Solving linear equations

3.1

Introduction

Many problems in engineering reduce to the solution of an equation or a set of equations. An equation is a type of mathematical expression which contains one or more unknown quantities which you will be required to find. In this Block we consider a particular type of equation which contains a single unknown quantity, and is known as a linear equation. Later Blocks will describe techniques for solving other types of equations.

Prerequisites

Before starting this Block you should ...

- be able to add, subtract, multiply and divide fractions
- be able to transpose formulae

Learning Outcomes

After completing this Block you should be able to ...

✓ recognise and solve a linear equation

Learning Style

To achieve what is expected of you ...

☞ allocate sufficient study time

☞ briefly revise the prerequisite material

☞ attempt *every* guided exercise and most of the other exercises

1. Linear equations

Key Point

A **linear equation** is an equation of the form

$$ax + b = 0 \qquad a \neq 0$$

where a and b are known numbers and x represents an unknown quantity which we must find.

In the equation $ax + b = 0$, the number a is called the **coefficient of** x, and the number b is called the **constant term**.

The following are examples of linear equations

$$3x + 4 = 0, \quad -2x + 3 = 0, \quad -\frac{1}{2}x - 3 = 0$$

Note that the unknown, x, appears only to the first power, that is as x, and not as x^2, $\sqrt{x}$, $x^{1/2}$ etc. Linear equations often appear in a nonstandard form, and also different letters are sometimes used for the unknown quantity. For example

$$2x = x + 1 \qquad 3t - 7 = 17, \qquad 13 = 3z + 1, \qquad 1 - \frac{1}{2}y = 3$$

are all examples of linear equations. Where necessary the equations can be rearranged and written in the form $ax + b = 0$. We will explain how to do this later in this Block.

Now do this exercise

Which of the following are linear equations and which are not linear?

(a) $3x + 7 = 0$, (b) $-3t + 17 = 0$, (c) $3x^2 + 7 = 0$, (d) $5x = 0$

The equations which can be written in the form $ax + b = 0$ are linear.

Answer

To solve a linear equation means to find the value of x that can be substituted into the equation so that the left-hand side equals the right-hand side. Any such value obtained is known as a **solution** or **root** of the equation and the value of x is said to satisfy the equation.

Example Consider the linear equation $3x - 2 = 10$.

(a) Check that $x = 4$ is a solution.

(b) Check that $x = 2$ is *not* a solution.

Solution

(a) To check that $x = 4$ is a solution we substitute the value for x and see if both sides of the equation are equal. Evaluating the left-hand side we find $3(4) - 2$ which equals 10, the same as the right-hand side. So, $x = 4$ is a solution. We say that $x = 4$ satisfies the equation.

(b) Substituting $x = 2$ into the left-hand side we find $3(2) - 2$ which equals 4. Clearly the left-hand side is not equal to 10 and so $x = 2$ is not a solution. The number $x = 2$ does not satisfy the equation.

Try each part of this exercise

Test which of the given values are solutions of the equation

$$18 - 4x = 26$$

(a) $x = 2$, (b) $x = -2$, (c) $x = 8$

Part (a) Substituting $x = 2$, the left hand side equals

Answer

Part (b) Substituting $x = -2$, the left-hand side equals

Answer

Part (c) Substituting $x = 8$, the left-hand side equals

Answer

More exercises for you to try

1. (a) Write down the general form of a linear equation.
   (b) Explain what is meant by the root or solution of a linear equation.

   In questions 2-8 verify that the given value is a solution of the given equation.

2. $3x - 7 = -28, \quad x = -7$
3. $8x - 3 = -11, \quad x = -1$
4. $2x + 3 = 4, \quad x = \frac{1}{2}$
5. $\frac{1}{3}x + \frac{4}{3} = 2, \quad x = 2$
6. $7x + 7 = 7, \quad x = 0$
7. $11x - 1 = 10, \quad x = 1$
8. $0.01x - 1 = 0, \quad x = 100.$

Answer

2. Solving a linear equation.

To solve a linear equation we try to make the unknown quantity the **subject** of the equation. This means we attempt to obtain the unknown quantity on its own on the left-hand side. To do this we may apply the same five rules used for transposing formulae given in Chapter 1 Block 7. These are given again here.

