Quantum Leader Election

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Abstract

A group of n individuals $A_1, \ldots A_n$ who do not trust each other and are located far away from each other, want to select a leader. This is the leader election problem, a natural extension of the coin flipping problem to n players. We want a protocol which will guarantee that an honest player will have at least $\frac{1}{n} - \epsilon$ chance of winning $(\forall \epsilon > 0)$, regardless of what the other players do (weather they are honest, cheating alone or in groups). It is known to be impossible classically. This work gives a simple algorithm that does it, based on the weak coin flipping protocol with arbitrarily small bias derived by Mochon [Moc00] in 2007, and recently published and simplified in [ACG⁺16]. A protocol with linear number of coin flipping rounds is quite simple to achieve; We further provide an improvement to logarithmic number of coin flipping rounds. This is a much improved journal version of a preprint posted in 2009; The first protocol with linear number of rounds, was achieved independently also by [NJ10] around the same time.

1 Introduction

In this paper we present a quantum protocol for the Leader election problem - a natural extension of coin flipping; in fact, our protocol uses as a black box a quantum solution to the coin flipping protocol ([Moc00, ACG+16]).

Thus, let us first review the coin flipping problem and what is known about it.

1.1 The coin flipping problem

A standard coin flipping is a game in which two parties, Alice and Bob, wish to flip a coin from a distance. The two parties do not trust each other, and would each like to win with probability of at least 0.5. A natural problem is to find good protocols - a protocol in which a player could not cheat and force the outcome of the game to his benefit.

There are two types of coin flipping - strong and weak. In strong coin flipping, each party might want to bias the outcome to any result, and the protocol has to protect against any such cheating. In weak coin flipping each party has a favorite outcome, and so the protocol has to protect only against cheating in that direction.

We denote the winning probability of Alice to win a weak coin flipping when both players are honest as P_A , and similarly P_B for Bob. The maximum winning probability of a cheating Alice (i.e. when she acts according to her optimal strategy, while Bob is honest) is denoted by P_A^* , and similarly P_B^* for a cheating Bob.

Let $\epsilon = max(P_A^*, P_B^*) - \frac{1}{2}$ be the *bias* of the protocol. The bias actually tells us how good the protocol is. The smaller the bias is, the better the protocol is.

It is well known that without computational assumptions, even weak coin flipping is impossible to achieve in the classical world (see [Cle87]. Note that impossibility of weak coin flipping implies impossibility of strong coin flipping). That is, one of the players can always win with probability 1. In the quantum setting, the problem is far more interesting.

1.2 Quantum coin flipping

Quantum strong coin flipping protocols with large but still non-trivial biases were first discovered by [ATSVY00] (with bias $\epsilon < 0.4143$). Kitaev then proved (see for example in [ABDR03]) that in strong coin flipping, every protocol must satisfy $P_0^* \cdot P_1^* \geq \frac{1}{2}$, hence $\epsilon \geq \frac{\sqrt{2}-1}{2}$. This result raised the question of whether weak coin flipping with arbitrarily small bias is possible. Protocols were found with smaller and smaller biases ([ABDR03] showed strong CF with bias $\frac{1}{4}$, [Moc04] showed weak CF with bias 0.192), until Mochon showed in his unpublished breakthrough paper [Moc00] that there are families of weak coin flipping protocols whose bias converges to zero. This result was simplified in [ACG⁺16]. It is also known that even in the quantum world, a perfect protocol (i.e. $\epsilon = 0$) is not possible ([MSCK99]).

1.3 The leader election problem

The leader election problem is the natural generalization of the weak coin flipping, to n players.

The bias of the problem, is defined to be the minimal ϵ such that, every honest player has a winning probability of at least $\frac{1}{n} - \epsilon$ (we do not have any limitation on the number of cheating players). We will denote leader election with bias ϵ by LE_{ϵ} .

As mentioned before, it is classically impossible to do a weak coin flipping with a bias < 0.5 without assumptions about the computation power ([Cle87]). The same argument will show that it is also impossible to solve the leader election problem in the classical setting, in the sense that there will always be a player who can guarantee getting elected.

Since leader election is not possible in the general sense, people tried to use

assumptions and conditions in order to make it possible. For example, [Fei00] presents a classical leader election protocol (given that a player can flip a coin by herself) such that given $\frac{(1+\delta)n}{2}$ honest players, the player whom the protocol chooses is one of the honest players, with probability $\Omega(\delta^{1.65})$. In addition there is a proof that every classical protocol has a success (electing an honest leader) probability of $O(\delta^{1-\epsilon})$, for every $\epsilon > 0$. Note that there are limitation on the number of cheaters.

Another variant is a protocol that chooses a processor randomly among n possibilities (In this case, there are no cheaters, the players are anonymous and run the same protocol, and we want a protocol with minimal running time / communication complexity, or one that works without knowing the number of processors. See [TKM07] for a quantum exact algorithm that solves this, but requires knowing a bound on n). This type is sometimes called fair leader election problem (because there are no cheaters).

In this paper we investigate the version of Leader election with cheaters, defined in the beginning of this subsection. We will refer to this type simply as the *leader election problem*.

Until a short time before the first version of this result was posted, there was no quantum result regarding this Leader election problem as it was defined here. We note however that there were some results on other types, such as [ABDR03] which considered our leader election problem, but allow penalty for cheaters that got caught (which is obviously a weaker version of the problem), and also the mentioned [TKM07] for the fair leader election problem.

1.4 Our result

If n is a power of 2, then a trivial solution exists, given a good weak coin flip protocol. We can do a knock-out tournament (the loser quits) of weak coin flipping, with log(n) rounds. In each round, all eligible players divide into pairs, and play weak coin flip, where the loser gets eliminated from the

tournament. The winner of the tournament will be elected as the leader.

Theorem 1. If $n = 2^k$ for some $k \in \mathbb{N}$, then for every $\epsilon > 0$ there exists a leader election protocol LE_{ϵ} , in which if all players are honest, then each has a winning probability of $\frac{1}{n}$, otherwise any honest player has a winning probability $> \frac{1}{n} - \epsilon$. With running time of $O(N_{\epsilon} \log n)$, and $O(\log n)$ rounds of balanced weak coin flipping, where N_{ϵ} is the number of rounds in a weak balanced coin flipping protocol P_{ϵ} of bias ϵ .

