QUESTIONS AND EXERCISES – 13

Q1. A car goes along a section of road shaped like the letter S. The speed of the car stays constant. Discuss how the acceleration changes, and make a rough plot.

Q2. All bodies dropped together fall to the ground in the same amount of time. But if two spheres roll down an inclined plane, one may take more time than the other. Explain why.

Q3. A solid cylinder of mass M, radius R, and moment of inertia I rolls without slipping down an inclined plane of length L and height h. Find the speed of its centre of mass when the cylinder reaches the bottom.

Q4. If the radius of the earth, assumed to be a perfect sphere, suddenly shrinks to half its present value, the mass of the Earth remaining unchanged, what will be the duration of one day?

Q5. A uniform solid cylinder of radius R = 12cm and mass M = 3.2kg is given an initial clockwise angular velocity ω_0 of 15rev/s and then lowered on to a flat horizontal surface. The coefficient of kinetic friction between the surface and the cylinder is $\mu = 0.21$. Initially, the cylinder slips as it moves along the surface, but after a time t pure rolling without slipping begins.

(a) What is the velocity v_{cm}?

(b) What is the value of t?

[Hint: find the acceleration, and hence the force. When slipping stops, the frictional force produces acceleration]

Summary of Lecture 14 – EQUILIBRIUM OF RIGID BODIES

1. A rigid body is one where all parts of the body are fixed relative to each other (for example, a pencil). Fluids and gases are non-rigid.

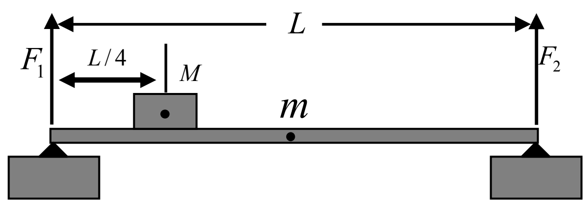
2. The translational motion of the centre of mass of a rigid body is governed by:

$$\frac{d\vec{P}}{dt} = \vec{F} \text{ where } \vec{F} = \sum \vec{F}_{ext} \text{ is the } net \text{ external force.}$$

Similarly, for rotational motion, $\frac{d\vec{L}}{dt} = \vec{\tau}$ where $\vec{\tau} = \sum \vec{\tau}_{ext}$ is the net external torque.

3. A rigid body is in mechanical equilibrium if **both** the linear momentum $\vec{P}$ and angular momentum $\vec{L}$ have a constant value. i.e., $\frac{d\vec{P}}{dt} = 0$ **and** $\frac{d\vec{L}}{dt} = 0$. **Static equilibrium** refers to $\vec{P} = 0$ and $\vec{L} = 0$.

4. As an example of static equilibrium, consider a beam resting on supports:



We want to find the forces F_1 and F_2 with which the supports push on the rod in the upwards direction. First, balance forces in the vertical y direction:

$$\sum F_y = F_1 + F_2 - Mg - mg = 0$$

Now demand that the total torque vanishes:

$$\sum \tau_y = (F_1)(0) + (F_2)(L) - (Mg)(L/4) - (mg)(L/2) = 0$$

From these two conditions you can solve for F_1 and F_2,

$$F_1 = \frac{(3M + 2m)g}{4}, \quad \text{and} \quad F_2 = \frac{(M + 2m)g}{4}.$$

5. Angular momentum and torque depend on where you choose the origin of your coordinates. However, I shall now prove that for a body in equilibrium, the choice of origin does not matter. Let's start with the origin O and calculate the torque about O,

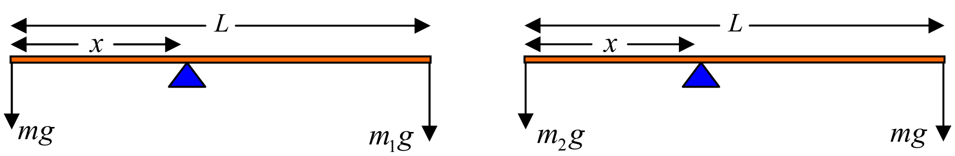
$$\vec{\tau}_O = \vec{\tau}_1 + \vec{\tau}_2 + \cdots + \vec{\tau}_N = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \cdots + \vec{r}_N \times \vec{F}_N$$

