N and the background level to increase by a factor of a
- if $a > 1$ the function models exponential growth
- if $0 < a < 1$ the function models exponential decay.

EXAM HINT

Your calculator may not display asymptotes, as they are not really part of the graph; you can only guess approximately where they are by looking at large values of x. This is why it is important to know how to find asymptotes directly from the equation.

EXAM HINT

Once you have plotted a graph on your calculator, you can use it to find x and y values, as well as check the value of the asymptote.

See Calculator Skills sheet 4 on the CD-ROM for how to analyse graphs.

Remember to round calculator answers to 3 significant figures, unless asked to do otherwise.

Worked example 2.4

The temperature T, in degrees Celsius, of a cooling liquid is modelled by the equation $T = 24 + 72 \times 0.6^{3t}$, where t is the time in minutes after the cooling begins.

(a) What was the initial temperature of the liquid?

(b) Find the temperature of the liquid after 2 minutes.

(c) How long does it take for the liquid to cool to 26°C?

(d) What temperature does the model predict the liquid will eventually reach?

The word 'initial' means when $t = 0$.

(a) When $t = 0$,
$T = 24 + 72 = 96°C$

continued . . .

When the answer is not exact, we should round it to 3 significant figures.

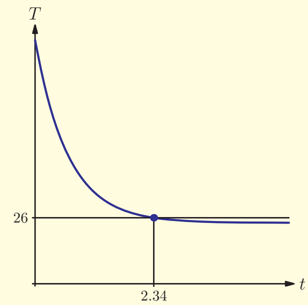
(b) When $t = 2$,

$T = 24 + 72 \times 0.6^6$

$= 27.4°C$ (3 SF)

We can answer this by looking at the graph on a calculator.

(c)



From GDC: $t = 2.34$ (3 SF)

It will take 2.34 minutes.

In the long term, the temperature approaches the asymptote.

(d) 24°C

In many applications, we first need to find the constants in the model using the information given.

Worked example 2.5

A population of bacteria in a culture medium doubles in size every 15 minutes. At 08:00 there are 1000 bacterial cells. Let N be the number of bacterial cells t hours after 08:00.

(a) Write down a model for N in terms of t.

(b) How many cells are there at

(i) 08:15?

(ii) 09:24?

There is a constant factor of increase, so we use an exponential growth model.

(a) $N = Ba^{\left(\frac{t}{k}\right)}$

Let N be the number of cells at time t hours after 08:00.

Every time t increases by 0.25 hours, N doubles.

Doubles every quarter hour $\Rightarrow$

$a = 2,\ k = 0.25$

$\therefore N = B \times 2^{4t}$

continued . . .

Initial value gives B.

(b) When $t=0$, $N=1000=B$, so
$N=1000\times 2^{4t}$

Remember to convert minutes to hours.

(i) When $t=0.25$, $N=1000\times 2=2000$ cells.
(ii) When $t=1.4$, $N=1000\times 2^{5.6}=48503$ cells.

When modelling exponential growth or decay, you may be given a percentage increase or decrease. This needs to be converted into a growth factor to be used in the exponential model.

We will meet similar questions again when we study geometric series in chapter 6.

Worked example 2.6

A car that cost \$17 500 initially loses value at a rate of 18% each year.

(a) Write a model for the value (V) of the car after n years in the form $V = ka^n$.

(b) Hence or otherwise find the value of the car after 20 years.

Find the growth factor.

(a) The growth factor is $1-\frac{18}{100}=0.82$

Use initial value information.

When $n=0$, $V=k=17500$, hence
$V=17500\times 0.82^n$

Substitute for n.

(b) After 20 years, $V=17500\times 0.82^{20}=\$330.61$

EXAM HINT

'Hence or otherwise' means that you can use any method you like, but the word 'hence' suggests that the answers from previous parts of the question might be helpful.

Exercise 2B

1. Using your calculator, sketch the following functions on the same set of axes, for $-5 \le x \le 5$ and $0 \le y \le 10$. Show all the axis intercepts and state the equation of the horizontal asymptote.

   (a) (i) $y = 1.5^x$ (ii) $y = 3^x$

   (b) (i) $y = 2 \times 3^x$ (ii) $y = 6 \times 1.4^x$

   (c) (i) $y = \left(\frac{1}{2}\right)^x$ (ii) $y = \left(\frac{2}{3}\right)^x$

   (d) (i) $y = 5 + 2^x$ (ii) $y = 8 + 3^x$

   (e) (i) $y = 6 - 2^x$ (ii) $y = 1 - 5^x$

2. An algal population on the surface of a pond grows by 10% every day. The area it covers can be modelled by the equation $y = k \times 1.1^t$, where t is measured in days, starting from 09:00 on Tuesday, when the algae covered $10\,\text{m}^2$. What area will it cover by 09:00 on Friday? *[4 marks]*

3. A tree branch is observed to bend as the fruit growing on it increase in size. By estimating the mass of the developing fruit and plotting the data over time, a student finds that the height h in metres of the branch end above the ground is closely approximated by the function

$$h = 2 - 0.2 \times 1.6^{0.2m}$$

   where m is the estimated mass, in kilograms, of fruit on the branch.

   (a) Sketch the graph of h against m.

   (b) What height above ground is the branch without fruit?

   (c) The total mass of fruit on the branch at harvest was 7.5 kg. Find the height of the branch immediately prior to harvest.

