The Logic of Entanglement

Bob Coecke

Oxford University Computing Laboratory,
Wolfson Building, Parks Road, OX1 3QD Oxford, UK.

coecke@comlab.ox.ac.uk

Abstract. We expose the information flow capabilities of pure bipartite entanglement as a theorem — which embodies the exact statement on the 'seemingly acausal flow of information' in protocols such as teleportation [14]. We use this theorem to redesign and analyze known protocols (e.g. logic gate teleportation [10] and entanglement swapping [15]) and show how to produce some new ones (e.g. parallel composition of logic gates). We also show how our results extend to the multipartite case and how they indicate that entanglement can be measured in terms of 'information flow capabilities'. Ultimately, we propose a scheme for automated design of protocols involving measurements, local unitary transformations and classical communication.

1 Introduction

Entanglement has always been a primal ingredient of fundamental research in quantum theory, and more recently, quantum computing. By studying it we aim at understanding the operational/physical significance of the use of the Hilbert space tensor product for the description of compound quantum systems. Many typical quantum phenomena are indeed due to compound quantum systems being described within the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ and not within a direct sum $\mathcal{H}_1 \oplus \mathcal{H}_2$.

In this paper we reveal a new structural ingredient of the supposedly well-understood pure bipartite entanglement, that is, we present a new theorem about the tensor product of Hilbert spaces. It identifies a 'virtual flow of information' in so-called entanglement specification networks. For example, it is exactly this flow of information which embodies teleporting [6] an unknown state from one physical carrier to another. Furthermore, our theorem (nontrivially) extends to multipartite entanglement. We also argue that it provides a new way of conceiving entanglement itself and hence of measuring entanglement:

entanglement \equiv information flow capabilities

Indeed, our result enables reasoning about quantum information flow without explicitly considering classical information flow — this despite the impossibility of transmitting quantum information through entanglement without the use of a classical channel.

Using our theorem we can evidently reconstruct protocols such as *logic gate teleportation* [10] and *entanglement swapping* [15]. It moreover allows smooth generation of new protocols, of which we provide an exam-

ple, namely the conversion of accumulation of inaccuracies causing 'sequential composition' into fault-tolerant 'parallel composition' [13]. Indeed, when combing our new insights on the flow of information through entanglement with a model for the flow of classical information we obtain a powerful tool for designing protocols involving entanglement.

An extended version of this paper is available as a research report [9]. It contains details of proofs, other/larger pictures, other references, other applications and some indications of connections with logic, proof theory and functional programming.

2 Classical information flow

By the spectral theorem any non-degenerated measurement on a quantum system described in a n-dimensional complex Hilbert space \mathcal{H} has the shape

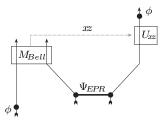
$$M = x_1 \cdot P_1 + \ldots + x_n \cdot P_n$$
.

Since the values x_1, \ldots, x_n can be conceived as merely being tokens distinguishing the projectors P_1, \ldots, P_n in the above sum we can abstract over them and conceive such a measurement as a set

$$M \simeq \{P_1, \dots, P_n\}$$

of n mutually orthogonal projectors which each project on a one-dimensional subspace of \mathcal{H} . Hence, by von Neumann's projection postulate, a measurement can be conceived as the system being subjected to an *action* P_i and the observer being informed about which action happened (e.g. by receiving the token x_i).

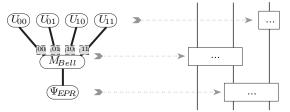
In most quantum information protocols the indeterminism of measurements necessitates a flow of classical information e.g. the 2-bit classical channel required for teleportation [6]. We want to separate this *classical information flow* from what we aim to identify as the *quantum information flow*. Consider a protocol involving local unitary operations, (non-local) measurements and classical communication e.g. teleportation:



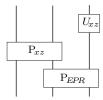
We can decompose such a protocol in

- a tree with the consecutive operations as nodes, and, in case of a measurement, the emerging branches being labeled by tokens representing the projectors;
- 2. the configuration of the operations in terms of the time when they are applied and the subsystem to which they apply.