Now, if we take a second point P, then all distances will be measured from P and each vector will be shifted by an amount $\vec{r}_P$. Hence the torque about P is,

$$\begin{aligned}\vec{\tau}_P &= \left(\vec{r}_1 - \vec{r}_P\right)\times\vec{F}_1 + \left(\vec{r}_2 - \vec{r}_P\right)\times\vec{F}_2 + \cdots + \left(\vec{r}_N - \vec{r}_P\right)\times\vec{F}_N \\ &= \left[\vec{r}_1\times\vec{F}_1 + \vec{r}_2\times\vec{F}_2 + \cdots + \vec{r}_N\times\vec{F}_N\right] - \left[\vec{r}_P\times\vec{F}_1 + \vec{r}_P\times\vec{F}_2 + \cdots + \vec{r}_P\times\vec{F}_N\right] \\ &= \vec{\tau}_O - \left[\vec{r}_P\times\left(\vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_N\right)\right] \\ &= \vec{\tau}_O - \left[\vec{r}_P\times\left(\sum\vec{F}_{ext}\right)\right]\end{aligned}$$

but $\sum\vec{F}_{ext} = 0,$ for a body in translational equilibrium $\therefore\quad \vec{\tau}_P = \vec{\tau}_O.$

6. Let us use the equilibrium conditions to do something of definite practical importance. Consider the balance below which is in equilibrium when two known weights are hung as shown. We want to know m in terms of m_1 and m_2.



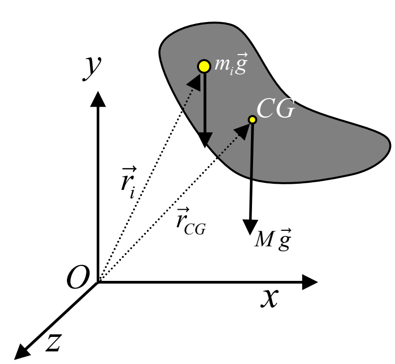
Taking the torques about the knife edge in the two cases, we have:

$$mgx = m_1 g\left(L - x\right) \text{ and } m_2 gx = mg\left(L - x\right)$$

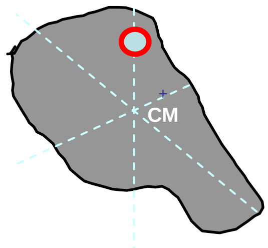
$$\Rightarrow \frac{m}{m_2} = \frac{m_1}{m} \text{ or } m = \sqrt{m_1 m_2}.$$

Remarkably, we do not need the values of x or L.

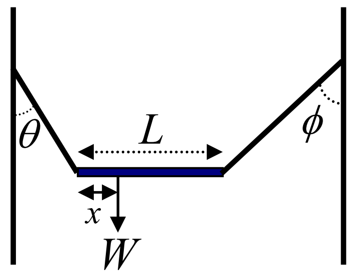
7. **Centre of Gravity.** The centre of gravity is the average location of the weight of an object. This is not quite the same as the centre of mass of a body (see lecture 12) but suppose the gravitational acceleration $\vec{g}$ has the same value at all points of a body. Then: 1) The weight is equal to $M\vec{g}$, and 2)the centre of gravity coincides with the centre of mass. Remember that weight is force, so the CG is really the centre of gravitational force acting on the body. The net force on the whole body = sum of forces over all individual particles, $\sum\vec{F} = \sum m_i\vec{g}$. If $\vec{g}$ has the same value at all points of the body, then $\sum\vec{F} = \vec{g}\sum m_i = M\vec{g}$. So the net torque about the origin O is $\sum\vec{\tau} = \sum\left(\vec{r}_i\times m_i\vec{g}\right) = \sum\left(m_i\vec{r}_i\times\vec{g}\right)$ Hence, $\sum\vec{\tau} = M\vec{r}_{cm}\times\vec{g} = \vec{r}_{cm}\times M\vec{g}$. So the torque due to gravity about the centre of mass of a body (i.e. at $\vec{r}_{cm} = 0$) is zero !!



8. In the demonstration I showed, you saw how to find the CG of an irregular object by simply suspending it on a pivot,



9. Let's solve a problem of static equilibrium: A non-uniform bar of weight W is suspended at rest in a horizontal position by two light cords. Find the distance x from the left-hand end to the center of gravity.



Solution : Call the tensions T_1 and T_2 . Put the forces in both directions equal to zero,

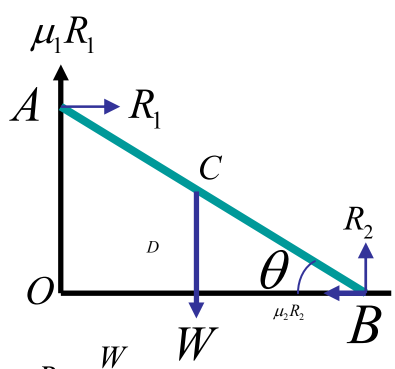
a) $T_2 \sin\phi - T_1 \sin\theta = 0$ (horizontal)

b) $T_2 \cos\phi + T_1 \cos\theta - W = 0$ (vertical) $\Rightarrow T_2 = \dfrac{W}{\sin(\theta+\phi)}$

The torque about any point must vanish. Let us choose that point to be one end of the bar,

$$-Wx + (T_2 \cos\phi)L = 0 \Rightarrow x = \frac{(T_2 \cos\phi)L}{W} = \frac{L\cos\phi}{\sin(\theta+\phi)}$$

10. Here is another problem of the same kind: find the least angle θ at which the rod can lean to the horizontal without slipping.