   (d) The student wishes to estimate what mass of fruit would cause the branch end to touch the ground. Why might his model not be suitable to assess this? *[10 marks]*

4. (a) Sketch the graph of $y = 1 + 16^{1-x^2}$. Label clearly the horizontal asymptote and the maximum value.

   (b) Find all values of x for which $y = 3$. *[6 marks]*

5. A bowl of soup is served at a temperature of 55°C, in a room with air temperature 20°C. Every 5 minutes, the

temperature difference between the soup and the room air decreases by 30%. Assuming that the room air temperature remains constant, at what temperature will the soup be 7 minutes after serving? *[7 marks]*

6. The speed V (in metres per second) of a parachutist t seconds after jumping from an aeroplane is modelled by the equation

$$V = 40(1-3^{-0.1t})$$

(a) Find the parachutist's initial speed.

(b) What speed does the model predict that the parachutist will approach eventually? *[6 marks]*

7. The air temperature T (in degrees Celsius) around a light bulb is given by the equation

$$T = A + B\times 2^{-\frac{x}{k}}$$

where x is the distance in millimetres from the surface of the light bulb. The background temperature in the room is a constant 25°C, and the temperature on the surface of the light bulb is 125°C.

(a) Suppose that the air temperature 3 mm from the surface of the bulb is 75°C. Find the values of A, B and k.

(b) Determine the air temperature 2 cm from the surface of the bulb.

(c) Sketch a graph of air temperature against distance from the surface of the bulb. *[10 marks]*

2C The number e

In this section we introduce a special mathematical constant, e, which will be used extensively in the rest of this chapter and throughout the course.

Consider the following three situations, which might describe early population growth of a cell culture, for example.

There is a 100% increase every 100 seconds.

There is a 50% increase every 50 seconds.

There is a 25% increase every 25 seconds.

At first glance these may appear to be equivalent statements, but they are subtly different because of the compounding nature of percentage increases. If we begin with a population of size P, then after 100 seconds the population becomes, in the three cases,

$$P \times (1+1) = 2P$$

$$P \times \left(1+\frac{1}{2}\right)^2 = 2.25P$$

$$P \times \left(1+\frac{1}{4}\right)^4 = 2.44P$$

Generalising, if we consider an increase of $\frac{100}{n}$% which occurs n times in the course of 100 seconds, the population after 100 seconds would be given by $P \times \left(1+\frac{1}{n}\right)^n$.

From the above it may seem that as n increases, the overall increase in population over 100 seconds will keep getting larger. This is indeed the case, but not without limit. By taking larger and larger values of n, it can be seen that the population increase factor $\left(1+\frac{1}{n}\right)^n$ tends towards a value of approximately 2.71828182849. This number, much like π, arises so often in mathematics and is so useful in applications that it has been given its own letter, e.

KEY POINT 2.12

$$\mathrm{e} = 2.71828182849\ldots$$

The numbers π and e have many similar properties. Both are irrational, meaning that they cannot be written as a fraction of two whole numbers. They are also both transcendental, which means that they cannot be the solution to any polynomial equation (an equation involving only powers of x). The proofs of these facts are intricate but beautiful.

In chapter 12 you will see that e plays a major role in studying rates of change.

EXAM HINT

In questions involving the number e, you may be asked either to give an exact answer (for example, in the form of e^2) or to use your calculator, in which case you should round the answer to 3 significant figures unless told otherwise.

Although e has many important properties, it is after all just a number. Therefore, the standard rules of arithmetic and exponents still apply.

Exercise 2C

1. Find the values of the following to 3 significant figures.

(a) (i) $e+1$ (ii) $e-4$

(b) (i) $3e$ (ii) $\frac{e}{2}$

(c) (i) e^2 (ii) e^{-3}

(d) (i) $5e^{0.5}$ (ii) $\frac{3}{e^7}$

2. Evaluate $\sqrt[6]{(\pi^4+\pi^5)}$. What do you notice about the result?

3. Expand $\left(e^2+\frac{2}{e^2}\right)^2$. *[4 marks]*

2D Introduction to logarithms

In this section we shall look at an operation which reverses the effect of exponentiating (raising to a power) and allows us to find an unknown power. If you are asked to solve

$$x^2 = 3 \text{ for } x \geq 0$$

then you can can either find a decimal approximation (for example by using a calculator or trial and improvement) or use the square root symbol to write

$$x = \sqrt{3}$$

This statement just says that 'x is the positive value which when squared gives 3'.

Similarly, to solve

$$10^x = 50$$

we could use trial and improvement to seek a decimal value:

$$10^1 = 10$$

$$10^2 = 100$$

See the Supplementary sheet 2 'Logarithmic scales and log-log graphs' on the CD-ROM if you are interested in discovering logarithms for yourself.

So x must be between 1 and 2:

$$10^{1.5} = 31.6$$

$$10^{1.6} = 39.8$$

$$10^{1.7} = 50.1$$

So the answer is around 1.7.

Just as we can use the square root to answer the question 'what is the number which when squared gives this value?', there is also a function that can be used to answer the question 'what is the number which when put as the exponent of 10 gives this value?' This function is called a base-10 **logarithm**, written $\log_{10}$.

In the above example, we can write the solution as $x = \log_{10} 50$. More generally, the equation $y = 10^x$ can be re-expressed as $x = \log_{10} y$. In fact, the base need not be 10, but could be any positive value other than 1.

