

5 - Pseudosphere and the hyperbolic plane

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1 Beltrami's Insight

- two areas of discovery were reaching a head at the same time. hyperbolic geometry by lobachevsky, and gauss' differential geometry. both around 1830.
- 1868, beltrami thought there might be a connection. hyperbolic geometry was 40 years old and obscure. [in some way, this makes sense to me- he was legit just denying one of the euclidean axioms and showing it produced cool stuff].
- beltrami reasoned via the local gauss-bonnet theorem that if he found a surface of constant negative curvature it would obey the laws of hyperbolic geometry.

2 The Tractrix and the pseudosphere

- pseudosphere has this constant negative curvature, but it has a boundary...which is something of a problem because the hyperbolic plane was conceived to be infinite
- *tractrix*, investigated by newton. imagine a table (axes X and Y), with a paper weight on it, and a string of length R attached to it. the other end of the string touches the end of the table. drag the string around the *edge* of the table. curve the weight makes is a tractrix. the weight will hypothetically never reach the 'Y-axis'.
- let σ represent the arc length, so $\sigma = 0$ would be before you start pulling the string at all. [nifty triangle similarity argument between $\Delta \sigma / \Delta X$ and X/R ...second triangle being the big one made by the string and distance from Y axis, first one obviously being infinitesimal]:
$$X = Re^{-\frac{\sigma}{R}}$$
- pseudosphere is the surface of revolution for a tractrix

3 A Conformal Map of the Pseudosphere

- need to create a conformal map from pseudosphere to \mathbb{C} . let x be the angle and σ the arc length of the tractrix. so (x, σ) will be sent to $x + yi$
- a few considerations lock you into the bounded circle approach.
- generators are orthogonal so the mapping must be orthogonal.
- take a circle around the pseudosphere of $\sigma = \text{constant}$. this circle will have a radius X . now take a triangle of two lines $Xdx, d\sigma$, because of the metric, this has to be shrunk to a triangle of lines $dx, d\sigma$.
- the map is $d\hat{s} = \frac{Rds}{y}$, where $ds = \sqrt{dx^2 + dy^2}$

4 the beltrami-poincare half-plane

- Gauss

5 using optics to find geodesics

- Gauss

6 the angle of parallelism

- Gauss

7 the beltrami-poincare disc

- Gauss