Summary of Lecture 10 – COLLISIONS

1. Collisions are extremely important to understand because they happen all the time - electrons collide with atoms, a bat with a ball, cars with trucks, star galaxies with other galaxies,...In every case, the sum of the initial momenta equals the sum of the final momenta. This follows directly from Newton's Second Law, as we have already seen.

2. Take the simplest collision: two bodies of mass m_1 and m_2 moving with velocities u_1 and u_2. After the collision they are moving with velocities v_1 and v_2. From momentum conservation,
$$m_1u_1 + m_2u_2 = m_1\mathrm{v}_1 + m_2\mathrm{v}_2 \Rightarrow m_1(u_1 - \mathrm{v}_1) = m_2(\mathrm{v}_2 - u_2)$$
This is as far as we can go. There are two unknowns but only one equation. However, if the collision is elastic then,
$$\tfrac{1}{2}m_1u_1^2 + \tfrac{1}{2}m_2u_2^2 = \tfrac{1}{2}m_1\mathrm{v}_1^2 + \tfrac{1}{2}m_2\mathrm{v}_2^2 \Rightarrow \tfrac{1}{2}m_1(u_1^2 - \mathrm{v}_1^2) = \tfrac{1}{2}m_2(\mathrm{v}_2^2 - u_2^2).$$
Combine the two equations above,
$$u_1 + \mathrm{v}_1 = \mathrm{v}_2 + u_2 \Rightarrow u_1 - u_2 = \mathrm{v}_2 - \mathrm{v}_1.$$
In words, this says that in an elastic collision the relative speed of the incoming particles equals the relative speed of the outgoing particles.

3. One can solve for v_1 and v_2 (please do it!) easily and find that:
$$\mathrm{v}_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)u_1 + \left(\frac{2m_2}{m_1 + m_2}\right)u_2$$
$$\mathrm{v}_2 = \left(\frac{2m_1}{m_1 + m_2}\right)u_1 + \left(\frac{m_2 - m_1}{m_1 + m_2}\right)u_2$$
Notice that if $m_1 = m_2$, then $\mathrm{v}_1 = u_2$ and $\mathrm{v}_2 = u_1$. So this says that after the collision, the bodies will just reverse their velocities and move on as before.

4. What if one of the bodies is much heavier than the other body, and the heavier body is at rest? In this case, $m_2 >> m_1$ and $u_2 = 0$. e can immediately see that $\mathrm{v}_1 = -u_1$ and $\mathrm{v}_2 = 0$. This makes a lot of sense: the heavy body continues to stay at rest and the light body just bounces back with the same speed. In the lecture, you saw a demonstration of this!

5. And what if the lighter body (rickshaw) is at rest and is hit by the heavier body (truck)? In this case, $m_2 << m_1$ and $u_2 = 0$. From the above equation we see that $\mathrm{v}_1 = u_1$ and $\mathrm{v}_2 = 2u_1$. So the truck's speed is unaffected, but the poor rickshaw is thrust in the direction of the truck at twice the truck's speed!

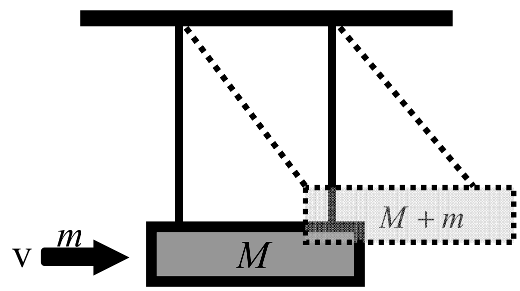
6. Sometimes we wish to slow down particles by making them collide with other particles.

In a nuclear reactor, neutrons can be slowed down in this way. let's calculate the fraction by which the kinetic energy of a neutron of mass m_1 decreases in a head-on collision with an atomic nucleus of mass m_2 that is initially at rest:

Solution: $\dfrac{K_i - K_f}{K_i} = 1 - \dfrac{K_f}{K_i} = 1 - \dfrac{v_f^2}{v_i^2}$

For a target at rest: $v_f = (\dfrac{m_1 - m_2}{m_1 + m_2})v_i$ $\quad\quad \therefore \dfrac{K_i - K_f}{K_i} = \dfrac{4m_1 m_2}{(m_1 + m_2)^2}.$

7. A bullet with mass m, is fired into a block of wood with mass M, suspended like a pendulum and makes a completely inelastic collision with it. After the impact, the block swings up to a maximum height y. What is the initial speed of the bullet?



Solution:

By conservation of momentum in the direction of the bullet,

$m\text{v} = (m+M)V, \;\Rightarrow\; \text{v} = \dfrac{(m+M)}{m}V$

The block goes up by distance y, and so gains potential energy. Now we can use the conservation of energy to give, $\dfrac{1}{2}(m+M)V^2 = (m+M)gy,$ where V is the velocity acquired by the block+bullet in the upward direction just after the bullet strikes. Now use $V = \sqrt{2gy}$. So finally, the speed of the bullet is: $\text{v} = \dfrac{(m+M)}{m}\sqrt{2gy}.$

8. In 2 or 3 dimensions, you must apply conservation of momentum in each direction separately. The equation $\vec{P}_i = \vec{P}_f$ looks as if it is one equation, but it is actually 3 separate equations: $p_{ix} = p_{fx}, p_{iy} = p_{fy}, p_{iz} = p_{fz}$. On the other hand, suppose you had an elastic collision. In that case you would have only one extra equation coming from energy conservation, not three.