Key Point

Operations which can be used in the process of solving a linear equation:

- add the same quantity to both sides
- subtract the same quantity from both sides
- multiply both sides by the same quantity
- divide both sides by the same quantity
- take functions of both sides; for example square both sides.

A useful summary of these rules is 'whatever we do to one side of an equation we must also do to the other'.

Example Solve the equation $x + 14 = 5$.

Solution

Note that by subtracting 14 from both sides, we leave x on its own on the left. Thus

$$\begin{aligned} x + 14 - 14 &= 5 - 14 \\ x &= -9 \end{aligned}$$

Hence the solution of the equation is $x = -9$. It is easy to check that this solution is correct by substituting $x = -9$ into the original equation and checking that both sides are indeed the same. You should get into the habit of doing this.

Example Solve the equation $19y = 38$.

Solution

In order to make y the subject of the equation we can divide both sides by 19:

$$\begin{aligned} 19y &= 38 \\ \frac{19y}{19} &= \frac{38}{19} \\ \text{cancelling 19's gives} \qquad y &= \frac{38}{19} \\ \text{so} \qquad y &= 2 \end{aligned}$$

Hence the solution of the equation is $y = 2$.

Example Solve the equation $4x + 12 = 0$.

Solution

Starting from $4x + 12 = 0$ we can subtract 12 from both sides to obtain

$$\begin{aligned} 4x + 12 - 12 &= 0 - 12 \\ \text{so that} \qquad 4x &= -12 \end{aligned}$$

If we now divide both sides by 4 we find

$$\begin{aligned} \frac{4x}{4} &= \frac{-12}{4} \\ \text{cancelling 4's gives} \qquad x &= -3 \end{aligned}$$

So the solution is $x = -3$.

Now do this exercise

Solve the linear equation $14t - 56 = 0$.

Answer

Example Solve the following equations:

(a) $x + 3 = \sqrt{7}$,

(b) $x + 3 = -\sqrt{7}$.

Solution

(a) Subtracting 3 from both sides gives $x = \sqrt{7} - 3$.

(b) Subtracting 3 from both sides gives $x = -\sqrt{7} - 3$.

Note that when asked to solve $x + 3 = \pm\sqrt{7}$ we can write the two solutions as $x = -3 \pm \sqrt{7}$. It is usually acceptable to leave the solutions in this form (i.e. with the $\sqrt{7}$ term) rather than calculate decimal approximations. This form is known as the **surd form**.

Example Solve the equation $\frac{2}{3}(t + 7) = 5$.

Solution

There are a number of ways in which the solution can be obtained. The idea is to gradually remove unwanted terms on the left-hand side to leave t on its own. By multiplying both sides by $\frac{3}{2}$ we find

$$\begin{aligned} \frac{3}{2} \times \frac{2}{3}(t+7) &= \frac{3}{2} \times 5 \\ &= \frac{3}{2} \times \frac{5}{1} \\ \text{and after simplifying and cancelling,} \qquad t+7 &= \frac{15}{2} \end{aligned}$$

Finally, subtracting 7 from both sides gives

$$\begin{aligned} t &= \frac{15}{2} - 7 \\ &= \frac{15}{2} - \frac{14}{2} \\ &= \frac{1}{2} \end{aligned}$$

So the solution is $t = \frac{1}{2}$.

Example Solve the equation $3(p-2) + 2(p+4) = 5$.

Solution

At first sight this may not appear to be in the form of a linear equation. Some preliminary work is necessary. Removing the brackets and collecting like terms we find the left-hand side yields $5p + 2$ so the equation is $5p + 2 = 5$ so that, finally, $p = \frac{3}{5}$.

Try each part of this exercise

Solve the equation $2(x-5) = 3 - (x+6)$.

Part (a) First remove the brackets on both sides.