Proof. We will have full knock-out tournament of balanced weak coin flipping P_{ϵ} between the players, and the winner of the tournament will be declared as the leader.

It is obvious that if all players are honest, then each player has a winning probability of $\frac{1}{n} = \left(\frac{1}{2}\right)^k$.

The only thing left to show is that each honest player has a winning probability of at least $\frac{1}{n} - \epsilon$.

This is true because we have $\log n$ rounds, and in each round an honest player has a winning probability of at least $\frac{1}{2} - \epsilon$, hence in total the winning probability is at least

$$\left(\frac{1}{2} - \epsilon\right)^{\log n} = \left(\frac{1 - 2\epsilon}{2}\right)^k \ge \frac{1 - 2\epsilon k}{2^k} = \frac{1}{n} - \epsilon \frac{2\log n}{n} \ge \frac{1}{n} - \epsilon$$

where we used the Bernoulli inequality $(1+x)^n \ge 1 + nx$ which is true for any

 $x \ge -1$, $n \in \mathbb{N}$, and in our case, $\forall \epsilon \le \frac{1}{2}$ (which is obviously satisfied). We prove a generalization for that inequality in lemma 7.

A problem arises when n is not a power of 2, then this is not possible, and putting in a dummy player involves some difficulties. If the cheaters could control the dummy player, they would increase their winning probability. This paper addresses this question.

Our first solution (which was also discovered in [NJ10], independently) is to let A_1 play against A_2 and then the winner of that will play A_3 and so on, as in a tournament, except we use unbalanced weak coin flips. These are known to be possible using the balanced weak coin flipping protocol (see in [CK09]) but are more expensive in terms of time.

As mentioned, in [Moc00] Mochon showed the existence of a weak coin flipping protocol with an arbitrarily small bias of at most ϵ . Let us denote this protocol by P_{ϵ} throughout this paper. This protocol assumes that if both players are honest, then each player has $\frac{1}{2}$ chance of winning (this is called a *balanced* coin flipping protocol). We denote by N_{ϵ} the running time of a balanced coin flipping protocol P_{ϵ} with bias ϵ , which is the same as the number of rounds in that protocol.

It is also possible to build an unbalanced weak coin flipping with an arbitrarily small bias ϵ , in which if both players are honest, then one honest player will have q winning probability, and the other player will have 1-q winning probability. If only the first player is honest, his winning probability is at least $q - \epsilon$ (similarly $1 - q - \epsilon$ for the second player, in case he is honest). We will denote this protocol as $P_{q,\epsilon}$. In [CK09] it was shown that such protocols can be approximated using repetition of P_{ϵ} , with a total of $O\left(N_{\epsilon} \cdot \log \frac{1}{\epsilon}\right)$ rounds (See corollary 5 for more details).

The leader will be the winner of the final $(n-1)^{th}$ step. We arrive at the following theorem:

Theorem 2. For every $\epsilon > 0$, there exists a quantum leader election protocol LE_{ϵ} , in which any honest player has a winning probability $\geq \frac{1}{n} - \epsilon$, with running time of $O\left(n \cdot \log(\frac{n}{\epsilon}) \cdot N_{\frac{\epsilon}{4n}}\right)$, and $O\left(n\right)$ rounds of coin flipping.

This theorem will be proven is section 2, but as mentioned before, this simple solution is inefficient. It uses a linear number of coin flipping rounds: n-1 of them.

Therefore we searched and found a better solution, that reduces the number of coin flipping rounds, and yields the following theorem: **Theorem 3.** For every $\epsilon > 0$, there exists a leader election protocol LE_{ϵ} , in which any honest player has a winning probability $\geq \frac{1}{n} - \epsilon$, with running time of $O\left(N_{\frac{\epsilon}{4}} \log n \log \frac{1}{\epsilon}\right)$, and $O\left(\log n\right)$ rounds of coin flipping.

This theorem provides improvement of parameters with respect to theorem 2. The number of coin flipping rounds improves from O(n) to $O(\log n)$, and the number of total unbalanced coin flipping from O(n) to $O(\log n)$ as well. So the running time complexity (players can play in parallel which does not increase the time complexity - we only count the longest coin flipping in each round. See 2.1 for exact definition) reduces from $O(n \cdot \log(\frac{n}{\epsilon}) \cdot N_{\frac{\epsilon}{4n}})$ to $O(N_{\frac{\epsilon}{4}} \cdot \log n \cdot \log \frac{1}{\epsilon})$ in the worst case, and in the best case, where $n = 2^k$, our protocol runs at $O(\log n \cdot N_{\epsilon})$, achieving the optimal complexity as that of Theorem 1.

The actual complexity improvement depends on N_{ϵ} . As of today, we only know a lower bound of $O\left(\log\log\frac{1}{\epsilon}\right)$ due to [ABDR03]. In this scenario, assuming $\epsilon = \Theta\left(\frac{1}{n}\right)$ we get time complexity of $O\left(\log^2 n \cdot \log\log n\right)$ instead of $O\left(n \cdot \log^2 n\right)$, which is exponentially better.

The only upper bound known is $\frac{1}{\epsilon}^{O\left(\frac{1}{\epsilon}\right)}$ due to [ACG⁺16]. Even if we only use $N_{\epsilon} = O\left(2^{\frac{1}{\epsilon}}\right)$, then assuming $\epsilon = \Theta\left(\frac{1}{n}\right)$ we get time complexity of $O\left(2^n \cdot \log n \cdot \log \log n\right)$ instead of $O\left(2^{n^2} \cdot n \log n\right)$, which is again exponentially better.

However if N_{ϵ} is linear in $\frac{1}{\epsilon}$, then assuming $\epsilon = \Theta\left(\frac{1}{n}\right)$ we get time complexity $O\left(n\log^2 n\right)$ instead of $O\left(n^3\log n\right)$.

Still in all cases theorem 3 provides a significant improvement in parameters.

To prove theorem 3, we use a knock-out tournament of weak coin flipping, in which the loser quits and the winner continues to the next round. Since n (the number of players) is not necessarily a power of 2, then one must adjust it.