Hence we abstract over spatial dynamics. The nodes in the tree are connected to the boxes in the *configuration* picture by their temporal coincidence. For teleportation we thus obtain



Classical communication is encoded in the tree as the dependency of operations on the labels on the branches below it e.g. the dependency of the operation U_{xz} on the variable xz stands for the 2-bit classical channel required for teleportation. We will also replace any initial state Ψ by the projector P_{Ψ} on it, which can be conceived as its preparation e.g. P_{EPR} is the preparation of an EPR-pair. It should be clear that for each path from the root of the tree to a leaf, by 'filling in the operations on the included nodes in the corresponding boxes of the configuration picture', we obtain a network involving only local unitary operations and (non-local) projectors e.g. one network



for each of the four values xz takes. It will be these networks (from which we extracted the classical information flow) for which we will reveal the quantum information flow. Hence each projector in it which is not a preparation is to be conceived *conditionally*.

3 Bipartite entanglement

Let \mathcal{H}_1 and \mathcal{H}_2 be two finite dimensional complex Hilbert spaces. The elements of $\mathcal{H}_1 \otimes \mathcal{H}_2$ are in bijec-

tive correspondence with those of $\mathcal{H}_1 \to \mathcal{H}_2$, the vector space of *linear maps* with domain \mathcal{H}_1 and codomain \mathcal{H}_2 , and also with those of $\mathcal{H}_1 \hookrightarrow \mathcal{H}_2$, the vector space of *anti-linear maps* with domain \mathcal{H}_1 and codomain \mathcal{H}_2 . Given a base $\{e_{\alpha}^{(1)}\}_{\alpha}$ of \mathcal{H}_1 and a base $\{e_{\beta}^{(2)}\}_{\beta}$ of \mathcal{H}_2 this can easily be seen through the correspondences

$$\sum_{\alpha\beta} m_{\alpha\beta} \langle e_{\alpha}^{(1)} | - \rangle \cdot e_{\beta}^{(2)} \stackrel{\mathcal{L}}{\simeq} \sum_{\alpha\beta} m_{\alpha\beta} \cdot e_{\alpha}^{(1)} \otimes e_{\beta}^{(2)}$$
$$\sum_{\alpha\beta} m_{\alpha\beta} \langle - | e_{\alpha}^{(1)} \rangle \cdot e_{\beta}^{(2)} \stackrel{\text{aL}}{\simeq} \sum_{\alpha\beta} m_{\alpha\beta} \cdot e_{\alpha}^{(1)} \otimes e_{\beta}^{(2)}$$

where $(m_{\alpha\beta})_{\alpha\beta}$ is the *matrix* of the corresponding function in bases $\{e_{\alpha}^{(1)}\}_{\alpha}$ and $\{e_{\alpha}^{(2)}\}_{\beta}$ and where by

$$\langle e_{\alpha}^{(1)} \mid - \rangle : \mathcal{H}_1 \to \mathcal{H}_2 \quad \text{and} \quad \langle - \mid e_{\alpha}^{(1)} \rangle : \mathcal{H}_1 \hookrightarrow \mathcal{H}_2$$

we denote the functionals which are respectively the linear and the anti-linear duals to the vector $e_{\alpha}^{(1)}$. While the second correspondence does not depend on the choice of $\{e_{\alpha}^{(1)}\}_{\alpha}$ the first one does since

$$\langle c \cdot e_{\alpha}^{(1)} | - \rangle = \bar{c} \cdot \langle e_{\alpha}^{(1)} | - \rangle \text{ and } \langle - | c \cdot e_{\alpha}^{(1)} \rangle = c \cdot \langle - | e_{\alpha}^{(1)} \rangle.$$

We can now represent the *states* of $\mathcal{H}_1 \otimes \mathcal{H}_2$ by functions in $\mathcal{H}_1 \to \mathcal{H}_2$ or in $\mathcal{H}_1 \hookrightarrow \mathcal{H}_2$, and vice versa, these functions represent states of $\mathcal{H}_1 \otimes \mathcal{H}_2$. Omitting normalization constants, an attitude we will abide by throughout this paper, examples of linear maps encoding states are:

$$id := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \stackrel{L}{\simeq} \quad |00\rangle + |11\rangle$$

$$\pi := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \stackrel{L}{\simeq} \quad |01\rangle + |10\rangle$$

$$id^* := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \stackrel{L}{\simeq} \quad |00\rangle - |11\rangle$$

$$\pi^* := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \stackrel{L}{\simeq} \quad |01\rangle - |10\rangle$$

