7 Binomial expansion

In this chapter you will learn:

- how to expand an expression of the form $(a+b)^n$ for a positive integer n
- how to find individual terms in the expansion of $(a+b)^n$ for a positive integer n.

Introductory problem

Without using a calculator, find the value of $(1.002)^{10}$ correct to 8 decimal places.

A **binomial** expression is an expression that contains two terms, for example $a+b$.

Expanding a power of a binomial expression could be performed laboriously by expanding brackets; for example, $(a+b)^7$ could be found by multiplying out, at length,

$$(a+b)(a+b)(a+b)(a+b)(a+b)(a+b)(a+b)$$

This is time-consuming and mistakes could easily be made, but fortunately there is a much quicker approach.

7A Introduction to the binomial theorem

To see how we might rapidly expand an expression of the form $(a+b)^n$ for an integer power n, let us first look at some expansions of $(a+b)^n$ done using the slow method of multiplying out brackets repeatedly. The table shows the results for $n=1,2,3$ and 4; in the rightmost column, the coefficients and powers in the expansions are coloured to highlight the pattern.

$(a+b)^0$	$=1$	$=1a^0b^0$
$(a+b)^1$	$=a+b$	$=1a^1b^0+1a^0b^1$
$(a+b)^2$	$=a^2+2ab+b^2$	$=1a^2b^0+2a^1b^1+1a^0b^2$
$(a+b)^3$	$=a^3+3a^2b+3ab^2+b^3$	$=1a^3b^0+3a^2b^1+3a^1b^2+1a^0b^3$
$(a+b)^4$	$=a^4+4a^3b+6a^2b^2+4ab^3+b^4$	$=1a^4b^0+4a^3b^1+6a^2b^2+4a^1b^3+1a^0b^4$

The red numbers form a famous mathematical structure called *Pascal's triangle*. There are many amazing patterns in Pascal's triangle – for example, by highlighting all the even numbers you can generate an ever-repeating pattern called a *fractal*.

EXAM HINT

See Calculator Skills sheet 3 on the CD-ROM for how to find $\binom{n}{r}$ using your calculator.

We can see several patterns in each expansion:

- The powers of a and b (coloured green) in each term always add up to n.
- Every power of a from 0 up to n is present in one of the terms, with the corresponding complementary power of b.
- The pattern of coefficients (coloured red) is symmetrical.

The red numbers are called **binomial coefficients**. The expansion of $(a+b)^n$ has $n+1$ binomial coefficients, and the one associated with the terms $a^{n-r}b^r$ and a^rb^{n-r} is denoted by $\binom{n}{r}$.

KEY POINT 7.1

Binomial theorem

The coefficient of the term containing $a^{n-r}b^r$ in the expansion of $(a+b)^n$ is $\binom{n}{r}$.

Worked example 7.1

Find the coefficient of x^5y^3 in the expansion of $(x+y)^8$.

Write down the required term in the form $\binom{n}{r}(a)^{n-r}(b)^r$.

Here $a=x$, $b=y$, $r=3$, $n=8$.

The required term is $\binom{8}{3}(x)^5(y)^3$

Calculate the coefficient and apply the powers to the bracketed terms.

$\binom{8}{3}=56$

$(x)^5=x^5$

$(y)^3=y^3$

The term is $56x^5y^3$

The coefficient is 56

Sometimes one of a and b is just a number, but we treat the expansion in exactly the same way as if both were variables.

Worked example 7.2

Find the y^3 term in $(2+y)^5$.

Write down the required term in the form $\binom{n}{r}(a)^{n-r}(b)^r$.

Here $a=2$, $b=y$, $r=3$, $n=5$.

The relevant term is $\binom{5}{3}(2)^2(y)^3$

Calculate the coefficient and apply the powers to the bracketed terms.

$\binom{5}{3}=10$

$(2)^2=4$

$(y)^3=y^3$

The term is $40y^3$

EXAM HINT

A question may ask for the whole term or just the coefficient. Make sure that your answer gives what was requested!

Exercise 7A

1. (i) Find the x^5y^7 term in the expansion of $(x+y)^{12}$.
   (ii) Find the a^7b^9 term in the expansion of $(a+b)^{16}$.
   (iii) Find the c^3d^2 term in the expansion of $(c+d)^5$.
   (iv) Find the a^2b^7 term in the expansion of $(a+b)^9$.
   (v) Find the x^2y^4 term in the expansion of $(x+y)^6$.