The symbol $\Leftrightarrow$ means that if the left-hand side is true then so is the right-hand side, and if the right-hand side is true then so is the left-hand side. When it appears between two statements, it means that the statements are equivalent and you can switch between them.

KEY POINT 2.13

$$b = a^x \Leftrightarrow x = \log_a b$$

It is worth noting that the two most common bases have abbreviations for their logarithms. Since we use a decimal system of counting, 10 is the default base for a logarithm, so $\log_{10} x$ is usually written simply as $\log x$ and is called the 'common logarithm'. Also, the number e that we met in section 2C is considered the 'natural' base, so the base-e logarithm is called the 'natural logarithm' and is denoted by $\ln x$.

EXAM HINT

Log x and ln x have a button on graphical calculators ('log' and 'ln') that you can use to evaluate. If the base is not 10 or e however, you will have to use the principles of Key point 2.15. Or, use the change-of-base rule in Key point 2.22.

KEY POINT 2.14

$\log_{10} x$ is often written as $\log x$

$\log_e x$ is often written as $\ln x$

Since taking a logarithm reverses the process of exponentiating, we have the following facts:

KEY POINT 2.15

$$\log_a(a^x) = x$$

$$a^{\log_a x} = x$$

These are referred to as the cancellation principles. This sort of 'cancellation', similar to stating that (for positive x) $\sqrt[n]{x^n} = x = \left(\sqrt[n]{x}\right)^n$, is often useful when simplifying logarithm expressions; but remember that you can only do such cancellations when the base of the logarithm and the base of the exponential match and are immediately adjacent in the expression.

The cancellation principles can be combined with the rules of exponents to derive an interesting relationship between the base-e exponential function and any other exponential function. From the second cancellation principle it follows that $e^{\ln a} = a$. By raising both sides to the power x and using the rule of exponents $\left(b^y\right)^x = b^{yx}$ (Key point 2.5), we obtain the following useful formula.

A related change-of-base rule for logarithms is given in Key point 2.22.

KEY POINT 2.16

$$e^{x\ln a} = a^x$$

When we study rates of change in chapter 12, we will need to use base e for exponential functions.

This says that we can always change the base of an exponential function to e.

Worked example 2.7

Evaluate

(a) $\log_5 625$ (b) $\log_8 16$

Express the argument of the logarithm in exponent form with the same base.

(a) $\log_5 625 = \log_5 5^4$

Apply the cancellation principle $\log_a(a^x) = x$.

$= 4$

The argument of the logarithm, 16, is not a power of the base 8, but both 8 and 16 are powers of 2.

(b) $\log_8(16) = \log_8(2^4)$

continued . . .

Using a rule of exponents, convert 2^4 to an exponent of $8 = 2^3$.

$$= \log_8\left(2^{3\times\frac{4}{3}}\right)$$

$$= \log_8\left(8^{\frac{4}{3}}\right)$$

Apply the cancellation principle $\log_a(a^x) = x$.

$$= \frac{4}{3}$$

Whenever you raise a positive number to a power, whether positive or negative, the result is always positive. Therefore a question such as 'to what power do you raise 10 to get −3?' has no answer.

KEY POINT 2.17

You cannot take the logarithm of a negative number or zero.

Exercise 2D

1. Evaluate the following:

(a) (i)	$\log_3 27$	(ii)	$\log_4 16$
(b) (i)	$\log_5 5$	(ii)	$\log_3 3$
(c) (i)	$\log_{12} 1$	(ii)	$\log_{15} 1$
(d) (i)	$\log_3 \frac{1}{3}$	(ii)	$\log_4 \frac{1}{64}$
(e) (i)	$\log_4 2$	(ii)	$\log_{27} 3$
(f) (i)	$\log_8 \sqrt{8}$	(ii)	$\log_2 \sqrt{2}$
(g) (i)	$\log_8 4$	(ii)	$\log_{81} 27$
(h) (i)	$\log_{25} 125$	(ii)	$\log_{16} 32$
(i) (i)	$\log_4 2\sqrt{2}$	(ii)	$\log_9 81\sqrt{3}$
(j) (i)	$\log_{25} 0.2$	(ii)	$\log_4 0.5$

2. Use a calculator to evaluate each of the following, giving your answer correct to 3 significant figures.

(a) (i)	$\log 50$	(ii)	$\log\left(\frac{1}{4}\right)$
(b) (i)	$\ln 0.1$	(ii)	$\ln 10$

3. Simplify the following expressions:

(a) (i) $7\log x - 2\log x$ (ii) $2\log x + 3\log x$

(b) (i) $(\log x - 1)(\log y + 3)$ (ii) $(\log x + 2)^2$

(c) (i) $\dfrac{\log a + \log b}{\log a \log b}$ (ii) $\dfrac{(\log a)^2 - 1}{\log a - 1}$

See Prior Learning section O on the CD-ROM if you need to brush up on simplifying fractions.

4. Make x the subject of the following:

(a) (i) $\log_3 x = y$ (ii) $\log_4 x = 2y$

(b) (i) $\log_a x = 1 + y$ (ii) $\log_a x = y^2$

(c) (i) $\log_x 3y = 3$ (ii) $\log_x y = 2$

EXAM HINT

Remember that 'log x' is just another value so can be treated the same way as any variable.