Our protocol is fairly simple and uses $\log n$ rounds of unbalanced weak protocols $P_{q'_i,\frac{\epsilon}{2}}$ as will be defined later, in section 3, where we will prove Theorem 3 (at most one at each round). This limitation is important, be-

cause at the moment we only know how to implement an unbalanced flip using a repetition of balanced coin flip, which influences the total message complexity.

If it were possible to improve the complexity of unbalanced coin flip, to that of a balanced coin flip (this is an open problem), then one can improve the complexity of the suggested leader election protocol to $O(N_{\epsilon} \log n)$. It is possible that this can also be achieved by finding an appropriate families of time independent point games (see [Moc00, ACG⁺16]) to derive more efficient protocols, but this has never been done before and will not be done in this paper.

1.5 Related work

This work was first posted in 2009. This paper is the journal version of that preprint, which is much improved.

A related work was published [NJ10] at the same time as the preprint. They refer to the leader election problem as weak dice rolling, and they use the same protocol as we did (independently) in 2.4, proving theorem 2. (However, they assume there exists $P_{q,\epsilon}$ for every $q \in [0,1]$, which is not known to be true, but only an approximation to such). As mentioned before, our work improves that result significantly.

[NJ10] also study the leader election problem in the strong scenario, under the name of strong dice rolling. Namely they consider the problem of $n \geq 2$ remote parties, having to decide on a number between 1 and $N \geq 3$, in which the parties want to avoid bias in any direction. They generalize Kitaev's bound (see [ABDR03]) to apply to n parties N sided strong dice-rolling. This was done by noting that strong dice rolling can always be used to implement strong imbalanced coin flipping. Note that this rules out Leader election in the strong version for n = N.

[NJ10] also extend the strong optimal coin flipping protocol in [CK09] and provide a family of strong dice rolling protocols which matches this bound,

for the case of the number of parties being $n=2\cdot M$ and the number of outcomes to be $N=T^M$ for any $M,T\in\mathbb{N}$.

1.6 Organization of the paper:

Section 2 gives formal definitions and proves Theorem 2.

Section 3 proves Theorem 3.

Section 4 is open questions.

In the Appendix, we gives formal definition of weak coin flipping.

2 Leader election protocol

We start by some standard definitions.

2.1 Definitions and requirements

By a weak coin flipping protocol we mean that Alice wins if the outcome is 0, and Bob wins if it is 1.

A weak coin flipping protocol with $P_A = q$, $P_B = 1 - q$, bias ϵ will be denoted by $P_{q,\epsilon}$.

A leader election protocol with n parties A_1, \ldots, A_n has an outcome $t \in \{1, \ldots, n\}$. We will denote by P_i the probability that the outcome is t = i.

We assume that each player has its own private space, untouchable by other players, a message space \mathcal{M} which is common to all (\mathcal{M} can include a space for the identification of the sender and receiver).

We use the existence of a weak coin flipping protocol with bias at most ϵ for every $\epsilon > 0$. This fact was proved in [Moc00] for $P_A = P_B = \frac{1}{2}$ and we will denote it as P_{ϵ} and by N_{ϵ} the number of its rounds (we define a round shortly). There is a proof that there is an approximation to such a protocol for every P_A , P_B (s.t. $P_A + P_B = 1$) in [CK09] by repetitions of P_{ϵ} , with

 $O\left(\log \frac{1}{\epsilon} \cdot N_{\epsilon}\right)$ rounds. (It seems possible to generalize the weak coin flipping protocol [9,2] directly to unbalanced coin flipping with any $P_A + P_B = 1$, without this $\log n$ increase in complexity, but this wasn't done yet.) Recall that we denoted such unbalanced protocol with $P_A = q$ and bias ϵ by $P_{q,\epsilon}$. We will denote the approximate protocol by $P_{q',\frac{\epsilon}{2}}$ See corollary 5 for details.

Remark. A round in a coin flipping protocol, consists of two steps: First where Alice does something on her space $\mathcal{A} \otimes \mathcal{M}$ and sends Bob the message space. Second when Bob receives the message, does something on his space $\mathcal{M} \otimes \mathcal{B}$, and sends Alice a reply. Hence, the running time of a coin flipping protocol, is the number of its rounds (see section 4 for more details).

A coin flipping round in our leader election protocol, is essentially a coin flipping, performed in parallel, between the pairs of players who are still eligible to be elected.

In our analogy to a knock-out tournament, a coin flipping round corresponds to a round in that tournament.

The running time of a coin flipping round is thus the running time of a single coin flipping (which is the number of communication rounds of the two players). Since different coin flipping in the same round might take different times, we consider the worst coin flipping time of each round.

Hence the running time of our leader election protocol, will be the sum of the running times of its coin flipping rounds. It can be bounded by the number of coin flipping rounds times the running time of the worst coin flipping (in all the tournament).

Proposition 4. [CK09] Let P_{ϵ} be a balanced weak coin flipping protocol with bias ϵ and N_{ϵ} rounds.

Then $\forall q \in [0,1]$ and $\forall k \in \mathbb{N}$, there exists an unbalanced weak coin flipping protocol $P_{q',\delta}$ with $k \cdot N_{\epsilon}$ rounds, such that $|q'-q| \leq 2^{-k}$ and $\delta = 2\epsilon$.

Assume we are interested in a protocol with $P_A=q\,,\ P_B=1-q$ for some q.

Let $k=1+\left\lceil\log\frac{1}{\epsilon}\right\rceil$. Then according to the proposition w.r.t $P_{\frac{\epsilon}{4}}$, there exist $P_{q',\delta}$ with $O\left(\log\frac{1}{\epsilon}\cdot N_{\frac{\epsilon}{4}}\right)$ rounds, such that $|q'-q|\leq \frac{\epsilon}{2}$ and $\delta=2\cdot \frac{\epsilon}{4}=\frac{\epsilon}{2}$. Hence an honest first player has a winning probability of at least $q'-\delta\geq q-\frac{\epsilon}{2}-\delta=q-2\delta$.