2. (a) Find the coefficient of x^3 in $(3+x)^4$.
   (b) Find the coefficient of y^2 in $(y+5)^6$.
   (c) Find the coefficient of z in $(5+z)^5$.

3. Find the coefficient of x^5y^3 in the expansion of $(x+y)^6$.

7B Binomial coefficients

$n = 0$						1						
$n = 1$					1		1					
$n = 2$				1		2		1				
$n = 3$			1		3		3		1			
$n = 4$		1		4		6		4		1		
$n = 5$	1		5		10		10		5		1	

The easiest way to find binomial coefficients is by using a calculator, but in some situations you may need to find them without a calculator. One method of doing this is to use Pascal's triangle – the pattern of red numbers in the table of binomial expansions from the previous section arranged in the shape of a pyramid.

The slanted sides are made up of 1s. Each of the other values is obtained by adding the two values above it. The binomial coefficients of $(a + b)^n$ are found in the $(n + 1)$th row from the top.

Although it is not difficult to generate Pascal's triangle, finding binomial coefficients in this way is impractical when n is large. In such cases, there is a formula that can be used. Before introducing this formula we need to define a new function, the factorial function $n!$ (pronounced 'n factorial').

KEY POINT 7.2

For a positive integer n,

$$n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n$$

0! is defined to be 1.

One reason that 0! = 1 was agreed on as a convention is because it is useful to have this in other formulas involving factorials. Is this a valid mathematical reason for assigning a certain value to an undefined expression?

Using the factorial function, it is possible to write down a simple formula for the binomial coefficients. You do not need to know the proof of this formula.

KEY POINT 7.3

$$\text{Binomial coefficient: } \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Worked example 7.3

Evaluate $\binom{6}{4}$.

Using the formula with $n = 6$, $r = 4$ and $n - r = 2$.

$$\binom{6}{4} = \frac{6!}{4! \times 2!}$$
$$= \frac{1 \times 2 \times 3 \times 4 \times 5 \times 6}{(1 \times 2 \times 3 \times 4) \times (1 \times 2)}$$
$$= \frac{5 \times 6}{2}$$
$$= 15$$

Using the formula, we can find expressions for binomial coefficients with general n and small values of r.

KEY POINT 7.4

$$\binom{n}{0} = 1 \quad \binom{n}{1} = n \quad \binom{n}{2} = \frac{n(n-1)}{2}$$

These expressions are useful when part of an expansion has been given.

Worked example 7.4

The expansion of $(1+x)^n$ up to the third term is given by $1 + 6x + ax^2$. Find the value of n and of a.

Write out the first three terms of the expansion of the left-hand side in terms of n.

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots$$
$$= 1 + nx + \frac{n(n-1)}{2}x^2 + \cdots$$

Compare this with the given expression $1 + 6x + ax^2$.

Comparing coefficients of x gives

$n = 6$

Comparing coefficients of x^2 gives

$$\frac{n(n-1)}{2} = a$$

$$\therefore \quad a = \frac{6 \times 5}{2} = 15$$

Exercise 7B

1. If we say that the first row of Pascal's triangle is 1,1, find:
   (a) the second row
   (b) the third row
   (c) the fifth row

2. Find the following binomial coefficients.
   (a) (i) $\binom{7}{3}$ (ii) $\binom{9}{2}$
   (b) (i) $\binom{9}{0}$ (ii) $\binom{6}{0}$
   (c) (i) $\binom{8}{7}$ (ii) $\binom{10}{8}$

3. (i) Find the coefficient of xy^3 in the expansion of $(x+y)^4$.
   (ii) Find the coefficient of x^3y^4 in the expansion of $(x+y)^7$.
   (iii) Find the coefficient of ab^6 in the expansion of $(a+b)^7$.
   (iv) Find the coefficient of a^5b^3 in the expansion of $(a+b)^8$.

4. Find the coefficient of x^4 in $(3+2x)^5$. *[4 marks]*

5. Suppose that $(2+x)^n = 32 + ax + \cdots$
   (a) Find the value of n.
   (b) Find the value of a. *[5 marks]*

6. Suppose that $(1+2x)^n = 1 + 20x + ax^2 + \cdots$
   (a) Find the value of n.
   (b) Find the value of a. *[5 marks]*

7C Applying the binomial theorem

In section 7A you learned the general pattern for expanding powers of a binomial expression $(a+b)$. Many expansions can be done using this method, if we substitute more complicated expressions for a and b.

Worked example 7.5

Find the term in x^6y^4 in the expansion of $(x+3y^2)^8$.