5. Find the value of x in each of the following:

(a) (i) $\log_2 x = 32$ (ii) $\log_2 x = 4$

(b) (i) $\log_5 25 = 5x$ (ii) $\log_{49} 7 = 2x$

(c) (i) $\log_x 36 = 2$ (ii) $\log_x 10 = \dfrac{1}{2}$

6. Solve the equation $\log_{10}(9x+1) = 3$. *[4 marks]*

7. Solve the equation $\log_8 \sqrt{1-x} = \dfrac{1}{3}$. *[4 marks]*

8. Find the exact solution to the equation $\ln(3x-1) = 2$. *[5 marks]*

9. Find all values of x which satisfy $(\log_3 x)^2 = 4$. *[5 marks]*

10. Solve the equation $3(1+\log x) = 6 + \log x$. *[5 marks]*

11. Solve the equation $\log_x 4 = 9$. *[4 marks]*

12. Solve the simultaneous equations

$$\log_3 x + \log_5 y = 6$$

$$\log_3 x - \log_5 y = 2$$

[6 marks]

13. The Richter scale is a way of measuring the strength of earthquakes. An increase of one unit on the Richter scale corresponds to an increase by a factor of 10 in the strength of the earthquake. What would be the Richter level of an earthquake which is twice as strong as a level 5.2 earthquake? *[5 marks]*

2E Laws of logarithms

Just as there are rules to follow when performing arithmetic with exponents, so there are corresponding rules which apply to logarithms.

See Fill-in proof 12 'Differentiating logarithmic functions graphically' on the CD-ROM for how these rules for logarithms can be derived from the laws of exponents.

KEY POINT 2.18

The logarithm of a *product* is the *sum* of the logarithms.

$$\log_a xy = \log_a x + \log_a y$$

For example, you can check that $\log_2 32 = \log_2 8 + \log_2 4$.

KEY POINT 2.19

The logarithm of a *quotient* is the *difference* of the logarithms.

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

For example, $\log 6 = \log 42 - \log 7$.

KEY POINT 2.20

The logarithm of an *exponent* is the *multiple* of the logarithm.

$$\log_a x^r = r \log_a x$$

For example, $\log_5 8 = \log_5 (2^3) = 3\log_5 2$.

A special case of this is the logarithm of a reciprocal:

$$\log x^{-1} = -\log x$$

We know that $a^0 = 1$ irrespective of a. We can express this in terms of logarithms:

KEY POINT 2.21

The logarithm of 1 is always 0, irrespective of the base.

$$\log_a 1 = 0$$

EXAM HINT

It is just as important to know what you *cannot* do with logarithms. One very common mistake is to rewrite $\log(x+y)$ as $\log x + \log y$ or as $\log x \times \log y$.

We can use the laws of logarithms to manipulate expressions and solve equations involving logarithms, as the next two examples illustrate.

Worked example 2.8

If $x = \log_{10} a$ and $y = \log_{10} b$, express $\log_{10} \frac{100a^2}{b}$ in terms of x, y and integers.

Use laws of logs to isolate $\log_{10} a$ and $\log_{10} b$ in the given expression. First, use the law about the logarithm of a fraction.

$$\log_{10} \frac{100a^2}{b} = \log_{10}(100a^2) - \log_{10} b$$

Then use the law about the log of a product.

$$= \log_{10} 100 + \log_{10} a^2 - \log_{10} b$$

For the second term use the law about the log of an exponent.

$$= \log_{10} 100 + 2\log_{10} a - \log_{10} b$$

Finally, simplify (by evaluating) $\log_{10} 100$.

$$= 2 + 2\log_{10} a - \log_{10} b$$
$$= 2 + 2x - y$$

Worked example 2.9

Solve the equation $\log_2 x + \log_2(x+4) = 5$.

Rewrite the left-hand side as a single logarithm.	$\log_2 x + \log_2(x+4) = 5$ $\Leftrightarrow \log_2(x(x+4)) = 5$
Undo the logarithm by exponentiating both sides; the base must be the same as the base of the logarithm, 2.	$2^{\log_2(x(x+4))} = 2^5$
Apply the cancellation principle to the left-hand side.	$x(x+4) = 32$ $x^2 + 4x = 32$
This is a quadratic equation, which can be factorised.	$x^2 + 4x - 32 = 0$ $(x+8)(x-4) = 0$ $x = -8$ or $x = 4$
Check the validity of the solutions by putting them into the original equation.	When $x = -8$: LHS $= \log_2(-8) + \log_2(-4)$, and since we cannot take logarithms of negative numbers, this solution does not work. When $x = 4$: LHS $= \log_2 4 + \log_2 8$ $= 2 + 3 = 5 =$ RHS So $x = 4$ is the only solution of the given equation.

EXAM HINT

Checking your answers is an important part of solving mathematical problems, and involves more than looking for arithmetic errors. As this example shows, false solutions may arise through correct algebraic manipulations.

Although we have discussed logarithms with a general base a, your calculator may only have buttons for the common logarithm and the natural logarithm ($\log x$ and $\ln x$). To use a calculator to evaluate a logarithm with base other than 10 or e (for example, $\log_5 20$), we use the following **change-of-base rule** for logarithms.