An honest second player has a winning probability of at least

$$1 - q' - \delta \ge 1 - q - \frac{\epsilon}{2} - \delta = 1 - q - 2\delta.$$

Note that this protocol ensures that when both players are honest, $P_A = q'$, hence $|P_A - q| \le \frac{\epsilon}{2}$.

Similarly
$$|P_B - (1-q)| \le \frac{\epsilon}{2}$$
, $P_A + P_B = 1$.

This proves:

Corollary 5. For Every $q \neq \frac{1}{2}$, $\delta > 0$ there exists a weak coin flipping protocol $P_{q',\delta}$ s.t. $|q'-q| \leq \delta$ with $O\left(N_{\frac{\delta}{2}} \cdot \log \frac{1}{\delta}\right)$ rounds.

When all players are honest, then $P_A = q'$ is guaranteed to satisfy

$$|P_A - q| \le \delta$$
, $P_B = 1 - q'$, $|P_B - (1 - q)| \le \delta$.

The winning probability is at least $q' - \delta \ge q - 2\delta$ for an honest A player, and $1 - q - 2\delta$ for an honest B player.

The proof of theorem 2 is basically a simple combinatorial manipulation of how to combine balanced coin flips to achieve the correct probability of winning. Corollary 5 enables to calculate the complexity due to using balanced coin flips.

There is one delicate point: we have to make sure that the cheaters can not increase their winning probability in a specific coin flip, by losing previous coin flip (say by creating entanglements). This will be discussed in subsection 2.2.

We will first present the simpler case of three parties, to show the basic idea. The general case is a natural generalization of this, and we will analyze it in details. Nevertheless the three party case captures the basic idea of the problem in its simple version, namely the proof of Theorem 2.

2.2 Group of cheaters

When we analyze coin flipping between two players, we assume one of them is honest and analyze the scenario that the other player is cheating and we then bound his winning probability. In multiparty protocol, such as the leader election, another possibility might occur. A group of cheating players $A_i \in C$ might try to increase their winning probability as a team.

For example, maybe it is possible that A_2 , A_3 , A_4 are a cheating team. They know that in the first round A_1 plays A_2 , and in the second round the winner of that encounter will play A_3 or A_4 . Maybe they can create some cheating strategy that will cause A_2 , A_3 to lose, but will increase significantly the winning chances of A_4 in the second round?

This might sound far fetched, but as [GS] show, it is a possible scenario in parallel coin flipping; it is the analogy of quantum hedging (see [MW12]). However, this is not possible to do in a sequential setting as ours. The point is that an honest player A_j plays one coin flipping round at a time, and the next coin flipping protocol he participates in starts after the previous one ends. In this setting, the following claim holds:

Claim 6. if A_j is honest, and is playing coin flipping protocols sequentially, where the i^{th} protocol is $P_{q'_i,\epsilon_i}$, against whoever may be, then her chance of winning the i^{th} round, even conditioned on whatever happened in previous rounds with other players, is at least $q'_i - \epsilon_i$.

Proof. Since A_j is honest, she starts her i^{th} round with a clean ancilla register. Assume in the worst case that all other players conspire against her together. Suppose they can collude to make her winning probability strictly smaller than $q'_i - \epsilon_i$. Then a cheater playing against A_j a single coin flipping protocol can also do this, by simulating what they have done in previous rounds to prepare the initial state of the protocol, and then simulating what they do in the current round. This is in contradiction to the fact that no matter what a cheating player does, A_j has at least $q'_i - \epsilon_i$ probability to win against her

opponent when playing a single round of a coin flipping protocol $P_{q'_i,\epsilon_i}$.

In our protocols, every round each eligible player plays a weak coin flipping $P_{q,\epsilon}$. This protocol guarantees the honest player a winning probability of $q - \epsilon$.

2.3 Three parties: A,B,C

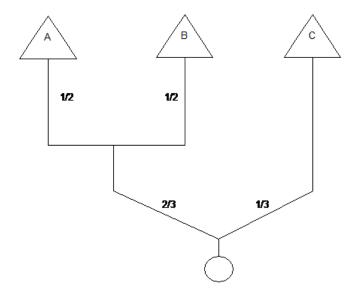
Alice, Bob and Charlie want to select a leader.

Let $\epsilon > 0$. We will show a leader election protocol LE_{ϵ} , such that an honest party has a winning probability of at least $\frac{1}{3} - \epsilon$.

Let
$$\epsilon' = \frac{\epsilon}{2}$$
.

2.3.1 Protocol

- 1. Alice plays Bob $P_{\epsilon'}$ (balanced coin flip with bias $\frac{\epsilon}{2}$).
- 2. The winner plays Charlie $P_{\frac{2}{3},\epsilon'}$, where by this we mean $P_{q',\epsilon'}$ such that $\left|q'-\frac{2}{3}\right|\leq\frac{\epsilon}{2}$ (the existence of this protocol is given by corollary 5).
- 3. The winner of that flip is declared as the leader.



2.3.2 Analysis

- If all players are honest, then A (same for B) has at least $\frac{1}{2} \cdot q' \geq \frac{1}{2} \cdot \left(\frac{2}{3} \frac{\epsilon}{2}\right) = \frac{1}{3} \frac{\epsilon}{4} > \frac{1}{3} \epsilon$ chance of winning. $(\frac{1}{2}$ the chance of winning against B, and q' to then win against C). C has just one game, so he obviously has at least $1 q' \geq \frac{1}{3} \frac{\epsilon}{2} > \frac{1}{3} \epsilon$ chance of winning.
- If A is honest, we can think of it as if A is the only honest player. Then the calculation is almost the same from her point of view: In the first game she has $\frac{1}{2} \epsilon'$ winning probability, and in the second (conditioned that she had won the first round) she has at least $q' \epsilon' \geq \frac{2}{3} \epsilon' \epsilon' = \frac{2}{3} \epsilon$ winning probability. So in the total she has $(\frac{1}{2} \frac{\epsilon}{2})(\frac{2}{3} \epsilon) \geq \frac{1}{3} \frac{5}{6}\epsilon + \frac{\epsilon^2}{2} > \frac{1}{3} \frac{5}{6}\epsilon > \frac{1}{3} \epsilon$.
- If B is honest then the calculation is the same, just replace A with B and vice versa.
- If C is honest again he has only one flip, so he has at least $1 q' \epsilon' \ge 1 \frac{2}{3} \epsilon' \epsilon' = \frac{1}{3} \epsilon$ chance of winning.
- Number of coin flips = 2.
- First coin flip is $P_{\epsilon'}$, hence involves $N_{\frac{\epsilon}{2}}$ rounds. Second coin flip is $P_{\frac{2}{3},\epsilon'}$, and by corollary 5, it involves $O\left(N_{\frac{\epsilon}{4}}\log\frac{1}{\epsilon}\right)$ rounds, hence the total running time is $O\left(N_{\frac{\epsilon}{4}}\log\frac{1}{\epsilon}\right)$.