Write down the required term in the form $\binom{n}{r}(a)^{n-r}(b)^r$, with $a=x,\ \ b=3y^2,\ \ r=2,\ \ n=8$.

The relevant term is $\binom{8}{2}(x)^6(3y^2)^2$

Calculate the coefficient and apply the powers to the bracketed terms.

$\binom{8}{2}=28$

$(x)^6=x^6$

$(3y^2)^2=9y^4$

The required term is $28\times x^6\times 9y^4=252x^6y^4$

EXAM HINT

A common mistake is to assume that the powers of each variable in the term you're asked to find correspond to the value of r in the expansion; but as you can see in this example, y is raised to the power 4 while r is 2.
Take care also to apply the power not only to the variable but also to its coefficient. In this example, $(3y^2)^2=9y^4$, not $3y^4$.

Although examination questions typically ask you to calculate just one term, you should also be able to find the entire expansion. To do this, repeat the calculation for each of the values that r can take (from 0 up to n) and then add together all these terms. This leads to the following formula given in the Formula booklet.

KEY POINT 7.5

Binomial theorem

$$(a+b)^n=a^n+\binom{n}{1}a^{n-1}b+\cdots+\binom{n}{r}a^{n-r}b^r+\cdots+b^n$$

Worked example 7.6

Use the binomial theorem to expand and simplify $(2x-3y)^5$.

Write down each term in the form $\binom{n}{r}(a)^{n-r}(b)^r$, with

$a=2x,\ b=-3y,\ n=5.$

Coefficients are 1,5,10,10,5,1

The expansion is

$1(2x)^5+5(2x)^4(-3y)^1+10(2x)^3(-3y)^2+$

$10(2x)^2(-3y)^3+5(2x)^1(-3y)^4+1(-3y)^5$

Apply the powers to the bracketed terms and multiply through.

$=32x^5-240x^4y+720x^3y^2-1080x^2y^3$

$+810xy^4-243y^5$

A question may ask for a term in a binomial expansion where 'a' and 'b' in the binomial expression both contain the same variable. You can use the rules of exponents to determine which term of the expansion is needed.

Worked example 7.7

Find the coefficient of x^5 in the expansion of $\left(2x^2-\frac{1}{x}\right)^7$.

Start with the form of a general term and simplify using the rules of exponents.

Each term will be of the form

$\binom{7}{r}(2x^2)^{7-r}(-x^{-1})^r$

$=\binom{7}{r}(2)^{7-r}x^{14-2r}(-1)^r x^{-r}$

$=\binom{7}{r}(2)^{7-r}(-1)^r x^{14-3r}$

continued . . .

The power of x in the required term is 5, so equate that to the power of x in the general term.	Need $14-3r=5$ $3r=9$ $r=3$
Write down the required term in the form $\binom{n}{r}(a)^{n-r}(b)^r$, with $a=2x^2$, $b=-x^{-1}$, $n=7$, $r=3$.	The relevant term is $\binom{7}{3}(2x^2)^4(-x^{-1})^3$
Calculate the coefficient and apply the powers to the bracketed terms.	$\binom{7}{3}=35$ $(2x^2)^4=16x^8$ $(-x^{-1})^3=-x^{-3}$
Combine the elements to calculate the coefficient.	The term is $35\times16x^8\times(-x^{-3})=-560x^5$ The coefficient is -560

EXAM HINT

If 'a' or 'b' has a negative sign, it must stay with each of the terms $a^{n-r}b^r$ in the expansion and needs to be acted upon by the corresponding power. Lots of people forget the negative signs!

You may need to work with the product of a binomial expression and a power of another binomial expression.

Worked example 7.8

Use the binomial theorem to expand and simplify $(5-3x)(2-x)^4$.

To expand $(2-x)^4$, write down each term in the form $\binom{n}{r}(a)^{n-r}(b)^r$.

Here $a=2,\ b=-x,\ n=4$

Coefficients are 1,4,6,4,1

The expansion is

$$(5-3x)[1(2)^4+4(2)^3(-x)+6(2)^2(-x)^2+4(2)^1(-x)^3+1(-x)^4]$$

5 multiplies the square bracket and $-3x$ multiplies the square bracket.

$$=5[16-32x+24x^2-8x^3+x^4]$$
$$-3x[16-32x+24x^2-8x^3+x^4]$$
$$=80-208x+216x^2-112x^3+29x^4-3x^5$$

When x is a small number (modulus less than 1), high powers of x will be very small, even after multiplying by a binomial coefficient. Thus, if we take a binomial expansion involving values of x very close to zero and throw away terms containing higher powers of x, this should have little impact on the overall value of the expansion. Truncated binomial expansions, which retain only a few terms containing low powers of x, are useful for calculating approximate values of powers of numbers.