9. What happens to energy in an inelastic collision? Let's say that one body smashes into another body and breaks it into 20 pieces. To create 20 pieces requires doing work against the intermolecular forces, and the initial kinetic energy is used up for this.

QUESTIONS AND EXERCISES – 10

Q.1 Give two examples of nearly elastic collisions, and two examples of nearly inelastic collisions that are not from the lecture.

Q.2 In the lecture demonstration you saw that there are 3 balls of equal mass placed in a straight line. The first strikes the second and comes to rest. The second moves and hits the third and comes to rest. The third one then moves off alone.

a) What would happen if the second one was twice the mass of the first or the third, and if the collision was elastic?

b) What would happen if the second was infinitely heavy, and the collision was elastic?

c) What would happen if the collision of the first and second resulted in these two getting stuck to each other? With what speed would they strike the third one?

Q.3 Suppose a ball loses 10% of its energy when it is bounced off the ground. If it dropped from a height h initially, find the maximum height it reaches after

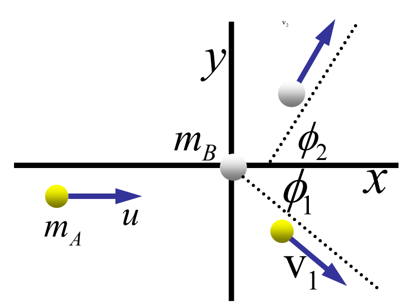
a) The first bounce

b) The second bounce

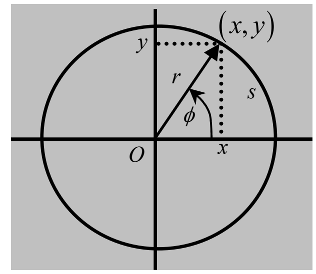
c) The $n'th$ bounce

Q.4 In the example in point 7 above, suppose that the bullet bounces back with speed v/2 instead of lodging itself into the wooden block. In that case, calculate how high the block will rise.

Q.5 The figure shows an elastic collision of two bodies A,B on a frictionless table. A has mass $m_A = 0.5$ kg, and B has mass $m_B = 0.3$ kg. A has an initial velocity of 4 m/s in the the positive x-direction and a final velocity of 2 m/s in an unknown direction. B is initially at rest. Find the final velocity of B and the angles in the figure.



Summary of Lecture 11 – ROTATIONAL KINEMATICS



1. Any rotation is specified by giving two pieces of information:
   a) The point about which the rotation occurs, i.e. the origin.
   b) The angle of rotation is denoted by ϕ in the diagram.

2. The arc length = radius × angular displacement, or $s = r\phi$. Here ϕ is measured in radians. The maximum value of ϕ is 2π radians, which corresponds to 360 degrees or one full revolution. From this it follows that 1 radian = 57.3^0 or 1 radian = 0.159 revolution. Obviously, if ϕ = 2π, then $s = 2\pi r$, which is the total circumference.

3. Suppose that there is a particle located at the tip of the radius vector. Now we wish to describe the rotational *kinematics* of this particle, i.e. describe its motion as goes around the circle. So, suppose that the particle moves from angle ϕ_1 to ϕ_2 in time $t_2 - t_1$. Then, the *average angular speed* $\bar{\omega}$ is defined as, $\bar{\omega} = \frac{\phi_2 - \phi_1}{t_2 - t_1} = \frac{\Delta\phi}{\Delta t}$. Suppose that we look at $\bar{\omega}$ over a very short time. Then, $\omega = \lim\limits_{\Delta t \to 0} \frac{\Delta\phi}{\Delta t} = \frac{d\phi}{dt}$ is called the instantaneous angular speed.

4. To familarize ourselves with the notion of angular speed, let us compute ω for a clock second, minute and hour hands:

$$\omega_{\text{second}} = \frac{2\pi}{60} = 0.105\ rad/s,$$

$$\omega_{\text{minute}} = \frac{2\pi}{60 \times 60} = 1.75 \times 10^{-3}\ rad/s,$$

$$\omega_{\text{hour}} = \frac{2\pi}{60 \times 60 \times 12} = 1.45 \times 10^{-4}\ rad/s.$$

5. Just as we defined accelaration for linear motion, we also define acceleration for circular motion:

$$\bar{\alpha} \equiv \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t} \qquad \text{(average angular speed)}$$

Hence, $\alpha = \lim\limits_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t}$ becomes $\alpha = \frac{d\omega}{dt} = \frac{d}{dt}\frac{d\phi}{dt} = \frac{d^2\phi}{dt^2}$ (angular acceleration). Let us see what this means for the speed with which a particle goes around. Now use $s = r\phi$. Differentiate with respect to time t: $\frac{ds}{dt} = r\frac{d\phi}{dt}$. The rate of change of arc length s is clearly what we should call the circular speed, v. So $v = r\omega$. Since r is held fixed, it follows that $\frac{dv}{dt} = r\frac{d\omega}{dt}$. Now define $a_T = \frac{dv}{dt}$. Obviously, $a_T = r\alpha$. Here T stands for tangential, i.e. in the direction of increasing s.