KEY POINT 2.22

Change-of-base rule for logarithms:

$$\log_a x = \frac{\log_b x}{\log_b a}$$

So, we can calculate $\log_5 20$ using the logarithm with base 10 as follows:

$$\log_5 20 = \frac{\log 20}{\log 5} = 1.86 \text{ (3 SF)}$$

The change-of-base rule is useful for more than just evaluating logarithms.

Worked example 2.10

Solve the equation $\log_3 x + \log_9 x = 2$.

We want to have logarithms involving just one base, so use the change-of-base rule to turn the log with base 9 into a log with base 3.

$$\log_9 x = \frac{\log_3 x}{\log_3 9}$$

$$= \frac{\log_3 x}{2}$$

Therefore the equation is

$$\log_3 x + \frac{\log_3 x}{2} = 2$$

Collect the logs together.

$$\Leftrightarrow \frac{3}{2}\log_3 x = 2$$

$$\Leftrightarrow \log_3 x = \frac{4}{3}$$

Exponentiate both sides with base 3.

Hence

$$x = 3^{\frac{4}{3}} = 4.33 \text{ (3SF)}$$

Exercise 2E

1. Given that $b>0$, simplify each of the following.

(a) (i) $\log_b b^4$ (ii) $\log_b \sqrt{b}$

(b) (i) $\log_{\sqrt{b}} b^3$ (ii) $\log_b b^2 - \log_{b^2} b$

2. If $x=\log a$, $y=\log b$ and $z=\log c$, express the following in terms of x, y and z.

(a) (i) $\log(bc)$ (ii) $\log\left(\frac{c}{a}\right)$

(b) (i) $\log a^3$ (ii) $\log b^5$

(c) (i) $\log cb^7$ (ii) $\log a^2 b$

(d) (i) $\log\left(\frac{ab^2}{c}\right)$ (ii) $\log\left(\frac{a^2}{bc^3}\right)$

(e) (i) $\log\left(\frac{100}{bc^5}\right)$ (ii) $\log(5b)+\log(2c^2)$

3. Solve the following equations for x.

(a) (i) $\log_3\left(\frac{2+x}{2-x}\right)=3$ (ii) $\log_2(7x+4)=5$

(b) (i) $\log_3 x - \log_3(x-6)=1$ (ii) $\log_8 x - 2\log_8\left(\frac{1}{x}\right)=1$

(c) (i) $\log_2 x + 1 = \log_4 x$ (ii) $\log_8 x + \log_2 x = 4$

(d) (i) $\log_4 x + \log_8 x = 2$ (ii) $\log_{16} x - \log_{32} x = 0.5$

(e) (i) $\log_3(x-7)+\log_3(x+1)=2$

(ii) $2\log(x-2)-\log(x)=0$

(f) (i) $\log(x^2+1)=1+2\log(x)$

(ii) $\log(3x+6)=\log(3)+1$

4. Find the exact solution to the equation $2\ln x + \ln 9 = 3$, giving your answer in the form Ae^B where A and B are rational numbers. *[5 marks]*

5. If $a=\ln 2$ and $b=\ln 5$, write the following in terms of a and b.

(a) $\ln 50$

(b) $\ln 0.16$ *[6 marks]*

6. Solve $\log_2 x = \log_x 2$. *[5 marks]*

7. If $x = \log a$, $y = \log b$ and $z = \log c$, express the following in terms of x, y and z.

(a) $\log a^3 - 2\log ab^2$

(b) $\log(4b) + 2\log(5ac)$ *[8 marks]*

8. Evaluate $\log\frac{1}{2} + \log\frac{2}{3} + \log\frac{3}{4} + \log\frac{4}{3} + \cdots + \log\frac{8}{9} + \log\frac{9}{10}$. *[4 marks]*

9. If $x = \log a$, $y = \log b$ and $z = \log c$, express the following in terms of x, y and z.

(a) $\log_a a^2 b$

(b) $\log_{ab} ac^2$ *[6 marks]*

10. If $x = \log a$, $y = \log b$ and $z = \log c$, express the following in terms of x, y and z.

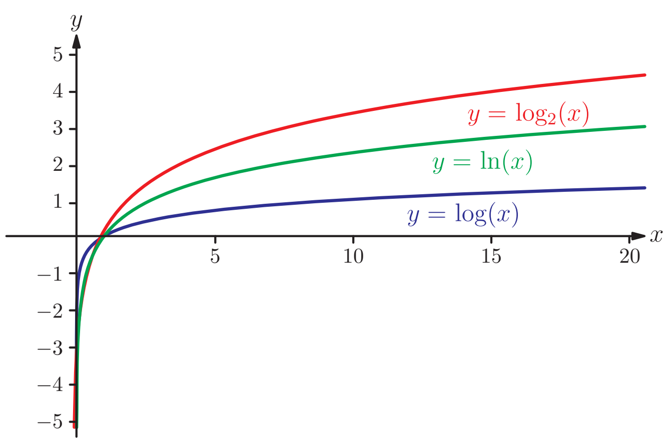
(a) $\log_b\left(\frac{a}{bc}\right)$

(b) $\log_{ab}(b^a)$ *[7 marks]*

2F Graphs of logarithms

Let us now look at the graph of the logarithm function and the various properties of logarithms that we can deduce from it. Here are the graphs of $y = \log x$, $y = \log_2 x$ and $y = \ln x$.

In chapter 5 you will see how this type of change in the function causes a vertical stretch of the graph.