Solution : Considering the translational equilibrium of the rod, $R_1 = \mu_2 R_2$ and $R_2 + \mu_1 R_1 = W$. This gives, $R_2 = \dfrac{W}{(1+\mu_1\mu_2)}$. Now consider rotational equilibrium about the point A: $R_2 \times OB = W \times OD + \mu_2 R_2 \times OA$ or, $R_2 \times AB\cos\theta = W \times \dfrac{AB\cos\theta}{2} + \mu_2 R_2 \times AB\sin\theta$.

This gives $\cos\theta\left(R_2 - \dfrac{W}{2}\right) = \mu_2 R_2 \sin\theta$ from which $\tan\theta = \dfrac{R_2 - \frac{W}{2}}{\mu_2 R_2}$ with $R_2 = \dfrac{W}{(1+\mu_1\mu_2)}$.

Using this value of R_2, we get $\tan\theta = \dfrac{1\text{-}\mu_1\mu_2}{2\mu_2}$.

10. **Types of Equilibrium.** In the lecture you heard about:

a) Stable equilibrium: object returns to its original position if displaced slightly.

b) Unstable equilibrium: object moves farther away from its original position if displaced slightly.

c) Neutral equilibrium: object stays in its new position if displaced slightly.

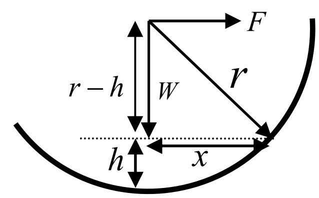
QUESTIONS AND EXERCISES – 14

Q.1 Give three examples of each of the following that are *not* given in either the lecture or these notes:

a) Static equilibrium
b) Dynamic equilibrium
c) Stable equilibrium
d) Neutral equilibrium
e) Unstable equilibrium

Q.2 Ship designers want to keep the CG of their ships as low as possible. Why? Discuss why there is a contradiction between this requirement, and other requirements.

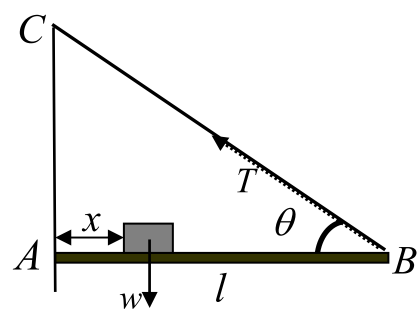
Q.3 Work through the example given in the lecture where you are asked to find the minimum force F applied horizontally at the axle of the wheel in order to raise it over an obstacle of height h.



Q.4 A thin bar AB of negligible weight is pinned to a vertical wall at A and supported by a thin wire BC. A weight w can be moved along the bar.

a) Find T as a function of x.
b) Find the horizontal and vertical components of the force exerted on the bar by the pin at A.

[Hint: Since the system is in rotational equilibrium, the net torque about A is zero.]

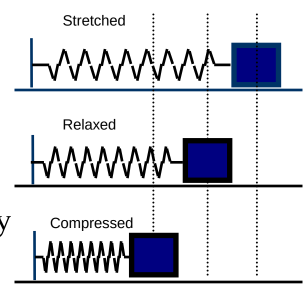


Summary of Lecture 15 – OSCILLATIONS: I

1. An oscillation is any self-repeating motion. This motion is characterized by:
   a) The period T, which is the time for completing one full cycle.
   b) The frequency $f = 1/T$, which is the number of cycles per second. (Another frequently used symbol is ν).
   c) The amplitude A, which is the maximum displacement from equilibrium (or the size of the oscillation).

2. Why does a system oscillate? It does so because a force is always directed towards a central equilibrium position. In other words, the force always acts to return the object to its equilibrium position. So the object will oscillate around the equilibrium position. The restoring force depends on the displacement $F_{\text{restore}} = -k\,\Delta x,$ where Δx is the distance away from the equilibrium point, the negative sign shows that the force acts towards the equilibrium point, and k is a constant that gives the strength of the restoring force.

