5 - Pseudosphere and the hyperbolic plane

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1 Belatrami's Insight

- two areas of discovery were reaching a head at the same time. hyperbolic geometry by lobachevsky, and gauss' differential geometry. both around 1830.
- 1868, belrimi thought there might be a connection. hyperbolic geometry was 40 years old and obscure. [in some way, this makes sense to me- he was legit just denying one of the euclidean axioms and showing it produced cool stuff].
- beltrimi reasoned via the local gauss-bonnet theorom that if he found a surface of constant negative curvature it would obey the laws of hyperbolic geometry.

2 The Tractrix and the pseudosphere

- psuedosphere has this constant negative curvature, but it has a boundary...which is something of a problem because the hyperbolic plane was conceived to be infinite
- *tratrix*, investigated by newton. imagine a table (axes X and Y), with a paper weight on it, and a string of length R attatched to it. the other end of the string touches the end of the table. drag the string around the *edge* of the table. curve the weight makes is a tractrix. the weight will hypothetically never reach the 'Y-axis'.
- let σ represent the arc length, so $\sigma = 0$ would be before you start pulling the string at all. [nifty triangle similarty argument between delta sigma/-delta X and X/R...second triangle being the big one made by the string and distance from Y axis, first one obviously being infinitesimal]:
  $X = Re^{-\frac{\sigma}{R}}$
- pseudosphere is the surface of revolution for a tratrix

3 A Conformal Map of the Pseudosphere

- need to create a conformal map from pseudosphere to $\mathbb{C}$. let x be the angle and sigma the arc length of the tractrix. so (x, σ) will be sent to $x + yi$
- a few considerations lock you into the bounded circle approach.
- generators are orthogonal so the mapping must be orthogonal.
- take a circle around the pseduosphere of sigma = constant. this circle will have a radius X. now take a triangle of two lines Xdx, $d\sigma$, because of the metric, this has to be shrunk to a trianle of lines dx dsigma.
- the map is $d\hat{s} = \frac{Rds}{y}$, where $ds = \sqrt{dx^2 + dy^2}$

4 the beltrami-poincare half-plane

- it is typical to set the radius of the pseudosphere to 1 so the curvature is -1.
- it was important to beltrami that the pseduosphere be generalized to be infinite, like real hyperbolic geometry. imagine a paint roller of radius 1. after one revolution on a wall you have painted a strip 2π thick. now we use our pseudosphere as a roller on the plane. horizontal strips represent lines of constantσ, going around the one circular strip of the pseduosphere.
- that removes the lack of inifinitude, but it still has a rim that must be removed. one can easily use the conformal mapping and the known metric to extend the rim further - you know exactly the scale factor thanks to the metric so you just keep going.
- standardized hyperbolic metric: $d\hat{s} = \frac{ds}{y}$

- this really makes for a half-plane. the tractrix's 'end' at y = 0. if you imagine a particle traveling in this space, it will slow down more and more the closer you get to y = 0.
- thus we now have *The Hyperbolic Plane* $\mathbb{H}^2$

5 using optics to find geodesics

- Gauss

6 the angle of parallelism

- Gauss

7 the beltrami-poincare disc

- Gauss