Given the change-of-base rule from section E (Key point 2.22), it is not surprising that these curves all have a similar shape: since $\log_2 x = \frac{\log x}{\log 2}$ and $\ln x = \frac{\log x}{\log e}$, each of the logarithm functions is a multiple of the common logarithm function.

EXAM HINT

Vertical asymptotes are even harder to detect accurately from a calculator display than horizontal ones. This is why it is essential that you know where the asymptote is for a logarithmic graph.

Note that a logarithm graph is the reflection of an exponential graph. You will see why this is in chapter 5.

From the graphs above, we can observe the following important properties of the logarithm function.

KEY POINT 2.23

If $y = \log_a x$ (for any positive value of a), then

- the graph of y against x crosses the x-axis at (1, 0), because $\log_a 1 = 0$
- $\log x$ is negative for $0 < x < 1$ and positive for $x > 1$
- the graph lies entirely to the right of the y-axis, since the logarithm of a negative value of x is not a real number
- the graph increases (slopes upward from left to right) throughout, and as x tends to infinity so does y
- the y-axis is an asymptote to the curve.

It is unlikely you will find exam questions testing just this topic, but you may have to sketch a graph involving a logarithm as part of another question.

Exercise 2F

1. Sketch the following graphs, labelling clearly the vertical asymptote and all axis intercepts.

(a) (i) $y = \log(x^2)$ (ii) $y = \log(x^3)$

(b) (i) $y = \log 4x$ (ii) $y = \log 2x$

(c) (i) $y = \log(x-2)$ (ii) $y = \log(x+1)$

2G Solving exponential equations

One of the main uses of logarithms is to solve equations that have the unknown in an exponent. By taking logarithms, the unknown becomes a factor, which is easier to deal with.

EXAM HINT

When taking logarithms of both sides of an equation, you can use any base. We usually choose log or ln so that we can easily find or check the answer with a calculator.

Worked example 2.11

Find the exact solution of the equation $3^{x-2} = 5$.

Take logs of both sides.

$\log(3^{x-2}) = \log 5$

The log of an exponent is the multiple of the log (Key point 2.20).

$\Leftrightarrow (x-2)\log 3 = \log 5$

$\Leftrightarrow x - 2 = \dfrac{\log 5}{\log 3}$

$\Leftrightarrow x = \dfrac{\log 5}{\log 3} + 2$

Note that we can use the rules of logarithms to make the answer more compact:

$$\frac{\log 5}{\log 3} + 2 = \frac{\log 5 + 2\log 3}{\log 3} = \frac{\log(5 \times 3^2)}{\log 3} = \frac{\log 45}{\log 3}$$

You need to be able to carry out this sort of simplification, but you do not *have* to do it unless the question explicitly asks you to – for instance, if the above example specified that you should write the answer in the form $\dfrac{\log a}{\log b}$ where a and b are integers.

If you are allowed to use a calculator, you can plot both sides of the equation and find the solution by looking for intersections of the graphs.

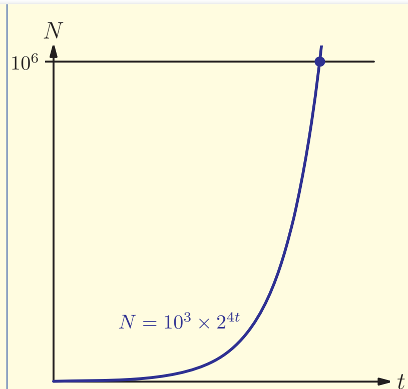
EXAM HINT

See Calculator skills sheet 5 on the CD-ROM for how to use graphs to solve equations.

Worked example 2.12

The number of bacteria in a culture medium is given by $N = 1000 \times 2^{4t}$, where t is the number of hours since 08:00. At what time will the population first reach one million?

We need to solve the equation $1000 \times 2^{4t} = 1000000$. First, sketch the graph of $N = 1000 \times 2^{4t}$ with t on the x-axis and N on the y-axis.



The solution is the intersection of the curve with the line $y = 1000000$.

From GDC: $t = 2.49$ (hours after 08:00)

Convert this t-value into hours and minutes.

$0.49 \times 60 = 29$ minutes

So the population will reach 1 million at 10:29.

The next example shows how to solve a more complicated exponential equation, where the unknown appears in the exponent on both sides. The method of taking logarithms of both sides works here as well, but a little more algebraic manipulation is needed.

There is another type of exponential equation that sometimes comes up: a disguised quadratic equation, for example $2^{2x} - 5 \times 2^x + 6 = 0$. Such equations are explored in section 3B.

Worked example 2.13

Solve the equation $10^x = 5 \times 2^{3x}$, giving your answer in terms of natural logarithms.

Take the natural logarithm of both sides.

$\ln(10^x) = \ln(5 \times 2^{3x})$

On the RHS, the log of a product is the sum of logs.

$\Leftrightarrow \ln(10^x) = \ln 5 + \ln(2^{3x})$

continued . . .