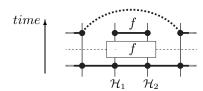
These four functions which encode the *Bell-base* states are almost the *Pauli matrices*

$$\sigma_x \equiv X := \pi$$
 $\sigma_y \equiv Y := i\pi^*$ $\sigma_z \equiv Z := id^*$

plus the *identity* which itself encodes the *EPR-state*. We can also encode each projector

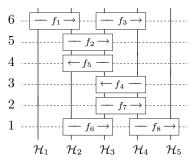
$$P_{\Psi}: \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2 :: \Phi \mapsto \langle \Psi \mid \Phi \rangle \cdot \Psi$$

with $\Psi \in \mathcal{H}_1 \otimes \mathcal{H}_2$ by a function either in $\mathcal{H}_1 \to \mathcal{H}_2$ or $\mathcal{H}_1 \hookrightarrow \mathcal{H}_2$. Hence we can use these (linear or antilinear) functional labels both to denotate the states of $\mathcal{H}_1 \otimes \mathcal{H}_2$ and the projectors on elements of $\mathcal{H}_1 \otimes \mathcal{H}_2$. We introduce a graphical notation which incarnates this.



The box f depicts the projector which projects on the bipartite state labeled by the (anti-)linear function f and the barbell f depicts that state itself. Hence the projector f acts on the multipartite state represented by ____ and produces a pure tensor consisting of (up to a normalization constant) f and some remainder. Hence this picture portrays 'preparation of the f-labeled state'.

By an entanglement specification network we mean a collection of bipartite projectors f'configured in space and time' e.g.



The arrows indicate which of the two Hilbert spaces in $\mathcal{H}_1 \otimes \mathcal{H}_i$ is the domain and which is the codomain of the labeling function. Such a network can also contain local unitary operations — which we will represent by a grey square box U. We will refer to the lines labeled by some Hilbert space \mathcal{H}_i (\simeq time-lines) as tracks.

Definition 3.1 A path is a line which progresses along the tracks either forward or backward with respect to the actual physical time, and, which: (i) respects the four possibilities

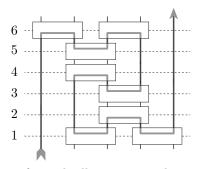


for entering and leaving a bipartite projector; passes local unitary operations unaltered, that is



in pictures; (iii) does not end at a time before any other time which it covers.

An example of a path is the grey line below.

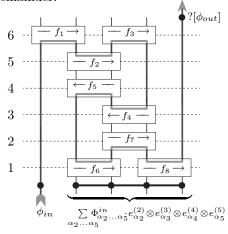


The notion of a path allows us to make certain predictions about the output Ψ_{out} of a network, that is, the cation network and a path. Assume that:

state of the whole system after all projectors have been effectuated. Before stating the theorem we illustrate it on our example. Let

$$\Psi_{\mathbf{in}} := \phi_{in} \otimes \sum_{\alpha_2 \dots \alpha_5} \Phi_{\alpha_2 \dots \alpha_5}^{in} \cdot e_{\alpha_2}^{(2)} \otimes e_{\alpha_3}^{(3)} \otimes e_{\alpha_4}^{(4)} \otimes e_{\alpha_5}^{(5)}$$

be its input state. This input state factors into the pure factor ϕ_{in} , which we call the input of the path, and a remainder.



It should be clear that after effectuating all projectors we end up with an output which factors in the bipartite state labeled by f_1 , the bipartite state labeled by f_2 and a remaining pure factor ϕ_{out} — which we call the output of the path. Our theorem (below) predicts that

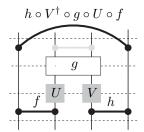
$$\phi_{out} = (f_8 \circ f_7 \circ f_6 \circ f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1)(\phi_{in}). \quad (1)$$

Be aware of the fact that the functions f_1, \ldots, f_8 are not physical operations but labels obtained via a purely mathematical isomorphism. Moreover, the order in which they appear in the composite (1) has no obvious relation to the temporal order of the corresponding projectors. Their order in the composite (1) is:

the order in which the path passes through them

— this despite the fact that the path goes both forward and backward in physical time. Here's the theorem.