6. Compare the formulae for constant linear and angular accelerations:

LINEAR	ANGULAR
$v = v_0 + at$	$\omega = \omega_0 + \alpha t$
$x = x_0 + v_0 t + \frac{1}{2}at^2$	$\phi = \phi_0 + \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = v_0^2 + 2a(x - x_0)$	$\omega^2 = \omega_0^2 + 2\alpha(\phi - \phi_0)$

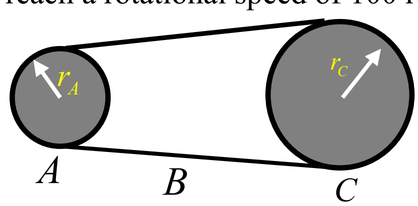
Why are they almost identical even though they describe two totally different physical situations. Answer: because the mathematics is identical!

7. The angular speed of a car engine is increased from 1170 rev/min to 2880 rev/min in 12.6 s. a)Find the average angular acceleration in rev/min^2. (b) How many revolutions does the engine make during this time?

SOLUTION: this is a straightforward application of the formulae in point 5 above.

$$\alpha = \frac{\omega_f - \omega_i}{t} = 8140\,\text{rev/min}^2, \qquad \phi = \omega_i t + \frac{1}{2}\alpha t^2 = 425\,\text{rev}.$$

8. Wheel A of radius $r_A =$ 10.0 cm is coupled by a chain B to wheel C of radius r_C = 25.0 cm. Wheel A increases its angular speed from rest at a uniform rate of 1.60 rad/s2. Determine the time for wheel C to reach a rotational speed of 100 rev/min.

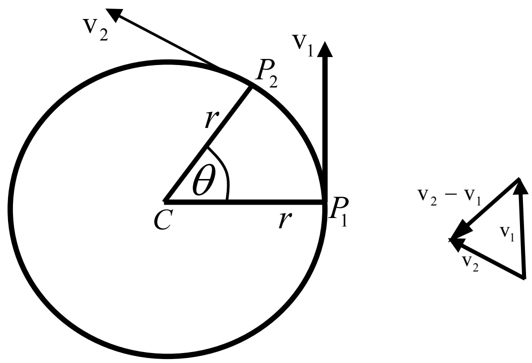


SOLUTION: Obviously every part of the chain moves with the same speed and so $v_A = v_C$. Hence $r_A\omega_A = r_C\omega_C \Rightarrow \omega_A = \frac{r_C\omega_C}{r_A}$. From the definition of acceleration,

$\alpha = \frac{\omega_A - 0}{t}$. From this, $t = \frac{\omega_A}{\alpha} = \frac{r_C\omega_C}{r_A\alpha} = 16.4\ s.$

9. Imagine a disc going around. All particles on the disc will have same 'ω' and 'α' but different 'v' and 'a'. Clearly 'ω' and 'α' are simpler choices !!

10. Now consider a particle going around a circle at constant speed. You might think that constant speed means no acceleration. Bu this is wrong! It is changing its direction and accelerating. This is called "centripetal acceleration", meaning acceleration directed towards the centre of the circle. Look at the figure below:



Note that the distance between points P_1 and P_2 is $\Delta r = \mathrm{v}\Delta t \approx r\theta$. Similarly, $\Delta \mathrm{v} \approx \mathrm{v}\theta \Rightarrow \bar{a} = \dfrac{\Delta \mathrm{v}}{\Delta t} \approx \dfrac{\mathrm{v}\theta}{r\theta/\mathrm{v}} = \dfrac{\mathrm{v}^2}{r}$. More generally, $a = \lim\limits_{\Delta t \to 0} \dfrac{\Delta \mathrm{v}}{\Delta t} = \dfrac{\mathrm{v}^2}{r}$. In vector form, $\vec{\mathrm{a}}_R = -\dfrac{\mathrm{v}^2}{r}\hat{r}$. The negative sign indicates that the acceleration is towards the centre.

12. Vector Cross Products: The vector crossproduct of two vectors is defined as: $\vec{A} \times \vec{B} = AB\sin\theta\,\hat{n}$ where $\hat{n}$ is a unit vector that is perpendicular to both $\vec{A}$ and $\vec{B}$.

Apply this definition to unit vectors in 3-dimensions: $\hat{i} \times \hat{j} = \hat{k}$, $\hat{k} \times \hat{i} = \hat{j}$, $\hat{j} \times \hat{k} = \hat{i}$.