3. Let us understand these matters in the context of a spring tied to a mass that can move freely over a frictionless surface. The force $F(x) = -kx$ (or $-k\Delta x$ because the extension Δx will be called x for short). In the first diagram x is positive, x is negative in the second, and zero in the middle one. The energy stored in the spring, $U(x) = \frac{1}{2}kx^2,$ is positive in the first and third diagrams and zero in the middle one. Now we will use Newton's second law to derive a differential euation that describes the motion of the mass: From $F(x) = -kx$ and $ma = F$ it follows that $m\frac{d^2x}{dt^2} = -kx,$ or $\frac{d^2x}{dt^2} + \omega^2 x = 0$ where $\omega^2 \equiv \frac{k}{m}$. This is the equation of motion of a simple harmonic oscillator (SHO) and is seen widely in many different branches of physics. Although we have derived it for the case of a mass and spring, it occurs again and again. The only difference is that $\omega,$ which is called the oscillator frequency, is defined differently depending on the situation.



4. In order to solve the SHO equation, we shall first learn how to differentiate some elementary trignometric functions. So let us first learn how to calculate $\frac{\mathrm{d}}{\mathrm{dt}}\cos\omega t$ starting from the basic definition of a derivative:

Start : $x(t) = \cos\omega t$ and $x(t+\Delta t) = \cos\omega(t+\Delta t)$. Take the difference:

$$\begin{aligned} x(t+\Delta t) - x(t) &= \cos\omega(t+\Delta t) - \cos\omega t \\ &= -\sin\omega\Delta t \sin(\omega t + \omega\Delta t/2) \\ &\approx -\omega\Delta t \sin\omega t \quad \text{as } \Delta t \text{ becomes very small.} \end{aligned}$$

$$\therefore \frac{d}{dt}\cos\omega t = -\omega\sin\omega t.$$

(Here you should know that $\sin\theta \approx \theta$ for small θ, easily proved by drawing triangles.)

You should also derive and remember a second important result:

$$\frac{d}{dt}\sin\omega t = \omega\cos\omega t \ .$$

(Here you should know that $\cos\theta \approx 1$ for small θ.)

5. What happens if you differentiate twice?

$$\frac{d^2}{dt^2}(\sin\omega t) = \omega\frac{d}{dt}\cos\omega t = -\ \omega^2\sin\omega t$$

$$\frac{d^2}{dt^2}(\cos\omega t) = -\omega\frac{d}{dt}\sin\omega t = -\ \omega^2\cos\omega t.$$

So twice differentiating either $\sin\omega t$ or $\cos\omega t$ gives the same function back!

6. Having done all the work above, now you can easily see that any function of the form $x(t) = a\cos\omega t + b\sin\omega t$ satisfies $\dfrac{d^2x}{dt^2} = -\omega^2 x$. But what do ω, a, b represent ?

a) The significance of ω becomes clear if your replace t by $t + \dfrac{2\pi}{\omega}$ in either $\sin\omega t$ or $\cos\omega t$. You can see that $\cos\omega\left(t + \dfrac{2\pi}{\omega}\right) = \cos(\omega t + 2\pi) = \cos\omega t$. That is, the function merely repeats itself after a time $2\pi/\omega$. So $2\pi/\omega$ is really the period of the motion T, $T = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{m}{k}}$. The frequency ν of the oscillator is the number of complete vibrations per unit time: $\nu = \dfrac{1}{T} = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{m}}$ so $\omega = 2\pi\nu = \dfrac{2\pi}{T} = \sqrt{\dfrac{k}{m}}$. Sometimes ω is also called the angular frequency. Note that $\dim[\omega] = T^{-1}$, from it is clear that the unit of ω is radian/second.

b) To understand what a and b mean let us note that from $x(t) = a\cos\omega t + b\sin\omega t$ it follows that $x(0) = a$ and that $\dfrac{\mathrm{d}}{\mathrm{dt}}x(t) = -\omega a\sin\omega t + \omega b\cos\omega t = \omega b$ (at $t = 0$). Thus, a is the initial position, and b is the initial velocity divided by ω.

c) To understand what a and b mean let us note that from $x(t) = a\cos\omega t + b\sin\omega t$ it follows that $x(0) = a$ and that $\frac{\mathrm{d}}{\mathrm{dt}}x(t) = -\omega a\sin\omega t + \omega b\cos\omega t = \omega b$ (at $t = 0$). Thus, a is the initial position, and b is the initial velocity divided by ω.

d) The solution can also be written as: $x(t) = x_m\cos(\omega t + \phi)$. Since cos and sin never beome bigger than 1, or less than -1, it follows that $-x_m \le x \le +x_m$. For obvious reason x_m is called the amplitude of the motion. The frequency of the simple harmonic motion is independent of the amplitude of the motion.

e) The quantity $\theta = \omega t + \phi$ is called the phase of the motion. The constant ϕ is called the *phase constant.* A different value of ϕ just means that the origin of time has been chosen differently.