Lemma 3.2 For f, g and h anti-linear maps and U and V unitary operations we have



Proof: Straightforward verification or see [9] $\S 5.1$. \square

Theorem 3.3 (i) Given are an entanglement specifi-

- 1. The order in which the path passes through the projectors is $f_1 \Rightarrow f_2 \Rightarrow \ldots \Rightarrow f_{k-1} \Rightarrow f_k$.
- 2. The input of the path is a pure factor ϕ_{in} .
- 3. $\Psi_{\mathbf{out}}$ has a non-zero amplitude.

Then the output of the path is (indeed) a pure factor ϕ_{out} which is explicitly given by

$$\phi_{out} = (f_k \circ f_{k-1} \circ \dots \circ f_2 \circ f_1)(\phi_{in}). \tag{2}$$

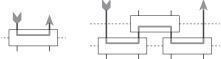
(ii) If the path passes forwardly through U then U will be part of the composite (2) and if it passes backwardly through U then U^{\dagger} will be part of the composite (2).

Proof: Lemma 3.2 is the crucial lemma for the proof. For a full proof see [9] §5.

It might surprise the reader that in the formulation of Theorem 3.3 we didn't specify whether f_1, \ldots, f_k are either linear or anti-linear, and indeed, we slightly cheated. The theorem is only valid for f_1, \ldots, f_k anti-linear. However, in the case that f_1, \ldots, f_k are linear, in order to make the theorem hold it suffices to conjugate the matrix elements of those functional labels for which the path enters (and leaves) the corresponding projector 'from below' (see [9] §4.1):



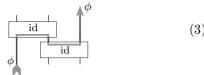
In most practical examples these matrix elements are real (see below) and hence the above theorem also holds for linear functional labels. One also verifies that if a path passes though a projector in the opposite direction of the direction of an anti-linear functional label f, then we have to use the adjoint f^{\dagger} of the anti-linear map f in the composite (2) — the matrix of the adjoint of an anti-linear map f^{\dagger} is the transposed of the matrix of f (see [9] §4.2). Finally note that we did not specify that at its input a path should be directed forwardly in physical time, and indeed, the theorem also holds for paths such as



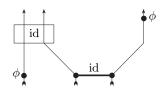
We discuss this in Section 5.

4 Re-designing teleportation

By Theorem 3.3 we have



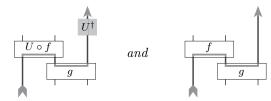
due to $(id \circ id)(\phi) = \phi$. When conceiving the first projector as the preparation of an EPR-pair while tilting the tracks we indeed obtain 'a' teleportation protocol.



However, the other projector has to 'belong to a measurement' e.g. $M_{Bell} := \{P_{id}, P_{\pi}, P_{id^*}, P_{\pi^*}\}$. Hence the above introduced protocol is a *conditional* one. We want to make it *unconditional*.

Definition 4.1 Paths are *equivalent* iff for each input ϕ_{in} they produce the same output ϕ_{out} .

Corollary 4.2 For U unitary and $g \circ U = U \circ g$



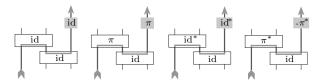
are equivalent paths.

Proof: Since $U^{\dagger} \circ g \circ (U \circ f) = g \circ f$ the result follows by Theorem 3.3.

Intuitively, one can move the box U^{\dagger} along the path and permute it with projectors whose functional labels commute with U (= commute with U^{\dagger}) until it gets annihilated by the U-factor of $U \circ f$. Applying Corollary 4.2 to

$$f, q := \mathrm{id}$$
 and $U \in \{\mathrm{id}, \pi, \mathrm{id}^*, \pi^*\}$,

since $\pi^{\dagger} = \pi$, $(id^*)^{\dagger} = id^*$ and $(\pi^*)^{\dagger} = -\pi^*$, we obtain four conditional teleportation protocols

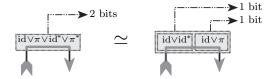


of which the one with U := id coincides with (3). These four together constitute an unconditional teleportation protocol since they correspond to the four paths 'from root to leaf' of the tree discussed in Section 2, from which then also the 2-bit classical channel emerges.

