8 - curvature of plane curves

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1 introduction

- two dimentional surface can be deformed in some limited ways while preserving intrisinc geometry, one dimentional surfaces (curves) can be arbitrarily deformed while preserving the intrinsic geometry....so to an ant on a curve, the shape of the curve has no meaning
- extrinsic curvature of 1-d curves will have direct meaning for intrinsic curvature of 2d surfaces
- if we imagine a bead on the 1d curve, and imagine the bead follows phyiscal laws, when a turn happens, the bead feels an acceleration because the curvature implies a change of velocity.

2 the circle of curvature

- newton was the first to examine curvature, and did so when he was 21. first he found a way to determine the 'tangent circle'. at a point to the curve. much like tangent line at a point, but instead it is to describe the curvature at a point. basically take two perpendicular lines near the point, note where they directionally meet, and take the limit as the two points come together...similar to tangent lines. where they meet is the center of the cicle. the radius of the circle is k.
- newton used that to find sigma, the amount the tangent lines diverage

3 newton's curvature formula

- if $y = f(x)$ is the curve, and the x-axis is parallel to the tangent at a point, the curvature is just the second derivative of the function (follows from taylors theorem).
- newton found that if you place the x axis at an arbitary point instead, you just have to patch the formula. looks interesting

4 curvature as rate of turning

- after newton, in 1761, someone came up with a more flexible definition of curvature. rate of turning of the tangent with respect to arc length. ϕ is angle of tangent change, so $\kappa = \frac{d\phi}{ds}$.
- totally local definition, don't have to shuffle around normals
- now look at things from the perspective of 2 unit normals, T and N. picture them wiggling in place, subtracting the difference. these are unit vectors so when they wiggle, they wiggle on the unit circle. i think here you are supposed to view it as just a trig thing, that the wiggle of one has a trig relationship to the wiggle of the other, and we also know the curvature of the circle so bring that in.
- take v to be $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$ and now we can do our calculations with just a parameterized curve instead of an explicity function, which is very useful. e.g. $\tan\phi = \frac{\dot{y}}{\dot{x}}$

5 example: newton's tractrix

- Gauss