7. **Energy of simple harmonic motion**. Put $\phi = 0$ for convenience, and so imagine a mass whose position oscillates like $x = x_m\cos\omega t$. Let us first calculate the potential energy:

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kx_m^2\cos^2\omega t.$$

Now calculate the kinetic energy:

$$K = \frac{1}{2}m\mathrm{v}^2 = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 = \frac{1}{2}m\omega^2x_m^2\sin^2\omega t = \frac{1}{2}kx_m^2\sin^2\omega t$$

The sum of potential + kinetic is:

$$\begin{aligned} E = K + U &= \frac{1}{2}kx_m^2\cos^2\omega t + \frac{1}{2}kx_m^2\sin^2\omega t \\ &= \frac{1}{2}kx_m^2\left(\cos^2\omega t + \sin^2\omega t\right) = \frac{1}{2}kx_m^2. \end{aligned}$$

Note that this is independent of time and energy goes from kinetic to potential, then back to kinetic etc.

8. From the above, you can see that $\mathrm{v} = \frac{dx}{dt} = \pm\sqrt{\frac{k}{m}\left(x_m^2 - x^2\right)}$. From this it is clear that the speed is maximum at $x = 0$ and that the speed is zero at $x = \pm x_m$.

9. Putting two springs in parallel makes it harder to stretch them, and $k_{eff} = k_1 + k_2$. In series they are easier to stretch, and $k_{eff} = \left(\frac{k_1k_2}{k_1 + k_2}\right)$. So a mass will oscillate faster in the first case as compared to the second.

QUESTIONS AND EXERCISES – 15

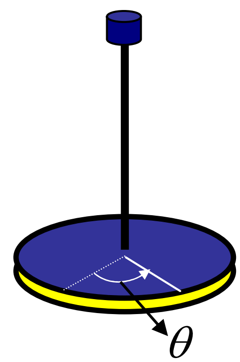
Q.1 A bottle is half filled with water (so that it floats upright) and then pushed a little into the water. As you can see, it oscillates up and down.

a) Where does the restoring force come from?

b) Suppose that you filled the bottle 3/4 full. What would happen to the oscillation frequency?

c) Why does the bottle eventually stop oscillating?

Q.2 A disc is suspended from a string. The equilibrium position is that for which there is no twist in the string, i.e. at $\theta = 0$. When it is slightly moved off from equilibrium, there is a restoring force $F = -\kappa\theta$.



a) Show that $\dfrac{d^2\theta}{dt^2} = -\left(\dfrac{\kappa}{I}\right)\theta$ where I is the moment of inertia.

b) If the disc is at $\theta = 0$ at $t = 0$ and suddenly given a twist so that $\dfrac{d\theta}{dt} = \omega_0$, find how long it takes to return to its initial position. Where will be at time t?

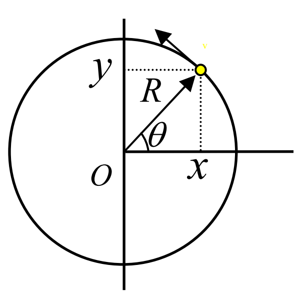
c) Why does the motion eventually cease? List all the ways in which this disc loses energy.

Q.3 Referring to the figure below, you can see that the coordinates of a particle going around a circle are given by $(x, y) = (R\cos\theta, R\sin\theta)$ where $\theta = \omega t$.

a) On the same axes, plot x and y as a function of time. Obviously, here is a case of two harmonic oscillations. What is the phase difference between the two?

b) Find $\dot{x}^2 + \dot{y}^2$, where $\dot{x}$ is $\dfrac{dx}{dt}$ (this is a very popular way of denoting time derivatives because it is short, so you should be familiar with it.

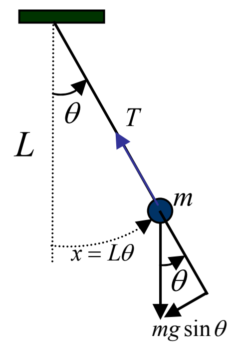
c) Repeat the above for $\ddot{x}^2 + \ddot{y}^2$, where we now have second derivatives instead.