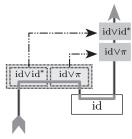
In order to obtain the teleportation protocol as it is found in the literature, observe that $\pi^* = \pi \circ id^*$, hence

0	id	id^*
id	id	id*
π	π	π^*

and thus we can factor — with respect to composition of functional labels — the 2-bit Bell-base measurement in two 1-bit 'virtual' measurements (\vee stands for 'or'):

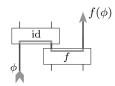


Note that such a decomposition of M_{Bell} does not exist with respect to \otimes nor does it exist with respect to composition of projector actions. All this results in



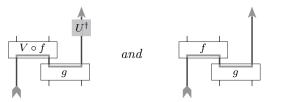
which is the standard teleportation protocol [6].

The aim of *logic gate teleportation* [10] is to teleport a state and at the same time subject it to the action of a gate f. By Theorem 3.3 we evidently have



We make this protocol unconditional analogously as we did it for ordinary teleportation.

Corollary 4.3 For U and V unitary and $g \circ V = U \circ g$



are equivalent paths.

Proof: Analogous to that of Corollary 4.3.

We apply the above to the case

$$f := id \otimes id$$
 and $q := CNOT$

that is, the first projector is now to be conceived as the preparation of the state

$$\Psi_{\text{cnot}} = |00\rangle \otimes |00\rangle + |01\rangle \otimes |01\rangle + |10\rangle \otimes |11\rangle + |11\rangle \otimes |10\rangle .$$

Let Ψ_f be defined either by $f \stackrel{\text{L}}{\simeq} \Psi_f$ or $f \stackrel{\text{aL}}{\simeq} \Psi_f$.

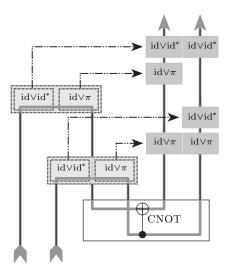
Proposition 4.4
$$\Psi_{f \otimes g} = \Psi_f \otimes \Psi_g$$
; $P_{f \otimes g} = P_f \otimes P_g$.

Proof: The first claim is verified straightforwardly. Hence $P_{f \otimes g} \equiv P_{\Psi_{f \otimes g}} = P_{\Psi_{f} \otimes \Psi_{g}} = P_{\Psi_{f}} \otimes P_{\Psi_{g}} \equiv P_{f} \otimes P_{g}$ what completes the proof.

Hence we can factor the 4-qubit measurement to which the second projector belongs in two Bell-base measurements, that is, we set

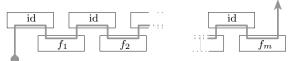
$$V \in \{U_1 \otimes U_2 \mid U_1, U_2 \in \{id, \pi, id^*, \pi^*\}\}$$
.

The resulting protocol



is the one to be found in [10] — recall that U^{\dagger} factors as a tensor since CNOT is a member of the Clifford group.

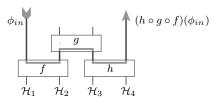
Our last example in this section involves the passage from sequential to parallel composition of logic gates. Due to the accumulation of inaccuracies in sequential composition [13] it would be desirable to have a fault-tolerant parallel alternative. This would for example be useful if we have a limited set of available gates from which we want to generate more general ones e.g. generating all Clifford group gates from CNOT gates, Hadamard gates and phase gates via tensor and composition. By Theorem 3.3 the network



realizes the composite $f_m \circ ... \circ f_1$ conditionally. Again this protocol can be made unconditional — an algorithm which captures the general case can be found in [9] §3.4. Note that by Theorem 3.3 it suffices to make unitary corrections only at the end of the path [9] §3.4.

5 Entanglement swapping

By Theorem 3.3 we have



However, Theorem 3.3 assumes ϕ_{in} to be a pure factor while it is part of the output $\Psi_{\mathbf{out}}$ of the network. This fact constraints the network by requiring that

$$h \circ q \circ f \stackrel{\mathrm{aL}}{\simeq} \phi_{in} \otimes \phi_{out}$$

for some ϕ_{in} and ϕ_{out} i.e. the state labeled by $h \circ g \circ f$ has to be disentangled — which is equivalent to the range of $h \circ g \circ f$ being one-dimensional [9] §5.3.