2.4 Simple solution

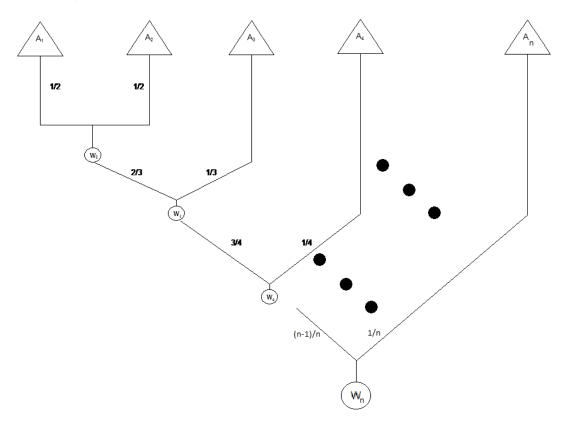
In the general case we have n parties A_1, \ldots, A_n .

Let $\epsilon > 0$. We will show a leader election protocol, such that an honest player has a winning probability of at least $\frac{1}{n} - \epsilon$.

Let
$$\epsilon' = \frac{\epsilon}{2n}$$
.

2.4.1 Protocol:

- 1. Let $W_1 = A_1$.
- 2. For i = 2 to n
 - (a) W_{i-1} plays A_i a $P_{\frac{i-1}{i},\epsilon'}$ unbalanced weak coin flipping protocol. Namely, $P_{q',\epsilon'}$ such that $\left|q'-\left(\frac{i-1}{i}\right)\right|\leq\epsilon'$. Except when i=2, then the protocol is simply $P_{\epsilon'}$.
 - (b) W_i is the winner.
- 3. W_n is declared as the leader.



2.4.2 Analysis

Note that player j enters the game in the j^{th} stage (i.e. when i=j on $\#2_a$ in the protocol), when he plays coin flip with winning probability at least $1-q'-\epsilon' \geq 1-\frac{i-1}{i}-2\epsilon' \geq \frac{1}{i}-2\epsilon'$ (The only exception in the protocol is A_1 that also plays for the first time when i=2, but then also $P_1=\frac{1}{2}$, so it has the same path as A_2).

Lemma 7. If $\forall i \ 0 \le a_i \le 1$, $0 \le x \le \frac{1}{n} \prod_{i=1}^n a_i$, then

$$\prod_{i=1}^{n} (a_i - x) \ge \left(\prod_{i=1}^{n} a_i\right) - nx$$

Proof. By induction on n.

If n = 2 then $(a - x)(b - x) = ab - x(a + b) + x^2 \ge ab - 2x$.

Assume correctness for n and prove for n + 1:

$$\prod_{i=1}^{n+1} (a_i - x) = \left(\prod_{i=1}^n (a_i - x)\right) (a_{n+1} - x) \ge \left(\left(\prod_{i=1}^n a_i\right) - nx\right) (a_{n+1} - x) =$$
because every multiplicand is non-negative (from our assumptions on x)
$$= \left(\prod_{i=1}^{n+1} a_i\right) - x \left(n \cdot a_{n+1} + \prod_{i=1}^n a_i\right) + nx^2 \ge \left(\prod_{i=1}^{n+1} a_i\right) - x \left(n+1\right) \qquad \square$$

- If A_i is honest $(i \geq 2)$, then his winning probability is at least: $\left(\frac{1}{i} 2\epsilon'\right) \cdot \left(\frac{i}{i+1} 2\epsilon'\right) \cdot \left(\frac{i+1}{i+2} 2\epsilon'\right) \cdot \dots \cdot \left(\frac{n-2}{n-1} 2\epsilon'\right) \cdot \left(\frac{n-1}{n} 2\epsilon'\right)$ $\geq \frac{1}{n} (n-i+1) 2\epsilon' \geq \frac{1}{n} \epsilon$ by lemma 7, with $x = 2\epsilon' = \frac{\epsilon}{n}, \ a_1 = \frac{1}{i}, \ a_j = \frac{i-2+j}{i-1+j} \ j = 2, \dots, n-i+1 \ \text{and indeed}$ $0 < \{a_j\}$ and $0 < x < \prod_{j=1}^{n-i+1} a_j = \frac{1}{n}$.
- If all are honest, then A_i winning probability is only bigger than that (we have $-\epsilon'$ instead of $-2\epsilon'$ at every multiplicand).
- Number of coin flips = n-1. The i^{th} coin flip is $P_{\frac{i-1}{i}',\epsilon'}$, hence by corollary 5, it will have $O\left(\log(\frac{n}{\epsilon})\cdot N_{\frac{\epsilon}{4n}}\right)$ rounds.

This proves theorem 2.

3 Improved protocol

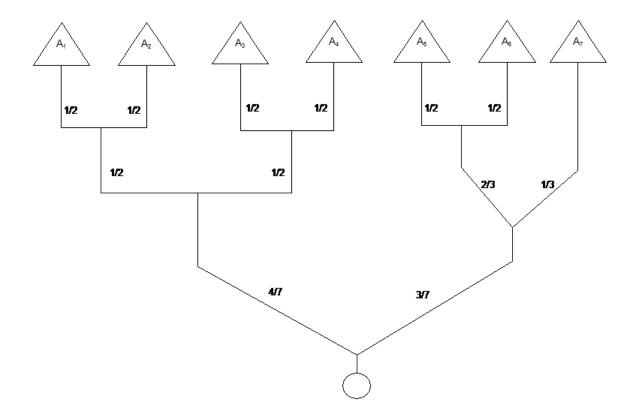
We mentioned in the beginning that if $n=2^m$, we can do a tournament with m rounds. In the last try we came up with a protocol of n-1 rounds, because each time only one couple played a coin flip. The problem with this simple solution is that it is quite inefficient in terms of number of rounds, and also almost all the coin flips are unbalanced, which implies an extra factor of at least $\log \frac{1}{\epsilon}$ per unbalanced coin flip to the running time. We can improve this protocol by combining it with the tournament idea.

