4 - Mapping Surfaces: The Metric

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1 introduction

- Gauss' big insight was to show that the intrinsic geometry of a surface is totally determined by having a rule for infinitesimal distances between two points; the metric. this can thus also determine length's of curves, and geodesics which are paths that minimize the distance
- in this context, map means 'cartographic' map, where 'mapping' will be the more typical mathematical usage.
- stragegy is to make a one to one function between:
  points $\hat{z}$ on $\mathcal{S}$, to points z on $\mathbb{C}$. this will inevitably cause some distortion
- take points $\hat{z}$, $\hat{q}$ on $\mathcal{S}$.
  represent them as z, q on $\mathbb{C}$ with z = $re^{i\theta}$, q = z + δz
  $\delta\hat{s}$ is distance between $\hat{z}$ and $\hat{q}$
  $\delta s = |\delta z|$
- rule giving $|\delta z|$ is the metric. obviously depends on both direction and length
- $d\hat{s} = \Lambda(z, \gamma)ds$
  given a point and a direction, how much do we have to locally expand $\mathbb{C}$ to preserve distances?

2 projective map of the sphere

- imagine the southern hemisphere of the sphere, a bowl shape. the south pole, the bottom of the bowl, rest on the origin of $\mathbb{C}$. the center of the sphere, shoots out light rays. where those light rays hit $\mathbb{C}$ is called the *projective map of the southern hemisphere*
- this map sends circles on $\mathcal{S}$ to ellipses on $\mathbb{C}$. this is true in general for any surface, if the radius of the circle is infinitesimal.
- projective map sends geodesics on $\mathcal{S}$ to lines on $\mathbb{C}$, but it does not preserve angles.
- formula for metric on the sphere, given polor coordiantes on $\mathbb{C}$

3 the metric of a general surface

- different maps have different different metrics even though they describe the same intrinsic geometry. for example, imagine a map of the sphere where longitude and latitude (θ, ϕ), we map to the carteasan coordiantes (θ, ϕ). the metric for this map will be $d\hat{s}^2 = R^2[\sin^2(\phi)d\theta^2 + d\phi^2]$, which is different from the projective map metric for a sphere.
- take a general surface and draw 2 families of curves, that both vary smoothly, and so any point on the surface can be uniquely represented by a point on each of the curves (let's call them U curves and V curves). let a point on the surface be labeled as $u + iv$ and a point can be labeled as $\hat{z} = U + iV$
- imagine a small movement away from z on the map. $\delta z = \delta u + i\delta v$
  but this is on the map, how do we project it back to the actual surface?
- by virtue of differentiability, we can say that some small movement on the v curve will produce some small movement on the surface. put another way, $\frac{\partial \hat{s_1}}{\partial u} \equiv A$, $\frac{\partial \hat{s_2}}{\partial v} \equiv B$
- we can see that A and B are the local scale factors that have to be applied to the map for distances to be preserved. A can be viewed as inversly proportional to the crowding of the u-curves. the greater the crowding, the greater result $\delta\hat{s_1}$ will have on u
- ω is angle between u-curves and v-curves, which depends on position

- general metric for the surface:
$d\hat{s}^2 = A^2 du^2 + B^2 dv^2 + 2F du\ dv$
- however, once the u-curves are chosen, it is always possible to select an orthoganal set of v-curves, which anniliates the last term.
- in general it is impossible to cover all of the surface with a single u-v set of curves - the curves will inevitably intersect on any closed surface

4 the metric curvature formula

- once handed a metric, you should in principle be able to derive the curvature at any point. what is not a given is that the formula is beautiful and simple. it will take most of the book to derive for real.
- *** $\mathcal{K} = -\frac{1}{AB}(\partial_v[\frac{\partial_v A}{B}] + \partial_u[\frac{\partial_u B}{A}])$
- can also be used to calculate areas. $dA = AB du\ dv$

5 conformal maps

- projective map preserves straight lines but usually it's better to preserve angles.
- map that preserves angles and sense is *conformal*, map that preserves angles and inverts sense ins *anti-conformal*
- angle between curves means the angle between their tangents
- with respect to metric formula, a map is conformal if the scale factor, Λ, depends on position but not direction. this way, *infinitesimal shapes on the map are the same shape, just a different size!*
- gauss proved that given any surface, you can find an orthogonal map that is also conformal, in the sense that A = B. since A is the same as B and the coordinates are orthogonal, we can express the general metric of the surface as $d\hat{s}^2 = \Lambda^2[du^2 + dv^2]$.
- this also beatifully simplifies the curvature formaula to:
$\mathcal{K} = -\frac{\nabla^2 \ln \Lambda}{\Lambda^2}$

6 some visual complex analysis

- every surface contains infinite variety of u-v curves and conformal maps
- let a conformal mapping be $F = \mathbb{C} \rightarrow \mathcal{S}$
- conformal $(\tilde{u}, \tilde{v})$-coordinates can be created by rotating/expanding/translating any given u-v curves.
- let $z = u + iv$ be in $\mathbb{C}$ and $\tilde{z} = \tilde{u} + i\tilde{v}$ be a seperate copy of $\mathbb{C}$ but under some function f.
- big idea is that before we had f going from S to C, now we also have a function F going from C to S. so all together we have a copy of C mapping to S, which then maps onto another C. C -¿ S -¿ C. both are conformal. we now have a freedom of rotating/expanding/translating the u-v lines in the first copy of C, which will give us new u-v coordinates in S, thus leading to an infinite variety of them.
- $f(z) = ae^{i\tau}z + w$. scale by a, twist by tau, shift by a complex contstant w.
- remember we have F as the map and now f as a function on the first copy of C. so our new u-v coordinates on S are given by $\tilde{F} = F \circ f$
- useful to think of the derivative which blows away the constant w, and is just the amplification + twist * (tiny move in z). "amplitwist"
- from some basic reasoning (?) we can see that differentiable complex mappings are all conformal.
- this procedure allows one to take any conformal mapping from C to itself and "pass it through" S.

7 the conformal stereographic map of the sphere

- f

8 stereographic formulas

- f

9 stereographic preservation of circles

- f