The log of an exponent is the multiple of the log.

$$\Leftrightarrow x\ln 10 = \ln\ 5 + 3x\ln 2$$

Collect all terms containing x.

$$\Leftrightarrow x\ln 10 - 3x\ln 2 = \ln 5$$

We want only one term with x, so take x out as a factor.

$$\Leftrightarrow x(\ln 10 - 3\ln 2) = \ln 5$$

$$\Leftrightarrow x = \frac{\ln 5}{\ln 10 - 3\ln 2}$$

Exercise 2G

1. Solve for x, giving your answer correct to 3 significant figures.

(a) (i) $3\times 4^x = 90$ (ii) $1000\times 1.02^x = 10000$

(b) (i) $6\times 7^{3x+1} = 1.2$ (ii) $5\times 2^{2x-5} = 94$

(c) (i) $3^{2x} = 4^{x-1}$ (ii) $5^x = 6^{1-x}$

(d) (i) $3\times 2^{3x} = 7\times 3^{3x-2}$ (ii) $4\times 8^{x-1} = 3\times 5^{2x+1}$

EXAM HINT

In question 1 above, where you are allowed to use a calculator, you need to judge whether it will be faster to plot graphs or to rearrange the equation and then evaluate the logarithms with your calculator, for example:

$$2^{x-1} = 5$$

$$\Rightarrow (x-1)\log\ 2 = \log\ 5$$

$$\Rightarrow x - 1 = \frac{\log\ 5}{\log\ 2}$$

$$\Rightarrow x = 1 + \frac{\log\ 5}{\log\ 2} = 3.32\ (3SF)$$

2. Solve the following equations, giving your answers in terms of natural logarithms.

(a) (i) $3^{2x} = 5$ (ii) $10^{3x} = 7$

(b) (i) $2^{x+1} = 5^x$ (ii) $5^{x-2} = 3^x$

(c) (i) $2^{3x} = 3e^x$ (ii) $e^{2x} = 5\times 2^x$

3. In a yeast culture, cell numbers are given by $N = 100e^{1.03t}$, where t is measured in hours after the cells are introduced to the culture.

(a) What is the initial number of cells?

(b) How many cells will be present after 6 hours?

(c) How long will it take for the population to reach one thousand cells? *[4 marks]*

4. A rumour spreads exponentially through a school, so that the number of people who know it can be modelled by the equation $N = Ae^{kt}$, where t is the time, in minutes, after 9 a.m. When school begins (at 9 a.m.), 18 people know the rumour. By 10 a.m. 42 people know it.

(a) Write down the value of A.

(b) Show that $k = 0.0141$, correct to three significant figures.

(c) How many people know the rumour at 10:30 a.m.?

(d) There are 1200 people in the school. According to the exponential model, at what time will everyone know the rumour? *[6 marks]*

5. The mass M of a piece of plutonium at time t seconds is given by

$$M = ke^{-0.01t}$$

(a) Write down the initial mass of the plutonium.

(b) Sketch the graph of M against t.

(c) How long will it take for the plutonium to reach 25% of its original mass? Give your answer in minutes. *[5 marks]*

6. Find the solution to the equation $15^{2x} = 3 \times 5^{x+1}$ in the form $\dfrac{\log a}{\log b}$ where a and b are positive integers. *[6 marks]*

7. Solve the equation, $\dfrac{1}{7^x} = 3 \times 49^{5-x}$, giving your answer in the form $a + \log_7 b$ where $a, b \in \mathbb{Z}$. *[6 marks]*

8. Solve the equation $5 \times 4^{x-1} = \dfrac{1}{3^{2x}}$, giving your answer in the form $x = \dfrac{\ln p}{\ln q}$ where p and q are rational numbers. *[6 marks]*

9. (a) Show that the equation $3^x = 3 - x$ has only one solution.

(b) Find the solution, giving your answer to 3 significant figures. *[6 marks]*

Summary

- In this chapter, we revisited the rules for exponents and explored the meaning of fractional and negative exponents. (These rules are not in the Formula booklet.)

$$a^m \times a^n = a^{m+n} \qquad (a^m)^n = a^{m\times n}$$

$$a^m \div a^n = a^{m-n} \qquad a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

$$a^n \times b^n = (ab)^n \qquad a^0 = 1 \quad (a \neq 0)$$

$$a^n \div b^n = \left(\frac{a}{b}\right)^n \qquad a^{-n} = \frac{1}{a^n} \quad (a \neq 0)$$

- Exponential functions can be used to model growth or decay of some simple real-life systems, taking as a general form the function: $N = Ba^{\left(\frac{t}{k}\right)} + c$
- $\log_a b$ represents the answer to the question 'to what exponent do I have to raise a to get b?'

$$b = a^x \Leftrightarrow x = \log_a b$$

- Logarithms undo the effect of exponentiating, and vice versa:

$$\log_a (a^x) = x = a^{\log_a x}$$

- The following properties of the logarithm function can be deduced from its graph, if $y = \log x$ then:
  - the graph of y against x crosses the x-axis at (1,0)
  - $\log x$ is negative for $0 < x < 1$ and positive for $x > 1$
  - the graph lies to the right of the y-axis; the y-axis is an asymptote to the curve
  - the logarithm graph increases throughout; as x tends to infinity so does y.
- Logarithms obey a number of rules, most of which are given in the Formula booklet:

$$\log_a xy = \log_a x + \log_a y$$

$$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$\log_a \left(\frac{1}{x}\right) = -\log_a x \text{ (not in Formula booklet)}$$

$$\log_a x^p = p\log_a x$$

$$\log_a 1 = 0 \text{ (not in Formula booklet)}$$

- e is the mathematical constant (Euler's number): e = 2.71828182849...
- Logarithms with base 10 (common logarithms, $\log_{10} x$), are often written simply as $\log x$. Logarithms with base e (natural logarithms) are usually written as $\ln x$.