13. Some key properties of the crossproduct:
- $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$
- $\vec{A} \times \vec{A} = 0$
- $\left(\vec{A} + \vec{B}\right) \times \vec{C} = \left(\vec{A} \times \vec{B}\right) + \left(\vec{A} \times \vec{C}\right)$
- $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_y B_z - A_z B_y)\hat{i} + (A_z B_x - A_x B_z)\hat{j} + (A_x B_y - A_y B_x)\hat{k}$

14. The cross product is only definable in 3 dimensions and has no meaning in 2-d. This is unlike the dot product which as a meaning in any number of dimensions.

QUESTIONS AND EXERCISES – 11

Q.1 Give an example of a physical situation where:

(a) $a_T = 0$ but $a_R \neq 0$.

(b) $a_R = 0$ but $a_T \neq 0$.

(c) $a_T \neq 0$ and $a_R \neq 0$.

Q.2 The Moon revolves about the Earth, making a complete revolution in 27.3 days. Assume that the orbit is circular and has a radius of 238,000 miles. What is the magnitude of the acceleration of the Moon towards the Earth?

Q.3 When a car engine is turned off, the angular speed of the flywheel decreases linearly with time from 1200 rev/min to zero.

(a) Find the average angular acceleration.

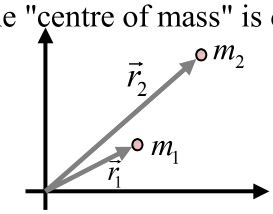
(b) How many revolutions does the flywheel make before topping?

Q.4 Calculate the acceleration of a car that is travelling at a constant speed of 100 km per hour on a straight road. The radius R of the Earth is 6370 km. Why do we not feel this acceleration?

Q.5 A boy ties a stone to a string and whirls it around. Just as the stone reaches the highest point, the string breaks. What will be the values of a_T and a_R at the very moment? What will be the path of the stone before it hits the ground?

Summary of Lecture 12 – PHYSICS OF MANY PARTICLES

1. A body is made of a collection of particles. We would like to think of this body having a "centre". For two masses the "centre of mass" is defined as: $\vec{r}_{cm} \equiv \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$.



In 2 dimensions (i.e. a plane) this is actually two equations:

$x_{cm} = \dfrac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$ and $y_{cm} = \dfrac{m_1 y_1 + m_2 y_2}{m_1 + m_2}$. These give the coordinates of the centre of mass of the two-particle system.

2. Example: one mass is placed at $x = 2cm$ and a second mass, equal to the first, is placed at $x = 6cm$. The cm position lies halfway between the two as you can see from:

$$x_{cm} = \frac{mx_1 + mx_2}{m + m} = \frac{2m + 6m}{2m} = 4cm.$$

Note that there is no physical body that is actually located at $x_{cm} = 4cm$! So the centre of mass can actually be a point where there is no matter. Now suppose that the first mass is three times bigger than the first:

$$x_{cm} = \frac{(3m)x_1 + mx_2}{3m + m} = \frac{2(3m) + 6m}{4m} = 3cm$$

This shows that the cm lies closer to the heavier body. This is always true.

3. For N masses the obvious generalization of the centre of mass position is the following:

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_N\vec{r}_N}{m_1 + m_2 + \cdots + m_N} = \frac{1}{M}\left(\sum_{n=1}^{n=N} m_n \vec{r}_n\right).$$

In words, this says that the following: choose any origin and draw vectors $\vec{r}_1, \vec{r}_2, \cdots \vec{r}_N$ that connect to the masses $m_1, m_2, \cdots m_N$. Heavier masses get more importance in the sum. So suppose that m_2 is much larger than any of the others. If so, $\vec{r}_{cm} \approx \dfrac{m_2\vec{r}_2}{m_2} = \vec{r}_2$. Hence, the cm is very close to the position vector of m_2.

4. For symmetrical objects, it is easy to see where the cm position lies: for a sphere or circle it lies at the centre; for a cylinder it is on the axis halfway between the two faces, etc.

5. Our definition of the cm allows Newton's Second Law to be written for entire collection

of particles:

$$\vec{v}_{cm} = \frac{d\vec{r}_{cm}}{dt} = \frac{1}{M}\left(\sum m_n \vec{v}_n\right)$$

$$\vec{a}_{cm} = \frac{d\vec{v}_{cm}}{dt} = \frac{1}{M}\left(\sum m_n \vec{a}_n\right)$$

$$\therefore \; M\,\vec{a}_{cm} = \sum \vec{F}_n = \sum\left(\vec{F}_{ext} + \vec{F}_{int}\right) \qquad \text{use } \sum \vec{F}_{int} = 0$$

$$\Rightarrow \; \sum \vec{F}_{ext} = M\,\vec{a}_{cm} \text{ (the sum of external forces is what causes acceleration)}$$

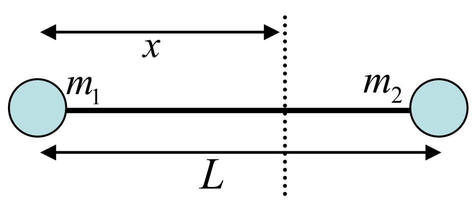
In the above we have used Newton's Third Law as well: $\vec{F}_{12} + \vec{F}_{21} = 0$ etc.

