6 - isometries and complex numbers

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1 introduction

- isometries preserve magnitude of angles. ones that preserve the sense of the angle are called *direct isometries* and ones that reverse the sense are called *opposite isometries.* direct are a special kind of conformal mapping and opposite are a special kind of anticonformal.
- isometries of a surface form a group $\mathcal{G}(\mathcal{S})$
- direct and opposite compose like + and - under multiplication, e.g. (-) (-) = +. thus, direct isometries are a subgroup of $\mathcal{G}(\mathcal{S})$, denoted $\mathcal{G}_+(\mathcal{S})$, but opposites don't (closure is not respected in the example above)
- let's take 2 opposite isometries. let one be fixed, denote it by ξ, and let another be the general opposite isometry, varying over all possible ones on the surface... call it ζ. the inverse of an opposite transformation is also opposite, so ξ^{-1} is opposite. via the (-)(-) = (+) example, $\xi^{-1} \circ \zeta \in \mathcal{G}_+(\mathcal{S})$. thus $\zeta \in \mathcal{G}_+(\mathcal{S}) \circ \xi$
- its clear that by fixing one opposite, and left multiplying by g+, we get the full group of opposite isometries. now we can construct the full group g by combining the direct isometries with the opposite isometries.
- $\mathcal{G}(\mathcal{S})$ = direct union opposite
  $\mathcal{G}(\mathcal{S}) = [\mathcal{G}_+(\mathcal{S})] \bigcup [\mathcal{G}_+(\mathcal{S}) \circ \xi]$
- not every surface has meaningful isometries
- **three geometries of constant curvature have symmetry groups $\mathcal{G}_+(\mathcal{S})$ that are subgroups of mobius transformations of complex plane, which are $\frac{az+b}{cz+d}$**

2 mobius transformations

- can be decomposed into 4 simple transformations:
  translate, rotate/scale, inversion, another translation. this can make proving and understanding certain properties much more straightforward. transformation is called singular when ad-bc=0
- $z \to \frac{1}{z}$, complex inversion. key to understanding mobius transformations. view in polar coordinates:
  $re^{i\theta} \to (\frac{1}{r})e^{-i\theta}$. invert the radius, negate the angle. can be viewed as two step process: take reciprical lengh, complex conjuate. order actually doesn't matter. inverting the length is called *geometric inversion.* unit circle is very important to the geometric inversion. sends points interior to exterior and vice a versa, leaves circle untouched. denote this transformation $\mathcal{J}_C$. we can generalize the transform to other circles of different center and radius, then denote it $\mathcal{J}_K$, or whatever.
- *inversion is just a rotation of the riemann sphere by π, about the real axis*
  inversion is anticonformal, preserves circles
  if a circle inside C passes through 0, it gets mapped to a line and its interior becomes like a half plane, split by the line. this correspondence between half-planes and interior circles is how half-plane model of hyperbolic geometry is mapped to the beltrami-pointcare disc(nice...).
- we can now see that mobius transformations are conformal and map circles to circles, region to the left of the circle is mapped to the left of the circle after the transform
- we've geometrically made it rigourous that $\frac{1}{\infty} = 0$ and $\frac{1}{0} = \infty$, to make it algebraically clear, let a point of the riemann sphere be defined by 2 complex numbers, [like spinors...cool]. z = one/the other. one point on the riemann sphere has one z in C, but one z in C has infinite pairs on riemann sphere. now infinity just equals [one of the pairs,0]
- mobius transformation can be thought of as a 2x2 matrix, linear transformation with complex constants a,b,c,d. this helps to find compositions of 2 mobius transformations (matrix multiplication), and inverse of a mobius transformation (inverse of matrix). these matrices are not unique, but we can normalize them. then they are unqiue up the the sign....so M will always be the same as -M

3 the main result

- Gauss

4 einstein's spacetime geometry

- Gauss

5 three dimentional hyperbolic geometry

- Gauss