KEY POINT 7.6

If the value of x is close to zero, large powers of x will be extremely small.

Worked example 7.9

Find the first 3 terms of the expansion of $(2-x)^5$ in ascending powers of x. By setting $x = 0.01$, use your answer to find an approximate value of 1.99^5.

Write down each term of the expansion in the form $\binom{n}{r}(a)^{n-r}(b)^r$, with $a=2,\ b=-x,\ n=5$.

The first 3 terms are
$1(2)^5 + 5(2)^4(-x)^1 + 10(2)^3(-x)^2$

Apply the powers to the bracketed terms and multiply through.

$= 32 - 80x + 80x^2$

Calculate the powers of the given value of x and hence the value of each term in the expansion.

$x^0 = 1 \quad \Rightarrow \quad 32x^0 = 32$
$x = 0.01 \quad \Rightarrow \quad -80x = -0.8$
$x^2 = 0.0001 \quad \Rightarrow \quad 80x^2 = 0.008$

Add up the values of the terms.

Hence, approximately,
$1.99^5 = 32 - 0.8 + 0.008 = 31.208$

Exercise 7C

1. (a) Find the coefficient of xy^3 in
   (i) $(2x+3y)^4$ (ii) $(5x+y)^4$
   (b) Find the term in x^3y^4 in
   (i) $(x-2y)^7$ (ii) $(y-2x)^7$
   (c) Find the coefficient of a^2b^3 in
   (i) $\left(2a-\frac{1}{2}b\right)^5$ (ii) $(17a+3b)^5$

2. (a) (i) Fully expand and simplify $(2-x)^5$.
   (ii) Fully expand and simplify $(3+x)^6$.
   (b) (i) Find the first three terms in the expansion of $(3x+y)^5$ in descending powers of x.
   (ii) Find the first three terms in the expansion of $(2c-d)^4$ in ascending powers of d.
   (c) Fully expand and simplify $(2x^2-3x)^3$.

3. (a) Find the first 3 terms in the expansion of $(3-5x)^4$.

(b) By choosing a suitable value of x, use your answer from part (a) to find an approximation for 2.995^4.

[7 marks]

4. Find the first 4 terms in the expansion of $(y+3y^2)^6$ in ascending powers of y. *[6 marks]*

5. Which term in the expansion of $(x-2y)^5$ has coefficient

(a) 40?

(b) -80? *[6 marks]*

6. Find the coefficient of x^3 in $(1-5x)^9$. *[4 marks]*

7. Find the x^2 term in $(3-2x)^7$. *[4 marks]*

8. Find the coefficient of x^2y^6 in $(3x+2y^2)^5$. *[5 marks]*

9. Find the coefficient of x^2 in the expansion of $\left(x+\frac{1}{x}\right)^8$.

10. (a) Find the first 3 terms in the expansion of $(2+3x)^7$.

(b) Hence find an approximation to

(i) 2.3^7 (ii) 2.03^7

(c) Which of your answers in part (b) provides a more accurate approximation? Justify your answer. *[6 marks]*

11. (a) Expand $\left(\mathrm{e}+\frac{2}{\mathrm{e}}\right)^5$.

(b) Simplify $\left(\mathrm{e}+\frac{2}{\mathrm{e}}\right)^5+\left(\mathrm{e}-\frac{2}{\mathrm{e}}\right)^5$. *[7 marks]*

12. The expansion of $(x+ay)^n$ contains the term $60x^4y^2$.

(a) Write down the value of n.

(b) Find the value of a. *[4 marks]*

13. Complete and simplify the expansion of $\left(2z^2+\frac{3}{z}\right)^4$, which begins with $16z^8+96z^5$. *[4 marks]*

14. (a) Write the expression $(1+x)^n(1-x)^n$ in the form $(f(x))^n$.

(b) Find the first three non-zero terms in the expansion of $(1-x)^{10}(1+x)^{10}$ in ascending powers of x. *[6 marks]*

15. The expansion of $\left(3x^2y+\frac{5x}{y}\right)^n$ begins with $27x^6y^3+135x^5y$.

(a) Write down the value of n

(b) Complete and simplify the expansion *[5 marks]*

16. Find the coefficient of x^2 in the expansion of $\left(2x+\frac{1}{\sqrt{x}}\right)^5$. *[4 marks]*

17. Find the constant coefficient in the expansion of $(x-2x^{-2})^9$. *[4 marks]*

EXAM HINT

The 'constant coefficient' is the term in x^0. It may also be described as 'the term independent of x'.