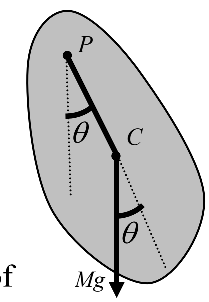
Summary of Lecture 16 – OSCILLATIONS: II

1. In this chapter we shall continue with the concepts developed in the previous chapter that relate to simple harmonic motion and the simple harmonic oscillator (SHO). It is really very amazing that the SHO occurs again and again in physics, and in so many different branches.

2. As an example illustrating the above, consider a mass suspended a string. From the diagram, you can see that $F = -mg\sin\theta$. For small values of θ we know that $\sin\theta \approx \theta$. Using $x = L\theta$ (length of arc), we have $F = -mg\theta = -mg\dfrac{x}{L} = -\left(\dfrac{mg}{L}\right)x$. So now we have a restoring force that is proportional to the distance away from the equilibrium point. Hence we have a SHO with $\omega = \sqrt{g/L}$. What if we had not made the small θ approximation? We would still have an oscillator (i.e. the motion would be self repeating) but the solutions of the differential equation would be too complicated to discuss here.



3. If you take a common object (like a piece of cardboard) and pivot it at some point, it will oscillate when disturbed. But this is not the simple pendulum discussed above because all the mass is not concentrated at one point. So now let us use the ideas of torque and angular momentum discussed earlier for many particle systems. You can see that $\tau = -Mgd\sin\theta$. For small θ, $\sin\theta \approx \theta$ and so $\tau = -Mgd\theta$. But we also know that $\tau = I\alpha$ where I is the moment of inertia and α is the angular acceleration, $\alpha = \dfrac{d^2\theta}{dt^2}$. Hence, we have $I\dfrac{d^2\theta}{dt^2} = -Mgd\theta$, or, $\dfrac{d^2\theta}{dt^2} = -\left(\dfrac{Mgd}{I}\right)\theta$. From this we immediately see that the oscillation frequency is $\omega = \sqrt{\dfrac{Mgd}{I}}$. Of course, we have used the small angle approximation over here again. Since all variables except I are known, we can use this formula to tell us what I is about any point. Note that we can choose to put the pivot at any point on the body. However, if you put the pivot exactly at the centre of mass then it will not oscillate. Why? Because there is no restoring force and the torque vanishes at the cm position, as we saw earlier.



4. Suppose you were to put the pivot at point P which is at a distance L from the centre of mass of the irregular object above. What should L be so that you get the same formula as for a simple pendulum?

   Answer: $T = 2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{I}{Mgd}} \Rightarrow L = \frac{I}{Md}$

   P is then called the centre of gyration - when suspended from this point it appears as if all the mass is concentrated at the cm position.

5. Sum of two simple harmonic motions of the same period along the same line:

$$x_1 = A_1 \sin\omega t \text{ and } x_2 = A_2 \sin(\omega t + \phi)$$

   Let us look at the sum of x_1 and x_2,

$$\begin{aligned} x = x_1 + x_2 &= A_1 \sin\omega t + A_2 \sin(\omega t + \phi) \\ &= A_1 \sin\omega t + A_2 \sin\omega t \cos\phi + A_2 \sin\phi \cos\omega t \\ &= \sin\omega t\left(A_1 + A_2 \cos\phi\right) + \cos\omega t\left(A_2 \sin\phi\right) \end{aligned}$$

   Let $A_1 + A_2\cos\phi = R\cos\theta$ and $A_2 \sin\phi = R\sin\theta$. Using some simple trigonometry, you can put x in the form, $x = R\sin(\omega t + \theta)$. It is easy to find R and θ:

   $R = \sqrt{A_1^2 + A_2^2 + A_1A_2\cos\phi}$ and $\tan\theta = \dfrac{A_2\sin\phi}{A_1 + A_2\cos\phi}$.

   Note that if $\phi = 0$ then $R = \sqrt{A_1^2 + A_2^2 + A_1A_2} = \sqrt{(A_1 + A_2)^2} = A_1 + A_2$ and $\tan\theta = 0$ $\Rightarrow \theta = 0$. So we get $x = (A_1 + A_2)\sin\omega t$. This is an example of *constructive interference*. If $\phi = \pi$ then $R = \sqrt{A_1^2 + A_2^2 - A_1A_2} = \sqrt{(A_1 - A_2)^2} = A_1 - A_2$ and $\tan\theta = 0$ $\Rightarrow \theta = 0$. Now we get $x = (A_1 - A_2)\sin\omega t$. This is *destructive interference.*

6. Composition of two simple harmonic motions of the same period but now at right angles to each other:

   Suppose $x = A\sin\omega t$ and $y = B\sin(\omega t + \phi)$. These are two independent motions. We can write $\sin\omega t = \dfrac{x}{A}$ and $\cos\omega t = \sqrt{1 - x^2/A^2}$.