6. Consider rotational motion now for a rigid system of N particles. Rigid means that all particles have a fixed distance from the origin. The kinetic energy is,

$$\begin{aligned} K &= \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + \frac{1}{2}m_3 v_3^2 + \cdots \\ &= \frac{1}{2}m_1 r_1^2\omega^2 + \frac{1}{2}m_2 r_2^2\omega^2 + \frac{1}{2}m_3 r_3^2\omega^2 + \cdots \\ &= \frac{1}{2}\left(\sum m_i r_i^2\right)\omega^2 \end{aligned}$$

Now suppose that we define the "moment of inertia" $I \equiv \sum m_i r_i^2$. Then clearly the kinetic energy is $K = \frac{1}{2} I\omega^2$. How similar this is to $K = \frac{1}{2}Mv^2$!

7. To familiarize ourselves with I, let us consider the following: Two particles m_1 and m_2 are connected by a light rigid rod of length L. Neglecting the mass of the rod, find the rotational inertia I of this system about an axis perpendicular to the rod and at a distance x from m_1.



Answer: $I = m_1x^2 + m_2\left(L - x\right)^2$. Of course, this was quite trivial. Now we can ask a more interesting question: For what x is I the largest? Now, near a maximum, the slope of a function is zero. So calculate $\frac{dI}{dx}$ and then put it equal to zero:

$$\therefore \quad \frac{dI}{dx} = 2m_1x - 2m_2\left(L - x\right) = 0 \quad \Rightarrow \quad x_{\max} = \frac{m_2L}{m_1 + m_2}.$$

8. Although matter is made up of discrete atoms, even if one takes small pieces of any body, there are billions of atoms within it. So it is useful to think of matter as being continuously distributed. Since a sum $\sum$ becomes an integral $\int$, it is obvious that the new definitions of I and $\vec{\mathrm{R}}_{\mathrm{cm}}$ becom:

$$I \equiv \int r^2 dm \quad \text{and} \quad \vec{\mathrm{R}}_{\mathrm{cm}} \equiv \frac{1}{M}\int \vec{r} dm.$$

9. A simple application: suppose there is a hoop with mass distributed uniformly over it. The moment of inertia is: $I = \int r^2 dm = R^2 \int dm = MR^2$.

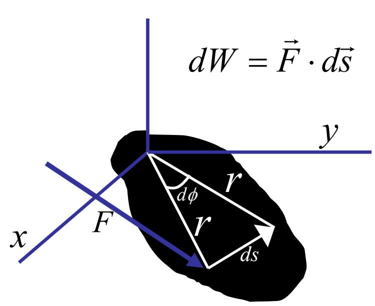
10. A less trivial application: instead of a hoop as above, now consider a solid plate:

$$I = \int r^2 dm \quad (dm = 2\pi r dr \rho_0)$$

$$= \int_0^R 2\pi r^3 dr \rho_0 = \frac{1}{2}(\pi R^2 \rho_0) R^2 = \frac{1}{2} M R^2$$

11. You have seen that it is easier to turn things (e.g. a nut, when changing a car's tyre after a puncture) when the applied force acts at a greater distance. This is because the *torque* τ is greater. We define $\vec{\tau} = \vec{r} \times \vec{F}$ from the magnitude is $\tau = rF\sin\theta$. Here θ is the angle between the radius vector and the force.

12. Remember that when a force $\vec{F}$ acts through a distance $d\vec{r}$ it does an amount of work equal to $\vec{F} \cdot d\vec{r}$. Now let us ask how much work is done when a torque acts through a certain angle as in the diagram below:



The small amount of work done is:

$$dW = \vec{F} \cdot d\vec{s} = F\cos\theta\, ds = \left(F\cos\theta\right)\left(r d\phi\right) = \tau d\phi$$

Add the contributions coming from from all particles,

$$dW_{net} = \left(F_1\cos\theta_1\right) r_1 d\phi + \left(F_2\cos\theta_2\right) r_2 d\phi + \cdots + \left(F_n\cos\theta_n\right) r_n d\phi$$

$$= \left(\tau_1 + \tau_2 + \cdots + \tau_n\right) d\phi$$

$$\therefore dW_{net} = \left(\sum \tau_{\text{ext}}\right) d\varphi = \left(\sum \tau_{\text{ext}}\right) \omega dt \qquad .....(1)$$

Now consider the change in the kinetic energy K,

$$dK = d\left(\frac{1}{2} I\omega^2\right) = I\omega d\omega = (I\alpha)\omega dt \qquad ......(2)$$

By conservation of energy, the change in K must equal the work done, and so:

$$dW_{net} = dK \quad \Rightarrow \quad \sum \tau_{ext} = I\alpha$$

In words this says that the total torque equals the moment of inertia times the angular acceleration. This is just like Newton's second law, but for rotational motion !