- A change-of-base formula for logarithms, and a related rule for exponents, are given in the formula book:

$$\log_b a = \frac{\log_c a}{\log_c b}$$

$$a^x = e^{x \ln a}$$

- Many exponential equations can be solved by taking logarithms of both sides.

Introductory problem revisited

A radioactive substance has a half-life of 72 years (this is the time it takes for half of the mass, and hence radioactivity, to decay). A 1 kg block of the substance is found to have a radioactivity of 25 million becquerels (Bq). How long, to the nearest 10 years, would it take for the radioactivity to fall to 10 000 Bq?

Write down the exponential equation.

Let R be the radioactivity after t years. Then

$$R = Ba^{\frac{t}{k}}$$

Initial condition gives *B*.

When $t = 0$, $R = 25 \times 10^6 = B$

Every time *t* increases by 72, *R* falls by 50%.

$a = 0.5,\ k = 72$

$\therefore R = (25 \times 10^6) \times 0.5^{\frac{t}{72}}$

We want to find the value of *t* for which $R = 10\,000$.

$R = 10^4$

$\Leftrightarrow 25 \times 10^6 \times 0.5^{\frac{t}{72}} = 10^4$

$\Leftrightarrow 0.5^{\frac{t}{72}} = 0.0004$

The unknown is in the exponent, so use logarithms.

$\log\left(0.5^{\frac{t}{72}}\right) = \log(0.0004)$

$\frac{t}{72}\log(0.5) = \log(0.0004)$

Now rearrange to find *t*.

$t = \frac{72\log(0.0004)}{\log(0.5)} = 812.7$

It takes around 810 years for the radioactivity to fall to 10 000 Bq.

Mixed examination practice 2

Short questions

1. Solve $\log_5\left(\sqrt{x^2+49}\right)=2$. *[4 marks]*

2. If $a=\log x$, $b=\log y$ and $c=\log z$ (where all logs are with base 10), express the following in terms of a, b and c.

(a) $\log\dfrac{x^2\sqrt{y}}{z}$ (b) $\log\sqrt{0.1x}$ (c) $\log_{100}\left(\dfrac{y}{z}\right)$ *[6 marks]*

3. Given that $B=4+12\mathrm{e}^{\frac{t}{3}}$, find the value of t for which $B=25$. *[3 marks]*

4. Find the exact solution of the equation $4\times 3^{2x}=5^x$, giving your answer in terms of natural logarithms. *[6 marks]*

5. Solve the simultaneous equations

$$\ln x+\ln y^2=8$$

$$\ln x^2+\ln y=6$$ *[6 marks]*

6. If $y=\ln x-\ln(x+2)+\ln(4-x^2)$, express x in terms of y. *[6 marks]*

7. Find the exact value of x which satisfies the equation

$$2\times 3^{x-2}=36^{x-1}$$

giving your answer in the simplified form $\dfrac{\ln p}{\ln q}$ where $p,q\in\mathbb{Z}$. *[5 marks]*

8. Given that $\log_a b^2=c$ and $\log_b a=c-1$ for some value c, where $0<a<b$, express a in terms of b. *[6 marks]*

9. Find the exact solutions of the equation $\ln x=4\log_x \mathrm{e}$. *[5 marks]*

Long questions

1. The speed V in metres per second of a parachutist t seconds after jumping is modelled by the expression

$$V=42(1-\mathrm{e}^{-0.2t})$$

(a) Sketch a graph of V against t.

(b) What is the initial speed?

(c) What is the maximum speed that the parachutist could approach?

The parachutist opens the parachute when his speed reaches 22 m s^{-1}.

(d) How long is he falling before opening the parachute? *[9 marks]*

2. Scientists think that the global population of tigers is falling exponentially. Estimates suggest that in 1970 there were 37 000 tigers but by 1980 the number had dropped to 22 000.

   (a) The number T of tigers n years after 1970 can be modelled by $T = ka^n$.

   (i) Write down the value of k.

   (ii) Show that $a = 0.949$ to three significant figures.

   (b) What does the model predict that the population will be in 2020?

   (c) When the population reaches 1000, the tiger population will be described as 'near extinction'. In which year will this happen?

   In the year 2000 a worldwide ban on the sale of tiger products was implemented, and it is believed that by 2010 the population of tigers had recovered to 10 000.

   (d) If the recovery has been exponential, find a model of the form $T = ka^m$ connecting the number of tigers (T) with the number of years after 2000 (m).

   (e) If each year since 2000 the rate of growth has been the same, find the percentage increase in the tiger population each year. *[12 marks]*

3. A sum of $2000 is paid into a bank account which pays 3.5% annual interest. Assume that no money is taken out of the account.

   (a) What is the amount of money in the account after

   (i) 1 year?

   (ii) 5 years?

   (b) How long, to the nearest year, does it take for the amount of money in the account to double?

   (c) Sketch a graph showing how the amount of money in the account varies with time over the first 20 years.

   (d) Another bank account pays only 2% annual interest, but the initial investment is $3000. After how many years, to the nearest year, will the amount of money in the two accounts be the same?

   (e) How much should be invested in the second account so that after 20 years the amount of money in the two accounts is the same? *[14 marks]*