3.1 Improved protocol for seven players

We will first start by creating an efficient protocol for seven players, before describing the general solution.

3.1.1 Protocol

- 1. The following couples play P_{ϵ} (balanced coin flip): $A_1-A_2,\ A_3-A_4,\ A_5-A_6.$
- 2. The winners of $A_1 A_2$, $A_3 A_4$ play between them P_{ϵ} . The winner of $A_5 A_6$ plays A_7 a $P_{\frac{2}{3},\frac{\epsilon}{2}}$. Namely, $P_{q',\frac{\epsilon}{2}}$ such that $|q' \frac{2}{3}| \leq \frac{\epsilon}{2}$.
- 3. The two winners of last stage play a $P_{\frac{4}{7},\frac{\epsilon}{2}}$.



3.1.2 Analysis

- If all are honest, then $A_1 A_4$ have the same steps, and they have winning probability (using similar calculations as before) of at least $\frac{1}{2} \cdot \frac{1}{2} \cdot \left(\frac{4}{7} \frac{\epsilon}{2}\right) \ge \frac{1}{7} \frac{1}{8}\epsilon \ge \frac{1}{7} \epsilon$. A_5, A_6 have a winning probability of at least $\frac{1}{2} \cdot \left(\frac{2}{7} \frac{\epsilon}{2}\right) \cdot \left(\frac{3}{7} \frac{\epsilon}{2}\right) \ge \frac{1}{7} \frac{1}{6}\epsilon \left(\frac{3}{7} + \frac{1}{7}\right) + \frac{1}{6}\epsilon^2 \ge \frac{1}{7} \epsilon$
 - $$\begin{split} &\frac{1}{2}\cdot\left(\frac{2}{3}-\frac{\epsilon}{2}\right)\cdot\left(\frac{3}{7}-\frac{\epsilon}{2}\right)\geq \frac{1}{7}-\frac{1}{2}\epsilon\left(\frac{3}{14}+\frac{1}{3}\right)+\frac{1}{8}\epsilon^2\geq \frac{1}{7}-\epsilon.\\ &A_7 \text{ has only two flips, so obviously } P_7\geq \left(\frac{1}{3}-\frac{\epsilon}{2}\right)\cdot\left(\frac{3}{7}-\frac{\epsilon}{2}\right)\geq \frac{1}{7}-\epsilon. \end{split}$$
- If A_1 is honest, he has winning probability of at least $(\frac{1}{2} \epsilon)(\frac{1}{2} \epsilon)(\frac{4}{7} \epsilon) = \frac{1}{7} \frac{23}{28}\epsilon + 1\frac{4}{7}\epsilon^2 \epsilon^3 \ge \frac{1}{7} \epsilon$ (since $0 < \epsilon < 1$). Same result for $A_2 A_4$.

If A_5 (or A_6) is honest then he has winning probability of at least $(\frac{1}{2} - \epsilon)(\frac{2}{3} - \epsilon)(\frac{3}{7} - \epsilon) \ge \frac{1}{7} - \frac{35}{42}\epsilon + 1\frac{25}{42}\epsilon^2 - \epsilon^3 \ge \frac{1}{7} - \epsilon$.

For A_7 it is obviously at least $\left(\frac{1}{3} - \epsilon\right) \left(\frac{3}{7} - \epsilon\right) \ge \frac{1}{7} - \frac{16}{21}\epsilon \ge \frac{1}{7} - \epsilon$.

- Number of coin flipping rounds = 3.
- The two unbalanced coin flips will have each $O\left(N_{\frac{\epsilon}{4}} \cdot \log \frac{1}{\epsilon}\right)$ rounds according to corollary 5, so the total protocol has running time of $O\left(N_{\frac{\epsilon}{4}} \cdot \log \frac{1}{\epsilon}\right)$.

3.2 Final protocol

In the general case we have n parties A_1, \ldots, A_n .

Let $\epsilon > 0$. We will show a leader election protocol LE_{ϵ} , with $\log(n)$ coin flipping rounds, and a running time of $O\left(N_{\frac{\epsilon}{4}} \cdot \log n \cdot \log \frac{1}{\epsilon}\right)$, s.t. $\forall i$ if party i is honest, he has winning probability of at least $\frac{1}{n} - \epsilon$.

3.2.1 Protocol

We will define the protocol recursively.

Let us call it $Leader(\{A_1,\ldots,A_n\},\epsilon)$.

Say it returns the leader selected.

Let $k \in \mathbb{N}$ s.t. $2^k \le n < 2^{k+1}$.

- $1. \ \,$ The following are done simultaneously:
 - A_1, \ldots, A_{2^k} plays a tournament (of k rounds) among themselves with $P_{\frac{\epsilon}{2}}$. Denote the winner as w_1 .
 - $w_2 = Leader(\{A_{2^k+1}, \dots, A_n\}, \epsilon).$
- 2. w_1 plays w_2 a $P_{\frac{2^k}{n},\frac{\epsilon}{2}}$. Namely, $P_{q',\frac{\epsilon}{2}}$ such that $\left|q'-\frac{2^k}{n}\right|\leq \frac{\epsilon}{2}$.

The winner of this is the leader.

Remark. In contrast to the protocol presented with seven players, here the 2^k first players play $P_{\frac{\epsilon}{2}}$ between themselves. If we had used weak coin flipping

protocol P_{ϵ} , this would give a Leader Election protocol for n players with bias a bit over ϵ for n = 3, 5. However, this small concession doesn't increase our running time.

3.2.2 Analysis

• Assume that all parties are honest. We shall prove by induction the following:

Claim 8. If all parties are honest, then $P_i \geq \frac{1}{n} - \epsilon$.

Proof. For n = 2 it is obvious. Assume correctness for all m < n, and we will prove it for n.