13.A comparison between linear and rotational motion quantities and formulae:

LINEAR	ROTATIONAL
x, M	ϕ, I
$\text{v} = \frac{dx}{dt}$	$\omega = \frac{d\phi}{dt}$
$\text{a} = \frac{d\text{v}}{dt}$	$\alpha = \frac{d\omega}{dt}$
$F = M\text{a}$	$\tau = I\alpha$
$K = \frac{1}{2} M\text{v}^2$	$K = \frac{1}{2} I\omega^2$
$W = \int F dx$	$W = \int \tau d\phi$

14. Rotational and translational motion can occur simultaneously. For example a car's wheel rotates and translates. In this case the total kinetic energy is clearly the sum of the energies of the two motions: $K = \frac{1}{2} M\, \text{v}_{\text{cm}}^2 + \frac{1}{2} I_{cm} \omega^2$.

15. It will take a little work to prove the following fact that I simply stated above: for a system of N particles, the total kinetic energy divides up neatly into the kinetic energy of rotation and translation. Start with the expression for kinetic energy and write $\vec{\text{v}}_{\text{cm}} + \vec{\text{v}}_i'$ where $\vec{\text{v}}_i'$ is the velocity of a particle with respect to the cm frame,

$$K = \sum \frac{1}{2} m_i \text{v}_i{}^2 = \sum \frac{1}{2} m_i \vec{\text{v}}_i \cdot \vec{\text{v}}_i$$

$$= \sum \frac{1}{2} m_i \left(\vec{\text{v}}_{\text{cm}} + \vec{\text{v}}_i'\right) \cdot \left(\vec{\text{v}}_{\text{cm}} + \vec{\text{v}}_i'\right)$$

$$= \sum \frac{1}{2} m_i \left(\text{v}_{\text{cm}}^2 + 2\vec{\text{v}}_{\text{cm}} . \vec{\text{v}}_{\text{i}}' + \text{v}_{\text{i}}'^2\right)$$

Now, $\sum m_i \vec{\mathrm{v}}_{\mathrm{cm}} \cdot \vec{\mathrm{v}}'_{\mathrm{i}} = \vec{\mathrm{v}}_{\mathrm{cm}} \cdot \sum m_i \vec{\mathrm{v}}'_i = \vec{\mathrm{v}}_{\mathrm{cm}} \cdot \sum \vec{p}'_i = 0$. Why? because the total momentum is xero in the cm frame! So this brings us to our result that,

$$K = \sum \frac{1}{2} m_i \mathrm{v}_{\mathrm{cm}}^2 + \sum \frac{1}{2} m_i \mathrm{v}_{\mathrm{i}}'^2$$

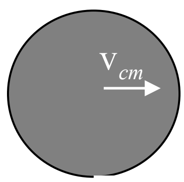
$$= \frac{1}{2} M \mathrm{v}_{\mathrm{cm}}^2 + \sum \frac{1}{2} m_i r_{\mathrm{i}}'^2 \omega^2.$$

QUESTIONS AND EXERCISES – 12

Q.1 A triangle has 3 equal masses placed at each of its vertices. Locate the centre of mass if: a)All 3 angles are equal, b)Two sides are equal and there is one right angle.

Q.2 Suppose that two concentric rings, each with equal mass per unit length are joined so that they rotate together. Find the moment of inertia.

Q.3 A wheel of mass M, radius R, and moment of inertia I is on a surface and moves towards the right as shown.

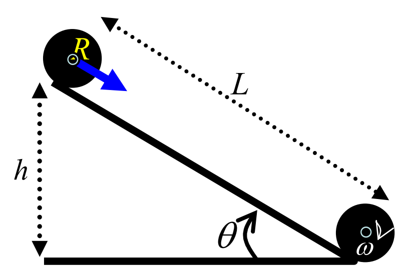


a) What is the kinetic energy if the wheel slips and does not rotate?

b) What is the kinetic energy if it does not slip? Write your answer only in terms of the quantities specified above.

Q.4 Find the moment of inertia of a solid cylinder that rotates about the (long) axis of symmetry.

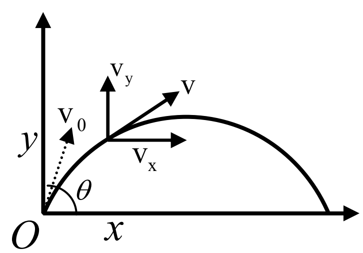
Q.5 A sphere rolls down the inclined plane shown here. The total kinetic energy is $K = \frac{1}{2} M \mathrm{v}_{\mathrm{cm}}^2 + \frac{1}{2} I_{cm} \omega^2$. Use this to show that the speed at which it reaches the lowest point is $\mathrm{v}_{\mathrm{cm}} = \sqrt{\frac{4}{3} gh}$. Assume that there is only rolling and no slipping.