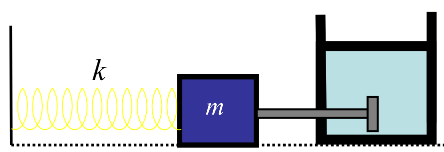
   From this, $\dfrac{y}{B} = \sin\omega t\cos\phi + \sin\phi\cos\omega t = \dfrac{x}{A}\cos\phi + \sin\phi\sqrt{1 - x^2/A^2}$. Now square and rearrange terms to find:

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} - 2\frac{xy}{AB}\cos\phi = \sin^2\phi$$

   This is the equation for an ellipse (see questions at the end of this section).

7. If two oscillations of different frequencies at right angles are combined, the resulting motion is more complicated. It is not even periodic unless the two frequencies are in the ratio of integers. This resulting curve are called Lissajous figures. Specifically, if $x = A\sin\omega_x t$ and $y = B\sin\left(\omega_y t + \phi\right)$, then periodic motion requires $\frac{\omega_x}{\omega_y} = \text{integers}$. You should look up a book for more details.

8. **Damped harmonic motion**: Typically the frictional force due to air resistance, or in a liquid, is proportional to the speed. So suppose that the damping force $= -b\frac{dx}{dt}$ (why negative sign?). Now apply Newton's law to a SHO that is damped: $-kx - b\frac{dx}{dt} = m\frac{d^2x}{dt^2}$ Rearrange slightly to get the equation for a damped SHO: $m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$.



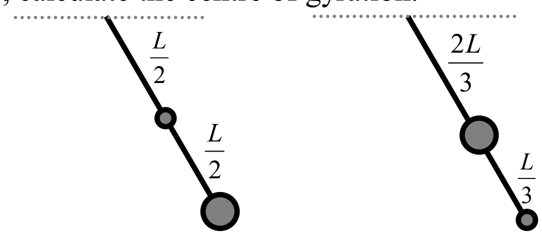
Its solution for $\frac{k}{m} \geq \left(\frac{b}{2m}\right)^2$ is $x = x_m e^{-bt/2m}\cos\left(\omega' t + \phi\right)$. The frequency is now changed: $\omega' = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$. The damping causes the amplitude to decrease with time and when $bt/2m = 1$, the amplitude is $1/e \approx\ 1/2.7$ of its initial value.

8. **Forced oscillation and resonance.** There is a characteristic value of the driving frequency ω at which the amplitude of oscillation is a maximum. This condition is called resonance. For negligible damping resonance occurs at $\omega = \omega_0$. Here ω_0 is the natural frequency of the system and is given by $\omega_0 = \sqrt{\frac{k}{m}}$. The equation of motion is: $m\frac{d^2x}{dt^2} + kx = F_0\cos\omega t$. You should check that this is solved by putting $x = \frac{F_0}{m\left(\omega_0^2 - \omega^2\right)}\cos\omega t$ (just subsitute into the equation and see!). Note that the amplitude "blows up" when $\omega \to \omega_0$. This is because we have no damping term here. With damping, the amplitude is large when $\omega \to \omega_0$ but remains finite.

QUESTIONS AND EXERCISES – 16

Q.1 For the equation derived in point 6, sketch the curves on an x-y plot for:
a)$\phi = 0$, and, b)$\phi = \pi / 2$. Take $A = 1,\ B = 2$.

Q.2 A light rod of length L has two masses M and $3M$ attached to it as shown in the diagrams below.
a) In each case, calculate the frequency of small oscillations.
b) In each case, calculate the centre of gyration.



Q.3 In each of the two cases below, eliminate the time t. In other words, find a relation between x and y which does not involve t.
a) $x = \sin t, y = 2\sin t$
b) $x = \sin t, y = \cos t$
c) $x = \cos t, y = \sin 2t$

Q.4 Verify that $x(t) = x_m e^{-bt/2m} \cos\left(\omega' t + \phi\right)$ is a solution of the damped SHO equation. Plot $x(t)$ from $t = 0$ to $t = 2$ for the following case: $x_m = 1,\ b = 2,\ m = 1,\ \phi = 0.$

Q.5 A SHO is driven by a force $F(t)$ that depends upon time and obeys the equation,
$m\dfrac{d^2x}{dt^2} + kx = F(t)$. Suppose that $F(t) = F_0 \cos 2\omega_0 t + 2F_0 \sin 3\omega_0 t,$ where $\omega_0 = \sqrt{\dfrac{k}{m}}.$
a) Find the general solution $x(t)$.
b) Find that particular solution which has $x(t = 0) = 0$ and $\dot{x}(t = 0) = 1.$

Summary of Lecture 17 – PHYSICS OF MATERIALS

1. **Elasticity :** the property by virtue of which a body tends to regain its original shape and size when external forces are removed. If a body completely recovers its original shape and size , it is called perfectly elastic. Quartz, steel and glass are very nearly elastic.

