4 - Mapping Surfaces: The Metric

June 11, 2022

1 introduction

- Gauss' big insight was to show that the intrinsic geometry of a surface is totally determined by having a rule for infinitesimal distances between two points; the metric. this can thus also determine length's of curves, and geodesics which are paths that minimize the distance
- in this context, map means 'cartographic' map, where 'mapping' will be the more typical mathematical usage.
- stragegy is to make a one to one function between:
  points $\hat{z}$ on $\mathcal{S}$, to points z on $\mathbb{C}$. this will inevitably cause some distortion
- take points $\hat{z}$, $\hat{q}$ on $\mathcal{S}$.
  represent them as z, q on $\mathbb{C}$ with z = $re^{i\theta}$, q = z + δz
  $\delta\hat{s}$ is distance between $\hat{z}$ and $\hat{q}$
  $\delta s = |\delta z|$
- rule giving $|\delta z|$ is the metric. obviously depends on both direction and length
- $d\hat{s} = \Lambda(z, \gamma)ds$
  given a point and a direction, how much do we have to locally expand $\mathbb{C}$ to preserve distances?

2 projective map of the sphere

- imagine the southern hemisphere of the sphere, a bowl shape. the south pole, the bottom of the bowl, rest on the origin of $\mathbb{C}$. the center of the sphere, shoots out light rays. where those light rays hit $\mathbb{C}$ is called the *projective map of the southern hemisphere*
- this map sends circles on $\mathcal{S}$ to ellipses on $\mathbb{C}$. this is true in general for any surface, if the radius of the circle is infinitesimal.
- projective map sends geodesics on $\mathcal{S}$ to lines on $\mathbb{C}$, but it does not preserve angles.
- formula for metric on the sphere, given polor coordiantes on $\mathbb{C}$

3 the metric of a general surface

- different maps have different different metrics even though they describe the same intrinsic geometry. for example, imagine a map of the sphere where longitude and latitude (θ, ϕ), we map to the carteasan coordiantes (θ, ϕ). the metric for this map will be $d\hat{s}^2 = R^2[\sin^2(\phi)d\theta^2 + d\phi^2]$, which is different from the projective map metric for a sphere.
- take a general surface and draw 2 families of curves, that both vary smoothly, and so any point on the surface can be uniquely represented by a point on each of the curves (let's call them U curves and V curves). let a point on the surface be labeled as $u + iv$ and a point can be labeled as $\hat{z} = U + iV$
- imagine a small movement away from z on the map. $\delta z = \delta u + i\delta v$
  but this is on the map, how do we project it back to the actual surface?
- by virtue of differentiability, we can say that some small movement on the v curve will produce some small movement on the surface. put another way, $\frac{\partial\hat{s_1}}{\partial u} \equiv A$, $\frac{\partial\hat{s_2}}{\partial v} \equiv B$
- we can see that A and B are the local scale factors that have to be applied to the map for distances to be preserved. A can be viewed as inversly proportional to the crowding of the u-curves. the greater the crowding, the greater result $\delta\hat{s_1}$ will have on u
- ω is angle between u-curves and v-curves, which depends on position

- general metric for the surface:
  $d\hat{s}^2 = A^2 du^2 + B^2 dv^2 + 2F du\ dv$
- however, once the u-curves are chosen, it is always possible to select an orthoganal set of v-curves, which anniliates the last term.
- in general it is impossible to cover all of the surface with a single u-v set of curves - the curves will inevitably intersect on any closed surface

4 the metric curvature formula

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5 conformal maps

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6 some visual complex analysis

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7 the conformal stereographic map of the sphere

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8 stereographic formulas

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9 stereographic preservation of circles

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