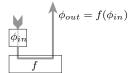
Using Lemma 3.2 this pathology can be overcome by conceiving the output state of the bipartite subsystem described in $\mathcal{H}_1 \otimes \mathcal{H}_4$ not as a pair (ϕ_{in}, ϕ_{out}) but as a function $\varphi : \mathcal{H}_1 \hookrightarrow \mathcal{H}_4$ which relates any input $\phi_{in} \in \mathcal{H}_1$ to an output $\phi_{out} := \varphi(\phi_{in}) \in \mathcal{H}_4$. Hence we conceive the above network as producing a function

$$\varphi := h \circ g \circ f \stackrel{\mathrm{aL}}{\simeq} \Psi_{\varphi}$$

where $\Psi_{\mathbf{out}} = \Psi_{\varphi} \otimes \Psi_g$ with

$$\Psi_{\varphi} \in \mathcal{H}_1 \otimes \mathcal{H}_4 \quad \text{and} \quad g \stackrel{\mathrm{aL}}{\simeq} \Psi_q \in \mathcal{H}_2 \otimes \mathcal{H}_3 \,.$$

To such a function produced by a network we can provide an input via a unipartite projector. The generic example (which can be easily verified) is

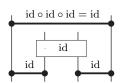


One can then conceive f as a λ -term $\lambda \phi. f \phi$ [5] and the process of providing it with an input via a unipartite projector embodies the β -reduction [5]

$$(\lambda \phi. f \phi) \phi_{in} \stackrel{\beta}{=} f(\phi_{in}).$$

As we will see below we can 'feed' such a function at its turn as an input of *function type* in another network. This view carries over to the interpretation of multipartite entanglement where it becomes crucial.

The entanglement swapping protocol [15] can now be derived analogously as the teleportation protocol by setting $f=g=h:=\mathrm{id}$ in the above. For this particular case Lemma 3.2 becomes



Details can be found in [9] §6.2.

6 Multipartite entanglement

The passage from states to functions as inputs and outputs enables to extend our functional interpretation of bipartite entanglement to one for multipartite entanglement. In general this involves higher order functions and hence the use of denotational tools from modern logic and proof theory such as λ -calculus [1, 5].

Whereas (due to commutativity of $-\otimes -$) a bipartite tensor $\mathcal{H}_1 \otimes \mathcal{H}_2$ admits interpretation as a function either of type $\mathcal{H}_1 \hookrightarrow \mathcal{H}_2$ or of type $\mathcal{H}_1 \hookrightarrow \mathcal{H}_2$, a tripartite tensor (due to associativity of $-\otimes -$) admits interpretation as a function of a type within the union of two (qualitatively different) families of types namely

$$\mathcal{H}_i \hookrightarrow (\mathcal{H}_j \hookrightarrow \mathcal{H}_k)$$
 and $(\mathcal{H}_i \hookrightarrow \mathcal{H}_j) \hookrightarrow \mathcal{H}_k$.

Explicitly, given

$$\sum_{\alpha\beta} M_{\alpha\beta\gamma} \cdot e_{\alpha}^{(1)} \otimes e_{\beta}^{(2)} \otimes e_{\gamma}^{(3)} \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$$

we respectively obtain

$$f_1: \mathcal{H}_1 \hookrightarrow (\mathcal{H}_2 \hookrightarrow \mathcal{H}_3)$$

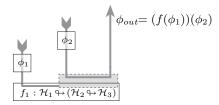
$$:: \sum_{\alpha} \psi_{\alpha} \cdot e_{\alpha}^{(1)} \mapsto \sum_{\beta \gamma} \left(\sum_{\alpha} \bar{\psi}_{\alpha} M_{\alpha \beta \gamma} \right) \langle - \mid e_{\beta}^{(2)} \rangle \cdot e_{\gamma}^{(3)}$$

and

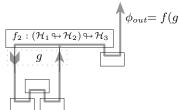
$$f_2: (\mathcal{H}_1 \hookrightarrow \mathcal{H}_2) \hookrightarrow \mathcal{H}_3$$

$$:: \sum_{\alpha\beta} m_{\alpha\beta} \langle - \mid e_{\alpha}^{(1)} \rangle \cdot e_{\beta}^{(2)} \mapsto \sum_{\gamma} \left(\sum_{\alpha\beta} \bar{m}_{\alpha\beta} M_{\alpha\beta\gamma} \right) \cdot e_{\gamma}^{(3)}$$