Answer

We may write this as

$$2x - 10 = -x - 3$$

We will try to rearrange this equation so that terms involving x appear only on the left-hand side, and constants on the right.

Part (b) Start by adding 10 to both sides.

Answer

Part (c) Now add x to both sides.

Answer

Part (d) Finally solve this to find x.

Answer

Example Solve the equation

$$\frac{6}{1-2x} = \frac{7}{x-2}$$

Solution

This equation appears in an unfamiliar form but it can be rearranged into the standard form of a linear equation. By multiplying both sides by $(1-2x)$ and $(x-2)$ we find

$$(1-2x)(x-2) \times \frac{6}{1-2x} = (1-2x)(x-2) \times \frac{7}{x-2}$$

Considering each side in turn and cancelling common factors:

$$6(x-2) = 7(1-2x)$$

Removing the brackets and rearranging to find x we have

$$6x - 12 = 7 - 14x$$

further rearrangement gives:

$$20x = 19$$

$$x = \frac{19}{20}$$

The solution is therefore $x = \frac{19}{20}$.

Example Consider Figure 1 which shows three branches of an electrical circuit which meet together at X. Point X is known as a **node**. As shown in Figure 1 the current in each of the branches is denoted by I, I_1 and I_2. Kirchhoff's current law states that the current entering any node must equal the current leaving that node. Thus we have the equation

$$I = I_1 + I_2$$

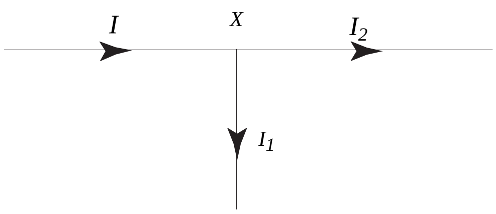

Figure 1.

(a) If $I_2 = 10$A and $I = 18$A calculate I_1.

(b) Suppose $I = 36$A and it is known that current I_2 is five times as great as I_1. Find the branch currents.

Solution

(a) Substituting the given values into the equation we find $18 = I_1 + 10$. Solving for I_1 we find

$$I_1 = 18 - 10 = 8$$

Thus I_1 equals 8 A.

(b) We are given that, from Kirchhoff's law, $I = I_1 + I_2$. We are told that I_2 is five times as great as I_1, and so we can write $I_2 = 5I_1$. Since $I = 36$ we have

$$36 = I_1 + 5I_1 = 6I_1$$

Solving this linear equation $36 = 6I_1$ gives $I_1 = 6$A. Finally, since I_2 is five times as great as I_1, we have $I_2 = 5I_1 = 30$ A.

More exercises for you to try

In questions 1-24 solve each equation:

1. $7x = 14$
2. $-3x = 6$
3. $\frac{1}{2}x = 7$
4. $3x = \frac{1}{2}$
5. $4t = -2$
6. $2t = 4$
7. $4t = 2$
8. $2t = -4$
9. $\frac{x}{6} = 3$
10. $\frac{x}{6} = -3$
11. $7x + 2 = 9$
12. $7x + 2 = 23$
13. $-7x + 1 = -6$
14. $-7x + 1 = -13$
15. $\frac{17}{3}t = -2$
16. $3 - x = 2x + 8$
17. $x - 3 = 8 + 3x$
18. $\frac{x}{4} = 16$
19. $\frac{x}{9} = -2$
20. $-\frac{13}{2}x = 14$
21. $-2y = -6$
22. $-7y = 11$
23. $-69y = -690$
24. $-8 = -4\gamma$.