18. Find the term in x^5 in $\left(x^2-\frac{3}{x}\right)^7$. *[6 marks]*

19. Find the term that is independent of x in the expansion of $\left(2x-\frac{5}{x^2}\right)^{12}$. *[6 marks]*

20. Find the coefficient of x^5 in the expansion of $(1+3x)(1+x)^7$.

21. If $(1+ax)^n = 1+10x+40x^2+\cdots$, find the values of a and of n.

Summary

- A binomial expression is one that contains two terms, e.g. $a+b$.
- The binomial coefficient, $\binom{n}{r}$, is the coefficient of the term containing $a^{n-r}b^r$ in the expansion of $(a+b)^n$.
- The expansion of $(a+b)^n$ can be accomplished directly by using the **binomial theorem**:

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + b^n$$

- The coefficient of individual terms in a **binomial expansion** can be found by looking at the powers of the algebraic components and using the formula for the rth term

$$\binom{n}{r}(a)^{n-r}(b)^r$$

with appropriate choices of n, r, a and b.

- Approximations to powers of numbers can be made by taking the first few terms of a binomial expansion $(a + bx)^n$ and substituting an appropriate small value for x, so that terms containing higher powers of x will be negligible.

Introductory problem revisited

Without using a calculator, find the value of $(1.002)^{10}$ correct to 8 decimal places.

The first step is to recognise that $(1.002)^{10}$ can be obtained by evaluating the binomial expansion $(1 + 2x)^{10}$ with $x = 0.001$.

To ensure accuracy to 8 decimal places, we need to include terms up to $x^3 = 10^{-9}$ at least; we can safely disregard terms in x^4 and greater powers, since these will be too small to affect the first 8 decimal places of the result.

Write down each term in the form $\binom{n}{r}(a)^{n-r}(b)^r$ with $a=1, b=2x, n=10$.

The first 4 terms are
$1(1)^{10} + 10(1)^9(2x)^1 + 45(1)^8(2x)^2 + 120(1)^7(2x)^3$

Apply the powers to the bracketed terms and multiply through.

The first 4 terms are
$1 + 20x + 180x^2 + 960x^3$

Calculate the powers of the appropriate value of x and thus the value of each term.

$x^0 = 1$	$1 = 1$
$x^1 = 0.001$	$20x^1 = 0.02$
$x^2 = 0.000\,001$	$180x^2 = 0.000\,18$
$x^3 = 0.000\,000\,001$	$960x^3 = 0.000\,000\,96$

Total the values of the terms.

Hence $1.002^{10} \simeq 1.020\,180\,96$

From calculator, $1.002^{10} = 1.020\,180\,963\,368\,08\ldots$
So approximation error is $3.30 \times 10^{-9} = 0.000\,000\,33\%$

Mixed examination practice 7

Short questions

1. Find the coefficient of x^5 in the expansion of $(2-x)^{12}$. *[5 marks]*

2. Fully expand and simplify $(2x^{-1}+5y)^3$. *[4 marks]*

3. Let $a=2-\sqrt{2}$. Using binomial expansion or otherwise, express a^5 in the form $m+n\sqrt{2}$. *[5 marks]*

4. Find the constant coefficient in the expansion of $(x^3-2x^{-1})^4$. *[4 marks]*

5. Fully expand and simplify $\left(x^2-\frac{2}{x}\right)^4$. *[6 marks]*

6. Find the coefficient of c^4d^{11} in the expansion of $(2c+5d)(c+d)^{14}$. *[6 marks]*

7. Find the coefficient of x^6 in the expansion of $(1-x^2)(1+x)^5$. *[5 marks]*

Long questions

1. (a) Sketch the graph of $y=(x+2)^3$.

(b) Find the binomial expansion of $(x+2)^3$.

(c) Find the exact value of 2.001^3.

(d) Solve the equation $x^3+6x^2+12x+16=0$. *[12 marks]*

2. The expansion of $(2x+ay)^n$ contains the term $20x^3y^2$.

(a) Write down the value of n.

(b) Find the value of a.

(c) Find the first four terms in ascending powers of y.

(d) Hence or otherwise, find 20.05^n correct to the nearest hundred. You do not need to justify the accuracy of your approximation. *[11 marks]*