Summary of Lecture 13 – ANGULAR MOMENTUM

1. Recall the definition of angular momentum: $\vec{L} = \vec{r} \times \vec{p}$. The magnitude can be written in several different but equivalent ways,
   (a) $L = rp\sin\theta$
   (b) $L = (r\sin\theta)\,p = r_\perp p$
   (c) $L = r(p\sin\theta) = r\,p_\perp$

2. Let us use this definition to calculate the angular momentum of a projectile thrown from the ground at an angle θ. Obviously, initial angular momentum is zero (why?).



We know what the projectile's coordinates will be at time t after launch,

$$x = (\mathrm{v}_0 \cos\theta)t, \quad y = (\mathrm{v}_0 \sin\theta)t - \frac{1}{2}gt^2$$

as well as the velocity components,

$$\mathrm{v}_x = \mathrm{v}_0 \cos\theta, \ \mathrm{v}_y = \mathrm{v}_0 \sin\theta - gt.$$

Hence, $\vec{L} = \vec{r} \times \vec{p} = \left(x\hat{i} + y\hat{j}\right) \times \left(\mathrm{v}_x\hat{i} + \mathrm{v}_y\hat{j}\right)m = m\left(x\mathrm{v}_x - y\mathrm{v}_y\right)\mathrm{v}_x\hat{k}$

$$= m\left(\frac{1}{2}gt^2\mathrm{v}_0\cos\theta - gt^2\mathrm{v}_0\cos\theta\right)\hat{k} = -\frac{m}{2}gt^2\mathrm{v}_0\cos\theta\,\hat{k}.$$

In the above, $\hat{k} = \hat{i} \times \hat{j}$ is a unit vector perpendicular to the paper. You can see here that the angular momentum increases as t^2.

2. Momentum changes because a force makes it change. What makes angular momentum change? Answer: torque. Here is the definition again: $\vec{\tau} = \vec{r} \times \vec{F}$. Now let us establish a very important relation between torque and rate of change of L.
   Begin:
   $\vec{L} = \vec{r} \times \vec{p}$. At a slightly later time, $\vec{L} + \Delta\vec{L} = (\vec{r} + \Delta\vec{r}) \times (\vec{p} + \Delta\vec{p})$

$$= \vec{r} \times \vec{p} + \vec{r} \times \Delta\vec{p} + \Delta\vec{r} \times \vec{p} + \Delta\vec{r} \times \Delta\vec{p}$$

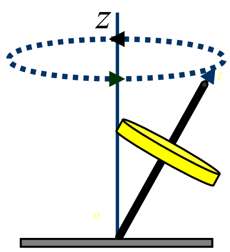
   By subtracting, $\Delta\vec{L} = \vec{r} \times \Delta\vec{p} + \Delta\vec{r} \times \vec{p}\ \frac{\Delta\vec{L}}{\Delta t} = \frac{\vec{r} \times \Delta\vec{p} + \Delta\vec{r} \times \vec{p}}{\Delta t} = \vec{r} \times \frac{\Delta\vec{p}}{\Delta t} + \frac{\Delta\vec{r}}{\Delta t} \times \vec{p}.$

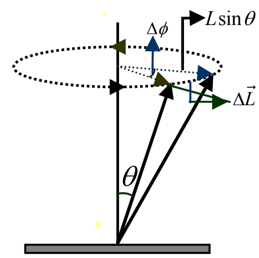
Now divide by the time difference and then take limit as $\Delta t \to 0$:

$$\because \quad \lim_{\Delta t\to 0}\frac{\Delta\vec{L}}{\Delta t}=\frac{d\vec{L}}{dt} \quad \therefore \quad \frac{d\vec{L}}{dt}=\vec{r}\times\frac{d\vec{p}}{dt}+\frac{d\vec{r}}{dt}\times\vec{p}$$

But $\frac{d\vec{r}}{dt}$ is $\vec{\mathrm{v}}$ and $\vec{p}=m\vec{\mathrm{v}}$! Also, $\frac{d\vec{r}}{dt}\times\vec{p}=\vec{\mathrm{v}}\times m\vec{\mathrm{v}}=m\left(\vec{\mathrm{v}}\times\vec{\mathrm{v}}\right)=0$. So we arrive at $\frac{d\vec{L}}{dt}=\vec{r}\times\frac{d\vec{p}}{dt}$. Now use Newton's Law, $\vec{F}=\frac{d\vec{p}}{dt}$. Hence we get the fundamental equation $\frac{d\vec{L}}{dt}=\vec{r}\times\vec{F}$, or $\frac{d\vec{L}}{dt}=\vec{\tau}$. So, just as a particle's momentum changes with time because of a force, a particle's *angular* momentum changes with time because of a torque.

3. As you saw in the lecture, the spinning top is an excelllent application of $\frac{d\vec{L}}{dt}=\vec{\tau}$.





Start from $\vec{\tau}=\vec{r}\times\vec{F}$ where $\vec{F}=m\vec{g}$ $\therefore$ $\tau=Mgr\sin\theta$. But $\vec{\tau}$ is perpendicular to $\vec{L}$ and so it cannot change the magnitude of $\vec{L}$. Only the direction changes. Since $\Delta\vec{L}=\vec{\tau}\ \Delta t$, you can see from the diagram that $\Delta\phi=\frac{\Delta L}{L\sin\theta}=\frac{\tau\ \Delta t}{L\sin\theta}$. So the precession speed ω_P is: $\omega_P=\frac{\Delta\phi}{\Delta t}=\frac{\tau}{L\sin\theta}=\frac{Mgr\sin\theta}{L\sin\theta}=\frac{Mgr}{L}$. As the top slows down due to friction and L decreases, the top precesses faster and faster.