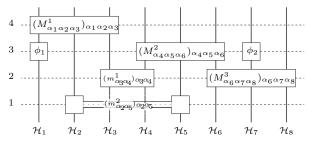
as the corresponding functions — the complex conjugation of the coefficients $\bar{\psi}_{\alpha}$ and $\bar{m}_{\alpha\beta}$ is due to the anti-linearity of the maps. The appropriate choice of an interpretation for a tripartite projector depends on the *context* i.e. the configuration of the whole network to which it belongs. A *first order function* f_1 enables interpretation in a configuration such as



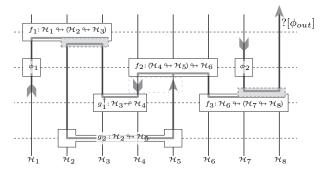
One can think of this tripartite projector as producing a bipartite one at its 'output'. A second order function f_2 — recall that a definite integral is an example of a second order function — enables interpretation in the configuration



We illustrate this in an example — we will not provide an analogue to Theorem 3.3 for the multipartite case since even its formulation requires advanced denotational tools. Consider the following configuration.



For 'good' types we can draw a 'compound' path.



If a multipartite analogue to Theorem 3.3 truly holds one would obtain

$$\phi_{out} = (f_3 \circ f_2)(g_2 \circ (f_1(\phi_1))^{\dagger} \circ g_1^{\dagger})(\phi_2).$$

Hence in terms of matrices we predict $\phi_{\alpha_8}^{out}$ to be

$$\sum_{\alpha_1...\alpha_7} \bar{\phi}^2_{\alpha_7} m^1_{\alpha_3\alpha_4} \phi^1_{\alpha_1} \bar{M}^1_{\alpha_1\alpha_2\alpha_3} m^2_{\alpha_2\alpha_5} \bar{M}^2_{\alpha_4\alpha_5\alpha_6} M^3_{\alpha_6\alpha_7\alpha_8}.$$

To verify this we explicitly calculate $\phi_{\alpha s}^{out}$. Set

$$\Psi^{\tau} = \sum_{i_1 \dots i_8} \Psi^{\tau}_{i_1 \dots i_8} \cdot e^{(1)}_{i_1} \otimes \dots \otimes e^{(8)}_{i_8}$$

where Ψ^0 is the (essentially arbitrary) input of the network and Ψ^{τ} for $\tau \in \{1, 2, 3, 4\}$ is the state at time $\tau + \epsilon$. For $I \subseteq \{1, \dots, 8\}$ and $I^c := \{1, \dots, 8\} \setminus I$ let P_{Φ}^I stipulate that this projector projects on the subspace

$$\Phi \otimes \bigotimes_{i \in I^c} \mathcal{H}_i$$
 for some $\Phi \in \bigotimes_{i \in I} \mathcal{H}_i$.

Lemma 6.1 If $\Psi^{\tau} = P_{\Phi}^{I}(\Psi^{\tau-1})$ then

$$\Psi_{i_1...i_8}^{\tau} = \sum_{i_{\alpha} \mid \alpha \in I} \Psi_{i_1...i_8[j_{\alpha}/i_{\alpha}\mid \alpha \in I]}^{\tau-1} \bar{\Phi}_{(j_{\alpha}\mid \alpha \in I)} \Phi_{(i_{\alpha}\mid \alpha \in I)}$$

where $i_1 \dots i_8[j_{\alpha}/i_{\alpha} \mid \alpha \in I]$ denotes that for $\alpha \in I$ we substitute the index i_{α} by the index j_{α} which ranges over the same values as i_{α} .

Proof: Straightforward verification or see [9] §6.4. \square

Using Lemma 6.1 one verifies that the resulting state $\Psi^4_{i_1...i_8}$ factors into five components, one in which no index in $\{i_1, ..., i_8\}$ appears, three with indices in $\{i_1, ..., i_7\}$ and one which contains the index i_8 namely

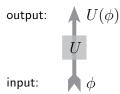
$$\sum_{\substack{l_4 l_5 l_6 l_7 \\ m_1 m_2 m_3}} \!\!\! m_{m_3 l_4}^2 M_{l_6 l_7 i_8}^3 \phi_{m_1}^1 \bar{M}_{l_4 l_5 l_6}^2 \bar{\phi}_{l_7}^2 \bar{M}_{m_1 m_2 m_3}^1 \,.$$

Substituting the indices m_1 , m_2 , m_3 , l_4 , l_5 , l_6 , l_7 , l_8 by $\alpha_1, \ldots, \alpha_8$ we exactly obtain our prediction for $\phi_{\alpha_8}^{out}$.