2. **Plasticity :** if a body has no tendency to regain its original shape and size , it is called perfectly plastic. Common plastics, kneaded dough, solid honey, etc are plastics.

3. **Stress** characterizes the strength of the forces causing the stretch, squeeze, or twist. It is defined usually as force/unit area but may have different definitions to suit different situations. We distinguish between three types of stresses:
   a) If the deforming force is applied along some linear dimension of a body, the stress is called *longitudinal stress* or *tensile stress* or *compressive stress.*
   b) If the force acts normally and uniformly from all sides of a body, the stress is called *volume stress*.
   c) If the force is applied tangentially to one face of a rectangular body, keeping the other face fixed, the stress is called tangential or shearing stress.

4. Strain: When deforming forces are applied on a body, it undergoes a change in shape or size. The fractional (or relative) change in shape or size is called the strain.

$$\text{Strain} = \frac{\text{change in dimension}}{\text{original dimension}}$$

Strain is a ratio of similar quantities so it has no units. There are 3 different kinds of strain:

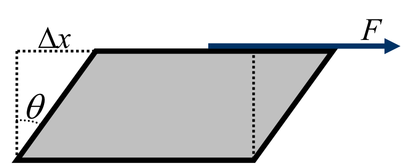
a) *Longitudinal (linear) strain* is the ratio of the change in length (ΔL) to original length (l), i.e., the linear strain $=\dfrac{\Delta l}{l}$.

b) *Volume strain* is the ratio of the change in volume (ΔV) to original volume (V)

$$\text{Volume strain} = \frac{\Delta V}{V}.$$

c) *Shearing strain :* The angular deformation (θ) in radians is called shearing stress.

For small θ the shearing strain $\equiv \theta \approx \tan\theta = \dfrac{\Delta x}{l}$.



5. Hooke's Law: for small deformations, stress is proportional to strain.

$$\text{Stress} = E \times \text{Strain}$$

The constant E is called the modulus of elasticity. E has the same units as stress because strain is dimensionless. There are three moduli of elasticity.

(a) Young's modulus (Y) for linear strain:

$$Y \equiv \frac{\text{longitudinal stress}}{\text{longitudinal strain}} = \frac{F/A}{\Delta l/l}$$

(b) Bulk Modulus (B) for volume strain: Let a body of volume V be subjected to a uniform pressure ΔP on its entire surface and let ΔV be the corresponding decrease in its volume. Then,

$$B \equiv \frac{\text{Volume Stress}}{\text{Volume Strain}} = -\frac{\Delta P}{\Delta V/V}.$$

$1/B$ is called the compressibility. A material having a small value of B can be compressed easily.

(c) Shear Modulus (η) for shearing strain: Let a force F produce a strain θ as in the diagram in point 4 above. Then,

$$\eta \equiv \frac{\text{shearing stress}}{\text{shearing strain}} = \frac{F/A}{\theta} = \frac{F}{A\tan\theta} = \frac{Fl}{A\Delta x}.$$

6. When a wire is stretched, its length increases and radius decreases. The ratio of the lateral strain to the longitudinal strain is called Poisson's ratio, $\sigma = \dfrac{\Delta r/r}{\Delta l/l}$. Its value lies between 0 and 0.5.

7. We can calculate the work done in stretching a wire. Obviously, we must do work against a force. If x is the extension produced by the force F in a wire of length l, then $F = \dfrac{YA}{l}x$. The work done in extending the wire through Δl is given by,

$$W = \int_0^{\Delta l} F dx = \frac{YA}{l}\int_0^{\Delta l} x dx = \frac{YA}{l}\frac{(\Delta l)^2}{2}$$

$$= \frac{YA}{l}\frac{(\Delta l)^2}{2} = \frac{1}{2}(Al)\left(\frac{Y\Delta l}{l}\right)\left(\frac{\Delta l}{l}\right) = \frac{1}{2} \times \text{volume} \times \text{stress} \times \text{strain}$$

Hence, Work / unit volume $= \dfrac{1}{2} \times \text{stress} \times \text{strain}$. We can also write this as,

$$W = \frac{1}{2}\left(\frac{YA\Delta l}{l}\right)\Delta l = \frac{1}{2} \times \text{load} \times \text{extension}.$$