4. Now consider the case of many particles. Choose any origin with particles moving with respect to it. We want to write down the total angular momentum,

$$\vec{L}=\vec{L}_1+\vec{L}_2+\cdots+\vec{L}_N=\sum_{n=1}^{N}\vec{L}_n$$

$$\frac{d\vec{L}}{dt}=\frac{d\vec{L}_1}{dt}+\frac{d\vec{L}_2}{dt}+\cdots+\frac{d\vec{L}_N}{dt}=\sum_{n=1}^{N}\frac{d\vec{L}_n}{dt}$$

Since $\frac{d\vec{L}_n}{dt}=\vec{\tau}_n$, it follows that $\frac{d\vec{L}}{dt}=\sum_{n=1}^{N}\vec{\tau}_n$. Thus the time rate of change of the total angular momentum of a system of particles equals the net torque acting on the system. I showed earlier that internal forces cancel. So also do internal torques, as we shall see.

5. The torque on a system of particles can come both from external and internal forces. For example, there could be charged particles which attract/repel each other while they are all in an external gravitational field. Mathematically, $\sum \vec{\tau} = \sum \vec{\tau}_{\text{int}} + \sum \vec{\tau}_{ext}$. Now, if the forces between two particles not only are equal and opposite but are also directed along the line joining the two particles, then can easily show that the total internal torque, $\sum \vec{\tau}_{\text{int}} = 0$. Take the case of two particles,

$$\sum \vec{\tau}_{\text{int}} = \vec{\tau}_1 + \vec{\tau}_2 = \vec{r}_1 \times \vec{F}_{12} + \vec{r}_2 \times \vec{F}_{21}$$

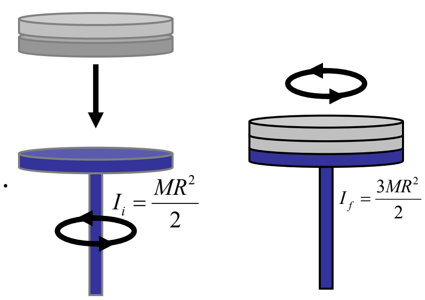
But $\vec{F}_{12} = -\vec{F}_{21} = F\hat{r}_{12}, \quad \therefore \sum \vec{\tau}_{\text{int}} = (\vec{r}_1 - \vec{r}_2) \times \vec{F}_{12} = \vec{r}_{12} \times (F\hat{r}_{12}) = F(\vec{r}_{12} \times \hat{r}_{12}) = 0$

Thus net external torque acting on a system of particles is equal to the time rate of change of the of the total angular momentum of the system.

6. It follows from $\dfrac{d\vec{L}}{dt} = \vec{\tau}$ that if no net external torque acts on the system, then the angular momentum of the system does not change with the time: $\dfrac{d\vec{L}}{dt} = 0 \quad \Rightarrow \quad \vec{L} = \text{a constant}$.

This is simple but extremely important. Let us apply this to the system shown here. Two stationary discs, each with $I = \frac{1}{2}MR^2$, fall on top of a rotating disc. The total angular momentum is unchanged so, $I_i\omega_i = I_f\omega_f \quad \Rightarrow \quad \omega_f = \omega_i\left(\dfrac{I_i}{I_f}\right)$.

Hence, $\omega_f = \omega_i\left(\dfrac{MR^2}{2} \times \dfrac{2}{3MR^2}\right) = \dfrac{1}{3}\omega_i$.



7. You shoud be aware of the similarities and differences between the equations for linear and rotational motion: $\vec{F} = \dfrac{d\vec{p}}{dt} \Leftrightarrow \vec{\tau} = \dfrac{d\vec{L}}{dt}$. One big difference is that for momentum, $\vec{p} = m\vec{\text{v}}$, it does not matter where you pick the origin. But $\vec{L}$ definitely depends on the choice of the origin. So changing $\vec{r}$ to $\vec{c} + \vec{r}$ changes $\vec{L}$ to $\vec{L}'$:

$$\vec{L}' = \vec{r}' \times \vec{p} = (\vec{c} + \vec{r}) \times \vec{p} = \vec{c} \times \vec{p} + \vec{L}.$$

8. Linear and angular acceleration:

$$\vec{\text{a}} = \frac{d\vec{\text{v}}}{dt} = \frac{d}{dt}(\vec{\omega} \times \vec{r}) = \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt}$$
$$= \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{\text{v}}$$

So the acceleration has a tangential and radial part, $\vec{\text{a}} = \vec{\text{a}}_{\text{T}} + \vec{\text{a}}_{\text{R}}$.