It should be clear from our discussion of multipartite entanglement that, provided we have an appropriate entangled state involving a sufficient number of qubits, we can implement arbitrary (linear) λ -terms [1, 5].

7 Discussion

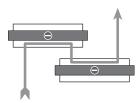
For a unitary operation $U: \mathcal{H} \to \mathcal{H}$ there is a flow of information from the input to the output of U in the sense that for an input state ϕ the output $U(\phi)$ fully depends on ϕ .



How does a projector P_{ψ} act on states? After renormalization and provided that $\langle \phi | \psi \rangle \neq 0$ the input state ϕ is not present anymore in the output $\psi = P_{\psi}(\phi)$. At first sight this seems to indicate that through projectors on one-dimensional subspaces there cannot be a flow of information cfr. the 'wall' in the picture below.



Theorem 3.3 provides a way around this obstacle.



While there cannot be a flow from the input to the output, there is a 'virtual flow' between the two inputs and the two outputs of a bipartite projector whenever it is configured within an appropriate context. And such a bipartite projector on a state in $\mathcal{H}_1 \otimes \mathcal{H}_2$ can act on this flow as any (anti-)linear function f with domain in \mathcal{H}_1 and codomain in \mathcal{H}_2 — which is definitely more general than unitary operations and also more general than actions by (completely) positive maps. This behavioral interpretation extends to multipartite entanglement, and, as is shown in [9] §6.6, it also enables interpretation of non-local unitary operations.

The wall within a projector incarnates the fact that

$$P_{\psi} \stackrel{L}{\simeq} \psi \otimes \psi$$
.

Indeed, one verifies that disentangled states $\psi \otimes \phi$ are in bijective correspondence with those linear maps which have a one-dimensional range [9] §5.3, that is, since states correspond to one-dimensional subspaces, disentangled states correspond to (partial) constant maps on states. Since constant maps incarnate the absence of information flow (cfr. 'the wall' mentioned above):

$$\frac{\text{entangled}}{\text{disentangled}} \simeq \frac{\text{information flow}}{\text{no information flow}}.$$

Pursuing this line of thought of conceiving entanglement in terms of its *information flow capabilities* yields

a proposal for measuring pure multipartite entanglement [9] §7.5 — given a measure for pure bipartite entanglement e.g. majorization [11].

The use of Theorem 3.3 in Sections 4 and 5 hints towards automated design of general protocols involving entanglement. We started with a simple configuration which conditionally incarnates the protocol we want to implement. Conceiving this conditional protocol as a pair consisting of (i) a single path 'from root to leaf' in a tree, and, (ii) a configuration picture, we can extend the tree and the configuration picture with unitary corrections in order to obtain an unconditional protocol. It constitutes an interesting challenge to produce an explicit algorithm which realizes this given an appropriate front-end design language.

Elaborating on the results in [2] S. Abramsky and the author have produced an axiomatic characterization of the in this paper exposed behavioral properties of quantum entanglement with respect to information flow. Remarkably, the additive feature of a vector space which gives rise to the notion of superposition and hence to that of entanglement itself seems not to be crucial with respect to the quantum-flow! In particular, we obtain a similar information-flow as the one enabled by quantum entanglement when replacing 'vector space' by 'set', 'linear map' by 'relation' and 'tensor product' by 'cartesian product' [3]. Replacing 'linear map' by 'function' in stead of 'relation' would not enable such an information flow. This is due to a different categorical status [4, 12] of the cartesian product in the category of sets and relations as compared to its status in the category of sets and functions. The category of relations does fail to have a full teleportation protocol because it has no four-vector Bell-base [3].

Recent proposals for fault-tolerant quantum computers of which the architecture is manifestly different from the circuit model require a different mathematical setting for programming them and reasoning about them [8]. We are convinced that the insights obtained in this paper provide the appropriate tool for doing so.

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