Look at A_1, \ldots, A_{2^k} . In the tournament everyone has a $\frac{1}{2^k}$ winning chance. Whoever wins, has another coin flip with at least $\left(\frac{2^k}{n} - \frac{\epsilon}{2}\right)$ winning chance, so in total they each have at least

$$\frac{1}{2^k} \cdot \left(\frac{2^k}{n} - \frac{\epsilon}{2}\right) = \frac{1}{n} - \frac{\epsilon}{2^{k+1}} \ge \frac{1}{n} - \epsilon$$
 winning chance.

From the induction hypothesis, we know that each one of A_{2^k+1},\ldots,A_n has at least $\frac{1}{n-2^k}-\epsilon$ winning chance in the recursive leader election procedure (to become w_2). Notice that $\frac{1}{n-2^k}$ is well defined since we can assume that n is strictly larger than 2^k , otherwise we are done. Then the winner has at least $\left(1-\frac{2^k}{n}-\frac{\epsilon}{2}\right)$ winning chance in the last step, so altogether he has a winning probability $P_i \geq \left(\frac{1}{n-2^k}-\epsilon\right)\cdot \left(1-\frac{2^k}{n}-\frac{\epsilon}{2}\right) =$

$$\frac{1}{n} - \epsilon \left(\frac{1}{2} \frac{1}{n - 2^k} + 1 - \frac{2^k}{n} \right) + \frac{\epsilon^2}{2}$$
since $2^k > \frac{n}{2}$ we get that $P_i \ge \frac{1}{n} - \epsilon \left(\frac{1}{2} + 1 - \frac{1}{2} \right)$
hence $P_i \ge \frac{1}{n} - \epsilon$.

- We have $\log(n)$ coin flipping rounds, and according to corollary (5) we will have up to $O(N_{\frac{\epsilon}{4}}\log(\frac{1}{\epsilon})\log(n))$ total rounds.
- The number of unbalanced coin flips is bounded by log(n).

Claim 9. #unbalanced coin flips = (# of 1's in the binary representation of n) - 1.

Proof. This can be proved easily by induction on n:

For n = 1, 2 it is clear. No unbalanced coin flipping protocols are used. If n is a power of 2, say $n = 2^k$, then it has 0 such.

Else $2^k < n < 2^{k+1}$, and the first 2^k players use again 0 unbalanced coin flips between them. The remaining $m = n - 2^k$ players use (from the induction hypothesis) the # of 1's in the binary representation of m, minus 1, in the appropriate rounds. When joining the two groups, we again use an unbalanced coin flip (in the last round) which corresponds to the MSB of n. As $n = 2^k + m$, #1's in m is exactly one less than the # of 1s in n, which proves the claim.

Remark. There can only be one unbalanced coin flip per round i, if the i^{th} bit in the binary representation of n is 1.

• The last thing needed to be proven is that an honest player has a winning probability of $\frac{1}{n} - \epsilon$.

A simple proof of that will be:

If A_i is honest, then he has a winning probability of at least

 $\prod_{i=1}^{\log(n)} (c_i - \epsilon) \ge \frac{1}{n} - \log(n)\epsilon \text{ where the } \{c_i\} \text{ are the winning probabilities of } A_i \text{ in the individual weak coin flipping rounds } (c_i < 1)$

Therefore, we can use $\frac{\epsilon}{\log n}$ as the weak unbalanced coin flipping bias in the protocol (instead of $\frac{\epsilon}{2}$).

Then we finish the proof, but we get running time of $O\left(N_{\frac{\epsilon}{2\log n}}\log n\log \frac{\log n}{\epsilon}\right)$ instead of $O\left(N_{\frac{\epsilon}{4}}\log n\log \frac{1}{\epsilon}\right)$.

While one might say that $\log \log n$ factor is insignificant, the difference between $N_{\frac{\epsilon}{4}}$ and $N_{\frac{\epsilon}{2\log n}}$ might be huge.

In fact, at the moment, the only known proof for existence of weak coin

flipping with arbitrarily small bias P_{ϵ} is analyzed in [ACG⁺16], giving a bound of $N_{\epsilon} \leq \frac{1}{\epsilon}^{O(\frac{1}{\epsilon})}$ rounds.

Therefore, we will make a more precise calculation, which will allow us to use our original $\frac{\epsilon}{2}$ in the unbalanced coin flip.

• Let us first do a precise calculation for n=3: $A_1 \text{ will play } P_{\frac{\epsilon}{2}} \text{ with } A_2, \text{ and the winner will play } P_{\frac{2}{3},\frac{\epsilon}{2}}.$ An honest A_1 (or A_2) will have a winning probability of $\left(\frac{1}{2} - \frac{\epsilon}{2}\right) \left(\frac{2}{3} - \epsilon\right) \geq \frac{1}{3} - \frac{5}{6}\epsilon \geq \frac{1}{3} - \epsilon.$ The running time of this protocol is $O\left(N_{\frac{\epsilon}{4}} \cdot \log \frac{1}{\epsilon}\right)$, by corollary (5).

We now prove for n players:

Lemma 10. An honest player A_i has a winning probability which is at least $\frac{1}{n} - \epsilon$.

Proof. We will prove it by induction on n.

If n = 1, 2, 3, 4 then we saw this is true.

Assume correctness for all n < N, and we shall prove it for n = N.

- We saw this is true if N is a power of 2.
- If $N = 2^k + 1$:

Assume A_N is honest. A_N only plays one unbalanced coin flip $P_{\frac{2^k}{N},\frac{\epsilon}{2}}$, having winning probability of at least $\frac{1}{N} - \epsilon$ by corollary (5).

 A_1, \ldots, A_{N-1} play a full tournament of 2^k players, with winning probability of $p \ge \left(\frac{1}{2} - \frac{\epsilon}{2}\right)^k = \left(\frac{1-\epsilon}{2}\right)^k = \frac{(1-\epsilon)^k}{N-1}$ for an honest player.

By Bernoulli inequality: $\forall x \geq -1, n \in \mathbb{N} \ (1+x)^n \geq 1+nx$, plugging $x = -\epsilon$, we get that $p \geq \frac{1-\epsilon k}{N-1}$.