In questions 25 - 47 solve each equation:

25. $3y - 8 = \frac{1}{2}y$
26. $7t - 5 = 4t + 7$
27. $3x + 4 = 4x + 3$
28. $4 - 3x = 4x + 3$
29. $3x + 7 = 7x + 2$
30. $3(x + 7) = 7(x + 2)$
31. $2x - 1 = x - 3$
32. $2(x + 4) = 8$
33. $-2(x - 3) = 6$
34. $-2(x - 3) = -6$
35. $-3(3x - 1) = 2$
36. $2 - (2t + 1) = 4(t + 2)$
37. $5(m - 3) = 8$
38. $5m - 3 = 5(m - 3) + 2m$
39. $2(y + 1) = -8$
40. $17(x - 2) + 3(x - 1) = x$
41. $\frac{1}{3}(x + 3) = -9$
42. $\frac{3}{m} = 4$
43. $\frac{5}{m} = \frac{2}{m + 1}$
44. $-3x + 3 = 18$
45. $3x + 10 = 31$
46. $x + 4 = \sqrt{8}$
47. $x - 4 = \sqrt{23}$

48. If $y = 2$ find x if $4x + 3y = 9$
49. If $y = -2$ find x if $4x + 5y = 3$
50. If $y = 0$ find x if $-4x + 10y = -8$
51. If $x = -3$ find y if $2x + y = 8$
52. If $y = 10$ find x when $10x + 55y = 530$
53. If $\gamma = 2$ find β if $54 = \gamma - 4\beta$

In questions 54-63 solve each equation:

54. $\frac{x - 5}{2} - \frac{2x - 1}{3} = 6$
55. $\frac{x}{4} + \frac{3x}{2} - \frac{x}{6} = 1$
56. $\frac{x}{2} + \frac{4x}{3} = 2x - 7$
57. $\frac{5}{3m + 2} = \frac{2}{m + 1}$
58. $\frac{2}{3x - 2} = \frac{5}{x - 1}$
59. $\frac{x - 3}{x + 1} = 4$
60. $\frac{x + 1}{x - 3} = 4$
61. $\frac{y - 3}{y + 3} = \frac{2}{3}$
62. $\frac{4x + 5}{6} - \frac{2x - 1}{3} = x$
63. $\frac{3}{2s - 1} + \frac{1}{s + 1} = 0$

64. Solve the linear equation $ax + b = 0$ to find x
65. Solve the linear equation $\frac{1}{ax + b} = \frac{1}{cx + d}$ to find x

Answer

End of Block 3.1

(a) linear

(b) linear; the unknown is t

(c) not linear because of the term x^2

(d) linear; here the constant term is zero

Back to the theory

10. But $10 \neq 26$ so $x = 2$ is not a solution.

Back to the theory

$18 - 4(-2) = 26$.   This is the same as the right-hand side, so $x = -2$ is a solution.

Back to the theory

$18 - 4(8) = -14$. But $-14 \neq 26$ and so $x = 8$ is not a solution.

Back to the theory

1. (a) The general form is $ax + b = 0$ where a and b are known numbers and x represents the unknown quantity.

   (b) A root is a value for the unknown which satisfies the equation.

Back to the theory

$t = 4$

Back to the theory

$2x - 10 = 3 - x - 6$

Back to the theory

$2x = -x + 7$

Back to the theory

$3x = 7$

Back to the theory

$\frac{7}{3}$

Back to the theory

1. 2	2. -2	3. 14	4. $\frac{1}{6}$	5. $-\frac{1}{2}$	6. 2
7. $\frac{1}{2}$	8. -2	9. 18	10. -18	11. 1	12. 3
13. 1	14. 2	15. $-6/17$	16. $-5/3$	17. $-11/2$	18. 64
19. -18	20. $-28/13$	21. $y = 3$	22. $-11/7$	23. $y = 10$	24. 2
25. $16/5$	26. 4	27. 1	28. $1/7$	29. $5/4$	30. $7/4$
31. -2	32. 0	33. 0	34. 6	35. $1/9$	36. $-7/6$
37. $23/5$	38. 6	39. -5	40. $37/19$	41. -30	42. $3/4$
43. $-5/3$	44. -5	45. 7	46. $\sqrt{8} - 4$	47. $\sqrt{23} + 4$	48. $3/4$
49. $13/4$	50. 2	51. 14	52. -2	53. -13	54. -49
55. $12/19$	56. 42	57. 1	58. $8/13$	59. $-7/3$	60. $13/3$
61. 15	62. $7/6$	63. $-2/5$	64. $-b/a$	65. $\frac{d-b}{a-c}$	

Back to the theory