5 - Pseudosphere and the hyperbolic plane

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1 Belatrami's Insight

- two areas of discovery were reaching a head at the same time. hyperbolic geometry by lobachevsky, and gauss' differential geometry. both around 1830.
- 1868, belrimi thought there might be a connection. hyperbolic geometry was 40 years old and obscure. [in some way, this makes sense to me- he was legit just denying one of the euclidean axioms and showing it produced cool stuff].
- beltrimi reasoned via the local gauss-bonnet theorom that if he found a surface of constant negative curvature it would obey the laws of hyperbolic geometry.

2 The Tractrix and the pseudosphere

- psuedosphere has this constant negative curvature, but it has a boundary...which is something of a problem because the hyperbolic plane was conceived to be infinite
- *tratrix*, investigated by newton. imagine a table (axes X and Y), with a paper weight on it, and a string of length R attatched to it. the other end of the string touches the end of the table. drag the string around the *edge* of the table. curve the weight makes is a tractrix. the weight will hypothetically never reach the 'Y-axis'.
- let σ represent the arc length, so $\sigma = 0$ would be before you start pulling the string at all. [nifty triangle similarty argument between delta sigma/-delta X and X/R...second triangle being the big one made by the string and distance from Y axis, first one obviously being infinitesimal]:
  $X = Re^{-\frac{\sigma}{R}}$
- pseudosphere is the surface of revolution for a tratrix

3 A Conformal Map of the Pseudosphere

- need to create a conformal map from pseudosphere to $\mathbb{C}$. let x be the angle and sigma the arc length of the tractrix. so (x, σ) will be sent to $x + yi$
- a few considerations lock you into the bounded circle approach.
- generators are orthogonal so the mapping must be orthogonal.
- take a circle around the pseduosphere of sigma = constant. this circle will have a radius X. now take a triangle of two lines Xdx, $d\sigma$, because of the metric, this has to be shrunk to a trianle of lines dx dsigma.
- the map is $d\hat{s} = \frac{Rds}{y}$, where $ds = \sqrt{dx^2 + dy^2}$

4 the beltrami-poincare half-plane

- it is typical to set the radius of the pseudosphere to 1 so the curvature is -1.
- it was important to beltrami that the pseduosphere be generalized to be infinite, like real hyperbolic geometry. imagine a paint roller of radius 1. after one revolution on a wall you have painted a strip 2π thick. now we use our pseudosphere as a roller on the plane. horizontal strips represent lines of constantσ, going around the one circular strip of the pseduosphere.
- that removes the lack of inifinitude, but it still has a rim that must be removed. one can easily use the conformal mapping and the known metric to extend the rim further - you know exactly the scale factor thanks to the metric so you just keep going.
- standardized hyperbolic metric: $d\hat{s} = \frac{ds}{y}$

- this really makes for a half-plane. the tractrix's 'end' at y = 0. if you imagine a particle traveling in this space, it will slow down more and more the closer you get to y = 0.
- thus we now have *The Hyperbolic Plane* $\mathbb{H}^2$, defined on y ¿ 0, with the metric above.
- to imagine how the metric affects things like arc length, we can imagine painting the half plane with circles of radius ϵ, and letting them scale. the length of the path is how many balls it goes through - way more balls towards the origin
- this all makes sense - but how can we explain how geodesics are semicircles that hit the original at a right angle?

5 using optics to find geodesics

- light takes the path that minimizes time. imagine points a and b. light starts at a and ends at b. a is above water, b is below. now imagine the point where the line will hit the water. changing the position will change the time spent on the path. because of fermat's principle, the light will in general change angles when it his the water...aka snells law.
- if you nudge the (stationary) point where light hits by ϵ, you increase the 'time in flight' by $\frac{\epsilon \sin\theta_1}{v_1}$ (extra time spent in air), but decrease it by $\frac{\epsilon \sin\theta_2}{v_2}$. noting that at a stationary point these two quantities must be equal, and cancelling the ϵ, we have :
  $\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$, a rearranged snells law.
- take $\hat{a}$ and $\hat{b}$ on pseudosphere, corresponding to a and b on $\mathbb{H}^2$. imagine a particle taking different paths between the points but always at unit speed. on hyperbolic plane, because of the metric, the particle will not travel at unit speed. bc of conformality, this change in speed doesn't depend on direction
- on pseudosphere, geodesic will be the shortest path, and given the constant velocity...also shortest time, thus obeying fermat's principle
- thus geodesics in the half plane follow snells law, which means they follow semicircle paths.

6 the angle of parallelism

- now it's obvious, visually, that there are infinitly many parallel lines given a point p and a line L. they are called *ultra-parallel*
- looking at same diagram, [5.9], for any point p and line L, there are also 2 lines that only hit L at the horizon, called 'asymytotic'
- given this point p and a line L, there is still only one point (q) that, when connected by a geodesic, will hit L at a right angle, we will call this line M. M lets us define a distance from a point to a line.
- take this setup as a whole - for any point p and line L, we have two asmptotic lines, and a third line M, hitting L at a right angle. M actually bisects the angle contained by the two asymtotics. angle between M and an asmyptotic is called the angle of parallelism, usually called Π.
- distance between p and L is denoted D. there is a relationship between D and Π, $\tan(\frac{\Pi}{2}) = e^{-D}$, Bolyai-Lobachevski Formula.
- as p approaches L, Π approaches $\frac{\pi}{2}$, looks euclidean. as p gets farther away from L, the proportion of rays eminating from p that hit L become smaller and smaller.
- to a tiny observer, no hyperbolic line is different from any other. to a global observer, it seems like some lines only have one endpoint and then go off to infinity. in actuality, those lines all meet at infinity. shows how important it is to switch between the different perspectives
- meaningful application of this: rotating something about a point. distance preserving transformations are called isometries. thinking about a transformation of an arc looking hyperbolic line, we can rotate it until it becomes a verticle line.

7 the beltrami-poincare disc

- half plane is one way of representing $\mathbb{H}^2$, but there are other (mathematically but not psychologically) equivelent representations. one is the beltrami-poincare disc.
- infinite distant horizon is a boundary disk, metric is:
  $d\hat{s} = \frac{2}{1-r^2} ds$, r is distance from center of disc