Then the honest winner plays $P_{\frac{2^k}{N},\frac{\epsilon}{2}}$ with A_N , hence having winning probability of at least

$$\frac{1 - \epsilon k}{N - 1} \left(\frac{N - 1}{N} - \epsilon \right) = \frac{1}{N} - \epsilon \left(\frac{k}{N} + \frac{1}{N - 1} \right) + \epsilon^2 \frac{NK}{N - 1} \ge \frac{1}{N} - \epsilon$$

Where the last inequality is true for every $N=2^k+1\geq 3$ (because $k=\lfloor \log N \rfloor$, and already for k=1 we get $\frac{k}{N}+\frac{1}{N-1}=\frac{1}{3}+\frac{1}{2}<1$).

• Otherwise $2^k + 2 \le N < 2^{k+1}$, for $k = \lfloor \log N \rfloor$, $k \ge 2$. For all $i \le 2^k$ by the induction hypothesis (for $n = 2^k$), we have that the probability for A_i to reach the final weak coin flipping round (round k + 1) is at least $\frac{1}{2^k} - \epsilon$.

Hence, his winning probability (to be elected) p is at least

$$p \ge \left(\frac{1}{2^k} - \epsilon\right) \left(\frac{2^k}{N} - \epsilon\right) = \frac{1}{N} - \epsilon \left(\frac{2^k}{N} + \frac{1}{2^k}\right) + \epsilon^2$$

Since $2^k + 2 \le N < 2 \cdot 2^k$ we get that $p > \frac{1}{N} - \epsilon \left(\frac{2^k + 2}{N}\right) \ge \frac{1}{N} - \epsilon$. For all $i > 2^k$ by the induction hypothesis (for $t = N - 2^k$), we have that the probability for A_i to reach the final weak coin flipping round is at least $\frac{1}{t} - \epsilon$.

Hence, his winning probability (to be elected) is at least

$$\left(\frac{1}{t} - \epsilon\right) \left(\frac{t}{N} - \epsilon\right) \ge \frac{1}{N} - \epsilon \left(\frac{t}{N} + \frac{1}{t}\right) + \epsilon^2 \ge \frac{1}{N} - \epsilon \left(\frac{t}{N} + \frac{1}{t}\right)$$

Since $\frac{t}{N} \le \frac{1}{2}$, $t \ge 2$ we get that $p \ge \frac{1}{N} - \epsilon$.

This completes the proof of theorem 3.

4 Open questions

Here we will present the open question rising from this paper.

• The first obvious open question, is to find a specific family of balanced weak coin flipping P_{ϵ} , hence getting a tighter bound on N_{ϵ} .

- The second open question, which was mentioned before, is to try and find unbalanced weak coin flipping directly, hence removing the $O\left(\log \frac{1}{\epsilon}\right)$ factor from corollary 5.
- A more specific question is: Can one improve the three party leader election protocol. Our solution (subsection 3.2) used running time of $O\left(N_{\frac{\epsilon}{4}} \cdot \log \frac{1}{\epsilon}\right)$, and the number of rounds in $P_{\frac{\epsilon}{4}}$ might be significantly greater than in P_{ϵ} .
- Can one find a better solution (in terms of running time) to the leader election problem? Maybe by finding a direct solution to the problem, and not via weak coin flipping.
- When we needed to use unbalanced weak coin flipping $P_{q,\epsilon}$, we used the approximate coin flip $P_{q',\frac{\epsilon}{2}}$, which implied two annoying consequences:
 - If all players are honest, they are not granted a $\frac{1}{n}$ winning probability, but only $\frac{1}{n} \frac{\epsilon}{2}$.

 Hence the open question is, can one find an arbitrarily small biased leader election, that guarantees exactly $\frac{1}{n}$ winning probability in the scenario where all players are honest?
 - Can one improve corollary 5 to use $O\left(N_{\delta} \cdot \log \frac{1}{\delta}\right)$ rounds, which will improve Theorem 3 running time to $O\left(N_{\frac{\epsilon}{2}} \cdot \log n \cdot \log \frac{1}{\epsilon}\right)$?

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Appendix: Weak coin flipping

Let P be a weak coin flipping protocol, with P_B^* the maximal cheating probability of Bob.

We want to run two instances of P, one after the other (not even at the same time).

We will define (see [Moc00, ACG $^+$ 16] for full details) for P:

- Let $\mathcal{H} = \mathcal{A} \otimes \mathcal{M} \otimes \mathcal{B}$ be the Hilbert space of the system.
- $|\psi_0>=|\psi_{A,0}>|\psi_{M,0}>|\psi_{B,0}>$ is the initial state of the system.
- Let there be n (even) stages, i denote the current stage.
- On the odd stages i, Alice will apply a unitary $U_{A,i}$ on $A \otimes M$.
- On the even stages, Bob will apply a unitary $U_{B,i}$ on $\mathcal{M} \otimes \mathcal{B}$.
- Let $|\psi_i\rangle$ be the state of the system in the i_{th} stage.
- Let $\rho_{A,i} = Tr_{\mathcal{M}\otimes\mathcal{B}}(|\psi_i> < \psi_i|)$ be the density matrix of Alice in the i_{th} stage.
- Alice's initial state (density matrix) is $^{(i)}\rho_{A,0} = |\psi_{A,0}> < \psi_{A,0}|$.
- For even state i we have $^{(ii)}\rho_{A,i}=\rho_{A,i-1}$.
- Let $\tilde{\rho}_{A,i}$ be the state of $A \otimes \mathcal{M}$ after Alice gets the i_{th} message.
- For odd i: ${}^{(iii)}\rho_{A,i} = Tr_{\mathcal{M}}(\tilde{\rho}_{A,i}), {}^{(iv)}\rho_{A,i} = Tr_{\mathcal{M}}(U_{A,i}\tilde{\rho}_{A,i-1}U_{A,i}^{\dagger}).$

We know that regardless of Bob's actions (see [Moc00, ACG⁺16] for full proof):

$$P_B^* \le \max Tr[\Pi_{A,1}\rho_{A,n}] \tag{1}$$

where the maximization is done over all density matrices ρ that satisfies the conditions (i) - (iv).

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